THE INTERACTION OF PERIODIC SURFACE GRAVITY WAVES WITH SLOWLY VARYING WATER CURRENTS

by

G. P. Bleach

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ABSTRACT

The governing equations for interactions between surface gravity wavetrains and slowly-varying water currents are derived and the incorporation of Vocoidal water wave theory into this framework is discussed. The emphasis throughout is on the derivation of the general form of the governing equations plus a detailed discussion of the qualitative physical behaviour implied by the equations. Particular solutions are usually given only where they serve to clarify the general method or some physical feature of the analysis.

The thesis proper is introduced by a derivation of wave kinematics on still water. A review of the kinematics and dynamics of an inviscid and irrotational fluid follows. The wave and fluid properties are then combined via the definition of wave integral properties. A derivation of the Airy and Stokes $O(a^2)$ wave theories is given and used to illustrate a number of points.

Water currents (following or opposing the waves) are introduced via their influence on the wave kinematics. The wave/current dynamics are then presented in two ways; firstly using a wave energy approach and secondly by introducing the wave action concept. Wave action is more convenient because it is a conserved quantity unlike wave energy. The general equations for two dimensional wave/current interactions are derived and discussed. At this point three topics are reconsidered; group velocity, momentum density in wave motion and Lagrangian mean forms of averaging.

The general equations for wave/current interaction are shown to be compatible with the Vocoidal water wave theory and applications of the theory to wave/current problems are discussed.

D E C L A R A T I O N

No portion of this thesis has been previously submitted in support of an application for any other degree or qualification in this or any other University.

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C O N T E N T S

1. INTRODUCTION

1.1. SOME FUNDAMENTALS OF SURFACE GRAVITY WATER WAVES

The range of phenomena regarded as waves is so broad that the precise definition of a wave is difficult to make. Whitham (1974, chapter 1.1) suggests a wave is any recognisable signal that is transferred from one part of the medium to another with a recognisable velocity of propagation, where the signal may be any feature of the disturbance (eg. a maximum of some quantity) which can be clearly located at any time. The signal may distort, change orientation and velocity of propagation as long as it remains recognisable. Such changes may be characteristic of the wave or (as is particularly relevant to this thesis) may be a result of variations in the medium through which the waves propagate. If there is a succession of similar signals, the waves are said to be periodic and the system of waves is known as a wavetrain. It is then possible to define the time delay between successive signals as the period of the wave and to define the spatial separation between signals as the wavelength.

Periodic waves that propagate across a water mass at the air/water interface (free surface) are usually categorised as follows. Capillary waves are periodic surface waves governed by surface tension which have wavelengths ≤ 0.02 m (Lighthill, 1978). Above this wavelength, gravity is the dominant force and longer waves are accordingly called periodic surface gravity waves. For extremely long surface waves (about 1km), the Coriolis force becomes significant. There are numerous similarities between waves dominated by surface tension and gravity waves and it is possible to incorporate them in one formulation (Crapper, 1979). The generation of gravity waves by wind also involves capillary waves because the wind energy is first transmitted to the short capillary waves and then by wave/wave interaction to the longer gravity waves (Phillips, 1980). Energy transfer to waves of different length also occurs in wave/current interactions (eg. reflection of waves by an opposing current (Peregrine 1976)) but there are few physically interesting cases involving capillary waves. As wave generation is not a part of this thesis, capillary waves are not discussed further. The long wavelength limitation (neglecting Coriolis effects) is convenient as it retains all wind generated waves but excludes tidal and seismic waves.

The structure of surface gravity waves is not confined to the surface oscillation, but influences water down to a depth one half of the wavelength. Changing water depth causes changes in wave characteristics provided the depth is less than half the wavelength; in particular, waves shorten, steepen and finally break as the water depth decreases. (Stiassnie and Peregrine 1980). These effects are also produced by wave/current interaction (Peregrine 1976).

The modification of the waves by changes in the medium of propagation (changes in water depth, current velocity) is the primary concern of this thesis. There has been considerable progress in this in recent years, beginning with the work of Longuet - Higgins and Stuart (1960,1961,1964) who were the first to solve the dynamics of wave/current interaction.

More recent work on the waves themselves (Longuet - Higgins 1975) has introduced new descriptions for mean wave properties and new qualitative features for steep water waves have been found. These features are now appearing in wave/current interaction work (Peregrine and Thomas 1979) and in wave shoaling.

1.2. AIMS OF THE THESIS

The aims of the thesis are:

(1) to discuss recent developments in the theory of periodic surface gravity waves. Emphasis is placed on the ray description of wave trajectories and on the definition of wave integral properties, since these are of value both in the development of wave theories and in the modification of waves by currents or bottom slope. Attention is also focussed on the problems of defining group velocities for steep waves and on the momentum density and flux in periodic surface waves.

(2) to discuss the energy/radiation stress approach to wave/current interaction and then to introduce the recent wave action approach to these problems. The discussion is limited to interaction with currents that follow or oppose the waves, ie. refraction of

waves and the creation of 'caustics' (regions of wave convergence) are not considered. (See Peregrine (1976,1981) for details).
The recent technique of Generalised Lagrangian Mean averaging is briefly described although there are as yet no water wave applications.

(3) to suggest applications of (1) and (2) to the Vocoidal wave theory developed by Swart and Loubser (1978). This requires in particular the definition of integral properties for Vocoidal waves and the use of the finite amplitude wave equations of Stiassnie and Peregrine (1979) to solve for the wave changes caused by bottom slope or water currents.

The approach taken here is to follow the mathematical development of each topic carefully, but to emphasise wherever possible the physical basis of the argument and the physical consequences of the results. In order to preserve continuity, detailed solutions and experimental verification are not emphasised, but extensive references are given.

1.3. CHOICE OF REFERENCE FRAMES AND NOTATION

Unless stated otherwise, the x, y plane is regarded as horizontal with z measured upwards from the undisturbed (no waves or currents) water surface. Wave propagation is chosen to be in the positive x direction in most cases. The bottom boundary is at $z = -h$ in undisturbed water. Changes in mean water depth due to waves or currents are usually indicated by the choice of $z = -d(x)$ representing the bottom boundary. The choice of tensor notation (ie. repeated subscript indices are summed) is made only when it serves to clarify the more traditional vector representation. The following convention is used for repeated indices:

Greek indices run from 1 to 2

ie. $\alpha = 1,2$

and represent components in the (x, y) plane

Roman indices run from 1 to 3

ie. $i = 1,2,3$

and represent components in the (x, y, z) planes.

$\delta^1_1 = \delta^2_2 = \delta^3_3 = 1 \; ; \; \delta^i_j = 0$ if $i \neq j$

(The only deviation from this occurs in a set of conservation equations (3.87) where the subscripts have a different interpretation which is made clear there).

2. KINEMATICS OF THE WAVES AND OF THE FLUID

2.1. A CONSERVATION EQUATION AND THE DERIVATION OF WAVE KINEMATICS

The aim of this section is to formalise the description of waves and wave propagation given in 1.1. (This is essentially a discussion of the wave kinematics; the wave dynamics are considered in 3.2 once the fluid dynamics has been given in 3.1).

A mathematical description is required for the propagation of a group of periodic waves on still water in a region of no wave dissipation or generation. In this case, the number of waves is a conserved quantity as waves are neither being created nor destroyed. Therefore the wave density (number of waves per unit distance in the direction of propagation) and wave flux (number of waves passing some point per unit time) can be related by a conservation equation. This conservation equation occurs in a number of contexts in the thesis. The equation is therefore derived in 2.1.1 for a <u>general</u> conserved property and the propagation of such a property is investigated.

The conservation equation is applied in 2.1.2 to a wavetrain on still water of constant depth. This requires the formal definition of elementary wave properties (phase, wavenumber, frequency). Analogies with the general case in 2.1.1 lead to the definition of the phase velocity and group velocity for the waves. The difference between these velocities is explained, as is their dependence on the dispersion relation (relation between wave density and wave flux). The kinematics of this simple case are completed by deriving the ray description for the wave trajectories. The Airy theory (derived in section 3.3) is used to illustrate various points.

Variations in the medium (such as a change in water depth) influence the wave propagation. These effects can be studied using the structure of 2.1 and 2.2 if the definitions of wavenumber and frequency are extended. This is discussed in 2.3. It is found that there are limitations on the rate at which the medium can vary if the extended definitions are to retain physical significance. The wave trajectories are also found, with Airy waves again used as an example. The wave kinematics developed in 2.1 forms the basis for the kinematics of waves on currents presented in 4.2.

A reference for the work in 2.1 is "Nonlinear Waves" (Leibovich and Seebass, editors, 1974; especially chapters 3 and 5). A second reference is "Linear and Non-linear Waves" (Whitham, 1974) which is a general reference for the thesis.

2.1.1. The conservation equation and the propagation of a conserved quantity

A continuous distribution of either a material property or of some state of the medium is assumed. The aim is to first define the density and flux of such a property or state, and then to investigate its propagation. (One spatial dimension is used for clarity; extensions to higher dimensions are simple).

density per unit length $P(x,t)$ (2.1)

flux per unit time $Q(x,t)$ (2.2)

A flow velocity can be defined as

$$v = \frac{Q}{P} \tag{2.3}$$

Because the material or state is conserved, the rate of change of it in any section $x_1 < x < x_2$ is balanced by the net inflow across x_1 and x_2, giving a conservation equation for the material or state in this region. (See Leibovich and Seebass, "Nonlinear Waves", chapter 3, 1974).

$$\frac{d}{dt}\int_{x_2}^{x_1} P(x,t)\,dx + Q(x_1,t) - Q(x_2,t) = 0 \tag{2.4}$$

Now if $P(x,t)$ has continuous derivatives, let $x_1 \to x_2$ and the "divergence form" (see John, (1978), p17) of the conservation is obtained:

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0 \tag{2.5}$$

The conservation equation in this form will be used repeatedly throughout this thesis. One now requires a function relating P and Q, ie.

$$Q = Q(P) \tag{2.6}$$

If this can be found (from theory or experiment), (2.5) can be expressed as

$$\frac{\partial P}{\partial t} + \left(\frac{\partial Q}{\partial P}\right) \cdot \frac{\partial P}{\partial x} = 0 \tag{2.7}$$

which is a 1st order (nonlinear in general) hyperbolic equation for the propagation of P, with the velocity of propagation of P given by v_g, where

$$v_g \equiv \frac{\partial Q}{\partial P} \tag{2.8}$$

The form of the function $Q(P)$ governs the propagation velocity through (2.8). This is now shown for the simple case where Q/P is constant, and secondly for a more complex function $Q(P)$.

(i) If the function $Q(P)$ is

$$Q = c\,P \qquad c = \text{constant} \tag{2.9}$$

then the propagation velocity is constant and equals the flow velocity (2.3).

$$\left.\begin{aligned} v_g &\equiv \frac{\partial Q}{\partial P} = c \\ v &\equiv \frac{Q}{P} = c \end{aligned}\right\} \tag{2.10}$$

The equation for the propagation of P (2.7) is now a linear first order hyperbolic equation (as c is constant)

$$\frac{\partial P}{\partial t} + c \cdot \frac{\partial P}{\partial x} = 0 \qquad (2.11)$$

and the solution is simply $P(x,t) = f(x - ct)$ (2.12)

This means the material or state P moves at speed c in the positive direction. The initial distribution $P = f(x,0)$ is not changed with time but simply moves a distance ct away after time t. There is therefore no dispersion with time.

The trajectories followed by particular values of P are given by the characteristic curves $\mathcal{C}$. If P is constant then:

$$\frac{dP}{dt} = 0$$

But

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \frac{dx}{dt} = 0$$

Comparing this with (2.7), the curves $\mathcal{C}$ must satisfy

$$\frac{dx}{dt} = v_g$$

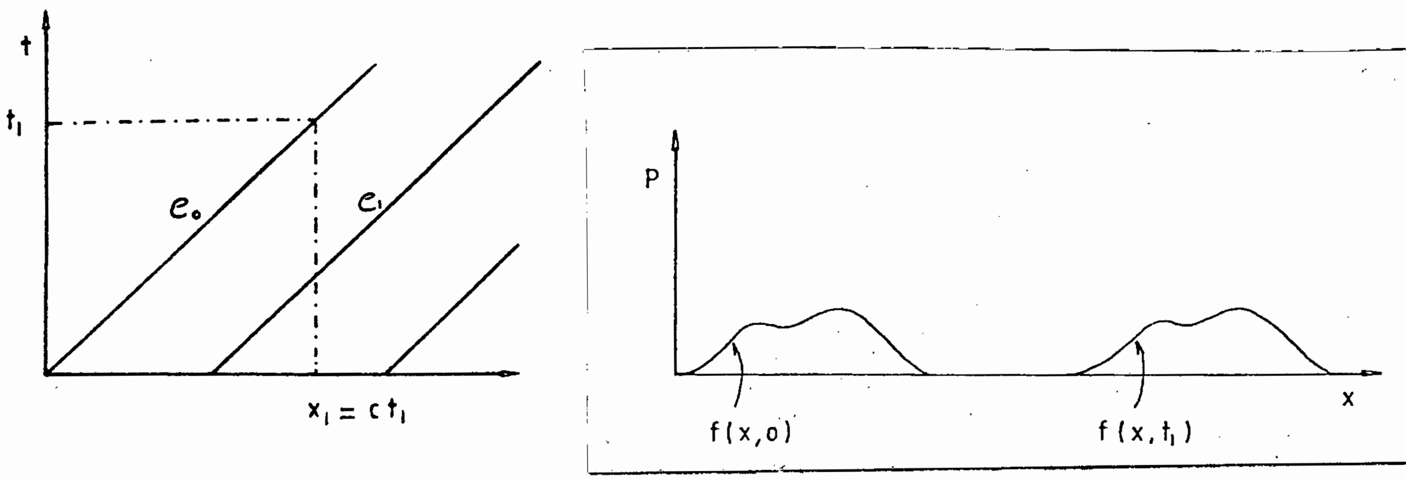
For the simple relation $Q = cP$ then, the characteristic curves $\mathcal{C}$ must all have the same slope

$$\frac{dx}{dt} = c$$

and the curves themselves are given by

$$x = ct$$

Figure 2-1.



(ii) If the function $Q(P)$ is more complicated than (2.9), the propagation equation for P is nonlinear, because the propagation velocity v_g is a function of P and is no longer the same as the flow velocity.

$$\frac{\partial P}{\partial t} + v_g(P) \cdot \frac{\partial P}{\partial x} = 0$$

$$v_g(P) \neq v \equiv \frac{Q}{P}$$

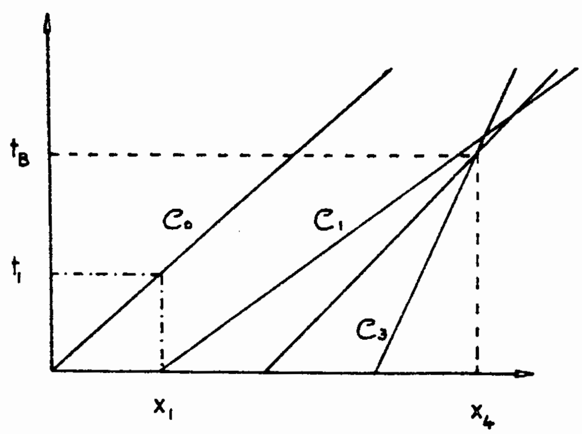
The trajectories followed by different parts of the disturbance are also more complicated. They are given as before by the characteristic curves C along which P is constant. On each curve C therefore:

$$\frac{dP}{dt} = 0 \qquad \frac{dx}{dt} = v_g(P) \tag{2.13}$$

Hence on the curve C, P is constant. If P is constant then $v_g(P)$ is constant, so $\frac{dx}{dt}$ is constant and the curve C

is a straight line, with slope $v_g(P)$. This differs from (i) because different values of P propagate at different speeds, (although as before the speed of each value P' does not change with time). The initial distribution therefore changes and dispersion occurs.

Figure 2-2.



possible curves $\mathcal{C}$ (compare previous case for $v_g = c$, constant)

For comparison with case (i) (linear), choose the initial distribution $P(x,0) = f(x,0)$. The value $P(x,t)$ is unchanged along the curves $\mathcal{C}_{1,2,..}$ passing through $x_{1,2,....}$. The curve $\mathcal{C}_1$ is given by

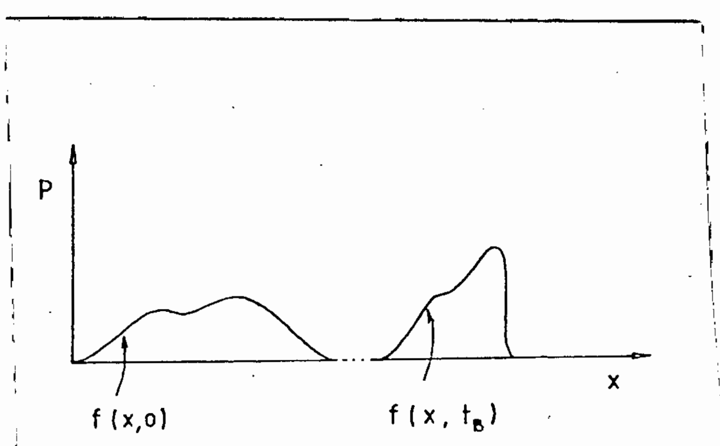
$$x(t) = x_1 + v_g\left(f(x_1,0)\right).\,t \tag{2.14}$$

with slope $\dfrac{dx}{dt} = v_g\left(f(x_1,0)\right)$

and with the value of $P = f(x_1,0)$ along it.

This completes the general introduction to the conservation equation (2.5) and the propagation (and dispersion) of the conserved property represented by the density distribution $P(x,t)$. To relate this to the propagation of waves, recall from 1.1 that for waves, "something propagates with a recognisable velocity". This can clearly be represented by the propagation of $P(x,t)$ at a speed of $v_g(P)$. The solution at time t is found by moving each point on the initial curve $P = f(x,0)$ by a distance $v_g(P)$ in the positive x direction. Since $v_g(P)$ is <u>not</u> constant, the distance moved will vary (as the slopes of curves $\mathcal{C}_i$ are different) and the shape $P = f(x,t)$ will change with time. The $P(x,t)$ curve corresponding to the diagram above is shown for $t = 0$; $t = 4$.

Figure 2-3.



There is a "wave" interpretation of the events shown in this diagram which is as follows. At x_4, P has infinite slope at time t_B. This is where the wave breaks and is caused by the greater speed on $\mathcal{C}_2$ than on $\mathcal{C}_3$; ie. $v_g(x_2,t) > v_g(x_3,t)$; $t \leq t_B$.

Breaking waves are not discussed further here; see Whitham (1974, p22) or John (1978, chapter 1, example 6) for this approach to wave breaking.

2.1.2. Propagation of wave properties obeying first order hyperbolic equations in a uniform time independent medium

A fundamentally important application of the general discussion above is the propagation of the wave phase χ. This is the quantity that governs the periodic variation of the physical wave parameters (such as surface elevation $\eta(x,t)$) through relations similar to

$$\varphi(\underset{\sim}{x},t) = a \cos \chi(\underset{\sim}{x},t) \qquad (2.15)$$

$a \equiv$ amplitude

$\varphi \equiv$ some physical parameter

The phase conservation equation using the formalism of 2.1.1 leads to the complete kinematic structure for wave propagation. The analysis also provides:

(i) the definition of propagation velocities for various wave parameters

(ii) the dispersion relation

(iii) a system of wave classification

(iv) the governing equation for wave trajectories

(v) the kinematic conservation equations for waves on uniform and non-uniform flows.

These results are obtained with an analysis of progressively more complex situations. The analysis will also clarify the the general discussion given in 2.1.1.

A "recognisable feature of the disturbance" ie. some wave parameter φ , follows the trajectory $\underset{\sim}{x}(t)$ along which

$$\varphi(\underset{\sim}{x},t) = \varphi(\underset{\sim}{x}_o,0) \tag{2.16}$$

where $\varphi(\underset{\sim}{x}_o,0)$ is the initial value of φ . This is essentially a requirement that

$$\chi(\underset{\sim}{x},t) = \chi(\underset{\sim}{x}_o,0) = \text{constant}$$

(see (2.15) above) and in this way the wave propagation is controlled by the phase.

The periodic nature of the wave (see (2.15)) in both $\underset{\sim}{x}, t$ requires:

$$\begin{aligned} \varphi(\underset{\sim}{x},t) &= \varphi(\underset{\sim}{x}+\lambda,t) \\ \varphi(\underset{\sim}{x},t) &= \varphi(\underset{\sim}{x},t+\tau) \end{aligned} \tag{2.17}$$

where $\lambda \equiv$ wavelength (2.18)

$\tau \equiv$ period (2.19)

Hence $\chi(\underset{\sim}{x},t)$ can be expressed in the following form:

$$\chi(\underset{\sim}{x},t) = k_\alpha x_\alpha - \omega t \qquad (2.20)$$

with the definitions:

wavenumber $\quad \underset{\sim}{k} \equiv \frac{2\pi}{\lambda} . \hat{k} \qquad \left(\hat{k} \equiv \frac{\underset{\sim}{k}}{|k|} \right)$

frequency $\quad \omega \equiv \frac{2\pi}{\tau} \qquad (2.20)$

(Initially, it is assumed that $\underset{\sim}{k}, \omega$ are constants).

Analogies with the general discussion earlier can now be made. $\underset{\sim}{k}$ is a measure of phase (or wave) density per unit length (number of waves/unit distance) and ω a measure of phase (wave) flux per unit time (rate at which waves pass a fixed point). These correspond to P and Q respectively (definitions (2.1) and (2.2)).

The flow velocity for the wave is Q/P and is given the name phase velocity, denoted by c :

ie. $$c \equiv \frac{Q}{P}\,(\text{phase}) = \frac{\omega}{k} = \frac{\lambda}{\tau} \qquad (2.23)$$

It represents the propagation velocity for the phase and hence for the wave profile. For example, in the Airy solution for surface gravity waves (derived in 3.3), the wave profile is given by

$$\eta(\underset{\sim}{x},t) = a\cos(kx - \omega t) \qquad (3.64)$$

$$a \equiv \text{amplitude}$$

and a particular crest $\eta(0,0)$ travels at speed $c = \omega/k$ along the x axis, and is found at x_1, at time $t_1 = \frac{x_1 k}{\omega}$

since $$\eta\left(x_1, \frac{x_1 k}{\omega}\right) = a\cos(0) = \eta(0,0) \qquad (2.24)$$

The conservation equation (2.5) for the phase takes the following form

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (2.5)$$

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

Equation (2.25) represents the "conservation of waves". It is of great value in situations where $\omega, \underset{\sim}{k}$ are functions of position and time.

Following the approach of 2.1.1, a kinematic relation is required that is of the form

$$Q = Q(P) \qquad (2.6)$$

This relation is known as the dispersion relation, and is expressed as:

$$\omega = W(k) \qquad (2.26)$$

(The use of $W(k)$ not $\omega(k)$ will clarify the evaluation of partial derivatives later).

This relation is of crucial importance for wave propagation.

Its use is now illustrated by an example, namely the dispersion relation for Airy waves in water of constant depth h .

$$\omega^2 = gk\tanh kh \qquad g \equiv \text{gravitational acceleration} \qquad (3.66)$$

For simplicity, special cases of (3.66) are considered; "shallow water" $kh \to 0$, which implies $\tanh kh \to kh$, and "deep water" for which $\tanh kh \to 1$. For these two cases, the Airy dispersion relation simplifies to:

shallow water $$\omega_1 = \sqrt{gh}\cdot k \qquad (2.27)$$

deep water $$\omega_2 = \sqrt{gk} \qquad (2.28)$$

The dispersion relation can be used to obtain the propagation velocity for the wavenumber k . First, recall the propagation velocity for P (2.8):

$$V_g \equiv \frac{\partial Q}{\partial P} \qquad (2.8)$$

When $P = k$ and $Q = W(k)$ this velocity is the propagation velocity denoted by c_g . It is therefore the propagation velocity for for k , and is the (kinematic) group velocity. Its importance is obvious, as it describes the movement of waves of any particular length. In later chapters, the group velocity reappears as the propagation velocity for wave energy and wave action. The various roles of the group velocity are discussed at length in chapter 8. A detailed comparison of the phase and group velocities is given later in this section.

(kinematic) group velocity: $c_{g_\alpha} \equiv \dfrac{\partial W(k)}{\partial k_\alpha}$ (2.29)

By analogy with (2.13), the group velocity is also given by

$$c_{g_\alpha} \equiv \frac{dx_\alpha}{dt}$$

An observer moving at the group velocity moves along a curve $\mathcal{C}$ in space-time known as a ray (recall (2.8)). This reference frame is important and will be distinguished later in this section from the frame in which the observer moves at the phase velocity.

For the special cases (2.27), (2.28), the group velocity is given by the following expressions:

$$c_{g_1} = \sqrt{gh} \tag{2.30}$$

$$c_{g_2} = \tfrac{1}{2}\sqrt{g/k} \tag{2.31}$$

and these can be compared to the corresponding phase velocities:

$$c_1 = \omega/k = \sqrt{gh} = c_{g_1} \tag{2.32}$$

$$c_2 = \omega/k = \sqrt{g/k} = 2c_{g_2} \tag{2.33}$$

In shallow water, the phase and group velocities are equal. This is because the dispersion relation (2.27) is simply proportional to k. (Compare (2.10) in the general case in 2.1.1). As $h\uparrow$, the complicated k dependence causes the phase and group velocities to diverge.

This example illustrates the significance of the dispersion relation, since it shows how the wave velocities are derived directly from the dispersion relation. Further consequences of the dispersion relation arise when the group velocity is used in the propagation equation for k.

The propagation equation for k comes from the wave conservation equation (2.25) by rewriting the frequency term $\frac{\partial \omega}{\partial x_\alpha}$ as $\frac{\partial \omega}{\partial k_\alpha} \cdot \frac{\partial k_\alpha}{\partial x}$ and using the dispersion relation $\omega = W(k)$ (2.24) to finally get the group velocity (2.29) into the equation. (Compare (2.5, 2.6, 2.7), in 2.1.1).

$$\frac{\partial k}{\partial t} + c_g(k) \cdot \frac{\partial k}{\partial x} = 0 \tag{2.34}$$

For shallow water Airy waves: $\frac{\partial k}{\partial t} + \sqrt{gh} \cdot \frac{\partial k}{\partial x} = 0$ (2.35)

For deep water Airy waves: $\frac{\partial k}{\partial t} + \sqrt{\frac{g}{k}} \cdot \frac{\partial k}{\partial x} = 0$ (2.36)

The propagation of k is crucially different in the two cases, as in (2.35) the velocity $\sqrt{gh}$ is the same for all wavenumbers whereas in (2.36) waves of different length $\lambda = \frac{2\pi}{k}$ will steadily separate from each other.

This example illustrates a general criterion for the classification of waves:

(a) waves for which the dispersion relation is of the form $W = ck$ are classed as <u>hyperbolic waves</u> (waves of different length move with the same speed)

(b) waves for which the dispersion relation is of a more complicated form $W(k) \neq ck$ are classed as dispersive waves. (different wavelengths disperse in time).

Hyperbolic waves have the qualitative behaviour of the general solution (2.12) to equation (2.7); ie., an initial disturbance comprising waves of different wavelengths propagates unchanged at speed $c\ (= c_g)$, each wavenumber k obeying the linear first order hyperbolic equation

$$\frac{\partial k}{\partial t} + c.\frac{\partial k}{\partial x} = 0 \tag{2.37}$$

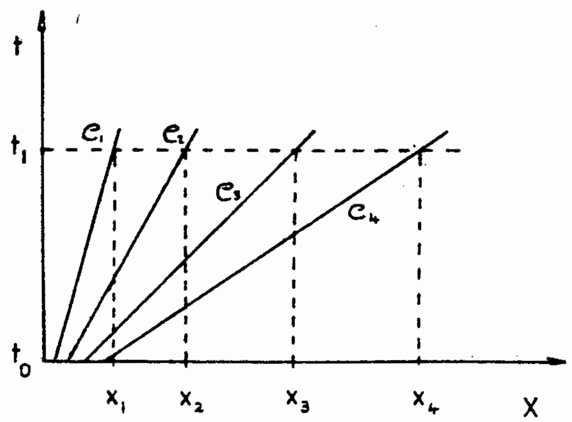
Dispersive waves have hyperbolic equations embedded in them. For instance, the wavenumber propagates at $c_g(k)$ according to the nonlinear order hyperbolic equation

$$\frac{\partial k}{\partial t} + c_g(k).\frac{\partial k}{\partial x} = 0 \tag{2.38}$$

The analysis leading to (2.13), (2.14) shows that each wavenumber k propagates at the group velocity (constant if the medium does not change), but that this velocity is dependent on k. Hence an initial disturbance separates into components and the profile of the disturbance changes.

For the deep water Airy waves obeying (2.36) it is clear that the longest waves ($\lambda\uparrow$, $k\downarrow$) propagate the fastest. Hence the characteristics $\mathcal{C}$ for waves generated at (say) a storm centre will spread as follows at some time $\geqslant t_o$ after generation: (At very early times, "conservation of waves" as expressed by equation (2.25) may not hold, as an initial disturbance will gradually separate into many wavetrains, each with a different wavenumber. Once this has happened, the present description applies).

Figure 2-4.



e_j has slope $c_g(k_j) = \sqrt{g/k_j}$

and $k_1 < k_2 \cdots < k_4$

Hence longer swells arrive first. The nature of the dispersion relation for Airy waves in water of arbitary depth quoted previously (3.66) shows that the dispersive nature is maintained in all depths except the shallowest. This dispersive behaviour $\left(W(k) \neq ck \right)$ is common to all periodic gravity wave theories; for instance the dispersion relation for Stokes waves is

$$\omega^2 = gk \tanh kh \left\{ 1 + gk \left(\frac{9 \tanh^4 kh - 10 \tanh^2 kh + 9}{8 \tanh^4 kh} \right) a^2k^2 \right\} + O(a^4k^4) \quad (3.76)$$

Comparison of the phase and group velocities

It is important to make clear the distinction between the phase and group velocities. This is done by using propagation equations in their characteristic form. The results are interpreted by comparing the waves as seen when moving with the phase velocity to what is seen when moving at the group velocity. Recall the definitions of:

phase velocity $\qquad c \equiv \dfrac{W(k)}{k}$ (2.23)

group velocity $\qquad c_g \equiv \dfrac{\partial W(k)}{\partial k}$ (2.29)

These are equal for hyperbolic waves, where

$$\omega(k) = ck$$

For dispersive waves, the two velocities are not equal, implying that different properties of the wave propagate with different velocities. The phase velocity is most easily identified, since it is the speed at which the wave profile moves. This is shown by choosing a wave crest (a surface satisfying $\chi = 0$ by (2.15)) and following it. Since χ will not change along this path,

$$\chi = 0 \quad ; \quad \frac{d\chi}{dt} = 0$$

$$\therefore \frac{d\chi}{dt} = 0 \quad \Rightarrow \quad \frac{\partial \chi}{\partial t} + \frac{\partial \chi}{\partial x} \cdot \frac{dx}{dt} = 0$$

This equation expressed in characteristic form is

$$\frac{dx}{dt} = \left\{ \frac{-\dfrac{\partial \chi}{\partial t}}{\dfrac{\partial \chi}{\partial x}} \right\} = \frac{\omega}{k} = c \qquad (2.39)$$

Hence in order to follow the crest, one must move at the phase speed c. One must move along the trajectory obtained by integrating (2.39)

$$x = \frac{\omega}{k} \cdot t + \text{constant} \tag{2.40}$$

Although the crests move at the phase velocity along the characteristics given by (2.40), the wavenumber k is governed by (2.34):

$$\frac{\partial k}{\partial t} + \frac{\partial W}{\partial k} \cdot \frac{\partial k}{\partial x} = 0 \tag{2.34}$$

The wavenumber k therefore propagates at the group velocity $c_g \equiv \frac{\partial W}{\partial k}$, and a particular wavenumber k_0 is found on rays satisfying

$$x = \frac{\partial W}{\partial k} \cdot t + \text{constant} \tag{2.41}$$

These are the characteristic curves $\mathcal{C}$ described by (2.14) in the general discussion in 2.1.1. Notice that the frequency is also constant along the rays (2.41) because $\omega = W(k)$ and k is constant. The comparison of phase and group velocities is discussed by Whitham (1960,1974) and Lighthill (1978).

The above results are illustrated by considering waves of different lengths as they propagate away from a region of initial wave generation (such as a stone dropped in a pond). A particular wavenumber k_0 (ie wavelength) is chosen and observations are compared for either:

(i) motion at the corresponding group velocity

(ii) motion at the corresponding phase velocity ie. following a particular crest in the group of waves of wavenumber k .

Airy waves on deep water are used for this example, with phase and group velocities given respectively by:

$$c = \sqrt{g/k} \qquad (2.33)$$

$$c_g = \frac{c}{2} \qquad (2.31)$$

The Airy wave group is now represented on an (x,t) diagram by lines of constant phase and lines of constant wavenumber (rays). Inserting the deep water Airy dispersion relation (2.28) into (2.40), (2.41) gives the following equations:

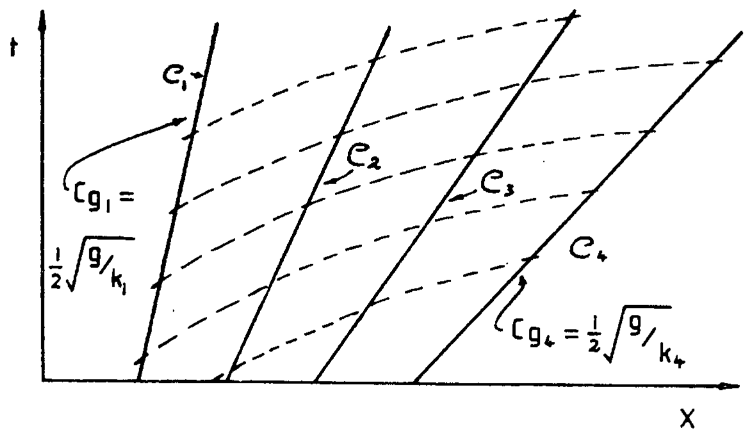
$$x = \sqrt{g/k}\,.\,T \qquad \text{for phase propagation} \qquad (2.42)$$

$$x = \tfrac{1}{2}\sqrt{g/k}\,.\,t \qquad \text{for wavenumber propagation} \qquad (2.43)$$

These are plotted on the diagram overleaf.

Figure 2-5.

Adapted from Lighthill (1978)



KEY:

/ characteristics $\mathcal{C}$ are group lines of constant k, ω

--- phase lines of constant χ

The straight group lines are the (x,t) trajectories of each wavenumber k, with the shortest waves of wavenumber k_1 moving the slowest.

From the diagram, the cases (i) and (ii) can be distinguished as follows:

(i) Moving along at the <u>group</u> velocity c_{g_i}, ie along a characteristic $\mathcal{C}_i$, an observer always sees waves of length $2\pi/k_i$ and frequency ω_i. The <u>crests</u> will continually pass him though, because $c_i = 2c_{g_i}$. (ie. the gradient of the broken lines is constant along the characteristic $\mathcal{C}_i$).

(ii) If the observer now starts at the back of the wave group (somewhere on $\mathcal{C}_1$) and follows a particular crest he will move along a phase line of constant χ . He will observe neighbouring crests getting steadily further away as the crest propagates into regions where waves of higher wavelengths (lower k) are found. Eventually the crest arrives at the front of the group and as there is no propagation of waves longer than λ_4 the crest loses energy and dies away. (loss of energy as it dies is replaced by the appearance of a new crest at the back of the group).

The propagation of the important quantities χ and k at two different velocities raises the question of the propagation velocity for the wave energy. It is shown later that the energy (for linear or near linear waves; see discussion in 8.1) is propagated at the group velocity c_g , ie. the energy of waves of wavenumber k moves slower than the wave crests.

This subsection has shown that the kinematics of surface gravity waves can be described in terms of the phase χ, wavenumber k and frequency ω. The propagation of these quantities has been investigated using the equations and results of 2.1.1. This has provided a basis for the classification of waves as dispersive or hyperbolic depending on the form of the dispersion relation (2.26). It has been found that periodic surface gravity waves are dispersive (except when the depth $h \to 0$) but that important properties of the waves propagate according to 1st. order hyperbolic wave equations; a linear equation for χ (propagation at phase velocity c) and nonlinear for k, ω (propagation at c_g). It is interesting to note that the nonlinear equation for k occurs despite the fact that in this example the overall system of the waves is linear.

The forgoing analysis holds for propagation through a homogeneous time independent medium. It is important to extend the analysis of the wave kinematics to include non-homogeneous dispersion; ie. to include variation in local properties of the medium. These could include changes in depth h and propagation over a non-uniform current. This is now discussed in 2.1.3.

2.1.3. Wave dispersion in a non-uniform medium

The previous subsection was based on the idea of "conservation of phase" χ, plus the ensuing definitions of wavenumber k and frequency ω. Propagation of these quantities in particular situations required the dispersion relation $\omega = W(k)$. In a non-homogeneous medium, variations in the medium influence the wave properties k, ω and their propagation. One expects that the concept of phase χ will remain, ie. there must still be "some recognisable feature of the disturbance", although distortions of the original wave are going to occur. Essentially this means that the "conservation of waves" (2.25) is still required, but this raises problems with the definitions of wavenumber k (2.21) and frequency ω (2.22), since these will now vary with position. New definitions are needed that will:

(i) preserve the concept of the phase χ

(ii) have some intuitive relationship to the elementary definitions used previously

(iii) vary with the local properties of the medium.

The elementary definitions of the phase, wavenumber and frequency are first recalled:

(2.20) $\chi(\underset{\sim}{x},t) = \underset{\sim}{k}.\underset{\sim}{x} - \omega t$ where k, ω are constants, defined by:

(2.21) $\underset{\sim}{k} \equiv 2\pi/\lambda . \hat{k}$

(2.22) $\omega \equiv 2\pi/\tau$

The form of (2.20) is preserved if k, ω are redefined as functions of x, t in the following way:

local wavenumber $\underset{\sim}{k}(\underset{\sim}{x},t) \equiv \nabla\chi(\underset{\sim}{x},t)$ (2.44)

local frequency $\omega(\underset{\sim}{x},t) \equiv -\dfrac{\partial\chi}{\partial t}(\underset{\sim}{x},t)$ (2.45)

Note that the eliminations of χ from these definitions implies

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

$$\frac{\partial k_\alpha}{\partial x_\beta} - \frac{\partial k_\beta}{\partial x_\alpha} = 0 \qquad (2.46)$$

(2.25) expresses the conservation of waves and (2.46) is a consistency condition for $\underset{\sim}{k}$.

These definitions (2.44), (2.45) will certainly satisfy (i) and (ii) but (iii) is only going to be true if there are restrictions on the rate at which $\underset{\sim}{k}$ and ω are allowed to vary. In fact these restrictions are implicit in the requirement that the conservation equation (2.25) (now expressed in terms of the <u>local</u> wavenumber and frequency) should hold:

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

The appropriate restrictions are that the variations in the medium must be on a scale that is long compared with the wavelength $\left(2\pi/k(x,t)\right)$ or long compared with the wave period. This will ensure that (2.25) holds and implies that $\underset{\sim}{k}$, , ω do not vary so rapidly that the intuitive ideas of wavelength and wave period are lost. (An example is the initial disturbance after a stone is thrown into a pond; only after a short while does the disturbance begin to propagate in such a way that crests (or phase) obey the conservation equation (2.25) and local wavenumbers and frequencies can be defined).

Propagation velocities and dispersion in a non-uniform medium

The slowly varying wave is specified by $\chi(\underset{\sim}{x},t)$, $k(\underset{\sim}{x},t)$, $\omega(\underset{\sim}{x},t)$ and to investigate the propagation of these quantities the dispersion relation must be obtained. It seems intuitively clear (Peregrine (1976),p 17) that the slow variation restriction means that the <u>form</u> of the dispersion relation for the <u>uniform</u> medium can be retained, but with the slow variation of the medium incorporated directly into the appropriate parameter. For example, the Airy wave dispersion relation is:

$$\omega^2 = gk \tanh kh \qquad (3.66)$$

so if $h = h(x)$, the local dispersion relation becomes

$$\omega^2 = W^2(k, x, t) = gk(x,t) \tanh[k(x,t) . h(x)] \qquad (2.47)$$

In general then, the local dispersion relation can be written

$$\omega = W(\underset{\sim}{k}, \underset{\sim}{x}, t) \qquad (2.48)$$

and so the local propagation velocities c, c_g can be defined as before, but with (2.48) replacing $\omega = W(k)$ (2.26).

$$c(\underset{\sim}{x}, t) = \frac{W(\underset{\sim}{k}, \underset{\sim}{x}, t)}{k(\underset{\sim}{x}, t)} \qquad (2.23)$$

$$c_{g_\alpha}(\underset{\sim}{x}, t) = \frac{\partial W(\underset{\sim}{k}, \underset{\sim}{x}, t)}{\partial k_\alpha(\underset{\sim}{x}, t)} \qquad (2.29)$$

The propagation of $k(x,t)$ and $W(k,x,t)$ can now be analysed from the conservation of phase equation (2.25) using (2.29). The advantage of using the notation $W(k,x,t)$ not $\omega(k,x,t)$ now becomes apparent.

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

ie. $$\frac{\partial k_\alpha}{\partial t} + \frac{\partial W}{\partial k_\beta} \cdot \frac{\partial k_\beta}{\partial x_\alpha} + \frac{\partial W}{\partial x_\alpha} = 0 \qquad (2.49)$$

Use of the consistency condition for k_α (2.46) means (2.49) can be rewritten in terms of $c_{g\alpha}$:

$$\frac{\partial k_\alpha}{\partial t} + \underbrace{\frac{\partial W}{\partial k_\alpha}}_{c_{g\alpha}} \cdot \frac{\partial k_\beta}{\partial x_\beta} + \frac{\partial W}{\partial x_\alpha} = 0 \qquad (2.50)$$

This equation for the propagation of $\underset{\sim}{k}$ can now be expressed in characteristic form:

$$\frac{dk_\alpha}{d\tau} \left(= \frac{\partial k_\alpha}{\partial t} + c_{g\alpha} \cdot \frac{\partial k_\beta}{\partial x_\alpha} \right) = - \frac{\partial W}{\partial x_\alpha} \qquad (2.51)$$

along curves $\mathcal{C}$ (rays) $\quad \frac{dx_\alpha}{d\tau} = \frac{\partial W}{\partial x_\alpha} = c_{g\alpha} \qquad (2.52)$

There is an analogy with the equations of classical mechanics that is worth noting. Equations (2.51, 2.52) are identical to Hamilton's equations if $\underset{\sim}{x}$ and $\underset{\sim}{k}$ are regarded as generalised coordinates and momenta respectively with $W(\underset{\sim}{k}, \underset{\sim}{x}, t)$ taken as the Hamiltonian. (Leibovich and Seebass, 1974, chapter 5; Whitham 1974, p383). If the dispersion relation is expressed in terms of χ rather than ω and $\underset{\sim}{k}$, it becomes

$$\frac{\partial \chi}{\partial t} + W\left(x, \frac{\partial \chi}{\partial x}, t\right) = 0$$

This is the Hamilton-Jacobi equation with the phase χ as the action. Returning to the analysis of (2.51, 2.52), recall that when $\omega = W(k)$ in the homogeneous medium, $c_{g\alpha}$ was constant and so $\frac{dk_\alpha}{d\tau} = 0$ on $\frac{dx_\alpha}{d\tau} = c_{g\alpha}$ ie. $\underset{\sim}{k}$ did not vary along the rays and the rays were straight.

Now however, $\frac{\partial W}{\partial x_\alpha}$ is non-zero and $c_{g\alpha}$ is not constant, so the rays (given by (2.52)) are curved in the $(\underset{\sim}{x},t)$ plane. In addition, the wavenumber $\underset{\sim}{k}$ varies along the rays.

So far $\underset{\sim}{k}, W$ have been regarded as functions of $\underset{\sim}{x}, t$. They will in general be functions of other properties of the medium (eg. the depth) which may also be functions of $\underset{\sim}{x}, t$. These <u>local</u> properties can be summarised in the parameter $\zeta(\underset{\sim}{x},t)$ which may have a number of components. The dispersion relation is now

$$\omega = W(\underset{\sim}{k}, \zeta) \tag{2.53}$$

and in a non-uniform medium, the previous equation for $\underset{\sim}{k}$ becomes

$$\frac{dk_\alpha}{dt} = -\frac{\partial W}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x_\alpha} \tag{2.54}$$

on rays

$$\frac{dx_\alpha}{dt} = \frac{\partial W}{\partial k_\alpha} = c_{g\alpha} \tag{2.52}$$

A useful example of this is the shoaling (moving into water of decreasing depth) of Airy waves, where the depth $h = h(x)$ is a slowly decreasing function of x. The dispersion relation is (waves in the x direction):

$$\omega = \sqrt{gk \operatorname{Tanh} k\,h(x)} \Rightarrow \zeta(\underset{\sim}{x},t) = h(x) \tag{3.66}$$

$$\therefore \frac{dk}{dt} = -\frac{\partial W}{\partial h} \cdot \frac{\partial h}{\partial x} \qquad \text{from} \quad (2.54)$$

$$\therefore \frac{dk}{dt} = -\frac{\frac{1}{2} g k^2 \operatorname{sech}^2 kh}{\sqrt{gk \tanh kh}} \cdot \frac{\partial h}{\partial x}$$

along rays $$\frac{dx_\alpha}{dt} = C_{g\alpha} \left(= \frac{\partial W(k, \xi)}{\partial k} \right) \qquad (2.52)$$

Since $\frac{\partial h}{\partial x} < 0$, this expresses the well known result that the waves shorten $\left(\frac{dk}{dt} > 0\right)$ and slow down $(C_g \downarrow)$ as the water depth decreases. The waves move on a straight line in space (along the x axis) but because C_g is a function of x, the rays in the x, t plane are curved.

The variation of $\omega(\underset{\sim}{k}, \underset{\sim}{x}, t)$ along the rays (2.52) is also of interest:

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} \cdot \frac{dx_\alpha}{dt}$$

but on the rays, $$\frac{dx_\alpha}{dt} = C_{g\alpha}$$

$$\therefore \frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + C_{g\alpha} \cdot \frac{\partial \omega}{\partial x_\alpha} \qquad (2.55)$$

The conservation equation (2.25) is now used to make the right hand side of (2.55) vanish:

$$\frac{\partial k_\alpha}{\partial t} = -\frac{\partial \omega}{\partial x_\alpha} \qquad (2.25)$$

Multiply (2.25) by $C_{g\alpha}$

$$\therefore -C_{g\alpha} \frac{\partial k_\alpha}{\partial t} = C_{g\alpha} \frac{\partial \omega}{\partial x_\alpha}$$

and $$-c_{g\alpha}\frac{\partial k_\alpha}{\partial t} = -\frac{\partial \omega}{\partial k_\alpha}\cdot\frac{\partial k_\alpha}{\partial t} = -\frac{\partial \omega}{\partial t} \qquad \left(c_{g\alpha} \equiv \frac{\partial \omega}{\partial k_\alpha}\right)$$

Hence equation (2.55) implies

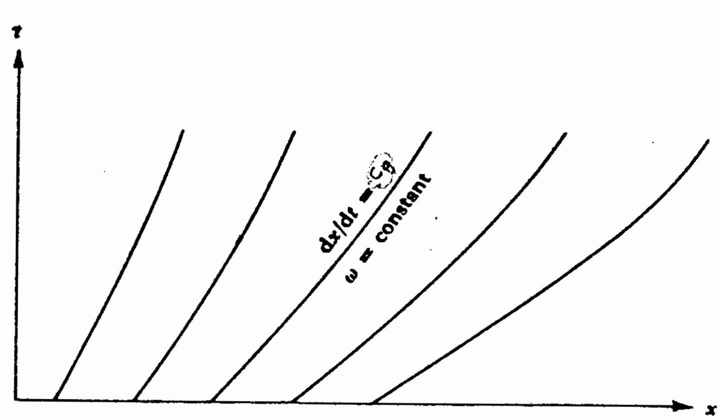
$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} - \frac{\partial \omega}{\partial t} = 0 \qquad (2.56)$$

along the rays $$\frac{dx_\alpha}{dt} = c_{g\alpha} \qquad (2.52)$$

For a time independent non-uniform medium, the above result (2.56) shows that the frequency is constant along the (curved) rays (2.52) but the wavenumber varies along these rays due to the inhomogeneities of the medium. For the earlier example of Airy waves approaching a beach, the wavelength decreases but the wave period is unchanged.

Figure 2-6.

Lighthill (1978)



Nonhomogeneous dispersion depicted on an (x, t) diagram. The paths (2.52) are curves along which the *frequency* ω remains constant.

The change of ω in a time dependent medium can be obtained from the dispersion relation in the form (2.53).

$$\omega = W(\underset{\sim}{k}, \zeta) \quad \text{and now} \quad \zeta = \zeta(\underset{\sim}{x}, t) \tag{2.53}$$

$$\therefore \quad \frac{d\omega}{d\tau} = \frac{\partial W}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial t} \tag{2.57}$$

$$\text{on curves} \qquad \frac{dx_\alpha}{d\tau} = \frac{\partial W}{\partial k_\alpha} = c_{g\alpha} \tag{2.52}$$

This indicates that the frequency ω varies along a ray <u>only</u> if the local properties of the medium (ζ) are <u>explicitly</u> time dependent, and the result ((2.56) above) for a time independent medium appears as a special case.

Summary of section 2.1

The structure of the wave kinematics that has been set up in this section rests on two essential features:

(i) One must be able to identify a particular wave ie. be able to define the "recognisable signal" referred to in 1.1 for a periodic wave. This was done by defining the phase χ and following the path of a particular value of χ .

(ii) The properties of the medium that influence the wave must be related to the phase. This appeared in the dispersion relation, which related the time and space derivatives of the phase to the relevant parameters of the medium.

(i) gave rise to the question of "conservation of waves/phase" and the conservation equation (2.25)

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

holds in a region where the number of waves remain constant. The general form of this relation derived in 2.1.1 is an important prototype of further conservation equations introduced later in this thesis. The conservation equation (2.25) places restrictions on the degree of inhomogeneity allowed in order for the analysis in terms of χ, k, ω to remain valid.

(ii) introduced the influence of the medium on the waves, so the propagation velocities of the parameters χ, k, ω were found to be intimately connected with the form of the dispersion relation. The propagation velocities were then inserted into the phase conservation equation (2.25) and the variation of k, ω obeyed the general 1st. order hyperbolic equations analysed in 2.1.1. The rays correspond to the characteristic curves of the wavenumber equation and represent the path followed by an observer moving at the group velocity c_g. Further discussion in 8.1 of the group velocity for nonlinear waves will compare a number of alternative definitions (including the kinematic definition given here).

This completes the general analysis of the kinematic behaviour of the waves. It is now followed in 2.2 by the kinematic description of the fluid through which the waves propagate. Later, in chapter 4, the wave kinematics are reintroduced when particular wave solutions are studied as they shoal and propagate on non-uniform currents.

2.2. KINEMATICS OF THE FLUID

There are two approaches to specifying the motion of the fluid. These involve choosing either :

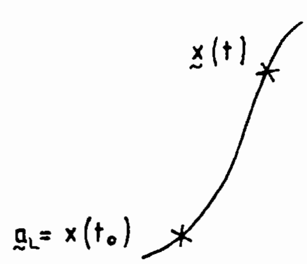
a reference position (Eulerian representation)

or

a reference particle or fluid element (Lagrangian representation)

This choice can be clarified by comparing the Euler and Lagrangian representation for the motion of a single fluid element : (Linn and Segel, 1974).

Figure 2-7.



Assume that a fluid element moves along the trajectory shown, being at $\underset{\sim}{a}_L$ at time t_o, and at $\underset{\sim}{x}$ at time t. Then $\underset{\sim}{x}$ is a function of $\underset{\sim}{a}_L$ and t :

ie. $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{a}_L, t)$ and in particular $\underset{\sim}{a}_L = \underset{\sim}{x}(\underset{\sim}{a}_L, t_o)$ initially (2.58)

hence

$\underset{\sim}{a}_L = \underset{\sim}{a}_L(\underset{\sim}{x}, t)$ and $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{a}_L, t)$ are inverse functions.

Now consider a fluid property such as pressure and let

$p(\underset{\sim}{x},t)$ be the pressure at time t at point $\underset{\sim}{x}$

$P(\underset{\sim}{x},t)$ be the pressure at time t, of the particle that was at $\underset{\sim}{a}_L$ at t_0 .

The link between the two descriptions is that the value of the pressure (or any other dependent variable) at the <u>point</u> $\underset{\sim}{x}$ is equal to the value of the pressure for the <u>particle located at</u> $\underset{\sim}{x}$.

ie.

Eulerian description

$$p(\underset{\sim}{x},t) = P[\underset{\sim}{a}(\underset{\sim}{x},t),t] \quad \text{(in terms of } \underset{\sim}{x},t\text{)} \qquad (2.59)$$

Lagrangian description

$$p[\underset{\sim}{x}(\underset{\sim}{a}_L,t),t] = P(\underset{\sim}{a}_L,t) \quad \text{(in terms of } \underset{\sim}{a}_L,t\text{)} \qquad (2.60)$$

The velocity of a fluid element can be expressed similarly :

Lagrangian: $$\underset{\sim}{V}(\underset{\sim}{a}_L,t) \equiv \frac{\partial \underset{\sim}{x}}{\partial t}(\underset{\sim}{a}_L,t) \qquad (2.61)$$

Eulerian: $$\underset{\sim}{v}(\underset{\sim}{x},t) \equiv \underset{\sim}{V}[\underset{\sim}{a}_L(\underset{\sim}{x},t),t] \qquad (2.62)$$

hence $\underset{\sim}{v}[\underset{\sim}{x}(\underset{\sim}{a}_L,t),t] = \underset{\sim}{V}(\underset{\sim}{a}_L,t)$ (2.63); now in <u>Lagrangian</u> form so (2.61) can be used:

$$\underset{\sim}{v}[\underset{\sim}{x}(\underset{\sim}{a}_L,t),t] = \frac{\partial \underset{\sim}{x}}{\partial t}(\underset{\sim}{a}_L,t) \qquad (2.64)$$

and in Euleria form,

$$\underset{\sim}{v}(\underset{\sim}{x},t) = \frac{\partial \underset{\sim}{x}}{\partial t}$$ (2.65) (dependence on initial conditions now <u>implicit</u>)

There are two further relations of importance that must be quoted. The Lagrangian one gives the trajectory in terms of $\underset{\sim}{a}_L,t_0$:

$$x = a_L + \int_{t_0} u(a_L, t)\, dt$$

and the Eulerian is the derivative operator following the particle path (initial position not explicitly stated).

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \qquad (2.67)$$

convective changes due to spatial gradients

rate of change at the chosen point

The above relations show that there is no inherent superiority of one or other specification. It is because of the difficulty in general of finding transformations between Euler and Lagrangian representations that a choice must be made between the two. The most common choice (which is used here) is to use the Eulerian approach, as one is usually more concerned with behaviour at a specified point than with that of an individual particle. Exceptions to this are the investigation of diffusion and mass transport phenomena, where particle paths are precisely what is required.

It is important to note that the averaging of Eulerian and Lagrangian representations lead to significant differences in results. Consequences of this are discussed later. (See 3.2; definitions of mean velocity (2.52) and implications). Shortcomings of both approaches are discussed in 8.3 where a hybrid averaging technique is introduced.

3. DYNAMICS OF THE FLUID AND THE WAVES

3.1. FLUID CONSERVATION EQUATIONS AND BOUNDARY CONDITIONS

3.1.1. Conservation equations for the fluid

A general reference for this section is Phillips (1980), chapter 2.2.

The conservation of mass for an incompressible fluid is simply,

$$\nabla \cdot \underset{\sim}{u} = 0 \qquad \text{or} \qquad \frac{\partial u_i}{\partial x_i} = 0 \qquad (3.1)$$

The conservation equation for momentum when viscous and Coriolis effects are neglected is :

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\left(u_i u_j\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\,\delta^i_3 \qquad (3.2)$$

p is the dynamic pressure.

The equation is arranged to show the forces on the right hand side and the responses on the left. The energy conservation equation is obtained from the scalar product of u_i and (3.2) :

$$\therefore \tfrac{1}{2}\frac{\partial}{\partial t}\left(\rho u_i^2\right) + u_i\frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) + u_i\frac{\partial p}{\partial x_i} - \rho g u_i \delta_3^i = 0$$

$$\therefore \tfrac{1}{2}\frac{\partial}{\partial t}\left(\rho u_i^2\right) + u_i^2 \overset{0}{\cancel{\frac{\partial}{\partial x_j}\left(\rho u_j\right)}} + u_i u_j \rho \frac{\partial u_i}{\partial x_j} - \rho g u_i \delta_3^i + u_i\frac{\partial p}{\partial x_i} = 0$$

$$\left.\begin{aligned} &\therefore \tfrac{1}{2}\left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}\right)\left(\rho \frac{u_i^2}{2}\right) + u_i\frac{\partial p}{\partial x_i} - \rho g u_i \delta_3^i = 0 \\ &\text{or: } \rho\frac{d}{dt}\left(\tfrac{1}{2}\underset{\sim}{u}^2\right) + \underset{\sim}{u}\cdot\nabla p - \rho\underset{\sim}{u}\cdot g = 0 \end{aligned}\right\} \qquad (3.3)$$

This can be clarified by writing $-\rho\underset{\sim}{u}\cdot\underset{\sim}{g} = \rho w = \rho\frac{d\zeta}{dt}$ where ζ is the vertical displacement of a fluid element. Then, using (2.68) and (3.1), (3.3) becomes

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho\underset{\sim}{u}^2 + \rho g\zeta\right\} + \nabla.\left\{\underset{\sim}{u}\left(p + \tfrac{1}{2}\rho\underset{\sim}{u}^2 + \rho g\zeta\right)\right\} = 0 \qquad (3.4)$$

This shows that the rate of change of kinetic and potential energy per unit volume is given by the divergence of the energy flux vector

$$\underset{\sim}{F} \equiv \underset{\sim}{u}\left(p + \tfrac{1}{2}\rho\underset{\sim}{u}^2 + \rho g\zeta\right) \qquad (3.5)$$

So far no reference has been made to vorticity, defined as :

$$\underset{\sim}{\omega} \equiv \nabla \times \underset{\sim}{u} \qquad (3.6)$$

This quantity is inherent in (3.2) which can be written as :

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\underset{\sim}{u}^2\right) + \underset{\sim}{w} \times \underset{\sim}{u} = -\frac{1}{\rho}\nabla p - g\hat{z} \tag{3.7}$$

$$\left(\hat{z} = \frac{\underset{\sim}{z}}{|z|}\right)$$

The governing equation for the vorticity is obtained by taking the curl of (3.7):

$$\frac{\partial \underset{\sim}{w}}{\partial t} + \nabla \times (\underset{\sim}{w} \times \underset{\sim}{u}) = 0$$

Helmholtz equation (3.8)

Using (3.1) in (3.8) $\Rightarrow$

$$\frac{d\underset{\sim}{w}}{dt} = \frac{\partial \underset{\sim}{w}}{\partial t} + (\underset{\sim}{u}.\nabla)\underset{\sim}{w} = (\underset{\sim}{w}.\nabla).\underset{\sim}{u} \tag{3.9}$$

One solution of this equation is $\underset{\sim}{w} = 0$ (which is a unique solution as long as $\nabla\underset{\sim}{u}$ is bounded). This equation therefore shows that the vorticity will remain zero everywhere if zero initially. This is assumed to be the case.

$$\underset{\sim}{w} = 0$$
$$\therefore \quad \nabla \times \underset{\sim}{u} = 0 \tag{3.10}$$

(It is possible to have the waves riding over a mean flow which has non zero vorticity; this is discussed later in 7.3).

The advantage of setting $\underset{\sim}{w} = 0$ is that $\nabla \times \underset{\sim}{u} = 0$ implies that $\underset{\sim}{u}$ can be represented as the gradient of a scalar potential function ϕ

$$\underset{\sim}{u} = \nabla\phi \tag{3.11}$$

$\phi \equiv$ velocity potential

and since the divergence of $\underset{\sim}{u}$ vanishes (3.1), ϕ obeys Laplace's equation :

$$\nabla^2 \phi = 0 \qquad (3.12)$$

One can use the velocity potential to reformulate the momentum equation (3.7) and then to integrate the equation. This first integral is one form of Bernoulli's equation.

$$(3.7) \qquad \nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{1}{2}(\nabla\phi)^2\right) = -\frac{\nabla p}{\rho} - \nabla(gz)$$

This equation has the first integral:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + \frac{p - p_0}{\rho} + gz = f(t) \qquad (3.13)$$

where $f(t)$ is an arbitrary function of time which can be absorbed by the transformation $\phi = \phi - \int f \, d\tau$. p_0 is a constant pressure which is separated from $f(t)$ to assist in applying the dynamic boundary condition.

The problem now is to solve Laplace's equation (3.12) for ϕ using the relevant boundary conditions, and then to use (3.11) and Bernoulli's equation (3.13) to find the interesting physical quantities $\underset{\sim}{u}$ and p.

This process does not immediately seem to involve waves because Laplace's equation is involved. A wave solution is possible because of the nature of the free surface boundary conditions.

3.1.2. Boundary conditions for the fluid

The boundary conditions required are kinematic conditions at the free surface and at the bottom, plus the dynamic condition satisfied by the pressure at the free surface. This derivation of the kinematic boundary conditions follows Whitham (1974, p433).

The free surface of the water is governed by the position of the interface. The defining property of an interface is that no fluid crosses it. Therefore the velocity of the fluid normal to the interface must equal the velocity of the interface normal to itself. Let the interface be defined by :

$$f(x,y,z,t) = 0$$

then the normal velocity of the surface defined by the above equation is :

$$\frac{-\frac{\partial f}{\partial t}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}$$

and the normal velocity of the fluid is :

$$\frac{u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}}$$

The two velocities are equal if:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u_i\frac{\partial f}{\partial x_i} = 0 \qquad (3.14)$$

Consequently, particles that are at the surface will remain there. Let the free surface be defined by :

$$\eta = \eta(x, y, t) \qquad (3.15)$$

and let the function f in (3.14) be chosen as $f \equiv \eta - z$

Now (3.14) $\Rightarrow$ $\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w$ (3.16)

This is the kinematic free surface boundary condition.

On the bottom, which is assumed solid and impermeable, the normal velocity vanishes.

ie. $$\hat{n} \cdot \underset{\sim}{u} = 0 \qquad (3.17)$$

$\hat{n}$ is the local inward normal to the solid boundary

Hence at the bottom the kinematic boundary condition is :

$$w + u\frac{dh}{dx} + v\frac{dh}{dy} = 0 \quad (\text{at } z = -h) \tag{3.18}$$

and for a horizontal bottom :

$$w(x, y, -h) = 0 \tag{3.19}$$

These are kinematic conditions, but in addition the boundary must satisfy a dynamic condition. The interface has no mass and so forces must balance on either side of it. Surface tension is assumed negligible compared to gravity, so the water pressure must equal that of the air. Although water motion will change the air pressure near the interface, the change is very small because of the low density of the air. Hence the air pressure can be regarded as constant and the dynamic free surface boundary condition is: (from (3.13), with p_0 : air pressure and with $f(t)$ absorbed into ϕ).

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right) + g\eta = 0 \tag{3.20}$$

It is these nonlinear surface boundary conditions applied at the unknown free surface which allow the wave solutions to Laplace's equation.

3.2. INTEGRAL PROPERTIES OF PERIODIC WAVES

Earlier sections have determined separate relationships for the fluid and for the waves. These are now related by defining wave quantities that are expressed in terms of fluid properties. The definitions involve integrals of the following form: (Le Blond & Mysak, 1977, p108)

$$\mathcal{I} \equiv \overline{\int_{-h}^{\eta} f(\underset{\sim}{u}, p, \rho, \cdots)\, dz} \tag{3.21}$$

where $\eta(x,y,t)$ is the function defining the free surface and $z=-h(x,y)$ is the position of the bottom. The integrands are functions of position, velocity, pressure, density and derivatives of these quantities, and the overbar denotes an average over the phase χ. In particular, at a given time t_0, the overbar represents:

$$\frac{1}{\lambda}\int_0^{\lambda}\left(\int_{-h}^{\eta} \ldots\, dz\right) dx \tag{3.22}$$

Note that the integrands are functions of variables that vary rapidly along the wave. The integral properties average out these fluctuations and so are slowly varying quantities as defined in 2.1. The integral properties and the relationships that can be established between can be used to investigate the wave properties themselves (Longuet - Higgins 1975) and the large scale modification of such properties. (Crapper, 1979; Phillips, 1980; Stiassnie & Peregrine, 1979).

The definitions and consequences of the integral properties are now considered, following Longuet - Higgins (1975), but with minor adjustments as in the above references.

3.2.1. Fundamental averaged properties and alternative forms of averaging

For convenience, the waves are assumed to propagate in the x direction only over a flat bottom (motion in the x,z plane) and are assumed to be exactly periodic. Extensions to more general cases are discussed with the introduction of water currents in chapter 4.

The choice of the z axis determines the mean elevation, defined by :

$$\bar{\eta} \equiv \frac{1}{\lambda}\int_0^{\lambda} \eta(x,t_0)\,dx = b \tag{3.23}$$

It is convenient to choose $b = 0$ until currents are introduced.

$$\bar{\eta} \equiv 0 \tag{3.24}$$

The mean velocity is defined analogously. It holds for all z below the wave <u>trough</u> and will be seen to determine the reference frame for the motion.

$$\bar{u}(z) \equiv \frac{1}{\lambda}\int_0^{\lambda} u(x,z,t_0)\,dx \tag{3.25}$$

$$(z \leq \eta_{min.})$$

In this form the definition seems quite innocuous, but in fact the irrotationality restriction on $\underset{\sim}{u}$ (3.10) introduces some unexpected features involving the z dependence of $\bar{u}$. In fact, all z dependence disappears irrespective of the vertical distribution of u ! This result is now shown in two ways:

(i) the restrictions imposed on the velocity potential by the wave and fluid boundary conditions are explored, followed by the use of in (3.25).

(ii) the irrotationality condition is used directly in (3.25), and (3.25) is then differentiated with respect to z.

(i) Rewriting (3.25) in terms of the velocity potential:

$$\bar{u}(z) = \frac{1}{\lambda}\int_0^{\lambda} \nabla\phi(x,z,t_0)\,dx = \frac{1}{\lambda}\Big[\phi(\lambda,z) - \phi(0,z)\Big]_{t_0} \qquad (3.26)$$

Now if ϕ is perfectly periodic, the expression in square brackets will vanish. Since periodic waves are assumed, $\underset{\sim}{u} = \nabla\phi$ must be periodic. Therefore the allowed form of ϕ is investigated by considering:

$$\phi(x,z,t_0) = \Phi(\chi,z) + A(x,z,t_0) \qquad (3.27)$$

$\chi \equiv$ phase as in (2.15)

where Φ is the periodic part of ϕ, so $\phi(\lambda,z) - \phi(0,z) = A(\lambda,z) - A(0,z)$

ϕ must satisify the following relations :

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(Laplace's equation)} \qquad (3.12)$$

$$\frac{\partial\phi}{\partial x}(x+\lambda,z) - \frac{\partial\phi}{\partial x}(x,z) = 0 \qquad \text{(periodicity)} \qquad (2.15)$$

$$\frac{\partial\phi}{\partial z}(x,-h) = 0 \qquad \text{(boundary condition)} \qquad (3.13)$$

Substituting (3.26) into the above equations, one obtains :
($a, b, c, \alpha, \beta, \gamma$ constants of integration)

(2.15) $\Rightarrow$ $\frac{\partial A}{\partial x}(x+\lambda,z) - \frac{\partial A}{\partial x}(x,z) = 0$

$\Rightarrow$ $A = ax f(z) + \gamma$ at most, since this is true $\forall x$. (3.28)

(3.12) $\Rightarrow$ $\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} = 0$ and so

(3.27) $\Rightarrow$ $ax\frac{\partial^2 f}{\partial x^2} = 0$ $\Rightarrow$ $A = ax(bz+c) + \gamma$ at most.

(3.13) $\Rightarrow$ $\frac{\partial}{\partial z}(\alpha x z + \beta x + \gamma)\Big|_{z=-h} = 0$ $\Rightarrow$ $\alpha = 0$

Hence finally $\quad \phi(x,z,t_0) = \Phi(x,z) + \beta x + \gamma \quad$ (3.29)

(these manipulations have taken $t = t_0$: if t is used as a variable, then $-\gamma t$ is used instead of γ . This term is an additional pressure term affecting the mean level of the water. Peregrine, 1976, p20). This term is discussed in detail in the derivation of the Stokes wave solution in 3.3; see equation

(3.55)).

Now (3.25) $\Rightarrow$ $$\bar{u} = \frac{1}{\lambda}\left[\beta\lambda - \beta.0\right] = \beta \qquad (3.30)$$

This proves the earlier assertion that $\bar{u}$ is depth independent and that all information about $u(z)$ is lost when averaged.

(ii) This result (3.30) can also be shown from (3.25) by using the irrotationality condition (3.10) directly :

$$\frac{\partial \bar{u}}{\partial z} = \frac{1}{\lambda}\frac{\partial}{\partial z}\int_0^\lambda u\,dx = \frac{1}{\lambda}\int_0^\lambda \frac{\partial u}{\partial z}\,dx = \frac{1}{\lambda}\int_0^\lambda \frac{\partial w}{\partial x}\,dx$$

$$= \frac{1}{\lambda}\left[w(\lambda,z) - w(0,z)\right] = 0 \quad \text{by periodicity of } \phi.$$

The choice of reference frame for the wave motion depends on the choice of β. The appropriate choice is :

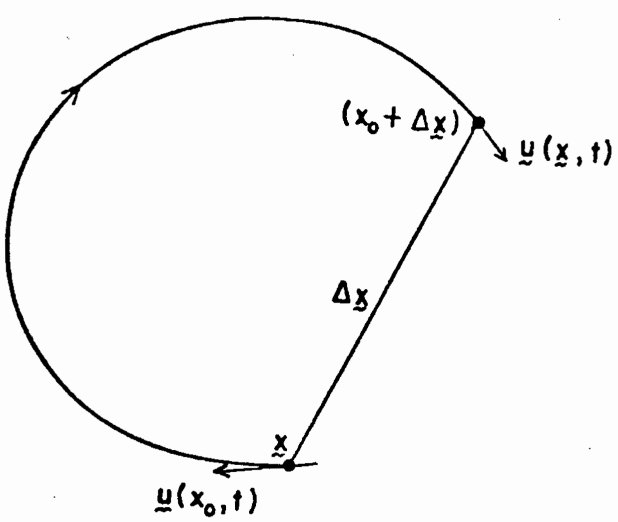
$$\beta = 0 \qquad (3.31)$$

This choice is discussed later in this section where it is shown that the above choice does <u>not</u> imply there is no momentum flux in the wave, but rather that the momentum flux in the <u>Eulerian</u> representation appears in the <u>surface region only</u>. $(\eta_{min.} \leq z \leq \eta_{max})$.

The conclusion to be drawn is that Eulerian averaging is not ideal as information can be lost in surprising ways. Here the loss of information is a consequence of a further restriction, that of irrotationality, but in general the problem arises if fluctuations about the mean are of the same order as the mean itself. In such situations, Lagrangian averaging and Eulerian averaging give different results. Since these effects are obviously significant in surface waves and hence in wave/current interactions, the Lagrangian mean is now defined and related to the Eulerian mean.

The derivation of the Lagrangian mean follows that of Longuet - Higgins (1969). A marked particle, at $\underset{\sim}{x}_o$ at time t_o with (Eulerian) velocity $\underset{\sim}{u}(\underset{\sim}{x}_o, t_o)$ moves to $\underset{\sim}{x}_o + \Delta\underset{\sim}{x}$ at time t. The displacement $\Delta\underset{\sim}{x}$ is assumed small compared to the local length scale of the velocity, ie. the particle is assumed to oscillate in the neighbourhood of its original position.

Figure 3-1.



The velocity at the new position at time t can then be written (to order $\Delta\underset{\sim}{x}$) :

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{u}(\underset{\sim}{x}_o, t) + \Delta\underset{\sim}{x} \cdot \underset{\sim}{\nabla}\underset{\sim}{u}(\underset{\sim}{x}_o, t) \qquad (3.32)$$

Since $\Delta\underset{\sim}{x}$ is small, it can be written as:

$$\Delta\underset{\sim}{x} = \int_{t_o}^{t} \underset{\sim}{u}(\underset{\sim}{x}_o, t)\, d\tau \qquad (3.33)$$

Use of this in (3.29) with mean value taken as in (3.21) gives :

$$\overline{\underset{\sim}{u}(\underset{\sim}{x},t)} = \overline{\underset{\sim}{u}(\underset{\sim}{x_0},t)} + \overline{\int_{t_0}^{t} \underset{\sim}{u}(\underset{\sim}{x_0},t)\,d\tau \cdot \nabla \underset{\sim}{u}(\underset{\sim}{x_0},t)} \qquad (3.34)$$

The left hand term is the mean velocity of a marked particle and is also known as the mass transport velocity. The first term on the right is the Eulerian mean (here expressed as an integral over a period at $\underset{\sim}{x_0}$ instead of an integral over a wavelength at t_0 as in (3.25)) and the remaining term is the difference, also known as the Stokes velocity.

$$u_L = \bar{u} + u_s \qquad (3.35)$$

The significance of large fluctuations is clear from the nature of the Stokes velocity term, and it is equally clear that in such circumstances the Lagrangian average can be quite different in magnitude and direction from the Eulerian. In fact, for double Kelvin waves (bottom hugging ocean waves) the two are in opposite directions! (Longuet - Higgins, 1969).

The inadequacies of the Eulerian averages compared to the Lagrangian ones led Andrews and McIntyre (1978 (a),(b)) to derive a new averaging technique known as the Generalised Lagrangian Mean description. In essence, it involves Eulerian averaging over neighbouring positions - a synthesis of the normal Eulerian average with the Lagrangian particle description.

The G.L. mean for a function φ: $\quad \bar{\varphi}^{L}(x,t) \equiv \overline{\varphi(x+\xi(x,t))} \quad (3.36)$

(overbar $\Rightarrow$ Eulerian average)

The G.L.M. description at x is given in terms of the function behaviour at neighbouring points $\xi(x,t)$. This is an extremely powerful technique which requires no essential changes when finite amplitude oscillations occur (as $\xi(x,t)$ is simply made larger) and provides equations corresponding to (3.34) in a much more elegant and generalised form. It is described further in chapter 8.3 but not applied in this thesis as it has not yet been formulated for water waves. It will undoubtedly have wide applications for water wave/current interactions, since it has had remarkable successes in atmospheric wave/mean flow problems.

3.2.2. Definitions of integral properties and the relations they satisfy

The five fundamental integral properties are now defined using the form of (3.21). All the following properties are defined for a wave propagating in the x direction and are therefore defined per unit width in the y direction. For this reason they are often referred to as densities ie. per unit horizontal surface area.

Mass flux per unit horizontal distance (ie per unit horizontal surface area)

$$I \equiv \overline{\rho \int_{-h}^{\eta} u \, dz} \tag{3.37}$$

The mass flux is sometimes referred to as the "momentum" or "momentum density" of the wave. It is shown in 8.2 that waves in a material medium do not possess momentum. There is a momentum flux associated with the wavetrain, but (for finite length wavetrains) the distribution of the momentum density is quite different to the distribution of the waves. The momentum is in fact associated with the propagation of $O(a^2)$ changes in the water level caused when the wavetrain is generated. The use of the terms "wave momentum" or "wave momentum density" is therefore avoided. (McIntyre 1981b). Mean kinetic energy per unit horizontal surface area:

$$T \equiv \frac{\rho}{2}\overline{\int_{-h}^{\eta}(u^2+w^2)\,dz} \tag{3.38}$$

Mean potential energy per unit horizontal surface area:
(measured from the level $z = 0$)

$$V \equiv \rho g\overline{\int_0^{\eta} z\,dz} = \tfrac{1}{2}\rho g\,\overline{\eta^2} \tag{3.39}$$

Radiation stress (excess momentum flux due to the waves);

$$S_{\alpha\beta} \equiv \rho\overline{\int_{-h}^{\eta}\left(\frac{p}{\rho}\delta^{\alpha}_{\beta} + u_\alpha u_\beta\right)dz} - \tfrac{1}{2}\rho g h^2 \tag{3.40}$$

Mean energy flux (recall (3.5)):

$$\mathcal{F} \equiv \overline{\int_{-h}^{\eta}F\,dz} \equiv \rho\overline{\int_{-h}^{\eta}u\left\{\frac{p}{\rho} + \tfrac{1}{2}(u^2+w^2) + gz\right\}dz} \tag{3.41}$$

The overbar is the average over phase χ which for a particular time t_0 becomes an average over wavelength (see (3.22)). eg. for the mass flux definition :

$$I \equiv \frac{1}{\lambda}\int_0^\lambda \int_{-h}^{\eta} u\,dz\,dx \tag{3.42}$$

These integral properties will play a fundamental role in the analysis of wave/current interactions. They are useful because the definitions are valid for finite amplitude waves and slowly varying wavetrains and because they are interlinked by a number of exact relations. The significance of some of the integral properties (eg. radiation stress), will be clarified in 3.3 where they are evaluated and discussed for Airy waves and in 5.2 where the dynamics of wave/current interactions are derived.

The consequences of the integral property definitions are now considered. These are firstly a clarification of the problems associated with the definition of the Eulerian mean velocity (recall (3.25) and secondly an investigation of the exact relations between the integral properties.

The mean velocity problems involving $\bar{u}$ are reviewed by using the mass flux (3.42). The reference frame used is that defined in (3.31) :

$$\bar{u} \equiv \int_0^\lambda u(x,z)\,dx = \beta \equiv 0 \qquad (3.25;3.31)$$

The mass flux is separated into two integrals with fixed and fluctuating limits respectively:

$$I = \frac{\rho}{\lambda}\int_0^\lambda \left\{ \int_{-h}^0 \frac{\partial\phi}{\partial x} dz + \int_0^\eta \frac{\partial\phi}{\partial x} dz \right\} dx \qquad \text{(from (3.42))}$$

Leibnitz's rule for differentiation under the integral sign is applied to each integral:

eg: $$\frac{\partial}{\partial x}\int_0^\eta \phi(x,z)\,dz - \phi(x,\eta)\frac{\partial\eta}{\partial x} = \int_0^\eta \frac{\partial\phi(x,z)}{\partial x} dz \qquad \left(\text{sim. for } \int_{-h}^0 dz\right)$$

$$\therefore \quad I = \frac{\rho}{\lambda}\int_0^\lambda \frac{\partial}{\partial x}\int_{-h}^0 \frac{\partial\phi}{\partial x} dz\,dx + \frac{\rho}{\lambda}\int_0^\lambda \frac{\partial}{\partial x}\int_0^\eta \frac{\partial\phi}{\partial x} dz\,dx - \frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx$$

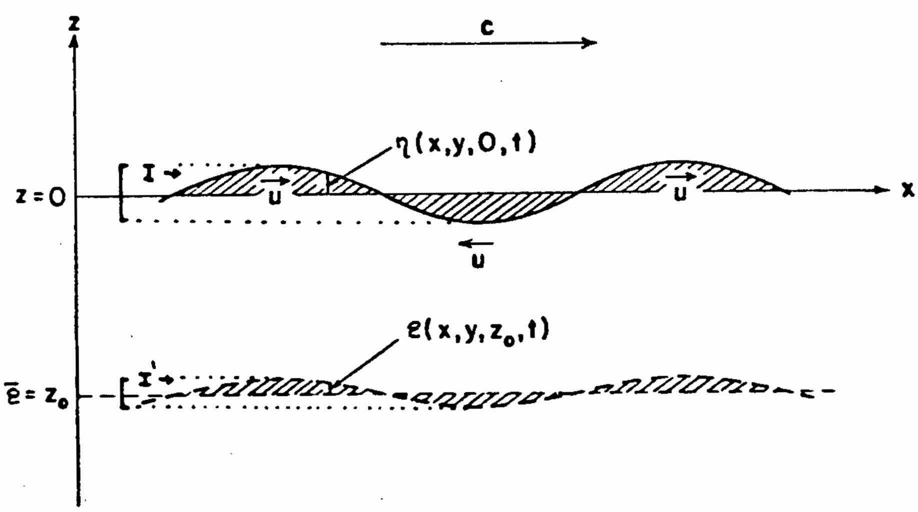
$$= \frac{\rho}{\lambda}\left[\int_{-h}^0 \frac{\partial\phi}{\partial x} dz\right]_0^\lambda + \frac{\rho}{\lambda}\left[\int_0^\eta \frac{\partial\phi}{\partial x} dz\right]_0^\lambda - \frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx$$

$$= \frac{\rho}{\lambda}\left[\beta\lambda h\right] - \frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx \;, \quad \text{and } \beta = 0 .$$

$$\therefore \quad I = -\frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx \qquad (3.43)$$

This proves the assertion following (3.31), namely that for irrotational flow, the Eulerian representation of the wave mass flux appears only in the fluctuating boundary region.

Figure 3-2.



Note that I gives the total mass flux for the wave. To see that there is in fact mass flux below $z = \eta_{min.}$, consider

$$I' = \frac{\rho}{\lambda}\int_0^{\lambda}\int_{-h}^{\varrho(x)} u \, dz \, dx \qquad (3.44)$$

where $\varrho(x)$ is a surface similar to the free surface but moving within the fluid. By (3.43), I' will give the total mass flux up to the level $\overline{\varrho(x)}$, but again it will appear concentrated in the fluctuating boundary region.

This completes the discussion of the two simplest velocity integrals, namely $\bar{u}$ and I . The relationship between the mass flux and the next velocity integral, the kinetic energy , is now derived. This relation was derived by Levi - Civita (1925) and subsequently discussed by Starr (1947a,b; 1958) and by Longuet - Higgins (1975), on whose work the following derivation is based.

The reference fame is now changed to that which propagates with the phase velocity c . The mass flux is defined in this frame:

$$\rho\int_{-h}^{\eta}(u-c)\,dz = -Q \qquad (3.45)$$

Integrating (3.45) over one wavelength:

$$\lambda I - \lambda c h \rho = -\lambda Q \qquad (3.46)$$

Now by definition of the kinetic energy (3.38) in the fixed reference frame :

$$2\lambda T = \rho \int_0^\lambda \int_{-h}^\eta \left\{ \left[(u-c)+c \right]^2 + w^2 \right\} dz\,dx$$

$$= \rho \int_0^\lambda \int_{-h}^\eta \left[(u-c)^2 + w^2 \right] dz\,dx + 2c\rho \int_0^\lambda \int_{-h}^\eta (u-c)\,dz\,dx + c^2\rho \int_0^\lambda \int_{-h}^\eta dz\,dx$$

$$= \rho \int_0^\lambda \int_{-h}^\eta \left[(u-c)^2 + w^2 \right] dz\,dx$$

$$- 2c\lambda Q + \rho c^2 \lambda h \tag{3.47}$$

Now if Φ, Ψ are the velocity potential and stream function in the moving reference frame;

ie.
$$\Phi \equiv \phi - cx$$
$$\Psi \equiv \psi - cz$$

where ϕ, ψ are the corresponding functions in the fixed frame of reference,

then
$$\frac{\partial(\Psi, \Phi)}{\partial(z, x)} = \begin{vmatrix} \frac{\partial \Psi}{\partial z} & \frac{\partial \Psi}{\partial x} \\ \frac{\partial \Phi}{\partial z} & \frac{\partial \Phi}{\partial x} \end{vmatrix}$$

$$= \frac{\partial \Psi}{\partial z} \cdot \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial x} \cdot \frac{\partial \Phi}{\partial z} = (u-c)(u-c) + w(-w)$$

$$= (u-c)^2 - w^2$$

Hence
$$\int_0^\lambda \int_{-h}^\eta \left[(u-c)^2 - w^2 \right] dz\,dx = \iint d\Psi\, d\Phi$$

Also
$$\Phi(\lambda) - \Phi(0) = \beta - \lambda c$$

$$\Psi(\eta) - \Psi(-h) = -\frac{Q}{\rho}$$

Hence (3.47) becomes :

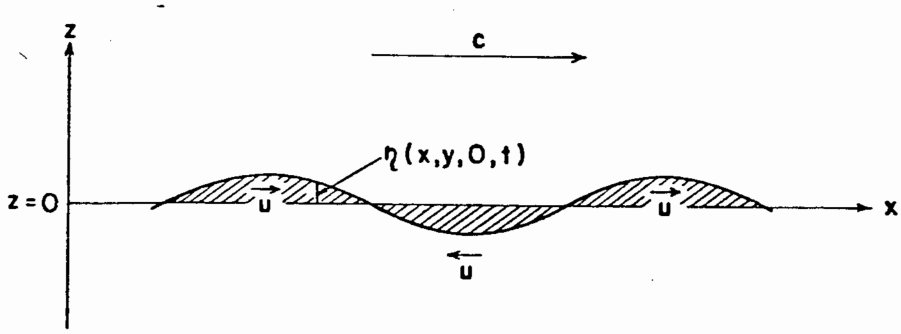
$$\begin{aligned} 2\lambda T &= -Q(\beta-\lambda c) - 2c\lambda Q + c^2\lambda h\rho \\ &= -\beta Q - \lambda c Q + c^2\lambda h\rho \\ &= -\beta Q + \lambda c I \end{aligned} \tag{3.48}$$

With $\beta = 0$:

$$2T = cI \tag{3.49}$$

This relationship has invoked some comment (Starr 1947 (a), (b), 1958)) as it is surprising to see that the phase velocity and not the group velocity links the kinetic energy to the mass flux. This can be clarified by looking at the water movement in the surface region of the waves :

Figure 3-3.



Above $z = 0$, there is mass carried forward by the crests. In the troughs, there is a mass deficiency carried backward. In fact the total mass flux can be written (Longuet - Higgins (1969)) as:

$$I = \rho \overline{u\eta} \tag{3.50}$$

Since u and η have a periodicity propagating at c, the presence of the phase velocity in the energy/momentum relation is explained.

Further relationships between the integral properties are derived by Crapper (1979), Longuet - Higgins (1975), and are quoted here in the form given in Stiassine and Peregrine (1979). (with $\bar{\eta} = 0$).

$$S_{11} = 4T - 3V + \overline{u^2(x_1, -h)} \tag{3.51}$$

$$F = (3T - 2V).c + \tfrac{1}{2}(ch + I)\overline{u^2(x_1,-h)} \tag{3.52}$$

Note that these relations simplify in deep water when the bottom velocity $u(x_1 - h)$ vanishes. It is again noteworthy that the energy flux expression involves the phase velocity and also that the radiation stress and energy flux can in fact be expressed in terms of I, T and V.

It is now possible to investigate wave current interactions using the kinematic laws governing wave motion, the dynamics of the fluid flow and the relationship between them as contained in the integral property representation. Before investigating the interactions, two particular solutions for surface gravity waves are derived in 3.3.

3.3. THE AIRY AND STOKES SOLUTIONS FOR PERIODIC SURFACE GRAVITY WAVES

The behaviour of periodic surface gravity waves is governed by the equations derived in 2.1 - 3.2. It is appropriate to consider a solution to these equations at this point (before the discussion of the governing equations for wave/current interactions) in order to clarify features of the equations, especially as many features of the interactions will be compared to the wave properties in the absence of a current.

The solution derived here is that found by Stokes (1847), which incorporates the Airy solution (see Lamb (1932)) as a special case for small amplitude waves. These solutions are chosen for a number of reasons. Firstly they have been in widespread use since their original derivation; due to some extent to their relatively simple analytical form. Secondly, they have been used in all the pioneering work on wave/current interactions (Longuet-Higgins & Stewart 1960, 1961, 1964; Bretherton & Garrett 1968) and provide a reference to which more recent work can be compared.

3.3.1. Derivation of the Stokes solution correct to $O(a^2)$ in the amplitude with the Airy solution as a special case

The second order Stokes solution (Stokes 1847; Lamb 1932 Article 250) is valid when the parameter a/k^2h^3 is small, implying a restriction to moderate or deep water conditions (Peregrine

1976). The special case of the Airy solution is obtained when ak and a/h are also required to be small, linearizing the problem and restricting the solution to waves of infinitesmal amplitude. The significance of the parameters a/h, ak, a/k^2h^3 is discussed in terms of the nonlinearities in the governing equations by Svendsen (1971) and Ursell (1953).

This derivation assumes two dimensional x, z motion, the wave propagating along the x axis with the motion inviscid and irrotational. A velocity potential $\phi(x,z,t)$ can therefore be defined (3.11), and the motion must satisfy Laplace's equation (3.12), with the pressure field obeying Bernoulli's equation (3.13).

Notation: the subscript "s" will be used to denote Stokes wave properties and subscript "A" for Airy waves, where necessary.

$$\underset{\sim}{u}(x,z,t) = \nabla\phi(x,z,t) \tag{3.11}$$

$$\nabla^2\phi = 0 \tag{3.12}$$

$$\frac{p}{\rho} + \frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] + gz = f(t) \tag{3.13}$$

The boundary conditions at the free surface and the bottom are:

free surface:

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}(x,\eta)\cdot\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial z}(x,\eta) = 0 \tag{3.16}$$

$$\frac{\partial\phi}{\partial t}(x,\eta) + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)_\eta^2 + \left(\frac{\partial\phi}{\partial z}\right)_\eta^2\right] + g\eta = 0 \tag{3.20}$$

bottom: $$\frac{\partial \phi}{\partial z}(x,-h) = 0 \tag{3.19}$$

A periodic solution is wanted so ϕ, η are required to be functions of the phase χ

$$\chi = kx - \sigma t \tag{2.20}$$

where σ is the (intrinsic) frequency as in chapter 2. The change of symbol from ω to σ is to allow use of ω to represent the absolute frequency in the presence of currents. (See chapter 4).

$$\phi = \phi(\chi, z) \tag{3.53}$$

$$\eta = \eta(\chi) \tag{3.54}$$

Recall from (3.29) and the ensuing discussion that the general expression for ϕ satisifying the boundary conditions and having a periodic component is

$$\phi(x, z, t) = \Phi(\chi, z) + \beta x - \gamma t \tag{3.55}$$

The value of the constants β, γ must be considered with care. β determines the motion of the reference frame and as noted by Stokes (1847) (see also Peregrine 1976 p20) there is ambiguity in defining "still water" for a finite amplitude wave. The two possibilities of:

(i) average velocity is zero for any submerged point ($z \leq \eta$ trough) ie. $\bar{u} = 0$ (see 3.25)

(ii) average flow of water through any vertical plane is zero ie. $I = 0$. (see 3.37)

were shown in 3.2 (see analysis and discussion preceding (3.43)) to give different results. Specifically,

(i) $\bar{u} = 0 \Rightarrow \beta = 0$; $I = -\frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x}dx$

(ii) $I = 0 \Rightarrow \beta \neq 0$ and $\rho\beta h = \frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x}dx$ (3.43)

Stokes chose the first of these possibilities and so the velocity potential is of the form

$$\phi(x,z,t) = \Phi(\chi,z) - \gamma t \qquad (3.56)$$

The value of γ is determined from the mean value of the dynamic free surface condition (3.20)

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{z=\eta} + g\eta = 0 \qquad (3.20)$$

Mean value: $$\overline{\frac{\partial\phi}{\partial t}} + \frac{1}{2}\left\{\overline{\left(\frac{\partial\phi}{\partial x}\right)^2} + \overline{\left(\frac{\partial\phi}{\partial z}\right)^2}\right\}_{z=\eta} + g\bar{\eta} = 0 \qquad (3.57)$$

The periodicity of $\Phi(\chi)$ means that

$$\overline{\frac{\partial\phi}{\partial t}} = -\gamma \qquad (3.58)$$

$$-\gamma + \frac{1}{2}\left\{\overline{\left(\frac{\partial\phi}{\partial x}\right)^2} + \overline{\left(\frac{\partial\phi}{\partial z}\right)^2}\right\}_{z=\eta} + g\bar{\eta} = 0 \qquad (3.59)$$

Since the term in curly brackets will be non-zero, it is clear that only <u>one</u> of $\bar{\eta}$ or γ can be chosen arbitrarily. The usual choice for a periodic wavetrain is (see also 3.23)

$$\bar{\eta} = 0 \tag{3.60}$$

In the case of a wavetrain being modulated by the presence of a slowly varying current, the local value of $\bar{\eta}$ and β $\left(\text{ie.} \ \overline{\frac{\partial \phi}{\partial x}} \right)$ will be non-zero and slowly varying. Here it is convenient to continue with the choices $\beta = 0$; $\bar{\eta} = 0$.

The particular solution due to Stokes is now derived. It is in essence a perturbation expansion for ϕ and η about the mean free surface $z = 0$.

$$\phi = -\gamma t + \varepsilon^1 \phi_1 (x,z) + \varepsilon^2 \phi_2 (x,z) + \cdots \tag{3.61}$$

$$\eta = \varepsilon^1 \eta_1 (x) + \varepsilon^2 \eta_2 (x) + \cdots \tag{3.62}$$

The parameter ε is taken as ak.

Such an expansion will work best for ak small. Since ak is of the order of the wave slope, this suggests that the expansion procedure will be cumbersome for the steepest waves, especially in shallow water. This approach is therefore restricted by the requirement that a/k^2h^3 be small, implying waves of finite height propagating over moderate to deep water. (Peregrine 1976, p20). Phillips (1980) notes that the success of the Stokes expansion is due to the fact that typical nonlinear terms in the governing equations at the free surface can be shown to be of the order of: [(wave steepness) × (typical linear term)]. Since the steepness is not large, the nonlinear effect can be regarded as imposing a regular perturbation on the linear solution.

Convergence of the Stokes expansions was shown in 1925 by Levi-Civita for a steady wave train on deep water (although the radius of convergence was not established) and convergence for the finite depth case was proved in 1926 by Struik.

The expansions for ϕ, η are inserted into the governing equations and the boundary conditions are applied.

Terms involving ε^1 lead to the following relations:

$$\left.\begin{aligned}
&\underset{\sim}{u}_1 = \nabla\phi_1 \\
&\nabla^2\phi_1 = 0 \\
&\frac{p}{\rho} + \frac{\partial\phi}{\partial t} + gz = 0 \\
&\frac{\partial\phi_1}{\partial z}(x_1,-h) = 0 \\
&\text{The surface boundary conditions are applied at } z=0\text{:} \\
&\text{eg.}\quad \left.\frac{\partial\eta_1}{\partial t}\right|_{z=0} - \left.\frac{\partial\eta_1}{\partial z}\right|_{z=0} = 0
\end{aligned}\right\} \tag{3.63}$$

The solution to these equations is the Airy theory, (Phillips 1980, p36,37) valid when the parameters ak, a/h and a/k^2h^3 are all small.

$$\eta = a\cos\chi \tag{3.64}$$

$$\phi = \frac{\sigma a \cosh k(z+h)\sin\chi}{k\sinh kh} \tag{3.65}$$

$$\sigma^2 = gk\tanh kh \tag{3.66}$$

(terms $O(a^2)$ neglected)

If this approach is continued for the terms in ϵ^2, the following equations are obtained. Here the expansion parameter ϵ has been replaced by νa or μa where ν, μ represent the constants to be determined. The form of the z dependence and periodic nature of ϕ_1, ϕ_2, η_1, η_2 have been specified. (Whitham, 1974 p474).

$$\eta = a\cos\chi + \mu_2 a^2 \cos 2\chi + O(a^3k^3) \quad (3.67)$$

$$\phi = -\gamma t + \nu_1 a \cosh k(z+h)\sin\chi + \nu_2 a^2 \cosh 2k(z+h)\sin 2\chi + O(a^4k^2) \quad (3.68)$$

It is also necessary to express the frequency σ (which is contained in χ) as an expansion

$$\sigma = \sigma_0(k) + a^2\sigma_2(k) + \dots \quad (3.69)$$

The terms σ_1 and σ_3 and their removal (to avoid secular terms) are discussed by Whitham (1974, p472).

The solutions for the constants $\mu_2, \gamma, \nu_1, \nu_2$ are eventually found to be:

$$\mu_2 = \frac{1}{2}k\coth kh\left(1 + \frac{3}{2\sinh^2 kh}\right) \quad (3.70)$$

$$\gamma = \frac{gka^2}{2\sinh 2kh} \quad (3.71)$$

$$\nu_1 = \frac{\sigma_0}{k\sinh kh} \quad (3.72)$$

$$\gamma_2 = \frac{3}{8}\cdot\frac{\sigma_0}{\sinh^4 kh} \qquad (3.73)$$

The Stokes second order solution for a periodic wave (with a/k^2h^3 small) is therefore given by:

$$\eta = a\cos\chi + \tfrac{1}{2}a^2\coth kh\left(1 + 3/(2\sinh^2 kh)\right)\cos 2\chi + O(a^3k^2) \qquad (3.74)$$

$$\phi = \underbrace{\frac{-gka^2t}{2\sinh 2kh}}_{-\gamma t} + \underbrace{\frac{\sigma_0 a\cosh k(z+h)\sin\chi}{k\sinh kh} + \frac{3\sigma_0 . a^2\cosh 2k(z+h)\sin 2\chi}{8\sinh^4 kh}}_{\Phi(\chi, z)} + O(a^3k^2) \qquad (3.75)$$

$$\sigma^2 = gk\tanh kh + gk\tanh kh\left[\frac{9\tanh^4 kh - 10\tanh^2 kh + 9}{8\tanh^4 kh}\right]a^2k^2 + O(a^3k^3) \qquad (3.76)$$

(This is the form of the equations given in Whitham (1974, p474); there are some minor errors in the corresponding equations given by Peregrine (1976, p21; his 2.16; 2.17)).

3.3.2. Comparison of the linear Airy solution and the nonlinear Stokes solution

(i) Dispersion relation

Since there are terms of $O(a^2)$ appearing in the Stokes expressions for wave profile η and the velocity potential ϕ it is clear that all the wave properties are going to be modified to some extent from the linear solution. There is one change that is <u>qualitatively</u> different and is therefore of most significance and that is the appearance of the amplitude in the dispersion relation. This is most easily discussed for the simplified deep water form:

$$\sigma^2 = gk(1 + a^2k^2) + O(a^2k^2) \qquad (3.77)$$

The physical interpretation is clearly that a group of waves of the same wavelength but different heights will disperse, since the phase speed σ/k (2.23) and group velocity $\partial\sigma/\partial k$ (2.29) are now proportional to a^2 . The complexities that this introduces are illustrated here by considering the equations governing modulations on a linear wavetrain (Whitham, 1974, p489). The kinematic equation is familiar; the dynamic equation will be derived in 6.3.1 but for the present illustration it is sufficient to note that it is an energy conservation equation expressed in terms of the wave amplitude. The two governing equations are:

kinematics: $$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial x} = 0 \qquad (2.25)$$

dynamics: $$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}\left(c_g a^2\right) = 0 \qquad (6.30)$$

and in deep water the <u>linear</u> dispersion relation is given by the Airy relation:

$$\sigma^2 = gk \qquad (3.66)$$

The kinematic equation is clearly independent of the amplitude variation in (6.29). If <u>nonlinear</u> waves of small amplitude are considered, the Stokes dispersion relation (3.69) is appropriate and the kinematic relation (2.38) becomes:

$$\frac{\partial k}{\partial t} + \frac{\partial \sigma}{\partial k} \cdot \frac{\partial k}{\partial x}$$

$$= \frac{\partial k}{\partial t} + \left\{ \underbrace{\frac{\partial}{\partial k}(gk)^{1/2}}_{\text{linear}} + \frac{\partial}{\partial k}(gk^3)^{1/2} \cdot a^2 \right\} \frac{\partial k}{\partial x} + (gk^3)^{1/2} \cdot \frac{\partial a^2}{\partial x} = 0 \qquad (3.78)$$

There are now two coupling terms involving a^2 (of which the $\frac{\partial a^2}{\partial x}$ term is the largest) which link this equation to the dynamic equation (6.29). For small amplitudes, the nonlinear corrections to the dynamic equation are small, and so for an initial assessment of nonlinear effects, the system of coupled equations can be taken as:

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial k}(gk)^{1/2} \cdot \frac{\partial k}{\partial x} + (gk^3)^{1/2} \cdot \frac{\partial a^2}{\partial x} = 0 \qquad (3.79)$$

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial k}(gk)^{1/2} \cdot \frac{\partial a^2}{\partial x} + a^2 \frac{\partial^2}{\partial k^2}(gk^3)^{1/2} \cdot \frac{\partial k}{\partial x} = 0 \qquad (3.80)$$

(using $c_g = \partial\sigma/\partial k$)

Analysis of the characteristic form of these equations (Whitham 1974, ch.5) leads to interesting results for Stokes waves. In fact, the waves are found to be unstable, in the sense that small perturbations will grow with time. (Whitham, 1974, p490). This does not neccessarily mean that the waves will break but the wavetrain may be modified (Peregrine 1976, p13). This effect is qualitatively different from that for Airy theory and a direct result of the amplitude dependence of σ.

As a final comment on the effect of wave amplitude on the dispersion relation (and hence on phase and group velocities) Longuet-Higgins (1975) shows by the use of the integral properties defined in 3.2 that phase speed increases with amplitude to a maximum, beyond which higher waves move slightly slower. This effect occurs for waves near the maximum possible height and so is outside the range for which the Stokes second order solution is applicable.

(ii) Mean water level and influence of the γ term

Despite the care taken in order to include the $-\gamma t$ term in the velocity potential (3.56), there was no sign of it in the linear velocity potential (3.65). The explanation is found from the nonlinear solution (3.68); the value of γ (3.71) is proportional to a^2 and so is neglected in the linear approximation.

3.3.3. Integral properties for the linear Airy solution and validity of quantities involving terms of $O(a^2)$

The Airy theory does not include terms of $O(a^2)$ in the expressions for η, ϕ and σ (3.64 - 3.66). This means that the fluid velocity $\underset{\sim}{u}$ will also exclude such terms (since $\underset{\sim}{u} = \nabla\phi$), and so will the pressure field p obtained from the linearised Bernoulli equation (3.63). When the integral properties ((3.37-3.42) are evaluated using the above linear solution, terms such as $u\eta$, u^2, η^2 appear in the integrands. These are clearly going to give rise to terms $O(a^2)$. This raises speculation that the improvement of the original solution to include $O(a^2)$ terms in η, ϕ will lead to further $O(a^2)$ terms in the integral properties, invalidating the original results. It is now shown that this is not possible by using the definition of the mean mass flux (3.42) and repeating the analysis preceding (3.43), ie. splitting the integral $\int_{-h}^{\eta} dz$ into $\int_{-h}^{0} dz + \int_{0}^{\eta} dz$. The reference frame is one in which $\beta = 0$ and $\bar{\eta} = 0$.

$$I \equiv \frac{\rho}{\lambda}\int_0^\lambda \int_{-h}^{\eta} u\,dz\,dx \tag{3.42}$$

$$\therefore\ I = \frac{\rho}{\lambda}\int_0^\lambda \left\{ \int_{-h}^{0} \frac{\partial\phi}{\partial x} dz + \int_0^\eta \frac{\partial\phi}{\partial x} dz \right\} dx$$

$$= \frac{\rho}{\lambda}\left[\beta\lambda h\right] + \frac{\rho}{\lambda}\int_0^\lambda \int_0^\eta \frac{\partial\phi}{\partial x} dz\,dx$$

$$\therefore\ I = 0 + \frac{\rho}{\lambda}\int_0^\lambda \int_0^\eta \frac{\partial\phi}{\partial x} dz\,dx \tag{3.81}$$

In this expression the influence of the products of $O(a)$ terms is begining to appear, since both the integrals and the upper limit are $O(a)$ terms. It is convenient now to take the derivative outside the z integration using Leibnitz' rule:

$$I = -\frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx + \frac{\rho}{\lambda}\int_0^\lambda \frac{\partial}{\partial x}\int_0^\eta \phi\,dz\,dx$$

$$= -\frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx + \frac{\rho}{\lambda}\left[\int_0^\lambda \frac{\partial\phi}{\partial x}\right]_0^\lambda$$

$$= -\frac{\rho}{\lambda}\int_0^\lambda \phi(x,\eta)\frac{\partial\eta}{\partial x} dx + 0$$

(3.43)

The appropriate expressions for Airy theory can now be inserted:

$$I = -\frac{\rho}{\lambda}\int_0^\lambda \frac{\sigma a}{k} \frac{\cosh k(a\cos\chi + h)}{\sinh kh} \sin\chi . (-ak\sin\chi)\, dx$$

$$= \frac{\rho a^2 \sigma}{\lambda} \int_0^\lambda \frac{\cosh k(a\cos\chi + h)}{\sinh kh} . \sin^2\chi \, dx$$

It is clear that the result will be proportional to a^2, as expected. It is consistent with the restrictions on Airy theory that $a\cos\chi$ can be neglected in the integrand since $a/h \ll 1$ giving a result correct to $O(a^2)$.

$$\therefore \quad I = \frac{\rho a^2 \sigma \coth kh}{2} \qquad (3.82)$$

Now consider the improvement of this result if for example the Stokes expressions for $\phi(x,\eta)$; $\frac{\partial\eta}{\partial x}$ were used in (3.43). It is clear that the second order correction terms would have influences only on higher order terms in the result and so the above result (3.82) is unaltered (to $O(a^2)$) by such changes.

The remaining integral properties are now obtained. In each case, the restriction $a/h \ll 1$ is invoked at some stage in an analagous manner to that used above. Often, alternative derivations of each property are possible.

Mean kinetic energy:

$$T = \frac{\rho}{2\lambda}\int_0^\lambda \int_{-h}^{\eta} \left\{ \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \right\} dz\, dx \qquad (3.38)$$

Rather than integrate the Airy expressions directly, the value of

T is most easily obtained from the value of I by use of the relationship (3.49), valid for $\beta = 0$:

$$2T = cI \qquad (3.49)$$

$$\therefore \quad T = \frac{\sigma I}{2k}$$

$$= \frac{1}{4} \cdot \frac{\rho \sigma^2}{k \tanh kh} \qquad \text{from (3.82)}$$

Since

$$\sigma^2 = gk \tanh kh, \qquad (3.66)$$

$$T = \frac{1}{4} \rho g a^2 \qquad (3.83)$$

Alternatively, Phillips (1961) gives an exact expression for T in terms of quantities evaluated at the free surface only, (analogous to (3.43) for I on the previous page) which can be used to derive (3.83):

$$T = \frac{1}{2} \frac{\rho}{\lambda} \int_0^\lambda \phi(x, \eta) \frac{\partial \eta}{\partial t} \, dx \qquad (3.84)$$

$$= \frac{1}{2} \frac{\rho}{\lambda} \int_0^\lambda \frac{\sigma a \cosh k(\eta + a)}{k \sinh kh} \sin \chi \, . \, (\sigma a \sin \chi) \, dx$$

$$= \frac{1}{2} \frac{\rho}{\lambda} \frac{a^2 \sigma^2}{k} \int_0^\lambda \frac{\cosh k(\eta + a)}{\sinh kh} \sin^2 \chi \, dx$$

The requirement $a/h \ll 1$ now appears directly, as it did for I.

$$\therefore \; T = \frac{1}{4} \rho g a^2 \qquad \text{correct to second order (3.83)}$$

Mean potential energy:

$$V \equiv \frac{\rho g}{2}\overline{\eta^2} \qquad (3.39)$$

An alternative to direct substitution is to use the equivalence of mean potential and kinetic energies for small oscillations of any conservative dynamical system: (Starr, 1959)

$$\therefore V = T = \frac{1}{4}\rho g a^2$$

correct to second order (3.84)

It is convenient to define the mean energy per unit distance (or "wave energy density") as the sum of the kinetic and potential energies.

$$E \equiv T + V \qquad (3.85)$$

This is easily evaluated for Airy theory since $T = V$:

$$E_{Airy} \equiv E = \frac{1}{2}\rho g a^2$$

correct to second order (3.86)

The integral properties discussed so far are extensions of well known concepts. Less well known is the radiation stress (3.40) which is the excess momentum flux due to the presence of waves. It was defined by Longuet-Higgins and Stewart (1960, 1961, 1964) in a number of contexts, one of which was for waves riding over slowly changing mean flows. Perhaps the radiation stress concept can be seen most naturally as a part of the following system:

(i) The integral properties are first listed as mean fluxes of wave mass, momentum, energy:

I : the mean mass flux associated with the waves.

S : the mean momentum flux associated with the waves.

F : the mean energy flux associated with the waves.

(ii) The conservation equations for mass, momentum and energy in the fluid are now written in conservation form, as in the general case discussed in 2.1, equation (2.5).

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (2.5)$$

$$\frac{\partial \bar{P}_i}{\partial t} + \frac{\partial \bar{Q}_i}{\partial x} = 0 \qquad (3.87)$$

$$(i = 0,1,2)$$

where the $\bar{P}_i$ represent mean density of mass $(\bar{P}_0)$, of momentum $(\bar{P}_1)$, and of energy $(\bar{P}_2)$; the $(\bar{Q}_i)$ representing the mean flux of mass, momentum and energy respectively. The terms $\bar{P}_i$ and $\bar{Q}_i$ will contain terms involving <u>both</u> the mean wave flux properties (the integral properties listed above) and properties of the main stream flow.

For uniform flows, the behaviour of such a system could equally well be expressed as (for instance) total energy = constant or wave energy = constant, but for non-uniform mean flows, the form (3.87) leads to the correct results and the correct interpretation of the roles played by the various wave properties. This system was proposed by Whitham (1962) and is discussed in detail in chapter 5. The results of the analysis shows that the presence of a "mean wave momentum flux" term (ie. the radiation stress S) and its derivatives in non-uniform flow situations is not surprising and is essentially the effect of the "fictious forces" required to satisfy conservation equations in an accelerating reference frame.

The above discussion indicates that the radiation stress concept does arise in a natural way when it is regarded as part of a hierarchy of wave integral properties used in the conservation equations (3.87). The radiation stress for the Airy wave solution is now derived.

For a wave propagating in the x direction, the component of the radiation stress in that direction is given by the S_{11} component of (3.40)

$$S_{11} = \frac{\rho}{\lambda}\int_0^{\lambda}\int_{-h}^{\eta}\left(\frac{p}{\rho} + u^2\right)dz\,dx - \frac{1}{2}\rho g h^2 \qquad (3.88)$$

The radiation stress is regarded as the excess momentum flux due to the waves (other interpretations will be made in chapter 5). The first term in (3.87) is the momentum flux crossing a plane (x = constant) in the presence of waves and the second term is the background flux with the waves absent. Terms of the form ρu^2

represent the transfer of momentum ρu at a rate u across the plane and the pressure p is the isotropic stress on the plane. The second term in (3.88) arises from a consideration of the stress on the plane in the absence of waves, which is given by:

$$\int_{-h}^{0} p_0 \, dz \tag{3.89}$$

Since the pressure distribution must be hydrostatic, $p_0 = -\rho g z$.

$$\therefore \quad \int_{-h}^{0} p_0(z) dz = \frac{1}{2} \rho g h^2 \tag{3.90}$$

When no waves are present, the stress on the plane x = constant is balanced by the stress on the plane $x + \delta x$ = constant, since the pressure gradient there is also hydrostatic.

The expression for S_{11} (3.88) is split into three parts which are then evaluated separately using the Airy theory. (Longuet-Higgins and Stewart, 1964, p533).

$$S_{11} = \frac{\rho}{\lambda} \int_0^{\lambda} \int_{-h}^{\eta} \left(\frac{p}{\rho} + u^2 \right) dz\, dx - \int_{-h}^{0} p_0 \, dz$$

$$= \frac{\rho}{\lambda} \int_0^{\lambda} \Big[\underbrace{\int_{-h}^{\eta} u^2 dz}_{(a)} + \underbrace{\frac{1}{\rho} \int_{-h}^{0} (p - p_0) dz}_{(b)} + \underbrace{\frac{1}{\rho} \int_0^{\eta} p\, dz}_{(c)} \Big] \tag{3.91}$$

(a) $$\frac{\rho}{\lambda} \int_0^{\lambda} \int_{-h}^{a\cos\chi} u^2 \, dz\, dx \simeq \frac{\rho}{\lambda} \int_0^{\lambda} \int_{-h}^{0} u^2 \, dz\, dx$$

(correct to second order; extend u analytically to $z = 0$ if η was < 0)

$$= \rho \int_{-h}^{0} \overline{u^2}\, dz \qquad \text{(overbar} \Rightarrow \frac{1}{\lambda}\int_0^{\lambda} dx \text{ as in (3.42))} \qquad (3.92)$$

This is essentially the Reynolds stress contribution and is always positive, since a negative flow ($-u$) of momentum ($-u.\rho$) is equivalent to a positive flow (u) of momentum (ρu).

(b)

$$\frac{1}{\lambda}\int_0^{\lambda}\int_{-h}^{0} (p - p_0)\, dz\, dx \simeq \frac{1}{\lambda}\int_0^{\lambda}\int_{-h}^{0} (p - p_0)\, dz\, dx \quad \text{to } O(a^2)$$

$$= \int_{-h}^{0} (\bar{p} - p_0)\, dz \qquad (3.93)$$

If the mean water level does not change, then $\bar{p}$ can be determined from the vertical balance of momentum flux (the water column must be supported)

$$\overline{p + \rho w^2} = -\rho g z = p_0$$

$$\Rightarrow \quad \bar{p} - p_0 = -\rho\overline{w^2} \qquad (3.94)$$

The mean pressure $\bar{p}$ is therefore less than the hydrostatic pressure p_0 and (3.94) in (3.93) implies

$$\int_{-h}^{0} (\bar{p} - p_0)\, dz = \int_{-h}^{0} (-\rho\overline{w^2})\, dz, \qquad (3.95)$$

and the contribution (b) is negative.

The terms (a) and (b) can be combined, using (3.92); (3.95) to give:

$$(a) + (b) = \rho \int_{-h}^{0} \overline{(u^2 - w^2)}\, dz \qquad (3.96)$$

This is ≥ 0 since $|u| \geq |w|$ for the particle velocities obtained from $\underset{\sim}{u} = \nabla\phi$. In fact, use of ϕ (3.65) leads to:

$$(a) + (b) = \frac{\rho g a^2 kh}{\sinh 2kh} \tag{3.97}$$

In deep water $|u| = |w|$ and (a) + (b) = 0, but in shallow water $u \gg w$ so the expression (3.96) becomes $\rho h \overline{u^2}$. The mean kinetic energy is given by $\frac{1}{2}\rho\overline{u^2}h$ in this situation , so the contribution (a) + (b) is equal to $2T$; ie. equal to E .

(c) $\quad \frac{\rho}{\lambda}\int_0^\lambda \int_0^\eta \frac{p}{\rho}\, dz\, dx \quad$ must be evaluated using the Airy expression for the pressure distribution p , which is obtained from the Bernoulli equation with the use of the Airy potential ϕ

$$p = -\rho g z + \frac{\rho \sigma^2 a \cosh k(z+d) \sin\chi}{k \sinh kd} \tag{3.98}$$

For $-a \leq z \leq a$, the pressure is essentially hydrostatic and so fluctuates in phase with the surface elevation η .

$$p = \rho g(\eta - z) \tag{3.99}$$

$$\therefore \int_0^\eta p\, dz = \tfrac{1}{2}\rho g \eta^2 ;$$

$$\therefore \quad (c) = \tfrac{1}{2}\rho g \overline{\eta^2}. \tag{3.100}$$

This term is positive and equal to the potential energy V (3.84) since

$$\overline{\eta^2} = \frac{1}{2}a^2$$

correct to $O(a^2)$

The radiation stress component S_{11} can now be written in terms of the mean energy $E = \frac{1}{2}\rho g a^2$ (3.86).

$$S_{11} = (a) + (b) + (c)$$

$$= \frac{\rho g a^2 \cdot kh}{\sinh kh} + \frac{1}{4}\rho g a^2 \quad \text{correct to } O(a^2)$$

$$= E\left[\frac{2kh}{\sinh 2kh} + \frac{1}{2}\right] \tag{3.101}$$

Since the radiation stress contains terms of the form $\overline{\rho u^2}$ it is not surprising to see the energy density E appearing in (3.101). The term in brackets can be expressed in terms of the phase and group velocities as follows:

(1) find an expression for $\frac{c_g}{c}$; ie. $\frac{\partial\sigma}{\partial k}\cdot\frac{k}{\sigma}$:

$$\sigma^2 = gk \tanh kh \tag{3.66}$$

$$\therefore 2\sigma\frac{\partial\sigma}{\partial k} = \frac{\partial}{\partial k}\left(gk\tanh kh\right)$$

$$\therefore \frac{\partial\sigma}{\partial k}\cdot\frac{k}{\sigma} = \frac{gk}{2\sigma^2}\cdot\left(\tanh kh + \frac{kh}{\cosh^2 kh}\right)$$

$$\therefore \frac{c_g}{c} = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{3.102}$$

This shows that $c_g = \frac{1}{2}c$ in deep water and $c_g = c$ in shallow water.

(2) use of (3.102) in (3.101) now leads to

$$S_{11} = E\left[\frac{2c_g}{c} - \frac{1}{2}\right] \tag{3.103}$$

The value of S_{11} in deep water $= E/2$ and in shallow water $= \frac{3E}{2}$; as is easily seen from (3.103). These results for waves of infinitesmal height can be compared with the general expression for S_{11} quoted in (3.51) valid for waves of all heights:

$$S_{11} = 4T - 3V + h\,\overline{u^2(x_1,-h)} \tag{3.51}$$

where the deep water case implies $u(x_1,-h) \to 0$ and for small amplitude, $T = V = E/2$, so (3.103) is recovered from (3.51) as a special case.

It does not seem possible to provide intuitive physical reasoning for the form of the term in square brackets in (3.103) apart from analysing the terms (a), (b) and (c) used in the derivation.

- The radiation stress component S_{22} represents the flow of momentum <u>parallel</u> to the crests (at right angles to wave direction) across a plane y = constant. Similar analysis to that given above leads to an expression for S_{22} in terms of Airy theory:

$$S_{22} = \frac{f}{\lambda}\int_0^{\lambda}\left[\int_{-h}^{0}(p-p_0)\,dz + \int_0^{\eta} p\,dz\right]dx \tag{3.104}$$

$$= E\left[\frac{kh}{\sinh kh}\right] = E\left[\frac{c_g}{c} - \frac{1}{2}\right] \tag{3.105}$$

Since there is no motion in the y direction, Reynolds stress type terms do not appear in a term corresponding to (a) in the analysis for S_{11}. S_{22} varies between 0 (deep water) and $\frac{E}{2}$ (shallow water).

The other components S_{12}, S_{21} vanish if the wave propagation is in the x direction because the pressure p is isotropic and has no shear component and also because the velocity terms are of the form $\rho\overline{uv}$, and $v = 0$.

The diagonal form of the radiation stress tensor S for Airy theory is therefore given by:

$$S = E \begin{bmatrix} \frac{2kh}{\sinh 2kh} + \frac{1}{2} & 0 \\ 0 & \frac{kh}{\sinh 2kh} \end{bmatrix} \qquad (3.106)$$

Tensor transformation laws can be used to adjust the form of S in the case of waves travelling in other directions in the (x, y) plane.

The above derivation of radiation stress completes the $O(a^2)$ wave properties to be used in an examination of the interaction of small amplitude waves with currents in 5.2, 5.3, 6.2, and 6.3. The derivation of these properties also clarified the concepts introduced in earlier sections and completes the general analysis of periodic surface gravity waves. The interaction of such waves with slowly varying currents is now described in the following chapters.

4. INTRODUCTION TO WAVE/CURRENT INTERACTIONS AND THE ANALYSIS OF WAVE/CURRENT KINEMATICS

4.1. INTRODUCTION TO INTERACTIONS BETWEEN PERIODIC GRAVITY WAVES AND NON-UNIFORM CURRENTS

Surface gravity waves interact with a wide variety of currents as they move from a region of wave generation through to final dissipation. Such currents can cause significant modification of the wave properties (and will in turn be modified by the waves if the waves are of finite amplitude). The interaction of water waves with currents is the subject of a major review by Peregrine (1976) and this is a general reference for the rest of the thesis.

Crucial to the analysis of the interactions is the length and time scale of the current variations.

Few currents have been studied whose length, depth or time variation is rapid on the scale of a wavelength; examples of such currents are surface drift currents generated by wind stress (Peregrine 1976, p11; Banner & Phillips 1974) and surface shear waves (Peregrine 1974) which are the small stationary waves generated on beaches in the backwash of breaking waves. Most studies of non-uniform currents have concentrated on steady large scale currents ie. currents for which the change in current velocity over one wavelength is much smaller than the wave velocity (Longuet-Higgins and Stewart (1961); Peregrine (1976, p17)).

$$k \gg \max\left|\frac{1}{\mathcal{U}}\cdot\frac{\partial \mathcal{U}}{\partial x}\right| \qquad \omega \gg \max\left|\frac{1}{\mathcal{U}}\cdot\frac{\partial \mathcal{U}}{\partial t}\right| \tag{4.1}$$

where $\mathcal{U}$ is the current velocity.

The reason for imposing this restriction is that locally the waves can be treated as if they were on a uniform current. Wave parameters such as ω, k will be slowly varying in the sense discussed in 2.1.3, (equations (2.44, 2.45). The interaction equations can therefore be used witn solutions for periodic waves on uniform currents to obtain the required solutions for non-uniform currents.

Since the current variations are on such a large scale, it proves convenient to think in terms of a wavetrain or wave "packet" propagating on the current. The wavetrain is a succession of a large number of waves (identical except for the few leading and trailing waves). The wavetrain is still much shorter than the scale of current variation.

This restriction to large scale currents is reasonable for many gravity wave/current interactions and is therefore assumed for this thesis. For example, ocean currents (such as the Agulhas) fulfill the criteria as in some cases do waves propagating up rivers. The restriction unfortunately limits the allowed beach slope in the analysis of wave shoaling, but does not preclude its investigation altogether. Finally, it is also possible to study the interaction of short waves riding on much longer waves (such as tides) by regarding the longer waves as a variable current (Longuet-Higgins and Stewart, 1960, Peregrine, 1976, section 2F).

Analysis of wave/current interactions can be split into a study of the kinematics (discussed here) and dynamics (see chapters 5 & 6). The kinematics is basically a study of the effects of a shift on the phase (and group) velocity of the wavetrain. This velocity shift has two major implications:

(i) a change in the phase velocity causes changes in the wavenumber and therefore, because of the dispersion relation (2.26), the intrinsic frequency is also altered.

$$\sigma = \sigma(k) \tag{2.26}$$

(ii) some wave properties propagate at the phase velocity (eg. wave profile) and others at the group velocity (eg. wavenumber; energy). This causes three different interactions for wavetrains propagating on opposing currents. These depend on the relative speeds of the current compared to the phase and group velocities of the waves.

(a) $0 \le |U| \le \frac{\partial\sigma}{\partial k}$ (b) $\frac{\partial\sigma}{\partial k} \le |U| \le \frac{\sigma}{k}$ (c) $|U| > \frac{\sigma}{k}$

A derivation of the governing kinematic equation and discussion of (i) and (ii) is given in 4.2.

4.2. KINEMATICS OF WAVETRAINS ON TWO DIMENSIONAL CURRENTS

4.2.1. THE RAY DESCRIPTION OF WAVES ON CURRENTS

The kinematics of waves on currents is based on the wave kinematics on still water introduced in 2.1. Some essential definitions are first recalled from 2.1, then modified to incorporate the current effects.

The definitions of local frequency and wavenumber, in terms of the phase function are:

$$\sigma(x,t) \equiv -\frac{\partial \chi}{\partial t} \qquad k(x,t) \equiv \frac{\partial \chi}{\partial x} \qquad (2.44,45)$$

The wavetrain propagates at the local group velocity c_g, as defined in (2.29):

$$c_{g\alpha} \equiv \frac{\partial W(k)}{\partial k_\alpha} \qquad (2.29)$$

$$\left(W = \sigma(\underset{\sim}{x},t)\right)$$

Also $$c_{g\alpha} \equiv \frac{dx_\alpha}{dt}$$

The path of an observer moving with the local group velocity follows a path in space time called a ray (recall discussion following (2.29)) and differentiation in this reference frame is given by (4.2). (See discussion preceding (2.13), with c_g replacing V_g).

$$\frac{d}{dt} = \frac{\partial}{\partial t} + c_{g\alpha} \cdot \frac{\partial}{\partial x_\alpha} \qquad (4.2)$$

The propagation of σ, k in this reference frame is therefore

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial t} + c_{g\alpha} \cdot \frac{\partial\sigma}{\partial x_\alpha} = 0 \tag{4.3}$$

$$\frac{dk_\alpha}{dt} = \frac{\partial k_\alpha}{\partial t} + c_{g\beta} \cdot \frac{\partial k_\alpha}{\partial x_\beta} = 0 \tag{4.4}$$

For a steady state situation in a homogeneous medium this implies that frequency and wavenumber are constant along a ray, and so by (2.41) the $\underset{\sim}{c_g}$ is constant and the ray is straight (recall the discussion preceding (2.41) which compared wave properties seen by an observer moving at the local phase velocity to those seen when travelling at the local group velocity).

Now assume a steady horizontal uniform current $\underset{\sim}{U}$ and consider differentiation with the absolute group velocity ($\underset{\sim}{U} + \underset{\sim}{c_g}$). The corresponding results (with characteristics given by: $\frac{d\underset{\sim}{x}}{dt} = \underset{\sim}{U} + \underset{\sim}{c_g}$)

are simply:

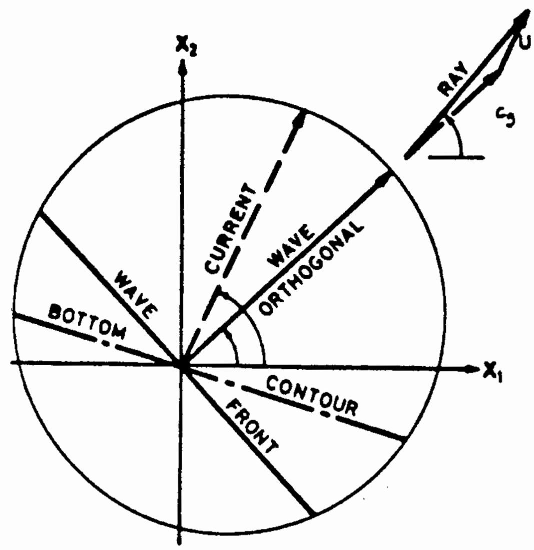
$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\underset{\sim}{U} + \underset{\sim}{c_g})\frac{\partial\sigma}{\partial t} \tag{4.5}$$

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial t} + (U_\alpha + c_{g\alpha})\frac{\partial\sigma}{\partial x_\alpha} = 0 \tag{4.6}$$

$$\frac{dk_\alpha}{dt} = \frac{\partial k_\alpha}{\partial t} + (U_\alpha + c_{g\alpha})\frac{\partial k_\alpha}{\partial t} = 0 \tag{4.7}$$

and the rays remain straight but their directions are altered.

If the current $U(\underset{\sim}{x},t)$ is slowly varying in the sense defined previously (4.1) then the wave frequency and wavenumber will change along rays and the rays will in general be curved. In addition, the rays may be at an angle to the wave orthogonal. (Jonsson, 1978, Fig1).



These features are often displayed in the water flow produced by waves breaking on a beach. The return flow down the beach after a large breaker can be regarded as a uniform current sheet. It will intersect small incoming waves (the tiny ones; amplitude 10cm.), usually at an angle $\neq 180°$. One can clearly see the wavefronts of the small waves being swept along at an angle to the apparent direction of propagation of the wavecrests ie. they move along the ray not the orthogonal in the above figure. These features can also be seen at a river mouth as well as more complex movements which arise from the current varying on the same scale as the wavelength of the incoming waves.

The kinematic changes in the wavetrain (ie. wavenumber, frequency) in the presence of the current are essentially the result of a Doppler shift in the phase velocity. The absolute phase velocity is the velocity relative to a fixed observer and is related to the intrinsic phase velocity σ/k (2.23) by:

$$c_{abs} = \frac{\sigma}{k} + \underset{\sim}{U} \cdot \hat{k} \qquad (4.7)$$

where $\hat{k}$ is a unit vector normal to the wave crest; $k = |\underset{\sim}{k}|$

This relation is more commonly written in terms of ω, the absolute or observed frequency (relative to a fixed observer) (Phillips 1980, p24; Whitham (1960))

$$\omega = \sigma + U_\alpha k_\alpha \qquad (4.8)$$

An important feature of this relation is that the changes in $U(\underset{\sim}{x}, t)$ cause changes in $\underset{\sim}{k}$ and therefore in the intrinsic frequency σ, since σ is related to $\underset{\sim}{k}$ by the dispersion relation. There is no change in ω for a steady situation, as can be shown by substituting (4.8) into the conservation equation for the waves (2.25):

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

"ω" in (2.13) must be replaced by "ω" not "σ" in (4.8) since both refer to the observed frequency in a fixed reference frame

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial}{\partial x_\alpha}\left[\sigma(k_{x_i}, x_i, t) + U_\beta k_\beta\right] = 0 \qquad (4.9)$$

If the flow is steady:

$$\frac{\partial k_\alpha}{\partial t} = 0 \Rightarrow \omega = \text{const.} = \sigma_0(\underset{\sim}{k_0}, \underset{\sim}{x}, \tau) \qquad (4.10)$$

ie. the value of ω is the intrinsic frequency of the wavetrain in still water. (Note that $\underset{\sim}{k_0}$ will not be the value of $\underset{\sim}{k}$ in the presence of the current).

4.2.2. Qualitative features of wavetrain kinematics on two dimensional currents in the (x,z) plane

The essential features of the kinematic relation (4.8) are investigated for the two dimensional case ($\alpha = 1$; x,z plane); the current moving either in the same direction as the waves or directly opposing them. The local relationship of $\omega, \sigma, \underset{\sim}{k}, \underset{\sim}{U}$ is investigated for various values of $\underset{\sim}{U}$. The current may vary slowly in the x direction only, and $\underset{\sim}{U}$ is positive if the current is in the direction of the waves. This discussion is extended in 4.2.3 by solving (graphically) the kinematics of Airy waves on two dimensional currents.

$$\alpha = 1 \Rightarrow \omega = \sigma \pm kU \qquad (4.11)$$

CASE 0: $U_0 = 0$

$$U_0 = 0 \Rightarrow \omega = \sigma_0 + k_0 \cdot 0 \qquad (4.12)$$

The absolute frequency is equal to the intrinsic frequency σ_0 of the wavetrain (wavenumber k_0).

CASE 1: $U_1 > 0$

$$\omega \, (= \sigma_0) = \sigma_1 + k_1 U_1 \qquad (4.13)$$

The intrinsic frequency must decrease since ω is constant and $k_1 U_1 > 0$. Since σ is $\sigma(k)$, there is a change in k from k_0 to k_1. For example, the Airy wave dispersion relation (3.66) is:

$$\sigma = \sqrt{gk \tanh kh} \quad \text{in water depth } h \qquad (4.14)$$

and so the change of wavelength when $U_1 > 0$ would be the solution k_1 of:

$$\omega \left(= \sigma_0 = \sqrt{gk_0 \tanh k_0 h} \right) = \sqrt{gk_1 \tanh k_1 h} + k_1 U_1 \qquad (4.15)$$

Clearly the reduction in σ from σ_0 to σ_1 is accompanied by a reduction in k from k_0 to k_1. As the dispersion relation for finite amplitude waves is not very different from (4.14), it is a general result that

$$U_1 > 0 \Rightarrow \sigma_1 < \omega \Rightarrow k_1 < k_0$$

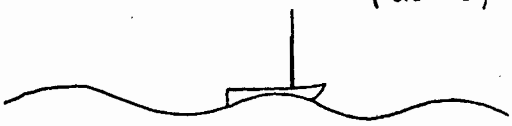
Hence the waves lengthen $(k_1 = 2\pi/\lambda_1)$ and an observer moving with the current would see the period increase $(\sigma_1 = 2\pi/\tau_1)$.

Figure 4-2.

Case 0:

observer moving with current sees τ_0, λ_0.

$(u_0 = 0)$



fixed observer sees: τ_0, λ_0

Case 1:

moving at u_1 sees $\tau_1 > \tau_0$; $\lambda_1 > \lambda_0$.

u_1

fixed observer sees: τ_0 ; $\lambda_1 > \lambda_0$

CASE 2: $u_2 < 0$

The opposition of the current to the wavetrain produces additional effects not present when the current follows the waves. This is because the current velocity is now subtracted from that of the wave and so the two velocities associated with the wave propagation (intrinsic phase velocity σ/k and intrinsic group velocity $\partial\sigma/\partial k$) divide the range of solutions into three:

2 (i) $0 < -u_3 < \frac{\partial\sigma_3}{\partial k_3}$

2 (ii) $\frac{\partial\sigma_4}{\partial k_4} \leq -u_4 < \frac{\sigma_4}{k_4}$

2 (iii) $\frac{\sigma_5}{k_5} \leq -u_5$

CASE 2 (i):

$$\omega = \sigma_3 - k_3 U_3 \tag{4.16}$$

$$\left(\omega > 0 \quad \text{as} \quad k_3 U_3 < k_3 \frac{\partial \sigma_3}{\partial k_3} < k_3 \frac{\sigma_3}{k_3} = \sigma_3 \right)$$

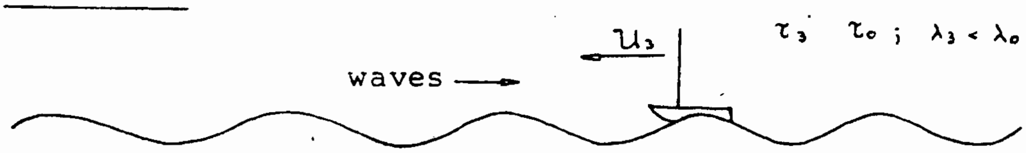
Because $k_3 U_3 > 0$, the change in σ must be $\sigma_3 > \sigma_o$, ie. the intrinsic period decreases. Use of the dispersion relation (4.10) shows that the associated wavenumber change is:

$$k_3 > k_o$$

The waves are shortened and for an observer drifting with the current, the frequency increases.

Figure 4-3.

Case 2 (i) moving at U_3; sees τ_3 τ_o ; $\lambda_3 < \lambda_o$.



fixed observer sees τ_o, λ_3 ;
wave crests still move to the
right as $\omega > 0$

CASE 2 (ii) $\frac{\partial \sigma_4}{\partial k_4} \leq -U_4 < \frac{\sigma_4}{k_4}$

The group velocity $\frac{\partial \sigma}{\partial k}$ does not appear explicitly in the relation (4.8) and so case 2 (i) is as for case 2 (ii) ie.

$$\omega > 0 \quad ; \quad \sigma_4 > \sigma_o \quad ; \quad k_4 > k_o$$

The group velocity has been shown to be the propagation velocity for the wavenumber. It is also the propagation velocity for the energy (Whitham 1974, section 11.6-11.8; Lighthill 1978, section 3.8. See also the discussion preceding (5.23) and section 8.1).

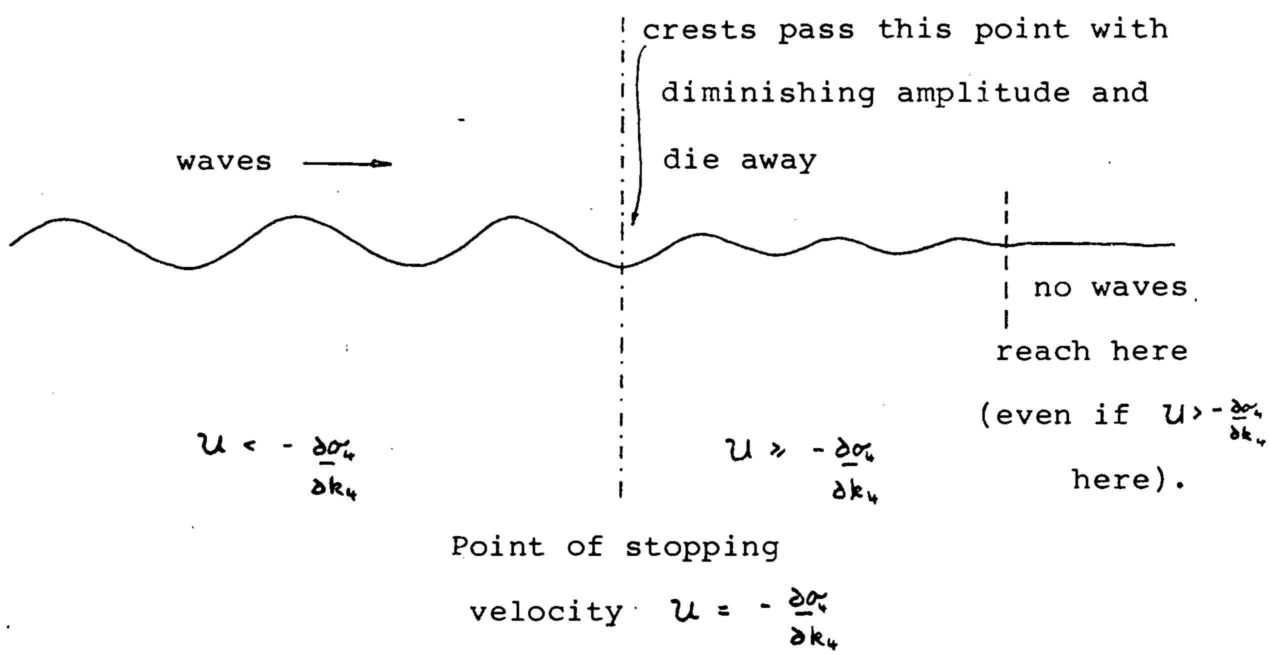
If the opposing current exceeds this speed but is $< \frac{\sigma}{k}$, wave crests will still propagate against it but will no longer be supplied with energy, so will dissipate away .

A current velocity equal and opposite to the local group velocity is known as a stopping velocity.

$$u = - \frac{\partial \sigma}{\partial k} \tag{4.17}$$

This velocity depends on the wavelength of the waves and this is exploited in the design of hydraulic breakwaters. (Evans, 1955 Taylor 1955). Problems arise with the relation between the stopping velocity and the group velocity for steep (nonlinear) waves; see 8.1 and Peregrine and Thomas (1979).

Figure 4-4.



CASE 2 (iii): $\frac{\sigma_s}{k_s} \leq -U_s$

$$\omega = \sigma_s - k_s U_s \tag{4.18}$$

The wave crests are now either stationary or swept downstream by the current. The intrinsic frequency and wavelength will be

$$\sigma_s >> \sigma_0; \quad k_s >> k_0$$

For gravity waves, no aspect of the wave can propagate against the stream. For capillary waves, (defined in section 1.1) the group velocity is higher than the phase velocity and the capillary waves can transmit energy upstream of an obstacle in a stream. (Lighthill, 1978, section 3.9)

Figure 4-5.

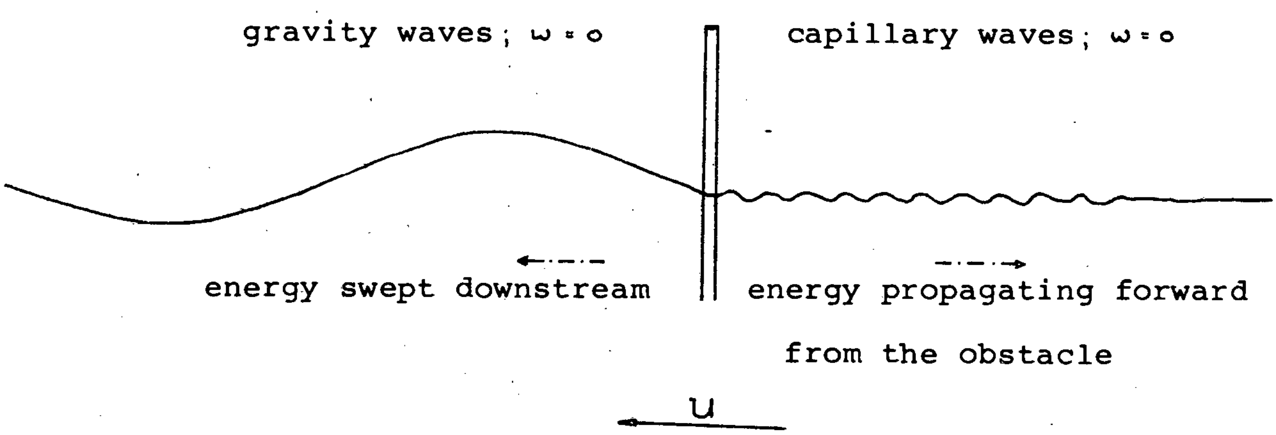


Figure 4-6.

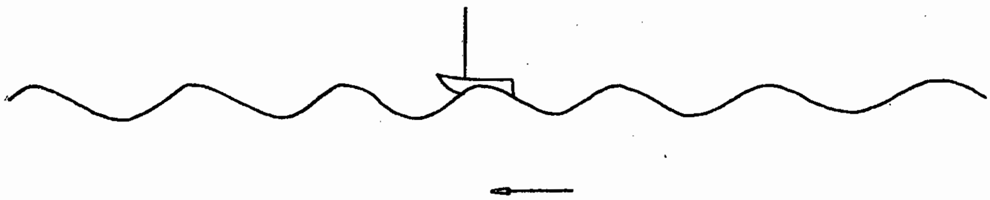
Case 2 (iii)

observer sees $\tau_s < \tau_o$; $\lambda_s < \lambda_o$

but waves still appear to move

to the right.

$-\mathcal{U} > \frac{\sigma}{k}$



for fixed observer $\omega < 0$; waves swept back to the left.

4.2.3. Airy wave kinematics on a 2 dimensional current

It is possible to give a graphical solution for Airy waves on currents that demonstrates succinctly the qualitative cases studied in 4.2.2. The solution given here is an extension of those given by Jonsson et al (1970) and by Peregrine (1976).

The waves are assumed to propagate over water of constant depth h. The current $\mathcal{U}$ is depth dependent and is slowly varying in the direction of wave propagation so that locally it can be considered constant. The influence of the current on the intrinsic frequency is given by the kinematic relation

$$\sigma = \omega - k\mathcal{U} \tag{4.19}$$

To find a solution for σ, k with a given absolute frequency ω and a given current $\mathcal{U}$, the dispersion relation $\sigma(k)$ (2.26) must be introduced into (4.19), which then becomes an equation for k.

$$\sigma(k) = \omega - k\mathcal{U} \qquad (4.20)$$

This equation must be completed by making a particular choice for the dispersion relation and the choice made here is the Airy relation:

$$\sigma(k) = \pm\sqrt{gk \tanh kh} \qquad (3.66)$$

This is a convenient choice for this illustration as the Airy dispersion relation does not involve the wave amplitude, so the wave/current kinematics are completely detatched from the dynamics (discussed in the following chapters).

The graphical solution is obtained by plotting the curve of (3.66) and the straight line (4.20) on a wavenumber/frequency graph for given values of $\omega, h, \mathcal{U}$. The intersection of the two represents a solution for σ, k ie.

$$\omega - k\mathcal{U} = \sigma(k) = \pm\sqrt{gk \tanh kh} \qquad (4.21)$$

For convenience, only one absolute frequency $\omega_\bullet$ will be used, plus three current values $\mathcal{U}_a, \mathcal{U}_b, \mathcal{U}_c$. This will be sufficient to illustrate all the cases discussed in 4.2.2, plus the stopping velocity (4.17). The graphical solution is first introduced in a simplified form, illustrating case 0 and case 1 for zero current and a following current respectively.

Case 0: $\mathcal{U}_a = 0$

The current strength $\mathcal{U}_a$ is represented by the slope of the line $\omega_0 - k\mathcal{U}_a$. For $\mathcal{U}_a = 0$, the line is horizontal and the solution (σ_0, k_0) (point Q on the graph below) is simply the wave solution in still water:

$$\omega_0 = \sigma_0 + k_0 . 0 \Rightarrow \omega_0 = \sigma_0 \qquad (4.22)$$

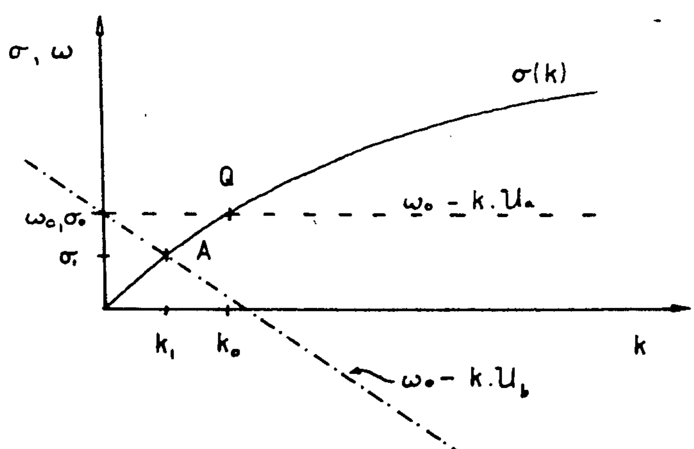
Case 1: $\mathcal{U}_b > 0$

The line $\omega_0 - k.\mathcal{U}_a$ now has negative slope and intersects the $\sigma(k)$ curve at (σ_1, k_1) , so the kinematic relation is

$$\omega_0 = \sigma_1 - k_1 \mathcal{U}_a \qquad (4.23)$$

This is represented by point A on the graph. The influence of the current is clearly to decrease the intrinsic wave frequency $(\sigma_1 < \sigma_0)$ and to lengthen the waves $(k_1 < k_0)$.

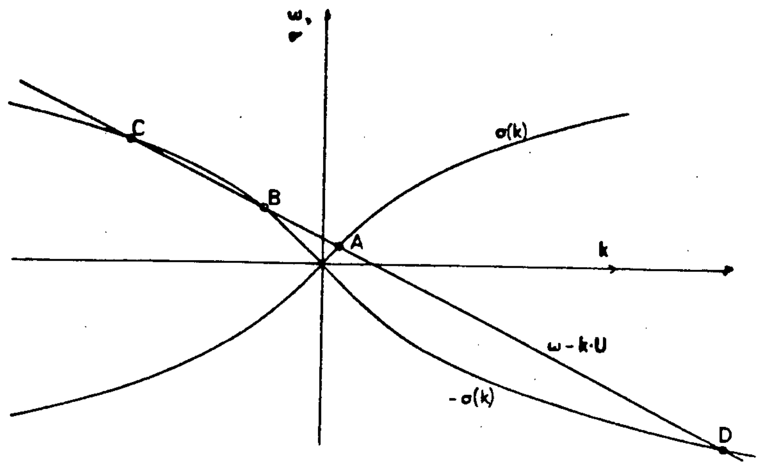
Figure 4-7.



The remaining cases 2 (i),(ii),(iii) require the wave direction to be <u>reversed</u> but the same ω_0 , $\mathcal{U}_a$ are retained. For the stopping velocity (4.17), a stronger current ($\mathcal{U}_c > \mathcal{U}_b$) is required.

Figure 4-8.

(Peregrine 1976)



Solution of the dispersion relation showing multiple values of **k** for given *ω, h,* and **U**.

Case 2(i): $0 < U_b < \frac{\partial \sigma_2}{\partial k_2}$

Waves in this quadrant of the graph $(\sigma < 0,\ k < 0)$ are moving in the opposite direction to the current. The solution point B shows that the local group velocity $\left(c_{g_2} = \frac{\partial \sigma}{\partial k_2}\right)$ (ie. the slope of the $-\sigma(k)$ curve at B) is greater than U_b (slope of the $\omega_0 - kU_b$ line). The opposing current causes an increase in intrinsic frequency $|\sigma_2| > \sigma_0$ and a decrease in wavelength $|k_2| > k_0$.

Case 2(ii): $\frac{\partial \sigma_3}{\partial k_3} < U_b < \frac{\sigma_3}{k_3}$

Here (point C) the relative slopes are different to those of Case 2 (i) and represent waves whose <u>crests</u> (moving at $c = \sigma_3/k_3$) still propagate upstream but whose <u>energy</u> is swept downstream, as $c_{g_3} < -U_b$. Such waves must be generated on the current, and this solution does not occur for non-dispersive waves.

Case 2(iii): $\mathcal{U}_b > -\frac{\sigma_4}{k_4}$

For this case (point D), even the crests are swept downstream and so the solution appears in the fourth quadrant (phase speed now negative, wavenumber positive). The waves in this situation have been considerably shortened $|k_4| \gg k_o$ and raised in frequency $|\sigma_4| \gg \sigma_o$.

Finally, the stopping velocity for the waves (4.17) is represented by a coalescing of the points B and C when the waves move on to a current of the correct strength - which means graphically that the curve slope and the line slope coincide:

$$\mathcal{U}_c = \frac{\partial\sigma}{\partial k} \tag{4.17}$$

Difficulties with this relationship for steep waves are discussed in 8.1 where it is seen that for steep waves the various group velocity definitions differ in magnitude, making the interpretation of the stopping velocity difficult. If the current is stronger than $\mathcal{U}_c$, the solutions B, C will not occur for waves of frequency ω_o.

This completes the present discussion of the wave/current kinematics. The topic reappears in chapter 5 as part of the analytic solution for Airy waves on currents varying along the flow, and the governing kinematic equations (especially the Doppler shifted dispersion relation (4.19)) reappear in the finite amplitude wave/current equations derived in chapter 7.

5. THE ENERGY/RADIATION STRESS APPROACH TO WAVE/CURRENT INTERACTION FOR SMALL AMPLITUDE WAVES

5.1. GENERAL INTRODUCTION TO WAVE/CURRENT DYNAMICS

The dynamics of the interaction of large scale currents and gravity waves has received considerable attention since the influential work of Longuet-Higgins and Stewart (1960, 1961, 1964) in which interaction between the waves and the current was described using a new concept, that of "radiation stress" (see 3.40, 3.91). Despite the success of this approach in wave shoaling, wave/current and wave/wave interactions for small amplitude wave theories, it has been superseded by the concept of "wave action", which arose from an investigation of wavetrains in inhomogeneous moving media by Bretherton and Garrett in 1969. They showed the equivalence of their wave action formulation to the energy/radiation stress approach of Longuet-Higgins and Stewart. Earlier work by Whitham (1965(a),1965(b), 1967, 1970) on variational methods introduced the definition and use of "averaged Lagrangians". The variational approach using wave action has subsequently been confirmed in its superiority over the energy approach involving radiation stress.

The energy and wave action solutions for small amplitude (linear) wavetrains are derived in chapters 5 and 6 respectively This lays the groundwork for the application of these approaches to finite amplitude waves in chapter 7. Attention is focussed in chapter 7 on the work of Stiassnie and Peregrine (1979), who show the equivalence of finite amplitude energy and wave action approaches with the use of wave integral properties. They then use the wave action equations to find solutions for waves shoaling over gently sloping beaches using Cokelet's wave theory (1977). Further support for the wave action work comes from the very general results of Andrews and McIntyre (1978 a,b) for wave/mean flow interactions which they obtained using their "Generalised Lagrangian Mean" formulation (3.36). This is discussed further in 8.3.

5.1.1. Advantages of the energy approach and an outline of chapter 5

The direct nature of the energy/radiation stress approach makes it valuable in introducing the dynamics of wave/current interaction, although the wave action approach has since proved to be superior. The energy approach is therefore discussed in detail, concentrating on the derivation and explanation given by Whitham (1962), and highlighting the features of the earlier work by Longuet-Higgins and Stewart where appropriate. The chapter splits conveniently into the following sections:

5.2 The conservation equations for mass, momentum and energy of the fluid are expressed in the form of (3.87) (Whitham, 1962):

$$\frac{\partial P_i}{\partial t} + \frac{\partial Q_i}{\partial x} = 0 \qquad (i = 0, 1, 2) \qquad (3.87)$$

where P_i ; Q_i ; are the density and flux of mass ($i = 0$), momentum ($i = 1$) and energy ($i = 2$) respectively. The mean value of each equation is calculated for Airy waves on a uniform current. The generalised equations for non-uniform flows are then derived.

5.3 The equations of 5.2 for non-uniform flows are otained by following the original derivations (perturbation analysis; averaged equations of motion) given by Longuet-Higgins and Stewart. The solutions correct to O(a^2) for Airy waves on non-uniform currents are found and difficulties in interpreting the wave energy equation are noted.

5.4 Whitham's analysis (given in 5.2) is used to explain the form of the wave energy equation derived by Longuet-Higgins and Stewart. This leads to a new interpretation of the radiation stress term and clarifies the form of the conservation equations. These equations are then used to investigate the propagation of various fluid properties.

5.2. CONSERVATION FORM OF THE MASS, MOMENTUM AND ENERGY EQUATIONS FOR THE FLUID; APPLICATION TO $O(a^2)$ WAVES ON CURRENTS

Interaction of waves and relatively large scale flows were studied by Longuet-Higgins and Stewart (1960, 1961) using perturbation techniques. The presence of $O(a^2)$ interaction terms in the solutions led to the definition of radiation stress (3.40, 3.88) and the derivation of an energy flux equation incorporating a radiation stress interaction term for waves on a uniform mean flow. The perturbation analysis also led to the correct form of the energy flux equation for non-uniform mean flows, but the form of the interaction term proved difficult to interpret physically.

Whitham (1962) investigated the mass, momentum and energy flux for two dimensional flows and showed that it is possible to express the mean values of the mass, momentum and energy conservation equations as follows (recall 3.87)

$$\frac{\partial \overline{P}_0}{\partial t} + \frac{\partial \overline{Q}_0}{\partial x} = 0 \qquad (5.1)$$

$$\frac{\partial \overline{P}_1}{\partial t} + \frac{\partial \overline{Q}_1}{\partial x} = 0 \qquad (5.2)$$

$$\frac{\partial \overline{P}_2}{\partial t} + \frac{\partial \overline{Q}_2}{\partial x} = 0 \qquad (5.3)$$

where P_0, P_1, P_2 are the mean (averaged over χ) densities of mass, momentum and energy for the <u>whole flow</u> (eg. wave plus current)

Q_0, Q_1, Q_2 are the mean (averaged over χ) fluxes of mass, momentum and energy for the whole flow

This analysis enables the results found by Longuet-Higgins and Stewart for small amplitude waves on currents to be easily obtained. In addition, the form of the equations is clarified. Whitham's approach is now described with references to the Longuet-Higgins and Stewart solution (discussed further in (5.3) where appropriate.

5.2.1. Derivation of the conservation equations for small amplitude waves on currents

The current is assumed two dimensional (x, z plane) with the waves propagating along the x axis. Since the whole flow is two dimensional and is assumed irrotational, a velocity potential $\Phi(x,z,t)$ can be defined. Choice of a particular wave theory and current profile means that the velocity potential Φ for the whole flow can be expressed in terms of velocity potentials for the wave theory and for the current. Once Φ has been found, it can be used in the conservations for mass, momentum and energy.

Here, Φ is derived for a Stokes wave (as derived in 3.3) interacting with a current which will (later) vary along the direction of travel. All terms are evaluated correct to $O(a^2)$. In 5.2.2, the conservation equations are solved. This approach follows Whitham (1962).

The velocity potential Φ satisfies Laplace's equation (3.12) and Bernoulli's equation (3.13), with the familiar boundary conditions (3.16, 3.19, 3.20). The fluid depth must now be denoted by d, as the original depth h will in general be modified by the current or the waves.

$$\nabla^2 \Phi = 0 \qquad (3.12)$$

$$\frac{p}{\rho} + \frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + gz = f(t) \qquad (3.13)$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x}(x,\eta,t)\cdot\frac{\partial \eta}{\partial x} - \frac{\partial \Phi}{\partial z}(x,\eta,t) = 0 \qquad (3.16)$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + gz = 0 \qquad (3.20)$$

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -d \qquad (3.19)$$

The general solution for the velocity potential for the whole flow will be

$$\bar{\Phi}(x,z,t) = \Phi(\chi,z) + \beta x - \gamma t \qquad (3.29;\ 3.55)$$

The solution is to be valid to $O(a^2)$ so the Stokes solution is used initially although the higher order terms finally required are correctly given by Airy theory. Recall the form of the Stokes velocity potential (3.56; 3.75):

$$\phi_s(\chi,z) = -\gamma_s t + \Phi_s(\chi,z) \qquad (3.56)$$

$$\phi_s(\chi, z) = \frac{-gka^2 t}{2\sinh 2kd} + \frac{\sigma_0 a \cosh k(z+d) \sin\chi}{k \sinh kh} + \frac{3}{8}\sigma_0 a^2 \frac{\cosh 2k(z+d) \sin 2\chi}{\sinh^4 kd} + O(a^3k^3) \quad (3.75)$$

Hence the general form can be re-expressed as:

$$\underset{\sim}{\Phi}(x, z, t) = \Phi_s(\chi, z) + \beta x - (\gamma_s + \gamma')t \quad (5.4)$$

The discussion following (3.30) noted that β represents a depth independent mean flow that could be associated with a periodic wave motion. Since the Stokes solution being used here has $\beta = 0$, the value of β can be used here to represent the uniform (with respect to x) depth independent current.

$$\beta \equiv \frac{\partial \underset{\sim}{\Phi}}{\partial x} = \bar{u} \text{ for the flow } \equiv \mathcal{U}_0$$

$$(5.5)$$

(The use of $\mathcal{U}_0$ not $\mathcal{U}$ departs from Whitham's notation but matches that of Longuet-Higgins and Stewart (1961) and helps maintain the distinction between this uniform stream and the non-uniform case denoted by $\mathcal{U}$). This non-zero value of β will influence γ' through the Bernoulli equation and the free surface dynamic boundary condition and so γ' will be a function of $\mathcal{U}_0$.

The solution (5.4) for $\underset{\sim}{\Phi}$ correct to $O(a^2)$ is found to be (Whitham, 1962):

$$\underset{\sim}{\Phi}(x, z, t) = \Phi_s(\chi, z) + \mathcal{U}_0 x - (\gamma_s + \gamma')t \quad (5.6)$$

$$\eta = a\cos\chi + a^2 k \coth kd \left(1 + \frac{3}{2\sinh^2 kd}\right)\cos 2\chi \quad (3.74)$$

$$\Phi_s = \frac{\sigma_0 a \cosh k(z+d) \sin\chi}{k \sinh kd} + \frac{3}{8}\sigma_0 a^2 \frac{\cosh(2k(z+d))\sin 2\chi}{\sinh^4 kd} \text{ from } (3.75)$$

$$\gamma_s = \frac{\frac{1}{2} gka^2}{\sinh 2kd} = \frac{\frac{1}{2}\rho g}{\rho}\frac{a^2}{\sinh 2kd} \qquad (3.71)$$

$$= \frac{E}{\rho}\left(\frac{c_g}{c} - \frac{1}{2}\right) = \frac{S_{22}}{\rho d}$$

see (3.103);(3.105) for similar manipulations

$$E = \tfrac{1}{2}\rho g a^2 \qquad (3.86)$$

$$\omega = \sigma_0 + \mathcal{U}.k \qquad (4.8)$$

$$\sigma_0^2 = gk \tanh kd \qquad (3.66)$$

$$c = \sigma_0 / k$$

$$c_g = \frac{\partial \sigma_0}{\partial k} = \frac{1}{2}c\left(1 + \frac{2kd}{\sinh 2kd}\right) \qquad (3.102)$$

Inserting the expression for $\tilde{\Phi}$ in (5.6) into Bernoulli's equation (3.13) gives:

$$\frac{p}{\rho} = \gamma_s - \frac{\partial \Phi_s}{\partial t} - \mathcal{U}_0 \frac{\partial \Phi_s}{\partial x} - \frac{1}{2}\left(\nabla \Phi_s\right)^2 - gz \qquad (5.7)$$

and the mean pressure (averaged over the phase χ) is given by:

$$\frac{\bar{p}}{\rho} = \gamma_s - \frac{1}{2}\overline{\left(\nabla \Phi_s\right)^2} - gz \qquad (5.8)$$

since Φ_s is periodic (5.4) and the mean of its derivatives vanish.

The above solution is used in the conservation equations for mass, momentum and energy, once these equations have first been expressed in terms of the velocity potential $\Phi(x,z,t)$ for the flow. The conservation equations are formulated in terms of the densities (P_0, P_1, P_2) and the fluxes (Q_0, Q_1, Q_2) of mass, momentum and energy respectively. It is instructive to recall the analogous formulation (2.13) derived for the kinematics of the waves which expresses the "conservation of phase" or "conservation of waves" in terms of a phase density (2.10) (or wavenumber) and a phase flux (2.11) (frequency)

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad (2.13)$$

where

$$k = \frac{\partial \chi}{\partial x} \qquad (2.10)$$

$$\omega = -\frac{\partial \chi}{\partial t} \qquad (2.11)$$

The general form for the dynamic conservation equations is therefore

$$\frac{\partial P_i}{\partial t} + \frac{\partial Q_i}{\partial x} = 0 \qquad \begin{array}{l} i=0 \quad (5.1) \\ i=1 \quad (5.2) \\ i=2 \quad (5.3) \end{array}$$

(The genesis of conservation equations in this form is discussed by Whitham (1974, p40) where it is stated that for a differential equation of the form

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$ (c the velocity of propagation of ρ)

(5.9)

there are an infinite number of conservation equations of the form

$$\frac{\partial f(\rho)}{\partial t} + \frac{\partial g(\rho)}{\partial x} = 0 \qquad (3.27)$$

as long as $$\frac{\partial g}{\partial \rho} = \frac{\partial f}{\partial \rho} \cdot c(\rho) . \qquad (5.11)$$

The particular forms of the conservation equations are now given in terms of depth averaged properties at a plane x = constant. The fluid velocities are obtained from $\underset{\sim}{u} = \nabla \Phi$.

Mass density: $$P_0 = \int_{-d}^{\eta} \rho \, dz \qquad (5.12)$$

flux $$Q_0 = \int_{-d}^{\eta} \rho \frac{\partial \Phi}{\partial x} dz \qquad (5.13)$$

Momentum density: $$P_1 = Q_0 = \int_{-d}^{\eta} \rho \frac{\partial \Phi}{\partial x} dz \qquad (5.14)$$

flux $$Q_1 = \int_{-d}^{\eta} \left[p + \rho \left(\frac{\partial \Phi}{\partial x} \right)^2 \right] dz \qquad (5.15)$$

Energy density: $$P_2 = \int_{-d}^{\eta} \left[\tfrac{1}{2} \rho \left(\nabla \Phi \right)^2 + \rho g z \right)\Big] dz \qquad (5.16)$$

flux $$Q_2 = \int_{-d}^{\eta} \left[p + \tfrac{1}{2} \rho \left(\nabla \Phi \right)^2 + \rho g z \right] \cdot \frac{\partial \Phi}{\partial x} dz \qquad (5.17)$$

where the zero level of potential energy has been chosen as $z = 0$.

These terms are now averaged over χ (ie. over a wavelength) to obtain the form of (5.1), (5.2) and (5.3):

$$\frac{\partial \overline{P_i}}{\partial t} + \frac{\partial \overline{Q_i}}{\partial x} = 0 \qquad (5.1,2,3)$$

where the average is defined (as before in (3.42)) as $\frac{1}{\lambda}\int_0^\lambda \ldots dx$.

This averaging process is familiar from the derivation of integral properties in 3.2 and their application in 3.3 to Stokes waves. Since the averaging process is being applied to the flow as a whole, the wave integral properties of 3.2 appear only in wave terms or wave/current interaction terms. Before evaluating the averaged quantities $\bar{P}_i$, $\bar{Q}_i$, note the following averaged properties of the Stokes solution, which will be used frequently:

(i) no mass flux below η_{min}:

$$\beta_s = 0 \qquad (3.56;3.75)$$

(ii) mean elevation is zero

$$\bar{\eta} = 0 \qquad (3.60;3.74)$$

(iii) periodicity

$$\overline{\Phi_s} = 0$$

$$\overline{\frac{\partial \Phi_s}{\partial x}} = 0$$

$$\overline{\frac{\partial \Phi_s}{\partial t}} = 0$$

where

$$\phi_s = -\gamma_s t + \Phi_s(x,z) \qquad (3.56;3.75)$$

The above mean values are exact. The solution of the conservation equations will be correct to $O(a^2)$ so all mean values of wave properties are only required to this accuracy. The effect of this requirement is that the integral properties of the Stokes waves (derived from the periodic potential $\Phi(x,z)$) are the same as the corresponding properties for Airy waves derived in 3.3, since the greater accuracy of the Stokes solution only affects $O(a^2)$ terms after averaging. (See discussion preceding (3.81) of the validity of $O(a^2)$ terms in the integral properties of Airy theory; also the example below for I_s, etc.).

eg.
$$\begin{aligned} I_s &\equiv \rho \int_{-d}^{\eta} \frac{\partial}{\partial x}\left[\Phi_s(x,z) - \gamma + \right] dz \\ &= \rho \int_{-d} \frac{\partial \Phi_s}{\partial x} dz + \rho \int^{\eta} \frac{\partial \Phi_s}{\partial x} dz \\ &= \quad 0 \quad + \eta \left. \frac{\partial \Phi_s}{\partial x} \right|_{z=0} + O(a^3) \\ &= I_{\text{Airy}} = \frac{E}{c}_{\text{Airy}} \quad (3.81;\ 3.82) \end{aligned}$$

Similarly $T_s = T_A$; $E_s = E_A$;

correct to $O(a^2)$. The only additional term required that arises directly from the η, $\tilde{\Phi}$ expressions (5.6, etc.) is the γ_s term which is non-zero for the Stokes wave (3.71). This γ_s term plays an important part in the solution. As it does not appear explicitly in the Airy solution (3.65), one may feel that the solutions found using this will not be those of Airy waves. The answer is found in the expression

$$\gamma_s = \frac{\frac{1}{2} g k a^2}{\sinh 2kd} = \frac{\frac{1}{2}\rho g d a^2}{\rho h \sinh 2kd} = \frac{E}{\rho d}\left(\frac{c_g}{c} - \frac{1}{2}\right) = \frac{S_{22}}{\rho d}$$

see (3.103) (3.105)

showing that the second order γ_s term (which affects mean pressures and hence mean water levels, see (5.8) appears as S_{22} and as a portion of S_{11} in Airy theory. Hence γ_s can be regarded as a valid second order term in Airy theory and of course its effect on mean water level is implied in its role as part of the radiation stress.

No other second order terms arising directly from the

$\eta, \underset{\sim}{\Phi}$ expressions are required.

The mean density and flux terms ($\overline{P_i}$, $\overline{Q_i}$) for the conservation equations are now evaluated correct to $O(a^2)$ for Stokes waves on uniform currents. Terms involving I_s, T_s, will be expressed in terms of $E_A \equiv E = \frac{1}{2}\rho g a^2$ (3.86).

Mass: mean density $\quad \overline{P_0} \equiv \overline{\int_{-d}^{\eta} \rho \, dz} = \rho(\bar{\eta} + \bar{d})$ (5.12)

$$= \rho d \qquad (5.18)$$

mean flux $\quad \overline{Q_0} \equiv \overline{\int_{-d}^{\eta} \rho \frac{\partial \underset{\sim}{\Phi}}{\partial x} dz}$ from (5.13)

$$= \rho \overline{\int_{-d}^{\eta} \left(U_0 + \frac{\partial \underset{\sim}{\Phi}_s}{\partial x} \right) dx} \quad \text{from (5.6)}$$

(Note that $I \equiv \overline{\int_{-d}^{\eta} \rho \frac{\partial \phi}{\partial x} dz}$ in 3.2 is the mean mass flux for any wave motion represented by $\phi(x,z,t)$; ie. $\overline{Q_0} \equiv$ current mass flux plus mean wave mass flux).

$$= \rho U_0 d + I_A$$

$$= \rho U_0 d + \frac{E}{c} \qquad (5.19)$$

Momentum: mean density $\quad \overline{P_1} \equiv \overline{Q_0}$ (5.20) mean flux $\quad \bar{Q}_1 \equiv \overline{\int_{-d}^{\eta}\left[p + \rho\left(\frac{\partial \Phi}{\partial x}\right)^2\right]dz}$ from (5.15)

It is possible to derive this result directly by inserting all the appropriate expressions from the Stokes solution and averaging as in Whitham (1962), but it will be recalled from 3.2, 3.3 that the radiation stress (3.40; 3.88) represents the excess momentum flux due to the waves. This means that it will appear in $\bar{Q}_1$ and to avoid rederiving it here, the result (3.103) can be used since it is correct to $O(a^2)$.

$$S_{11} \equiv \overline{\int_{-d}^{\eta}\left[p + \rho\left(\frac{\partial \Phi_s}{\partial x}\right)^2\right]dz} - \tfrac{1}{2}\rho g d^2 \qquad (3.88)$$

$$= E\left(2\frac{c_g}{c} - \tfrac{1}{2}\right) \qquad (3.103)$$

$$\bar{Q}_1 \equiv \overline{\int_{-d}^{\eta}\left\{p + \rho\left[U_0^2 + 2U_0\frac{\partial \Phi_s}{\partial x} + \left(\frac{\partial \Phi_s}{\partial x}\right)^2\right]\right\}dz}$$

$$= \rho d U_0^2 + S_{11} + \tfrac{1}{2}\rho g d^2 + \rho\overline{\int_{-d}^{\eta}\left(2U_0\frac{\partial \Phi_s}{\partial x}\right)dz}$$

$$= \rho d U_0^2 + S_{11} + \tfrac{1}{2}\rho g d^2 + 2U_0 I_A,$$

$$= \rho d\left[U_0^2 + 2E/c\rho d\right] + S_{11} + \tfrac{1}{2}\rho g d^2 \qquad (5.21)$$

Energy: mean density: from (5.16)

$$\bar{P}_2 \equiv \overline{\int_{-d}^{\eta}\left\{\tfrac{1}{2}\rho(\nabla\Phi)^2 + \rho g z\right\}dz}$$

$$= \overline{\int_{-d}^{\eta}\left\{\tfrac{1}{2}\rho\left(U_0 + \frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_s}{\partial z}\right)^2 + \rho g z\right\}dz}$$

$$= \overline{\int_{-d}^{\eta} \left\{ \tfrac{1}{2}\rho U_0^2 + \rho U_0 \left(\frac{\partial \Phi_s}{\partial x} + \frac{\partial \Phi_s}{\partial z} \right) + \tfrac{1}{2}\rho \left(\nabla \Phi_s \right)^2 + \rho g z \right\} dz}$$

$$= \tfrac{1}{2}\rho d U_0^2 + \rho U_0 \overline{\int_{-d}^{\eta} \frac{\partial \Phi_s}{\partial x} dz} + \rho U_0 \cancelto{0}{\overline{\int_{-d}^{\eta} \frac{\partial \Phi_s}{\partial z} dz}} + T_A + V_A$$

mean level unchanged

$$= \tfrac{1}{2}\rho d U_0^2 + \rho U_0 I_A + E$$

$$= \tfrac{1}{2}\rho d \left[U_0^2 + \frac{2 U_0 E}{\rho d c} \right] + E \qquad (5.22)$$

Energy: mean flux from (5.17)

$$\bar{Q}_2 \equiv \overline{\int_{-d}^{\eta} \left\{ p + \tfrac{1}{2}\rho \left(\frac{\partial \Phi}{\partial x} \right)^2 + \rho g z \right\} \cdot \frac{\partial \Phi}{\partial x} dz}$$

$$= \overline{\int_{-d}^{\eta} \left\{ p + \tfrac{1}{2}\rho \left(U_0^2 + 2U_0 \frac{\partial \Phi_s}{\partial x} + \left(\frac{\partial \Phi_s}{\partial x} \right)^2 \right) + \rho g z \right\} \left\{ \frac{\partial \Phi_s}{\partial x} + U_0 \right\} dz}$$

This integral is most easily evaluated by expanding it as follows (see Longuet-Higgins and Stewart, 1960, p574)

$$\bar{Q}_2 \equiv R_0 + R_1 + R_2 + R_3$$

where

$$R_0 \equiv \overline{\int_{-d}^{\eta} \left\{ p + \tfrac{1}{2}\rho \left(\frac{\partial \Phi_s}{\partial x} \right)^2 + \rho g z \right\} \cdot \frac{\partial \Phi_s}{\partial x} . dz}$$

$$R_1 \equiv \overline{\int_{-d}^{\eta} \left\{ p + \tfrac{3}{2}\rho \left(\frac{\partial \Phi_s}{\partial x} \right)^2 + \rho g z \right\} dz}$$

$$R_2 \equiv \overline{\int_{-d}^{\eta} \tfrac{3}{2}\rho\left(\frac{\partial\Phi_s}{\partial x}\right) dz} \, . \, U_0^2$$

$$R_3 \equiv \overline{\int_{-d}^{\eta} \tfrac{1}{2}\rho\, dz} \, . \, U_0^3$$

Note that R_0 = wave energy flux in the absence of the current (the term in brackets is simply the wave energy). This is precisely the definition of F' in (3.41): ie. R_0 is F' for Stokes waves.

Hence analysis of R_0 to $O(a^2)$ gives

$$F' = R_0 = \tfrac{1}{4}\rho g a^2 c \left(1 + 2kd / (\sinh 2kd)\right) = E c_g$$

The remaining integrals are now evaluated correct to $O(a^2)$:

$$R_1 = \overline{\int_{-d}^{\eta} \left\{ p + \rho\left(\frac{\partial\Phi_s}{\partial x}\right)^2 \right\} . U_0 \, dz} - \tfrac{1}{2}\rho g d^2 U_0$$

$$+ \overline{\int_{-d}^{\eta} \left[\tfrac{1}{2}\rho\left(\frac{\partial\Phi_s}{\partial x}\right)^2 + \rho g z \right] U_0 \, dz} + \tfrac{1}{2}\rho g d^2 U_0$$

$$= S_{11} U_0 + (T + V)_{Airy} \, . \, U_0$$

$$= U_0 (S_{11} + E)$$

$$R_2 = \tfrac{3}{2}\rho U_0^2 \overline{\int_{-d}^{\eta} \frac{\partial\Phi_s}{\partial x} dz}$$

$$= \tfrac{3}{2} U_0^2 I_A = \tfrac{3}{2} U_0^2 \frac{E}{c}$$

$$R_3 = \tfrac{1}{2}\rho d U_0^3$$

The value of $\bar{Q}_2$ is therefore

$$\bar{Q}_2 = F' + R_1 + R_2 + R_3$$

$$= E(u_0 + g) + u_0 S_{11} + \frac{1}{2}\rho d\, u_0^2\left(u_0 + \frac{3E}{\rho d}\right) \qquad (5.23)$$

The expressions for $\bar{P}_i$, $\bar{Q}_i$ can be simplified by the introduction of a mass transport velocity for the flow. Since $\bar{P}_0$ is the mean mass density and $\bar{Q}_0$ the mean mass flux, the mass transport velocity is simply

$$u_m \equiv \frac{\bar{Q}_0}{\bar{P}_0} = u_0 + \frac{E}{\rho d c} = u_0 + \frac{I}{\rho d} \qquad (5.24)$$

(Note that for non-uniform flow, u_m is replaced by in (5.24)).

$$u_m^2 = u_0^2 + \frac{2u_0 E}{\rho d c} + \frac{E^2}{\rho^2 d^2 c^2} \simeq u_0^2 + \frac{2u_0 E}{\rho d c} \text{ to } O(a^2) \qquad (5.25)$$

Use of $\bar{u}_m$ in the expressions for the $\bar{P}_i$ (mean densities) and the $\bar{Q}_i$ (mean fluxes) leads to:

$$\bar{P}_0 = \rho d \qquad (5.26)$$

$$\bar{Q}_0 = \rho d u_m \qquad (5.27)$$

$$\bar{P}_1 = \bar{Q}_0 \qquad (5.28)$$

$$\bar{Q}_1 = \rho d u_m^2 + \frac{1}{2}\rho g d^2 + S_{11} \qquad (5.29)$$

$$\bar{P}_2 = \frac{1}{2}\rho d U_m^2 + \frac{1}{2}\rho g d^2 + E \qquad (5.30)$$

$$\bar{Q}_2 = \frac{1}{2}\rho d U_m^3 + U_m S_{11} + (U_m + c_g)E \qquad (5.31)$$

These quantities are the ones that will be used in the governing conservation equations for the mass, momentum and energy of the whole flow.

$$\frac{\partial \bar{P}_i}{\partial t} + \frac{\partial \bar{Q}_i}{\partial x} = 0 \qquad (5.1,2,3)$$

5.2.2. Solution of the conservation equations

for $O(a^2)$ waves on steady non-uniform currents

The expressions (5.26) - (5.31) for the constants $\bar{P}_i$, $\bar{Q}_i$ mean that the conservation equations for <u>the flow as a whole</u> become:

$$\frac{\partial}{\partial t}(\rho d) + \frac{\partial}{\partial x}(\rho d U_m) = 0 \qquad (5.32)$$

$$\frac{\partial}{\partial t}(\rho d U_m) + \frac{\partial}{\partial x}\left(\rho d U_m^2 + \frac{1}{2}\rho g d^2 + S_{11}\right) = 0 \qquad (5.33)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho d U_m^2 + \frac{1}{2}\rho g d^2 + E\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho d U_m^3 + U_m S_{11} + (U_m + c_g)E\right) = 0 \qquad (5.34)$$

One is usually most concerned with the <u>wave</u> properties rather than those of the current. The wave properties can be extracted from the expressions above as follows:

$$\bar{P}_3 \equiv \bar{P}_2 - U_m \bar{P}_1 + \left[\tfrac{1}{2} U_m^2 + \tfrac{1}{2} gd\right] \bar{P}_0$$

$$= \tfrac{1}{2} \rho d U_m^2 + \tfrac{1}{2} \rho g d^2 + E - U_m \rho d U_m + \tfrac{1}{2} \rho d U_m^2 + \tfrac{1}{2} \rho g d^2$$

$$= E \tag{5.35}$$

$$\bar{Q}_3 \equiv \bar{Q}_2 - U_m \bar{Q}_1 + \left[\tfrac{1}{2} U_m^2 + \tfrac{1}{2} gd\right] \bar{Q}_0$$

$$= \tfrac{1}{2} \rho d U_m^3 + U_m S_{11} + (U_m + c_g)E - U_m \rho d U_m^2 - U_m \tfrac{1}{2} \rho g d^2 - U_m S_{11}$$

$$+ \tfrac{1}{2} U_m^2 \rho d U_m + \tfrac{1}{2} \rho g d^2 U_m$$

$$= E\left(U_m + c_g\right) \tag{5.36}$$

Here $\bar{P}_3$ and $\bar{Q}_3$ can be regarded as density and flux of "excess energy" or "wave energy", respectively, and it is important to find out under what circumstances these quantities obey a conservation equation:

$$\frac{\partial \bar{P}_3}{\partial t} + \frac{\partial \bar{Q}_3}{\partial x} = 0 \tag{5.37}$$

The form of the flow conservation equations is first considered for a <u>uniform</u> current. Since the flow is steady,

$$U_m = \text{constant}$$

and

$$\frac{\partial \bar{P}_i}{\partial t} = 0 \qquad (i = 0, 1, 2) \tag{5.38}$$

and the conservation equations simplify to:

Mass flux $\bar{Q}_0 = \rho d U_m$ = constant (5.39)

Momentum flux $\bar{Q}_1 = \rho d U_m^2 + \frac{1}{2}\rho g d^2 + S_{11}$ = constant (5.40)

Energy flux $\bar{Q}_2 = \frac{1}{2}\rho d U_m^3 + U_m S_{11} + (U_m + c_g)E$ = constant (5.41)

Since all the terms in the equations are constant for the steady uniform flow, the corresponding wave energy equation

Wave energy flux $\bar{Q}_3 = (U_m + c_g)E$ = constant (5.42)

will also describe energy conservation.

For non-uniform flows, it is to be expected that the conservation equations (5.32) - (5.34) will still hold for slowly varying flows. Their precise form will now be derived. It is clear though that the quantities $\bar{P}_3$, $\bar{Q}_3$ will no longer satisify the conservation equation (5.37), because by their definitions (5.36), (5.35) they depend on U_m, d and these quantities will vary in the non-uniform flow.

$$\bar{P}_3 \equiv \bar{P}_2 - U_m \bar{P}_1 + \left[\tfrac{1}{2}U_m^2 + \tfrac{1}{2}g d\right]\bar{P}_0 \qquad (5.35)$$

$$\bar{Q}_3 \equiv \bar{Q}_2 - U_m \bar{Q}_1 + \left[\tfrac{1}{2}U_m^2 + \tfrac{1}{2}g d\right]\bar{Q}_0 \qquad (5.36)$$

Hence any attempt to express the effect of a steady non-uniform flow on <u>wave</u> energy will be of the form

$$\frac{d}{dx}\left(\overline{Q}_3\right) + \text{correction terms} = 0 \qquad (5.43)$$

where the correction terms compensate for the effects of $\frac{dU_m}{dx}$ and $\frac{d(d)}{dx}$ in (5.35).

The correct form of the wave energy equation (5.43) was first derived by Longuet-Higgins and Stewart in 1961 by perturbation analysis of the whole flow (see section 5.3.1). An alternative discussion given by them expressed (5.43) in terms of $\overline{Q}_3$ plus a correction term involving the radiation stress S_{11}, but difficulties arose in explaining the particular form of the correction term.

Their result is now derived directly from the conservation equations for the flow and in 5.4 the reason for its form is explained.

The current is assumed to be of the form ($U_a(x), 0, 0$) and the depth $h_a(x)$ with the waves <u>absent</u>. The quantities U_a, h_a are related since the pressure must vanish at the free surface. This can be seen by writing the dynamic free surface boundary condition (3.20) in the form

$$\frac{1}{2}U_a^2(x) + g h_a(x) = \text{constant} \qquad (5.44)$$

This condition still holds when the waves propagate across the surface of the current, causing a change in the current velocity and depth to $U(x)$ and $d(x)$, respectively.

Since the current is non-uniform, inflow of water is required to satisfy continuity of mass. Two situations are considered by Longuet-Higgins and Stewart (1961); namely inflow from below or inflow from the sides. Only the first of these is considered here.

A solution is required for the following conservation equations:

kinematics: $$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad (2.25)$$

dynamics: $$\frac{\partial \bar{P}_i}{\partial t} + \frac{\partial \bar{Q}_i}{\partial x} = 0 \qquad (i = 0, 1, 2)(5.1,\ 5.2,\ 5.3)$$

Since the flow is steady, the changes in wavenumber for the Airy wave theory are given by:

$$\frac{d}{dx}\left\{\omega\right\} = 0$$

$$\therefore \frac{d}{dx}\left\{Uk + \sigma\right\} = 0$$

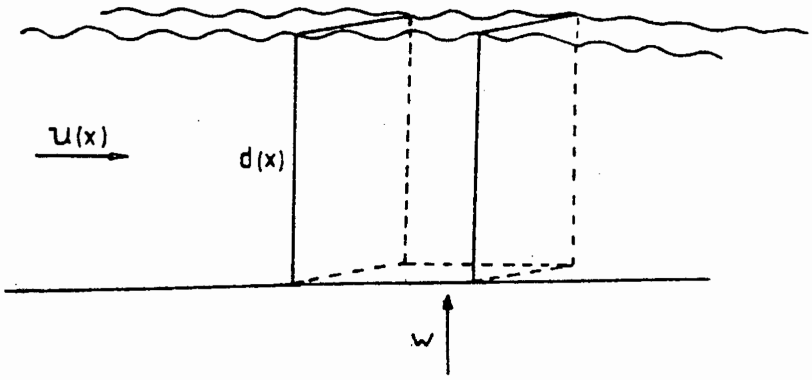
ie. $$\frac{d}{dx}\left\{Uk + \sqrt{gk \tanh kd}\right\} = 0 \qquad (5.45)$$

This expresses the <u>kinematics</u> of the interaction.

The solution for this equation in deep water is covered in 5.3.2. (see (5.55) and following derivation).

The dynamics of the interaction are derived by finding expressions for the rate of change of mass, momentum and energy flux, ie. for $\frac{d\bar{Q}_0}{dx}$, $\frac{d\bar{Q}_1}{dx}$ and $\frac{d\bar{Q}_2}{dx}$ respectively.

Figure 5-1.



rate of inflow

Change in mass flux at x :

$$\frac{d}{dx}\bar{Q}_0 = \frac{d}{dx}\left[\rho d U_m\right] \qquad \text{from} \quad (5.39)$$

This must be provided by inflow from below and the velocity w must therefore have the value:

$$w = \frac{d}{dx}\left[d U_m\right]$$

ie.

$$\rho w = \frac{d}{dx}\bar{Q}_0 \qquad (5.46)$$

Change in momentum flux at x :

$$\frac{d}{dx}\bar{Q}_1 = \frac{d}{dx}\left\{\rho d U_m^2 + \frac{1}{2}\rho g d^2 + S_{11}\right\} \qquad \text{from} \quad (5.41)$$

This must equal the rate of inflow of momentum from below. An assumption must be made about the change in horizontal velocity $\mathcal{U}^*$ of the incoming fluid (originally at rest) and $\mathcal{U}^* = \mathcal{U}$ is the appropriate choice. (It is found that the final result is unchanged if $\mathcal{U}^* = \mathcal{U}_0$, $\mathcal{U}$, or $\mathcal{U}_m$ ie. the result is unchanged for definitions of $\mathcal{U}^*$ that vary by $O(E)$ ie. $O(a^2)$).

$$\frac{d}{dx}\bar{Q}_1 = \mathcal{U}^*\frac{d}{dx}\bar{Q}_0$$

$$\frac{d}{dx}\left\{\rho d\,\mathcal{U}_m^2 + \tfrac{1}{2}\rho g d^2 + S_{11}\right\} = \mathcal{U}^*\frac{d}{dx}\left[\rho d\,\mathcal{U}_m\right] \qquad (5.47)$$

Change in energy flux at x :

$$\frac{d}{dx}\bar{Q}_2 = \frac{d}{dx}\left\{\tfrac{1}{2}\rho d\,\mathcal{U}_m^3 + \mathcal{U}_m S_{11} + E(\mathcal{U}_m + c_g)\right\} \text{ from } \qquad (5.42)$$

This must come from change in energy of the fluid inflow. As the vertical inflow velocity is small, only the horizontal velocity change $\mathcal{U}^*$ appears in the kinetic energy flux. The potential energy flux has a negative sign as the zero level for potential energy is taken as $z = 0$.

$$\frac{d}{dx}\left\{\tfrac{1}{2}\rho d\,\mathcal{U}_m^3 + \mathcal{U}_m S_{11} + E(\mathcal{U}_m + c_g)\right\} = \tfrac{1}{2}\mathcal{U}^{*2}\frac{d}{dx}\left[\rho d\,\mathcal{U}_m\right] - \rho g d\,\mathcal{U}_m\frac{d}{dx}(d) \qquad (5.48)$$

These dynamic conservation equations for the non-uniform flow are now used to determine the correction terms in (5.43) which appear due to the non-uniform mean flow.

$$\frac{d}{dx}\left[E\left(\mathcal{U}_m + c_g\right)\right] \quad + \text{ correction terms } = 0 \qquad (5.43)$$

(5.43) was obtained from (5.35), and the correction terms are found by analysis of the non-uniform flow relation that is equivalent to (5.36)

$$\bar{Q}_3 \equiv \bar{Q}_2 - \mathcal{U}_m \bar{Q}_1 + \tfrac{1}{2}\left[\mathcal{U}_m^2 + \tfrac{1}{2} g d\right] \bar{Q}_0 \qquad (5.36)$$

The analogous relation is:

$$\frac{d}{dx}\left(\bar{Q}_2\right) - \mathcal{U}_m \frac{d}{dx}\left(\bar{Q}_1\right) + \left(\tfrac{1}{2}\mathcal{U}_m^2 + \tfrac{1}{2} g d\right) \cdot \frac{d\bar{Q}_0}{dx} = \frac{d(I)}{dx} - \mathcal{U}_m \frac{d(II)}{dx} + \left[\tfrac{1}{2}\mathcal{U}_m^2 + \tfrac{1}{2} g d\right] \frac{d(III)}{dx} \qquad (5.49)$$

where I, II, III are the expressions for $\bar{Q}_2$, $\bar{Q}_1$, $\bar{Q}_0$ for this situation, obtained from equations (5.48), (5.47) and (5.46) respectively. Equation (5.46) shows in fact that the expressions in (5.49) above for $\bar{Q}_0$ and III are identical, so it is only necessary to consider

$$\frac{d}{dx}\left(\bar{Q}_2\right) - \mathcal{U}_m \frac{d}{dx}\left(\bar{Q}_1\right) = \frac{d}{dx}(I) - \mathcal{U}_m \frac{d}{dx}(II) \qquad (5.50)$$

The expressions for $\bar{Q}_2$ and II from (5.48) and for $\bar{Q}_1$ and II from (5.47) are inserted into (5.49) and the derivatives evaluated as follows: ($\mathcal{U}^* \equiv \mathcal{U}$).

$$\frac{d}{dx}\left\{\tfrac{1}{2}\rho d\,\mathcal{U}_m^2 + \mathcal{U}_m S_{11} + E\left(\mathcal{U}_m + c_g\right)\right\} - \mathcal{U}_m \cdot \frac{d}{dx}\left\{\rho d\,\mathcal{U}_m^2 + \tfrac{1}{2}\rho g d^2 + S_{11}\right\}$$

$$= \tfrac{1}{2}\rho\,\mathcal{U}^2 \frac{d}{dx}\left[\mathcal{U}_m\right] - \rho g d\,\mathcal{U}_m \frac{d}{dx}(d) - \rho\,\mathcal{U}\,\mathcal{U}_m \frac{d}{dx}\left[d\,\mathcal{U}_m\right]$$

$$\therefore \quad \frac{3}{2}\rho d \cancel{U_m^2} \frac{dU_m}{dx} + \frac{1}{2}\rho \cancel{U_m^3 \frac{d}{dx}}(d) + \cancel{U_m \frac{dS_{11}}{dx}} + S_{11}\frac{dU_m}{dx} + \frac{d}{dx}\left[E(U+c_g)\right]$$

$$- \cancel{U_m^3 \rho \frac{d}{dx}}(d) - 2\rho d \cancel{U_m^2} \frac{dU_m}{dx} - \rho g U_m \frac{d}{dx}(d) - \cancel{U_m \frac{dS_{11}}{dx}}$$

$$= \frac{1}{2}\rho (U - U_m)^2 \frac{d}{dx}(dU_m) - \frac{1}{2}\rho \cancel{U_m^3 \frac{d}{dx}}(d) - \frac{1}{2}\rho \cancel{U_m^2} \frac{d}{dx}(U_m).d - \rho g d U_m \frac{d}{dx}(d),$$

$$\therefore \quad \frac{d}{dx}\left\{E(U_m + c_g)\right\} + S_{11}\frac{dU_m}{dx} = \frac{1}{2}\rho (U - U_m)^2 \frac{d}{dx}(dU_m) \qquad (5.51)$$

The right hand term will be $O(a^4)$ since $U - U_m$ is $O(a^2)$ from (5.24), and so can be ignored. Similarly, the velocity in the left hand terms can be replaced by either $U_a(x)$ or $U(x)$ since the difference will involve terms $O(a^4)$ because E and S_{11} are themselves $O(a^2)$ (see (3.86); (3.106). The final form for equation (5.51), (the particular form of (5.43)), is now given in terms of $U(x)$ as this is the original form derived by Longuet-Higgins and Stewart (1961), (see (5.73)).

$$\frac{d}{dx}\left\{E(U + c_g)\right\} + S_{11}\frac{dU}{dx} = 0 \qquad (5.52)$$

Compare the uniform current result:

$$\frac{d}{dx}\left\{E(U + c_g)\right\} = 0 \qquad (5.42)$$

Particular solutions of a given situation require first a solution of the kinematic relation (5.45) for k and then use of this to solve (5.52) for E and hence for a.

$$\frac{d}{dx}\left\{Uk + \sqrt{gk \tanh kd}\right\} = 0 \qquad (5.45)$$

$$\frac{d}{dx}\left\{ E\left(U_m + \frac{d\sigma}{dk}\right)\right\} + S_{11}\frac{dU_m}{dx} = 0 \qquad (5.52)$$

The variation of k and a with U for waves propagating on deep water is derived in 5.3.1 from the above relations and so is not given here. (see (5.83), (5.88))

The purpose of section 5.2 has been to show that the governing equation for waves on non-uniform currents ie. (5.52) can be obtained directly from the conservation equations for the flow as a whole.

The perturbation analysis method for deriving (5.52) is now outlined in 5.3.1. This is the original method used by Longuet - Higgins and Stewart. The particular form of (5.52), especially the form of the additional term involving the current variation, is explained in 5.4.

5.3. SOLUTION BY PERTURBATION ANALYSIS FOR O(a^2) WAVES ON STEADY NON-UNIFORM CURRENTS.

Before introducing the perturbation analysis, the governing equations for the flow are recalled and the effects of the 'small amplitude wave' and 'large scale current' restrictions are noted. The simplifications introduced by these restrictions will show that perturbation analysis is a natural technique to adopt to study the interactions.

The flow is incompressible and inviscid so the governing equations are (motion in x, y plane): ($u_f \equiv$ flow velocity) incompressibility: $\nabla . \underset{\sim}{u}_f = 0$ (3.1)

momentum conservation: $\frac{\partial u_f}{\partial t} + (\underset{\sim}{u}_f . \nabla) \underset{\sim}{u}_f = - \frac{\nabla p}{\rho}$ (3.2)

The small amplitude requirement means that the wave velocity u can be regarded as a perturbation of the current $\mathcal{U}$:

$$u_f = \mathcal{U} + u \tag{5.53}$$

The restriction of large scale variation for the current implies that

$$\frac{1}{\sigma} \frac{\partial \mathcal{U}}{\partial x} << 1 \tag{4.1}$$

Substitution of (5.53) into (3.1), (3.2) and use of (4.1) leads to the following equations:

incompressibility: $\nabla . \underset{\sim}{u} = 0$ (5.54)

momentum conservation: $\frac{\partial \underset{\sim}{u}}{\partial t} + (\mathcal{U} . \nabla) u = - \frac{\nabla p}{\rho}$ (5.55)

This shows that to a first approximation the current appears in the equations only as a uniform velocity, as terms involving $\nabla \mathcal{U}$ vanish.

The analysis of such a system can be approached by choosing a plane wave solution on a uniform current (eg. an Airy wave solution) and then regarding the parameters defining the wave (eg. $\underset{\sim}{k}$ and a) to be slowly varying functions of the non-uniform current u . The wave parameters can then be described by Taylor series expansions and perturbation analysis is used to give the solution for the waves on the non-uniform current to the required degree of accuracy.

This approach was the one used by Longuet-Higgins and Stewart (1961) to extend their work (1960) in which they analysed the changes in gravity waves due to long gravity waves or uniform currents. The purpose of this section (5.3) is to outline their analysis for deep water Airy waves on non-uniform currents and quote the solutions. The same solution for the dynamics was derived in 5.2.2 following Whitham's method (1962). It will be interpreted using the conservation equations in 5.4. This approach will clarify problems of interpretation raised by Longuet-Higgins and Stewart.

5.3.1. The perturbation method of Longuet-Higgins and Stewart

The solution must obey the following equations for the fluid: (x, z plane:)

Irrotationality: $$u_f = \nabla \Phi \qquad (3.11)$$

Incompressibility: $$\nabla^2 \Phi = 0 \qquad (3.12)$$

Bernoulli's equation: $$\frac{p}{\rho} + \frac{\partial \Phi}{\partial t} + \frac{1}{2}(u_f)^2 + gz = C \qquad (3.13)$$

where $\Phi(x,z,t)$ is the velocity potential for the whole flow and where C is a constant, as the analysis will be for small amplitude waves in deep water.

The boundary conditions are the familiar kinematic and dynamic conditions at the free surface and the bottom. The free surface conditions are expressed here as Taylor series about $z=0$:

$$\frac{\partial\eta}{\partial t} + \left(\frac{\partial\Phi}{\partial x}\cdot\frac{\partial\eta}{\partial x} - \frac{\partial\Phi}{\partial z}\right)_{z=0} + \eta\left(\frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial x}\cdot\frac{\partial\eta}{\partial x} - \frac{\partial\Phi}{\partial z}\right)\right)_{z=0} + \cdots = 0 \qquad (5.56)$$

$$g\eta + \left(\frac{1}{2}(u_f)^2 + \frac{\partial\Phi}{\partial t}\right)_{z=0} + \eta\left(\frac{\partial}{\partial z}\left(\frac{1}{2}(u_f)^2 + \frac{\partial\Phi}{\partial t}\right)\right)_{z=0} + \cdots = 0 \qquad (5.57)$$

Bottom condition: periodic part of motion vanishes (5.58)

The kinematic equation for the wave frequency completes the set of equations required.

$$\omega = \sigma + U k \qquad (3.8)$$

The form of the solution is to be a periodic wave motion on a non-uniform steady flow. It is appropriate to express Φ, η in the following form:

$$\Phi = U_0 x + (\alpha\phi_{10} + \beta\phi_{01}) + (\alpha^2\phi_{20} + \alpha\beta\phi_{11} + \beta^2\phi_{02}) + \cdots \qquad (5.59)$$

$$\eta = (\alpha\eta_{10} + \beta\eta_{01}) + (\alpha^2\eta_{20} + \alpha\beta\eta_{11} + \beta^2\eta_{02}) + \cdots \qquad (5.60)$$

with the following definitions:

$U_0 \equiv$ steady uniform velocity; the current at $x=0$

ϕ_{01} : potential for a steady non-uniform current which is zero at $x=0$

ϕ_{10} : potential for an undisturbed surface wave (Airy theory)

α : small parameter proportional to wave steepness

β : small parameter proportional to current velocity gradient

Higher order terms:

ϕ_{20} ; ϕ_{02} : correction terms for free surface boundary conditions

ϕ_{11} : interaction potential between waves and current

It is sufficient for the small amplitude large scale current solution to neglect the ϕ_{20}, ϕ_{02} terms, but $\alpha\beta\phi_{11}$ is the second order interaction term and must be retained. With these simplifications, (5.58) and (5.59) lead to:

Irrotationality: $$\nabla\Phi = (u_0, 0, 0) + \alpha\nabla\phi_{10} + \beta\nabla\phi_{01} + \alpha\beta\nabla\phi_{11} \quad (5.61)$$

$$\frac{\partial\Phi}{\partial t} = \alpha\frac{\partial\phi_{10}}{\partial t} + \alpha\beta\frac{\partial\phi_{11}}{\partial t} \quad (5.62)$$

$$\eta = \alpha\eta_{10} + \beta\eta_{01} + \alpha\beta\eta_{11} \quad (5.63)$$

$$\frac{1}{2}(u_f)^2 = \frac{1}{2}u_0^2 + u_0\left(\alpha\frac{\partial\phi_{10}}{\partial x} + \beta\frac{\partial\phi_{01}}{\partial x}\right) + \alpha\beta\left(u_0\frac{\partial\phi_{11}}{\partial x} + \frac{\partial\phi_{10}}{\partial x}\cdot\frac{\partial\phi_{01}}{\partial x} + \frac{\partial\phi_{01}}{\partial z}\frac{\partial\phi_{10}}{\partial z}\right) + \cdots \quad (5.64)$$

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}(u_s)^2\right] = u_0\left(\alpha\frac{\partial^2\phi_{10}}{\partial x\partial t} + \beta\frac{\partial\phi_{01}}{\partial x\partial t}\right)+\cdots \qquad (5.65)$$

Use of these expressions in the free surface boundary conditions implies

$$C_1 = \tfrac{1}{2}u_0^2 \qquad (5.66)$$

as in 5.2.1; equation (5.6).

The equation for ϕ_{10} is now obtained from the free surface boundary conditions after eliminating η_{10} :

$$\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)^2\phi_{10} + g\frac{\partial\phi_{10}}{\partial z} = 0 \quad \text{at } z=0. \qquad (5.67)$$

The same conditions lead to an expression for ϕ_{01} :

$$u_0^2\frac{\partial^2\phi_{01}}{\partial x^2} + g\frac{\partial\phi_{01}}{\partial z} = 0 \quad \text{at } z=0 \qquad (5.68)$$

Finally, use of intermediate expressions involving ϕ_{10}, ϕ_{01} in the boundary condition gives an expression for the interaction potential ϕ_{11}

$$\frac{1}{g}\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)^2\phi_{11} + \frac{\partial\phi_{11}}{\partial z} = 2\left(\frac{\partial\phi_{10}}{\partial x}\frac{\partial\eta_{01}}{\partial x} + \frac{\partial\phi_{01}}{\partial x}\frac{\partial\eta_{10}}{\partial x} + k_0\eta_{01}\frac{\partial\phi_{01}}{\partial z}\right) - \eta_{10}\frac{\partial^2\phi_{01}}{\partial z^2} \qquad (5.69)$$

$$\text{at } z=0.$$

The solutions for ϕ_{10}, ϕ_{01} and ϕ_{11} can be found once the required form of each has been chosen. The forms used by Longuet-Higgins and Stewart are given below, each including constants to be determined in order to specify the solution.

Wave potential: $$\phi_{10} = A e^{k_0 z} . e^{i(k_0 x - \omega t)} \tag{5.70}$$

where k_0 is wavenumber at $x=0$ and ω is the corresponding frequency. (The use of "ω" not "σ" will be discussed following (5.76)).

Potential for steady non-uniform flow in the x direction vanishing at $x=0$:

$$\phi_{01} = c_0 k_0 (x^2 - z^2) + D c_0 z \tag{5.71}$$

where c_0 is phase velocity at $x=0$ (see (5.57)) and D a constant

Interaction potential: $$\phi_{11} = i\left(k_1(z+ix) + \ell_1^2 (z+ix)^2\right)\phi_{10} \tag{5.54}$$

where k_1 and ℓ_1 are constants

The values of these constants are not derived here, but it is worth noting the use of the kinematic conservation equation (4.8) in order to find a relation between ω and k_0 :

$$\omega = \sigma + Uk \tag{4.8}$$

For deep water small amplitude waves in the absence of a current, the dispersion relation is given by (3.66):

$$\sigma^2 = gk \tanh kd \tag{3.66}$$

$$\therefore \quad \sigma^2 = gk \quad \text{(deep water)} \tag{5.73}$$

The current is equal to U_0 at $x = 0$ (5.40) and the wavenumber there is k_0. The value of the constant ω is therefore given by:

$$\omega = \sqrt{gk_0} + U_0 k_0 \qquad (5.74)$$

and the velocity c_0 at $x = 0$ is:

$$c_0 = \frac{\omega}{k_0} = \sqrt{\frac{g}{k_0}} \qquad (5.75)$$

$$\therefore\ \omega = c_0 k_0 \left(1 + \frac{U_0}{c_0}\right) \qquad (5.76)$$

(Note that this notation replaces the " σ " of Longuet-Higgins by " ω " since " σ " has been reserved for the intrinsic frequency as in (3.66) and (5.73) above).

5.3.2. Solutions for the wave kinematics and for wave energy changes on the current

The solution for the variation of the kinematic parameters (eg. k, ω) is obtained from the kinematic conservation equation (4.8) plus the use of the Airy deep water dispersion relation (5.73). This derivation is included here, but not the alternative derivation using ϕ_{10}, ϕ_{01}, ϕ_{11} . The solution is expressed in terms of the wavenumber k and also in terms of the phase and group velocities.

The dynamics are expressed as an equation for wave amplitude

a, using the expressions for $\phi_{10}, \phi_{01}, \phi_{11}$ obtained after solving for the remaining constants D_1, k_1, ℓ_1. The solutions for the constants and the ϕ's are in Longuet-Higgins and Stewart (1961, p534) and are not reproduced here.

The kinematic and dynamic solutions are now given. It is interesting to note that the fundamental requirement of large scale current variation appears in the solution as follows:

$$\frac{\partial U}{\partial x} = \beta \frac{\partial^2 \phi_{01}}{\partial x^2} = 2\beta\omega\left(\frac{1}{1+\frac{U_0}{c_0}}\right) \qquad (5.77)$$

Since β was assumed small, the large-scale restriction (4.1) is satisfied.

(a) Wavelength solution

Recall the kinematic equation (4.8) and the phase velocity expression for small amplitude waves in deep water (2.23); (5.75):

$$\omega = k_0(c_0 + U_0) = k(c + U) = \text{constant} \quad (4.8)$$

$$c \equiv \omega / k \qquad (2.23)$$

$$c = \sqrt{g/k} \qquad \text{(deep water)} \, (5.75)$$

$$\therefore \quad \frac{k}{k_0} = \frac{c_0 + U_0}{c + U} \qquad (5.78)$$

$$\therefore \quad \frac{c^2}{c_0^2} = \frac{k}{k_0} = \left(\frac{1}{1+\frac{U_0}{c_0}}\right) \cdot \left(\frac{c}{c_0} + \frac{U}{c_0}\right) \qquad (5.79)$$

Differentiation with respect to x gives:

$$\frac{2c}{c_o^2}\frac{\partial c}{\partial x} = \frac{1}{\left(1+\frac{u_o}{c_o}\right)} \cdot \frac{1}{c_o} \cdot \left(\frac{\partial c}{\partial x} + \frac{\partial u}{\partial x}\right) \qquad (5.80)$$

At $x = 0$ (5.79) becomes:

$$\frac{\partial c}{\partial x} = \frac{1}{\left(1+2\frac{u_o}{c_o}\right)} \cdot \frac{\partial u}{\partial x} \qquad (5.81)$$

Replacing c by k using (5.79), followed by logarithmic differentiation leads to:

$$\frac{1}{k}\cdot\frac{\partial k}{\partial x} = -\frac{2}{c}\cdot\frac{\partial c}{\partial x} = \frac{-2}{\left(1+2\frac{u_o}{c_o}\right)}\cdot\frac{1}{c}\cdot\frac{\partial u}{\partial x} \quad \text{at } x=0. \qquad (5.82)$$

This is the same result as obtained via the solution for ϕ_{10}, ϕ_{01} and ϕ_{11}.

The solution is clarified by treating (5.73) as a quadratic in $\frac{c}{c_o}$:

$$\frac{c}{c_o} = \frac{1}{2\left(1+\frac{u_o}{c_o}\right)}\cdot\left\{1+\sqrt{\left(1+4\left(1+u_o/c_o\right)\frac{u}{c_o}\right)}\right\} \qquad (5.83)$$

and in terms of $k/k_o = (c/c_o)^{-2}$ (from (5.79)):

$$\frac{k}{k_o} = \left\{\frac{2\left(1+u_o/c_o\right)}{1+\sqrt{\left(1+4\left(1+u_o/c_o\right)u/c_o\right)}}\right\}^2 \qquad (5.84)$$

There is no real solution for the quadratic in c/c_0 when

$$1 + \frac{4}{c_0}\left(1 + U_0/c_0\right)U < 0$$

$$\Rightarrow \quad \frac{c}{c_0} = 1\Big/\left(2\left(1 + U_0/c_0\right)\right)$$

$$\Rightarrow \quad U/c = -\tfrac{1}{2} \qquad (5.85)$$

This corresponds to the "stopping velocity" introduced in 4.2 (4.17) since the current is equal and opposite to the local group velocity of the waves:

$$c_g = \frac{\partial\sigma}{\partial k} = \frac{\partial}{\partial k}\left(\sqrt{gk}\right) = \tfrac{1}{2}\sqrt{g/k} \qquad (5.86)$$

$$= c/2$$

$$\therefore \quad U = -c_g \qquad (5.87)$$

The wave energy can no longer be propagated against the stream and so there must be large changes in the wave energy at this point. This is clearly seen in the wave amplitude solution.

(b) Wave amplitude solution

The wavelength solution was obtained from physical quantities (ω, k, U, c), although it could be found from the full perturbation analysis and solution. By contrast, Longuet-Higgins and Stewart found that the perturbation solution for the wave energy (ie. wave amplitude, since $E = \frac{1}{2}\rho g a^2$ from (3.86)) was <u>crucial</u> in assisting the formulation of the energy equation in terms of physical quantities (E, U, c_g, S). Since this formulation hinges on the interpretation of (for example) the radiation stress it is exactly this interpretative point that Whitham discussed in his 1962 paper and which is to be dealt with in 5.4. Here it is sufficient to quote the results of Longuet-Higgins and Stewart and postpone the discussion of them to 5.4.

Solution for D, k_1, l_1 and hence $\phi_{10}, \phi_{01}, \phi_{11}$ implies that the variation of wave amplitude a at $x = 0$ is given by:

$$\left(\frac{1}{a} \cdot \frac{\partial a}{\partial x}\right) = \frac{-2 + 3(U_0/c_0)}{(1 + 2U_0/c_0)^2} \cdot \frac{1}{c} \cdot \frac{\partial U}{\partial x} \tag{5.88}$$

The problem is to relate this to an expression for wave energy in the presence of the current $U(x)$. Recall the energy flux equation for waves on a <u>uniform</u> flow (5.23).

$$\bar{Q}_2 = E(U_0 + c_g) + U_0 S_{11} + \frac{1}{2}\rho d U_0^3 + \frac{3}{2} U_0^2 \frac{E}{c} = \text{constant.} \tag{5.23}$$

Longuet-Higgins and Stewart split $\bar{Q}_2$ into expressions for "wave energy" and "mean flow energy" and then generalised the "wave energy" part to apply to non-uniform flows. If there is no dissipation or reflexion of wave energy, then

$$\frac{\partial}{\partial x}(\bar{Q}_2) = 0$$

The amplitude relation (5.88) is crucial in selecting the correct possibility for a "wave energy" equation (each possibility is obtained by choosing portions of (5.23)), and the correct choice is the one that is consistent with the amplitude relation given above (5.88).

The possibilities are:

$$\frac{\partial}{\partial x}\left[E(U+c_g)\right] = 0 \qquad (5.89)$$

$$\frac{\partial}{\partial x}\left[E(U+c_g) + S_{11}U + \frac{3}{2}E\frac{U^2}{c}\right] = 0 \qquad (5.90)$$

$$\frac{\partial}{\partial x}\left[E(U+c_g) + S_{11}U\right] = 0 \qquad (5.91)$$

$$\frac{\partial}{\partial x}\left[E(U+c_g)\right] + S_{11}\frac{\partial U}{\partial x} = 0 \qquad (5.92)$$

The last choice is the correct one, as in deep water it becomes: (see (3.103); (5.86))

$$\frac{\partial}{\partial x}\left[E\left(U+\frac{c}{2}\right)\right] + \frac{E}{2}\frac{\partial U}{\partial x} = 0 \qquad (5.93)$$

At $x=0$, $U = \frac{U_0}{c_0}.c$ and (5.81) used in the derivative gives

$$\frac{\partial E}{\partial x}\left[\frac{1}{2}c\left(1+2U_0/c_0\right)\right] + E\left[1/(2+4U_0/c_0) + \frac{3}{2}\right]\frac{\partial U}{\partial x} = 0$$

$$\therefore \quad \frac{1}{E}\cdot\frac{\partial E}{\partial x}\bigg|_{x=0} = \frac{-4+6U_0/c_0}{(1+2U_0/c_0)^2}\cdot\frac{1}{c}\cdot\frac{\partial U}{\partial x}$$

$E \propto a^2 \Rightarrow$

$$\left(\frac{1}{a}\cdot\frac{\partial a}{\partial x}\right)_{x=0} = \frac{2+3U_0/c_0}{(1+2U_0/c_0)^2}\cdot\frac{1}{c}\cdot\frac{\partial U}{\partial x}$$

This is exactly the result of the perturbation analysis.

The problems of interpretation revolve around the reasons why (5.92) is the correct choice out of (5.89) - (5.92) for the wave energy. This is discussed in 5.4.

The solution is completed by integrating (5.93):

$$E\left(\frac{c}{2} + U\right) . c \quad = \text{constant} \tag{5.93}$$

(since differentiation and division by c gives

$$\frac{\partial}{\partial x}\left[E\left(\frac{c}{2} + U\right)\right] + E\left(\frac{c}{2} + U\right) \cdot \frac{1}{c} \cdot \frac{\partial c}{\partial x} = 0 ,$$

and (5.81) can then be used to retrieve (5.93).

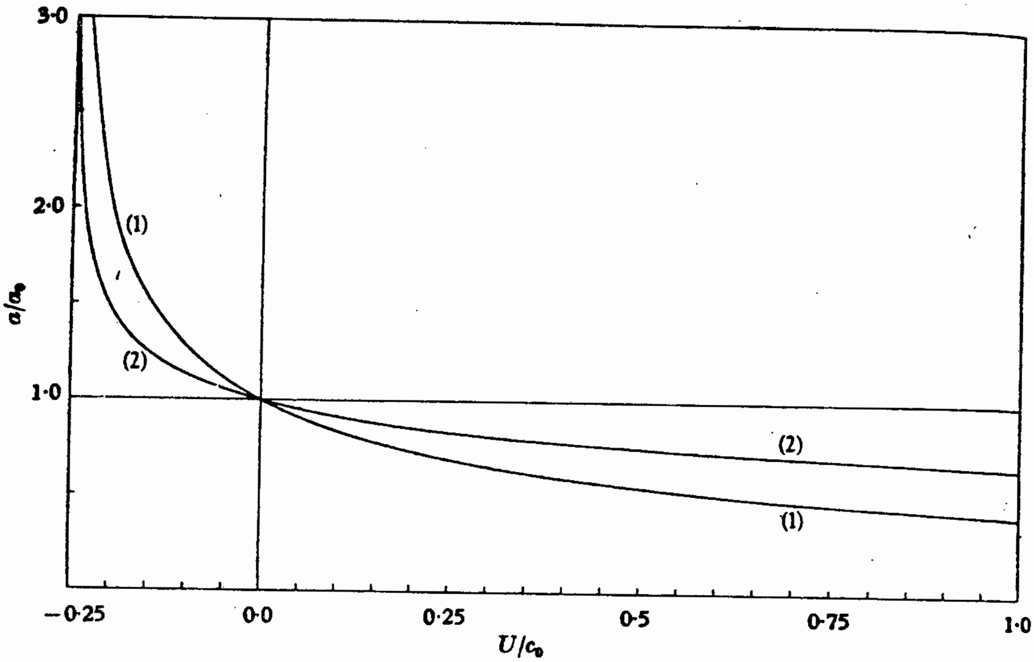
Hence (5.94) $\Rightarrow$

$$\frac{E}{E_o} = \frac{c_o}{c} \frac{(c_o + 2U_o)}{(c + 2U)} \tag{5.95}$$

$$\frac{a}{a_o} = \left\{ \frac{c_o}{c} \frac{(c_o + 2U_o)}{(c + 2U)} \right\}^{1/2} \tag{5.96}$$

This solution is illustrated below and it is clear that $a \to \infty$ as $U \to -c/2$ (the "stopping velocity" noted in the solution for the wave kinematics). The small amplitude limitation means that the solution (5.95) breaks down when $a \to \infty$. Difficulties in extending this analysis to finite amplitude waves are discussed in 8.1 where it is found that the "stopping velocity" is not related to any of the finite amplitude definitions of the group velocity.

Figure 5-2.



The amplification factor a/a_0 for waves on a current U in the direction of wave propagation: (1) with vertical upwelling from below; (2) with horizontal inflow from the sides. [a_0 and c_0 denote the values of a and c when $U = 0$.]

Longuet-Higgins and Stewart (1961).

(Case 2 not evaluated in the text)

5.4. INTERPRETATION OF THE WAVE ENERGY EQUATION AND DISCUSSION OF PROPAGATION VELOCITIES FOR MASS, MOMENTUM AND ENERGY

5.4.1. The original interpretations of the energy equation by Longuet - Higgins and Stewart

It has been shown that the energy equation for the whole flow can be written in the following form if the current is <u>uniform</u> and steady:

$$\bar{Q}_2 = \text{constant}$$

where
$$\bar{Q}_2 = \tfrac{1}{2}\rho d U_m^3 + U_m S_{11} + E(U_m + c_g) \qquad (5.41)$$

or, equivalently, to the same $O(a^2)$ accuracy, as given by Longuet-Higgins and Stewart (1960)

$$\bar{Q}_2 = \tfrac{1}{2}\rho d U_m^3 + U_0 S_{11} + E(U_0 + c_g) \qquad (5.34)$$

where U_0 is the uniform current in the absence of waves and U_m is the "mass transport velocity" of the current plus waves.

$$U_m \equiv \frac{\bar{Q}_0}{\bar{\bar{P}}_0} = U_0 + \frac{E}{\rho d c} \qquad (5.24)$$

Longuet-Higgins and Stewart interpret the energy flux equation (5.34) by splitting it into "main stream energy flux" ($\tfrac{1}{2}\rho d U_m^3$) plus "wave energy flux" (E $(U_0 + c_g)$) plus "work done by the current on the waves" ($U_0 S_{11}$). This interpretation is not entirely successful, since to class the first term as a "main stream energy flux" term is to assume that

$$\left\{\tfrac{1}{2}\rho d U_m^2\right\} \cdot U_m = \tfrac{1}{2}\rho d U_m^3 \qquad (5.97)$$

stream kinetic energy . transport velocity = energy flux

The definition (5.24) of U_m , however shows that one <u>cannot</u> regard terms involving U_m as simply "main stream" since U_m involves the waves through $E/\rho d c$.

The generalisation of the energy equation to non-uniform steady currents $\mathcal{U}(x)$ was shown to be (Longuet-Higgins and Stewart, (1961).

$$\frac{d}{dx}\left\{\left(\mathcal{U}_m + c_g\right)E\right\} + S_{11}\frac{d\mathcal{U}_m}{dx} = 0 \qquad (5.52, 5.92)$$

where again the current velocity $\mathcal{U}$ can replace $\mathcal{U}_m$ to accuracy $O(a^2)$. This result was derived by perturbation analysis (see (5.88) and ensuing discussion) and then interpreted as an equation for "wave energy". However it will be recalled that the energy flux equation

$$\frac{\partial}{\partial x}\left(\bar{Q}_2\right) = 0$$

led to the following possibilities if various terms involving the wave energy E were retained:

$$\frac{\partial}{\partial x}\left[E\left(\mathcal{U}+c_g\right) + S_{11}\mathcal{U} + \underbrace{\tfrac{3}{2}\mathcal{U}^2\tfrac{E}{c}}_{\text{from } \frac{1}{2}\rho d\,\mathcal{U}_m^3}\right] = 0 \qquad (5.91)$$

$$\frac{\partial}{\partial x}\left[E\left(\mathcal{U}+c_g\right) + S_{11}\mathcal{U}\right] = 0 \qquad (5.92)$$

$$\frac{\partial}{\partial x}\left[E\left(\mathcal{U}+c_g\right)\right] + S_{11}\frac{\partial\mathcal{U}}{\partial x} = 0 \qquad (5.52;\ 5.93)$$

Since the correct equation (5.93) does <u>not</u> have all the possible "wave energy" terms (compare (5.91), it seems that it should <u>not</u> be regarded as an equation for wave energy changes.

5.4.2. Whitham's interpretation of the energy equation

Whitham (1962) derived the correct equation directly from the conservation equations for the whole flow, as discussed in 5.2.2. This suggested that it was the division into "main stream energy flux" and "wave energy flux" that was causing the interpretation difficulties rather than the form of the equations.

To explain the form of the two equations (5.34) and (5.52), the corresponding equations for a mechanical system of particles of mass m, each moving at u relative to the centre of mass moving at U are discussed.

First consider the energy flux of the whole (fluid) flow:

$$U_0 = \text{constant}$$

$$\frac{1}{3}\rho d U_m^3 + U_0 S_{11} + E(U_0 + c_g) = \text{constant} \tag{5.34}$$

wave energy flux: $U = U(x)$

$$\frac{\partial}{\partial x}\left[E(U + c_g)\right] + S_{11}\frac{\partial U}{\partial x} = 0 \tag{5.52}$$

Particle analogy: energy relation for U constant:

$$\frac{1}{2}U^2\Sigma m + U\Sigma mu + \frac{1}{2}\Sigma mu^2 = \text{constant} \tag{5.98}$$

This is clearly analogous to (5.34) (except that it refers to energy not energy flux) and shows that the momentum (Σmu) in the moving frame multiplied by the reference frame velocity $\mathcal{U}$ must appear in the energy equation for a moving system. Hence the term $\mathcal{U}_o S_{11}$ is to be expected in (5.33) since $\mathcal{U}_o S_{11}$ is simply the wave momentum flux (S_{11}) multiplied by the velocity of the frame of reference.

Now in the case of an accelerated frame of reference, the situation is similar except that in the momentum terms, the acceleration appears instead of the velocity of the reference frame.

For the particles: (whole system; external forces F)

$$\frac{d}{dt}\left\{ \frac{1}{2}\mathcal{U}^2 \Sigma m + \mathcal{U}\Sigma mu + \frac{1}{2}\Sigma mu^2 \right\} = \Sigma F.(\mathcal{U}+u) \qquad (5.99)$$

An analogy is needed to (5.52) which is for "wave energy", not for the whole flow. Therefore a subtraction of the form (recall (5.49) will be made to retain terms relative to the moving reference frame. This requires a particle momentum flux equation.

Particle momentum flux: $$\Sigma m . \frac{d}{dt}(\mathcal{U}+u) = \Sigma F \qquad (5.100)$$

Now $\mathcal{U}$.(5.100) ⇒ $$\mathcal{U}.\Sigma m\left(\frac{d\mathcal{U}}{dt}+\frac{du}{dt}\right) = \Sigma F.\mathcal{U} \qquad (5.101)$$

and (5.99) - (5.101) ⇒ particle energy relative to accelerating reference frame: $\frac{d}{dt}\left\{\frac{1}{2}mu^2\right\} + (\Sigma mu)\frac{dU}{dt} = \Sigma F_u$ (5.102)

This equation shows that in an accelerating reference frame, the particle kinetic energy is not conserved and the momentum (relative to the accelerating reference frame) appears, multiplied by the acceleration (not velocity as in the previous case) of the frame. Comparing this to (5.52), the analogies are apparent.

$$\frac{d}{dx}\left\{E(U+c_g)\right\} + S_{11}\frac{dU}{dx} = 0 \qquad (5.52)$$

The change in wave energy flux in the accelerated motion, (due to current non-uniformity) is again influenced by the momentum flux (S_{11}) as in the uniform current case, but now multiplied by an acceleration term for the current. This shows why the alternative possibilities (5.90), (5.91) for this equation proved to be incompatible with the result of the perturbation analysis, and shows that there are analogies to the role of the radiation stress in other physical situations.

One can conclude that terms such as $S_{11}\frac{dU}{dx}$ <u>may</u> be regarded as "rate of work done on the waves by the current" but that an <u>alternative</u> explanation as "rate of work done by the fictitious forces due to the accelerated reference frame" leads more directly to the form of equations such as (5.52). The corresponding term in (5.102) can be explained similarly; either as $u.\Sigma m\frac{dU}{dt}$, the rate of working of fictitious forces, or as $\Sigma mu.\frac{dU}{dt}$, the rate of working of the "radiation stress".

5.5. PROPAGATION OF CHANGES IN WAVENUMBER, MASS, MOMENTUM AND ENERGY

It is interesting to obtain an estimation of the propagation speeds of the four basic variables k, d, E and U_m (or U). These are obtained from the kinematic and dynamic equations for the flow. The time dependence must now be included, and the equations are:

kinematics: $$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad (2.13)$$

mass: $$\frac{\partial d}{\partial t} + \frac{\partial}{\partial x}\left(d U_m\right) = 0 \qquad (5.32)$$

momentum: $$\frac{\partial}{\partial t}\left(d U_m\right) + \frac{\partial}{\partial x}\left(d U_m^2 + \frac{1}{2} g d^2 + S\right) = 0 \qquad (5.33)$$

energy: $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho d U_m^2 + \frac{1}{2}\rho g d^2 + E\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho d U_m^3 + U_m S_{11} + \left(U_m + c_g\right)E\right) = 0 \qquad (5.34)$$

A solution can be obtained for the relatively simple case where the water is initially undisturbed and of uniform depth h, implying that U_m and $d - h$ are due solely to the waves and so are each $O(a^2)$, ie. $O(E)$. Linearising the governing equations to $O(E)$ gives the following equations:

$$\frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = 0 \qquad (2.38)$$

$$\frac{\partial d}{\partial t} + h\frac{\partial U_m}{\partial x} = 0 \tag{5.103}$$

$$h\frac{\partial U_m}{\partial t} + gh\frac{\partial d}{\partial x} + \frac{\partial S_{11}}{\partial x} = 0 \tag{5.104}$$

$$\frac{\partial E}{\partial t} + c_g.\frac{\partial E}{\partial x} = 0 \tag{5.105}$$

Equation (2.38) for the propagation of k is uncoupled from the dynamic equations as $c_g = \frac{\partial \sigma}{\partial k}$ is <u>independent of the amplitude</u> of the Airy wave solution considered here. (Recall discussion of Airy and Stokes dispersion relations eg. (3.77)). Clearly changes in k propagate with the group velocity c_g, as do changes in E from (5.105). This is to be expected from the previous discussion of the group velocity (section 2.1, also section 4.1; discussion following (4.17)).

Since the momentum flux S_{11} is proportional to E, changes in S_{11} must also propagate at c_g. In other words, S_{11} must obey

$$S_{11} = f_1(x - c_g.t) \tag{5.106}$$

The general solution to (5.103), (5.104) is then: (Whitham, 1962):

$$d - h = -1/(gh - c_g^2) \cdot f_1(x - c_g t) + f_2(x \pm \sqrt{gh}\,t) \tag{5.107}$$

$$U_m = -c_g/(gh - c_g^2) \cdot f_1(x - c_g t) \pm \sqrt{g/h}\, f_2(x \mp \sqrt{gh}\,t) \tag{5.108}$$

This indicates that direct changes in water depth $d-h$ and current U_m propagate at the long wave velocities $\pm\sqrt{gh}$ (from the ($x \pm \sqrt{gh}.t$) terms) but that the S_{11} term introduces changes in $d-h$ and in U_m propagating at c_g ; these changes being caused by changes in wave energy. The long wave component arises because the mass and momentum equations (5.31) and (5.32) are exactly the nonlinear shallow water wave equations except for the addition of $\frac{\partial S}{\partial x}$; the long wave solution being the limit of the Airy group velocity as $h \to 0$ ie. "long wave" or "shallow water". The presence of shallow water velocities for $d-h$, U_m is a natural consequence of the restriction on $\frac{dU}{dx}$:

$$\frac{1}{\sigma}\frac{dU}{dx} << 1$$

and therefore on $\frac{d(d)}{dx}$ to variation on a scale large compared to the wavelength $2\pi/k$ of the waves. This long wave propagation of changes in mean level is related to the momentum flux associated with a finite length wavetrain and is discussed further in 8.2.

Inclusion of higher order terms should introduce further coupling terms and hence complicate the propagation velocities for d, U_m etc. especially as higher order solutions for the waves introduce an amplitude dependence for $\omega(k)$ (see (3.77) for Stokes waves). Analysis would then require the evaluation of $\overline{P_i}$, $\overline{Q_i}$ to greater accuracy.

Conclusion

In chapter 5 solutions to the dynamics of wave/current interaction problems have been obtained, using averaged equations for mass, momentum and energy flux. The interaction between the waves and the current has been described in terms of radiation stress, the excess momentum flux due to the waves. Careful analysis of the flow as a whole has clarified the form of the wave energy equation and has provided an alternative explanation for the presence of radiation stress terms. Finally, the analysis has made it clear that wave energy is <u>not</u> a conserved quantity but has given no indication of any property that is conserved apart from the obvious ones of mass, momentum and energy for the whole flow. In the next chapter it is shown that a conserved wave property does exist, that of wave action. Interaction equations involving wave action that are equivalent to those of this section are derived and are shown to be more convenient. However, the greater sophistication involved in the derivation of wave action concepts makes the preceding analysis in terms of energy and radiation stress useful as a means of comparison.

6. WAVE ACTION AND VARIATIONAL METHODS FOR SURFACE GRAVITY WAVES

6.1. INTRODUCTION

In chapter 5 the dynamics of wave/current interactions were expressed in an equation for wave energy. An alternative equation is proposed and shown in 6.2 to be equivalent to the wave energy equation (5.52). The new equation introduces a conserved quantity for the waves, namely wave action.

The origin of the wave action concept is discussed in 6.3. This involves a variational approach to water wave problems, in which an "averaged Lagrangian" is obtained for the waves by averaging the wave Lagrangian over the phase χ . One of the consequences of this approach is that wave action equations arise naturally instead of equations for wave energy. The averaged Lagrangian was introduced by Whitham (1965) and the significance of wave action in this context was first recognised by Bretherton and Garrett (1969).

6.2. EQUIVALENCE OF WAVE ACTION AND ENERGY FORMULATIONS FOR SMALL AMPLITUDE WAVE/CURRENT INTERACTIONS

The conservation equation for wave action for small amplitude waves follows the familiar form of (3.87) ie.

$$\frac{\partial \overline{P}}{\partial t} + \frac{\partial \overline{Q}}{\partial x} = 0 \tag{3.87}$$

with the following definitions:

$$\text{wave action density} \equiv \frac{E}{\sigma} \tag{6.1}$$

$$\text{wave action flux} \equiv (u_\alpha + c_{g\alpha}) \cdot \frac{E}{\sigma} \tag{6.2}$$

The wave action conservation equation for surface gravity waves on a current $\underset{\sim}{u}$ is one form of the general wave action equation introduced by Bretherton and Garrett (1969, equation 1.9)

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \frac{\partial}{\partial x_\alpha}\left(u_\alpha + c_{g\alpha}\right)\frac{E}{\sigma} = 0 \tag{6.3}$$

where σ, E, c_g are as defined previously in 3.3. This equation will be substantiated in 6.3 .

It is now shown that (6.3) is equivalent to the generalised form of the wave energy equation (5.52) proposed by Longuet-Higgins and Stewart (1961) for small amplitude gravity waves on slowly varying currents. The wave energy equation is:

$$\frac{\partial E}{\partial t} + \left(u_\alpha + c_{g\alpha}\right)\frac{\partial E}{\partial x_\alpha} + E\left(\frac{\partial u_\beta}{\partial x_\beta} + \frac{\partial c_{g\beta}}{\partial x_\beta}\right) + S_{\gamma\delta}\frac{\partial u_\gamma}{\partial x_\delta} = 0 \tag{6.4}$$

To show the equivalence, (6.3) is first expanded:

$$\frac{1}{\sigma}\frac{\partial E}{\partial t} + E\frac{\partial}{\partial t}\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}\left(u_\alpha + c_{g\alpha}\right)\frac{\partial E}{\partial x_\alpha} + \left(u_\beta + c_{g\beta}\right)E\frac{\partial}{\partial x_\beta}\left(\frac{1}{\sigma}\right) + \frac{E}{\sigma}\left(\frac{\partial}{\partial x_\gamma}\left(u_\gamma + c_{g\gamma}\right)\right) = 0$$

$$\Rightarrow \quad \frac{\partial E}{\partial t} + E\sigma\frac{\partial}{\partial t}\left(\frac{1}{\sigma}\right) + (u_\alpha + c_{g\alpha})\frac{\partial E}{\partial x_\alpha} + (u_\beta + c_{g\beta})E\sigma\frac{\partial}{\partial x_\beta}\left(\frac{1}{\sigma}\right) + E\frac{\partial}{\partial x_\gamma}(u_\gamma + c_{g\gamma})$$

$$= 0 \qquad (6.5)$$

Comparision of (6.4) amd (6.5) shows that the following must hold if (6.3) and (6.4) are to be equal:

$$E\sigma\left\{\frac{\partial}{\partial t} + (u_\beta + c_{g\beta})\frac{\partial}{\partial x_\beta}\right\}\left[\frac{1}{\sigma}\right] = S_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \qquad (6.6)$$

The left hand side of (6.6) involves the time and spatial variation of the kinematic quantity σ and so can be analysed in terms of the ray theory introduced in 2.1. Recall that an observer moving with the local group velocity will always move along a ray and that the following relationships hold for propagation in a non-uniform time dependent medium:

dispersion relation: $\omega \equiv W(\underset{\sim}{k}, \zeta) \qquad \zeta = \zeta(\underset{\sim}{x}, t)$ (2.53)

group velocity: $c_{g_{*}\alpha} \equiv \dfrac{\partial W}{\partial k_\alpha}$ (2.29)

(the subscript "$*$" distinguishes this from the c_g appearing in (6.2), (6.3) etc. as (2.29) is the absolute group velocity in a fixed reference frame).

differentiation along a ray:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + c_{g_{*}\alpha}\frac{\partial}{\partial x_\alpha} \qquad (6.7)$$

variation of $\underset{\sim}{k}$ along a ray: $$\frac{dk_\alpha}{dt} = -\frac{\partial W}{\partial \xi} \cdot \frac{\partial \xi}{\partial x_\alpha} \tag{2.54}$$

variation of ω along a ray: $$\frac{d\omega}{dt} = \frac{\partial W}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} \tag{2.57}$$

rays in characteristic form: $$\frac{dx_\alpha}{dt} = \underset{*}{Cg_\alpha} \tag{2.52}$$

The dispersion relation (2.53) for the waves on the current is

$$W(k, \xi) = \sigma + U_\alpha k_\alpha \tag{4.8}$$

$$\therefore \underset{*}{Cg_\alpha} = \frac{\partial \sigma}{\partial k_\alpha} + U_\alpha = Cg_\alpha + U_\alpha \tag{6.8}$$

This, together with (6.7) implies that (6.6) can be expressed as:

$$E\sigma \frac{d}{dt}\left(\frac{1}{\sigma}\right) = S_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta}$$

$$\therefore -\frac{E}{\sigma}\frac{d\sigma}{dt} = S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} \tag{6.9}$$

The derivative $\frac{d\sigma}{dt}$ is now evaluated, using the dispersion relation (4.8) and the derivatives $\frac{d\omega}{dt}$, $\frac{dk}{dt}$, $\frac{dU}{dt}$ (Here $\underset{\sim}{\xi} = \underset{\sim}{d}$, and represents the water depth).

(4.8) $\Rightarrow$ $$\frac{d\sigma}{dt} = \frac{d\omega}{dt} - \frac{d}{dt}(U_\alpha k_\alpha) \tag{6.10}$$

(2.54) ⇒

$$\frac{dk_\alpha}{dt} = -k_\alpha \frac{\partial U_\alpha}{\partial x_\beta} - \frac{\partial \sigma}{\partial d_\alpha} \cdot \frac{\partial d_\alpha}{\partial x_\beta} \tag{6.11}$$

(2.55) ⇒

$$\frac{d\omega}{dt} = k_\alpha \frac{\partial U_\alpha}{\partial t} + \frac{\partial \sigma}{\partial d_\alpha} \cdot \frac{\partial d_\alpha}{\partial t} \tag{6.12}$$

(6.7) ⇒

$$\frac{dU_\alpha}{dt} = \frac{\partial U_\alpha}{\partial t} + U_\beta \frac{\partial U_\alpha}{\partial x_\beta} + c_{g\beta} \frac{\partial U_\alpha}{\partial x_\beta} \tag{6.13}$$

Hence (6.10) ⇒

$$\frac{d\sigma}{dt} = \left(k_\alpha \frac{\partial U_\alpha}{\partial t} + \frac{\partial \sigma}{\partial d_\alpha} \cdot \frac{\partial d_\alpha}{\partial t} \right) - k_\alpha \frac{\partial U_\alpha}{\partial t} + \left(U_\beta k_\alpha \frac{\partial U_\alpha}{\partial x_\beta} + U_\beta \frac{\partial \sigma}{\partial d_\alpha} \frac{\partial d_\alpha}{\partial x_\beta} \right)$$

$$= \cancel{k_\alpha \frac{\partial U_\alpha}{\partial t}} + \frac{\partial \sigma}{\partial d_\alpha} \cdot \frac{\partial d_\alpha}{\partial t} - \cancel{k_\alpha \frac{\partial U_\alpha}{\partial t}} - k_\alpha c_{g\beta} \frac{\partial U_\alpha}{\partial x_\beta} - \cancel{k_\alpha U_\beta \frac{\partial U_\alpha}{\partial x_\beta}} + \cancel{k_\alpha U_\beta \frac{\partial U_\alpha}{\partial x_\beta}} +$$

$$= \frac{\partial \sigma}{\partial d_\alpha} \left(\frac{\partial U_\alpha}{\partial t} + U_\beta \frac{\partial d_\alpha}{\partial x_\beta} \right) - k_\alpha c_{g\beta} \frac{\partial U_\alpha}{\partial x_\beta} \tag{6.14}$$

The current must satisify the continuity equation

$$\frac{\partial d_\alpha}{\partial t} + U_\beta \frac{\partial d_\alpha}{\partial x_\beta} + d_\alpha \cdot \frac{\partial U_\beta}{\partial x_\beta} = 0 \tag{6.15}$$

so finally the derivative is: $\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial d_\alpha} \cdot d_\alpha \cdot \frac{\partial U_\beta}{\partial x_\beta} - k_\alpha c_{g\beta} \frac{\partial U_\alpha}{\partial x_\beta}$

Now (6.9) becomes: $-\frac{E}{\sigma} \cdot \frac{d\sigma}{dt} = \left(-\frac{d_\alpha}{\sigma} \cdot \frac{\partial \sigma}{\partial d_\alpha} + \frac{c_{g\beta}}{c_\gamma} \right) \frac{\partial U_\gamma}{\partial x_\beta} = S_{\alpha\beta} \frac{\partial U_\alpha}{\partial x_\beta}$

This is now evaluated for the S_{11} component, assuming the waves and current are in the positive x direction. From (3.103),

$$S_{11} = \left\{ \frac{2c_g}{c} - \frac{1}{2} \right\} \qquad (3.103)$$

and so if $\frac{\partial d}{\partial x} < 0$, $\left(d : d(x) \text{ only} \right)$ it must now be shown that

$$-\frac{d}{\sigma} \cdot \frac{\partial \sigma}{\partial d} = \frac{c_g}{c} - \frac{1}{2} \,, \qquad \text{if (6.9) is to hold.}$$

Recall the Airy dispersion relation:

$$\sigma^2 = gk \tanh kd \qquad (3.66)$$

$$-\frac{d}{\sigma} \cdot 2\sigma \cdot \frac{\partial \sigma}{\partial d} = \frac{d}{\sigma^2} \cdot gk \cdot \frac{\partial}{\partial d} \left(\tanh kd \right) \quad \text{from} \quad -\frac{d}{\sigma} \cdot \frac{\partial}{\partial \sigma} \left(\sigma^2 \right)$$

$$= \frac{dgk^2 \operatorname{sech}^2 kd}{gk \tanh kd}$$

$$= \left\{ \frac{c_g}{c} - \frac{1}{2} \right\} \qquad \text{by (3.102)}$$

$\therefore$ (6.9) becomes (for the x component)

$$-\frac{E}{\sigma} \cdot \frac{d\sigma}{dt} = \left\{ \frac{c_g}{c} - \frac{1}{2} + \frac{c_g}{c} \right\} \frac{\partial U}{\partial x} = S_{11} \frac{\partial U}{\partial x} \,, \quad \text{as required.}$$

Other components follow similarly and the general form for (6.9) is given in Bretherton and Garrett (1969, appendix).

Hence the wave action conservation equation (6.3) is exactly equivalent to the wave energy equation (6.4). The greater simplicity of (6.3) is immediately apparent; for example in a time independent medium the growth of wave energy is contained in the wave action equation:

$$\left(U_\alpha + c_{g\alpha} \right) \cdot \frac{E}{\sigma} = \text{constant} \qquad (6.16)$$

Compare $$(U_\alpha + c_{g\alpha})\frac{\partial E}{\partial x_\alpha} + E\left(\frac{\partial U_\beta}{\partial x_\beta} + \frac{\partial c_{g\beta}}{\partial x_\beta}\right) + S_{\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta} = 0 \qquad (6.4)$$

which is the corresponding wave energy/radiation stress equation.

Since these equations are equivalent, wave action can be used to derive the results of chapter 5, but this is not pursued here. The wave action concept is now investigated via the variational approach to water waves and their modulation by slowly varying properties of the medium. (Whitham, 1974).

6.3. VARIATIONAL METHODS FOR SURFACE GRAVITY WAVES

Variational methods have in the past had limited application to surface water waves, although such methods have long been used in the study of Laplace's equation. One problem has been to generalise the Lagrangian for Laplace's equation in order to give the boundary conditions that are crucial for the wave solution. This was achieved by Luke (1967), whose work was presumably stimulated by Whitham's work on nonlinear wave dispersion (1965a) and the use of an "averaged variational principle" in linear and nonlinear dispersion (1965b). Later papers by Whitham (1967a), Seliger and Whitham (1968), Bretherton and Garrett (1969), Bretherton (1970), Hayes (1970, 1973), have steadily increased the scope of the variational approach to waves in fluids and to continuum mechanics in general. Much of this work is discussed in Leibovich and Seebass (1974), and by Whitham (1974). In this section, the variational methods for water waves are dealt with in three parts:

6.3.1 introduces the variational approach and the averaged variational principle. These are used to derive governing equations for the waves.

6.3.2 derives Luke's Lagrangian (1967) for surface gravity waves.

6.3.3 justifies the averaged variational principle and applies the variational methods to nonlinear (Stokes) waves.

It should be emphasised that the variational methods described here are easily applied to nonlinear waves; in fact they were derived specifically for such applications. Further discussion of the nonlinear (finite amplitude) aspects is given in chapter 7.

6.3.1. The variational principle for linear waves and the averaged Lagrangian concept

The variational principle for a function $\varphi(\underset{\sim}{x}, t)$ in a finite region R is

$$\delta J = \iint_R L(\varphi_t, \varphi_x, \varphi)\, dt\, dx = 0 \qquad (6.17)$$

where the subscripts denote differentiation with respect to t

and x respectively. (Gelfand and Fomin, (1963)). The integral J is stationary to small changes of φ by (6.17), since the first variation $\delta J = 0$. It follows that (if the boundaries of R are fixed):

$$\frac{\partial}{\partial t} L_{\varphi_t} + \frac{\partial}{\partial x_\alpha} L_{\varphi,\alpha} - L_\varphi = 0 \qquad (6.18)$$

where $L_{\varphi_t} \equiv \frac{\partial L}{\partial(\varphi_t)}$; $L_{\varphi,\alpha} \equiv \frac{\partial L}{\partial(\varphi_{x_\alpha})}$

If L includes higher derivatives in φ, the corresponding variational equation is

$$L_\varphi - \frac{\partial}{\partial t} L_{\varphi_t} - \frac{\partial}{\partial x_\alpha} . L_{\varphi,\alpha} + \frac{\partial^2}{\partial t^2} L_{\varphi_{tt}} + \frac{\partial^2}{\partial x_\alpha \partial t} L_{\varphi_{t,\alpha}} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta} L_{\varphi_{,\alpha\beta}} - \cdots = 0 \qquad (6.19)$$

and if there are a number of functions $\varphi^{(1)}(\underset{\sim}{x},t)$, $\varphi^{(2)}(\underset{\sim}{x},t)$ etc. there is an equation of this form for variations of each function.

If a system of particles is considered and L is defined as the kinetic energy minus the potential energy of the system, L is known as the Lagrangian of the system. The time integral in (6.17) is then known as the action integral and the solution of (6.17) will give the trajectories of the particles. (Gelfand and Fomin, (1963), p84).

One wishes to extend the use of (6.17) to the fluid continuum and in particular to the study of wave properties.

The problems are that:

(i) the rapid wave oscillations and slow integral property variations must be distinguished;

(ii) a Lagrangian must be found corresponding to the normal Eulerian specification of the wave and fluid motion. Unfortunately this is more difficult than with a Lgrangian approach, since individual fluid elements are distinguished in the Lagrangian specification of fluid motion.

Problem (i) is considered here; analysis of the precise form of the Lagrangian is deferred to 6.3.2.

A slowly varying linear wave has a velocity potential of the form

$$\phi_{Airy} = a \sin \chi \qquad (3.65)$$

where a , χ are each slowly varying functions of $(\underset{\sim}{x}, t)$, and for the present it is assumed that the Lagrangian will be a depth integrated function of the velocity potential and of local properties of the fluid (eg. of depth d)

$$L = L(\phi_A, d) \qquad (6.20)$$

This Lagrangian can be averaged over the phase χ as in the derivation of the wave integral properties in 3.2, with derivatives of a , ω , $\underset{\sim}{k}$ neglected as being small.

The resulting function is known as the averaged Lagrangian.

$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L(\phi_A, h)\, d\chi \qquad (6.21)$$

The crucial extension of the previous variational approach is to propose the "<u>averaged variational principle</u>" for the functions $\chi(\underset{\sim}{x},t)$, $a(\underset{\sim}{x},t)$, corresponding to (6.17) for $\varphi(\underset{\sim}{x},t)$.

$$\delta \iint \mathcal{L}(-\chi_t, \chi_x, a)\, dt\, dx = 0 \qquad (6.22)$$

(The justification of this is postponed to 6.3.3).

The implications of this approach are immediately seen from the variation equations (6.19) for $\chi(\underset{\sim}{x},t)$ and $a(\underset{\sim}{x},t)$. There are no derivatives of $a(\underset{\sim}{x},t)$ in $\mathcal{L}$, so variation of a leads to:

$$\mathcal{L}_a = 0 \qquad (6.23)$$

Variation of χ gives

$$\frac{\partial}{\partial t}\mathcal{L}_{\chi_t} + \frac{\partial}{\partial x_\alpha}\mathcal{L}_{\chi,\alpha} = 0 \qquad (6.24)$$

Only <u>derivatives</u> of χ appeared in (6.22), and so only derivatives of χ appear in (6.24). This can be used to reintroduce ω, $\underset{\sim}{k}$ (2.44, 2.45) to obtain the following set of equations for ω, $\underset{\sim}{k}$, a; equations (2.25) and (2.46) being the consistency conditions for the existence of χ.

$$\mathcal{L}_a = 0 \qquad (6.23)$$

$$\frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x_\alpha}\mathcal{L}_{k_\alpha} = 0 \qquad (6.25)$$

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

$$\frac{\partial k_\alpha}{\partial x_\beta} - \frac{\partial k_\beta}{\partial x_\alpha} = 0 \qquad (2.46)$$

The equation (6.23) is a relation between ω, $\underset{\sim}{k}$, a containing no derivatives. It is therefore the <u>dispersion</u> relation. In addition, for linear problems, the Lagrangian L is quadratic in φ and its derivatives (recall that the Lagrangian is usually kinetic minus potential energy; see also the examples in Whitham (1974), p392). Hence by (3.65), $\mathcal{L}$ (for waves on a uniform flow) will be of the form

$$\mathcal{L} = f(\omega, \underset{\sim}{k})\, a^2 \qquad (6.26)$$

Hence by (6.23) the dispersion relation must be simply

$$f(\omega, \underset{\sim}{k}) = 0 \qquad (6.27)$$

The averaged variational principle therefore includes the kinematics of the waves discussed in 2.2, although it was devised in order to find the amplitude variation. This will be obtained from the equation (6.25), using the form of (6.26) for $\mathcal{L}$.

$$\frac{\partial}{\partial t}\left(f_\omega a^2\right) + \frac{\partial}{\partial x_\alpha}\left(f_{k_\alpha} a^2\right) = 0 \qquad (6.28)$$

Since $\omega = W(k)$ (the dispersion relation in its original form), (6.27) becomes:

$$f\left(W(k), k\right) = 0 \Rightarrow f_\omega \underbrace{\frac{\partial W}{\partial k_\alpha}}_{c_{g\alpha}} + f_{k,\alpha} = 0 \qquad (6.29)$$

Use of this plus the consistency relations (2.25), (2.46) allow

(6.28) to be rearranged in "conservation equation" form for the wave amplitude. (This was the equation used in 3.3 in the discussion of wave stability):

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(c_{g_\alpha} a^2\right) = 0 \qquad (6.30)$$

This result illustrates that the averaged variational principle does provide the dynamics of linear waves in a slowly varying situation. It is the form of (6.25) that is interesting though, as one might assume that it is an energy equation (and (6.30) the expression of it in terms of amplitude). The energy equation is now derived, and shown to differ from (6.25). Since (6.25) is derived from a variational principle, Noether's theorem (Gelfand and Fomin, (1963) p177) will give the energy conservation equation if the variational principle is invariant under translation with respect to time. This is so, and the energy equation is found to be

$$\frac{\partial}{\partial t}\left(\omega \mathcal{L}_\omega - \mathcal{L}\right) + \frac{\partial}{\partial x_\alpha}\left(-\omega \mathcal{L}_{k_\alpha}\right) = 0 \qquad (6.31)$$

This equation is clearly not the same as (6.25). The comparison between these equations can be carried further. Noting that the stationary value of $\mathcal{L}$ is zero (corresponding to equipartition of energy for linear systems) and since the energy equation (6.31) is in the typical conservation form of (3.87), the energy density E and energy flux $\underset{\sim}{F}$ must be

$$E = \omega \mathcal{L}_\omega \qquad F_\alpha = -\omega \mathcal{L}_{k_\alpha} \qquad (6.32)$$

and (6.31) is:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(F\right) = 0 \qquad (6.33)$$

(Recall from the analysis preceding (5.23) that $F_\alpha = E c_{g_\alpha}$; see also Whitham (1974), p 389).

From (6.32) it is seen that

$$\mathcal{L}_\omega = \frac{E}{\sigma} \qquad (6.34)$$

and that (6.25) can be written as

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega}\right) + \frac{\partial}{\partial x_\alpha}\left(c_{g_\alpha}\frac{E}{\omega}\right) = 0 \qquad (6.35)$$

Compare the energy equation:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(c_{g_\alpha} E\right) = 0 \qquad (6.36)$$

In the simplest case of propagation through a uniform medium, ω is constant and the equations are essentially the same. The variational approach emphasises the quantity $\mathcal{L}_\omega$ $(= E/\omega)$ rather than $\omega\mathcal{L}_\omega$ $(= E)$ and it is in this way that "wave action" (wave action density $\equiv E/\omega$) was recognised as a significant quantity. Equation (6.25) (or (6.35)) is the wave action conservation equation introduced in 6.2.

A wave momentum flux equation can also be obtained using Noether's theorem since the averaged variational principle is also invariant under translation in space. The equation is very similar in form to the energy equation (6.31) but with ω and

k_α swopping roles:

$$\frac{\partial}{\partial t}\left(k_\alpha \mathcal{L}_\omega\right) + \frac{\partial}{\partial x_\beta}\left(-k_\alpha \mathcal{L}_{k_\beta} + \mathcal{L}_{\delta_{\alpha\beta}}\right) = 0 \qquad (6.37)$$

The interesting aspect here is that the momentum density I is easily related to the energy density E by this equation, since by the form of the equation:

$$\text{momentum density} = k_\alpha \mathcal{L}_\omega \qquad \text{and} \qquad \mathcal{L}_\omega = \frac{E}{\omega}$$

$$\therefore \; k_\alpha \mathcal{L}_\omega = k_\alpha \frac{E}{\omega} = \frac{E}{c}$$

$$\therefore \; E = 2T = cI \qquad (3.49)$$

The averaged variational principle again includes previously obtained results in a very elegant way.

The particular form of $\mathcal{L}$ (see (6.25)) used to illustrate equations (6.23, (6.25) is valid for a uniform medium only. The equations (6.23), (6.25) still hold for the non-uniform situation (if variation is slow) but $\mathcal{L}$ now depends <u>explicitly</u> on $\underset{\sim}{x}, t$ in addition to its dependence on $a(\underset{\sim}{x},t)$, $\chi(\underset{\sim}{x},t)$. This causes changes in the wave energy and wave momentum flux equations, which become respectively:

$$\frac{\partial}{\partial t}\left(\omega \mathcal{L}_\omega - \mathcal{L}\right) + \frac{\partial}{\partial x_\alpha}\left(-\omega \mathcal{L}_{k_\alpha}\right) = -\mathcal{L}_t \qquad (6.28)$$

$$\frac{\partial}{\partial t}\left(k_\alpha \mathcal{L}_\omega\right) + \frac{\partial}{\partial x_\beta}\left(k_\alpha \mathcal{L}_{k_\beta} + \mathcal{L}_{\delta_{\alpha\beta}}\right) = \mathcal{L}_{x_\alpha} \qquad (6.29)$$

The value of the wave action conservation equation (6.25) is now clear, since wave action alone is conserved for propagation in non-uniform media. It is shown in 6.3.3 that the averaged variational principle and the wave action concept are easily extended to nonlinear wavetrains and their modification in slowly varying media.

6.3.2. Luke's Lagrangian for periodic surface gravity waves and justification of the averaged variational principle

The difficulties with the variational principle for water waves arise from the Eulerian specification of the fluid and from the free surface boundary conditions. The Lagrangian function is usually taken as the kinetic energy minus potential energy, but it turns out that it must be taken as the pressure in this case. This approach has been followed by Clebsch (1859) and Bateman (1944) but Luke's Lagrangian (1967) was the first to provide the free surface boundary conditions.

Whitham ((1974), chapter 13.2) points out that Laplace's equation follows from the variational principle: ($\underset{\sim}{x}$ is the horizontal two dimensional vector)

$$\delta \iiint \tfrac{1}{2} \left(\nabla \phi \right)^2 d\underline{x}\, dz\, dt = 0 \qquad (6.30)$$

What is needed in addition to Laplace's equation are the fluid boundary conditions. These come from the modifications made by Luke (1967) who chose the Lagrangian L to be (using a slight notation change of Whitham's, (1974)):

$$L = -\rho \int_{-d}^{\eta} \left[\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz \right] dz \qquad (6.31)$$

in the variational principle $\delta \iint L \, d\underset{\sim}{x} \, dt = 0$ (6.32)

The additional terms (compared to (6.30)) integrate out everywhere except on the boundary.

If a small change $\delta\phi$ is considered:

$$-\delta \iint \frac{L}{\rho} \, d\underset{\sim}{x} \, dt = \iint \left\{ \int_{-d}^{\eta} \left(\delta\phi_t + \nabla\phi \cdot \nabla\delta\phi \right) dz \right\} d\underset{\sim}{x} \, dt$$

$$= \iint \overset{①}{\left\{ \frac{\partial}{\partial t} \int_{-d}^{\eta} \delta\phi \, dz + \frac{\partial}{\partial x_\alpha} \int_{-d}^{\eta} \phi_{x_\alpha} \delta\phi \right\}} d\underset{\sim}{x} \, dt - \iint \overset{②}{\left\{ \int_{-d}^{\eta} \left(\phi_{x_\alpha x_\alpha} + \phi_{zz} \right) \delta\phi \, dz \right\}} d\underset{\sim}{x} \, dt$$

$$- \iint \overset{③}{\left[\left(\eta_t + \phi_{x_\alpha}\eta_{x_\alpha} - \phi_z \right) \delta\phi \right]_{z=\eta}} d\underset{\sim}{x} \, dt + \iint \overset{④}{\left[\left(\phi_{x_\alpha} h_{x_\alpha} + \phi_z \right) \delta\phi \right]_{z=-d}} d\underset{\sim}{x} \, dt \qquad (6.33)$$

Term ① integrates out to be a boundary term and vanishes if $\delta\phi$ is chosen to vanish on the boundaries. For (6.33) to vanish for all such $\delta\phi$, then the integrands in ②, ③, ④ must vanish. Choice of $\delta\phi = 0$ leads to the following equation for ②. Now one can choose $\delta\phi > 0$ at $z = \eta$; $\delta\phi = 0$ at $z = -d$ to give ③ below and choosing $\delta\phi > 0$ at $z = -d$; $\delta\phi = 0$ at $z = \eta$ gives ④.

$$② : \quad \phi_{x_\alpha x_\alpha} + \phi_{zz} = 0 \quad -d < z < \eta \qquad (3.12)$$

$$③ : \quad \eta_t + \phi_{x_\alpha}\eta_{x_\alpha} - \phi_z = 0 \quad z = \eta \qquad (3.16)$$

$$4: \quad \phi_{x_\alpha} d_{x_\alpha} + \phi_z = 0 \qquad z = -d \tag{3.18}$$

Laplace's equation for the fluid and the kinematic boundary conditions are thereby obtained. The dynamic boundary condition (3.20) comes from a variation $\delta\eta$ in (6.32):

$$\delta \iint L \, d\underline{x} \, dt = -\rho \iint \left\{ \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\eta \right\}_{z=\eta} \delta\eta \cdot d\underline{x} \, dt$$

and the normal variational method then requires

$$\left\{ \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz \right\}_{z=\eta} = 0 \tag{3.20}$$

The form of the Lagrangian in (6.31) is simply the pressure (integrated over depth) as can be seen by comparing it to the Bernoulli equation (3.13). One can compare the more common choice for a Lagrangian of kinetic minus potential energy, which for the water wave situation would be

$$\mathcal{L} = \rho \int_{-d}^{\eta} \left[\tfrac{1}{2}(\nabla\phi)^2 - gz \right] dz \tag{6.34}$$

This would give the Laplace equation within the fluid, but would not contribute all the boundary conditions. If the kinematic boundary conditions are assumed to start with, $\mathcal{L}$ can be used to find (3.20), but this is obviously not as satisfactory as the use of L from the outset. Luke (1967) suggests that the extension of L to rotational flows is best approached by using Clebsch potentials for the velocity of the form:

$$\underset{\sim}{u} = \nabla\phi + \alpha\nabla\bar{\phi} \tag{6.35}$$

$\phi, \alpha, \bar{\phi}$ are potentials and each are functions of x, z, t. (Lamb, 1932)

This has been taken up by Jonsson (1978) for waves on a current with vertical shear.

Justification of the averaged variational principle

The averaged variational principle (6.22) provided in a compact way the governing equations for the modulation of linear waves in a slowly varying medium. It also drew attention to a conserved quantity (wave action) that is not apparent in alternative approaches. It is important to justify this variational principle and also to investigate its [illegible] use for nonlinear wavetrains.

The justification of the averaged variational principle is discussed by Whitham in a number of papers (1965(b), 1967(a), 1967(b), 1968, 1970) and also in Whitham (1974), chapter 14.4. An outline of the last reference is given here, followed by an application to nonlinear (Stokes) waves.

It is assumed that the waves are one dimensional and described by a variational principle

$$\delta \iint L(\phi_t, \phi_x, \phi)\, d\underset{\sim}{x}\, dt = 0 \qquad (6.17)$$

with the corresponding Euler equation

$$\frac{\partial}{\partial t} L_1 + \frac{\partial}{\partial x} L_2 - L_3 = 0 \qquad (6.18)$$

where $$L_1 = L_{\phi_t}\, ; \quad L_2 = L_{\phi_x}\, ; \quad L_3 = L_{\phi} \qquad (6.19)$$

The essential feature of slow modulations is the presence of two time and length scales, one being a typical period or wavelength of the wave oscillation and the second being the typical time or length scale of the modulation or variation in the medium. In order to describe the modulated wavetrain precisely, slowly varying parameters such as k, ω are regarded as functions of ϵx and ϵt respectively, where x and t are of the scale of the wavelength or period and ϵ is a small parameter. No restriction on the magnitude of the amplitude is made but it is also slowly varying.

The variation of the potential ϕ on both the slow and rapid time scales is important, so it is written explicitly as a function of χ and of ϵx, ϵt. Choosing $\chi = \frac{1}{\epsilon}\Theta(\epsilon x, \epsilon t)$ will give the fast oscillations plus the correct dependence of $k(\equiv \chi_x)$ and $\omega(\equiv -\chi_t)$ on $\epsilon x, \epsilon t$. The above choices are listed as:

$$X = \epsilon x \tag{6.36}$$

$$T = \epsilon t \tag{6.37}$$

$$\phi = \Phi(\chi, X, T; \epsilon) \tag{6.38}$$

$$\chi = \frac{1}{\epsilon}\Theta(X, T) \tag{6.39}$$

These choices, together with the following two definitions ensure that in the expressions for ϕ_t and ϕ_x below, the two scales of motion appear separately.

$$k(X,T) = \Theta_x \tag{6.40}$$

$\nu(X,T) = -\omega(X,T)$ (change of sign in ω for symmetry with Θ_x)

$$= \Theta_t \tag{6.41}$$

$$\frac{\partial \phi}{\partial x} = k\frac{\partial \Phi}{\partial \chi} + \frac{\partial \Phi}{\partial X} \tag{6.42}$$

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial \Phi}{\partial \chi} + \epsilon\frac{\partial \Phi}{\partial T} \tag{6.43}$$

The aim of setting up the above system is to be able to use $\phi = \Theta(\chi, X, T)$ instead of the original form $\phi = \phi(x,t)$ in various parts of the analysis, although finally the original form with two independent variables is retrieved.

Substitution of (6.38), (6.42), (6.43) into the Euler equation (6.18) gives

$$\nu\frac{\partial L_1}{\partial \chi} + \epsilon\frac{\partial L_1}{\partial T} + k\frac{\partial L_2}{\partial \chi} + \epsilon\frac{\partial L_2}{\partial X} - L_3 = 0 \tag{6.44}$$

where the L_i are in terms of

$$L_i = L_i\Big(\overset{\phi_t}{\nu\Phi_\chi + \epsilon\Phi_T},\ \overset{\phi_x}{k\Phi_\chi + \epsilon\Phi_X},\ \overset{\phi}{\Phi}\Big) \tag{6.45}$$

Although the expression $\chi = \frac{1}{\epsilon}\Theta(X,T)$ was used to obtain (6.44), it is now replaced by regarding χ as an independent variable and not as a function of (X,T). The dependence of ν, k through $\Theta(X,T)$ on χ is also removed.

One now looks for solutions for $\Phi(\chi,X,T)$ and $\Theta(X,T)$ and if these are found, then $\Phi(\epsilon^{-1}\Theta,X,T)$ will be a solution of the original problem. It is found that in the process of solving the two equations, analogies with the "average variational principle" appear. There is some flexibility in the choice of $\Theta(X,T)$ and this is used to ensure that $\Phi(\chi,X,T)$ has the correct behaviour (removal of secular terms, etc.). The solution for (6.44) is obtained by requiring Φ and its derivatives to be periodic in χ and normalising the period to 2π. Equation (6.44) can be written as

$$\frac{\partial}{\partial \chi}\left\{\left(\nu L_1 + k L_2\right)\Phi_\chi - L\right\} + \epsilon\frac{\partial}{\partial T}\left(\Phi_\chi L_1\right) + \epsilon\frac{\partial}{\partial X}\left(\Phi_\chi L_2\right) = 0 \quad (6.45)$$

Integration of this equation over one period will remove the first term by the periodicity requirement, leaving

$$\frac{\partial}{\partial T}\frac{1}{2\pi}\int_0^{2\pi}\Phi_\chi L_1\,d\chi + \frac{\partial}{\partial X}\frac{1}{2\pi}\int_0^{2\pi}\Phi_\chi L_2\,d\chi = 0 \quad (6.46)$$

The equations (6.44) and (6.46) are the two equations for $\Phi(\chi,X,T)$ and $\Theta(X,T)$. These equations can be rewritten (using (6.45)) respectively as

$$\frac{\partial}{\partial \chi}L_{\Phi_\chi} + \frac{\partial}{\partial X}L_{\Phi_X} + \frac{\partial}{\partial T}L_{\Phi_T} - L_\Phi = 0 \quad (6.47)$$

$$\frac{\partial}{\partial T}\bar{L}_\nu + \frac{\partial}{\partial X}\bar{L}_k = 0 \quad (6.48)$$

where $$\bar{L} = \frac{1}{2\pi}\int_0^{2\pi} L\left(\nu\Phi_\chi + \epsilon\Phi_T,\ k\Phi_\chi + \epsilon\Phi_X,\ \Phi\right)d\chi \tag{6.49}$$

In this form, (6.44) and (6.46) are in fact the variational equations (corresponding to $\delta\Phi$ and $\delta\Theta$ variations respectively) for the variational principle

$$\delta\iint \frac{1}{2\pi}\int_0^{2\pi} L\left(\nu\Phi_\chi + \epsilon\Phi_T,\ k\Phi_\chi + \epsilon\Phi_X,\ \Phi\right)d\chi\, dX\, dT = 0 \tag{6.50}$$

and this variational principle is the <u>exact</u> form of the average variational principle! In this way the variational approach is justified and it is seen to provide a concise framework for the perturbation analysis.

The lowest order approximation to (6.50) is as follows:

$$\delta\iint \mathcal{L}\, dX\, dT = 0 \tag{6.51}$$

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L\left\{\nu\Phi_\chi^{(0)},\ k\Phi_\chi^{(0)},\ \Phi^{(0)}\right\}d\chi \tag{6.52}$$

with variational equations:

$$\delta\Phi^{(0)}:\quad \frac{\partial}{\partial\chi}\left\{\nu L_1^{(0)} + k L_2^{(0)}\right\} - L_3^{(0)} = 0 \tag{6.53}$$

$$\delta\Theta:\quad \frac{\partial}{\partial T}\mathcal{L}_\nu + \frac{\partial}{\partial X}\mathcal{L}_k = 0 \tag{6.54}$$

As (6.53) contains no X, T derivatives of $\Phi^{(0)}$, it can be regarded as an ordinary differential equation for $\Phi^{(0)}$ as a function of χ, and integrated directly.

$$\left\{ \nu L_1^{(0)} + L_2^{(0)} \right\} \Phi_\chi^{(0)} - L^{(0)} = \underbrace{A(X,T)}_{\text{an amplitude-related function}} \tag{6.55}$$

This equation (or (6.53)) is the ordinary differential equation describing a uniform periodic wavetrain, except that it is now described using ν, k, A which are functions of (X,T). This (X,T) dependence controls the slow variation of the wavetrain, whereas the χ dependence (explicitly separated from (X,T)) is the same as for the periodic wavetrain.

The solution of (6.55) combined with the equations (6.51), (6.52) gives the variational approach proposed earlier, and is seen to be the first approximation in a perturbation scheme.

This completes the justification of the averaged variational principle. There are numerous points still to be considered; the best use of (6.55) (as it can be used to solve for $\Phi^{(0)}$ or for the dispersion relation between ν, k, A); the extensions to more spatial variables and the extension to more dependent variables. These and other related topics are covered in Whitham (1974) chapter 14.5-14.10. Additional dependent variables introduce features of particular significance to the propagation of mean flow quantities (such as changes in mean water level). These aspects appear in 6.3.3 where the variational principle is aplied to Stokes waves. The variational method will again provide a framework that produces all the governing equations (kinematic and dynamic), unlike other approaches that consider the various elements of the motion separately.

6.3.3. The averaged variational principle applied to Stokes waves on a mean current

The averaged Lagrangian for Stokes waves is first derived. It is then used to find the governing equations for the waves (in terms of ω, k, a) and for the variation of mean flow parameters (γ, $\mathcal{U}$, depth d). The variational method separates these two groups (ω, k, a); $(\gamma, \mathcal{U}, d)$ in a particularly interesting way and the final equations show wave, mean flow and interaction terms in a clear and concise framework.

Derivation of the averaged Lagrangian for Stokes waves

The general form for a periodic irrotational wavetrain is given by

$$\phi = \Phi(\chi, z) + \beta x - \gamma t \tag{3.55}$$

$$\chi = kx - \omega t \tag{2.25}$$

This was the basis for the derivation of the Stokes and Airy theories in 3.3.

In order to derive the averaged Lagrangian $\mathcal{L}$ (6.21) for Stokes waves, (3.55) and (2.25) are inserted into Luke's Lagrangian (6.31) and the Stokes approximations for Φ, η are applied. The result is averaged over χ and approximated to retain the lowest order nonlinear interaction terms. This finally gives the form for $\mathcal{L}$. (Whitham, (1974) chapter 16.6).

averaged Lagrangian:

$$\mathcal{L} \equiv \frac{1}{2\pi}\int_0^{2\pi} L \, d\chi \tag{6.21}$$

Luke's Lagrangian:
$$L = -\rho\int_{-d}^{\eta(x)} \left[\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz\right] dz \tag{6.31}$$

Use of (3.55), (2.25):

$$L = \rho\int_{-d}^{\eta} \left[\gamma + \omega\Phi_\chi - \tfrac{1}{2}\left(\beta + k\Phi_\chi\right)^2 - \tfrac{1}{2}\Phi_z^2 - gz\right] dz$$

$$= \rho\left[\left(\gamma - \tfrac{1}{2}\beta^2\right)d - \tfrac{1}{2}gd^2 + (\omega - \beta k)\int_{-d}^{\eta}\Phi_\chi dz\right. \tag{6.56}$$

The nonlinear interaction effects first appear as terms of $O(a^4)$ in $\mathcal{L}$ and the solution for $\mathcal{L}$ to this order of accuracy is found to be

$$\mathcal{L} = \rho\left(\gamma - \tfrac{1}{2}\beta^2\right)d - \tfrac{1}{2}\rho g d^2 + \tfrac{1}{2}E\left(\frac{(\omega-\beta k)^2}{gk\tanh kd} - 1\right)$$

$$- \frac{1}{2}\frac{k^2E^2}{\rho g}\left\{\frac{9T^4 - 10T^2 + 9}{8T^4}\right\} + O(E^3) \tag{6.57}$$

where: mean depth $d \equiv h + b$ (6.58)

mean surface level $b \equiv$ height of mean profile $\bar{\eta} = 0$ above undisturbed water level $z = 0$.

$$T \equiv \tanh kd \tag{6.59}$$

$$E = \frac{1}{2}\rho g a^2 \qquad (3.86)$$

The retention of β (representing a depth independent current (3.30)) in the averaged Lagrangian may be surprising since it was identically zero in the derivation of the Stokes and Airy theories (3.56). It is found that the restriction on β in the variational approach is that changes in β must be slow, but that the magnitude of β is not limited to (say) $O(a^2)$ values associated with the waves. is therefore used to represent the slowly varying main stream, which in the notation of chapter 5 corresponds to choosing

$$\beta = \mathcal{U}(x) \qquad (6.60)$$

Although the depth d is used in (6.57), it can be replaced in the I terms by h, (the depth in the absence of currents) since this will not affect the final solution to $O(a^2)$. Elsewhere d must be used.

Variational equations for the averaged Lagrangian

The averaged Lagrangian is a function $\mathcal{L}(\omega, k, E, \gamma, \mathcal{U}, d)$. The first triad (ω, k, E) is essentially derived from derivatives of the periodic potential $\Phi(\chi, z)$ (3.55); (2.25)

$$\phi = \Phi(\chi, z) + \mathcal{U}x - \gamma t \qquad (3.55)$$

$$\chi(x,t) = kx - \omega t \qquad (2.25)$$

and the effect as far as the averaged Lagrangian is concerned (Whitham (1974) chapter 14.7) is to extend the average variational principle to

$$\delta \iint \mathcal{L}(\omega, k, E\,;\, \gamma, \mathcal{U}, d) = 0 \qquad (6.67)$$

There are now four variational equations for $\delta E, \delta\chi, \delta d, \delta\psi$ with a further two consistency conditions: (compare the linear wave case with only ω, k, a ; equations (6.22), (6.23), (6.25)).

$$\mathcal{L}_E = 0 \qquad \text{equivalent to (6.22)}$$

$$\frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x}\mathcal{L}_k = 0 \qquad \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad (6.25)$$

$$\mathcal{L}_d = 0 \qquad (6.68)$$

$$\frac{\partial}{\partial t}\mathcal{L}_\gamma - \frac{\partial}{\partial x}\mathcal{L}_\mathcal{U} = 0 \qquad \frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \gamma}{\partial x} = 0 \qquad (6.69)$$

These variational equations contain the kinematic and dynamic conservation equations for the flow. The particular form of the equations for Stokes waves is obtained by taking the appropriate derivatives of the Stokes averaged Lagrangian (6.57).

(1) <u>Dispersion relation:</u>

$$\mathcal{L}_E = 0 \Rightarrow \frac{(\omega - \mathcal{U}k)^2}{gk\tanh kd} = 1 + \frac{9T^4 - 10T^2 + 9}{4T^4}\cdot\frac{k^2E}{\rho g} + O(\epsilon)^2 \qquad (3.76)$$

This is precisely the form quoted in 3.3, equation (3.76), and is the nonlinear equivalent of (6.22).

(2) $\mathcal{L}_d = 0 \Rightarrow \gamma = \frac{1}{2}\mathcal{U}^2 + gd + \frac{1}{2}\left(1 - \Sigma^2\right)\frac{kE}{\rho} + O(E^2)$

and substitution of the linear theory results for c, c_g gives:

$$\gamma = \frac{1}{2}\mathcal{U}^2 + gd + \underbrace{\left(\frac{c_g}{c} - \frac{1}{2}\right)\cdot\frac{E}{\rho d}}_{S_{22}/\rho d} + O(E^2) \tag{6.70}$$

This equation is a Bernoulli style of equation for the mean flow potential Ψ (instead of Φ) with the waves influencing the mean surface level through the "excess pressure" component of the wave momentum flux S.

(3) Wave action conservation (6.25):

$$\mathcal{L}_\omega = \frac{E\overbrace{(\omega - \mathcal{U}k)}^{\sigma}}{\underbrace{gk\tanh kd}_{\sigma^2_{A\,iry}}} = \frac{E}{\sigma_A} + O(E^2) \tag{6.71}$$

$$-\mathcal{L}_k = \frac{E\,\mathcal{U}(\omega - \mathcal{U}k)}{\sigma_A^2} + \frac{1}{2}\frac{E(\omega - \mathcal{U}k)^2}{\sigma_A^4}\frac{d}{dk}(\sigma_A)^2 + O(E^2)$$

$$= \frac{E}{\sigma_A}\left(\mathcal{U} + c_g\right) + O(E^2) \tag{6.72}$$

$$\frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x}\mathcal{L}_k = 0 \Rightarrow \frac{\partial}{\partial t}\left(\frac{E}{\sigma_A}\right) - \frac{\partial}{\partial x}\left(\frac{E}{\sigma_A}\cdot(\mathcal{U} + c_g)\right) = 0 \tag{6.3}$$

This is the original form of the terms in the wave action equation

(6.25) derived by Bretherton and Garrett (1969).

(4) Mass conservation (6.69):

mass density $\mathcal{L}_\gamma = \rho d$

mass flux $-\mathcal{L}_\mathcal{U} = \rho d \mathcal{U} + \frac{E}{cd} + O(\epsilon^1)$ (6.73)

The general "conservation form" (3.87) suggests (as in chapter 5) the reconstruction of the mass transport velocity $\mathcal{U}_m$ (5.24):

$$\frac{\partial \bar{P}_o}{\partial t} + \frac{\partial \bar{Q}_o}{\partial x} = 0 \qquad (3.87)$$

$$\mathcal{U}_m \equiv \frac{\bar{Q}_o}{\bar{P}_o}$$

$$= \frac{\mathcal{L}_\gamma}{-\mathcal{L}_\mathcal{U}} = \frac{E}{c} + \mathcal{U} \qquad (5.24)$$

$$\therefore \quad \frac{\partial}{\partial t}\mathcal{L}_\gamma + \frac{\partial}{\partial x}\left(-\mathcal{L}_\mathcal{U}\right) = 0 \Rightarrow \frac{\partial}{\partial t}\left(\rho d\right) + \frac{\partial}{\partial x}\left(\rho d \mathcal{U}_m\right) = 0 \qquad (5.32)$$

Since the variations have been completed, this is the full set of equations required for the flow (plus the two consistency conditions). It is noteworthy that the wave action equation plus the mean flow consistency condition have appeared instead of momentum and energy conservation equations. The latter pair can be obtained from the averaged Lagrangian; an application of Noether's theorem is required as in 6.3.2. The results are:

$$\text{momentum density} = k\mathcal{L}_\omega + \mathcal{U}\mathcal{L}_\gamma$$

$$\text{momentum flux} = -k\mathcal{L}_k - \mathcal{U}\mathcal{L}_u + \mathcal{L}$$

$$\text{energy density} = \omega\mathcal{L}_\omega + \gamma\mathcal{L}_\gamma - \mathcal{L}$$

$$\text{energy flux} = -\omega\mathcal{L}_k - \gamma\mathcal{L}_u$$

These terms are easily calculated to $O(\varepsilon^2)$ using the Stokes averaged Lagrangian. The results are identical to those obtained for $\bar{P}_1, \bar{Q}_1, \bar{P}_2, \bar{Q}_2$ in chapter 5, and hence the conservation equations (5.33), (5.34) are obtained. (see below)

Comparison of the two sets of governing equations

The averaged Lagrangian leads naturally to the following form of the variational equations (6.25), (3.192) with ω and γ as in (1) and (2)

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \frac{\partial}{\partial x}\left((\mathcal{U}+g)\frac{E}{\sigma}\right) = 0 \qquad \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \qquad (6.3)$$

$$\frac{\partial}{\partial t}(\rho d) + \frac{\partial}{\partial x}\underbrace{\left(\rho d \mathcal{U} + \frac{E}{c}\right)}_{\rho d\, \mathcal{U}_m} = 0 \qquad \frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \gamma}{\partial x} = 0 \tag{5.32}$$

The equivalent energy equations, derived either from Noether's theorem plus the above variational equations, or from chapter 5 are:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(\rho d) + \frac{\partial}{\partial x}(\rho d\, \mathcal{U}_m) = 0 \tag{5.32}$$

$$\frac{\partial}{\partial t}(\rho d\, \mathcal{U}_m) + \frac{\partial}{\partial x}\left(\rho d\, \mathcal{U}_m^2 + \tfrac{1}{2}\rho g d^2 + S_{11}\right) = 0 \tag{5.33}$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho d\, \mathcal{U}_m^2 + \tfrac{1}{2}\rho g d^2 + E\right) + \frac{\partial}{\partial x}\left(\rho d \mathcal{U}\left(\tfrac{1}{2}\mathcal{U}_m^2 + g d\right) + \mathcal{U} S_{11} + (\mathcal{U} + c_g)E\right) = 0 \tag{5.34}$$

The relative simplicity of the wave action formulation is immediately apparent, particularly for steady flows ($\frac{\partial}{\partial t} = 0$). One apparent disadvantage of the variational approach is that it appears to be restricted to irrotational flow in both the wave and curent motion whereas the energy equations can be generalised to cope with rotational currents. It is shown in chapter 7 that the wave action approach can be extended for rotational currents with some adjustment to the consistency conditions for the mean flow. (Stiassine and Peregrine, 1979).

7. CONSERVATION EQUATIONS FOR FINITE AMPLITUDE WAVES ON SLOWLY VARYING ROTATIONAL CURRENTS

The previous chapter has established the variational approach to water waves and has compared the wave action equations to the energy approach of chapter 5. Crapper (1979) discusses this comparison for the finite amplitude form of the two sets of equations (the variational equations of chapter 6 and the energy equation set given by Phillips (1980)). He shows that the wave action formulation can be manipulated into the form of the finite amplitude energy equations. He points out that the reverse procedure is not generally possible as the variational equations specifically require the main flow to be irrotational (6.64), whereas large scale current vorticity is allowed in the averaged equations.

Stiassine and Peregrine (1979) start from Phillips' set of averaged equations and derive extended forms of Whitham's variational equations valid for rotational currents. Wave action still appears as a conserved quantity but the consistency conditions for the current are extended.

In this chapter, Stiassnie and Peregrine are followed and the generalised wave action formulation is derived. This is done in two stages:

(i) the averaged equations for mass, momentum and energy for finite amplitude waves in slowly varying media are derived. These are similar to the corresponding equations of chapter 5 but are expressed in terms of the integral properties defined in 3.2 and retain all nonlinearities.

(ii) the averaged equations of (i) are modified until a wave action equation is found. The accompanying consistency conditions are then generalised to allow rotational currents.

7.1. DERIVATION OF MASS, MOMENTUM AND ENERGY EQUATIONS FOR FINITE AMPLITUDE WAVETRAINS IN SLOWLY VARYING MEDIA

The equations governing the propagation of slowly varying finite amplitude wavetrains are expressed in terms of the following dependent variables (with their usual interpretation)

local properties: $h \equiv$ depth in absence of current

$d \equiv$ depth in presence of current

$d = b + h$ where $b \equiv \bar{\eta}$

kinematic variables: $\underset{\sim}{k}$, ω

dynamic variables: a, $\underset{\sim}{u}$

The kinematic variables obey the governing equations of 2.2, 4.2; namely consistency conditions for the existence of the phase function χ and the wave dispersion relation $\sigma(a, k, d)$ modified by the presence of a current.

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

$$\frac{\partial k_1}{\partial x_2} = \frac{\partial k_2}{\partial x_1} \qquad (2.46)$$

$$\omega = \sigma + u_\alpha k_\alpha \qquad (4.8)$$

These equations are common to both the Phillips and Whitham approaches. The relation (4.8) links the kinematics to the dynamics for finite amplitude waves since the dispersion relation involves the amplitude a . The dynamic equations for mass, momentum and energy conservation are obtained by the following standard procedure:

Procedure for deriving averaged mass,momentum and energy equations

(a) The differential form of the fluid conservation equations from chapter 3 form the starting point for the derivations:

mass conservation: $$\frac{\partial u_i}{\partial x_i} = 0 \qquad (3.1)$$

momentum conservation:

$$\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij}\right) - \rho g \delta_3^i = 0 \qquad (3.2)$$

energy conservation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\, \mathcal{U}_i \mathcal{U}_i + \rho g z\right) + \frac{\partial}{\partial x_j}\left(\mathcal{U}_j\left(p + \frac{1}{2}\rho\, \mathcal{U}_k \mathcal{U}_k + \rho g z\right)\right) = 0 \qquad (3.4)$$

(b) The fluid velocity $\underset{\sim}{\mathcal{U}}$ is split into $\underset{\sim}{U} + \underset{\sim}{u}$, the current and wave velocities respectively.

(c) The equations are averaged over depth and the x, t derivatives taken outside the z integration. The boundary terms that appear are simplified using the kinematic boundary conditions (3.16), (2.46)

$$w(\eta) = \frac{\partial \eta}{\partial t} + \mathcal{U}_\alpha(\eta)\frac{\partial \eta}{\partial x_\alpha} \qquad (3.16)$$

$$w(-h) = -\mathcal{U}_\alpha(-h)\frac{\partial h}{\partial x_\alpha} \qquad (2.46)$$

(d) The integral properties (3.37)-(3.41) are used and the equations are manipulated into the "conservation form" (3.87).

$$I_\alpha \equiv \rho \overline{\int_{-h}^{\eta} u_\alpha dz} \qquad (3.37)$$

$$T \equiv \frac{\rho}{2} \overline{\int_{-h}^{\eta} \left(u_\beta u_\beta + w^2\right) dz} \qquad (3.38)$$

$$V \equiv \frac{1}{2}\rho g\left[\overline{\eta^2} - (d-h)^2\right] \qquad (3.39)$$

$$S_{\alpha\beta} \equiv \overline{\int_{-h}^{\eta} \left(\rho u_\alpha u_\beta + p\delta_\beta^\alpha\right) dz} - \frac{1}{2}\rho g d^2 \delta_{\alpha\beta} \qquad (3.40)$$

$$F_\alpha \equiv \overline{\int_{-h}^{\eta} u_\alpha\left[\left(\tfrac{1}{2}\rho u_\beta u_\beta + w^2\right) + p + \rho g(z+h-d)\right]dz} \qquad (3.41)$$

Conservation of total mass:

From (3.1):
$$\int_{-h}^{\eta}\left(\frac{\partial U_\alpha}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_\alpha} + \frac{\partial w}{\partial z}\right)dz = 0$$

$$\therefore \int_{-h}^{\eta}\left(\frac{\partial U_\alpha}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_\alpha}\right)dz = -w(\eta) + w(-h)$$

$$\therefore \frac{\partial}{\partial x_\alpha}\int_{-h}^{\eta} U_\alpha dz + \frac{\partial}{\partial x_\alpha}\int_{-h}^{\eta} u_\alpha dz - (U_\alpha + u_\alpha)\frac{\partial \eta}{\partial x_\alpha} - (U_\alpha + u_\alpha)\frac{\partial h}{\partial x_\alpha} = -w(\eta) + w(-h)$$

Use of the kinematic free surface and bottom conditions (3.16), (3.18) shows that

$$\frac{\partial}{\partial x_\alpha}\left[U_\alpha(\eta + h)\right] + \frac{\partial}{\partial x_\alpha}\int_{-h}^{\eta} u_\alpha dz = -\frac{\partial \eta}{\partial t}$$

This equation is averaged over a wavelength, introducing the integral properties of 3.2

$$\frac{\rho}{\lambda}\int_0^\lambda \frac{\partial}{\partial x_\alpha}\left(U_\alpha(\eta+h)\right)dx + \frac{\partial}{\partial x_\alpha}\left(\frac{I_\alpha}{\lambda}\right) = -\rho\frac{\partial}{\partial t}\int_0^\lambda \eta\, dx$$

$$\frac{\partial}{\partial x_\alpha}\left(\rho U_\alpha d\right) + \frac{\partial I_\alpha}{\partial x_\alpha} = -\rho\frac{\partial d}{\partial t}$$

The mass conservation equation is

$$\frac{\partial}{\partial x_\alpha}\left\{\rho U_\alpha d + I_\alpha\right\} + \rho\frac{\partial d}{\partial t} = 0 \qquad (7.1)$$

Conservation of total momentum:

The horizontal momentum equations from the full equations (3.2) are

$$\frac{\partial}{\partial t}\left(\rho(U_\alpha + u_\alpha)\right) + \frac{\partial}{\partial x_\beta}\left(\rho(U_\alpha + u_\alpha)(U_\beta + u_\beta) + p\delta_{\alpha\beta}\right) + \frac{\partial}{\partial z}\left(\rho(U_\alpha + u_\alpha)w\right) = 0$$

$$\therefore \frac{\partial}{\partial t}\int_{-h}^{\eta} \rho(U_\alpha+u_\alpha)\,dz + \frac{\partial}{\partial x_\beta}\int_{-h}^{\eta}\left[\rho(U_\alpha+u_\alpha)(U_\beta+u_\beta)+p\delta^\alpha_\beta\right]dz + \rho(U_\alpha+u_\alpha)w\Big|_{z=\eta}$$

$$- \rho\left[(U_\alpha+u_\alpha)\frac{\partial\eta}{\partial t} + (U_\alpha+u_\alpha)(U_\beta+u_\beta)\frac{\partial\eta}{\partial T}\right] - p(\eta)\frac{\partial\eta}{\partial x_\alpha} - \rho\left[(U_\alpha+u_\alpha)(U_\beta+u_\beta)\frac{\partial h}{\partial x_\beta}\right] - p\frac{\partial h}{\partial x_\alpha}$$

$$= 0$$

(3.16), (3.18) $\Rightarrow$ cancellation of all boundary terms except the pressure terms, and the surface pressure term is removed by assuming atmospheric pressure to be negligible.

$$\therefore \frac{1}{\lambda}\int_0^\lambda\left\{\frac{\partial}{\partial t}\int_{-h}^{\eta}\rho(U_\alpha+u_\alpha)\,dz + \frac{\partial}{\partial x_\beta}\int_{-h}^{\eta}\left[\rho(U_\alpha+u_\alpha)(U_\beta+u_\beta)+p\delta^\alpha_\beta\right]dz\right\}dx = \frac{1}{\lambda}\int_0^\lambda p\frac{\partial h}{\partial x_\alpha}\,dx$$

$$\therefore \frac{\partial}{\partial t}\left(\rho U_\alpha d + I_\alpha\right) + \frac{\partial}{\partial x_\beta}\left[\rho(dU_\alpha U_\beta + U_\alpha I_\beta + U_\beta I_\alpha\right] + \frac{\partial}{\partial x_\beta}\left[\frac{1}{\lambda}\int_0^\lambda\int_{-h}^{\eta}\left(\rho u_\alpha u_\beta + p\right)\right)dz\,dx = \rho g d\frac{\partial h}{\partial x_\alpha}$$

$$\therefore \frac{\partial}{\partial t}\left(\rho U_\alpha d + I_\alpha\right) + \frac{\partial}{\partial x_\beta}\left[\left(\rho dU_\alpha + I_\alpha\right)\left(\frac{I_\beta}{\rho d} + U_\beta\right) - \frac{I_\alpha I_\beta}{\rho d}\right] + \frac{\partial}{\partial x_\beta}\left[S_{\alpha\beta} + \tfrac{1}{2}\rho g d^2\delta^\alpha_\beta\right] = \rho g d\frac{\partial h}{\partial x_\alpha}$$

The averaged momentum equation is therefore

$$\frac{\partial}{\partial t}\left(\rho U_\alpha d + I_\alpha\right) + \frac{\partial}{\partial x_\beta}\left[\left(\rho dU_\alpha + I_\alpha\right)\left(\frac{I_\beta}{\rho d} + U_\beta\right) - \frac{I_\alpha I_\beta}{\rho d} + S_{\alpha\beta} + \tfrac{1}{2}\rho g d^2\delta^\alpha_\beta\right] = \rho g d\frac{\partial h}{\partial x_\alpha} \qquad (7.2)$$

<u>Conservation of energy:</u>

Expanding (3.4) to separate the horizontal components U_α from w gives:

$$\frac{\partial}{\partial t}\left[\rho\left(\tfrac{1}{2}U_\alpha^2 + \tfrac{1}{2}w^2 + gz\right)\right]$$

$$+ \frac{\partial}{\partial x_\beta}\left[U_\beta\left(\tfrac{1}{2}\rho U_\gamma U_\gamma + \tfrac{1}{2}\rho w^2 + p + \rho g z\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[w\left(\tfrac{1}{2}\rho U_\gamma U_\gamma + \tfrac{1}{2}\rho w^2 + p + \rho g z\right)\right] = 0$$

$$\therefore \frac{\partial}{\partial t}\int_{-h}^{\eta}\left[\rho\left(\tfrac{1}{2}\mathcal{U}_\alpha^2 + \tfrac{1}{2}w^2 + gz\right)\right]dz + \frac{\partial}{\partial x_\beta}\int_{-h}^{\eta}\left[\mathcal{U}_\beta\left(\tfrac{1}{2}\rho\,\mathcal{U}_\gamma^2 + \tfrac{1}{2}\rho w^2 + p + \rho g z\right)\right]dz + \frac{\partial}{\partial z}\int_{-h}^{\eta}\left[w\left(\tfrac{1}{2}\rho\,\mathcal{U}_\gamma^2 + \ldots\right)\right]dz$$

$$\overset{*}{-\left[\rho\left(\tfrac{1}{2}\mathcal{U}_\alpha^2 + \tfrac{1}{2}w^2 + g\eta\right)\right]_{z=\eta}\frac{\partial\eta}{\partial t}} \;\overset{*}{-\left[\mathcal{U}_\mu\left(\tfrac{1}{2}\rho\,\mathcal{U}_\gamma^2 + \tfrac{1}{2}\rho w^2 + \rho g\eta + p\right)\right]_{z=\eta}\frac{\partial\eta}{\partial x_\mu}} \;\overset{*}{-\left[w\left(\tfrac{1}{2}\rho\,\mathcal{U}_\delta^2 + \tfrac{1}{2}\rho w^2 + p + \rho g z\right)\right]_{z=\eta}}$$

$$-\left[\rho\left(\tfrac{1}{2}\mathcal{U}_\alpha^2 + \tfrac{1}{2}w^2 + gz\right)\right]_{z=-h}\cancel{\frac{\partial h}{\partial t}}^{\,0} - \left[\mathcal{U}_\mu\left(\tfrac{1}{2}\rho\,\mathcal{U}_\gamma^2 + \tfrac{1}{2}\rho w^2 + \rho g z + p\right)\right]_{z=-h}\frac{\partial h}{\partial x_\mu} - \left[w\left(\tfrac{1}{2}\rho\,\mathcal{U}_\delta^2 + \tfrac{1}{2}\rho w^2 + p + \rho g z\right)\right]_{z=-h} = 0$$

The kinematic free surface condition will eliminate the three terms marked * as the terms can be collected together as follows:

$$\rho\left(\tfrac{1}{2}\mathcal{U}_\alpha^2 + \tfrac{1}{2}w^2 + g\eta\right)\Big(\underbrace{w - \mathcal{U}_\mu\frac{\partial\eta}{\partial x_\mu} - \frac{\partial\eta}{\partial t}}_{0}\Big) = 0$$

The bottom boundary terms are removed similarly, leaving the equation as $(\mathcal{U}_\alpha = U_\alpha + u_\alpha)$:

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}\left[\rho\left(\tfrac{1}{2}U_\alpha^2 + U_\alpha u_\alpha + \frac{u_\alpha^2}{2} + \frac{w^2}{2} + gz\right)\right]dz + \frac{\partial}{\partial x_\gamma}\int_{-h}^{\eta}\rho\,(1)\,(U_\gamma + u_\gamma)\,dz = 0$$

This is now averaged over a wavelength and the appropriate integral properties (3.37)-(3.41) are inserted.

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho\,U_\alpha^2 d + U_\beta I_\beta + T + \frac{1}{\lambda}\int_0^{\lambda}\int_{-h}^{\eta}\rho g z\,dz\right\} + \frac{\partial}{\partial x_\gamma}\left\{U_\gamma\left[\tfrac{1}{2}\rho d U_\alpha^2 + U_\beta I_\beta + T + \bar{p} + \tfrac{1}{2}\rho g(\bar{\eta}^2 - h^2)\right]\right\}$$
$$+ \overline{u_\gamma\left(\tfrac{1}{2}\rho(u_\alpha^2 + w^2) + p + \rho g z\right)} + \overline{\tfrac{1}{2}\rho\,U_\alpha U_\gamma u_\alpha + \rho\,U_\alpha u_\alpha u_\gamma} = 0$$

$$\therefore \frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho\,U_\alpha^2 d + U_\beta I_\beta + T + \tfrac{1}{2}\rho g(d-h)^2 - \frac{\rho g h^2}{2}\right\} + \frac{\partial}{\partial x_\gamma}\left\{U_\gamma\left[\tfrac{1}{2}\rho d U_\alpha^2 + U_\beta I_\beta + T + V + \tfrac{1}{2}\rho g(d-h)^2 - \tfrac{1}{2}\rho g h^2\right]\right\}$$
$$+ E_\gamma - g I_\gamma d(h-d) + \tfrac{1}{2}U_\beta^2 I_\gamma + S_{\gamma\beta}U_\beta + \tfrac{1}{2}\rho g d^2 U_\gamma = 0$$

Since $\frac{\partial}{\partial t}\left(\frac{\rho g h^2}{2}\right) = 0$ the energy equation is finally:

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho d U_\alpha^2 + U_\beta I_\beta + T + V + \tfrac{1}{2}\rho g(d-h)^2\right\}$$

$$+\frac{\partial}{\partial x_\alpha}\left\{U_\alpha\left[\tfrac{1}{2}\rho d U_\beta^2 + \rho g d(d-h) + T + V + U_\beta I_\beta\right] + E_\alpha + I_\alpha\left[g(d-h) + \tfrac{1}{2}U_\gamma^2 + S_{\alpha\beta}U_\beta\right\} = 0 \tag{7.3}$$

This completes the derivation of the averaged equations given in Stiassine and Peregrine (1979) for the total mass, momentum and energy in the fluid. The left hand side of the equations are arranged in the conservation form

$$\frac{\partial \bar{P}_i}{\partial t} + \frac{\partial \bar{Q}_i}{\partial x_\alpha} = 0 \tag{3.87}$$

and within each expression the order is: current, wave and interaction terms. The analogy with the equivalent equations for small amplitude waves (5.32-5.34) is apparent if one regards $U_\alpha + \dfrac{I_\alpha}{\rho d}$ as the mass transport velocity U_{m_α} (5.24).

7.2. EXTENSION OF THE AVERAGED LAGRANGIAN EQUATIONS FOR ROTATIONAL CURRENTS

In place of the momentum and energy equations, Whitham's averaged Lagrangian formulation has consistency relations (6.64, 6.65) for the pseudophase ψ (6.61) plus the wave action conservation equation (6.25). For these equations to be valid for wave propagation on rotational flows, the averaged Lagrangian $\mathcal{L}$ is first expressed in terms of the integral properties. From this, variation of the appropriate parameters gives the correct form of the equations and consistency conditions. This was done by Crapper (1979) and it is sufficient here to sketch his approach and to quote the final equations used by Stiassine and Peregrine (1979).

The wave potential ϕ and surface elevation η are defined as before, but with $b(x,t)$ explicitly included. The waves propagate in the x_1 direction.

$$\phi = \Phi(kx_1 - \omega t) + U_\alpha x_\alpha - \gamma t \qquad (3.55)$$

$$b(x,t) = \overline{\eta(x)} \qquad (7.4)$$

Substitution into Luke's Lagrangian (6.31) and averaging over a wavelength gives the average Lagrangian

$$L = -\rho \int_{-h}^{\eta} \left(\phi_t + \tfrac{1}{2}\left(\phi_{x_\alpha}\phi_{x_\alpha} + \phi_z^2\right) + gz \right) dz \qquad (6.31)$$

$$\therefore \quad \mathcal{L} = \rho\left(\gamma - \tfrac{1}{2}U^2\right)d - \rho\overline{\int_{h}^{\eta}\left(\Phi_t + U_1\Phi_{x_1}\right)dz} - T - V - \tfrac{1}{2}\rho g\left(b^2 - h^2\right)$$

But $$\Phi_t = -\frac{\omega}{k}\Phi_{x_1} = -(c + U_1)\,\Phi_{x_1} \qquad \left(c = \frac{\sigma}{k}\right)$$

$$\therefore \quad -\rho\overline{\int_{-h}^{\eta}\left(\Phi_t + U_1\Phi_{x_1}\right)dz} = \rho c\overline{\int_{-h}^{\eta}\Phi_{x_1}} = cI_1 = 2T \qquad (3.49)$$

The averaged Lagrangian $\mathcal{L}$ is therefore given by

$$\mathcal{L} = \rho\left(\gamma - \tfrac{1}{2}U_\alpha^2\right)d - \tfrac{1}{2}\rho g\left(b^2 - h^2\right) + T - V \qquad (7.5)$$

This can now be used in the "averaged variational principle" to derive the kinematic and dynamic equations governing the flow, as chapter 6. The equations now contain integral properties and can be considerably rearranged using the exact relations between the integral properties (3.49-3.51). The most important equations here are the expression for γ and the wave action equation, which are respectively: $\left(u_b = u(x, -h) \right)$

$$\gamma = g(d-h) + \tfrac{1}{2} U_\alpha U_\alpha + \tfrac{1}{2} \overline{u_b^2} \qquad (7.6)$$

$$\frac{\partial}{\partial t}\left(\frac{I}{k}\right) + \frac{\partial}{\partial x_\alpha}\left\{ U_\alpha \frac{I}{k} + \left(3T - 2V + \tfrac{1}{2}\rho d\, \overline{u_b^2}\right)\frac{k_\alpha}{k^2}\right\} = 0 \qquad (7.7)$$

The γ term appears in Whitham's main-flow consistency conditions, which are as follows, and require an irrotational current.

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial \gamma}{\partial x_\alpha} = 0 \qquad (6.65)$$

$$\frac{\partial U_1}{\partial x_2} = \frac{\partial U_2}{\partial x_1} \qquad (6.64)$$

A method is now outlined whereby the wave action equation in the form (7.7) is shown to hold for <u>rotational</u> currents. New consistency conditions are found which replace (6.64), (6.65) above.

Stiassine and Peregrine (1979) start with the mass and energy equations (7.1, 7.3) and transform the energy equation into a form similar to the wave action equation (7.7):

$$\frac{\partial}{\partial t}\left(\frac{I}{k}\right) + \frac{\partial}{\partial x_\alpha}\left\{\frac{I}{k}\, U_\alpha + \left(3T - 2V + \tfrac{1}{2}\rho d\overline{u_b^2}\right)\frac{k_\alpha}{k^2}\right. \qquad (7.8)$$

$$\left. + \frac{1}{W}\left(\rho d U_\alpha + I_\alpha\right)\left(\frac{\partial U_\alpha}{\partial t} + \frac{\partial \gamma}{\partial x_\alpha}\right) = 0\right\}$$

A similar manipulation of the momentum equation leads to:

$$\frac{k_\alpha}{\rho d}\left\{\frac{\partial}{\partial t}\left(\frac{I}{k}\right) + \frac{\partial}{\partial x_\beta}\left[\frac{I}{k} U_\beta + \left(3T + 2V + \tfrac{1}{2}\rho d\overline{u_b^2}\right)\frac{k_\alpha}{k^2}\right]\right\}$$

$$+ \frac{\partial \gamma}{\partial x_\alpha} + \left(\frac{\partial U_\alpha}{\partial x_\beta} - \frac{\partial U_\beta}{\partial x_\alpha}\right)\left(U_\beta + \frac{I_\beta}{\rho d}\right) = 0 \qquad (7.9)$$

The $\{\ \}$ bracketed term is the left hand side of the wave action equation and the remaining terms in (7.8), (7.9) have similarities to the original consistency conditions (6.64), (6.65). The $\{\ \}$ bracketed term is eliminated from these equations by subtraction and this yields:

$$\left(\rho d U_\alpha + I_\alpha\right)\left(\frac{\partial U_\alpha}{\partial t} + \frac{\partial \gamma}{\partial x_\alpha}\right) = 0 \qquad (7.10)$$

Using (7.10) in (7.8) leaves just the wave action equation (7.7) and proves it to be valid on a rotational current.

Returning to (7.9) and using (7.7), the following equations remain:

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial \gamma}{\partial x_\alpha} + \left(\frac{\partial U_\alpha}{\partial x_\beta} - \frac{\partial U_\beta}{\partial x_\alpha}\right)\left(U_\beta + \frac{I_\beta}{\rho d}\right) = 0 \qquad (7.11)$$

These equations (7.10), (7.11) therefore <u>replace</u> Whitham's (6.64), (6.65) and make the wave action approach fully consistent with Phillips' energy approach.

(7.11) can be rearranged as

$$\frac{\partial}{\partial t}(U_\alpha) + U_\beta \frac{\partial U_\alpha}{\partial x_\beta} + g \frac{\partial}{\partial x_\alpha}(d-h) + \left(\frac{\partial U_\alpha}{\partial x_\beta} - \frac{\partial U_\beta}{\partial x_\alpha}\right)\frac{I_\beta}{\rho d} + \frac{1}{2}\frac{\partial}{\partial x_\alpha}\left(\overline{u_b^2}\right) = 0 \qquad (7.12)$$

In this form, the wave influences are contained in the three right hand terms.

The full wave action set of equations governing the wave/current interaction is as follows:

kinematics

consistency for $\underset{\sim}{k}$ $$\frac{\partial k_\alpha}{\partial x_\beta} - \frac{\partial k_\beta}{\partial x_\alpha} = 0 \qquad (2.46)$$

wave conservation $$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \qquad (2.25)$$

Doppler shift $$\omega = \sigma + k_\alpha U_\alpha \qquad (4.8)$$

dynamics

mass conservation $$\rho \frac{\partial d}{\partial t} + \frac{\partial}{\partial x_\beta}\left(\rho d U_\beta + I_\beta\right) = 0 \qquad (7.1)$$

consistency $$\frac{\partial U_\alpha}{\partial t} + U_\beta \frac{\partial U_\alpha}{\partial x_\beta} + g \frac{\partial}{\partial x_\alpha}(d-h) + \left(\frac{\partial U_\alpha}{\partial x_\beta} - \frac{\partial U_\beta}{\partial x_\alpha}\right)\frac{I_\beta}{\rho d} + \frac{1}{2}\frac{\partial}{\partial x_\alpha}\left(\overline{u_b^2}\right) = 0 \qquad (7.12)$$

$$\gamma = g(d-h) + \frac{1}{2} U_\alpha U_\alpha + \frac{1}{2}\overline{u_b^2} \qquad (7.6)$$

wave action conservation (7.7)

$$\frac{\partial}{\partial t}\left(\frac{I}{k}\right) + \frac{\partial}{\partial x_\alpha}\left\{ \frac{U_\alpha I}{k} + \left(3T - 2V + \frac{1}{2}\rho d \overline{u_b^2}\right)\frac{k_\alpha}{k}\right\} = 0$$

The system of equaions can be further simplified for many applications, and an example of this is now given.

Consider a steady flow ($\partial/\partial t = 0$) with no variation in the x_2 direction ($\partial/\partial x_2 = 0$) and assume the waves to be propagating in the x_1 direction. The above equations then simplify considerably and some can be integrated immediately, giving:

Doppler shift $$\omega = \sigma + U_1 k_1 \qquad (7.13)$$

mass conservation $$q = I_1 + \rho d U_1 = \text{constant} \qquad (7.14)$$

consistency $$q \frac{d\gamma}{dx_1} = 0 \qquad (7.15)$$

$$\frac{d\gamma}{dx_1} - \frac{dU_1}{dx_1}\left\{ U_2 + \frac{I_2}{\rho d} \right\} = 0 \qquad (7.16)$$

$$\frac{dU_2}{dx_1} = 0 \qquad (7.17)$$

wave action $$B = \frac{1}{k}\left\{ UI + 3T - 2V + \frac{1}{2}\rho d\left(\overline{u_b^2}\right) \right\} = \text{constant} \qquad (7.18)$$

If the main stream flow is irrotational, then $\frac{dU_2}{dx_1} = 0$ and the consistency equations simplify further to

$$\gamma = \text{constant} \qquad (7.19)$$

Stiassnie and Peregrine note that if there is vorticity, it may be more convenient to use the momentum equation (7.2) for the x_1 direction than the consistency conditions given above.

This set of equations (7.13 - 7.19) is essentially the set used by Stiassnie and Peregrine (1980) to investigate the shoaling of finite amplitude periodic waves. They use u to represent the return flow due to the waves, ie. q in (7.14) is taken as zero, since there is no water flow into the beach.

A further application of these equations is discussed in 8.5. The Vocoidal water wave theory can be used within the reference frames required for these equations to solve for the wave shoaling or for wave interaction with following or opposing currents.

8. SOME FUNDAMENTALS REVIEWED AND A NEW APPLICATION OF FINITE AMPLITUDE WAVE/CURRENT EQUATIONS

The central theme of this thesis has been to develop the mathematical description of nonlinear periodic gravity waves, and to derive the equations governing changes in such waves due to inhomogeneities in the medium. Because of the variety of approaches used in certain sections, there have inevitably been some changes of perspective on various topics. Three fundamental aspects are now reconsidered, namely group velocity, momentum density and averaging processes. These are dealt with in 8.1-8.3 respectively. A general review of the thesis and of likely research trends in water wave studies is given in 8.4. The thesis is concluded in 8.5 with an application of the finite amplitude interaction equations of chapter 7 to the Vocoidal wave theory derived by Swart and Loubser (1978).

8.1. GROUP VELOCITIES FOR LINEAR AND NONLINEAR WAVES

The group velocity is the propagation velocity for important kinematic and dynamic properties of the wave. For the simplest case of linear waves, the governing kinematic and dynamic equations are completely decoupled and analysis of each of these aspects shows that the group velocity for the wavenumber propagation is identical to the group velocity for the wave energy (or amplitude). When a similar investigation is attempted for nonlinear waves, the kinematics is coupled to the dynamics through the dispersion relation because the nonlinear dispersion relation depends on the wave amplitude. The consequence of this is that the various definitions of the group velocity no longer lead to a unique result and different aspects of the nonlinear wave appear to propagate at quite different speeds. There is unfortunately little to indicate which of these velocities is the most significant. The following discussion is devoted to the definition of the various nonlinear group velocities, using the linear case as a reference. Comparisons between the definitions are made and some comments are given on the results for deep water, following Peregrine and Thomas (1979).

Linear waves

The following section is a brief resume of section 2.2 and 6.3, deriving the various aspects of the group velocity for linear waves. The kinematic conservation equation satisfied by all wave solutions is (2.25), ie. the "conservation of phase" or "conservation of waves".

$$\frac{\partial \underline{k}_\alpha}{\partial t} + \frac{\partial \underline{\omega}}{\partial x_\alpha} = 0 \qquad (2.25)$$

The linear wave restriction is introduced into (2.25) by the choice of the linear dispersion relation (2.26). The crucial property of the linear relations is that the frequency is independent of the amplitude. (Local variations in the medium such as changes in depth are neglected here).

$$\omega = W(k) \qquad (2.26)$$

Using (2.26) in (2.25), the equation governing the propagation of the wavenumber k is obtained and the kinematic definition of the group velocity appears:

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial W}{\partial k_\beta} \cdot \frac{\partial k_\beta}{\partial x_\alpha} = 0 \qquad (2.34)$$

$$c_{g\beta} \equiv \frac{\partial W(k)}{\partial k_\beta} \qquad (2.29)$$

The equation (2.34) also contains information about the trajectory followed by a particular value of $\underset{\sim}{k}$, as $\underset{\sim}{c_g}$ is also the characteristic velocity for the equation and $\underset{\sim}{k}$ propagates along rays specified by

$$c_{g\beta} \equiv \frac{dx_\beta}{dt} \qquad (2.52)$$

For the second role of the group velocity, namely the propagation velocity for wave action/energy/amplitude, a differential equation corresponding to (2.34) is required. The geneal form of this equation is the wave action conservation equation (6.25) derived using variational methods in 6.3.1.

$$\frac{\partial}{\partial t} \mathcal{L}_\omega - \frac{\partial}{\partial x_\alpha} \mathcal{L}_{k_\alpha} = 0 \qquad (6.25)$$

It is convenient to use:

$$A \equiv \mathcal{L}_\omega \qquad \text{for wave action density} \qquad (8.1)$$

$$\underset{\sim}{B} \equiv -\mathcal{L}_{\underset{\sim}{k}} \qquad \text{for wave action flux} \qquad (8.2)$$

Now (6.25) becomes

$$\frac{\partial \underline{A}}{\partial t} + \frac{\partial \underset{\sim}{B}}{\partial x} = 0 \qquad (8.3)$$

The simplifications introduced in linearising this lead to the following alternatives, with the group velocity performing the same function as in (2.34).

wave action: $$\frac{\partial}{\partial t}\left(\frac{E}{\omega}\right) + \frac{\partial}{\partial x_\alpha}\left(c_{g_\alpha} \cdot \frac{E}{\omega}\right) = 0 \qquad (6.35)$$

wave energy: $$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(c_{g_\alpha} \cdot E\right) = 0 \qquad (6.36)$$

amplitude: $$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(c_{g_\alpha} \cdot a^2\right) = 0 \qquad (6.30)$$

These quantities also propagate along the rays specified by (2.52) above.

Nonlinear waves

The <u>nonlinear</u> group velocities are derived from the characteristic form of the governing kinematic and dynamic equations. This emphasises the effect of the dispersion relation on c_g for nonlinear waves, and apart from defining (two) characteristic group velocities also gives a criterion for wave stability. The definition of a number of <u>alternative</u> group velocities follows. They are compared for deep water waves, following Peregrine and Thomas (1979). It is found that no one definition is conclusively superior to the rest.

Nonlinear wave solutions have a dispersion relation that is dependent on the wave amplitude:

$$\omega = W(\underset{\sim}{k}, a) \tag{8.4}$$

This means that the kinematic equation (2.25) can no longer be separated from the dynamics (8.3) and instead one must deal with the equations as a coupled pair.

$$\frac{\partial \underline{k}_\alpha}{\partial t} + \frac{\partial \underline{\omega}}{\partial x_\alpha} = 0 \tag{2.25}$$

$$\frac{\partial A}{\partial t} + \frac{\partial B_\alpha}{\partial x_\alpha} = 0 \tag{8.3}$$

The <u>linear</u> group velocity turned out to be the characteristic velocity for both of (2.25), (6.25), so it is reasonable to regard the characteristic velocities for the <u>nonlinear</u> case as group velocities too. The characteristic velocities can be obtained from (8.3) in a number of ways. One way is to express it as a second order equation in χ (rather than in $\omega, \underset{\sim}{k}$) using (2.25) and (8.4) (Whitham, 1974, p513):

$$p\,\chi_{tt} - 2r\,\chi_{tx} + q\,\chi_{xx} = 0 \tag{8.5}$$

where p, q, r involve second derivatives of the averaged Lagrangian $\mathcal{L}$. The characteristics are given by

$$\frac{dx}{dt} = \frac{-r \pm \sqrt{r^2 - pq}}{p} \tag{8.6}$$

If $r^2 > pq$ is assumed for the moment, the two group velocities correspond to the two solutions of (8.6) and are denoted by:

$$\frac{dx}{dt} = c_+ \text{ or } c_- \tag{8.7}$$

depending on the sign of the $\sqrt{\ }$ in (8.6). The <u>linear</u> result quoted previously is the solution to (8.6) with $r^2 = pq$ and the linear group velocity is simply the (double) characteristic velocity

$$\frac{dx}{dt} = c_g = -\frac{r}{p} \tag{8.8}$$

The importance of the characteristic group velocities is that they represent the speed at which modulations of the wavetrain will propagate. The determining factor is obviously the form of equation (8.6) and this is directly related to the stability of the wavetrain. The system is <u>hyperbolic</u> if

$$r^2 - pq > 0 \tag{8.9}$$

in which case there are two characteristic group velocities, c_+ and c_- (8.7). If the system is <u>elliptic</u>, the characteristic velocities are imaginary and the wavetrain is unstable. The question of stability was discussed in 3.3 for a nonlinear Stokes wave in deep water, which was found to be unstable. This instability persists for fully nonlinear situations and is discussed at length by Peregrine and Thomas (1979). They suggest on the basis of experiments by Lake et al (1977) that despite these instabilities, even strong modulations may settle back into a uniform (but modified) wavetrain.

For the hyperbolic wave (8.9) the two velocities c_+, c_- have the same sign but quite different magnitudes; $c_+ \to \infty$ for the steepest waves whereas c_- is relatively constant. (see graph at the end of this section). These different velocities imply the eventual splitting of any modulation into two separate disturbances, propagating at c_+ and c_- respectively.

The characteristic group velocities just defined show the difficulties that appear in the nonlinear extensions of linear concepts. It is difficult to attach physical significance to a propagation velocity such as c_+ that tends to infinity for steep waves!

A straightforward approach to the group velocity for uniform wavetrains (not modulated;all waves identical) is to define the group velocity in the following way:

$$c_g^{(1)} \equiv Q/P \tag{8.10}$$

where Q is the flux of the integral property and P the corresponding density. At first sight this definition may seem incorrect since it follows the form of the phase velocity definition rather than that of the linear group velocity: compare (2.3), (2.9), (2.23), (2.29)

$$v \equiv Q/P \tag{2.3}$$

$$\therefore c \equiv \frac{Q(x)}{P(x)} \tag{2.23}$$

$$v_g \equiv \frac{\partial Q}{\partial P} \tag{2.9}$$

$$c_g \equiv \frac{\partial Q(x)}{\partial P(x)} \tag{2.29}$$

The integral properties used in (8.7) however, are those that propagate at the group velocity in linear waves and so in each case the general relation

$$Q = Q(P) \qquad (2.6)$$

(that is used in the derivation of the group velocity (2.10)) is simply

$$Q = c_g . P \qquad \left(\text{cf. } F = c_g . E\right) \qquad (8.11)$$

and so for a uniform wavetrain

$$\frac{\partial Q}{\partial P} = \frac{Q}{P} = c_g \qquad (8.12)$$

Two obvious choices for the integral property are the wave action and the wave energy ie.

$$c_{g_\alpha}^{(A)} \equiv \frac{B_\alpha}{A} \equiv \frac{\text{action flux}}{\text{action density}} \qquad (8.13)$$

$$c_{g_\alpha}^{(E)} \equiv \frac{F_\alpha}{E} \equiv \frac{\text{energy flux}}{\text{energy density}} \qquad (8.14)$$

These are evaluated for deep water waves by Peregrine and Thomas (1979) using approximations to the accurate results tabulated by Longuet-Higgins (1975). Unfortunately, the two definitions lead to different values for $c_g^{(A)}$, $c_g^{(E)}$. This is immediately evident from the form of $c_g^{(A)}$, $c_g^{(E)}$ quoted by Peregrine and Thomas, expressed in terms of L , E which are approximations to the $\mathcal{L}$ and $\mathcal{E}$ respectively obtained from Longuet-Higgins (1975).

$$c_g^{(A)} = \frac{\omega}{2k} \cdot \frac{(E + 5L)}{(E + L)} \hat{k} \qquad (8.15)$$

$$c_g^{(E)} = \frac{\omega}{2k} \cdot \frac{(E + 5L)}{E} \hat{k} \qquad (8.16)$$

$c_g^{(E)}$ and $c_g^{(A)}$ are compared graphically at the end of this section. One test of these velocities has been made by Peregrine and Thomas, who looked at deep water current stopping velocities (4.17) hoping to find that either

$$c_g^{(A)} + U_{max} = 0$$

or

$$c_g^{(E)} + U_{max} = 0$$

would give the maximum velocity U_{max} against which the waves can propagate. They found that this maximum velocity is not obviously related to either $c_g^{(A)}$ or $c_g^{(E)}$.

A group velocity derived from a slowly varying wavetrain is described by Hayes (1973) and called the "basic group velocity" c_B . It is obtained by eliminating ω in favour of A , the action density, which is used as the measure of wave amplitude.

$$c_B \equiv \left.\frac{\partial \underset{\sim}{B}}{\partial A}\right|_k \qquad (8.17)$$

Hayes shows that this velocity is the mean of the characteristic velocities c_+ and c_- defined earlier. This means that c_B also tends to infinity for the steepest waves.

The final possibility considered here is the natural extension of the kinematic group velocity, with some measure of the amplitude kept constant.

$$c_g^{(k)} \equiv \frac{\partial W(k,a)}{\partial k_\alpha} \tag{8.18}$$

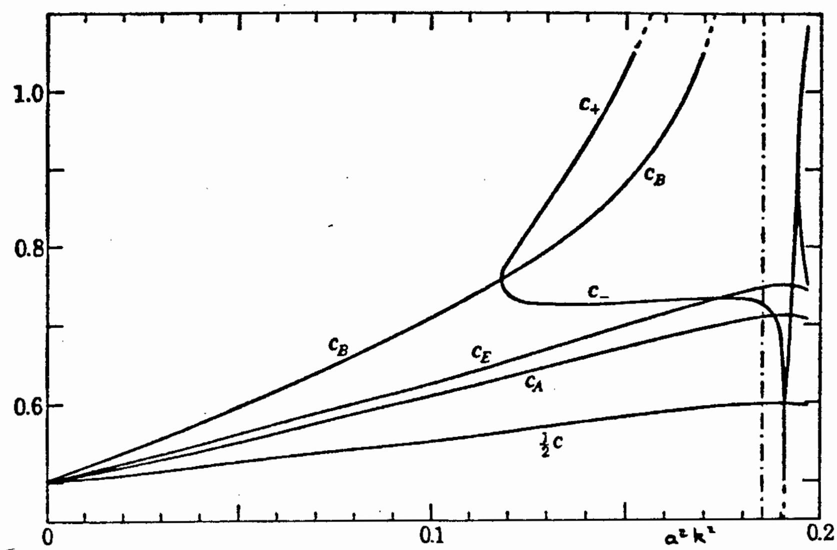
The value of $c_g^{(k)}$ depends on which amplitude measue is kept constant. For example, the basic group velocity just defined in (8.16) is equal to $c_g^{(k)}$ if A is constant. Lighthill (1965) suggested $\mathcal{L}/\omega$, which means that $c_g^{(k)}$ takes the same value as $c_g^{(E)}$ in (8.16). Willebrand (1975) gives explicit expressions for the near-linear value of $c_g^{(k)}$ for different measures of wave amplitude, and finds also that for a <u>spectrum</u> of near-linear waves, unique propagation velocities for the component waves are obtained. This is because the nonlinear effects are overshadowed by interactions between different components.

Conclusions

. The degree of disparity between the various group velocities presented here is made clear in the following graph, taken from Peregrine and Thomas (1979). The notation follows that used in 8.1 except for $c_A \equiv c_g^{(A)}$; $c_E \equiv c_g^{(E)}$. "$\frac{1}{2}c$" is half the value of the wave phase velocity.

There seem few grounds for preference of any one definition over the others, apart from the obvious disadvantages of c_+ and c_B. Perhaps the consistency offered by $c_g^{(E)}$ and $c_g^{(k)}$ (with the choice of $\mathcal{L}/\omega$ constant) is an advantage, but the immediate discrepancy with the stopping velocity for opposing currents prevents one from attatching too much significance to this choice.

Figure 8-1.



Possible extensions of group velocity for deep water waves. The ratio of the velocity to the velocity of linear waves of the same frequency is plotted for the wave-action velocity c_A, the wave-energy velocity c_E, the basic group velocity c_B and the characteristic velocities c_+ and c_-. Note, the dot–dash line is at the steepness where c_B and c_+ are singular.

8.2. MOMENTUM DENSITY AND MOMENTUM FLUX OF SURFACE GRAVITY WAVES

The value of quantities such as momentum and energy lies in their conservation properties. Such properties have a close correspondence with symmetry operations, eg.

energy conservation requires the governing laws of motion for the dynamical system to be time independent

momentum conservation requires the governing laws of motion to be invariant under spatial translation

These correspondences are unambiguous, but in discussing wave motion difficulties can arise as one is often concerned with properties associated with the <u>wave</u> and not the entire system of the wave plus the material medium. There is a tendency to treat such a system <u>as if</u> the medium were absent and to associate quantities such as momentum with the waves themselves. This procedure leads in many cases to correct quantitive results but for the wrong reasons. It ignores the fact that waves in material media are fundamentally different from waves in vacuo. Momentum is one property that demonstrates this difference; waves in vacuo possess momentum but those in material media do not. What does occur in material media is a momentum flux associated with the wave generation but distributed quite differently from the wave distribution. An analysis of the "wave momentum" as if the medium were absent will lead to the correct value for the wave momentum flux, but <u>not</u> for momentum density. Since the flux (ie. the radiation stress) is usually the significant quantity this approach is apparently successful in many cases. The purpose of this section is to examine the fallacy of "wave momentum" and to give definitions of two quantities, pseudoenergy and pseudomomentum, which remove the ambiguities inherent in the use of terms such as "wave momentum". This section is based on the paper by McIntyre (1981), entitled "On the 'wave momentum' myth".

8.2.1. Momentum and surface gravity waves

Wavetrains of infinite length

Consider an infinitely long train of periodic surface gravity waves generated from rest in an irrotational fluid of large depth h by a travelling periodic surface pressure distribution. Analysis shows that all the momentum appears in the fluctuating boundary region (3.43) and is known as the Stokes drift. This drift is generally defined as the difference between mean particle velocity and Eulerian-mean velocity (3.35), and is a <u>wave</u> property. Here it accounts for <u>all</u> the momentum, since the Eulerian-mean velocity is zero and will not change during wave generation (since there are no horizontal pressure gradients to change the velocity and irrotationality prevents changes moving in from the boundaries). In this situation then, the mean momentum is coincident with the waves.

Finite length wavetrains

One might expect that the momentum density would remain coincident with the waves for wavetrains of finite length. In fact there are additional contributions to the momentum besides the Stokes drift. Brooke Benjamin (1970) analysed the case of a wavetrain generated by a towed body and found that there are $O(\alpha^2)$ changes in the height of the free surface that propagate ahead of and behind the wavetrain at the long wave speed (2.28):

$$c = \sqrt{gh} \qquad (2.28)$$

These results are derived using the equations for the propagation of mass, momentum and energy (5.102, 5.103, 5.104) proposed by Whitham (1962). The solution is essentially contained in equation (5.106) which shows that changes in the mean water depth propagate at either long wave velocities $\sqrt{gh}$ or at the group velocity of the periodic wavetrain

$$d - h = -\left\{ \frac{1}{gh - c_g^2} \right\} f_1(x - c_g t) + f_2(x - \sqrt{gh}.t) - f_3(x + \sqrt{gh}.t) \qquad (5.106)$$

There are slight complications since the periodic waves are generated at phase speed u by the obstacle ($u < \sqrt{gh}$) whereas the tail of the wavetrain is moving at $c_g < u$, but these do not affect the important conclusion that the long wave changes in water level will steadily move <u>away</u> from the area of wave generation, with an associated change in the momentum density of the system.

Figure 8-2.

Brooke Benjamin (1970)

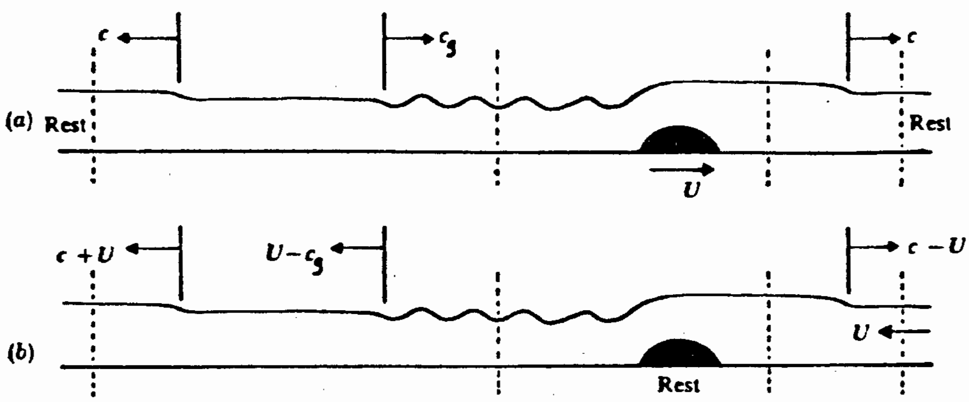
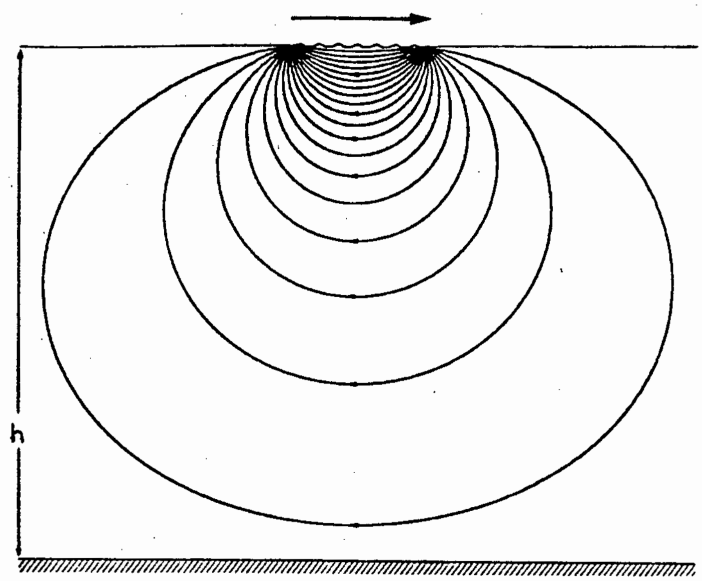


Illustration of water-wave problem: (a) obstacle propelled at constant velocity U in water originally at rest; (b) obstacle fixed in stream approaching with velocity U.

If wave generation is stopped, the wavetrain will move at its group velocity c_g , and the local water level changes will move steadily away from the waves. Brooke Benjamin found that the momentum I associated with the Stokes drift is cancelled by a deep irrotational return flow with momentum I. There is no further <u>local</u> contribution to the mean motion if h is large compared to the wavelength and the wavetrain is slowly varying (Longuet-Higgins and Stewart (1962), their equation 3.26). The propagating level changes however, contain a further contribution to the momentum of magnitude I, and so the final result has exactly the magnitude I of the Stokes drift.

McIntyre (1981a)

Figure 8-3.



The irrotational, $O(a^2)$ return flow underneath a packet of surface gravity waves propagating to the right. (The streamlines, plotted at equal intervals, are quantitatively correct for a two-dimensional wave packet whose amplitude is constant except near its ends.)

In shallow water, the wavetrain itself no longer has zero momentum, but the water level changes still separate from the wavetrain.

This example demonstrates conclusively that the momentum <u>density</u> is not coincident with the wavetrain, and shows that there is an unambiguous flux of momentum associated with the wavetrain. The more general question of why one obtains the correct answers by assigning to the waves all the momentum locally coincident with the waves and neglecting the medium, is now considered.

8.2.2. Pseudomomentum and pseudoenergy

The symmetry conditions quoted earlier for conservation of energy and momentum are conditions for the complete system of the waves plus the medium. It is possible to define conserved quantities corresponding to energy and momentum that are <u>wave</u> properties rather than properties of the whole system. These conservation laws are:

$$\left.\begin{array}{l}\text{conservation of}\\ \text{pseudoenergy:}\end{array}\right\}\quad \begin{array}{l}\text{which requires the } \underline{\text{medium}} \text{ to}\\ \text{be time independent (ie.}\\ \text{restricted to steady flows)}\end{array}$$

$$\left.\begin{array}{l}\text{conservation of}\\ \text{pseudomomentum:}\end{array}\right\}\quad \begin{array}{l}\text{which requires translational}\\ \text{invariance of the } \underline{\text{medium}}\end{array}$$

These definitions have wide acceptance in plasma and solid state physics (McIntyre 1981)).

Since pseudoenergy and pseudomomentum are wave properties, it is interesting to compare their densities and fluxes with those of energy and momentum respectively. McIntyre (1981b) shows that there is no general relation between the densities but a fairly close relation between the fluxes.

For the case of small amplitude slowly modulated wavetrains though, the densities of both pseudoenergy and pseudomomentum show a close correspondence with energy and momentum density respectively.

$$\text{pseudoenergy density} \quad \simeq \quad E \, . \, \frac{\omega}{\sigma} \qquad (8.19)$$

$$\text{pseudomomentum density} \approx E \cdot \frac{k}{\omega} \qquad (8.20)$$

Here, as before, ω is the observed frequency in a fixed reference frame and σ the intrinsic frequency. If there is no mean flow, $\omega = \sigma$ and the pseudoenergy density equals the wave energy E and the pseudomomentum density E/c is equal to the momentum density (for an infinitely long wavetrain).

The close relationship between the fluxes of the pseudomomentum and momentum is what has caused the confusion between these quantities.

Conditions under which correct answers are obtained by (wrongly) neglecting the medium are succinctly expressed by the "pseudomomentum rule" (McIntyre 1981). For systems obeying this rule, correct results are obtained by treating the system as if:

(i) the waves have momentum equal to their pseudomomentum

(ii) the medium is absent

This rule certainly holds for the finite length wavetrains considered earlier and McIntyre (1981) gives the necessary conditions for it to hold in general. These conditions are essentially restrictions on the rate of change of stresses (eg. ambient pressure) unconnected with the pseudomomentum flux. The "pseudomomentum rule" turns out to be correct in many experimental situations, for example, when wavetrains are scattered from immersed obstacles. This appears to be one good reason why the difference between momentum and pseudomomentum has often been obscured. An example of a flow which does not obey the "rule" is that of surface gravity waves on a weir. In this situation, the speed of the mean flow is too high for equilibrium to occur between mean pressures and the radiation stress. Further exceptions to the "rule" occur for waves in a stratified fluid. (McIntyre, 1973, 1981 b).

Conclusions

It is interesting that pseudoenergy and pseudomomentum can be related to wave action. The reason is that the symmetry condition for wave action conservation is that the medium be invariant to a phase shift in the wave field. This requirement involves mean quantities defined by averaging over phase. This averaging process degenerates to an average over a wavelength or over a wave period if done at one particular time or position respectively, so the link with pseudomomentum (spatial wave average) or pseudoenergy (time average) is not too surprising. Certainly, all three properties are definitely wave properties.

The purpose of 8.2 has not been to introduce new results (although the solutions for finite length wavetrains have not been given earlier in the thesis) but to indicate the value of the new quantities pseudomomentum and pseudoenergy. This value is that they are both wave properties and so provide a less ambiguous picture of the dynamics of wave behaviour than do momentum and energy, which require the entire system to be considered and which require some care in interpretation.

8.3. A LAGRANGIAN MEAN DESCRIPTION OF WAVE/MEAN FLOW INTERACTION

Throughout this thesis there have been instances where the Eulerian mean equations have had limitations as a description of wave properties or of wave/mean flow interactions. In 2.2 the alternative Lagrangian description of fluid particle motion was introduced. Lagrangian averaging has not been developed and used as widely as the simpler Eulerian approach, partly because in many situations a detailed knowledge of particle trajectories is not required. It seems that an "ideal" averaging method should combine aspects of both the Eulerian and Lagrangian descriptions. This approach has been followed by Andrews and McIntyre (1978a, 1978b) who developed a "generalised Lagrangian mean" description that appears to unify many of the approaches to wave/mean flow problems, and has achieved some considerable success in atmospheric problems. Their approach has not yet been applied to water wave situations in any detail although it has been used as a reference for water wave results (Stiassine and Peregrine 1979). Because of this limited application to water waves, this discussion is restricted to an outline of the essentials of the theory and an indication of its advantages for wave/mean flow work.

Definition of the Generalised Lagrangian Mean flow

The basic concept of a Lagrangian mean is that of a "mean following a single fluid particle". This has its limitations; for instance, ideas such as "steady mean flow" are difficult to express compared with the Eulerian mean description. By extending the Lagrangian mean to a hybrid Eulerian-Lagrangian description, these difficulties are overcome, although the simple idea of the "mean following a single fluid particle" is lost. What is gained is an exact definition of a mean velocity ($\bar{u}^L$), and also finite amplitude versions of the basic theorems on mean flow evolution.

The first stage in the development of the generalised Lagrangian mean (GLM) description is to define an exact Lagrangian mean operator $\overline{(\ )}^L$ corresponding to any given Eulerian mean operator $\overline{(\ )}$, where the Eulerian mean operator denotes averaging over $\underset{\sim}{x}$ or t or over an ensemble. (Andrews and McIntyre 1978a). Since the Lagrangian mean will depend on particle positions, one must first define an exact, disturbance associated particle displacement field $\underset{\sim}{\xi}(\underset{\sim}{x},t)$. Once this is done, the Lagrangian mean operator can be defined for a function $\varphi(\underset{\sim}{x},t)$ as:

$$\overline{\varphi(\underset{\sim}{x},t)}^L = \overline{\varphi\{\underset{\sim}{x} + \underset{\sim}{\xi}(\underset{\sim}{x},t),\, t\}} \tag{8.21}$$

This shows the hybrid Eulerian-Lagrangian nature of the GLM description; the Eulerian aspect is contained in the use of $(\underset{\sim}{x},t)$ as the independent variables rather than labelling each particle, and the Lagrangian aspect appears through the average over the particle displacement field $\underset{\sim}{\xi}(\underset{\sim}{x},t)$. The actual averaging proceedure is the usual Eulerian one, denoted by the overbar.

The definition of $\underset{\sim}{\xi}(\underset{\sim}{x},t)$ as a disturbance quantity requires firstly:

$$\overline{\underset{\sim}{\xi}(\underset{\sim}{x},t)} = 0 \tag{8.22}$$

and it also requires a condition to be satisified by velocity field at $\underset{\sim}{x}$ and $\underset{\sim}{x}+\underset{\sim}{\xi}$. A convenient notation for the function at displaced positions is:

$$\varphi^{\xi}(\underset{\sim}{x},t) \equiv \varphi(\underset{\sim}{x}+\underset{\sim}{\xi},t) \qquad (8.23)$$

Now it can be shown that if the <u>actual</u> velocity is $\underset{\sim}{u}(\underset{\sim}{x},t)$, then there is a <u>unique associated velocity</u> field $\underset{\sim}{v}(\underset{\sim}{x},t)$ such that $\underset{\sim}{x}$ moves at $\underset{\sim}{v}$ if $\underset{\sim}{x}+\underset{\sim}{\xi}$ moves at $\underset{\sim}{u}^{\xi}$, ie.

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{v}.\nabla\right)\left[\underset{\sim}{x}+\underset{\sim}{\xi}\right] = \underset{\sim}{u}^{\xi} \qquad (8.24)$$

To ensure that $\underset{\sim}{\xi}(\underset{\sim}{x},t)$ <u>is</u> a disturbance associated quantity, $\underset{\sim}{v}$ must be a mean quantity, ie.

$$\overline{\underset{\sim}{v}(\underset{\sim}{x},t)} = \underset{\sim}{v}(\underset{\sim}{x},t) \qquad (8.25)$$

Now finally, use of standard proceedures of <u>Eulerian</u> averaging on (8.24) shows that the associated velocity $\underset{\sim}{v}(\underset{\sim}{x},t)$ is actually $\bar{u}^{L}(\underset{\sim}{x},t)$. Equation (8.24) can now be written in terms of a "Lagrangian mean material derivative" $\bar{D}^{L}$:

$$\left(\frac{\partial}{\partial t} + \bar{u}^{L}.\nabla\right)\left(\underset{\sim}{x}+\underset{\sim}{\xi}\right) = u^{\xi}$$

or

$$\bar{D}^{L}\left(\underset{\sim}{x}+\underset{\sim}{\xi}\right) = u^{\xi} \qquad (8.26)$$

This expression of the velocity at $\underset{\sim}{x}+\underset{\sim}{\xi}$ as a material rate of change of the position $\underset{\sim}{x}+\underset{\sim}{\xi}$ can be used to define the "Lagrangian disturbance velocity" $\underset{\sim}{u}^{\ell}$ where

$$\underset{\sim}{u}^{\ell} \equiv \underset{\sim}{u}^{\xi} - \bar{u}^{L} \qquad (8.27)$$

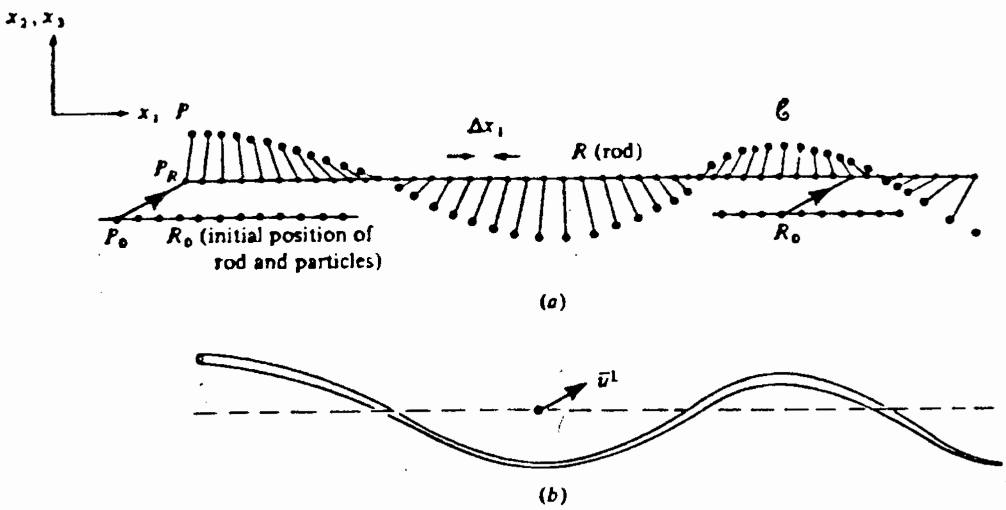
and

$$\overline{u^{\ell}} = 0 \qquad (8.28)$$

and (8.26) becomes $\bar{D}^L \underset{\sim}{\xi} = \underset{\sim}{u}^\ell$ (8.29)

The result of these definitions of $\underset{\sim}{\xi}$ and $\bar{u}^L$ is that "wave" and "mean" effects are separated as neatly as possible, and in such a way that extensions to finite amplitude are easily made. (Simply allow $\underset{\sim}{\xi}$ to get larger!). Clearly the Lagrangian mean velocity $\bar{u}^L$ is not the "mean following a single particle" and one wonders about its physical relevance. The following diagrams show two possible interpretations.

Figure 8-4.



Two ways of visualizing ξ and $\bar{\mathbf{u}}^L$, in the case where $\overline{(\ )}$ is a spatial average with respect to x_1. (a) Mechanical analogy in which a rigid rod R moves with velocity $\bar{\mathbf{u}}^L$ under the pull of a large number of elastic bands whose lengths and orientations give ξ at each point on R. (b) A material tube γ^τ of fluid whose centre of mass moves with velocity $\bar{\mathbf{u}}^L$, in the limit of infinite tube length, as conjectured by Matsuno

In (a), the light rigid rod R is initially at P_0. Marked particles P are joined to R by "elastic bands" that pull P to R with a force proportional to the distance PR. The rod is released at P_0 and the particles P follow the fluid. The rod moves to P_R under the influence of the elastic forces, and if $\underset{\sim}{x}$ is the position P_R then $\underset{\sim}{\xi}(\underset{\sim}{x}, \tau)$ is the elastic band vector $P_R P$.

Now if there are infinitely many particles P_i ; the rod will be moving with velocity $\bar{u}^L$. (The averaging here is along x_1 , and not along the curve $\mathcal{C}$).

The second example is less contrived; the centre of mass of a long thin tube of fluid initially in the x_1 direction moves with the velocity $\bar{u}^L$.

Some Properties of the GLM Description

One significant advantage of the GLM operator $\overline{(\ )}^L$ is that it leads to simple results when the material derivative is involved. A consequence of (8.24) is (8.30) below and this is now used to show the simplicity of GLM descriptions through two corollaries of (8.30), namely (8.31,32)

$$\left(\frac{d\varphi}{dt}\right)^{\xi} = \bar{D}^L\left(\varphi^{\xi}\right) \tag{8.30}$$

Eulerian GLM

$$\Rightarrow \overline{\left(\frac{d\varphi}{dt}\right)}^L = \bar{D}^L\,\bar{\varphi}^L \tag{8.31}$$

and $$\left(\frac{d\varphi}{dt}\right)^{\ell} = \bar{D}^L\,\bar{\varphi}^{\ell} \qquad \left(\varphi^{\ell} = \varphi^{\xi} - \bar{\varphi}^L\right) \tag{8.32}$$

These corollaries must be compared with their Eulerian equivalents to see the simplicity of form of the GLM descriptions:

(Compare (8.31))

$$\overline{\frac{d\varphi}{dt}} = \bar{D}\bar{\varphi} + \overline{\underset{\sim}{u}'\cdot\nabla\varphi'} \qquad \text{(Eulerian)} \tag{8.33}$$

Compare (8.32)

$$\left(\frac{d\varphi}{dt}\right)' = \bar{D}\varphi' + \underset{\sim}{u}'\cdot\nabla\bar{\varphi} + \underset{\sim}{u}\cdot\nabla\varphi' - \overline{\underset{\sim}{u}'\cdot\nabla\varphi'} \text{ (Eulerian)} \qquad (8.34)$$

Notation: $\bar{D} \equiv \frac{\partial}{\partial t} + \bar{u}\cdot\nabla$; $\varphi' \equiv \varphi - \bar{\varphi}$; $\underset{\sim}{u} \equiv \bar{u} + u'$

The simplicity is apparent in that there are no products of fluctuating terms such as $\overline{\underset{\sim}{u}'\cdot\nabla\varphi'}$, since the fluctuations are contained in the GLM averaging over the disturbed positions $\underset{\sim}{\xi}$.

A consequence of this is that Lagrangian mean equations exactly follow the form of equations such as:

$$\frac{dZ}{dt} + Q = 0 \qquad (8.35)$$

The form of (8.35) immediately implies

$$\bar{D}^L \bar{Z} + \bar{Q}^L = 0 \qquad (8.36)$$

Eulerian equivalent: $\bar{D}\bar{Z} + \bar{Q} = -\overline{\underset{\sim}{u}'\cdot\nabla Z'}$ (8.37)

This approach has particular significance when Z is either entropy or potential vorticity, when $Q = 0$. The disadvantage of the GLM formulation compared to the Eulerian is that the operator $\overline{(\ )}^L$ does not commute with $\partial/\partial t$ or $\partial/\partial x$, whereas $\overline{(\ )}$ does.

As a final example of the elementary comparisions of GLM and Eulerian means, the "Stokes correction" to each mean field can be <u>exactly</u> defined as:

$$\bar{\varphi}^S(\underset{\sim}{x},t) \equiv \bar{\varphi}^L(\underset{\sim}{x},t) - \bar{\varphi}(\underset{\sim}{x},t) \qquad (8.38)$$

and the Stokes drift: $\quad \bar{u}^S(\underset{\sim}{x},t) \equiv \bar{u}^L - \bar{u}$ (8.39)

In this sense the GLM description contains well known approximate results in an exact framework.

The significance of the GLM description is not restricted to these simplifications of standard Eulerian results for the <u>constraints</u> on mean flow evolution. It also provides the exact equation of motion for the mean flow, containing wave dissipation, rotational effects and if necessary, thermodynamic effects as well. The Eulerian equation of motion for the <u>total</u> flow is the Navier-Stokes equation. (See the simplified version (3.7))

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \nabla\left(\tfrac{1}{2}\underset{\sim}{u}^2\right) + \underset{\sim}{\omega} \times \underset{\sim}{u} = -\frac{\nabla p}{\rho} - g\hat{z} \qquad (3.7)$$

$$\text{N.S:} \qquad \frac{du_j}{dt} + 2(\underset{\sim}{\Omega} \times \underset{\sim}{u})_j + \Phi_{,j} + \frac{1}{\rho}\cdot p_{,j} + X_j = 0 \qquad \left({}_{,j} \Rightarrow \frac{\partial}{\partial x_j}\right)$$

where Φ is a gravitational or centrifugal potential, $\underset{\sim}{\Omega}$ is the constant angular velocity and X represents any dissipative forces. (Thermodynamic changes can be introduced via the pressure term p). The GLM description of the mean flow is derived from an evaluation of (N.S.) at $\underset{\sim}{x} + \underset{\sim}{\xi}$, and is finally found to be:

$$\bar{D}^L(\bar{u}_i^L - \not{p}_i) + (\bar{u}_k^L)_{,i}(\bar{u}_k^L - \not{p}_k) + 2(\underset{\sim}{\Omega} \times \underset{\sim}{\bar{u}}^L)_i + \bar{X}_i^L - \bar{\Phi}_{,i}^L \qquad (8.40)$$

$$= -\xi_{j,i} X_j^{\ell} + \frac{1}{\bar{\rho}^L}\cdot \bar{P}_{,i}^L$$

One interesting feature of this equation is the presence of the $\not{p}$ terms, since $\not{p}$ is the vector "pseudomomentum per unit mass"

$$\not{p}_i(\underset{\sim}{x},t) = -\overline{\xi_{j,i}\left\{u_j^{\ell} + (\underset{\sim}{\Omega} \times \xi)_j\right\}} \qquad (8.41)$$

This term represents the <u>nonlinear</u> forcing of the mean flow by the waves. In contrast to the Eulerian representation there is no anisotropic wave stress tensor (ie. radiation stress term), because it has been replaced by the pseudomomentum vector. Since there are close links between pseudomomentum, pseudoenergy and wave action (Andrews and McIntyre 1978b) the very general GLM results such as (8.40) again demonstrate the natural way in which wave action appears in conservation equations rather than radiation stress.

Conclusions

This has been a summary of the GLM description of wave/mean flow interaction. As such it has not considered many ramifications of the theory, but has been confined to a comparison of the basis of the theory with the Eulerian description. Stiassine and Peregrine (1979) comment on the use of the GLM description for slowly varying water waves, and suggest that it will be applicable if the averaging process is taken as an average over phase but not over depth. The GLM description also draws attention to the reference frame in which there is zero mass flow associated with the waves, unlike the more common choice of zero flow beneath the wave troughs. It seems reasonable to conclude that the GLM description will be used to advantage in slowly varying water wave theory, but until such applications are made, it will at least provide a general guide for analysis by other methods.

8.4. AN OVERVIEW OF THE THESIS

The essence of this thesis has been to present recent developments in the mathematical description of nonlinear periodic surface gravity waves and their modulation by variations in the medium. The approach used here has been to emphasise the physical aspects of the wave motion, while giving a reasonably rigorous mathematical derivation of the equations. This approach seems valid since some commonly used procedures (eg Eulerian averaging for irrotational flows) lead to mathematical results that are unexpected from a intuitive point of view. The analysis has usually been taken as far as governing equations and their general solutions; there has not been an emphasis on detailed final solutions. This has improved the continuity of most arguments and the salient features of the final solutions have been pointed out from the equations.

Future developments in surface gravity waves

Although periodic surface gravity waves have been investigated for well over one hundred years, there has been considerable recent progress. For instance, Benjamin and Feir (1967) demonstrated the instability of finite amplitude deep water wavetrains, a fact that had gone unnoticed through the long controversy over the validity of the finite amplitude Stokes solution. Also, the discovery that the highest waves were neither the fastest nor the most energetic (Longuet-Higgins 1975) introduced another qualitative change that was quite unexpected. Unlike the instability results, this discovery was a consequence of the availability of improved numerical methods. An extension of Longuet-Higgins work led to the Cokelet wave theory (1977) which gave the wave integral properties to high accuracy in all water depths. Longuet-Higgins has since shown (1978 I,II) that there is a connection between the instability of finite amplitude waves and the fact that the maximum phase velocity occurs for waves of less than maximum steepness. This instability is different to that found by Benjamin and Feir.

It seems inevitable that it is the finite amplitude shallow water (highly nonlinear) wave regime that will provoke the most interest in the near future. With the availability of more accurate wave solutions for irrotational waves in shallow water, one expects that attention will be given to the modifications introduced by bottom friction and vorticity generation. Such modifications are likely to be substantial and to be especially significant as a cause of wave breaking, (although models of breaking waves have been studied that generate wave breaking from irrotational flows; see Longuet -Higgins and Cokelet, 1976).

Even in deep water there are many unresolved topics of considerable interest. The ambivalence of the group velocity concept for water waves of near maximum steepness is one example. The fundamental importance of the group velocity for linear waves makes these problems disturbing as it is difficult to interpret physically the result that wave action, energy and wavenumber all proagate at quite different speeds.

As far as the modification of waves by currents is concerned, the subject does not have a long history and most of the significant work dates from the work of Longuet-Higgins and Stewart (1960). (See for example the survey by Peregrine (1976)). A certain amount of the recent work has involved the nonlinear effects near caustics (points or lines along which rays intersect) and have not been discussed here. The most recent results (apart from those specifically involving caustics) have been those of Peregrine and Thomas (1979) for finite amplitude waves on deep water currents using the accurate wave results of Longuet-Higgins (1975). Stiassnie and Peregrine (1980) applied their general wave/current equations (1979) to shoaling waves, using either Cokelet's theory or, for very shallow water, a train of solitary waves (see equations 7.13-7.19).

Many wave/current problems remain. Virtually all the present work relies on the restriction to large scale current variation, which eliminates many coastal flows of interest. There have also been few attempts to model flows with significant vertical structure, an obvious requirement for many shallow water situations.

In the recent work quoted above, the improved results available from the wave theories of Longuet-Higgins and of Cokelet have been used. THis trend is likely to continue, perhaps with wave /current problems in finite depth water. The possibility of a completely new approach is provided by Andrews and McIntyre with their GLM description of waves on mean flows, discussed in 8.3 This holds promise for a number of reasons; firstly it enables clean distinction to be made between "wave" and "mean" parts of the motion, secondly it incorporates very general theorems on mean-flow evolution and thirdly it has been successfully used on previously intractable problems in the atmosphere. Stiassnie and Peregrine (1979) suggest that the GLM theory will require the retention of the z dependence (no averaging over depth, only over the phase) which may prove an advantage for shear flows. As yet though, no water wave applications have been published.

It seems reasonable to conclude that the study of surface gravity waves and their interactions with currents holds considerable promise for the forseeable future, since there are numerous recent advances that have not yet been fully exploited.

8.5. AN APPLICATION TO VOCOIDAL GRAVITY WAVE THEORY

Introduction to Vocoidal theory

A significant disadvantage of the recent periodic wave theories such as Cokelet's is their complexity. In addition, only the integral properties (apart from a deep water profile in Cokelet's theory) are available. This hinders the wider application of such theories and there is a need for a wave theory that is easy to apply, accurate in all water depths and complete in the sense that all commonly used wave properties are available.

A theory developed to provide this is the Vocoidal theory of Swart and Loubser (1978); see also Swart (1978,1979a,b and 1981). The theory has analytical expressions for all required properties and wave properties are determined by the choice of the three parameters τ (period), d (water depth) and H (trough to crest height). As far as accuracy is concerned, Vocoidal theory provided a better fit to ~600 data sets than twelve other theories (Swart et al 1979a,b). The recent wave theories such as that of Cokelet were not compared with Vocoidal theory because of the disadvantages mentioned previously. It should be emphasised that the objective in deriving Vocoidal theory was not to strive for absolute accuracy at all costs but to provide a theory that could be used with confidence and ease in all water depths, and is therefore not invalidated by the existence of the extremely accurate results of Cokelet's theory for the integral properties.

Integral properties for Vocoidal theory

It would be valuable to analyse Vocoidal theory from an integral property viewpoint, as this would provide a check on the consistency of the theory and would facilitate comparisons with Cokelet's results. A further advantage is that the integral properties are used by Stiassnie and Peregrine in their equations (7.13-7.19) for wave evolution during shoaling and / or current variation. This means that this approach could be used to check the Vocoidal results given by Swart (1981) for Vocoidal wave shoaling and to derive new results for Vocoidal wave/current interaction.

This integral property approach has been formulated for Vocoidal waves. The approach used is outlined here, with comments on the problems encountered and the future possibilities. The integral properties introduced in 3.2 require in essence:

(i) a choice of horizontal and vertical reference frames, ie. a definition of "still water" and water level

(ii) a solution for the phase velocity

(iii) for a "complete" theory, the integral properties are obtained once the wave profile $\eta(x)$ and velocities are known. (Cokelet and Longuet-Higgins avoid specifying these, except in some special cases).

For Vocoidal theory the main difficulty is in (i), namely the choice of the reference frame for horizontal motion (still water definition). The problem is that Vocoidal waves are rotational and a direct consequence is that one cannot find a unique horizontal mean velocity $\bar{u}$ that is independent of depth, as discussed in 3.2 (see 3.43).

$$\nabla \times \underset{\sim}{u} = 0 \Rightarrow \exists \quad \bar{u} \equiv \frac{1}{\lambda}\int_0^\lambda u(x,z)\,dx = \frac{1}{\lambda}\left(\phi(\lambda) - \phi(0)\right) = \text{const.} \; \forall z \leq \eta_{\text{trough}}$$

Although one can evaluate $\bar{u}(z)$ for Vocoidal theory, it is difficult to quantify what is an acceptable variation of $\bar{u}(z)$ with depth, ie. to decide whether the rotation is negligible and to choose a reference frame for the horizontal motion. Analysis by Swart and Goncalves of the rotation by comparing it to the effects of bottom friction and viscosity (effects neglected in all irrotational theories) indicates that the rotation is negligible. Nevertheless it is important to compare this result with values of $\bar{u}(z)$ for Vocoidal waves in all relative water depths, and this is an area of current investigation.

A second (simpler) problem is that the choice of reference frame used in Vocoidal theory is not the one used by most authors - and in particular, is not the one used by Peregrine and Stiassnie. Vocoidal theory uses the reference frame favoured by Jonsson (1978) among others in which the "still water" definition is made by choosing $I = 0$ rather than the more common choice of

$\bar{U} = 0$. (recall the discussion following (3.55)).

$$\bar{u} = 0 \Rightarrow I \neq 0$$

As this is essentially just an $O(a^2)$ shift of reference frame, it does not pose as many problems as the question of the z dependence of the mean horizontal velocity.

Thirdly, it is worth noting that a comparison of the deep water phase velocity c for the Longuet-Higgins solution (1975) and the Vocoidal solution shows that the Vocoidal result is essntially constant for all wave steepnesses, whereas the L-H and Cokelet solutions all vary considerably with wave steepness. One should add that Vocoidal theory is no worse in this respect than any other theory in common use.

If the rotation question can be resolved, then the remaining reference frame problems are unlikely to be serious. It will certainly be possible to construct the integral properties and evaluate them for all wave steepnesses. It can also be shown that Vocoidal theory can be written in a form suitable for use in the equations (7.13-7.19) of Stiassnie and Peregrine (1980). The form of the governing equations suitable for the investigation of wave shoaling or modification by opposing or following currents are as follows:

kinematics: $$\omega = k(c + U) \qquad (4.8)$$

mass conservation: $$q = 0 = \rho U d + I \qquad (8.42)$$

consistency condition: $$\gamma = g(d-h) + \tfrac{1}{2}U^2 + \tfrac{1}{2}\overline{(u_b)}^2 \qquad (8.43)$$

wave action flux: $$B = \frac{1}{k}\left\{ \mathcal{U} I + 3T - 2V + \tfrac{1}{2}\rho d(\overline{u_b})^2 \right\} \qquad (8.44)$$

$h \equiv$ undisturbed water depth; $\mathcal{U} \equiv$ return current; $u_b \equiv$ bottom vel.

These equations can then be solved for Vocoidal theory (essentially by using a secant method on the wave action flux equation). The solution method requires the choice of a deep water wave (τ, H_∞) and the choice of the undisturbed water depth h at which the properties are required. The equations will then give the solution for the wave height H and the actual water depth d.

Extensions of this approach are easily made. Jonsson (1970) uses Airy theory to solve two dimensional channel flow for various current strengths using a similar set of equations to (4.8); (8.42-44). The only modification required in order to do this for the equation set (4.8, 8.42-44) for Vocoidal theory is to make the value of q (the mass flux) equal to the current strength. The equations will give the modified wave properties and actual water depth at a chosen original depth h.

This concludes the outline of Vocoidal integral properties and their applications. The outline has shown the relative simplicity of the wave action set of equations in this particular case and in this way has linked the theoretical basis of the thesis with present and future applications.

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