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Masters Dissertation

Strict Extensions in Pointfree Topology

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Dedication

Dedicated with love to Zev and Yakov Apfel. Without you, this dissertation would have been finished a lot sooner, but it would not have been as much fun.

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Contents

1 Introduction

1.1 Introduction and summary

Extensions of spaces have been constructed and used since the 19th century, for example, to form the complex sphere from the complex plane by adding a point at infinity. Once topological spaces were invented in the 20th century, completions and compactifications became important examples of extensions. Banaschewski wrote in [2] that extension problems have a "philosophical charm" in that they seem to ask the question: "What possibilities in the unknown are determined by the known?"

Strict extensions were first defined for topological spaces by Stone in [30]. The idea was initially translated into the pointfree setting by Hong, in [21], and has since been extensively studied. Just recently, interest has been shown in studying strict extensions in the asymmetric setting of biframes, for example, by Frith and Schauerte in [17].

The intention of this dissertation is to provide a systematic and detailed exposition of strict extensions of frames and nearness frames, which can be used as a reference on this topic. For instance, someone interested in pursuing strict extensions of biframes might obtain the relevant background from reading this text, although the topic of strict extensions of biframes itself will not be discussed here.

We now provide an overview of this dissertation. After some preliminary definitions and facts, we see how strict extensions were defined for spaces, and we present some results which will later be generalised for frames. The main result here is that the strict extensions of a given space are exactly the subspaces of its space of filters.

In chapter 2 we explore strict extensions in the pointfree context. We begin with the definition of strict extensions for frames, and show that the strict extensions of a given frame are exactly the frame homomorphisms that are quotients of the join map. We show that this result is in fact a generalisation of a result for spaces.

Next we show how to construct strict extensions using sets of filters, and see how the same strict extensions can be produced using nuclei. Then we look at trace filters of strict extensions, and see in what sense they define the strict extensions that they come from. We show that every relatively spatial strict extension can be constructed from a set of classical filters, while any strict extension can be constructed from a general filter.

In the final section of that chapter, we look at two applications of our constructions of strict extension: we construct a compactification from strong filters, and we construct a realcompactification from the relatively spatial reflection of a Lindelöfication.

In chapter 3, we add structure to our frames and consider strict extensions of nearness frames. In particular, we are interested in completions. First we introduce two notions of completeness, which are completeness and Cauchy completeness, and we show how they relate. We also see how to construct completions and Cauchy completions using the methods of the previous chapter.

Then we introduce the notion of weak completions, and show how to construct them and how they relate to completions. We show that completion is not a coreflection of the category of nearness frames and uniform homomorphisms, but we give three different categories in which completion, Cauchy completion and weak completion are coreflections, respectively. We conclude the chapter with a discussion of complete versus compact in the pointfree setting.

We end this dissertation with a brief overview of possible further work. We outline some of the results that have been proven in the asymmetric setting, and mention what has not yet been done. We also mention some other notions of completion that could be described using the constructions discussed here. We leave the reader with a problem that we were unable to resolve during the course of this work.

It is my hope that the reader will accompany me on this meander through pointfree topology without getting lost or bored. There are many sights to see along the way.

1.2 Definitions and preliminary results

We will now define the notions in pointfree topology that will be needed in what follows. For terminology and definitions regarding classical topology, refer to [31]. Throughout this dissertation, all topological spaces will be T_0. For details regarding what is mentioned below, as well as further background in pointfree topology, refer to [29].

Definition 1.2.1. *A set L together with a partial order $\leq$ is a* **lattice** *if every finite set has a greatest lower bound (a* **meet***, written $\wedge$) and a least upper bound (a* **join***, written $\vee$). In particular, the empty join exists, so there is a bottom, 0, and the empty meet exists, so there is a top, e. In some contexts, a lattice is not required to have empty joins and meets, and then one that does would be called a* **bounded lattice***, but in the present work all lattices will be bounded, and the term "finite set" will include the empty set. If all finite meets exist, but not all finite joins, then L is called a* **meet-semilattice***. A meet-semilattice is* **bounded** *if it* *has a bottom. If all joins and meets exist, including infinite ones, then L is called a* **complete lattice***. A lattice L is called a* **distributive** *lattice if for any a, b and $c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, which is called the* **distributive law***. If L is a distributive lattice such that for every element $a \in L$ there is an element $a' \in L$ called the* **complement** *of a, such that $a \wedge a' = 0$ and $a \vee a' = e$, then L is a* **Boolean algebra***.*

Lemma 1.2.2. *The join of an arbitrary set S in a complete lattice L can be expressed as the join of all the finite joins of finite subsets of S. A set T in a partially ordered set L is an* **updirected** *set if for any a and b in T, there is a $c \in T$ such that $c \geq a$ and $c \geq b$. The join of an updirected set in an* **updirected join***. Therefore any join can be expressed as the updirected join of finite joins.*

Definition 1.2.3. *A* **frame** *is a complete lattice L such that for any point $x \in L$ and any set $S \subseteq L$,*

$$x \wedge \bigvee S = \bigvee \{x \wedge s | s \in S\}.$$

This equation is called the **frame law***. A map $h : M \to L$ between frames M and L is a* **frame homomorphism** *if h preserves all finite meets, including e, and all arbitrary joins, including 0. The map h is* **dense** *if whenever $h(x) = 0$, then $x = 0$, and h is* **onto** *if for each $y \in L$ there is an $x \in M$ such that $h(x) = y$.*

Example 1.2.4. The two element chain, consisting of 0 and 1, is a frame, which will be denoted $\mathbb{2}$.

For a partially ordered set L, a subset $A \subseteq L$ is called a **downset** if whenever $a \leq b$ and $b \in A$, then $a \in A$. If L is a bounded meet-semilattice, then the set $\mathcal{D}L$ consisting of all non-empty downsets of L is a frame, with set inclusion as the partial order. If L is a frame, the join map $\bigvee : \mathcal{D}L \to L$ is a frame homomorphism.

Categorical concepts will be needed throughout this dissertation, but we will not provide a formal definition of a category here, as we assume that the reader is familiar with category theory. Rather, we define below those concepts that will be needed for this dissertation, and refer the reader to [1] or [24] for more information.

Informally, a **category** is a pair $\mathbf{A} = (\mathbf{A}_0, \mathbf{A}_1)$, where $\mathbf{A}_0$ is a class of **objects** and $\mathbf{A}_1$ is a class of **morphisms** of those objects, which should include identity morphisms, and which should be able to compose with each other. Categories of particular interest will be the category **Frm** of frames and frame homomorphisms, and the category **Top**, of topological spaces and continuous maps.

Definition 1.2.5. *A morphism $f : A \to B$ in a category $\mathbf{A}$ is an* **isomorphism** *if there is another morphism $g : B \to A$ such that $fg = \mathrm{id}_A$ and $gf = \mathrm{id}_B$, where id_X*

is the identity morphism on the object X. If there is an isomorphism $f : A \to B$, then we say that A and B are **isomorphic**, *and we write $A \cong B$.*

A **covariant functor** $F : \mathbf{A} \to \mathbf{B}$ *between categories* $\mathbf{A}$ *and* $\mathbf{B}$ *is a function that assigns to each object A of* $\mathbf{A}$, *an object FA of* $\mathbf{B}$, *and to each morphism $h : A_1 \to A_2$ of* $\mathbf{A}$ *it assigns a morphism $Fh : FA_1 \to FA_2$ of* $\mathbf{B}$. *A* **contravariant functor** *is the same except that $Fh : FA_2 \to FA_1$. Any functor must preserve identity morphisms, and composition. Explicitly, if F is a functor from* $\mathbf{A}$ *to* $\mathbf{B}$ *and $i : A \to A$ is an identity morphism in* $\mathbf{A}$, *then $Fi : FA \to FA$ is an identity morphism in* $\mathbf{B}$. *If f and g are morphisms in* $\mathbf{A}$, *then for a covariant functor we require that $F(f \circ g) = Ff \circ Fg$, and for a contravariant functor we need $F(f \circ g) = Fg \circ Ff$.*

We say that $\eta : F \to G$ is a **natural transformation** *of the functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{A} \to \mathbf{B}$ if for each object A of* $\mathbf{A}$, *$\eta_A : FA \to GA$ is a morphism in* $\mathbf{B}$, *and for each morphism $h : A_1 \to A_2$ in* $\mathbf{A}$, *the following square commutes:*

$$\begin{tikzcd}
FA_1 \arrow[r, "\eta_{A_1}"] \arrow[d, "Fh"'] & GA_1 \arrow[d, "Gh"] \\
FA_2 \arrow[r, "\eta_{A_2}"'] & GA_2
\end{tikzcd}$$

If the morphisms η_A are isomorphisms in $\mathbf{B}$ *for each object A of* $\mathbf{A}$, *then η is a* **natural isomorphism**.

Two contravariant functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$ are **adjoint** *functors if there exist natural transformations $\eta : \mathrm{id}_{\mathbf{A}} \to GF(\mathbf{A})$ and $\varepsilon : \mathrm{id}_{\mathbf{B}} \to FG(\mathbf{B})$, called the* **unit** *and* **counit** *respectively, which satisfy the following* **triangle identities**:

$$\begin{tikzcd}
FA \arrow[r, "\varepsilon_{FA}"] \arrow[dr, equal] & FGFA \arrow[d, "F\eta_A"] \\
& FA
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
GB \arrow[r, "\eta_{GB}"] \arrow[dr, equal] & GFGB \arrow[d, "G\varepsilon_B"] \\
& GB
\end{tikzcd}$$

If the functors F and G are covariant, the counit of adjunction is $\varepsilon : FG(\mathbf{B}) \to \mathrm{id}_{\mathbf{B}}$, and the arrows in the first triangle identity are reversed. In this case, F is the **left adjoint** *of the functor G.*

A subcategory $\mathbf{B}$ *of a category* $\mathbf{A}$ *is a* **reflective** *subcategory if for every object A of* $\mathbf{A}$ *there is a* **reflection map** $h_A : A \to hA$ *in* $\mathbf{A}$, *where hA is an object in* $\mathbf{B}$ *(called the* **reflection** *of A), such that for every object B of* $\mathbf{B}$ *and morphism*

$f : A \to B$ in $\mathbf{A}$, there is a unique morphism $hA \to B$ in $\mathbf{B}$ such that the triangle below commutes.

$$\begin{tikzcd} A \arrow[rr, "h_A"] \arrow[dr, "f"'] & & hA \arrow[dl, dotted, "!"] \\ & B & \end{tikzcd}$$

The dual of a reflective subcategory is a **coreflective** *subcategory, that is, all the arrows in the above triangle are reversed.*

If there is an object A of $\mathbf{A}$ such that for any object B of $\mathbf{A}$ there is exactly one morphism $A \to B$, then A is an **initial** *object of the category $\mathbf{A}$. If a category has an initial object, then that initial object must be unique up to isomorphisms.*

Remark 1.2.6. In what follows we will be discussing strict extensions, which are morphsms in their respective categories. However, the concept of strict extensions is not related to the categorical concept of strict monomorphisms, as defined in [1] Definition 7D. Indeed, strict extensions are not necessarily monomorphisms, and strict monomorphisms are not all extensions.

Definition 1.2.7. *A non-empty subset F of a frame L is a* **filter** *if it is up-closed and closed under finite meets. That is, if $a \in F$ and $b \geq a$, then $b \in F$, and if a and b are in F, then $a \wedge b \in F$. A filter F is a* **proper** *filter if $0 \notin F$. Throughout this dissertation it will be assumed that all filters are proper. A filter F* **converges** *if whenever $S \subseteq L$ such that $\bigvee S = e$, then $F \cap S \neq \emptyset$. If F does not converge, then F is* **free**. *A filter F is* **prime** *if whenever $a \vee b \in F$, then either $a \in F$ or $b \in F$, and it is* **completely prime** *if whenever $\bigvee S \in F$, then $S \cap F \neq \emptyset$. A filter F is called an* **ultrafilter** *if whenever $F \subseteq G$ and G is a proper filter, then $F = G$.*

Remark 1.2.8. If F is a completely prime filter, then F converges.

Lemma 1.2.9. *If F is a completely prime filter on a frame L, and $h : M \to L$ is a frame homomorphism, then $h^{-1}[F]$ is a completely prime filter on M.*

Proof: Firstly, F is not empty, so at least $e \in F$, and then $h(e) = e$, so $e \in h^{-1}[F]$. Therefore $h^{-1}[F] \neq \emptyset$.

Now, if a and b are elements of M such that a and b are in $h^{-1}[F]$, then $h(a)$ and $h(b)$ are in F, so $h(a) \wedge h(b) \in F$. Then since h is a frame homomorphism, we have that $h(a \wedge b) \in F$, so $a \wedge b \in h^{-1}[F]$. Therefore $h^{-1}[F]$ is closed under finite meets.

Next, if $a \in h^{-1}[F]$ and $b \geq a$, then $h(a) \in F$. Since h preserves order, $h(a) \leq h(b)$, so $h(b) \in F$ too. Therefore $b \in h^{-1}[F]$, and so $h^{-1}[F]$ is up-closed.

Finally, if $\bigvee S \in h^{-1}[F]$ for some set $S \subseteq M$, then $h(\bigvee S) \in F$. Now h is join preserving, so $\bigvee\{h(s)|s \in S\} \in F$. Then since F is completely prime, $h(s) \in F$ for some $s \in S$, and then $s \in h^{-1}[F]$, so we have that $h^{-1}[F]$ is completely prime. $\square$

Definition 1.2.10. *For a frame L, let ΣL be the set of all completely prime filters on L. For $a \in L$, let $\Sigma_a = \{F \in \Sigma L | a \in F\}$. Then $\{\Sigma_a | a \in L\}$ is a topology on ΣL. For a frame homomorphism $h : M \to L$, $\Sigma h : \Sigma L \to \Sigma M$ where $\Sigma h(F) = h^{-1}[F]$ is a continuous map. So $\Sigma :$ **Frm** $\to$ **Top** is a contravariant functor.*

*For a topological space X, let $\mathcal{O}X$ be the set of open sets of X. Then $\mathcal{O}X$ is a frame, with set inclusion being the partial order. For a continuous map $f : X \to Y$, the map $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ where $\mathcal{O}f(U) = f^{-1}[U]$ is a frame homomorphism. So $\mathcal{O} :$ **Top** $\to$ **Frm** is a contravariant functor.*

The two functors $\mathcal{O}$ and Σ are adjoint functors, with unit $\varepsilon_X : X \to \Sigma\mathcal{O}X$ such that $\varepsilon_X(x) = \mathcal{N}_x$, the filter consisting of open neighbourhoods of x, and co-unit $\eta_L : L \to \mathcal{O}\Sigma L$ such that $\eta_L(a) = \Sigma_a$.

If $\mathcal{O}\Sigma L$ is isomorphic to L, then L is a **spatial frame***, and if $\Sigma\mathcal{O}X$ is homeomorphic to X, then X is a* **sober space***.*

Lemma 1.2.11. *For any space X, $\mathcal{O}X$ is a spatial frame, and for any frame L, ΣL is a sober space. Every sober space is T_0. A frame L is spatial if and only if its elements are separated by points, that is, for any $a \neq b \in L$, there is a completely prime filter F that contains one and not the other. If M is a subframe of L, and L is spatial, then M is also spatial. For any frame L, the frame $\mathcal{D}L$ is spatial.*

Example 1.2.12. Consider $\mathbb{R}$, the set of real numbers with the usual topology. A set $U \subseteq \mathbb{R}$ is a **regular open** set if $U = \operatorname{int} \overline{U}$. The collection of regular open sets of $\mathbb{R}$ forms a frame, which is not spatial.

Definition 1.2.13. *For a frame L, an element $p \in L$ is called a* **prime** *element if whenever $a \wedge b = p$, either $a = p$ or $b = p$. Equivalently, whenever $a \wedge b \leq p$, either $a \leq p$ or $b \leq p$.*

Lemma 1.2.14. *The space ΣL of* **points** *of L, whose topology is given by the sets $\{\Sigma_a | a \in L\}$, can be described in a number of different, but equivalent ways:*

1. *ΣL is the set of completely prime filters on L, and $\Sigma_a = \{F \in \Sigma L | a \in F\}$.*

2. *ΣL is the set of prime elements in L, and $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$.*

3. *ΣL is the set of all frame homomorphisms $\xi : L \to \mathbb{2}$, and $\Sigma_a = \{\xi \in \Sigma L | \xi(a) = 1\}$.*

Definition 1.2.15. *For a lattice L, a subset $J \subseteq L$ is an* **ideal** *if it is the dual of a filter, that is, it is a non-empty downset that is closed under finite joins. A* **σ-ideal** *is an ideal that is closed under countable joins. The set of all σ-ideals of L will be denoted $\mathfrak{h}L$.*

Definition 1.2.16. *For a and b in a lattice L, a is* **rather below** *b, written $a \prec b$, if there is a* **separating element** *$c \in L$ such that $a \wedge c = 0$ and $c \vee b = e$. If there exists a set of elements $\{c_q | q \in \mathbb{Q} \cap [0, 1]\}$ such that $a = c_0$, $b = c_1$ and $c_q \prec c_r$ whenever $q < r$, then a is* **completely below** *b, written $a \prec\!\!\prec b$.*

Lemma 1.2.17. *If $a \prec b$, then $a \leq b$. If $x \leq a \prec b \leq y$, then $x \prec y$. If $x \prec b$ and $a \prec b$, then $x \vee a \prec b$, and if $a \prec b$ and $a \prec y$, then $a \prec b \wedge y$.*

Definition 1.2.18. *If a frame L satisfies the property that for each $a \in L$,*

$$a = \bigvee \{x \in L | x \prec a\},$$

then L is a **regular** *frame. Similarly, if for each $a \in L$,*

$$a = \bigvee \{x \in L | x \prec\!\!\prec a\},$$

then L is a **completely regular** *frame. These correspond to the definitions of (complete) regularity for spaces, in that the space X is (completely) regular if and only if $\mathcal{O}X$ is a (completely) regular frame.*

Lemma 1.2.19. *If L is a regular frame, then if p is a prime element of L, p is* ***maximal****, meaning that if $p < a$ for some $a \in L$, then $a = e$. In the category of regular frames, if $h : M \to L$ is a dense homomorphism then h is* ***monic****, which means that if $hu = hv$ for frame homomorphisms $u : N \to M$ and $v : N \to M$, then $u = v$.*

Definition 1.2.20. *For an element a in a frame L, the* **pesudocomplement** *of a is*

$$a^* = \bigvee \{x \in L | x \wedge a = 0\}.$$

Lemma 1.2.21. *For a and b in L, $a \prec b$ if and only if $a^* \vee b = e$. If $a \prec b$, then $b^* \prec a^*$. For any $x \in L$, $x \leq x^{**}$, and $x^* = x^{***}$. For any x and y in L, $x^* \wedge y^* = (x \vee y)^*$ and $x^{**} \wedge y^{**} = (x \wedge y)^{**}$, and if $x \leq y$, then $x^{**} \leq y^{**}$.*

Definition 1.2.22. *A filter F is a* **regular** *filter if for each $a \in F$, there is a $b \in F$ such that $b \prec a$.*

Lemma 1.2.23 ([7] before Section 1). *A regular filter converges if and only if it is completely prime.*

Proof: See the proof of Proposition 2.4.1, where we show that a regular filter is free if and only if it is not completely prime. $\square$

Remark 1.2.24. For a filter F on a frame L, the characteristic function of F is a bounded meet-semilattice homomorphism. That is, the function $\varphi_F : L \to \mathbb{2}$ such that $\varphi_F(a) = 1$ if and only if $a \in F$, preserves the bottom and finite meets, including the top.

Definition 1.2.25. *A* **general filter** *on a frame L is a bounded meet-semilattice homomorphism $\varphi : L \to T_\varphi$, where T_φ is any frame, called the* **truth frame** *of the filter φ. In contrast to these, the filters that we defined previously will be called* **classical filters***.*

Definition 1.2.26. *If $h : M \to L$ is a frame homomorphism, then h has a* **right adjoint** *$h_* : L \to M$ satisfying the property that $x \leq h_*(y)$ in M if and only if $h(x) \leq y$ in L. For $a \in L$,*

$$h_*(a) = \bigvee\{x \in M | h(x) \leq a\}.$$

When a frame homomorphism is written with a subscript, for example, γ_L, the right adjoint will be written γ_{L*}. The right adjoint is usually not a frame homomorphism.

Remark 1.2.27. The concept of a right adjoint defined here is the same as the categorical right adjoint of a functor, defined in Definition 1.2.5, because a partially ordered set can be thought of as a category. This equivalence will not be discussed further.

Lemma 1.2.28. *If $h : M \to L$ is any frame homomorphism, then for a and b in L, $h_*(a \wedge b) = h_*(a) \wedge h_*(b)$. If p is a prime element of L, then $h_*(p)$ is a prime element of M. If $f : N \to M$ is another frame homomorphism, then $(hf)_* = f_*h_*$. If h is dense, then h_* is a general filter. If h is onto, then $hh_*(a) = a$ for all $a \in L$.*

Lemma 1.2.29. *For any frame homomorphism $h : M \to L$, $hh_*h = h$.*

Proof: If $a \in L$, then

$$hh_*(a) = h\left(\bigvee\{x \in M | h(x) \leq a\}\right) = \bigvee\{h(x) | h(x) \leq a\} \leq a.$$

Therefore if $a = h(b)$, then $hh_*h(b) \leq h(b)$. On the other hand,

$$h_*h(b) = \bigvee\{x \in M | h(x) \leq h(b)\} \geq b,$$

since b itself is such an x, so $hh_*h(b) \geq h(b)$. Therefore for all $b \in M$, $hh_*h(b) = h(b)$, as required. $\square$

Definition 1.2.30. *For a frame M, a* **closure operator** *on M is a function $n : M \to M$ such that for each $x \in M$, $x \leq n(x)$, $n(n(x)) = n(x)$, and if $x \leq y$, then $n(x) \leq n(y)$. The function $n : M \to M$ is a* **nucleus** *on M if it is a closure operator that preserves binary meets. For a nucleus n,* $\mathrm{fix}\ n = \{x \in M | n(x) = x\}$.

Example 1.2.31. For a frame homomorphism $h : M \to L$, $h_* h$ is a nucleus on M, which is the nucleus **associated** with h.

Lemma 1.2.32. *For a nucleus $n : M \to M$,* $\mathrm{Fix}\ n$ *is a frame where meet is the same as the meet in M, but for join, if $S \subseteq \mathrm{Fix}\ n$, the join is $\bigsqcup S = n(\bigvee S)$. If we restrict $n : M \to \mathrm{Fix}\ n$, then n is an onto frame homomorphism. On the other hand, if $h : M \to L$ is an onto frame homomorphism, let*

$$n_h(a) = \bigvee \{x \in M | h(x) = h(a)\}.$$

Then $n_h(a)$ is a nucleus on M such that $\mathrm{Fix}\ n_h$ *is isomorphic to L.*

Definition 1.2.33. *If $f : L \to M$ is a frame homomorphism, the* **kernel** *of f is* $\ker f = \{(x, y) \in L \times L | f(x) = f(y)\}$.

Lemma 1.2.34 (Kernel Factorisation Lemma)**.** *Suppose $f : L \to M$ is an onto frame homomorphism, and $g : L \to N$ is a frame homomorphism such that* $\ker f \subseteq \ker g$*. Then there exists a unique frame homomorphism $h : M \to N$ such that $hf = g$. Specifically, $h = g f_*$.*

$$\begin{tikzcd}
L \arrow[rr, "g"] \arrow[dr, "f", "onto"'] & & N \\
& M \arrow[ur, dotted, "!h"'] &
\end{tikzcd}$$

Lemma 1.2.35 (Image Factorisation)**.** *If $h : M \to L$ is a frame homomorphism, then it has a factorisation fg where g is onto and f is one-one. Specifically, $g : M \to h[M]$ acts like h, and $f : h[M] \to L$ is an identical embedding.*

$$\begin{tikzcd}
M \arrow[rr, "h"] \arrow[dr, "g"'] & & L \\
& h[M] \arrow[ur, hook, "f"'] &
\end{tikzcd}$$

Definition 1.2.36. *A frame L is* **compact** *if whenever $S \subseteq L$ such that $\bigvee S = e$, there is a finite set $T \subseteq S$ such that $\bigvee T = e$. Similarly, L is* **Lindelöf** *if whenever*

$\bigvee S = e$, *there is a countable set* $T \subseteq S$ *such that* $\bigvee T = e$. *A dense, onto frame homomorphism* $h : M \to L$ *is a* **compactification** *of* L *if* M *is a compact, regular frame. A* **Lindelöfication** *is a dense, onto frame homomorphism from a completely regular Lindelöf frame.*

Definition 1.2.37. *For a binary relation* $\lhd$ *on a frame* L, $\lhd$ *is a* **strong inclusion** *on* L *if it satisfies all of the following:*

1. *If* $x \leq a \lhd b \leq y$, *then* $x \lhd y$.
2. *The set* $\{(a, b) \in L \times L | a \lhd b\}$ *is a sublattice of* $L \times L$.
3. *If* $a \lhd b$, *then* $a \prec b$.
4. *If* $a \lhd b$, *then there is a* $c \in L$ *such that* $a \lhd c \lhd b$.
5. *If* $a \lhd b$, *then* $b^* \lhd a^*$.
6. *For each* $a \in L$, $a = \bigvee\{x \in L | x \lhd a\}$.

A filter F *on* L *is a* **strong** *filter (or* $\lhd$*-filter) if for each* $a \in F$, *there is a* $b \in F$ *such that* $b \lhd a$. *Dually, an ideal* J *on* L *is a* **strong** *ideal (or* $\lhd$*-ideal) if for each* $a \in J$ *there is a* $b \in J$ *such that* $a \lhd b$. *Let* $\mathcal{S}_\lhd L$ *be the collection of all* $\lhd$*-ideals on* L.

Lemma 1.2.38. *If* $h : M \to L$ *is a compactification of* L, *then the relation* $\lhd_h$ *given by* $a \lhd_h b$ *in* L *if and only if* $h_*(a) \prec h_*(b)$ *in* M *is a strong inclusion on* L, *for which if* $x \prec y$ *in* M, *then* $h(x) \lhd_h h(y)$ *in* L. *On the other hand, if* $\lhd$ *is a strong inclusion on* L, *then* $\bigvee : \mathcal{S}_\lhd L \to L$ *is a compactification of* L, *called the* $\lhd$**-compactification** *of* L. *These two operations are inverses of each other.*

Definition 1.2.39. *The* **Axiom of Countable Choice** *states that for every countable family of non-empty sets there is a choice function, that is, a function that selects one element from each set in the family. The* **Boolean Ultrafilter Theorem** *(abbreviated* BUT*) states that every non-trivial Boolean algebra contains an ultrafilter.*

Lemma 1.2.40. *The Boolean Ultrafilter Theorem is equivalent to the statement that every compact regular frame is spatial. If the Boolean Ultrafilter Theorem holds, then every filter is contained in a prime filter.*

Definition 1.2.41. *A* σ**-frame** *is a distributive lattice where all countable joins exist, and the frame law holds for countable subsets* S. *A* σ*-frame* A *is* **regular** *if for each* $a \in A$,

$$a = \bigvee\{x_n | x_n \prec a\}$$

for some countable set $\{x_n\} \subseteq A$. *Similarly, a* σ*-frame is* **completely regular** *if each* $a \in A$ *can be written as*

$$a = \bigvee \{x_n | x_n \prec\!\!\prec a\}$$

for some countable set $\{x_n\} \subseteq A$.

In the category of σ*-frames, the morphisms are* σ**-frame homomorphisms**, *which preserve finite meets, including* e, *and countable joins, including* 0.

The set of real numbers, $\mathbb{R}$, with its usual topology, gives a frame, $\mathcal{O}\mathbb{R}$, that should be called the frame of reals. However, this frame requires some prior knowledge of the real numbers. Instead, the real numbers can be built up from the rational numbers by generating a frame from ordered pairs (p, q) for $p, q \in \mathbb{Q}$, subject to certain relations. This frame, $\mathcal{L}(\mathbb{R})$, is called the **frame of reals**, and is indeed isomorphic to $\mathcal{O}\mathbb{R}$. For a precise definition and more details about the frame of reals, see [29] Chapter 4.

Definition 1.2.42. *For a frame* L, *a frame homomorphism* $\alpha : \mathcal{L}(\mathbb{R}) \to L$ *is called a* **continuous real function** *on* L. *For a continuous real function* $\alpha : \mathcal{L}(\mathbb{R}) \to L$, *let* $\mathrm{coz}(\alpha) = \bigvee\{\alpha((p, q)) | p > 0 \text{ or } q < 0\}$. *Such an element is called a* **cozero** *element of* L. *The set of all cozero elements of a frame* L *will be denoted by* $\mathrm{Coz}L$.

Lemma 1.2.43. *For every* σ*-frame* A, $\mathfrak{H}A$ *(see Definition 1.2.15) is a Lindelöf frame, and* $\mathfrak{H}$ *is a functor from the category of completely regular* σ*-frames, to the category of completely regular frames. For every frame* L, $\mathrm{Coz}L$ *is a* σ*-frame, and* Coz *is a functor from the category of completely regular frames, to the category of completely regular* σ*-frames. The category of completely regular Lindelöf frames is coreflective in the category of completely regular frames, with the coreflection maps given by* $\bigvee : \mathfrak{H}\mathrm{Coz}L \to L$.

We now introduce the basic notions of structured frames. For more details, refer to chapter 8 of [29].

Definition 1.2.44. *For a frame* L, *if* $S \subseteq L$ *such that* $\bigvee S = e$, *then* S *is a* **cover** *of* L. *Let* $\mathrm{Cov}L$ *be the set of all covers of* L. *For two covers* C *and* D, *we say that* C **refines** D, *written* $C \leq D$, *if for each* $c \in C$, *there is a* $d \in D$ *such that* $c \leq d$. *The* **meet** *of two covers* C *and* D *is* $C \wedge D = \{c \wedge d | c \in C, d \in D\}$, *which is also a cover. For* $a \in L$ *and* $C \in \mathrm{Cov}L$,

$$Ca = \bigvee \{c \in C | c \wedge a \neq 0\},$$

which is called the **C-star** *of* a. *Note that for each* $a \in L$, $a \leq Ca$. *For a cover* C, *the* **star** *of* C *is* $CC = \{Cc | c \in C\}$, *which is also a cover.*

If $\mathcal{N}$ is a non-empty subset of $\mathrm{Cov}L$*, then for a and $b \in L$, a is* **uniformly below** *b, written $a \lhd_{\mathcal{N}} b$ if there is a $C \in \mathcal{N}$ such that $Ca \leq b$. When it is not necessary to be explicit about the set $\mathcal{N}$, we will simply write $a \lhd b$. A set $\mathcal{N} \subseteq \mathrm{Cov}L$ is* **admissible** *if for each $a \in L$,*

$$a = \bigvee\{x \in L | x \lhd_{\mathcal{N}} a\}.$$

If $\mathcal{N}$ is a filter in $\mathrm{Cov}L$ *with respect to meet and refinement, and $\mathcal{N}$ is admissible, then $\mathcal{N}$ is a* **nearness** *structure on L. If $\mathcal{N}$ has the further property that for every $C \in \mathcal{N}$ there is a $B \in \mathcal{N}$ such that $BB \leq C$, then $\mathcal{N}$ is a* **uniform** *structure on L, or a* **uniformity***. This property, called the* **star refinement** *property, is normally written $B \leq^* C$, and we say B* **star refines** *C.*

A **structured frame** *is frame together with a structure (nearness or uniform) on it. The pair $(L, \mathcal{N})$ is called a* **nearness frame** *if $\mathcal{N}$ is a nearness on L. If $\mathcal{N}$ is a uniformity, then $(L, \mathcal{N})$ is a* **uniform frame***. When explicit reference to the nearness structure is unnecessary or clumsy, a nearness frame will be referred to just as L. When a nearness frame is called L, its nearness structure will be called $\mathcal{N}L$. If $\mathcal{N}$ is generated by its finite members, then $(L, \mathcal{N})$ is* **totally bounded***. A frame homomorphism $h : M \to L$ between nearness frames is a* **uniform** *homomorphism if for each $C \in \mathcal{N}M$, $h[C] = \{h(c) | c \in C\} \in \mathcal{N}L$.*

Lemma 1.2.45. *If $(M, \mathcal{N}M)$ is a nearness frame and L is an unstructured frame, then an onto frame homomorphism $h : M \to L$* **induces** *a nearness on L generated by $\{h[C] | C \in \mathcal{N}M\}$. If $\mathcal{N}M$ is uniform, then $\mathcal{N}L$ is uniform, and if $\mathcal{N}M$ is totally bounded, $\mathcal{N}L$ is totally bounded.*

Lemma 1.2.46. *If $\mathcal{N}$ is a uniform structure on a frame L, then $\lhd_{\mathcal{N}}$ is a strong inclusion on L. If $\mathcal{N}$ is a nearness structure that is not uniform, then $\lhd_{\mathcal{N}}$ may not be a strong inclusion because $\lhd_{\mathcal{N}}$ does not necessarily interpolate, but all the other conditions still hold.*

Proof: We will only prove that the third property of strong inclusions holds, that is, if $a \lhd_{\mathcal{N}} b$, then $a \prec b$.

If $a \lhd_{\mathcal{N}} b$, then there exists a $C \in \mathcal{N}$ such that $Ca \leq b$. Let $y = \bigvee\{s \in C | s \wedge a = 0\}$. Then $a \wedge y = 0$ by the frame law, and

$$\begin{aligned} y \vee b &\geq y \vee Ca \\ &= \bigvee\{s \in C | s \wedge a = 0\} \vee \bigvee\{s \in C | s \wedge a \neq 0\} \\ &= \bigvee C \\ &= e. \end{aligned}$$

So $a \prec b$, with y the separating element. $\square$

Lemma 1.2.47 ([3] Proposition 1(2))**.** *If L is a compact regular frame, then L has exactly one nearness structure on it, given by all of* $\mathrm{Cov}L$*, which is a uniformity.*

1.3 Strict extensions in spaces

Before we consider strict extension in the pointfree setting, we will discuss their predecessors in classical topology. Recall that all topological spaces under consideration are T_0.

Definition 1.3.1. *For a topological space X, the pair (f, Y) is an* **extension** *of X if $f : X \to Y$ is a continuous function that maps X homeomorphically onto a dense subspace of Y.*

The concept of a strict extension was first introduced by Stone in [30] Definition 14. There he states that a strict extension Y of a space X is one that has a base of open sets such that if G is an open set in Y and H is a basic open set contained in G, then if a nowhere dense set of points from $Y \setminus X$ were added to H, the interior of the resulting set would still be contained in G.

In terms of closed sets, this condition says that there is a base for the closed sets of Y such that if A is a closed set in Y, and F is a basic closed set containing A, then removing a nowhere dense set of points of $Y \setminus X$ from F would result in a set that, when closed, still contains A.

Note that when F is intersected with $f[X]$ it is a closed set in $f[X]$, and removing points from $Y \setminus X$ does not change that. Therefore another way to view the basic closed sets F are as the closures in Y of the closed sets in $f[X]$. This is the definition used in [14], and it is the one we will use here.

Definition 1.3.2. *An extension $f : X \to Y$ is* **strict** *if $\{\overline{f(A)}^Y | A$ closed in $X\}$ is a base for the closed sets in Y.*

Remark 1.3.3. Although this definition uses only closed $A \subseteq X$, the definition in [14] uses all subsets of X. In fact this does not effect the definition at all. To see that, take a set A, not necessarily closed. We have $A \subseteq \overline{A}$, so $f(A) \subseteq f(\overline{A})$. But $f(\overline{A}) \subseteq \overline{f(A)}^Y$ because f is a continuous function. So we have

$$\begin{aligned}\overline{f(A)}^Y &\subseteq \overline{f(\overline{A})}^Y \\ &\subseteq \overline{\overline{f(A)}}^Y \\ &= \overline{f(A)}^Y\end{aligned}$$

which means that $\overline{f(A)}^Y = \overline{f(\overline{A})}^Y$. Therefore it would not change anything to use only closed sets from the start.

We will be studying strict extensions in terms of open filters on the original space. The following filters will be significant.

Definition 1.3.4. *For a space X and an extension $f : X \to Y$, for each $y \in Y$ the* **trace filter** *of y is*

$$\mathcal{T}(y) = \{f^{-1}(V) | y \in V \in \mathcal{O}Y\},$$

which is a filter on $\mathcal{O}X$. In the case where X is a subspace of Y,

$$\mathcal{T}(y) = \{V \cap X | y \in V \in \mathcal{O}Y\}.$$

The set $\{\mathcal{T}(y) | y \in Y\}$ is called the **filter trace** *of f.*

Remark 1.3.5. Although we are assuming that all filters are proper, note that trace filters of strict extensions are proper by necessity. This is because if $\emptyset \in \mathcal{T}(y)$ for some $y \in Y$, then $\emptyset = f^{-1}(V)$ for some $V \in \mathcal{O}Y$, where $y \in V$. But this contradicts the fact that strict extensions are dense, which implies that every non-empty open set in Y must contain points of X.

Note also that for an extension f, if $y = f(x)$ for some $x \in X$, then $\mathcal{T}(y) = \mathcal{N}_x$, the open neighbourhood filter of x. This follows because f is a both continuous and open.

The topology on a strict extension can be described in terms of the filter trace of the strict extension.

Lemma 1.3.6 ([2] Section 2)**.** *Let $f : X \to Y$ be an extension. For each $V \in \mathcal{O}X$, let $V^* = \{y \in Y | V \in \mathcal{T}(y)\}$. Then f is a strict extension if and only if $\{V^* | V \in \mathcal{O}X\}$ generates $\mathcal{O}Y$.*

Proof: The extension f is a strict extension if and only if the sets $\overline{f(A)}^Y$ for A closed in X form a base for the closed sets of Y, which is the case if and only if the sets $\overline{f(X \setminus V)}^Y$ for $V \in \mathcal{O}X$ form a base for the closed sets of Y. We will show that for each $V \in \mathcal{O}X$, $\overline{f(X \setminus V)} = Y \setminus V^*$. Then f is a strict extension if and only if the sets $Y \setminus V^*$ for $V \in \mathcal{O}X$ form a base for the closed sets of Y, which is equivalent to saying that the sets $\{V^* | V \in \mathcal{O}X\}$ form a base for $\mathcal{O}Y$, as required.

If $y \in \overline{f(X \setminus V)}$, then for every $G \in \mathcal{O}Y$ such that $y \in G$, $G \cap f(X \setminus V) \neq \emptyset$. Now f is an open map, so $f(V) \in \mathcal{O}Y$. However, $f(V) \cap f(X \setminus V) = \emptyset$, so $y \notin f(V)$.

Then $f^{-1}f(V) \notin \mathcal{T}(y)$, which means that $V \notin \mathcal{T}(y)$ because f is an embedding. Therefore $y \notin V^*$, which means that $y \in Y \setminus V^*$, and $\overline{f(X \setminus V)} \subseteq Y \setminus V^*$.

For the other inclusion, we will show that $Y \setminus \overline{f(X \setminus V)} \subseteq V^*$. First note that $Y \setminus \overline{f(X \setminus V)} \in \mathcal{O}Y$, so $f^{-1}\left(Y \setminus \overline{f(X \setminus V)}\right) \in \mathcal{O}X$. Now

$$\begin{aligned} & x \in f^{-1}\left(Y \setminus \overline{f(X \setminus V)}\right) \\ \Rightarrow\ & f(x) \in Y \setminus \overline{f(X \setminus V)} \\ \Rightarrow\ & f(x) \notin \overline{f(X \setminus V)} \\ \Rightarrow\ & f(x) \notin f(X \setminus V) \\ \Rightarrow\ & x \notin X \setminus V \\ \Rightarrow\ & x \in V. \end{aligned}$$

So $f^{-1}\left(Y \setminus \overline{f(X \setminus V)}\right) \subseteq V$. Now if $y \in Y \setminus \overline{f(X \setminus V)}$, then $f^{-1}\left(Y \setminus \overline{f(X \setminus V)}\right) \in \mathcal{T}(y)$, and $\mathcal{T}(y)$ is a filter in $\mathcal{O}Y$, so $V \in \mathcal{T}(y)$ also. But this means that $y \in V^*$, so $Y \setminus \overline{f(X \setminus V)} \subseteq V^*$, as required.

□

We will now see how to construct all the strict extensions of a given space. What follows is from [2] Section 3.

Definition 1.3.7. *For a topological space X, let Φ_X be the set of all filters on $\mathcal{O}X$. For an open set V of X, let $\Phi_V = \{\mathcal{F} \in \Phi_X | V \in \mathcal{F}\}$. Let $j : X \to \Phi_X$ be a map taking each element x to its open neighbourhood filter $\mathcal{N}_x$.*

Lemma 1.3.8. *The map $j : X \to \Phi_X$ is a strict extension, when $\{\Phi_V | V \in \mathcal{O}X\}$ forms a base for the topology on Φ_X.*

Proof: We first need to check that the sets $\{\Phi_V | V \in \mathcal{O}X\}$ do form a base for a topology. For open sets U and V of X, we must check that $\Phi_U \cap \Phi_V$ contains Φ_W for some open set W of X, and that $\{\Phi_V | V \in \mathcal{O}X\}$ covers Φ_X. For the former,

$$\begin{aligned} \Phi_U \cap \Phi_V &= \{\mathcal{F} \in \Phi_X | U \in \mathcal{F} \text{ and } V \in \mathcal{F}\} \\ &= \{\mathcal{F} \in \Phi_X | U \cap V \in \mathcal{F}\} \text{ because } \mathcal{F} \text{ is a filter} \\ &= \Phi_{U \cap V} \end{aligned}$$

and $U \cap V \in \mathcal{O}X$. For the latter, the fact that $\{\Phi_V | V \in \mathcal{O}X\}$ is a cover of Φ_X follows from the fact that filters in Φ_X are not empty, so each contains some element of $\mathcal{O}X$. Therefore, we do indeed have a base for a topology.

The map $j : X \to \Phi_X$ is one-one because X is a T_0 space. It is also dense, because for a basic open set Φ_V in Φ_X, if $x \in V$, $V \in \mathcal{N}_x$, so $\mathcal{N}_x \in \Phi_V$. Therefore Φ_V contains $\mathcal{N}_x = j(x)$, the image of a point in X. Since all filters in Φ_X are proper, $\Phi_\emptyset = \emptyset$, so we do not need to check the case where there are no points in V.

To check that j is continuous, take a basic open set Φ_V. Then

$$\begin{aligned} j^{-1}(\Phi_V) &= \{x \in X | j(x) \in \Phi_V\} \\ &= \{x \in X | \mathcal{N}_x \in \Phi_V\} \\ &= \{x \in X | V \in \mathcal{N}_x\} \\ &= V. \end{aligned}$$

We need to check that j is a homeomorphism onto a subset of Φ_X. That is, we must check that $j^{-1} : j[X] \to X$ is continuous, or that $j : X \to j[X]$ is open. To do this, take $V \in \mathcal{O}X$, then

$$\begin{aligned} j(V) &= \{\mathcal{N}_x | x \in V\} \\ &= \{\mathcal{N}_x | V \in \mathcal{N}_x\} \\ &= \{\mathcal{N}_x | X \in \mathcal{N}_x\} \cap \{\mathcal{F} | V \in \mathcal{F}\} \\ &= j[X] \cap \Phi_V. \end{aligned}$$

Since Φ_V is an open set in Φ_X, $j(V)$ is an open set in $j[X]$.

Finally, we must check that j is strict. To do this we will use the characterisation in Lemma 1.3.6. We will show that $V^* = \Phi_V$, so that $\{V^* | V \in \mathcal{O}X\}$ forms a base for the topology of Φ_X. In this context, $V^* = \{\mathcal{F} \in \Phi_X | V \in \mathcal{T}(\mathcal{F})\}$. Now for $\mathcal{F} \in \Phi_X$,

$$\begin{aligned} \mathcal{T}(\mathcal{F}) &= \{j^{-1}(\Phi_V) | \mathcal{F} \in \Phi_V\} \\ &= \{V \in \mathcal{O}X | \mathcal{F} \in \Phi_V\} \text{ as shown above} \\ &= \{V \in \mathcal{O}X | V \in \mathcal{F}\} \\ &= \mathcal{F}. \end{aligned}$$

Therefore $V^* = \{\mathcal{F} \in \Phi_X | V \in \mathcal{F}\} = \Phi_V$. $\square$

Corollary 1.3.9. *If $\Psi \subseteq \Phi_X$, and $j[X] \subseteq \Psi$, then $j : X \to \Psi$ is a strict extension.*

Proof: The proof above works in this case too, by simply replacing Φ_X with Ψ, and the basic open sets Φ_V with $\Phi_V \cap \Psi$. $\square$

We have seen that if we identify an element with its neighbourhood filter, we can create a strict extension by simply adding any other open filters of the space. But in fact, all strict extensions of a given space are essentially of this form.

Proposition 1.3.10. *If $f : X \to Y$ is a strict extension, then Y is homeomorphic to a subspace of Φ_X.*

Proof: Let Ψ be the subspace of Φ_X consisting of the filter trace of f, that is, $\Psi = \{\mathcal{T}(y) | y \in Y\}$, and let $t : Y \to \Psi$ be the map sending each $y \in Y$ to its trace filter $\mathcal{T}(y)$. It is clear that t is onto, and we claim that t is one-one.

Suppose for contradiction that for $y \neq z$, we have $t(y) = t(z)$, so $\mathcal{T}(y) = \mathcal{T}(z)$. Now Y is T_0, so there is an open set V that separates y and z. Suppose that $y \in V$ and $z \notin V$. Then since f is a strict extension, by Lemma 1.3.6 there is an open set $U \in \mathcal{O}X$ such that $y \in U^* \subseteq V$. But if $y \in U^*$, then $f^{-1}(U^*) \in \mathcal{T}(y)$, and so by assumption, $f^{-1}(U^*) \in \mathcal{T}(z)$. Now

$$\begin{aligned} f^{-1}(U^*) &= \{x \in X | f(x) \in U^*\} \\ &= \{x \in X | U \in \mathcal{T}(f(x))\} \\ &= \{x \in X | U \in \mathcal{N}_x\} \text{ from Remark 1.3.5} \\ &= U. \end{aligned}$$

So we have that $U \in \mathcal{T}(z)$, which means that $z \in U^* \subseteq V$. But this contradicts the choice of V, so in fact no such y and z are possible, and t is one-one.

Now to show that t is continuous, consider $\Phi_V \cap \Psi$, a basic open set of Ψ.

$$\begin{aligned} \Phi_V \cap \Psi &= \{\mathcal{F} \in \Phi_X | V \in \mathcal{F}\} \cap \{\mathcal{T}(y) | y \in Y\} \\ &= \{\mathcal{T}(y) | V \in \mathcal{T}(y)\} \end{aligned}$$

So $t^{-1}(\Phi_V \cap \Psi) = \{y \in Y | V \in \mathcal{T}(y)\} = V^*$, because t is one-one.

To show that t^{-1} is continuous is now trivial, because for V^*, a basic open set of Y, $t(V^*) = \{\mathcal{T}(y) | V \in \mathcal{T}(y)\} = \Phi_V \cap \Psi$, a basic open set of Ψ. Therefore we have shown that t is a homeomorphism of Y onto a subspace of Φ_X. $\square$

We have seen that the filter trace of a strict extension is homeomorphic to the strict extension itself. Therefore a strict extension is completely determined by its filter trace (these are just the new filters that we must add to make the extension) and conversely, the filter trace is completely determined by the set of filters used to construct a strict extension. We will see that this is not the case in the pointfree setting. There it is possible to construct a strict extension from a set of filters, and have the filter trace be something different. But to understand this, we will need to see how strict extensions are defined in the pointfree setting, and how to construct them. This is the subject for the next chapter.

2 Strict Extensions in Frames

2.1 Generalising the space concept

In the previous chapter, (Definition 1.3.2) we defined an extension $f : X \to Y$ to be strict if $\{\overline{f(A)}^Y | A \text{ closed in } X\}$ forms a base for the closed sets in Y. This is equivalent to saying that $\{Y \setminus \overline{f(X \setminus U)} | U \in \mathcal{O}X\}$ forms a base for the open sets of Y.

Lemma 2.1.1 ([9] before Definition 1)**.** *For a continuous function $f : X \to Y$, $(\mathcal{O}f)_*(U) = Y \setminus \overline{f(X \setminus U)}$.*

Proof: We need to show that $Y \setminus \overline{f(X \setminus U)} = \bigvee\{G \in \mathcal{O}Y | \mathcal{O}f(G) \subseteq U\}$. Now $\mathcal{O}f = f^{-1}$, so we must show that $f^{-1}\left(Y \setminus \overline{f(X \setminus U)}\right) \subseteq U$, and that if $f^{-1}(G) \subseteq U$ for some $G \in \mathcal{O}Y$, then $G \subseteq Y \setminus \overline{f(X \setminus U)}$.

We already showed towards the end of the proof of Lemma 1.3.6 that for $U \in \mathcal{O}X$, $f^{-1}\left(Y \setminus \overline{f(X \setminus U)}\right) \subseteq U$, so it remains to show that if $f^{-1}(G) \subseteq U$ for some $G \in \mathcal{O}Y$, then $G \subseteq Y \setminus \overline{f(X \setminus U)}$. Now if $f^{-1}(G) \subseteq U$, then

$$\begin{aligned}
& X \setminus U \subseteq X \setminus f^{-1}(G) \\
\Rightarrow\ & f(X \setminus U) \subseteq f\left(X \setminus f^{-1}(G)\right) = f\left(f^{-1}(Y \setminus G)\right) \subseteq Y \setminus G \\
\Rightarrow\ & \overline{f(X \setminus U)} \subseteq \overline{Y \setminus G} = Y \setminus G \\
\Rightarrow\ & G \subseteq Y \setminus \overline{f(X \setminus U)}.
\end{aligned}$$

□

This motivates the following definition:

Definition 2.1.2. *A frame homomorphism $h : M \to L$ is called* **strict** *if $h_*[L]$ generates M. If h is dense and onto, then h is an* **extension***. If h is strict and onto, then it is called a* **strict extension***.*

Remark 2.1.3. The term "strict extension" implies a dense, onto frame homomorphism that is also strict. However, provided that we use the term "generates" to refer to non-empty joins only, a frame homomorphism h is automatically dense if it is strict and onto.

Proof: If h is strict, then $0 = \bigvee\{h_*(b)|b \in S\}$ for some non-empty set $S \subseteq M$. Now $h(0) = 0$, so $h\left(\bigvee\{h_*(b)|b \in S\}\right) = 0$, and so $\bigvee\{hh_*(b)|b \in S\} = 0$, since h is a frame homomorphism. Now h is onto, so $hh_*(b) = b$ for each $b \in S$, and therefore $\bigvee\{b|b \in S\} = 0$. But this implies that $S = \{0\}$, so $0 = h_*(0)$. Since $h_*(0) = \bigvee\{a \in L|h(a) = 0\}$, this implies that h is dense. $\square$

Lemma 2.1.4. *A continuous map $f : X \to Y$ is a strict extension in* **Top** *if and only if $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is a strict extension in* **Frm**.

Proof: First note that $f : X \to Y$ is dense if and only if $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ dense. We will show that if $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is onto, then $f : X \to Y$ is one-one.

Suppose that $x \neq y \in X$ such that $f(x) = f(y)$. Recall that X is T_0, so there is a $U \in \mathcal{O}X$ that separates x and y. Suppose that $x \in U$ and $y \notin U$. Since $\mathcal{O}f$ is onto, there is an open set $V \in \mathcal{O}Y$ such that $\mathcal{O}f(V) = f^{-1}(V) = U$. So we have $f(x) \in V$ but $f(y) \notin V$, which contradicts the assumption that $f(x) = f(y)$.

Now if $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ is a strict extension in **Frm**, then $(\mathcal{O}f)_*[\mathcal{O}X]$ generates $\mathcal{O}Y$. From Lemma 2.1.1, this means that $\left\{Y \setminus \overline{f(X \setminus U)}\middle|U \in \mathcal{O}X\right\}$ forms a base for the open sets of Y. Therefore $f : X \to Y$ is a strict extension in **Top**.

On the other hand, if $f : X \to Y$ is a strict extension in **Top**, then from Lemma 1.3.6, the topology on Y is generated by $\{V^*|V \in \mathcal{O}X\}$. Now if $V \in \mathcal{O}X$, then $V \subseteq V^*$, since if $x \in V$, then $V \in \mathcal{N}_x = \mathcal{T}(x)$, so $x \in V^*$. Now

$$\begin{aligned} f^{-1}(V^*) &= \{x \in X|f(x) \in V^*\} \\ &= \{x \in X|V \in \mathcal{T}(f(x))\} \\ &= \{x \in X|V \in \mathcal{N}_x\} \\ &= V. \end{aligned}$$

Therefore $\mathcal{O}f$ is onto.

For strictness, we must show that $(\mathcal{O}f)_*[\mathcal{O}X]$ generates $\mathcal{O}Y$. But from Lemma 2.1.1, $(\mathcal{O}f)_*(U) = Y \setminus \overline{f(X \setminus U)}$ for each $U \in \mathcal{O}X$, and these do generate $\mathcal{O}Y$ because f is a strict extension in **Top**.

$\square$

We saw in Remark 2.1.3 that strict, onto frame homomorphisms are dense. Under certain circumstances, dense homomorphisms are strict.

Lemma 2.1.5 ([7] after Definition 1)**.** *If M is a regular frame, and $h : M \to L$ is a dense frame homomorphism, then it is strict.*

Proof. We claim that if $x \prec a$ in M, then $x \leq h_*h(x) \leq a$. Then since each $a \in M$ can be written as $a = \bigvee\{x \in M | x \prec a\}$, we get $a = \bigvee\{h_*h(x) | x \prec a\}$, and so a is the join of elements of $h_*[L]$, implying that h is strict. So it remains to prove the claim.

It is always true that $x \leq h_*h(x)$, because $h_*h(x) = \bigvee\{y \in M | h(y) \leq h(x)\}$, and of course x is such a y. To show that $h_*h(x) \leq a$, we show that $h_*h(x) \prec a$ with x^* as the separating element. Firstly, $x^* \vee a = e$ because $x \prec a$. Secondly

$$\begin{aligned} h_*h(x) \wedge x^* &= \bigvee\{y \in M | h(y) \leq h(x)\} \wedge \bigvee\{z \in M | z \wedge x = 0\} \\ &= \bigvee\{y \wedge z | h(y) \leq h(x) \text{ and } z \wedge x = 0\} \text{ because } M \text{ is a frame} \\ &= 0 \end{aligned}$$

because if $z \wedge x = 0$, then $h(z) \wedge h(x) = 0$, but $h(y) \leq h(x)$, so $h(z) \wedge h(y) = 0$, and then $h(z \wedge y) = 0$, which implies that $z \wedge y = 0$ because h is dense. $\square$

We see from this that the notion of a strict extension is a narrowing down of the idea of an extension, which has been very well studied. However, it is only slightly narrower, because as soon as you have regularity, there is no distinction between an extension and a strict extension. However, without regularity, there is certainly a difference.

Example 2.1.6. Let M be $\mathcal{O}Y$, where Y is the set $\mathbb{R}$ of real numbers with the cofinite topology. That is, the closed sets of Y are the finite sets and the whole space. Let L be $\mathcal{O}X$, where X is the set $\mathbb{Q}$ of rational numbers with the cofinite topology. If $f : X \to Y$ is the identical embedding map, then f is continuous because any cofinite set in $\mathbb{R}$ restricted to $\mathbb{Q}$ is still cofinite (since only finitely many rational numbers have been excluded). Then $h = \mathcal{O}f : M \to L$ is a frame homomorphism. Furthermore, it is clearly onto, as any cofinite $U \in \mathcal{O}X$ is the image of $U \cup (\mathbb{R} \setminus \mathbb{Q}) \in \mathcal{O}Y$, and it is dense, because no open set in $\mathcal{O}Y$, besides the empty set, can exclude all of $\mathbb{Q}$. Therefore h is an extension. However, for $U \in \mathcal{O}X$, $h_*(U) = U \cup (\mathbb{R} \setminus \mathbb{Q})$, so, for example, $\pi \in h_*(U)$ for all $U \in \mathcal{O}X$. But then $\mathbb{R} \setminus \{\pi\}$ is an element of M which does not contain $h_*(U)$ for any $U \in \mathcal{O}X$, and therefore cannot be written as the union of such sets. So h is not a strict extension.

We saw in the last chapter that the embedding $j : X \to \Phi_X$ of a given space X into its filter space is the universal strict extension of X, in that any other strict extension Y of X is essentially a subspace of Φ_X. For the remainder of this section, we will show that there is corresponding universal strict extension for frames, and that this result for spaces is just a special case of that result for frames. The proof of this is based on [17].

Firstly, for any frame L, $\bigvee : \mathcal{D}L \to L$, is a strict extension, because its right adjoint is $\downarrow: L \to \mathcal{D}L$, and any downset D can be expressed as $D = \bigvee\{\downarrow a | a \in D\}$. This is the fundamental example of a strict extension, and it has a universal property:

Lemma 2.1.7 ([7] before Lemma 1)**.** *If $h : M \to L$ is a strict extension, then there is an onto frame homomorphism $\tilde{h} : \mathcal{D}L \to M$ such that $\bigvee = h\tilde{h}$:*

$$\begin{tikzcd}
\mathcal{D}L \arrow[rr, "\bigvee"] \arrow[dr, dotted, "\tilde{h}"'] & & L \\
& M \arrow[ur, "h"'] &
\end{tikzcd}$$

Proof: For $U \in \mathcal{D}L$, define $\tilde{h}(U) = \bigvee\{h_*(x)|x \in U\}$, so that $\tilde{h}(\downarrow a) = \bigvee\{h_*(x)|x \leq a\}$ $= h_*(a)$ because h_* is order-preserving. We show that $\tilde{h}$ is a frame homomorphism. Firstly, $\tilde{h}(\downarrow 0) = h_*(0) = 0$ because h is dense, and secondly, $\tilde{h}(\downarrow e) = h_*(e) = e$ because $h(e) \leq e$. Next, for U and V in $\mathcal{D}L$,

$$\begin{aligned}
\tilde{h}(U) \wedge \tilde{h}(V) &= \bigvee\{h_*(x)|x \in U\} \wedge \bigvee\{h_*(y)|y \in V\} \\
&= \bigvee\{h_*(x) \wedge h_*(y)|x \in U, y \in V\} \text{ by the frame law.} \\
&= \bigvee\{h_*(x \wedge y)|x \in U, y \in V\} \text{ since } h_* \text{ preserves meets.} \\
&= \bigvee\{h_*(z)|z \in U \cap V\} \text{ because } U \text{ and } V \text{ are downsets.} \\
&= \tilde{h}(U \cap V).
\end{aligned}$$

Lastly, $\tilde{h}$ preserves joins because

$$\tilde{h}\left(\bigcup_{i\in I} U_i\right) = \bigvee\left\{h_*(x)\middle|x \in \bigcup_{i\in I} U_i\right\} = \bigvee_{i\in I}\left(\bigvee\{h_*(x)|x \in U_i\}\right) = \bigvee_{i\in I}\tilde{h}(U_i).$$

Now this homomorphism is onto because h is strict, so any $a \in M$ can be written as

$$a = \bigvee\{h_*(b)|h_*(b) \leq a\} = \bigvee\left\{\tilde{h}(\downarrow b)\middle|\tilde{h}(\downarrow b) \leq a\right\} = \tilde{h}\left(\bigvee\left\{\downarrow b\middle|\tilde{h}(\downarrow b) \leq a\right\}\right).$$

Finally, $h\tilde{h}(\downarrow a) = hh_*(a) = a$, because h is onto, and $\bigvee \downarrow a = a$, so $\tilde{h}$ provides the required factorisation because the principal downsets generate $\mathcal{D}L$. $\square$

So we have seen that any strict extension of L is a quotient of $\bigvee : \mathcal{D}L \to L$. But in fact, any frame homomorphism $f : M \to L$ that is a factor of $\bigvee : \mathcal{D}L \to L$, where the other factor $g : \mathcal{D}L \to M$ is onto, is a strict extension. This a particular case of the following lemma:

Lemma 2.1.8 ([7] Lemma 1)**.** *If $h : M \to L$ is a strict extension such that $h = fg$, where $g : M \to N$ is onto and $f : N \to L$ is any frame homomorphism, then $f_* = gh_*$, and f is a strict extension:*

$$\begin{tikzcd}
M \arrow[rr, "h", "strict"'] \arrow[dr, "g", "onto"'] & & L \\
& N \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof: First we show that $f_* = gh_*$, and we do this by showing that for $x \in N$ and $y \in L$, $x \leq f_*(y)$ if and only if $x \leq gh_*(y)$. So suppose that $x \leq gh_*(y)$. Then $f(x) \leq fgh_*(y)$, so $f(x) \leq hh_*(y)$ because $fg = h$, and so $f(x) \leq y$ because h is onto. This gives that $x \leq f_*(y)$.

On the other hand, if we have that $x \leq f_*(y)$, then $f(x) \leq y$. Now since g is onto, $x = gg_*(x)$, so $fgg_*(x) \leq y$. Then $hg_*(x) \leq y$ because $h = fg$, but this means that $g_*(x) \leq h_*(y)$. Then $gg_*(x) \leq gh_*(y)$, and so $x \leq gh_*(y)$, again because $gg_*(x) = x$.

Now we show that f is a strict extension. For any $a \in N$, there exists a point $x \in M$ such that $g(x) = a$ because g is onto. Then since h is strict, $x = \bigvee\{h_*(y)|h_*(y) \leq x\}$. So

$$\begin{aligned}
a = g(x) &= g\left(\bigvee\{h_*(y)|h_*(y) \leq x\}\right) \\
&= \bigvee\{gh_*(y)|h_*(y) \leq x\} \\
&= \bigvee\{f_*(y)|h_*(y) \leq x\} \text{ since } gh_* = f_*.
\end{aligned}$$

Therefore f is strict. Finally, if $y \in L$, then, since h is onto, there is an $x \in M$ such that $h(x) = y$. But then $fg(x) = y$, that is, $f(g(x)) = y$, and so f is onto. $\square$

Taking these two lemmas together, we see that the strict extensions of a frame L are precisely the homomorphisms $M \to L$ obtained by factoring $\bigvee : \mathcal{D}L \to L$ with onto maps $\mathcal{D}L \to M$. We will now show that this generalises the situation we had in spaces, that the strict extensions of a space X are precisely the continuous functions $X \to Y$ obtained by factoring $X \to \varphi_X$ with embeddings $Y \to \varphi_X$. To establish this connection precisely, we need to define some functors.

Definition 2.1.9. *Let* **SLat** *be the category of bounded meet-semilattices, and bounded meet-semilattice homomorphisms. That is, maps in this category preserve the bottom, and finitary meets, including the top.*

Definition 2.1.10. *Recall from Example 1.2.4 that for a bounded meet-semilattice L, $\mathcal{D}L$ is the frame of non-empty downsets of L, with join and meet being set union and intersection. If $h : L \to M$ is a bounded meet-semilattice homomorphism, then let $\mathcal{D}h : \mathcal{D}L \to \mathcal{D}M$ be the map such that for every $U \in \mathcal{D}L$,*

$$\mathcal{D}h(U) = \bigcup\{\downarrow h(x)|x \in U\}.$$

Lemma 2.1.11. *The map $\mathcal{D}h : \mathcal{D}L \to \mathcal{D}M$ is a frame homomorphism, and $\mathcal{D} :$* **SLat** $\to$ **Frm** *is a functor.*

Proof: We first show that $\mathcal{D}h$ preserves finite meets, starting with the top. If $U = L$, then $e \in U$, so $\downarrow h(e) \subseteq \mathcal{D}h(L)$, and $h(e) = e$, so $\mathcal{D}h(L) = M$.

Now if U and V are downsets of L, then we must show that $\mathcal{D}h(U \cap V) = \mathcal{D}h(U) \cap \mathcal{D}h(V)$, that is, that $\bigcup\{\downarrow h(x)|x \in U \cap V\} = \bigcup\{\downarrow h(x)|x \in U\} \cap \bigcup\{\downarrow h(y)|y \in V\}$. One inclusion is obvious because if $x \in U \cap V$, then $x \in U$ and $x \in V$. For the other inclusion, if $z \in \bigcup\{\downarrow h(x)|x \in U\} \cap \bigcup\{\downarrow h(y)|y \in V\}$, then $z \leq h(x)$ for some $x \in U$ and $z \leq h(y)$ for some $y \in V$. But then $z \leq h(x) \wedge h(y) = h(x \wedge y)$, and $x \wedge y \in U \cap V$, because U and V are downsets. So $z \in \bigcup\{\downarrow h(x)|x \in U \cap V\}$.

Next we show that Dh preserves arbitrary joins. If $U = \downarrow 0 = \{0\}$, then $\mathcal{D}h(U) = \downarrow h(0) = \{0\}$, so $\mathcal{D}(h)$ preserves the bottom. Also,

$$\begin{aligned}
\mathcal{D}h\left(\bigcup_{i \in I} U_i\right) &= \bigcup\left\{\downarrow h(x)\Big|x \in \bigcup_{i \in I} U_i\right\} \\
&= \bigcup\{\downarrow h(x)|x \in U_i \text{ for some } i \in I\} \\
&= \bigcup_{i \in I}\left(\bigcup\{\downarrow h(x)|x \in U_i\}\right) \\
&= \bigcup_{i \in I} \mathcal{D}h(U_i).
\end{aligned}$$

Now to show that $\mathcal{D}$ is a functor, we must show that it preserves identities and composition. If $i : L \to L$ is an identity meet-semilattice homomorphism, then for every $U \in \mathcal{D}L$, $\mathcal{D}i(U) = \bigcup\{\downarrow x|x \in U\} = U$, because U is a downset.

If $h : L \to M$ and $k : M \to N$ are meet-semilattice homomorphisms, then we must show that $\mathcal{D}(kh)(U) = \mathcal{D}k \circ \mathcal{D}h(U)$ for all $U \in \mathcal{D}L$, that is, that

$\bigcup\{\downarrow kh(x)|x \in U\}= \bigcup\{\downarrow k(y)|y \in \mathcal{D}h(U)\}$. If $a \leq kh(x)$, where $x \in U$, then if $y = h(x)$, $y \in \mathcal{D}h(U)$ and $a \leq k(y)$. In the other direction, if $a \leq k(y)$ for some $y \in \mathcal{D}h(U)$, then $y \leq h(x)$ for some $x \in U$, so $k(y) \leq kh(x)$, and then $a \leq kh(x)$ for this $x \in U$. $\square$

Corollary 2.1.12. *$\mathcal{D}'$: **Frm** $\to$ **Frm**, the functor $\mathcal{D}$ restricted to the category **Frm**, is the downset functor on frames.*

The functor $\mathcal{D}$ will only be needed in the next section. For the remainder of this section, we will use the notation $\mathcal{D}$ instead of $\mathcal{D}'$ to refer to the downset functor on frames, as no confusion will arise.

Definition 2.1.13. *If L is a frame, let ΦL be the set of proper filters on L. For $a \in L$, let $\Phi_a = \{F \in \Phi L|a \in F\}$. If $h : L \to M$ is a frame homomorphism, then let $\Phi h : \Phi M \to \Phi L$ be the map such that for $F \in \Phi M$, $\Phi h(F) = h^{-1}[F]$, which we showed in Lemma 1.2.9 is a filter on L.*

Lemma 2.1.14. *For a frame L, the sets $\{\Phi_a|a \in L\}$ form a base for a topology on ΦL. Then for any frame homomorphism $h : L \to M$, $\Phi h : \Phi M \to \Phi L$ is a continuous function, and Φ : **Frm** $\to$ **Top** is a contravariant functor.*

Proof: To show that $\{\Phi_a|a \in L\}$ is a base for a topology, we must show that it is a cover, and that the set it generates is closed under finite meets. Now $\bigcup\{\Phi_a|a \in L\} = \{F \in \Phi L|a \in F \text{ for some } a \in L\}$. But since each $F \in \Phi L$ is not empty, it must contain some point of L, and so $\bigcup\{\Phi_a|a \in L\} = \Phi L$.

Now consider Φ_a and Φ_b for a and b in L. We have that $\Phi_{a\wedge b} \subseteq \Phi_a \cap \Phi_b$, because if $F \in \Phi_{a\wedge b}$, then $a \wedge b \in F$, so both $a \in F$ and $b \in F$ because F is up-closed, and so $F \in \Phi_a$ and $F \in \Phi_b$. Therefore $\Phi_{a\wedge b} \subseteq \Phi_a \cap \Phi_b$, as required.

To show that Φh is continuous, we will show that for any $a \in L$, $(\Phi h)^{-1}[\Phi_a] = \Phi_{h(a)}$.

$$\begin{aligned}F \in (\Phi h)^{-1}[\Phi_a] &\Leftrightarrow \Phi h(F) \in \Phi_a\\ &\Leftrightarrow a \in \Phi h(F)\\ &\Leftrightarrow a \in h^{-1}[F]\\ &\Leftrightarrow h(a) \in F\\ &\Leftrightarrow F \in \Phi_{h(a)}.\end{aligned}$$

Next we must show that Φ preserves identities and composition. If $i : L \to L$ is an identity frame homomorphism, then for $F \in L$, $\Phi i(F) = i^{-1}[F] = F$. If $h : L \to M$ and $k : M \to N$ are frame homomorphisms, then $\Phi kh(F) = (kh)^{-1}[F] = h^{-1}k^{-1}[F] = h^{-1}(\Phi k(F)) = \Phi h \circ \Phi k(F)$. $\square$

We are trying to prove that the fact that the embedding $j : X \to \Phi_X$ is the universal strict extension for T_0 spaces, is just a special case of $\bigvee : \mathcal{D}L \to L$, the universal strict extension for frames. Using the functor Φ, we can write Φ_X as $\Phi\mathcal{O}X$, the space made up of filters of open sets of X, so $j : X \to \Phi\mathcal{O}X$. We need to relate the functors Φ and $\mathcal{D}$ in order to show how the two results are connected.

Lemma 2.1.15. *There is a bijection between the set of filters on a frame L and the set of completely prime filters on $\mathcal{D}L$.*

Proof: If F is a filter on L, then $\{\downarrow a | a \in F\}$ is a filter base, because if a and b are in F, then $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$, and $a \wedge b \in F$. Let $\downarrow[F]$ be the filter on $\mathcal{D}L$ generated by $\{\downarrow a | a \in F\}$. To show that $\downarrow[F]$ is completely prime, consider $\mathcal{U}$, a set of downsets of L such that $\bigcup \mathcal{U} \in \downarrow[F]$. This means that for some $a \in F$, $\downarrow a \subseteq \bigcup \mathcal{U}$. But then $a \in \bigcup \mathcal{U}$, so $a \in U$ for some $U \in \mathcal{U}$. But since U is a downset, $\downarrow a \subseteq U$, so $U \in \downarrow[F]$.

To go the other way, let $\mathcal{P}$ be a completely prime filter on $\mathcal{D}L$, and let $\bigvee[\mathcal{P}] = \{\bigvee U | U \in \mathcal{P}\}$. This is up-closed because if $a \in L$ such that $\bigvee U \leq a$ for some $U \in \mathcal{P}$, then $U \subseteq \downarrow a$, and so $\downarrow a \in \mathcal{P}$, which means that $a = \bigvee \downarrow a \in \bigvee[\mathcal{P}]$. Also, if U and W are in $\mathcal{P}$, then $\bigvee U \wedge \bigvee W = \bigvee\{u \wedge w | u \in U, w \in W\} = \bigvee\{U \cap W\}$ since U and W are downsets. Now $U \cap W \in \mathcal{P}$, so $\bigvee U \wedge \bigvee W \in \bigvee[\mathcal{P}]$, and so $\bigvee[\mathcal{P}]$ is a filter on L.

We must now show that these functions are inverses of each other. Firstly, if F is a filter on L, and $a \in F$, then $\downarrow a \in \downarrow[F]$, so $a = \bigvee \downarrow a \in \bigvee[\downarrow[F]]$. On the other hand, if $a \in \bigvee[\downarrow[F]]$, then $a = \bigvee U$ for some $U \in \downarrow[F]$, so there is a $b \in F$ such that $\downarrow b \subseteq U$. But then $b = \bigvee \downarrow b \leq \bigvee U = a$, so $a \in F$. Therefore $F = \bigvee[\downarrow[F]]$.

Composing the other way, if $\mathcal{P}$ is a completely prime filter on $\mathcal{D}L$, and $U \in \mathcal{P}$, then $U = \bigcup\{\downarrow u | u \in U\}$ since U is a downset, so $\bigcup\{\downarrow u | u \in U\} \in \mathcal{P}$. Now $\mathcal{P}$ is completely prime, so $\downarrow u \in \mathcal{P}$ for some $u \in U$. Then $u = \bigvee \downarrow u \in \bigvee[\mathcal{P}]$, and $\downarrow u \subseteq U$, so $U \in \downarrow[\bigvee[\mathcal{P}]]$. On the other hand, if $U \in \downarrow[\bigvee[\mathcal{P}]]$, then there is $a \in \bigvee[\mathcal{P}]$ such that $\downarrow a \subseteq U$. Then $a = \bigvee D$ for some $D \in \mathcal{P}$. But D is a downset, so $D \subseteq \downarrow a \subseteq U$, so $U \in \mathcal{P}$. Therefore $\mathcal{P} = \downarrow[\bigvee[\mathcal{P}]]$. $\square$

So we see that there is a bijection between ΦL and $\Sigma\mathcal{D}L$. This correspondence is in fact a categorical isomorphism, and to prove that, we need to first show that the bijection given above is in fact a homeomorphism.

Lemma 2.1.16. *For any frame L, let $\lambda_L : \Phi L \to \Sigma\mathcal{D}L$ be the function defined by $\lambda_L(F) = \downarrow[F]$, where $\downarrow[F]$ is the completely prime filter generated by $\{\downarrow a | a \in F\}$. Then λ_L is a homeomorphism.*

Proof: Since there is no ambiguity regarding the frame L, we will write λ instead of λ_L for this proof. It was shown above that λ is a bijection. Now we must show that λ is continuous and open.

We show that λ is continuous by showing that $\lambda^{-1}[\Sigma_U] = \bigcup\{\Phi_a | a \in U\}$. Now it was shown in Lemma 2.1.15 above that for $\mathcal{P} \in \Sigma\mathcal{D}L$, $\lambda^{-1}(\mathcal{P}) = \bigvee[\mathcal{P}]$, so $\lambda^{-1}[\Sigma_U] = \{\bigvee[\mathcal{P}] | \mathcal{P} \in \Sigma_U\} = \{\bigvee[\mathcal{P}] | U \in \mathcal{P}\}$, and we therefore want to show that $\{\bigvee[\mathcal{P}] | U \in \mathcal{P}\} = \bigcup\{\Phi_a | a \in U\}$.

Now if $U \in \mathcal{P}$, then $U = \bigvee\{\downarrow a | a \in U\} \in \mathcal{P}$. Now $\mathcal{P}$ is completely prime, so $\downarrow a \in \mathcal{P}$ for some $a \in U$. But then $\bigvee(\downarrow a) = a \in \bigvee[\mathcal{P}]$, and so $\bigvee[\mathcal{P}] \in \Phi_a$ for this $a \in U$. On the other hand, if $F \in \Phi_a$ for some $a \in U$, then $a \in F$ for this $a \in U$. Now $F = \bigvee[\downarrow F]$, and $\downarrow a \in \downarrow F$. But $a \in U$, so $\downarrow a \subseteq U$ because U is a downset, so $U \in \downarrow F$. Therefore $F = \bigvee[\mathcal{P}]$, where $U \in \mathcal{P}$.

Next we show that λ is open by showing that $\lambda[\Phi_a] = \Sigma_{\downarrow a}$ for all $a \in L$. We have $\lambda[\Phi_a] = \lambda[\{F \in \Phi L | a \in F\}] = \{\downarrow[F] | a \in F\}$, and $\Sigma_{\downarrow a} = \{\mathcal{P} \in \Sigma\mathcal{D}L | \downarrow a \in \mathcal{P}\}$. Now if $a \in F$ for some $F \in \Phi L$, then $\downarrow a \in \downarrow[F]$, so $\downarrow[F] \in \Sigma_{\downarrow a}$. On the other hand, if $\downarrow a \in \mathcal{P}$ for some $\mathcal{P} \in \Sigma\mathcal{D}L$, then since λ is onto, $\mathcal{P} = \downarrow[F]$ for some $F \in \Phi L$. Now $\downarrow a \in \downarrow[F]$ means that there exists an $x \in F$ such that $\downarrow x \subseteq \downarrow a$, but then $x \leq a$, and so $a \in F$. So $\mathcal{P} \in \lambda[\Phi_a]$, and therefore $\lambda[\Phi_a] = \Sigma_{\downarrow a}$. $\square$

Corollary 2.1.17. *For any topological space X, there is a frame isomorphism $\alpha_{\mathcal{O}X}$ making this triangle commute:*

$$\begin{tikzcd}
\mathcal{D}\mathcal{O}X \arrow[rr, "\bigvee"] \arrow[dr, "\alpha_{\mathcal{O}X}"'] & & \mathcal{O}X \\
& \mathcal{O}\Phi\mathcal{O}X \arrow[ur, "\mathcal{O}j"'] &
\end{tikzcd}$$

Proof: Firstly, given any frame L, $\mathcal{D}L$ is a spatial frame, as mentioned in Lemma 1.2.11. Therefore, the spatial reflection, $\eta_{\mathcal{D}L} : \mathcal{D}L \to \mathcal{O}\Sigma\mathcal{D}L$ is a frame isomorphism. Secondly, since we just showed that $\lambda_L : \Phi L \to \Sigma\mathcal{D}L$ is a homeomorphism, $\mathcal{O}\lambda_L : \mathcal{O}\Sigma\mathcal{D}L \to \mathcal{O}\Phi L$ is a frame isomorphism. Therefore if $\alpha_L = \mathcal{O}\lambda_L \circ \eta_{\mathcal{D}L}$, then $\alpha_L : \mathcal{D}L \to \mathcal{O}\Phi L$ is a frame isomorphism. We just need to show that the triangle above commutes.

If $S \in \mathcal{D}L$, $\eta_{\mathcal{D}L}(S) = \Sigma_S = \{\mathcal{P} \in \Sigma\mathcal{D}L | S \in \mathcal{P}\}$. Then

$$\begin{aligned}
\alpha_L(S) &= (\mathcal{O}\lambda_L \circ \eta_{\mathcal{D}L})(S) \\
&= \lambda_L^{-1}[\Sigma_S] \\
&= \{F \in \Phi L | \lambda_L(F) \in \Sigma_S\} \\
&= \{F \in \Phi L | \downarrow[F] \in \Sigma_S\} \\
&= \{F \in \Phi L | S \in \downarrow[F]\} \\
&= \{F \in \Phi L | \downarrow a \subseteq S \text{ for some } a \in F\} \\
&= \{F \in \Phi L | a \in S \text{ for some } a \in F\} \\
&= \{F \in \Phi L | F \cap S \neq \emptyset\}.
\end{aligned}$$

Now if $\mathcal{S} \in \mathcal{D}\mathcal{O}X$, then

$$\begin{aligned}
(\mathcal{O}j \circ \alpha_{\mathcal{O}X})(\mathcal{S}) &= j^{-1}[\{\mathcal{F} \in \Phi\mathcal{O}X | \mathcal{F} \cap \mathcal{S} \neq \emptyset\}] \\
&= \{x \in X | j(x) \cap \mathcal{S} \neq \emptyset\} \\
&= \{x \in X | x \in U \text{ for some } U \in \mathcal{S}\} \\
&= \bigcup \mathcal{S} = \bigvee \mathcal{S}.
\end{aligned}$$

□

Lemma 2.1.18. *The map $\lambda : \Phi \to \Sigma\mathcal{D}$ is a natural isomorphism.*

Proof: We must show that whenever $h : L \to M$ is a frame homomorphism, then the following square commutes:

$$\begin{tikzcd}
\Phi M \arrow[r, "\lambda_M"] \arrow[d, "\Phi h"'] & \Sigma\mathcal{D}M \arrow[d, "\Sigma\mathcal{D}h"] \\
\Phi L \arrow[r, "\lambda_L"'] & \Sigma\mathcal{D}L
\end{tikzcd}$$

Take $F \in \Phi M$. Then

$$\lambda_L(\Phi h(F)) = \downarrow[h^{-1}[F]] = \{U \in \mathcal{D}L | \downarrow a \subseteq U \text{ for some } a \in h^{-1}[F]\}.$$

Going the other way,

$$\begin{aligned}
\Sigma\mathcal{D}h(\lambda_M(F)) &= (\mathcal{D}h)^{-1}[\{V \in \mathcal{D}M | \downarrow b \subseteq V \text{ for some } b \in F\}] \\
&= \{U \in \mathcal{D}L | \mathcal{D}h(U) = V \text{ where } \downarrow b \subseteq V \text{ for some } b \in F\} \\
&= \{U \in \mathcal{D}L | \downarrow b \subseteq \mathcal{D}h(U) \text{ for some } b \in F\} \\
&= \{U \in \mathcal{D}L | \downarrow b \subseteq \bigcup\{\downarrow h(y) | y \in U\} \text{ for some } b \in F\}.
\end{aligned}$$

Now we must show that these two sets are equal. Suppose that $U \in \lambda_L(\Phi h(F))$, so that $\downarrow a \subseteq U$ for some $a \in h^{-1}[F]$. Now $\downarrow a \subseteq U$ implies that $a \in U$, and so $\downarrow h(a) \subseteq \mathcal{D}h(U)$. Further, $h(a) \in F$ because $a \in h^{-1}[F]$, so $U \in \Sigma\mathcal{D}h(\lambda_M(F))$, using $b = h(a)$.

On the other hand, if $U \in \Sigma\mathcal{D}h(\lambda_M(F))$, then for some $b \in F$,

$$\begin{aligned} \downarrow b &\subseteq \bigcup\{\downarrow h(y) | y \in U\} \\ \Rightarrow b &\in \bigcup\{\downarrow h(y) | y \in U\} \\ \Rightarrow b &\in \downarrow h(y) \text{ for some } y \in U \\ \Rightarrow b &\leq h(y) \text{ for some } y \in U. \end{aligned}$$

Now $y \in U$, so $\downarrow y \subseteq U$. Also, $b \in F$, so $h(y) \in F$ also, and so $y \in h^{-1}[F]$. Therefore $U \in \lambda_L(\Phi h(F))$, using $a = y$. $\square$

Proposition 2.1.19. *The universal strict extension $j : X \to \Phi\mathcal{O}X$ for spaces can be derived from $\bigvee : \mathcal{D}L \to L$, the universal strict extension for frames.*

Proof: Firstly, from Corollary 2.1.17, $\mathcal{O}j : \mathcal{O}\Phi\mathcal{O}X \to \mathcal{O}X$ is a strict extension in **Frm**, using Lemma 2.1.8, and the fact that $\alpha_{\mathcal{O}X}$, being an isomorphism, is onto. Then from Lemma 2.1.4, $j : X \to \Phi\mathcal{O}X$ is a strict extension. To show that it is universal, we must show that if $f : X \to Y$ is another strict extension of X, then we can find a continuous function $\bar{f} : Y \to \Phi\mathcal{O}X$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \Phi\mathcal{O}X \\ Y \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

Now consider the diagram below.

$$\begin{tikzcd} \mathcal{O}X & \mathcal{O}\Phi\mathcal{O}X \arrow[l, "\mathcal{O}j"'] \\ \mathcal{O}Y \arrow[u, "\mathcal{O}f"] & \mathcal{D}\mathcal{O}X \arrow[l, dotted, "\bar{h}"] \arrow[u, "\alpha_{\mathcal{O}X}"'] \arrow[ul, "\bigvee_{\mathcal{O}X}"'] \end{tikzcd}$$

The top triangle commutes by Corollary 2.1.17. Since $\mathcal{O}f$ is a strict extension of $\mathcal{O}X$, and $\bigvee_{\mathcal{O}X}$ is the universal such, there is a frame homomorphism $\bar{h} : \mathcal{D}\mathcal{O}X \to \mathcal{O}Y$ that makes the bottom triangle commute. Let $h = \bar{h} \circ (\alpha_{\mathcal{O}X})^{-1}$. Then h is a frame homomorphism making this triangle commute:

$$\begin{tikzcd}
\mathcal{O}X & \mathcal{O}\Phi\mathcal{O}X \arrow[l, "\mathcal{O}j"'] \arrow[dl, "h"] \\
\mathcal{O}Y \arrow[u, "\mathcal{O}f"] &
\end{tikzcd}$$

Now apply the functor Σ to this triangle to get the small triangle below. The two quadrilaterals commute because $\varepsilon : I \to \Sigma\mathcal{O}$ is a natural transformation, where $I : \mathbf{Top} \to \mathbf{Top}$ is the identity functor.

$$\begin{tikzcd}
X \arrow[rrr, "j"] \arrow[dr, "\varepsilon_X"] \arrow[ddd, "f"'] & & & \Phi\mathcal{O}X \arrow[dl, "\varepsilon_{\Phi\mathcal{O}X}"] \\
& \Sigma\mathcal{O}X \arrow[r, "\Sigma\mathcal{O}j"] \arrow[d, "\Sigma\mathcal{O}f"'] & \Sigma\mathcal{O}\Phi\mathcal{O}X & \\
& \Sigma\mathcal{O}Y \arrow[ur, "\Sigma h"'] & & \\
Y \arrow[ur, "\varepsilon_Y"'] & & &
\end{tikzcd}$$

Now using the fact, from Lemma 2.1.18, that Φ and $\Sigma\mathcal{D}$ are naturally isomorphic,

$$\begin{aligned}
\Sigma\mathcal{O}\Phi\mathcal{O}X &\cong \Sigma\mathcal{O}\Sigma\mathcal{D}\mathcal{O}X \\
&\cong \Sigma\mathcal{D}\mathcal{O}X \text{ because } \mathcal{D}\mathcal{O}X \text{ is spatial} \\
&\cong \Phi\mathcal{O}X.
\end{aligned}$$

So $\Phi\mathcal{O}X$ is a sober space, making $\varepsilon_{\Phi\mathcal{O}X} : \Phi\mathcal{O}X \to \Sigma\mathcal{O}\Phi\mathcal{O}X$ a homeomorphism. Therefore, let $\bar{f} = (\varepsilon_{\Phi\mathcal{O}X})^{-1} \circ \Sigma h \circ \varepsilon_Y$, and then the original triangle is just the big triangle in the above diagram, which commutes. $\square$

2.2 Hong's Construction

We have now seen in two different ways that strict extensions of spaces can be described as embeddings into filter spaces. Another way to view this is that the trace filters which were not already neighbourhood filters, were added to the space as new points. We will now describe a pointfree equivalent of this idea of adding filters to make strict extensions. This idea was originally introduced by Hong in [21], and so I call it "Hong's Construction".

Let L be a frame, and let X be a set of filters on L. Let $s_X L$ be the subset of $L \times \mathcal{P}(X)$ consisting of pairs $(a, \mathcal{S})$, where $a \in F$ for every $F \in \mathcal{S}$. Then let $s : s_X L \to L$ be the restriction of the first projection map, so that $s(a, \mathcal{S}) = a$. Then s is a frame map because it is essentially the identity map on the first co-ordinate. Let s_* be the right adjoint of s, so $s_*(a) = (a, X_a)$, where $X_a = \{F \in X | a \in F\}$. Then let $t_X L$ be the subframe of $s_X L$ that is generated by $s_*[L]$, and call the restriction of s to $t_X L$, t. Then t is strict by definition, and it is onto because $t(s_*(a)) = a$, so t is a strict extension.

Definition 2.2.1. *The map $t : t_X L \to L$ is called the* **strict extension of** L **with respect to** X.

Remark 2.2.2. Note that the first component of $t_X L$ is still all of L, because for each $a \in L$, $s_*(a) = (a, X_a)$, and so a is in the first component of $t_X L$.

Remark 2.2.3. This is a generalisation of the way strict extensions were constructed in spaces. For a topological space Y and a set X of filters on Y, we constructed a strict extension such that X is the filter trace of the extension. We saw in Lemma 1.3.8 that for each filter $\mathcal{F} \in X$, the trace filter $\mathcal{T}(\mathcal{F})$ associated with $\mathcal{F}$ is simply $\mathcal{F}$ itself. Now the topology on X is generated by the sets $V^* = \{\mathcal{F} \in X | V \in \mathcal{T}(\mathcal{F})\} = \{\mathcal{F} \in X | V \in \mathcal{F}\} = X_V$. So if $L = \mathcal{O}Y$, then $t_X L$ is the subframe of $\mathcal{O}Y \times \mathcal{P}(X)$ that is generated by $s_*(V) = (V, X_V)$. But as we have seen, this is $\mathcal{O}Y \times \mathcal{O}X$. So Hong's Construction in this case gives $\mathcal{O}Y \times \mathcal{O}X \to \mathcal{O}Y$, which corresponds exactly to the continuous map $Y \to X$ in the second coordinate, and the first coordinate is just the identity map.

This construction is fairly intuitive - we started with the frame we were interested in, we added all the filters we wanted by sticking them into the second component of a pair, and then restricted this to the frame generated by the right adjoint of the first projection, to make a strict extension. However, this construction is not really practical to use because it involves so many steps, so in [7] it is streamlined, based on the universal properties of $\mathcal{D}L$ established in the previous section.

Let L be a frame and X a set of filters on L. Let $\mathcal{O}X$ be the topology on X generated by the sets X_a where $a \in L$. Then we claim that the map $h : \mathcal{D}L \to \mathcal{O}X$ taking each $U \in \mathcal{D}L$ to $X_U = \bigcup_{a \in U} X_a = \{F \in X | F \cap U \neq \emptyset\}$ is a frame homomorphism:

- First note that $h(\downarrow 0) = \emptyset$ because no filters contain 0, and $h(\downarrow e) = X$, because every filter contains at least e.

- For U and V in $\mathcal{D}L$,

$$\begin{aligned} h(U \cap V) &= \{F \in X | F \cap (U \cap V) \neq \emptyset\} \\ &= \{F \in X | F \cap U \neq \emptyset\} \cap \{F \in X | F \cap V \neq \emptyset\} \\ &= h(U) \cap h(V). \end{aligned}$$

  This is because on the one hand, if $a \in F \cap (U \cap V)$, then clearly $a \in F \cap U$ and $a \in F \cap V$. On the other hand, if $a \in F \cap U$ and $b \in F \cap V$, then $a \wedge b \in U \cap V$, since U and V are downsets, and so $a \wedge b \in F \cap (U \cap V)$.

- For $\mathcal{S} \subseteq \mathcal{D}L$,

$$\begin{aligned} h\left(\bigvee\{U | U \in \mathcal{S}\}\right) &= h\left(\bigcup\{U | U \in \mathcal{S}\}\right) \\ &= \left\{F \in X | F \cap \bigcup\{U | U \in \mathcal{S}\} \neq \emptyset\right\} \\ &= \bigcup_{U \in \mathcal{S}} \{F \in X | F \cap U \neq \emptyset\} \\ &= \bigcup\{h(U) | U \in \mathcal{S}\}. \end{aligned}$$

So we have a frame homomorphism $h : \mathcal{D}L \to \mathcal{O}X$. We also have the join map $\bigvee : \mathcal{D}L \to L$, which is a frame homomorphism. Therefore $\bigvee \times h : \mathcal{D}L \to L \times \mathcal{O}X$ is a frame homomorphism, where $U \to (\bigvee U, X_U)$. Now let $\tau_X L$ be the image of this map, and let $\tau : \tau_X L \to L$ be the restriction to $\tau_X L$ of the first projection map. Then the composition $\mathcal{D}L \to \tau_X L \to L$ is a factorisation of the join map, since for $U \in \mathcal{D}L$, $U \to (\bigvee U, X_U) \to \bigvee U$. Therefore, by Lemma 2.1.8, $\tau_X L \to L$ is a strict extension.

Lemma 2.2.4. *For any frame L and set of filters X on L, $t_X L = \tau_X L$.*

Proof: If $(x, \mathcal{S}) \in t_X L$, then there exists a set $U \subseteq L$ such that $(x, \mathcal{S}) = \bigvee\{(a, X_a) | a \in U\}$. This means that $x = \bigvee U$, and $\mathcal{S} = \bigcup\{X_a | a \in U\} = X_U$. So $(x, \mathcal{S}) = (\bigvee U, X_U)$. However, to show that $(x, \mathcal{S}) \in \tau_X L$, we must show that

$(x, \mathcal{S}) = (\bigvee U, X_U)$ for some downset $U \in \mathcal{D}L$. We sill show that $(\bigvee U, X_U) = (\bigvee {\downarrow} U, X_{\Downarrow U})$

Clearly, $\bigvee U = \bigvee {\downarrow} U$. For the second component, if $F \in X_U$, then $F \cap U \neq \emptyset$, so then $F \cap {\downarrow} U \neq \emptyset$, and so $F \in X_{\Downarrow U}$. On the other hand, if $F \in X_{\Downarrow U}$, then $F \cap {\downarrow} U \neq \emptyset$, so there must be some $x \in F$ such that $x \leq u$ for some $u \in U$. But filters are up-closed, so $u \in F$ also, and so $F \cap U \neq \emptyset$, which means that $F \in X_U$.

Going the other way, if $(x, \mathcal{S}) \in \tau_X L$, then there is some set $U \in \mathcal{D}L$ such that $x = \bigvee U$ and $\mathcal{S} = X_U$. But $X_U = \bigvee\{X_a | a \in U\}$, so $(x, \mathcal{S}) = \bigvee\{(a, X_a) | a \in U\}$, and so $(x, \mathcal{S}) \in t_X L$. □

Therefore, either of these definitions suffice for the strict extension of L with respect to X, but we will use the notation $\tau_X L \to L$.

Recall that a general filter φ on a frame L is any bounded meet-semilattice homomorphism $\varphi : L \to T_\varphi$, where T_φ is the truth frame associated with φ. In [9], Hong's Construction is generalised for the case where X is a set of general filters on L. This improves some of the properties of $\tau_X L$, which we will discuss in the next section.

In this case, the first component of $\tau_X L$ is unchanged, but for the second component, it is no longer sufficient to simply take sets of filters. The second component needs to reflect the fact that each filter has its own codomain. This is done by taking the product of all the codomains of the filters in X, and then instead of taking subsets of X, we use subsets of this product. Note that since the domain of each filter is L, which is fixed, the product under consideration is not a product over a proper class.

Now in order to generalise the construction that we had for classical filters, we need a frame homomorphism $\mathcal{D}L \to \prod_{\varphi \in X} T_\varphi$. We have at hand meet-semilattice homomorphisms from L to T_φ, and these can be used to make the required frame homomorphism, by using a categorical property of the category of bounded meet-semilattices.

Definition 2.2.5. *Recall that* **SLat** *is the category of bounded meet-semilattices and bounded meet-semilattice homomorphisms. Recall the functor* $\mathcal{D} : \mathbf{SLat} \to \mathbf{Frm}$ *which was defined in Definition 2.1.10. Since* **Frm** *is a (non-full) subcategory of* **SLat***, there is an embedding functor* $E : \mathbf{Frm} \to \mathbf{SLat}$*, where* $E(L) = L$ *on objects, and* $E(h : L \to M) = h : L \to M$ *on morphisms.*

Lemma 2.2.6. *The functor* $\mathcal{D}$ *is the left adjoint of the functor* E*, with unit of adjunction* ${\downarrow}_A : A \to E\mathcal{D}A$*, and co-unit* $\bigvee_L : \mathcal{D}EL \to L$*, where* A *is a bounded meet-semilattice and* L *is a frame.*

Proof: We need to check that the two triangle identities hold. These are

$$\begin{tikzcd} \mathcal{D}A \arrow[r, "\mathcal{D}\downarrow_A"] \arrow[dr, equal] & \mathcal{D}E\mathcal{D}A \arrow[d, "\bigvee_{\mathcal{D}A}"] \\ & \mathcal{D}A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} EL \arrow[r, "\downarrow_{EL}"] \arrow[dr, equal] & E\mathcal{D}EL \arrow[d, "E\bigvee_L"] \\ & EL \end{tikzcd}$$

The functor E makes no change to the frame in question, so we can simplify these diagrams by removing the explicit reference to it:

$$\begin{tikzcd} \mathcal{D}A \arrow[r, "\mathcal{D}\downarrow_A"] \arrow[dr, equal] & \mathcal{D}\mathcal{D}A \arrow[d, "\bigvee_{\mathcal{D}A}"] \\ & \mathcal{D}A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \arrow[r, "\downarrow_{L}"] \arrow[dr, equal] & \mathcal{D}L \arrow[d, "\bigvee_L"] \\ & L \end{tikzcd}$$

Now the second diagram is trivially commutative: for $a \in L$, $\downarrow a \in \mathcal{D}L$, and $\bigvee \downarrow a = a$. To show that the first diagram also commutes, take $U \in \mathcal{D}A$. Then $\mathcal{D}\downarrow_A(U) = \bigcup\{\downarrow(\downarrow a)|a \in U\} = \{V \in \mathcal{D}A|V \subseteq\downarrow a \text{ for some } a \in U\}$. Now the join of this is just the union, and since for each $a \in U$, $\downarrow a$ is one such V, and each such V is contained in $\downarrow a$ for some $a \in U$, the join is just $\bigcup\{\downarrow a|a \in U\} = U$, as expected. □

Remark 2.2.7. This adjoint situation means that the bounded meet-semilattice homomorphism $\downarrow_A : A \to E\mathcal{D}A$ has a universal property. Explicitly, if A is a bounded meet-semilattice and L is a frame, and $\varphi : A \to EL$ is a bounded meet-semilattice homomorphism, then there exists a unique frame homomorphism $f : \mathcal{D}A \to L$ making the diagram commute:

$$\begin{tikzcd} A \arrow[r, "\downarrow_A"] \arrow[d, "\varphi"'] & E\mathcal{D}A \arrow[dl, dotted, "Ef"] \\ EL & \end{tikzcd}$$

Now we have a set of general filters, X, which are bounded meet-semilattice homomorphisms $\varphi : L \to T_\varphi$. Then this universal property gives, for each $\varphi \in X$, a unique frame homomorphism $\overline{\varphi} : \mathcal{D}L \to T_\varphi$ such that the diagram below commutes:

$$\begin{tikzcd} L \arrow[r, "\downarrow"] \arrow[d, "\varphi"'] & \mathcal{D}L \arrow[dl, dotted, "\overline{\varphi}"] \\ T_\varphi & \end{tikzcd}$$

We can now use these frame homomorphisms to make the required frame homomorphism $\mathcal{D}L \to \prod_{\varphi\in X} T_\varphi$. For $U \in \mathcal{D}L$, simply take $(\overline{\varphi}(U))_{\varphi\in X}$, which gives a frame homomorphisms because of the categorical definition of a product. This is exactly the frame homomorphism that we need to generalise Hong's Construction.

Definition 2.2.8. *Let L be a frame, and let X be a set of general filters on L. Let $g' : \mathcal{D}L \to L \times \prod_{\varphi\in X} T_\varphi$ be the frame homomorphism $g'(U) = (\bigvee U, (\overline{\varphi}(U))_{\varphi\in X})$. Let the image of g' be $\tau_X L$, and call $\tau : \tau_X L \to L$ the* **the strict extension of** L **with respect to** X*, where τ is the restriction of the first projection to $\tau_X L$. Let g be the corestriction of g' onto $\tau_X L$.*

Remark 2.2.9. For $U \in \mathcal{D}L$, $\tau g(U) = \tau(\bigvee U, (\overline{\varphi}(U))_{\varphi\in X}) = \bigvee U$, so $\tau g = \bigvee$, and g is onto, so τ is indeed a strict extension by Lemma 2.1.8. From the same lemma we have that $\tau_* = g \bigvee_* = g\downarrow$. Therefore for $a \in L$,

$$\begin{aligned}\tau_*(a) &= g(\downarrow a)\\ &= \left(\bigvee \downarrow a, \overline{\varphi}(\downarrow a)_{\varphi\in X}\right)\\ &= (a, (\varphi(a))_{\varphi\in X}),\end{aligned}$$

since $\overline{\varphi}\downarrow = \varphi$ by definition.

Recall that a classical filter can be expressed as a general filter with truth frame $\mathbb{2}$. For a classical filter F on L, the corresponding meet-semilattice homomorphism is $\varphi_F : L \to \mathbb{2}$, where $\varphi_F(a) = 1$ if and only if $a \in F$. So for a classical filter F on L, $\overline{\varphi_F}$ is a frame homomorphism $\mathcal{D}L \to \mathbb{2}$, and so if X is a set of classical filters on L and $U \in \mathcal{D}L$, then $(\overline{\varphi_F}(U))_{F\in X}$ can be considered as $\{F \in X | \overline{\varphi_F}(U) = 1\}$.

Lemma 2.2.10. *If X is a set of classical filters on a frame L, and $U \in \mathcal{D}L$, then $X_U = \{F \in X | \overline{\varphi_F}(U) = 1\}$.*

Proof: If $F \in X_U$, then $F \cap U \neq \emptyset$, and so there exists $a \in U$ such that $a \in F$. Now this implies that $\varphi_F(a) = 1$, which means that $\overline{\varphi_F}(\downarrow a) = 1$. Then since $a \in U$, $\downarrow a \subseteq U$ because U is a downset, and so $\overline{\varphi_F}(U) \geq \overline{\varphi_F}(\downarrow a) = 1$, since $\bar{\varphi}_F$ is a frame homomorphism. Therefore $\overline{\varphi_F}(U) = 1$.

On the other hand, if $F \notin X_U$, then $F \cap U = \emptyset$, and so $\varphi_F(a) = 0$ for all $a \in U$. But then $\overline{\varphi_F}(\downarrow a) = 0$ for all $a \in U$. Now $\overline{\varphi_F}$ is a frame homomorphism, so $\overline{\varphi_F}(\bigcup\{\downarrow a | a \in U\}) = 0$, which means that $\overline{\varphi_F}(U) = 0$. $\square$

In the case where X is a set of classical filters, $\tau_X L$ can either consist of pairs $(\bigvee U, X_U)$ or pairs $(\bigvee U, (\overline{\varphi_F}(U))_{F\in X})$. But we have seen that both of these are essentially the same. So the construction for the general case is really a generalisation of Hong's Construction for the classical case, which justifies the use of the same notation $\tau_X L$.

We now present one final version of Hong's Construction, found in [6]. The method used here is completely different, because instead of starting with a frame L and a set of filters X, we start with a strict extension $h : M \to L$, and form Hong's Construction $\tau_X L \to L$ for some set X. To clarify this, we first need some definitions.

Definition 2.2.11. *If L and M are frames, and $h : M \to L$ is a frame homomorphism, then h is* **relatively spatial** *if each fiber in M is spatial. This means that for each $a \in L$, if x and y are distinct elements in the fiber $h^{-1}\{a\}$, then there must be a point of M that separates them. Using completely prime filters as points, we need a completely prime filter of M that contains one point and not the other. Using primes as points, we need a prime $p \in M$ greater than or equal to one but not the other.*

In [7], relatively spatial frame homomorphisms $h : M \to L$, are called *spatial over* L, but we will not use that terminology.

Remark 2.2.12 ([7] After Definition 3)**.** If M is spatial, then $h : M \to L$ is relatively spatial. On the other hand, if $h : M \to L$ is relatively spatial and L is spatial, then M is spatial.

Proof: If M is spatial, then any two elements are separated by completely prime filters, including those in the same fiber, so $h : M \to L$ is relatively spatial. For the other direction, suppose $h : M \to L$ is relatively spatial and L is spatial, and take two points $x \neq y$ in M. If $h(x) = h(y)$, then x and y are in the same fiber, and so are separated because $h : M \to L$ is relatively spatial. If not, then $h(x) \neq h(y)$, and so $h(x)$ and $h(y)$ are separated in L, because L is spatial. So there exists a completely prime filter F on L such that, say, $h(x) \in F$ and $h(y) \notin F$. Then from Lemma 1.2.9, $h^{-1}[F]$ is a completely prime filter on M that contains x but not y. $\square$

It is possible to have $h : M \to L$ relatively spatial without M being spatial.

Example 2.2.13 ([7] after Definition 3)**.** For all frames L, $\tau_X L \to L$ is relatively spatial if X is a set of classical filters on L, but if L is not spatial, then $\tau_X L$ is not spatial.

Proof: We mentioned in Remark 2.2.2 that the first component of $\tau_X L$ is L. So $\tau_X L$ can be expressed as $L \times N$, where N is a subframe of $\mathcal{O}X$. Since $\mathcal{O}X$ is obviously spatial, N is also spatial. We will show in general that whenever N is spatial, the projection map $L \times N \to L$ is relatively spatial, but if L is not spatial, then $L \times N$ is not spatial.

For the first part, if $a \in L$, the fiber of a is $\{a\} \times N$. Now N is spatial, so if (a, x) and (a, y) are distinct elements in $\{a\} \times N$, then there exists a completely prime filter P separating x and y in N. But then (L, P) is a completely prime filter on $L \times N$ separating (a, x) and (a, y). Therefore $L \times N \to L$ is relatively spatial, and in particular, $\tau_X L \to L$ is relatively spatial for any frame L.

For the second part, we will show that if $L \times N$ is spatial, then L is spatial. So take $L \times N$ which is spatial and consider $x \neq y$ in L. Now (x, e) and (y, e) are distinct elements in $L \times N$, and so are separated by a completely prime filter P. Let $F = \{a \in L | (a, n) \in P \text{ for some } n \in N\}$, and we claim that F is a completely prime filter that separates x and y.

- If $a \in F$, and $a \leq b$, then $(a, n) \in P$ for some $n \in N$, and since $(a, n) \leq (b, n)$, $(b, n) \in P$, and so $b \in F$.
- If $a \in F$ and $b \in F$, then $(a, n) \in P$ and $(b, m) \in P$ for some n and m in N, but then $(a \wedge b, n \wedge m) \in P$, and so $a \wedge b \in F$.
- If $\bigvee S \in F$, then $(\bigvee S, n) \in P$ for some $n \in N$. So $\bigvee\{(s, n) | s \in S\} \in P$, but P is completely prime, so $(s, n) \in P$ for some $s \in S$. Therefore $s \in F$ for this $s \in S$.
- Say $(x, e) \in P$ and $(y, e) \notin P$. Then $x \in F$, and if $y \in F$, then $(y, n) \in P$ for some $n \in N$. But $(y, e) \geq (y, n)$, so $(y, e) \in P$, which is a contradiction, so in fact $y \notin F$.

So we see that if $L \times N$ is spatial, then L is spatial, and so if L is not spatial, then $L \times N$ is not spatial, and in particular, $\tau_X L$ is not spatial. $\square$

Definition 2.2.14. *For a frame L, the category* **Frm**$\downarrow L$ *has as objects all frame homomorphisms $h : M \to L$. A morphism f from object $h : M \to L$ to $k : N \to L$ is a frame homomorphism $f : M \to N$ making the diagram commute:*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "h"'] & & N \arrow[dl, "k"] \\ & L & \end{tikzcd}$$

The category **Strict** $\downarrow L$ *is the full subcategory of* **Frm** $\downarrow L$ *where objects are strict extensions of* L*, but morphisms are the same as in* **Frm** $\downarrow L$*. The category* **RelSpStrict** $\downarrow L$ *of relatively spatial strict extensions of* L *is a full subcategory of* **Strict** $\downarrow L$*.*

We will show below that the category **RelSpStrict** $\downarrow L$ is a reflective subcategory of the category **Strict** $\downarrow L$, and we will construct the reflection of a strict extension $h : M \to L$. Then later, in Proposition 2.3.8, we will show that the reflection of a strict extension $h : M \to L$ is $\tau_X L \to L$, for an appropriate set X. This means that the reflection that we will construct now is isomorphic to $\tau_X L \to L$, because reflections are unique up to isomorphisms. But before we give the construction, we need a few lemmas.

Lemma 2.2.15. *For a frame* M*, let* $r_M : M \to M$ *be the map given by* $r_M(a) = \bigwedge\{p \in M | a \leq p \text{ and } p \text{ is prime}\}$*. Then* r_M *is a nucleus on* M*.*

Proof: We need to show that r_M is a closure operator that preserves finite meets.

- For any $a \in M$, $r_M(a) \geq a$, because a is a lower bound for the set being met, and $r_M(a)$ is the greatest lower bound.
- If $a \leq b$, then any prime bigger than b is also bigger than a. Then $r_M(b) \geq r_M(a)$, because the meet of a smaller set is bigger.
- Since $a \leq r_M(a)$, we have that $r_M(a) \leq r_M(r_M(a))$. Then in order to show that $r_M(a) = r_M(r_M(a))$, we need to show that $r_M(r_M(a)) \leq r_M(a)$. Now $r_M(a) = \bigwedge\{p \in M | a \leq p \text{ and } p \text{ is prime}\}$, and for such p, $r_M(a) \leq r_M(p)$ because $a \leq p$. But $r_M(q) = q$ for any prime q, so we have that $r_M(a) \leq p$. So $r_M(a) \leq p$ and p is prime, so p is part of the set being met to form $r_M(r_M(a))$. Therefore $r_M(r_M(a)) \leq p$, for all such p, and so $r_M(r_M(a)) \leq r_M(a)$, as required.
- For a and b in M, if p is a prime that is bigger than $a \wedge b$, then by definition, $p \geq a$ or $p \geq b$. So, assuming that p is always prime in what follows,

$$\begin{aligned}
&\{p \in M | a \wedge b \leq p\} = \{p \in M | a \leq p\} \cup \{p \in M | b \leq p\} \\
\Rightarrow\ &\bigwedge\{p \in M | a \wedge b \leq p\} = \bigwedge\{p \in M | a \leq p\} \wedge \bigwedge\{p \in M | b \leq p\} \\
\Rightarrow\ &r_M(a \wedge b) = r_M(a) \wedge r_M(b).
\end{aligned}$$

□

Lemma 2.2.16 ([6] Lemma 2)**.** *If $h : M \to N$ is a frame homomorphism, then $r_M \le h_* r_N h$.*

Proof: Take $a \in M$. Then

$$\begin{aligned} h_* r_N h(a) &= h_* \left(\bigwedge \{ p \in N | h(a) \le p,\, p \text{ prime} \} \right) \\ &= \bigwedge \{ h_*(p) | h(a) \le p,\, p \text{ prime} \} \text{ since right adjoints preserve meets} \\ &= \bigwedge \{ h_*(p) | a \le h_*(p),\, p \text{ prime} \} \text{ by the definition of right adjoints.} \end{aligned}$$

Now if p is prime in N, then $h_*(p)$ is prime in M, so $r_M(a) = \bigwedge \{ q \in M | a \le q,\, q \text{ prime} \}$ $\le \bigwedge \{ h_*(p) | a \le h_*(p),\, p \text{ prime} \}$, because the first meets all primes in M greater than a, and the second only those of the form $h_*(p)$. Therefore $r_M(a) \le h_* r_N h(a)$. □

Recall that for any frame homomorphism $h : M \to L$, $h_* h$ is a nucleus on M.

Lemma 2.2.17 ([6] in the proof of Lemma 3)**.** *If $l : K \to L$ is a relatively spatial strict extension, then $(l_* l) \wedge r_K = \mathrm{id}_K$.*

Proof: First note that since nuclei are closure operators, $\mathrm{id}_K \le (l_* l) \wedge r_K$. Now assume for contradiction that $\mathrm{id}_K \ne (l_* l) \wedge r_K$. Then there is an $a \in K$ such that $a < l_* l(a) \wedge r_K(a)$. Now

$$\begin{aligned} l((l_* l) \wedge r_K)(a) &= l l_* l(a) \wedge l(r_K(a)) \\ &= l(a) \wedge l(r_K(a)) \text{ from Lemma 1.2.29} \\ &= l(a) \text{ because } a \le r_K(a). \end{aligned}$$

Now l is relatively spatial, and we have assumed that $a < l_* l(a) \wedge r_K(a)$, so we can find a prime $p \in K$ such that $a \le p$, but $l_* l(a) \wedge r_K(a) \not\le p$. But this is impossible, because if $a \le p$, then p is one of the primes to be met for $r_K(a)$, so $r_K(a) \le p$, and then $l_* l(a) \wedge r_K(a) \le p$ also. So in fact, no such a is possible, and $(l_* l) \wedge r_K = id_K$. □

Now we are ready to present our final version of Hong's Construction.

Proposition 2.2.18 ([6] Lemma 3)**.** *The category* **RelSpStrict** $\downarrow L$ *is a reflective subcategory of the category* **Strict** $\downarrow L$*, and if $h : M \to L$ is a strict extension, the relatively spatial reflection of h is given by $\tilde{h} : Fix(n_M) \to L$, where $n_M = (h_* h) \wedge r_M$.*

Proof: Let $\tilde{M} = \mathrm{Fix}(n_M)$. Then we have the following situation:

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "n_M"'] & & L \\ & \tilde{M} \arrow[ur, dotted, "\tilde{h}"'] & \end{tikzcd}$$

First we must show that there exists a frame homomorphism $\tilde{h} : \tilde{M} \to L$. We can use Lemma 1.2.34, because $n_M : M \to \tilde{M}$ is onto, so we need to show that $\ker n_M \subseteq \ker h$. Take $(x, y) \in M \times M$ such that $n_M(x) = n_M(y)$. Then

$$\begin{aligned} & h_*h(x) \wedge r_M(x) = h_*h(y) \wedge r_M(y) \\ \Rightarrow\ & hh_*h(x) \wedge hr_M(x) = hh_*h(y) \wedge hr_M(y) \\ \Rightarrow\ & h(x) \wedge hr_M(x) = h(y) \wedge hr_M(y) \text{ from Lemma 1.2.29} \\ \Rightarrow\ & h(x \wedge r_M(x)) = h(y \wedge r_M(y)) \\ \Rightarrow\ & h(x) = h(y) \text{ because } x \leq r_M(x) \text{ and } y \leq r_M(y). \end{aligned}$$

So $\ker n_M \subseteq \ker h$, and then Lemma 1.2.34 gives a unique frame homomorphism $\tilde{M} \to L$, which we call $\tilde{h}$. In particular, $\tilde{h} = hn_{M*} = h|_{\tilde{M}}$ because n_M is a nucleus. (See Remark 2.4.6 for more details.) Now since h is strict and n_M is onto, $\tilde{h}$ is also strict, from Lemma 2.1.8.

Next we must show that $\tilde{h}$ is relatively spatial. Take $x \neq y$ in $\tilde{M}$ such that $\tilde{h}(x) = \tilde{h}(y)$, then we must find a point of $\tilde{M}$ that separates them. Since x and y are in $\tilde{M} = \mathrm{Fix}(n_M)$, $x = n_M(x) = h_*h(x) \wedge r_M(x)$, and $y = n_M(y) = h_*h(y) \wedge r_M(y)$. Now

$$h(x) = \tilde{h}(n_M(x)) = \tilde{h}(x) = \tilde{h}(y) = \tilde{h}(n_M(y)) = h(y)$$

and therefore $h_*h(x) = h_*h(y)$. But we have that $x \neq y$, so either $x \not\leq y$ or $y \not\leq x$. Suppose that $y \not\leq x$. Then $h_*h(y) \wedge r_M(y) \not\leq h_*h(x) \wedge r_M(x)$, and so $r_M(y) \not\leq r_M(x)$. Now if $\{p \in M | p \geq x, p \text{ prime}\} \subseteq \{p \in M | p \geq y, p \text{ prime}\}$, then we would have $r_M(y) \leq r_M(x)$, which is a contradiction. Therefore there must be a prime $p \in M$ such that $x \leq p$ and $y \not\leq p$. Now p separates x and y in M, but we need a point of $\tilde{M}$ that separates x and y, so let $F = \{a \in \tilde{M} | a \not\leq p\}$. Then $y \in F$ and $x \notin F$, and we will show that F is a completely prime filter on $\tilde{M}$:

- If $b \geq a$ and $b \leq p$, then $a \leq p$, so if $a \not\leq p$, then $b \not\leq p$.
- If $a \wedge b \leq p$, then because p is prime, either $a \leq p$ or $b \leq p$. So if $a \not\leq p$ and $b \not\leq p$, then $a \wedge b \not\leq p$.

- If $S \subseteq \tilde{M}$ such that $s \leq p$ for each $s \in S$, then $\bigvee S \leq p$. So if $\bigvee S \not\leq p$, then $s \not\leq p$ for some $s \in S$.

So now we have a relatively spatial strict extension of L, but we must still show that it is the reflection of $h : M \to L$. We need to show that given any other relatively spatial strict extension $l : K \to L$ and a frame homomorphism $f : M \to K$ such that $lf = h$, then there exists a frame homomorphism $\tilde{f} : \tilde{M} \to K$ satisfying $l\tilde{f} = \tilde{h}$, and it is unique in making the big triangle below commute.

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "h"] \arrow[ddr, "n_M"'] & & K \arrow[dl, "l"'] \\
& L & \\
& \tilde{M} \arrow[u, "\tilde{h}"] \arrow[uur, dotted, "!\tilde{f}"'] &
\end{tikzcd}$$

Now

$$\begin{aligned}
f_*f &= f_*((l_*l) \wedge r_K)f \text{ from Lemma 2.2.17} \\
&= (f_*l_*lf) \wedge (f_*r_Kf) \\
&= (h_*h) \wedge (f_*r_Kf) \\
&\geq (h_*h) \wedge r_M \text{ from Lemma 2.2.16} \\
&= n_M
\end{aligned}$$

So if $m \in M$, $f(n_M(m)) \leq f(f_*f(m)) = f(m)$ from Lemma 1.2.29. But also, $m \leq n_M(m)$, so $f(m) \leq f(n_M(m))$, and so $f(n_M(m)) = f(m)$ for all $m \in M$. Now if $(x, y) \in M \times M$ such that $n_M(x) = n_M(y)$, then $f(n_M(x)) = f(n_M(y))$, so $f(x) = f(y)$, and so $\ker n_M \subseteq \ker f$. Then Lemma 1.2.34 gives a unique frame homomorphism $\tilde{f} : \tilde{M} \to K$ making the big triangle diagram commute.

Now $\tilde{f}n_M = f$, so $l\tilde{f}n_M = lf = h$, and $h = \tilde{h}n_M$. So $l\tilde{f}n_M = \tilde{h}n_M$. But $\tilde{M}$ is Fix n_M, so every $a \in \tilde{M}$ is in fact $n_M(a)$, and so $l\tilde{f}(a) = l\tilde{f}n_M(a) = \tilde{h}n_M(a) = \tilde{h}(a)$. So $\tilde{f}$ satisfies all the required properties. □

Remark 2.2.19. In the last paragraph above, we have shown in general that onto maps are right cancellable, because if f is onto and $hf = gf$ for some maps h and g, then any element x in the domain of h can be written as $f(a)$ for some element a in the domain of f, and then $h(x) = hf(a) = gf(a) = g(x)$, so $h = x$.

We have constructed a relatively spatial strict extension $\tilde{h} : \tilde{M} \to L$ that is the reflection of a strict extension $h : M \to L$. In Proposition 2.3.8, we will show that this reflection can also be expressed as $\tau_X L \to L$ for an appropriate set of filters X, which will be called the filter trace of h. Therefore, even though this construction looks very different from $\tau_X L \to L$, it is in fact a form of Hong's Construction. In the next section, we will explore these filter traces in detail.

2.3 Filter traces

In the last section, we were able to construct strict extensions using any set X of filters on the frame in question. However, certain filters, called trace filters, provide some insight into strict extensions. The definition of a trace filter is dependent on the definition of a filter, so we encounter three different definitions of trace filters, corresponding to the three different types of filters that we have. We have already seen the first one: if $f : X \to Y$ is a strict extension of the space X, then the trace filters of f are the preimages of the neighbourhood filters of $\mathcal{O}Y$. This concept was originally extended to frames in [7].

Definition 2.3.1. *If $h : M \to L$ is a strict extension, then a filter F on L is a* **trace filter** *of h if $F = h[P]$ for some completely prime filter P on M. The set of all trace filters of h is called the* **filter trace** *of h.*

Remark 2.3.2. To reconcile this definition with the one for spaces, suppose that $X \subseteq Y$ for sober spaces X and Y, and $h : \mathcal{O}Y \to \mathcal{O}X$ given by $h(U) = U \cap X$ is a strict extension. The completely prime filters of $\mathcal{O}Y$ are the points in its spectrum, and since Y is sober, there is an exact correspondence between the points y in Y and the completely prime filters, which is that the completely prime filters on $\mathcal{O}Y$ are exactly the open neighbourhood filters of points of Y. So, if a filter $\mathcal{F} = h[\mathcal{P}]$ for some neighbourhood filter $\mathcal{P}$, then $\mathcal{F} = \{U \cap X | y \in U \in \mathcal{O}Y\}$ for some $y \in Y$. That is, $\mathcal{F} = \mathcal{T}(y)$ for some $y \in Y$, where $\mathcal{T}(y)$ is the trace filter corresponding to the point y.

In [7], the definition of a trace filter was restricted to exclude any filter that was itself completely prime. This is equivalent to excluding all $\mathcal{T}(y)$ for $y \in X$. In fact this is an unnecessary complication, because adding completely prime filters to X does not change $\tau_X L \to L$.

Lemma 2.3.3 ([7])**.** *If X and Y are sets of filters on L such that $X \subseteq Y$ and $Y \setminus X$ consists only of completely prime filters, then $\tau_X L \cong \tau_Y L$.*

Proof: Since $\mathcal{D}L \to \tau_X L$ is an onto frame homomorphism, it corresponds to a nucleus n_X on $\mathcal{D}L$, where for $W \in \mathcal{D}L$, $n_X(W) = \bigvee\{U \in \mathcal{D}L | (\bigvee U, X_U) = (\bigvee W, X_W)\}$. Let $\tilde{W} = \{s \in L | s \leq \bigvee W, X_s \subseteq X_W\}$. We claim that $n_X(W) = \tilde{W}$.

- To show that $\tilde{W}$ is a downset, take $s \in \tilde{W}$ and $t \leq s$. Firstly, $s \leq \bigvee W$, so $t \leq \bigvee W$. Secondly, if $F \in X_t$, then $t \in F$, so $s \in F$ because filters are up-closed, and so $F \in X_s \subseteq X_W$, so $X_t \subseteq X_W$. Therefore $t \in \tilde{W}$.

- We have that $(\bigvee \tilde{W}, X_{\tilde{W}}) = (\bigvee W, X_W)$: Firstly, $W \subseteq \tilde{W}$ because if $s \in W$, then $s \leq \bigvee W$ and if $F \in X_s$, then $s \in F$, so $F \cap W \neq \emptyset$ and so $F \in X_W$. This means that $\bigvee W \leq \bigvee \tilde{W}$. On the other hand, for each $s \in \tilde{W}$, $s \leq \bigvee W$, so $\bigvee \tilde{W} \leq \bigvee W$. Therefore $\bigvee \tilde{W} = \bigvee W$.

  Secondly, if $F \in X_W$, then $F \cap W \neq \emptyset$, and so $F \cap \tilde{W} \neq \emptyset$, because $W \subseteq \tilde{W}$, so $F \in X_{\tilde{W}}$. On the other hand, if $F \in X_{\tilde{W}}$, then $F \cap \tilde{W} \neq \emptyset$, and so there is an $s \in F$ such that $s \leq \bigvee W$ and $X_s \subseteq X_W$, but since $s \in F$, this means that $F \in X_s \subseteq X_W$, so $F \in X_W$. Therefore $X_W = X_{\tilde{W}}$.

- If U is another set in $\mathcal{D}L$ such that $(\bigvee U, X_U) = (\bigvee W, X_W)$, then for each $s \in U$, firstly, $s \leq \bigvee U = \bigvee W$, and secondly, if $F \in X_s$, then $s \in F$, so $F \cap U \neq \emptyset$, so $F \in X_U = X_W$, and so $X_s \subseteq X_W$. Therefore $U \subseteq \tilde{W}$.

So we have that $n_X(W) = \tilde{W}$. Now from Lemma 1.2.32 we know that $\tau_X L$ is isomorphic to $\operatorname{Fix} n_X$. Specifically, $\operatorname{Fix} n_X = \{W \in \mathcal{D}L | n_X(W) = W\} = \{n_X(W) | W \in \mathcal{D}L\}$ because if W is such that $n_X(W) = W$, then $n_X(W)$ is in the last set, and if we have $n_X(W)$ for some $W \in \mathcal{D}L$, then $n_X(n_X(W)) = n_X(W)$, so $n_X(W)$ is in the second set. Therefore $\tau_X L$ is isomorphic to $\{n_X(W) | W \in \mathcal{D}L\}$. Now to show that $\tau_X L$ is isomorphic to $\tau_Y L$, we will show that for each $W \in \mathcal{D}L$, $n_X(W) = n_Y(W)$.

If $s \in n_Y(W)$, then $s \leq \bigvee W$ and $Y_s \subseteq Y_W$. So if $F \in X_s$, then $s \in F \in X$, and so $s \in F \in Y$ because $X \subseteq Y$. Then $F \in Y_s \subseteq Y_W$, so $F \cap W \neq \emptyset$, and so $F \in X_W$. Therefore $s \leq \bigvee W$ and $X_s \subseteq X_W$, so $s \in n_X(W)$.

On the other hand, if $s \in n_X(W)$, then $s \leq \bigvee W$ and $X_s \subseteq X_W$. Then if $F \in Y_s$, $s \in F$, so either $F \in X$, in which case $F \in X_s \subseteq X_W \subseteq Y_W$, or $F \in Y \setminus X$, in which case F is completely prime. But then since $s \in F$ and $s \leq \bigvee W$, we have $\bigvee W \in F$, and so $F \cap W \neq \emptyset$. Therefore $F \in Y_W$, and so in either case, $Y_s \subseteq Y_W$. Since we also have that $s \leq \bigvee W$, $s \in n_Y(W)$. $\square$

We already saw in the context of spaces that if you construct a strict extension from a set of filters, the filter trace of that extension corresponds exactly to that set of filters that you use. This lemma shows that that cannot be true in the pointfree context, because in the case above, $\tau_X L \to L$ and $\tau_Y L \to L$ have the same filter trace, since they are isomorphic, but they are constructed from different sets X and Y. However, one inclusion still holds:

Lemma 2.3.4. *For a frame L and a set of filters X on L, each $F \in X$ is a trace filter of $\tau_X L \to L$.*

Proof: Take $F \in X$. We must show that $F = \tau[P]$ for some completely prime filter P on $\tau_X L$. Let $P = \{(\bigvee W, X_W) \in \tau_X L | F \in X_W\}$.

Firstly, if $(\bigvee W, X_W) \in P$ and $(\bigvee V, X_V) \geq (\bigvee W, X_W)$, then $F \in X_W \subseteq X_V$, so $(\bigvee V, X_V) \in P$. Secondly, if $(\bigvee W, X_W)$ and $(\bigvee U, X_U)$ are in P, then $F \in X_W$ and $F \in X_U$, so $F \in X_W \cap X_U = X_{W \cap U}$, as shown before Lemma 2.2.4, and so $(\bigvee W, X_W) \wedge (\bigvee U, X_U) = (\bigvee (W \cap U), X_{W \cap U}) \in P$. Therefore P is a filter.

Next we show that P is completely prime. If $\bigvee\{(\bigvee W_s, X_{W_a})|s \in S\} \in P$, then $F \in \bigcup\{X_{W_s}|s \in S\}$. But then $F \in X_{W_s}$ for some $s \in S$, and so $(\bigvee W_s, X_{W_S}) \in P$.

Finally, we show that $\tau[P] = F$. If $a \in F$, then $F \in X_a \subseteq X_{\downarrow a}$, so $(a, X_{\downarrow a}) \in P$. Then $\tau(a, X_{\downarrow a}) = a$, so $a \in \tau[P]$. On the other hand, if $a \in \tau[P]$, then $a = \bigvee W$ for some $W \in \mathcal{D}L$ such that $F \in X_W$. This means that $F \cap W \neq \emptyset$, so let $b \in F \cap W$. Now $a = \bigvee W \geq b$, and $b \in F$, so $a \in F$ also. $\square$

So if Y is the filter trace of $\tau_X L \to L$, then $X \subseteq Y$, but we need not have $X = Y$, due to Lemma 2.3.3.

Recall Definition 2.2.14, that the category **Strict** $\downarrow L$ is a full subcategory of the category **Frm** $\downarrow L$. Therefore two strict extensions are equivalent if there is an isomorphism between the extension frames that makes the resulting triangle commute.

Lemma 2.3.5 ([7] Lemma 3)**.** *Let $h : M \to L$ be a relatively spatial strict extension with filter trace X. Then h is equivalent to $\tau_X L \to L$.*

Proof: For $a \in M$, let $\widehat{h} : M \to L \times \mathcal{P}X$ be the map

$$\widehat{h}(a) = (h(a), \{h[P]|a \in P, \text{ a completely prime filter on } M\}).$$

Then we must show that $\widehat{h}$ is a frame isomorphism from M onto $\tau_X L$ such that $\tau\widehat{h} = h$. In what follows, we will only use P to refer to a completely prime filter on M.

- $\tau\widehat{h} = h$: For $a \in M$, $\tau\widehat{h}(a) = \tau((h(a), \{h[P] | a \in P\})) = h(a)$, since τ is a first projection map.

- $\widehat{h}$ preserves the top: $\widehat{h}(e) = (h(e), \{h[P] | P \text{ completely prime}\})$, because every filter contains e. Therefore $\widehat{h}(e) = (e, X)$, because by definition, every element of X is the image of a completely prime filter on M.

- $\widehat{h}$ preserves the bottom: $\widehat{h}(0) = (h(0), \{h[P] | 0 \in P\}) = (0, \emptyset)$ because all filters are proper.

- $\widehat{h}$ preserves binary meets:

$$\begin{aligned}
\widehat{h}(a) \wedge \widehat{h}(b) &= (h(a) \wedge h(b), \{h[P] | a \in P\} \cap \{h[P] | b \in P\}) \\
&= (h(a) \wedge h(b), \{h[P] | a \in P \text{ and } b \in P\}) \\
&= (h(a \wedge b), \{h[P] | a \wedge b \in P\}) \\
&= \widehat{h}(a \wedge b)
\end{aligned}$$

  because if $a \in P$ and $b \in P$, then $a \wedge b \in P$, and if $a \wedge b \in P$ then both a and b are in P.

- $\widehat{h}$ preserves arbitrary joins:

$$\begin{aligned}
\widehat{h}\left(\bigvee S\right) &= \left(h\left(\bigvee S\right), \left\{h[P] \Big| \bigvee S \in P\right\}\right) \\
&= \left(\bigvee_{s \in S} h(s), \{h[P] | s \in P \text{ for some } s \in S\}\right) \\
&= \bigvee_{s \in S} \widehat{h}(s)
\end{aligned}$$

  because each P is completely prime, so $\bigvee S \in P$ implies that $s \in P$ for some $s \in S$, and if an $s \in P$, then $\bigvee S \in P$ because P is up-closed.

- $\widehat{h}(a) \in \tau_X L$ for each $a \in M$: For $a \in M$, $a = \bigvee\{h_*(s) | h_*(s) \leq a\}$, because h is strict. So we just need to show that for each $s \in L$, $\widehat{h}(h_*(s)) \in \tau_X L$, and the rest follows because $\widehat{h}$ is a frame homomorphism and $\tau_X L$ is a frame. Now

$$\begin{aligned}
\widehat{h}(h_*(s)) &= (hh_*(s), \{h[P] | h_*(s) \in P\}) \\
&= (s, \{h[P] | s \in h[P]\}) \\
&= (s, X_s) \in \tau_X L
\end{aligned}$$

because $h_*(s) \in P$ if and only if $s \in h[P]$: Firstly, $hh_*(s) = s$ because h is onto, so $h_*(s) \in P$ implies that $s \in h[P]$. On the other hand, if $s \in h[P]$, then $s = h(x)$ for some $x \in P$, but then $h(x) \leq s$ which implies that $x \leq h_*(s)$, so $h_*(s) \in P$ also.

- $\widehat{h}$ maps M onto $\tau_X L$: For any $a \in L$, $\tau_*(a) = (a, X_a) = \widehat{h}(h_*(a))$, and so $\tau_*[L] \subseteq \text{Im } \widehat{h}$. But $\tau_*[L]$ generates $\tau_X L$, so all of $\tau_X L$ is in the image of $\widehat{h}$.

- $\widehat{h}$ is one-one: If $\widehat{h}(a) = \widehat{h}(b)$, then $(h(a), \{h[P] | a \in P\}) = (h(b), \{h[P] | b \in P\})$, so $h(a) = h(b)$ and if $a \in P$, then there exists a completely prime filter P' on M such that $b \in P'$ and $h[P] = h[P']$.

  Now suppose that $a \neq b$. Then since $h(a) = h(b)$ and h is relatively spatial, we can find a completely prime filter P containing one and not the other. So suppose that $a \in P$ and $b \notin P$. Now $b = \bigvee\{h_*(s) | h_*(s) \leq b\}$ because h is strict, and $b \in P'$ for some completely prime filter P' such that $h[P] = h[P']$, so $h_*(s) \in P'$ for some s such that $h_*(s) \leq b$. But then $hh_*(s) \in h[P'] = h[P]$, so $s \in h[P]$, and we showed above that this implies that $h_*(s) \in P$. Therefore, since $h_*(s) \leq b$, $b \in P$ also, and this contradicts our choice of P. Therefore $a = b$.

  Note that this is the only part of the proof that requires h to be relatively spatial.

$\square$

Remark 2.3.6. For a set X of filters on L, we can consider $[X]$ to be all those sets of filters which produce the same strict extension as X. This is an equivalence relation on the sets of filters on L. Then for a given X, if Y is the filter trace of $\tau_X L \to L$, then from Lemma 2.3.5 above, $\tau_X L = \tau_Y L$, and so $Y \in [X]$. From Lemma 2.3.4, Y is a maximal element in $[X]$, and so $Y = \bigcup[X]$. Also, since the filter trace of $\tau_X L \to L$ is Y, Y is the filter trace of all the strict extensions formed from members of $[X]$, including $\tau_Y L \to L$. Therefore, for any set X of filters on L, the filter trace of $\tau_X L \to L$ is $Y = \bigcup[X]$. We saw in Lemma 2.3.3 that members of the class $[X]$ can differ by completely prime filters, but it is not clear whether there can be non-completely prime filters that are not common to all members of $[X]$. This question is elaborated on in Chapter 4.

Proposition 2.3.7. *The strict extensions that can be constructed using Hong's Construction with classical filters are exactly the relatively spatial strict extensions.*

Proof: We showed in Example 2.2.13 that Hong's Construction always produces a relatively spatial strict extension. On the other hand, we just saw in Lemma

2.3.5 that every relatively spatial strict extension can be expressed as Hong's Construction for an appropriate set X. $\square$

In spaces, we saw that any strict extension can be exactly determined from its filter trace. We see now from the proposition above that that can only be done for relatively spatial strict extensions in frames. However, this is an important subcategory of the category of strict extensions, because we saw before that the relatively spatial strict extensions are reflective in the category of strict extensions. We will show now that in fact $\tau_X L \to L$ is the relatively spatial reflection of the strict extension $h : M \to L$, when X is the filter trace of h.

Proposition 2.3.8 ([7] Proposition 3). *If $h : M \to L$ is a strict extension with filter trace X, then $\tau_X L \to L$ is its reflection in the category* **Strict** $\downarrow L$ *to the category* **RelSpStrict** $\downarrow L$*, with reflection map $\widehat{h} : M \to \tau_X L$, defined in Proposition 2.3.5.*

Proof: We need to show that given a relatively spatial strict extension $l : N \to L$, with a frame map $f : M \to N$ such that $lf = h$, there exists a frame homomorphism $\tilde{f} : \tau_X L \to N$ such that $\tau = lk$, which is unique in making the big triangle below commute.

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "h"] \arrow[ddr, "\widehat{h}"'] & & N \arrow[dl, "l"'] \\
& L & \\
& \tau_X L \arrow[u, "\tau"] \arrow[uur, dotted, "!k"'] &
\end{tikzcd}$$

We only need to show that this $\tilde{f}$ exists for the case where f is onto, because if f is not onto, then it can be factorised using the image factorisation describe in Lemma 1.2.35.

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "h"'] & & N \arrow[dl, "l"] \\
& L &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
M \arrow[r, "f'"] \arrow[dr, "h"'] & \mathrm{Im}\ f \arrow[r, "i"] \arrow[d, "l'"] & N \arrow[dl, "l"] \\
& L &
\end{tikzcd}$$

We get that $f = if'$, where f' is onto and i is one-one. Let $li = l'$, then $l'f' = h$, and so by Lemma 2.1.8, l' is a strict extension because h is strict. We will show that

l' is also relatively spatial, so that l' can take the place of l in the original diagram, when f' takes the place of f. Then if we can show that $\widehat{h}$ factors through f', we can use the same factorisation composed with i to get a factorisation through f.

Now to show that l' is relatively spatial, take $a \in L$ and $x \neq y \in l'^{-1}(a)$. Then $l'(x) = l'(y) = a$, so $li(x) = li(y) = a$, and so $i(x)$ and $i(y)$ are in $l^{-1}(a)$. Now i is one-one, so $x \neq y$ means that $i(x) \neq i(y)$, and so $i(x)$ and $i(y)$ are separated by a completely prime filter P on N, because l is relatively spatial. But then $i^{-1}[P]$ is a completely prime filter on $\mathrm{Im}\, f$ that separates x and y.

Note that if we find a frame homomorphism k such that $k\widehat{h} = f$, then that will be unique because $\widehat{h}$ is onto, which makes it right cancellable by Remark 2.2.19. So now we must just show that there is such a factorisation when f is onto.

Let Y be the filter trace of l, and we have that X is the filter trace of h. For $F \in Y$, $F = l[S]$ for some completely prime filter S on N. But then $P = f^{-1}[S]$ is a completely prime filter on M, by Lemma 1.2.9, and $f[P] = ff^{-1}[S] = S$ because f is onto. So $F = l[S] = lf[P] = h[P]$, and so $F \in X$, so we have that $Y \subseteq X$. Therefore, we have a frame homomorphism $\mathcal{O}X \to \mathcal{O}Y$ which sends $U \in \mathcal{O}X$ to $U \cap Y$. In particular, for each $a \in L$, X_a gets mapped to Y_a. Using this map, we get a frame homomorphism $L \times \mathcal{O}X \to L \times \mathcal{O}Y$ sending $(a, X_a) = \tau_*^X(a)$ to $(a, Y_a) = \tau_*^Y(a)$. Then since τ^X and τ^Y are strict, these are generating elements of $\tau_X L$ and $\tau_Y L$ respectively, and so this frame homomorphism generates a frame homomorphism $\bar{f} : \tau_X L \to \tau_Y L$.

Now since $l : N \to L$ is relatively spatial, we have an isomorphism $\widehat{l} : N \to \tau_Y L$. Let $k = \widehat{l}^{-1}\bar{f}$. To show that k is the required factorisation, we need to show that $k\widehat{h} = f$ and $lk = \tau$. For the first equation, we just need to show that $k\widehat{h}(h_*(s)) = f(h_*(s))$ for each $s \in L$, and the other points in M follow because h is strict.

$$\begin{aligned}
k\widehat{h}(h_*(s)) &= k(s, X_s) \text{ because } \widehat{h}(h_*(s)) = (s, X_s), \text{ from Lemma 2.3.5} \\
&= \widehat{l}^{-1}(\bar{f}(s, X_s)) \\
&= \widehat{l}^{-1}(s, Y_s) \\
&= l_*(s) \text{ because } \widehat{l}(l_*(s)) = (s, Y_s) \text{ and } \widehat{l} \text{ is an isomorphism} \\
&= f(h_*(s)) \text{ by Lemma 2.1.8, because } h = lf, \text{ and } f \text{ is onto.}
\end{aligned}$$

So $k\widehat{h} = f$, as required.

For the second equation, since $k\widehat{h} = f$, we have $lk\widehat{h} = lf = h = \tau\widehat{h}$. Now $\widehat{h}$ is onto, and so it is right cancellable by Remark 2.2.19, which gives that $lk = \tau$. Therefore k satisfies all of the required properties. □

The situation that we have now is that any relatively spatial strict extension can be constructed using Hong's Construction. This means that for relatively spatial strict extensions, the extension can be determined precisely using only information internal to the original frame. We saw that in spaces, all strict extensions have this property, and in fact that is also true in the pointfree context, but for strict extensions that are not relatively spatial, we need to use general filters.

Definition 2.3.9. *If $h : M \to L$ is a strict extension and n is a nucleus on M, then nh_* is a* **(general) trace filter** *of h, and the set of all these is the* **(general) filter trace**. *When necessary, we will refer to the filter trace for classical filters as the* classical filter trace.

Remark 2.3.10. For a strict extension $h : M \to L$, any nucleus n on M determines an onto frame homomorphism $n : M \to \mathrm{Fix}\, n$. Now h_* preserves finite meets, including the top, and $h_*(0) = 0$ because h is dense, so $h_* : M \to L$ is a bounded meet-semilattice homomorphism. Therefore nh_* is a bounded meet-semilattice homomorphism $L \to \mathrm{Fix}\, n$, that is, nh_* is a general filter on L.

As a particular case, if n is the identity nucleus on M, then $\mathrm{Fix}\, n = M$, and so $h_* : L \to M$ is a trace filter of h for any strict extension $h : M \to L$.

For classical filters, a trace filter was a filter that was the image of a completely prime filter, so in order to compare our new definition with that one, we need to understand what is meant by the image of a filter when the filter is considered as a meet-semilattice homomorphism to $\mathbb{2}$.

Lemma 2.3.11 ([8], Lemma 3, part 1)**.** *If $h : L \to M$ is an onto frame homomorphism and $\varphi_F : L \to \mathbb{2}$ is the frame homomorphism corresponding to a filter F on L, then*

$$\varphi_{h[F]} = \varphi_F h_*.$$

Proof: Let $a \in L$. Then

$$\varphi_{h[F]}(a) = 1 \Leftrightarrow a \in h[F] \Leftrightarrow h_*(a) \in F \Leftrightarrow \varphi_F(h_*(a)) = 1$$

where the middle implication was shown in the proof of Lemma 2.3.5. □

Corollary 2.3.12. *If F is a classical trace filter of a strict extension $h : M \to L$, then φ_F is a general trace filter of h. Conversely, if $\varphi : L \to \mathbb{2}$ is a general trace filter of h, then $\varphi = \varphi_F$ for some classical trace filter F of h.*

Proof: If F is a classical trace filter of a strict extension $h : M \to L$, then $F = h[P]$ for some completely prime filter P. Now for completely prime P, $\varphi_P : M \to \mathbb{2}$ is an onto frame homomorphism, because completely prime filters correspond to frame homomorphisms to $\mathbb{2}$, as mentioned in Lemma 1.2.14. Now $\varphi_P : M \to \mathbb{2}$ corresponds to a nucleus n_P on M, such that Fix $n_P \cong \mathbb{2}$. Then from the above lemma, $\varphi_F = \varphi_P h_*$, where φ_P corresponds to the nucleus n_P, and so φ_F is a general trace filter of h.

On the other hand, if $\varphi : L \to \mathbb{2}$ is a general trace filter of h, then $\varphi = nh_*$ for some nucleus n on M. But the codomain of φ is $\mathbb{2}$, so the onto frame homomorphism corresponding to the nucleus n must also have codomain $\mathbb{2}$. Now a frame homomorphism from M to $\mathbb{2}$ corresponds to a completely prime filter P on M, so we can write $n = \varphi_P$. Also, $\varphi = \varphi_F$, where F is the filter corresponding to φ, so we have $\varphi_F = \varphi_P h_*$. But from the previous lemma we have $\varphi_{h[P]} = \varphi_P h_*$, so in fact $\varphi_F = \varphi_{h[P]}$, which implies that $F = h[P]$, and so F is a trace filter of h. $\square$

Recall that for a general filter φ, T_φ is the codomain of φ. We will now see how a set X of general filters on L compares to the general filter trace of $\tau_X L \to L$.

Lemma 2.3.13 ([9] Proposition 1)**.** *For any $\varphi \in X$, there exists a unique frame homomorphism $\tilde{\varphi} : \tau_X L \to T_\varphi$ such that $\tilde{\varphi}\tau_* = \varphi$.*

Proof: Let $j : \tau_X L \to L \times \prod_{\varphi\in X} T_\varphi$ be the embedding map, and let $\mathrm{pr}_\varphi : L \times \prod_{\varphi\in X} T_\varphi \to T_\varphi$ be the φth projection map, and let $\tilde{\varphi} = \mathrm{pr}_\varphi j$. Then $\tilde{\varphi}$ is a frame homomorphism, and

$$\begin{aligned}
\tilde{\varphi}\tau_*(a) &= \tilde{\varphi}(a, (\varphi(a))_{\varphi\in X}) \text{ from Remark 2.2.9} \\
&= \mathrm{pr}_\varphi j(a, (\varphi(a))_{\varphi\in X}) \\
&= \varphi(a).
\end{aligned}$$

Now if f were another such homomorphism, then we would have $f\tau_* = \tilde{\varphi}\tau_*$. But τ is strict, so f and $\tilde{\varphi}$ agree on a generating set, and so they must agree everywhere. Therefore $\tilde{\varphi}$ is unique.

Note that if φ generates the frame T_φ, then $\tilde{\varphi}$ is onto, because if $y \in T_\varphi$, then $y = \bigvee\{\varphi(x)|\varphi(x) \leq y\} = \bigvee\{\tilde{\varphi}\tau_*(x)|\varphi(x) \leq y\} = \tilde{\varphi}(\bigvee\{\tau_*(x)|\varphi(x) \leq y\})$ because $\tilde{\varphi}$ is a frame homomorphism. $\square$

Corollary 2.3.14. *If Y is the general filter trace of $\tau_X L \to L$, then $X \subseteq Y$.*

Proof: For $\varphi \in X$, we must show that $\varphi = n\tau_*$ for some nucleus n on $\tau_X L$. Now $\tilde{\varphi} : \tau_X L \to \text{Im } \tilde{\varphi}$ is an onto frame homomorphism, which corresponds to a nucleus n on $\tau_X L$. From the previous lemma, $\tilde{\varphi}\tau_* = \varphi$, so also $n\tau_* = \varphi$, so $\varphi \in Y$.

$\square$

As a consequence of the following result, we will see that in the general case, any strict extension can be constructed from a single element of the filter trace. This is in contrast to the classical result, that only relatively spatial strict extensions could be constructed from trace filters. This also implies that, just as in the classical case, the inclusion $X \subseteq Y$ above cannot be improved to an equality, as it was in the space case. This is because X can have just one element, but Y will have more as long as the frame M has more than the trivial nucleus on it.

Lemma 2.3.15 ([9] Proposition 2)**.** *If X is a subset of the filter trace of the strict extension $h : M \to L$, then there is a unique onto frame homomorphism $\tilde{h} : M \to \tau_X L$, such that $\tau_* = \tilde{h}h_*$.*

Proof: If $X = \emptyset$, then $\tau_X L = L$, and $\tilde{h} = h : M \to L$, since h is onto and $hh_* = \text{id}_L = \tau_*$.

If $X \neq \emptyset$, then take $\varphi \in X$. Since φ is a trace filter of h, it can be written as nh_*, where n is a nucleus on M. Now n corresponds to an onto frame homomorphism f from M to a subframe of T_φ which is isomorphic to Fix n. So $\varphi = fh_*$. Let $\tilde{\varphi} : M \to T_\varphi$ be f with codomain extended to T_φ, then $\tilde{\varphi}h_* = \varphi$.

Now let $m : M \to L \times \prod_{\varphi \in X} T_\varphi$ be such that $m(x) = (h(x), (\tilde{\varphi}(x))_{\varphi \in X})$. Since m is the product of frame homomorphisms, it is also a frame homomorphism. Then for $a \in L$,

$$\begin{aligned} m(h_*(a)) &= (hh_*(a), (\tilde{\varphi}h_*(a))_{\varphi \in X}) \\ &= (a, \varphi(a)_{\varphi \in X}) \\ &= \tau_*(a) \text{ from Remark 2.2.9} \end{aligned}$$

So $m(h_*(a)) \in \tau_X L$ for all $a \in L$. But also, h is strict, so $m(x) \in \tau_X L$ for all $x \in M$. Further, since τ is strict, m maps onto $\tau_X L$. Now let $\tilde{h}$ be the corestriction of m onto $\tau_X L$. Then $\tilde{h}$ is an onto frame homomorphism, and we have shown that $\tau_* = \tilde{h}h_*$.

Now if f were another such homomorphism, then we would have $fh_* = \tilde{h}h_*$. But h is strict, so f and $\tilde{h}$ agree on a generating set, and so they must agree everywhere. Therefore $\tilde{h}$ is unique. $\square$

Remark 2.3.16. The condition that $\tau_* = \tilde{h}h_*$ is equivalent to the condition that $\tau\tilde{h} = h$, for any strict extensions h and τ and onto frame homomorphism $\tilde{h}$. For the reverse implication we have Lemma 2.1.8, because $\tilde{h}$ is onto and h is a strict extension. For the forward implication, if $\tau_* = \tilde{h}h_*$, then because τ is onto, $\mathrm{id}_L = \tau\tau_* = \tau\tilde{h}h_*$. But h is also onto, so $hh_* = \mathrm{id}_L$ also, so $hh_* = \tau\tilde{h}h_*$. Now h is strict, so this implies that $h = \tau\tilde{h}$, as required.

Corollary 2.3.17 ([9] Corollary 1)**.** *Every strict extension $h : M \to L$ is equivalent to the strict extension $\tau_X L \to L$ determined by $X = \{h_*\}$.*

Proof: Let $X = \{h_*\}$, then the above lemma gives an onto frame homomorphism $\tilde{h} : M \to \tau_X L$ such that $\tau_* = \tilde{h}h_*$. But also, from Lemma 2.3.13, there is a frame homomorphism $\widetilde{h_*} : \tau_X L \to T_{h_*} = M$ such that $\widetilde{h_*}\tau_* = h_*$. It remains to show that $\tilde{h}$ and $\widetilde{h_*}$ are inverses of each other.

If $s \in L$, then $\tilde{h}\widetilde{h_*}\tau_*(s) = \tilde{h}h_*(s) = \tau_*(s)$, so using the strictness of τ, $\tilde{h}\widetilde{h_*}$ is the identity on $\tau_X L$. On the other hand, $\widetilde{h_*}\tilde{h}h_*(s) = \widetilde{h_*}\tau_*(s) = h_*(s)$, so $\widetilde{h_*}\tilde{h}$ is the identity on M, using the strictness of h. □

Another corollary of the two previous lemmas is the generalisation of Lemma 2.3.3.

Corollary 2.3.18 ([9] Corollary 2)**.** *If X and Y are sets of general filters on L such that $X \subseteq Y$ and $Y \setminus X$ consists only of frame homomorphisms, then $\tau_X L = \tau_Y L$.*

Proof: For each $\varphi \in X$, $\varphi \in Y$ also, so by Corollary 2.3.14, φ is a trace filter of $\tau_Y : \tau_Y L \to L$. So X is a subset of the filter trace of $\tau_Y : \tau_Y L \to L$, and so by Lemma 2.3.15 , there is an onto frame homomorphism $\tilde{\tau}_Y : \tau_Y L \to \tau_X L$ such that $\tau_{X*} = \tilde{\tau}_Y \tau_{Y*}$.

For a map going the other way, consider $\varphi \in Y$. If $\varphi \in X$ also, then it is in the filter trace of $\tau_X : \tau_X L \to L$, by Corollary 2.3.14. However, if $\varphi \notin X$, then φ is a frame homomorphism, so let $\tilde{\varphi} = \varphi\tau_X$, which is a frame homomorphism from $\tau_X L$, and $\tilde{\varphi}\tau_{X*} = \varphi\tau_X\tau_{X*} = \varphi$, so again φ is a trace filter of τ_X, because $\tilde{\varphi}$ corresponds to a nucleus on $\tau_X L$. Then using Lemma 2.3.15 again, we get an onto frame homomorphism $\tilde{\tau}_X : \tau_X L \to \tau_Y L$ such that $\tau_{Y*} = \tilde{\tau}_X \tau_{X*}$.

Now $\tilde{\tau}_Y \tilde{\tau}_X \tau_{X*} = \tilde{\tau}_Y \tau_{Y*} = \tau_{X*}$, so $\tilde{\tau}_Y \tilde{\tau}_X$ is the identity on $\tau_X L$ using the strictness of τ_X. Similarly, $\tilde{\tau}_X \tilde{\tau}_Y$ is the identity on $\tau_Y L$. Therefore $\tilde{\tau}_X$ and $\tilde{\tau}_Y$ are inverses of each other, and so $\tau_X L$ and $\tau_Y L$ are isomorphic. □

We conclude this section with one more example of the usefulness of filter traces.

Corollary 2.3.19 ([9] Remark 2 after Definition 5)**.** *If $h : M \to L$ and $k : N \to L$ are two strict extensions having the same general filter trace, then h and k are equivalent.*

Proof: We must find an isomorphism $f : M \to N$ such that $kf = h$. Let X be the filter trace of both h and k. Then from Lemma 2.3.15 we have homomorphisms $\tilde{k} : N \to \tau_X L$ and $\tilde{h} : M \to \tau_X L$ such that $\tau_* = \tilde{k}k_*$ and $\tau_* = \tilde{h}h_*$. Also, k_* and h_* are in X, so from Lemma 2.3.13, there are frame homomorphisms $\widetilde{k_*} : \tau_X L \to N$ and $\widetilde{h_*} : \tau_X L \to M$ such that $\widetilde{k_*}\tau_* = k_*$ and $\widetilde{h_*}\tau_* = h_*$. We claim that $\widetilde{h_*}\tilde{k}$ and $\widetilde{k_*}\tilde{h}$ are inverses of one another, so that we can define $f = \widetilde{k_*}\tilde{h}$.

Now

$$\begin{aligned}
\widetilde{k_*}\tilde{h}\widetilde{h_*}\tilde{k}k_* &= \widetilde{k_*}\tilde{h}\widetilde{h_*}\tau_* \\
&= \widetilde{k_*}\tilde{h}h_* \\
&= \widetilde{k_*}\tau_* \\
&= k_*
\end{aligned}$$

and similarly, $\widetilde{h_*}\tilde{k}\widetilde{k_*}\tilde{h}h_* = h_*$. So the two maps are inverses of each other by the fact that h and k are strict.

Finally,

$$\begin{aligned}
kfh_* &= k\widetilde{k_*}\tilde{h}h_* \\
&= k\widetilde{k_*}\tau_* \\
&= kk_* = \mathrm{id}_L \\
&= hh_*
\end{aligned}$$

so by the strictness of h, $kf = h$. Therefore f is the required isomorphism. $\square$

2.4 Applications of strict extensions

Recall that a compactification of a frame L is a dense onto frame homomorphism $h : M \to L$, where M is a compact regular frame.

We saw in Lemma 2.1.5 that any dense frame homomorphism from a regular frame M is strict, so compactifications are an important example of strict extensions.

Under the assumption of the Boolean Ultrafilter Theorem, every compact regular frame is spatial, and we saw in Remark 2.2.12 that any strict extension with a spatial frame as its domain is relatively spatial. Therefore if we assume BUT, then every compactification is relatively spatial, and so, by Lemma 2.3.5, every compactification can be expressed as $\tau_X L \to L$, where X is the classical filter trace of the compactification. But in this case, we can describe the set X more precisely. Refer to Definition 1.2.37 and Lemma 1.2.38 for the concepts used below.

Proposition 2.4.1 ([7] Proposition 4)**.** *For any strong inclusion $\triangleleft$ on the frame L, if X is the set of all free maximal $\triangleleft$-filters on L, then $\tau_X L \to L$ is the $\triangleleft$-compactification of L.*

Proof: Let $h : M \to L$ be the $\triangleleft$-compactification of L. Then from Lemma 2.3.5, h is equivalent to $\tau_Y L \to L$, where Y is the filter trace of h. Now from Lemma 2.3.3, Hong's Construction is not affected by the addition or removal of completely prime filters, so let X be the set Y with all the completely prime filters removed. Then h is equivalent to $\tau_X L \to L$. We will show that X is precisely the set of free maximal $\triangleleft$-filters on L.

Take $F \in X$, so F is a trace filter of h and F is not completely prime. Then $F = h[P]$, where P is a completely prime filter on M.

We first show that F is a $\triangleleft$-filter. For any $a \in F$, $hh_*(a) = a$, and $a \in h[P]$, so $h_*(a) \in P$, as shown in the proof of Lemma 2.3.5. Now P is a regular filter because M is regular, so if $y \in P$, then $y = \bigvee\{x \in M | x \prec y\} \in P$, and so there is an $x \prec y$ such that $x \in P$, because P is completely prime. In particular, there is an $x \in P$ such that $x \prec h_*(a)$. But also, h is strict, so $x = \bigvee\{h_*(b)|h_*(b) \leq x\}$, and using the fact that P is completely prime again, there is a $b \in L$ such that $h_*(b) \in P$. Now $h_*(b) \leq x \prec h_*(a)$, so $b \triangleleft a$ and $b \in h[P] = F$, so F is a $\triangleleft$-filter.

Next we show that F is a maximal $\triangleleft$-filter. Suppose that G is another $\triangleleft$-filter with $F \subseteq G$, and let H be the filter in M generated by $h_*[G]$. Now h is onto, so $G = hh_*[G]$, and $F = h[P]$, so since $F \subseteq G$, $h[P] \subseteq hh_*[G]$. But $h_*[G] \subseteq H$, so $hh_*[G] \subseteq h[H]$, so we have $h[P] \subseteq h[H]$.

Now if $z \in h[H]$, then $z = h(a)$ for some $a \in H$. Then there is an $x \in G$ such that $h_*(x) \leq a$, since H is generated by $h_*[G]$. But G is a $\triangleleft$-filter, so there is a $y \in G$ such that $y \triangleleft x$. This means that $h_*(y) \prec h_*(x) \leq a$ and so $(h_*(y))^* \vee a = e \in P$. Now P is prime, so either $(h_*(y))^* \in P$ or $a \in P$. If $a \in P$, then $z = h(a) \in h[P]$, so we have that $h[H] \subseteq h[P]$. Then $G = hh_*[G] \subseteq h[H] \subseteq h[P] = F$, so $G = F$.

Now consider the other possibility, that $(h_*(y))^* \in P$. Then $h((h_*(y))^*) \in h[P] \subseteq h[H]$. But $y \in G$, so $h_*(y) \in h_*[G] \subseteq H$, and so $h(h_*(y)) \in h[H]$. But then $h((h_*(y))^*) \wedge h(h_*(y)) = h((h_*(y))^* \wedge h_*(y)) = h(0) = 0 \in h[H]$. Now if $0 \in h[H]$ it means that $0 \in H$ because h is dense, so there is some $g \in G$ such that $h_*(g) \leq 0$.

But then $h_*(g) = 0$, so $g = hh_*(g) = h(0) = 0$, so $0 \in G$. Therefore we conclude that either $F = G$, or G is not proper, so F is maximal.

Finally, we must show that F is free. Suppose that F is not free, so that it converges. Then if $S \subseteq L$ such that $\bigvee S \in F$, we can find $x \in F$ such that $x \prec \bigvee S$ because F is regular. So $x^* \vee (\bigvee S) = e$, and so $S \cup \{x^*\}$ is a cover of L. But F meets every cover of L because F converges, so $F \cap (S \cup \{x^*\}) \neq \emptyset$. But $x^* \notin F$ because F is proper and $x \in F$, so $F \cap S \neq \emptyset$. This is true for any subset $S \subseteq L$ such that $\bigvee S \in F$, which implies that F is completely prime. But we assumed that F was not completely prime, so F must be free.

Now for the other direction, we must show that if F is a free maximal $\triangleleft$-filter, then F is a trace filter of h that is not completely prime. If F was completely prime, then for any cover $S \subseteq L$, $\bigvee S = e \in P$, and so $S \cap P \neq \emptyset$. But this means that F converges. Since F is free, it must not be completely prime.

To show that F is a trace filter of h, we must show that $F = h[P]$ for some completely prime filter P on M. Now let G be the filter on M generated by $h_*[F]$. Then since F is a $\triangleleft$-filter, G is a regular filter. Now we are assuming BUT, so by Lemma 1.2.40, G is contained in a prime filter Q on M. Let $P = \{a \in M | x \prec a \text{ for some } x \in Q\}$. We claim that P is a completely prime filter and that $h[P] = F$.

If $a \in P$ and $a \leq b$, then $x \prec a$ for some $x \in Q$. But then $x \prec b$ also, so $b \in P$. Also, if $a \in P$ and $b \in P$, then there exist x and y in Q such that $x \prec a$ and $y \prec b$. But then $x \wedge y \prec a \wedge b$, and since $x \wedge y \in Q$, $a \wedge b \in P$. So P is a filter.

If $a \vee b \in P$, then $z \prec a \vee b$ for some $z \in Q$, and so $z^* \vee a \vee b = e$. Then by regularity, $z^* \vee \bigvee\{x \in M | x \prec a\} \vee \bigvee\{y \in M | y \prec b\} = e$. Now M is compact, so this cover has a finite subcover $\{z^*, x_1, ..., x_n, y_1, ..., y_m\}$. But since $x_i \prec a$ for each $i = 1, ..., n$, if $x = \bigvee\{x_1, ..., x_n\}$, then $x \prec a$, and similarly, if $y = \bigvee\{y_1, ..., y_m\}$, then $y \prec b$, and $z^* \vee x \vee y = e$. But then $z \prec x \vee y$, where $x \prec a$ and $y \prec b$. Now $z \in Q$, so $x \vee y \in Q$, and Q is prime, so either $x \in Q$ or $y \in Q$. Therefore either $a \in P$ or $b \in P$, so P is prime.

To show that P is completely prime, it only remains to show that $\bigvee S \in P$ implies that $P \cap S \neq \emptyset$ for any updirected set S, because arbitrary joins can be expressed as the updirected join of all the finite subjoins. So if $\bigvee S \in P$ for an updirected set $S \subseteq M$, then there is an $x \in Q$ such that $x \prec \bigvee S$. This means that $x^* \vee \bigvee S = e$, so by compactness, $x^* \vee t_1 \vee ... \vee t_n = e$ for some finite subset $\{t_1, ..., t_n\}$ of S. But S is updirected, so if $t = t_1 \vee ... \vee t_n$ then $t \in S$ and $x^* \vee t = e$, so $x \prec t$ which means that $t \in P$. Therefore $S \cap P \neq \emptyset$ and so P is completely prime.

Finally, we must show that $h[P] = F$. If $a \in G$, then since G is regular, there is a

$b \in G$ such that $b \prec a$. But then $b \in Q$ because $G \subseteq Q$, so $a \in P$, which gives that $G \subseteq P$. Therefore $F = hh_*[F] \subseteq h[G] \subseteq h[P]$, since G is generated by $h_*[F]$. Now if $a \in P$, then $x \prec a$ for some $x \in Q$, so $x^* \vee a = e$. Now M is compact regular, so by a previous argument, $x^* \vee y = e$ for some $y \prec a$. This means that $x \prec y$, and $x \in Q$, so $y \in P$. Therefore we have found $y \in P$ such that $y \prec a$, and so P is regular. But then $h[P]$ is a $\triangleleft$-filter because $y \prec a$ implies that $h(y) \triangleleft h(a)$, and we showed that $F \subseteq h[P]$, so $h[P] = F$ by the maximality of F. $\square$

The next application is regarding realcompactness, which is a weakening of the idea of compactness. For the remainder of this section we will restrict our attention to completely regular frames, and assume the Axiom of Countable Choice. This is so that we can benefit from the properties of the σ-frame Coz$L$ that depend on these two things.

Definition 2.4.2. *If J is an ideal in a sublattice A of a frame L, then J is called* **σ-proper** *if $\bigvee S \neq e$ for any countable $S \subseteq J$, where the join is taken in L. J is called* **completely proper** *if $\bigvee J \neq e$.*

Definition 2.4.3. *A completely regular frame L is called* **realcompact** *if every maximal ideal of Coz$L$ that is σ-proper is completely proper.*

The following lemma makes the above definition easier to work with. Refer to Definition 1.2.15 for relevant definitions.

Lemma 2.4.4 ([6] Lemma 1)**.** *For any regular σ-frame A, the σ-proper maximal ideals of A are exactly the maximal σ-ideals of A.*

Proof: First, if P is a σ-proper maximal ideal, then we need only show that it is a σ-ideal, because every σ-ideal is an ideal, so if Q is a σ-ideal such that $P \subset Q \subset \downarrow e$, then Q contradicts the maximality of P as an ideal. So we must just show that for every countable $S \subseteq P$, $\bigvee S \in P$.

Consider $Q = \{\bigvee S | S \subseteq P,\ S \text{ countable}\}$. Then we claim that Q is an ideal:

- If $a \in Q$ and $b \leq a$, then $a = \bigvee S$ for some countable $S \subseteq P$ and $b = b \wedge a = b \wedge (\bigvee S) = \bigvee\{b \wedge s | s \in S\}$. Now $S \subseteq P$, so for each $s \in S$, $b \wedge s \leq s \in P$, and so $b \wedge s \in P$. Therefore $b = \bigvee S'$ where $S' \subseteq P$ and S' is countable, so $b \in Q$.
- If $a \in Q$ and $b \in Q$, then $a = \bigvee S$ and $b = \bigvee T$ for some countable sets S and T in P. Then $a \vee b = \bigvee S \vee \bigvee T = \bigvee(S \cup T)$, and $S \cup T$ is a countable subset of P, so $a \vee b \in Q$.

Now $P \subseteq Q$ because each $a \in P$ can be written as $\bigvee\{a\}$, and $\{a\}$ is clearly a countable subset of P. Also, $Q \neq \downarrow e$ because for each countable $S \subseteq P$, $\bigvee S \neq e$ because P is σ-proper, so $e \notin Q$. Therefore we have an ideal Q such that $P \subseteq Q \subset \downarrow e$, and so by the maximality of P, $Q = P$. Therefore, for every countable $S \subseteq P$, $\bigvee S \in P$, so P is a σ-ideal.

For the other direction, suppose that P is a maximal σ-ideal. Then P is σ-proper because if $S \subseteq P$ and S is countable, then $\bigvee S \in P$, and since P is a proper ideal, $\bigvee S \neq e$. So it remains to show that P is a maximal ideal. We will do this by showing that any ideal containing P and another element of A must be $\downarrow e$.

Let $a \in A \setminus P$. Since A is a regular σ-frame, $a = \bigvee_n \{b_n \in A | b_n \prec a\}$. Now if all these b_n's were in P, then their join, a, would also be in P because P is a σ-ideal. But $a \notin P$, so there is a $b \prec a$ such that $b \notin P$. Now take c such that $c \wedge b = 0$ and $c \vee a = e$. Then $\downarrow b \cap \downarrow c = \downarrow(b \wedge c) = \downarrow 0 \subseteq P$, because P is a downset. Also, $P \in \mathfrak{h}A$, which is a regular frame because A is a regular σ-frame, so P being a maximal σ-frame implies that P is prime in $\mathfrak{h}A$. So either $\downarrow b$ or $\downarrow c$ is contained in P. Now $b \notin P$, so $\downarrow b \not\subseteq P$, so we must have that $\downarrow c \subseteq P$, and in particular, $c \in P$. But we know that $c \vee a = e$, so any ideal containing P and the element a must also contain e. Since this is true for any $a \notin P$, the only ideal greater than P is $\downarrow e$, and so P is maximal. $\square$

It was shown in [25] that the realcompact frames are coreflective in the category of completely regular frames. For a frame L, the universal Lindelöfication $\bigvee : \mathfrak{h}\mathrm{Coz}L \to L$ of L is a strict extension because for $a \in L$, $\bigvee_*(a) = \downarrow a \cap \mathrm{Coz}L$, and so for any $H \in \mathfrak{h}\mathrm{Coz}L$, $H = \bigcup\{\bigvee_*(a) | a \in H\}$. It was pointed out in [6] that the coreflection map for a frame L given in [25] is in fact the relatively spatial reflection (see Proposition 2.2.18) of the universal Lindelöfication of L. This insight allows the proof to be simplified significantly, but before we can show it, we need one more lemma.

Lemma 2.4.5 ([6] Lemma 4)**.** *Any commuting square*

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "f"'] & L \arrow[d, "g"] \\ N \arrow[r, "k"'] & K \end{tikzcd}$$

where h and k are strict extensions, determines a commuting diagram

$$\begin{tikzcd}
M \arrow[r, "n_M"] \arrow[d, "f"'] & \tilde{M} \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{f}"] & L \arrow[d, "g"] \\
N \arrow[r, "n_N"'] & \tilde{N} \arrow[r, "\tilde{k}"'] & K
\end{tikzcd}$$

where the maps $\tilde{h}$ and $\tilde{k}$ are the relatively spatial reflections of the strict extensions h and k, which were described in Proposition 2.2.18, and the maps $n_M : M \to \tilde{M}$ and $n_N : N \to \tilde{N}$ are their respective nuclei.

Proof: We will show that ker $n_M \subseteq$ ker $n_N f$, so that we can use Lemma 1.2.34 to provide a unique frame homomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$ making the left square commute, because n_M is onto since it is a nucleus. Then the right square will also commute, because $gh = kf$, so $g\tilde{h}n_M = \tilde{k}n_N f = \tilde{k}\tilde{f}n_M$, if the left square commutes, and so $g\tilde{h} = \tilde{k}\tilde{f}$ because n_M is right-cancelable since it is onto. (See Remark 2.2.19). So it remains to show that ker $n_M \subseteq$ ker $n_N f$.

Note that

$$\begin{aligned}
f_* n_N f &= f_*((k_* k) \wedge r_N) f \text{ by the definition of } n_N \\
&= (f_* k_* k f) \wedge (f_* r_N f) \\
&= (h_* g_* g h) \wedge (f_* r_N f) \text{ since } kf = gh \\
&\geq (h_* h) \wedge (f_* r_N f) \text{ because } g_* g \text{ is a nucleus} \\
&\geq (h_* h) \wedge r_M \text{ by Lemma 2.2.16} \\
&= n_M.
\end{aligned}$$

So we have that $n_M \leq f_* n_N f$, which implies that $f n_M \leq n_N f$, and so $n_N f n_M \leq n_N n_N f = n_N f$. On the other hand, $n_N f \leq n_N f n_M$ because n_M is a nucleus, and so $n_N f = n_N f n_M$. Therefore, if $(x, y) \in M \times M$ such that $n_M(x) = n_M(y)$, then $n_N f n_M(x) = n_N f n_M(y)$, and so $n_N f(x) = n_N f(y)$. Therefore ker $n_M \subseteq$ ker $n_N f$, as required. □

Remark 2.4.6. The frame homomorphism $\tilde{f}$ obtained from Lemma 1.2.34 is $\tilde{f} = n_N f (n_M)_*$. Now for any element $a \in \tilde{M}$, $(n_M)_*(a) = \bigvee\{x \in M | n_M(x) = a\} = a$, because $\tilde{M} = \text{fix } n_M$, so $n_M(a) = a$, and if $n_M(x) = a$, then $x \leq a$ because n_M is a nucleus. Therefore $f(n_M)_* = f|_{\tilde{M}}$, and so $\tilde{f} = n_N f|_{\tilde{M}}$.

We will now show the promised coreflection.

Proposition 2.4.7 ([6] Proposition 3)**.** *The realcompact frames are coreflective in the category of completely regular frames, with coreflection maps $\upsilon_L : \upsilon L \to L$ given by the relatively spatial reflection of $\bigvee : \mathfrak{h}CozL \to L$.*

Proof. For a completely regular frame L, and a frame homomorphism $h : M \to L$, where M is a realcompact frame, we need to find a unique frame homomorphism $f : M \to \upsilon L$ making the triangle commute.

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, dotted, "!f"'] & & L \\ & \upsilon L \arrow[ur, "\upsilon_L"'] & \end{tikzcd}$$

We will show that υ is a functor, that υL is realcompact for any frame L, and that $\upsilon L \to L$ is an isomorphism whenever L is realcompact. Then given the frame map $h : M \to L$, we have a frame homomorphism $\upsilon h : \upsilon M \to \upsilon L$, because υ is a functor. Also $\upsilon_M : \upsilon M \to M$ is an isomorphism because M is realcompact, so it has an inverse $\upsilon_M^{-1} : M \to \upsilon M$.

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "\upsilon_M^{-1}"'] & L \\ \upsilon M \arrow[r, "\upsilon h"'] & \upsilon L \arrow[u, "\upsilon_L"'] \end{tikzcd}$$

The square above commutes because υ is a functor, so if $f = \upsilon h \circ \upsilon_M^{-1}$, then f is the required factorisation. The function f is unique because if $g : M \to \upsilon L$ was another map satisfying $\upsilon_L g = h$, then $\upsilon_L g = \upsilon_L f$, but υ_L is dense because it is a strict extension, so it is monic because all the frames above are regular, which means that $g = f$.

So first we must show that υ is a functor. Suppose that $h : M \to L$ is a frame homomorphism, then we must find a frame homomorphism $\upsilon h : \upsilon M \to \upsilon L$. Now $\mathfrak{h}\mathrm{Coz}$ is a functor, so we have the following commutative diagram:

$$\begin{tikzcd} \mathfrak{h}\mathrm{Coz}M \arrow[r, "\bigvee_M"] \arrow[d, "\mathfrak{h}\mathrm{Coz}\, h"'] & M \arrow[d, "h"] \\ \mathfrak{h}\mathrm{Coz}L \arrow[r, "\bigvee_L"'] & L \end{tikzcd}$$

Recall that $\upsilon_L : \upsilon L \to L$ is the relatively spatial reflection of $\bigvee_L : \mathfrak{h}\mathrm{Coz}L \to L$, so $\upsilon L = \widetilde{\mathfrak{h}\mathrm{Coz}L}$. The diagram therefore splits according to Lemma 2.4.5:

$$\begin{tikzcd}
\mathfrak{h}\mathrm{Coz}M \arrow[r, "n_{\mathfrak{h}\mathrm{Coz}M}"] \arrow[d, "\mathfrak{h}\mathrm{Coz}\ h"'] & \upsilon M \arrow[r, "\widetilde{\bigvee_M}"] \arrow[d, "\upsilon h"'] & M \arrow[d, "h"] \\
\mathfrak{h}\mathrm{Coz}L \arrow[r, "n_{\mathfrak{h}\mathrm{Coz}L}"'] & \upsilon L \arrow[r, "\widetilde{\bigvee_L}"'] & L
\end{tikzcd}$$

From Remark 2.4.6, $\upsilon h = n_{\mathfrak{h}\mathrm{Coz}L} \circ \mathfrak{h}\mathrm{Coz}\ h|_{\upsilon M}$. Now if $i : L \to L$ is an identity frame homomorphism, then for any $a \in \upsilon L$,

$$\begin{aligned}
\upsilon i(a) &= n_{\mathfrak{h}\mathrm{Coz}L} \circ \mathfrak{h}\mathrm{Coz}\ i|_{\upsilon L}(a) \\
&= n_{\mathfrak{h}\mathrm{Coz}L} \circ \mathfrak{h}\mathrm{Coz}\ i(a) \text{ since } a \in \upsilon L \\
&= n_{\mathfrak{h}\mathrm{Coz}L}(a) \text{ because } \mathfrak{h}\mathrm{Coz}\ i \text{ is an identity} \\
&= a \text{ because } a \in \upsilon L = \mathrm{Fix}\ n_{\mathfrak{h}\mathrm{Coz}L}.
\end{aligned}$$

So υ preserves identities. Also, if $h : M \to L$ and $k : L \to N$ are frame homomorphisms, then

$$\begin{aligned}
\upsilon k \circ \upsilon h &= n_{\mathfrak{h}\mathrm{Coz}N} \circ \mathfrak{h}\mathrm{Coz}\ k|_{\upsilon L} \circ n_{\mathfrak{h}\mathrm{Coz}L} \circ \mathfrak{h}\mathrm{Coz}\ h|_{\upsilon M} \\
&= n_{\mathfrak{h}\mathrm{Coz}N} \circ \mathfrak{h}\mathrm{Coz}\ k \circ \mathfrak{h}\mathrm{Coz}\ h|_{\upsilon M} \text{ because } \mathfrak{h}\mathrm{Coz}\ k|_{\upsilon L} \circ n_{\mathfrak{h}\mathrm{Coz}L} = \mathfrak{h}\mathrm{Coz}\ k \\
&= n_{\mathfrak{h}\mathrm{Coz}N} \circ \mathfrak{h}\mathrm{Coz}\ kh|_{\upsilon M} \text{ because } \mathfrak{h}\mathrm{Coz} \text{ is a functor} \\
&= \upsilon kh.
\end{aligned}$$

So υ preserves compositions, and therefore υ is a functor.

The second thing to show is that υL is realcompact for any completely regular frame L. For this we must show that any σ-proper maximal ideal of $\mathrm{Coz}\upsilon L$ is completely proper. From Lemma 2.4.4, this means that we must show that for any maximal $\mathcal{P} \in \mathfrak{h}\mathrm{Coz}\ \upsilon L$, $\mathcal{P}$ is completely proper. Now υL consists of σ-ideals of $\mathrm{Coz}L$, so $\mathcal{P}$ is a σ-ideal of σ-ideals of cozero elements of L, so we must show that $\bigvee \mathcal{P} \neq \mathrm{Coz}L$.

The map $\bigvee_{\upsilon L} : \mathfrak{h}\mathrm{Coz}\ \upsilon L \to \upsilon L$ is the universal Lindelöfication of υL, which is a coreflection, so for any frame homomorphism $h : M \to \upsilon L$ from a Lindelöf frame M, there is a unique frame homomorphism $M \to \mathfrak{h}\mathrm{Coz}\ \upsilon L$ making the diagram below commute:

$$\begin{tikzcd}
M \arrow[rr, "h"] \arrow[dr, dotted] & & \upsilon L \\
& \mathfrak{h}\mathrm{Coz}\ \upsilon L \arrow[ur, "\bigvee_{\upsilon L}"'] &
\end{tikzcd}$$

Now $\mathfrak{h}\mathrm{Coz}L$ is a Lindelöf frame, and $n_{\mathfrak{h}\mathrm{Coz}L} : \mathfrak{h}\mathrm{Coz}L \to \upsilon L$ is a frame homomorphism, which we will abbreviate as n for the remainder of this proof, so we get a unique frame homomorphism $f : \mathfrak{h}\mathrm{Coz}L \to \mathfrak{h}\mathrm{Coz}\ \upsilon L$ such that $\bigvee_{\upsilon L} f = n$.

Similarly, $\bigvee_L : \mathfrak{h}\mathrm{Coz}L \to L$ is the universal Lindelöfication of L, so for any frame homomorphism $k : N \to L$, from a Lindelöf frame N, there is a unique frame homomorphism $N \to \mathfrak{h}\mathrm{Coz}L$ making the diagram below commute:

$$\begin{tikzcd}
N \arrow[rr, "k"] \arrow[dr, dotted] & & L \\
& \mathfrak{h}\mathrm{Coz}L \arrow[ur, "\bigvee_L"'] &
\end{tikzcd}$$

Now $\bigvee_{\upsilon L} : \mathfrak{h}\mathrm{Coz}\ \upsilon L \to \upsilon L$ and $\upsilon : \upsilon L \to L$ are frame homomorphisms, so the map $\upsilon \bigvee_{\upsilon L}$ is a frame homomorphism from $\mathfrak{h}\mathrm{Coz}\ \upsilon L$, a Lindelöf frame, to L, and so there is a unique frame homomorphism $g : \mathfrak{h}\mathrm{Coz}\ \upsilon L \to \mathfrak{h}\mathrm{Coz}L$ such that $\bigvee_L g = \upsilon \bigvee_{\upsilon L}$.

Also, $\bigvee_L : \mathfrak{h}\mathrm{Coz}L \to L$ is a frame homomorphism from a Lindelöf frame to L, so there is a unique frame homomorphism $\mathfrak{h}\mathrm{Coz}L \to \mathfrak{h}\mathrm{Coz}L$ making the corresponding triangle commute. However, there are two options for this map, one being the identity map, and the other being gf, the composition of the two maps that we just found. So by uniqueness, $gf = \mathrm{id}_{\mathfrak{h}\mathrm{Coz}L}$. Similarly, using uniqueness in the top triangle, $fg = \mathrm{id}_{\mathfrak{h}\mathrm{Coz}\upsilon L}$. Therefore g is the inverse of f, and so f is an isomorphism.

We are trying to show that for any maximal $\mathcal{P} \in \mathfrak{h}\mathrm{Coz}\ \upsilon L$, $\bigvee \mathcal{P} \neq \mathrm{Coz}L$. Now for such a $\mathcal{P}$, there exists a $Q \in \mathfrak{h}\mathrm{Coz}L$ such that $\mathcal{P} = f(Q)$ because f is onto. If $Q \subseteq R$ for some $R \in \mathfrak{h}\mathrm{Coz}L$, then $f(Q) \subseteq f(R)$, but $f(Q) = \mathcal{P}$, which is maximal, so $f(Q) = f(R)$. But this means that $Q = R$, because f is one-one. Also, $Q \neq \mathrm{Coz}L$, because if $Q = \mathrm{Coz}L$ then $f(Q) = e$, but $\mathcal{P} \neq e$, because it is maximal. Therefore Q is maximal in $\mathfrak{h}\mathrm{Coz}L$.

Now $\bigvee \mathcal{P} = \bigvee f(Q) = n(Q)$, by the way we defined f. Then using the definition of n from Proposition 2.2.18 and Lemma 2.2.15, $n(Q) = \bigvee_* \bigvee Q \cap \bigwedge\{J \in \mathfrak{h}\mathrm{Coz}L | Q \subseteq J, J \text{ prime}\}$. Since $\mathfrak{h}\mathrm{Coz}L$ is regular, and $\bigvee_*(a) = \downarrow a \cap CozL$, this can be written as

$$n(Q) = \downarrow \left(\bigvee Q \right) \cap \mathrm{Coz}L \cap \bigwedge\{J \in \mathfrak{h}\mathrm{Coz}L | Q \subseteq J, J \text{ maximal}\}.$$

Now Q is maximal, so $\bigwedge\{J \in \mathfrak{h}\mathrm{Coz}L | Q \subseteq J, J \text{ maximal}\} = Q$, and $Q \subseteq \downarrow \bigvee Q$ and $Q \subseteq \mathrm{Coz}L$, so $n(Q) = Q$, and so $\bigvee \mathcal{P} = Q \neq \mathrm{Coz}L$, as required.

Finally, we must show that if L is already realcompact, then $\upsilon_L : \upsilon L \to L$ is an isomorphism. Now for any $P \in \mathfrak{h}\mathrm{Coz}L$, $P \subseteq \downarrow(\bigvee P) \cap \mathrm{Coz}L$, which is a σ-ideal. Then if P is maximal in $\mathfrak{h}\mathrm{Coz}L$, $\bigvee P \neq e$ if L is realcompact, so this σ-ideal is not $\mathrm{Coz}L$. Therefore $P \subseteq \downarrow(\bigvee P) \cap \mathrm{Coz}L \subset \mathrm{Coz}L$, which implies that $P = \downarrow(\bigvee P) \cap \mathrm{Coz}L$ by the maximality of P.

For any $J \in \mathfrak{h}\mathrm{Coz}L$, $n(J) = \downarrow(\bigvee J) \cap \mathrm{Coz}L \cap \bigwedge\{P \in \mathfrak{h}\mathrm{Coz}L | J \subseteq P, P \text{ maximal}\}$. But if $J \subseteq P$, then $\downarrow(\bigvee J) \cap \mathrm{Coz}L \subseteq \downarrow(\bigvee P) \cap \mathrm{Coz}L = P$ if P is maximal. So $\downarrow(\bigvee J) \cap \mathrm{Coz}L \subseteq \bigwedge\{P \in \mathfrak{h}\mathrm{Coz}L | J \subseteq P, P \text{ maximal}\}$. Therefore $n(J) = \downarrow(\bigvee J) \cap \mathrm{Coz}L = \bigvee_*(\bigvee J)$.

Recall that $\upsilon L = \mathrm{Fix}\, n$, and $\upsilon_L : \upsilon L \to L$ is just $\bigvee : \mathfrak{h}\mathrm{Coz}L \to L$ restricted to $\mathrm{Fix}\, n$, as described in Proposition 2.2.18. The map υ_L is an onto homomorphism because it is a strict extension, so we just need to show that it is one-one. Let J and K be elements of υL such that $\bigvee J = \bigvee K$. Then $\bigvee_*(\bigvee J) = \bigvee_*(\bigvee K)$, that is, $n(J) = n(K)$, and so $J = K$ because they are elements of $\mathrm{Fix}\, n$. This proves that $\upsilon_L : \upsilon L \to L$ is one-one, and so υL is isomorphic to L. $\square$

In Proposition 14 in the appendix to [6], the zero-dimensional counterpart of the proposition above is discussed. A frame L is **zero-dimensional** if it is generated by its Boolean part, BL, which is the Boolean algebra of all complemented elements of L. Here, BL plays the role of $\mathrm{Coz}L$, and instead of realcompactifications, we get $\mathbb{N}$-compactifications. That is, a frame L is **$\mathbb{N}$-compact** if every maximal ideal of BL that is σ-proper is completely proper, corresponding exactly to the definition of realcompactness. Then the $\mathbb{N}$-compact frames are coreflective in the category of zero dimensional frames, with coreflection maps given by the relatively spatial reflection of $\bigvee : \mathfrak{h}BL \to L$. This is exactly the counterpart of the above proposition, and the proof follows the same argument.

At the beginning of this section, we saw that any compactification of a frame L can be expressed as $\tau_X L \to L$ for an appropriate set of filters X. For nearness frames, if X is a certain set of Cauchy filters on the nearness frame L, then the completion of L can be expressed as $\tau_X L \to L$. But for this we need to discuss how strict extensions behave in the presence of nearness structure, which is the topic of the next chapter.

3 Strict Extensions in Structured Frames

3.1 Completions and Cauchy completions of nearness frames

In this chapter we will consider strict extensions of nearness frames, so we will first see which frames admit a nearness structure.

Lemma 3.1.1 ([3] or [5] Proposition 1(a))**.** *A frame L admits a nearness structure if and only if L is regular.*

Proof: Suppose that L admits a nearness structure $\mathcal{N}$. Then for each $a \in L$, $a = \bigvee\{x \in L | x \triangleleft_{\mathcal{N}} a\}$. Now we saw in Lemma 1.2.46 that if $x \triangleleft_{\mathcal{N}} a$, then $x \prec a$, and also, if $x \prec a$ then $x \leq a$, so we have

$$\begin{aligned} a &= \bigvee\{x \in L | x \triangleleft_{\mathcal{N}} a\} \\ &\leq \bigvee\{x \in L | x \prec a\} \\ &\leq a. \end{aligned}$$

So $a = \bigvee\{x \in L | x \prec a\}$, and therefore L is regular.

For the other direction, we will show that if L is regular, then $\mathcal{N} = \mathrm{Cov}L$ is a nearness on L. The set $\mathcal{N}$ is a filter because the meet of two covers is a cover by the frame law, and anything that a cover refines is clearly still a cover. For admissibility, take $a \in L$, and we have $a = \bigvee\{x \in L | x \prec a\}$. But if $x \prec a$, then $x^* \vee a = e$, so $C = \{x^*, a\} \in \mathcal{N}$. Now $Cx = \bigvee\{s \in C | s \wedge x \neq 0\}$, and $x^* \wedge x = 0$, so $Cx \leq a$, meaning that $x \triangleleft_{\mathcal{N}} a$. Then

$$\begin{aligned} a &= \bigvee\{x \in L | x \prec a\} \\ &\leq \bigvee\{x \in L | x \triangleleft_{\mathcal{N}} a\} \\ &\leq a. \end{aligned}$$

So $a = \bigvee\{x \in L | x \triangleleft_{\mathcal{N}} a\}$, and $\mathrm{Cov}L$ is an admissible nearness on L. $\square$

As a consequence of this and Lemma 2.1.5, any dense onto frame homomorphism between nearness frames is a strict extension. We will see that completions of nearness frames are dense onto frame homomorphisms, and therefore, completions are important examples of strict extensions.

Before we consider completions in the pointfree setting, we should recall how completions work for uniform spaces. Since we saw in the previous lemma that all nearness frames are regular, the corresponding spaces that will be of interest are those that are regular and T_0. Regular T_0 spaces are Hausdorff, so we now focus on Hausdorff uniform spaces.

We say that a uniform space is complete if every Cauchy filter on it converges. A completion is a dense embedding of a uniform space into a complete space. It is known that every Hausdorff uniform space has a unique Hausdorff completion consisting of its minimal Cauchy filters. A complete uniform space can also be defined as a Hausdorff space for which every dense embedding from it is an isomorphism. These two notions of completeness coincide for uniform spaces.

In moving from spaces to frames, completeness can be defined in terms of either of the two notions mentioned above. We will consider each one in turn, and we will see that even for uniform frames, they give two different concepts of completeness.

Definition 3.1.2. 1. *A filter F on a nearness frame L is called* **Cauchy** *if for every $C \in \mathcal{N}L$, $C \cap F \neq \emptyset$.*

2. *A uniform frame L is* **Cauchy complete** *if every Cauchy filter converges.*

Remark 3.1.3 ([13] before Definition 1)**.** In uniform spaces, Cauchy filters have two properties that we need to maintain when we generalise to the pointfree setting. The first is the Cauchy property itself, that the filter has arbitrarily small members. Here "small" means members that can be contained in any given uniform cover, and it is generalised as above, by defining Cauchy filters to be those that meet every uniform cover.

The second property is that every member of the filter contains another one that is significantly smaller that it, with respect to the uniformity. This ensures that the members of the filter get smaller in a uniform way, and is intrinsically related to the star-refinement property of uniformities. When we move to nearness frames, we no longer have star refinements, so we need to impose a new condition that governs the way members of a filter shrink.

Definition 3.1.4. 1. *A Cauchy filter F on a nearness frame $(L, \mathcal{N})$ is called* **regular** *if for each $a \in F$ there exists $b \in F$ such that $b \triangleleft_{\mathcal{N}} a$. Recall that we might abbreviate this to $b \triangleleft a$.*

2. *A nearness frame L is* **Cauchy complete** *if every regular Cauchy filter converges.*

Remark 3.1.5. A regular Cauchy filter is a regular filter in the original sense of the word, because we saw in Lemma 1.2.46 that if $a \triangleleft b$ then $a \prec b$. In this chapter we will only use the term "regular filter" to mean regular with respect to the nearness structure.

The following lemma shows that the notion of regular is exactly what we need to generalise the idea of Cauchy completeness to nearness frames.

Lemma 3.1.6 ([11] Lemma 11)**.** *For a uniform frame $(L, \mathcal{N})$, the regular Cauchy filters are exactly the minimal Cauchy filters.*

Proof: Suppose that F is a regular Cauchy filter, and let G be a Cauchy filter such that $G \subseteq F$. We will show that $F \subseteq G$ to show that in fact $F = G$. Take $x \in F$. Then since F is regular, there exists $y \in F$ such that $y \lhd_{\mathcal{N}} x$, and so for some $C \in \mathcal{N}$, $Cy \leq x$. Now G is a Cauchy filter, so $C \cap G \neq \emptyset$, so let $a \in C \cap G$. Since $a \in G \subseteq F$, and $y \in F$, we have $a \wedge y \in F$, and since F is a proper filter, $a \wedge y \neq 0$. Then $Cy = \bigvee\{s \in C | s \wedge y \neq 0\}$, and $a \in C$, so $a \leq Cy$. But $Cy \leq x$, so $a \leq x$, and $a \in G$, so $x \in G$. Therefore $F = G$, and so F is a minimal Cauchy filter. Note that this part of the proof did not require star refinements.

To show that every minimal Cauchy filter is regular, we will show that every Cauchy filter contains a regular one. Then if F is a minimal Cauchy filter, the regular Cauchy filter that it contains must be equal to itself, and so F is a regular Cauchy filter. We will show that $F^\circ = \{x \in L | a \lhd_{\mathcal{N}} x \text{ for some } a \in F\}$ is a regular Cauchy filter contained in F.

- F° is a filter: If $x \in F^\circ$ and $y \geq x$, then $a \lhd_{\mathcal{N}} x$ for some $a \in F$, so $a \lhd_{\mathcal{N}} y$, and so $y \in F^\circ$. If x and y are in F°, then $a \lhd_{\mathcal{N}} x$ and $b \lhd_{\mathcal{N}} y$ for some a and b in F, but then $a \wedge b \lhd_{\mathcal{N}} x \wedge y$, and $a \wedge b \in F$, so $x \wedge y \in F^\circ$.

- F° is Cauchy: Take any $C \in \mathcal{N}$, and we must show that $C \cap F^\circ \neq \emptyset$. There is a $B \in \mathcal{N}$ such that $B \leq^* C$, and F is a Cauchy filter, so $F \cap B \neq \emptyset$. Let $b \in F \cap B$, and then since $B \leq^* C$, $Bb \leq a$ for some $a \in C$. But then $b \lhd_{\mathcal{N}} a$ and $b \in F$, so $a \in F^\circ$. This means that $a \in C \cap F^\circ$, and so F° is Cauchy.

- F° is regular: Take $a \in F^\circ$, so there is an $x \in F$ such that $x \lhd_{\mathcal{N}} a$. Let C be a cover in $\mathcal{N}$ such that $Cx \leq a$. Then there is a cover $B \in \mathcal{N}$ such that $B \leq^* C$, so $B(Bx) \leq Cx \leq a$. Now let $b = Bx$, so $Bx \leq b$, which means that $x \lhd_{\mathcal{N}} b$. Then $x \in F$, so $b \in F^\circ$, and $Bb \leq a$, so $b \lhd_{\mathcal{N}} a$. This shows that F° is regular.

- $F^\circ \subseteq F$: Take $x \in F^\circ$, so that $a \lhd_{\mathcal{N}} x$ for some $a \in F$. From Lemma 1.2.46, we know that $a \prec x$, and so $a \leq x$, which means that $x \in F$ because $a \in F$. So $F^\circ \subseteq F$.

$\square$

We now consider the second generalisation, of the fact that for complete Hausdorff uniform spaces, every dense embedding is an isomorphism.

Definition 3.1.7. *1. For nearness frames L and M, a uniform homomorphism $h : M \to L$ is a* **surjection** *if it is onto in terms of both the nearness structure and the underlying set. That is, each $y \in L$ can be expressed as $y = h(x)$ for some $x \in M$, and each $B \in \mathcal{N}L$ can be expressed as $B = h[A]$ for some $A \in \mathcal{N}M$.*

2. A uniform frame L is **complete** *if every dense surjection onto it is an isomorphism.*

Remark 3.1.8. To show that an onto uniform homomorphism $h : M \to L$ is a surjection, it suffices to show that the set $\{h[A] | A \in \mathcal{N}M\}$ generates $\mathcal{N}L$. Then for each $B \in \mathcal{N}L$, there is an $A \in \mathcal{N}M$ such that $h[A] \leq B$. Now if $a \in A$, $h(a) \leq b$ for some $b \in B$, so $a \leq h_*(b)$, and we get $A \leq h_*[B]$. Since $A \in \mathcal{N}M$, this means that $h_*[B] \in \mathcal{N}M$, and $B = hh_*[B]$ because h is onto. So $B = h[C]$, where $C \in \mathcal{N}M$.

As in the case of Cauchy completeness, we need to add an extra condition to compensate for the loss of star refinements when moving from uniform to nearness frames. In order to form a completion of a nearness frame, it is not enough that the uniform structure of the completion is compatible with that of the original frame, in the sense that a surjection requires, but the structure of the frame that is to be completed should also generate the structure of the completion. This property is automatic for uniform frames, as we will see in the next section (Corollary 3.2.4), but when we remove star refinements it needs to be imposed separately.

Definition 3.1.9. *1. For nearness frames L and M, a frame homomorphism $h : M \to L$ is a* **strict surjection** *if it is a dense surjection that is strict in terms of both the nearness structure and the underlying set. That is, $h_*[L]$ generates M, and $\{h_*[C] | C \in \mathcal{N}L\}$ is a set of covers that generates $\mathcal{N}M$.*

2. A nearness frame L is **complete** *if every strict surjection onto it is an isomorphism.*

Remark 3.1.10. If $h : M \to L$ is a dense, onto uniform homomorphism such that $\{h_*[C] | C \in \mathcal{N}L\}$ is a set of covers that generates $\mathcal{N}M$, then h is a strict surjection. To see that it is a surjection, take $C \in \mathcal{N}L$, then $h_*[C]$ is a uniform cover of M, and $hh_*[C] = C$ because h is dense. Therefore it is not necessary to check the surjection property separately.

Note that all dense surjections are strict extensions, because they are dense onto frame homomorphisms between regular frames. Therefore strictness on the level of frames is automatic. The extra condition here is strictness on the structure level. We will see in the next section what happens when this condition is not imposed.

We now have two definitions of completeness, and we need a definition for the corresponding completions. In uniform spaces, a completion is a dense embedding into a complete uniform space. This translates to uniform frames as a dense surjection from a complete frame. For nearness frames, we need to use strict surjections instead of dense surjections.

Definition 3.1.11. *For nearness frames M and L, $h : M \to L$ is*

1. *a* **completion** *of L if it is a strict surjection and M is complete.*
2. *a* **Cauchy completion** *of L if it is a strict surjection and M is Cauchy complete.*

We will now construct completions of nearness frames. We begin with the Cauchy completion, because that follows the same strategy that was used for uniform spaces. Recall that a uniform space is complete if every Cauchy filter converges, which is equivalent to the condition that every minimal Cauchy filter converges, and so the completion of a Hausdorff uniform space is given by the set of minimal Cauchy filters on that space. We have a construction that mimics this in the pointfree setting — Hong's construction. We should construct the strict extension of a nearness frame L with respect to the set X that corresponds to minimal Cauchy filters.

We saw in Remark 3.1.5 that in the non-uniform case, we need to use regular Cauchy filters in place of all Cauchy filters. We also saw in the proof of Lemma 3.1.6 that all regular Cauchy filters are minimal (although minimal Cauchy filters need not be regular in the non-uniform case). Therefore, the set X that corresponds to minimal Cauchy filters on a uniform space is the set of regular Cauchy filters on a nearness frame.

Before we can use this set to construct a Cauchy completion of a nearness frame, we need a nearness structure on the frame $\tau_X L$ that will turn the strict extension $\tau_X L \to L$ into a strict surjection.

Definition 3.1.12. *For a nearness frame $(L, \mathcal{N}L)$, and any set X of filters on L, let $\mathcal{N}^*$ be the set generated (with respect to refinement) by set $\{\tau_*[C] | C \in \mathcal{N}L\}$.*

The set $\mathcal{N}^*$ is not necessarily a nearness structure on $\tau_X L$, and it does not even necessarily consist of covers, but we will see below under which conditions it is a nearness structure. First, we need a lemma that links the nearness structure of L to $\mathcal{N}^*$.

Lemma 3.1.13 ([3] Lemma 2)**.** *If $h : M \to L$ is a dense and onto frame homomorphism, then*

1. *for any cover C of L and elements x and a in L, $Cx \leq a$ if and only if $h_*[C]h_*(x) \leq h_*(a)$.*
2. *for any cover C of M and elements x and a in M, $Cx \leq a$ implies that $h[C]h(x) \leq h(a)$.*

Proof: For part 1, if $Cx \leq a$, then

$$\begin{aligned} h_*[C]h_*(x) &= \bigvee\{z \in h_*[C] | z \wedge h_*(x) \neq 0\} \\ &= \bigvee\{h_*(s) | s \in C, h_*(s) \wedge h_*(x) \neq 0\} \\ &= \bigvee\{h_*(s) | s \in C, h_*(s \wedge x) \neq 0\} \text{ because } h_* \text{ preserves meets.} \end{aligned}$$

Now we show that $h_*(s \wedge x) \neq 0$ if and only if $s \wedge x \neq 0$. On the one hand, if $s \wedge x = 0$, then $h_*(s \wedge x) = 0$ because h is dense. On the other hand, if $s \wedge x \neq 0$, then because h is onto, there is a $z \in M$ such that $h(z) = s \wedge x$, and $z \neq 0$ because $h(0) = 0$. Now $h_*(s \wedge x)$, being the join of such z, must be bigger than or equal to z, and z is strictly bigger than 0. This means that $\bigvee\{h_*(s) | s \in C, h_*(s \wedge x) \neq 0\} = \bigvee\{h_*(s) | s \in C, s \wedge x \neq 0\}$.

Now we have that $Cx = \bigvee\{s \in C | s \wedge x \neq 0\} \leq a$, so for each such s, $s \leq a$. But then $h_*(s) \leq h_*(a)$, since h_* preserves order, and so also $\bigvee\{h_*(s) | s \in C, s \wedge x \neq 0\} \leq h_*(a)$. But $\bigvee\{h_*(s) | s \in C, s \wedge x \neq 0\} = h_*[C]h_*(x)$, and so $h_*[C]h_*(x) \leq h_*(a)$, as required.

For the other direction, if $h_*[C]h_*(x) \leq h_*(a)$, then

$$\begin{aligned}
Cx &= \bigvee\{s \in C | s \wedge x \neq 0\} \\
&= \bigvee\{hh_*(s) | s \in C, hh_*(s) \wedge hh_*(x) \neq 0\} \text{ since } h \text{ is onto} \\
&= h\left(\bigvee\{h_*(s) | s \in C, h(h_*(s) \wedge h_*(x)) \neq 0\}\right) \\
&\leq h\left(\bigvee\{h_*(s) \in h_*[C] | h_*(s) \wedge h_*(x) \neq 0\}\right) \text{ because } h(0) = 0 \\
&= h(h_*[C]h_*(x)) \\
&\leq hh_*(a) \\
&= a \text{ because } h \text{ is onto.}
\end{aligned}$$

For part 2, if $Cx \leq a$, then

$$\begin{aligned}
h[C]h(x) &= \bigvee\{z \in h[C] | z \wedge h(x) \neq 0\} \\
&= \bigvee\{h(s) | s \in C, h(s) \wedge h(x) \neq 0\} \\
&= h\left(\bigvee\{s \in C | h(s \wedge x) \neq 0\}\right) \\
&= h\left(\bigvee\{s \in C | s \wedge x \neq 0\}\right) \text{ because } h \text{ is dense} \\
&= h(Cx) \\
&\leq h(a).
\end{aligned}$$

□

Now we can construct the Cauchy completion of a nearness frame.

Definition 3.1.14. *For a nearness frame $(L, \mathcal{N}L)$, let X be the set of regular Cauchy filters on L. Let $\tau_X L$ be called cL, and let $\tau : cL \to L$ be called c_L, or c.*

Lemma 3.1.15. *For a nearness frame $(L, \mathcal{N}L)$, $(cL, \mathcal{N}^*)$ is a nearness frame.*

Proof: First we need to show that $c_*[C]$ is a cover of cL for each $C \in \mathcal{N}L$. Now for any $C \in \mathcal{N}L$

$$\begin{aligned}
\bigvee c_*[C] &= \bigvee\{c_*(s) | s \in C\} \\
&= \bigvee\{(s, X_s) | s \in C\} \text{ from Remark 2.2.2} \\
&= \left(\bigvee_{s\in C} s, \bigcup_{s \in C} X_s\right) \\
&= \left(\bigvee C, X_C\right).
\end{aligned}$$

Now $\bigvee C = e$ because C is a cover of L, and $X_C = \{F \in X | F \cap C \neq \emptyset\} = X$, because X consists only of Cauchy filters, which meet every cover in $\mathcal{N}L$, including C.

Next we must show that $\mathcal{N}^*$ is a filter in CovcL with respect to refinement. We only need to show that $\{c_*[C] | C \in \mathcal{N}L\}$ is not empty, and that it is closed under finite meets, so that it is a filter base. Then $\mathcal{N}^*$ is a filter because it is generated by this filter base. We do not need to show that $\mathcal{N}^*$ is a proper filter, because the improper filter CovcL is a valid nearness on cL. For the first point, $\{c_*[C] | C \in \mathcal{N}L\}$ is not empty because $\mathcal{N}L$ is not empty. For the second, take A and B in $\mathcal{N}L$, then

$$\begin{aligned} c_*[A] \wedge c_*[B] &= \{c_*(a) | a \in A\} \wedge \{c_*(b) | b \in B\} \\ &= \{c_*(a) \wedge c_*(b) | a \in A, b \in B\} \\ &= \{c_*(a \wedge b) | a \in A, b \in B\} \text{ because } c_* \text{ preserves meets} \\ &= c_*[A \wedge B]. \end{aligned}$$

Finally, we must show that $\mathcal{N}^*$ is admissible. That is, for each $a \in cL$, we must show that $a = \bigvee\{x \in cL | x \triangleleft_{\mathcal{N}^*} a\}$. We will show that this is true for each $a \in c_*[L]$, and then it follows for the rest of cL because $c_*[L]$ generates cL. Now if $a \in c_*[L]$, then $a = c_*(s) = (s, X_s)$ for some $s \in L$. We know that $s = \bigvee\{x \in L | x \triangleleft_{\mathcal{N}L} s\}$ because $\mathcal{N}L$ is a nearness on L. We will show that $X_s = \bigcup\{X_x | x \triangleleft_{\mathcal{N}L} s\}$.

If $F \in X_s$, then $s \in F$, and each $F \in X$ is regular, so there is an $x \in F$ such that $x \triangleleft_{\mathcal{N}L} s$, and so $F \in X_x$ for this x. So $X_s \subseteq \bigcup\{X_x | x \triangleleft_{\mathcal{N}L} s\}$. For the other inclusion, if $F \in X_x$ for some x such that $x \triangleleft_{\mathcal{N}L} s$, then $x \in F$, and $x \leq s$. So $s \in F$, which means that $F \in X_s$. Therefore $a = (s, X_s) = \bigvee\{(x, X_x) | x \triangleleft_{\mathcal{N}L} s\} = \bigvee\{c_*(x) | x \triangleleft_{\mathcal{N}L} s\}$.

Now if $x \triangleleft_{\mathcal{N}L} s$, then $Cx \leq s$ for some $C \in \mathcal{N}L$, and from Lemma 3.1.13, this means that $c_*[C]c_*(x) \leq c_*(s)$. But $c_*[C] \in \mathcal{N}^*$, so we have that $c_*(x) \triangleleft_{\mathcal{N}^*} c_*(s)$. Therefore

$$\begin{aligned} a &= \bigvee\{c_*(x) | x \triangleleft_{\mathcal{N}L} s\} \\ &\leq \bigvee\{c_*(x) | c_*(x) \triangleleft_{\mathcal{N}^*} c_*(s)\} \\ &\leq a \text{ because } c_*(s) = a, \end{aligned}$$

and so $a = \bigvee\{c_*(x) | c_*(x) \triangleleft_{\mathcal{N}^*} a\}$. This is enough to show that $a = \bigvee\{z \in cL | z \triangleleft_{\mathcal{N}^*} a\}$, as required. □

Proposition 3.1.16 ([22] Theorem 10)**.** *For any nearness frame L, the map $c : cL \to L$ is a Cauchy completion of L.*

Proof: We need to show that cL is Cauchy complete, and that c is a strict surjection. The fact that c is a strict surjection is straightforward. We have constructed c to be a strict extension, so it is dense and onto. It is a uniform homomorphism because if $C \in \mathcal{N}^*$, then $C \geq c_*[A]$ for some $A \in \mathcal{N}L$, and then $c[C] \geq cc_*[A] = A$, so $c[C] \in \mathcal{N}L$. The strictness of c on the structured level follows from the definition of $\mathcal{N}^*$, and this is sufficient, by Remark 3.1.10.

To show that cL is Cauchy complete, we must show that every regular Cauchy filter on it converges. Let F be a regular Cauchy filter on cL, and we must show that for any cover C of cL, $F \cap C \neq \emptyset$. It is enough to show this for every basic uniform cover C, that is, the C such that $C = c_*[A]$ for some $A \in \mathcal{N}L$. In that case, $C = \{(a, X_a) | a \in A\}$.

We will first show that $c[F]$ is a regular Cauchy filter on L. Firstly, $c[F]$ is a filter:

- $F \neq \emptyset$, so $c[F] \neq \emptyset$.
- If $0 \in c[F]$, this would imply that $0 \in F$ because c is dense, but F is a proper filter, so $0 \notin c[F]$.
- If $c(a)$ and $c(b)$ are in $c[F]$, then a and b are in F, so $a \wedge b \in F$, and so $c(a \wedge b) = c(a) \wedge c(b) \in c[F]$.
- If $c(a) \in c[F]$ and $s \geq c(a)$, then $s = c(b)$ for some $b \in L$ because c is onto, and $c(b) \geq c(a)$. Now $s = c(b) = c(b) \vee c(a) = c(b \vee a)$, and $b \vee a \geq a$, so the fact that $a \in F$ means that $a \vee b \in F$, and so $s \in c[F]$.

Secondly, $c[F]$ is a Cauchy filter: Take any uniform cover B of L, then $B = c[c_*[B]]$ because c is onto. Now F is a Cauchy filter and $c_*[B] \in N^*$ by definition, so $F \cap c_*[B] \neq \emptyset$. But then $c[F] \cap c[c_*[B]] \neq \emptyset$, that is, $c[F] \cap B \neq \emptyset$.

Thirdly, $c[F]$ is a regular Cauchy filter: For $x \in c[F]$, $x = c(a)$ for some $a \in F$. Since F is regular, there is a $b \in F$ such that $b \triangleleft_{\mathcal{N}^*} a$. This means that there is a cover $B \in \mathcal{N}^*$ such that $Bb \leq a$. Now $c[B]$ is a cover in $\mathcal{N}L$ because c is a uniform map, and $c[B]c(b) \leq c(a)$, from Lemma 3.1.13. So if $c(b) = y$, we have $y \in c[F]$ since $b \in F$, and $c[B]y \leq x$, so $y \triangleleft_{\mathcal{N}L} x$. Therefore $c[F]$ is a regular Cauchy filter on L.

Now X is the set of all regular Cauchy filters on L, so $c[F] \in X$. We want to show that for a basic uniform cover $C = c_*[A]$, $C \cap F \neq \emptyset$. Since C is a cover, $\bigvee C = (e, X)$, but $C = \{(a, X_a) | a \in A\}$, so $\bigvee C = (\bigvee A, X_A)$. Therefore $X_A = X$, and so $c[F] \in X_A$, which means that $A \cap c[F] \neq \emptyset$. Let $a \in A \cap c[F]$. Then $c_*(a) \in C$ because $C = c_*[A]$. Also, $a \in c[F]$ means that $a = c(s)$ for some $s \in F$, and then $c_*(a) = c_*c(s) \geq s$, so $c_*(a) \in F$ also. Therefore $c_*(a) \in F \cap C$, showing that $F \cap C \neq \emptyset$, as required. $\square$

We have constructed a Cauchy completion of a frame by adding all its regular Cauchy filters, analogous to the way the completion of a uniform space is constructed. In fact, we can construct a completion of a nearness frame in a similar way — by adding all the general regular Cauchy filters.

Definition 3.1.17. *For a nearness frame $(L, \mathcal{N}L)$, a general filter $\varphi : L \to T$ is*

1. **Cauchy** *if $\varphi[C]$ is a cover of T for each $C \in \mathcal{N}L$.*
2. **regular** *if $\varphi(a) = \bigvee\{\varphi(x) | x \triangleleft a\}$ for each $a \in L$.*

Remark 3.1.18. These definitions are consistent with the definitions that we had for classical filters. For Cauchy filters, if φ_F is the characteristic function for the classical filter F, then $F \cap C \neq \emptyset$ means that there exists a $c \in C$ such that $\varphi_F(c) = 1$, and so $\bigvee \varphi_F[C] = 1$, which is the top of the frame $\mathbb{2}$, so $\varphi_F[C]$ is a cover. On the other hand, if $\varphi_F[C]$ is a cover, then there is a $c \in C$ such that $\varphi_F(c) = 1$, so $c \in F \cap C$.

For regularity, take $a \in L$, and we must consider two cases. If $a \notin F$, then $\varphi_F(a) = 0$. Now if $x \triangleleft a$, then $x \leq a$, and so $x \notin F$, which means that $\varphi_F(x) = 0$. Then $\varphi_F(a) = \bigvee\{\varphi_F(x) | x \triangleleft a\}$ in this case. In the second case, if $a \in F$, then $\varphi_F(a) = 1$, and then $\varphi_F(a) = \bigvee\{\varphi_F(x) | x \triangleleft a\}$ if and only if $\varphi_F(x) = 1$ for some $x \triangleleft a$. That is, $\varphi_F(a) = \bigvee\{\varphi_F(x) | x \triangleleft a\}$ if and only if there is an $x \in F$ such that $x \triangleleft a$, which is equivalent to F being regular.

Note that if $\varphi : L \to T$ is a frame homomorphism, then φ is a regular Cauchy filter. It is Cauchy because the image of any cover under a frame homomorphism is a cover, and it is regular because for all $a \in L$, $a = \bigvee\{x \in L | x \triangleleft a\}$ by admissibility, and frame homomorphisms preserve arbitrary joins.

The next lemma shows how these concepts relate to the homomorphisms we are interested in.

Lemma 3.1.19 ([3] Corollary 1 of Lemma 4 and Corollary of Lemma 2)**.** *For nearness frames M and L, if $h : M \to L$ is a strict surjection, then $h_* : L \to M$ is a regular Cauchy filter. On the other hand, if M is a frame and L is a nearness frame, and $h : M \to L$ is a strict extension such that $h_* : L \to M$ is a regular Cauchy filter, then $\{h_*[C] | C \in \mathcal{N}L\}$ generates a nearness on M that makes h into a strict surjection.*

Proof: If h is a strict surjection, then for each $C \in \mathcal{N}L$, $h_*[C] \in \mathcal{N}M$. But this means that $h_*[C]$ is a cover, and therefore h_* is a Cauchy filter. To show that h_* is regular, we must show that for each $a \in L$, $h_*(a) = \bigvee\{h_*(x)|x \triangleleft_{\mathcal{N}L} a\}$. Now $\mathcal{N}M$ is admissible, so for each $a \in L$, $h_*(a) = \bigvee\{z \in M|z \triangleleft_{\mathcal{N}M} h_*(a)\}$. But if $z \triangleleft_{\mathcal{N}M} h_*(a)$, we have that $Cz \leq h_*(a)$ for some $C \in \mathcal{N}M$, and so from Lemma 3.1.13, $h[C]h(z) \leq hh_*(a) = a$ because h is onto. Now $h[C] \in \mathcal{N}L$ because h is uniform, so $h(z) \triangleleft_{\mathcal{N}L} a$. So we have

$$\begin{aligned}
h_*(a) &= \bigvee\{z \in M|z \triangleleft_{\mathcal{N}M} h_*(a)\} \\
&\leq \bigvee\{h_*(h(z))|z \in M, z \triangleleft_{\mathcal{N}M} h_*(a)\} \text{ because } z \leq h_*(h(z)) \\
&\leq \bigvee\{h_*(h(z))|z \in M, h(z) \triangleleft_{\mathcal{N}L} a\} \text{ as shown above} \\
&\leq \bigvee\{h_*(x)|x \in L, x \triangleleft_{\mathcal{N}L} a\} \\
&\leq \bigvee\{h_*(x)|x \in L, x \leq a\} \text{ because } x \triangleleft_{\mathcal{N}L} a \text{ implies } x \leq a \\
&\leq h_*(a) \text{ because } x \leq a \text{ implies that } h_*(x) \leq h_*(a).
\end{aligned}$$

So all these inequalities are in fact equalities, and in particular, $h_*(a) = \bigvee\{h_*(x)|x \triangleleft_{\mathcal{N}L} a\}$, so h_* is a regular Cauchy filter.

Now for the other direction, if h_* is a Cauchy filter, then $h_*[C]$ is a cover of M for each $C \in \mathcal{N}L$, and so the idea of using $\{h_*[C]|C \in \mathcal{N}L\}$ as a base for the nearness on M makes sense. To show that it is in fact a nearness structure on M we must show that it is a filter base in Cov$M$, and that it is admissible. The fact that it is a filter base in Cov$M$ follows as in the proof of Lemma 3.1.15. Therefore it remains to show admissibility.

Now h_* is a regular filter, so for each $a \in L$, $h_*(a) = \bigvee\{h_*(x)|x \triangleleft_{\mathcal{N}L} a\}$. But if $x \triangleleft_{\mathcal{N}L} a$, then $Cx \leq a$ for some $C \in \mathcal{N}L$, and we know from Lemma 3.1.13 that $Cx \leq a$ if and only if $h_*[C]h_*(x) \leq h_*(a)$. By definition, $h_*[C] \in \mathcal{N}M$, and if $A \in \mathcal{N}M$ then there is a $C \in \mathcal{N}L$ such that $h_*[C] \leq A$, so we have that $x \triangleleft_{\mathcal{N}L} a$ if and only if $h_*(x) \triangleleft_{\mathcal{N}M} h_*(a)$. Therefore $h_*(a) = \bigvee\{h_*(x)|h_*(x) \triangleleft_{\mathcal{N}M} h_*(a)\}$, which is enough to prove admissibility because h is a strict extension.

Finally, the fact that h is a strict surjection follows from the definition of the nearness structure on M, as shown in the proof of Proposition 3.1.16. □

Now we can generalise Lemma 3.1.15.

Lemma 3.1.20 ([9] proposition 5)**.** *If X is a set of general regular Cauchy filters on a nearness frame $(L, \mathcal{N}L)$, then $(\tau_X L, \mathcal{N}^*)$ is a nearness frame, and $\tau_X L \to L$ is a strict surjection.*

Proof: We will show that if X is a set of general regular Cauchy filters, then $\tau_* : L \to \tau_X L$ is a regular Cauchy filter, and so the result follows from Lemma 3.1.19.

In Remark 2.2.9, we saw that for $a \in L$, $\tau_*(a) = (a, (\varphi(a))_{\varphi \in X})$. So for $C \in \mathcal{N}L$,

$$\begin{aligned}
\bigvee \tau_*[C] &= \bigvee \{\tau_*(a) | a \in C\} \\
&= \bigvee \{(a, (\varphi(a))_{\varphi \in X}) | a \in C\} \\
&= \left(\bigvee_{a \in C} a, \left(\bigvee_{a \in C} \varphi(a) \right)_{\varphi \in X} \right) \\
&= \left(\bigvee C, \left(\bigvee \varphi[C] \right)_{\varphi \in X} \right).
\end{aligned}$$

Now $\bigvee C = e$ because C is a cover, so $\bigvee \tau_*[C] = e$ if and only if $\bigvee \varphi[C] = e$ for each $\varphi \in X$. So τ_* is a Cauchy filter if and only if each $\varphi \in X$ is a Cauchy filter.

Now for each $a \in L$, $a = \bigvee \{x \in L | x \lhd_{\mathcal{N}L} a\}$, because $\mathcal{N}L$ is admissible. Further, for $\varphi \in X$, $\varphi(a) = \bigvee \{\varphi(x) | x \lhd_{\mathcal{N}L} a\}$ for all $a \in L$ if and only if φ is a regular filter. So $\tau_*(a) = (a, (\varphi(a))_{\varphi \in X}) = \bigvee \{(x, (\varphi(x))_{\varphi \in X}) | x \lhd_{\mathcal{N}L} a\} = \bigvee \{\tau_*(x) | x \lhd_{\mathcal{N}L} a\}$ for all $a \in L$ if and only if each $\varphi \in X$ is a regular filter. That is, τ_* is a regular filter, if and only if each $\varphi \in X$ is a regular filter.

Therefore, in the case where each $\varphi \in X$ is a regular Cauchy filter, $\tau_* : L \to \tau_X L$ is a regular Cauchy filter, and so $\tau : \tau_X L \to L$ is a strict surjection, from Lemma 3.1.19. $\square$

Remark 3.1.21. In the proof above, we showed that if X is a set of Cauchy filters on L, then $\tau_* : L \to \tau_X L$ is a Cauchy filter. Suppose that $h : M \to L$ is a uniform homomorphism, and let X_L be the set of all general Cauchy filters on L, and similarly for X_M. Consider $\tau_L : \tau_{X_L} L \to L$ and $\tau_M : \tau_{X_M} M \to M$. The map $\tau_{L*} h : M \to \tau_{X_L} L$ is a Cauchy filter on M, because if $C \in \mathcal{N}M$, then $h[C] \in \mathcal{N}L$ since h is uniform, and so $\tau_{L*} h[C]$ is a cover of $\tau_{X_L} L$ because τ_{L*} is a Cauchy filter. So $\tau_{L*} h$ is in X_M. Then from Lemma 2.3.13, there is a frame homomorphism $\tilde{h} : \tau_{X_M} M \to \tau_{X_L} L$ such that $\tilde{h} \tau_{M*} = \tau_{L*} h$. But

$$\begin{aligned}
\tau_L \tilde{h} \tau_{M^*} &= \tau_L \tau_{L*} h \\
&= h \text{ because } \tau_L \text{ is onto} \\
&= h \tau_M \tau_{M*} \text{ because } \tau_M \text{ is onto.}
\end{aligned}$$

Now τ_M is a strict extension, so this implies that $\tau_L \tilde{h} = h \tau_M$, or the square below commutes:

$$\begin{tikzcd}
\tau_{X_M} M \arrow[r, "\tilde{h}"] \arrow[d, "\tau_M"'] & \tau_{X_L} L \arrow[d, "\tau_L"] \\
M \arrow[r, "h"'] & L
\end{tikzcd}$$

Now we can construct a completion of a nearness frame.

Proposition 3.1.22 ([9] Proposition 7, [3] Proposition 2)**.** *If X is the set of all general regular Cauchy filters on a nearness frame $(L, \mathcal{N}L)$, then $(\tau_X L, \mathcal{N}^*) \to (L, \mathcal{N}L)$ is a completion of L, which is unique up to isomorphism.*

Proof: From the previous lemma we know that the map $(\tau_X L, \mathcal{N}^*) \to (L, \mathcal{N}L)$ is a strict surjection, so to show that it is a completion, we just need to show that $\tau_X L$ is complete. We must show that if $f : M \to \tau_X L$ is a strict surjection for some nearness frame M, then f is a uniform isomorphism.

Let $h = \tau f : M \to L$. We will show that h is a strict surjection.

- h is dense: If $h(x) = 0$, then $\tau f(x) = 0$. We have that $f(x) = 0$ because τ is dense, and so $x = 0$ because f is dense.

- h is onto: If $x \in L$, then $x = \tau(y)$ for some $y \in \tau_X L$, because τ is onto. But then $y = f(z)$ for some $z \in M$ because f is onto, so $x = \tau f(z) = h(z)$.

- h is uniform: For any cover $A \in \mathcal{N}M$, $f[A] \in \mathcal{N}^*$ because f is uniform, so $\tau f[A] \in \mathcal{N}L$ because τ is uniform. So $h[A] \in \mathcal{N}L$.

- h is a surjection: If $C \in \mathcal{N}L$, then $C = \tau[B]$ for some $B \in \mathcal{N}^*$ because τ is a surjection. Then $B = f[A]$ for some $A \in \mathcal{N}M$ because f is a surjection. So $C = \tau f[A] = h[A]$, and $\mathcal{N}L = \{h[A] | A \in \mathcal{N}M\}$.

- h is a strict surjection: For $C \in \mathcal{N}L$, $h_*[C] = (\tau f)_*[C] = f_*[\tau_*[C]]$. Now $\tau_*[C] \in \mathcal{N}^*$ by definition, so $f_*[\tau_*[C]] \in \mathcal{N}M$ because f is a strict surjection, and so $h_*[C] \in \mathcal{N}M$.

  Any cover $A \in \mathcal{N}M$ is refined by $f_*[B]$ for some cover $B \in \mathcal{N}^*$ because f is a strict surjection. In turn, B is refined by $\tau_*[C]$ for some cover $C \in \mathcal{N}L$ because τ is a strict surjection. So $f_*\tau_*[C]$ refines A, which means that A is refined by $(\tau f)_*[C] = h_*[C]$. So $\{h_*[C] | C \in \mathcal{N}L\}$ generates $\mathcal{N}M$.

We see that in general, the composition of two strict surjections is a strict surjection, and that this is also true for surjections which are not strict. In particular, we have from Lemma 3.1.19 that $h_* : L \to M$ is a regular Cauchy filter, and so $h_* \in X$.

Now from Lemma 2.3.13, there exists a unique frame homomorphism $\tilde{h} : \tau_X L \to M$ such that $\tilde{h}\tau_* = h_*$, which is onto because h is a strict extension. Then using Remark 2.3.16, this is equivalent to saying that $h\tilde{h} = \tau$. Now $h = \tau f$, so $\tau = \tau f\tilde{h}$, and τ is a dense homomorphism between regular frames, which means that it is monic, so $f\tilde{h} = \mathrm{id}_{\tau_X L}$. Then $f\tilde{h}f = f$, and f is also a dense homomorphism between regular frames, so f is also monic, and so $\tilde{h}f = \mathrm{id}_M$. So we see that $\tilde{h}$ is the inverse of f, and so f is a frame isomorphism. Then f is a uniform isomorphism because it is a uniform surjection.

Now to show that this completion $(\tau_X L, \mathcal{N}^*) \to (L, \mathcal{N}L)$ is unique, we must show that whenever $h : M \to L$ is a strict surjection with complete M, then M is isomorphic to $\tau_X L$. For a strict surjection $h : M \to L$, we saw in the previous paragraph that there is an onto frame homomorphism $\tilde{h} : \tau_X L \to M$ such that $\tilde{h}\tau_* = h_*$, or equivalently, $h\tilde{h} = \tau$. We will show that $\tilde{h}$ is a strict surjection, so that $\tilde{h}$ is an isomorphism by the completeness of M.

- $\tilde{h}$ is dense: If $\tilde{h}(x) = 0$, then $h\tilde{h}(x) = 0$, and so $\tau(x) = 0$. Now τ is dense, so $x = 0$, and so we have that $\tilde{h}$ is dense.

- $\tilde{h}$ is uniform: If $A \in \mathcal{N}^*$, then $A \geq \tau_*[B]$ for some $B \in \mathcal{N}L$, by definition of $\mathcal{N}^*$. So $\tilde{h}[A] \geq \tilde{h}\tau_*[B] = h_*[B]$. Now h is a strict surjection, so $h_*[B] \in \mathcal{N}M$. Therefore $\tilde{h}[A]$, which is refined by $h_*[B]$, is in $\mathcal{N}M$, and $\tilde{h}$ is uniform.

- $\tilde{h}$ is a strict surjection: If $C \in \mathcal{N}M$, then $C \geq h_*[B]$ for some $B \in \mathcal{N}L$, since h is a strict surjection. Then
$$\tau_*[B] \leq \tilde{h}_*\tilde{h}\tau_*[B] = \tilde{h}_* h_*[B] \leq \tilde{h}_*[C].$$
So $\tilde{h}_*[C] \in \mathcal{N}^*$ for each $C \in \mathcal{N}M$.

  Now, if $A \in \mathcal{N}^*$, then $A \geq \tau_*[B]$ for some $B \in \mathcal{N}L$. But $\tau_* = (h\tilde{h})_* = \tilde{h}_* h_*$, so $A \geq \tilde{h}_*[h_*[B]]$. Now $h_*[B] \in \mathcal{N}M$ because h is a strict surjection, so A is refined by $\tilde{h}_*[C]$ where $C \in \mathcal{N}M$.

For Remark 3.1.10, this is enough to see that the onto frame homomorphism $\tilde{h} : \tau_X L \to M$ is a strict surjection, and so it is an isomorphism by the completeness of M. Therefore the completion of a frame L is unique. $\square$

Definition 3.1.23. *For a nearness frame L, and X the set of general regular Cauchy filters on L, $\tau : \tau_X L \to L$ is* **the completion** *of L, and will be written as $\gamma_L : \gamma L \to L$.*

Notice the similarity between this construction and the Cauchy completion that we described previously — the Cauchy completion was constructed using regular Cauchy filters, and this completion uses general regular Cauchy filters. Now just as Cauchy complete nearness frames are those where all regular Cauchy filters converge, we can describe complete nearness frames as those for which all general regular Cauchy filters converge.

Definition 3.1.24 ([5])**.** *A general filter $\varphi : L \to T$* **converges** *if $\varphi[C]$ is a cover of T for every cover C of L.*

Remark 3.1.25. This definition of convergence is consistent with the definition of convergence for classical filters. There we had that a filter F converges if it meets every cover C of L. But this means that $\varphi_F(c) = 1$ for some $c \in C$, or equivalently, $\bigvee \varphi_F[C] = 1$. Hence $\varphi_F[C]$ is a cover of $\mathbb{2}$.

In [9], there is another concept of convergence. There, a general filter φ is said to be **strongly convergent** if there is a frame homomorphism $h \leq \varphi$. In older papers, such as [3] and [22], strong convergence is called convergence. In general, if a filter is strongly convergent than it is convergent, but if the filter is regular, then the converse also holds.

Lemma 3.1.26 ([5] Lemma 4.5)**.** *In a nearness frame L, a general regular filter $\varphi : L \to T$ converges if and only if it is a frame homomorphism.*

Proof: If φ is a frame homomorphism, then for any cover C of L, $\bigvee \varphi[C] = \varphi\left(\bigvee C\right) = \varphi(e) = e$. Therefore every frame homomorphism converges. For the other direction, we need to show that if φ converges, then for any $S \subseteq L$, $\bigvee \varphi[S] = \varphi(\bigvee S)$. Now if $s \in S$, $s \leq \bigvee S$, so $\varphi(s) \leq \varphi(\bigvee S)$, and so also $\bigvee \varphi[S] \leq \varphi(\bigvee S)$. So it remains to show the other inequality.

Now $\varphi(\bigvee S) = \bigvee\{\varphi(x) | x \triangleleft \bigvee S\}$ because φ is a regular filter. If $x \triangleleft \bigvee S$, then from Lemma 1.2.46, $x \prec \bigvee S$, and so $x^* \vee \bigvee S = e$. So $\{x^*\} \cup S$ is a cover of L, and so $\varphi(x^*) \vee \bigvee \varphi[S] = e$, since φ converges. But also, φ preserves finite meets, so $\varphi(x^*) \wedge \varphi(x) = \varphi(x^* \wedge x) = \varphi(0) = 0$. This shows that $\varphi(x) \prec \bigvee \varphi[S]$, so $\varphi(x) \leq \bigvee \varphi[S]$. This is true for all $x \triangleleft \bigvee S$, so it is also true for $\bigvee\{\varphi(x) | x \triangleleft \bigvee S\}$, that is, $\varphi(\bigvee S) \leq \bigvee \varphi[S]$. □

Proposition 3.1.27 ([5] Proposition 4.5)**.** *A nearness frame is complete if and only if every general regular Cauchy filter on it converges.*

Proof: Suppose that L is a complete nearness frame, so that L is isomorphic to $\gamma L = \tau_X L$, where X is the set of general regular Cauchy filters on L. Let φ be a general regular Cauchy filter on L, so that $\varphi \in X$. Then from Lemma 2.3.13, there is a unique frame homomorphism $\tilde{\varphi} : \tau_X L \to T_\varphi$ such that $\varphi = \tilde{\varphi}\tau_*$. But since τ is an isomorphism, τ_* is its inverse, which means that τ_* is a frame homomorphism. Therefore φ is also a frame homomorphism, and so it converges.

On the other hand, if every general regular Cauchy filter on L converges, then from the lemma above, every general regular Cauchy filter on L is a frame homomorphism. But then from Corollary 2.3.18, the completion of L, which is $\gamma L = \tau_X L$ where X is the set of general regular Cauchy filters on L, is isomorphic to the strict extension $\tau_Y L$ where $Y = \emptyset$. But $\tau_Y L \subseteq L \times \prod_{\varphi \in Y} T_\varphi = L \times \{\emptyset\} \cong L$. So $\tau_Y L \cong L$, which means that L is already a complete frame. $\square$

We saw in Proposition 2.3.7 that the strict extensions that can be obtained by Hong's construction using classical filters are those that are relatively spatial. By moving from classical to general filters, we move from being able to obtain only relatively spatial strict extensions, to all strict extensions. So the Cauchy completion of a nearness frame is relatively spatial, while the completion need not be. However, due to the similarity between the sets of filters used to construct these two, they remain related.

Proposition 3.1.28 ([7] Lemma 4)**.** *For a nearness frame L, the Cauchy completion $c_L : cL \to L$ is the relatively spatial reflection of the completion $\gamma_L : \gamma L \to L$.*

Proof: From Proposition 2.3.8, we just need to show that the classical filter trace of γ_L is the set of classical regular Cauchy filters on L. On the one hand, if F is a classical filter on L that is regular and Cauchy, then it is shown in Remark 3.1.18 that φ_F is a regular and Cauchy general filter. So φ_F is an element of X, where $\gamma L = \tau_X L$, and then from Corollary 2.3.14, φ_F is a general trace filter of γ_L. Now Corollary 2.3.12 says that since φ_F is a filter trace of γ_L with codomain $\mathbb{2}$, F is a classical trace filter of γ_L. So it remains to show the other direction, that if F is a classical trace filter of γ_L, then F is a regular Cauchy filter.

If F is a classical trace filter of γ_L, then $F = \gamma_L[P]$ for some completely prime filter P on γL. Take $a \in F$, so $a = \gamma_L(s)$ for some $s \in P$. Now by admissibility, $s = \bigvee\{t \in \gamma L | t \triangleleft_{\mathcal{N}(\gamma L)} s\}$, and P is completely prime, so there is a $t \in \gamma L$ such that $t \triangleleft_{\mathcal{N}(\gamma L)} s$ and $t \in P$. Now from Lemma 3.1.13, this implies that $\gamma_L(t) \triangleleft_{\mathcal{N}L} \gamma_L(s)$, because γ_L is a uniform map, and so we have that $\gamma_L(t) \triangleleft a$ and $\gamma_L(t) \in F$. So we have that F is regular.

To show that F is Cauchy, take $C \in \mathcal{N}L$, and we must show that $C \cap F \neq \emptyset$. We have that $\gamma_{L*}[C] \in \mathcal{N}(\gamma L)$, because γ_L is strict, so $\gamma_{L*}[C]$ is a cover of γL, that is, $\bigvee \gamma_{L*}[C] = e$. Now P is non-empty, so $e \in P$, and so $\bigvee \gamma_{L*}[C] \in P$. Then the fact that P is completely prime means that $\gamma_{L*}(c) \in P$ for some $c \in C$. But then $\gamma_L(\gamma_{L*}(c)) = c \in \gamma_L[P] = F$, so $c \in F \cap C$. Therefore F is Cauchy. $\square$

Now we can explore the relationship between theses two notions of completeness.

Lemma 3.1.29 ([23] Corollary 2.4)**.** *If a nearness frame is complete, then it is Cauchy complete.*

Proof: Take a complete nearness frame L, and consider $c_L : cL \to L$, the Cauchy completion of L. We saw in Proposition 3.1.16 that cL is Cauchy complete and c_L is a strict surjection. Now L is complete, so any strict surjection onto it is an isomorphism, and in particular, c_L is an isomorphism. So L is Cauchy complete. $\square$

However, the converse is false. Cauchy completeness is a distinct concept to completeness, even in the uniform context.

Example 3.1.30 ([4] Appendix 3)**.** The set $\mathbb{R}$ of real numbers is complete as a metric space with its usual metric, and so, as shown in [31] Theorem 39.4, it is a complete uniform space, when equipped with its metric uniformity. Now the product of complete uniform spaces is complete, as shown in [31] Theorem 39.6, so the uniform space $\mathbb{R}^m$, with its usual uniformity, is complete for any power m.

Now if a uniform space X is complete, then we will show that the open set frame $\mathcal{O}X$, with the uniformity generated by the uniform open covers of X, is a Cauchy complete uniform frame. Firstly, the uniform covers of $\mathcal{O}X$ have the same base as the uniform covers of X, and the definition of a Cauchy filter is the same in both cases, so a filter $\mathcal{F}$ is a Cauchy filter on $\mathcal{O}X$ if and only if it is a Cauchy filter on X. If X is complete, then a Cauchy filter $\mathcal{F}$ converges to a point $x \in X$. This means that the open neighbourhood filter $\mathcal{N}_x$ is contained in $\mathcal{F}$. But neighbourhood filters are completely prime, so if $\mathcal{S}$ is a cover of $\mathcal{O}X$, then $\bigvee \mathcal{S} = X \in \mathcal{N}_x$, and so there is a set $S \in \mathcal{S}$ such that $S \in \mathcal{N}_x$. But then $S \in \mathcal{F}$ also, so $\mathcal{F}$ meets the cover $\mathcal{S}$. So we see that every Cauchy filter on $\mathcal{O}X$ converges, showing that $\mathcal{O}X$ is Cauchy complete. It follows from this that since $\mathbb{R}^m$ is a complete uniform space for any power m, $\mathcal{O}(\mathbb{R}^m)$ is a Cauchy complete uniform frame for any power m.

In [28], Corollary 2.3, it is shown that the product of uncountably many copies of a separable T_1 topological space Y is normal if and only if Y is a compact T_2 space. The space $Y = \mathbb{R}$ is a separable T_1 space, but it is not compact, and so the product of uncountably many copies of it is not normal. Therefore $\mathbb{R}^m$ is not normal for uncountable m. Now [15] Theorem 5.1.5 shows that any paracompact topological space is normal, but $\mathbb{R}^m$ is not normal, and so it is not paracompact.

Now a topological space is paracompact if every open cover has a locally finite refinement, and a frame is paracompact if every cover has the same property. Therefore a space is paracompact if and only if its open set frame is. Since $\mathbb{R}^m$ is not paracompact, $\mathcal{O}(\mathbb{R}^m)$ is also not paracompact. But Proposition 1 of [12] shows that if a regular frame has a complete uniformity, it must be paracompact. So since $\mathcal{O}(\mathbb{R}^m)$ is not paracompact, it cannot have a complete uniformity. Therefore the uniform frame $\mathcal{O}(\mathbb{R}^m)$ with the uniformity generated by the open covers of the product uniformity of the metric uniformity of $\mathbb{R}$ is a uniform frame that is Cauchy complete, but not complete.

Remark 3.1.31. The above example also shows that it is possible to have a strict extension that is not relatively spatial. This is because the uniform frame $\mathcal{O}(\mathbb{R}^m)$ considered above is Cauchy complete, but not complete, and we saw in Proposition 3.1.28 that the Cauchy completion of a nearness frame is the relatively spatial reflection of the completion. In this case, since the Cauchy completion is not complete, the completion of L is different to its relatively spatial reflection, showing the completion was not already relatively spatial.

Another point that comes from the example above is the fact that Cauchy completions are not unique. For example, $\mathcal{O}(\mathbb{R}^m)$ considered above is Cauchy complete, so the identity map is a Cauchy completion. But it also has a completion, and we saw in Lemma 3.1.29 that any complete nearness frame is Cauchy complete, so the completion of $\mathcal{O}(\mathbb{R}^m)$ is also a Cauchy completion of it.

Since we have multiple Cauchy completions of a given nearness frame L, we can consider the set of all Cauchy completions of it. We can form a category whose objects are the Cauchy completions of L, and morphisms are uniform homomorphisms that form commuting triangles with the Cauchy completions. We conclude this section with one final connection between the completion of a nearness frame and its Cauchy completions, with respect to this category.

Lemma 3.1.32. *In the category of Cauchy completions of a nearness frame L, the completion of L is the initial object.*

Proof: We must show that there is a unique morphism from the initial object to any other object in the category. That is, if $h : M \to L$ is a Cauchy completion of L, we must find a uniform homomorphism $g : \gamma L \to M$ such that $\gamma_L = hg$. If such a g exists, then it is unique because h is a dense homomorphism between regular frames, and is therefore monic.

$$
\begin{tikzcd}
 & M \arrow[d, "h"] \\
\gamma L \arrow[ur, dotted, "g"] \arrow[r, "\gamma_L"'] & L
\end{tikzcd}
$$

Now $\gamma_M : \gamma M \to M$ is the completion of M, which is a strict surjection. Also, $h : M \to L$ is a strict surjection, and we saw in the proof of Proposition 3.1.22 that the composition of two strict surjections is a strict surjection. Therefore $h\gamma_M$ is a strict surjection.

$$
\begin{tikzcd}
\gamma M \arrow[r, "\gamma_M"] & M \arrow[d, "h"] \\
\gamma L \arrow[u, "i"] \arrow[ur, dotted, "g"] \arrow[r, "\gamma_L"'] & L
\end{tikzcd}
$$

Now γM is complete and $h\gamma_M : M \to L$ is a strict surjection, so $h\gamma_M$ is a completion of L. Then by uniqueness of the completion, there is a uniform isomorphism $i : \gamma L \to \gamma M$ such that $\gamma_L = h\gamma_M i$. Let $g = \gamma_M i$. Then $\gamma_L = hg$, as required. $\square$

3.2 Completions and weak completions of nearness frames

We have defined a nearness frame to be complete if every strict surjection onto it is an isomorphism. However, we saw in the previous section that for a uniform frame, it is complete if every dense surjection onto it is an isomorphism. When we moved from uniform to nearness frames, we needed the additional condition of strict surjections in order to ensure that the structure of the completion was based on the structure of the original frame. However, we did not show that this was really a restriction, that is, that there even exist dense surjections that are not strict. Since these will be important in this section, we present such an example now.

Example 3.2.1 ([10] Proposition 3.3)**.** Let L be the frame $\mathcal{O}\mathbb{Q}$, the frame consisting of the usual open sets of the set of rational numbers. Equip L with the uniformity $\mathcal{N}L$ consisting of the open covers of the metric uniformity of $\mathbb{Q}$. Let M be the frame $\mathcal{O}\mathbb{R}$, the usual open sets of the set of real numbers. Equip M with the nearness $\mathcal{N}M$ generated by the metric uniformity of $\mathbb{R}$ as well as the cover $\mathcal{C}$ consisting of the following open sets for each $m \in \mathbb{Z}$:

$$U_m = (2m, 2m+2),\ W_m = (2m-1, 2m+1) \setminus \left\{2m + \frac{\lambda}{2|m|+1}\right\}$$

where $\lambda \in (0,1)$ is a fixed irrational number. Finally, let $h : M \to L$ be the map that takes $U \in \mathcal{O}\mathbb{R}$ to $U \cap \mathbb{Q}$.

The map h is dense because if $h(U) = \emptyset$, then $U \cap \mathbb{Q} = \emptyset$, but this means that $U = \emptyset$ because the rationals are dense in the reals. In addition, h is onto because every open set of rational numbers can be expressed as $U \cap \mathbb{Q}$ for some set $U \in \mathcal{O}\mathbb{R}$.

The image of any cover in the metric uniformity of $\mathcal{O}\mathbb{R}$ is clearly a cover in $\mathcal{N}L$, but to see that h is a uniform homomorphism we must check that $h[\mathcal{C}] \in \mathcal{N}L$. Now λ is irrational, so $2m + \frac{\lambda}{2|m|+1}$ is irrational, and so $h(W_m) = h(2m-1, 2m+1)$ for each $m \in \mathbb{Z}$. So if $\mathcal{B}_1 \cap \mathbb{Q}$ is the uniform cover in $\mathcal{N}L$ consisting of all open unit intervals, $\mathcal{B}_1 \cap \mathbb{Q}$ refines $h[\mathcal{C}]$, so that $h[\mathcal{C}] \in \mathcal{N}L$. So we see that h is a uniform homomorphism.

For every uniform cover $\mathcal{A}$ in $\mathcal{N}L$, there exists an $\varepsilon > 0$ such that $\mathcal{A} \geq \mathcal{B}_\varepsilon \cap \mathbb{Q}$, where $\mathcal{B}_\varepsilon$ is the basic ε-cover in $\mathcal{N}M$. Since $h_*[\mathcal{B}_\varepsilon \cap \mathbb{Q}] = \mathcal{B}_\varepsilon$, and $\mathcal{B}_\varepsilon \cap \mathbb{Q} \leq \mathcal{A}$, $\mathcal{B}_\varepsilon \leq h_*[\mathcal{A}]$, so $h_*[\mathcal{A}] \in \mathcal{N}M$. Now $hh_*[\mathcal{A}] = \mathcal{A}$, so we see that h is a dense surjection.

However, this map h is not a strict surjection, because the cover $\mathcal{C}$ cannot be refined by $h_*[\mathcal{A}]$ for any $\mathcal{A} \in \mathcal{N}L$. We show that $h_*[\mathcal{A}]$ cannot refine $\mathcal{C}$ for any basic cover $\mathcal{A}$, and then the right adjoint of any cover refined by $\mathcal{A}$ will also not be able to refine $\mathcal{C}$, since right adjoints preserve order.

If $\mathcal{A}$ is a basic cover, then $\mathcal{A} = \mathcal{B}_\varepsilon \cap \mathbb{Q}$ for some $\varepsilon > 0$, so that $h_*[\mathcal{A}] = \mathcal{B}_\varepsilon$. Now if $\mathcal{B}_\varepsilon \leq \mathcal{C}$, then for each $m \in \mathbb{Z}$, the interval $(2m - \frac{\varepsilon}{2}, 2m + \frac{\varepsilon}{2})$ is contained in some member of $\mathcal{C}$. It cannot be contained in any U_m, since it overlaps an even number, which the U_m do not. If $|m| > \frac{\lambda}{\varepsilon} - \frac{1}{2}$, then $\frac{\lambda}{2|m|+1} < \frac{\varepsilon}{2}$, and then the singleton $\{2m + \frac{\lambda}{2|m|+1}\} \subseteq (2m - \frac{\varepsilon}{2}, 2m + \frac{\varepsilon}{2})$. This means that it is not possible that $(2m - \frac{\varepsilon}{2}, 2m + \frac{\varepsilon}{2}) \subseteq W_m$ either, and so $(2m - \frac{\varepsilon}{2}, 2m + \frac{\varepsilon}{2})$ is not contained in any member of $\mathcal{C}$. So $\mathcal{C}$ is not refined by $\mathcal{B}_\varepsilon = h_*[\mathcal{A}]$.

We have therefore found a map $h : M \to L$ that is a dense surjection, but is not a strict surjection.

In the uniform case, it was not necessary to consider only strict surjections, because in that environment, strictness on the structured level is automatic. To show that every dense surjection between uniform frames is in fact a strict surjection, we will need some definitions and an important lemma.

Definition 3.2.2. *1. If $(L, \mathcal{N}L)$ is a nearness frame, and $C \in \mathcal{N}L$, define*

$$\widehat{C} = \{x \in L | x^{**} \leq a \text{ for some } a \in C\}.$$

Then L is **smooth** *if $\widehat{C} \in \mathcal{N}L$ for every $C \in \mathcal{N}L$.*

2. *For a nearness frame L, let $\beta L = \{x^{**}|x \in L\}$, let $\beta_L : L \to \beta L$ be the map sending x to x^{**}, and let βL have the nearness generated by $\{\beta_L[C]|C \in \mathcal{N}L\}$. Then βL is a Boolean algebra, and $\beta_L : L \to \beta L$ is the* **Booleanisation** *of L.*

Lemma 3.2.3 ([10] Proposition 3.5)**.** *For any nearness frame L, the following are equivalent:*

1. *L is smooth.*
2. *For any nearness frame M, if $h : L \to M$ is a dense surjection, then it is a strict surjection.*
3. *$\beta_L : L \to \beta L$ is a strict surjection.*

Proof: *1.* $\Rightarrow$ *2.*: Suppose that L is a smooth nearness frame, and let $h : L \to M$ be a dense surjection for some nearness frame M. For $B \in \mathcal{N}M$, $B = h[C]$ for some $C \in \mathcal{N}L$ because h is a surjection. Now $\widehat{C} \leq C$ because if $x \in \widehat{C}$, then $x \leq x^{**} \leq a$ for some $a \in C$, so $h[\widehat{C}] \leq h[C] = B$. Now $\widehat{C} \leq h_*h[\widehat{C}] \leq h_*[B]$, and $\widehat{C} \in \mathcal{N}L$ because L is smooth, so $h_*[B] \in \mathcal{N}L$ for all $B \in \mathcal{N}M$

Now take $C \in \mathcal{N}L$. Since $\widehat{C} \in \mathcal{N}L$, $h[\widehat{C}] \in \mathcal{N}M$, because h is a uniform homomorphism. We claim that $h_*[h[\widehat{C}]] \leq C$, so that h a strict surjection.

Take $x \in \widehat{C}$. Then

$$\begin{aligned} h(h_*h(x) \wedge x^*) &= hh_*h(x) \wedge h(x^*) \\ &= h(x) \wedge h(x^*) \\ &= h(x \wedge x^*) \\ &= h(0) \\ &= 0. \end{aligned}$$

Now h is dense, so this means that $h_*h(x) \wedge x^* = 0$. But $x^{**} = \bigvee\{y \in L|y \wedge x^* = 0\}$, so $h_*h(x) \leq x^{**}$. But $x \in \widehat{C}$, so $x^{**} \leq a$ for some $a \in C$, so also $h_*h(x) \leq a$ for this $a \in C$. Therefore $h_*h[\widehat{C}] \leq C$, as claimed.

2. $\Rightarrow$ *3.*: We will show that $\beta_L : L \to \beta L$ is a dense surjection, so that it is a strict surjection by the assumption.

- β_L is onto, because for any $x^{**} \in \beta L$, x^{**} is an element of L, and $(x^{**})^{**} = x^{**}$.
- If $x^{**} = 0$, then since $x \leq x^{**}$, $x = 0$, so β_L is dense.

- If $A \in \mathcal{N}L$, then $\beta_L[A] \in \mathcal{N}\beta L$, by definition, so β_L is uniform.
- The uniformity on βL is generated by $\{\beta_L[C] | C \in \mathcal{N}L\}$, so from Remark 3.1.8, β_L is a surjection.

3. $\Rightarrow$ *1.*: Take $C \in \mathcal{N}L$. Then $C \geq \beta_{L*}[B]$ for some $B \in \mathcal{N}\beta L$, because β_L is a strict surjection, by assumption. But β_L is a dense surjection, so $B = \beta_L[A]$ for some $A \in \mathcal{N}L$. So $C \geq \beta_{L*}\beta_L[A]$, for some $A \in \mathcal{N}L$. Note that for $x \in L$, $\beta_{L*}(x^{**}) = x^{**}$, since $(x^{**})^{**} = x^{**}$, and if $y^{**} = x^{**}$, then $y \leq y^{**} = x^{**}$, so x^{**} is the biggest element of L that gets sent to x^{**}. So for $x \in A$, $\beta_L(x) = x^{**}$, and $\beta_{L*}(x^{**}) = x^{**}$, so $\beta_{L*}\beta_L[A] = \{x^{**} | x \in A\}$. Now $\beta_{L*}\beta_L[A] \leq C$, so for any $x \in A$, $x^{**} \leq c$ for some $c \in C$. But then $x \in \widehat{C}$. So $A \leq \widehat{C}$, and $A \in \mathcal{N}L$, so $\widehat{C} \in \mathcal{N}L$, proving that L is smooth. $\square$

Corollary 3.2.4. *If L and M are uniform frames and $h : L \to M$ is a dense surjection, then h is a strict surjection.*

Proof: We show that L is smooth, so that the result follows from *1.* $\Rightarrow$ *2.* in the previous lemma. Take $C \in \mathcal{N}L$, then there is a cover $A \in \mathcal{N}L$ such that $A \leq^* C$ because L is uniform. Now if $a \in A$, then $Aa \leq c$ for some $c \in C$, so we have $a \lhd c$. But then from Lemma 1.2.46, $a \prec c$, so $a^* \vee c = e$. Now $a^{**} \wedge a^* = 0$, so a^* is a separating element for $a^{**} \prec c$, which implies that $a^{**} \leq c$. This means that $a \in \widehat{C}$. So $A \leq \widehat{C}$, and $A \in \mathcal{N}L$, so $\widehat{C} \in \mathcal{N}L$ also, proving that L is smooth. $\square$

We see from this that the definition of completeness, that every strict surjection is an isomorphism, is consistent with the definition of completeness of uniform frames, that every dense surjection is an isomorphism. In [10] however, the concept of completeness for uniform frames is transported into the nearness setting, without the addition of strictness. This results in a new concept of completeness for nearness frames, which we now explore.

Definition 3.2.5. *A nearness frame L is* **weakly complete** *if every dense surjection onto it is an isomorphism. A map $h : M \to L$ is a* **weak completion** *of L if it is a dense surjection and M is weakly complete.*

Note that the definition of a weak completion is consistent with the definition of a weakly complete nearness frame. That is, strict surjections have been replaced with dense surjections in both instances. It is clear from the definition that if L is weakly complete then it is complete, because if every dense surjection onto L is an isomorphism, then every strict surjection, which is a special dense surjection, is an isomorphism. This makes the terminology seem rather unfortunate, but we will see that the converse also holds, that is, every complete nearness frame is weakly complete, so the two concepts are equivalent. However, in order to show that, we first need another construction.

Lemma 3.2.6 ([10] Lemma 3.1)**.** *Let $(L, \mathcal{N})$ be a nearness frame, and let l be a nucleus on L such that $l(0) = 0$. Then*

$$\mathcal{N}^l = \{C \in \text{Cov}L | l[A] \leq C \textit{ for some } A \in \mathcal{N}\}$$

is also a nearness on L.

Proof: First we show that $\mathcal{N}^l$ is a filter in Cov$L$. If C and D are covers in $\mathcal{N}^l$, there are covers A and B in $\mathcal{N}$ such that $l[A] \leq C$ and $l[B] \leq D$. Then

$$\begin{aligned} l[A \wedge B] &= \{l(a \wedge b) | a \in A, b \in B\} \\ &= \{l(a) \wedge l(b) | a \in A, b \in B\} \text{ because nuclei preserve meets} \\ &= l[A] \wedge l[B]. \end{aligned}$$

So $l[A \wedge B] = l[A] \wedge l[B] \leq C \wedge D$. Now $A \wedge B \in \mathcal{N}$, so $C \wedge D \in \mathcal{N}^l$. Further, if $C \in \mathcal{N}^l$ and $C \leq D$, then there is a cover $A \in \mathcal{N}$ such that $l[A] \leq C \leq D$, so $D \in \mathcal{N}^l$ also. Therefore $\mathcal{N}^l$ is a filter.

Now for admissibility, take $a \in L$, and then since L is regular, $a = \bigvee\{y \in L | y \prec a\}$. For each y, $y = \bigvee\{x \in L | x \lhd_{\mathcal{N}} y\}$ by the admissibility of $\mathcal{N}$. Therefore $a = \bigvee\{x \in L | x \lhd_{\mathcal{N}} y \text{ for some } y \prec a\}$. We show that if $x \lhd_{\mathcal{N}} y$ and $y \prec a$, then $x \lhd_{\mathcal{N}^l} a$, so that $a = \bigvee\{x \in L | x \lhd_{\mathcal{N}^l} a\}$, as required.

Take x and y in L such that $x \lhd_{\mathcal{N}} y$ and $y \prec a$. Then there is a cover $A \in \mathcal{N}$ such that $Ax \leq y$. Suppose there is an $s \in A$ with the property that $l(s) \wedge x \neq 0$. Then $l(l(s) \wedge x) \geq l(s) \wedge x \neq 0$, by the closure property of nuclei. So $l(l(s)) \wedge l(x) \neq 0$ because l preserves meets, and so $l(s) \wedge l(x) \neq 0$, because nuclei are idempotent. But then $l(s \wedge x) \neq 0$, which means that $s \wedge x \neq 0$, since $l(0) = 0$.

Now $Ax = \bigvee\{t \in A | t \wedge x \neq 0\}$, so $s \leq Ax$. But $Ax \leq y$, so $s \leq y$, and then since $y \wedge y^* = 0$, $s \wedge y^* = 0$ also. Now $l(s \wedge y^*) = l(0) = 0$, so $l(s) \wedge l(y^*) = 0$, and then $l(s) \wedge y^* = 0$ because $y^* \leq l(y^*)$.

Now $y \prec a$, so $y^* \vee a = e$, and we just saw that $l(s) \wedge y^* = 0$, so we have that $l(s) \prec a$, with y^* the separating element. This shows that $l(s) \leq a$ for all $s \in A$ such that $l(s) \wedge x \neq 0$. Therefore

$$\begin{aligned} \bigvee\{l(s) | s \in A \text{ and } l(s) \wedge x \neq 0\} &= \bigvee\{l(s) \in l[A] | l(s) \wedge x \neq 0\} \\ &= l[A]x \\ &\leq a. \end{aligned}$$

Finally, we must show that $l[A] \in \mathcal{N}^l$. Firstly, $l[A]$ is a cover because for each $a \in A$, $l(a) \geq a$, so $\bigvee l[A] \geq \bigvee A = e$. Secondly, $l[A] \leq l[A]$, where $A \in \mathcal{N}$, so $l[A] \in \mathcal{N}^l$. Therefore we have that $x \triangleleft_{\mathcal{N}^l} a$, as required. $\square$

Corollary 3.2.7. *Let $h : M \to L$ be a dense surjection, and let K be a nearness frame with the same underlying frame as M, and nearness given by $\mathcal{N}M^{h_*h}$. Then if $f : K \to M$ acts identically, $hf : K \to L$ is a strict surjection.*

Proof: Firstly, h_*h is a nucleus on M, and $h_*h(0) = \bigvee\{x \in M | h(x) = h(0)\} = 0$ because $h(0) = 0$, and h is dense, so if $h(x) = 0$, then $x = 0$. Therefore h_*h satisfies the requirements for l in the previous lemma, and the notation $\mathcal{N}M^{h_*h}$ is appropriate.

Now hf is dense and onto because h is dense and onto, and f acts identically. For a cover $C \in \mathcal{N}M^{h_*h}$, $C \geq h_*h[A]$ for some $A \in \mathcal{N}M$. But $A \leq h_*h[A] \leq C$, so $C \in \mathcal{N}M$ also. Then $hf[C] = h[C] \in \mathcal{N}L$ because h is uniform. Therefore hf is a uniform homomorphism.

Using Remark 3.1.10, it remains to show that $\{(hf)_*[B] | B \in \mathcal{N}L\}$ is a set of covers generating $\mathcal{N}M^{h_*h}$. If $B \in \mathcal{N}L$, there is an $A \in \mathcal{N}M$ such that $h[A] = B$ because h is a surjection, so $h_*h[A] \leq h_*[B]$, and so $h_*[B] \in \mathcal{N}M^{h_*h}$. Now take $C \in \mathcal{N}M^{h_*h}$, so $C \geq h_*h[A]$ for some $A \in \mathcal{N}M$. Then $h[A] \in \mathcal{N}L$, because h is a uniform map, so $C \geq h_*[B]$, where $B = h[A] \in \mathcal{N}L$. Since f acts identically, f_* also does, so $h_*[B] = f_*h_*[B] = (hf)_*[B]$, and so we have $C \geq (hf)_*[B] \in \mathcal{N}M^{h_*h}$, where $B \in \mathcal{N}L$. This shows that $\{(hf)_*[B] | B \in \mathcal{N}L\}$ is a set of covers generating $\mathcal{N}M^{h_*h}$, and so hf is indeed a strict surjection. $\square$

Definition 3.2.8. *For a dense surjection $h : M \to L$, call $f : K \to M$ defined above the* **strict reduct** *of h.*

Proposition 3.2.9 ([10] Proposition 4.1)**.** *If a nearness frame is complete, then it is weakly complete.*

Proof: We need to show that if a nearness frame L is complete, then any dense surjection $h : M \to L$ is an isomorphism. If h is a dense surjection, then to show that h is a uniform isomorphism, it is only necessary to show that it is a frame isomorphism. But h is onto, so it only remains to show that it is one-one. So take x and y in M such that $h(x) = h(y)$, and consider the strict reduct $f : K \to M$ of h. Since K and M have the same underlying frames, x and y are elements of K, and $f(x) = x$ and $f(y) = y$. So $hf(x) = hf(y)$. Now $hf : K \to L$ is a strict surjection onto L, so it is an isomorphism, because L is complete. Therefore hf is one-one, so $x = y$, as required. $\square$

We see from this that a nearness frame is complete if and only if it is weakly complete. We will therefore only use the term complete from now on. However, the concept of completion is still different in the two cases. It is clear that a completion of a nearness frame L, being a strict surjection from a complete frame, is also a weak completion, since strict surjections are dense surjections. However, it is not the case that a weak completion of a nearness frame is necessarily a completion. To show this, we need another result about smooth nearness frames.

Lemma 3.2.10 ([19] Lemma 4.8(a)). *If M and L are nearness frames and $h : M \to L$ is a dense surjection, then if M is a smooth nearness frame, L is also smooth.*

Proof: For $C \in \mathcal{N}L$, we must show that $\widehat{C} \in \mathcal{N}L$. Since h is a surjection, there is a cover $A \in \mathcal{N}M$ such that $h[A] = C$. Now M is smooth, so $\widehat{A} \in \mathcal{N}M$, and so $h[\widehat{A}] \in \mathcal{N}L$. We will show that $h[\widehat{A}] \le \widehat{C}$, so that $\widehat{C} \in \mathcal{N}L$.

If $y \in \widehat{A}$, then $y \in M$ and $y^{**} \le a$ for some $a \in A$. So $h(y^{**}) \le h(a)$. Now

$$\begin{aligned}
h(y^*) &= h\left(\bigvee\{w \in M | w \wedge y = 0\}\right) \\
&= \bigvee\{h(w) | w \wedge y = 0\} \\
&= \bigvee\{h(w) | h(w \wedge y) = 0\} \text{ because } h \text{ is dense} \\
&= \bigvee\{h(w) | h(w) \wedge h(y) = 0\} \\
&= \bigvee\{z \in L | z \wedge h(y) = 0\} \text{ because } h \text{ is onto} \\
&= (h(y))^*.
\end{aligned}$$

So $h(y^{**}) = (h(y^*))^* = ((h(y))^*)^* = (h(y))^{**}$. Therefore $(h(y))^{**} \le h(a) \in h[A]$. This means that $h(y) \in \widehat{h[A]}$. Now $h[A] = C$, so $\widehat{h[A]} = \widehat{C}$, and so $h(y) \in \widehat{C}$. Therefore $h[\widehat{A}] \le \widehat{C}$, as required. $\square$

Now we can present an example of a weak completion that is not a completion.

Example 3.2.11. Consider the nearness frame M defined in Example 3.2.1. Since we were able to find a map h from M that is a dense surjection but is not a strict surjection, it follows from Lemma 3.2.3 *2.* $\Rightarrow$ *1.* that M is not smooth. Now $\gamma_M : \gamma M \to M$ is a dense surjection, and so if γM was smooth, then M would also be smooth, from the lemma above. But since M is not smooth, γM is not smooth either. Then Lemma 3.2.3 *3.* $\Rightarrow$ *1.* says that $\beta_{\gamma M} : \gamma M \to \beta\gamma M$ is not a strict surjection, even though we saw in the proof of Lemma 3.2.3 *2.* $\Rightarrow$ *3.* that it is a dense surjection. Now γM is complete, so $\beta_{\gamma M} : \gamma M \to \beta\gamma M$ is a weak completion of $\beta\gamma M$, but it is not a completion, because $\beta_{\gamma M}$ is not a strict surjection.

We see from this that there is no unique weak completion of a given nearness frame, just as there was no unique Cauchy completion. For example, the nearness frame $\beta\gamma M$ discussed above has at least two weak completions — $\beta_{\gamma M} : \gamma M \to \beta\gamma M$, and $\gamma_{\beta\gamma M} : \gamma\beta\gamma M \to \beta\gamma M$. Just as we did with Cauchy completions, we can consider the category of weak completions of a given nearness frame L, where the morphisms are uniform homomorphisms that form commuting triangles with the weak completions. Then as was the case for Cauchy completions, the completion of L is the initial object in this category of weak completions.

Lemma 3.2.12 ([10] proposition 4.2)**.** *In the category of weak completions of a nearness frame L, the completion of L is the initial object.*

Proof. We must show that there is a unique morphism from the initial object to any other object in the category. That is, if $h : M \to L$ is a weak completion of L, we must find a uniform homomorphism $g : \gamma L \to M$ such that $\gamma_L = hg$. Since h is a dense homomorphism between regular frames, it is monic, and so if such a g exists, it is unique.

$$\begin{tikzcd}
 & M \arrow[d, "h"] \\
\gamma L \arrow[ur, dotted, "g"] \arrow[r, "\gamma_L"'] & L
\end{tikzcd}$$

Let $f : K \to M$ be the strict reduct of h. Then $hf : K \to L$ is a strict surjection, and $\gamma_K : \gamma K \to K$, the completion of K, is a strict surjection. So $hf\gamma_K$ is a strict surjection, as we saw in the proof of Proposition 3.1.22.

$$\begin{tikzcd}
\gamma K \arrow[r, "\gamma_K"] & K \arrow[r, "f"] & M \arrow[d, "h"] \\
 & \gamma L \arrow[ul, "i"] \arrow[ur, dotted, "g"] \arrow[r, "\gamma_L"'] & L
\end{tikzcd}$$

Now γK is complete and $hf\gamma_K : \gamma K \to L$ is a strict surjection, so $hf\gamma_K$ is a completion of L. Then by uniqueness of the completion, there is a uniform isomorphism $i : \gamma L \to \gamma K$ such that $\gamma_L = hf\gamma_K i$. If we let $g = f\gamma_K i$, then $\gamma_L = hg$, as required. $\square$

Remark 3.2.13. Note that the uniform homomorphism g found in this lemma is dense and onto, because it is composed of dense, onto uniform homomorphisms, f, γ_K and i.

So far, for a given nearness frame L, we have constructed a Cauchy completion cL, and a completion γL. We will now describe a weak completion of L that is different to the completion.

Definition 3.2.14. *For a nearness frame L, let wL have the same underlying frame as γL, and let $\mathcal{N}^w = \{A \in \mathrm{Cov}\gamma L | \gamma_L[A] \in \mathcal{N}L\}$. Let $w_L : wL \to L$ act the same as $\gamma_L : \gamma L \to L$.*

Remark 3.2.15. The nearness structure $\mathcal{N}(\gamma L)$ is contained in $\mathcal{N}^w$. This is because if $A \in \mathcal{N}(\gamma L)$, then $A = \gamma_{L*}[C]$ for some $C \in \mathcal{N}L$. But then $\gamma_L[A] = \gamma_L\gamma_{L*}[C] = C \in \mathcal{N}L$, so $A \in \mathcal{N}^w$.

Proposition 3.2.16 ([10] Lemma 4.3)**.** *For a nearness frame L, $(wL, \mathcal{N}^w)$ is a complete nearness frame, and $w_L : wL \to L$ is a dense surjection, so that $w_L : wL \to L$ is a weak completion of L.*

Proof: First we show that $\mathcal{N}^w$ is a nearness on wL.

- If $A \in N^w$ and $A \leq B$, then $\gamma_L[A] \leq \gamma_L[B]$, and $\gamma_L[A] \in \mathcal{N}L$, so $\gamma_L[B] \in \mathcal{N}L$, which means that $B \in \mathcal{N}^w$.
- If A and B are in $\mathcal{N}^w$, then $\gamma_L[A]$ and $\gamma_L[B]$ are in $\mathcal{N}L$, and so $\gamma_L[A] \wedge \gamma_L[B] = \gamma_L[A \wedge B] \in \mathcal{N}L$, which means that $A \wedge B \in \mathcal{N}^w$.
- For admissibility, take any $a \in wL = \gamma L$. Then $a = \bigvee\{x \in \gamma L | x \triangleleft_{\mathcal{N}(\gamma L)} a\}$, since $\mathcal{N}(\gamma L)$ is a nearness on γL. But if $x \triangleleft_{\mathcal{N}(\gamma L)} a$, then there exists a $C \in \mathcal{N}(\gamma L)$ such that $Cx \leq a$, and we saw in Remark 3.2.15 that $\mathcal{N}(\gamma L) \subseteq \mathcal{N}^w$, so $C \in \mathcal{N}^w$. This means that $x \triangleleft_{\mathcal{N}^w} a$, and so $a = \bigvee\{x \in \gamma L | x \triangleleft_{\mathcal{N}^w} a\}$.

Next we show that w_L is a dense surjection.

- Since w_L is a strict extension by construction, it is dense and onto.
- For $A \in \mathcal{N}^w$, $w_L[A] = \gamma_L[A] \in \mathcal{N}L$, by definition, so w_L is uniform.
- For $C \in \mathcal{N}L$, $C = w_L[w_{L*}[C]]$, and $w_{L*}[C] = \gamma_{L*}[C] \in \mathcal{N}(\gamma L) \subseteq \mathcal{N}^w$. So $C = w_L[A]$ for $A \in \mathcal{N}^w$, showing that w_L is a surjection.

Now it remains to show that wL is complete. Consider the following diagram:

$$\begin{tikzcd}
K \arrow[r, "j"] \arrow[d, "l"'] & \gamma wL \arrow[d, "\gamma_{wL}"] \\
\gamma L \arrow[ur, "g"] \arrow[r, "i"] \arrow[dr, "\gamma_L"'] & wL \arrow[d, "w_L"] \\
 & L
\end{tikzcd}$$

Since γL and wL have the same underlying frames, the map $i : \gamma L \to wL$ acts identically, and it is a uniform homomorphism because $\mathcal{N}(\gamma L) \subseteq \mathcal{N}^w$. Also, w_L was defined to be identical to γ_L, so $\gamma_L = w_L i$, and the bottom triangle commutes.

Regarding the middle triangle, γwL is a complete nearness frame and $w_L \gamma_{wL} : \gamma wL \to L$ is the composition of two dense surjections, and so is a dense surjection. Therefore, $w_L \gamma_{wL} : \gamma wL \to L$ is a weak completion of L, and so by Lemma 3.2.12 there is a uniform homomorphism $g : \gamma_L \to \gamma wL$ such that $\gamma_L = w_L \gamma_{wL} g$. But $\gamma_L = w_L i$, so $w_L i = w_L \gamma_{wL} g$. Now w_L is a dense homomorphism between regular frames, so it is monic, which means that $i = \gamma_{wL} g$, and the middle triangle commutes.

Let K be the same underlying frame as γwL, and define $\mathcal{N}K = \{g[A] | A \in \mathcal{N}(\gamma L)\}$. Since g is onto by Remark 3.2.13, $\mathcal{N}K$ is the image of a nearness structure under an onto frame homomorphism, which is a nearness structure on K by Lemma 1.2.45. Now g is a uniform homomorphism, so if $C \in \mathcal{N}K$, then $C = g[A]$ for some $A \in \mathcal{N}(\gamma L)$, so $g[A] \in \mathcal{N}(\gamma wL)$. Therefore the frame isomorphism $j : K \to \gamma wL$ is a uniform homomorphism.

For $a \in K$, let $l(a) = \gamma_{wL}(a)$. This is well defined because K and γwL have the same underlying frames, as do γL and wL. Now l is a uniform homomorphism because for $C \in \mathcal{N}K$, $C = g[A]$ for some $A \in \mathcal{N}(\gamma L)$, so $l[C] = l[g[A]] = \gamma_{wL}[g[A]] = i[A] = A \in \mathcal{N}(\gamma L)$. In addition, the top triangle commutes precisely because the middle one does.

Now γ_{wL} is a dense surjection because it is a completion, and l acts the same as γ_{wL}, so l is also dense and onto. Further, for $A \in \mathcal{N}(\gamma L)$, $g[A] \in \mathcal{N}K$, and then $l[g[A]] = j[A] = A$ because the top triangle commutes, so l is a dense surjection. But γL is complete, and we saw from Proposition 3.2.9 that this means that γL is weakly complete, so l is an isomorphism. But l acts the same as γ_{wL}, so γ_{wL} must also be an isomorphism. Then since wL is isomorphic to its completion, wL is complete. $\square$

Remark 3.2.17. The weak completion of a complete nearness frame is an isomorphism, because if L is complete, then $\gamma_L : \gamma L \to L$ is an isomorphism, and so

$w_L : wL \to L$ is also an isomorphism because γL and wL have the same underlying frames, and w_L acts like γ_L.

3.3 Completions as coreflections

It may be somewhat surprising that the completion of a nearness frame is not a coreflection of the category of nearness frames and uniform homomorphisms. We will show this below, and also present some coreflection results that do hold. In order to do this we will need to define some objects intermediate between nearness frames and uniform frames.

Definition 3.3.1. 1. *A nearness frame $(L, \mathcal{N}L)$ is called* **strong** *if for each $C \in \mathcal{N}L$, the cover*

$$\check{C} = \{x \in L | x \triangleleft a \text{ for some } a \in C\}$$

is also in $\mathcal{N}L$.

2. *A nearness frame $(L, \mathcal{N}L)$ is called* **erc**, *which stands for having* enough regular Cauchy *filters, if for each general Cauchy filter $\varphi : L \to T$ there is a general regular Cauchy filter $\psi : L \to T$ such that $\psi(a) \leq \varphi(a)$ for every $a \in L$.*

Remark 3.3.2. Smooth nearness frames were defined in Definition 3.2.2, and we saw that every uniform frame is smooth. Strong nearness frames are intermediate between uniform and smooth frames.

Proof: Suppose that L is a uniform frame and $C \in \mathcal{N}L$. Then C has a star refinement B, that is, there is a $B \in \mathcal{N}L$ such that for each $b \in B$, $Bb \leq c$ for some $c \in C$. This means that $b \triangleleft c$, so that $b \in \check{C}$, and so $B \leq \check{C}$. Therefore $\check{C} \in \mathcal{N}L$, and L is strong.

On the other hand, if L is a strong nearness frame and $C \in \mathcal{N}L$, then $\check{C} \in \mathcal{N}L$, and if $b \in \check{C}$, then $b \triangleleft c$ for some $c \in C$. Now from Lemma 1.2.46, $b \triangleleft c$ implies that $b \prec c$, so $b^* \vee c = e$. But $b^* \wedge b^{**} = 0$, so $b^{**} \prec c$, with b^* as a separating element. So we see that $b \in \widehat{C}$, which gives that $\check{C} \leq \widehat{C}$, so $\widehat{C} \in \mathcal{N}L$, and L is smooth. □

Lemma 3.3.3 ([10] Lemma 2.2)**.** *Every strong nearness frame is erc.*

Proof: Suppose that L is a strong nearness frame and $\varphi : L \to T$ is a Cauchy filter. Let $\varphi^\circ : L \to T$ be defined by $\varphi^\circ(a) = \bigvee\{\varphi(x) | x \triangleleft a\}$ for each $a \in L$. We claim that φ° is a regular Cauchy filter such that $\varphi^\circ \leq \varphi$, which would show that L is erc.

- $\varphi^\circ \leq \varphi$: For each $a \in L$, if $x \lhd a$, then $x \leq a$, so $\varphi(x) \leq \varphi(a)$. Then

$$\varphi^\circ(a) = \bigvee\{\varphi(x)|x \lhd a\} \leq \varphi(a).$$

- φ° is a filter (that is, a bounded meet-semilattice homomorphism):
  - $\varphi^\circ(0) = \bigvee\{\varphi(x)|x \lhd 0\} = \bigvee\{\varphi(0)\} = \varphi(0) = 0$.
  - $\varphi^\circ(e) = \bigvee\{\varphi(x)|x \lhd e\} \geq \varphi(e)$ because $e \lhd e$. But $\varphi(e) = e$, so $\varphi^\circ(e) = e$.
  - φ° preserves finite meets:

$$\begin{aligned}
\varphi^\circ(a) \wedge \varphi^\circ(b) &= \bigvee\{\varphi(x)|x \lhd a\} \wedge \bigvee\{\varphi(y)|y \lhd b\} \\
&= \bigvee\{\varphi(x) \wedge \varphi(y)|x \lhd a, y \lhd b\} \\
&= \bigvee\{\varphi(x \wedge y)|x \lhd a, y \lhd b\} \text{ because } \varphi \text{ is a filter.} \\
&= \bigvee\{\varphi(z)|z \lhd a \wedge b\} \\
&= \varphi^\circ(a \wedge b).
\end{aligned}$$

- φ° is Cauchy: For $C \in \mathcal{N}L$, $\check{C} \in \mathcal{N}L$ because L is strong, so

$$\begin{aligned}
\bigvee \varphi^\circ[C] &= \bigvee\left\{\bigvee\{\varphi(x)|x \lhd a\}\middle|a \in C\right\} \\
&= \bigvee\{\varphi(x)|x \lhd a \text{ for some } a \in C\} \\
&= \bigvee \varphi[\check{C}] \\
&= e
\end{aligned}$$

  because φ is Cauchy. Therefore $\varphi^\circ[C]$ is a cover, and so φ° is Cauchy.

- φ° is regular: We need to show that for each $a \in L$, $\varphi^\circ(a) = \bigvee\{\varphi^\circ(x)|x \lhd a\} = \varphi^{\circ\circ}(a)$. We have already seen that $\varphi^\circ \leq \varphi$, so similarly, $\varphi^{\circ\circ} \leq \varphi^\circ$. So it remains to show that $\varphi^\circ \leq \varphi^{\circ\circ}$.

  Take $a \in L$ and let $x \lhd a$, so that $Cx \leq a$ for some $C \in \mathcal{N}L$. Take $s \in C$. If $s \wedge x \neq 0$, then $s \leq a$, because $Cx = \bigvee\{s \in C|s \wedge x \neq 0\} \leq a$. If $s \wedge x = 0$, then $s \leq x^* = \bigvee\{z \in L|z \wedge x = 0\}$. Therefore $C \leq \{a, x^*\}$, which is a cover because $x \lhd a$, which implies that $x \prec a$. Since $C \in \mathcal{N}L$, $\{a, x^*\} \in \mathcal{N}L$ also.

Now φ° is a Cauchy filter, so $\varphi^{\circ\circ}$ is also one, and so we have that $\varphi^{\circ\circ}(a) \vee \varphi^{\circ\circ}(x^*) = e$. But $\varphi^{\circ\circ} \leq \varphi^\circ \leq \varphi$, so $\varphi^{\circ\circ}(a) \vee \varphi(x^*) = e$. Now $x \wedge x^* = 0$, and φ is a filter, so $\varphi(x) \wedge \varphi(x^*) = 0$, and so $\varphi(x^*)$ is a separating element to show that $\varphi(x) \prec \varphi^{\circ\circ}(a)$. So we have that $\varphi(x) \leq \varphi^{\circ\circ}(a)$ whenever $x \lhd a$, which means that $\varphi^\circ(a) = \bigvee\{\varphi(x) | x \lhd a\} \leq \varphi^{\circ\circ}(a)$. Since this holds for all $a \in L$, $\varphi^\circ \leq \varphi^{\circ\circ}$, as required.

□

Next we want to give an example of a nearness frame that is erc but not strong. We will do that in Example 3.3.7 below, but in order to do so we first need a characterisation of erc nearness frames.

Definition 3.3.4. *Let L be a nearness frame and let X be the set of all general Cauchy filters on L. Then $\sigma_L : \tau_X L \to \gamma L$ is the projection map onto the general regular Cauchy filters. That is, for $(\bigvee U, (\bar{\varphi}(U))_{\varphi \in X}) \in \tau_X L$, if Y is the set of general regular Cauchy filters on L, then $\sigma_L(\bigvee U, (\bar{\varphi}(U))_{\varphi \in X}) = (\bigvee U, (\bar{\varphi}(U))_{\varphi \in Y})$. Since σ_L is a projection, it is an onto frame homomorphism. Since the first component is unchanged by σ_L, the map $\tau : \tau_X L \to L$ factors through σ_L, that is, $\tau = \gamma_L \sigma_L$.*

Lemma 3.3.5 ([10] Proposition 2.9)**.** *A nearness frame L is an erc nearness frame if and only if $\sigma_L : \tau_X L \to \gamma L$ has a right inverse, where X is the set of general Cauchy filters on L.*

Proof: Suppose that $\sigma_L : \tau_X L \to \gamma L$ has a right inverse, $r : \gamma L \to \tau_X L$, and let $\varphi : L \to T$ be a Cauchy filter. Then $\varphi \in X$, so by Lemma 2.3.13, there is a frame homomorphism $h : \tau_X L \to T$ such that $\varphi = h\tau_*$. Let $\psi = hr\gamma_{L*}$, which is a regular Cauchy filter because h and r are frame homomorphisms, and γ_{L*} is a regular Cauchy filter from Lemma 3.1.19. Therefore it remains to show that $\psi \leq \varphi$, to prove that L is erc.

Now $\psi = hr\gamma_{L*}$, and $\varphi = h\tau_*$, so we will show that $r\gamma_{L*} \leq \tau_*$. For $a \in L$, $\tau_*(a) = \bigvee\{x \in \tau_X L | \tau(x) = a\}$, so if we can show that $\tau(r\gamma_{L*}(a)) = a$, then $r\gamma_{L*}(a) \leq \tau_*(a)$. Now

$$\begin{aligned} \tau r \gamma_{L*} &= \gamma_L \sigma_L r \gamma_{L*} \text{ because } \gamma_L \sigma_L = \tau \\ &= \gamma_L \gamma_{L*} \text{ because } \sigma_L r = \mathrm{id}_{\gamma L} \\ &= \mathrm{id}_L \text{ because } \gamma_L \text{ is onto.} \end{aligned}$$

So $\tau(r\gamma_{L*}(a)) = a$, as required.

On the other hand, suppose that L is erc, and consider $\tau_*^\circ : L \to \tau_X L$ (using the notation of Lemma 3.3.3). We will show that τ_*° is a regular Cauchy filter on L, and then use Lemma 2.3.13 to find a candidate for r, the right inverse of σ_L.

$$\begin{aligned}\tau_*^\circ(a) &= \bigvee\{\tau_*(x) | x \lhd a\} \\ &= \bigvee\{(x, (\varphi(x))_{\varphi\in X}) | x \lhd a\} \text{ from Remark 2.2.9} \\ &= \left(\bigvee_{x\lhd a} x, \left(\bigvee_{x\lhd a} \varphi(x)\right)_{\varphi\in X}\right) \\ &= (a, (\varphi^\circ(a))_{\varphi\in X}).\end{aligned}$$

Now for $\varphi \in X$, φ is a Cauchy filter on L, and L is erc, so there is a regular Cauchy filter $\psi \leq \varphi$. Then $\psi^\circ \leq \varphi^\circ$, and ψ is regular, so by definition, $\psi^\circ = \psi$. Therefore $\psi \leq \varphi^\circ$, which implies that φ° is a Cauchy filter because ψ is. Similarly, $\psi \leq \varphi^{\circ\circ}$, so $\varphi^{\circ\circ}$ is a Cauchy filter.

Now in the proof of Lemma 3.3.3, in order to show that φ° is regular, we needed to show that φ° and $\varphi^{\circ\circ}$ are Cauchy, which we did using the fact that L was strong. Since we have now shown that in our case as well, we can use that proof to show that φ° is regular, because the rest of the proof did not depend on L being strong. Therefore φ° is a regular Cauchy filter for each $\varphi \in X$.

Then for $C \in \mathcal{N}L$,

$$\begin{aligned}\bigvee \tau_*^\circ[C] &= \bigvee\{(a, (\varphi^\circ(a))_{\varphi\in X}) | a \in C\} \text{ as shown above} \\ &= \bigvee(C, (\varphi^\circ[C])_{\varphi\in X}) \\ &= (e, (e)_{\varphi\in X}) \text{ because each } \varphi^\circ \text{ is Cauchy,}\end{aligned}$$

and for each $a \in L$,

$$\begin{aligned}\bigvee\{\tau_*^\circ(x) | x \lhd a\} &= \bigvee\{(x, (\varphi^\circ(x))_{\varphi\in X}) | x \lhd a\} \\ &= (a, (\varphi^{\circ\circ}(a))_{\varphi\in X}) \\ &= (a, (\varphi^\circ(a))_{\varphi\in X}) \text{ because each } \varphi^\circ \text{ is regular} \\ &= \tau_*^\circ(a).\end{aligned}$$

Therefore τ_*° is a regular Cauchy filter on L. Now using Lemma 2.3.13 for γL, there is a frame homomorphism $r : \gamma L \to \tau_X L$ such that $\tau_*^\circ = r\gamma_{L*}$. We claim that

r is the right inverse of σ_L.

$$
\begin{aligned}
\sigma_L r \gamma_{L*}(a) =& \sigma_L \tau_*^\circ(a) \\
&= \sigma_L \bigvee \{\tau_*(x) | x \lhd a\} \\
&= \bigvee \{\sigma_L \tau_*(x) | x \lhd a\} \text{ because } \sigma_L \text{ is a frame homomorphism} \\
&= \bigvee \{\gamma_{L*}(x) | x \lhd a\} \text{ from Remark 2.3.16, using } \tau = \gamma_L \sigma_L \\
&= \gamma_{L*}(a) \text{ because } \gamma_{L*} \text{ is a regular filter.}
\end{aligned}
$$

Therefore $\sigma_L r \gamma_{L*} = \gamma_{L*}$, which implies that $\sigma_L r = \mathrm{id}_{\gamma L}$, because γ_L is a strict extension. So r is a right inverse of σ_L, as required. $\square$

We can now use this characterisation to describe a situation in which a frame is erc.

Lemma 3.3.6 ([10] Proposition 2.10)**.** *If L is a complete, erc nearness frame, and $h : L \to M$ is a uniform homomorphism that is a frame isomorphism, then M is also erc.*

Proof: Firstly, $\gamma_{M*}h : L \to \gamma M$ is a Cauchy filter, because γ_{M*} is one, from Lemma 3.1.19, and h is a uniform homomorphism. Then from the proof of the lemma above, $(\gamma_{M*}h)^\circ : L \to \gamma M$ is a regular Cauchy filter, because L is erc. Then Lemma 2.3.13 gives a frame homomorphism $\bar{h} : \gamma L \to \gamma M$ such that $\bar{h}\gamma_{L*} = (\gamma_{M*}h)^\circ$. We claim that $h\gamma_L = \gamma_M \bar{h}$.

$$
\begin{tikzcd}
\gamma L \arrow[r, "\bar{h}"] \arrow[d, "\gamma_L"'] & \gamma M \arrow[d, "\gamma_M"] \\
L \arrow[r, "h"'] & M
\end{tikzcd}
$$

For $a \in L$,

$$\begin{aligned}
\gamma_M \bar{h} \gamma_{L*}(a) &= \gamma_M (\gamma_{M*} h)^\circ (a) \\
&= \gamma_M \left(\bigvee \{ \gamma_{M*} h(x) | x \lhd a \} \right) \\
&= \bigvee \{ \gamma_M \gamma_{M*} h(x) | x \lhd a \} \\
&= \bigvee \{ h(x) | x \lhd a \} \text{ since } \gamma_M \text{ is onto} \\
&= h(a) \text{ since } h \text{ is a frame homomorphism} \\
&= h \gamma_L \gamma_{L*}(a) \text{ since } \gamma_L \text{ is onto.}
\end{aligned}$$

This implies that $\gamma_M \bar{h} = h \gamma_L$ as claimed, because γ_L is a strict extension.

Now in addition, L is complete, so γ_L is a uniform isomorphism. Less formally, we can say that $L = \gamma L$, and we can write $\bar{h} : L \to \gamma M$, and $\gamma_M \bar{h} = h$. But h is a frame isomorphism, so it has an inverse h^{-1}, and $\gamma_M \bar{h} h^{-1} = h h^{-1} = \mathrm{id}_M$, which means that $\bar{h} h^{-1}$ is a right inverse of γ_M. On the other hand, $\bar{h} h^{-1} \gamma_M \bar{h} = \bar{h} h^{-1} h = \bar{h}$, and $\bar{h}$ is dense, because if $\bar{h}(a) = 0$, then $\gamma_M \bar{h}(a) = 0$, so $h(a) = 0$, and h is an isomorphism, so $a = 0$. Dense homomorphisms between regular frames are monic, so we have that $\bar{h} h^{-1} \gamma_M = \mathrm{id}_{\gamma M}$. This means that $\bar{h} h^{-1}$ is also the left inverse of γ_M, so γ_M is a frame isomorphism. Since γ_M is also a surjection, it is a uniform isomorphism, and so M is complete.

We have a uniform homomorphism $h : L \to M$, so by Remark 3.1.21, there is a frame homomorphism $\tilde{h} : \tau_X L \to \tau_X M$ such that $\tau_M \tilde{h} = h \tau_L$. But $\tau_M = \gamma_M \sigma_M$, and $\tau_L = \gamma_L \sigma_L$, so $\gamma_M \sigma_M \tilde{h} = h \gamma_L \sigma_L$. However, M and L are both complete, so γ_M and γ_L are isomorphisms, and so we can write $\sigma_M \tilde{h} = h \sigma_L$.

Now L is erc, so from the previous lemma, σ_L has a right inverse $r : \gamma L \to \tau_X L$. Since L is complete, we can write $r : L \to \tau_X L$. Then

$$\begin{aligned}
\sigma_M \tilde{h} r h^{-1} &= h \sigma_L r h^{-1} \\
&= h h^{-1} \text{ because } r \text{ is the right inverse of } \sigma_L \\
&= \mathrm{id}_M
\end{aligned}$$

and so $\tilde{h} r h^{-1}$ is a right inverse of σ_M. Since σ_M has a right inverse, M is erc, from the previous lemma. □

Example 3.3.7. Consider the nearness frame M in Example 3.2.1. It is shown in [5] Proposition 6.4 that the frame of reals $\mathcal{O}\mathbb{R}$ with its metric uniformity is complete, and since it is uniform, $\mathcal{O}\mathbb{R}$ is erc from Remark 3.3.2 and Lemma 3.3.3. Now the frame isomorphism $h : \mathcal{O}\mathbb{R} \to M$ is a uniform homomorphism because the uniformity on $\mathcal{O}\mathbb{R}$ is contained in $\mathcal{N}M$, so by the lemma above, M is also erc. However, we saw in Example 3.2.11 that M is not smooth, and this means that M is not strong, by Remark 3.3.2.

We now introduce a new property of homomorphisms, which will be needed in order to discuss coreflections.

Definition 3.3.8. *A frame homomorphism $h : L \to M$ is called* **completable** *if there exists a uniform homomorphism $\gamma h : \gamma L \to \gamma M$ making the square commute:*

$$\begin{tikzcd} \gamma L \arrow[r, "\gamma h"] \arrow[d, "\gamma_L"'] & \gamma M \arrow[d, "\gamma_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

Since γ_M is dense, such a uniform homomorphism would be unique.

Lemma 3.3.9 ([10] Lemma 2.3)**.** *If $h : L \to M$ is a uniform homomorphism and L is a strong nearness frame, then h is completable.*

Proof: In the proof of Lemma 3.3.6, we saw that if L is an erc nearness frame, then there is a frame homomorphism $\bar{h} : \gamma L \to \gamma M$ making the above square commute. Now in our case, L is strong, so it is also erc by Lemma 3.3.3, and therefore it only remains to show that the frame homomorphism $\bar{h}$ is a uniform homomorphism.

If $A \in \mathcal{N}(\gamma L)$, then A is refined by $\gamma_{L*}[C]$ for some $C \in \mathcal{N}L$, since γ_L is a strict surjection. It is enough to show that $\bar{h}\gamma_{L*}[C] \in \mathcal{N}(\gamma M)$ in order to show that $\bar{h}$ is uniform. Now $C \in \mathcal{N}L$, and L is strong, so $\check{C} \in \mathcal{N}L$. Also, h is uniform, so $h[\check{C}] \in \mathcal{N}M$ and γ_M is a strict surjection, so $\gamma_{M*}h[\check{C}] \in \mathcal{N}(\gamma M)$. Recall from Lemma 3.3.6 that $\bar{h}$ was constructed so that $\bar{h}\gamma_{L*} = (\gamma_{M*}h)^\circ$, which means that

$$\bar{h}\gamma_{L*}[C] = \left\{ \bigvee \{\gamma_{M*}h(x) | x \triangleleft a\} \middle| a \in C \right\}.$$

Now

$$\gamma_{M*}h[\check{C}] = \{\gamma_{M*}h(x) | x \triangleleft a \text{ for some } a \in C\},$$

so $\gamma_{M*}h[\check{C}] \leq \bar{h}\gamma_{L*}[C]$, and so $\bar{h}\gamma_{L*}[C] \in \mathcal{N}(\gamma M)$, as required. $\square$

Before we prove another result about completability, we need a transfer result for strongness.

Lemma 3.3.10 ([19] Lemma 4.20)**.** *If $h : L \to M$ is a strict surjection, and M is strong, then L is also strong.*

Proof: Take $C \in \mathcal{N}L$. Then because h is a strict surjection, there is a cover $A \in \mathcal{N}M$, such that $h_*[A] \leq C$. Now M is strong, so $\check{A} \in \mathcal{N}M$, and so $h_*[\check{A}] \in \mathcal{N}L$, by strictness. Take $s \in \check{A}$, so $s \triangleleft_{\mathcal{N}M} a$ for some $a \in A$. Then Lemma 3.1.13 shows that $h_*(s) \triangleleft_{\mathcal{N}L} h_*(a)$, because h is a strict surjection. But $h_*[A] \leq C$, so $h_*(a) \leq c$ for some $c \in C$, and so $h_*(s) \triangleleft_{\mathcal{N}L} c$, which means that $h_*(s) \in \check{C}$. So $h_*[\check{A}] \leq \check{C}$, and so $\check{C} \in \mathcal{N}L$, as required. $\square$

Lemma 3.3.11 ([10] Lemma 3.2)**.** *If a dense surjection $h : L \to M$ is completable and M is a strong nearness frame, then h is a strict surjection.*

Proof: Let $h : L \to M$ be a completable dense surjection, with strong M. Let $f : K \to L$ be the strict reduct of h, as defined in Definition 3.2.8, so $hf : K \to M$ is a strict surjection. Now since M is strong, K is also, from Lemma 3.3.10. Then from Lemma 3.3.9, f is also completable. So we have the following commutative diagram:

$$\begin{tikzcd}
\gamma K \arrow[r, "\gamma f"] \arrow[d, "\gamma_K"'] & \gamma L \arrow[r, "\gamma h"] \arrow[d, "\gamma_L"] & \gamma M \arrow[d, "\gamma_M"] \\
K \arrow[r, "f"'] & L \arrow[r, "h"'] & M
\end{tikzcd}$$

Since hf is a strict surjection and γ_K is a strict surjection, $hf \circ \gamma_K : \gamma K \to M$ is a completion of M. But the completion is unique, so there is a uniform isomorphism $i : \gamma K \to \gamma M$ such that $\gamma_M i = hf \circ \gamma_K$. But $hf \circ \gamma_K = \gamma_M \circ \gamma h \circ \gamma f$, so $\gamma_M i = \gamma_M \circ \gamma h \circ \gamma f$. Now γ_M is a dense frame homomorphism between regular frames, so it is monic, and so $i = \gamma h \circ \gamma f$. Then let $l = (\gamma h \circ \gamma f)^{-1}$.

By definition, $\gamma h \circ (\gamma f \circ l) = \mathrm{id}_{\gamma M}$. So $\gamma_M \circ (\gamma h \circ \gamma f \circ l) \circ \gamma h = \gamma_M \circ \gamma h$. But $\gamma_M \circ \gamma h = h \circ \gamma_L$, which is dense because h and γ_L are dense, so $\gamma_M \circ \gamma h$ is monic. Therefore $(\gamma f \circ l) \circ \gamma h = \mathrm{id}_{\gamma L}$. So in fact γh is a frame isomorphism, with $(\gamma h)^{-1} = \gamma f \circ l$. Then since γh is a surjection, it is a uniform isomorphism.

Now we have $(l \circ \gamma h) \circ \gamma f = \mathrm{id}_{\gamma K}$ because $l = (\gamma h \circ \gamma f)^{-1}$, and $\gamma f \circ (l \circ \gamma h) = \mathrm{id}_{\gamma L}$ because $(\gamma h)^{-1} = \gamma f \circ l$, so γf, being a surjection, is also a uniform isomorphism. Then $(\gamma f)^{-1}$ is a uniform homomorphism.

We claim that f is a surjection. Take $C \in \mathcal{N}L$. Then $\gamma_{L*}[C] \in \mathcal{N}(\gamma L)$ because γ_L is a strict surjection. So $(\gamma f)^{-1} \circ \gamma_{L*}[C] \in \mathcal{N}(\gamma K)$ because $(\gamma f)^{-1}$ is a uniform homomorphism. So then $\gamma_K \circ (\gamma f)^{-1} \circ \gamma_{L*}[C] \in \mathcal{N}K$, and

$$\begin{aligned} f[\gamma_K \circ (\gamma f)^{-1} \circ \gamma_{L*}[C]] &= (\gamma_L \circ \gamma f) \circ (\gamma f)^{-1} \circ \gamma_{L*}[C] \\ &= \gamma_L \gamma_{L*}[C] \\ &= C. \end{aligned}$$

This means that $C = f[A]$ for a cover $A \in \mathcal{N}K$, proving that f is indeed a surjection.

Now f is a frame isomorphism, by definition, and f is a surjection, so f is a uniform isomorphism. Then we conclude from the fact that hf is a strict surjection and f is an isomorphism that h is already a strict surjection. $\square$

Example 3.3.12. Consider the map $h : M \to L$ described in Example 3.2.1. It is a dense surjection and L is strong because it is uniform. However, it is not a strict surjection, so from the lemma above, it is not completable. This proves that the completion is not a coreflection of nearness frames and uniform homomorphisms, because if it was, there would be a uniform homomorphism $f : M \to \gamma L$ such that $h = \gamma_L f$, and then $f\gamma_M$ would be a uniform homomorphism making the square commute.

$$\begin{tikzcd} \gamma M \arrow[r, dashed] \arrow[d, "\gamma_M"'] & \gamma L \arrow[d, "\gamma_L"] \\ M \arrow[r, "h"'] \arrow[ur, dotted, "f"] & L \end{tikzcd}$$

Corollary 3.3.13 ([10] Lemma 2.7)**.** *Completion is not a coreflection in the category of erc nearness frames and uniform homomorphisms.*

Proof: We claim that all the frames in the example above are erc, so that it also provides a counterexample for this case. Firstly, L is erc because it is uniform, and secondly, we showed in Example 3.3.7 that M is erc. To show that γL and γM are erc, we show that the completion of any erc nearness frame is erc.

Let M be an erc nearness frame, and let $\varphi : \gamma M \to T$ be a Cauchy filter on γM. For a uniform cover $C \in \mathcal{N}M$, $\gamma_{M*}[C] \in \mathcal{N}(\gamma M)$ because γ_M is a strict surjection. Then $\varphi\gamma_{M*}[C]$ is a cover of T because φ is a Cauchy filter. Therefore $\varphi\gamma_{M*}$ is a Cauchy filter on M. Now M is erc, so there exists a regular Cauchy filter $\psi : M \to T$ such that $\psi \leq \varphi\gamma_{M*}$. Then using Lemma 2.3.13, we get a frame homomorphism $h : \gamma M \to T$ such that $h\gamma_{M*} = \psi$. Now a frame homomorphism is a regular Cauchy filter, so we will show that $h \leq \varphi$, so that h is the required regular Cauchy filter.

Take $a \in \gamma M$. Since γ_M is a strict extension, $a = \bigvee \gamma_{M*}[S]$ for some non-empty set $S \subseteq M$. Then

$$\begin{aligned} h(a) &= h\left(\bigvee \gamma_{M*}[S]\right) \\ &= \bigvee h\gamma_{M*}[S] \\ &= \bigvee \psi[S] \\ &\leq \bigvee \varphi\gamma_{M*}[S] \\ &\leq \varphi(a) \text{ because for } s \in S,\ \gamma_{M*}(s) \leq a. \end{aligned}$$

So $h \leq \varphi$, as required. □

One way to obtain a positive result is to further restrict the objects under consideration.

Proposition 3.3.14 ([10] Proposition 2.4)**.** *In the category of strong nearness frames and uniform homomorphisms, completion is a coreflection to the subcategory of complete strong nearness frames.*

Proof: From Lemma 3.3.10, we see that the completion of a strong nearness frame is strong, because completion is a strict surjection. Therefore we can follow the strategy used in the proof of Proposition 2.4.7. That is, we need to show that the completion of a strong nearness frame is complete, that completion is functorial in the category of strong nearness frames and uniform homomorphisms, and that if a strong nearness frame is complete, then the completion is an isomorphism.

The last point follows from the definition of completeness and completions, because if M is complete, then $\gamma_M : \gamma M \to M$ is a strict surjection onto M, and so is an isomorphism. The fact that a completion is complete also follows from the definition of completion, which is, a strict surjection from a complete nearness frame. The fact that completion is functorial in the category of strong nearness frames and uniform homomorphisms was proved in Lemma 3.3.9. Therefore, we do have a coreflection in the category of strong nearness frames and uniform homomorphisms. □

A second way to obtain a positive result corresponding to Corollary 3.3.13 is to change the coreflection map. Instead of using the completion as the coreflection, we use the weak completion $w_L : wL \to L$, constructed in Proposition 3.2.16, as the coreflection of an erc nearness frame L.

Proposition 3.3.15 ([10] Proposition 4.5)**.** *The complete nearness frames form a coreflective subcategory of the category of erc nearness frames and uniform homomorphisms, with the coreflection map given by the weak completion* $w_L : wL \to L$*.*

Proof: Following the same proof strategy as in the proposition above, it is enough to show that the weak completion is erc, and functorial in erc nearness frames and uniform homomorphisms. Now we saw in Corollary 3.3.13 that for an erc nearness frame M, γM is erc. By Remark 3.2.15, $\mathcal{N}(\gamma M) \subseteq \mathcal{N}(wM)$, so we can use Lemma 3.3.6 to show that wM is also an erc nearness frame, using the frame isomorphism $\gamma M \to wM$ as h.

For functoriality, take a uniform homomorphism $h : M \to L$, where M and L are erc. Then from the proof of Lemma 3.3.6, there is a frame homomorphism $\overline{h} : \gamma M \to \gamma L$ such that $h\gamma_M = \gamma_L \overline{h}$. Since γM and wM have the same underlying frames, the frame homomorphism $\overline{h} : \gamma M \to \gamma L$ corresponds to a frame homomorphism $wh : wM \to wL$, also satisfying $h\gamma_M = \gamma_L wh$. It remains to show that wh is a uniform homomorphism.

Take $C \in \mathcal{N}(wM)$, that is, $\gamma_M[C] \in \mathcal{N}M$. So $h\gamma_M[C] \in \mathcal{N}L$, because h is a uniform homomorphism. This means that $\gamma_L wh[C] \in \mathcal{N}L$, which proves that $wh[C] \in \mathcal{N}(wL)$. So $wh : wM \to wL$ is a uniform homomorphism. Furthermore, the square below commutes, that is, $hw_M = w_L wh$, because γ_M and w_M act the same, as do γ_L and w_L.

$$\begin{tikzcd} wM \arrow[r, "wh"] \arrow[d, "w_M"'] & wL \arrow[d, "w_L"] \\ M \arrow[r, "h"'] & L \end{tikzcd}$$

$\square$

We have one more result corresponding to Corollary 3.3.13. We can obtain a coreflection on a category with all nearness frames as objects, but then we must change the morphisms in the category, and we use Cauchy completions as the coreflection maps.

Definition 3.3.16. *A frame homomorphism $h : M \to L$ between nearness frames M and L is called a* **Cauchy homomorphism** *if for every (classical) regular Cauchy filter F on L there is a regular Cauchy filter G on M such that $G \subseteq h^{-1}[F]$.*

It should be pointed out that the term Cauchy homomorphism is defined in [10] to mean something different. An example is given in [13] Remark 3 after Proposition 8 of a Cauchy homomorphism in our sense which is not one in the other sense. In addition, it is mentioned in [10] in the remark after the definition of Cauchy homomorphisms that there is yet another type of homomorphism which has also been called Cauchy. In what follows we will only use the term "Cauchy homomorphism" in the sense defined above.

Remark 3.3.17 ([13] Lemma 4(1))**.** If $h : M \to L$ is a uniform homomorphism between erc nearness frames, then h is a Cauchy homomorphism.

Proof: If F is a Cauchy filter on L, then $h^{-1}[F]$ is a filter on M, as shown in Lemma 1.2.9. Then if $C \in \mathcal{N}M$, $h[C] \in \mathcal{N}L$, because h is a uniform homomorphism. So $h[C] \cap F \neq \emptyset$ because F is a Cauchy filter, and so there exists $a \in h[C] \cap F$. This means that $a = h(c)$ for some $c \in C$, and $a \in F$, so $c \in h^{-1}[F] \cap C$. Therefore $h^{-1}[F]$ is a Cauchy filter on M.

Now M is erc, and we saw in the proof of Lemma 3.3.5 that for any general Cauchy filter φ on an erc nearness frame, φ° is a regular Cauchy filter smaller than φ. In particular, the characteristic function φ of the Cauchy filter $h^{-1}[F]$ is a general Cauchy filter on M, and so φ° is a regular Cauchy filter, with $\varphi^\circ \leq \varphi$.

Now φ° is also the characteristic function of a classical filter G on M, which is regular and Cauchy by Remark 3.1.18. Further, if $a \in G$, then $\varphi^\circ(a) = 1$, which means that $\varphi(x) = 1$ for some $x \triangleleft_{\mathcal{N}M} a$. But then $\varphi(a) = 1$ because φ is order preserving, so $a \in h^{-1}[F]$. This means that $G \subseteq h^{-1}[F]$, showing that h is a Cauchy homomorphism, as claimed. □

Example 3.3.18 ([13] Remark after Lemma 4)**.** Not every Cauchy homomorphism is a uniform homomorphism, even between uniform frames. An example is given in the remark cited here, which we mention without details and without defining all the relevant terms. Let B be an infinite atomic Boolean algebra whose set of atoms is uncountable and of non-measurable cardinality. Let M be the uniform frame $(B, \mathrm{Cov}B)$, and let L be the uniform frame $(B, \mathrm{CountCov}B)$ consisting of all countable covers of B, and define $h : M \to L$ to be the identity map. Then h is clearly not uniform because there are more uniform covers in the domain than in the codomain, but h can still be shown to be a Cauchy homomorphism.

We will show that the Cauchy completion is a coreflection, not just in the category of erc nearness frames, where all uniform homomorphisms are Cauchy, but in the category of all nearness frames. We need to therefore check that Cauchy completions are in fact Cauchy homomorphisms, when the frames in question are not necessarily erc. This is a particular case of the following lemma:

Lemma 3.3.19 ([13] Lemma 4(3)). *If $h : M \to L$ is a strict surjection, then h is a Cauchy homomorphism.*

Proof: Let P be a regular Cauchy filter on L, and let F be the filter generated by $h_*[P]$. Since $hh_*[P] = P$, we have that $h_*[P] \subseteq h^{-1}[P]$. Then since $h^{-1}[P]$ is also a filter, $F \subseteq h^{-1}[P]$. We claim that F is a regular Cauchy filter, which would prove that h is a Cauchy homomorphism.

We must first check that F is a proper filter. Suppose that $0 \in F$, then $0 \geq h_*(x)$ for some $x \in P$, so $0 = h_*(x)$. But then $x = hh_*(x) = h(0) = 0$, so $x = 0$. But this means that $0 \in P$, which contradicts the fact that P is a proper filter. So F is a proper filter.

To show that F is Cauchy, take $C \in \mathcal{N}M$, and we must show that $C \cap F \neq \emptyset$. Now h is a strict surjection, so there is an $A \in \mathcal{N}L$ such that $h_*[A] \leq C$, and $A \cap P \neq \emptyset$ because P is a Cauchy filter. If $x \in A \cap P$, then $h_*(x) \in h_*[P] \subseteq F$, and $h_*(x) \in h_*[A] \leq C$, so $h_*(x) \leq c$ for some $c \in C$. This means that $c \in F \cap C$, so $F \cap C \neq \emptyset$, as required.

For regularity, take $s \in F$, so that $s \geq h_*(a)$ for some $a \in P$. Since P is regular, there is some $x \in P$ and $B \in \mathcal{N}L$ such that $Bx \leq a$. Now from Lemma 3.1.13, $h_*[B]h_*(x) \leq h_*(a)$, and since h is a strict surjection, this implies that $h_*(x) \triangleleft h_*(a) \leq s$. So we have that $h_*(x) \triangleleft s$, and $h_*(x)$ is in F, proving that F is regular. □

Now we can prove the coreflection result.

Proposition 3.3.20 ([22] Theorem 15). *The category of Cauchy complete nearness frames is coreflective in the category of nearness frames and Cauchy homomorphisms, with the coreflection of a nearness frame L given by the Cauchy completion $c_L : cL \to L$.*

Proof: We need to show that if $h : M \to L$ is a Cauchy homomorphism and M is Cauchy complete, then there is a unique Cauchy homomorphism $\bar{h} : M \to cL$ such that $c_L\bar{h} = h$.

$$\begin{tikzcd} cL \arrow[r, "c_L"] & L \\ & M \arrow[ul, dotted, "!"] \arrow[u, "h"'] \end{tikzcd}$$

Define $\bar{h} : M \to cL$ such that $\bar{h}(a) = \bigvee\{(h(x), X_{h(x)}) | x \lhd a\}$, where X is the set of classical regular Cauchy filters on L. We first show that $\bar{h}$ is a frame homomorphism, and we do this by considering each component separately. If both components are frame homomorphisms, then $\bar{h}$ is a frame homomorphism, by the definition of a product.

For the first component, recall that c_L is the restriction of the first projection map to cL, so the first component of $\bar{h}$ is just $c_L \circ \bar{h}$. Then for any $a \in M$,

$$\begin{aligned} c_L \circ \bar{h}(a) &= c_L\left(\bigvee\{(h(x), X_{h(x)}) | x \lhd a\}\right) \\ &= \bigvee\{h(x) | x \lhd a\} \\ &= h\left(\bigvee\{x | x \lhd a\}\right) \\ &= h(a) \text{ by admissibility.} \end{aligned}$$

So $c_L \circ \bar{h} = h$, which is indeed a frame homomorphism, and further, this shows that the required triangle commutes.

For the second component, let $h_1 = p_2 \circ \bar{h}$, where p_2 is the restriction of the second projection map to cL. So for $a \in M$, $h_1(a) = \bigvee\{X_{h(x)} | x \lhd a\} = \bigcup\{X_{h(x)} | x \lhd a\}$. We must show that h_1 is a frame homomorphism.

For finite meets, consider the top first: $h_1(e) = \bigvee\{X_{h(x)} | x \lhd e\}$. But $e \lhd e$, so $h_1(e) \geq X_{h(e)} = X_e = X$ because all filters must contain the top, since filters are non-empty and up-closed. Therefore $h_1(e) = X$.

For binary meets, take a and b in M. We have

$$\begin{aligned}
h_1(a) \wedge h_1(b) &= \bigcup\{X_{h(x)} | x \lhd a\} \cap \bigcup\{X_{h(y)} | y \lhd b\} \\
&= \bigcup\{X_{h(x)} \cap X_{h(y)} | x \lhd a, y \lhd b\} \\
&= \bigcup\{X_{h(x) \wedge h(y)} | x \lhd a, y \lhd b\} \text{ because } X \text{ consists of filters} \\
&= \bigcup\{X_{h(x \wedge y)} | x \lhd a, y \lhd b\} \text{ because } h \text{ is a frame homomorphism} \\
&= \bigcup\{X_{h(z)} | z \lhd a \wedge b\} \\
&= h_1(a \wedge b).
\end{aligned}$$

For arbitrary joins, consider the bottom first. We have $h_1(0) = \bigvee\{X_{h(x)} | x \lhd 0\} = X_{h(0)}$ since only $0 \lhd 0$. But $X_{h(0)} = X_0 = \emptyset$ because no proper filters contain 0. So $h_1(0) = \emptyset$, as required.

Now for non-empty joins, take $S \subseteq M$ such that $S \neq \emptyset$. If $s \in S$, then $s \leq \bigvee S$, so $s \wedge \bigvee S = s$. Now h_1 preserves meets, so $h_1(s) \wedge h_1(\bigvee S) = h_1(s)$, so $h_1(s) \leq h_1(\bigvee S)$, that is, $h_1(s) \subseteq h_1(\bigvee S)$. Therefore $\bigvee\{h_1(s) | s \in S\} \subseteq h_1(\bigvee S)$, and it remains to show the other inclusion.

Take $F \in h_1(\bigvee S)$. This means that $F \in X_{h(x)}$ for some $x \lhd \bigvee S$, that is, $h(x) \in F$ for some $x \lhd \bigvee S$, and F is a regular Cauchy filter on L. Now h is a Cauchy homomorphism by assumption, so there is a regular Cauchy filter G on M such that $G \subseteq h^{-1}[F]$. Now $x \lhd \bigvee S$, so there is a $C \in \mathcal{N}M$ such that $Cx \leq \bigvee S$, and $G \cap C \neq \emptyset$ because G is Cauchy. So let $c \in G \cap C$. If $c \wedge x = 0$, then $c \leq x^*$, so $x^* \in G$. However, if $c \wedge x \neq 0$, then $c \leq Cx \leq \bigvee S$, so $\bigvee S \in G$. So either $x^* \in G$ or $\bigvee S \in G$. However, if $x^* \in G \subseteq h^{-1}[F]$, then $h(x^*) \in F$, but we already have that $h(x) \in F$, so then $h(x^*) \wedge h(x) = h(x^* \wedge x) = h(0) = 0 \in F$, which is not possible since F is proper. So $x^* \notin G$, and therefore $\bigvee S \in G$.

Now G is a regular Cauchy filter, and M is Cauchy complete, so G converges. We mentioned in Lemma 1.2.23 that any regular filter that converges is completely prime, so since $\bigvee S \in G$, $G \cap S \neq \emptyset$. Let $s \in G \cap S$. Since G is regular, there is a $y \in G$ such that $y \lhd s$. Since $y \in G \subseteq h^{-1}[F]$, $h(y) \in F$, meaning that $F \in X_{h(y)}$, and $y \lhd s$. Therefore $F \in \bigcup\{X_{h(y)} | y \lhd s\} = h_1(s)$, and so $F \in \bigvee\{h_1(s) | s \in S\}$. This shows that $h_1(\bigvee S) \subseteq \bigvee\{h_1(s) | s \in S\}$, and so h_1 preserves arbitrary joins. Therefore h_1 is a frame homomorphism, and so $\bar{h}$ is also a frame homomorphism.

It remains to show that $\bar{h}$ is a Cauchy homomorphism, and that it is unique. Uniqueness follows because c_L is a dense frame homomorphism between regular frames, so it is monic. Then if g were another frame homomorphism satisfying $c_L g = h$, then $c_L g = c_L \bar{h}$, and so $g = \bar{h}$. So we just need to show that $\bar{h}$ is Cauchy.

Let P be a regular Cauchy filter in cL. Since cL is Cauchy complete, P converges, so as mentioned above, P is completely prime. Then it was shown in Lemma 1.2.9 that $\bar{h}^{-1}[P]$ is also a completely prime filter. We claim that $\bar{h}^{-1}[P]$ is itself a regular Cauchy filter contained in $\bar{h}^{-1}[P]$. For regularity, if $a \in \bar{h}^{-1}[P]$, then $a = \bigvee\{b \in M | b \triangleleft a\} \in \bar{h}^{-1}[P]$ by admissibility, so there is a $b \in \bar{h}^{-1}[P]$ such that $b \triangleleft a$, because $\bar{h}^{-1}[P]$ is completely prime. To show that $\bar{h}^{-1}[P]$ is Cauchy, take $C \in \mathcal{N}M$, and then $\bigvee C = e \in \bar{h}^{-1}[P]$, so $C \cap \bar{h}^{-1}[P] \neq \emptyset$, again because $\bar{h}^{-1}[P]$ is completely prime. So $\bar{h}^{-1}[P]$ is the regular Cauchy filter required to show that $\bar{h}$ is a Cauchy homomorphism. □

3.4 Completions and compactifications of nearness frames

We conclude this chapter by returning to the concept of compactifications, described at the end of the previous chapter. We will see the relationship between compact nearness frames and the complete ones that we have been studying.

Recall that a nearness frame is called **totally bounded** if its nearness structure is generated by its finite members.

Lemma 3.4.1 ([3] Proposition 1(2))**.** *A compact nearness frame is totally bounded and uniform.*

Proof. If L is a compact nearness frame, then L is a compact regular frame, and so it has a unique nearness structure, Cov$L$. So $\mathcal{N}L = \text{Cov}L$, which is a uniformity, as mentioned in Lemma 1.2.47. It is also totally bounded because every cover of L has a finite subcover, since L is compact, and so the finite uniform covers of L generate all the uniform covers. □

As a result of this lemma, all nearness frames considered in this section will be totally bounded and uniform.

We saw in the previous chapter, in Proposition 2.4.1, that for a strong inclusion $\triangleleft$, the filter trace of the $\triangleleft$-compactification consists of maximal $\triangleleft$-filters. In the next few lemmas, we will discuss results about these specific filters.

Lemma 3.4.2 ([23] Remark 1.9)**.** *Given a strong inclusion $\triangleleft$ on a frame L, a $\triangleleft$-filter F is maximal if and only if whenever $x \triangleleft y$ in L, either $x^* \in F$ or $y \in F$.*

Proof: Suppose that F is a $\triangleleft$-filter with the above mentioned property, and let G be another $\triangleleft$-filter such that $F \subset G$. Let $a \in G \setminus F$, then since G is a $\triangleleft$-filter, there is a $b \in G$ such that $b \triangleleft a$. By assumption, either $b^* \in F$ or $a \in F$. However,

a was defined such that $a \notin F$, so $b^* \in F \subseteq G$. Now $b \in G$ also, so $b^* \wedge b = 0 \in G$, making G not proper. Therefore F is maximal.

On the other hand, suppose that F is a maximal $\triangleleft$-filter and let $x \triangleleft y$. We showed in the proof of Proposition 2.4.1 that $F = h[P]$ for some completely prime filter P on M, where $h : M \to L$ is the $\triangleleft$-compactification of L. Recall that the $\triangleleft$-compactification h has the property that $a \triangleleft b$ in L if and only if $h_*(a) \prec h_*(b)$ in M. So if $x \triangleleft y$, then $h_*(x) \prec h_*(y)$, and so $(h_*(x))^* \vee h_*(y) = e$. Since $e \in P$ and P is a prime filter, either $(h_*(x))^* \in P$ or $h_*(y) \in P$.

If $h_*(y) \in P$, then $hh_*(y) = y \in h[P] = F$. So $y \in F$. On the other hand, if $y \notin F$, then $h_*(y) \notin P$, so $(h_*(x))^* \in P$, and so $h((h_*(x))^*) \in F$. But we showed in the proof of Lemma 3.2.10 that for a dense, onto frame homomorphism f, $f(z^*) = (f(z))^*$, so $h((h_*(x))^*) = (h(h_*(x)))^* = x^* \in F$. Therefore either $y \in F$ or $x^* \in F$, as required.

$\square$

Lemma 3.4.3 ([23] Lemma 1.10)**.** *If $(L, \mathcal{N}L)$ is a totally bounded uniform frame and F is a classical filter on L, then F is a regular Cauchy filter if and only if F is a maximal $\triangleleft_{\mathcal{N}L}$-filter.*

Proof: If F is a regular Cauchy filter, then F is a $\triangleleft_{\mathcal{N}L}$-filter, by definition. To show that F is maximal, we can use Lemma 3.4.2, because $\triangleleft_{\mathcal{N}L}$ is a strong inclusion on L when L is uniform. So take $x \triangleleft_{\mathcal{N}L} y$, then there is a $C \in \mathcal{N}L$ such that $Cx \leq y$. Now $F \cap C \neq \emptyset$ because F is a Cauchy filter, so let $c \in F \cap C$. If $c \wedge x = 0$, then $c \leq x^*$, so $x^* \in F$. On the other hand, if $c \wedge x \neq 0$, then $c \leq Cx \leq y$, so $y \in F$. So we have shown that either $x^* \in F$ or $y \in F$, and so F is a maximal $\triangleleft_{\mathcal{N}L}$-filter.

On the other hand, if F is a maximal $\triangleleft_{\mathcal{N}L}$-filter, then it is regular by definition, so we just need to show that it is Cauchy. Assume for contradiction that F is not Cauchy, so there is a $C \in \mathcal{N}L$ such that $C \cap F = \emptyset$. Now L is uniform, so from Remark 3.3.2, L is strong, and so $\check{C} = \{x \in C | x \triangleleft_{\mathcal{N}L} c \text{ for some } c \in C\} \in \mathcal{N}L$. Also, L is totally bounded, so there is a finite $D \in \mathcal{N}L$ such that $D \leq \check{C}$.

For $x \in D$, there is a $y \in \check{C}$ such that $x \leq y$, which means that there is a $c \in C$ such that $y \triangleleft_{\mathcal{N}L} c$. So $x \triangleleft_{\mathcal{N}L} c$, and by assumption, $C \cap F = \emptyset$, so $c \notin F$. Then from Lemma 3.4.2, we must have that $x^* \in F$, since F is a maximal $\triangleleft_{\mathcal{N}L}$-filter. So we see that for each $x \in D$, $x^* \in F$, and so $\bigwedge\{x^*|x \in D\} \in F$, since D is finite. Now

$$\begin{aligned}\bigwedge\{x^*|x \in D\} &= \left(\bigvee\{x|x \in D\}\right)^* \\ &= \left(\bigvee D\right)^* \\ &= e^* \\ &= 0,\end{aligned}$$

so $0 \in F$. But this is not possible because F is a proper filter, so F must be a regular Cauchy filter. □

We now use the fact that these two sets of filters are the same to describe the set of all compactifications of a given frame.

Proposition 3.4.4 ([3] Propositions 6)**.** *The compactifications of a frame L are exactly the Cauchy completions of its totally bounded uniformities.*

Proof: For a frame L, let $\mathcal{N}$ be a totally bounded uniformity on L. The Cauchy completion of $(L, \mathcal{N})$ is constructed using the set of classical regular Cauchy filters on L. But from the lemma above, this is the same as the set of maximal $\triangleleft_{\mathcal{N}}$-filters on L. So the Cauchy completion can be constructed using the maximal $\triangleleft_{\mathcal{N}}$-filters on L, or equivalently, using the free maximal $\triangleleft_{\mathcal{N}}$-filters on L, which is the set used to construct the $\triangleleft_{\mathcal{N}}$-compactification. Therefore the Cauchy completion of $(L, \mathcal{N})$ is a compactification.

To show that every compactification of L is the Cauchy completion of a totally bounded uniformity on L, we will show that every strong inclusion $\triangleleft$ on L is $\triangleleft_{\mathcal{N}}$ for some totally bounded uniformity $\mathcal{N}$ on L. Then from Lemma 3.4.3, the $\triangleleft$-compactification of L is the Cauchy completion of $(L, \mathcal{N})$.

So suppose that $\triangleleft$ is a strong inclusion on L, with $h : M \to L$ the $\triangleleft$-compactification of L. Since M is a compact regular frame, it has a unique uniformity $\mathcal{N}M$ which is totally bounded, so h induces a totally bounded uniformity $\mathcal{N}L$ on L. We will show that $\triangleleft = \triangleleft_{\mathcal{N}L}$.

Take $a \triangleleft b$ in L, which means that $h_*(a) \prec h_*(b)$ in M. So there exists a $c \in M$ such that $h_*(a) \wedge c = 0$ and $h_*(b) \vee c = e$. Then $C = \{h_*(b), c\} \in \mathrm{Cov}M$ and $\mathrm{Cov}M = \mathcal{N}M$. Now $h[C] = \{b, h(c)\} \in \mathcal{N}L$, and $h(c) \wedge a = h(c \wedge h_*(a)) = h(0) = 0$. Therefore $h[C]a \leq b$, so $a \triangleleft_{\mathcal{N}L} b$.

On the other hand, if $a \triangleleft_{\mathcal{N}L} b$, then there is a $C \in \mathcal{N}L$ such that $Ca \leq b$. From Lemma 3.1.13, $h_*[C]h_*(a) \leq h_*(b)$. Now $C = h[A]$ for some $A \in \mathcal{N}M$, because h

is a surjection. So we have $Ah_*(a) \leq h_*[h[A]]h_*(a) = h_*[C]h_*(a) \leq h_*(b)$, which means that $\bigvee\{s \in A | s \wedge h_*(a) \neq 0\} \leq h_*(b)$. Then

$$\begin{aligned} h_*(b) \vee (h_*(a))^* &\geq \bigvee\{s \in A | s \wedge h_*(a) \neq 0\} \vee (h_*(a))^* \\ &= \bigvee\{s \in A | s \wedge h_*(a) \neq 0\} \vee \bigvee\{s \in M | s \wedge h_*(a) = 0\} \\ &\geq \bigvee\{s \in A | s \wedge h_*(a) \neq 0\} \vee \bigvee\{s \in A | s \wedge h_*(a) = 0\} \\ &= \bigvee A = e. \end{aligned}$$

In addition, $h_*(a) \wedge (h_*(a))^* = 0$, so $h_*(a) \prec h_*(b)$, which means that $a \triangleleft b$. □

We have seen that Cauchy completions and compactifications are essentially the same things for any frame L. We can make this result more precise, but in order to do that we will need a characterisation of compact frames.

Lemma 3.4.5 ([7] Proposition 5)**.** *Let L be a frame with a strong inclusion $\triangleleft$ on it. Then L is compact if and only if every maximal $\triangleleft$-filter on L converges.*

Proof: If every maximal $\triangleleft$-filter on the frame L converges, then the set X consisting of all free maximal $\triangleleft$-filters on L is empty. So the $\triangleleft$-compactification of L, given in Proposition 2.4.1 by $\tau_X L$, is just L. This means that L itself is compact.

On the other hand, suppose that L is compact, and for a strong inclusion $\triangleleft$ on L, let F be a maximal $\triangleleft$-filter on L. Let C be any cover of L, and we want to show that $C \cap F \neq \emptyset$, so suppose for contradiction that $C \cap F = \emptyset$. For the cover C, $\check{C} = \{x \in L | x \triangleleft a \text{ for some } a \in C\}$ is also a cover of L by the admissibility property of $\triangleleft$. Since L is compact, $\check{C}$ has a finite subcover, D.

Now for $x \in D$, $x \in \check{C}$, so $x \triangleleft c$ for some $c \in C$. Then from Lemma 3.4.2, either $x^* \in F$ or $c \in F$, but since $C \cap F = \emptyset$, $c \notin F$. Therefore $x^* \in F$ for each $x \in D$, which means that $\bigwedge D^* \in F$ since D is finite. Now $\bigwedge D^* = (\bigvee D)^* = e^* = 0$, so $0 \in F$. But this is impossible because F is a proper filter, so we must have that $C \cap F \neq \emptyset$ for each $C \in \mathrm{Cov}L$, meaning that F converges. □

We now have the main result for this section.

Proposition 3.4.6 ([23] Proposition 1.11)**.** *A totally bounded uniform frame is compact if and only if it is Cauchy complete.*

Proof: By definition, L is a Cauchy complete nearness frame if and only if every regular Cauchy filter converges. From Lemma 3.4.3, every regular Cauchy filter converges if and only if every maximal $\triangleleft_{\mathcal{N}L}$-filter converges. But then from Lemma 3.4.5 above, every maximal $\triangleleft_{\mathcal{N}L}$-filter converges if and only if L is compact. So L is Cauchy complete if and only if it is compact. $\square$

Corollary 3.4.7. *A totally bounded, uniform frame is compact if and only if it is complete.*

Proof: If a totally bounded uniform frame L is complete, then from Lemma 3.1.29, L is Cauchy complete, and so from the proposition above, L is compact. On the other hand, suppose that L is compact, and consider γL. Since γL is complete, it is Cauchy complete from Lemma 3.1.29, and so it is compact by the proposition above. But γL is also regular, so assuming the Boolean Ultrafilter Theorem, γL is spatial. Then from Remark 2.2.12, $\gamma_L : \gamma L \to L$ is relatively spatial.

Now $c_L : cL \to L$ is the relatively spatial reflection of $\gamma_L : \gamma L \to L$, as shown in Proposition 3.1.28, so if $\gamma_L : \gamma L \to L$ is already relatively spatial, then $c_L = \gamma_L$. Now L is compact, so it is Cauchy complete, from the proposition above, and so every classical regular Cauchy filter on L converges. But then $cL = \tau_\emptyset L \cong L$, so $c_L : cL \to L$ is an isomorphism. But then $\gamma_L : \gamma L \to L$ is also an isomorphism, so L is complete. $\square$

4 Further Work

In this dissertation we have seen how to use Hong's construction to construct strict extensions of frames. In particular, we constructed compactifications of frames and different kinds of completions of structured frames. There are a number of ways in which the ideas discussed here can be pursued further.

Recently, strict extensions have been studied in the asymmetric setting. A **biframe** is a triple (L_0, L_1, L_2) where L_0 is a frame and L_1 and L_2 are subframes of L_0 such that $L_1 \cup L_2$ generates L_0. This is the pointfree version of a **bitopological space**, which is a space on which two topologies are defined. A number of papers by Frith and Schauerte explore strict extensions of biframes. In [17], the result presented in Section 2.1 of this dissertation was generalised to the asymmetric setting.

An asymmetric nearness structure, called a **quasi-nearness**, has been defined on biframes. The appropriate concept of completeness for quasi-nearness biframes, called **quasi-completeness**, was defined in [16], where a **quasi-completion** was also constructed. Asymmetric filters, called **bifilters**, as well as their general counterparts, were explored in [20], together with their relationship to the quasi-completion of a given biframe. A category in which quasi-completion is a coreflection is given in [18].

One of the things that has not yet been done in this setting is a generalisation of Hong's construction for biframes. We found that Hong's construction simplified the construction of the completion of a nearness frame in the symmetric setting (compared, for example, to the construction given in [3] Proposition 2). We also found that the proofs of some results, such as Lemma 3.3.5, could be simplified using Hong's construction. We wonder whether such a simplification could be achieved in the asymmetric setting as well.

Another possible avenue of exploration is the E-completions discussed by Marcus in [26]. For a regular frame E, an extension $h : M \to L$ of a frame L is called a **C_E-extension** if for any frame homomorphism $f : E \to L$ there is another frame homomorphism $\bar{f} : E \to M$ making the triangle commute:

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, dotted, "\bar{f}"'] & & L \\
& M \arrow[ur, "h"'] &
\end{tikzcd}$$

If every C_E-extension of L is an isomorphism, then L is said to be E**-complete**, and an E**-completion** of L is a C_E-extension of L where the domain frame M is E-complete.

In [9], E-completions of zero-dimensional frames are described using Hong's construction for two particular cases of E. First, for the case where E is the four element Boolean algebra, denoted $\mathbf{D}$, and second, for the power set of the natural numbers $\mathbb{N}$, which is denoted $\mathbf{N}$. It would be interesting to see if this could be done for general E, and for frames L that are not zero-dimensional.

Anther kind of completion for uniform frames was studied by Naidoo in, for example, [27]. He calls a filter F on a uniform frame L **weakly Cauchy** if $\sec F = \{y \in L | x \wedge y \neq 0 \text{ for all } x \in F\}$ meets every uniform cover. A uniform frame is **strongly Cauchy complete** if every weakly Cauchy filter F **clusters**, that is, $\sec F$ meets every cover. He also discusses **uniform paracompactness**, which is the property that every cover has a uniform locally finite refinement.

It is possible that strong Cauchy completions and uniform paracompactifications can be constructed in a similar manner to the way Cauchy completions and compactifications were constructed in this dissertation. This is an avenue that could be explored further.

There is also a question remaining from this dissertation. When we constructed strict extensions of spaces in Section 1.3, we added filters to the original space, which then became points in the extended space. We remarked there that there is an equivalence between the set of filters used to construct an extension, and the filter trace of that extension. If we add new filters to a space (which are not already neighbourhood filters) then we get a space with extra points, and the filter trace of that extension is bigger.

In the case of frames, it is not clear whether this equivalence holds or not. We saw in Lemma 2.3.3 that members of the class $[X]$, that is, the class of sets of filters that all produce the same strict extension, can differ by completely prime filters, which correspond to neighbourhood filters in the space case. What is not clear is whether adding additional filters to X which are not completely prime, necessarily produces a different strict extension.

As an example of where this might be useful, consider Lemma 3.4.5. We showed that a frame is compact if and only if every maximal $\lhd$-filter on it converges. The proof that convergence of maximal $\lhd$-filters implies compactness was almost trivial using the compactification given in Proposition 2.4.1. However the converse needed work to prove.

If the conjecture that only completely prime filters can be added to a set of filters if it is to remain in the same class is correct, then the proof of the second part of Lemma 3.4.5 could be simplified to something like this: Since L is compact, its $\triangleleft$-compactification can be given by $\tau_X L$ where $X = \emptyset$ for any strong inclusion $\triangleleft$. However, we know that the $\triangleleft$-compactification is also given by $\tau_Y L$ where Y is the set of free maximal $\triangleleft$-filters on L. Since $X \subseteq Y$, they can only differ by completely prime filters, which all converge. Therefore it must be true that all the maximal $\triangleleft$-filters on L converge.

The question is also applicable in the general case, and can be stated like this: If there are sets of general filters X and Y such that $X \subseteq Y$ and $\tau_X L = \tau_Y L$, then $Y \setminus X$ must consist only of frame homomorphisms. My feeling is that both of these results are true, although it would not be surprising if it is true in the classical case but not in the general case.

There is also a more general question, when X is not necessarily a subset of Y. If X and Y are any two sets of filters producing the same strict extension, then they might differ by filters that are not completely prime, even if that can be shown to be impossible for the case $X \subseteq Y$. For example, we could have $X = Z \cup \{F\}$ and $Y = Z \cup \{G\}$ for some non-completely prime filters F and G and a set of filters Z. The resulting strict extensions could well be the same, although it is also possible that one can prove that they are not.

As with the previous question, this question can be stated for general filters as well. However, unlike the previous question, the answer to this one is also unknown in the case of classical topology.

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