Topics in 2-categorical Algebra

Matthew Sutton

Department of Mathematics and Applied Mathematics

University of Cape Town

Supervisor: Professor George Janelidze
Co-supervisor: Dr Tamar Janelidze-Gray

May 20, 2023

Acknowledgement

I would like to thank Sanjiv Ranchod and David Broodryk for engaging discussions and helping me with both the format and content of this thesis, as well as Yaseen Mowzer for helping me with LaTeX issues. I would also like to thank my friends Andrew McGregor, Hannah Clayton, Bronson Rudner, and Renato Vieira, as well as my brother Steven, for their moral support, and my parents, Anne and Chris, for always looking after my best interests. Finally, I would like to thank my supervisors, Professor George Janelidze and Doctor Tamar Janelidze-Gray for taking care of me for the duration of the writing process of this thesis and aiding my growth. All diagrams were created using the website q.uiver.app, and I am deeply indebted to the creator(s) of it.

Abstract

In this thesis we will examine 2-categories and higher categorical structures and formulate 1-categorical theorems in the language of higher categories as well as formulate some internal definitions of these base structures in finitely complete categories. We will begin by defining the relevant 2-categorical structures, such as 2-categories, double categories, bicategories and enriched categories, as well as examples of all. Following this, we will show first how these structures relate to each other (for instance, a 2-category is a special case of a double category) and then demonstrate that the category of **V**-enriched categories forms a 2-category.

Chapter 2 begins with the definition of internal categories in a category **C** with pullbacks, as well as internal functors and internal natural transformations, after which we will demonstrate that the category of internal categories forms a 2-category. We will then show that in **C** with pullbacks and terminal object, one can define an internal 2-category and an internal bicategory , and show that these are the same as small 2-categories and small bicategories in the case of $\mathbf{C} = \mathbf{Set}$.

In the final chapter, we demonstrate that some of the familiar constructions of 1-category theory can actually be defined in a 2-category, and certain theorems about these structures proven using only 2-categorical methods.

Contents

1. Introduction

In the first chapter, we define the foundational objects of this thesis, namely 2-categories (under which we also consider double categories), the weaker notion, bicategories, and finally, enriched categories. For 2-categories, we only write the abstract definition and show how it is a special case of a double category, before moving onto bicategories. In the section considering bicategories, we have reasonably detailed examples to show why we would want to consider a 2-category with a weaker definition of associativity.

For the section on enriched categories, we necessarily begin with the definition of monoidal categories. Though it is not expanded upon further, we also mention the case of symmetric monoidal categories and give a few examples and constructions within monoidal categories. When this is completed, we are ready to define categories enriched over a given monoidal category **V**. What follows is to then define "**V**-functors" between these enriched categories and "**V**-natural transformations" between these enriched functors. Having these definitions in hand, the bulk of the section is to show that **V**-enriched categories, together with their enriched functors and enriched natural transformations satisfy the definition of a 2-category, therefore furnishing us with plenty of examples of 2-categories in the wild. Finally, we then show that a 2-category is a category enriched over the monoidal category **Cat** and likewise, a category enriched over **Cat** is a 2-category.

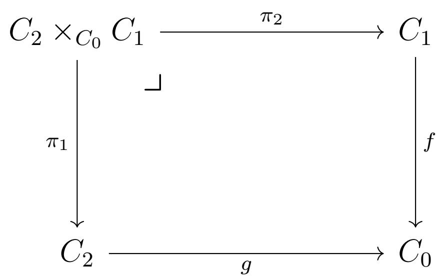
In the second chapter on internal structures, we begin with the definition of an internal category in a category **C** with pullbacks and terminal object. Following the structure of the section on enriched categories, we define internal functors and internal natural transformations, and show that considered together, the category of internal categories in **C** is in fact a 2-category, yet again, furnishing us with examples of 2-categories and giving us further motivation for why defining this structure makes sense as a natural generalisation of the definition of a category.

In the next section of chapter 2, we push the idea of internalisation further: Given a category **C** with pullbacks and terminal object, we define an internal 2-category, and in fact go even further by defining an internal bicategory. These definitions are rather involved and show the limits of internalizing structures, as well as why size issues may appear at this level, restricting examples. We also compare our internalizations, and show how they are related and where they differ critically.

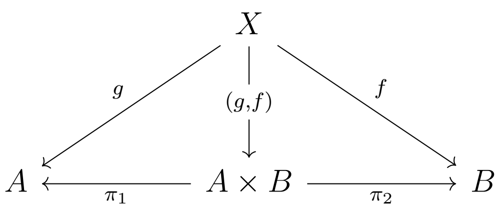
In our final chapter, we define notions that we are familiar with from the 2-category **Cat** in a general 2-category **C**. This allows us to see which familiar 1-categorical theorems and definitions are actually 2-categorical in nature. These include definitions of adjunction, equivalence, monads, and extensions. We then prove some of the standard results that one would have in **Cat** in a general 2-category using only the definitions within the general 2-category. After this, we provide examples of these structures, first in **Cat** as a sort of verification, and then in other 2-categories.

1.0.1 Notation

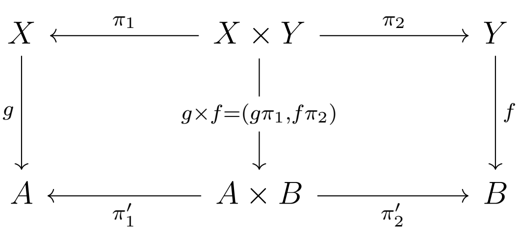
Perhaps the only notational oddity in this thesis is the following: Given a pullback as in



note that we use $\pi_2 : C_2 \times_{C_0} C_1 \to C_1$ to denote the first projection and $\pi_1 : C_2 \times_{C_0} C_1 \to C_2$ to denote the second. We will use the bracket notation for certain maps, that is, those ones formed from diagrams of the following shape:



where $(g, f) : X \to A \times B$ is the unique map induced by the universal property of the product. We will use the product notation for maps of the following form:



We use similar notation for arrows induced by pullbacks, their projections, and the universal morphisms induced by them. It should be noted that this notation is for the sections dealing with internal categories only, and the other sections may have different notation which should hopefully be clear from the context.

2. Foundations

In the introduction we cover the basic definitions of 2-categories, monoidal categories, bicategories, and enriched categories and show some equivalences of definitions as well as provide examples of these structures.

2.1 Preliminaries

2.1.1 2-categories

We begin by introducing the definition of a 2-category. Intuitively a 2-category consists of objects, arrows between objects, and arrows between these arrows which satisfy the expected properties. The archetypal 2-category is the 2-category of categories **Cat** consisting of small categories as objects, functors as morphisms or 1-cells, and natural transformations as 2-cells. From here, we define a general 2-category.

Definition 2.1.1 (2-category).

1. A 2-category **C** consists of a triple of sets (C_0, C_1, C_2), which we call the objects, the 1-cells, and the 2-cells respectively. We also have maps between these sets $s_0, t_0 : C_1 \to C_0$ and $s_1, t_1 : C_2 \to C_1$ which specify the domain and codomain (source and target) of 1-cells and 2-cells respectively. We write $f : a \to b$ for a 1-cell f with $s(f) = a$ and $t(f) = b$. We write $\beta : f \Rightarrow g$ for a 2-cell with $s_1(\beta) = f$ and $t_1(\beta) = g$.

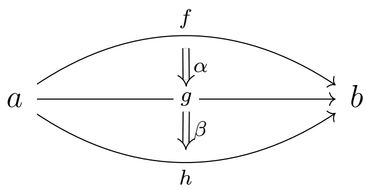
2. We have a composition of 1-cells, defined as a map $C_1 \times_{C_0} C_1 \to C_1$ with domain the pullback over the maps s_0 and t_0. Given 1-cells $f : a \to b$ and $g : b \to c$, we write this map as $(g, f) \mapsto gf : a \to c$, and it is associative (i.e. $h(gf) = (hg)f$.)

$$a \xrightarrow{f} b \xrightarrow{g} c$$

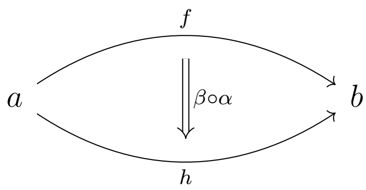
3. For each object a in C_0 there is a 1-cell $1_a : a \to a$, such that given any 1-cells $f : a \to b$ and $g : b \to a$ we have $f1_a = f$ and $1_a g = g$. We call this the identity 1-cell of a.

4. We have a vertical composition of 2-cells which we define as a map $\circ : C_2 \times_{C_1} C_2 \to C_2$ with domain the pullback over s_1 and t_1. Given

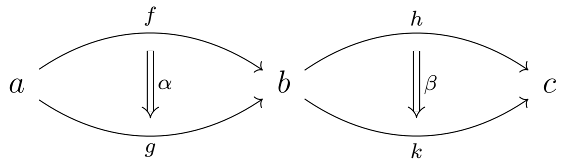
1-cells $f, g, h : a \to b$ and 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ we write v as $(\beta, \alpha) \mapsto \beta \circ \alpha : f \Rightarrow h$. This vertical composition is associative. We visualize it as follows:



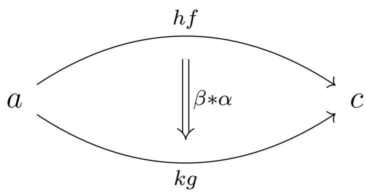
which becomes



5. We have a horizontal composition which we define as a map $C_2 \times_{C_0} C_2 \to C_2$ with domain the pullback over s_0t_1 and t_0s_1. Given 1-cells $f, g : a \to b$ and $h, k : b \to c$ and 2-cells $\alpha : f \Rightarrow g$ and $\beta : h \Rightarrow k$,we write h as $(\beta, \alpha) \mapsto \beta * \alpha : hf \Rightarrow kg$. This horizontal composition is associative, and we visualize it as follows:



which becomes



Note that we could just as well choose t_0t_1 and s_0s_1 for our domain and codomain maps, since $t_0t_1 = t_0s_1$ and $s_0s_1 = s_0t_1$.

6. To each 1-cell $f : a \to b$ there exists a 2-cell $1_f : f \Rightarrow f$ such that given a 2-cell $\alpha : f \Rightarrow g$ we have $\alpha \circ 1_f = \alpha$ and given a 2-cell $\beta : h \Rightarrow f$ we have $1_f \circ \beta = \beta$.

7. Given 1-cells $f : a \to b$ and $g : b \to c$, we have that $1_g * 1_f = 1_{gf}$.

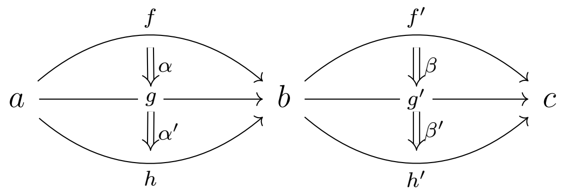
8. For each object a there is a 2-cell $1_{1_a} : 1_a \to 1_a$ such that given 1-cells $f, g : a \to b$ and a 2-cell $\alpha : f \Rightarrow g$ we have $\alpha * 1_{1_a} = \alpha$ and given

1-cells $h, k : c \to a$ and a 2-cell $\beta : h \Rightarrow k$ we have $1_{1_a} * \beta = \beta$. These are called the horizontal identities of 2-cells.

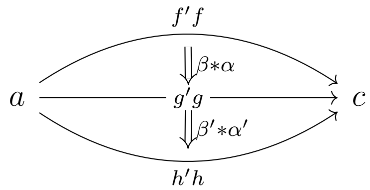
9. There is a law relating horizontal and vertical composition called the middle-four exchange. Given 1-cells $f, g, h : a \to b$ and $f', g', h' : b \to c$ and 2-cells $\alpha : f \Rightarrow g$ and $\alpha' : g \Rightarrow h$, as well as $\beta : f' \Rightarrow g'$ and $\beta' : g' \Rightarrow h'$ we have that the following is satisfied:

$$(\beta' * \alpha') \circ (\beta * \alpha) = (\beta' \circ \beta) * (\alpha' \circ \alpha)$$

i.e that this diagram:



becomes the following:



It is conceptually important to note that given a 2-category **K** one can form three distinct 1-categories. First, there is the 1-category consisting of the objects and the 1-cells, forgetting about the 2-cells entirely. Second, there is the category consisting of the 1-cells as objects and the 2-cells as arrows, with the composition being the vertical composition. Finally, there is the category consisting of the objects of **K** as objects and the arrows as 2-cells with the composition being the horizontal composition of 2-cells. Before we proceed, we make a brief notational comment. Given 1-cells $f, g : a \to b$ and a 2-cell $\alpha : f \to g$, as well as 1-cells $f' : a' \to a$ and $g' : b \to c$, we write $g' * \alpha = 1_{g'} * \alpha$ and $\alpha * f' = \alpha * 1_{f'}$ to denote composition of the 2-cell α with the respective identity 1-cells. This is called whiskering, and will be used in later sections.

The composition of 1-cells and horizontal composition of 2-cells can be encoded as a bifunctor: Let **K** be a 2-category, and let $\mathbf{K}(a, b)$ be the category whose objects are 1-cells with domain a and codomain b and whose arrows are the 2-cells between these 1-cells. We can then define a map $M : \mathbf{K}(b, c) \times \mathbf{K}(a, b) \to \mathbf{K}(a, c)$ for any ordered triple of objects, as follows. For 1-cells $(g, f) \in \mathbf{K}(b, c) \times \mathbf{K}(a, b)$, we let $M(g, f) = gf$ and for 2-cells $(\beta, \alpha) \in \mathbf{K}(b, c) \times \mathbf{K}(a, b)$ we let $M(\beta, \alpha) = \beta * \alpha$. The map is well defined since $t_0(f) = s_0(g)$ for all $(g, f) \in \mathbf{K}(b, c) \times \mathbf{K}(a, b)$, and likewise for 2-cells. For this map to be bifunctorial, we require that the identity and composition are preserved and that the domain and codomain of the map on 2-cells are consistent with its action on 1-cells. Preservation of identity is given by the condition:

$$M(1_g, 1_f) = 1_g * 1_f = 1_{gf} = 1_{M(g,f)}$$

For preservation of composition, we have:

$$M(\beta' \circ \beta, \alpha' \circ \alpha) = (\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha) = M(\beta', \alpha') \circ M(\beta, \alpha)$$

and for consistency of domain and codomain we have that

$$s_1(M(\beta, \alpha)) = s_1(\beta * \alpha) = s_1\beta s_1\alpha = M(s_1\beta, s_1\alpha)$$

and likewise for t_1, and so we see that our composition operation encodes a bifunctor for each ordered triple of objects.

2.1.1.1 Duality in 2-categories

Given a 2-category $\mathbf{K}$, we will define in passing the notions of $\mathbf{K}^{op}$, $\mathbf{K}^{co}$, and $\mathbf{K}^{coop}$. $\mathbf{K}^{op}$ is the 2-category with the same objects and 2-cells as $\mathbf{K}$, but with the direction of the 1-cells reversed, that is to say, that considering $\mathbf{K}(a, b)$ as a category, we have $\mathbf{K}^{op}(a, b) = \mathbf{K}(b, a)$. $\mathbf{K}^{co}$ is the 2-category with the same objects and 1-cells as $\mathbf{K}$, but with 2-cells reversed, which is to say that when $\mathbf{K}(a, b)$ is considered with its category structure, $\mathbf{K}^{co}(a, b) = \mathbf{K}(a, b)^{op}$. $\mathbf{K}^{coop}$ is the 2-category that has the same objects as $\mathbf{K}$, but with both 1-cells and 2-cells reversed. This means that when we consider the category structure on the homs, we have $\mathbf{K}^{coop}(a, b) = \mathbf{K}(b, a)^{op}$. The same duality principles that hold for 1-categories hold in this case too.

2.1.2 Double categories

We will now define the notion of a double category, which is similar to that of a 2-category. A double category consists of objects, vertical arrows between these objects, horizontal arrows between these objects, and squares between vertical and horizontal arrows. More precisely:

Definition 2.1.2 (Double Category). A double category $\mathbf{D}$ consists of a quadruple of sets (D_0, D_1, D_2, D_3), which we call the set of objects, the set of horizontal arrows, the set of vertical arrows, and the set of squares. To each of D_1, D_2 we have arrows specifying domain and codomain as follows:

$s_0, t_0 : D_2 \to D_0$, $s_1, t_1 : D_1 \to D_0$, and for D_3 we have $s_h, t_h : D_3 \to D_1$ and $s_v, t_v : D_3 \to D_2$ (for the horizontal and vertical domain/codomain respectively). A double category can be shown diagrammatically (following [6]) as in the following: it consists of the objects a, b, c etc.
The horizontal arrows,

$$c \xrightarrow{g} d$$

$$a \xrightarrow{f} b$$

the vertical arrows,

$$\begin{array}{cc} c & d \\ \uparrow{\scriptstyle f'} & \uparrow{\scriptstyle g'} \\ a & b \end{array}$$

and the squares between these arrows

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ c & \xrightarrow[g]{} & d \end{array}$$

where we use the same notational conventions for domain and codomain as in the definition of a 2-category. Also note that all pullbacks in this definition are defined over the appropriate source and target maps. Then, the following are required:

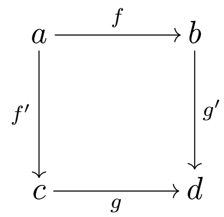
1. We have a map $h : D_1 \times_{D_0} D_1 \to D_1$ called the horizontal composition. Given horizontal arrows $f : a \to b$ and $g : b \to c$, we write h as $(g, f) \mapsto gf : a \to c$. This composition of horizontal arrows is associative, i.e $k(gf) = (kg)f$.

2. For every object a in D_0 there is a horizontal arrow $1_{ha} : a \to a$ such that given horizontal arrows $f : a \to b$ and $g : b \to a$, we have that $f1_{ha} = f$ and $1_{ha}g = g$. We call these arrows the horizontal identities.

3. We have a map $v : D_2 \times_{D_0} D_2 \to D_2$ called vertical composition. Given vertical arrows $f' : a \to b$ and $g' : b \to c$ we write v as $(g', f') \mapsto g' \circ f'$. This composition of vertical arrows is associative, i.e $h' \circ (g' \circ f') = (h' \circ g') \circ f'$

4. For every object a in D_0 there is a vertical arrow $1_{va} : a \to a$ such that given vertical arrows $f : a \to b$ and $g : b \to a$, we have that $f \circ 1_{va} = f$ and $1_{va} \circ g = g$. We call this the vertical identity.

5. Given a square α as in

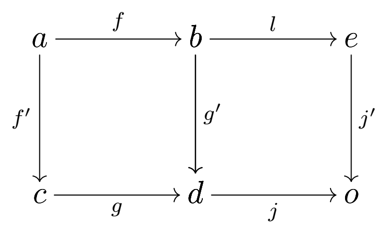


we require that the following hold:

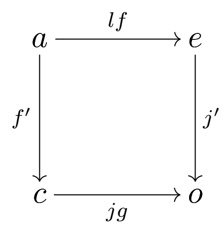
$$\begin{aligned} t_1 s_h &= s_0 t_v \\ s_1 s_h &= s_0 s_v \\ t_1 t_h &= t_0 t_v \\ s_1 s_h &= t_0 s_v \end{aligned}$$

as in the diagram given above. These conditions allow us to define vertical and horizontal composition of squares later. We note that it has $s_v(\alpha) = f', t_v(\alpha) = g'$ respectively, and $s_h(\alpha) = f, t_h(\alpha) = g$. In this representation, horizontal domain will always be the top arrow, and the vertical domain will always be the left arrow.

6. We have a horizontal composition of squares given by the map $h_{sq} : D_3 \times_{D_2} D_3 \to D_3$. Given squares α and β with the vertical codomain of α equal to the vertical domain of β we write h_{sq} as $(\beta, \alpha) \mapsto \beta\alpha$ with vertical domain equal to that of α and vertical codomain equal to that of β. The horizontal domain and codomain are the horizontal composites of the horizontal domain and codomain of β and α. Pictorially, we have:

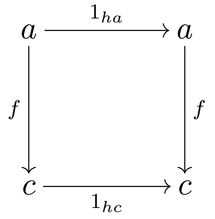


which is equal to



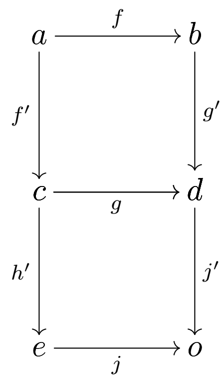
This horizontal composition of squares is associative.

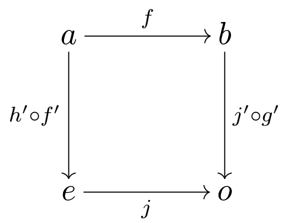
7. For every vertical arrow $f : a \to c$, there is a square 1_{hsq_f}, pictured below:



such that for a square α with $s_v(\alpha) = f$ we have $\alpha 1_{hsq_f} = \alpha$ and for a square β with $t_v(\beta) = f$ we have $1_{hsq_f}\beta = \beta$. We call these squares the horizontal identity squares.

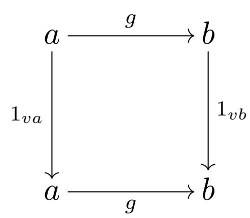
8. We have a vertical composition of squares given by the map $v_{sq} : D_3 \times_{D_1} D_3 \to D_3$. Given squares α and β with the horizontal codomain of α equal to the horizontal domain of β, we write v_{sq} as $(\beta, \alpha) \mapsto \beta \circ \alpha$, pictured as follows:





9. This vertical composition of squares is associative.

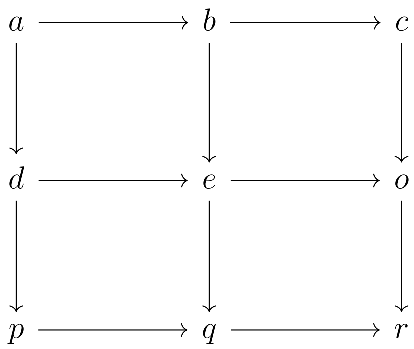
10. For every horizontal arrow $g : a \to b$, there is a square 1_{vsq_g}:



such that for a square α with $s_h(\alpha) = g$ we have $\alpha \circ 1_{vsq_g} = \alpha$ and for a square β with $t_h(\alpha) = g$ we have $1_{vsq_g} \circ \beta = \beta$

11. Given horizontal arrows $f : a \to b$ and $g : b \to c$, we have that $1_{vsq_g} 1_{vsq_f} = 1_{vsq_{gf}}$.

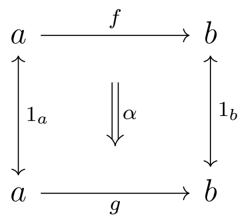
12. Given vertical arrows $f' : a \to b$ and $g' : b \to c$, we have $1_{hsq_g} \circ 1_{hsq_f} = 1_{hsq_{g \circ f}}$
13. The vertical and horizontal composition of squares satisfies the middle-four exchange (i.e. composing the following squares first vertically and then horizontally yields the same square as composing them first horizontally and then vertically)



2.1.3 2-categories and double categories

Given a 2-category **C** we can show that it also satisfies the definition of a double category. The converse is, however, not true.

Example 2.1.1. Every 2-category **C** is a double category. Given a 2-category **C** with objects C_0, 1-cells C_1, and 2-cells, C_2, define a subobject of the 1-cells given by the map $e : C_0 \to C_1$ which maps to each object its identity 1-cell. Denote the set of these 1-cells as C_1'. From the fact that they are a subset of C_1, they can be composed, and the only composite that can be yielded is the identity itself. Now, setting these as our vertical arrows, we note that our 2-cells have the structure of squares (pictured below):



There exists a horizontal and vertical composition of these squares, which is just the horizontal and vertical composition of 2-cells, and so has identity, is associative, and satisfies the middle-four exchange. Finally, the identity is preserved under horizontal composition, and so we can see that a 2-category has the structure of a double category.

To demonstrate that not all double categories are 2-categories, consider the following example, as in [7]:

Example 2.1.2 (Double Category of squares). Given a category $\mathbf{C}$, define the double category of squares in $\mathbf{C}$ to have objects those of $\mathbf{C}$, vertical and horizontal arrows as the arrows of $\mathbf{C}$, and the composition is also taken from $\mathbf{C}$. As squares we have the following:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ c & \xrightarrow[g']{} & d \end{array}$$

where $gf = g'f'$. The composition is simply given by pasting squares together (where we can define this of course), since we know that this will also give us a commutative square. It is not, however, a 2-category, because a horizontal identity is not always a vertical identity, which is always the case in a 2-category.

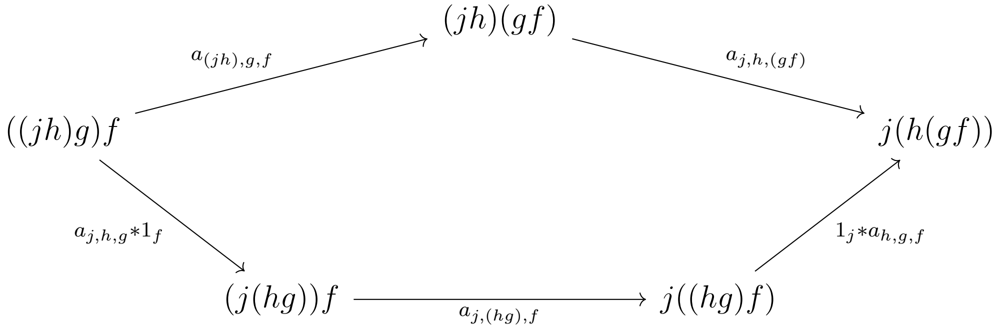
2.2 Bicategories

We now define a bicategory, which is very similar to a 2-category, but which has 1-cells which are only associative and with identity up to isomorphism.

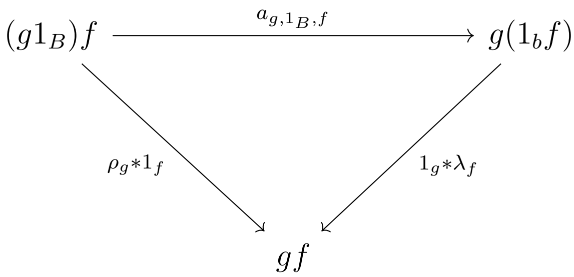
Definition 2.2.1 (Bicategory). A bicategory $\mathbf{C}$ consists of the following:

1. A class of objects C_0
2. For each ordered pair of objects A, B, a category $\mathbf{C}(A, B)$, whose objects we call 1-cells and whose arrows we call 2-cells.
3. For each triple of objects A, B, C, a bifunctor $M : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$ which we will call composition (for convenience, we will write $M(g, f)$ as gf for 1-cells g, f and $M(\beta, \alpha)$ as $\beta * \alpha$ for 2-cells β, α)
4. For every object A, a distinguished 1-cell 1_A in $\mathbf{C}(A, A)$
5. A natural isomorphism $a : M(M \times 1) \cong M(1 \times M)$, such that the

following diagram commutes:



6. A pair of natural isomorphisms $\lambda_{A,B} : M(1_B, 1_{\mathbf{C}(A,B)}) \cong 1_{\mathbf{C}(A,B)}$ and $\rho_{A,B} : M(1_{\mathbf{C}(A,B)}, 1_A) \cong 1_{\mathbf{C}(A,B)}$, shown above as components, such that the following diagram commutes:



We note that in the case where a, ρ, λ are identities, the above definition will yield a 2-category $\mathbf{C}$. To demonstrate this, given a bicategory $\mathbf{C}$ with a, ρ, λ equal to identity, we let the objects of the 2-category be those of $\mathbf{C}$. Given a pair of objects (A, B), we let the objects of $\mathbf{C}(A, B)$ be the 1-cells with domain A and codomain B, and the arrows of $\mathbf{C}(A, B)$ be the 2-cells. The 1-cells and 2-cells of the 2-category are then the collection of all of these for each pair of objects $(A, B) \in C_0 \times C_0$. Vertical composition is given by composition within the hom-category $\mathbf{C}(A, B)$, and as such is associative and with identity. Composition of 1-cells and horizontal composition of 2-cells is given by the bifunctor $M : \mathbf{C}(B, C) \times \mathbf{C}(A, B) \to \mathbf{C}(A, C)$. This is well defined because the bifunctor is defined for triples (A, B, C) ensuring that composition is only defined where domain and codomain are consistent, i.e. $\pi_1(B, C) = \pi_2(A, B)$. Given 1-cells $f, f' \in \mathbf{C}(A, B)$ and $g, g' \in \mathbf{C}(A, B)$, and 2-cells $\alpha : f \to f'$ and $\beta : g \to g'$ bifunctorality gives us that the domain of $M(\beta, \alpha)$ is $M(g, f)$ and the codomain is $M(g', f')$. Since a, ρ, λ are identities, we have $M(M(h, g), f) = M(h, M(g, f))$, $M(f, 1_A) = f$ and $M(1_B, f) = f$, and so M is associative and with identity. That the middle-four exchange is satisfied follows from the functorality of M. Specifically:

$$M(\beta'\beta, \alpha'\alpha) = M(\beta', \alpha')M(\beta, \alpha)$$

We also have:

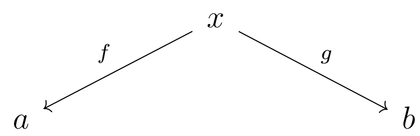
$$M(1_g, 1_f) = 1_{M(g,f)}$$

as a consequence of functorality, and so we see that the notion of a bicategory and a 2-category are closely related, since, as we have already established, for any 2-category $\mathbf{K}$, we have that the composition and horizontal composition can be shown as bifunctors from the hom-category $\mathbf{K}(b, c) \times \mathbf{K}(a, b) \to \mathbf{K}(a, c)$. This also allows us to have a more detailed breakdown of the data of a bicategory, which will be useful in defining an internal bicategory.

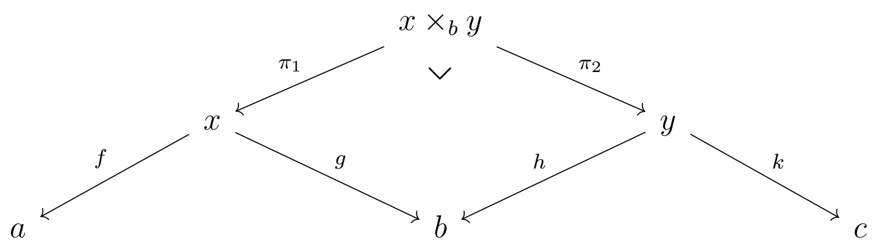
2.2.1 Examples of bicategories

Now, we look at a few examples of bicategories:

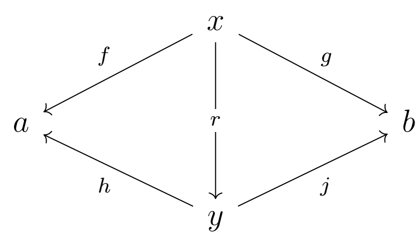
Example 2.2.1 (Bicategory of spans). Consider a category $\mathbf{A}$ which has pullbacks. Define $Span(\mathbf{A})$ as having objects those of $\mathbf{A}$, and 1-cells $(f, g, x) : a \to b$ as spans (pictured below):



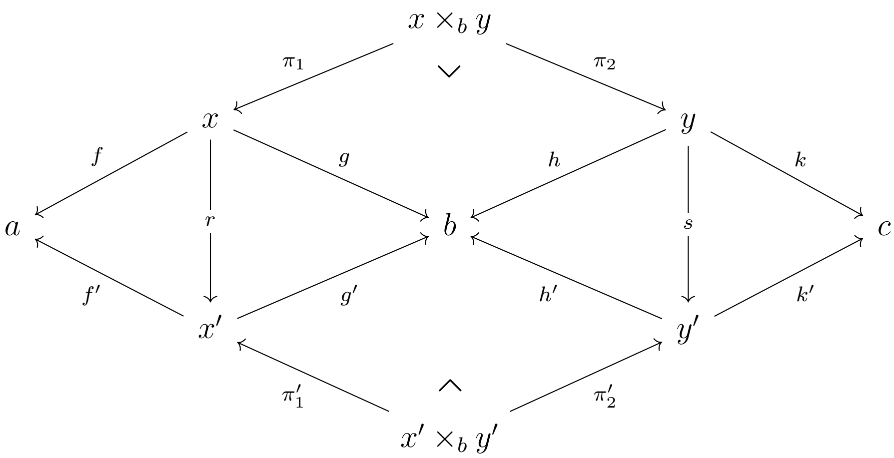
Composition of one cells is done by taking pullbacks, i.e.:



For 2-cells, we have morphisms of $\mathbf{A}$ between the vertices of spans which make both triangles in the following diagram commute:



Our 2-cells have vertical composition as that of **A**, with identity given by the identity arrow on the vertex of the span. Vertical composition is associative, since composition in **A** is associative. Note that there exists a canonical isomorphism of pullbacks $a : x \times_b (y \times_c z) \to (x \times_b y) \times_c z$, and this satisfies the pentagonal axiom. To define horizontal composition of 2-cells, consider the following diagram:



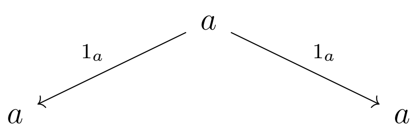
the horizontal composition $s * r : x \times_b y \to x' \times_b y'$ is given by $s * r = r \times s : x \times_b y \to x' \times_b y'$. To see that this is indeed a map of spans, note that

$$f'\pi_1'(r \times s) = f'r\pi_1$$
$$f'r\pi_1 =$$
$$f\pi_1$$

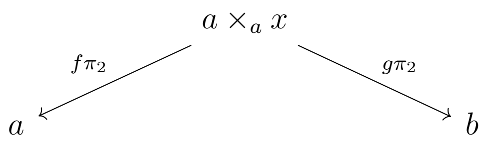
and

$$k'\pi_2'(r \times s) = k's\pi_2$$
$$k's\pi_2 =$$
$$k\pi_2$$

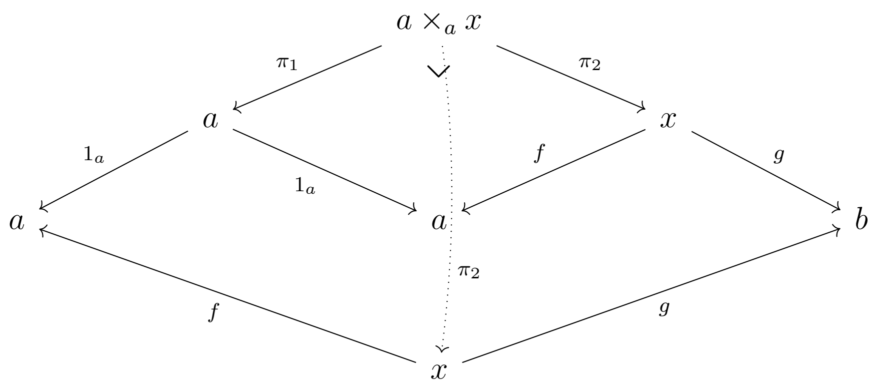
and so we have a map of spans. Now, if we define the left identity of a span $(f, g, x) : a \to b$ to be the following:



Taking the pullback along this gives us a span



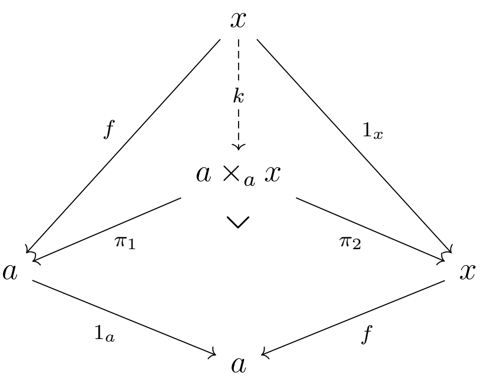
where the vertex is isomorphic to x and gives a map of spans, both of which we will show using the following diagrams:
We begin with the diagram below



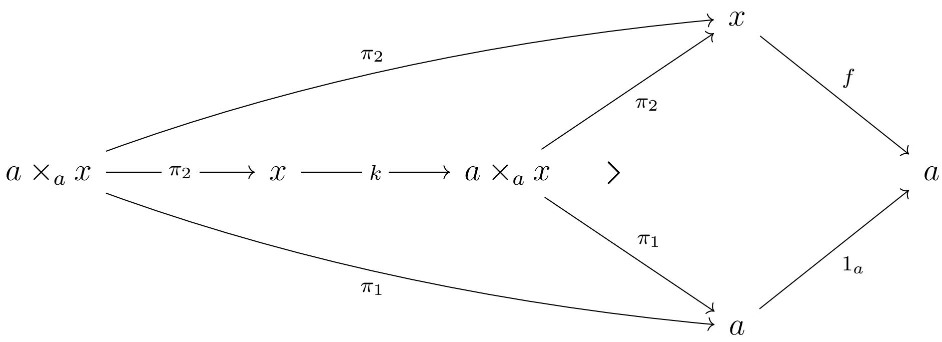
which commutes since

$$f\pi_2 = 1_a\pi_1 = \pi_1$$

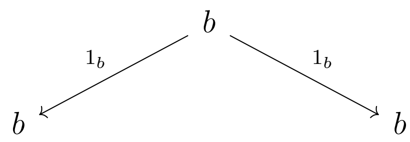
and so we have a map of spans. We then use the pullback property to give us:



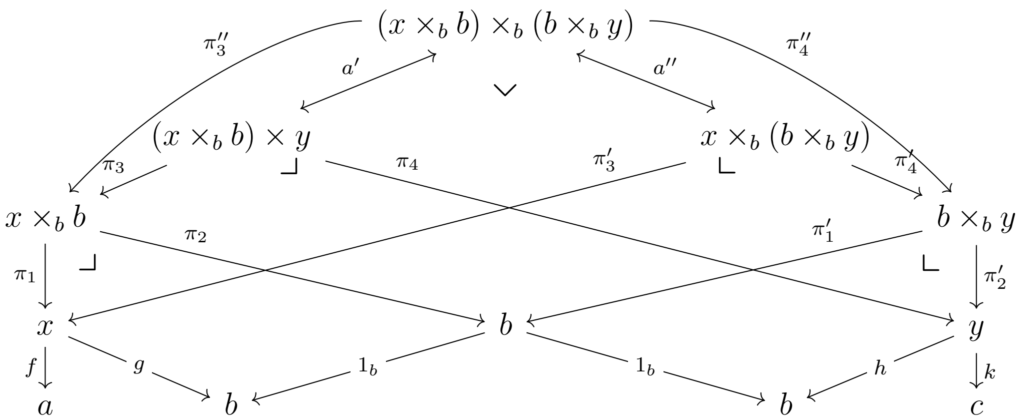
where we have $\pi_2 k = 1_x$ by the universal property of pullbacks. Finally, we have the following:



which is a pullback diagram and by the uniqueness of pullbacks we must have $k\pi_2 = 1_{a\times_a x}$ and so we have an isomorphism between the vertices of the spans. We then have that $r = \pi_2$.
The right identity for b is of course



and can be shown to produce an isomorphism in the same way. For the coherence of the identities to be satisfied, we must have $r \times 1_y : (x \times_b b) \times_b y \to x \times_b y$ is equal to $(1_x \times l)a : (x \times_b b) \times_b y \to x \times_b y$. To show this, consider the following diagram:



In this diagram we have both pullbacks, and the relevant associativity isomorphism displayed as constructed in the topmost vertex. We now see that showing the coherence condition is equivalent to showing that $\pi_1\pi_3 \times \pi_4 = (\pi_3' \times \pi_2'\pi_4')a''a'$. To demonstrate this, note that:

$$\pi_4 a'^{-1} = \pi_2'\pi_4''$$

by property of the pullback, but we also have

$$\pi_4'' = \pi_4' a''$$

and so

$$\pi_4 = \pi_2' \pi_4' a'' a'$$

and so we have the first part of coherence. The second part is done similarly.

Example 2.2.2 (Bimodules as Bicategories). Recall that given rings R, S with unit, an (R, S)-bimodule A is an abelian group that has both the structure of a left R-module and a right S-module, with the additional condition that for all $r \in R$, $s \in S$, and $a \in A$, the following is satisfied:

$$(ra)s = r(as)$$

Given (R, S)-bimodules A, B, we define an (R, S)-bimodule homomorphism $\phi : A \to B$ as a map which is both an R-module homomorphism and an S-module homomorphism, i.e. an abelian group homomorphism that satisfies the following:

$$\phi(ra) = r\phi(a) \tag{2.1}$$
$$\phi(1_R a) = \phi(a) \tag{2.2}$$
$$\phi(as) = \phi(a)s \tag{2.3}$$
$$\phi(a1_S) = \phi(a) \tag{2.4}$$

Now, consider a structure with objects the rings with unit, as arrows $A : R \to S$ the (R, S)-bimodules, and as 2-cells the (R, S)-bimodule homomorphisms. We will show that this structure satisfies the definition of a bicategory. First, we will note that the set of (R, S)-bimodules with (R, S)-bimodule homomorphisms form a category. To see this, note that given (R, S)-bimodules A, B, C and homomorphisms $\psi : A \to B$ and $\phi : B \to C$, we can define their composite $\phi\psi : A \to C$ as $\phi\psi(ras) = r\phi\psi(a)s$. This composition is clearly associative. Also, given an (R, S)-bimodule A we have an identity (R, S)-bimodule homomorphism $1_A : A \to A$ where we define $1_A(ras) = ras$. This clearly acts as an identity for composition of (R, S)-bimodule homomorphisms, and so we have a category whose objects are (R, S)-bimodules and whose arrows are (R, S)-homomorphisms. We take this composition to be our vertical composition.

It now remains to define horizontal composition of 1-cells and 2-cells in our structure. Given rings R, S, T, and 1-cells (bimodules) $A : R \to S$ and $B : S \to T$, we define the composite of B with A as the S-tensor product of A and B. Explicitly, A carries the structure of a right S-module, and B the structure of a left S-module. This means that taking the tensor product $A \otimes_S B$ gives us the structure of an (R, T)-bimodule. This (R, T)-bimodule structure is given by $r(a \otimes b) = ra \otimes b$ and $(a \otimes b)t = a \otimes bt$. We obtain this from the universal property of tensor products which gives us a homomorphism for each r derived from the bilinear map $r : A \times B \to A \otimes B$ defined as $(a, b) \mapsto ra \otimes b$. Likewise for each t we have $t : A \times B \to A \otimes B$ defined as $(a, b) \mapsto a \otimes bt$. We have

$$(r(a \otimes b))t = (ra \otimes b)t = ra \otimes bt = r((a \otimes b)t)$$

and so we have a bimodule structure.

We note that this construction satisfies the pentagon axiom on 1-cells, take objects R, S, T, U and 1-cells $A : R \to S$, $B : S \to T$, and $C : T \to U$. Taking the 1-cells $(A \otimes_S B) \otimes_T C$ and $A \otimes_S (B \otimes_T C)$, we note that there is an isomorphism $\tilde{a}_{A,B,C} : (a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$, shown on elements. The pentagon axiom then yields from a simple chase of elements.

To show left identity of (R, S)-bimodules, note that given a ring R, it can be equipped with the structure of an (R, R)-bimodule with the action as left and right multiplication. Taking the identity to be the (R, R)-bimodule which is R itself, we know from the definition of tensor products, that given an (R, S)-bimodule A, taking the tensor product $R \otimes_R A$ yields an isomorphism $\lambda : R \otimes_R A \cong A$ defined by $r \otimes a \mapsto ra$. For the right identity, we simply take the tensor product $A \otimes_S S$ and the isomorphism $\rho : A \otimes_S S \cong A$ defined by $a \otimes s \mapsto as$. Once again, one can show the coherence conditions in the following simple chase of elements:

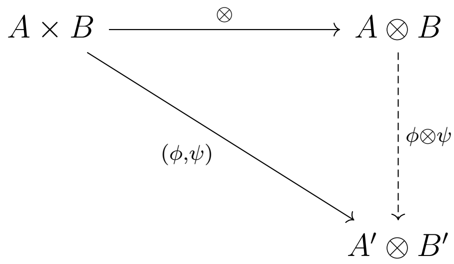
$$\begin{aligned}(1 \times \lambda)\tilde{a}((a \otimes r) \otimes b) &= (1 \times \lambda)(a \otimes (r \otimes b)) \\ (1 \times \lambda)(a \otimes (r \otimes b)) &= \\ a \otimes rb\end{aligned}$$

On the other side we have:

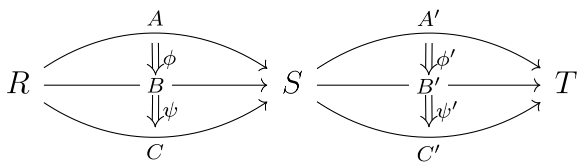
$$(\rho \times 1)((a \otimes r) \otimes b) = ar \otimes b$$

but of course we have $ar \otimes b = a \otimes rb$ by definition of the tensor product, and so we have that our coherence is satisfied.

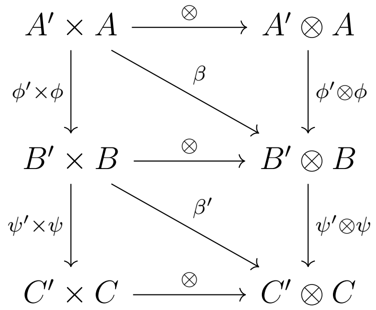
Now, we move on to define the vertical and horizontal composition of the 2-cells. The vertical composition is just the regular composition of bimodule homomorphisms, which as established, has identities and is associative. For the horizontal composition, we take the tensor product of bimodule homomorphisms, defined as in the following diagram:



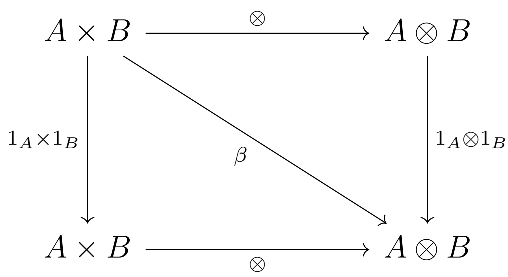
where $(\phi, \psi) : A \times B \to A' \otimes B'$ is the bilinear map which maps $(a, b) \in A \times B$ to $\phi(a) \otimes \psi(b)$. We will write this map simply as β. To show that the middle four exchange holds, we need to show that given a diagram:



we have $(\psi' \otimes \psi)(\phi' \otimes \phi) = \psi'\phi' \otimes \psi\phi$. To do this, note that the following diagram is commutative:



where the top and right arrows give the left hand side of the equation and the bottom and left arrows give the right hand side. For preservation of identity, we have the following commutative diagram:

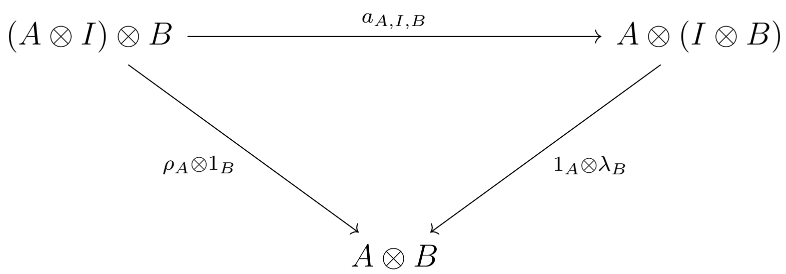


which gives $(1_A \otimes 1_B)(a \otimes b) = a \otimes b$ and so $1_A \otimes 1_B = 1_{A \otimes B}$.

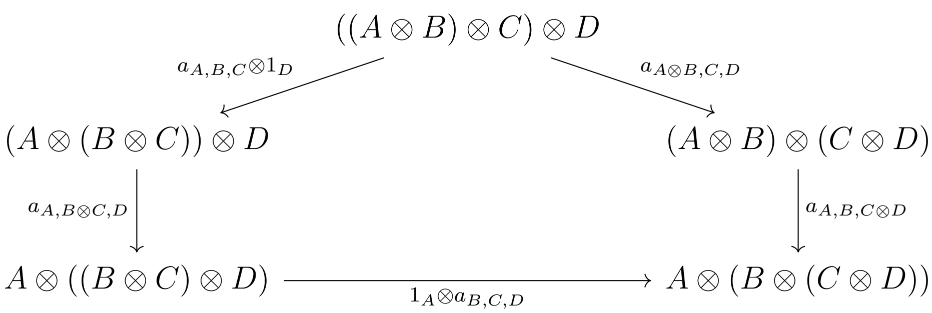
2.3 Monoidal and Enriched Categories

There is also another way to look at 2-categories, namely by considering them as a category enriched over the category of categories **Cat**. To define the concept of enrichment, first we must define monoidal categories, state some basic results about them, and provide examples and context. A monoidal category is a category equipped with a tensor product like structure, together with natural isomorphisms, and a unit object, such that certain diagrams commute. More formally:

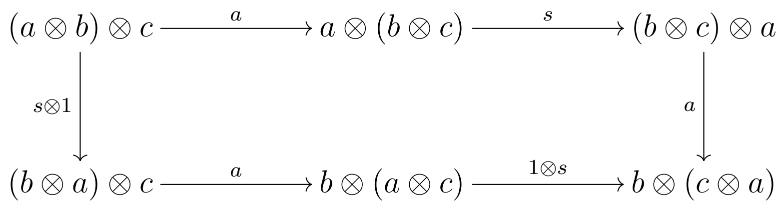
Definition 2.3.1 (Monoidal Category). A monoidal category $\mathbf{V}$, written as $\mathbf{V} = (V_0, \otimes, a, I, \lambda, \rho)$ consists of a category V_0, a bifunctor $\otimes : V_0 \times V_0 \to V_0$, a natural isomorphism $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, a unit object I of V_0, and unit natural isomorphisms $\lambda : I \otimes - \to -$ and $\rho : - \otimes I \to -$ (note that we will usually look at these natural isomorphisms componentwise, and so by abuse of notation will also write $\lambda : I \otimes A \to A$, omitting the subscript when it is understood which objects it is acting on), such that the following diagrams commute for all objects:
First, the identity



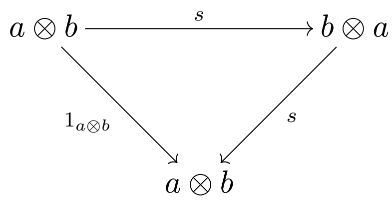
expressing that the maps λ and ρ act as identities in a consistent manner, and then the pentagonal identity:



Monoidal categories will provide the base for our enriched categories and we will use their tensor products to define composition of our objects in **V-Cat**, the category of enriched categories over the monoidal category **V**. Certain familiar categories have monoidal structures given by the products in these categories. These are called cartesian monoidal categories, and include **Set** and **Cat**. There is also the case where in addition to the usual natural isomorphisms, there exists another natural isomorphism $s_{a,b} : a \otimes b \to b \otimes a$ for each pair of objects $(a, b) \in V_0 \times V_0$ called the symmetry isomorphism, which must satisfy the following commutative diagrams:



expressing the relation between symmetry and associativity, and the diagram



In this case, we call **V** a symmetric monoidal category.

2.3.1 Structure and Coherence in Monoidal Categories

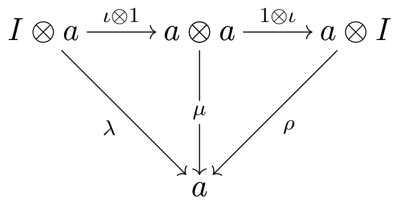
It is the case for monoidal categories that all diagrams of a certain type will commute, meaning that we can speak of arrows of this form between objects without confusion. This result can be stated as below, following [4]:

Theorem 2.3.1 (Coherence Theorem for Monoidal Categories). Given an ordered collection of objects $A_1, A_2, .., A_n$ of a monoidal category **V**, and let P_1, P_2 be products made up of all of these objects (in order), as well as the unit object I, arbitrarily inserted, with any parenthesising. Then, given maps $f, g : P_1 \to P_2$ made up of the composition of associativity and unit isomorphisms as well as their inverses, possibly tensored with the identity 1, we have that $f = g$.

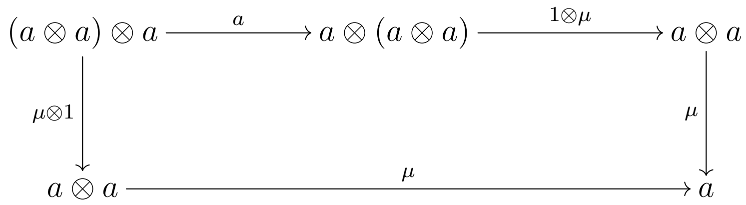
We can also examine structures internal to the category itself. For instance, we can define a monoid in a monoidal category:

Example 2.3.1 (Monoid in a Monoidal Category). A monoid in a monoidal category **V** consists of an object a in **V** and arrows $\mu : a \otimes a \to a$ and

$\iota : I \to a$ such that the following diagrams commute:



which shows that the identity acts as such, and then the following diagram:



which expresses associativity. We can then define the notion of a monoid transformation in a monoidal category $\mathbf{V}$. A monoid transformation between monoids (a, μ, ι) and (b, μ', ι') is a map $f : a \to b$ such that $f\mu = \mu'(f \otimes f)$ and $f\iota = \iota'$.

A monoid in a monoidal category corresponds exactly to the standard definition of a monoid when the category is **Set** with product, and monads when the category is $\mathbf{C}^C$, the category of endofunctors of $\mathbf{C}$ with tensor product given by composition. There is also a special monoidal category, called the simplex category, which appears widely in mathematics. We can define it as follows:

Example 2.3.2 (Simplex Category). The simplex category is a monoidal category Δ consisting of objects the ordinal numbers i.e. $\mathbf{0} = \emptyset$, $\mathbf{1} = \{0\}$, etc, and morphisms order-preserving functions $f : n \to m$ with $m \leq n$ implies $f(m) \leq f(n)$, where $\leq$ is the standard ordering on natural numbers. The monoidal structure $+$ is given by ordinal addition on objects and on functions $f : m \to n$ and $g : m' \to n'$ as $f + g : m + m' \to n + n'$ defined by $(f + g)(k) = f(k)$ if $k \leq m - 1$ and $(f + g)(k) = m + g(k - n)$ if $n \leq k \leq n + n' - 1$.

As an example that we will extend to our consideration of enriched categories, consider the following:

Example 2.3.3 (Category of Abelian Groups). The category of Abelian Groups **Ab** has the structure of a monoidal category, given by the standard tensor product $\otimes_{\mathbb{Z}}$ (which we will write as $\otimes$ for the rest of this example). We define our map $a_{A,B,C} : (A \otimes B) \otimes C :\to A \otimes (B \otimes C)$ as acting on elements of the group as follows $a_{A,B,C} : (a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$. We define our unit object to be $\mathbb{Z}$ and our unit maps $\lambda_A : \mathbb{Z} \otimes A \to A$ and $\rho_A : A \otimes \mathbb{Z} \to A$ as:

$$\lambda(n \otimes b) = nb$$
$$\rho(b \otimes n) = bn$$

respectively. To see that the relevant diagrams commute, note that we have:

$$\begin{aligned}(1 \otimes \lambda)a((a' \otimes n) \otimes b) &= (1 \otimes \lambda)(a' \otimes (n \otimes b))\\ (1 \otimes \lambda)(a' \otimes (n \otimes b)) &= a' \otimes nb\\ a' \otimes nb &= a'n \otimes b\\ a'n \otimes b\end{aligned}$$

but we also have:

$$\rho \otimes 1((a' \otimes n) \otimes b) = (\rho(a' \otimes n) \otimes b) = a'n \otimes b$$

and so the coherence for identity holds. The case for the pentagon axiom is as follows:

$$\begin{aligned}a(a(((a' \otimes b) \otimes c) \otimes d) &= a((a' \otimes b) \otimes (c \otimes d))\\ a((a' \otimes b) \otimes (c \otimes d)) &= a' \otimes (b \otimes (c \otimes d))\end{aligned}$$

Following the bottom 3 arrows, we obtain:

$$\begin{aligned}(1 \otimes a)(a)(a \otimes 1)((a' \otimes b) \otimes c) \otimes d) &=\\ (1 \otimes a)(a)((a' \otimes (b \otimes c)) \otimes d) &=\\ (1 \otimes a)(a' \otimes ((b \otimes c) \otimes d)) &=\\ a' \otimes (b \otimes (c \otimes d))\end{aligned}$$

and so we have that the category of abelian categories is a monoidal category.

Finally, to relate to our definition of bicategories in the previous section, we have the following example:

Example 2.3.4 (One object bicategory). Consider a bicategory $\mathbf{V}$ with only one object, which we will call V_0. Now, we will show that this is the same as a monoidal category. Let the 1-cells of $\mathbf{V}$ be objects and the composition of 1-cells be the tensor product. Then, because $\mathbf{V}$ is a bicategory, this assignment is bifunctorial and we have a natural isomorphism defined on 1-cells h, g, f as $a_{h,g,f} : (hg)f \mapsto h(gf)$, as well as a distinguished 1-cell I_{V_0}, equipped with natural isomorphisms $l : I_{V_0}f \to f$ and $r : fI_{V_0} \to f$

for any f in $\mathbf{V}$. These are of course subject to the coherence conditions, which are the same as those on a monoidal category. Since this is also true for the 2-cells with horizontal composition, we have that a one object bicategory $\mathbf{V}$ has the structure of a monoidal category, with objects the one cells, composition defining the tensor product, and 2-cells as the arrows.

2.3.2 Enriched Categories defined

Now that we have the definition of a monoidal category we can define an enriched category and show that a 2-category is simply a **Cat**-category, that is, a category enriched over **Cat**. Our main references for this section are [5] and [3].

Definition 2.3.2 (Enriched Category). A category $\mathbf{A}$ is said to be enriched over the monoidal category $\mathbf{V} = (V_0, \otimes, I, \rho, \lambda)$ if for the set of objects $\text{ob}(\mathbf{A}) = A_0$ we have for each pair $(a, b) \in A_0 \times A_0$ an object $\mathbf{A}(a, b)$ of $\mathbf{V}$, which we call the hom object of a and b. Then, for each triple $(a, b, c) \in A_0 \times A_0 \times A_0$, we have a composition arrow in $\mathbf{V}$ defined on the tensor product as $M : \mathbf{A}(b, c) \otimes \mathbf{A}(a, b) \to \mathbf{A}(a, c)$ and an identity for each object a of $\mathbf{A}$ defined as $id_a : I \to \mathbf{A}(a, a)$ such that the following diagrams commute:

$$\begin{array}{ccc}
(\mathbf{A}(c,d) \otimes \mathbf{A}(b,c)) \otimes \mathbf{A}(a,b) & \xrightarrow{a} & \mathbf{A}(c,d) \otimes (\mathbf{A}(b,c) \otimes \mathbf{A}(a,b)) \\
 & & \downarrow{\scriptstyle 1\otimes M} \\
\downarrow{\scriptstyle M\otimes 1} & & \mathbf{A}(c,d) \otimes \mathbf{A}(a,c) \\
 & & \downarrow{\scriptstyle M} \\
\mathbf{A}(b,d) \otimes \mathbf{A}(a,b) & \xrightarrow[M]{} & \mathbf{A}(a,d)
\end{array}$$

which expresses associativity of the composition, and then

$$\begin{array}{ccccc}
I \otimes \mathbf{A}(a,b) & & & & \mathbf{A}(a,b) \otimes I \\
\downarrow{\scriptstyle id_b\otimes 1} & \searrow^{\lambda} & & \swarrow^{\rho} & \downarrow{\scriptstyle 1\otimes id_a} \\
\mathbf{A}(b,b) \otimes \mathbf{A}(a,b) & \xrightarrow[M]{} & \mathbf{A}(a,b) & \xleftarrow[M]{} & \mathbf{A}(a,b) \otimes \mathbf{A}(a,a)
\end{array}$$

which expresses the identity of the composition

Before we proceed, we add a brief example here of categories enriched over the category **Ab** of abelian groups.

Example 2.3.5 (Ab-category). The category **Ab** comes with a monoidal structure given by the standard tensor product of abelian groups, as shown in the previous section. A category enriched over **Ab** would then consist of a class of objects A_0, and to each pair $(a, b) \in A_0 \times A_0$, an abelian group $\mathbf{A}(a, b)$, and for each triple $(a, b, c) \in A_0 \times A_0 \times A_0$, an abelian group homomorphism $M : \mathbf{A}(b, c) \otimes \mathbf{A}(a, b) \to \mathbf{A}(a, c)$. Our homomorphism is associative, since we have $M(M\otimes 1)((h\otimes g)\otimes f) = M(M(h\otimes g)\otimes f)$ which is equal to $M(1 \otimes M)a((h \otimes g) \otimes f) = M(1 \otimes M)(h \otimes (g \otimes f)) = M(h \otimes M(g \otimes f))$. The object I in **Ab** is of course Z, and so by definition we have that $id_a : \mathbb{Z} \to \mathbf{A}(a, a)$ is the map that makes $M(id_a(n)\otimes f) = \lambda(n\otimes f) = nf$ for $f \in \mathbf{A}(b, a)$. Taking $n = 1$, we then have that $M(id_a(1)\otimes f) = \lambda(1\otimes f) = f$. Likewise, for $g \in \mathbf{A}(a, b)$ we have $M(g \otimes id_a(n)) = \rho(g \otimes n) = gn$ and again for $n = 1$ we have $M(g \otimes id_a(1)) = \rho(g \otimes 1) = g1 = g$, and so $id_a(1)$ gives us a two-sided identity for M. In total, taking A_0 to be our class of objects, $\mathbf{A}(a, b)$ to be our arrows with domain a and codomain b, M our composition, and $id_a(1) \in \mathbf{A}(a, a)$ to be our identity arrow, we see that we have a category. For M we have that our composition arrow is bilinear: Given $g, g' \in \mathbf{A}(b, c)$ and $f, f' \in \mathbf{A}(a, b)$, we have $M((g' + g) \otimes f) = M(g' \otimes f + g \otimes f) = M(g' \otimes f) + M(g \otimes f)$ by the properties of the tensor product and homomorphism M. From this, we see that we have a pre-additive category. Then, if this category we have has finite products, we have an additive category.

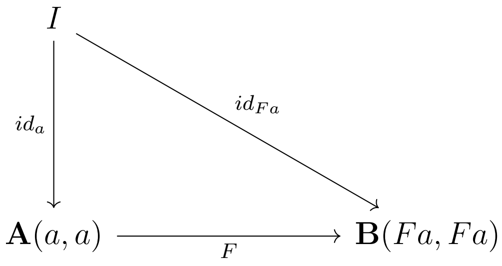
2.3.3 V-Functors

Given two **V**-categories, we can define a map between them, which we call a **V**-functor.

Definition 2.3.3 (V-functor). A **V**-functor F between **V**-categories **A** and **B** consists of a function $F_0 : A_0 \to B_0$ between the objects of **A** and **B** (we will write this as $F_0 : a \mapsto Fa$), and a collection of maps in the category **V**, which we write as $F_1 : \mathbf{A}(a, b) \to \mathbf{B}(Fa, Fb)$ indexed by pairs of objects (a, b) such that the following diagrams commute:

$$\begin{array}{ccc}
\mathbf{A}(b,c)\otimes\mathbf{A}(a,b) & \xrightarrow{\quad M \quad} & \mathbf{A}(a,c) \\
\Big\downarrow{\scriptstyle F\otimes F} & & \Big\downarrow{\scriptstyle F} \\
\mathbf{B}(Fb,Fc)\otimes\mathbf{B}(Fa,Fb) & \xrightarrow[\quad M' \quad]{} & \mathbf{B}(Fa,Fc)
\end{array}$$

which expresses what we call **V**-functorality, and

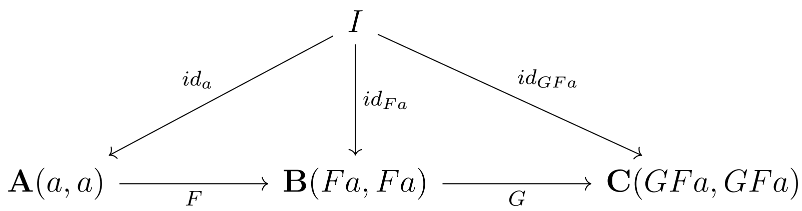


which expresses preservation of identity.

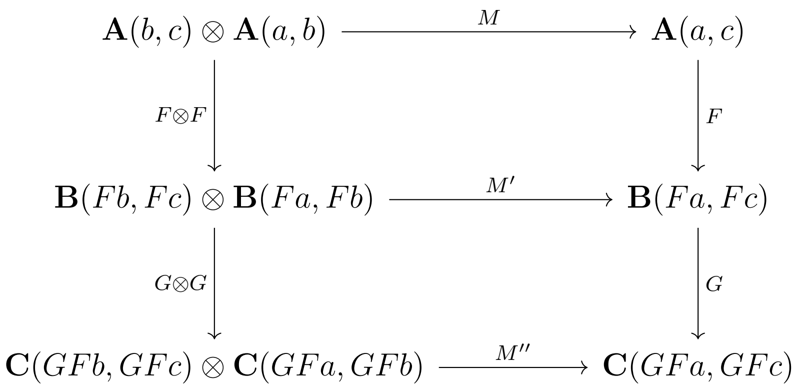
We can also compose **V**-functors as follows: Given $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$, we define the composite as sending objects a of $\mathbf{A}$ to the object GFa in $\mathbf{C}$, and acting on hom-objects as illustrated below:

$$\mathbf{A}(a,b) \xrightarrow{F} \mathbf{B}(Fa,Fb) \xrightarrow{G} \mathbf{C}(GFa,GFb)$$

This is a valid **V**-functor, as the diagrams below demonstrate:



for preservation of identity, and



for preservation of composition, and so **V**-functors behave as one might expect normal functors to.

2.3.4 V-Natural Transformations

Given that we now have a definition of **V**-Functors, we would like to define some notion of natural transformation between them. This leads us to the following definition:

Definition 2.3.4 (V-Natural Transformations). Given **V**-Functors $G, F : \mathbf{A} \to \mathbf{B}$, we define a **V**-natural transformation $\alpha : F \to G$ as a class of arrows $\alpha_a : I \to \mathbf{B}(Fa, Ga)$ in **V** for each object a in **A**, such that the following diagram commutes:

$$\begin{array}{ccc}
I \otimes \mathbf{A}(a,b) & \xrightarrow{\alpha_b \otimes F} & \mathbf{B}(Fb,Gb) \otimes \mathbf{B}(Fa,Fb) \\
\uparrow{\scriptstyle \lambda^{-1}} & & \downarrow{\scriptstyle M'} \\
\mathbf{A}(a,b) & & \mathbf{B}(Fa,Gb) \\
\downarrow{\scriptstyle \rho^{-1}} & & \uparrow{\scriptstyle M'} \\
\mathbf{A}(a,b) \otimes I & \xrightarrow[G \otimes \alpha_a]{} & \mathbf{B}(Ga,Gb) \otimes \mathbf{B}(Fa,Ga)
\end{array}$$

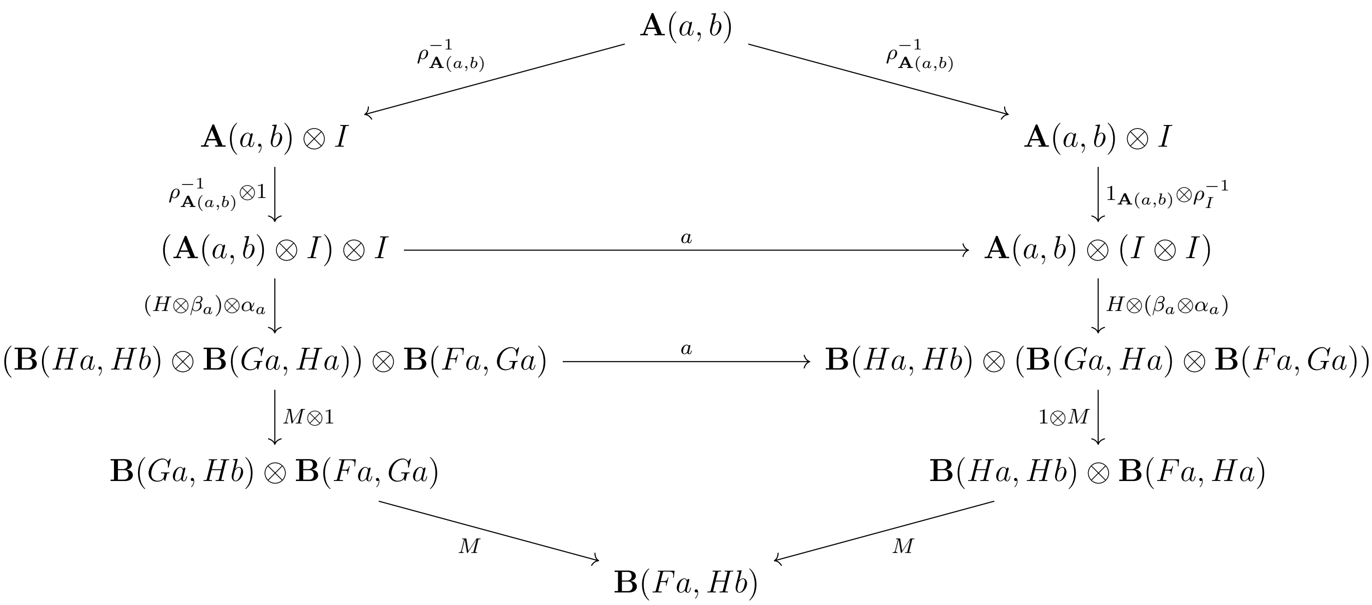
Now, given **V**-functors $H, G, F : \mathbf{A} \to \mathbf{B}$ and **V**-natural transfomations $\alpha : F \to G$ and $\beta : G \to H$, a composition $\beta\alpha : F \to H$ can be defined, as in the following diagram:

$$I \xrightarrow{\lambda^{-1}} I \otimes I \xrightarrow{\beta_a \otimes \alpha_a} \mathbf{B}(Ga, Ha) \otimes \mathbf{B}(Fa, Ga) \xrightarrow{M'} \mathbf{B}(Fa, Ha)$$

To show that this composite does in fact satisfy the definition of a **V**-Natural Transformation, we must show that $M'(\beta\alpha_b \otimes F)\lambda^{-1} = M'(H \otimes \beta\alpha_a)\rho^{-1}$. To do this we first examine the following diagram, as in [4]:

$$\begin{array}{ccc}
 & A \otimes B & \\
\swarrow{\scriptstyle 1 \otimes \lambda^{-1}} & & \searrow{\scriptstyle \rho^{-1} \otimes 1} \\
A \otimes (I \otimes B) & \xrightarrow[a^{-1}]{} & (A \otimes I) \otimes B
\end{array}$$

which commutes because it is the inverse of the diagram expressing coherence for identity in a monoidal category. Setting $A = B = I$ in the above diagram also shows that $\lambda_I^{-1} = \rho_I^{-1}$. Now, before we proceed with this calculation, we note that verifying these equalities involves the verification of some variety of very large commutative diagram. We will usually have only part of the diagram and sometimes not at all, and hope that the reader trusts that the author has done due diligence. We now begin with the first diagram:



Note that this only displays one side of the diagram for naturality, however, we can still use this to aid us. We now begin with $M(H \otimes \beta\alpha_a)\rho^{-1}$.

$$M(H \otimes \beta\alpha_a)\rho^{-1} = M(H \otimes M(\beta_a \otimes \alpha_a)\lambda^{-1})\rho^{-1}$$
$$M(H \otimes M(\beta_a \otimes \alpha_a)\lambda^{-1})\rho^{-1} =$$
$$M(H \otimes M(\beta_a \otimes \alpha_a))(1 \otimes \lambda^{-1})\rho^{-1}$$

Now, note that $M(H \otimes M(\beta_a \otimes \alpha_a)) = M(1 \otimes M)(H \otimes (\beta_a \otimes \alpha_a))$ and so we have:

$$M(H \otimes M(\beta_a \otimes \alpha_a)) = M(1 \otimes M)(H \otimes (\beta_a \otimes \alpha_a))$$
$$M(1 \otimes M)(H \otimes (\beta_a \otimes \alpha_a)) =$$
$$M(M \otimes 1)a^{-1}(H \otimes (\beta_a \otimes \alpha_a)) =$$
$$M(M \otimes 1)((H \otimes \beta_a) \otimes \alpha_a)a^{-1}$$

by associativity of M and naturality of a and a^{-1}. Now, returning this into our larger expression gives us:

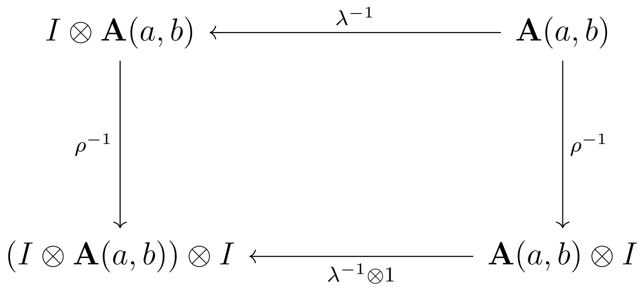
$$M(M \otimes 1)((H \otimes \beta_a) \otimes \alpha_a)a^{-1}(1 \otimes \lambda^{-1})\rho^{-1} =$$
$$M(M \otimes 1)((H \otimes \beta_a) \otimes \alpha_a)(\rho^{-1} \otimes 1)\rho^{-1} =$$
$$M(M(H \otimes \beta_a)\rho^{-1} \otimes \alpha_a)\rho^{-1}$$

however, we know that $M(H \otimes \beta_a)\rho^{-1}_{\mathbf{A}(a,b)} = M(\beta_b \otimes G)\lambda^{-1}_{\mathbf{A}(a,b)}$ by **V**-

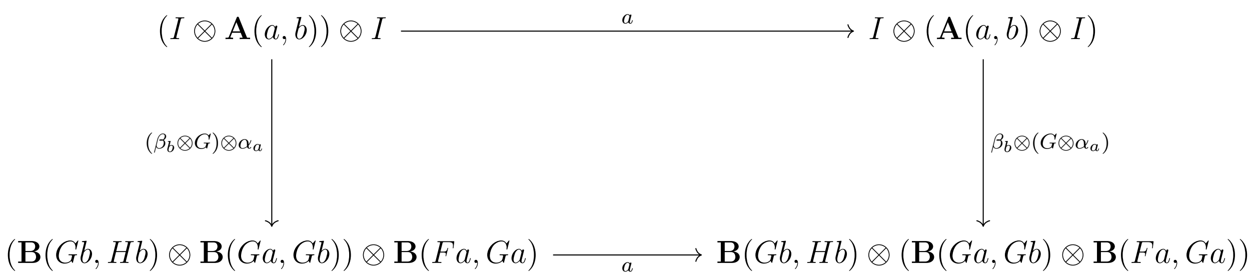
naturality of β. Now, we rewrite our expression as

$$\begin{aligned} M(M(H \otimes \beta_a)\rho^{-1} \otimes \alpha_a)\rho^{-1} &= M(M(\beta_b \otimes G)\lambda^{-1} \otimes \alpha_a)\rho^{-1} \\ &= M(M(\beta_b \otimes G) \otimes \alpha_a)(\lambda^{-1} \otimes 1)\rho^{-1} \end{aligned}$$

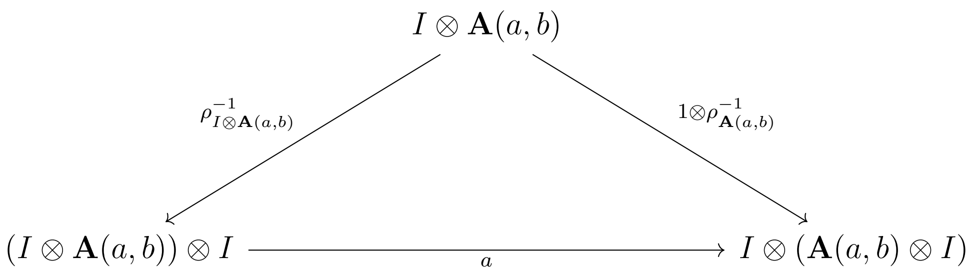
Now, consider the following two diagrams:



and



which commute by coherence and naturality of a respectively, and so we have $M(M(\beta_b \otimes G) \otimes \alpha_a)(\lambda^{-1} \otimes 1)\rho^{-1} = M(\beta_b \otimes M(G \otimes \alpha_a))a\rho^{-1}\lambda^{-1}$. However, we also have the following commutative diagram:



and so we have

$$\begin{gathered} M(\beta_b \otimes M(G \otimes \alpha_a))a\rho^{-1}\lambda^{-1} = \\ M(\beta_b \otimes M(G \otimes \alpha_a))(1 \otimes \rho^{-1})\lambda^{-1} = \\ M(\beta_b \otimes M(G \otimes \alpha_a)\rho^{-1})\lambda^{-1} \end{gathered}$$

but by naturality of α, $M(G \otimes \alpha_a)\rho^{-1} = M(\alpha_b \otimes F)\lambda^{-1}$. Putting this into our expression gives us

$$\begin{aligned} M(\beta_b \otimes M(G \otimes \alpha_a)\rho^{-1})\lambda^{-1} &= \\ M(\beta_b \otimes M(\alpha_b \otimes F)\lambda^{-1}_{\mathbf{A}(a,b)})\lambda^{-1} &= \\ M(\beta_b \otimes M(\alpha_b \otimes F))(1 \otimes \lambda^{-1}_{\mathbf{A}(a,b)})\lambda^{-1} \end{aligned}$$

This leads us to our final calculation:

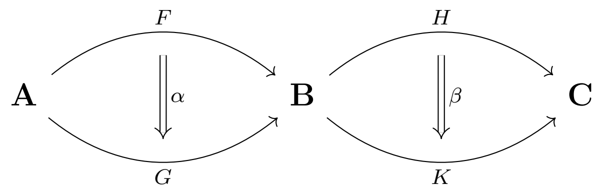
$$\begin{aligned} M(\beta_b \otimes M(\alpha_b \otimes F))(1 \otimes \lambda^{-1}_{\mathbf{A}(a,b)})\lambda^{-1} &= \\ M(M(\beta_b \otimes \alpha_b) \otimes F))a^{-1}(1 \otimes \lambda^{-1}_{\mathbf{A}(a,b)})\lambda^{-1} \end{aligned}$$

and by our diagram at the very top, we have

$$\begin{aligned} M(M(\beta_b \otimes \alpha_b) \otimes F))a^{-1}(1 \otimes \lambda^{-1}_{\mathbf{A}(a,b)})\lambda^{-1} &= \\ M(M(\beta_b \otimes \alpha_b) \otimes F))(\rho^{-1} \otimes 1)\lambda^{-1} \end{aligned}$$

but this is equal to $M(M(\beta_b \otimes \alpha_b)\rho^{-1} \otimes F)\lambda^{-1}$, which is exactly what we needed to show, and so the vertical composition of **V**-natural transformations is natural.

We would now like to define the horizontal composition of **V**-natural transformations. Given **V**-functors $G, F : \mathbf{A} \to \mathbf{B}$, and $H, K : \mathbf{B} \to \mathbf{C}$ along with **V**-natural transformations $\alpha : F \to G$ and $\beta : H \to K$, we let $\beta * \alpha$, depicted pictorially below,



be defined as the following composite, which is commutative:

$$\begin{array}{ccccc} I \otimes I & \xrightarrow{\alpha_a \otimes \beta_{Fa}} & \mathbf{B}(Fa, Ga) \otimes \mathbf{C}(HFa, KFa) & \xrightarrow{K \otimes 1} & \mathbf{C}(KFa, KGa) \otimes \mathbf{C}(HFa, KFa) \\ \uparrow{\scriptstyle \lambda^{-1}} & & & & \downarrow{\scriptstyle M''} \\ I & & & & \mathbf{C}(HFa, KGa) \\ \downarrow{\scriptstyle \lambda^{-1}} & & & & \uparrow{\scriptstyle M''} \\ I \otimes I & \xrightarrow[\beta_{Ga} \otimes \alpha_a]{} & \mathbf{C}(HGa, KGa) \otimes \mathbf{B}(Fa, Ga) & \xrightarrow[1 \otimes H]{} & \mathbf{C}(HGa, KGa) \otimes \mathbf{C}(HFa, HGa) \end{array}$$

To show that this is in fact a natural transformation, we need it to satisfy $M''(\beta * \alpha_b \otimes HF)\lambda^{-1} = M''(KG \otimes \beta * \alpha_b)\rho^{-1}$. This is done by noting that by definition $\beta * \alpha = M''(\beta_{Ga} \otimes H\alpha_a)\lambda^{-1}$, which is how we define vertical composition of natural transformations, and so because $H\alpha$ and $\beta G = \beta_G$ are natural, $\beta * \alpha$ will be as well. To show that $H\alpha$ is natural, we only need to note that the following diagram is commutative:

$$\begin{array}{ccccc}
I \otimes \mathbf{A}(a,b) & \xrightarrow{\alpha_b \otimes F} & \mathbf{B}(Fb,Gb) \otimes \mathbf{B}(Fa,Fb) & \xrightarrow{H \otimes H} & \mathbf{C}(HFb,HGb) \otimes \mathbf{C}(HFa,HFb) \\
\uparrow{\scriptstyle \lambda^{-1}} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M'} \\
\mathbf{A}(a,b) & & \mathbf{B}(Fa,Gb) & \xrightarrow{H} & \mathbf{C}(HFa,HGb) \\
\downarrow{\scriptstyle \rho^{-1}} & & \uparrow{\scriptstyle M} & & \uparrow{\scriptstyle M'} \\
\mathbf{A}(a,b) \otimes I & \xrightarrow{G \otimes \alpha_a} & \mathbf{B}(Ga,Gb) \otimes \mathbf{B}(Fa,Ga) & \xrightarrow{H \otimes H} & \mathbf{C}(HGa,HGb) \otimes \mathbf{C}(HFa,HGa)
\end{array}$$

by naturality of α and functorality of H. To show that β_G is natural is done similarly.

2.3.5 The 2-category of V-categories

Now that we have a definition of **V**-functors and **V**-natural transformations, for which we have notions of composition (and in the latter case, vertical composition and horizontal composition), it is natural to ask whether the collection of all the above forms a 2-category. The answer to this question is yes, and to show this, we need to demonstrate identities and associativity for these compositions, as well as show that for **V**-natural transformations, the middle-four exchange holds.

We begin by noting that associativity of composition for **V**-functors follows from the fact that they are defined by composing arrows in **V**, where the composition is associative. We now continue by defining a **V**-functor for each object a of **A** which we will show to be the identity. Define $1_A : \mathbf{A} \to \mathbf{A}$ as consisting of the identity function on the objects of **A**, and for each hom-object $\mathbf{A}(a,b)$ as consisting of the identity $1_{\mathbf{A}(a,b)} : \mathbf{A}(a,b) \to \mathbf{A}(a,b)$ in **V**. This satisfies the definition of a **V**-functor, since we have that $1_{\mathbf{A}(a,a)} id_a = id_a$ and

$$M(1_{\mathbf{A}(a,a)} \otimes 1_{\mathbf{A}(a,a)}) = M(1_{\mathbf{A}(a,a)\otimes\mathbf{A}(a,a)}) = M = 1_{\mathbf{A}(a,a)} M$$

and from the following arrow we show that it acts as the identity on **V**-functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{A}$. Recall that $F1_A : \mathbf{A} \to \mathbf{B}$ is defined as

$$\mathbf{A}(a,b) \xrightarrow{1_{\mathbf{A}(a,b)}} \mathbf{A}(a,b) \xrightarrow{F} \mathbf{B}(Fa,Fb)$$

which is of course equal to $F : \mathbf{A}(a,b) \to \mathbf{B}(Fa,Fb)$. The situation is the same for $G : \mathbf{B} \to \mathbf{A}$, and so we have defined an identity for composition of **V**-functors.

Now, for each **V**-functor $F : \mathbf{A} \to \mathbf{B}$, we will define a **V**-natural transformation $1_F : F \to F$ that will act as an identity for the vertical composition of **V**-natural transformations.
Given **V**-functors $G, F : \mathbf{A} \to \mathbf{B}$, let $1_F : F \to F$ be the following arrow in **V** for each object a in **A**,

$$I \xrightarrow{id_{Fa}} \mathbf{B}(Fa, Fa)$$

This satisfies the definition of a **V**-natural transformation, as demonstrated in the following diagram:

$$\begin{array}{ccc}
I \otimes \mathbf{A}(a,b) & \xrightarrow{id_{Fb} \otimes F} & \mathbf{B}(Fb,Fb) \otimes \mathbf{B}(Fa,Fb) \\
\uparrow{\scriptstyle \lambda^{-1}} & \searrow^{F\lambda} & \downarrow{\scriptstyle M'} \\
\mathbf{A}(a,b) & & \mathbf{B}(Fa,Fb) \\
\downarrow{\scriptstyle \rho^{-1}} & \nearrow^{F\rho} & \uparrow{\scriptstyle M'} \\
\mathbf{A}(a,b) \otimes I & \xrightarrow[F \otimes id_{Fa}]{} & \mathbf{B}(Fa,Fb) \otimes \mathbf{B}(Fa,Fa)
\end{array}$$

where the internal triangles commute and so the outer arrows of the rectangle have a composite equal to $F_1 : \mathbf{A}(a,b) \to \mathbf{B}(Fa, Fb)$. Given another **V**-natural transformation $\alpha : F \to G$, we have $\alpha 1_F$ as equal to the following:

$$I \xrightarrow{\lambda^{-1} = \rho^{-1}} I \otimes I \xrightarrow{\alpha_a \otimes id_{Fa}} \mathbf{B}(Fa, Ga) \otimes \mathbf{B}(Fa, Fa) \xrightarrow{M'} \mathbf{B}(Fa, Ga)$$

Now, because of the naturality of ρ^{-1}, we have the following commutative diagram:

$$\begin{array}{ccc}
I & \xrightarrow{\rho_I^{-1} = \lambda_I^{-1}} & I \otimes I \\
\downarrow{\scriptstyle \alpha_a} & & \downarrow{\scriptstyle \alpha_a \otimes 1} \\
\mathbf{B}(Fa, Ga) & \xrightarrow[\rho^{-1}_{\mathbf{B}(Fa,Ga)}]{} & \mathbf{B}(Fa, Ga) \otimes I
\end{array}$$

We then note that $M'(\alpha_a \otimes id_{Fa})\rho^{-1} = M'(1_{\mathbf{B}(Fa,Fb)} \otimes id_{Fa})(\alpha_a \otimes 1_I)$. However, by the identity property of id_a, we have

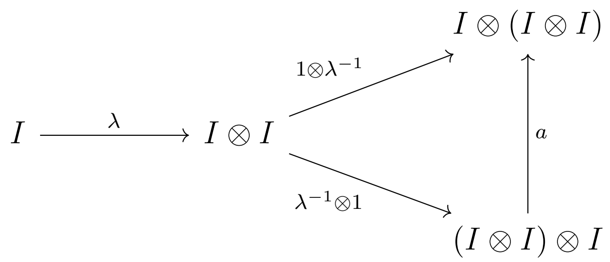
$$M'(1_{\mathbf{B}(Fa,Fb)} \otimes id_{Fa})(\alpha_a \otimes 1_I)\rho^{-1} = \\ \rho_{\mathbf{B}(Fa,Fb)}(\alpha_a \otimes 1_I)\rho^{-1}$$

and by the above diagram, we have that this is equal to α_a. To show the other side of the identity is done in the same way, and so id_{Fa} is an identity under composition of **V**-natural transformations

Now, to show associativity of vertical composition of **V**-natural transformations, we need to show that for **V**-functors $K, H, G, F : \mathbf{A} \to \mathbf{B}$ and **V**-natural transformations $\alpha : F \to G$, $\beta : G \to H$, and $\gamma : H \to K$, $\gamma(\beta\alpha) = (\gamma\beta)\alpha$. This follows from the observation that

$$(\gamma\beta)\alpha = M'(M'(\gamma_a \otimes \beta_a)\lambda^{-1} \otimes \alpha_a)\lambda^{-1}$$

and so from the following diagram and naturality of a:



we have

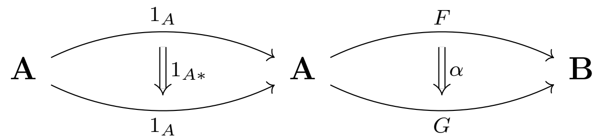
$$\gamma(\alpha\beta)_a = M'(\gamma_a \otimes M'(\beta_a \otimes \alpha_a)\lambda^{-1})\lambda^{-1}$$

$$M'(\gamma_a \otimes M'(\beta_a \otimes \alpha_a)\lambda^{-1})\lambda^{-1} = \\ M'(\gamma_a \otimes M'(\beta_a \otimes \alpha_a))(1 \otimes \lambda^{-1})\lambda^{-1} = \\ M'(M'(\gamma_a \otimes \beta_a) \otimes \alpha_a)a^{-1}(1 \otimes \lambda^{-1})\lambda^{-1} = \\ M'(M'(\gamma_a \otimes \beta_a) \otimes \alpha_a)(\rho^{-1} \otimes 1)\lambda^{-1}$$

which is exactly $\gamma(\beta\alpha)$.

Finally, we will demonstrate identity and associativity of the horizontal composition of **V**-natural transformations.

For identity, we begin with the following diagram:



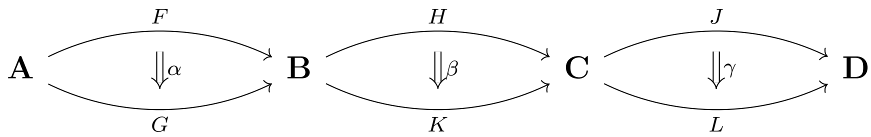
This indicates that we would like to define our horizontal identity as a **V**-natural transformation $1_{A*} : 1_A \to 1_A$. The obvious candidate for this is the map $id_a : I \to A(a,a)$ for each a in $Ob(A)$. To see that it is natural, note that:

$$M(id_b \otimes 1_A)\lambda^{-1} = 1_{\mathbf{A}(a,b)} = M(1_A \otimes id_a)\rho^{-1}$$

by the identity property of the id map, and so we have a legitimate **V**-natural transformation. To see that is is an identity, consider **V**-functors $F, G : A \to B$ with a **V**-natural transformation $\alpha : F \to G$, and consider $\alpha * 1_{A*} : F \to G$. We have that

$$\alpha * 1_{A*} = M'(\alpha_a \otimes Fid_a)\lambda^{-1} = M'(\alpha_a \otimes id_{Fa})\lambda^{-1} = \alpha_a$$

The case for the other side is done similarly, and so we have an identity for horizontal composition of **V**-natural transformations.
To show associativity, we need to demonstrate that given the following diagram of **V**-functors and **V**-natural transformations:



that $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$. To do this, note that we have:

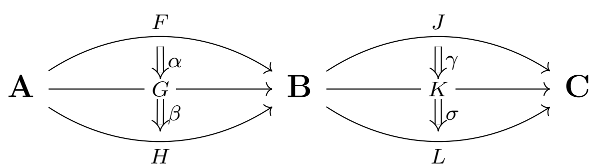
$$(\gamma * \beta)_a = M'''(\gamma_{Ka} \otimes J\beta_a)\lambda^{-1}$$

so that by the diagram used for vertical composition and naturality of a we can write:

$$
\begin{aligned}
(\gamma * \beta) * \alpha = \\
M'''(\gamma * \beta_{Ga} \otimes JH\alpha_a)\lambda^{-1} = \\
M'''(\gamma * \beta_{Ga} \otimes JH\alpha_a)\lambda^{-1} = \\
M'''(M'''(\gamma_{KGa} \otimes J\beta_{Ga})\lambda^{-1} \otimes JH\alpha_a)\lambda^{-1} = \\
M'''(M'''(\gamma_{KGa} \otimes J\beta_{Ga}) \otimes JH\alpha_a)(\lambda^{-1} \otimes 1)\lambda^{-1} = \\
M'''(\gamma_{KGa} \otimes M'''(J\beta_{Ga} \otimes JH\alpha_a))a(\lambda^{-1} \otimes 1)\lambda^{-1} = \\
M'''(\gamma_{KGa} \otimes M'''(J\beta_{Ga} \otimes JH\alpha_a))a(\rho^{-1} \otimes 1)\lambda^{-1} = \\
M'''(\gamma_{KGa} \otimes M'''(J\beta_{Ga} \otimes JH\alpha_a))(1 \otimes \lambda^{-1})\lambda^{-1} = \\
M'''(\gamma_{KGa} \otimes M'''(J\beta_{Ga} \otimes JH\alpha_a)\lambda^{-1})\lambda^{-1} = \\
M'''(\gamma_{KGa} \otimes JM''(\beta_{Ga} \otimes H\alpha_a)\lambda^{-1})\lambda^{-1} = \\
M'''(\gamma_{KGa} \otimes J(\beta * \alpha)_a)\lambda^{-1}
\end{aligned}
$$

But we have that $\gamma * (\beta * \alpha) = M'''(\gamma_{KGa} \otimes J(\beta * \alpha)_a)\lambda^{-1}$ and so we have that our horizontal composition is associative.

We will now show that the horizontal composition preserves vertical identity. Given **V**-functors $F : A \to B$ and $G : B \to C$, along with $1_F : F \to F$ and $1_G : G \to G$, we would like to show that $M(id_{GFa} \otimes Gid_{Fa})\lambda^{-1} = id_{GFa}$, but we have that $M(id_{GFa} \otimes Gid_{Fa})\lambda^{-1} = M(id_{GFa} \otimes id_{GFa})\lambda^{-1} = id_{GFa}$ by the identity property of id, and so we are done. The final step to show that the category of **V**-categories is a 2-category is to show that the middle four exchange holds. To do this, consider the following diagram of **V**-functors and **V**-natural transformations:



Given this, we need to show that $(\sigma\gamma) * (\beta\alpha) = (\sigma * \beta)(\gamma * \alpha)$. To aid in readability, we will employ the visual aid of parts of the diagram once again. First, consider the following diagrams:

$$
\begin{array}{ccccc}
 & & I & & \\
 & & \downarrow{\scriptstyle \lambda^{-1}} & & \\
 & & I \otimes I & & \\
 & \swarrow{\scriptstyle \lambda^{-1}\otimes 1} & & \searrow{\scriptstyle 1\otimes\lambda^{-1}} & \\
(I \otimes I) \otimes I & & \xrightarrow{\quad a \quad} & & I \otimes (I \otimes I) \\
\downarrow{\scriptstyle 1\otimes 1\otimes\lambda^{-1} = 1_{I\otimes I}\otimes\lambda^{-1}} & & & & \downarrow{\scriptstyle 1\otimes\lambda^{-1}_{I\otimes I} = 1\otimes 1\otimes\lambda^{-1}} \\
(I \otimes I) \otimes (I \otimes I) & & \xrightarrow{\quad a \quad} & & I \otimes (I \otimes (I \otimes I))
\end{array}
$$

and

$$
\begin{array}{ccc}
(\mathbf{C}(KHa, LHa) \otimes \mathbf{C}(KGa, KHa)) \otimes (\mathbf{C}(JGa, KGa) \otimes \mathbf{C}(JFa, JGa)) & \xrightarrow{\quad a \quad} & \mathbf{C}(KHa, LHa) \otimes (\mathbf{C}(KGa, KHa) \otimes (\mathbf{C}(JGa, KGa) \otimes \mathbf{C}(JFa, JGa))) \\
\downarrow{\scriptstyle (1\otimes 1)\otimes M''} & & \downarrow{\scriptstyle 1\otimes(1\otimes M'')} \\
(\mathbf{C}(KHa, LHa) \otimes \mathbf{C}(KGa, KHa)) \otimes \mathbf{C}(JFa, KGa) & \xrightarrow{\quad a \quad} & \mathbf{C}(KHa, LHa) \otimes (\mathbf{C}(KGa, KHa) \otimes \mathbf{C}(JFa, KGa)) \\
\downarrow{\scriptstyle M''\otimes 1} & & \downarrow{\scriptstyle 1\otimes M''} \\
\mathbf{C}(KGa, LHa) \otimes \mathbf{C}(JFa, KGa) & & \mathbf{C}(KHa, LHa) \otimes \mathbf{C}(JFa, KHa) \\
\searrow{\scriptstyle M''} & & \swarrow{\scriptstyle M''} \\
 & \mathbf{C}(JFa, LHa) &
\end{array}
$$

which commute by naturality of a and associativity of M. Beginning, we expand $(\sigma * \beta)(\gamma * \alpha)$:

$$
\begin{aligned}
(\sigma * \beta)(\gamma * \alpha) &= \\
M''((\sigma * \beta_a) \otimes (\gamma * \alpha_a))\lambda^{-1} &= \\
M''(M''(\sigma_{Ha} \otimes K\beta_a)\lambda^{-1} \otimes M''(\gamma_{Ga} \otimes J\alpha_a)\lambda^{-1})\lambda^{-1} &= \\
M''(M''(\sigma_{Ha} \otimes K\beta_a) \otimes M''(\gamma_{Ga} \otimes J\alpha_a))(\lambda^{-1} \otimes \lambda^{-1})\lambda^{-1} &= \\
M''(\sigma_{Ha} \otimes M''(K\beta_a \otimes M''(\gamma_{Ga} \otimes J\alpha_a)\lambda^{-1}))a(\lambda^{-1} \otimes 1)\lambda^{-1} &
\end{aligned}
$$

To simplify, we will drop the term $a(\lambda^{-1} \otimes 1)\lambda^{-1}$ as we will not be working with it at present. Now consider the following commutative diagram:

$$
\begin{array}{ccc}
I \otimes (I \otimes (I \otimes I)) & \xrightarrow{1 \otimes a^{-1}} & I \otimes ((I \otimes I) \otimes I) \\
\downarrow{\scriptstyle \sigma_{Ha} \otimes (K\beta_a \otimes (\gamma_{Ga} \otimes J\alpha_a))} & & \downarrow{\scriptstyle \sigma_{Ha} \otimes ((K\beta_a \otimes \gamma_{Ga}) \otimes J\alpha_a)} \\
\mathbf{C}(KHa, LHa) \otimes (\mathbf{C}(KGa, KHa) \otimes (\mathbf{C}(JGa, KGa) \otimes \mathbf{C}(JFa, JGa))) & \xrightarrow{1 \otimes a^{-1}} & \mathbf{C}(KHa, LHa) \otimes ((\mathbf{C}(KGa, KHa) \otimes \mathbf{C}(JGa, KGa)) \otimes \mathbf{C}(JFa, JGa)) \\
\downarrow{\scriptstyle 1 \otimes (1 \otimes M)} & & \downarrow{\scriptstyle 1 \otimes (M \otimes 1)} \\
\mathbf{C}(KHa, LHa) \otimes (\mathbf{C}(KGa, KHa) \otimes \mathbf{C}(JFa, KGa)) & & \mathbf{C}(KHa, LHa) \otimes (\mathbf{C}(JGa, KHa) \otimes \mathbf{C}(JFa, JGa)) \\
\downarrow{\scriptstyle 1 \otimes M} & & \downarrow{\scriptstyle 1 \otimes M} \\
\mathbf{C}(KHa, LHa) \otimes \mathbf{C}(JFa, KHa) & \xleftrightarrow{1} & \mathbf{C}(KHa, LHa) \otimes \mathbf{C}(JFa, KHa)
\end{array}
$$

from which we derive the following:

$$
\begin{aligned}
M''(\sigma_{Ha} \otimes M''(K\beta_a \otimes M''(\gamma_{Ga} \otimes J\alpha_a)\lambda^{-1})) &= \\
M''(\sigma_{Ha} \otimes M''(K\beta_a \otimes M''(\gamma_{Ga} \otimes J\alpha_a))(1 \otimes \lambda^{-1})) &= \\
M''(\sigma_{Ha} \otimes M''(M''(K\beta_a \otimes \gamma_{Ga}) \otimes J\alpha_a)a^{-1}(1 \otimes \lambda^{-1})) &= \\
M''(\sigma_{Ha} \otimes M''(M''(K\beta_a \otimes \gamma_{Ga}) \otimes J\alpha_a)(\rho^{-1} \otimes 1))
\end{aligned}
$$

Now, we have that:

$$M''(K\beta_a \otimes \gamma_{Ga})\rho^{-1} = \gamma * \beta = M''(\gamma_{Ha} \otimes J\beta_a)\rho^{-1}$$

by definition, and so returning this to our equation and inspecting the above diagrams gives:

$$
\begin{aligned}
M''(\sigma_{Ha} \otimes M''(M''(K\beta_a \otimes \gamma_{Ga}) \otimes J\alpha_a)(\rho^{-1} \otimes 1)) &= \\
M''(\sigma_{Ha} \otimes M''(M''(\gamma_{Ha} \otimes J\beta_a) \otimes J\alpha_a)(\rho^{-1} \otimes 1)) &= \\
M''(\sigma_{Ha} \otimes M''(\gamma_{Ha} \otimes M''(J\beta_a \otimes J\alpha_a))a(\rho^{-1} \otimes 1)) &= \\
M''(\sigma_{Ha} \otimes M''(\gamma_{Ha} \otimes M''(J\beta_a \otimes J\alpha_a))(1 \otimes \lambda^{-1})) &= \\
M''(M''(\sigma_{Ha} \otimes \gamma_{Ha}) \otimes M''(J\beta_a \otimes J\alpha_a)\lambda^{-1})a^{-1}.
\end{aligned}
$$

Returning the term $a(\lambda^{-1} \otimes 1)\lambda^{-1}$ to the equations gives us:

$$
\begin{aligned}
M''(M''(\sigma_{Ha} \otimes \gamma_{Ha}) \otimes M''(J\beta_a \otimes J\alpha_a)\lambda^{-1})(\lambda^{-1} \otimes 1)\lambda^{-1} &= \\
M''(M''(\sigma_{Ha} \otimes \gamma_{Ha})\lambda^{-1} \otimes M''(J\beta_a \otimes J\alpha_a)\lambda^{-1})\lambda^{-1} &= \\
M''((\sigma\gamma_{Ha}) \otimes J(\beta\alpha_a))\lambda^{-1}
\end{aligned}
$$

but this final term $M''((\sigma\gamma_{Ha}) \otimes J(\beta\alpha_a))\lambda^{-1}$ is precisely equal to $(\sigma\gamma) * (\beta\alpha)$ and so we are done.

2.3.6 2-categories as Cat-categories

With this definition in hand we can show that a 2-category is a **Cat**-category and vice versa.

Theorem 2.3.2 (2-category and Cat-category equivalence). Any 2-category **C** is also a **Cat**-category, and likewise a **Cat**-category is a 2-category

Proof. To start with, consider a 2-category **C**. For any two objects a, b in **C**, we have a category $\mathbf{C}(a, b)$ with objects those 1-cells with domain a and codomain b, and as arrows, the 2-cells between these 1-cells. We now take the cartesian monoidal structure on **Cat** and begin to define additional structure. Taking $\times$, the product in **Cat**, we can define for any ordered triple of objects in **C**, the map, $M : \mathbf{C}(b, c) \times \mathbf{C}(a, b) \to \mathbf{C}(a, c)$ which is the normal composition of 1-cells in **C**, i.e. $(g, f) \mapsto g \circ f$ and on 2-cells $\alpha \in \mathbf{C}(a, b)$ and $\beta \in \mathbf{C}(b, c)$ as $M(\beta, \alpha) = \beta * \alpha$ where $*$ is the horizontal composition of 2-cells defined in **C**. This is well defined, since the condition of triples ensures domain and codomain are consistent. We also have a map for each object a in **C**, $id_a : 1 \to \mathbf{C}(a, a)$ which is a functor taking the unique object in 1 to 1_a and the identity map to 1_{1_a}. The map $M : \mathbf{C}(b, c) \times \mathbf{C}(a, b) \to \mathbf{C}(a, c)$ is bifunctorial as shown earlier, and so what is left to verify is that $M(M \times 1) = M(1 \times M)\tilde{a}$. To do this, note that given 1-cells $h \in \mathbf{C}(c, d)$, $g \in \mathbf{C}(b, c)$, and $f \in \mathbf{C}(a, b)$, we have

$$\begin{aligned} M(1 \times M)\tilde{a}((h, g), f) &= M(1 \times M)(h, (g, f)) \\ M(1 \times M)(h, (g, f)) &= \\ M(h, M(g, f)) &= \\ h \circ (g \circ f) & \end{aligned}$$

but

$$M(M \times 1)((h, g), f) = M(M(h, g), f) = (h \circ g) \circ f = h \circ (g \circ f)$$

and likewise for 2-cells. Finally, given a 1-cell $f \in \mathbf{C}(a, b)$ we need to have $M(f, id_a) = \rho(f, 1)$, which is trivial since $\rho(f, 1) = f$ and $M(f, id_a) = f \circ 1_a = f$, and similarly for 2-cells and the other side of the diagram, and so a 2-category very clearly has the structure of a **Cat**-category.

Now, consider a **Cat**-category $\mathbf{A} = (A_0, \mathbf{Cat}, \circ, id)$. To show that this is a 2-category, we begin with the following: To each pair $(a, b) \in A_0 \times A_0$ we have a category $\mathbf{A}(a, b)$. We let the objects of this category be 1-cells with domain a and codomain b. The 2-cells are then the arrows of this category, with vertical composition being given by the composition in $\mathbf{A}(a,b)$. For each object $a \in A_0$ we have a map (in this case, a functor) $id_a : \mathbf{1} \to \mathbf{A}(a,a)$, which assigns to the unique object in $\mathbf{1}$ an object 1_a which will be our identity, as well as the identity map of this object $1_{1_a} : 1_a \Rightarrow 1_a$. For each triple of objects $(a,b,c) \in A_0 \times A_0 \times A_0$, we have a functor $M : \mathbf{A}(b,c) \times \mathbf{A}(a,b) \to \mathbf{A}(a,b)$. We take the action of this functor on objects to be composition for 1-cells, and its action on arrows to be horizontal composition of 2-cells. To show that it satisfies the conditions necessary to make $\mathbf{A}$ into a 2-category, we use the definitions we have from the enrichment. We begin, with identity: in **Cat** we have

$$M(1_b, f) = \lambda(1, f) = f$$

and

$$M(f, 1_a) = \rho(f, 1) = f$$

on 1-cells $f : a \to b$. On 2-cells $\alpha : f \to g$ with $f, g \in \mathbf{A}(a,b)$ we have

$$M(1_{1_b}, \alpha) = \lambda(1_1, \alpha) = \alpha$$

and

$$M(\alpha, 1_{1_a}) = \rho(\alpha, 1_1) = \alpha$$

and so if M is our composition, $id_a : \mathbf{1} \to \mathbf{A}(a,a)$ specifies a 1-cell and a 2-cell that act as identities for each object $a \in A_0$. Given arrows $(\beta, \alpha) \in \mathbf{A}(b,c) \times \mathbf{A}(a,b)$, with $\alpha : f \to g$ and $\beta : f' \to g'$, we have that the arrow $M(\beta, \alpha) \in \mathbf{C}(a,c)$ has domain $M(g,f)$ and codomain $M(g', f')$ by functorality of M, and so this composition operation is consistent with domain and codomain. To see that this composition is associative, note that we have a natural isomorphism in **Cat** $\tilde{a} : (\mathbf{A}(c,d) \times \mathbf{A}(b,c)) \times \mathbf{A}(a,b) \to \mathbf{A}(c,d) \times (\mathbf{A}(b,c) \times \mathbf{A}(a,b))$ given by the cartesian monoidal structure. By the definition of enrichment, this behaves as follows with our composition bifunctor M:

$$\begin{aligned} M(M(h,g),f) &= M(M \times 1)((h,g),f) \\ M(M \times 1)((h,g),f) &= \\ M(1 \times M)\tilde{a}((h,g),f) &= \\ M(1 \times M)(h,(g,f)) &= \\ M(h, M(g,f)) & \end{aligned}$$

for 1-cells, and likewise for 2-cells, and so our composition of 1-cells and horizontal composition of 2-cells is associative. Our vertical composition of 2-cells is of course also associative and with identity, since it is composition in a hom-category. Finally, we have to verify the middle-four exchange and horizontal preservation of vertical identity. This follows from the bifunctorality of M. In detail:

$$M(\beta' \circ \beta, \alpha' \circ \alpha) = M(\beta', \alpha') \circ M(\beta, \alpha)$$

from preservation of composition, and

$$M(1_g, 1_f) = 1_{M(g,f)}$$

by preservation of identity. □

This is tremendously convenient, because we can use the notions of **Cat**-functors and **Cat**-natural transformations to give us appropriate definitions for functors (called 2-functors) between 2-categories and natural transformations between these 2-functors.

3. Internal Versions of Structures

3.1 Introduction

We will now work with the notion of an internal category in a category **K** with pullbacks and a terminal object. This notion generalizes the definition of a category. We will also define internal functors and internal natural transformations, and demonstrate that we have a 2-category of internal categories.

Both 2-categories and bicategories can be defined internally in a category **K** with pullbacks and terminal object. In fact, some structures, such as double categories emerge easily when considered in this way (a double category is the same as an internal category in **Cat**). After the section on internal categories, we will show how to define these structures internally in this chapter.

3.2 Internal categories

To start, we will define the notion of an internal category, as well as the notions of internal functor and internal natural transformation and show that these constitute a 2-category.

Definition 3.2.1 (Internal category). Given a category **K** with pullbacks and terminal object, we define an internal category $K = (K_0, K_1, e, s, t, m)$ in **K** as the following data:

1. An object K_0, which we call the object of objects.
2. An object K_1 which we call the object of arrows.
3. Morphisms $t, s : K_1 \to K_0$, and $e : K_0 \to K_1$, such that $se = te = 1_{K_0}$
4. A composition map $m : K_1 \times_{K_0} K_1 \to K_1$ where $K_1 \times_{K_0} K_1$ is the pullback of $s, t : K_1 \to K_0$ (pictured below):

$$\begin{array}{ccc} K_1 \times_{K_0} K_1 & \xrightarrow{\pi_2} & K_1 \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle t} \\ K_1 & \xrightarrow[s]{} & K_0 \end{array}$$

5. The following commutative diagram:

$$\begin{array}{ccc} K_1 \times_{K_0} K_1 \times_{K_0} K_1 & \xrightarrow{1_{K_1} \times m} & K_1 \times_{K_0} K_1 \\ {\scriptstyle m \times 1_{K_1}} \downarrow & & \downarrow {\scriptstyle m} \\ K_1 \times_{K_0} K_1 & \xrightarrow[m]{} & K_1 \end{array}$$

This expresses ”associativity” of composition

6. The following commutative diagram:

$$\begin{array}{ccccc} & \nearrow^{(et, 1_{K_1})} & K_1 \times_{K_0} K_1 & \nwarrow^{(1_{K_1}, es)} & \\ & & \downarrow {\scriptstyle m} & & \\ K_1 & \xrightarrow[1_{K_1}]{} & K_1 & \xleftarrow[1_{K_1}]{} & K_1 \end{array}$$

which expresses that the identity behaves as we would like it to under composition.

7. The following commutative diagram:

$$\begin{array}{ccccc} K_1 & \xleftarrow{\pi_1} & K_1 \times_{K_0} K_1 & \xrightarrow{\pi_2} & K_1 \\ {\scriptstyle t} \downarrow & & \downarrow {\scriptstyle m} & & \downarrow {\scriptstyle s} \\ K_0 & \xleftarrow[t]{} & K_1 & \xrightarrow[s]{} & K_0 \end{array}$$

which expresses that the domain and codomain of the composite are consistent with our expectations.

In the case where our category $\mathbf{K} = \mathbf{Set}$ we simply recover the definition of a small category. In the case where $\mathbf{K} = \mathbf{Grp}$, an internal category is equivalent to a crossed module. We can see though, that the idea of an internal category generalizes that of a small category, and so it makes sense to want to define maps between internal categories in the same ambient category $\mathbf{K}$, which leads us to the following:

Definition 3.2.2 (Internal Functor). Given internal categories $K = (K_0, K_1, e, s, t, m)$ and $K' = (K'_0, K'_1, e', s', t', m')$ in $\mathbf{K}$, an internal functor $f : K \to K'$ consists of two morphisms in $\mathbf{K}$ between the objects of objects and the objects of arrows, which we write as $f_0 : K_0 \to K_0'$ and $f_1 : K_1 \to K_1'$ respectively, and such that:

1. The following diagrams commute:

$$\begin{array}{ccc} K_1 & \xrightarrow{f_1} & K_1' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ K_0 & \xrightarrow[f_0]{} & K_0' \end{array} \qquad \begin{array}{ccc} K_1 & \xrightarrow{f_1} & K_1' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ K_0 & \xrightarrow[f_0]{} & K_0' \end{array}$$

2. The following diagram commutes:

$$\begin{array}{ccc} K_0 & \xrightarrow{f_0} & K_0' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ K_1 & \xrightarrow[f_1]{} & K_1' \end{array}$$

3. The following diagram commutes:

$$\begin{array}{ccc} K_1 \times_{K_0} K_1 & \xrightarrow{f_1 \times f_1} & K_1' \times_{K_0'} K_1' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ K_1 & \xrightarrow[f_1]{} & K_1' \end{array}$$

Now that we have the definition of an internal functor, we note that we have a composition operation on internal functors: given $f : K \to K'$ and $g : K' \to K''$, we define $gf : K \to K''$ as the composites $(gf)_0 = g_0 f_0 : K_0 \to K_0''$ and $(gf)_1 = g_1 f_1 : K_1 \to K_1''$. That this itself forms an internal functor comes from the fact that the relevant diagrams are formed by pasting commutative diagrams, and so commute themselves. Now, with this composition, we would like to define a suitable notion of identity.

Example 3.2.1 (Identity internal functor). Given an internal category K, define $1_K : K \to K$ as consisting of the identity arrows $1_{K_0} : K_0 \to K_0$ and $1_{K_1} : K_1 \to K_1$. That this forms an internal functor is trivial. Composing this with another internal functor $f : K \to K'$ gives us the arrows $f_0 : K_0 \to K_0'$ and $f_1 : K_1 \to K_1'$ by the identity property of the ambient category $\mathbf{K}$ and so we have an identity for composition of internal functors.

Now that we have a suitable notion of internal functors, we move on to the definition of internal natural transformations between internal functors.

Definition 3.2.3 (Internal Natural Transformations). An internal natural transformation between internal functors $f, g : K \to K'$, which we will write as $\alpha : f \to g$, consists of a morphism $\alpha : K_0 \to K_1'$ such that the following commute:

1.

$$\begin{array}{ccc} K_0 & \xrightarrow{\alpha} & K_1' \\ & \searrow^{f} & \downarrow{\scriptstyle s'} \\ & & K_0' \end{array} \qquad \begin{array}{ccc} K_1' & \xleftarrow{\alpha} & K_0 \\ {\scriptstyle t'}\downarrow & \swarrow_{g} & \\ K_0' & & \end{array}$$

which expresses consistency of domain and codomain.

2.

$$\begin{array}{ccc} K_1 & \xrightarrow{(g_1, \alpha s)} & K_1' \times_{K_0'} K_1' \\ {\scriptstyle (\alpha t, f_1)}\downarrow & & \downarrow{\scriptstyle m'} \\ K_1' \times_{K_0'} K_1' & \xrightarrow[m']{} & K_1' \end{array}$$

which expresses naturality.

Now, we would like to define both vertical and horizontal composition of internal natural transformations. We shall begin with the vertical case. Given internal functors $f, g, h : K \to K'$ and internal natural transformations $\alpha : f \to g$ and $\beta : g \to h$, we define the vertical composite $\beta \circ \alpha : f \to h$ as the following:

$$K_0 \xrightarrow{(\beta, \alpha)} K_1' \times_{K_0'} K_1' \xrightarrow{m} K_1'$$

This is well defined since $s\beta = g_0 = t\alpha$. We have that the first two diagrams with domain and codomain are satisfied easily by the diagrams specifying domain and codomain of m'.
Now, we need to show that $m'(h_1, (\beta \circ \alpha)s) = m'((\beta \circ \alpha)t, f_1)$. To do this,

note that we have:

$$\begin{aligned} m'((\beta \circ \alpha)t, f_1) &= m'(m'(\beta t, \alpha t), f_1) \\ m'(m'(\beta t, \alpha t), f_1) &= \\ m'(\beta t, m'(\alpha t, f_1)) &= \\ m'(\beta t, m'(g_1, \alpha s)) &= \\ m'(m'(\beta t, g_1), \alpha s) &= \\ m'(m'(h_1, \beta s), \alpha s) &= \\ m'(h_1, m'(\beta s, \alpha s)) &= \\ m'(h_1, m'(\beta, \alpha)s) &= \\ m'(h_1, (\beta \circ \alpha)s) & \end{aligned}$$

Finally, to demonstrate associativity, note that given internal functors $f, g, h, k : K \to K'$ and internal natural transformations $\alpha : f \to g$, $\beta : g \to h$, and $\gamma : h \to k$, we have that $\gamma \circ (\beta \circ \alpha) = m'(\gamma, m'(\beta, \alpha)) = m'(m'(\gamma, \beta), \alpha) = (\gamma \circ \beta) \circ \alpha$ by the associativity of m' itself, and so our vertical composition is associative.

Example 3.2.2 (vertical identity natural transformation). Given internal functors $h, f, g : K \to K'$, and internal natural transformations $\alpha : f \to g$ and $\alpha' : h \to f$ we can define a (left and right) identity $1_f : f \to f$ for these as follows: $1_f = e' f_0 : K_0 \to K'_1$. This satisfies the first triangles since $s'e'f_0 = f_0 = t'e'f_0$ because $t'e' = s'e' = 1_{K'_0}$. It satisfies the naturality square since

$$m'(f_1, e' f_0 s) = m'(f_1, e's'f_1) = m'(1_{K'_1}, e's')f_1$$

and

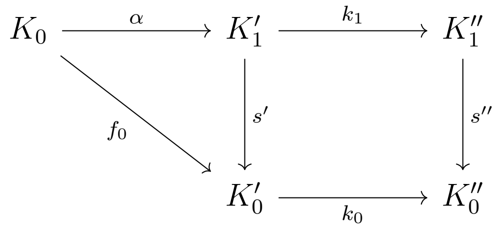
$$m'(e' f_0 t, f_1) = m'(e't'f_1, f_1) = m(e't', 1_{K'_1})f_1 = m'(1_{K'_1}, e's')f_1$$

and so both sides commute. To show that it is the identity, note that $\alpha \circ 1_f = m(\alpha, e' f_0) = m(\alpha, e's\alpha) = \alpha$ by the identity property of e'. The case for left sided identity is done similarly.

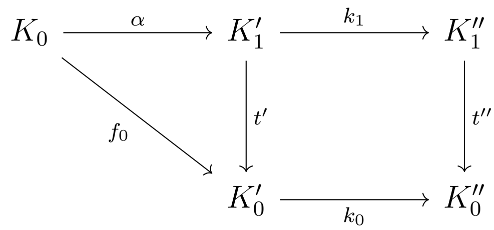
Having defined this, we now move on to horizontal composition of internal natural transformations. Given internal functors $f, g : K \to K'$ and $h, k : K' \to K''$ along with internal natural transformations $\alpha : f \to g$ and $\beta : h \to k$, we would like to define an internal natural map $\beta * \alpha : hf \to kg$. We do this as in the following composite:

$$K_0 \xrightarrow{(k_1\alpha, \beta f_0)} K''_1 \times_{K''_0} K''_1 \xrightarrow{m''} K''_1$$

We can now see that the first two diagrams are easily satisfied again. We now need to show that $m''((kg)_1, (\beta * \alpha)s) = m''((\beta * \alpha)t, (hf)_1)$. To do this, we show that both $k_1\alpha$ and βf_0 are internal natural transformations. To see this, note that the following diagrams commute:



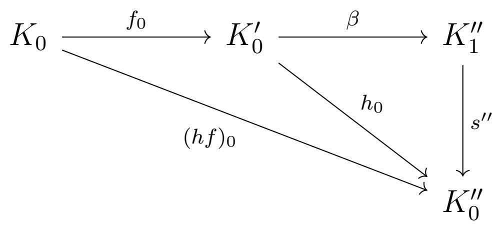
and



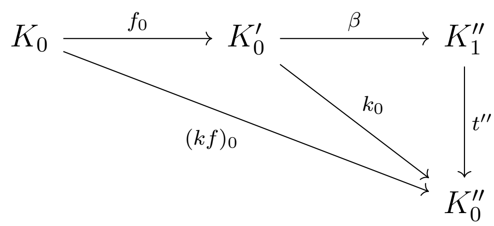
for identity. For naturality, note that

$$\begin{aligned} m''(k_1 g_1, k_1 \alpha s) &= m''((k_1 \times k_1)(g_1, \alpha s)) \\ m''((k_1 \times k_1)(g_1, \alpha s)) &= \\ k_1 m'(g_1, \alpha s) &= \\ k_1 m'(\alpha t, f_1) &= \\ m''(k_1 \alpha t, k_1 f_1) & \end{aligned}$$

by functorality of k and naturality of α. For βf_0 note that the following commute:



and



For naturality, note that:

$$
\begin{aligned}
m''(k_1 f_1, \beta f_0 s) &= m''(k_1 f_1, \beta s' f_1) \\
m''(k_1 f_1, \beta s' f_1) &= \\
m''(k_1, \beta s') f_1 &= \\
m''(\beta t', h_1) f_1 &= \\
m''(\beta t' f_1, h_1 f_1) &= \\
m''(\beta f_0 t, h_1 f_1)
\end{aligned}
$$

by functorality of f and naturality of β. We now see that $m''(k_1\alpha, \beta f_0)$ is a vertical composite and is therefore natural, and satisfies the diagrams. The final result to be derived from this is that the following diagram is commutative:

$$
\begin{array}{ccc}
K_0 & \xrightarrow{(k_1\alpha, \beta f_0)} & K_1'' \times_{K_0''} K_1'' \\
\Big\downarrow{\scriptstyle (\beta g_0, h_1\alpha)} & & \Big\downarrow{\scriptstyle m''} \\
K_1'' \times_{K_0''} K_1'' & \xrightarrow[m'']{} & K_1''
\end{array}
$$

This can be seen by noting that naturality of β

$$m''(\beta t', h_1) = m''(k_1, \beta s')$$

allows us to derive the following:

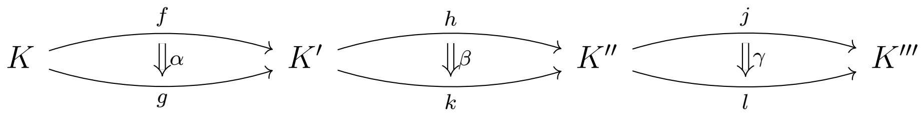
$$
\begin{aligned}
m''(k_1, \beta s')\alpha &= m''(k_1\alpha, \beta s'\alpha) \\
m''(k_1\alpha, \beta s'\alpha) &= \\
m''(k_1\alpha, \beta f_0)
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
m''(\beta t', h_1)\alpha &= m''(\beta t'\alpha, h_1\alpha) \\
m''(\beta t'\alpha, h_1\alpha) &= \\
m''(\beta g_0, h_1\alpha)
\end{aligned}
$$

and so we are done.

To show that our horizontal composition is associative, we need to show that given the diagram below:



we have $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$. To do this, note that:

$$\begin{aligned} \gamma * (\beta * \alpha) &= m'''(l_1(\beta * \alpha), \gamma h_0 f_0) \\ m'''(l_1(\beta * \alpha), \gamma h_0 f_0) &= \\ m'''(l_1 m''(k_1\alpha, \beta f_0), \gamma h_0 f_0) &= \\ m'''(m'''((l_1 \times l_1)(k_1\alpha, \beta f_0)), \gamma h_0 f_0) &= \\ m'''(m'''(l_1 k_1 \alpha, l_1 \beta f_0), \gamma h_0 f_0) &= \\ m'''(l_1 k_1 \alpha, m'''(l_1 \beta f_0, \gamma h_0 f_0)) &= \\ m'''(l_1 k_1 \alpha, m'''(l_1 \beta, \gamma h_0) f_0) & \end{aligned}$$

but we have that:

$$\begin{aligned} (\gamma * \beta) * \alpha &= m'''((\gamma * \beta) g_0, j_1 h_1 \alpha) \\ m'''((\gamma * \beta) g_0, j_1 h_1 \alpha) &= \\ m'''(l_1 k_1 \alpha, (\gamma * \beta) f_0) &= \\ m'''(l_1 k_1 \alpha, m'''(l_1 \beta, \gamma h_0) f_0) & \end{aligned}$$

and so we have shown that $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$.

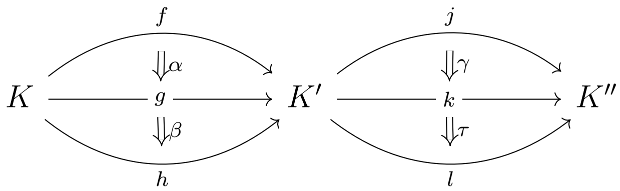
We will now show the existence of identities for horizontal composition of internal natural transformations. Given internal functors $f, g : K \to K'$ and an internal natural transformation $\alpha : f \to g$, recall the identity internal functor $1_K : K \to K$ and define $1h_\alpha : 1_K \to 1_K$ as $1h_\alpha = 1_{1_K} = e1_{K_0} = e : K_0 \to K_1$. This satisfies the definition of an internal natural transformation, since $se = te = 1_{K_0}$ and $m(1_{K_1}, es) = m(et, 1_{K_1})$. To show that it is the identity for horizontal composition, note that $\alpha * 1h_\alpha = \alpha * e = m'(g_1 e, \alpha) = m'(e' g_0, \alpha) = \alpha$. The case for the left inverse is done similarly.

We move onto showing that given internal functors $f : K \to K'$ and $g : K' \to K''$, we have the horizontal composite of $1_g : g \to g$ with $1_f : f \to f$ is equal to the 1_{gf}. To do this, note that

$$m''(g_1 e' f_0, e'' g_0 f_0) = m''(e'' g_0 f_0, e'' g_0 f_0) = e'' g_0 f_0 = 1_{gf}$$

by the identity property of e''.

Finally, we need to show that the middle-four exchange holds, and then we are done. To do this, we show the following diagram of internal functors and natural transformations for clarity:



From this, we need to show that $(\tau \circ \gamma) * (\beta \circ \alpha) = (\tau * \beta)(\gamma * \alpha)$. This can be shown with the following calculations. We begin with:

$$\begin{aligned}
(\tau \circ \gamma) * (\beta \circ \alpha) &= m''(l_1(\beta \circ \alpha), (\tau \circ \gamma) f_0) \\
m''(l_1(\beta \circ \alpha), (\tau \circ \gamma) f_0) &= \\
m''(l_1 m'(\beta, \alpha), m''(\tau, \gamma) f_0) &= \\
m''(m''((l_1 \times l_1)(\beta, \alpha)), m''(\tau, \gamma) f_0) &= \\
m''(m''(l_1\beta, l_1\alpha), m''(\tau f_0, \gamma f_0))
\end{aligned}$$

Now, we have that:

$$\begin{aligned}
(\tau * \beta)(\gamma * \alpha) &= m''(m''(l_1\beta, \tau g_0), m''(k_1\alpha, \gamma f_0)) \\
m''(m''(l_1\beta, \tau g_0), m''(k_1\alpha, \gamma f_0)) &= \\
m''(l_1\beta, m''(\tau g_0, m''(k_1\alpha, \gamma f_0)) &= \\
m''(l_1\beta, m''(m''(\tau g_0, k_1\alpha), \gamma f_0))
\end{aligned}$$

Extracting the term $m''(\tau g_0, k_1\alpha)$ from the above gives us:

$$m''(\tau g_0, k_1\alpha) = (\tau * \alpha) = m''(l_1\alpha, \tau f_0)$$

Returning this back into our equation above gives:

$$\begin{aligned}
m''(l_1\beta, m''(m''(\tau g_0, k_1\alpha), \gamma f_0)) &= \\
m''(l_1\beta, m''(m''(l_1\alpha, \tau f_0), \gamma f_0)) &= \\
m''(l_1\beta, m''(l_1\alpha, m''(\tau f_0, \gamma f_0))) &= \\
m''(m''(l_1\beta, l_1\alpha), m''(\tau f_0, \gamma f_0))
\end{aligned}$$

which is exactly what we had at the beginning, and so the middle-four exchange holds. From all of the above, we can conclude that the category of internal categories in some ambient **K** is in fact a 2-category.

3.3 Internal 2-categories

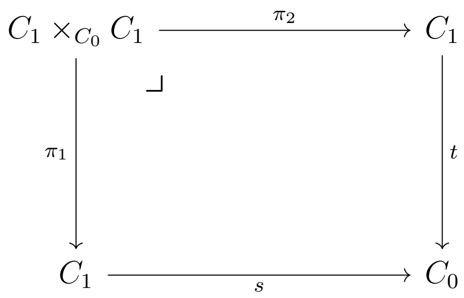
To begin, we define the notion of an internal 2-category. This definition is considerably more involved than that of an internal 1-category, as there is a lot more data required to define a structure that will produce desirable results in the case of $\mathbf{C} = \mathbf{Set}$ for instance.

Definition 3.3.1 (Internal 2-category). Given a category $\mathbf{C}$ with pullbacks and terminal object, an internal 2-category C consists of the following data:

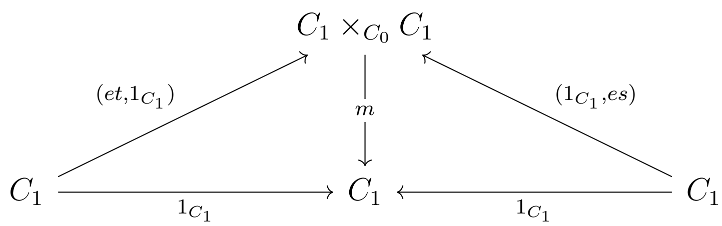
1. Objects C_0, C_1, C_2 of $\mathbf{C}$, which we call the object of objects, the object of 1-cells, and the object of 2-cells respectively.

2. Maps $s, t : C_1 \to C_0$, which intuitively suggest domain and codomain, and a map $e : C_0 \to C_1$ which acts as an identity, and satisfies the equality $se = 1_{C_0} = te$

3. An object of composable arrows $C_1 \times_{C_0} C_1$ defined on the following pullback:



4. A composition map $m : C_1 \times_{C_0} C_1 \to C_1$ subject to the following commutative diagrams:



and

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{1_{C_1} \times m} & C_1 \times_{C_0} C_1 \\ {\scriptstyle m \times 1_{C_1}} \downarrow & & \downarrow {\scriptstyle m} \\ C_1 \times_{C_0} C_1 & \xrightarrow[m]{} & C_1 \end{array}$$

and finally;

$$\begin{array}{ccccc} C_1 & \xleftarrow{\pi_1} & C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle t} \downarrow & & \downarrow {\scriptstyle m} & & \downarrow {\scriptstyle s} \\ C_0 & \xleftarrow[t]{} & C_1 & \xrightarrow[s]{} & C_0 \end{array}$$

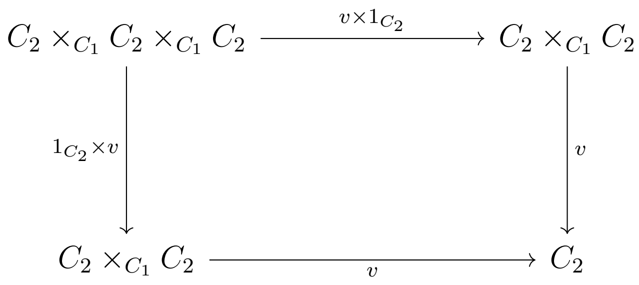
which express that our composition acts as such under identities, is associative, and respects domain and codomain.

5. We have maps $s_1, t_1 : C_2 \to C_1$, as well as $e_v : C_1 \to C_2$ and which satisfy $s_1 e_v = 1_{C_1} = t_1 e_v$.

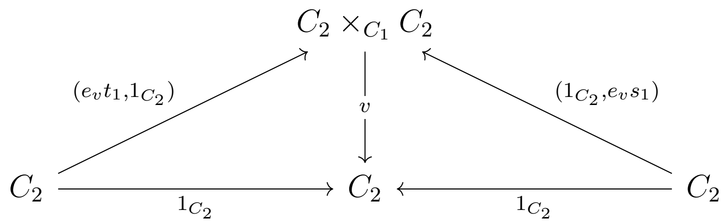
6. We have a composition map $v : C_2 \times_{C_1} C_2 \to C_2$ defined on the following pullback:

$$\begin{array}{ccc} C_2 \times_{C_1} C_2 & \xrightarrow{\pi_2} & C_2 \\ {\scriptstyle \pi_1} \downarrow & \lrcorner & \downarrow {\scriptstyle t_1} \\ C_2 & \xrightarrow[s_1]{} & C_1 \end{array}$$

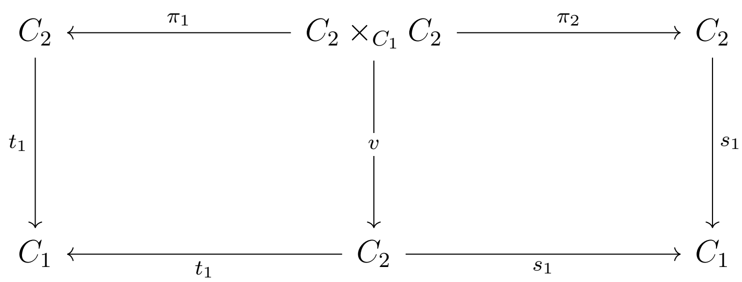
and subject to the commutativity of the following:



and

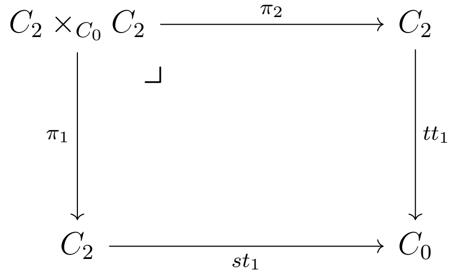


and finally



7. The maps s_1, t_1 also satisfy the additional equations $ss_1 = st_1$ and $ts_1 = tt_1$.

8. We can define a horizontal composition map of 2-cells $h : C_2 \times_{C_0} C_2 \to C_2$ over the pullback:



the map $e_v e : C_0 \to C_2$ plays the role of e_h, and the composition h is subject to the same diagrams as v, with the appropriate replacements of maps, and the following additional condition:

$$\begin{array}{ccccc}
C_1 \times_{C_0} C_1 & \xleftarrow{t_1 \times t_1} & C_2 \times_{C_0} C_2 & \xrightarrow{s_1 \times s_1} & C_1 \times_{C_0} C_1 \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m} \\
C_1 & \xleftarrow[t_1]{} & C_2 & \xrightarrow[s_1]{} & C_1
\end{array}$$

9. We require that h preserves e_v with respect to m as in the following commutative diagram:

$$\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{e_v \times e_v} & C_2 \times_{C_0} C_2 \\
\downarrow{\scriptstyle m} & & \downarrow{\scriptstyle h} \\
C_1 & \xrightarrow[e_v]{} & C_2
\end{array}$$

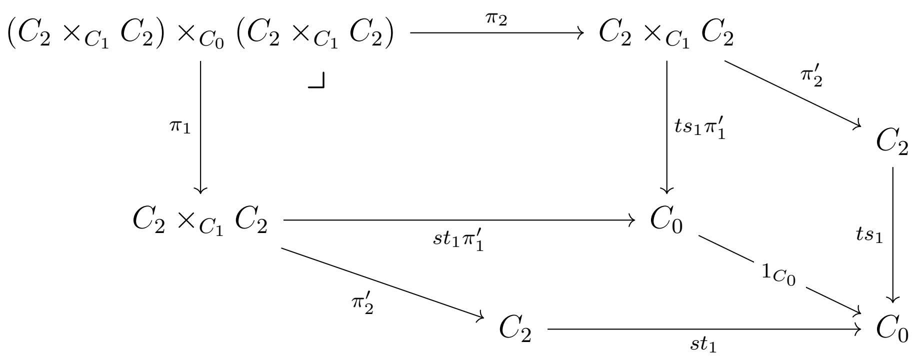
10. Our compositions h and v are subject to the middle-four exchange. To express this we define a map Mf as follows. Consider the following pullbacks:

$$\begin{array}{ccc}
(C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) & \xrightarrow{\pi_2} & C_2 \times_{C_1} C_2 \\
\downarrow{\scriptstyle \pi_1} & \lrcorner & \downarrow{\scriptstyle ts_1\pi_1'} \\
C_2 \times_{C_1} C_2 & \xrightarrow[st_1\pi_1']{} & C_0
\end{array}$$

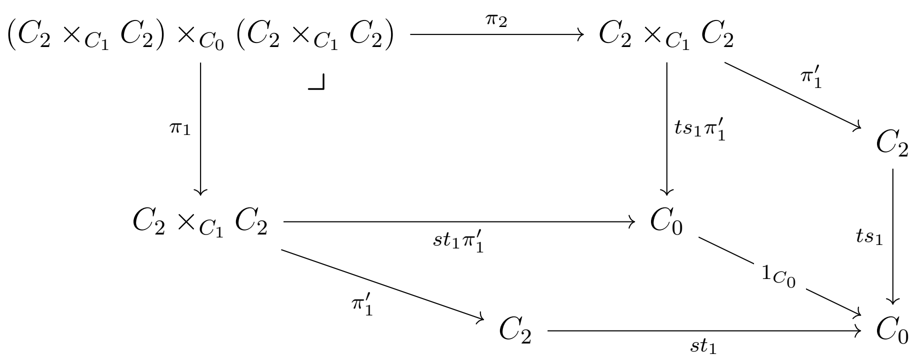
and

$$\begin{array}{ccc}
(C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2) & \xrightarrow{\pi_4} & C_2 \times_{C_0} C_2 \\
\downarrow{\scriptstyle \pi_3} & \lrcorner & \downarrow{\scriptstyle t_1 \times t_1} \\
C_2 \times_{C_0} C_2 & \xrightarrow[s_1 \times s_1]{} & C_1 \times_{C_0} C_1
\end{array}$$

from which we derive the following diagrams:



and

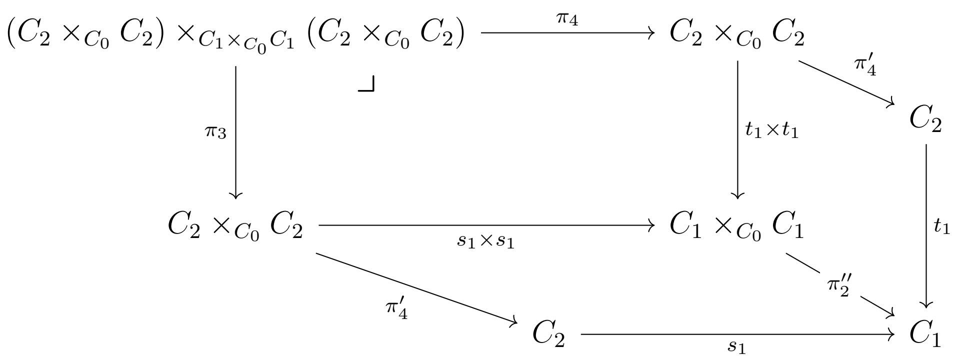


Note that by the commutativity of the previous diagrams, we have maps $f_0 = \pi_2' \times \pi_2' : (C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) \to C_2 \times_{C_0} C_2$ and $f_1 = \pi_1' \times \pi_1' : (C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) \to C_2 \times_{C_0} C_2$. Now, note that:

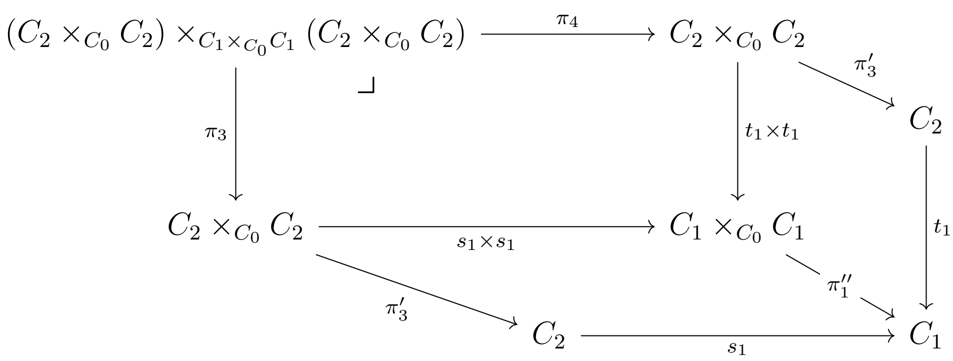
$$
\begin{aligned}
(t_1 \times t_1)(\pi_2' \times \pi_2') &= (t_1\pi_2' \times t_1\pi_2') \\
(t_1\pi_2' \times t_1\pi_2') &= \\
(s_1\pi_1' \times s_1\pi_1') &= \\
(s_1 \times s_1)(\pi_1' \times \pi_1')
\end{aligned}
$$

and so we construct a map $(f_1, f_0) : (C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) \to (C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2)$.

For the other half, we have:



and



Note that by the commutativity of the diagrams, we can define maps $f_0' = \pi_4' \times \pi_4' : (C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2) \to C_2 \times_{C_1} C_2$ and $f_1' = \pi_3' \times \pi_3' : (C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2) \to C_2 \times_{C_1} C_2$. Note that we have

$$\begin{aligned} st_1\pi_1'(\pi_3' \times \pi_3') &= st_1\pi_3' \\ st_1\pi_3' &= \\ s\pi_1''(t_1 \times t_1) &= \\ t\pi_2''(t_1 \times t_1) &= \\ tt_1\pi_4' &= \\ ts_1\pi_4' & \end{aligned}$$

And so we have a map $(f_1', f_0') : (C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2) \to (C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2)$. Furthermore we have that $(f_1', f_0')(f_1, f_0) = 1_{(C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2)}$ and $(f_1, f_0)(f_1', f_0') = 1_{(C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2)}$ and so $Mf = (f_1, f_0)$ is an isomorphism, although we will omit this calculation since it is not instructive.

Having now constructed an isomorphism $Mf : (C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1}$

$C_2) \to (C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2)$, we require that the following diagram is commutative:

$$\begin{array}{ccc}
(C_2 \times_{C_1} C_2) \times_{C_0} (C_2 \times_{C_1} C_2) & \xrightarrow{Mf} & (C_2 \times_{C_0} C_2) \times_{C_1 \times_{C_0} C_1} (C_2 \times_{C_0} C_2) \\
\downarrow{\scriptstyle v \times v} & & \downarrow{\scriptstyle h \times h} \\
C_2 \times_{C_0} C_2 & & C_2 \times_{C_1} C_2 \\
& \searrow^{h} \quad C_2 \quad {}^{v}\swarrow &
\end{array}$$

This gives a structure which has all the properties of a 2-category in any ambient category **C** with pullbacks and terminal object.

Example 3.3.1 (C = Set). If the ambient category is **Set**, we have a set of objects C_0, a set of one cells C_1 with specified domain, codomain, and identity, and a set of 2-cells C_2 with specified domain, codomain, and identity for both vertical and horizontal composition, and an interchange law. To expand on this further, in the tuple (C_0, C_1, e, t, s, m), we have for each element f in the set C_1 unique elements x, y in C_0 specified by $x = s(f), y = t(f)$, and an operation $m : C_1 \times_{C_0} C_1 \to C_1$ which takes elements $g, f \in C_1$ with $t(f) = s(g)$ to an element $m(g, f) \in C_1$ with $s(m(g, f)) = s(f)$ and $t(m(g, f)) = t(g)$, with the condition that $m(m(h, g), f) = m(h, m(g, f))$ and $m(et(f), f) = m(f, es(f)) = f$. We can see that this is an internal category, and we note that replacing (C_0, C_1, e, t, s, m) with $(C_1, C_2, e_v, t_1, s_1, v)$ or $(C_0, C_2, e_v e, tt_1, ss_1, h)$ gives us the same, since the conditions on the above data are the same as those required on the tuple (C_0, C_1, e, t, s, m), and give us two additional internal categories. Furthermore, we have that $s_1 h(\beta, \alpha) = m(s_1(\beta), s_1(\alpha))$ and $t_1 h(\beta, \alpha) = m(t_1(\beta), t_1(\alpha))$ relating h and m as composition operations. This allows us to note that the only conditions left that remain in order for this structure to be a 2-category are the middle-four exchange and the preservation of identity in h, but these are given since we have $h(e_v(g), e_v(f)) = e_v(m(g, f))$ and $h(v \times v) = v(h \times h)Mf$ where the action of Mf on $((\beta', \beta), (\alpha', \alpha))$ is $Mf : ((\beta', \beta), (\alpha', \alpha)) \mapsto ((\beta', \alpha'), (\beta, \alpha))$ and so we have $h(v(\beta', \beta), v(\alpha', \alpha)) = v(h(\beta', \alpha'), h(\beta, \alpha))$. From the above, we can conclude that if we take C_0 as the objects, C_1 as the 1-cells, and C_2 as the 2-cells with their source and targets given by the maps t, s etc, identity by $e, e_v, e_v e$, and composition as the maps m, v, h, we will have a 2-category.

3.4 Internal Bicategories

We now move on to the definition of an internal bicategory in an ambient category **C** with pullbacks and terminal object. Since bicategories have weaker associativity and identity laws for horizontal composition, the definition is more involved, but still similar to the case of 2-categories, since these are after all, special cases of bicategories in the non-internal case. The crux of the definition relies on replicating the coherence diagrams for associativity and identity in composition of 1-cells. When we have defined an internal bicategory, we will discuss the similarities between them and internal 2-categories in more detail.

Definition 3.4.1 (Internal Bicategory). Given a category **C** with pullbacks and terminal object, we define an internal bicategory as consisting of the following:

1. We have objects C_0, C_1, C_2 of **C**, which we call the object of objects, the object of 1-cells, and the object of 2-cells respectively

2. We have an arrow $e : C_0 \to C_1$ which we call the (weak) identity 1-cell, and arrows $s, t : C_1 \to C_0$ which we call the domain and codomain of the 1-cells. These arrows are subject to the condition $se = 1_{C_0} = te$. We also have a composition arrow $m : C_1 \times_{C_0} C_1 \to C_1$ defined on the following pullback square:

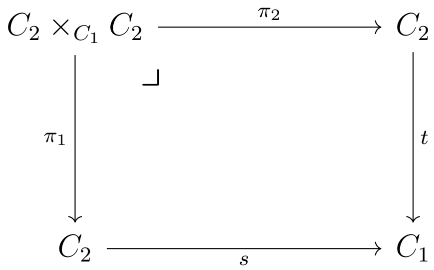
$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ \Big\downarrow{\scriptstyle \pi_1} & \lrcorner & \Big\downarrow{\scriptstyle t} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array}$$

and subject to the commutativity of:

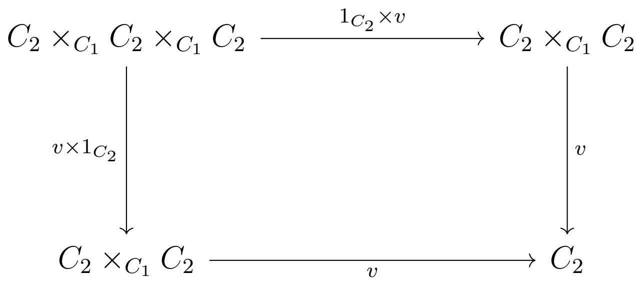
$$\begin{array}{ccccc} C_1 & \xleftarrow{\pi_1} & C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle s} \\ C_0 & \xleftarrow[t]{} & C_1 & \xrightarrow[s]{} & C_0 \end{array}$$

3. We have an arrow $e_v : C_1 \to C_2$ which we call the identity vertical 2-cell, and arrows $s_1, t_1 : C_2 \to C_1$ which we call the vertical domain

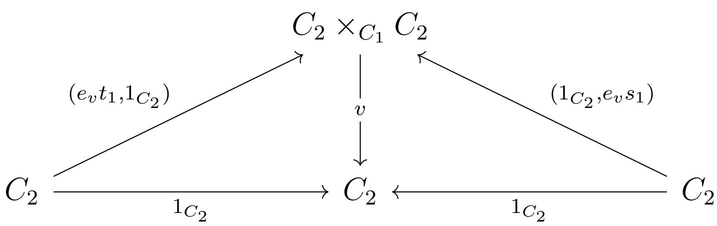
and codomain of 2-cells. These arrows are subject to the condition $s_1 e_v = 1_{C_1} = t_1 e_v$. We also have a vertical composition arrow $v : C_2 \times_{C_1} C_2 \to C_2$, defined on the following pullback:



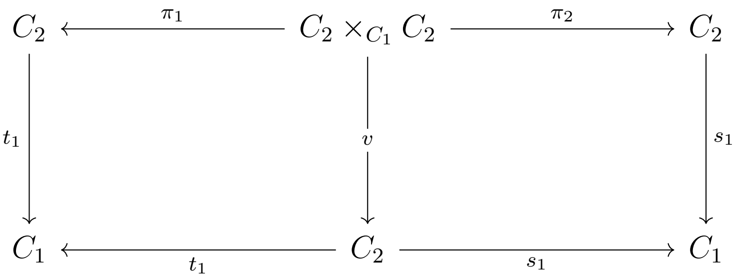
and subject to the following commutative diagrams:



and



as well as:



In addition to this, we have that $tt_1 = ts_1$ and $ss_1 = st_1$

4. We have a horizontal composition arrow $h : C_2 \times_{C_0} C_2 \to C_2$ defined

on the pullback:

$$\begin{array}{ccc} C_2 \times_{C_0} C_2 & \xrightarrow{\pi_2} & C_2 \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle ts_1} \\ C_2 & \xrightarrow[st_1]{} & C_0 \end{array}$$

and an identity preserving property as in the following commutative diagram:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{e_v \times e_v} & C_2 \times_{C_0} C_2 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle h} \\ C_1 & \xrightarrow[e_v]{} & C_2 \end{array}$$

5. The following diagram is commutative

$$\begin{array}{ccccc} C_2 & \xleftarrow{\pi_1} & C_2 \times_{C_0} C_2 & \xrightarrow{\pi_2} & C_2 \\ {\scriptstyle ts_1}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle st_1} \\ C_0 & \xleftarrow[ts_1]{} & C_2 & \xrightarrow[st_1]{} & C_0 \end{array}$$

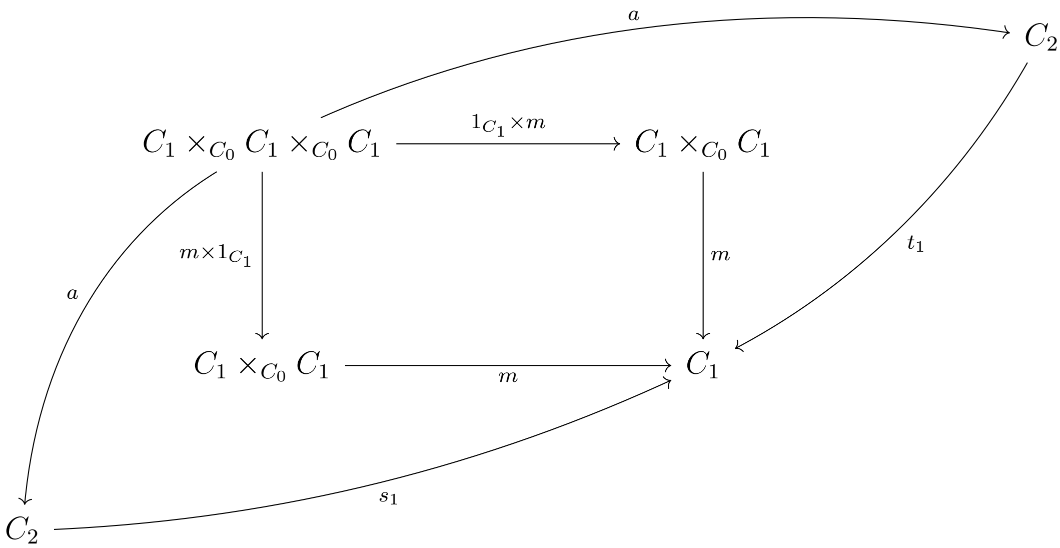
and moreover the following is commutative

$$\begin{array}{ccccc} C_1 \times_{C_0} C_1 & \xleftarrow{t_1 \times t_1} & C_2 \times_{C_0} C_2 & \xrightarrow{s_1 \times s_1} & C_1 \times_{C_0} C_1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m} \\ C_1 & \xleftarrow[t_1]{} & C_2 & \xrightarrow[s_1]{} & C_1 \end{array}$$

6. The middle-four exchange as in the definition of an internal 2-category is satisfied.

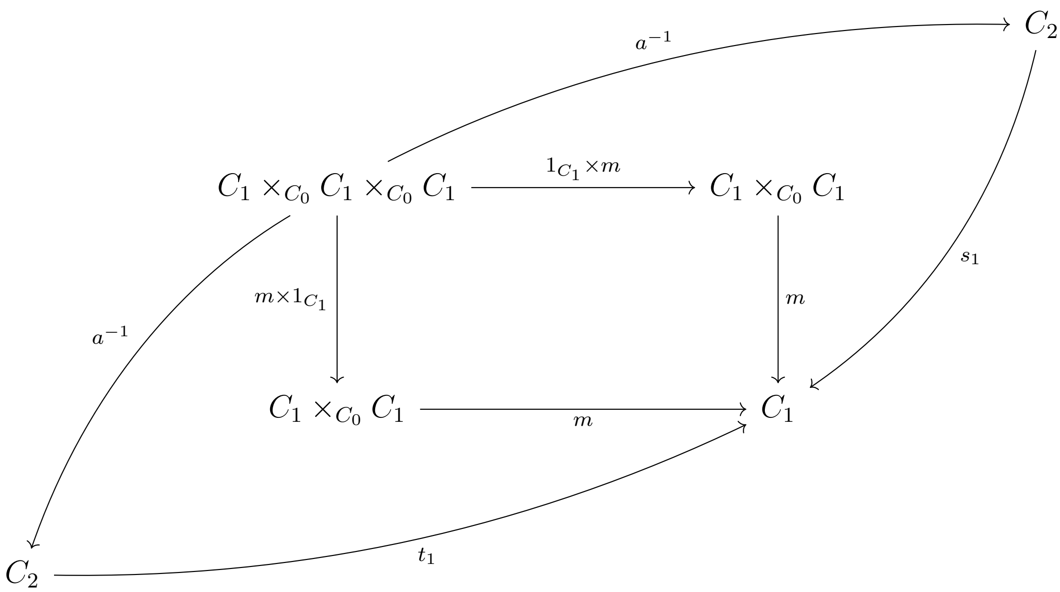
7. We have an arrow $a : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_2$ such that in the following

diagram:



the outer quadrilaterals commute.

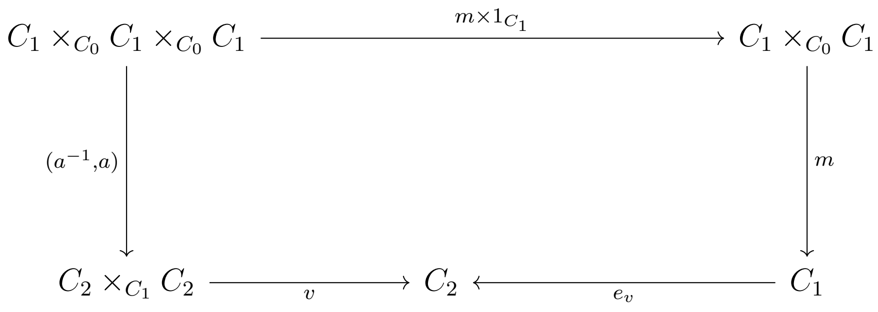
8. We have an arrow $a^{-1} : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_2$ such that in the following diagram:



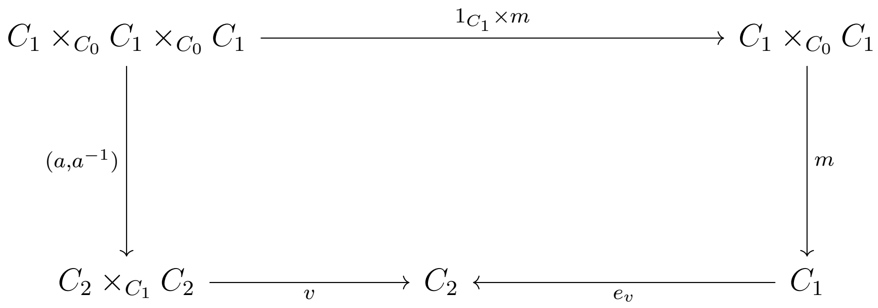
the outer quadrilaterals commute.

9. Given the diagrams as above, we have that the following two diagrams

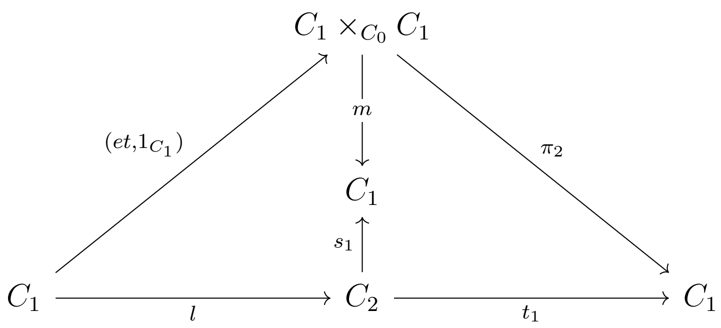
are both commutative:



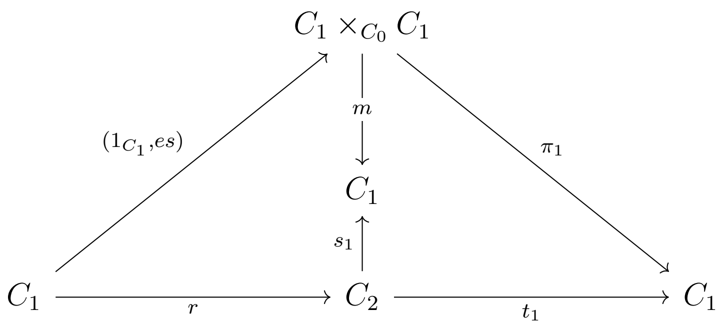
and



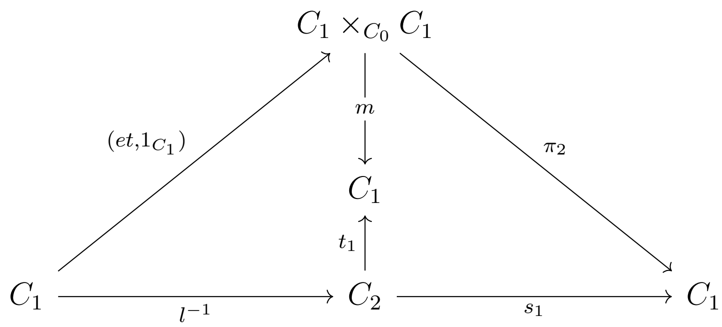
10. We have arrows $l : C_1 \to C_2$ and $r : C_1 \to C_2$ such that the following diagrams commute:



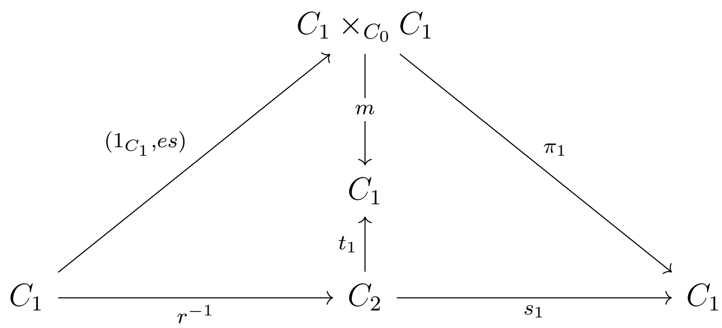
and



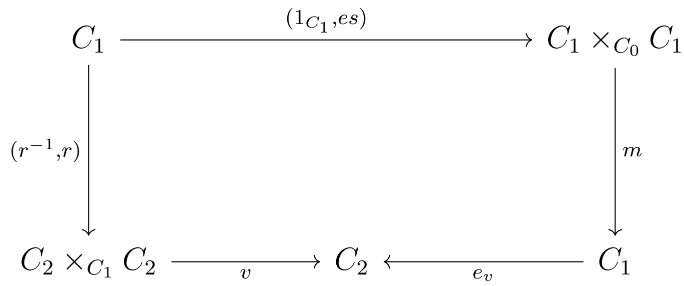
11. We have arrows $l^{-1} : C_1 \to C_2$ and $r^{-1} : C_1 \to C_2$ such that the following diagrams commute:



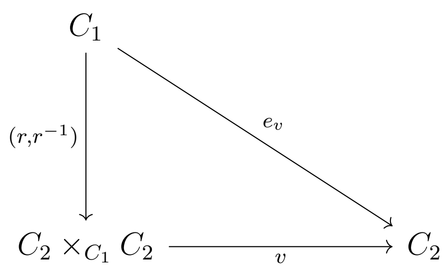
and



12. We have that the following diagrams commute:



and



The analogous diagrams involving l and l^{-1}, as shown below, are also required to commute:

$$\begin{array}{ccccc}
C_1 & & \xrightarrow{(et, 1_{C_1})} & & C_1 \times_{C_0} C_1 \\
\Big\downarrow{\scriptstyle (l^{-1}, l)} & & & & \Big\downarrow{\scriptstyle m} \\
C_2 \times_{C_1} C_2 & \xrightarrow[v]{} & C_2 & \xleftarrow[e_v]{} & C_1
\end{array}$$

and

$$\begin{array}{ccc}
C_1 & & \\
\Big\downarrow{\scriptstyle (l, l^{-1})} & \searrow{\scriptstyle e_v} & \\
C_2 \times_{C_1} C_2 & \xrightarrow[v]{} & C_2
\end{array}$$

13. We require a naturality condition on a, for which we define two arrows $C_2 \times_{C_0} C_2 \times_{C_0} C_2 \to C_2 \times_{C_1} C_2$ as in the following commutative diagrams:

$$\begin{array}{ccccc}
C_2 \times_{C_0} C_2 & \xleftarrow{1_{C_2} \times h} & C_2 \times_{C_0} C_2 \times_{C_0} C_2 & \xrightarrow{s_1 \times s_1 \times s_1} & C_1 \times_{C_0} C_1 \times_{C_0} C_1 \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle (h_1, a(s_1^3))} & & \Big\downarrow{\scriptstyle a} \\
C_2 & \xleftarrow[\pi_1]{} & C_2 \times_{C_1} C_2 & \xrightarrow[\pi_2]{} & C_2
\end{array}$$

where $h_1 = h(1_{C_2} \times h)$ and $a(s_1^3) = a(s_1 \times s_1 \times s_1)$. We also have:

$$\begin{array}{ccccc}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xleftarrow{t_1 \times t_1 \times t_1} & C_2 \times_{C_0} C_2 \times_{C_0} C_2 & \xrightarrow{h \times 1_{C_2}} & C_2 \times_{C_0} C_2 \\
\Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle (a(t_1^3), h_2)} & & \Big\downarrow{\scriptstyle h} \\
C_2 & \xleftarrow[\pi_1]{} & C_2 \times_{C_1} C_2 & \xrightarrow[\pi_2]{} & C_2
\end{array}$$

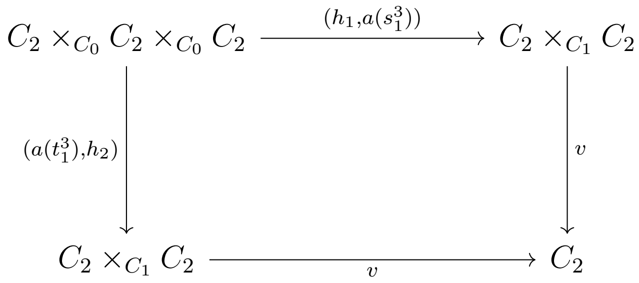
where we have $h_2 = h(h \times 1_{C_2})$ and $a(t_1^3) = a(t_1 \times t_1 \times t_1)$. These maps are well defined because

$$s_1(h_1) = s_1(h(1_{C_2} \times h))$$

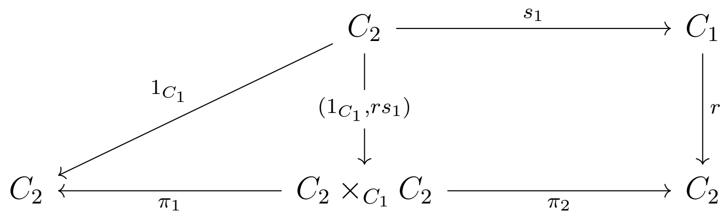
$$s_1(h(1_{C_2} \times h)) =$$

$$m(1_{C_1} \times m)(s_1 \times s_1 \times s_1)$$

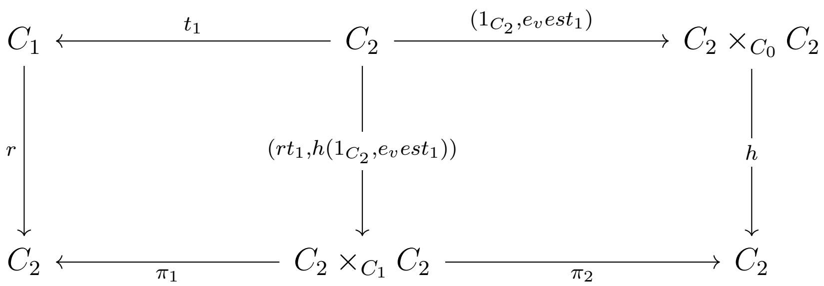
but we have $t_1(a(s_1^3)) = m(1_{C_1} \times m)(s_1 \times s_1 \times s_1)$ and likewise for the second diagram. The naturality condition is then given by requiring the following diagram to commute:



14. We require a naturality condition on r and l which we write diagrammatically. First, we construct the following commutative diagrams:



and



which are well defined because

$$t_1 r s_1 = s_1 = s_1 1_{C_2}$$

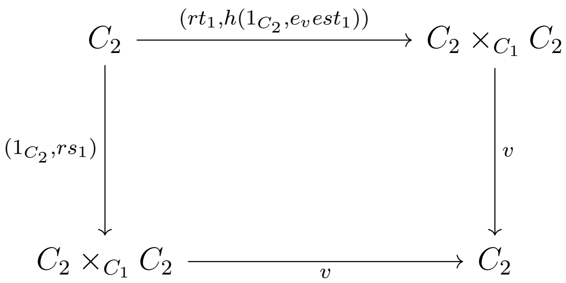
for the first diagram and for the second we have

$$\begin{aligned} t_1(h(1_{C_2}, e_v e s t_1)) &= m(t_1 \times t_1(1_{C_2}, e_v e s t_1)) \\ m(t_1 \times t_1(1_{C_2}, e_v e s t_1)) &= \\ m(t_1, t_1 e_v e s t_1) &= \\ m(t_1, e s t_1) &= \\ m(1_{C_1}, es)t_1 & \end{aligned}$$

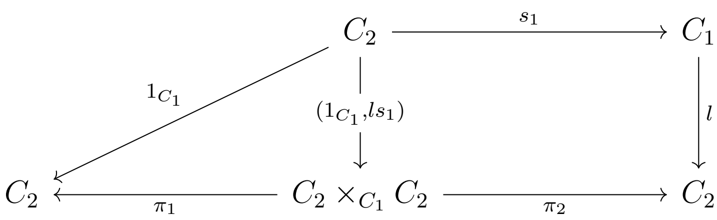
but we also have

$$s_1 r t_1 = m(1_{C_1}, es)t_1$$

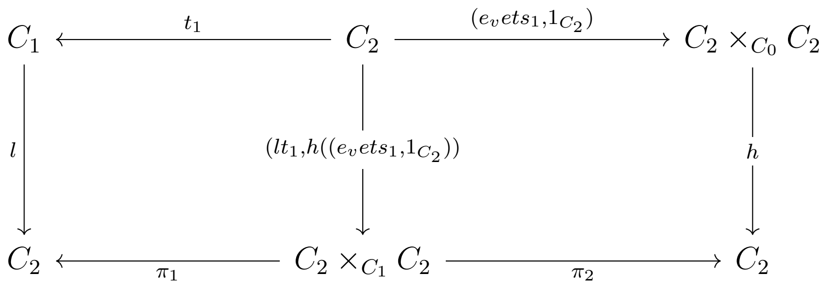
and so our arrows as constructed exist. Then from the above, we require that the following must commute:



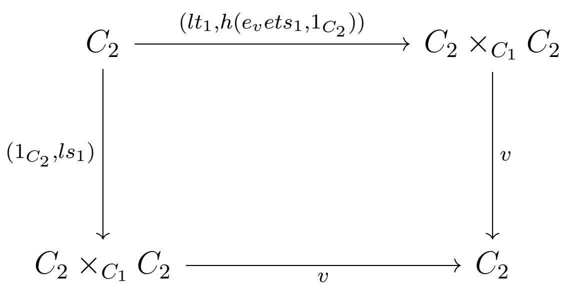
for r. In the case of l we construct the following commutative diagrams:



and



which can be checked to be well defined in the same way as the case for r, and from which we require the following to commute:



15. We require that the following diagram expressing coherence for l and r must commute. First define an arrow $C_1 \times_{C_0} C_1 \to C_2 \times_{C_1} C_2$ as follows:

$$\begin{array}{ccccc}
C_2 \times_{C_0} C_2 & \xleftarrow{e_v \times l} & C_1 \times_{C_0} C_1 & \xrightarrow{1_{C_1} \times (et, 1_{C_1})} & C_1 \times_{C_0} C_1 \times_{C_0} C_1 \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (h(e_v \times l), a(1_{C_1} \times (et, 1_{C_1})))} & & \downarrow{\scriptstyle a} \\
C_2 & \xleftarrow[\pi_1]{} & C_2 \times_{C_1} C_2 & \xrightarrow[\pi_2]{} & C_2
\end{array}$$

which is well defined because

$$s_1(h(e_v \times l)) = m((s_1 \times s_1)(e_v \times l)) = m(s_1 e_v \times s_1 l) = m(1_{C_1} \times m(et, 1_{C_1}))$$

and

$$\begin{aligned}
t_1(a(1_{C_1} \times (et, 1_{C_1})) &= m(1 \times m)(1_{C_1} \times (et, 1_{C_1})) \\
m(1 \times m)(1_{C_1} \times (et, 1_{C_1})) &= \\
m(1_{C_1} \times m(et, 1_{C_1}))
\end{aligned}$$

so both legs of the pullback commute. Then for our coherence, we require the following to commute:

$$\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \xrightarrow{(h(e_v \times l), a((1_{C_1} \times (et, 1_{C_2})))} & C_2 \times_{C_1} C_2 \\
\downarrow{\scriptstyle r \times e_v} & & \downarrow{\scriptstyle v} \\
C_2 \times_{C_0} C_2 & \xrightarrow[h]{} & C_2
\end{array}$$

16. We have a coherence condition, for which we define the following five arrows with domain $C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} C_1$ and codomain C_2:

$$\begin{aligned}
a_1 &= h(a \times e_v) \\
a_2 &= a(1_{C_1} \times m \times 1_{C_1}) \\
a_3 &= h(e_v \times a) \\
a_4 &= a(m \times 1_{C_1} \times 1_{C_1}) \\
a_5 &= a(1_{C_1} \times 1_{C_1} \times m)
\end{aligned}$$

It can be checked that $t_1a_4 = s_1a_5$, $t_1a_1 = s_1a_2$, $t_1a_2 = s_1a_3$, $s_1a_4 = s_1a_1$, and $t_1a_5 = t_1a_3$, which allows us to write the following diagram, which we require to commute:

$$\begin{array}{ccc} C_1^4 & \xrightarrow{(a_5,a_4)} & C_2^2 \\ {\scriptstyle (a_3,a_2,a_1)}\downarrow & & \downarrow{\scriptstyle v} \\ C_2^3 & \xrightarrow[v(v\times 1_{C_2})]{} & C_2 \end{array}$$

One can note that the definition has much of the same data that an internal 2-category has, but that there are three internal categories that can be formed from the definition of an internal 2-category (namely with internal composition on C_1 with identity e, internal vertical composition on C_2 with identity e_v and internal horizontal composition on C_2 with identity e_ve), but only one from the definition of an internal bicategory. This is because our arrows defining composition of internal 1-cells and horizontal composition of 2-cells are not strictly associative, nor with strict identity. One does however note that the ”base” diagrams used to construct these internal isomorphisms are the same as those in the strict case.

To show that we can actually bear fruit with this long definition, we consider the case where $\mathbf{C} = \mathbf{Set}$. Since the definition mirrors that of a 2-category, we use that example as a starting point.

Example 3.4.1 (Internal Bicategory in Set). Recall that in an internal 2-category we had three separate internal 1-categories which made it easy to verify that an internal 2-category was in fact a small 2-category. This fails in the case of internal bicategories because the only internal 1-category we can extract from the definition is $(C_1, C_2, e_v, v, t_1, s_1)$. However in the case of **Set**, we can show that we extract the data of a bifunctor from this small category and in doing so, show that an internal bicategory in **Set** is in fact a small bicategory. To each pair of elements $a, b \in C_0$ associate the small category $C_1(a, b)$. This has as objects the 1-cells with $s(f) = a, t(f) = b$, and as arrows the 2-cells with $st_1(\alpha) = a, tt_1(\alpha) = b$ for $\alpha \in C_1(a, b)$. As composition, we have v and the identity for this is given by e_v. Now, for each triple $(a, b, c) \in C_0 \times C_0 \times C_0$ assign the small category $C_1(b, c) \times C_1(a, b)$. Note that we have two maps from $C_1(b, c) \times C_1(a, b)$ to the category $C_1(a, c)$, which are $m : C_1 \times_{C_0} C_1 \to C_1$ and $h : C_2 \times_{C_0} C_2 \to C_2$ appropriately restricted. In order for the maps m, h to constitute a bifunctor, we need preservation of identity and composition in h. We also need consistency of domain and codomain. This is given by the condition that $m(s_1(\beta), s_1(\alpha)) = s_1 h(\beta, \alpha)$ and $m(t_1(\beta), t_1(\alpha)) = t_1 h(\beta, \alpha)$. Preservation of identity is given by the condition which specifies that $e_v m(g, f) = h(e_v(g), e_v(f))$ and preservation of the composition v is given by the internal middle four exchange property, specifically $h(v(\beta', \beta), v(\alpha', \alpha)) = v(h(\beta', \alpha'), h(\beta, \alpha))$ as in the case for internal 2-categories. Finally, we have for each element $a \in C_0$ a distinguished 1-cell $e(a)$ in $C_1(a, a)$ that will act as a pseudo-identity.

With these satisfied, we can now safely say that we for each triple a, b, c we have categories $C_1(a, b)$ and $C_1(b, c)$ and a bifunctor $h, m : C_1(b, c) \times C_1(a, b) \to C_1(a, c)$. We now require for each triple k, g, f of elements in C_1, a natural $a_{k,g,f} : m(m(k, g), f) \cong m(k, m(g, f))$ such that the pentagonal identity is satisfied. We posit that this is $a : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_2$, which assigns to each composable triple (k, g, f) an element in C_2 which we write as $a_{k,g,f}$ with $s_1(a_{k,g,f}) = m(m(k, g), f)$ and $t_1(a_{k,g,f}) = m(k, m(g, f))$. This a is natural since for any triple (σ, β, α) of 2-cells in $C_2 \times_{C_0} C_2 \times_{C_0} C_2$ with $(s_1)^3(\sigma, \beta, \alpha) = (k, g, f)$ and $(t_1)^3(\sigma, \beta, \alpha) = (k', g', f')$, we have the condition

$$v(a_{k',g',f'}, h(h(\sigma, \beta), \alpha)) = v(h(\sigma, h(\beta, \alpha)), a_{k,g,f})$$

In addition to this, a is an isomorphism, since there exists an $a^{-1} : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_2$ such that $v(a, a^{-1}) = e_v(t_1(a))$ and $v(a^{-1}, a) = e_v(t_1(a^{-1}))$, and so a^{-1} is a two sided inverse.

Now, as well as a, we require two natural arrows $C_1 \to C_2$ which take $m(e(b), f)$ to f and $m(f, e(a))$ to f for a 1-cell f with $s(f) = a$ and $t(f) = b$. These are of course $l, r : C_1 \to C_2$ respectively. By definition, these assign to each 1-cell f, the 2-cell which maps $m(et(f), f)$ to f for any f, in the case of l and $m(f, es(f))$ to f in the case of r. To see that they are natural, consider 1-cells f, g with $s(f) = t(f) = a$ and $t(f) = t(g) = b$ and a 2-cell α with $s_1(\alpha) = f$ and $t_1(\alpha) = g$. Now, from the definition of internal bicategory, we have

$$v(r, h(\alpha, e_v e(st_1(\alpha)))) = v(r(g), h(\alpha, e_v e(a))) = v(\alpha, r(f))$$

which shows that r is natural. The case for l is much the same. Moreover, we have that both l, r are isomorphisms, because there exist l^{-1}, r^{-1} such that $v(l^{-1}, l) = e_v(s_1(l))$ and $v(l, l^{-1}) = e_v(t_1(l))$. Similar equalities hold in the case of r.

Now, to see that the pentagonal identity is satisfied, note that we have $v(v(a_3, a_2), a_1) = v(a_5, a_4) = v(a_3, v(a_2, a_1))$, where the arrows a_n for $n =$

$1, 2, .., 5$ assign to each quadruple of composable 1-cells (k, j, g, f) the following:

$$
\begin{aligned}
a_1 &= h(a_{k,j,g}, e_v(f)) \\
a_2 &= a_{k,m(j,g),f} \\
a_3 &= h(e_v(k), a_{j,g,f}) \\
a_4 &= a_{m(k,j),g,f} \\
a_5 &= a_{k,j,m(g,f)}
\end{aligned}
$$

and so the definition of internal bicategory shows us that the pentagon identity is satisfied. The final condition to be satisfied is coherence of identity. For this, note that we have $h(r(g), e_v(f)) = v(h(e_v, l), a_{g,e_{s(g)},f})$ by definition again, and because of how we defined our composition operations and bifunctor, this is equal to the coherence condition for identity, and so an internal bicategory in **Set** is a small bicategory (that is, a bicategory whose objects form a set and with small hom-categories).

4. Structures in a 2-category

4.1 Adjunction and equivalence in a 2-category

We now move onto the subject of what can be done internally in an arbitrary 2-category. Many of the notions which we are familiar with in the 2-category of categories exist more generally in any 2-category. First, we begin with the idea of an adjunction between objects a, b of a 2-category $\mathbf{K}$. We say that a is left adjoint to b if there exist 1-cells $f : a \to b$ and $g : b \to a$ such that 2-cells $\epsilon : fg \Rightarrow 1_b$ and $\eta : 1_a \Rightarrow gf$ exist and satisfy the triangle equalities.

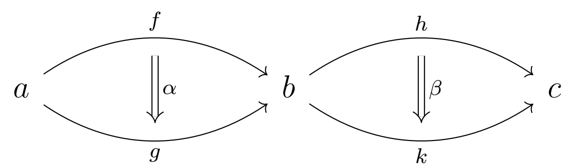
Definition 4.1.1 (Adjunction(1)). In a 2-category $\mathbf{K}$ we can define an adjunction between objects a and b as consisting of 1-cells $f : a \to b$ and $g : b \to a$ and 2-cells $\eta : 1_a \Rightarrow gf$ and $\epsilon : fg \Rightarrow 1_b$, such that $(\epsilon * f) \circ (\eta * f) = 1_f$ and $(g * \epsilon) \circ (\eta * g) = 1_g$.

We follow [7] in referring to this as the global definition. Often, we will write ηg instead of $\eta * g$ where there is no confusion. There is a second definition which we will call the local definition, following [7] again, and it can be stated as:

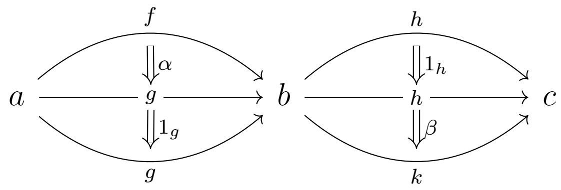
Definition 4.1.2 (Adjunction(2)). In a 2-category $\mathbf{K}$ we say that A and B are adjoint if there exist 1-cells $f : A \to B$ and $g : B \to A$ such that for arbitrary 1-cells $a : X \to A$ and $b : X \to B$ we have an isomorphism of 2-cells $\mathbf{K}(fa, b) \cong \mathbf{K}(a, gb)$, which is natural with respect to 2-cells $\alpha : a \Rightarrow a'$ and $\beta : b \Rightarrow b'$, as well as one cells $y : Y \to X$.

The second definition, however, is not desirable for our purposes, and we will not make use of it in this chapter.

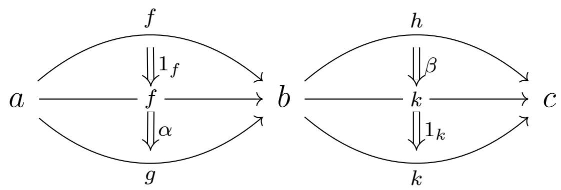
We now begin with the first of our theorems about adjoints in general 2-categories. However, before we do this, we establish the following important fact about the horizontal composition of 2-cells. Given the following diagram which depicts the horizontal composition of β and α:



we note that it is equal to



and



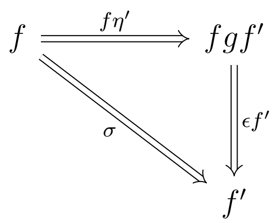
by the properties of identity. Now, by the middle-four exchange we can write:

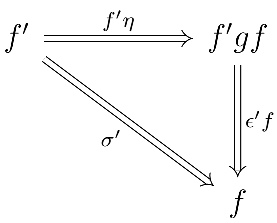
$$\beta * \alpha = k\alpha \circ \beta f = \beta g \circ h\alpha$$

This will be an important calculational tool in the following section.

Theorem 4.1.1. In a 2-category $\mathbf{K}$, if $f : a \to b$ is left adjoint to $g : b \to a$, and $f' : a \to b$ is also left adjoint to g, then f and f' are isomorphic

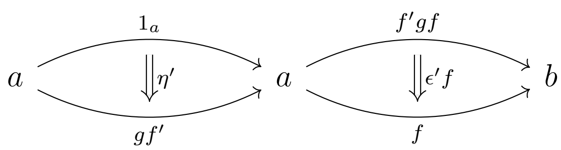
Proof. We write our unit and counit pair for $f \dashv g$ as $\eta : 1_a \Rightarrow gf$ and $\epsilon : fg \Rightarrow 1_b$ and for $f' \dashv g$ as $\eta' : 1_a \Rightarrow gf'$ and $\epsilon' : f'g \Rightarrow 1_b$ and use these to define the maps as in the diagrams:





We would like to show that $\sigma' \circ \sigma = 1_f$ and $\sigma \circ \sigma' = 1_{f'}$. To do this, we write $\sigma \circ \sigma' = (\epsilon f' \circ f\eta') \circ (\epsilon' f \circ f'\eta) = \epsilon f' \circ (f\eta' \circ \epsilon' f) \circ f'\eta$.

From this, we examine the term $f\eta' \circ \epsilon' f$, and note that we can write $f\eta' \circ \epsilon' f = \epsilon' f * \eta'$ as in the following diagram:



We can also write this horizontal composite as $\epsilon' f * \eta' = \epsilon' fgf' \circ f'gf\eta'$.

Now, we have the following:

$$\sigma \circ \sigma' = \epsilon f' \circ f\eta' \circ \epsilon' f \circ f'\eta$$
$$\epsilon f' \circ f\eta' \circ \epsilon' f \circ f'\eta =$$
$$\epsilon f' \circ \epsilon' fgf' \circ f'gf\eta' \circ f'\eta =$$

Extracting the left two terms gives us:

$$\epsilon f' \circ \epsilon' fgf' = (\epsilon \circ \epsilon' fg)f'$$
$$(\epsilon \circ \epsilon' fg)f' = (\epsilon' \circ f'g\epsilon)f'$$

while extracting the right two terms gives us:

$$f'gf\eta' \circ f'\eta = f'(gf\eta' \circ \eta)$$
$$f'(gf\eta' \circ \eta) = f'(\eta gf' \circ \eta')$$

since $\epsilon \circ \epsilon' fg = \epsilon' * \epsilon = \epsilon' \circ f'g\epsilon$ and $gf\eta' \circ \eta = \eta * \eta' = \eta gf' \circ \eta'$. Now, writing the original $\sigma \circ \sigma'$ and replacing with the expressions we derived above gives the following:

$$\sigma \circ \sigma' = \epsilon' f' \circ f'g\epsilon f' \circ f'\eta gf' \circ f'\eta'$$

extracting the middle term again gives us:

$$f'g\epsilon f' \circ f'\eta gf' = f'(g\epsilon \circ \eta g)f'$$
$$f'(g\epsilon \circ \eta g)f' = f'(1_g)f'$$
$$f'(1_g)f' = 1_{f'gf'}$$

Finally, substituting this into our expression $\sigma \circ \sigma'$ gives:

$$\epsilon' f' \circ 1_{f'gf'} \circ f'\eta' = \epsilon' f' \circ f'\eta'$$
$$\epsilon' f' \circ f'\eta' = 1_{f'}$$

and so we have shown that we have at least a one sided inverse. To show that $\sigma' \circ \sigma = 1_f$ is done in the same way. □

Proceeding, we would like to define the notion of equivalence of objects for a 2-category **K**.

Definition 4.1.3 (equivalence of objects). In a 2-category **K**, objects a and b are said to be equivalent if there exist 1-cells $f : a \to b$ and $g : b \to a$ such that we have invertible 2-cells $\alpha : 1_a \cong gf$ and $\beta : fg \cong 1_b$

If in addition to this, the 2-cells satisfy the triangle equations, there is said to be an adjoint equivalence between a and b. This leads to the following:

Theorem 4.1.2. If there is an equivalence of objects a and b in $\mathbf{K}$, there is also an adjoint equivalence between a and b

Proof. Given an equivalence of objects a, b in $\mathbf{K}$ with 1-cells $f : a \to b$ and $g : b \to a$ along with 2-cell isomorphisms $\epsilon : fg \cong 1_b$ and $\eta : 1_a \cong gf$, we want to show that there exists a 2-cell $\eta' : 1_a \Rightarrow gf$ such that $g\epsilon \circ \eta' g = 1_g$ and $\epsilon f \circ f\eta' = 1_f$. To start with, note that $g\epsilon : gfg \Rightarrow g$ and $\eta g : g \Rightarrow gfg$ are isomorphisms (being the composites of two isomorphisms), and so there exists a 2-cell $\sigma^{-1} : g \Rightarrow g$ such that $g\epsilon \circ \eta g \circ \sigma^{-1} = 1_g$. We now want to define a 2-cell $\eta' : 1_a \Rightarrow gf$ such that $g\epsilon \circ \eta' g = 1_g$. To this end, we define $\eta' = \sigma^{-1} f \circ \eta : 1_a \Rightarrow gf$. Now, we write $g\epsilon \circ \eta' g = g\epsilon \circ (\sigma^{-1} f \circ \eta) g = g\epsilon \circ \sigma^{-1} fg \circ \eta g$. We then examine the term $g\epsilon \circ \sigma^{-1} fg$. We note that $g\epsilon \circ \sigma^{-1} fg = \sigma^{-1} * \epsilon$ which is also equal to $\sigma^{-1} 1_b \circ g\epsilon$, and when we return this to our original equation we obtain $\sigma^{-1} 1_b \circ g\epsilon \circ \eta g = \sigma^{-1} \circ \sigma = 1_g$.

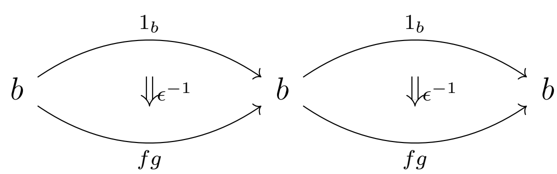
What remains is to verify that η' satisfies $\epsilon f \circ f\eta' = 1_f$. To do this we write

$$\epsilon f \circ f\eta' = \epsilon f \circ f(\sigma^{-1} f \circ \eta)$$

which we simplify as $\epsilon f \circ f\sigma^{-1} f \circ f\eta$. Now, note that we can write $\sigma^{-1} = \eta^{-1} g \circ g\epsilon^{-1}$. When we return this to our equation we get

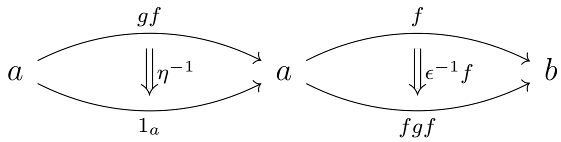
$$\epsilon f \circ f\sigma^{-1} f \circ f\eta = \epsilon f \circ f(\eta^{-1} g \circ g\epsilon^{-1}) f \circ f\eta$$

which we can again simplify to $\epsilon f \circ f(\eta^{-1} g \circ g\epsilon^{-1}) f \circ f\eta = \epsilon f \circ (f\eta^{-1} gf \circ fg\epsilon^{-1} f) \circ f\eta$. Looking only at the middle term $f\eta^{-1} gf \circ fg\epsilon^{-1} f$ and examining $fg\epsilon^{-1} f$ leads us to $fg\epsilon^{-1} f = \epsilon^{-1} fgf$ because $\epsilon^{-1} * \epsilon^{-1}$ can be written as in the following diagram:



so that $fg\epsilon^{-1} \circ \epsilon^{-1} = \epsilon^{-1} fg \circ \epsilon^{-1} = \epsilon^{-1} * \epsilon^{-1}$. Now, because the 2-cells are isomorphisms, we have that $fg\epsilon^{-1} = \epsilon^{-1} fg$ (by post-composing with ϵ). Similarly, we have that $\eta^{-1} gf = gf\eta^{-1}$, and so we have that the middle term is equal to $fgf\eta^{-1} \circ \epsilon^{-1} fgf$. Now, we can write that $fgf\eta^{-1} \circ \epsilon^{-1} fgf =$

$\epsilon^{-1}f * \eta^{-1}$ as in the following:

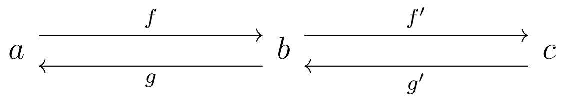


We can then write the second expression for $\epsilon^{-1}f*\eta^{-1}$, which is $\epsilon^{-1}f\circ f\eta^{-1}$. Returning this to our initial equation $\epsilon f \circ f\eta'$ gives us $\epsilon f \circ f\eta' = \epsilon f \circ (f\eta^{-1}gf \circ fg\epsilon^{-1}f) \circ f\eta = \epsilon f \circ \epsilon^{-1}f \circ f\eta^{-1} \circ f\eta$ which is equal to 1_f and so the second triangle equality is satisfied and we have an adjunction from the equivalence. Since all maps used in defining η' were isomorphisms, η' is an isomorphism itself and so we also have an adjoint equivalence. □

From here, we would like to define the composition of adjunctions in **K**. Given an adjunction $f : a \to b$, $g : b \to a$ with $f \dashv g$, with unit $\eta : 1_a \Rightarrow gf$ and counit $\epsilon : fg \Rightarrow 1_b$, and a second adjunction $f' : b \to c$, $g' : c \to b$ with $f' \dashv g'$ with unit $\eta' : 1_b \Rightarrow g'f'$ and counit $\epsilon' : f'g' \Rightarrow 1_c$, we would like to be able to compose the arrows in such a way as to yield an adjunction $f'f \dashv gg'$. To do this, we mimic the method of doing this in **Cat**.

Theorem 4.1.3. Given the adjunctions above, we can define 2-cells $\eta'' : 1_a \Rightarrow gg'f'f$ and $\epsilon'' : f'fgg' \Rightarrow 1_c$, such that these will satisfy the triangle equalities, yielding an adjunction $f'f \dashv gg'$

Proof.



Because of the existing adjunctions, we have 2-cells $\eta : 1_a \Rightarrow gf$, $\epsilon : fg \Rightarrow 1_b$, $\eta' : 1_b \Rightarrow g'f'$, $\epsilon' : f'g' \Rightarrow 1_c$. We contend that the unit and counit for this new adjunction will be given by $\eta'' = g\eta'f \circ \eta : 1_a \Rightarrow gg'f'f$ and $\epsilon'' = \epsilon' \circ f'\epsilon g' : f'fgg' \Rightarrow 1_c$ respectively, For this, we must show that the triangle equalities are satisfied.
To start, we have the following:

$$\begin{aligned}(\epsilon''f'f) \circ (f'f\eta'') = \\ [(\epsilon') \circ (f'\epsilon g')]f'f \circ [f'f(g\eta'f) \circ (\eta)] = \\ (\epsilon'f'f) \circ (f'\epsilon g'f'f) \circ (f'fg\eta'f) \circ (f'f\eta) = \\ (\epsilon'f'f) \circ f'[(\epsilon g'f') \circ (fg\eta')]f \circ (f'f\eta)\end{aligned}$$

examining the two middle terms gives us:

$$\begin{aligned} f'[(\epsilon g' f') \circ (f g \eta')]f &= f'(\epsilon * \eta')f \\ f'(\epsilon * \eta')f &= f'(\eta' \circ \epsilon)f \end{aligned}$$

putting this back into our equation gives us:

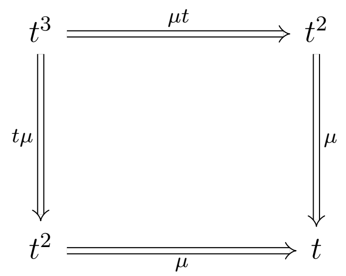
$$\begin{aligned} (\epsilon' f' f) \circ f'[(\epsilon g f') \circ (f g \eta')]f \circ (f' f \eta) &= \\ (\epsilon' f' f) \circ (f' \eta' f) \circ (f' \epsilon f) \circ (f' f \eta) &= \\ (\epsilon' f' \circ f' \eta')f \circ f'(\epsilon f \circ f \eta) &= \\ 1_{f'f} \end{aligned}$$

The case for the other triangle equality is done similarly. □

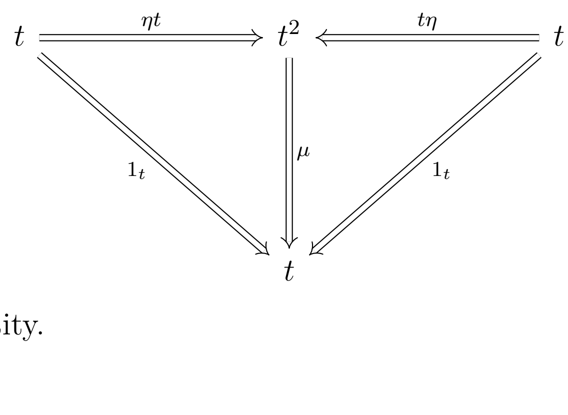
4.2 Monads in a 2-category

In **Cat**, monads are defined using the properties of functors (specifically endofunctors) and natural transformations, and since 1-cells and 2-cells generalize these, it stands to reason that we can define a monad in a 2-category **K** which is a usual monad when **K** = **Cat**.

Definition 4.2.1 (Monads in a 2-category). Given a 2-category **K** we can define a monad as a triple (t, μ, η) where $t : x \to x$ is a 1-cell, $\mu : t^2 \Rightarrow t$ and $\eta : 1_x \Rightarrow t$ are 2-cells, and the following diagrams commute:



expressing associativity, and

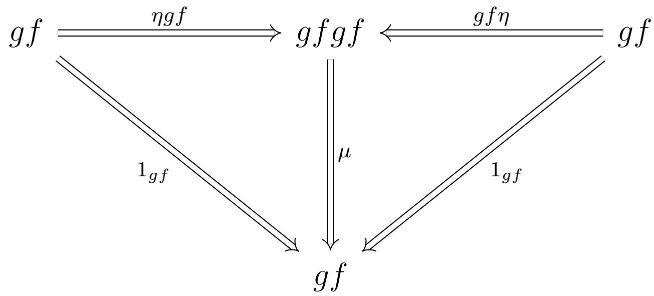


expressing identity.

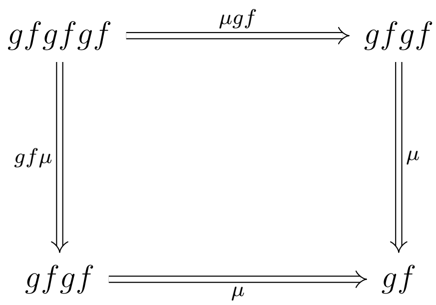
From this, we can see that in the case $\mathbf{K} = \mathbf{Cat}$, t is an endofunctor, and μ and η are natural transformations, so our monad (t, μ, η) is the same as a monad in **Cat**. Following this, we generalize a fundamental result from **Cat** to **K**.

Theorem 4.2.1 (Monads from adjoints). Given an adjunction $f \dashv g : b \to a$, with unit $\eta : 1_a \Rightarrow gf$ and counit $\epsilon : fg \Rightarrow 1_b$ we can form a monad $(gf, \eta, g\epsilon f = \mu)$ on a.

Proof. We need to show that the diagrams



and



commute. For identity, note that:

$$
\begin{aligned}
\mu \circ (gf\eta) &= (g\epsilon f) \circ (gf\eta) \\
(g\epsilon f) \circ (gf\eta) &= g[(\epsilon f) \circ (f\eta)] \\
g[(\epsilon f) \circ (f\eta)] &= 1_{gf}
\end{aligned}
$$

following the other side of the triangle, we have:

$$
\begin{aligned}
\mu \circ (\eta gf) &= (g\epsilon f) \circ (\eta gf) \\
(g\epsilon f) \circ (\eta gf) &= [(g\epsilon) \circ (\eta g)]f \\
[(g\epsilon) \circ (\eta g)]f &= 1_{gf}
\end{aligned}
$$

so both sides of our diagram commute, and the unit law is satisfied. For the associativity, we have:

$$\mu \circ (\mu g f) = (g\epsilon f) \circ (g\epsilon f g f)$$

considering the other side gives:

$$\mu \circ (g f \mu) = (g\epsilon f) \circ (g f g \epsilon f)$$
$$(g\epsilon f) \circ (g f g \epsilon f) = g[(\epsilon) \circ (f g \epsilon)]f$$

examining the term in square brackets gives:

$$(\epsilon) \circ (f g \epsilon) = \epsilon * \epsilon = (\epsilon) \circ (\epsilon f g)$$

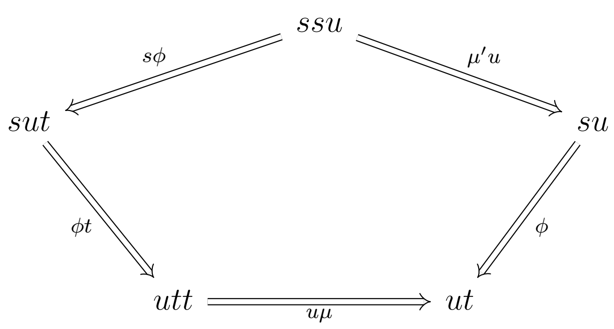
putting this back into our equation gives us:

$$g[(\epsilon) \circ (\epsilon f g)]f = g[(\epsilon) \circ (f g \epsilon)]f$$

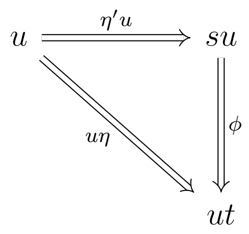
and so our square commutes and we have a monad. □

Given monads (t, μ, η) and (s, μ', η') we can define a map of monads.

Definition 4.2.2 (Monad morphism). Given monads (t, μ, η) and (s, μ', η') with $t : a \to a$ and $s : b \to b$, a morphism of monads from t to s is a pair $U = (u, \phi)$ consisting of a 1-cell $u : a \to b$ and a 2-cell $\phi : su \Rightarrow ut$ such that the following diagrams commute:



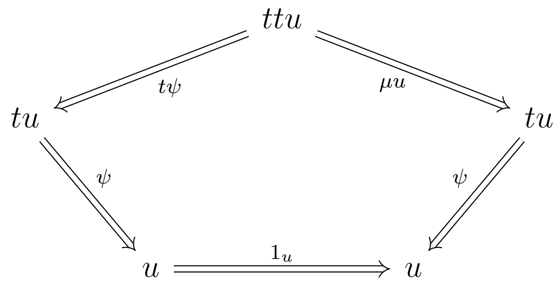
and



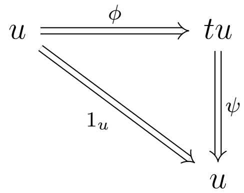
At this point, we note that for each object a of $\mathbf{K}$ we can define a monad, called the identity monad $(1_a, \mu, \phi)$, where $\mu : 1_a^2 \Rightarrow 1_a$ and $\phi : 1_a \Rightarrow 1_a$

are both the identity 2-cell on 1_a. This obviously satisfies the conditions needed to define a monad. Given a monad map $u : a \to b$ from the identity to a given monad (t, μ, ϕ), we have the following data:

1. A 1-cell $u : a \to b$ and a 2-cell $\psi : tu \Rightarrow u$
2. A diagram of the form



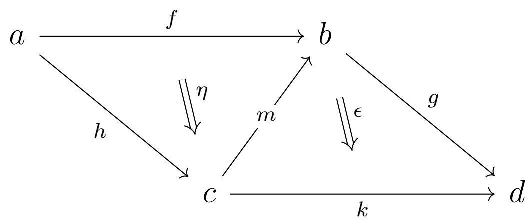
3. And a diagram of the form



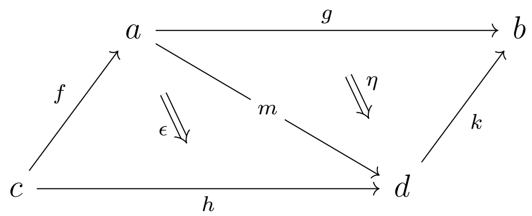
Examining the diagrams, we see that that this mimics the definition for a t-algebra, except that instead of an object, it is defined on a general object of **K**. In this way, we will say that a monad morphism from the identity to a monad (t, μ, ϕ) is a t-algebra in **K**.

4.3 Kan extensions in a 2-category K

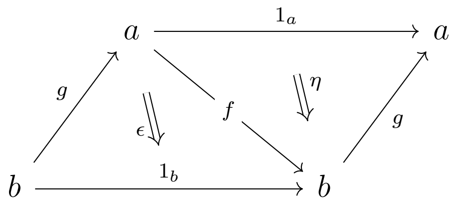
Another categorical concept that is defined using natural transformations and functors, and thus seems to be a good candidate for a 2-categorical generalisation is the concept of Kan Extensions. However, before we approach this subject we will define a helpful diagrammatic tool often used in the theory of 2-categories, called pasting diagrams. Our reference for this is the paper [6]. First, consider the following diagrams. First, we have:



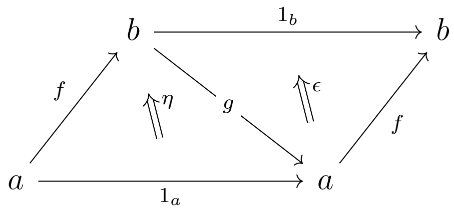
which depicts 2-cells $\eta : f \Rightarrow mh$ and $\epsilon : gm \Rightarrow k$. The diagram suggests that we would like to compose these 2-cells, but we cannot because the codomain of η is not the domain of ϵ. However, if we pre-compose ϵ with h and post-compose η with g, we have 2-cells $\epsilon h : gmh \to kh$ and $g\eta : gf \to gmh$, that are composable, with the source of the composite being the outer upper arrows of the square (composed, of course) and the target being the outer lower arrows. The other type of basic diagram is of the form:



utilizing the same method as in the past diagram, we have composable 2-cells $k\epsilon : kmf \Rightarrow kh$ and $\eta f : gf \Rightarrow kmf$, and the diagram will represent their composite $k\epsilon \circ \eta f$. The choice of notation for our 2-cells is also deliberate, since if we are given an adjunction $f \dashv g : b \to a$ with $\epsilon : fg \Rightarrow 1_b$ and $\eta : 1_a \Rightarrow gf$, we can write our triangle equalities as:



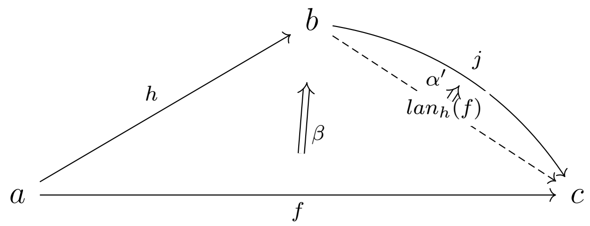
and



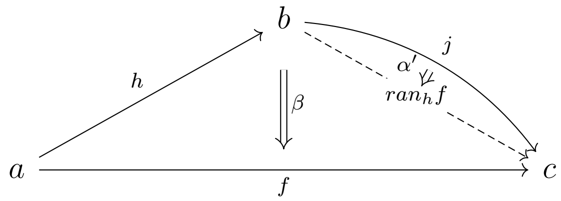
respectively. The operation of pasting can be done more generally, but that is not in the scope of this work and so we will refer the reader to [6] again. With this out of the way, we move on to our main definition:

Definition 4.3.1 (Left and Right extensions). Given 1-cells $f : a \to c$ and $h : a \to b$, a (left) extension of f along h is a pair (k, β) where $k : b \to c$ is a 1-cell and $\beta : f \Rightarrow kh$ is a 2-cell such that given another 1-cell $j : b \to c$ and a 2-cell $\alpha : f \Rightarrow jh$, there is a unique 2-cell $\alpha' : k \Rightarrow j$ such that $\alpha' h \circ \beta = \alpha$. If the direction of the 2-cells is reversed (i.e. if there is a left extension in $\mathbf{K}^{co}$), then we call (k, β) a right extension of f along h. Generally, if k is a part of a left extension we write $k = lan_h(f)$ and if it is a part of a right extension we write $k = ran_h(f)$.

To illustrate the definition, we provide the following diagram:



and for a right extension we have:



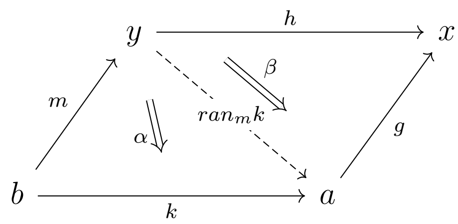
In the case of $\mathbf{K} = \mathbf{Cat}$, we recover the usual definition of Kan extensions. As is also the case in **Cat**, we can define other structures, such as adjunctions using our left extensions. Before we do this, we must first define what it means for a 1-cell to preserve a left (or right) extension.

Definition 4.3.2 (preservation of extension). A 1-cell $g : c \to d$ is said to preserve a left extension $(lan_h(f), \beta)$ if $(lan_h(gf), \beta')$ is equal to $(glan_h(f), g\beta)$.

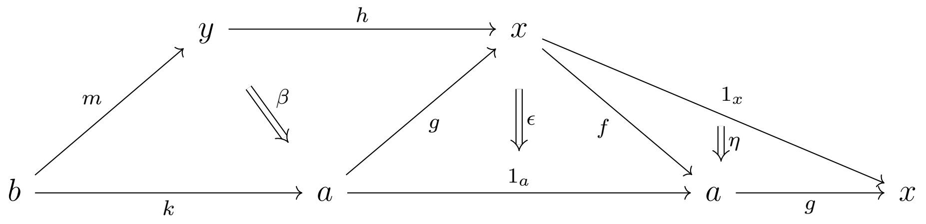
We now proceed to state the first theorem relating adjoints and extensions:

Theorem 4.3.1. In a 2-category $\mathbf{K}$, if $g : a \to x$ has a left adjoint $f : x \to a$, then g preserves all right extensions that exist for the object a.

Proof. Consider the following diagram:



where we have the 2-cell $\beta : hm \Rightarrow gk$ and the right extension $(ran_m k, \alpha)$. We need to show that there exists a unique 2-cell $\sigma : h \Rightarrow gran_m k$ such that $\beta = g\alpha \circ \sigma m$. To this end, we examine the following diagram:



In the first tetrahedron and triangle we have the composite $\epsilon k \circ f\beta : fhm \to k$. From this, because of the right extension property, we obtain a unique 2-cell $\sigma' : fh \Rightarrow ran_k m$ such that $\epsilon k \circ f\beta = \alpha \circ \sigma' m$. Finally, pasting the rightmost triangle in the diagram to this expression gives us the identity 2-cell on $g : a \to b$ because of the triangle equality, and so we get $\beta : hm \Rightarrow gk$ in return. More explicitly, note that we can write the expression in the pasting diagram as

$$g(\epsilon k \circ f\beta) \circ \eta hm = g\epsilon k \circ gf\beta \circ \eta hm$$

Examining the term $gf\beta \circ \eta hm$ gives us

$$gf\beta \circ \eta hm = \eta * \beta$$
$$\eta * \beta =$$
$$\eta gk \circ \beta$$

Returning this to the original expression gives us

$$g\epsilon k \circ gf\beta \circ \eta hm = g\epsilon k \circ \eta gk \circ \beta$$
$$g\epsilon k \circ \eta gk \circ \beta =$$
$$(g\epsilon \circ \eta g)k \circ \beta =$$
$$gk \circ \beta =$$
$$\beta$$

but we also have

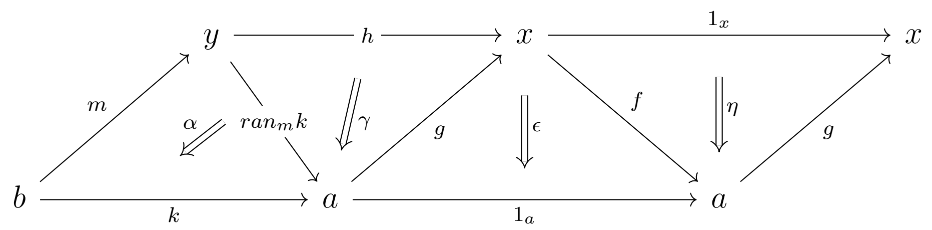
$$g(\epsilon k \circ f\beta) \circ \eta hm = g(\alpha \circ \sigma' m) \circ \eta hm$$
$$g(\alpha \circ \sigma' m) \circ \eta hm =$$
$$g\alpha \circ g\sigma' m \circ \eta hm =$$
$$g\alpha \circ (g\sigma' \circ \eta h)m$$

and so we have a factorization of β through $g\alpha$. To show uniqueness, consider another factorization of $\beta : hm \Rightarrow gk$ through $g\alpha$, so that $\beta = g\alpha \circ \gamma m$. We need to show that $\gamma = g\sigma' \circ \eta h$. To do this, consider $\epsilon k \circ f\beta$.

We already have an expression for this as $\alpha \circ \sigma' m$. However we also have:

$$\epsilon k \circ f\beta = \epsilon k \circ f(g\alpha \circ \gamma m)$$

Putting this into our diagram above gives us:



Calculating the composite of the three left triangles gives us:

$$\begin{aligned} \epsilon * \alpha \circ f\gamma m &= \\ \alpha \circ \epsilon ran_m k m \circ f\gamma m &= \\ \alpha \circ (\epsilon ran_m k \circ f\gamma) m & \end{aligned}$$

but this gives us a factorization of $\epsilon k \circ f\beta$ through α, and so uniqueness of right extensions gives us $\epsilon ran_m k \circ f\gamma = \sigma'$. We then have that:

$$\begin{aligned} g\epsilon ran_m k \circ (\eta * \gamma) &= g\epsilon ran_m k \circ \eta g ran_m k \circ \gamma \\ g\epsilon ran_m k \circ \eta g ran_m k \circ \gamma &= \\ (g\epsilon \circ \eta g) ran_m k \circ \gamma &= \\ g ran_{mk} \circ \gamma &= \\ \gamma & \end{aligned}$$

however, if we take $\eta * \gamma = gf\gamma \circ \eta h$ we get:

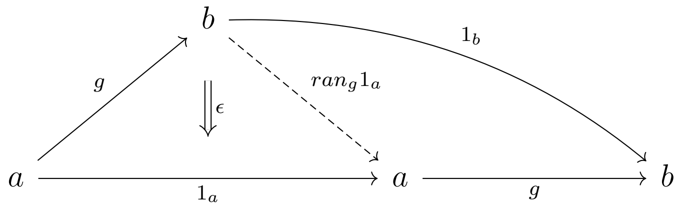
$$\begin{aligned} g\epsilon ran_m k \circ (\eta * \gamma) &= g\epsilon ran_m k \circ gf\gamma \circ \eta h \\ g\epsilon ran_m k \circ gf\gamma \circ \eta h &= \\ g(\epsilon ran_m k \circ f\gamma) \circ \eta h &= \\ g(\sigma') \circ \eta h & \end{aligned}$$

and so we have that $\gamma = g\sigma' \circ \eta h$ □

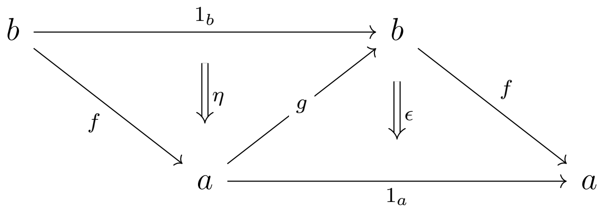
We also have the following relating adjoints and extensions:

Theorem 4.3.2 (Existence of adjoints). In a 2-category **K**, we have that $g : a \to b$ has a left adjoint if and only if the right extension $ran_g 1_a$ exists and is preserved by g. In this case, this right extension is a left adjoint for g, which we write as $f = ran_g 1_a$, and the counit 2-cell is given by $\epsilon : ran_g 1_a g \Rightarrow 1_a$ as defined by the extension.

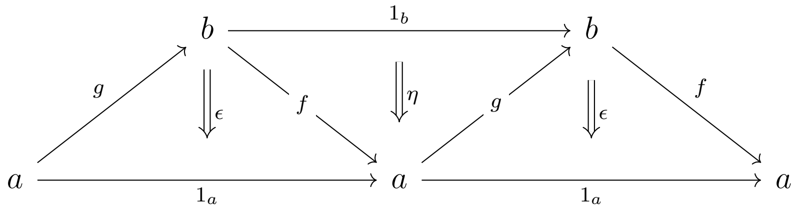
Proof. For the case where the right extension of 1_a along g, $(ran_g 1_a, \epsilon)$ exists and is preserved by g, we have the following diagram:



In particular, because g preserves $ran_g 1_a$, we have a factorisation of the identity 2-cell $1_g : g \Rightarrow g$ through the preserved extension $(gran_g 1_a, g\epsilon)$, which gives a 2-cell $\eta : 1_b \Rightarrow gran_g 1_a$ such that we can write $1_g = g\epsilon \circ \eta g$, which is our first triangle equality. The second follows from the following pasting diagrams, where $f = ran_g 1_a$:



this 2-cell $\epsilon f \circ f\eta$ maps f to f and so we have a unique 2-cell $\epsilon \circ (\epsilon f \circ f\eta)g$. Pasting it to find this factorization yields the following:



When we evaluate this diagram we see that the left 2 diagrams are the identity, from our work above, and that it is equal to the 2-cell $\epsilon : fg \Rightarrow 1_b$. In more detail, we have that the expression in the pasting diagram is equal to:

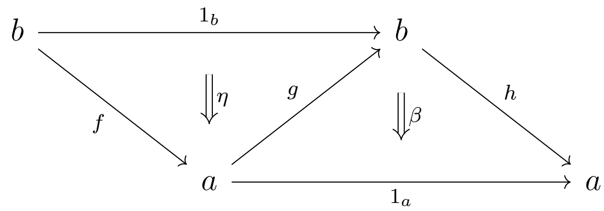
$$\epsilon \circ fg\epsilon \circ f\eta g$$

which can be reduced to

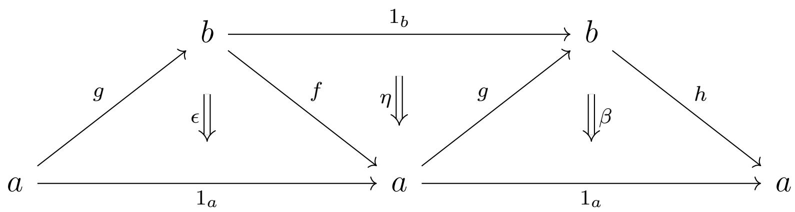
$$\epsilon \circ f(g\epsilon \circ \eta g) = \epsilon \circ fg = \epsilon$$

By our Kan extension bijection, this is the same as $\epsilon f \circ f\eta = 1_f$, and so we have shown both triangle identities.

For the other direction, suppose we are given a 1-cell $g : a \to b$ with a left adjoint $f : b \to a$, and we contend that the counit $\epsilon : fg \Rightarrow 1_a$ is the right extension and is preserved by g. To show this, suppose that we have a 1-cell $h : b \to a$ and a 2-cell $\beta : hg \Rightarrow 1_a$, and then consider the following diagram:



In it, we have defined an arrow $\beta f \circ h\eta : h \Rightarrow f$. Now, consider, the pasting diagram:



which we can see is equal to β and gives a factorization of $\beta : hg \Rightarrow 1_a$ through $\sigma = \beta f \circ h\eta : h \Rightarrow f$. More specifically, we have that the pasting diagram is equal to

$$\epsilon \circ (\beta f \circ h\eta)g = \epsilon \circ \beta fg \circ h\eta g$$

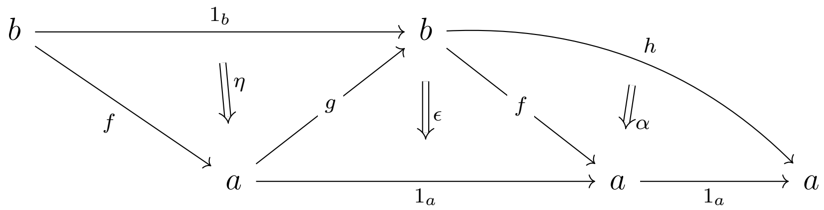
Now, looking specifically at the term $\epsilon \circ \beta fg$ we have:

$$\begin{aligned} \epsilon \circ \beta fg &= \beta * \epsilon \\ \beta * \epsilon &= \\ \beta \circ hg\epsilon & \end{aligned}$$

Returning this to the original equation then gives us:

$$\begin{aligned} \epsilon \circ \beta fg \circ h\eta g &= \beta \circ hg\epsilon \circ h\eta g \\ \beta \circ hg\epsilon \circ h\eta g &= \\ \beta \circ h(g\epsilon \circ \eta g) &= \\ \beta \circ hg &= \\ \beta & \end{aligned}$$

From here, all that is left to show is the uniqueness of the factorization. To do this, consider another factorization of $\beta : hg \Rightarrow 1_a$ through $\epsilon : fg \Rightarrow 1_a$, so we have an $\alpha : h \to f$ such that $\epsilon \circ \alpha g = \beta$. Now, we can examine the following pasting diagram:



We note that the right two triangles form β and the left two triangles are the identity by the triangle equality, and so we have that $\alpha = \beta f \circ h\eta$ and so our factorization is unique. In detail, we have that the above pasting diagram is equal to:

$$\epsilon f \circ \alpha g f \circ h\eta$$

Extracting the term $\alpha g f \circ h\eta$ gives us the following:

$$\begin{aligned} \alpha g f \circ h\eta &= \alpha * \eta \\ \alpha * \eta &= \\ f\eta \circ \alpha \end{aligned}$$

Returning this to the original equation gives us:

$$\begin{aligned} \epsilon f \circ \alpha g f \circ h\eta &= \epsilon f \circ \eta f \circ \alpha \\ \epsilon f \circ f\eta \circ \alpha &= \\ f \circ \alpha &= \\ \alpha \end{aligned}$$

Then, note that:

$$\alpha = \epsilon f \circ \alpha g f \circ h\eta = (\epsilon \circ \alpha g) f \circ h\eta = \beta f \circ h\eta$$

Finally, note that since this extension is made from an adjoint pair, by 3.5.1 it is preserved by g. □

4.4 Examples in Concrete 2-categories

Having developed results for 2-categories that generalize the notions we find in the study of 1-categories, we would like to demonstrate the validity of these constructions by considering them in various 2-categories. To begin we consider the 2-category **Cat** of categories, functors and natural transformations as a sort of base case. We have already stated many times that the constructions in this section lead to familiar notions in **Cat**, and now we will show this.

Example 4.4.1 (K = Cat). If we consider the above results in **Cat**, we find that we obtain the same standard definitions that we have in 1-categories. For example, an adjunction (in the 2-categorical sense) in **Cat** consists of categories A, B, functors $F : A \to B$ and $G : B \to A$, and natural transformations $\eta : 1_A \to GF$ and $\epsilon : FG \to 1_B$ such that $\epsilon F \circ F\eta = 1_F$ and $G\epsilon \circ \eta G = 1_G$, which is of course just one of the standard definitions of a pair of adjoint functors. Similarly, equivalence in **Cat** will give us functors $F : A \to B$ and $G : B \to A$ with natural isomorphisms $\epsilon : FG \cong 1_B$ and $\eta : 1_A \cong GF$, which again, gives us the definition of an equivalence of categories.

We can repeat this again with extensions and monads (2-categorical) to find that a monad in **Cat** consists of a category A and an endofunctor $T : A \to A$ as well as natural transformations $\mu : T^2 \to T$ and $\eta : 1_A \to T$ subject to the diagrams for all objects $x \in A$:

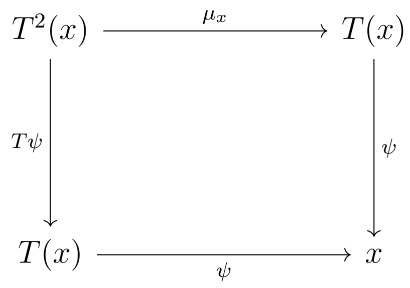
$$\begin{array}{ccc} T^3(x) & \xrightarrow{\mu T(x)} & T^2(x) \\ {\scriptstyle T\mu_x}\downarrow & & \downarrow{\scriptstyle \mu_x} \\ T^2(x) & \xrightarrow[\mu_x]{} & T(x) \end{array}$$

and

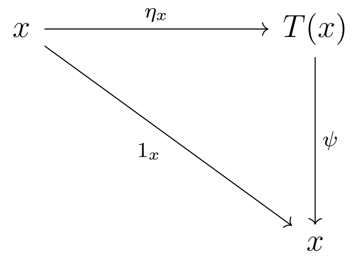
$$\begin{array}{ccccc} T(x) & \xrightarrow{\eta_{T(x)}} & T^2(x) & \xleftarrow{T\eta_x} & T(x) \\ & {\scriptstyle 1_{T(x)}}\searrow & \downarrow{\scriptstyle \mu_x} & \swarrow{\scriptstyle 1_{T(x)}} & \\ & & T(x) & & \end{array}$$

which as we see express the monadicity conditions for an endofunctor T, and so a monad in **Cat** is simply a normal monad in the category of categories. Of course, we can lift the definition of monad transformations again and obtain the notion of a T-algebra as well, in the case where the source monad is the identity monad on the category **1** with one object and one arrow. To see this, consider a monad transformation from 1_1 to T. This consist of a functor $X : \mathbf{1} \to A$ (note that each functor $X : \mathbf{1} \to A$ specifies one object in A) and a natural transformation $FX \to X$ i.e an arrow

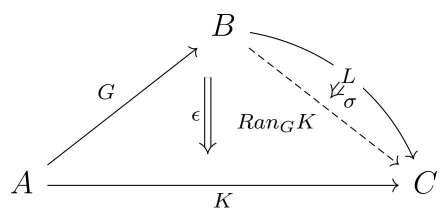
$\psi : F(x) \to x$. Then, the following diagrams are required to commute:



and

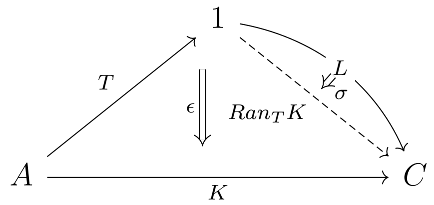


however, from this, we can see that these are simply the laws which define a T-algebra. Regarding extensions, we see that a right extension in **Cat** consists of a natural transformation $\beta : Ran_G KG \to K$ and a diagram of categories, functors, and natural transformations as follows:



such that ϵ provides a unique factorization of β through σG. This is of course the standard definition of a right Kan extension of the functor K through G. The case is the same for left extensions.

This suggests that these constructions which we find in 1-categories are actually manifestations of structures that are naturally defined in 2-categories. One can note though, that there are definitions which are standard in 1-categories which are harder to replicate internally in a 2-category, such as limits and colimits. In some sense, we then have to rely on the existence of extensions to give us a sense of ”object free” completeness, because of course, in the case **Cat** we can exhibit a limit of a functor as a right Kan extension:



as in the diagram above, where 1 is the terminal category with one object and one arrow. This is because we have natural arrows $\epsilon_a : Ran_T KT(a) \to K(a)$ and such that given any other object in C with natural arrows $\alpha_a : LT(a) \to K(a)$, we have a unique factorization $\alpha_a = \epsilon_a \sigma(T(a))$, which is of course the standard definition of a limit. Colimits emerge when we consider left extensions instead of right extensions.

Now that we have shown these notions in the obvious case of **Cat**, we move a more interesting example. Consider the 2-category of ordered sets, which we will write as **Ord**. The objects of **Ord** are ordered sets, with 1-cells being order-preserving maps. To define 2-cells, note that given 1-cells $f, g : X \to Y$, that we can define an ordering on f and g by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. This gives our hom-categories $\mathbf{Ord}(X, Y)$ the structure of an ordered set, and we can consider **Ord** as a category enriched over the category of orders (with objects just elements of a set, and if a map is defined, it is unique, meaning that there is at most one 2-cell between any 1-cells). We would like to consider some of our above defined structures in this 2-category.

To begin, consider an adjunction in **Ord**. This consists of ordered sets X, Y and order preserving maps $f : X \to Y$, $g : Y \to X$ and orders $\epsilon : fg \to 1_Y$ and $\eta : 1_X \to gf$. Order theoretically this gives us $fg(y) \leq y$ for all $y \in Y$ and $x \leq gf(x)$ for all $x \in X$. Furthermore we have that the triangle equalities are satisfied, which gives us

$$g(y) \leq gfg(y) \leq g(y)$$

and

$$f(x) \leq fgf(x) \leq f(x)$$

and so we have $fg(y) = y$ and $gf(x) = x$ and so g and f are isomorphisms. Setting $g(y) = a$ we get $y \leq g(a)$ if and only if $f(y) \leq a$, which is of course a Galois connection in the usual theory of orders.

Now, we consider monads. A monad in **Ord** consists of an ordered set X, an endomorphism of ordered sets $t : X \to X$ and 2-cells $\mu : t^2 \Rightarrow t$ and $\eta 1_X \Rightarrow t$. Order theoretically, this gives us $t^2(x) \leq t(x)$ and $x \leq t(x)$ for all $x \in X$. Furthermore we have that $t^3(x) \leq t^2(x) \leq t(x)$ and

$t(x) \leq t^2(x) \leq t(x)$ for all $x \in X$. This gives us that $t^2(x) = t(x)$ by anti-symmetry. Altogether, we see that the fact that $t(x) \leq t(y)$ for $x \leq y$, $x \leq t(x)$ and $t^2(x) = t(x)$ gives us a closure operator in the usual theory of orders.

Finally, we examine extensions in this 2-category. We begin with right extensions. For this, we have ordered sets X, Y, Z together with maps $k : X \to Z$ and $h : X \to Y$. Assuming that the right extension $ran_h(k) : Y \to Z$ exists, we have a 2-cell $\alpha : ran_h kh \Rightarrow k$. Order-theoretically, this gives us $ran_h kh(x) \leq k(x)$ for all $x \in X$. The universal property of right extensions then states that for any other order preserving map $m : Y \to Z$ with a 2-cell $mh \leq k$, then there is a 2-cell $\alpha' : m \leq ran_h k$ (unique by both the universal property of right extensions and the properties of **Ord**). Therefore, a right extension gives an upper bound on 1-cells $m : Y \to Z$ with the property that there exists a 2-cell $\beta' : mh \Rightarrow k$.

Bibliography

[1] J Benabou. *An introduction to Bicategories* Lecture Notes in Mathematics. Vol. 47. Berlin: Springer, 1967.

[2] F Borceux. *Handbook of Categorical Algebra Volume 1, Basic Category Theory*, Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[3] F Borceux. *Handbook of Categorical Algebra Volume 2, Basic Category Theory*, Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[4] P Etingof, S Gelaki, D Nikshych, V Ostrik. *Tensor Categories*, Lecture Notes, 2009.

[5] G.M Kelly. *Basic Concepts of Enriched Category Theory*, Lecture Notes in Mathematics. Cambridge University Press, 1982.

[6] G.M Kelly, R Street. *Review of the Elements of 2-Categories*, Lecture Notes in Mathematics. Springer, 1974.

[7] S Lack. *A 2-categories companion*, Towards Higher Categories, eds. John C. Baez and J. Peter May, Springer, 2009

[8] S MacLane. *Categories for the Working Mathematician*, Graduate Texts in Mathematics. Springer, 1998.

[9] E Riehl. *Category Theory in Context*, Courier Dover Publications, 2017.

[10] R Street. *The Formal Theory of Monads* J. Pure Appl. Algebra 2, 1972 149–168.

[11] T Tom-Dieck. *Algebraic Topology* European Mathematical Society, 2008.