CONTROLLING THE WALRASIAN TATONNEMENT PROCESS

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Abstract

In this thesis I examine a discrete-time Walrasian tatonnement process. The criterion for stability is examined in a two good tatonnement process. It is shown that the stability of the system depends upon the speed of adjustment and holdings of endowments as well as preferences. It is then shown that periodic solutions as well as aperiodic or chaotic trajectories occur. The analysis is then extended to multiple agents.

Having established the results for the one-dimensional system, the analysis is extended to the case of three goods in which one of the goods is a numeraire. It is shown that similar dynamics to the one dimensional case exist. It is found that if one market acts in a chaotic manner then both markets act in a chaotic manner. Such that markets do not act in a chaotic manner, certain restrictions on the speed of adjustment and the holding of the non-numeraire good with respect to the numeraire good need to be enforced.

Following in the footsteps of Uzawa [26], exchange out of equilibrium is examined for the case of one traded good and one numeraire as well as two traded goods and one numeraire. It is found that if any good can be exchanged for any other good there is a direct parallel between the tatonnement process and the non-tatonnement process. If the numeraire is treated as a primitive currency then the policy implications differ significantly due to the amount of liquidity in the system.

Contents

Introduction

This dissertation is organised as follows. In section one the stability of competitive equilibria are reviewed. Work by Arrow and Hurwicz [2], [3], [4], Arrow, Block and Hurwicz [5], Uzawa [22], [23] and Uzawa and Nikaido [24] are reviewed as well as Scarf's [18] counter-examples of global instability. Work by Day [8] is then reviewed in which global stability breaks down in the face of chaotic trajectories if the tatonnement process is discrete time in nature. Parallels and differences between aspects of these articles are highlighted.

Section two proposes the tatonnement process that will be used in this analysis. This section closely mirrors Day [8]. Price dynamics are examined for a two person, two good economy in which one good is treated as the numeraire. It is shown that the price sequences in the discrete tatonnement process can be complex.

Having established that complex price dynamics exist, section three proposes differing methods of controlling these dynamics in the one dimensional case. It is in this section that the fundamental argument of this paper is made; if there are complex movements in prices, such that the market clearing price is never reached, then, by suitable adjustment in the holdings of endowments and speed of adjustment, the system can be rendered stable. The liet motif of this section is "stability good, chaos even better" as an analysis of the complexity of the system strengthens the understanding of the stability conditions. Policy implications can be inferred from this maxim and further lines of enquiry necessarily arise.

Section four includes a three good tatonnement process. Stability and instability conditions are examined. It is noted that the two dimensional case is a natural extension of the one dimensional case but more complicated to analyse.

Section five examines the case in which exchange is permitted out of equilibrium. This section is based on the analysis by Uzawa [26]. Two approaches to exchange out of equilibrium are considered. The first approach is that in which any good can be exchanged for any other good. The second approach is that in which exchange for any good can only take place through the medium of the numeraire. In the first case the tatonnement process (T) and the non-tatonnement process (NT) have strong equivalence relations. In the second case, liquidity constraints drive a wedge in between T and NT, for which the dynamics of the latter differ fundamentally to those of the former.

The appendix contains the various theorems used in this paper as well as providing subsidiary explanations when needed.

Literature Review

In order to treat the subject of this thesis two strains of literature need to be considered. The first type is that which concerns the stability properties of general equilibrium, especially general equilibrium under gross substitution. The second is the application of chaos theory to such general equilibrium analysis. There is an abundant literature in both.

Negishi [13] provides a succinct proof that, if the tatonnement process is continuous time in nature, and for the case in which one of the goods is elected as a numeraire, then, if all goods are gross substitutes (GS), the tatonnement process is globally stable.

This proof relies on the construction of a Jacobian matrix whose principal minors alternate in sign. From this, Negishi is able to conclude that if the own price effect dominates all off-price effects, global stability is ensured in which there is convergence to a unique set of points. Gross substitution implies diagonal dominance which in turn guarantees global stability.

In his survey article Negishi [14] delves further into the idea of gross substitutability and global stability[1], naming this result as one of the most important results obtained in the study of the stability of the tatonnement process. Negishi notes that this result can be obtained by various means.

The starting point in [14] is the use of a strictly quasi-concave utility function[2] such as $U_i(X_{i1},...,X_{im}) = \sum_j \alpha_{ij} \log X_{ij}, \ \forall i$, where $\sum_j \alpha_{ij} = 1$ and $\alpha_{ij} > 0$. The demand for any good by any individual is given by;

$X_{ij} = \alpha_{ij} \dfrac{I_i}{P_j}$, where income, $I_i = \sum_j P_j \overline{X}_{ij}$, is a function of prices and endowments. Gross substitution implies that, $\dfrac{\partial X_{ij}}{\partial P_k} > 0, \ \forall P, \overline{X} > 0, \ j \neq k$ and $\dfrac{\partial X_{ij}}{\partial P_k} < 0, \ \forall P, \overline{X} > 0, \ j = k$

By means of such a utility function, it is shown that the equilibrium price vector is uniquely determined up to a scalar multiple. From the demonstration of this last point, the tatonnement process can be shown to be stable by considering that, if the equilibrium is unique, quasi-stability[3] implies global stability. To prove quasi-stability, some function needs to be defined such that this function is decreasing through time if there is a state of disequilibria[4]. If such a function exists, then by gross-substitutability, which implies the weak axiom of revealed preference, global stability is ensured. Different types of functions can be constructed that have this desired property. Negishi [14] notes that different choices of such a function offer different proofs of the same theorem; differing functions can be the Euclidean distance in the price space[5] or the distance in terms of maximum norm in the price space. Both types of proofs arrive at the same conclusion; the tatonnement process is globally stable under certain mild assumptions.

Nikaido and Uzawa [15] note a similar result. If the aggregate excess demand function is single valued and homogenous of degree zero, if Walras' law holds and if the weak axiom of revealed preference holds such that for any equilibrium price vector p^*, and any other price vector p not proportional to p^*, the inequality $\sum_{i=0}^{n} p_i^* Z_i(p) > 0$ prevails. If the speed of adjustment is small, the distance of any price to the unique price equilibrium falls shrinks monotonically to zero over time.

[1] This result is due to Arrow, Block and Hurwicz [5]
[2] As used by Arrow and Hurwicz [2], p. 550
[3] Convergence to a set of equilibrium points that are not necessarily unique
[4] i.e. a Lyapunov function
[5] The sum of squares of the difference between prices and equilibrium prices

At this point one may point out that global stability may be deemed the domain of continuous time price adjustment mechanisms. This would be erroneous as Uzawa [23] demonstrates the case in which global stability can take place in a tatonnement process that is of a discrete time formulation[6];

$$
\begin{aligned}
&p_i(t+1) = \max\{0, p_i(t) + f_i(t)\} \\
&t = 0,1,2,.....; \qquad (1) \\
&i = 0,1,2,...,n,
\end{aligned}
$$

In the same vein as Arrow, Block and Hurwicz [5], Uzawa shows that the Lyapunov stability theorem can be applied to the above set discrete time price adjustment mechanisms. He proposes the following stability theorem[7]:

Let $p(t; p_0)$ *be the solution to (1) with initial price vector* p_0. *If*

(a) the solution path $p(t; p_0)$ *is bounded for any initial price vector* p_0 *; and*

(b) there exists a continuous function $\Phi(p)$ *defined for all price vectors p such that* $\varphi(t) = \Phi[p(t; p_0)]$ *is strictly decreasing*[8] *unless* $p(t; p_0)$ *is an equilibrium, then the process is quasi-stable.*

The simultaneous adjustment process (1) is defined as quasi-stable if, for any initial price vector p_0, every limiting point of the solution $p(t; p_0)$ is an equilibrium. If the set of all equilibria is finite, the quasi-stability of the process (1) implies global stability. Uzawa goes on to show that if the speed of adjustment is small *and* if the weak axiom holds, then (1) is globally stable. Uzawa's stability theorem is of prime import in the context of this thesis as it is the transgression of this stability theorem that allows complex dynamics to occur. Strictly speaking, Uzawa's stability theorem cannot be violated; as if the distribution of endowments is such that the system (1) is not stable then Uzawa's theorem implies that stability can be induced by a reduction in the speed of adjustment. Whilst this is true and incontrovertible, this author asserts that a continual reduction in the speed of adjustment is a penurious line to follow; there are other factors that allow stability to be induced in the presence of instability, or that allow the auctioneer to pre-empt the occurrence of complex dynamics. In Uzawa's defence, one must note that the presence of complex dynamics are not a strict counter-example to global stability as Uzawa and his peers wrote at a time when chaos theory and its adjunct fields of interest had not yet been properly formulated. To evoke chaos as a counter argument to the stability properties of a discrete time tatonnement process simply does not hold as the latter was constructed when the former was absent[9].

Scarf [18] provides several counter-examples to stability which do not make recourse to complex behaviour. Scarf's counter-example may be deemed more appropriate in the sense that it was written contemporaneously and in response to Uzawa and his peers. One of Scarf's counter-examples considers the case in which there are three goods and three individual's, each of whom has taste for only two of the goods. The holdings of initial endowments are cyclic in the sense that each individual only holds one unit of a good; the endowment matrix is equivalent to a three dimensional identity matrix. The specific form of the utility function is of a Leontief type in which the two commodities desired by each of the three

[6] Saari [17] p. 1119 also notes that correct dynamical process for the tatonnement process is an iterative one, at least in the present context in which an auctioneer calls out prices

[7] Uzawa [23], p 185

[8] i.e. is a Lyapunov function

[9] Mathematical interest in deterministic dynamical systems that generate apparently random trajectories has dated back to at least Poincare's work in the late 1800s. Interest in the natural sciences was piqued by a paper by Ruelle and Takens in 1971 who argued that the traditional model of fluid flow turbulence was structurally unstable and that a dynamical system that converged to a low dimensional deterministic system was a better model of certain types of fluid flow turbulence than the traditional one

individuals are perfectly complementary to each other. It is shown that any other price vector other than the market clearing price vector will create a trajectory in which prices circle around the fixed points. The unique fixed points are thus never reached. It can be noted that, if trade can only ever be effectuated *once* the market clearing price is attained, then Scarf's counter-example provides a case in point in which trade will never take place. It is further noted that the instability[10] in this case is a product of the income effect dominating the substitution effect.

In light of [18], Negishi [14] concludes his survey article by stating that *"the tatonnement process is not perfectly reliable as a computing device to solve the system of equations for general equilibrium"*

On the one hand, a counterpoise to Negishi's conclusion may simply be that the weak axiom need hold or that all goods be gross substitutes. This would circumvent counter-examples such as those constructed by Scarf. On the other hand, as alluded to previously, work subsequent to Scarf has demonstrated that even in the presence of the weak axiom or revealed preference and gross substitutability, there is no guarantee that stability in the large need hold.

Morishima [12] does not appear so optimistic about any such counter-argument. He demonstrates, by means of a simple example, a case in which even in the presence of gross substitutability, periodic price dynamics can be generated. He concludes that the tatonnement process can be disappointing as a price adjustment mechanism.

Day [8] considers a pure exchange economy in which supply does not change as price changes. This corresponds to Uzawa's [23] in that endowments are static and there is a change in goods only once the market clearing price has been reached. The precise form of the tatonnement process that he considers uses a utility function of the Cobb-Douglas form as in Arrow, Block and Hurwicz [5], but uses a discrete time tatonnement mechanism as in Uzawa [23]. Day's analysis focuses on the fact that if endowments are cyclic[11], and there are two goods, two market participants and one of the goods is treated as a numeraire, then the discrete tatonnement process has the capacity to generate complex dynamics. Such complex dynamics encompass the case in which the unique fixed point is reached as well as the cases of the occurrence of periodic and chaotic trajectories. In the latter two cases the market clearing price is never reached. This occurs even if gross substitutes and the weak axiom of revealed preference hold. Day's result is therefore a caveat to Uzawa's stability theorem, as it constitutes a sub-set of that which can occur if the stability theorem does not hold.

Day states his results as[12]:

1. A unique competitive equilibrium exists and is the unique stationary state of the process.
2. If the eigenvalue falls within the unit circle, then the competitive equilibrium is globally stable.
3. The tatonnement process generates prices that are bound to a finite interval.
4. If the eigenvalue falls outside the unit circle then fluctuations persist almost surely.
5. For some set of parameters, chaotic trajectories exist robustly over a significant parameter interval.
6. Given any speed of adjustment, these results occur robustly.

Point 6 is of especial interest as, for a given distribution of endowments, the speed of adjustment may need to be altered such that either monotonic or oscillatory converge of price to equilibrium. Generally stated, if the holdings of the endowment of the numeraire are scarce relative to the non-numeraire, then, for a given speed of adjustment, price dynamics may be complex. The negation of this last point is subsumed into Uzawa's stability theorem as, in this theorem; the upper bound for the speed of adjustment is directly related to the precise value of aggregate excess demands, which in turn are dependent on the distribution of the holdings of the endowments. Day therefore appears to be providing a specific type of dynamic which

[10] Scarf's example is one in which global stability is not possible, but stability in the sense defined by Lyapunov is possible. Lyapunov's stability requires that the orbit tends to a neighbourhood of the fixed point and remain there or simply circle about it

[11] This is not a necessary condition to obtain the results that he does, but a simplifying factor

[12] Bearing in mind that Day's example is one-dimensional in nature

lends itself more to an examination of trajectories that are chaotic in nature. Furthermore, stability in [23] is rigorously demonstrated for a system of any dimension. Day falls short in his caveat in that he considers a one dimensional system.

We are therefore left with the following problem: it need not necessarily follow that global stability does not hold if any such caveat is restricted to a one-dimensional system. A strict caveat would have to be extended to a system of any arbitrary dimension simply by dint that (1) is stable for any dimension if the speeds of adjustment are small enough and WARP holds. A strict caveat would also have to show that holdings of relative endowments in a system of any dimension have a bearing on the stability of that system, given that Day's result is premised in part on the robust occurrence of chaos for any speed of adjustment.

Goree at al. [9] note that under the assumption of gross substitutability, the price process may not converge to a point, yet the price process is stable in the sense that prices converge to a bounded region. It is this fact that prices are bound to fluctuate within a certain region that permits them to posit that an out of equilibrium exchange is beneficial to no trade at all. They note that within this region, in the case of three goods, as the speed of adjustment changes, the attracting set of the system changes from (i) the unique fixed point to (ii) a period two orbit, to (iii) a quasi-periodic attractor, to (iv) a strange attractor on which the dynamics are completely chaotic.

Goree et al. state the following proposition for the type of economy considered here:

For all economies there exists a finite λ_0 such that for $\lambda > \lambda_0$ all fixed points of the discrete process ((1) above) are repellers[13].

They note the following:

1. Under gross substitutability the unique equilibrium is globally stable when the speed of adjustment is sufficiently small.
2. The tatonnement process becomes unstable at some finite value of the speed of adjustment.
3. There exists an attracting set to which all initial price vectors converge (for all values of the speed of adjustment)
4. The dynamics inside the attracting set are quite complicated for sufficiently high values of the speed of adjustment.

It is also concluded that 1 to 3 are independent of the precise choice of the type of aggregate excess demand function or the dimensionality of the system. However, the specific type of bifurcation occurs (periodic-doubling, Hopf, saddle-node, or global bifurcations) and for what values of the speed of adjustment, depends upon the particular system studied. It is also pointed out that if the system is stable, then the eigenvalues of the Jacobian of the aggregate excess demand function evaluated at the fixed point are less than one in absolute value and that this condition is dependent upon the value of the speed of adjustment (amongst other parameters). Goree at al.'s results accord entirely with the results of this thesis and indeed provide a generalisation of Day's caveat to higher (two) dimensions. Goree et al. however, do not consider that their results could be modified by a change in the holdings of relative endowments. In other words, points 1 – 4 hold for a given endowment matrix. If this endowment matrix changes 1 – 4 may be altered.

Tuinstra [21] considers a discrete price adjustment mechanism in which there are three agents each of whom have a CES utility function and three goods. He argues that the appropriate price normalisation rule is that prices sum to one. Tuinstra thus considers chaotic dynamics on the unit simplex rather than arbitrarily choosing one good as the numeraire. This analysis is especially instructive as the author notes that, in the three dimensional case, there are simply too many free parameters for a simple analysis. To this end symmetry is introduced into the tatonnement process and is imposed on the endowment matrix and the preference matrix. Such symmetry can be rotational, reflective or both. The imposition of some form of

[13] Where λ_0 is the speed of adjustment

symmetry allows the study of the global dynamics by reducing the number of free parameters from 20 to 8. Symmetry not only produces spectacular bifurcations on the simplex but also has the function that "*the symmetric tatonnement process can serve as an 'organising centre' in the analysis of the global dynamics in general tatonnement process*". Symmetry also yields the neat property that the Jacobian matrix has an immediate symmetry to it as well. Like Goree et al., Tuinstra states that in order to study the bifurcations in the system, the eigenvalues of the Jacobian, need to be examined. If the market clearing price vector is stable and the eigenvalues lie inside the unit circle then the system is locally stable[14]. If at least one of the eigenvalues lie outside the unit circle then the market clearing price vector is unstable. At the point at which an eigenvalue does cross the unit circle as a parameter is varied (for example the speed of adjustment), a bifurcation point occurs and the characteristics of the tatonnement process fundamentally change.

Whilst the literature reviewed is unequivocal in the conclusions reached regarding stability, there does appear to be some gaps in the analysis of general equilibrium. This author asserts that not only can instability be induced by shifts in the speeds of adjustment but endowments play an equally important role and have much more interesting implications for policy especially if trade in disequilibrium is permitted. The examination of the former, to date has been done at the expense of the latter and the latter cannot be neglected as both types of parameters find their way into the tatonnement process and will therefore generate stable or complex price dynamics.

Uzawa [26] examines the transaction rules for which an out of equilibrium exchange is allowed. If exchange increases the utility of at least one individual and reduces the utility of no one, then trade will take place. If this rule is abided by then the holdings of endowments converge to a Pareto optimum over time. Uzawa establishes this result for both the discrete and continuous time tatonnement process noting that a strict preference relation must be defined as well as that individual, and hence global, constraints of endowments remain unchanged. Under these conditions, there is an equilibrium price vector that supports the Pareto optimal holdings of endowments.

It must be noted that Pareto optimal holdings will be determined not only by the rigorous enforcement of the trading rule but also by the choice of initial prices and initial holdings of endowments. These starting conditions may favour one individual over the others. In this sense, for a different starting price and different holdings of endowments, the Pareto optimal may be highly skewed in the favour of one individual over the other. Uzawa is silent on this point.

Trade at disequilibrium also starts to allow various other questions to be asked. If individuals are allowed to trade at any price vector, providing that the trading rules are adhered to, such trade may start to encompass a speculative component, especially if one starts to build into the model the role of expectations. If prices are, for example, falling monotonically over time, one individual may withhold an amount that he would otherwise exchange even though, by the trading rules, it would be beneficial to trade. Such holdings of goods that are withheld would be speculative in nature and may either stabilise or destabilise the Pareto holdings of endowments with respect to the case in which no speculative trading were permitted. A confounding factor in the context of the non-tatonnement processes presented in this paper may be that speculative holdings either work in accordance or in discord with complex dynamics in the reaching of Pareto optimal holdings of goods. No statement can be made out of hand without a thorough analysis of both the effect of speculative holdings of goods and the effect of non-speculative holdings of goods in the presence and absence of complex dynamics. For the purposes of this thesis speculative holdings are disallowed and the trading rules are strict.

[14] Locally stability implies global stability in the case of gross substitutes at all price vectors

Section Two: The Tatonnement Process

In this section the tatonnement process is defined. The type of economy that is assumed is Cobb-Douglas. The specification of this type of economy ensures that gross-substitutability and the weak axiom of revealed preference hold and therefore, for a stable system, there is a unique fixed point.

This section is organised as follows. The specific tatonnement process is first derived. Conditions for price to remain bound to a particular interval are formulated. Conditions for stability are then introduced. A discussion of the importance of the relative abundance of the endowment of the numeraire with respect to the non-numeraire is noted. A specific cyclic endowment matrix is then considered, for which the stability of the tatonnement process is examined as the speed of adjustment is varied. Stable trajectories, periodic trajectories and chaotic trajectories are all found to exist for different values of the speed of adjustment. Whilst this analysis has been carried out elsewhere (Day [8]), the need for such a thorough examination in this section is justified by the contents of section three, in which solutions to the periodic or chaotic nature of the tatonnement process are presented. In light of what is presented below and in subsequent sections, we find that a sufficient condition for the attainment of the unique fixed points is that the eigenvalue of the system lies within the unit circle. It will be shown that this latter condition is directly related to the relative values of endowments and the speed of adjustment. This finding is an extension to the work named in the literature review in which *only* the speed of adjustment is considered as having the potential to generate complex price paths.

The Cobb-Douglas Economy and Walras' Tatonnement

Since GS holds then the demand for any good is given by[15] $x_i^j = \frac{\alpha_i^j m_i}{p^j}$. Since the income for any individual is dependent upon the price prevailing at any point in time, the following identity holds:

$m_i \equiv \sum_j p^j \omega_i^j, \ \forall i$, where ω_i^j is the endowment of good j for individual i. The aggregate excess demand is the summation of any particular good for all individuals, less the endowment of that good for all individuals:

$$Z^j = \sum_{i=1}^{n} x_i^j - \sum_{i=1}^{n} \omega_i^j, \ \ \forall j \tag{2.1}$$

Since the demand for any good j depends upon individual i's income, and this in turn depends upon prices then (2.1) becomes;

$$Z^j(p) = \sum_i x_i^j(\alpha, p, \omega) - \sum_i \omega_i^j, \ \ \forall j \tag{2.2}$$

The price adjustment mechanism or tatonnement process is

$$p_{t+1}^j = f^j(p_t^j) = \max\{0, \ p_t^j + \lambda^j Z^j(p_t)\}, \ \ j = 1, 2, \ldots \tag{2.3}$$

where λ^j is the speed of adjustment for market j.

[15] Index i refers to the individual and j refers to the good

This type of tatonnement process follows the mechanism described in detail in Uzawa[16] in which there are j markets running concurrently for the j commodities that are desired (but not necessarily held) by the i individuals.

We note that if the market clearing price vector is reached then $Z^j(p) = 0,\ \forall j$, which in turn implies that

$$\sum_{i=1}^{n} x_i^j - \sum_{i=1}^{n} \omega_i^j = 0 \Leftrightarrow \sum_{i=1}^{n} x_i^j = \sum_{i=1}^{n} \omega_i^j,\ \forall j; \tag{2.4}$$

Denote the market clearing holding as;

$$\sum_{i=1}^{n} x_i^j * = \sum_{i=1}^{n} \omega_i^j \tag{2.5}$$

which is the equilibrium exchange of goods[17]. Such that there has been the exchange of at least one good (between at least two agents) then;

$$\sum_{i=1}^{n} x_i^j \neq \sum_{i=1}^{n} x_i^j * = \sum_{i=1}^{n} \omega_i^j \tag{2.6}$$

(2.6) simply states such that the tatonnement process is not vacuous or trivial in the sense that individuals come to the market already holding the distribution of endowments that would come about by means of the price adjustment mechanism (that is the pre- and post-trade holdings of commodities are the same), then the initial holdings of at least one good for at least two people must differ to the optimal distribution of resources[18] at the market clearing price. Or alternatively, some trade has to take place. If gross substitution, the weak axiom of revealed preference and Walras' law hold, there is then some unique price vector that allows (2.6) to be attained.

In order to render the tatonnement process more concrete, the highly simplified case in which there are only two goods and two market participants is considered.

By (2.3) we have two simultaneous adjustment mechanisms:

$$\begin{aligned} p_{t+1}^1 &= f^1(p_t^{1,2}) = \max\{0,\ p_t^1 + \lambda^1 Z^1(p_t^{1,2})\} \\ p_{t+1}^2 &= f^2(p_t^{1,2}) = \max\{0,\ p_t^2 + \lambda^2 Z^2(p_t^{1,2})\} \end{aligned} \tag{2.7}$$

Commodity two is arbitrarily chosen as the numeraire good so that $p^2 = 1,\ \forall t$. (2.7) therefore reduces to a tatonnement process for good one only;

$$p_{t+1} = f(p_t) = \max\{0,\ p_t + \lambda Z(p_t)\} \tag{2.8}$$

Substituting the demand for good one and the endowments for all consumers into (2.8) and rearranging;

[16] See literature review for an outline of the details of this type of mechanism

[17] Exchange only takes place if the market clearing price has been reached. This is tantamount to the same price being called for at least two consecutive periods. No exchange takes place beforehand. This mechanism differs fundamentally to that proposed by Uzawa [26] in which trade can take place out of equilibrium. The use of the word "traded" in the context of sections two, three and four is slightly incorrect as no trade actually takes place throughout the price discovery process

[18] This condition is not strictly needed for what follows. It can be shown that even if individuals enter the market holding the optimal (post-exchange) distribution of endowments, the tatonnement process can still be chaotic

$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\sum_{i=1}^{n} x_i^1 - \sum_{i=1}^{n} \omega_i^1\right]\right\}$$
$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\sum_{i=1}^{n} \frac{\alpha_i^1\omega_i^2}{p_t} - \sum_{i=1}^{n} \omega_i^1(1-\alpha_i^1)\right]\right\} \tag{2.9}$$

If the market clearing price is reached then the same price is called by the auctioneer for at least two consecutive periods. If such a price is reached, and is positive, then (2.9) becomes;

$$\begin{aligned} p^* &= p^* + \lambda Z(p^*) \Rightarrow \\ \lambda Z(p^*) &= 0 \qquad \Rightarrow \\ Z(p^*) &= 0 \end{aligned} \tag{2.10}$$

If $Z(p^*) = 0$ (absence of excess demand and supply for good one), then;

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\alpha_i^1\omega_i^2}{p^*} - \sum_{i=1}^{n} \omega_i^1(1-\alpha_i^1) = 0 \Rightarrow \\ &\sum_{i=1}^{n} \frac{\alpha_i^1\omega_i^2}{p^*} = \sum_{i=1}^{n} \omega_i^1(1-\alpha_i^1) \qquad \Rightarrow \\ &p^* = \frac{\sum_{i=1}^{n} \alpha_i^1\omega_i^2}{\sum_{i=1}^{n} \omega_i^1(1-\alpha_i^1)} > 0 \end{aligned} \tag{2.11}$$

which is the market clearing price for good one. The market clearing price is thus simply the ratio of the endowment for the numeraire by the preference for the traded good to the endowment of the traded good by the preference for the numeraire.

It must be noted that if (2.11) holds, then the market for good two is also in a state of equilibrium in which there is no excess demand or supply[19];

$$(Z^1(p^*) = 0) \Leftrightarrow (0 = Z^2(p^*)) \tag{2.12}$$

It must furthermore be noted that the speed of adjustment has no bearing on the market clearing price; λ only has a bearing on the rate at which that price is attained (or whether such a price is attained at all).

We now want to establish the stability properties of the tatonnement process (2.9). By the stability property (A9);

[19] Conversely if $Z^1(p) \neq 0 \Leftrightarrow 0 \neq Z^2(p)$. Moreover, any particular dynamic that occurs in market one also occurs for the aggregate excess demand for good two. By (2.12), it is sufficient to consider only the market for which price has not been normalised in the two good case

$$\left|\frac{dp_{t+1}}{dp_t}\right|_{p=p^*} = |f'(p^*)| < 1 \tag{2.13}$$

Applying (2.11) to (2.13);

$$|f'(p)| = \left|1 - \lambda \sum_{i=1}^{n} \frac{\alpha_i^1 \omega_i^2}{(p_t)^2}\right|, \quad |f'(p^*)| = \left|1 - \lambda \frac{\left(\sum_{i=1}^{n} \omega_i^1 (1-\alpha_i^1)\right)^2}{\sum_{i=1}^{n} \alpha_i^1 \omega_i^2}\right| < 1 \tag{2.14}$$

Define[20] κ as; $\kappa \equiv \lambda \dfrac{\left(\sum_{i=1}^{n} \omega_i^1 (1-\alpha_i^1)\right)^2}{\sum_{i=1}^{n} \alpha_i^1 \omega_i^2}$ (2.15)

The stability condition is therefore;

$$|f'(p^*)| = |1-\kappa| < 1 \tag{2.16}$$

(2.16) is simply a statement that the modulus of the eigenvalue or multiplier of the discrete map is less than one, or lies within the unit circle, and for this particular type of map the eigenvalue can be decomposed into $|1-\kappa|$.

Such that (2.16) holds we can make a further statement about κ;

If $\kappa \in (0,2)$ (2.17)

then (2.16) holds[21].

If the tatonnement process is unstable the inequality in (2.16) is reversed such that;

$$|f'(p^*)| = |1-\kappa| > 1 \tag{2.18}$$

Definition (2.15) provides an immediate way of evaluating the conditions under which (2.16) is attained and as a consequence (2.17) holds. Given that κ depends on the speed of adjustment, preferences and endowments for the numeraire and the non-numeraire, the factors that enter into the tatonnement process (2.9) are precisely those factors that determine the value of the eigenvalue and therefore the stability of that process (2.8).

In order to ensure that prices are strictly positive, that is to say so that $p_t > 0$ at any point in time, then let there be a $\tilde{p}$ that minimises the map f, such that[22];

$$f'(\tilde{p}) = 0 \tag{2.19}$$

[20] This methodology is used by Day [8]
[21] The interval is open as if $\kappa = 0$, then by (2.15) and (2.9), there would be no actual tatonnement process
[22] The zero price boundary is avoided

From (2.9);

$$f'(\tilde{p}) = 1 - \lambda \sum_{i=1}^{n} \frac{\alpha_i^1 \omega_i^2}{(\tilde{p})^2} = 0 \quad \Rightarrow \quad \tilde{p} = \left[\lambda \sum_{i=1}^{n} \alpha_i^1 \omega_i^2 \right]^{\frac{1}{2}} \tag{2.20}$$

Substituting the expression for $\tilde{p}$ into (2.8) and noting[23];

$$f(\tilde{p}) = \tilde{p} + \lambda Z(\tilde{p}) > 0 \tag{2.21}$$

Rearranging (2.21) and simplifying the inequality

$$\kappa < 4 \tag{2.22}$$

is obtained. Combining (2.17) and (2.22) the following interval for κ is obtained;

$$\kappa \in (0,4) \tag{2.23}$$

An expression for the optimal speed of adjustment, λ^* can also be derived. This is the value for which the market clearing price (or a solution to the tatonnement process) is reached in the shortest time possible (i.e. the system is super-stable).

$$f'(p^*) = 1 - \lambda \sum_{i=1}^{n} \frac{\alpha_i^1 \omega_i^2}{(p^*)^2} = 0$$
$$\Rightarrow \lambda^* = \frac{(p^*)^2}{\left[\sum_{i=1}^{n} \alpha_i^1 \omega_i^2 \right]^{\frac{1}{2}}} \tag{2.24}$$

where p^* is defined as in (2.11)[24].

The following general properties are noted;

- For $\kappa \in (0,4)$ prices are positive and bounded.
- For $\kappa \in (0,2)$, the eigenvalue is less than one in absolute value; stability is ensured as in (2.16).
- For $\kappa \in (2,4)$, the eigenvalue is greater than one in absolute value; instability in the sense of (2.18) occurs for which (2.16) is necessarily ruled out.

[23] (2.19), (2.20) and (2.21) takes the minimum point in the domain for p and maps it onto the range by (2.9). By the inequality in (2.21), the minimum value for p ($f(\tilde{p})$) can be obtained. This constitutes the lower bound of the interval for which price is feasible. Since price is not bound from above, the domain of price is $p \in (f(\tilde{p}), \infty)$. (2.23) therefore imposes that prices exist on a non-degenerate interval of the real line

[24] The optimal speed of adjustment can also be seen as that speed of adjustment for which the eigenvalue is zero, i.e. $f'(p^*) = 0$,which would entail, by (2.16) $\kappa = 1$. At this point the system is super-stable

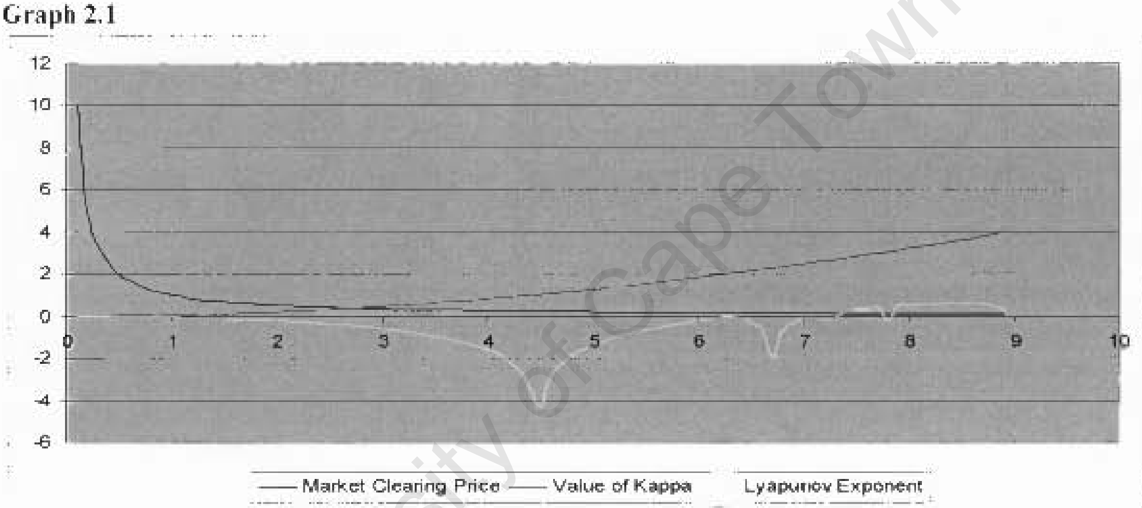
Changes in Relative Endowments

As a first pass in examining the properties of the tatonnement process, the relative value of one commodity vis-à-vis the other commodity is examined. The case in which there is a numeraire and a non-numeraire is examined. We assume that there are only two market participants[25]. Each market participant has equal preferences both for the numeraire and the non-numeraire, i.e. $\alpha_i^j = 0.5,\ i, j = 1, 2$. Furthermore, the endowments for both goods are described by the following matrix:

$$\varpi = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \tag{2.25}$$

Holdings of good one for individual one is denoted v. This value of v varies from 0.1 to 9 in steps of 0.1. This change captures the relative scarcity of the holding of the non-numeraire with respect to the numeraire. The market clearing price, the Lyapunov exponent and the value of κ are plotted against the relative holding of good one in graph 2.1.

Graph 2.1



As the relative abundance of the non-numeraire increases, the price falls. This is as theory would dictate. However, as the non-numeraire becomes relatively less scarce with respect to the numeraire, other dynamics occur. The following relations are noted:

- If $v \in (0, 6.3)$ then $\kappa \in (0, 2)$ [26]. Stability is ensured in the sense of (2.16)
- If $\kappa \in (0, 2)$ then the Lyapunov exponent is less than zero[27]; $h(p) < 0$. Chaos is absent
- If $v \in (6.3, 9)$ $\kappa \in (2, 4)$. The system is unstable in the sense of (2.18).
- If $\kappa \in (2, 4)$, there are certain values of v for which $h(p) > 0$. Chaos is present. The Lyapunov exponent is both positive and negative implying that chaos both does and does not occur[28] yet starts to occur more robustly as the non-numeraire becomes less scarce.

The presence of chaos indicates that Walras' idea about scarcity may be subject to the need for further qualifications. Walras notes that if a good becomes increasingly plentiful, it will cease to have a value in

[25] $\lambda = 0.1$ is assumed throughout this section.
[26] This can be read off the vertical axis
[27] See Appendix for discussion of Lyapunov exponents
[28] The absence of chaos does not imply the presence of a unique fixed point

exchange. It is noted here that both goods will cease to have a value in exchange as holdings of the non-numeraire increase relative to the numeraire simply as there is no convergence to the fixed point. Define;

$$y = \frac{\omega_1^1}{\omega_2^2} = \frac{v}{1} = v \tag{2.26}$$

As the value of y tends to zero, the holdings of the numeraire increase in terms of the non-numeraire. The tatonnement process is however stable in the sense that[29];

$$\begin{aligned} &\lim_{v \to 0^+} h(p) = 0^- \\ &\lim_{v \to 0^+} L(p) = 1^- \\ &\lim_{v \to 0^+} p = \infty \\ &\lim_{v \to 0^+} \lambda^* = \infty \end{aligned} \tag{2.27}$$

(2.27) states that as the relative amount of good one becomes small vis-à-vis good two, in the sense defined by endowment matrix (2.25)[30], then the Lyapunov exponent tends to zero from below. Similarly, the Lyapunov number tends asymptotically to one from below. This implies that average contraction of the map is zero at the limit and that the system is stable up until that point. Since the price of commodity one becomes infinitely large and it would take infinitely long to reach such a point (the point at which the system is super-stable tends towards infinity), Walras appears to be correct in that if one good becomes plentiful (good two), then that good ceases to have any value in terms of trade.

However, suppose now that the holding of good one becomes large. Recalling that the imposition of global stability requires that v be bound by a value approximately equal to nine[31], the converse to (2.27) can be examined. It would be expected that the same results as (2.27) hold; if commodity one becomes increasingly abundant then its price would become zero and at the limit, the market ceases to function. This is, however, not that which is demonstrated in graph 4.1 as for certain quantities of commodity one, we have;

$$\begin{aligned} &\lim_{v \to 9^-} h(p) > 0 \\ &\lim_{v \to 9^-} L(p) > 1 \end{aligned} \tag{2.28}$$

As the holdings of good one increases relative to that of good two, since $\kappa \in (2, 4)$ for certain values of v, and by (2.28), chaotic sequences are observed. The tatonnement process becomes unstable. Whilst Walras was correct in the assumption that the relative value of commodity one would become negligible as the supply of that commodity becomes large, his intuition needs to be corrected in the present context to the extent that as the non-numeraire becomes more abundant relative to the numeraire good, the tatonnement process becomes first unstable[32] and then chaos occurs. It can therefore be stated that[33]:

---

[29] 0^- means just below zero and 1^- means just below one, at the limit

[30] $v \to 0^+$ is equivalent to $\kappa \to 0^+$ as is $v \to 9^-$ equivalent to $\kappa \to 4^-$

[31] As this ensures that $\kappa \in (0, 4)$

[32] The specific dynamics in which the tatonnement process becomes first unstable in the sense of periodic orbits and then chaotic sequences

[33] This result is a direct consequence of the fact that goods one and two are treated asymmetrically in the tatonnement process

As the holdings of the non-numeraire increase, the tatonnement process is at first stable in the sense that $\kappa \in (0,2)$; the eigenvalue is within the unit circle, then unstable in the sense that $\kappa \in (2,4)$ or the eigenvalue is outside the unit circle. Furthermore for $\kappa \in (2,4)$ a positive Lyapunov exponent is observed indicative of the presence of chaos[34].

This result is not confined to the cyclic endowment of (2.25). Suppose that there are n market participants. The endowment matrix can be defined generically as;

$$\varpi = \begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \\ \dots & \dots \\ \omega_n^1 & \omega_n^2 \end{pmatrix} \qquad (2.29)$$

If the values in the first column of (2.29) become large relative to those in the second column, and all agents are identical and have equal preferences for both goods[35], then the tatonnement process becomes unstable and chaotic price sequences are observed[36]. This occurs as the numerator in (2.15) becomes large relative to the denominator. Of course by (2.15), if there were a commensurate shift in preferences away from the numeraire towards the non-numeraire, then it is conceivable that κ remains static so stability can occur even if the holdings of the non-numeraire become large relative to the numeraire. This however, cannot occur if all individuals are alike and have equal preferences for both goods.

Changes in the Speed of Adjustment

Consider the case of two market participants. Each market participant can hold either the non-numeraire or the numeraire but not both. The endowment matrix is cyclic;

$$\varpi = \begin{pmatrix} w_1^1 & 0 \\ 0 & w_2^2 \end{pmatrix}$$

and all preferences are equal[37]. Accordingly, in the two good case, preference for each individual and for each good is 0.5.

An arbitrary initial price of $p_0 = 4$ is chosen. Any initial price can be chosen provided that this initial price does not coincide with the market clearing price and is not zero.

In order to understand the stability of the system, the eigenvalue is considered. It is noted that the value of the eigenvalue is a function of the speed of adjustment, the parameters that capture preferences and the endowments. We recall that the stability condition is:

$|f'(p^*)| = |1-\kappa| < 1$, that is, the eigenvalue lies within the unit circle. If instability occurs then the eigenvalue lies without the unit circle; $|f'(p^*)| = |1-\kappa| > 1$. For the latter case the system will never

[34] This is the same result that Day obtains but by means of a change in the speed of adjustment not in relative holdings of endowments

[35] This assumption is not necessary but merely simplifies the analysis

[36] An increase in the number of individuals holding a certain good is equivalent to an increase in the holdings of that good by one individual, provided all market participants are the same

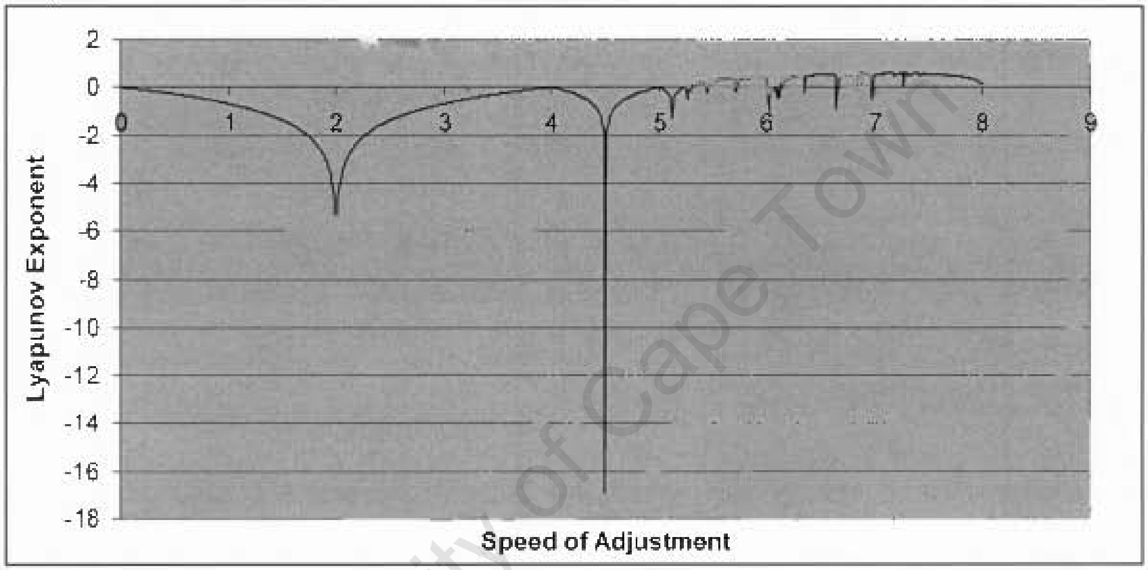
[37] By imposing these conditions, the number of free parameters falls from 9 to 1; the speed of adjustment

converge to the unique fixed point irrespective of the number of iterations over which the system is integrated.

In order to quantify the stability of the system with respect to the speed of adjustment, given that endowments are cyclic and that preferences are equal for both individuals and for both goods, the Lyapunov exponent is calculated for

$\varpi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} \alpha_1^1 & \alpha_1^2 \\ \alpha_2^1 & \alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, in which λ increases in steps of $\Delta\lambda = 0.01$ such that the open interval $\lambda \in (0,8)$ (corresponding to $\kappa \in (0,4)$)[38, 39] is ensured. The value of the speed of adjustment against the Lyapunov exponent is shown in graph 2.2.

Graph 2.2



The dynamics underlying graph 2.2 are outlined in table 2.1.

Table 2.1.

Value of λ	Value of κ	Dynamics
$\lambda \in (0,2)$	$\kappa \in (0,1)$	• Monotonic convergence towards the fixed point[40] • As $\kappa \to 0^+$, the eigenvalue remains just inside the boundary of the unit circle. The rate of contraction is low and the time until the fixed point is reached is long[41]. As $\kappa \to 1^-$, the eigenvalue falls from just inside the unit circle to just above zero. The system thus tends towards being super-stable[42 43]

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[38] If $\kappa > 4$ then the zero price boundary is hit and tatonnement process breaks down such that $p = 0, \ \forall t > t'$

[39] For the preferences and endowments given, $\frac{\lambda}{2} = \kappa$ by (2.15)

[40] See graph G.2.1

[41] See graph G.2.2, as well as discussion below regarding time until market clears

[42] See Strogatz, p. 350 for a brief discussion of super-stability

[43] See graph G.2.3

		• The Lyapunov exponent is negative and falling. The map contracts at an increasing rate as κ tends towards one
$\lambda = 2$	$\kappa = 1$	• Monotonic convergence of prices towards fixed point • Eigenvalue is zero. The system is super-stable (see (2.24)) • Super-stability coincides with, and is defined by the optimal speed of adjustment • The Lyapunov exponent is negative and large in magnitude (minus infinity)
$\lambda \in (2,4)$	$\kappa \in (1,2)$	• Oscillatory convergence of price towards fixed point[44] • For $\kappa \to 1^{+}$, the eigenvalue tends to just below zero and is thus negative. For $\kappa \to 2^{-}$, the Eigenvalue oscillates from just below zero to just above negative one. Such oscillations can take the eigenvalue outside the unit circle as price converges to the fixed point, but at the limit the eigenvalue still lies within the unit circle. Accordingly the modulus of the eigenvalue is still less than one.. The system is thus moving away from super-stability as κ increases to the upper end of the open interval[45] • The Lyapunov exponent is negative
$\lambda = 4$	$\kappa = 2$	• Onset of periodic orbit of period two. A bifurcation point is present • Fundamental shift in the nature of the dynamics, occurring at the point at which the Lyapunov exponent tends towards zero • The Eigenvalue fluctuates just above and below negative one • The system is stable in the sense of equation (A5), but unstable in the sense of (2.18). Stability in the sense of (2.16) will not be re-obtained for any value of κ equal to or greater than 2. There is therefore no convergence to the unique market clearing price and stability in the large breaks down. The system is however stable in Lyapunov's sense, but not asymptotically so[46].
$\lambda \in (4,5)$	$\kappa \in (2,2.5)$	• Periodic orbits of period two occur[47] • The product of the eigenvalues falls from positive one for $\lambda = 4$ to negative one for $\lambda = 5$. This is the same change in the values of the eigenvalues for $\lambda \in (0,4)$, but in this case the domain is one quarter that of the domain for which there is a unique (non-periodic) fixed point of the system. This indicates that the

[44] See Graph G.2.1
[45] See graph G.2.4
[46] That is to say, the system converges to a stable (non-expanding or non-contracting) area in the vicinity of the fixed point. See LST in the appendix
[47] See graph G.2.5

		domain over which each period doubling takes place gets shorter and shorter (see Feigenbaum's constant below). This implies that once the period doubling route to chaos starts, chaos quickly ensues. • The Lyapunov exponent is less than zero as the product of the first derivatives about the two period points is less than zero (see A5). • As $\kappa \to 2.25^-$ from 2^+, the oscillations of the eigenvalue become increasingly large and fall further and further outside the unit circle[48].
$\lambda = 5$	$\kappa = 2.5$	• Onset of periodic trajectory of period four[49]
$\lambda \in (5, 5.235)$	$\kappa \in (2.5, 2.6175)$	• Periodic orbit of period four • Product of eigenvalue for each of the unique periodic orbits is less than one • The Lyapunov exponent is negative
$\lambda = 5.235$	$\kappa = 2.6175$	• Periodic orbit of period eight first occurs (approximately) at this value
$\lambda \in (5.235, 5.299)$	$\kappa \in (2.6175, 2.6495)$	• Period doubling occurs. The system is stable in the sense of (A5)
$\lambda = 5.299$	$\kappa = 2.6495$	• Lyapunov exponent is positive for the first time; chaos first occurs at this point • The system is no longer stable in the sense of (A5)
$\lambda \in (6, 6.07)$	$\kappa \in (3, 3.035)$	• Trajectory of period three occurs
$\lambda \in (5.299, 8)$	$\kappa \in (2.6495, 4)$	• Chaos occurs robustly interspersed with windows of periodic orbits (see graphs 2.3 and 2.4)

Table 2.1 points to several interesting dynamics occurring. Given that stability occurs if $\kappa \in (0,2)$, then the Lyapunov exponent is negative. The tatonnement process is therefore energy dissipative. For this open interval in which the eigenvalue lies in the unit circle, stability in the large is guaranteed. However, the dynamics within this interval are not identical to each other. For $\kappa \in (0,1)$[50] prices (and the Lyapunov function) fall monotonically towards the fixed point. This open interval corresponds to the interval that lies to the left of the optimal speed of adjustment, or the super-stable point. For $\kappa \in (1,2)$[51] prices converge in an oscillatory manner towards the fixed point. Furthermore $\kappa \in (0,1)$, the eigenvalue lies in $(0^+, 1^-)$, whilst for $\kappa \in (1,2)$, the eigenvalue lies within $(-1^-, 0^-)$ at the fixed point. Once $\kappa > 2$ (corresponding to the eigenvalue lying outside the unit circle), stability in the large breaks down. We also note that for $\kappa \in (0,1)$ the dynamics that are generated by a continuous time tatonnement are equivalent to the discrete time case as there is monotonic convergence in the price and hence the Lyapunov function for this interval. Once $\kappa \in (1,2)$ occurs, there is a break in the equivalency of the dynamics of the continuous and discrete time tatonnement.

Graph 2.1 also shows that there are several open intervals for which trajectories of a given period occur. The midpoint of that open interval corresponds to the case in which $h(p) = -\infty$. Since the open interval for which a trajectory of a given period occurs becomes shorter and shorter as periods of higher order

[48] See graph G.2.6 and G.2.7
[49] See graph G.2.9
[50] $\lambda \in (0,2)$
[51] $\lambda \in (2,4)$

occur. I observe that periods of $\{2,4,8,16,...,2^k\}$, $\forall k$ occur. By Sharkovski's Theorem (see Appendix), trajectories of all periods occur. The point at which a 2^k period trajectory occurs, where $k = \infty$, denoted by κ_∞, is intimately related to Feigenbaum's constant.

I note that κ_∞ has a value of approximately 2.6495 which is equivalent to an approximate value of $\lambda_\infty = 5.299$. At this value of κ, a positive Lyapunov exponent is observed for the first time. The tatonnement process therefore has the ability to generate chaotic trajectories[52] and does so at this point only after period doubling has occurred. Furthermore, since period three orbits are observed then the second result of this section can be stated:

For $\kappa \in (3, 3.035)$, a three period trajectory is observed[53]. By LYT[54], if:

$f^3(p) \leq p < f(p) < f^2(p)$, or $f^3(p) \geq p > f(p) > f^2(p)$, then there is an uncountable set which contains no periodic points such that for *every* initial condition contained in that set, the solution of $p_{n+1} = f(p_n)$ is erratic, i.e., the solution is aperiodic and remains bounded in that set.

I note that for $\kappa = 3$, then $\lambda = 6$, the following prices are generated;

$p_{19,996} = 3.879414$

$p_{19,997} = 1.652727$

$p_{19,998} = 0.467909$

$p_{19,999} = 3.879414$

$p_{20,000} = 1.652727$

A three period cycle is present in the sense that $f^3(p) = p > f(p) > f^2(p)$. Trajectories of all periods as well as chaotic trajectories are present. The three period orbit can be seen in graph 2.4. Furthermore, this occurs for any initial price. It can be concluded that for $\kappa \in [\kappa_\infty, 4)$, chaos occurs robustly. Since the existence of chaos occurs for a given endowment matrix as the speed of adjustment changes, for a given speed of adjustment, changes in the endowment matrix will generate the same type of dynamics.

Feigenbaum's Constant and the Period Doubling Cascade

From table 2.1 it can be discerned that at first there is a unique fixed point towards which all trajectories approach. At a certain point there is a period doubling and a periodic orbit of period two is observed. After a certain interval, periodic orbits of period four are observed after which, periodic orbits of period eight are present. From this point, orbits of sixteen, thirty two and so on occur until at the limit chaos is present.

The period doubling route to chaos is intimately related to Feigenbaum's constant. (A25) can be used to get an approximate value of the point at which chaos will occur once the first few period doubling points are observed.

Noting that Feigenbaum's Constant is approximately given as[55];

[52] See graph G.2.11
[53] See graph G.2.14
[54] See appendix for a discussion of this theorem

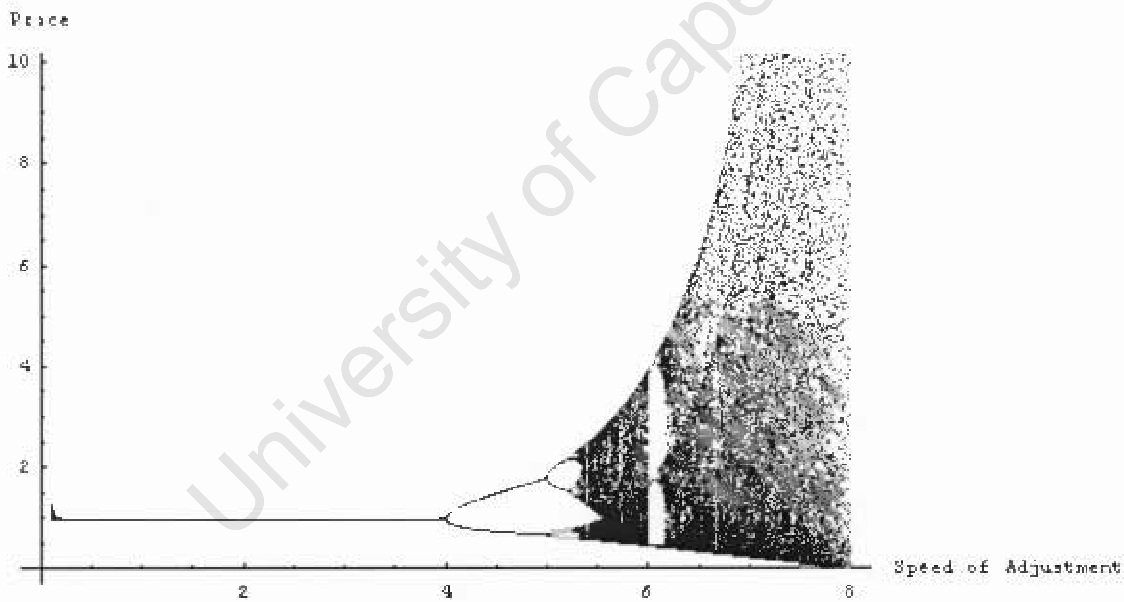
$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.6692016... \tag{2.30}$$

where r_n is the period doubling point. (2.30) can be rearranged to give an approximate value for r_{n+1} as follows;

$$r_{n+1} \approx \frac{r_n - r_{n-1}}{\delta} + r_n \tag{2.31}$$

Table 2.1 indicates that the first three period doubling points occur at $\lambda = 4$, $\lambda = 5$ and $\lambda = 5.235$, which correspond[56] to periods of two, four and eight respectively. Using the latter two values in (2.35) allows an approximation of a sixteen period orbit. This value is $\lambda = 5.285329$. Repeating the same process for a thirty two period orbit yields a value of $\lambda = 5.2961$. If the same procedure is carried out a few more time (corresponding to periodic orbits of 64, 128, 256 and so on), a value of $\lambda = 5.29901777$ is quickly converged upon. This is an approximation of the limit point in (2.30). Once the limit point has been reached chaos occurs. Graph 2.3 shows the bifurcation diagram as the value of the speed of adjustment is varied in steps of 0.01 from 0.01 to 7.99. Graph 2.4[57] shows a magnification of the bifurcation for $\lambda \in (5.75, 6.4)$ in steps of 0.001.

Graph 2.3

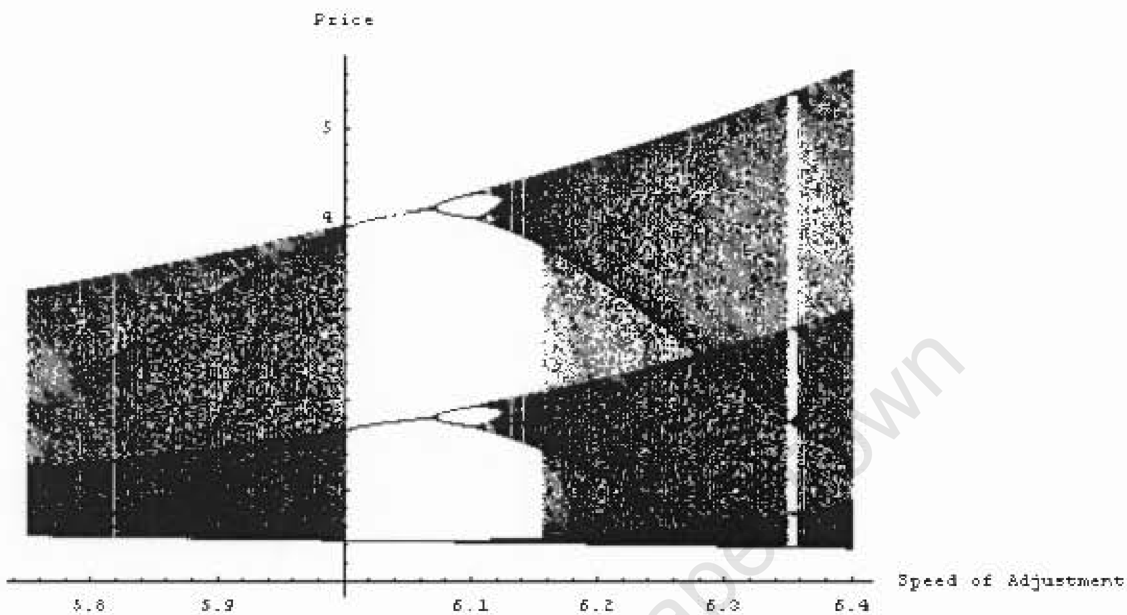


[55] Feigenbaum's constant is an irrational number

[56] Or alternatively $\kappa = 2$, $\kappa = 2.5$ and $\kappa = 2.6175$ respectively

[57] Graph 2.3 also shows that as the speed of adjustment increases the bounded interval within which price fluctuates increases. This corresponds to a frequency plot as in G.2.14

Graph 2.4



Discussion

Stability has been quantified in this section by means of the Lyapunov exponent. The value of the Lyapunov exponent has been plotted against both the relative value of the non-numeraire and the speed of adjustment. It was found that prices converge to the unique equilibrium point or follow complex paths as the values of both the former and the latter vary. In the literature review it was mentioned that the ability of the speed of adjustment to generate complex time paths has long been established and indeed analysed. However, this author has yet to find any mention of the relevance of the distribution of endowments in the setting of complex dynamics. The question thus arises of why a variation in both the speed of adjustment and the distribution of endowments allow the generation of complex dynamics.

Consider the first derivative of the aggregate excess demand evaluated at the fixed point (i.e. the one dimensional Jacobian matrix),

$$\left(\frac{\partial Z}{\partial p}\right)_{p=\bar{p}} = -\frac{\left(\omega_1^1\right)^2}{2\left(\omega_2^2\right)} \equiv s .$$

The stability criterion can be stated as: $|1+\lambda s|<1$. Since the eigenvalue of the aggregate excess demand matrix, s, is a function of relative endowments, it is clear that a change in the holdings of relative endowments will dictate whether the stability condition is satisfied or not. Similarly a change in the speed of adjustment will dictate whether the stability condition is met. The fact that a cyclic endowment matrix has been used does not have any bearing on the robustness of these results; both individuals could hold both goods yet, for a given speed of adjustment, stability may or may not be present depending on the relative ratios of those goods. The implication of this result is that, for a given speed of adjustment, if there is a scalar multiple of the economy, eventually, that economy will become unstable. For example, if

$\lambda = 0.2$ and 10 units of both good one and good two are held then $\kappa = 1$. If 35 units of both goods are held then $\kappa = 3.5$. Increasing the size of the economy by a factor of 3.5 has induced a chaotic price trajectory. A larger economy will generate complex price dynamics for a given speed of adjustment, until at the limit the tatonnement process breaks and the zero price boundary is reached.

Where does this result leave the assumptions of gross substitution and the weak axiom of revealed preference? It may be argued that if stability breaks down even in the presence of GS and WARP, then such assumptions are superfluous and do nothing to create conditions of stability. Such an argument would however be erroneous. By GS and WARP, there always exists a unique price that clears the market, whether these prices are reached or not. In other words there is a unique stable manifold. These assumptions permit the creation of a basin of attraction that is populated by single points (or singletons) towards which, in the presence of stability all points converge. As the speed of adjustment changes or the relative endowments change (or both), this basin of attraction becomes a repelling fixed point or a source, and periodic and chaotic dynamics are observed. In the presence of these latter dynamics, the auctioneer cannot call out a price that allows trade to take place. This occurs for any initial starting price. The Walrasian auctioneer is therefore charged with ensuring that the unique fixed point of the system is an attracting fixed point rather than a repelling point. In the presence of GS, he needs to modify the speed of adjustment such that convergence takes place. By the fact that GS implies WARP which in turn implies that the fixed points are unique or global stability occurs, then the auctioneer does not need to worry whether prices fall on a stable or unstable manifold. Very simply, the auctioneer needs only to consider that a fall in the speed of adjustment will put prices on a path that guarantees that the market clearing price vector is reached. The extent to which the speed of adjustment is modified will in turn depend on the holdings of the non-numeraire relative to the numeraire. If the auctioneer abides by this rule then stability in the large holds. The next section demonstrates this point in a more rigorous manner.

Section Three: Controlling Chaos in the One-Dimensional Case

In section two it was seen that for a one-dimensional discrete tatonnement process with two identical market participants differing only in their initial holdings of the two goods, both stable and unstable price sequences are observed. Whether the former was present and the latter absent depended entirely upon whether

$$|f'(p^*)| = |1-\kappa| < 1 \tag{3.1}$$

was observed. If the inequality switched directions such that

$$|f'(p^*)| = |1-\kappa| > 1 \tag{3.2}$$

then both periodic orbits of all periods and chaotic sequences were observed.

It was also noted that $\kappa \in (0,2)$ was associated with (3.1) and $\kappa \in (2,4)$ was associated with (3.2).

The possibility of the existence of an eigenvalue greater than one (i.e. instability) is ruled out in the continuous time one-dimensional tatonnement process. Uzawa [23] circumvents the possibility of this occurring in his discrete-time version by assuming that β (the equivalent to λ in the tatonnement process specified here) is small in value and positive and so (3.2) is avoided; Walras' price adjustment mechanism remains valid. In the absence of such an assumption, the question naturally arises of how to induce stability if the system is unstable.

Three methods are proposed:

1. The introduction of market participants who hold the numeraire good or simply a larger holding of the numeraire good by each individual[58]
2. A decrease in the speed of adjustment
3. An approximation to the continuous time system as presented in AH and ABH

Differing Market Participants

Consider the case in which there are initially two market participants who have the following preferences and endowments:

$$\varpi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

$$\mathrm{A} = \begin{pmatrix} \alpha_1^1 & \alpha_1^2 \\ \alpha_2^1 & \alpha_2^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \tag{3.4}$$

These dynamics were examined in section two. Now suppose that a third participant enters the market[59]. Furthermore assume that all the third participant has identical preferences to the incumbent market participants and holds an initial amount identical to either one or the other of the incumbents.

[58] These two approaches are equivalent if all agents are identical

[59] As mentioned previously, if all agents are equal, an increase in the number of individuals holding the numeraire is equivalent to an increase in the holding of the numeraire by only one individual

Denote type A market participant as those who hold one unit of good one; the traded good and type B market participant as those who hold one unit of good two; the numeraire good.

The introduction of another market participant alters (3.3) and (3.4) respectively to

$$\varpi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.5}$$

or

$$\varpi = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

and

$$\mathrm{A} = \begin{pmatrix} \alpha_1^1 & \alpha_1^2 \\ \alpha_2^1 & \alpha_2^2 \\ \alpha_3^1 & \alpha_3^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \tag{3.7}$$

Suppose now that (3.4) holds; there are only two market participants. Consider the tatonnement process for the first 200 periods with a speed of adjustment $\lambda = 6.5$, corresponding to $\kappa = 3.25$. For this speed of adjustment, preferences and initial endowments, and market clearing price of 1, the Lyapunov exponent is 0.51745606; chaos is present and (3.2) holds.

Suppose now that one type B participant enters the market precisely at $t = 200$. The introduction of this market participant changes the market clearing price to 2 and $\kappa = 1.625$ whilst the speed of adjustment remains unchanged at 6.5.

Graphs G.3.1 to G.3.6 reports the change in the price dynamics when a type B market participant is introduced into the market at $t = 200$

It is immediately apparent that the introduction of a type B participant induces stability precisely as the eigenvalue changes from 2.25 in (3.2) to 0.625 in (3.1). The Lyapunov exponent accordingly changes from being positive to being negative[60].

Suppose now that a type A market participant is introduced in the stead of a type B such that (3.6) occurs over (3.5).

In this case $\kappa = 13$ which is associated with an eigenvalue of 12. The tatonnement process becomes undefined:

$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\sum_i^n \frac{\alpha_i^1 \omega_i^2}{p_t} - \sum_i^n \omega_i^1 (1-\alpha_i^1)\right]\right\} = 0 \tag{3.8}$$

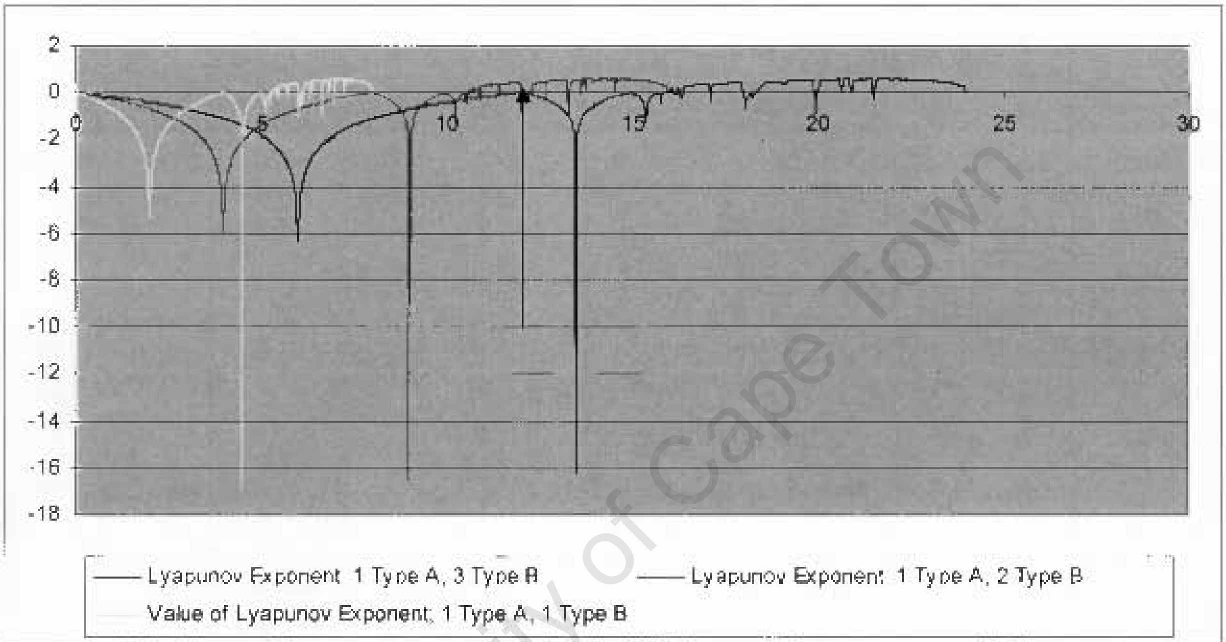
[60] There is no reason that the endowment needs to be an integer; any value could be considered

as $\kappa > 4$. The introduction of a type A market participant induces instability as the adjustment mechanism hits the zero price boundary[61].

The above line of reasoning only holds if an individual is allowed to enter the market once the price discovery mechanism has started. If this is not the case, it equivalently holds that a market with fewer individuals holding the numeraire or simply less holdings of the numeraire will, for a given speed of adjustment, generate periodic or complex dynamics, if such holdings are sufficiently small.

Graph 3.1 plots the Lyapunov exponent against changes in the speed of adjustment for differing numbers of market participants for preferences as represented as in (3.7) and cyclic endowments as in (3.5) or (3.6)

Graph 3.1.



From graph 3.1 it is noted that, if the Lyapunov exponent is positive, or if periodic orbits are present (corresponding to $\kappa \in (2,4)$), the addition of a type B participant will, for a given speed of adjustment, shift the map such that the Lyapunov exponent becomes negative. In other words, for a given λ, there is a unique κ. The introduction of a market participant will keep the speed of adjustment constant but shift the value of κ[62]. If more holdings of the numeraire good are introduced then $\kappa \to \kappa \in (0,2)$ and stability is induced. This allows the following result:

An unstable or chaotic system as well as one that has periodic orbits can be rendered stable if the holdings of the numeraire good vis-à-vis the non-numeraire are increased. Conversely, if chaotic or periodic trajectories are to be avoided, the auctioneer needs to note how much of the numeraire is held with respect to the non-numeraire and vary the speed of adjustment accordingly. To the extent that the speed of adjustment needs to be large or small is inversely proportional to the ratio of the holding of the non-numeraire with respect to the numeraire.

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[61] This result may not occur if the holding of the non-numeraire was less than one for the new market participant

[62] For example, if there are only two participants, the relevant plot of the Lyapunov exponent is the left most. If $\kappa \in (2,4)$, as shown by the arrow, then the trajectory occurs in the chaotic region. By introducing a type B market participant, the relevant plot of the Lyapunov exponent is the middle one. For a given value of the speed of adjustment now $\kappa \in (0,2)$ so stability is induced (trace the arrow down to the plot of the middle and right most Lyapunov exponent if one or two type Bs are introduced

Graph 3.1 also shows that the properties of the tatonnement process remain constant for the open interval $\kappa \in (0, 4)$. The three plots in graph 3.1 are replicas of each other but merely stretched or shrunk with respect to holdings of the numeraire as the speed of adjustment is altered. There is therefore a one-to-one correspondent between the property of the map and the value of κ[63]. It can be asserted that stability or instability cannot be defined by the speed of adjustment alone, and that any such consideration needs to examine the value of the relative endowments. Furthermore, the larger the value of the holdings of endowments of the numeraire, the larger the interval over which the speed of adjustment can vary in which the system is stable.

Change in the speed of adjustment

In the literature review it was noted that much of Uzawa's stability theorem was dependent upon β being small, where β is equivalent to λ. It has been shown numerically that if λ (β) is small, stability is guaranteed. There is thus an equivalence between that which Uzawa states and that which is found in this work.

However, to the extent that β is small then, following Uzawa's proof;

$\varphi(t+1) \le \varphi(t) - \beta \left\{ 2[z_0(t) + \sum_{i=1}^{n} \overline{p}_i z_i(t)] - \beta \sum_{i=1}^{n} z_i^2(t) \right\}$ becomes $\varphi(t+1) < \varphi(t)$. This argument rests crucially on β being a number smaller than

$$\frac{\varepsilon}{2} \le \inf \Phi(p) \le \Phi[p(0)] \frac{2\left[z_0(p) + \sum_{i=1}^{n} \overline{p}_i z_i(p) \right]}{\sum_{i=1}^{n} z_i^2(p)} \tag{3.9}$$

By the converse of the above result, it is conceivable that if there is a large holding of the traded good with respect to the numeraire good, the numerator in (3.9) would be large with respect to the denominator. This would entail a smaller β such that convergence and stability are ensured. The difficulty then becomes how small does β have to be.

For example, suppose there are 450 type A individuals and 30 type B individuals. If $\lambda = \beta = 0.001$, then, $\kappa = 3.375$ and the Lyapunov exponent has a value of 0.5037. The trajectory is chaotic even though the speed of adjustment is low. Now suppose that the speed of adjustment changes such that $\lambda = \beta = 0.0001$, then $\kappa = 0.3375$. The system is stable and the speed of adjustment is sufficiently small such that Uzawa's proof for stability holds.

This type of argument however presents a problem for Uzawa inasmuch as if there are increasingly more type A individuals in the market then

$z_0(p) + \sum_{i=1}^{n} \overline{p}_i z_i(p)$ becomes increasingly large. Such that $\beta \left\{ 2[z_0(t) + \sum_{i=1}^{n} \overline{p}_i z_i(t)] - \beta \sum_{i=1}^{n} z_i^2(t) \right\}$ becomes small then β must become smaller and smaller so that $\varphi(t+1) < \varphi(t)$ holds. It is conceivable that at the

[63] This point allows Day (see literature review) to make the statement that chaos can occur robustly for any speed of adjustment

limit and with the appropriate mix of type A and B participants, β must tend to zero. Relative holdings of endowments therefore play a role that is just as important as the speed of adjustment

Discussion

This section has demonstrated that chaotic trajectories can exist for any holdings of the numeraire and the non-numeraire and for various values of the speed of adjustment. Graph 3.1 showed that the *properties* of the one-dimensional tatonnement process remain invariant as the relative holdings of the non-numeraire are changed with respect to the numeraire.

Recalling from the last section that a stable system can be defined by;

$|f'(p^*)| = |1-\kappa| < 1$, where

$$\kappa \equiv \lambda \frac{\left(\sum_i^n \omega_i^1 (1-\alpha_i^1)\right)^2}{\sum_i^n \alpha_i^1 \omega_i^2} .$$

It holds that if $\lambda \to 0^+$, then $\kappa \to 0^+$. Similarly if $\omega_i^1 \to 0, \forall i$ then $\kappa \to 0^+$. Conversely, if $\omega_i^2 \to \infty$, for at least one individual, then the stability condition is satisfied. There is an asymmetry to the above criterion; for a given speed of adjustment, all individuals need to hold sufficiently small amounts of the non-numeraire relative to the numeraire, or at least one individual needs to hold a large amount of the numeraire relative to the non-numeraire such that stability is ensured. It can also be seen that if the preference for the non-numeraire is large[64] ($\alpha_i^1 \to 1$), then κ becomes large relative to a smaller value of α_i^1. This implies that, for a given speed of adjustment, if aggregate holdings of the non-numeraire are small vis-à-vis the numeraire and preferences for the non-numeraire are large relative to the numeraire, then the tatonnemnet process is stable. At first, this may seem a paradoxical result; one may expect that a market is well defined if individuals bring large amounts of the non-numeraire to the market and have a strong desire to exchange. By the stability criterion this is not the case as the tatonnement process will be destabilised to the extent that the non-numeraire is held in an amount that is too large relative to the numeraire. An intuitive explanation of why this should be the case is related to the idea of scarcity. If endowments of the non-numeraire are small and preferences for that good are large, that commodity is accordingly scarce; the market is stable and price sequences are well behaved. If the converse holds; aggregate holdings of the non-numeraire are large and there is little preference for holding the non-numeraire, there is then a lack of scarcity. Scarcity thus generates price sequences in the tatonnement process that are stable and converge to a single point. The opposite necessarily holds.

On the one hand, it is a strong result that a market that attempts to reallocate scarce resources is well-defined. On the other hand, the strength of this result is precisely its shortcoming; intuitively it is unclear why a market that does not exhibit scarcity is rendered stable if the speed of adjustment of that market is low, nor why a market that does not exhibit scarcity is able to generate chaotic price sequences.

Irrespective of why this is the case, the Walrasian auctioneer is able to rely on these properties in ensuring that the market clearing price is reached. If it is assumed that it is preferable to reach the market clearing price in as little time as possible, then the Walrasian auctioneer needs to consider the holdings of the endowment of the non-numeraire with respect to the numeraire when setting the speed of adjustment ((2.24) gives the value of the optimal speed of adjustment). If the auctioneer cannot know the holdings of

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[64] By homogeneity of degree one of the utility function, if $\alpha_i^1 \to 1$, then $(1-\alpha_i^1) \to 0$

the numeraire then the safest course of action, which would exclude the occurrence of periodic and chaotic price sequences, is to set the speed of adjustment as small as possible.

If the speed of adjustment is small there is then a coincidence between the properties of the discrete time tatonnement process and the continuous time tatonnement process. It follows that if there is to be convergence then the discrete time map should replicate properties of the continuous time process inasmuch as prices converge monotonically towards the fixed point. Scarcity and abundance have no bearing on whether the tatonnement process reaches the market clearing price if the speed of adjustment is made sufficiently small.

Section Four: The Two Dimensional Case

That the one dimensional case is both problematic in the sense of section two and has solutions to those problems as demonstrated in section three, one may question whether the introduction of another good changes the fundamental dynamics.

The case in which there are three market participants and three goods is now considered. The price of good three is normalised to one yielding two simultaneous discrete price adjustment mechanisms.

For the sake of completeness, the tatonnement process is presented in the appendix.

Consider the endowment matrix as;

$\varpi = \begin{pmatrix} \omega_1^1 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^3 \end{pmatrix}$ and $\alpha_i^j = \left(\frac{1}{3}\right)$, $\forall i, j$, so the initial holdings are cyclic in the sense of sections two and three and all agents are identical and have equal preference for all goods. We note that the market clearing prices are given as;

$p_1^* = \left(\frac{\omega_3^3}{\omega_1^1}\right)$ and $p_2^* = \left(\frac{\omega_3^3}{\omega_2^2}\right)$ as well as $p_3^* = 1$. The stability criteria can now be analysed.

Consider the Jacobian matrix of the linearised system.

$$\begin{pmatrix} p_{t+1}^1 - (p^1)^* \\ p_{t+1}^2 - (p^2)^* \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f^1}{\partial p^1} & \frac{\partial f^1}{\partial p^2} \\ \frac{\partial f^2}{\partial p^1} & \frac{\partial f^2}{\partial p^2} \end{pmatrix}}_{J} \begin{pmatrix} p_t^1 - (p^1)^* \\ p_t^2 - (p^2)^* \end{pmatrix}$$

Stability depends on the roots of the Jacobian, J, evaluated at the fixed points;

$$J = \begin{pmatrix} \frac{\partial f^1}{\partial p^1} & \frac{\partial f^1}{\partial p^2} \\ \frac{\partial f^2}{\partial p^1} & \frac{\partial f^2}{\partial p^2} \end{pmatrix}_{\vec{p}=\vec{p}^*} = \begin{pmatrix} 1 - \lambda^1 \frac{2}{3} \frac{(\omega_1^1)^2}{\omega_3^3} & \lambda^1 \frac{1}{3} \frac{(\omega_1^1)(\omega_2^2)}{\omega_3^3} \\ \lambda^2 \frac{1}{3} \frac{(\omega_1^1)(\omega_2^2)}{\omega_3^3} & 1 - \lambda^2 \frac{2}{3} \frac{(\omega_2^2)^2}{\omega_3^3} \end{pmatrix}$$

Stability will be assured if several conditions as satisfied. The first of these is that the map be dissipative. For a dissipative map, the modulus of the determinant needs to be less than one[65]. If the determinant satisfies this condition then the two dimensional map will be contracting about the fixed points (p^{1*}, p^{2*}) in the (p_t^1, p_t^2) space. By GS, the fixed points are unique so if the map contracts about this point, then the map

[65] In two dimensions, the tatonnement process maps an infinitesimal rectangle at (p^1, p^2), with area $dp^1 dp^2$, into an infinitesimal parallelogram with area $|\det J(p^1, p^2)|\, dp^1 dp^2$. Therefore if $|\det J(p^1, p^2)| < 1$ everywhere, the map is area-contracting or dissipative. Since the map has the property of GS, the map needs to be considered only about the market clearing fixed points; if the map is dissipative about these points it is dissipative everywhere

will contract for $p^1, p^2 \in (0,\infty)$. In other words, a contractionary map has a basin of attraction that is R^2_{++}; there is no smaller set of prices for which convergence will take place.

An alternative but related concept of stability is that both eigenvalues lie within the unit circle[66]. If one or both of the eigenvalues lies outside the unit circle, then $|\det J(p^1, p^2)| > 1$ and the system is unstable.

Attractors and strange attractor are defined as follows[67]:

An attracting set is a closed set; A with the following properties:

1. A is invariant; any trajectory $\vec{p}_t$, that starts in A remains in A for all t.
2. A attracts an open set of initial conditions: there is an open set U containing A such that if $\vec{p}_0 \in U$ then the distance from $\vec{p}_t$ to A tends to zero as $t \to \infty$. The largest U is the basin of attraction of A.
3. A is minimal; there is no proper subset of A that satisfies 1 and 2.

A strange attractor is an attractor that exhibits sensitive dependence to initial conditions.

As mentioned above, since $\vec{p} \in (0,\infty)$, then the set of feasible initial conditions is R^2_{++}. The set U that contains A is therefore $U \in (0,\infty)$. For any $\vec{p}$ starting in U, there is convergence to A. Now, if $|r_i| < 1$, $\forall i$, then $A \sim \vec{p}^*$; the invariant set A is precisely the set of fixed points or market clearing prices of the tatonnement process. All trajectories starting in R^2_{++} converge to this set. Furthermore A is a minimal set as this set is populated by two single points (or singletons) in R^2_{++}. If this were not so then at least one fixed point would be either equal to zero or infinity or there would be many local equilibria for at least one price sequence[68]. The tatonnement process would be ill defined and GS would be contravened leading to a contradiction that, by GS, a stable system is globally stable.

If $|r_i| > 1$ for one of the eigenvalues[69], then two possibilities can occur; periodic trajectories occur or a strange attractor is present. A strange attractor can be defined by considering the forward limit set[70] of the

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[66] In the two-dimensional case, the roots can be obtained by the following relationship;

$$r_1, r_2 = \frac{tr(J) \pm \sqrt{(tr(J))^2 - 4\det J}}{2} \qquad (4.1)$$

The condition for stability is that $\{|r_1|, |r_2|\} < 1$ or the spectral radius of the system is less than one in absolute value. Accordingly (4.1), once rearranged, gives the stability criterion as:

1. $1 - tr(J) + \det J > 0$ (4.2)
2. $1 + tr(J) + \det J > 0$ (4.3)
3. $\det J < 1$ (4.4)

Conditions (4.2) to (4.4) give three conditions which, when satisfied, will ensure that the market clearing price vector is reached. It can be noted that conditions (4.2) and (4.3) encompass the stability condition for which the modulus of the determinant is less than one. If one or more of these conditions are violated then the system is unstable in that either periodic or chaotic trajectories are present. If chaotic trajectories are observed there is then a strange attractor.

[67] This definition applies to a system of any dimension

[68] Strictly speaking, since the price sequences are coupled, if anything other than a single market clearing price is observed in one market, the same dynamic is observed in the other market

[69] If $|r_i| > 1$ for both eigenvalues the zero price boundary is reached

price sequences and observing whether this set exhibits sensitive dependence to initial conditions. If sensitive dependence is observed then chaos is present.

For a given endowment matrix, the characteristics of the system are analysed as the speed of adjustments change. Consider the case in which individual one holds one unit of commodity one, individual two holds one unit of commodity two and individual three holds one unit of the numeraire (as outlined above). The speed of adjustment of market one is fixed at $\lambda^1 = 0.75$ and the speed of adjustment for market two is allowed to vary by $\Delta\lambda^2 = 0.01$. Fixed, periodic and chaotic trajectories were searched for as the speed of adjustment of market two varied[71]. The results are presented in table 4.1.

Table 4.1

Value of λ^2	**Dynamics**
$\lambda^2 \in (0, 2.76)$	The unique market clearing price is reached for both goods.
$\lambda^2 = 2.76$	A period doubling bifurcation occurs and a two period trajectory emerges for *both* goods. This corresponds precisely to the point at which one of the eigenvalues lies on the unit circle and is negative.
$\lambda^2 \in [2.76, 4.3)$	A two period orbit is present for both prices. The amplitude of the two period is larger for p^2 than for p^1 due to $\lambda^2 > \lambda^1$. As λ^2 increases, one of the eigenvalues moves outside the unit circle and becomes increasingly negative.
$\lambda^2 = 4.3$	A period doubling bifurcation occurs and orbits of period four emerge for both price sequences. As in the case of a two period orbit, the amplitudes of the four period trajectory are larger for p^2 than for p^1
$\lambda^2 \in [4.3, 4.58)$	A four period trajectory is present in both markets.
$\lambda^2 = 4.58$	An eight period trajectory emerges.
$\lambda^2 \in [4.58, 4.65)$	An eight period trajectory is present in both markets
$\lambda^2 = 4.65$	A sixteen period trajectory is emerges. Between this point and the next periodic orbit, complex (chaotic) sequences are observed for both prices.
$\lambda^2 = 4.69$	A twenty period trajectory is present for both price movements. Between this point and the next periodic orbit, complex sequences are observed for both prices.
$\lambda^2 = 4.7$	A twenty eight period is observed. Between this point and the next periodic orbit, complex sequences are observed for both prices.
$\lambda^2 = 4.74$	A twelve period is observed. Between this point and the next periodic orbit, complex sequences are observed for both prices.
$\lambda^2 = 5.6$	A five period orbit emerges for both markets.
$\lambda^2 \in [5.6, 5.64)$	A five period is present in both markets. Between this point and the next periodic orbit, complex sequences are observed for both prices.
$\lambda^2 = 5.64$	A ten period emerges.
$\lambda^2 = 5.87$	The zero price boundary is hit (for good two first), and the tatonnement process ceases to function.

Table 4.1 demonstrates that, as in the case of the one dimensional tatonnement process, chaotic trajectories are interspersed with periodic trajectories. Unlike the case of the one dimensional map, the two dimensional tatonnement process is not as ordered inasmuch as there is no multi-dimensional analogue of Feigenbaum's constant, Sharkovski's natural ordering system or Li-Yorke's theorem. Instead of being able to evoke such theorems which can quantify and qualify the nature of a chaotic sequence, recourse has to be made to

[70] The forward limit set is equivalent to the invariant set A as $t \to \infty$

[71] The starting prices of goods one and two are $p_0^1 = 4.2$ and $p_0^2 = 3.95$ respectively

numerical evaluation. What is consistent between the one and two dimensional maps is that as the speed of adjustment of one of the markets increases, periodic trajectories emerge which are then followed by the emergence of chaotic trajectories amongst which periodic trajectories are also observed. Graphs G.4.1 and G.4.2 examine the sudden shift in dynamics as the speed of adjustment is altered for market two.

Despite the fact that the nature of the trajectories, be they stable, periodic or chaotic, are more complicated to evaluate in the two dimensional case, several points emerge.

- If one of the eigenvalues lies outside the unit circle, a saddle point is present. The saddle point can generate periodic or chaotic trajectories.
- If the market clearing price vector is reached, then the modulus of both eigenvalues need to lie inside the unit circle. The invariant set A is minimal and consists of only two points.
- If one price sequence is periodic or chaotic then both price sequences are periodic or chaotic. The reason for this is due to the way in which the tatonnement process is constructed; both simultaneous equations are coupled. Feed back between both equations occurs in both directions. Therefore, if one market does not converge towards its market clearing price then the other does not. This is not surprising given that the market clearing vector is determined simultaneously for all markets.
- By the results of table 4.1, the market that exhibits periodic or chaotic movements within a larger bounded interval is the perpetrating market in the sense that its speed of adjustment is the larger of the two.
- It naturally holds; such that $\left(p^1\right)^*$ and $\left(p^2\right)^*$ are reached then λ^2 needs to fall. If λ^2 falls sufficiently then $\{|r_1|,|r_2|\} < 1$, or the modulus of the spectral radius is less than one.

Graph G.4.3 shows how the eigenvalues calculated from the Jacobian above varies as λ^2 varies given $\lambda^1 = 0.75$.

The question now arises; can the auctioneer induce stability by shifting the speed of adjustment of market one, holding the speed of adjustment for market two constant?

Consider the same distribution of endowments as above and $\lambda^1 = 0.75$, $\lambda^2 = 2.79$. A two period trajectory is present. The Jacobian is;

$J = \begin{pmatrix} 0.5 & 0.25 \\ 0.93 & -0.86 \end{pmatrix}$ and the eigenvalues are $r_1 = 0.6536$ and $r_2 = -1.013$. Graph G.4.4 shows the eigenvalues over 20,000 iterates in the (r_1, r_2) space. One of the eigenvalues oscillates in and outside of the unit circle. Now if λ^1 falls to $\lambda^1 = 0.6$, the Jacobian becomes;

$J = \begin{pmatrix} 0.6 & 0.2 \\ 0.93 & -0.86 \end{pmatrix}$, and the corresponding eigenvalues are $r_1 = 0.7178$ and $r_2 = -0.9778$. Graph G.4.5 shows that both roots lie within the unit circle. A fall in the speed of adjustment for market one has shifted the eigenvalues such that they both lie within the unit circle for all iterates. If at the limit $\lambda^1 \to 0^+$, then the Jacobian becomes[72];

72 1^- means just below one and 0^+ means just above zero

$J = \begin{pmatrix} 1^{-} & 0^{+} \\ 0.93 & -0.86 \end{pmatrix}$. The corresponding eigenvalues are $r_1 = 1^{-}$ and $r_2 = -0.86$. The system is stable.

The auctioneer can induce stability or avoid instability by reducing *either* λ^1 or λ^2.

Such that this occurs then;

$$\left|\frac{\partial f^2}{\partial p^1}\right|_{\vec{p}=\vec{p}^*} < 1 \tag{4.5}$$

$$\left|\frac{\partial f^2}{\partial p^2}\right|_{\vec{p}=\vec{p}^*} < 1 \tag{4.6}$$

(4.5) and (4.6) imply that the present period change in both p^1 and p^2 have a less than unitary increase on the price in market two, and that this increase or decrease falls in magnitude over time.

In order to demonstrate this, assume that (4.5) and (4.6) do not hold and

$$\left|\frac{\partial f^2}{\partial p^1}\right|_{\vec{p}=\vec{p}^*} > 1 \tag{4.7}$$

$$\left|\frac{\partial f^2}{\partial p^2}\right|_{\vec{p}=\vec{p}^*} > 1 \tag{4.8}$$

A given p^1 and p^2 in this period will have a more than unitary increase or decrease on the price of good two in the next period. Chaotic or periodic trajectories *cannot* be rendered stable by reducing λ^1. The auctioneer must ensure that λ^2 is therefore sufficiently small such that (4.7) and (4.8) switch to (4.5) and (4.6) respectively. The case of (4.7) and (4.8) is illustrated in graphs G.4.6 and G.4.7.

In the case of one numeraire and one non-numeraire in section two, the ratio of the former to the latter had a bearing the degree to which the speed of adjustment could be large or small and stability present or absent. This concept can be extended to the case in which there are two non-numeraires. If the distribution of the endowments is:

$$\varpi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For $\lambda^1 = 0.75$, λ^2 varies in steps of 0.01. Fixed points, periodic and chaotic trajectories are searched for. Table 4.2 summarises the results.

Table 4.2

Value of λ^2	**Dynamics**
$\lambda^2 \in (0, 5.79)$	The unique market clearing price is reached for both goods. The Lyapunov function is well defined for this interval.
$\lambda^2 = 5.79$	A period doubling bifurcation occurs and a two period trajectory emerges for *both* goods. This corresponds precisely to the point at which one of the

	eigenvalues lies on the unit circle and is negative.
$\lambda^2 \in [5.79, 8.3)$	A two period orbit is present for both prices. The amplitude of the two period is larger for p^2 than for p^1 due to $\lambda^2 > \lambda^1$. As λ^2 increases, one of the eigenvalues moves outside the unit circle and becomes increasingly negative.
$\lambda^2 = 8.3$	A period doubling bifurcation occurs and orbits of period four emerge for both price sequences. As in the case of a two period orbit, the amplitudes of the four period trajectory are larger for p^2 than for p^1
$\lambda^2 \in [8.3, 8.8)$	A four period trajectory is present in both markets.
$\lambda^2 = 8.8$	An eight period trajectory emerges.
$\lambda^2 \in [8.8, 8.94)$	An eight period trajectory is present in both markets
$\lambda^2 = 8.94$	A sixteen period trajectory is emerges. Between this point and the next periodic orbit, complex (chaotic) sequences are observed for both prices.
$\lambda^2 = 9.11$	A twelve period trajectory is present for both price movements. Between this point and the next periodic orbit, complex (chaotic) sequences are observed for both prices.
$\lambda^2 = 11.22$	A seven period is observed. Between this point and the next periodic orbit, complex (chaotic) sequences are observed for both prices.
$\lambda^2 = 12.35$	The zero price boundary is hit (for good two first), and the tatonnement process ceases to function.

A comparison between tables 4.1 and 4.2 indicates that, if the holdings of the numeraire with respect to the non-numeraires increase, then the occurrence of periodic orbits and chaotic trajectories takes place but at higher levels of the speed of adjustment of market two. There is a direct parallel between the qualitative characteristics of the one dimensional case and the two dimensional case. The inclusion of another good into the tatonnement process does not therefore fundamentally alter the basic results of the one dimensional case. The only aspect that is altered in the two dimensional case is that the dynamics become harder to quantify and complexity occurs at higher dimensions for which numerical analysis becomes increasingly difficult unless more sophisticated techniques need to be employed.

A caveat needs to however be attached to the last statement: in the two dimensional case a stable manifold is present for a symmetric system that would otherwise generate periodic or chaotic trajectories if there were some asymmetry in the system. Consider the case in which $p_0^1 = p_0^2 = 1000$, $\lambda^1 = \lambda^2 = 4$ and each individual holds only one unit if one good and no individual holds the same good as any other individual. The eigenvalues are $r_1 = -0.333$ and $r_2 = -3$. The system is unstable by the definitions above. Graph G.4.8 shows prices in the (p_1, p_2) space. Prices converge over time to the market clearing vector. This occurs as the system is symmetrical. The symmetry generates price sequences that are also symmetric (seen by the straight line in the price space). Now suppose that there is an asymmetry in the initial prices; $p_0^1 = 1000$ and $p_0^2 = 1000.1$. Graph G.4.9 shows the prices in the price space and G.4.10 shows the limit sequence of these price trajectories. Contrasting G.4.8 and G.4.10 demonstrates that, for a small perturbation in initial conditions prices "fall off" the stable manifold and onto a strange attractor.

Discussion

This section has demonstrated that the inclusion of a second non-numeraire into the tatonnement process does not alter the fundamental results of sections two and three. It however does become necessary to evaluate the two dimensional case in a slightly different manner as the auctioneer now has two speeds of adjustment that can be altered. By altering the speed of adjustment whose price fluctuates within the smaller bounded interval, if prices are periodic or chaotic, then stability can be induced only if the own price effect and cross price effect on the next period price is less than unity for the market that has the higher speed of adjustment. If the own price and cross price effects for the market with the higher speed of adjustment is larger than unity, then stability can be induced only by a reduction in the speed of adjustment for that market. An implication of this last point is that if the auctioneer wishes to ensure that the unique market clearing price vector is reached then this can be achieved by reducing the speed of adjustment of the price that fluctuates within the larger bounded interval. By focusing on speed of adjustment for the market that has prices that are bound within the smaller interval, there is no guarantee that stability can be attained. This result extends to a system of any dimensions; the auctioneer can make sure that for an economy of any arbitrary size, stability can be guaranteed by fixing the speed of adjustment so that it is small in *all* markets. The largest speed of adjustment will be dictated by the market that has the smallest level of scarcity. Furthermore, this holds for *any* distribution of endowments. However, for significantly small holdings of the numeraire with respect to all non-numeraires, the speeds of adjustment have to be increasingly small.

The concept of stability and scarcity can also be extended to the two dimensional case. The results of table 4.1 indicate that if the holdings of goods one and two relative to the numeraire are sufficiently scarce, for a given speed of adjustment, the system will be stable. Similarly for sufficient degrees of scarcity of both goods, the range of values that both speeds of adjustment take such that the entire system is stable becomes larger. This is backed up by the results of table 4.2 compared to those of table 4.1.

Section Five: The Case of Trade out of Equilibrium

In sections two to four, the tatonnement process did not permit any trade or exchange at any other point other than the unique market clearing price vector. It was shown that periodic and complex price dynamics can occur for which no market clearing price was reached and, as a consequence, no equilibrium price vector was ever reached no matter how long the period over which the system was integrated.

The question now arises of whether exchange out of equilibrium can stabilise prices. The methodology of this section follows that of Uzawa [26]. The rule that determines whether exchange is optimal at each point in time is that if, by means of exchange, at least one individual's utility increases and the utility of no one decreases then, at that point, exchange takes place. If exchange takes place on this basis then the holdings of endowments will converge to a Pareto optimal distribution which, when reached, no individual can effectuate any exchange with another whilst none lose or gain in terms of utility.

For the one dimensional case, the price adjustment mechanism for the non-tatonnement process is exactly the same as in the tatonnement process but, if exchange is optimal at any point in time there will be a redistribution of endowments. Prices in the next period will be dependent on the same speed of adjustment and preferences but the distribution of endowments will have changed.

The price adjustment mechanism, in the case in which there are two individuals that have the same preferences for both goods but differ in the initial holdings of endowments is given as;

$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\alpha\sum_{i=1}^{n}\frac{\omega_i^2}{p_t} - (1-\alpha)\sum_{i=1}^{n}\omega_i^1\right]\right\} \qquad (5.1)$$

It is noted that if the global constraint for the traded and numeraire good hold, then *any* distribution of endowments does not alter the price sequence for a given speed of adjustment, initial endowments and staring price[73] [74] .

That the price sequence is unique and remains unchanged for any redistribution of endowments implies that there is a one-to-one correspondence between the tatonnement process and the non-tatonnement process in that, for a given speed of adjustment and starting price, the tatonnement and non-tatonnement process share the same price sequence irrespective of the distribution of resources, provided the global constraint on such resources is the same in both cases. The question is therefore whether a given speed of adjustment allows there to be a unique Pareto optimal distribution of endowments as $t \to \infty$.

It was noted in section two that for $\kappa \in (0,1)$ there was monotonic convergence in prices towards the market clearing price and for $\kappa \in (1,2)$ oscillatory convergence takes place. The same methodology can be used in the non-tatonnement case. The case in which initial starting price is $p_0 = 4$, both individuals have the same preferences for both goods (equal to a half) and good one is held entirely by individual one and good two by individual two was examined in section two and is reproduced here for the non-tatonnement process.

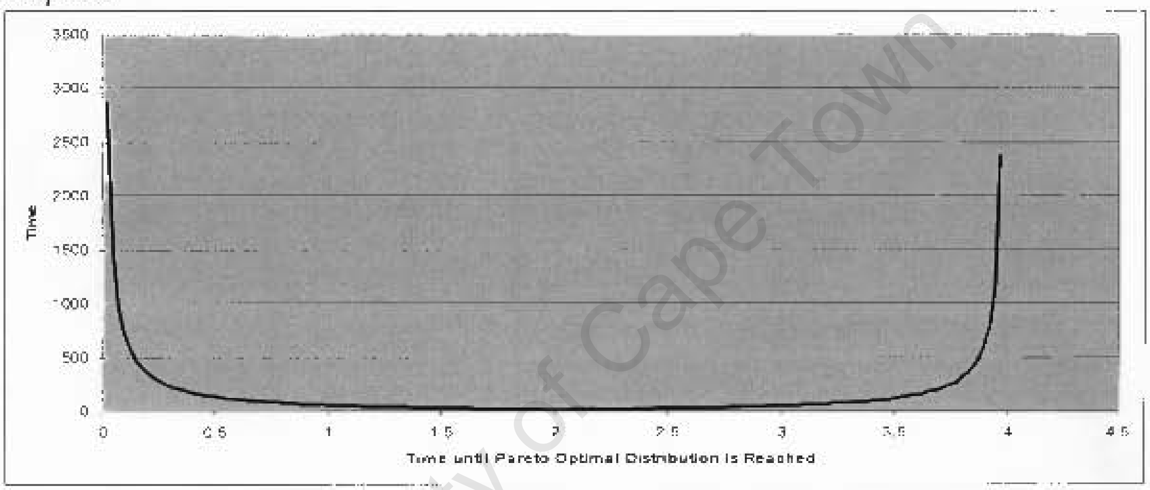
By the author's calculations for $\kappa \in (0,1)$, since prices fall monotonically towards $p^* = 1$, exchange can take place at each point in time[75] such that utility continually increases (by decreasing amounts) up until the point at which p^* is reached. At the market clearing price, no more exchange can take place as the Pareto optimal point has been reached. Furthermore, the number of iterations over which exchange takes place until the Pareto optimal distribution has been reached is exactly the same in the non-tatonnement process as the tatonnement process where, in the latter case, exchange only ever takes place once the market clearing price has been reached[76]. The reason for this is due to the price sequence remaining unchanged for *any* distribution of endowments.

[73] This is not the case is both individuals have differing preferences for both goods
[74] See Appendix for an explanation of why this is the case
[75] i.e. at each iterate of (5.1)
[76] This point cannot be stressed enough and is directly dependent upon each individual having the same preference for both the numeraire and traded good.

For $\kappa \in (1,2)$, the situation is slightly different. Since prices oscillate towards the fixed point yet do so in a manner that is not symmetrical about the fixed point, exchange, for the particular initial holdings of endowments and starting price mentioned above, is only beneficial on the up movement of the price. If the price oscillated in a manner that was symmetrical about p^*, it would be possible to envision that in one period, individual one is a net supplier of good one and net demander of good two whilst individual two is a net demander of good one and net supplier of good two. In the next period, individual one would be a net demander of good one and net supplier of good two and individual two would similarly have reversed his position. This situation cannot occur as the oscillations about the fixed point are not symmetrical. Since this is the case then the upper envelope of the price sequence over time can be considered as the points at which exchange takes place. Such exchange takes place every other period. As the upper envelope of the price sequence falls monotonically over time, due to the fact that the amplitude in the oscillations in price fall over time, then a Pareto optimal point of exchange is reached[77].

Graph 5.1 shows the time it takes until, by means of exchange, the Pareto optimal point is reached.

Graph 5.1



The time it takes until the Pareto optimal distribution of resources in the non-tatonnement is precisely the same as the time it takes until the market clears in the tatonnement process. For speeds of adjustment in which the system is stable in the tatonnement process, the system is stable in the non-tatonnement process and a Pareto optimal exchange is achieved. The minimum time it takes to achieve the Pareto optimal distribution of resources is that at which the system is super-stable; the eigenvalue about the fixed point is zero. An interesting result occurs for $\kappa \in (0,1)$. As the speed of adjustment increases in this interval then the Pareto optimal holding of resources is less for individual one at successively higher values of λ, and higher for individual two for successively higher values of λ.

It was noted in section two that a period doubling occurs at $\kappa = 2$ and for larger values of the speed of adjustments orbits of higher periods occur followed by chaotic trajectories interspersed with windows of (periodic) stability (see bifurcation diagram 2.3).

For the case in which a trajectory of period two is observed, it is again noted that once the system has settled down, the periodic fluctuations are not symmetrical about the fixed point. This can be seen by noting that in the bifurcation diagram, one side of the orbit occurs at a point that is further away from the fixed point (the upside of the price movement) and the other side of the orbit falls closer to the fixed point but below it (the downside of the price movement). This implies that the Pareto optimal point will always

[77] This situation would change if the starting price were different. The asymmetry in the fluctuations does imply that reversal of positions at each iterate would be difficult, if not impossible, to construct

favour one individual over the other[78]. There can therefore be no quid pro quo exchange in which utility remains unaltered. The price envelope at which trade takes place will either encompass the up movements of the periodic trajectories or the down movement depending on the initial endowment matrix and starting price[79]. However, even though periodic trajectories are observed, a Pareto optimal holding may be attained. For highly uneven initial holdings of the endowments this situation is more likely to arise than the case in which initial holdings are very similar for both of the individuals.

From the bifurcation diagram it can be seen that as the speed of adjustment increases, the amplitude of the two period orbit increases asymmetrically about the fixed point. As the speed of adjustment increases and as the amplitude of the price movements increase, the Pareto optimal endowments may favour one of the individuals over the other by increasing amounts and final holdings of endowments become increasingly skewed in favour of one of the individuals. As the speed of adjustment increases further, periodic orbits of higher periods are observed. As this takes place the Pareto optimal distribution of endowments may become even more skewed in favour of one of the individuals over the other.

The non-tatonnement process may therefore generate Pareto optimal holdings of the endowments that are increasingly skewed in favour of one or the other of the individuals. If this is the case then one of the individuals will be better off if the speed of adjustment is lower. However, that this is the case will depend upon the initial holdings of the endowments. It is therefore difficult to generalise such a statement. Furthermore, the initial starting price has a strong bearing on the Pareto optimal distribution.

It is however possible to state that for a large class of initial holdings, period trajectories still permit convergence to a Pareto optimal distribution of resources. The non-tatonnement process is an improvement on the tatonnement process as, in the case of the latter, if periodic trajectories are present then there can be no exchange and accordingly each individual cannot improve upon the initial distribution of endowments. Despite this improvement, exchange at disequilibrium does not stabilise prices if prices are periodic.

At a certain point chaotic trajectories are observed. For chaotic trajectories, several very different possibilities are observed. Certain speeds of adjustment may create sequences of exchange that allow the distribution of resources to move very close to the type of distribution of endowments that are present in the case in which the system is stable in the sense discussed above. However, given that a chaotic trajectory is often characterised by its "visiting" many points in a bounded interval (see G.2.14), the market participants may face a price vector for which exchange is beneficial, yet this price vector may not occur in finite time[80]. Accordingly no Pareto optimal point of exchange can take place as there may always exist another Pareto optimal point of exchange at an indeterminate point in the future.

Another potential difficulty is that, for a chaotic trajectory, there may be no exchange that can take place at any price. Whilst this will not be the case when initial holdings are[81]:

$$\varpi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

[78] The degree to which the Pareto optimal holdings are more inequitable depends not only upon the initial holdings but on the starting price. If the initial price is close to the price vector that supports the optimal holdings then there will be a lower degree of inequality in these holdings. However, that such a price can be chosen implies that the Walrasian auctioneer has a notion of what such a price is. There would need to be a strong justification that he can actually have such a notion

[79] The Pareto optimal holding will be reached once one fluctuation of the periodic orbit has taken place. If there is no quid pro quo exchange that leaves utility unaltered then utility cannot be increased at any subsequent point

[80] G.2.14 shows the frequency for 20,000 iterates. It is conceivable that as the number of iterates increases and as the intervals become increasingly fine grained then at some future point there will be a price such that exchange can take place. Since the sequence is chaotic, it is not possible to know beforehand whether there exists such a price or when such a price would take place

[81] For this initial holding of endowments some exchange can be effectuated for at least one element in any price sequence

if the initial holdings are different to this, then it is conceivable that for a chaotic trajectory, no exchange at any price can take place (see G.2.12)[82]. The same argument can be applied for periodic trajectories but not for stable trajectories. The sub-optimality of the Pareto optimal distribution of endowments for periodic and chaotic price with respect to stable price sequences poses a problem for both the auctioneer and the market participants. The nature of sub-optimality in the former two cases may take the form of either no trade taking place or trade taking place such that there is a much more inequitable distribution of endowments or trade not taking place in finite time. Similarly, in the presence of complex dynamics, the Pareto holdings may be less for both individuals with respect to the case in which the system is stable in that the Pareto optimal distribution of resources is reached precisely at the point at which prices cease to change. Whilst the occurrence of inequitable distributions of endowments is a function of the initial price and holdings of endowments, the presence of the case in which no trade can take place at any point is a more serious problem and cannot be overlooked.

In section three it was noted that if the tatonnement process is unstable in the sense of periodic or chaotic trajectories then there are two solutions; a reduction in the speed of adjustment or an increase in the holding of the numeraire vis-à-vis the non-numeraire. The same reasoning applies to the case in which trade out of equilibrium occurs.

Suppose that for a given speed of adjustment, periodic or chaotic price sequences are observed and no exchange at any price can take place. It naturally follows that there is too much demand for one of the goods and too much supply of the other good. In other words, there is pent up demand in the system; the ratio of the endowment of the traded good with respect to the numeraire is too high. If the numeraire is treated as a primitive form of money through which all exchange must take place[83], then it is natural to interpret periodic and chaotic orbits as situations in which the non-tatonnement process is liquidity constrained. For a given speed of adjustment if the system is sufficiently liquidity constrained then periodic or chaotic trajectories will occur for which either no exchange can take place or any exchange that does take place would be inferior to the exchange that would take place if the non-tatonnement process were stable or there were more liquidity in the system.

For the one-dimensional case in which all agents are alike in their preferences for both goods and in which trade is permitted out of equilibrium, it can be inferred that if the global constraint for the numeraire or "monetary good" is too small with respect to the non-numeraire or traded good then there will be pent up demand or supply of the traded good[84]. Periodic and chaotic trajectories may occur for which there may either be no feasible trade at any point over a price sequence or, if trade does occur it, may entail a Pareto optimal exchange that is inferior to the case in which there is a greater degree of liquidity. The auctioneer thus becomes charged with ensuring that there is sufficient liquidity in the system or that the speed of adjustment is low. By ensuring this, he also ensures that there is a price sequence that supports a sequence of exchanges that converges on a Pareto optimal distribution as time tends to infinity. It is tempting to assume that a periodic or chaotic trajectory can be rendered stable by the injection of more of the numeraire into the system. This line of reasoning would be erroneous, as at least one of the individuals would move to a utility contour that is higher than that before the point at which the injection of liquidity takes place. Accordingly, the market clearing price changes and the non-tatonnement process is disrupted. There is no guarantee that at the prevailing price or any other subsequent price, the injection of more of the numeraire into the system would permit a sequence of exchanges that converge towards a Pareto optimal distribution. Instead the auctioneer has to ensure from the *outset* that there is sufficient liquidity such that there is asymptotic convergence to the Pareto optimal distribution.

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[82] For G.2.12, prices occur in two intervals. There is a large sub-set of prices that are never reached irrespective of whether the number of iterates considered is large or small. If the initial endowment matrix is such that exchange could only take place within the price interval that is not visited, there can then never be any exchange. The Pareto optimal distribution is therefore the initial distribution. If the speed of adjustment is smaller, such that $\kappa \in (0,1)$, then some exchange can be effectuated. The presence of complex dynamics may therefore create a situation in which the Pareto optimal holdings are sub-optimal with respect to the absence of complex dynamics

[83] This is a trivial statement in the one dimensional case but not so in higher dimensions

[84] Whether there is pent up demand or supply will depend upon the preferences for the traded good with respect to the numeraire

The extension of these results to systems of higher dimensions is problematic. A system in which there are at least two traded goods and a numeraire can be interpreted in two different ways: At any price, exchange can take place either:

1. Between *any* of the goods in the system, or
2. Exchange can only take place through the medium of the numeraire which is treated as a primitive form of money

The first scenario is an extension of the one dimensional case for which the same type of reasoning can apply. In section four it was shown that for a given endowment matrix and given (distinct) starting prices, as one of the speeds of adjustment increases, periodic orbits as well as chaotic orbits occur. It was also noted that if the holding of the numeraire is increased then, for a speed of adjustment for which periodic or chaotic trajectories occur, the system may become stable. Since any distribution of endowments in a system of higher dimensions leaves the price sequences unaltered for given speeds of adjustment and starting values[85], higher dimensional systems, in the case of 1 above, are extensions of the one dimensional system.

If 2 is observed the dynamics are fundamentally altered. This author finds that if an initial endowment matrix of the form

$$\varpi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is assumed then, irrespective of whether the system exhibits stability or complex dynamics, (i.e. irrespective of the values of the speeds of adjustment) it becomes difficulty to carry out *any* trade that increases the utility of at least one individual without reducing that of no other. As in the first case, it is natural to assume that the system is liquidity constrained. However, unlike the first case, it is possible that if exchange can only take place through the medium of the numeraire, an impasse to trade can occur independently to whether the price sequences are stable or not (the speeds of adjustment are low). This implies that if trade can only ever take place through the numeraire, any higher dimensional system quickly becomes more liquidity constrained than the case in which trade can take place amongst any of the goods. For such a scenario, complex price dynamics are a confounding factor but of little import; the first difficulty faced is that the system is liquidity constrained. The second difficulty is that complex dynamics can occur.

Very different policy implications arise for case one or case two. In the first scenario, the auctioneer can ensure that there is convergence to a Pareto optimal holding of endowments by lowering the speed of adjustment in one or both of the markets (see section four for a discussion of which speed of adjustment can lead to stability and the conditions under which it can do so). In the second scenario it is conceivable that if the system is liquidity constrained then it is so for *any* speed of adjustment. The only solution in this case is to ensure that there is sufficient liquidity in the system. Clearly the second scenario is much more problematic as there is no way to gauge, prior to the event, whether a given amount of liquidity will ensure convergence to a Pareto optimal endowment distribution. The first scenario is less problematic in this regard as the speeds of adjustment and holdings of endowments need to be such that the modulus of the eigenvalues falls within the unit circle.

Conclusion

This thesis has attempted to analyse various aspects of both the discrete tatonnement and discrete non-tatonnement process in the presence of gross substitution and the weak axiom of revealed preference. It was found that, for the case of one non-numeraire and one numeraire that stable, periodic and chaotic price

[85] This statement necessarily relies on each individual having the same preference arrangement for all goods.

sequences were possible. The occurrence of one over the other depended entirely on the value of the speed of adjustment as well as the relative holdings of endowments. These dynamics were readily quantifiable by means of a Lyapunov exponent and bifurcation diagram. It was also noted that as the endowment matrix changed, the qualitative nature of the price sequences with respect to the speed of adjustment did not change. This implies that a periodic or chaotic trajectory can be avoided if the holdings of the numeraire with respect to the non-numeraire are sufficiently large or if the speed of adjustment is sufficiently low. To this author's knowledge, work to date has focussed almost exclusively on the analysis of the speed of adjustment and not on the role of endowments.

The tatonnement process was then extended to a two dimensional system, in which there were two non-numeraire goods and one numeraire. The quantification of chaos for this system was more difficult than in the one dimensional case. Concepts such as Feigenbaum's constant or the Li-Yorke theorem used to define chaos in a one dimensional map have no analogue in higher dimensions. Instead, it was found that once one of the eigenvalues moved outside the unit circle a saddle node was present and periodic orbits occurred. In the search for orbits of differing periods it was found that various periods occur as well as the presence of speeds of adjustment for which there is sensitive dependence to initial conditions. The situation in two dimensions is therefore more complex and less easy to quantify than in the one dimensional case but the results do point towards there being a natural extension in the qualitative nature of the analysis to higher dimensions.

Having examined the tatonnement process, Uzawa's non-tatonnement process was then considered. For the case in which there is one traded good and one numeraire, it was found that there is a direct parallel between the tatonnement process and the non-tatonnement process on the basis that, if all agents have the same preference relations but differ in their initial holdings of endowments, the price sequence remains unchanged if exchange does take place. This allowed there to be a unique correspondence between the tatonnement process and the non-tatonnement process if any good is allowed to be exchanged for any other good. If the tatonnement process was stable then the non-tatonnement process was stable in the sense that there was convergence to a unique distribution of endowments as time tends to infinity and that that distribution of endowments is Pareto optimal. In the presence of periodic and chaotic trajectories exchange becomes more complicated. Much depends upon the initial holding of endowments and the initial prices. It is conceivable that if prices were to settle into a periodic trajectory of any order there may or may not be a Pareto optimal exchange depending on whether the initial holding of endowments permits such an exchange.

If the numeraire has the function of acting as a primitive currency or medium of exchange, then it is possible that there be insufficient holdings of the currency. The system would then be liquidity constrained and there may be no feasible solution to such a constraint as there would be no Pareto optimal exchange for any set of speeds of adjustment. This case is much more complicated and difficult to diagnose than the case in which exchange can take place between any of the goods.

The policy implications in the case of the non-tatonnement are more interesting than those of the tatonnement process. A serious difficulty in the former case is that the system needs to have sufficient levels of liquidity, but such levels of liquidity are difficult to determine on an a priori basis and are difficult to determine irrespective of whether complex dynamics are present. The auctioneer or the equivalent policy making institution thus becomes charged with managing a market or an economy whose liquidity needs are uncertain. Only once the liquidity needs have been determined can attention be focused on the nature of the price sequences and how best to render a complex system stable, for which a Pareto optimal exchange can take place. Again such a statement cannot be justified prior to numerical investigation. Analysis in this direction becomes very complicated very quickly as even with only three agents, the number of exchanges at any point, such that the convergence of endowments lies on the Pareto optimal route, can be large. This number increases very quickly as the economy becomes large or the number of individuals wanting to exchange increases.

Not only does analysis in the direction of policy become difficult in the case of non-tatonnement but even in the case of tatonnement, the presence of chaos renders analysis difficult. The analysis of most chaotic systems does not go beyond systems of three dimensions. Since the general equilibrium framework is formulated as an economy of any dimension, the presence of chaos in such a system makes analysis almost impossible, if not at a qualitative level. This holds even for a two dimensional system. Due to the complexity of the system even at low dimensions, it is tempting to impose further simplifications on the discrete time general equilibrium framework. However, this direction is fraught with difficulties. For example, assume that symmetry is imposed upon the system. It was seen in section four that prices would fall on a stable manifold. Such a manifold yields predictions about the set of parameters that ensure convergence that are inconsistent with an asymmetric system. Whilst convenient to analyse, symmetric systems are but a small subset of all the systems that can be actually observed.

One may argue that general equilibrium is the incorrect framework in which to analyse the role of liquidity, even if this liquidity is deemed primitive in nature. However, many of the difficulties faced by more sophisticated models are captured in this general equilibrium framework.

There are however various unavoidable shortcomings to the analysis in this thesis other than the ones mentioned. The type of price generating algorithm used in this thesis is simplistic in nature; a more sophisticated algorithm may preclude complex dynamics. In the case of trade in disequilibrium, it would be natural to posit that market participants consider the direction in which prices are moving. If a price vector is increasing over time, a rational market participant that holds the good for which the price is increasing may withhold exchanges until some price threshold had been reached at which point he would exchange. The Pareto optimal distribution of endowments would thus have to incorporate speculative holdings. Questions such as adaptive expectations in the presence of stable or complex systems then arise.

Putting aside these shortcomings, the abstraction from the theoretical confines of the GE framework to actual observed behaviour leaves one wondering whether the predictions of the T and NT process as well as the caveat presented in this work, are applicable.

If one objects to the caveat, then the challenge of GE is explain certain facets of an economy given a model that predicts stability. Not an easy feat. If no such objection is put forward then the challenge is to find aspects of an economy that are both stable and unstable and that align with the GE framework as well as any caveat to this framework. The recent bailout of AIG and the re-absorption of Fannie Mae and Freddie Mac into the US Government are a tentative step in this direction.

It was noted in the last chapter, that if there is sufficient scarcity present in both the T and NT process then the system is stable. The present state of the US financial markets illustrates the case in which various institutions have become "bloated"; for example, AIG held large amounts of mortgage backed securities. Holdings of such securities were accordingly abundant with respect to cash holdings. As these assets changed to having a dubious value, AIG, along with other institutions, found that their position could not be reserved; the market was liquidity constrained. A direct parallel emerges between the present state of flux in the US and the predictions of the NT process in a GE framework. In both cases it is seen that a lack of liquidity in the system creates a block to the smooth functioning of the market; the price mechanism ceases to work either in a coherent manner or at all.

The US Federal Deposit Insurance Corporation deemed that AIG was simply too large to fail; lines of credit were opened up to AIG. The extension of lines of credit is akin to an increase in the quantity of the numeraire in the NT model in which there is an impasse to any trade taking place. An increase in the numeraire[86] in the NT process "loosens" the system, for a given speed of adjustment such that trade can take place at subsequent points in time. In this regard, the actions of the US Federal Reserve are consistent with the predictions of NT.

Whilst NT does not capture dynamics such as causal factors of bank failures (i.e. the banking sector searching for higher and higher earnings without giving proper weight to the assuming of increasing amounts of risk) or the aspects of moral hazard involved in the bail out of an institution (especially an institution as large as AIG), the cessation of a market's smooth functioning can be traced to factors that are

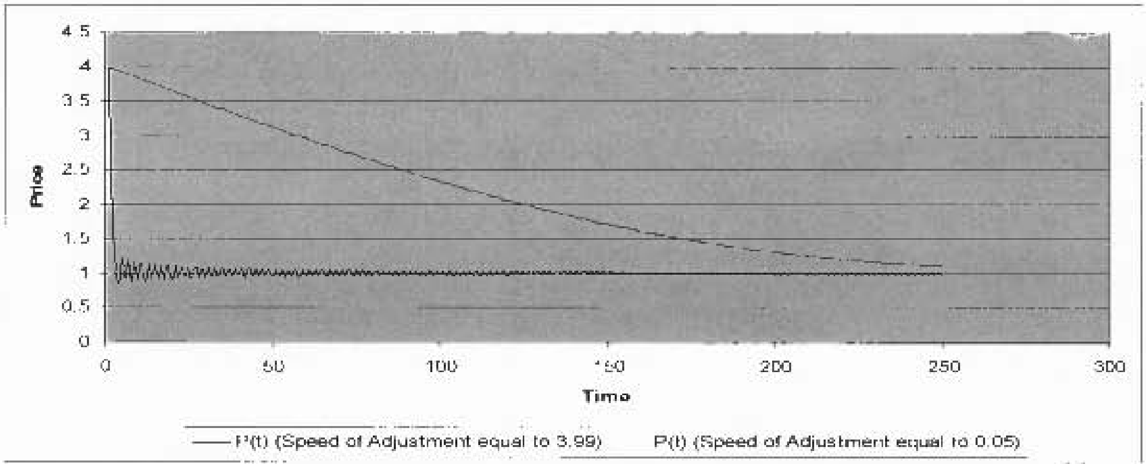
[86] For the case in which any good can be exchanged for any other good and the case in which any good can be exchanged only by means of the numeraire

present in the NT framework. For example, the "too-big-to-fail" policy implicit in the bail out of AIG, as well as Fortris bank, has an analogue in the NT framework; by the fact that all markets or price sequences are coupled, the failure of any single market will cause *all* markets to fail (i.e. if the zero price boundary is reached in one market, the zero price boundary is reached in all markets). Analogously, by the fact that the price sequence that hits the zero price boundary first is precisely the market that has the largest ratio of traded good to the numeraire good. Accordingly a crash in one price sequence causes a crash in all price sequences and the genesis of that crash emanates from the market (firm, sector or institution) that is most liquidity constrained. In the NT framework, AIG and similar institutions are the perpetrators. Similar reasoning premised on the NT framework points to the solution to the avoidance of the crash of all price sequences as the injection of liquidity into the whole economy or a reduction of holdings of assets (or endowments) away from the perpetrating sector; indeed it would be an odd situation if any set of institutions or sectors whose structure and behaviour was coherent with the smooth functioning of a market were treated as culpable of the failure of the price system. NT strongly predicts that any misplaced culpability will simply not resolve the improper functioning of the price system. Instead such that NT works properly and Pareto optimal holdings are reached, the problematic price sequence needs to be identified and that market modified accordingly.

NT (and also T) framework highlights only a few characteristics of any market, yet those characteristics are fundamental in nature. It is reassuring that the fundamental aspects captured in GE encompass the irregularities that a market may experience. Chaotic price sequences and the poor functioning of a market due to constrained amounts of liquidity are two cases in point. Yet a word of caution needs to be stated; any such reassurance must be recognised as being didactic in nature and wrested on abstractions from real world examples, but after all, wasn't the GE framework constructed as an abstraction from that which is actually observed.

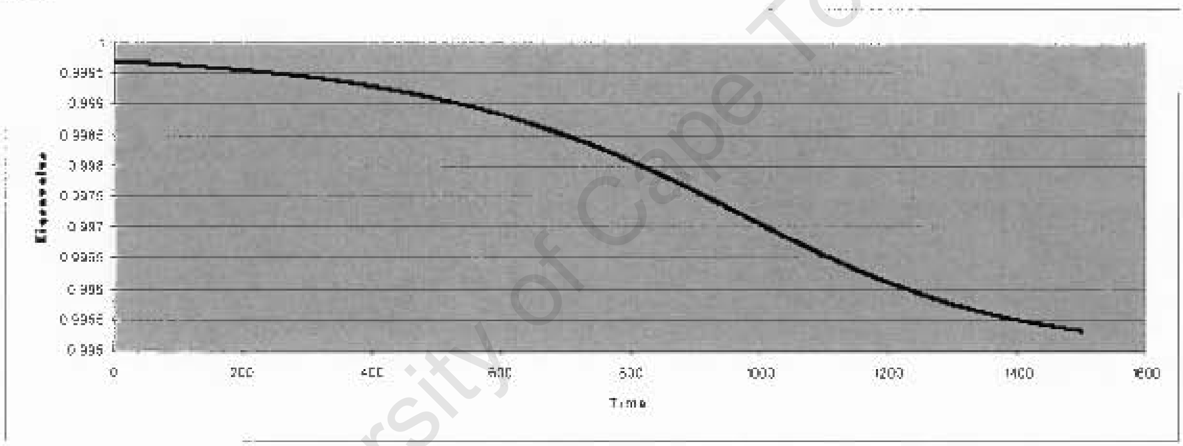
Graphs: Section Two

G.2.1



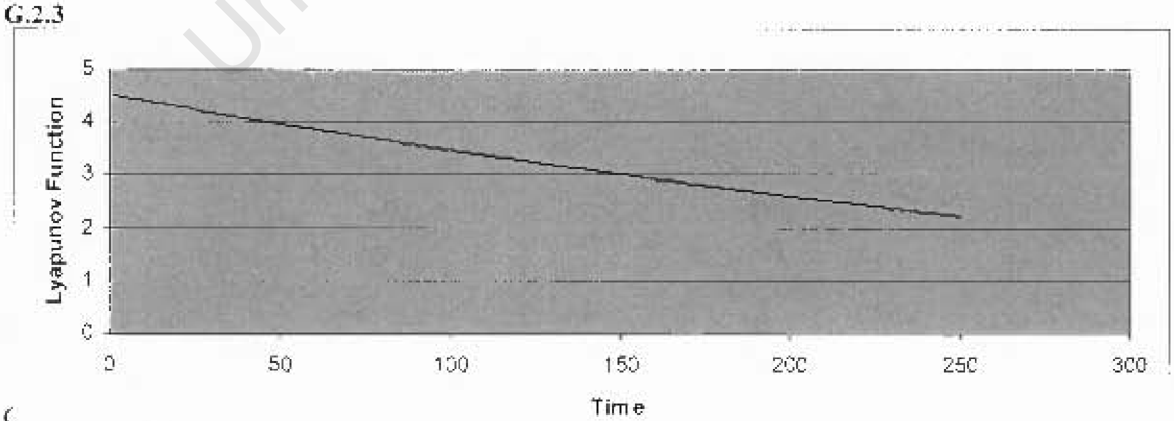
G.2.1 shows that the price dynamics differ depending on the speed of adjustment. If $\lambda = 0.05$ ($\kappa = 0.025$), convergence is monotonic, whilst if $\lambda = 3.99$ ($\kappa = 1.995$), convergence is oscillatory. That the latter dynamics occur is a precursor to a two period trajectory occurring.

G.2.2



G.2.2 shows that the eigenvalue falls within the unit circle for $\lambda = 0.01$ ($\kappa = 0.005$). Stability as well as monotonic convergence is ensured

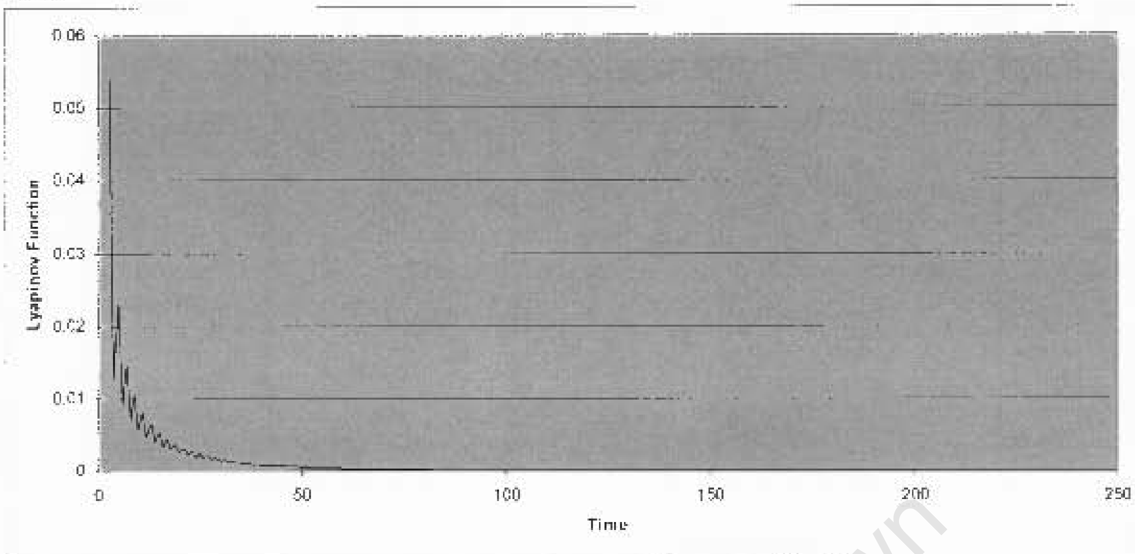
G.2.3



(

that the Lyapunov function falls in an oscillatory manner for $\lambda = 3.95$ ($\kappa = 1.975$).

G.2.4



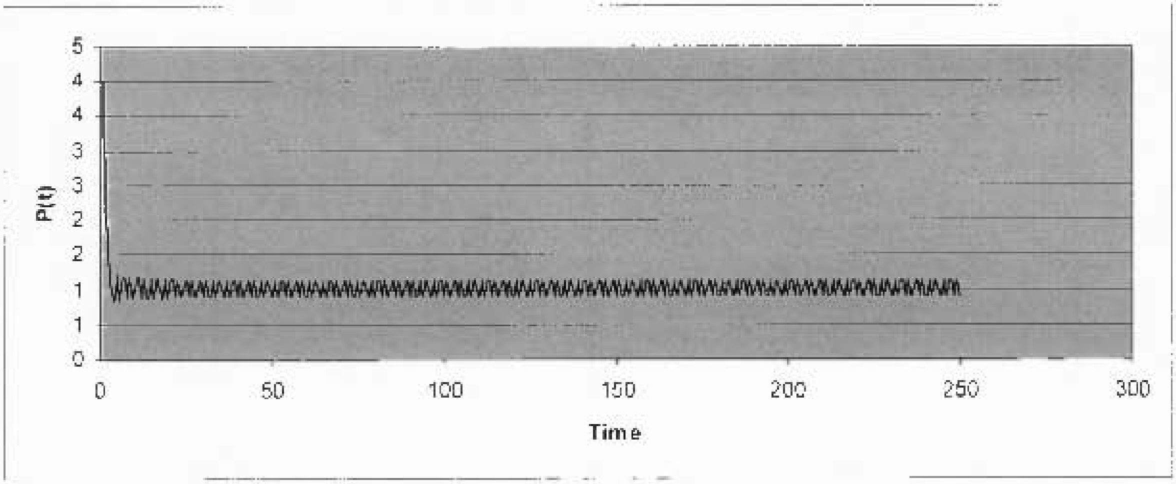
G.2.3 and G.2.4 show that the Lyapunov function falls over time. This decline is however not monotonic in the latter. To this end Uzawa[87] seems too generous in his statement that if the Weak Axiom of Revealed Preference holds (which necessarily does for the Cobb-Douglas function) then the function;

$$V(p_t) = \frac{1}{2}\sum_{j=1}^{k}(p_t^j - (p^j)^*)^2 \qquad (1)$$

is a Lyapunov function and the stability theorem holds. Recalling that the Lyapunov function must converge monotonically towards the fixed point, such monotonic convergence is not observed for $\kappa \in (1,2)$, yet the fixed point is reached nonetheless. If convergence does not have to be monotonic then (1) remains a Lyapunov function for the tatonnement process. This latter assumption is equivalent to the assumption that the tatonnement process be energy-dissipative. A dissipative system would mean that the price sequence falls monotonically towards the fixed point at which all energy has been dissipated, or if oscillations occur, the amplitude of the fluctuations diminish over time until the fixed point is reached at which all fluctuations and energy in the system would have vanished.

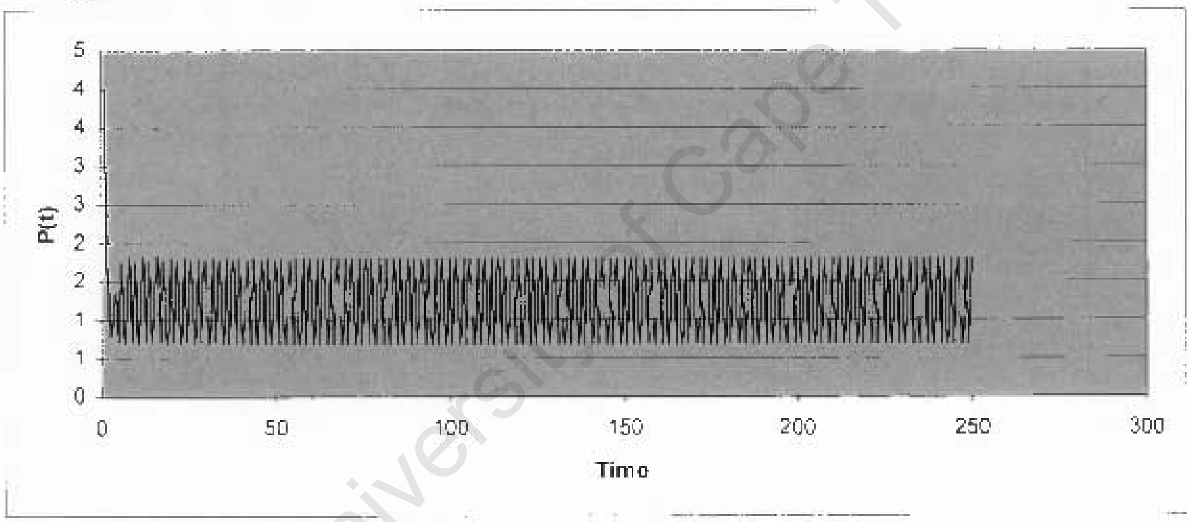
[87] Uzawa [23] p.185

G.2.5



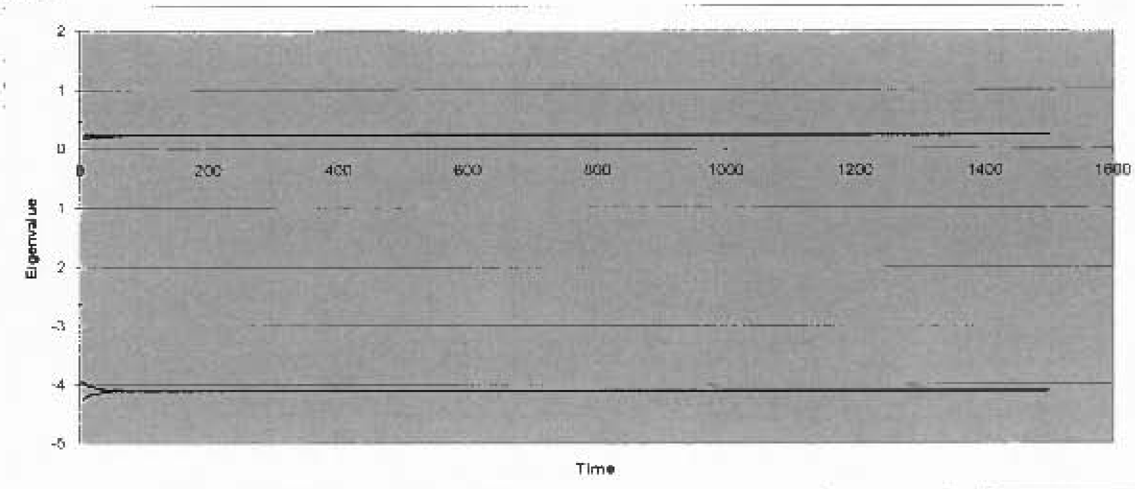
G.2.5. shows that for $\lambda = 4.05$ or $\kappa = 2.025$, two period oscillations occur. These do not die out over time.

G.2.6



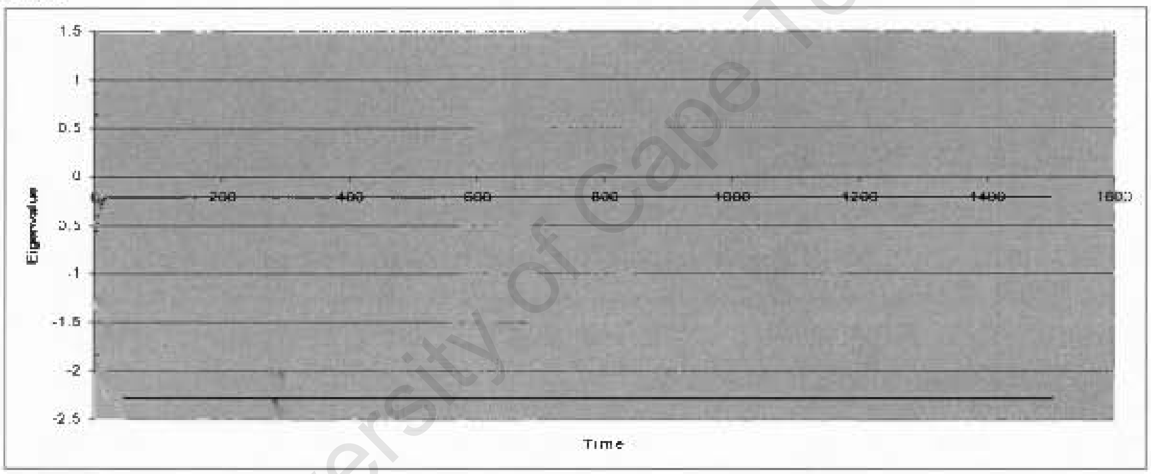
G.2.6. shows that for $\lambda = 4.95$ or $\kappa = 2.475$, two period trajectories persist but are larger than those shown in G.2.5.

G.2.7



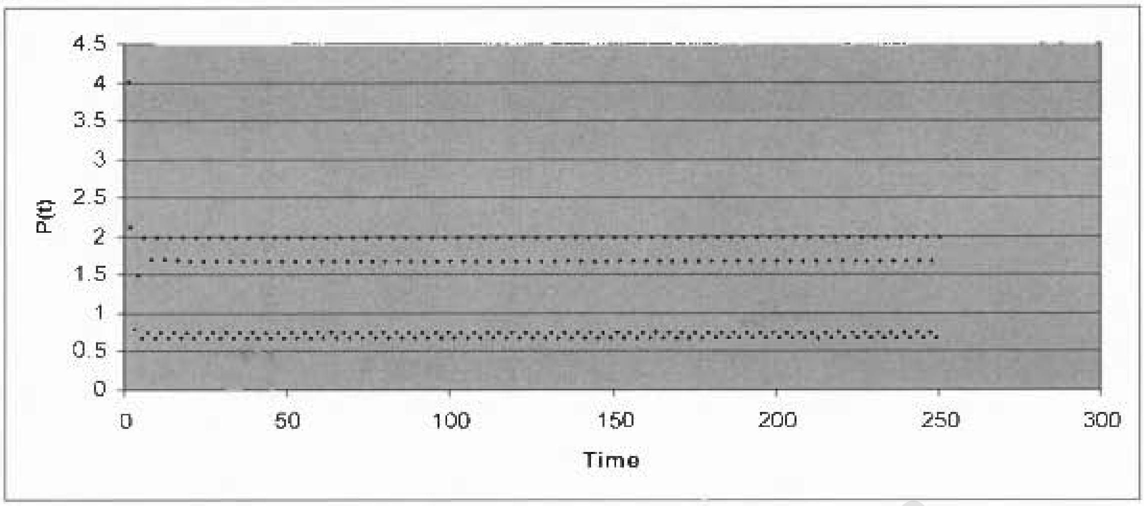
G.2.7. shows that the eigenvalue fluctuates in and out of the unit circle indicative of a two period trajectory occurring for $\lambda = 4.95$ or $\kappa = 2.475$.

G.2.8



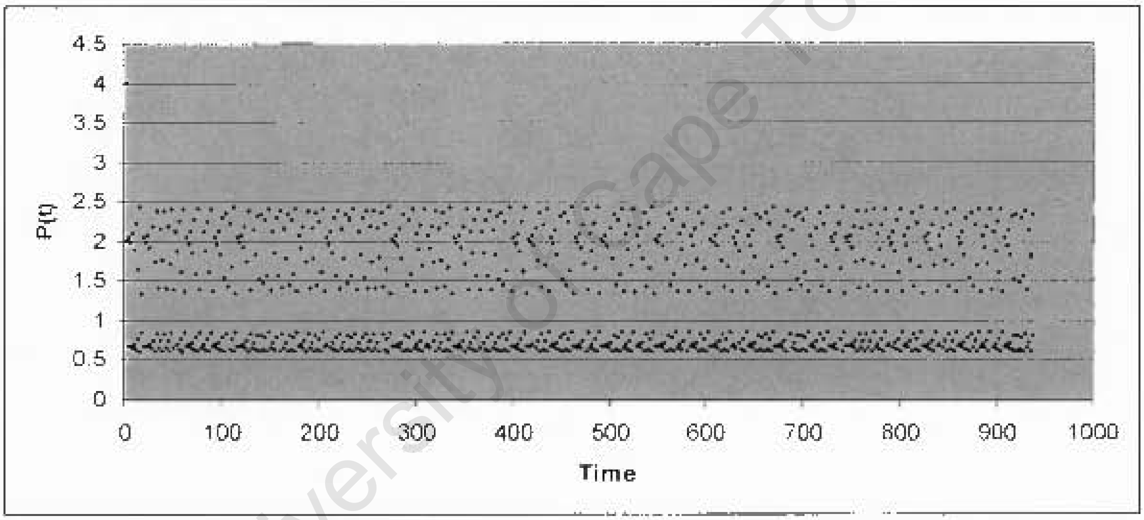
G.2.8. shows that for $\lambda = 4.25$ or $\kappa = 2.125$, the eigenvalue falls between $(-3^{+}, 0^{-})$. This is to be contrasted with G.2.7 above.

G.2.9



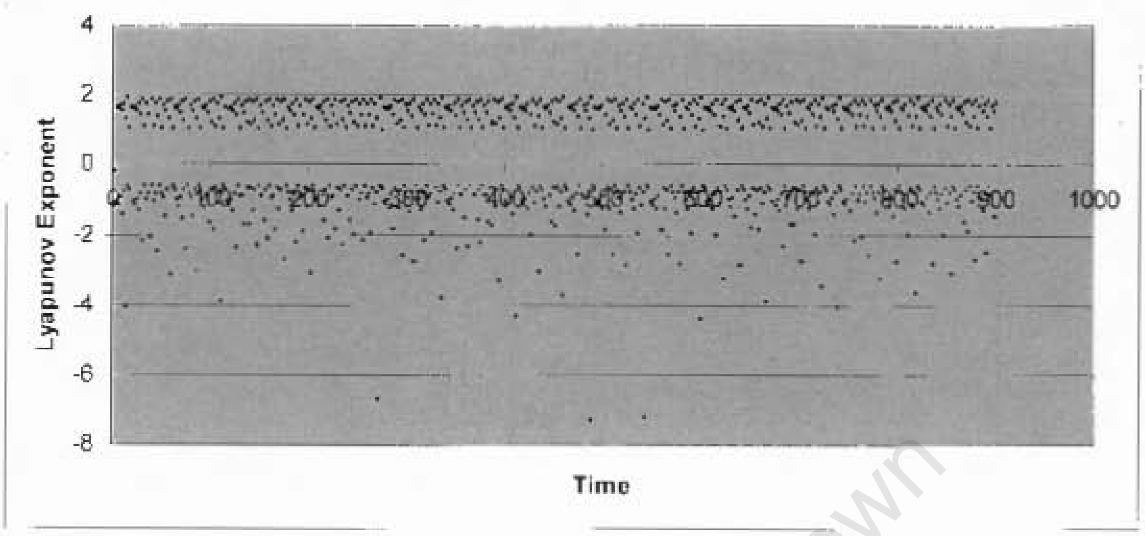
G.2.9. shows that for $\lambda = 5.05$ or $\kappa = 2.525$, a four period trajectory occurs.

G.2.10



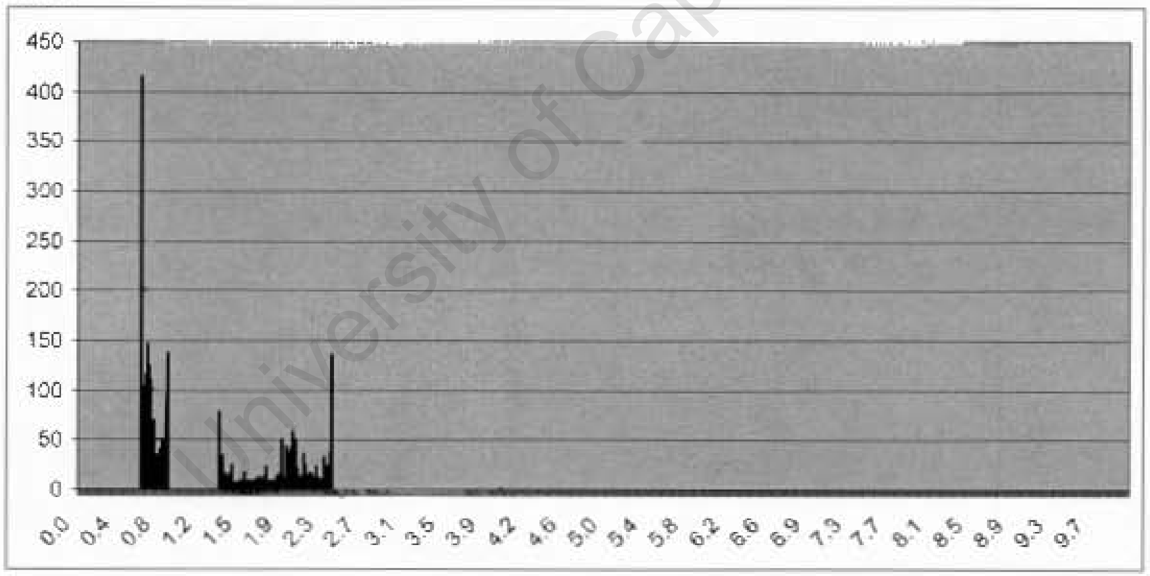
G.2.10. has values $\lambda = 5.35$, $\kappa = 2.675$ and $h(p) = 0.1485$, indicative of the presence of chaos. G.2.11 shows that the Lyapunov exponent fluctuates both above zero and below zero. The former fluctuations dominate as $h(p) = 0.1485$.

G.2.11

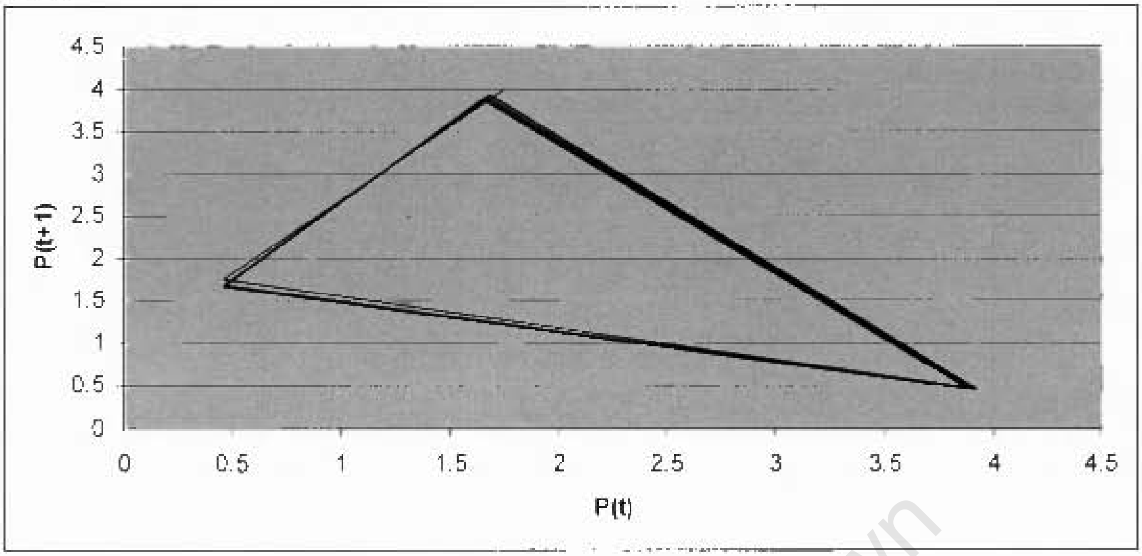


G.2.12 shows that for $\kappa = 2.675$, the frequency of the trajectories over 20.000 iterates start to fill the spectrum of permissible prices. In other words, a chaotic map is often synonymous with the trajectory visiting "many" points.

G.2.12

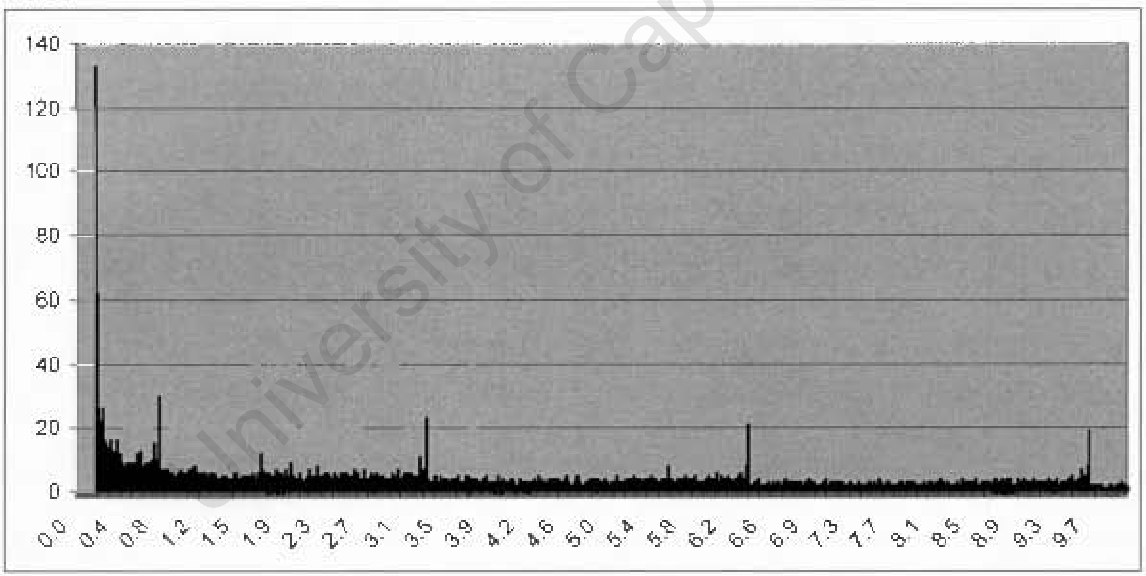


G.2.13



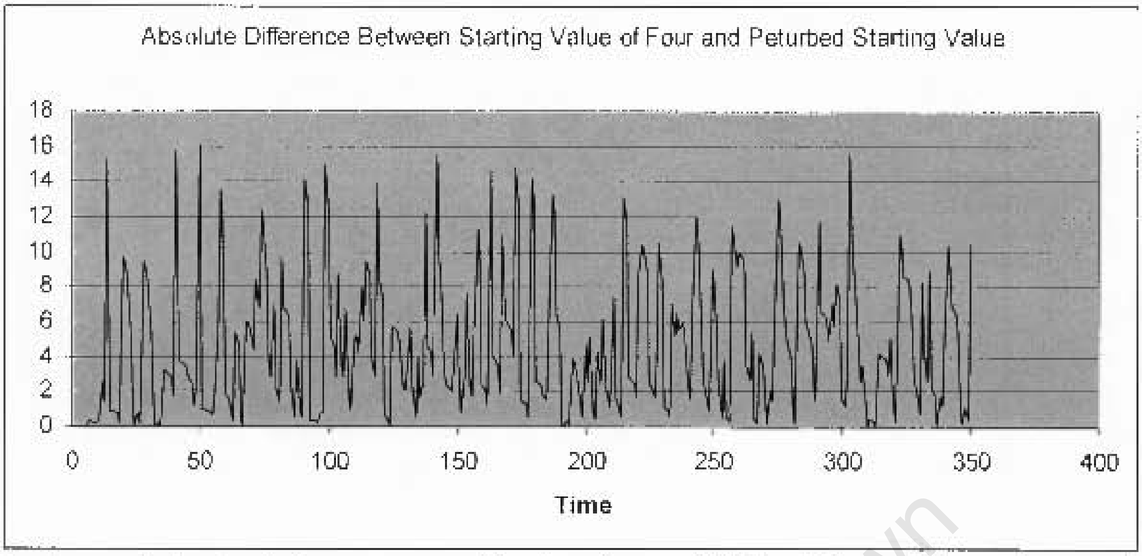
G.2.13 shows the delay plot in which the price one period forward is plotted against present period price. It is clear that a three period orbit is present. LYT can thus be evoked and the map can be said to have the ability to generate chaotic price sequences.

G.2.14



G.2.14 shows the frequency plot for $\kappa = 3.625$ or $\lambda = 7.25$, which generates the Lyapunov exponent of $h(p) = 0.57522$. The map is clearly chaotic for this speed of adjustment. The absolute difference between the price trajectory of that with a starting value of 4 and that with a starting value of 4.001

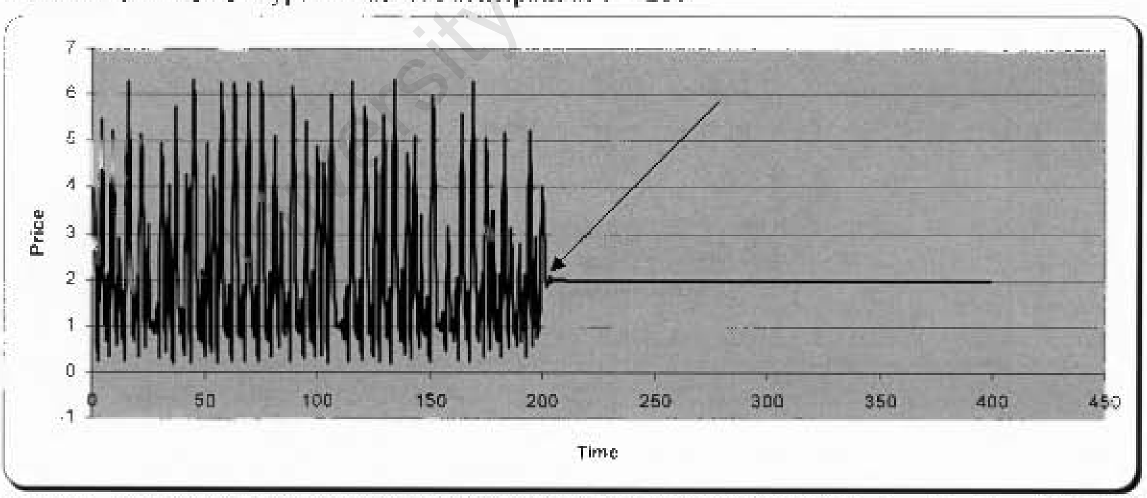
G.2.15



A trajectory that is chaotic demonstrates sensitive dependence to initial conditions. This is seen in G.2.15 in which the difference in the absolute values over 350 iterates of the map are plotted. The two trajectories are products of a set of identical parameters in all but that the starting values differ by a small amount; that is if the starting value is perturbed and the trajectory is chaotic, then two trajectories will wander arbitrarily close to each other and arbitrarily far away from each other within a bounded space. The former is observed by points close to zero and the latter by points close to 16.

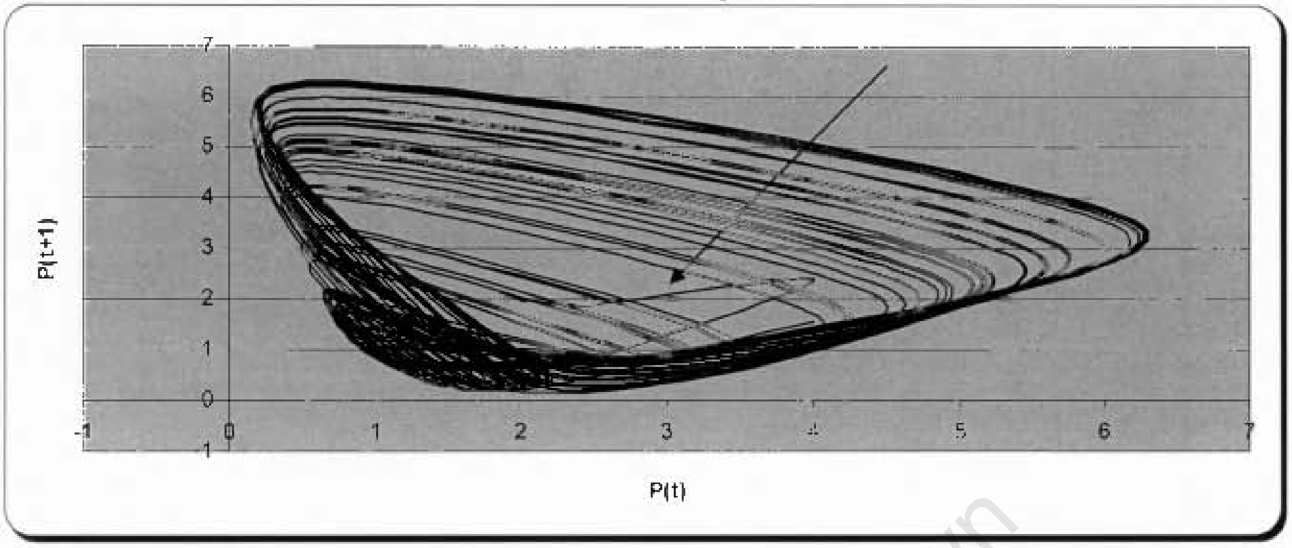
Graphs: Section Three

G.3.1. Introduction of Type B Market Participant at $t = 200$



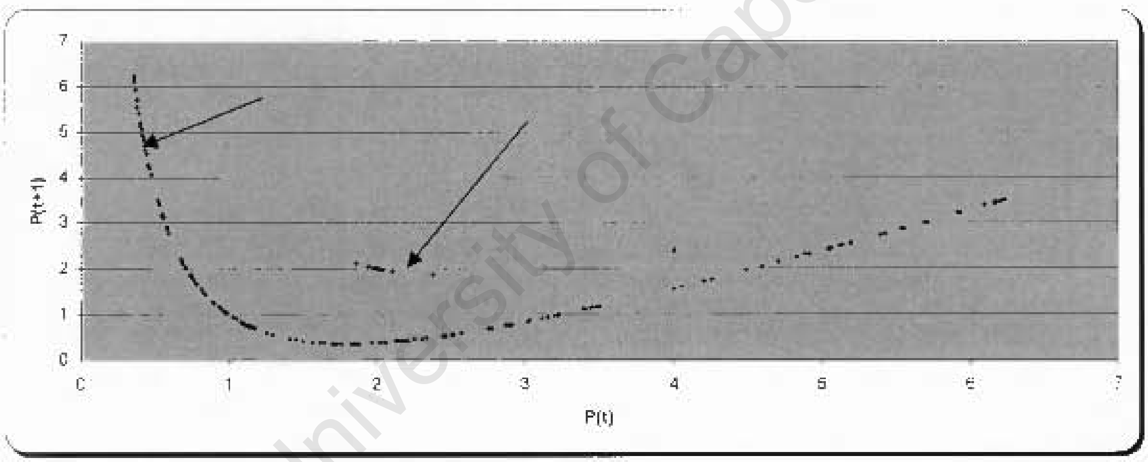
For $0 < t < 200$ there are two market participants and a speed of adjustment of $\lambda = 6.5$, $\kappa = 3.25$ so the trajectory is chaotic for this interval of time. At $t = 200$, a market participant that holds one unit of the numeraire (type B) enters the market. The fixed point of the system is no longer a repeller and increases from one to two. Stability is induced.

G.3.2 Delay Plot for Introduction of Type B Market Participant at $t = 200$



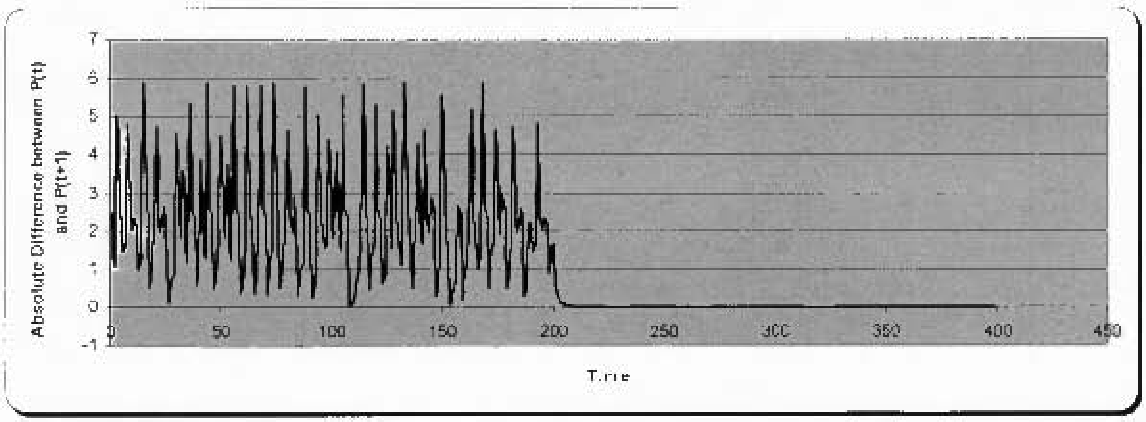
G.3.2 shows the delay plot for the case in which there is a switch at $t = 200$ by the introduction of a type B market participant. The arrow shows how the delay plot is modified when a type B individual is introduced; the line of the delay plot overlaps another part of the delay plot.

Graph 3.3. Delay Plot for Introduction of Type B Market Participant at $t = 200$



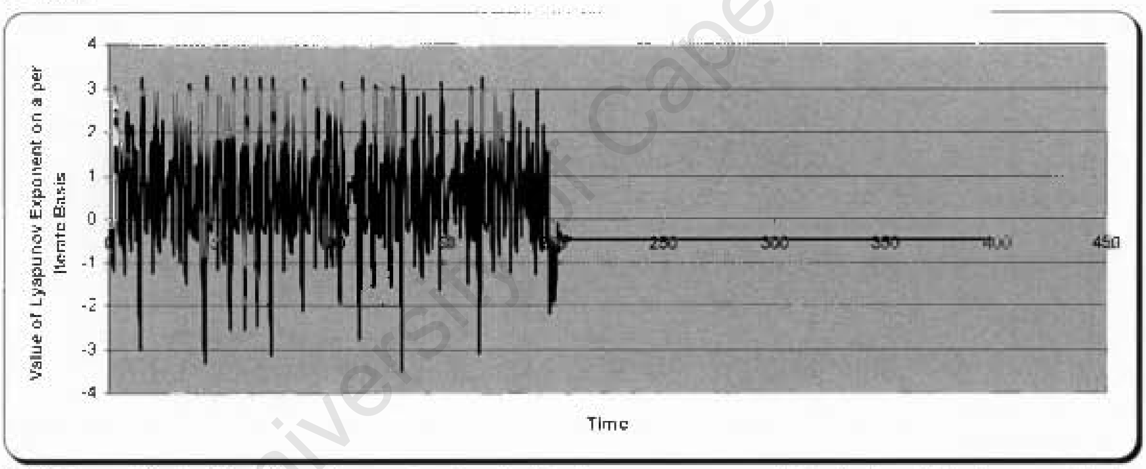
G.3.3 is a replication of G.3.2 but differs in that only points in the (p_t, p_{t+1}) space are considered. The first 200 iterates are associated with the trace of a smooth curve. The trajectory is chaotic here. The introduction of a type B market participant shifts the trajectory so that it is clustered around the market clearing price $p^* = 2$.

G.3.4. Absolute Difference between P(t) and P(t+1) for Introduction of Type B Market Participant at $t = 200$



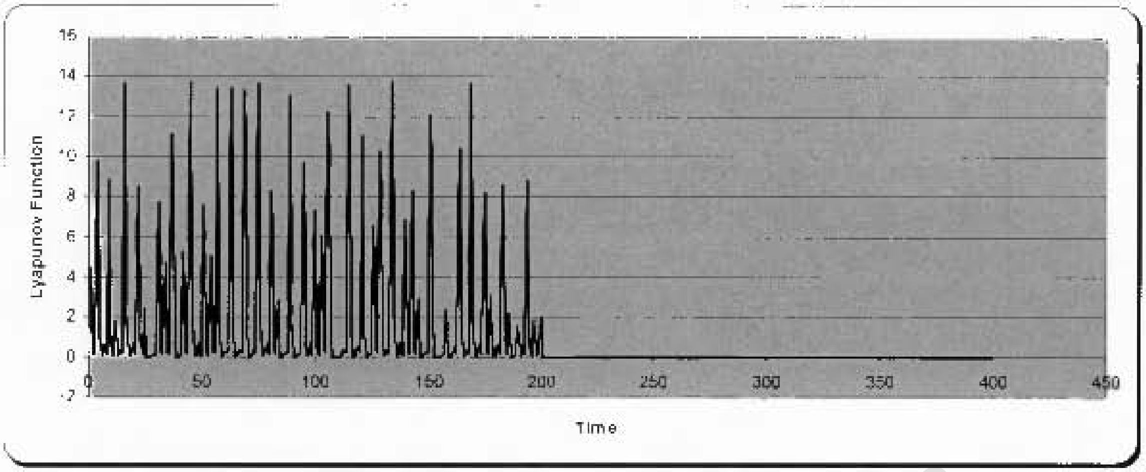
G.3.4 shows the absolute difference between the present value of price and the price one period forward both before the introduction and after the introduction of a type B market participant. For the first 200 periods there is clearly no convergence whilst at $t = 200$ convergence takes place. There is a move away from the case in which there is sensitive dependence to the initial condition.

G.3.5. Lyapunov Exponent on a per Iterate Basis for Introduction of Type B Market Participant at $t = 200$



G.3.5 shows that whilst the trajectory is chaotic, the Lyapunov exponent fluctuates both above and below zero (but the average of those fluctuations is greater than zero) whilst once stability is attained by the introduction of a type B market participant, the Lyapunov exponent stays below zero; the system is stable.

G.3.6. Value of Lyapunov Function for Introduction of Type B Market Participant at $t = 200$

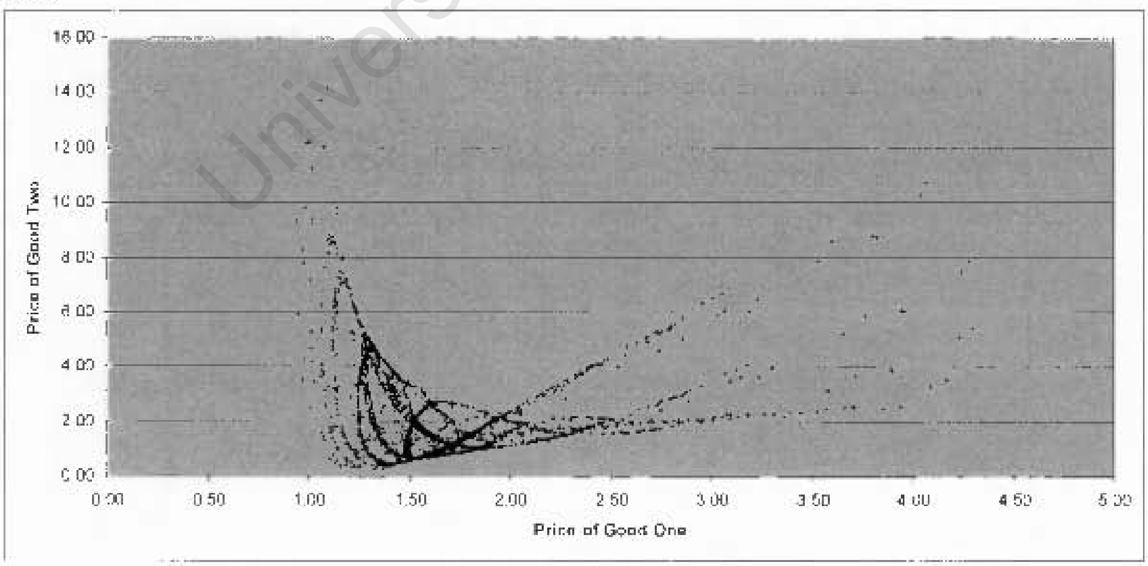


G.3.6 shows that the system is not stable for the first two hundred iterates as the Lyapunov function is ill-defined. Once a type B market participant is introduced then the Lyapunov function converges to a point. G.3.7 shows the change in the Lyapunov function described in G.3.6.

Graphs Section 4

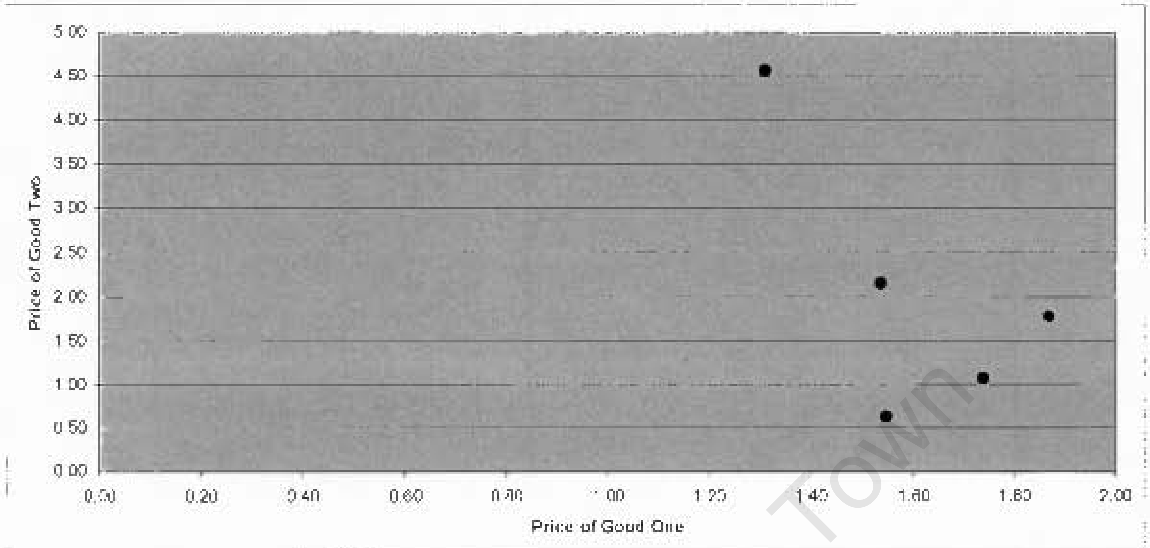
G.4.1 shows the prices of both goods in the phase space for holdings of endowments $\omega_i^j = 1$, $i = j$, $i, j = 1,2,3$ and speeds of adjustment $\lambda^1 = 0.75$ and $\lambda^2 = 5.59$. For this value of endowments and speeds of adjustment, the eigenvalues are $r_1 = 0.6384$ and $r_2 = -2.8506$. A saddle point is present. G.4.1 shows a strange attractor; for any initial set of conditions, the limit set of the prices tends towards this bounded shape.

G.4.1



It is apparent that since $\lambda^2 > \lambda^1$ then p^2 is bound to an interval that is larger than that of p^1. Graph G.4.2 shows the price phase space but λ^2 has increase marginally to $\lambda^2 = 5.6$.

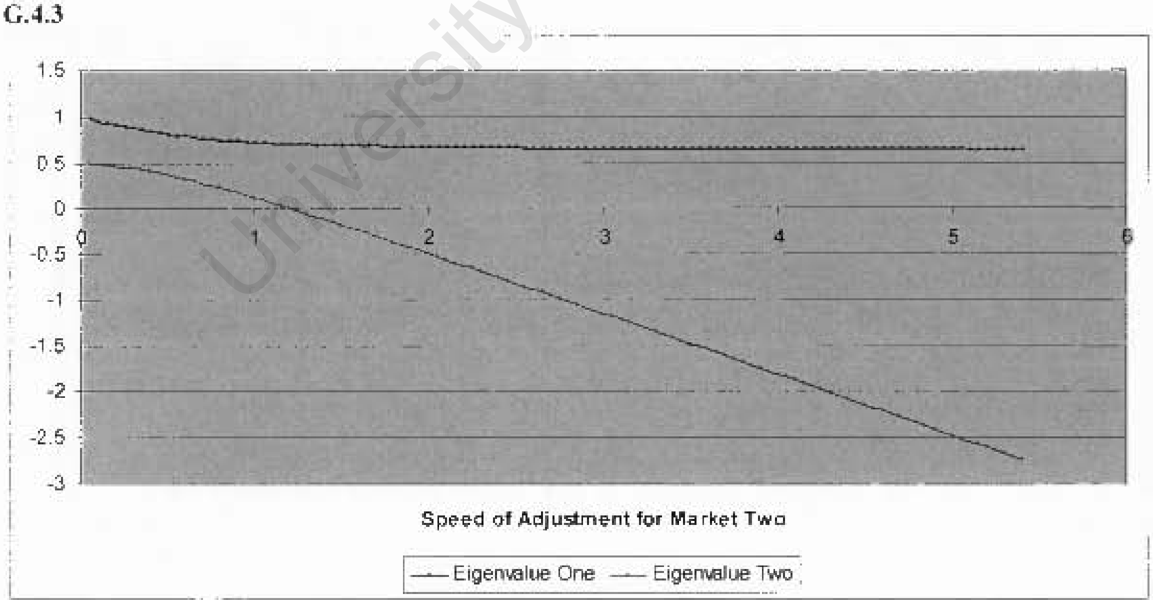
G.4.2.



G.4.2 shows that a five period trajectory is present. That the price space can exhibit such drastic changes in the qualitative nature of the price dynamics is a product of chaotic trajectories being interspersed with periodic orbits in both markets.

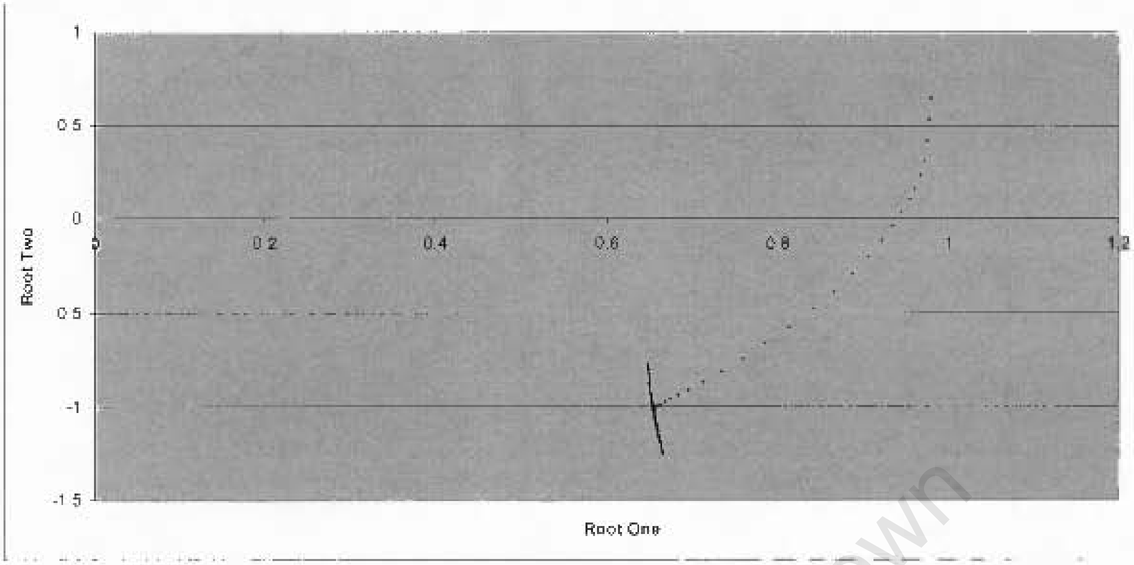
Graph G.4.3 shows the values of both of the eigenvalues as the speed of adjustment for market two is varied whilst the speed of adjustment for market one is kept constant at $\lambda^1 = 0.75$. The distribution of endowments is the same as graphs G.4.1 and G.4.2.

G.4.3



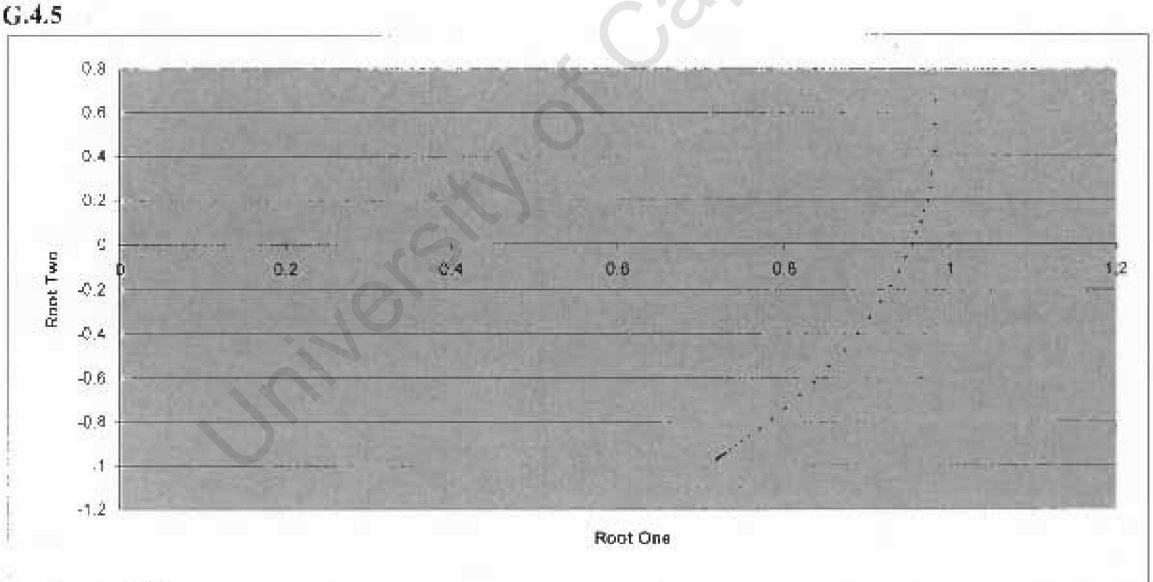
As the speed of adjustment λ^2 increases, one of the eigenvalues moves outside the unit circle. This corresponds to periodic orbits and chaotic trajectories starting to emerge. The other eigenvalue lies inside the unit circle. A saddle point emerges.

G.4.4



Graph G.4.4 shows that for all 20,000 iterates of the tatonnement process, for $\lambda^1 = 0.75$ and $\lambda^2 = 2.79$ for some iterations, the second eigenvalue falls outside the unit circle. Graph G.4.5 shows that for $\lambda^1 = 0.6$ and $\lambda^2 = 2.79$, both roots lie inside the unit circle for all 20 000 iterates.

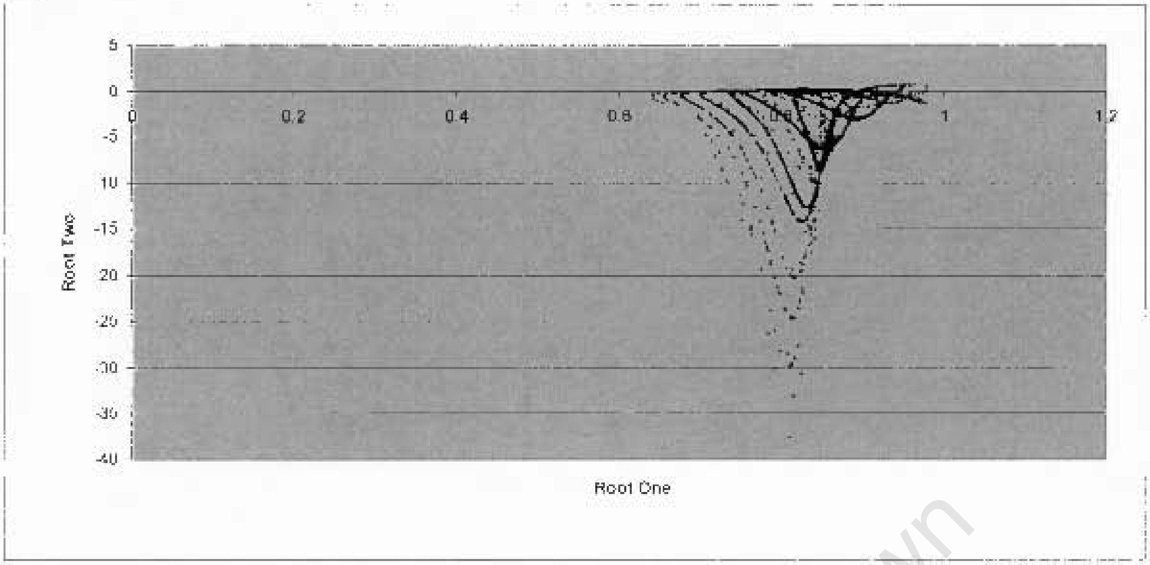
G.4.5



Graph G.4.6 shows the values of the eigenvalues for 20,000 iterates when $\lambda^1 = 0.75$ and $\lambda^2 = 5$. The Jacobian about the fixed point is

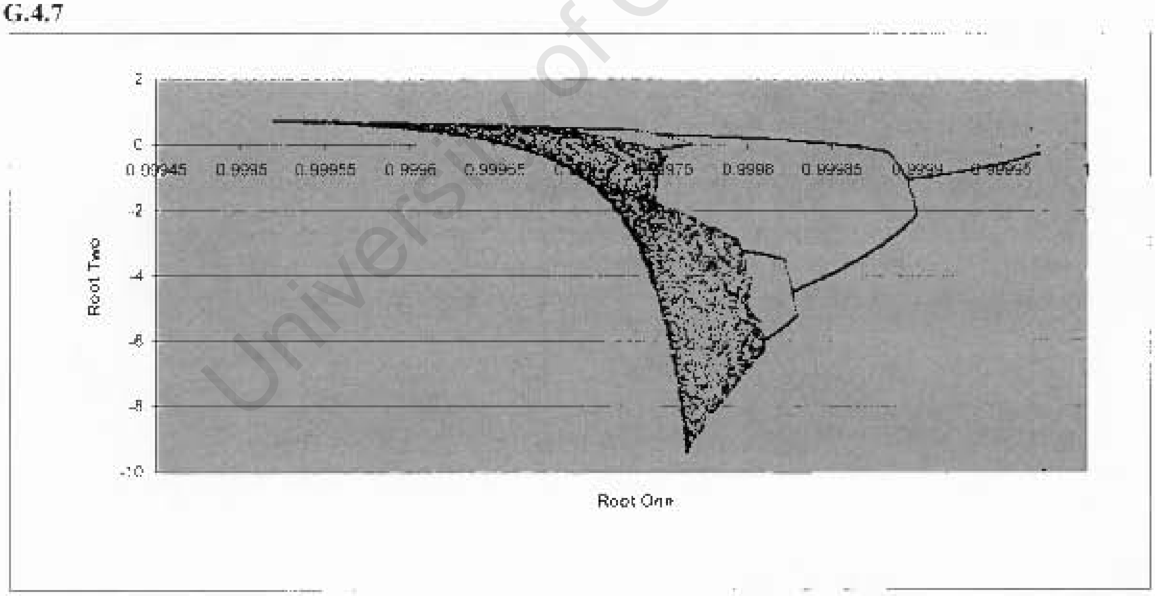
$J = \begin{pmatrix} 0.5 & 0.25 \\ 1.666 & -2.333 \end{pmatrix}$ and the eigenvalues are $r_1 = 0.640128$ and $r_2 = -2.473$

G.4.6

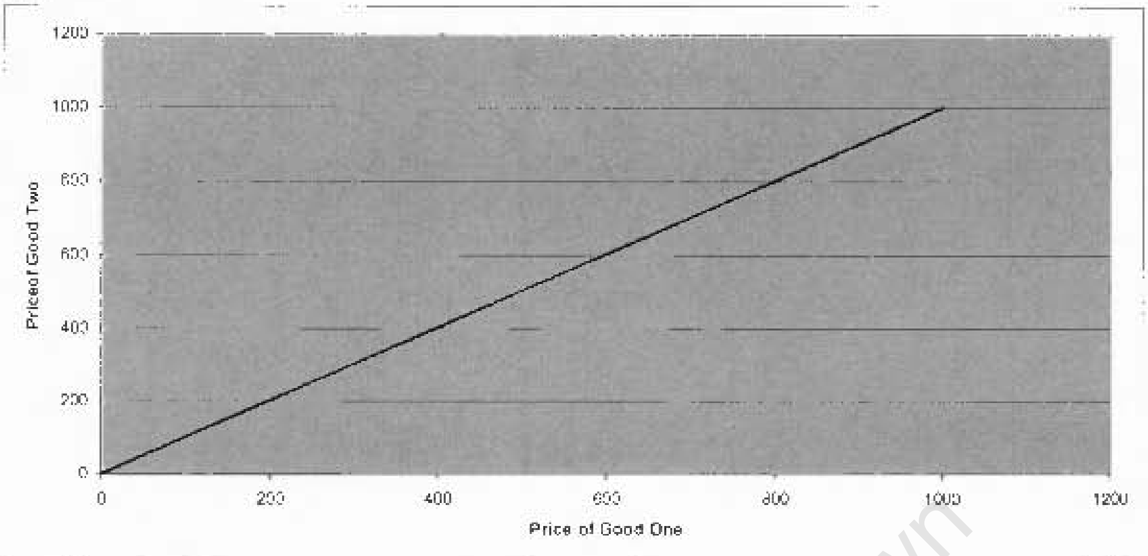


Graph G.4.7 shows the eigenvalues when $\lambda^1 = 0.001$ and $\lambda^2 = 5$. By decreasing the speed of adjustment for market one, stability cannot be induced as, about the fixed point, there is one eigenvalue whose modulus is greater than zero one ($r_1 = 0.9995$ and $r_2 = -2.335$) and this eigenvalue cannot be induced to fall within the unit circle as λ^1 falls.

G.4.7

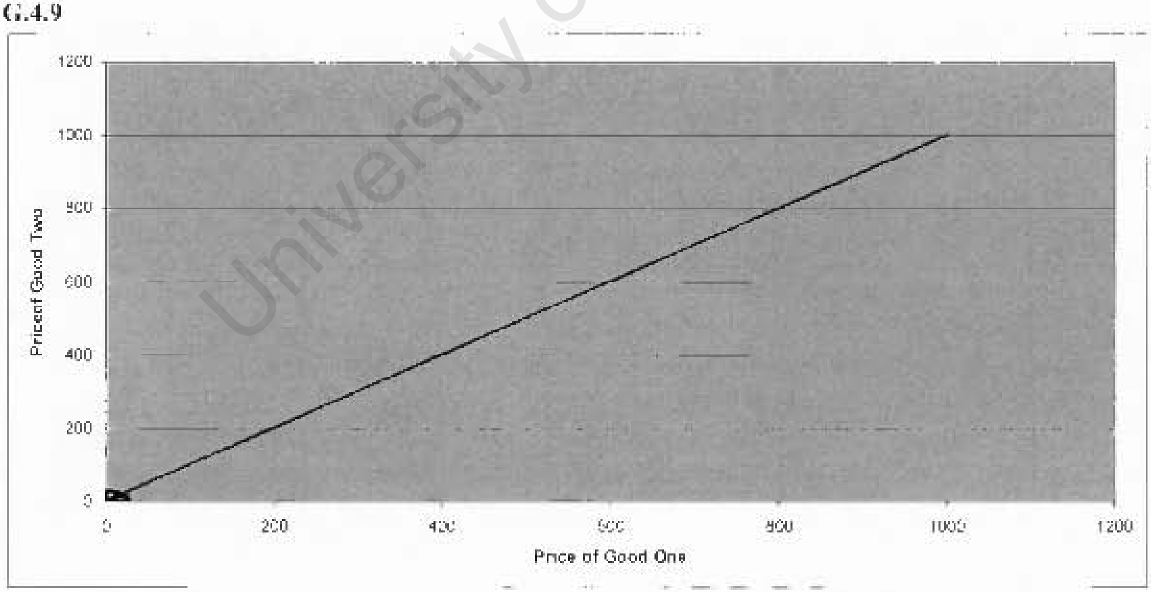


G.4.8

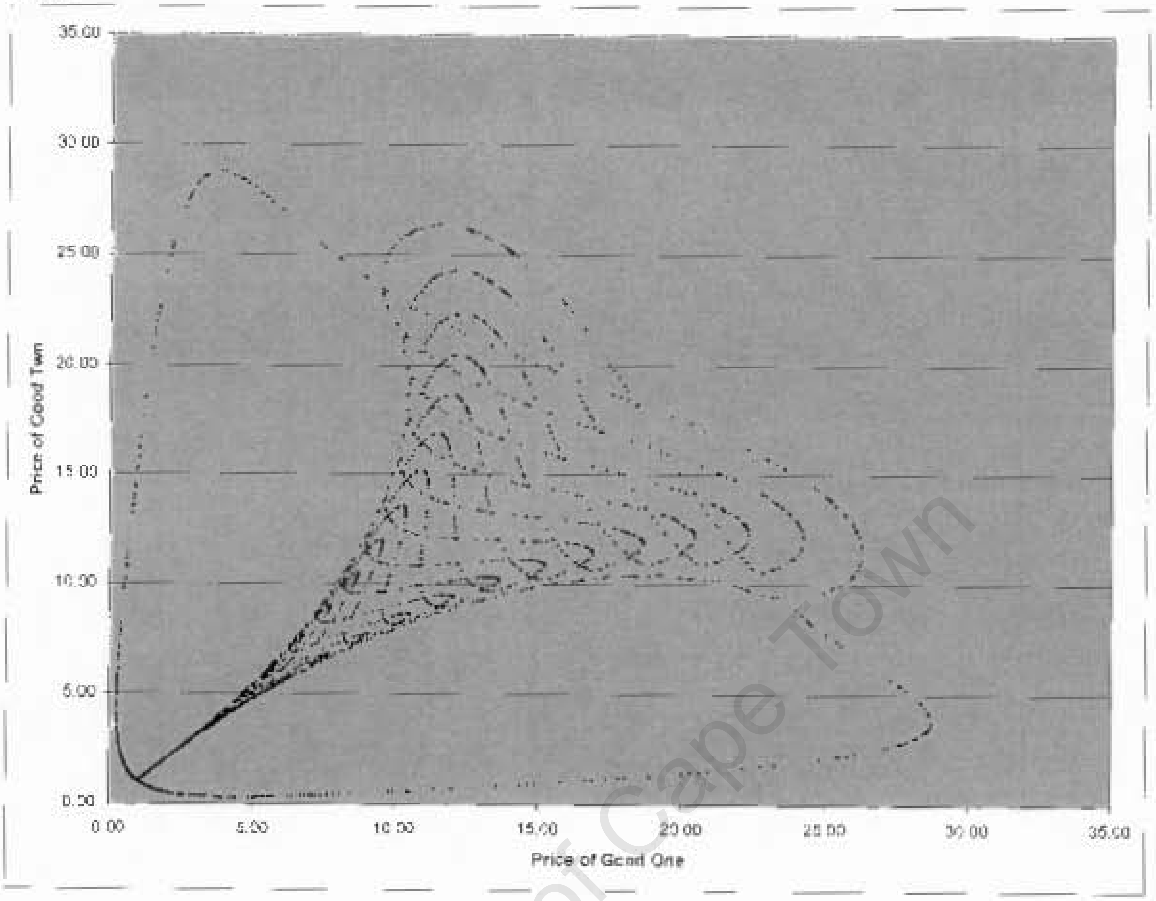


Graph G.4.8 shows the prices of both goods in the case in which the system is symmetric in the speeds of adjustment, holdings of endowments and starting prices. There is convergence to the unique market clearing prices even though one of the eigenvalues lies outside the unit circle. The reason that this occurs is that prices fall on the stable manifold of the system. If asymmetry is introduced into the system then a strange attractor is present. Graph G.4.9 shows prices over all 20,000 iterates whilst graph G.4.10 shows the forward limit set of the price sequences. G.4.10 exhibits trajectories that are chaotic and fall within a limit set which is invariant. The symmetry in the price space is due to the symmetry in the system. The strange attractor is present as prices "fall off" the stable manifold and "fall onto" a strange attractor (the modulus of one of the eigenvalues is greater than zero).

G.4.9



G.4.10



35.00
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Price of Good Two
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Price of Good One

Appendix

Stability

Let a be any number between $f'(p^*)$ and one. Since

$$\lim_{x \to p^*} \frac{|f(p)-f(p^*)|}{|p-p^*|} = |f'(p^*)| \tag{A1}$$

there is a neighbourhood $N_\varepsilon(p^*)$ for some $\varepsilon > 0$

$$\frac{|f(p)-f(p^*)|}{|p-p^*|} < a \tag{A2}$$

for p in $N_\varepsilon(p^*)$. This means that $f(p)$ is closer to p^* than p is, by at least a factor of a (which is less than one); if $p \in N_\varepsilon(p^*)$ then so is $f(p) \in N_\varepsilon(p^*)$ so that if p is within ε of p^* then so is $f(p)$. Repeating this argument over many iterations we may obtain:

$$|f^k(x) - p^*| < a^k |x - p^*| \tag{A3}$$

Thus for $a < 1$, $f^k(p)$ converges exponentially to p^* as $k \to \infty$.

The above considers only an epsilon area around the fixed point p. That is an area equivalent to the radius $(p^* - \varepsilon, p^* + \varepsilon)$. This radius could in theory be very small and at the limit be zero. A radius that is small in dimensions will rule out the consideration of other orbits that have nearby initial conditions. Such a set of initial conditions may be large in number and not necessarily converge to the same fixed point. All the fixed points of the system must thus be evaluated and initial conditions classified in terms of whether they converge to a given a given fixed point or are repelled away from it. Fortunately in the case of GS, there will only ever be one such set of unique points.

To recapitulate:

1. If $|f'(p)| < 1$, then p is an attracting fixed point
2. If $|f'(p)| > 1$, then p is a repelling fixed point.

Periodic Points along a Trajectory

Suppose that the map f has a periodic orbit of period k and that $\{p_1, \ldots, p_k\}$ are the k points along this orbit. If f is a smooth map that is at least once differentiable then, by the chain rule,

$$\begin{aligned}(f^k)'(p_1) &= (f((f^{k-1}))'(p_1) \\ &= f'(f^{k-1}(p_1))(f^{k-1})'(p_1) \\ &= f'((f^{k-1}(p_1))f'((f^{k-2}(p_1)) \ldots . f'(p_1) \\ &= f'(p_k)f'(p_{k-1}) \ldots f'(p_1)\end{aligned} \tag{A4}$$

This states that the derivative of the kth iterate f^k of the map at a point of an orbit that has k periods, is the product of the derivatives of the map at the k points of the orbit.

An orbit of period k is an attracting point (or a sink) if

$$|f'(p_k)f'(p_{k-1})...f'(p_1)|<1 \qquad \text{(A5)}$$

The orbit is a repelling point (or a source) if

$$|f'(p_k)f'(p_{k-1})...f'(p_1)|>1 \qquad \text{(A6)}$$

It must be noted that if there are infinitely many periodic orbits (i.e. $k \to \infty$), the existence of chaos is not guaranteed as chaos is associated with non-periodic orbits as well as sensitive dependence upon initial conditions.

Li-Yorke Theorem (LYT)

Let f be a difference equation that is continuous and for which two numbers $\underline{a}$ and $\underline{b}$ exist such that if $a \le y_t < b$ then $a \le y_{t+1} < b$. If a y_t can be found such that when y_t rises for two successive periods it will fall back to below its initial position in the next period, that is,

$$y_{t+1} = f(y_t) \ and \ y_{t+2} = f^2(y_t) > y_{t+1} \ but \ y_{t+3} = f^3(y_t) \le y_t,$$

then,

(a) For any integer k > 1 there is at least one initial point y_0 between $\underline{a}$ and $\underline{b}$ such that the subsequent time path y_t is characterised by cycles of period k.

(b) There exists an uncountable set, S, of initial points between $\underline{a}$ and $\underline{b}$ such that if initial points $x_0 \ and \ y_0$ both lie in S then,

i. At some time t in the future the difference $(x_t - y_t)$ will come arbitrarily close to zero and the two paths will temporarily move as close to one another as desired.

ii. After some interval of close proximity as in i. the two time paths will diverge again.

iii. No such time path will ever converge asymptotically to any periodic time path and a time path originating in S will not converge to any time path that originates outside S.

$$\liminf_{t\to\infty} |x_t - y_t| = 0 \qquad \text{(A7)}$$

$$\limsup_{t\to\infty} |x_t - y_t| > 0 \qquad \text{(A8)}$$

where x and y have two distinct starting values, one of which is in an epsilon neighbourhood vicinity of the other.

Therefore, if a periodic orbit of period three can be shown to exist, then it follows that there are a large set of initial points that demonstrate sensitive dependence. Chaos is thus present. Chaos in the sense of LYT is thus an orbit that is bounded, non-periodic and that is sensitive to initial conditions. LYT is not a very tractable theorem and is limited in its numerical applicability. Instead, a more utilisable definition of chaos can be applied in which the rate of convergence or divergence of one orbit to another orbit is compared and for which the beginning points of which are in epsilon neighbourhoods of each other. That which is to be determined is thus the degree to which two trajectories with near initial points either converge to each other exponentially fast or depart from each other exponentially fast. The Lyapunov exponent gives a measure of this.

Sharkovski's Theorem

Sharkovski's Theorem gives a scheme for ordering the natural numbers in a manner such that for each natural number n, the existence of a period-n point implies the existence of periodic orbits of all the periods in the ordering higher than n. Sharkovski's ordering is;

$$3 \prec 5 \prec 7 \prec 9 \prec \ldots \prec 2\times 3 \prec 2\times 5 \prec \ldots$$

$$\prec 2^2\times 3 \prec 2^2\times 5 \prec \ldots \prec 2^3\times 3 \prec 2^3\times 5 \ldots$$

$$\prec 2^4\times 3 \prec 2^4\times 5 \ldots \prec 2^3 \prec 2^2 \prec 2 \prec 1$$

On the basis of this ordering, the following theorem can be stated;

Sharkovski's Theorem:

Assume that f is a continuous map on an interval and has a period p orbit. If $p \prec q$, then f has a period q orbit.

This theorem states that if a period three orbit is observed then periodic orbits of *all* periods are observed as well as chaotic orbits[88].

The Lyapunov Number and the Lyapunov Exponent

The Lyapunov exponent provides a single value for an iterative map that summarises the extent to which a trajectory expands or contracts on average.

Consider $f'(p^*) = a$ where p^* is the fixed point of the map. We note that if

$$|f'(p^*)| = |a| < 1 \qquad \text{(A9)}$$

then the orbit of any p in the vicinity of p^* will converge towards p^*. If

$$|f'(p^*)| = |a| > 1 \qquad \text{(A10)}$$

then the orbit of each p near p^* will diverge away from p at a rate that is approximately equal to a per iteration. In other words, the distance between $f^n(p)$ and $f^n(p^*) = p^*$ is increased by an amount approximately equal to $a > 1$ upon each iteration. Thus for a periodic point of period k, the derivative of the kth iterate of the map needs to be evaluated. Using the chain rule (A4), the Lyapunov number is defined as:

Lyapunov Number

Let f be a smooth map of the real line R. The Lyapunov number $L(p)$ of the orbit $\{p_1, p_2, \ldots\}$ is defined as

[88] See [1], pp135 -138 for a comprehensive discussion of Sharkovski's ordering

$$L(p) = \lim_{n\to\infty}\left(|f'(p_1)|\ldots|f'(p_n)|\right)^{\frac{1}{n}} \qquad \text{(A11)}$$

if this limit exists.

The Lyapunov exponent is defined as

$$h(p) = \lim_{n\to\infty}\left(\frac{1}{n}\right)\left(\ln|f'(p_1)|\ldots\ln|f'(p_n)|\right) \qquad \text{(A12)}$$

if this limit exists.

The rationale of this number is as follows. Consider an initial point p_0 and a nearby point (or a perturbation of the initial point); $p_0 + \eta_0$. Let η_n be the value of the separation after n iterates. Consider:

$$|\eta_t| \approx |\eta_0|\, e^{th(p_1)} \qquad \text{(A13)}$$

Now take the logarithm of (A13);

$$\begin{aligned} h(p) &\approx \frac{1}{t}\ln\left|\frac{\eta_t}{\eta_0}\right| \\ &= \frac{1}{t}\ln\left|\frac{f'(p_0+\eta_0) - f'(p_0)}{\eta_0}\right| \\ &= \frac{1}{t}\ln\left|(f')'(p_0)\right| \end{aligned} \qquad \text{(A14)}$$

Now by the chain rule; (A4), (A13) is equivalent to (A12).

(A12) can be actualised by noting that for a trajectory of period k, the Lyapunov exponent can be defined as:

$$h(p) = \frac{\ln|f'(p_1)|\ldots\ln|f'(p_k)|}{k} \qquad \text{(A15)}$$

Or alternatively (A14) can be written as;

$$h(p) = \frac{1}{T}\sum_{i=1}^{T}\ln|f'(p_i)| \qquad \text{(A15')}$$

It is clear from (A12) that $h(p)$ is a limit argument. In order to use (A12) in a form such as (A14) then sufficiently large t needs to be applied. Whilst (A14) suggests that $h(p)$ is calculated for periodic points up to and including the smallest integer k such that the periodic orbit repeats itself, there will be a repetition of periodic k orbit in the calculation of (A14). This is deemed unimportant and in fact desirable as a standard number of iterations for the calculation of (A14), for different parameter values, facilitates the comparison between these different parameter values. Furthermore, the inclusion of repeated periodic orbits in the calculation of (A14) does not alter the results in any meaningful way especially if t is large enough; if a map is convergent for 1000 iterates it will be so for 20,000. Upon the reading of various articles, 20,000 iterations for a map are sufficiently large for the properties of that map to be discernible.

It must be noted that $h(p)$ depends on p_0 (the starting value). For that reason p_0 is taken as the same value for all comparisons for the change in parameters.

Fundamentally, $h(p) > 0$ is a hallmark of chaos as this is the condition for which a map expands as $t \to \infty$. Conversely, $h(p) < 0$ is a signature of the stability of the system as the map contracts as $t \to \infty$.

Lyapunov Stability Theorem (LST)

Consider a one dimensional system of the form $\dot{p} = f(p)$ that has a fixed point at p^*. Let E be an open subset of R that contains an isolated critical point p^*. Suppose that f is continuously differentiable and there exists a continuously differentiable function; $V(p)$ which satisfies the following properties:

- $V(p^*) = 0$ (A16)
- $V(p) > 0 \; if \; p \neq p^*$ (A17)

where $p \in R$. Then

LST 1. $\dot{V}(p) \leq 0$ for all $p \in E$, then p^* is stable
LST 2. $\dot{V}(p) < 0$ for all $p \in E$, then p^* is asymptotically stable
LST 3. $\dot{V}(p) > 0$ for all $p \in E$, then p^* is unstable

Stability in LST 1. is defined as:

A critical point such as p^* of a system such as $\dot{p} = f(p)$ is called stable if given $\varepsilon > 0$, there is a $\delta > 0$ such that for all $t \geq t_0$, then

$$\|p(t) - p^*(t)\| < \varepsilon \quad \text{(A18)}$$

whenever $\|p(t_0) - p^*(t_0)\| < \delta$

Stability in LST 2 is defined as:

A critical point is asymptotically stable if it is stable (LST 1 holds) and there is an $\eta > 0$ such that;

$$\lim_{t \to \infty} \|p(t) - p^*(t)\| = 0 \quad \text{(A19)}$$

whenever $\|p(t_0) - p^*(t_0)\| < \eta$

LST stability states that for a trajectory that starts within a distance δ of the fixed point, over time the Euclidean distance fall to within an ε distance of the fixed point. (A18) does not necessitate that the Euclidean distance shrink any further than this and accordingly the fixed point may not be reached. In other words a trajectory that starts in the vicinity of the critical point will remain close to that point[89]. Obviously, close is defined arbitrarily but arbitrarily small.

LST asymptotically stability requires that if the system starts at a distance η from the fixed point then over time the Euclidean distance converges monotonically to the fixed point of the system.

It must be noted that the LST pertains to continuous time functions. Since discrete time maps are under examination, it is necessary make an equivalent relation between (A16), (A17) and results 1.-3. of LST. We note first that, assuming that asymptotic stability holds then:

$$\frac{\dot{V}(p)}{V(p)} < 0 \tag{A20}$$

Since,

$$\frac{\dot{V}(p)}{V(p)} \approx \ln\left(\frac{V(p_{t+1})}{V(p_t)}\right) = \ln V(p_{t+1}) - \ln V(p_t) \tag{A21}$$

Then if;

$$\begin{aligned} &\ln V(p_{t+1}) - \ln V(p_t) < 0 \Leftrightarrow \ln V(p_{t+1}) < \ln V(p_t) \Leftrightarrow V(p_{t+1}) < V(p_t), \\ &V(p_{t+i}) > 0, \forall i \end{aligned} \tag{A22}$$

If (A20) is observed in a continuous time setting, then a discrete time analogue of LST asymptotic stability is implied which in turn implies that there is a function described in discrete time such that it shares the same properties as the Lyapunov function in continuous-time.

An equivalent relation between LST stability and (A29) can be made by changing the strict equality of (A20) to that of a weak one. Similarly, for a system that is unstable, the inequality in (A22) would be reversed such that:

$$\begin{aligned} &\ln V(p_{t+1}) - \ln V(p_t) > 0 \Leftrightarrow \ln V(p_{t+1}) > \ln V(p_t) \Leftrightarrow V(p_{t+1}) > V(p_t), \\ &V(p_{t+i}) > 0, \forall i \end{aligned} \tag{A23}$$

The rate of change of the Lyapunov function in discrete time is thus assumed to be

$$\frac{dV(p)}{dt} \approx V(p_{t+1}) - V(p_t) \tag{A24}$$

Feigenbaum's Constant

Section two showed period doubling of cycles or bifurcations occurring, before chaos becomes present in a system. Feigenbaum's constant is a measure of the ratios of these points between successive periodic doublings. Formally;

[89] Scarf's counter-example is an example of this type of stability

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.6692016... \qquad \text{(A25)}$$

(A25) is Feigenbaum's universal constant, so called as this is universal for all one dimensional maps, provided such maps satisfy the Schwarzian derivative.

Feigenbaum's constant is useful as, if several period doubling points have been observed, it is possible, by manipulation of (A25) to approximate the next period doubling point. Whilst such an approach may be prone to errors, given that (A25) is a limit argument, it is instructive to observe that the occurrence of period doubling points occur at intervals closer and closer together; that is to say, once period doubling has started to occur, the onset of chaos can occur quite quickly for certain types of maps.

Section Four

For the sake of completeness, the derivation of the two dimensional tatonnement process is presented here.

The tatonnement process for good one is given as:

$$\begin{aligned} p_{t+1}^1 = f^1(p_t^{1,2}) &= p_t^1 + \lambda^1[x_1^1 + x_2^1 + x_3^1 - \omega_1^1 - \omega_2^1 - \omega_3^1] \\ &= p_t^1 + \lambda^1\left[\frac{\alpha_1^1(p_t^1\omega_1^1 + p_t^2\omega_1^2 + \omega_1^3)}{p_t^1} + \frac{\alpha_2^1(p_t^1\omega_2^1 + p_t^2\omega_2^2 + \omega_2^3)}{p_t^1} + \frac{\alpha_3^1(p_t^1\omega_3^1 + p_t^2\omega_3^2 + \omega_3^3)}{p_t^1} - \omega_1^1 - \omega_2^1 - \omega_3^1\right] \end{aligned}$$

And that for good two:

$$\begin{aligned} p_{t+1}^2 = f^2(p_t^{1,2}) &= p_t^2 + \lambda^2[x_1^2 + x_2^2 + x_3^2 - \omega_1^2 - \omega_2^2 - \omega_3^2] \\ &= p_t^2 + \lambda^2\left[\frac{\alpha_1^2(p_t^1\omega_1^1 + p_t^2\omega_1^2 + \omega_1^3)}{p_t^2} + \frac{\alpha_2^2(p_t^1\omega_2^1 + p_t^2\omega_2^2 + \omega_2^3)}{p_t^2} + \frac{\alpha_3^2(p_t^1\omega_3^1 + p_t^2\omega_3^2 + \omega_3^3)}{p_t^2} - \omega_1^2 - \omega_2^2 - \omega_3^2\right] \end{aligned}$$

Rearranging and simplifying for both goods:

$$p_{t+1}^1 = f^1(p_t^{1,2}) = p_t^1 + \lambda^1\left[\left(\frac{p_t^2}{p_t^1}\right)\left(\alpha_1^1\omega_1^2 + \alpha_2^1\omega_2^2 + \alpha_3^1\omega_3^2\right) + \left(\frac{1}{p_t^1}\right)\left(\alpha_1^1\omega_1^3 + \alpha_2^1\omega_2^3 + \alpha_3^1\omega_3^3\right) - \left[\omega_1^1\left(1-\alpha_1^1\right) + \omega_2^1\left(1-\alpha_2^1\right) + \omega_3^1\left(1-\alpha_3^1\right)\right]\right]$$

$$p_{t+1}^2 = f^2(p_t^{1,2}) = p_t^2 + \lambda^2\left[\left(\frac{p_t^1}{p_t^2}\right)\left(\alpha_1^2\omega_1^1 + \alpha_2^2\omega_2^1 + \alpha_3^2\omega_3^1\right) + \left(\frac{1}{p_t^2}\right)\left(\alpha_1^2\omega_1^3 + \alpha_2^2\omega_2^3 + \alpha_3^2\omega_3^3\right) - \left[\omega_1^2\left(1-\alpha_1^1\right) + \omega_2^2\left(1-\alpha_2^2\right) + \omega_3^2\left(1-\alpha_3^2\right)\right]\right]$$

Section Five

It is asserted in section five that if individuals share the same preferences for both goods then the price sequence is invariant to the manner in which endowments are held. This implies that exchange at any point in time does not alter the price sequence and this price sequence is the same in the case of the tatonnement process and the non-tatonnement process. The demonstration of this is as follows;

By $\alpha_1^1 = \alpha_2^1 = \alpha$, the tatonnement process can be written as;

$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\alpha\sum_{i=1}^{n}\frac{\omega_i^2}{p_t} - (1-\alpha)\sum_{i=1}^{n}\omega_i^1\right]\right\} \qquad (5.1)$$

That initial endowments are;

$\varpi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. (5.1) thus takes the form of;

$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\alpha\frac{1}{p_t} - (1-\alpha)\right]\right\} \tag{5.2}$$

Suppose now that there is an exchange such that;

$\varpi' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This exchange is equivalent to the endowment matrix being rotated. Such a rotation encompasses every exchange permissible for a range of prices. (5.2) now has the form of;

$$p_{t+1} = f(p_t) = \max\left\{0,\ p_t + \lambda\left[\alpha\frac{1}{p_t} - (1-\alpha)\right]\right\} \tag{5.3}$$

(5.2) and (5.3) are equivalent. As long as $\sum_{i=1}^{n}\omega_i^2 = \sum_{i=1}^{n}\omega_i^1 = 1,\ \forall t$, *any* exchange will not alter the price sequence. The price sequence therefore remains invariant as exchange is effectuated and there is an equivalence between the price sequence in T and in NT. Clearly this result holds only for the case in which utility is of the Cobb-Douglas form and for which $\alpha_1^1 = \alpha_2^1 = \alpha$. If this were not so, then this result would not be able to be established.

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