THE UNIVERSITY OF CAPE TOWN

A Study of Vortex Lattices and Pulsar Glitches

by

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I dedicate this work to my parents: Phindile and Linda Nkomozake

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Abstract

In this project we study the three fundamental theories that explain the phenomenon of superconductivity: The London theory, the Ginzburg-Landau theory and the BCS theory. We review works by several authors who utilized these theories as the basis for their investigation. In our literature review we study the behavior of single and multivortex states in mesoscopic thin superconducting discs whose dimensions are comparable to the penetration depth λ and the coherence length ξ of a superconductor. We learn about the types of phase transitions that the vortex configurations undergo and the stability of the resulting states. Our aim is to investigate how vortex configurations reorganize after phase transitions and whether their reorganization releases any energy into the system of vortices in the disc. If so, then what is the precise mechanism through which the released energy is transferred into the disc? We aim to answer this question and generalize the results to neutron star interiors in order to explain and predict the behavior of pulsar glitches.

Contents

Introduction

Shortly after the discovery of the neutron by James Chadwick in 1932, Lev Landau was the first to speculate about the possible existence of the astrophysical objects called neutron stars [1]. Neutron stars form one of three end products of stellar evolution, with the less massive white dwarfs and more massive black holes comprising the other two. Neutron stars are composed predominantly of neutrons and have a mass of approximately $1.4M_\odot$, where $M_\odot = 2\times10^{30}\ kg$ is the solar mass. They have a radius of between 10 km and 12 km. Their densities range between $5\rho_0$ and $10\rho_0$, where $\rho_0 = 3 \times 10^{14}\ g\ cm^{-3}$ is the density of nuclear matter. They have magnetic field strengths that are in the range $10^8\ G$ and $10^{12}\ G$ [2]. In 1934, Walter Baade and Fritz Zwicky proposed that supernovae represent the transition of an ordinary star into a body of considerably smaller mass and consisting mainly of neutrons. They called such a body a neutron star. Neutron stars are created by the gravitational collapse of a sufficiently massive star in the absence of radiation pressure provided by thermonuclear reactions to balance the force of gravity. This happens once the star's core is converted into iron by a long cycle of nuclear fusion reactions, after which no more thermonuclear reactions occur spontaneously, since fusion reactions involving iron absorb rather than release energy.

In 1939, Robert Oppenheimer and George Volkoff constructed the first detailed models of neutron star structure using the equation of state of a cold Fermi gas and general relativity. They calculated that stars more massive than $3M_\odot$ would collapse into black holes. In 1966, John Wheeler published an article stating that due to the small radii and low luminosities of neutron stars it would be very difficult, if at all possible, to detect them [3]. In 1967, a Cambridge research group was investigating interplanetary scintillation of compact radio sources. A member of this group by the name of Jocelyn Bell, then a graduate student in astronomy, observed radio pulses arriving with a period of 1.3373011 seconds. She and her advisor Anthony Hewish-who was the head of the group-initially thought they might have detected a signal from an extraterrestrial civilization. It turned out that they had discovered the first pulsar. They announced their discovery in February 1968. Thomas Gold proposed that pulsars might be rotating neutron stars spinning at the pulsation frequency, because no other theoretically known astronomical objects would possess the short and accurate periodicity as those observed. His proposal was subsequently confirmed by observations [3].

It was pointed out by Gold that the rotational energy of a pulsar would decrease due to magnetic dipole radiation, resulting in the pulsar slowing down. The spin down is gradual and largely predictable. However, pulsars have a timing irregularity, where they experience a sudden and spectacular increase in rotation velocity, which is called a pulsar glitch. Up until the present day, there has been hundreds of pulsar glitches that have been detected, and their spin frequency Ω have relative increases that lie between $\Delta\Omega/\Omega \approx 10^{-11}$ and $\Delta\Omega/\Omega \approx 10^{-5}$. After the first glitches were observed in the Vela pulsar, around 1969, it was said that a superfluid component in the interior of the pulsar was responsible and that a weak coupling exists between the normal fluid component in the interior and the electromagnetic emission of the pulsar. It was suggested that such a system stores up the angular momentum of the vortices and then releases it to the crust to cause the glitch [3]. This mechanism is called vortex pinning and was first suggested by Anderson and Itoh. It states that: in order for a superfluid to rotate, it forms a configuration of quantized vortices that carry the circulation of the superfluid. In the neutron star crust the vortices are strongly attracted and pinned to the nuclear lattice and cannot move outward. This behaves like a reservoir for angular momentum. Due to electromagnetic emission, the crust of the neutron star spins down and as a result of that there's a lag that develops between the normal component and the superfluid. This lag leads to a hydrodynamical force that is called the magnus force, which acts on the vortices. When this lag reaches a critical point, the pinning force is no longer able to counteract the magnus force. At this point the vortices unpin and they transfer their angular momentum to the crust which causes the glitch. This theory is generally accepted but is still unsatisfactory. It doesn't give us the location of the angular momentum reservoir in the star and doesn't tell us what percentage of the star it constitutes. It also doesn't tell us what is responsible for the angular momentum transfer and where in the star this coupling occurs. Several mechanisms have been considered in an attempt to answer these questions and all have been unsatisfactory [3].

In chapter 1 we introduce and discuss the concepts of a topological vortex and the Bradlow bound. In chapter 2 we give a brief history of superconductivity and discuss two of its fundamental theories: the London theory and the Ginzburg-Landau theory. We also discuss magnetic vortices. In chapter 3 we discuss the Bardeen-Cooper-Schrieffer(BCS) theory of superconductivity and the formation of magnetic vortex configurations on mesoscopic discs, due to an applied magnetic field. In chapter 4 we discuss the effects that decreasing the applied field has on a vortex configuration. We also derive an "absorption formula" that connects an applied field value to the corresponding vortex location. In chapter 5 we do a numerical study of Bogomolnyi vortices on compact regions that are bounded and unbounded. In chapter 6 we conclude by summarizing the findings of the thesis.

Chapter 1

Topological Vortices

1.1 Introduction

Most of this section is a review that follows reference [4] closely. A topological vortex is defined as a topological soliton in two-dimensional space, with the following features:

1. It has a finite core size.
2. It has finite energy.
3. When you traverse a circle around the vortex in an anti-clockwise direction, its phase changes by $2\pi N$, where N is called its topological charge or winding number.

We work in the context of three-dimensional space-time. The coordinates are denoted by $x = (t, \vec{x})$, where $\vec{x} \in \mathbb{R}^2$. We also identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and we express a point $(x^1, x^2) \in \mathbb{R}^2$ as $z = (x^1 + ix^2) \in \mathbb{C}$. The field theory containing vortices has a scalar field $\phi(x)$ with two real components $(\phi_1(x), \phi_2(x))$, and is invariant under the following transformation [4],

$$\phi_a(x) \to R_{ab}\ \phi_b(x), \tag{1.1}$$

where $R_{ab}(\alpha) \in SO(2)$ are rotations about the origin of $\mathbb{R}^2$ given by,

$$R_{ab}(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \tag{1.2}$$

The field $\phi(x)$ may also be expressed as a complex field,

$$\phi(x) = \phi_1(x) + i\phi_2(x), \tag{1.3}$$

so that the group $SO(2)$ is replaced by the group $U(1)$ of phase rotations given by,

$$\phi(x) \to e^{i\alpha}\, \phi(x). \tag{1.4}$$

A field configuration has a vortex centered at an isolated point $X = (t, \vec{X})$, if ϕ vanishes at X and if when a circle enclosing X is traversed in an anti-clockwise sense, the phase of ϕ changes by $2\pi N$. Where N is a non-zero integer called the multiplicity/topological charge/winding number of the vortex. There's two types of field theories with vortices: global and gauged. Their solutions are called global vortices and gauged vortices respectively. A global field theory has only the complex/real scalar field $\phi(x)$. A gauged field theory contains a complex scalar field $\phi(x)$ coupled to a gauge field $a_\mu = (a_0(x), \vec{a}(x))$. We now consider time-independent (static) fields. The energy of global static fields is given by the Ginzburg-Landau (GL) energy functional,

$$V = \int \left(\frac{1}{2} \nabla\bar{\phi} \cdot \nabla\phi + U(|\phi|^2) \right) d^2x, \tag{1.5}$$

where V is invariant under (1.1). We vary V with respect to $\bar{\phi}$ and we get,

$$\nabla^2\phi - 2U'(|\phi|^2)\phi = 0. \tag{1.6}$$

Then we vary V with respect to ϕ to get,

$$\nabla^2\bar{\phi} - 2U'(|\phi|^2)\bar{\phi} = 0. \tag{1.7}$$

We have that $|\phi|^2 = \bar{\phi}\phi$. An example of a case where U is quadratic is given by,

$$U = \mu + \nu\bar{\phi}\phi + \frac{\lambda}{8}(\bar{\phi}\phi)^2, \tag{1.8}$$

where μ, ν and λ are real constants. The stability of the theory is determined by λ. If $\lambda > 0$ then the theory is stable. We tune μ such that the minimum value of U is zero. So if $\nu < 0$ then we can write,

$$U = \frac{\lambda}{8}(m^2 - \bar{\phi}\phi)^2, \tag{1.9}$$

where $m > 0$. When we expand (1.9) we see that $\mu = \lambda m^4/8$ and $\nu = -\lambda m^2/4$. The vacuum manifold is the locus of lowest energy solutions and is denoted by $\mathcal{V}$. In this case the vacuum manifold is the circle $|\phi| = m$. We now substitute (1.9) into (1.5) to get,

$$V = \frac{1}{2}\int \left(\nabla\bar{\phi}\cdot\nabla\phi + \frac{\lambda}{4}(m^2 - \bar{\phi}\phi)^2\right) d^2x. \tag{1.10}$$

Therefore (1.6) becomes,

$$\nabla^2\phi + \frac{\lambda}{2}(m^2 - \bar{\phi}\phi)\phi = 0, \tag{1.11}$$

which is the complex Ginzburg-Landau equation in two dimensions. The Ginzburg-Landau energy functional for the gauged theory is,

$$V = \frac{1}{2}\int \left(B^2 + \overline{D_i\phi}\ D^i\phi + \frac{\lambda}{4}(m^2 - |\phi|^2)^2\right) d^2x, \tag{1.12}$$

where $D_i\phi = \partial_i\phi + i a_i\phi$ is a gauge-covariant derivative and B is an external magnetic field. The energy functional V is invariant under the following gauge transformation,

$$\phi(\vec{x}) \to e^{i\alpha(\vec{x})}\ \phi(\vec{x}), \tag{1.13}$$

$$a_i(\vec{x}) \to a_i(\vec{x}) + \partial_i\alpha(\vec{x}), \tag{1.14}$$

where the phase rotation $e^{i\alpha(\vec{x})}$ varies in space. We vary the energy functional (1.12) with respect to $\bar{\phi}$, a_1 and a_2 to get the following equations of motion,

$$D_iD^i\phi + \frac{\lambda}{2}(m^2 - |\phi|^2)\phi = 0, \tag{1.15}$$

$$\epsilon_{ij}\partial_j B + \frac{i}{2}\left(\bar{\phi}D_i\phi - \phi\overline{D_i\phi}\right) = 0. \tag{1.16}$$

The vacuum for the gauged GL theory requires that $|\phi| = m$, $D_i\phi = 0$ and $B = 0$ everywhere. The conditions $|\phi| = m$ and $B = 0$ respectively imply that,

$$\phi(\vec{x}) = me^{i\chi(\vec{x})}, \tag{1.17}$$

$$a_i(\vec{x}) = \partial_i\alpha(\vec{x}). \tag{1.18}$$

Then condition $D_i\phi = 0$ requires that,

$$im(\partial_i\chi - \partial_i\alpha)e^{i\chi} = 0, \tag{1.19}$$

which implies that $\partial_i(\chi - \alpha) = 0$ and $\alpha = \chi + constant$. All of the above conditions are met if $\phi = me^{i\chi}$ and $a_i = \partial_i\chi$, which then defines the vacuum. This is gauge equivalent to the choice $\chi = constant$ and hence the vacuum is: $\phi = me^{i\chi}$, $a_i = 0$. In polar coordinates, the gauged GL energy takes the form,

$$V = \frac{1}{2}\int_0^\infty \int_0^{2\pi} \left(\frac{1}{\rho^2}f_{\rho\theta}^2 + \overline{D_\rho\phi}\ D_\rho\phi + \frac{1}{\rho^2}\overline{D_\theta\phi}\ D_\theta\phi + \frac{\lambda}{4}(m^2 - |\phi|^2)^2\right)\rho d\rho d\theta, \tag{1.20}$$

where $f_{\rho\theta} = \partial_\rho a_\theta - \partial_\theta a_\rho = \rho B$, $D_\theta\phi = \partial_\theta\phi - ia_\theta\phi$ and $D_\rho\phi = \partial_\rho\phi - ia_\rho\phi$. For the global theory we have,

$$V = \frac{1}{2}\int_0^\infty \int_0^{2\pi} \left(\partial_\rho\bar{\phi}\partial_\rho\phi + \frac{1}{\rho^2}\partial_\theta\bar{\phi}\partial_\theta\phi + \frac{\lambda}{4}(m^2 - |\phi|^2)^2\right)\rho d\rho d\theta. \tag{1.21}$$

1.2 Global GL-Vortices

Working in polar coordinates, we suppose $\phi(\vec{x})$ is a field configuration in the global GL theory whose energy density goes to zero rapidly as $|\vec{x}| \to \infty$. From (1.21), we get that $|\phi| \to m$ and $\partial_\rho\phi \to 0$ as $\rho \to \infty$. Let us therefore assume that $\lim_{\rho\to\infty}\phi(\rho,\theta)$ exists and,

$$\lim_{\rho\to\infty}\phi(\rho,\theta) = \phi_\infty(\theta) = me^{i\chi_\infty(\theta)}, \tag{1.22}$$

where $\phi_\infty(\theta)$ is the value of ϕ on the circle at infinity S^1_∞. We have that ϕ_∞ satisfies,

$$\phi_\infty : S^1_\infty \to S^1, \tag{1.23}$$

meaning that it's a map from the circle at infinity to the vacuum manifold $\mathcal{V} = S^1$. Since $\phi_\infty(\theta)$ must be single valued , that implies that,

$$\chi_\infty(2\pi) - \chi_\infty(0) = 2\pi N. \tag{1.24}$$

If we consider the contribution of the angular gradient of ϕ to the total energy given by,

$$\frac{1}{2}m^2\int_{\rho_0}^\infty \int_0^{2\pi} \frac{1}{\rho}(\partial_\theta\chi_\infty)^2 d\rho d\theta \tag{1.25}$$

$$= \frac{1}{2}m^2\Big(\int_{\rho_0}^\infty \frac{1}{\rho}d\rho\Big)\Big(\int_0^{2\pi}(\partial_\theta\chi_\infty)^2 d\theta\Big), \tag{1.26}$$

where ρ_0 is the radius of a sufficiently large circle at which one can take $\phi \sim me^{i\chi}$. As we can see above, the radial integral is log divergent unless χ_∞ is a constant or equivalently $N = 0$. From this we conclude that the vacuum manifold plays no significant role for finite energy fields in the global theory.

1.3 Gauged GL-Vortices

In the gauged GL theory, we suppose that $\{\phi(\vec{x}), a_i(\vec{x})\}$ is a finite energy configuration. The finiteness of energy implies that $|\phi| \to m$ as $\vec{x} \to \infty$. From (1.20) we see that $D_\rho\phi \to 0$ as $|\vec{x}| \to \infty$. Also, the finiteness of the integrals of $\frac{1}{\rho^2}\overline{D_\theta\phi}D_\theta\phi$ and $\frac{1}{\rho^2}f_{\rho\theta}^2$ require that, $D_\theta\phi \to 0$ and $f_{\rho\theta} \to 0$ when $|\vec{x}| \to \infty$. In the radial gauge, $f_{\rho\theta} = 0 \implies \partial_\rho a_\theta = 0$. Therefore,

$$\lim_{\rho\to\infty} a_\theta(\rho,\theta) = a_\theta^\infty(\theta). \tag{1.27}$$

The vanishing of $D_\theta\phi$ implies the following,

$$\partial_\theta\chi_\infty - a_\theta^\infty = 0. \tag{1.28}$$

The χ_∞ need not be a constant. Here we also have that,

$$\phi_\infty : S^1_\infty \to S^1, \tag{1.29}$$

with integer winding number,

$$\begin{aligned} N &= \frac{1}{2\pi}\int_0^{2\pi} \partial_\theta\chi_\infty(\theta)\ d\theta && (1.30)\\ &= \frac{1}{2\pi}(\chi_\infty(2\pi) - \chi_\infty(0)) && (1.31)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} B\ d^2x && (1.32)\\ &= \frac{1}{2\pi}\int_0^{2\pi} a_\theta^\infty(\theta)\ d\theta, && (1.33)\end{aligned}$$

where by Stokes' theorem we express (1.32) as a line integral along the circle at infinity (1.33). Therefore equations (1.30)-(1.33) imply that in the gauged GL theory the total magnetic flux Φ_{tot} through $\mathbb{R}^2$ is quantized and is given by ,

$$\Phi_{tot} = 2\pi N. \tag{1.34}$$

1.4 Vortex Solutions

We are interested in vortex solutions that have finite energy, are localized, axi-symmetric, parity symmetric, topologically stable and characterized by some topological charge N. Where $N = 1$ is a vortex, $N > 1$ are multi-vortices and $N < 0$ are anti-vortices. We consider the following vortex ansatz,

$$\phi(\rho, \theta) = \phi(\rho) e^{iN\theta}, \tag{1.35}$$

where a_ρ and $a_\theta(\rho, \theta) = a_\theta(\theta)$ are just functions of theta. We also impose the following boundary conditions,

$$\phi(\infty) = 1,\ \phi(0) = 0, \tag{1.36}$$

$$a_\theta(\infty)\Big|_{S^1_\infty} = N,\ a_\theta(0) = 0, \tag{1.37}$$

$$D_\theta \phi\Big|_\infty = 0, \tag{1.38}$$

where ϕ is single valued and $\vec{a}$ is non-singular at the vortex location. We also have that N is the winding number of ϕ at ∞, is proportional to the total magnetic flux and is a count of the zero (with multiplicity) of ϕ at the origin. We then substitute the above ansatz (1.35) subject to the corresponding boundary conditions (1.36)-(1.38) into the gauged Ginzburg-Landau system (1.15)-(1.16) to get,

$$\frac{d^2\phi}{d\rho^2} + \frac{1}{\rho}\frac{d\phi}{d\rho} - \frac{1}{\rho^2}(N - a_\theta)\phi + \frac{\lambda}{2}(1 - \phi^2)\phi = 0, \tag{1.39}$$

$$\frac{d^2 a_\theta}{d\rho^2} - \frac{1}{\rho}\frac{da_\theta}{d\rho} + (N - a_\theta)\phi^2 = 0. \tag{1.40}$$

The behavior of the solutions of (1.39)-(1.40) is such that, near $\rho = 0$ we have,

$$\phi(\rho) = \rho^N F(\rho^2), \tag{1.41}$$

$$a_\theta(\rho) = \rho^2 G(\rho^2), \tag{1.42}$$

where $F(\rho^2)$ and $G(\rho^2)$ are series in ρ^2 with non-zero constants as leading terms. We also have that near $\rho = \infty$,

$$\phi(\rho) = 1 - \sigma(\rho), \tag{1.43}$$

$$a_\theta(\rho) = N - \psi(\rho), \tag{1.44}$$

and substituting these into (1.39)-(1.40) gives us,

$$\frac{d^2\sigma}{d\rho^2} + \frac{1}{\rho}\frac{d\sigma}{d\rho} - \lambda\sigma = 0, \tag{1.45}$$

$$\frac{d^2\psi}{d\rho^2} - \frac{1}{\rho}\frac{d\psi}{d\rho} - \psi = 0. \tag{1.46}$$

The equations (1.45)-(1.46) are the modified Bessel equations whose solutions are the modified Bessel functions $K_0(\rho)$ and $K_1(\rho)$ respectively. That is,

$$\phi(\rho) = 1 - \frac{A_s}{2\pi}K_0(\sqrt{\lambda}\,\rho), \tag{1.47}$$

$$a_\theta(\rho) = N - \frac{A_m}{2\pi}\rho K_1(\sqrt{\lambda}\,\rho), \tag{1.48}$$

where A_s and A_m are constants to be determined numerically. We have the following conditions for λ:

1. $\lambda < 1 \implies Vortices\ attract.$

2. $\lambda > 1 \implies Vortices\ repel.$

3. $\lambda = 1 \implies Neutral\ vortices.$

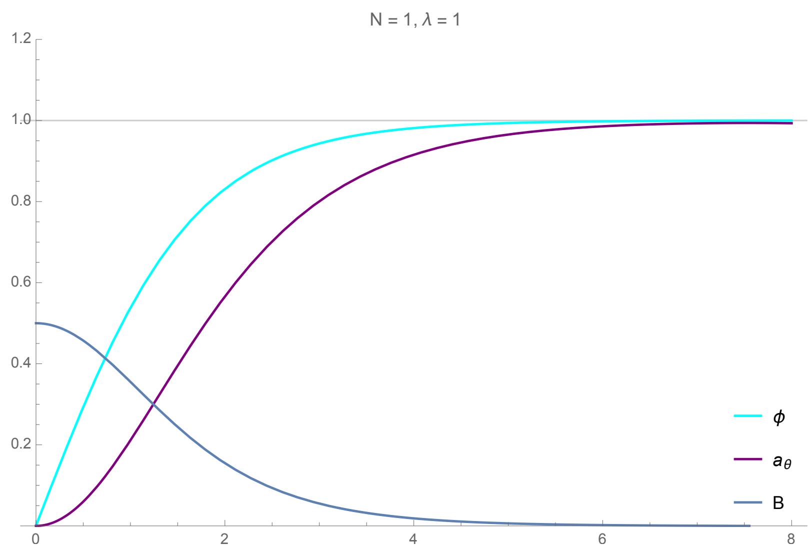

Figure 1.1: Solving equations (1.39) and (1.40) numerically gives the solution plotted in this figure. The magnetic field expressed as a function of ρ, $B = (1/\rho)\partial_\rho a_\theta$, is also plotted in this figure.

1.5 The Bogomolnyi Equations

Now we consider the gauged Ginsburg-Landau energy functional at critical coupling ($\lambda = 1$),

$$E = \frac{1}{2}\int_{\mathbb{R}^2}\left(B^2 + \overline{D_i\phi}D^i\phi + \frac{1}{4}(1-|\phi|^2)^2\ d^2x\right). \tag{1.49}$$

We then complete the square and rewrite (1.49) as,

$$E = \frac{1}{2}\int_{\mathbb{R}^2}\left[\left(B\mp\frac{1}{2}(1-|\phi|^2)\right)^2 + \left(\overline{D_1\phi \pm iD_2\phi}\right)\left(D_1\phi\pm iD_2\phi\right)\pm B - i\left(\partial_1(\bar\phi D_2\phi)-\partial_2(\bar\phi D_1\phi)\right)\right] d^2x. \tag{1.50}$$

The last term of (1.50) is a boundary term that vanishes on S^1_∞, since for finite energy we need $\overline{D\phi} \to 0$ as $\rho \to \infty$. We also know that,

$$\int_{\mathbb{R}^2} B\ d^2x = \Phi = 2\pi N. \tag{1.51}$$

This then implies that ,

$$E = \frac{1}{2}\int_{\mathbb{R}^2} d^2x\left[\left(B \mp \frac{1}{2}(1-|\phi|^2)\right)^2 + \left|D_1\phi \pm iD_2\phi\right|^2\right] \pm \pi N. \tag{1.52}$$

Since the term in the square bracket of (1.52) is a sum of two positive pieces, we get the following bound on the vortex energy,

$$E \geq \pi|N|, \tag{1.53}$$

called the Bogomolnyi bound. When $E = \pi|N|$ we say that "the bound saturates" and this occurs when,

$$D_1\phi \pm iD_2\phi = 0, \tag{1.54}$$

$$B \mp \frac{1}{2}(1-|\phi|^2) = 0. \tag{1.55}$$

These equations (1.54)-(1.55) are called the Bogomolnyi equations. In the Bogomolnyi equations, the plus sign corresponds to a vortex and the minus sign corresponds to an anti-vortex. So solutions of the Bogomolnyi equations are critically coupled vortices or anti-vortices. When working on a compact Riemann surface X with a metric that takes the following form,

$$ds^2 = \Omega(x^1, x^2)((dx^1)^2 + (dx^2)^2), \tag{1.56}$$

where $\Omega(x^1, x^2)$ is a conformal factor, the second Bogomolnyi equation (1.55) takes the following form,

$$B - \frac{\Omega}{2}(1 - |\phi|^2) = 0. \tag{1.57}$$

Bradlow [5] noticed that when you integrate (1.57) over X you get the following results,

$$2\int_X B\ d^2x + \int_X |\phi|^2\ \Omega\ d^2x = \int_X \Omega\ d^2x. \tag{1.58}$$

Which implies that,

$$4\pi N + \int_X |\phi|^2\ \Omega\ d^2x = A. \tag{1.59}$$

Since $\Omega\ d^2x$ is the area element of X, the second term on the left-hand side of equation (1.59) is the total area of X. This then implies that,

$$A \geq 4\pi N, \tag{1.60}$$

which is called the Bradlow inequality and $4\pi N$ the Bradlow bound. So this means that the number of vortices on a surface of area A cannot exceed $\frac{A}{4\pi}$.

1.6 The Bradlow Bound in AdS_3

In this section we present new material. We observe the Bradlow bound in three-dimensional anti-de Sitter space AdS_3. We start with the general Lorentz invariant Abelian Higgs action [6],

$$S = \int d^3x \left[\frac{1}{4}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} + \frac{1}{2}g^{\mu\nu}\overline{D_\mu\phi}D_\nu\phi + \frac{\lambda}{8}(1 - |\phi|^2)^2\right], \tag{1.61}$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and here we use a convention in which the volume element d^3x includes the determinant of the metric tensor. We then imbed (1.61) into AdS_3, which has the following metric,

$$ds^2 = -\left(\frac{r^2}{l^2} + 1\right)dt^2 + \left(\frac{r^2}{l^2} + 1\right)^{-1} dr^2 + r^2 d\theta^2, \tag{1.62}$$

where $l^2 = -\frac{1}{\Lambda}$, Λ is the cosmological constant. Due to the anti-symmetry of $F_{\mu\nu}$ from

(1.61) we get,

$$S = \int d^3x \Bigg(\frac{1}{2}\Bigg[g^{rr}g^{tt}F_{tr}^2 + g^{rr}g^{\theta\theta}F_{r\theta}^2 + g^{tt}g^{\theta\theta}F_{t\theta}^2 \Bigg] + \frac{1}{2}\Bigg[g^{tt}\overline{D_t\phi}D_t\phi \ + \ g^{rr}\overline{D_r\phi}D_r\phi \ + \ g^{\theta\theta}\overline{D_\theta\phi}D_\theta\phi \Bigg] + \frac{\lambda}{8}(1-|\phi|^2)^2 \Bigg),$$

and we only consider static solutions so this reduces to,

$$S = \int d^2x \Bigg(\frac{1}{2}\Bigg[g^{rr}g^{\theta\theta}F_{r\theta}^2 \Bigg] + \frac{1}{2}\Bigg[\ g^{rr}\overline{D_r\phi}D_r\phi \ + \ g^{\theta\theta}\overline{D_\theta\phi}D_\theta\phi \Bigg] + \frac{\lambda}{8}(1-|\phi|^2)^2 \Bigg). \tag{1.63}$$

So from (1.62) we get the following metric tensor,

$$[g] = \begin{bmatrix} \left(\frac{r^2}{l^2}+1\right)^{-1} & 0 \\ 0 & r^2 \end{bmatrix} \implies [g]^{-1} = \begin{bmatrix} \left(\frac{r^2}{l^2}+1\right) & 0 \\ 0 & r^{-2} \end{bmatrix}. \tag{1.64}$$

So the area element d^2x is given by [7],

$$d^2x = \left(\frac{r^2}{l^2}+1\right)^{-\frac{1}{2}} r dr d\theta. \tag{1.65}$$

This then implies the following,

$$S = \int r dr d\theta \ \Omega^{-\frac{1}{2}} \Bigg(\frac{1}{2}\Omega r^{-2}F_{r\theta}^2 + \frac{1}{2}\Bigg[\Omega\overline{D_r\phi}D_r\phi + r^{-2}\overline{D_\theta\phi}D_\theta\phi \Bigg] + \frac{\lambda}{8}(1-|\phi|^2)^2 \Bigg), \tag{1.66}$$

where we let $\Omega = \left(\frac{r^2}{l^2}+1\right)$. At critical coupling $\lambda = 1$ we have that,

$$r^{-2}\Omega^{\frac{1}{2}}B^2 + \frac{\Omega^{-\frac{1}{2}}}{4}(1-|\phi|^2)^2 = \left(r^{-1}\Omega^{\frac{1}{4}}B \pm \frac{\Omega^{-\frac{1}{4}}}{2} \right)^2 \mp r^{-1}B(1-|\phi|^2), \tag{1.67}$$

where $B = F_{r\theta}$. We also have,

$$\Omega^{\frac{1}{2}}\overline{D_r\phi}D_r\phi + r^{-2}\Omega^{-\frac{1}{2}}\overline{D_\theta\phi}D_\theta\phi = \left| \Omega^{\frac{1}{4}}D_r\phi \pm ir^{-1}\Omega^{-\frac{1}{4}}D_\theta\phi \right|^2 \mp \frac{i}{r}\left(\overline{D_r\phi}D_\theta\phi - \overline{D_\theta\phi}D_r\phi \right). \tag{1.68}$$

To obtain the Bogomolnyi equations the following condition must be met,

$$\int r dr d\theta \Bigg[r^{-1}B(1-|\phi|^2) + \frac{i}{r}\left(\overline{D_r\phi}D_\theta\phi - \overline{D_\theta\phi}D_r\phi \right) \Bigg] = constant. \tag{1.69}$$

The Bogomolnyi equations are then given by,

$$r^{-1}\Omega^{\frac{1}{4}}B - \frac{\Omega^{-\frac{1}{4}}}{2}(1-|\phi|^2) = 0, \tag{1.70}$$

$$\Omega^{\frac{1}{4}}D_r\phi - ir^{-1}\Omega^{-\frac{1}{4}}D_\theta\phi = 0. \tag{1.71}$$

We integrate equation (1.70) over a compact subset M of the two-dimensional time-slice of AdS_3 from (1.62) and we get,

$$2\int_M rdrd\theta\ r^{-1}B + \int_M rdrd\theta\left(\frac{r^2}{l^2}+1\right)^{-\frac{1}{2}}|\phi|^2 = \int_M rdrd\theta\left(\frac{r^2}{l^2}+1\right)^{-\frac{1}{2}}. \qquad (1.72)$$

Since AdS_3 is a Riemannian manifold with negative curvature, M is a compact Riemann surface. The right-hand side of (1.72) is the total area of M. The first term on the left-hand side is double the magnetic flux through M and since M is a compact Riemann surface, this flux is given by $2\pi N$, where N is a positive integer and in this case means the number of vortices on the surface M. The second term on the left-hand side is positive so we get the Bradlow inequality,

$$A_M \geq 4\pi N. \qquad (1.73)$$

At any time-slice, the two-dimensional background space of the metric (1.62) is the same because it doesn't depend on time. This implies that the Bradlow bound is the same for any time-slice of AdS_3. So if we take the same action (1.61) on a different representation of AdS_3,

$$ds^2 = -\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2}\left(dr^2 + r^2 d\theta^2\right), \qquad (1.74)$$

where the two-dimensional time-slice of (1.74) is the hyperbolic plane. We find that the Bradlow inequality for this case is also given by (1.73), where in this case M is a compact subset of the hyperbolic plane. Therefore this brings us to the following conclusions:

1. For the action (1.61), any representation of AdS_3 we use will give the same vortices with the same Bradlow bound $4\pi N$.

2. For any representation of AdS_3, the two-dimensional background space is the same at any time-slice. Therefore the Bradlow bound for full AdS_3 is $4\pi N$.

3. Since AdS_3 is infinite, the interpretation we give to the Bradlow inequality in this case is that: The area of a single vortex in AdS_3 is at least 4π, or in AdS_3, within a region of area 4π, you can only fit one vortex or none.

Chapter 2

Introduction To Superconductivity

2.1 Background

Superconductivity was discovered in 1911 by the Dutch physicist H. Kamerlingh Onnes [8]. During his investigations on the conductivity of metals at low temperature he found that the electrical resistance of a mercury sample dropped to zero at $4.2K$. This temperature at which the transition took place was called the critical temperature T_c. The phenomenon of superconductivity exists in other materials as well. The critical temperature is dependent on the material. Some of these materials and their critical temperatures are shown in the table below [9].

ELEMENT	SYMBOL	$T_c(K)$
Aluminium	Al	1.75
Beryllium	Be	0.03
Cadmium	Cd	0.52
Hafnium	Hf	0.13
Mercury	Hg	4.15

When a superconductor is placed in an external magnetic field B_0 and is cooled below its transition temperature, the magnetic field is ejected. This phenomenon is known as the Meissner effect, shown in Figure 2.1 below.

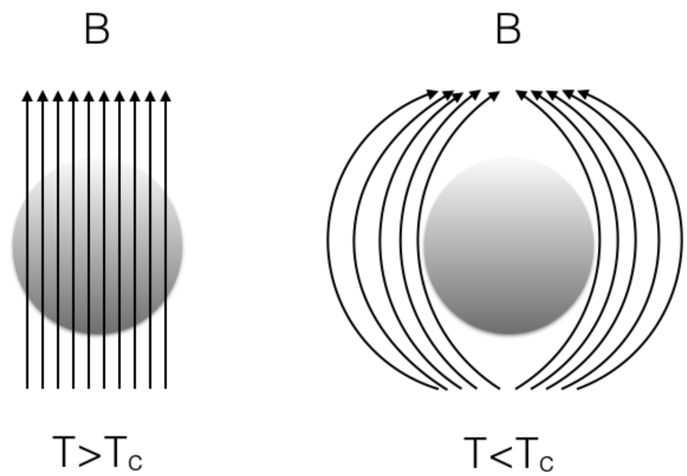

Figure 2.1: The Meissner effect.

The Meissner effect does not cause the external magnetic field to be completely ejected. Instead the field penetrates the superconductor only to a very small distance characterized by a parameter λ called the London penetration depth, decaying exponentially to zero within the bulk of the material. The expression that describes this is given by,

$$B(z) = B_0 \exp\left(-\frac{z}{\lambda}\right). \tag{2.1}$$

The graphical behavior of (2.1) is given in Figure 2.2 . For most superconductors the London penetration depth is on the order of $100nm$. The London penetration depth is also dependent on temperature and the formula for its temperature dependence is given by [10],

$$\left[\frac{\lambda(0)}{\lambda(T)}\right]^2 = 1 - \left(\frac{T}{T_c}\right)^4. \tag{2.2}$$

The Meissner effect is a defining characteristic of superconductivity. The total magnetic flux $\vec{B}$(in SI units) is given by,

$$\vec{B} = \mu_0\big(\vec{B}_0 + \vec{M}\big), \tag{2.3}$$

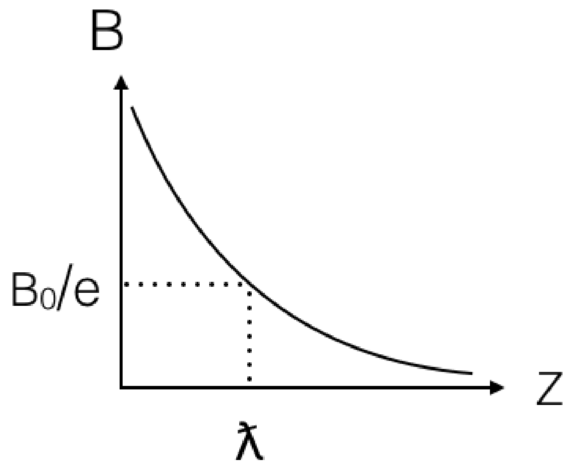

Figure 2.2: The penetration depth.

where $\vec{B}_0$ is the applied magnetic field and $\vec{M}$ is the magnetization. For superconductors we have that $\vec{B} = 0$, which implies that,

$$\vec{M} = -\vec{B}_0. \tag{2.4}$$

Diamagnets are classified based on the expression,

$$\vec{M} = \chi \vec{B}_0, \tag{2.5}$$

where χ is called the volume magnetic susceptibility. If $\chi < 0$ then the magnetization is in the opposite direction to the applied magnetic field, which makes the material a diamagnet. For superconductors we have that $\chi = -1$ and this means that superconductors are perfect diamagnets. There are two types of superconductors: Type I and Type II, as shown in Figure 2.3 and 2.4.

2.2 London Theory of Superconductivity

In 1935, brothers Fritz and Heinz London developed the London equations [8]. First they postulated that the current density $\vec{J}_s$ in the superconductor is directly proportional to the vector potential $\vec{A}$, which is related to the magnetic field through,

$$\vec{B} = \nabla \times \vec{A}. \tag{2.6}$$

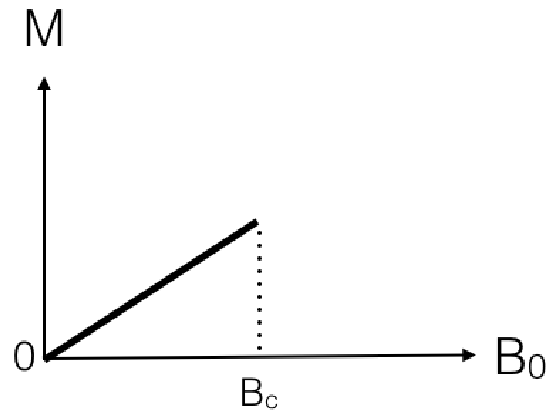

Figure 2.3: For Type I, the magnetization stops abruptly at the critical magnetic field B_c. The critical magnetic field also depends on temperature.

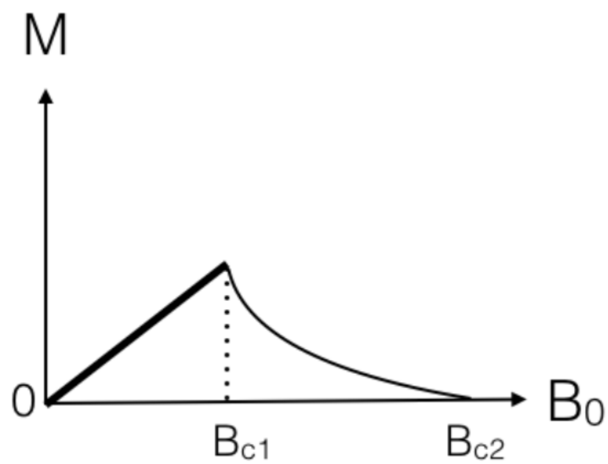

Figure 2.4: For Type II, the magnetization decreases gradually between the lower critical field B_{c1} and the upper critical field B_{c2}, allowing magnetic flux lines to penetrate. These flux lines are called magnetic vortices.

The current density in the superconductor is given by,

$$\vec{J_s} = n_s q_s \vec{v}_s, \tag{2.7}$$

where n_s is the number of charge carriers, q_s is the charge and $\vec{v}_s$ is the average velocity of the charge carriers. By Newton's second law we have,

$$m_s \frac{d\vec{v}_s}{dt} = q_s \vec{E}, \tag{2.8}$$

where m_s is the mass of the charge carriers. We differentiate (2.7) and substitute (2.8) to get,

$$\frac{d\vec{J}_s}{dt} = \frac{n_s q_s^2}{m_s} \vec{E}, \tag{2.9}$$

which is the first London equation. We then apply the curl on both sides of (2.9), using $\nabla \times \vec{E} = -\dot{\vec{B}}$, in order to get the second London equation,

$$\nabla \times \vec{J}_s = -\frac{n_s q_s^2}{m_s} \vec{B}. \tag{2.10}$$

By using (2.6) we combine (2.9) and (2.10) to get the single London equation,

$$\vec{J}_s = -\frac{n_s q_s^2}{m_s} \vec{A}. \tag{2.11}$$

From the London equations we get a precise expression for the London penetration depth [8],

$$\lambda = \left(\frac{m_s}{\mu_0 n_s q_s^2} \right)^{\frac{1}{2}}, \tag{2.12}$$

where μ_0 is the permeability of free space. Therefore from the London equations we can infer that the superconductor is a quantum state acting over macroscopic distances and can be described by the order parameter $|\psi|^2 = n_s$. We have that ψ denotes the wave function of the superconducting state.

2.3 Ginzburg-Landau Theory of Superconductivity

The London theory provided us with the order parameter in the form of $|\psi|^2 = n_s$, which is used to measure changes in the density of superconducting electrons following a phase transition. In order to derive the Ginzburg-Landau(GL) equations we first expand the free energy density in powers of the order parameter. Secondly, we minimize the free energy density with respect to variations in the order parameter, and then we expand the free energy density in the following way,

$$F_s = F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\big(-i\hbar\vec{\nabla} - q_s\vec{A}\big)^2|\psi|^2}{2m_s} - \int_0^{B_c} \vec{M}\cdot d\vec{B}_0, \tag{2.13}$$

where F_s is the free energy density of the superconducting state and F_n is the free energy density of the normal state. The second and third terms represent the expansion of the free energy density in the order parameter $|\psi|^2$ with constant coefficients α and β. The fourth term includes the kinetic momentum and the field momentum that contribute to the increase in free energy. The last term represents the increase in free energy due to the expulsion of magnetic flux. The next step is to minimize the free energy density with respect to variations in the order parameter. We ignore the last two terms in the above expansion and get,

$$F_s - F_n = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4. \tag{2.14}$$

We then differentiate (2.14) with respect to ψ^* (or ψ) where $|\psi|^2 = \psi^*\psi$, and equate the result to zero to get following,

$$\psi(\alpha + \beta|\psi|^2) = 0. \tag{2.15}$$

We then solve (2.15) to get $|\psi_0|^2 = -\alpha/\beta = |\alpha|/\beta$, where we choose $\beta > 0$. We then substitute $|\psi_0|^2$ into (2.14) to get the minimum,

$$(F_s - F_n)_{min} = -\frac{\alpha^2}{2\beta}. \tag{2.16}$$

We have that the following holds,

$$F_s(B_c) - F_s(0) = -\int_0^{B_c} \vec{M}\cdot d\vec{B}_0. \tag{2.17}$$

In Gaussian units (2.3) may be expressed as,

$$\vec{B} = \vec{B_0} + \mu_0\vec{M}. \tag{2.18}$$

Therefore since $\vec{B} = 0$ for a superconductor, we have that $\vec{M} = (-1/\mu_0)\vec{B_0}$, which we substitute into (2.17) to get,

$$F_s(B_c) - F_s(0) = \frac{B_c^2}{2\mu_0}. \tag{2.19}$$

In the normal conducting state we have that,

$$F_n(B_c) = F_n(0), \tag{2.20}$$

which means that the normal conducting state is not dependent on the applied magnetic field. We also have that,

$$F_s(B_c) = F_n(B_c). \tag{2.21}$$

Therefore,

$$F_s(B_c) = F_n(B_c) = F_n(0). \tag{2.22}$$

So substituting (2.22) into (2.19) we get,

$$F_n(0) - F_s(0) = \frac{B_c^2}{2\mu_0}. \tag{2.23}$$

We then equate (2.23) with (2.16) to get an expression for the critical applied field for a Type I superconductor in terms of α and β given by,

$$B_c = \left(\frac{\mu_0 \alpha^2}{\beta}\right)^{\frac{1}{2}}. \tag{2.24}$$

We can also express the London penetration depth in terms of α and β by evaluating it at $|\psi_0|^2$,

$$\lambda = \left(\frac{m_s \beta}{\mu_0 q_s^2 |\alpha|}\right)^{\frac{1}{2}}. \tag{2.25}$$

We now look at what happens at the surface of the superconductor. We first express the fourth term of (2.13) in one dimension with zero magnetic field to get,

$$\frac{\big(-i\hbar\vec{\nabla} - q_s\vec{A}\big)^2 |\psi|^2}{2m_s} = -\frac{\hbar^2}{2m_s}\frac{d^2|\psi|^2}{dx^2}. \tag{2.26}$$

We then add (2.26) to (2.14) to get,

$$F_s - F_n = -\frac{\hbar^2}{2m_s}\frac{d^2|\psi|^2}{dx^2} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4. \tag{2.27}$$

We minimize (2.27) by first differentiating it with respect to ψ^* (or ψ) and then setting it to zero to get,

$$-\frac{\hbar^2}{2m_s}\frac{d^2\psi}{dx^2} + \alpha\psi + \beta\psi|\psi|^2 = 0. \tag{2.28}$$

We then let $\psi = \psi_0 f$, where f depends on x and of course $|\psi_0|^2 = -\alpha/\beta$. Substituting these into (2.28) gives us the following equation,

$$-\frac{\hbar^2}{2m_s}\frac{d^2 f}{dx^2} + \alpha f - \alpha f^3 = 0. \tag{2.29}$$

Now within the superconductor we have that $f = 1 \implies \psi = \psi_0$. On the surface we can assume that $f = 1 + g$, where g is some small deviation from 1. We then substitute this into (2.29) to get,

$$\frac{d^2 g}{dx^2} + 2\left(\frac{2m\alpha}{\hbar^2}\right)g = 0. \tag{2.30}$$

We let $2m\alpha/\hbar^2 = 1/\xi^2$ in order to get the equation,

$$\frac{d^2 g}{dx^2} + \frac{2}{\xi}g = 0, \tag{2.31}$$

whose solution is,

$$g \approx \exp\left(-\frac{x\sqrt{2}}{\xi}\right). \tag{2.32}$$

Therefore the Ginzburg-Landau theory also provides us with a length scale in the form of ξ which is called the Ginzburg-Landau coherence length. The coherence length measures the "stiffness" of the superconductor.

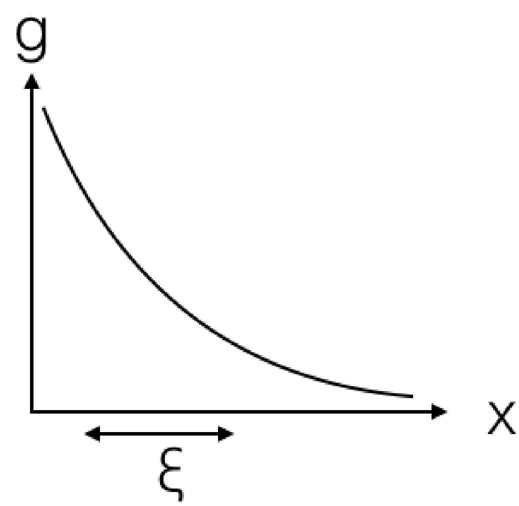

Figure 2.5: The behavior of g.

From the Ginzburg-Landau theory we also get a criterion from which we can determine whether a superconductor is Type I or Type II. First we let $\lambda/\xi = \kappa$ then we have that $\kappa < \frac{1}{\sqrt{2}}$ implies that the superconductor is of Type I and $\kappa > \frac{1}{\sqrt{2}}$ implies that the superconductor is of Type II.

Now we consider a situation where we ignore the third and the fifth term of (2.13) and get,

$$F_s - F_n = \alpha|\psi|^2 + \frac{\big(-i\hbar\vec{\nabla} - q_s\vec{A}\big)^2|\psi|^2}{2m_s}. \tag{2.33}$$

This tells us that there is magnetic flux penetrating the superconductor. We consider the surface of the superconductor where,

$$\frac{\big(-i\hbar\vec{\nabla} - q_s\vec{A}\big)^2}{2m_s}\ \psi + \alpha\psi = 0. \tag{2.34}$$

If we assume that $\vec{B} = \big(0, 0, B\big)$, the relation $\vec{B} = \vec{\nabla} \times \vec{A}$ implies that $\vec{A} = \big(0, Bx, 0\big)$. Working in cartesian coordinates, we expand (2.34) to get,

$$-\frac{\hbar^2}{2m_s}\bigg(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\bigg)\psi + \frac{i\hbar q_s}{m_s}\big(\vec{A}\cdot\vec{\nabla}\psi\big) + \frac{q_s^2A^2}{2m_s}\psi + \alpha\psi = 0. \tag{2.35}$$

Taking a seperable ansatz,

$$\psi = \psi(x, y, z) = \exp(ik_z z)\ \exp(ik_y y)\ \widetilde{\psi}(x), \tag{2.36}$$

where of course $\vec{k} = (k_x, k_y, k_z)$, we substitute (2.36) along with all our previous assumptions into (2.35) to get the following,

$$-\frac{\hbar^2}{2m_s}\frac{d^2\widetilde{\psi}(x)}{dx^2} + \bigg(\frac{\hbar^2k_y^2}{2m_s} - \frac{\hbar q_s k_y Bx}{m_s} + \frac{q_s^2B^2x^2}{2m_s}\bigg)\widetilde{\psi}(x) + \alpha\widetilde{\psi}(x) + \frac{\hbar^2k_z^2\widetilde{\psi}(x)}{2m_s} = 0. \tag{2.37}$$

We now let $\omega = q_sB/m_s$ and $x_0 = \hbar k_y/q_sB$, substitute into (2.37) and factorize to get the following,

$$\frac{\hbar^2}{2m_s}\frac{d^2\widetilde{\psi}(x)}{dx^2} - \frac{1}{2}m_s\omega^2\big(x - x_0\big)^2\widetilde{\psi}(x) = \epsilon'\widetilde{\psi}(x), \tag{2.38}$$

where $\epsilon' = \alpha + \hbar^2k_z^2/2m_s$. As we can see (2.38) takes the form of the Schrodinger equation for a one-dimensional harmonic oscillator. Therefore from the eigenvalues of (2.38) we derive a formula for the upper critical field of a Type II superconductor, which is given by, $B_{c2} = \phi_0/2\pi\xi^2$. The eigenfunctions of (2.38) take the form,

$$\widetilde{\psi} \approx \exp\left(\frac{i\,(-i\hbar\vec{\nabla} - q_s\vec{A})\cdot\vec{r}}{\hbar}\right), \tag{2.39}$$

where $\vec{r} = (x, y, z)$. From (2.39) we can see that $q_s\vec{A}/\hbar$ represents a phase shift. We then integrate this phase shift around a closed loop C within the superconductor and this gives us,

$$\frac{q_s}{\hbar}\oint_C \vec{A}\cdot d\vec{l} = \frac{q_s}{\hbar}\phi_0, \tag{2.40}$$

where ϕ_0 is the magnetic flux through the surface that is bounded by C. From (2.40) we get $\phi_0 = h/q_s$, which is called the flux quantum. The eigenfunctions of (2.38) tell us that between the lower and upper critical fields the applied magnetic field penetrates the superconductor in flux tubes each with magnetic flux equal to ϕ_0. These flux tubes are called magnetic vortices.

2.4 Phase Transitions

We may also use,

$$F_s - F_n = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\big(-i\hbar\vec{\nabla} - q_s\vec{A}\big)^2|\psi|^2}{2m_s}, \tag{2.41}$$

to classify first order and second order phase transitions in superconductors. For the sake of this classification we assume that the temperature (T) is constant and that we vary the applied magnetic field (B_0). We also assume that ψ and $\vec{A}$ depend on B_0 and are approximately constant with respect to position within the superconductor. We first integrate (2.41) with respect to the applied field B_0 to get,

$$\mathcal{F}(\psi, \vec{A}) = \int_0^{B_0} dB_0'\left[\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\big(-i\hbar\vec{\nabla} - q_s\vec{A}\big)^2}{2m_s}|\psi|^2\right]. \tag{2.42}$$

Then we vary the functional (2.42) with respect to ψ and $\vec{A}$ to get first order and second order functional derivatives, which we use in the following classification: At a point $(\psi, \vec{A})$ where $\mathcal{F}$ is continuous and $\frac{\delta\mathcal{F}}{\delta\psi}, \frac{\delta\mathcal{F}}{\delta\vec{A}}$ are discontinuous, we have a first order phase transition.

At a point $(\psi, \vec{A})$ where $\mathcal{F}$, $\frac{\delta \mathcal{F}}{\delta \psi}$, $\frac{\delta \mathcal{F}}{\delta \vec{A}}$ are continuous and $\frac{\delta^2 \mathcal{F}}{\delta \psi^2}$, $\frac{\delta^2 \mathcal{F}}{\delta \vec{A}^2}$, $\frac{\delta^2 \mathcal{F}}{\delta \psi \delta \vec{A}}$ are discontinuous, we have a second order phase transition. The above classification is known as the Ehrenfest classification. The point across which a phase transition occurs is called a phase boundary. For Type I superconductors, the point $(\psi, \vec{A})$ at which the material goes from superconducting to normal conducting is a phase boundary. For Type II superconductors, the point $(\psi, \vec{A})$ at which the number of magnetic vortices in the superconductor changes is a phase boundary.

Chapter 3

Theory of Superconductivity

3.1 BCS Theory of Superconductivity

One Cooper-Pair System

The BCS theory named after Bardeen, Cooper and Schrieffer, gives us a detailed microscopic description of how superconductivity occurs within the superconductor. We first start by considering a system of just two electrons interacting via an attractive potential $V(\vec{r_1} - \vec{r_2})$. The Schrodinger equation for such a system is given by [11],

$$\left[-\frac{\hbar^2 \vec{\nabla}^2_{\vec{r}_1}}{2m} - \frac{\hbar^2 \vec{\nabla}^2_{\vec{r}_2}}{2m} + V(\vec{r_1} - \vec{r_2}) \right] \Psi(\vec{r_1}, \vec{r_2}) = E\Psi(\vec{r_1}, \vec{r_2}), \tag{3.1}$$

where $\Psi(\vec{r_1}, \vec{r_2})$ is the wave function for the two-particle system and E is the energy. The center of mass of the system is given by $\vec{R} = \frac{1}{2}(\vec{r_1} + \vec{r_2})$. We let $\vec{r} = \vec{r_1} - \vec{r_2}$ and substitute $\vec{r}$ and $\vec{R}$ into (3.1) to get,

$$\left[-\frac{\hbar^2 \vec{\nabla}^2_{\vec{R}}}{2m^*} - \frac{\hbar^2 \vec{\nabla}^2_{\vec{r}}}{2\mu} + V(\vec{r}) \right] \Psi(\vec{r}, \vec{R}) = E\Psi(\vec{r}, \vec{R}), \tag{3.2}$$

where $m^* = 2m$ is the total mass of the system and $\mu = m/2$ is the reduced mass. The fact that the interaction potential only depends on $\vec{r}$ leads us to assume the following form of the wave function,

$$\Psi(\vec{r}, \vec{R}) = \psi(\vec{r}) \exp(i\vec{K} \cdot \vec{R}), \tag{3.3}$$

where $\vec{K}$ is the wave vector of the center of mass. We substitute (3.3) into (3.2) and get,

$$\left[-\frac{\hbar^2 \vec{\nabla}_{\vec{r}}^2}{2\mu} + V(\vec{r}) \right] \psi(\vec{r}) = \tilde{E}\psi(\vec{r}), \tag{3.4}$$

where $\tilde{E} = E - \hbar^2 K^2/2m^*$. We now have that $K = 0$ implies that $\tilde{E} = E$ and this corresponds to the lowest energy state of the center of mass. In this case, the two electrons have equal but opposite momenta which cancel each other out. The next step is that we Fourier transform (3.4) in order to get,

$$\int \frac{d^3k'}{(2\pi)^3} V(\vec{k} - \vec{k}')\psi(\vec{k}') = (E - 2\epsilon_k)\psi(\vec{k}), \tag{3.5}$$

where $\epsilon_k = \hbar^2 k^2/2m$ is defined as the energy of an independent electron with wave vector $\vec{k}$. Because V is attractive, we must have that $(E - 2\epsilon_k) < 0$, which implies that the total energy of the bound state is smaller than the energy of two free electrons. We let the right-hand side of (3.5) be the wave function,

$$\Theta(\vec{k}) = (E - 2\epsilon_k)\psi(\vec{k}). \tag{3.6}$$

We also rewrite the left-hand side of (3.5) in terms of (3.6) to get the following form of the Schrodinger equation,

$$\Theta(\vec{k}) = \int \frac{d^3k'}{(2\pi)^3} \frac{V(\vec{k} - \vec{k}')}{(E - 2\epsilon_{k'})} \Theta(\vec{k}'). \tag{3.7}$$

If we consider the following attractive potential for the interaction,

$$V(\vec{k} - \vec{k}') = \begin{cases} -V_0 & \epsilon_{k'}, \epsilon_k < \epsilon_F + \hbar\omega_D \\ 0 & otherwise, \end{cases} \tag{3.8}$$

for two electronic states above the Fermi surface, which corresponds to the Fermi energy ϵ_F, which is the energy difference between the highest occupied single particle state and the lowest. We have that ω_D is the Debye frequency, which is the highest angular frequency with which the crystal lattice of the superconductor can vibrate. We have that $\hbar\omega_D << \epsilon_F$. We suppose that $\Theta(\vec{k}) = \Theta(\vec{k}') = \Theta$, which is a constant. We denote the density of states at the Fermi surface with $\rho(\epsilon_F)$. We make the energies of the two electrons equal, $\epsilon_k = \epsilon_{k'} = \epsilon$. We then substitute all the above into the integral (3.7) while also replacing $d^3k'/(2\pi)^3$ by $\rho(\epsilon_F)d\epsilon$ in order to get the following integral,

$$\Theta = V_0\ \rho(\epsilon_F)\ \Theta \int_{\epsilon}^{\epsilon_F+\hbar\omega_D} \frac{d\epsilon}{2\epsilon - E}. \tag{3.9}$$

We solve (3.9) to get,

$$\frac{2}{V_0\ \rho(\epsilon_F)} = \ln\left(\frac{2\epsilon_F - E + 2\omega_D}{2\epsilon_F - E}\right). \tag{3.10}$$

If $V_0\ \rho(\epsilon_F) << 1$ then we have that $2\epsilon_F - E \approx 2\omega_D$. We define the binding energy which is given by,

$$E_b = 2\epsilon_F - E, \tag{3.11}$$

and substitute (3.10) into (3.11) to get,

$$E_b = 2\omega_D \exp\left(-\frac{2}{V_0\ \rho(\epsilon_F)}\right). \tag{3.12}$$

Equation (3.11) tells us that if the superconductor has a well defined Fermi surface then a bound state will be formed regardless of how small the attractive interaction V_0 is. The bound state is called a Cooper pair, see Fig 3.1.

Then using $\tilde{E} = E - \hbar^2K^2/2m^*$ and letting $\tilde{E} = 2\epsilon_F - E_b$ we arrive at the following formula for current density,

$$J_s = en_s\frac{\hbar K}{m}. \tag{3.13}$$

So this tells us that even for a non-zero center of mass momentum, the pair still produces a finite supercurrent.

Many Cooper-Pair system

A system of many Cooper pairs is described by the following effective Hamiltonian,

$$H = \sum_{k\sigma} \xi_k\ c_{k\sigma}^\dagger\ c_{k\sigma} + \frac{1}{N}\sum_{kk'} V_{kk'}\ c_{k\uparrow}^\dagger\ c_{-k\downarrow}^\dagger\ c_{-k'\downarrow}\ c_{k'\uparrow}. \tag{3.14}$$

The first term of (3.14) contains $c_{k\sigma}$ which destroys an electron with wave vector $\vec{k}$ and spin σ. The operator $c_{k\sigma}^\dagger$ creates such an electron and $\xi_k = \epsilon_k - \mu$. The quantity μ is the chemical potential, which gives the rate of change of the free energy with respect

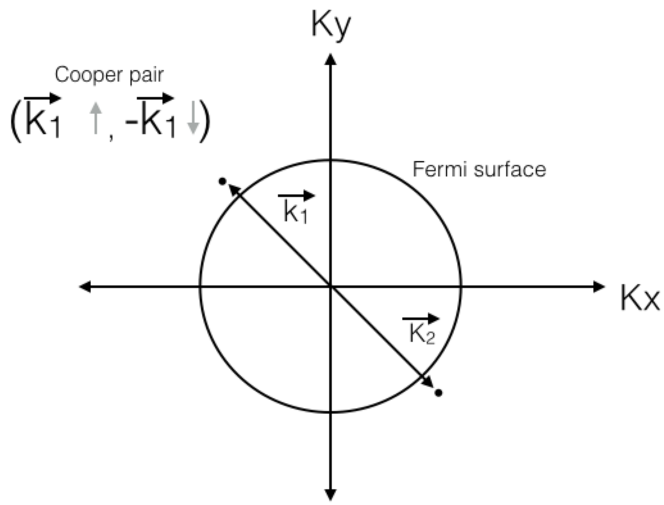

Figure 3.1: This is on the wave vector plane. The Cooper pair forms a singlet state. This means that the pair's net spin is zero. Note that the electrons that form the Cooper pair are located above the Fermi surface.

to the change in the number of Cooper pairs in the system. In the second term, the quantity $c_{-k'\downarrow}\ c_{k'\uparrow}$ destroys a Cooper pair and $c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}$ creates one. The quantity $V_{kk'}$ is the interaction potential of the two electrons that form the Cooper pair and N is the normalization constant. The fluctuations of the quantity $c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\ c_{-k'\downarrow}\ c_{k'\uparrow}$ are close to its mean, therefore we replace it with its mean in the second term of (3.14). Then we expand the mean using the following identity [11],

$$\langle c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\ c_{-k'\downarrow}\ c_{k'\uparrow}\rangle \approx \langle c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\rangle\ c_{-k'\downarrow}\ c_{k'\uparrow} + c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\ \langle c_{-k'\downarrow}\ c_{k'\uparrow}\rangle - \langle c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\rangle\ \langle c_{-k'\downarrow}\ c_{k'\uparrow}\rangle. \tag{3.15}$$

We have that, $\langle c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\rangle = 0$ as it corresponds to a Cooper pair. We then let,

$$\Delta_k = -\frac{1}{N}\sum_{k'} V_{kk'}\ \langle c_{-k'\downarrow}\ c_{k'\uparrow}\rangle, \tag{3.16}$$

which is called a gap function and is a complex quantity. Then we substitute (3.15) and (3.16) into (3.14) in order to get,

$$H = \sum_{k\sigma} \xi_k\ c^{\dagger}_{k\sigma}\ c_{k\sigma} - \sum_{k}\left(\Delta_k\ c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow} + \Delta^{*}_k\ c_{-k\downarrow}\ c_{k\uparrow}\right) + \sum_{k}\Delta_k\langle c^{\dagger}_{k\uparrow}\ c^{\dagger}_{-k\downarrow}\rangle, \tag{3.17}$$

where the asterisk on Δ_k^* denotes the complex conjugate. We now define "new" fermionic operators $\gamma_{k\sigma}$ in the form of the Bogoliubov transformation, which is given by,

$$c_{k\uparrow} = u_k^* \ \gamma_{k\uparrow} + v_k \ \gamma_{-k\downarrow}^\dagger,$$
$$c_{-k\downarrow}^\dagger = u_k \ \gamma_{-k\downarrow}^\dagger - v_k^* \ \gamma_{k\uparrow},$$

where u_k and v_k are complex coefficients. Now we're going to express the Hamiltonian (3.17) in terms of these new operators. The normalization condition,

$$|u_k|^2 + |v_k|^2 = 1, \tag{3.18}$$

is necessary in order for the new operators to satisfy the standard fermionic anti-commutation relations. These anti-commutation relations are given by,

$$\{c_{k\sigma}, c_{k'\sigma'}^\dagger\} = \delta_{kk'} \ \delta_{\sigma\sigma'},$$
$$\{c_{k\sigma}, c_{k'\sigma'}\} = \{c_{k\sigma}^\dagger, c_{k'\sigma'}^\dagger\} = 0.$$

We then substitute the Bogoliubov transformation, defined above, into the Hamiltonian (3.17) and diagonalize it. This results in the following expressions for u_k and v_k,

$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}\right),$$
$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}}\right).$$

We have that,

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}, \tag{3.19}$$

is called the excitation energy. The Hamiltonian (3.17) then reduces to the following,

$$H = \sum_{k\sigma} E_k \ \gamma_{k\sigma}^\dagger \ \gamma_{k\sigma} + E_0, \tag{3.20}$$

where,

$$E_0 = \sum_k \left(\xi_k - E_k + \Delta_k \langle c_{k\uparrow}^\dagger \ c_{-k\downarrow}^\dagger\rangle\right), \tag{3.21}$$

is the ground state energy. The first term of (3.20) represents excitations beyond the ground state, and $\xi_k = 0$ corresponds to the Fermi level, the energy of the highest occupied single particle state. Consequently, (3.20) shows us that even at the Fermi level the energy spectrum of the superconductor is still non-zero and equal to $|\Delta_k|$. This justifies why (3.16) is called a gap function. This means that within the superconductor even when all the single particle states are occupied the electrons are still excited beyond the Fermi surface. Therefore we get that the minimum energy that the electrons must have in order to form a Cooper pair is $2|\Delta_k|$, which is the same energy that is required to excite a quasiparticle. This quasiparticle is called a Bogoliubon [11] and is a superposition of an electron above the Fermi surface and an electron hole below the Fermi surface. The operator $\gamma^\dagger_{k\sigma}$ creates a Bogoliubon and $\gamma_{k\sigma}$ destroys it. Of course,

$$\gamma^\dagger_{-k\downarrow} = u^*_k \ c^\dagger_{-k\downarrow} + v^*_k \ c_{k\uparrow}, \tag{3.22}$$

and,

$$\gamma_{k\uparrow} = u_k \ c_{k\uparrow} - v_k \ c^\dagger_{-k\downarrow}. \tag{3.23}$$

We then conclude that the ground state that the Cooper pairs occupy, which is called the BCS ground state, corresponds to the ground state of Bogoliubons. The wave function of the BCS ground state is denoted by $|\Psi_{BCS}\rangle$. We have that $\gamma_{k\sigma}\,|\Psi_{BCS}\rangle = 0$, which, after substituting (3.23) implies the following,

$$u_k \ c_{k\uparrow}\,|\Psi_{BCS}\rangle = v_k \ c^\dagger_{-k\downarrow}\,|\Psi_{BCS}\rangle\,. \tag{3.24}$$

Therefore we may express the BCS ground state as a combination of Cooper pairs in the following way,

$$|\Psi_{BCS}\rangle = \mathcal{N}\prod_q \exp(\alpha_q \ c^\dagger_{q\uparrow} \ c^\dagger_{-q\downarrow}) \ |0\rangle\,, \tag{3.25}$$

where $|0\rangle$ is the electron ground state, $q = k - k'$, $\mathcal{N}$ is a normalization constant and α_q is a function to be determined. Then after some manipulation we arrive at the final form for the BCS ground state given by,

$$|\Psi_{BCS}\rangle = \prod_k \left(u_k + v_k \ c^\dagger_{k\uparrow} \ c^\dagger_{-k\downarrow}\right) \ |0\rangle\,. \tag{3.26}$$

Using the above results we arrive at the following general formula for the gap function Δ_k at absolute zero. Where we supposed that the gap function is k-independent and real,

$\Delta_k = \Delta_0 = \Delta(T = 0)$ and is given by,

$$\Delta_0 \approx 1.76\ k_B\ T_c, \tag{3.27}$$

where T_c is the critical temperature and k_B is the Boltzmann constant. This formula (3.27) holds in general for superconductors. We also note that the energy $2\Delta_0$ also corresponds to the amount of energy needed to break up a Cooper pair.

3.2 Fetter's Theory for Flux Penetration

In this section we review the work of Alexander Fetter [12]. Within superconductors there are two causes for Cooper pair breaking: impurities and thermal lattice vibrations. Magnetic flux penetration or magnetic vortices in a superconductor is the result of the breaking of Cooper pairs. Here Fetter considered the structure of a single quantized flux line in a thin superconducting disc of radius R and the corresponding interaction between pairs of such flux lines in order to derive a formula for the lower critical field of a type II superconductor.

We start from a disc-shaped superconducting film of thickness d and radius R lying in the xy-plane, and work in cylindrical coordinates (r, θ, z) centered at the disc's origin. Because of the circular geometry of the superconductor, we let the order parameter of this superconductor take the form,

$$\psi = |\psi_0| \exp(i\varphi), \tag{3.28}$$

where φ is the phase and $|\psi_0|$ represents the radius or amplitude of the order parameter. We neglect the spatial dependence of $|\psi_0|$. We therefore arrive at the following Ginzburg-Landau equations,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \begin{cases} \frac{4\pi \vec{J}_s}{c} & |z| < \frac{1}{2}d \ and \ r < R \\ 0 & otherwise, \end{cases} \tag{3.29}$$

where,

$$\frac{4\pi \vec{J}_s}{c} = \frac{1}{\lambda^2}\left[\frac{\phi_0}{2\pi}\vec{\nabla}\varphi - \vec{A}\right]. \tag{3.30}$$

Because we assume that the ratio d/λ is small we are able to average the above equations over the thickness d to get,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \frac{d}{\lambda^2}\delta(z)\left[\frac{\phi_0}{2\pi}\vec{\nabla}\varphi - \vec{A}\right]\eta(R-r), \tag{3.31}$$

where $\delta(z)$ is the Dirac delta function and $\eta(R-r)$ is the unit step function. If we interpret the order parameter ψ to be describing a single magnetic flux line penetrating the center of the disc then the phase φ is the azimuthal angle θ. This implies that the vector potential $\vec{A}$ has a purely azimuthal direction $\vec{A} = A\hat{\theta}$ and substituting this into (3.31) gives us,

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]A = \frac{2}{\Lambda}\delta(z)\left[A - \frac{\phi_0}{2\pi r}\right]\eta(R-r), \tag{3.32}$$

where $\Lambda = 2\lambda^2/d$. We introduce dimensionless variables, $x = r/R$ and $y = z/R$, which we then substitute into (3.32) to get,

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} - \frac{1}{x^2}\right]A(x,y) = \frac{2R}{\Lambda}\delta(y)\left[A - \frac{1}{x}\right]\eta(1-x). \tag{3.33}$$

If we let $A(x,y) = A(x,0)$ and substitute into (3.33) we derive the supercurrent density formula,

$$\vec{J_s} = \left(\frac{c\phi_0}{4\pi^2\Lambda R d}\right)\tilde{J_s}, \tag{3.34}$$

where $\tilde{J_s} = \Big(1/x - A(x)\Big)$ is the screening current caused by the circulation of the vortex. Now we're going to introduce another magnetic flux line into the system. When we bring this magnetic flux line from the boundary of the disc to a point x, it experiences a Lorentz force given by $\Big(\phi_0 d/c\Big)\vec{J_s} \times \hat{z}$, which we integrate to obtain the following interaction energy between the two vortices,

$$U = \left(\frac{\phi_0^2}{4\pi^2\Lambda}\right)\tilde{U}(x), \tag{3.35}$$

where,

$$\tilde{U}(x) = \ln\left(\frac{1}{x}\right) - \int_x^1 dx' A(x'). \tag{3.36}$$

The first term of (3.36) represents the interaction of the two unscreened vortex lines. We then write,

$$\tilde{U}(x) = \ln\left(\frac{1}{x}\right) + \tilde{U}_{sc}(x), \tag{3.37}$$

where $\tilde{U}_{sc}(x) = -\int_x^1 dx' A(x')$ represents the attractive effect of the screening currents. The self energy of a single vortex line at the center of the disc is given by,

$$\epsilon = \left(\frac{\phi_0^2}{4\pi^2\Lambda}\right)\tilde{\epsilon}, \tag{3.38}$$

where,

$$\tilde{\epsilon} = \frac{1}{2}\tilde{U}\left(\frac{r_c}{R}\right) \approx \frac{1}{2}\ln\left(\frac{R}{r_c}\right) + \frac{1}{2}\tilde{U}_{sc}(0), \tag{3.39}$$

where r_c is the core radius of the vortex line. The circulating supercurrent produces a magnetic moment given by,

$$\vec{m} = \frac{1}{2c}\int d^3\vec{r}\left(\vec{r} \times \vec{J}_s\right), \tag{3.40}$$

which lies along the z-axis, the axis of symmetry. The magnetic moment (3.40) has magnitude,

$$m = \left(\frac{\phi_0 R}{2\pi^2}\right)\tilde{m}, \tag{3.41}$$

where,

$$\tilde{m} = \frac{\pi R}{2\Lambda}\int_0^1 x dx \left(1 - xA(x)\right). \tag{3.42}$$

Therefore the lower critical field B_{c1} for magnetic flux penetration into the disc is given by,

$$B_{c1} = \left(\frac{\phi_0}{2R\Lambda}\right)\tilde{B}_{c1}, \tag{3.43}$$

where $\tilde{B}_{c1} = \tilde{\epsilon}/\tilde{m}$.

With the Ginzburg-Landau theory in section 2.3 we were able to derive the formula (2.24) for the critical field of a bulk type I superconductor. We were also able to derive the formula $B_{c2} = \phi_0/2\pi\xi^2$ for the upper critical field of a bulk type II superconductor. Now with Fetter's theory we have derived the formula (3.43) for the lower critical field of a type II superconducting disc .

As we stated before, magnetic flux penetration is due to Cooper pair breaking in the superconductor. Therefore the lower critical field B_{c1} basically gives the value at which the applied magnetic field causes enough Cooper pairs to break within the superconductor in order to allow a magnetic flux line to penetrate. Between the lower critical field and the upper critical field more flux lines will penetrate and form multivortex states, which we will discuss in the next section.

3.3 Giant and Multi-Vortex States

Here we discuss the results of Yampolski et.al, given in [13]. We study vortex configurations on mesoscopic discs. A mesoscopic sample is such that its size is comparable to the coherence length (ξ) and penetration depth (λ), which are measured in nanometers. Studies have shown that in mesoscopic discs surrounded by a vacuum or an insulator, we get two kinds of superconducting states: the circular symmetric state with a fixed value of angular momentum, which is called a giant vortex, and multivortices in discs with a sufficiently large radius.

We consider a Type II mesoscopic superconducting disc with radius R and thickness d placed in a medium that enhances superconductivity on the surface of the sample. The applied magnetic field here is given by $\vec{H} = (0, 0, H)$, is uniform within the superconductor and is directed normal to the disc plane. We use the Ginzburg-Landau equations in this investigation. We get the first equation by ignoring the first and the last term on the right-hand side of equation (2.13) and setting the left-hand side equal to zero. This gives us the following,

$$\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m_s}\left(-i\hbar\vec{\nabla} - q_s\vec{A}\right)^2|\psi|^2 = 0. \tag{3.44}$$

We let $q_s = 2e$, $m_s = m$, $c \neq 1$ and we use β instead of $\beta/2$ to get,

$$\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{2e\vec{A}}{c}\right)^2\psi = -\alpha\psi - \beta\psi|\psi|^2. \tag{3.45}$$

Combining the London equation and the Schrodinger equation we are able to derive the second equation,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \frac{4\pi}{c}\vec{j}, \tag{3.46}$$

where $\vec{j} = \vec{J}_s$, the supercurrent given by,

$$\vec{j} = \frac{e\hbar}{im}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{4e^2}{mc}|\psi|^2\vec{A}. \tag{3.47}$$

We will work in cylindrical coordinates where $\vec{r} = (\vec{\rho, z}) = (\rho, \varphi, z)$ is the three-dimensional position in space. The disc lies between $z = -d/2$ and $d/2$. The boundary conditions are,

$$\vec{n} \cdot \left(-i\hbar\vec{\nabla} - \frac{2e\vec{A}}{c} \right)\psi\Big|_S = \frac{i}{b}\psi\Big|_S, \tag{3.48}$$

where $\vec{n}$ is the unit vector normal to the surface of the disc, S represents an arbitrary point on the surface and b gives the effective penetration depth of the order parameter into the surrounding medium. This boundary condition shows that the supercurrent also flows out of the superconductor through the surface into the surrounding medium. This corresponds to the Neumann boundary condition. If there was no surrounding medium or the superconductor was placed in an insulator then $b \to \infty$, which would make the right-hand side of (3.48) zero. The second boundary condtion is,

$$\vec{A}\Big|_{\rho\to\infty} = \frac{1}{2}H\rho\vec{e}_\varphi, \tag{3.49}$$

where $\vec{e}_\varphi$ is a unit vector in the azimuthal direction. Using $\vec{\nabla} \cdot \vec{A} = 0$ in (3.45)-(3.48) we get,

$$\left(-i\vec{\nabla} - \vec{A} \right)^2 \psi = \psi - \psi|\psi|^2, \tag{3.50}$$

and,

$$-\kappa^2\vec{\nabla}^2\vec{A} = \frac{1}{2i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^* \right) - |\psi|^2\vec{A}, \tag{3.51}$$

with boundary condition,

$$\vec{n} \cdot \left(-i\vec{\nabla} - \vec{A} \right)\psi\Big|_S = \frac{i}{b}\psi\Big|_S, \tag{3.52}$$

where κ is the Ginzburg-Landau parameter. To get the free energy of the superconducting state, we combine equations (3.50)-(3.52) and integrate over the disc and get,

$$F = \frac{2}{V}\left[\int dV\left(-|\psi|^2 + \frac{1}{2}|\psi|^4 + |-i\vec{\nabla}\psi - \vec{A}\psi|^2 + \kappa^2[\vec{\nabla} \times \vec{A} - \vec{H}]^2 \right) + \frac{1}{b}\oint dS\ |\psi|^2 \right]. \tag{3.53}$$

We choose $d << \xi, \lambda$, which then implies that,

$$\vec{A} = \vec{A}_0 = \big(0, \frac{H\ \rho}{2}, 0\big). \tag{3.54}$$

We only have to solve the equation (3.50). We then expand $\psi(\vec{r})$ in its Fourier series to get,

$$\psi(\vec{r}) = \sum_k \psi_k(\vec{\rho})\exp(ikz), \tag{3.55}$$

which suits the circular geometry of the disc. This then allows us to average equation (3.50) over the disc thickness to get,

$$\langle(-i\vec{\nabla}-\vec{A})^2\rangle\psi(\vec{\rho}) = \left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+iH\frac{\partial}{\partial\varphi}+\left(\frac{H\rho}{2}\right)^2-k^2\right]\psi(\vec{\rho}) = \psi(\vec{\rho})-\psi(\vec{\rho})|\psi(\vec{\rho})|^2, \tag{3.56}$$

with boundary condition,

$$\frac{\partial\psi(\vec{\rho})}{\partial\rho}\bigg|_{\rho=R} = -\frac{1}{b}\psi(\vec{\rho})\bigg|_{\rho=R}. \tag{3.57}$$

The problem is now reduced to a two-dimensional problem for $\psi(\vec{\rho})$.

Giant Vortex States

The giant vortex state has cylindrical symmetry, therefore the order parameter can be written as,

$$\psi(\vec{\rho}) = f(\rho)\exp(iL\varphi), \tag{3.58}$$

where L is the winding number of the vortex. We linearize the equation (3.56) and substitute (3.58) into the linearized equation to get,

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\left(\frac{L}{\rho}-\frac{H\rho}{2}\right)^2-1-k^2\right]f(\rho) = 0, \tag{3.59}$$

where the operator in the square bracket is denoted $\hat{L}$. The negative eigenvalues of $\hat{L}$ correspond to the superconducting state. We solve,

$$\hat{L}f_{L,n}(\rho) = \Lambda f_{L,n}(\rho), \tag{3.60}$$

with boundary conditions,

$$\rho\left(\frac{\partial f}{\partial\rho}\right)\bigg|_{\rho=0} = 0, \tag{3.61}$$

in order to get the eigenvalues and eigenfunctions of $\hat{L}$. Here Λ denotes the eigenvalues of $\hat{L}$. The index n gives the number of states that share the same R and L.

When the applied field is increasing, the transition from the Meissner state (which corresponds to $L = 0$) to the normal state, goes through consecutive first order phase transitions between the L and the $L+1$ giant vortex states and ends with a second order phase transition to the normal conducting state.

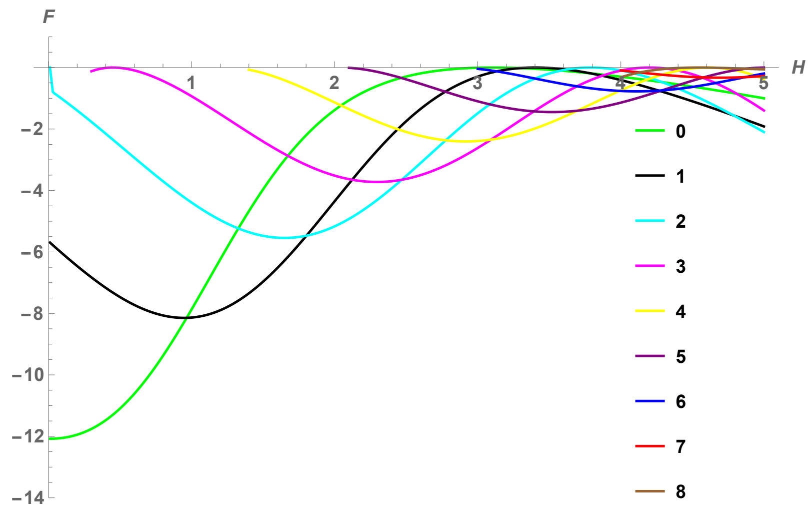

Figure 3.2: This plot is for the free energy F that we obtain from solving equation (3.60). It's for giant vortex states with various angular momenta L, as a function of the applied field H. The legends at the bottom-right of this figure give the color of a curve with the corresponding L value. These legends are consistent with the Figure 3.3 and Figure 3.4 as well.

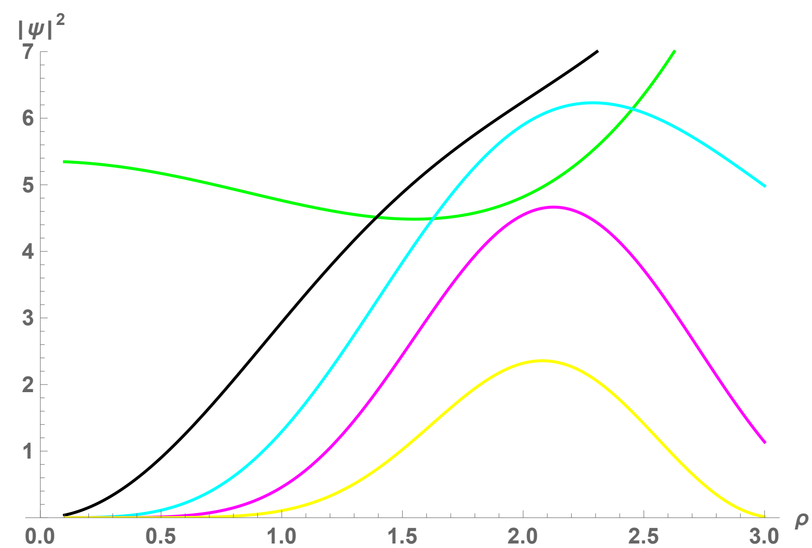

Figure 3.3: This shows the Cooper pair density $|\psi|^2$ that we obtain from solving equation (3.60), for the giant vortex states with various angular momenta L, as a function of ρ.

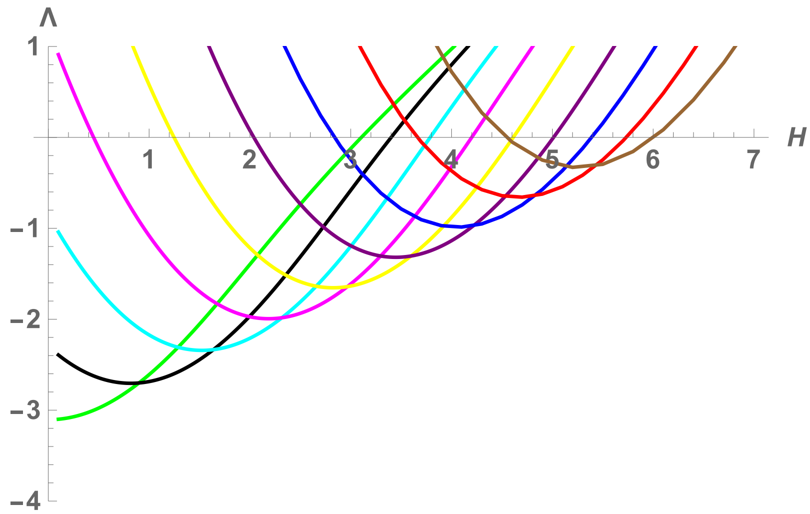

Figure 3.4: This plot shows the behavior of Λ for $\xi/b = -0.1$, $R = 2.0\xi$ and for L values from zero to nine.

Multivortex States

If the disc is large enough, then the giant vortex state can break up into multivortices. The order parameter of the multivortex state is written as a linear combination of the order parameters of giant vortex states and is given by,

$$\psi(\vec{\rho}) = \sum_{L_j=0}^{L} \sum_{n} C_{L_j,n}\; f_{L_j,n}(\rho) \exp(iL_j\varphi), \tag{3.62}$$

where L is the number of vortices in the disc (the effective angular momentum), L_j is the winding number for the j-th vortex in the configuration and n is the number of vortex states with the same L_j. In the results that follow we only consider $n = 0$, so we omit the index in (3.62). We substitute (3.62) into (3.53), which expresses F as a function of $\{C_{L_j}\}$. Then in order to determine the stability of the multivortex configurations, we minimize F with respect to these parameters $\{C_{L_j}\}$.

To carry this out we solve the following equations for the solutions $\{C_{L_j}^{(0)}\}$,

$$\frac{\partial F}{\partial C_{L_j}} = 0,\;\; L_j = 0, ..., L. \tag{3.63}$$

Then we calculate the Hessian matrix,

$$\frac{\partial^2 F}{\partial C_{L_j} \partial C_{L_k}}\Big|_{C_{L_j}=C_{L_j}^{(0)},\; C_{L_k}^{(0)}=C_{L_k}}, \tag{3.64}$$

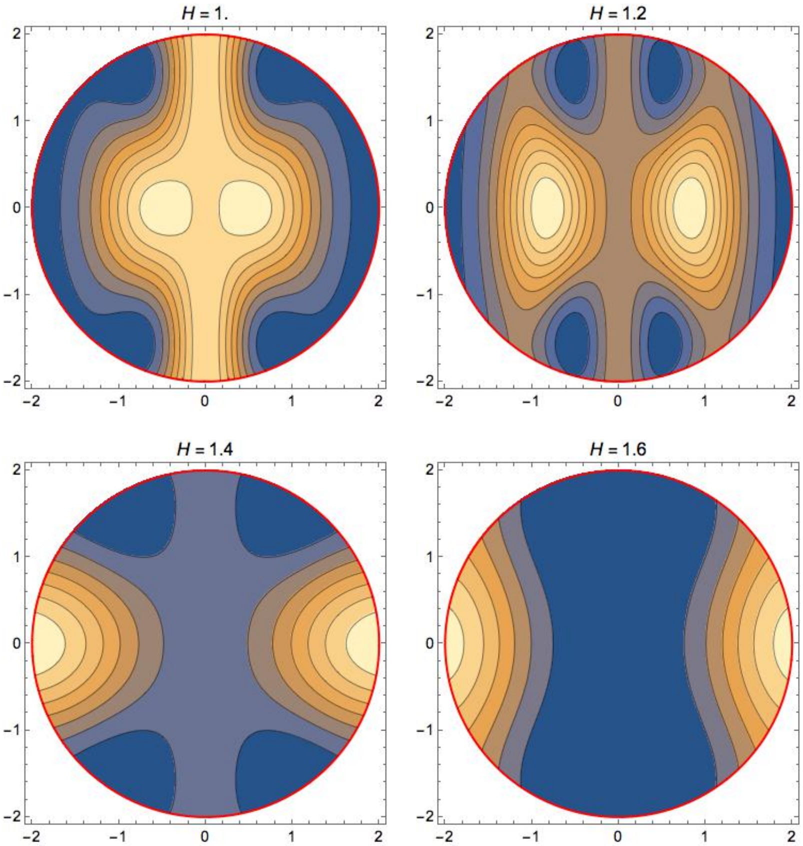

Figure 3.5: A contour plot of the superconducting density $|\psi|^2$ for the $(0 : 2)$ state, where $(L_1 : L_2)$ denotes the angular momenta that compose the multivortex state. As shown in the figure, for increasing values of the applied field H, the vortices move away from each other and approach the boundary.

which we use to determine the stable vortex states. By considering a simpler case of states only built up by two components in (3.62), we are able to determine how the system generally behaves. We obtain the free energy expression for such states and minimize it according to (3.63) to get all the possible equilibrium states.

Yampolski et.al showed here that for the radius $R/\xi = 2.0$ and $\xi/b = -0.1$, the states with $L = 0, 1, 2$ transition through a first order phase transition to states that have a different L value. The transition between the giant and the multivortex state with the same vorticity are of second order. We also find that in this disc, when we increase the applied field we get the following transitions occurring,

$$0 \rightarrow 2 \rightarrow 4 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow \text{“}normal\ state\text{”}. \tag{3.65}$$

When we decrease the applied field then the following transitions occur,

$$\text{“}normal\ state\text{”} \rightarrow 8 \rightarrow 7 \rightarrow 5 \rightarrow (0:5) \rightarrow 3 \rightarrow 0. \tag{3.66}$$

In (3.65) we see that for an increasing applied field the disc transitions between giant vortex states of higher and higher vorticity and finally becomes normal conducting. We also see in (3.66) that for a decreasing applied field the disc transitions from a normal conducting state to a giant vortex state of vorticity 8 and continues to decrease in vorticity. When the giant vortex state has vorticity 5 it splits into a multivortex state with a ring-like structure of five vortices with no vortex in the middle. Finally the disc becomes superconducting, with zero vortices. When we make the disc radius larger, more complicated vortex structures arise.

Chapter 4

Vortex Lattice Reorganization

4.1 The London Approximation

The results presented in this section were obtained by Cabral et.al [14; 15]. Here, stable vortex states are investigated in large Type II superconducting discs. Since vortices can be created or destroyed, interact with each other and with the interfaces, they can be treated as particles. The London approximation allows us to treat vortices as particles and is a good approximation of the Ginzburg-Landau theory, which becomes even better for higher values of κ.

Here, we consider a thin disc of radius $R = 50\xi$ and thickness d, where $d \leq \xi \leq \Lambda$. The Λ is the effective penetration depth we encountered in section 3.2 and is given by $\Lambda = \lambda^2/d$. Here we've ignored the factor of 2. In this case the disc is surrounded by a vacuum and is in the presence of a uniform perpendicular magnetic field $\vec{H}_0$.

We investigate vortex states in mesoscopic thin discs in the London approximation using the London gauge $\vec{\nabla} \cdot \vec{A} = 0$. We start with the same Ginzburg-Landau equations (3.50) and (3.51) that we encountered in chapter 3. However this time the disc is surrounded by a vacuum. Hence our boundary conditions take the following form,

$$\vec{n} \cdot \Big(- i\vec{\nabla} - \vec{A} \Big) \psi \Big|_S = 0, \tag{4.1}$$

and,

$$\vec{A} = \vec{A}_0 = \frac{1}{2} H_0 \rho \hat{\phi}, \tag{4.2}$$

where $\hat{\phi}$ denotes the unit vector in the azimuthal direction here. So the boundary condition (4.1) shows that the supercurrent within the disc does not flow out of the disc through the surface. We get the second boundary condition (4.2) because in thin mesoscopic discs the total magnetic flux through the disc $\vec{H}$ is uniform throughout the disc and is equal to the applied magnetic field $\vec{H}_0$. This is called demagnetization because of the form of (2.18). We write the order parameter in the following way,

$$\psi = |\psi| \exp(i\theta). \tag{4.3}$$

In this case we denote the Ginzburg-Landau free energy of the superconducting state with a $\mathcal{G}$ and express it in the following way,

$$\mathcal{G} = \frac{1}{V} \int \Big[- 2|\psi|^2 + |\psi|^4 + 2(\vec{\nabla}|\psi|)^2 + 2|\psi|^2 |(\vec{\nabla}\theta - \vec{A})|^2 + 2\kappa^2 (\vec{H} - \vec{H}_0)^2 \Big] dV, \tag{4.4}$$

where we ignore the fact that $\vec{H} = \vec{H}_0$, so we ignore demagnetization effects. In the London approximation we consider the order parameter to be constant throughout the disc. In particular, we let $|\psi|^2 = 1$. Therefore the first three terms in (4.4) fall away and we get,

$$\mathcal{G}_L = \frac{1}{V} \int \Big[2|(\vec{\nabla}\theta - \vec{A})|^2 + 2\kappa^2 (\vec{H} - \vec{H}_0)^2 \Big] dV, \tag{4.5}$$

which is the London free energy of the system of vortices. We've of course let $z = 0$ because we have a mesoscopic thin disc. So we integrate (4.5) along the thin film plane $z = 0$. In the presence of L vortices located at ρ_i, where $i = 1, 2, ..., L$, the equation (2.11) can be written as,

$$\vec{J}_s = \frac{d}{\kappa^2} (\vec{\nu} - \vec{A}), \tag{4.6}$$

where,

$$\vec{\nu} = \sum_{i=1}^{L} \Big[\Phi(|\vec{\rho} - \vec{\rho}_i|) - \Phi(|\vec{\rho} - (R/\rho_i)^2 \vec{\rho}_i|) \Big], \tag{4.7}$$

and $\vec{\rho}_i = (x_i, y_i)$ is the position of the i-th vortex in the disc. In this case we get an additional boundary condition,

$$\vec{J}_s(R) \cdot \hat{\rho} = 0, \tag{4.8}$$

where $\hat{\rho}$ is the unit vector in the radial direction. The boundary condition (4.8) shows that at the radial boundary of the disc, the supercurrent flows at a tangent to the disc. Through this approach, we ultimately are able to express the London energy functional as follows,

$$\begin{aligned}\mathcal{G}_L = \sum_{i=1}^{L} \left(\left(\frac{2}{R}\right)^2 \ln(1-r_i^2) - 2H_0(1-r_i^2) + \sum_{j\neq i} \left(\frac{2}{R}\right)^2 \ln\left[\frac{(r_i r_j)^2 - 2\vec{r}_i \cdot \vec{r}_j + 1}{r_i^2 - 2\vec{r}_i \cdot \vec{r}_j + r_j^2}\right]\right) \\ + \left(\frac{2}{R}\right)^2 N \ln\left(\frac{R}{a}\right) + \frac{R^2 H_0^2}{4},\end{aligned} \tag{4.9}$$

where $\vec{r}_i = \vec{\rho}_i / R$. Also for $i \neq j$ we have that $|\vec{\rho}_i - \vec{\rho}_j| = a\xi$ where a is a constant. This of course puts a restriction/cut-off on how close the vortices can be to one another. The first term in the bracket in (4.9) is the interaction energy of the i-th vortex with the radial boundary of the disc. The second term is the interaction energy between the i-th vortex and the screening currents. The third term is the repulsive energy between the i-th and the j-th vortex. The fourth term, which is outside the bracket, is the combined core energies of the L vortices in the disc. The last term is the energy of the external magnetic field.

By using the above results, Cabral et.al showed that for small vorticity, a ring-like structure dominates the disc. This ring-like structure is due to the circular geometry of the disc. For large vorticity, it is shown that an Abrikosov lattice of vortices forms at the center of the disc, with two rings of vortices forming near the boundary. The Abrikosov lattice is due to the repulsive interaction between the vortices.

4.2 The Vortex Absorption Formula and Free-Energy Release

In the paper of Cabral et.al that we discussed in the previous section, the fourth term in (4.9), which is the combined core energies of all the vortices in the disc is given as,

$$\epsilon_{core} = \left(\frac{2}{R}\right)^2 N \ln\left(\frac{R}{a}\right), \tag{4.10}$$

as can be seen in (4.9). The N denotes the total number of vortices in the disc. Instead of N we use L [1]. We interpret $(2/R)^2 \ln(R/a)$ as the core energy of a single vortex in the disc. As a result of this, we are able to write (4.9) in the following form,

$$\mathcal{G}_L = \sum_{i=1}^{L} \left[\left(\frac{2}{R}\right)^2 \ln(1-r_i^2) - 2H_0(1-r_i^2) + \sum_{j\neq i} \left(\frac{2}{R}\right)^2 \ln\left[\frac{(r_i r_j)^2 - 2\vec{r}_i\cdot\vec{r}_j + 1}{r_i^2 - 2\vec{r}_i\cdot\vec{r}_j + r_j^2}\right] + \left(\frac{2}{R}\right)^2 \ln\left(\frac{R}{a}\right)\right] + \frac{R^2 H_0^2}{4}, \tag{4.11}$$

For each i, the expression in the bracket gives us the total free energy that the i-th vortex contributes to the system of vortices in the disc, let's denote it E_i. If $E_i = 0$ for the i-th vortex in the disc, then this vortex does not contribute to the overall free energy of the system anymore. This then allows us to write an absorption formula which takes the form,

$$H_{0i} = \frac{\left(\frac{2}{R}\right)^2 \ln(1-r_i^2) + \sum_{j\neq i} \left(\frac{2}{R}\right)^2 \ln\left[\frac{(r_i r_j)^2 - 2\vec{r}_i\cdot\vec{r}_j + 1}{r_i^2 - 2\vec{r}_i\cdot\vec{r}_j + r_j^2}\right] + \left(\frac{2}{R}\right)^2 \ln\left(\frac{R}{a}\right)}{2(1-r_i^2)}, \tag{4.12}$$

where this formula gives us the value of the applied field at which the i-th vortex will be absorbed into the disc. In this context, by absorbed we simply mean that the vortex ceases to be part of the vortex configuration. Therefore where there was L vortices in the configuration, we now have $L-1$ vortices. The vorticity of the configuration has decreased by one. The energy that this vortex had when it was part of the configuration is given by E_i , and formed part of the free energy of the system. Free energy is defined as the energy

[1] They don't specify what N means in this expression. So we think that this is an error. It makes more sense to replace N with L, where L is of course the total number of vortices in the disc.

that the system stores up and keeps available to do work in the system. When the i-th vortex is absorbed, the free energy that it contributed to the system, which of course is given by,

$$E_i = \left(\frac{2}{R}\right)^2 \ln(1-r_i^2) - 2H_0(1-r_i^2) + \sum_{j\neq i} \left(\frac{2}{R}\right)^2 \ln\left[\frac{(r_i r_j)^2 - 2\vec{r}_i \cdot \vec{r}_j + 1}{r_i^2 - 2\vec{r}_i \cdot \vec{r}_j + r_j^2}\right] + \left(\frac{2}{R}\right)^2 \ln\left(\frac{R}{a}\right), \tag{4.13}$$

is now released into the system to do work.

Chapter 5

Bogomolnyi Vortices

5.1 Vortices with Dirichlet Boundary Conditions

The results presented in this chapter are derived from the work of Nasir [16]. In addition to reviewing [16] we also present numerical work that we did in this chapter. Nasir investigated vortices on a disc with Dirichlet boundary conditions and showed that solutions exist. In this section we study Bogomolnyi vortices on a disc in $\mathbb{R}^2$, which is a compact two-dimensional region with a boundary. In section 5.2 we study Bogomolnyi vortices on the surface of a sphere, which is a compact two-dimensional region without a boundary. We choose Dirichlet boundary conditions because they give the winding necessary for the topological existence of vortices.

We begin with a complex scalar field, ϕ, which is coupled to a $U(1)$ gauge potential A_μ in $(2+1)$ dimensions. We consider a two-dimensional region, M, which is topologically a disc. The metric of the spacetime $\mathbb{R} \times M$, is taken to be of the form,

$$ds^2 = dx_0^2 - \Omega(x_1, x_2)\ (dx_1^2 + dx_2^2), \tag{5.1}$$

where $\Omega = 1$ for a flat disc. The Lagrangian density for the Abelian-Higgs model is given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\nu\phi\,\overline{D^\mu\phi} - \frac{\lambda}{8}(|\phi|^2-1)^2, \tag{5.2}$$

where of course $D_\mu = \partial_\mu - iA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Working in the gauge $A_0 = 0$, the Lagrangian is given by $L = T - V$ where,

$$T = \frac{1}{2}\int_M d^2x\ (\dot{A}_i\ \dot{A}_i + \Omega\ \dot{\phi}\ \dot{\bar{\phi}}), \tag{5.3}$$

and,

$$V = \frac{1}{2}\int_M d^2x\ \Omega\bigg(\frac{1}{2}F_{ij}F^{ij} + D_i\phi\ \overline{D^i\phi} + \frac{1}{4}(|\phi|^2-1)^2\bigg), \tag{5.4}$$

which are the kinetic and potential energies respectively and where $i, j = 1, 2$. If we consider the static case then the total energy E is equal to just V and is given by,

$$E = \frac{1}{2}\int_M d^2x\bigg[(D_1 \pm iD_2)\,\phi\,\overline{(D_1 \pm D_2)\phi} + \Omega^{-1}\bigg(F_{12} \pm \frac{\Omega}{2}(|\phi|^2-1)\bigg)^2 \pm i\bigg(\partial_2(\bar{\phi}D_1\phi) - \partial_1(\bar{\phi}D_2\phi)\bigg) \pm F_{12}\bigg]. \tag{5.5}$$

The Bogomolnyi equations are then given by,

$$D_1\phi \pm iD_2\phi = 0, \tag{5.6}$$

$$F_{12} \pm \frac{\Omega}{2}(|\phi|^2 - 1) = 0. \tag{5.7}$$

We now rewrite the Bogomolnyi equations in terms of a new field h, where $h = 2\log|\phi|$. We then express equations (5.6) and (5.7) in terms of h and we get the Taubes equation which is given by,

$$\Delta h + \Omega(e^h - 1) = 4\pi\sum_{i=1}^{N}\ \delta^2(\vec{x} - \vec{x}_i), \tag{5.8}$$

where $\vec{x}_i$ denotes the i-th vortex and Δ is the Laplacian operator in $\mathbb{R}^2$. For these Bogomolnyi vortices we take the Dirichlet boundary conditions,

$$|\phi| = 1, \tag{5.9}$$

which implies that $E = \pi N$, which therefore means that the energy of the vortices is finite and quantized. We now solve equation (5.8) numerically. In order to do this we have to write equation (5.8) in an equivalent form which makes the numerical computations simpler to execute. In order to do this we write, $h = f + u$, where,

$$u = \sum_{i=1}^{N}\log|\vec{x} - \vec{x}_i|^2. \tag{5.10}$$

We substitute (5.10) into (5.8) and get the equation for f which is given by,

$$\Delta f = e^{f+u} - 1. \tag{5.11}$$

The corresponding Dirichlet boundary conditions are given by,

$$f\Big|_{\partial M} = -u\Big|_{\partial M}. \tag{5.12}$$

We solve equation (5.11) using the Dirichlet relaxation method. Here we solve it on a square grid of size 4, where in Nasir [16; 17; 18], they solve it on a square grid of size 0.5, 1 and 1.5. We choose to work on a square grid because it simplifies the numerical computation and it's topologically equivalent to a disc. When we express the modulus of the Higgs field in terms of u we get,

$$|\phi| = e^{f+u}. \tag{5.13}$$

When we consider a case of one vortex and do the list plot of equation (5.13) we get Figure 5.1. The corresponding contour plot for the one vortex case is given by Figure 5.2. We do the same for the two vortex case. As we know, the Bradlow inequality for this case is given by,

$$N \leq \frac{A}{4\pi}, \tag{5.14}$$

where A is the area of the square grid of size 4. Theoretically, in this case, the Bradlow inequality is expressed in terms of the first Chern number, which is usually a fraction and therefore no longer represents the vortex number. Hence, in this case, there is no Bradlow bound. However if we examine the code in Figure 5.5 we notice the function,

$$Sum\left[Log\left[\left(Norm\left[\{r,s\} - pt[[p]]\right]\right)^2\right], \{p, 1, 20\}\right], \tag{5.15}$$

which corresponds to u in (5.10). The vector $pt[[p]]$ in (5.15) corresponds to the p-th vortex and we're summing from 1 to 20, which means we're solving for a case of 20 vortices. For this code, we find that there is a maximum N value for which it gives a solution. We therefore conclude that, for a topological disc with a boundary, there is theoretically no Bradlow bound but practically, there is an upper bound on how many vortices you can place on the disc.

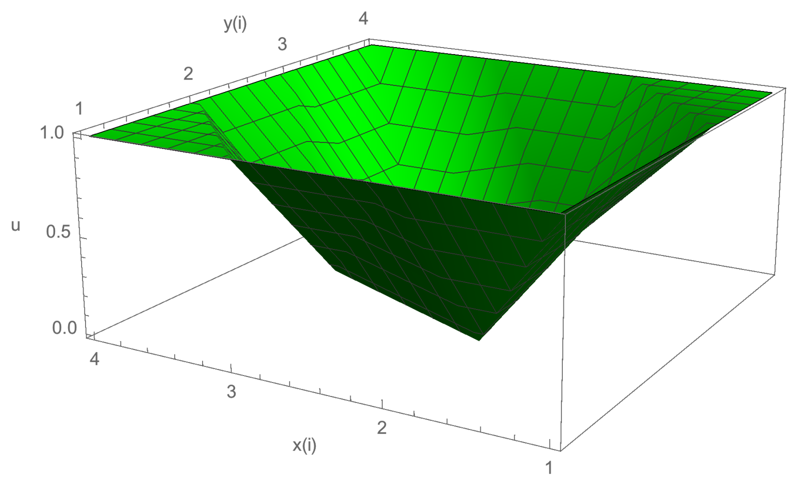

Figure 5.1: The plot of $|\phi|$ for the case of one vortex on a square grid of size 4.

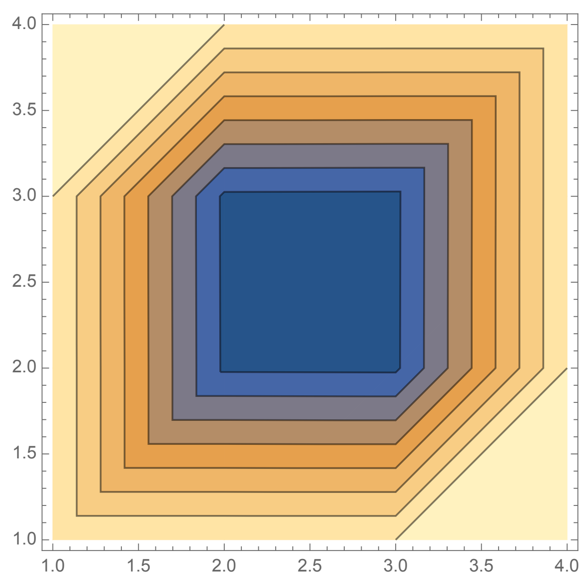

Figure 5.2: The corresponding contour plot for the case of one vortex on a square grid of size 4.

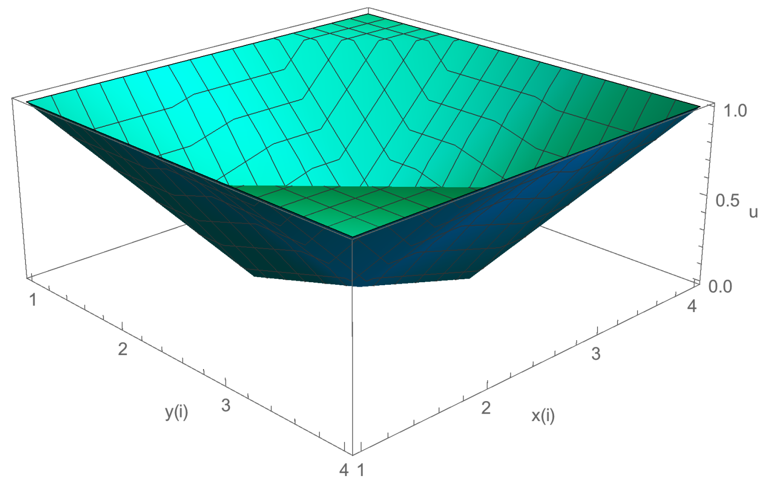

Figure 5.3: The plot of $|\phi|$ for the two-vortex case on a square grid of size 4.

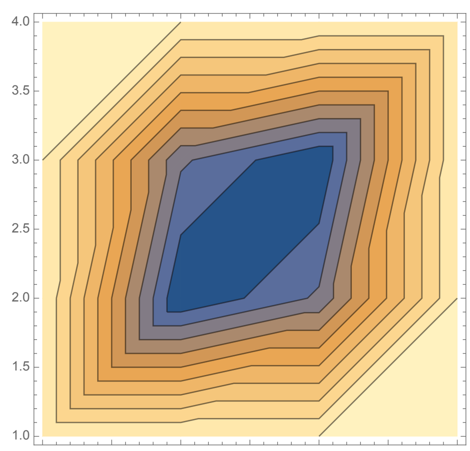

Figure 5.4: The corresponding contour plot for the two-vortex case on a square grid of size 4.

```
Off[General::spell1];
Clear[Dirichlet, q, n, m, w];
Dirichlet[q_List, n_, m_, w_] :=
 Module[{r, s, k},
 Err = 1;
   pt = Table[RandomPoint[Rectangle[{1, 1}, {4, 4}], 1][[1]], {l, 100}];
   k = 0;
 While[(Err > 0.001) && (k <= 4),
  Module[{},
   Err = 0.0;
   For[s = 2, s <= m - 1, s++,
    Module[{},
     For[r = 2, r <= n - 1, r++,
      Module[{},

          Relax =
           w / 4.0 (u[[r, s + 1]] + u[[r, s - 1]] + u[[r + 1, s]] + u[[r - 1, s]] - 4.0 u[[r, s]]) -
            (1)^2/4 ((Exp[u[[r, s]] + Sum[Log[(Norm[{r, s} - pt[[p]]])^2], {p, 1, 20}]]) - 1);
       u[[r, s]] = u[[r, s]] + Relax;
       Err = Max[Err, Abs[Relax]];
      ]]]];
   Print["Max grid change = ", Err];
   k = k + 1]]];
On[General::spell1];
Clear[u]; u = Table[4, {4}, {4}];
With[{n = 4}, For[r = 1, r ≤ (n) 1, r += 1,
  u[[r, 1]] = -(Sum[Log[(Norm[{r, 1} - pt[[p]]])^2], {p, 1, 20}])]]
With[{n = 4}, For[r = 1, r ≤ (n) 1, r += 1,
  u[[r, 4]] = -(Sum[Log[(Norm[{r, 4} - pt[[p]]])^2], {p, 1, 20}]) ]]
With[{n = 4}, For[s = 1, s ≤ (n) 1, s += 1,
  u[[1, s]] = -(Sum[Log[(Norm[{1, s} - pt[[p]]])^2], {p, 1, 20}])]]
With[{n = 4}, For[s = 1, s ≤ (n) 1, s += 1,
  u[[4, s]] = -(Sum[Log[(Norm[{4, s} - pt[[p]]])^2], {p, 1, 20}])]]
w = 4 / (2 + Sqrt[4 - (Cos[Pi / 3] + Cos[Pi / 3]) ^ 2]) // N;
duval = Dirichlet[u, 4, 4, w];
u1 = N[u, 4]
f = Table[Sum[Log[(Norm[{r, s} - pt[[p]]])^2], {p, 1, 20}], {r, 4}, {s, 4}]
```

Figure 5.5: This is the numerical code that we use to solve the elliptic equation (5.11) subject to the boundary conditions (5.12) for an N-vortex system.

5.2 Vortices without Boundary Conditions

On the surface of a sphere, which is a compact region without a boundary, the Bradlow inequality holds. For this case, in order to simplify numerical calculation, we solved the equation (5.11) on the surface of a cube with a maximum cell size of 0.01. A vortex here is represented by a sphere of diameter 0.05, as seen in figures 5.6, 5.7 and 5.8. Our code doesn't give solutions for vortex numbers that exceed 300, which of course corresponds to the Bradlow bound in this case.

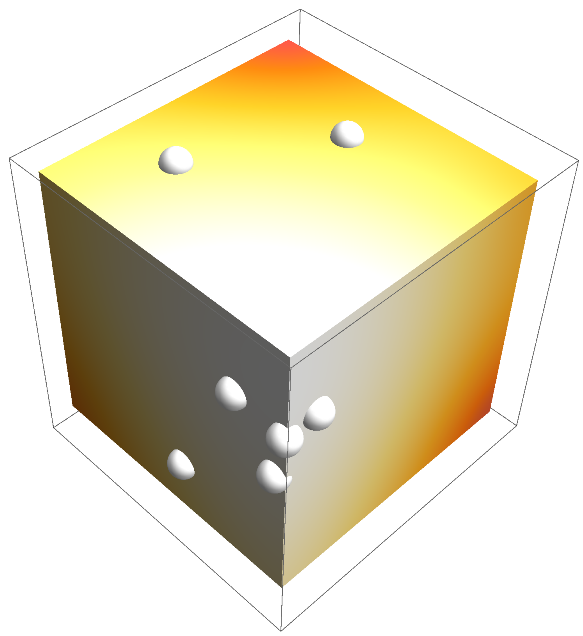

Figure 5.6: A plot for 10 vortices

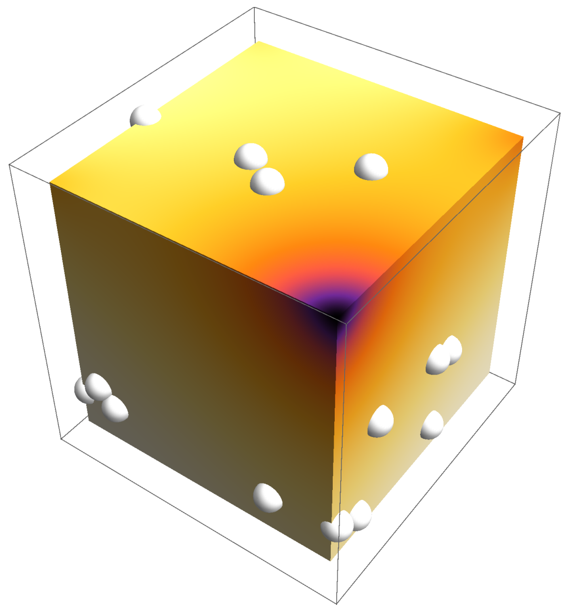

Figure 5.7: A plot for 30 vortices

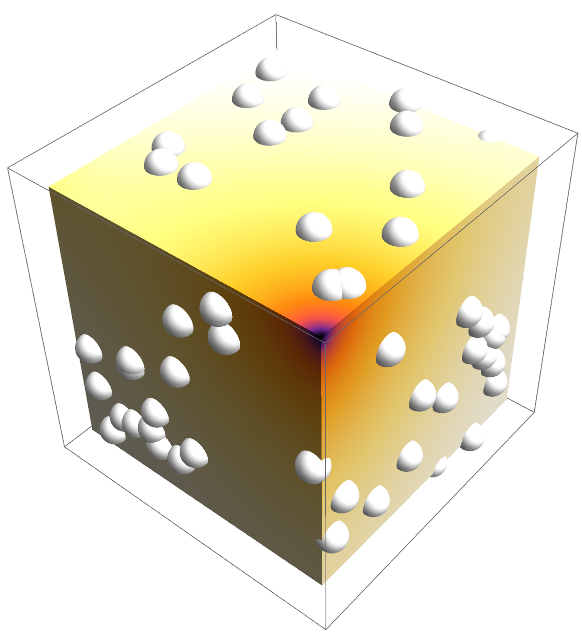

Figure 5.8: A plot for a 100 vortices

Chapter 6

Conclusion

6.1

Pulsars generate extremely high magnetic fields and observations made by the Chandra X-ray telescope confirm this [19]. X-rays are emitted from the Crab Nebula and we know that there's a pulsar at its core. These X-rays are emitted as a result of a centrally located strong magnetic source affecting the ionized plasma around it. This strong magnetic field is generated by charged particles in the neutron superfluid within the core of the pulsar. These charged particles are highly conductive and come in the form of protons and other unknown exotic particles.

The protons inside the core of a neutron star behave like a Type II superconductor with an upper critical field of approximately 10^{16} G [20; 21]. In this case, the magnetic field exists in the form of Abrikosov fluxoids. Recently, the magneto-hydrodynamic behavior of such superconductors has been investigated and it appears that the time-scales of field evolution are extremely long. Therefore, the processes responsible for magnetic field evolution can only be effective if the currents supporting the magnetic field are located in the crustal region which has metal-like transport properties. The basic observed quantities of a pulsar are the angular frequency Ω, the spin period P and the period derivative $\dot{P}$. The formula for the surface magnetic field of a pulsar is given by [20; 21],

$$B_s = 3.2 \times 10^{19} \left(P \ \dot{P} \right)^{\frac{1}{2}} \ G, \tag{6.1}$$

where,

$$P = \frac{2\pi}{\Omega}. \tag{6.2}$$

If we assume that $\dot{P}$ is constant, we can see from equations (6.1) and (6.2) that during the spin down of a pulsar, P increases and therefore B_s increases as well. We therefore make the following conclusions:

1. The surface of the outer core of a pulsar is a Type II superconductor, with an upper critical field of approximately 10^{16} G.

2. As the pulsar spins, magnetic vortices penetrate the outer core surface, similar to our discussion in chapter 2, see Figure 2.4.

3. As the pulsar spins down, its surface magnetic field B_s increases which corresponds to an increase in the number of magnetic vortices that penetrate the outer core surface. This process continues until the upper critical field is reached, at which point the outer core surface becomes normal conducting and the energy of the magnetic vortices is transferred to the outer core surface, which causes the pulsar glitch. We also postulate that this process is related to the Bradlow bound of the outer core surface.

6.2

Through reviewing the work of Cabral et.al [14], where we work in the London approximation, we were able to do the following:

1. Derive a formula (4.12) for vortex absorption in mesoscopic discs. We gave a precise explanation of what we mean by vortex absorption in this context. With this formula, we showed that for every vortex in the system, there is an associated applied magnetic

field value at which the vortex ceases to be part of the configuration, thereby releasing its free energy contribution into the disc to do work.

2. We also gave an expression (4.13) for the amount of free energy an absorbed vortex will release.

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