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Department of Mathematics

THE POWER FUNCTION

BY

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A thesis prepared under the supervision of
Dr. H. Rose in fulfilment of the requirements
for the degree of Master of Science in Mathematics

1993

ACKNOWLEDGEMENTS

This thesis was written at the Mathematics Department of the University of Cape Town under the supervision of Dr. Henry Rose. Dr. Rose above all others has been responsible for my mathematical education, and I will always be indebted to him for his encouragement and help.

I also wish to thank Dr. Peter Bruyns for support while Dr. Rose was on sabbatical.

Finally, I thank the head of the Mathematics Department, Professor Chris Brink, for providing me with a teaching assistentship and further help towards obtaining the financial support I needed in order to complete this thesis.

SYNOPSIS

The axioms of ZFC provide very little information about the possible values of the power function (i.e. the map $\kappa \longrightarrow 2^{\kappa}$). In this dissertation, we examine various theorems concerning the behaviour of the power function inside the formal system ZFC , and we shall be particularly interested in results which provide constraints on the possible values of the power function. Thus most of the results presented here will be consistency results.

A theorem of Easton (Theorem 2.3.1) shows that, when restricted to regular cardinals, the power function may take on any *reasonable* value, and thus a considerable part of this thesis is concerned with the power function on singular cardinals.

We also examine the influence of various strong axioms of infinity, and their generalization to smaller cardinals, on the possible behaviour of the power function.

CONTENTS

INTRODUCTION

The generalized continuum problem is one of the longest standing open problems in set theory, and attempts at its resolution can safely be said to be the prime cause of progress in that subject. Our main aim is to give an overview of the work done towards the resolution of this problem. Thus many of the results proved here may be found in textbooks such as [Jech 1978] or [Kunen 1980], and no original results are presented. Because of the sheer volume of the work on the generalized continuum problem, we have had to be quite choosy concerning what to include, and in how much detail. Thus regrettably the best results are not always proved because their inclusion would make the length of this dissertation unmanageable.

We have given the name "power function" to what is usually referred to as the "generalized continuum function" (i.e. the map $\kappa \longrightarrow 2^{\kappa}$) as we feel that this is a better description. After all, $2^{\aleph_1}$ has nothing to do with the continuum (the set of real numbers), but everything with the cardinality of the power set of $\aleph_1$.

In an attempt to make this dissertation fairly self–contained, we have included 4 appendices which deal with standard topics in set theory, such as large cardinals and forcing. We refer to these appendices as needed.

We begin in Chapter 1 with some results on the power function that can be proven inside ZFC. After a quick review of the axioms of ZFC and the introduction of the necessary notation in Section 1.1, we look at some elementary general results concerning the power function and the Singular Cardinals Hypothesis in Section 1.2. Theorems proved here include that of Cantor ($2^{\kappa} > \kappa$ for any cardinal κ) and Bukovsky (If the power function is eventually constant below a singular cardinal, it takes that constant value at the singular cardinal as well). The gimel function is also introduced, and it it shown that the power function may be computed from it. In Section 1.3 we look at the behaviour of the power function at singular cardinals of uncountable cofinality. The main theorems proved here are the Theorem 1.3.3 and Theorem 1.3.4. Theorem 1.3.3 states that the GCH cannot fail for the first time at a singular cardinal of uncountable cofinality (due to [Silver 1974]). Theorem 1.3.4 states that if κ is a singular strong limit cardinal of uncountable cofinality such that $\kappa < \aleph_\kappa$, then also $2^{\aleph_\kappa} < \aleph_\kappa$ (due to [Galvin–Hajnal 1975]). We also prove a result of [Magidor 1977] which gives a bound on $2^{\aleph_{\omega_1}}$ if $\aleph_{\omega_1}$ is strong limit and the Chang Conjecture holds (Theorem 1.3.5), as this is easily proven with the tools developed in this section.

In Chapter 2 we discuss the GCH. In Section 2.1 we prove that the GCH is consistent with ZFC by showing that it holds in the constructible universe. In Section 2.2 we prove that the GCH is independent of ZFC using Cohen extensions to manipulate the power function at regular cardinals. The main result of this section is Theorem 2.2.6, due to [Cohen 1963–1964]. In Section 2.3 we also prove a result of Easton[1964] which solves the generalized continuum problem for regular cardinals (Theorem 2.3.1). We also prove that the SCH is consistent with ¬SCH (Theorem 2.3.6).

Chapter 3 deals with the bearing of large cardinals on the GCH. In Section 3.1 we show that GCH is consistent with the existence of a measurable cardinal by exhibiting the model $L[\mathcal{U}]$ (Theorem 3.1.3 due to [Silver 1971a]). Theorem 3.1.4 states that the GCH cannot fail for the first time at a measurable cardinal. In Section 3.2 we show that the failure of the GCH at a measurable cardinal is, consistency–wise, a stronger assumption than the existence of a measurable cardinal: If the GCH does fail at a measurable cardinal, then there are inner models of ZFC with arbitrarily large numbers of measurable cardinals (Theorem 3.2.5, due to [Kunen 1971a]). We then go on to Section 3.3 to discuss the uniformizing effect of a compact cardinal on the behaviour of the power function. In particular, if a compact cardinal exists, then the SCH holds above it (Theorem 3.3.1, due to [Solovay 1974]. In Section 3.4 we introduce reverse Easton extensions, and use it to obtain models in which the GCH fails at a measurable cardinal (Theorem 3.4.1, due to Silver). The material presented here is important for some results of Section 4.1

In Chapter 4 we discuss the failure of the SCH. In Section 4.1 we prove that it is consistent, modulo the existence of a cardinal with a certain degree of supercompactness, for the SCH to fail (Theorem 4.1.5, due to Silver). We introduce Prikry forcing in order to accomplish this. The question then arises whether the SCH can fail at $\aleph_\omega$, the smallest singular cardinal. We also prove that the gimel function is not determined by the power function (Theorem 4.1.6). In Section 4.2 we introduce a forcing method, due to [Magidor 1977b,c], that was the first to make any progress towards the resolution of the singular cardinals problem. We then prove that it is consistent, modulo some degree of supercompactness, that the SCH fails at $\aleph_\omega$ (Theorem 4.2.1). In Section 4.3 we describe further applications of Magidor's forcing method: Theorem 4.3.1 (due to [Magidor 1977c]) states that the GCH can fail for the first time at $\aleph_\omega$ (although a huge cardinal is needed) and Theorem 4.3.4. (due to [Apter 1984]) states that it is consistent for the the SCH to fail on an unbounded class of cardinals. We also state a theorem of Shelah[1983] (Theorem 4.3.21) which improves the degree of the failure of the GCH at a strong limit $\aleph_\omega$ given by Theorem 4.2.1. Many of the results of this section are stated without proof, although the general arguments are outlined. In Section 4.4 we discuss, largely without proofs, the failure of the SCH as a large cardinal axiom. We introduce the covering property, and discuss results in [Devlin–Jensen 1975] and [Dodd–Jensen 1982] on Covering Lemmas and their relation to the SCH (Theorems 4.4.5 and 4.4.6). We end this chapter with a statement of a theorem of Gitik[1991] which provides the exact strength of the failure of the SCH (Theorem 4.4.8), namely ¬SCH is equiconsistent with to $o(\kappa) = \kappa^{++}$.

Chapter 5 is mainly concerned with the assumption of several "large cardinal–like" hyptheses on small cardinals (especially ω_1). We discuss work done by Jech and Prikry on the effect of saturated ideals on the power function ([Jech–Prikry 1979]). In Section 5.1 we present another proof of Theorem 1.3.3, and show similarly that the GCH cannot fail for the first time at a regular κ which carries a κ^+–saturated ideal (Theorem 5.1.3). We also introduce the notion of a nice cardinal function and indicate how it may be used to give upper bounds for the power function at cardinals which carry saturated ideals (Theorem 5.1.17). In Section 5.2 we look again at the power function at singular cardinals, particularly $\aleph_{\omega_1}$. Theorem 5.2.2 states that if ω_1 carries an ω_2–saturated ideal and $\aleph_{\omega_1}$ is strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$. These result are all reminiscent of Theorem 1.3.4. We also indicate how to obtain bounds for $2^{\aleph_\kappa}$ even if $\kappa = \aleph_\kappa$, which Theorem 1.3.4 could not do. In Section 5.3 we prove that if there is an η–saturated ideal over $[\lambda]^{<\kappa}$ (where κ is inaccessible, and $\eta < \lambda$) then the SCH hold on the interval of cardinals between η and λ (Theorem 5.3.4 due to [Matsubara 1992]). We also present another proof of Theorem 3.3.1.

In Chapter 6 we introduce pcf–theory as a tool to study cardinal arithmetic. This work, due to Shelah, is related to the material in Section 1.3, as it is concerned only with what is provable in ZFC. The first three sections of this chapter are very technical in nature and build up the necessary machinery required for the study of the power function. They say nothing about the behaviour of the power function themselves. In Section 6.4, we apply pcf theory to prove some results on the power function at singular strong limit cardinals: In particular, Theorem 6.4.9 states that if $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{(2^\omega)^+}$. Next we give outlines of the proofs of various other results: Theorem 6.4.10 gives a different bound for the value of $2^{\aleph_\delta}$ for strong limit $\aleph_\delta$. We end with a statement of Theorem 6.4.13, which states that if $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Finally, in the epilogue we attempt to provide a very brief and easy–reading history of the generalized continuum problem, and we look at its possible future.

Chapter 1: CARDINAL ARITHMETIC

Since Cohen first applied forcing to set theory [Cohen 1963–1964] it has been known that many statements about the power function (that is, the map $\kappa \longrightarrow 2^\kappa$) are neither provable nor disprovable in ZFC. This chapter is concerned with what we can prove about the power function. In Section 1.1 we make some general remarks about ZFC and introduce standard set theoretical notation. Section 1.2 consists of very elementary results from cardinal arithmetic and concludes with a brief discussion of various hypotheses on the behaviour of the power function, namely the *Generalized Continuum Hypohesis* (GCH) and the *Singular Cardinals Hypothesis* (SCH). In Section 1.3 we use machinery from combinatorial set theory to prove several fascinating (and perhaps unexpected) results concerning the behaviour of the power function at singular cardinals of uncountable cofinality.
The main results of this chapter are Theorems 1.3.3 and 1.3.4.

§ 1.1 ZFC

Throughout we shall work in one particular set theory only, namely *Zermelo–Fraenkel* set theory with the *Axiom of Choice* (ZFC). This theory was formulated by Zermelo (1908) and Fraenkel (1922) using a countable first order language $\mathcal{L}$ which has the standard logical symbols $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\longrightarrow$ (implies), $\forall$ (for all) and $\exists$ (there exists), the usual variables v_i and two binary predicates $\in$ (is element of) and $=$ (is equal to). Moreover we define the binary symbol $\subseteq$ (is a subset of) in $\mathcal{L}$ by: $x \subseteq y \longleftrightarrow \forall z(z \in x \longrightarrow z \in y)$,
as well as a constant symbol $\emptyset$ (empty set): $y = \emptyset \longleftrightarrow \forall x(x \notin y)$.

The axioms of ZFC are:
(1) Extensionality: $\forall x \forall y[\forall z(z \in x \longleftrightarrow z \in y) \longrightarrow x = y]$
(2) Union: $\forall x \exists y \forall z[z \in y \longleftrightarrow \exists r \in x(z \in r)]$
(3) Infinity: $\exists x[x \neq \emptyset \wedge \forall y(y \in x \longrightarrow \exists z(z \in x \wedge y \in z))]$
(4) Power Set: $\forall x \exists y \forall z(z \subseteq x \longrightarrow z \in y)$
(5) Foundation: $\forall x[x \neq \emptyset \longrightarrow \exists y(y \in x \wedge \forall z(z \in y \longrightarrow z \notin x))]$
(6) Comprehension Schema: $\forall \vec{a} \forall x \exists y \forall z[z \in y \longleftrightarrow (z \in x \wedge \Psi(z,\vec{a})]$ for any first order formula Ψ of $\mathcal{L}$.

(7) Replacement Schema: $\forall\vec{a}\forall x\forall y\forall z[(\Psi(x,y,\vec{a}) \wedge \Psi(x,z,\vec{a}) \longrightarrow y = z) \longrightarrow$

$\forall X\exists Y\forall y(y \in Y \longleftrightarrow \exists x \in X(\Psi(x,y,\vec{a})))]$ for every formula $\Psi(x,y,\vec{a})$ of $\mathcal{L}$.

(8) Choice: $\forall x[\forall y(y \in x \longrightarrow y \neq \emptyset) \wedge \forall y\forall y'(y \in x \wedge y' \in x \wedge y \neq y' \longrightarrow \forall w(w \in y \longrightarrow w \notin y'))$
$\longrightarrow \exists z\forall y(y \in x \longrightarrow \exists v\forall w(w \in y \wedge w \in z \longleftrightarrow v = w))]$

The Comprehension Schema guarantees that $\emptyset$ exists in any model of ZFC.
We allow the formation of *classes*, although these are strictly speaking not objects which ZFC deals with. If Ψ is a formula, and $\vec{a}$ is a set, then $C = \{x: \Psi(x,\vec{a})\}$ is a class. Clearly any set is a class. Classes that are not sets are termed *proper classes*. The class $\{x: x = x\}$ of all sets is denoted by V. The class of all ordinals is denoted by **On**, and the class of cardinals is denoted $\aleph$. **On** is a well–ordered class, and $\aleph$ inherits this well–ordering.
A class **C** is said to be *transitive* provided whenever $x \in C$, also $x \subseteq C$. We shall generally be interested in *transitive* models of set theory only. An *inner model* is a transitive model of ZFC which includes **On**.

We now introduce notation used throughout this dissertation:
The *cardinality* of a set X is denoted by $|X|$, and this is the least ordinal that can be placed in a one–to–one onto correspondence with X. The usual symbols $\cup$, $\cap$, $-$ and $\times$ will be used for *union*, *intersection*, *difference* and *Cartesian product*.
If X is a set then axioms (4) and (6) guarantee the existence of the set of all subsets of X. This set is called the *power set* of X and denoted $\mathcal{P}(X)$. Certain subsets of $\mathcal{P}(X)$ will often play an important role. $[X]^{\alpha}$ is the set of all subsets of X of order type α if α is an ordinal, and it represents the set of all subsets of X of cardinality exactly α if α is a cardinal. Similarly, $[X]^{<\alpha}$ and $[X]^{\leq\alpha}$ represent the sets of all subsets of X of order type (cardinality) $< \alpha$ and $\leq \alpha$ repectively. The order type of a set of ordinals X is denoted by otp(X).
The *ordered n–tuples* are defined following the usual convention: $(x,y) = \{\{x,y\},\{x\}\}$, and $(x_1, \ldots, x_{n+1}) = ((x_1, \ldots, x_n), x_{n+1})$. An n–ary *relation* is just a set of ordered n–tuples.
A *function* is a binary relation Rxy such that for all x, if Rxy and Rxz, then $y = z$. The unique y such that Rxy is then denoted R(x).
If R is a binary relation, then $\mathrm{dom}(R) = \{x: \exists yRxy\}$ denotes the *domain* of R, and $\mathrm{ran}(R) = \{y: \exists xRxy\}$ denotes the *range* of R. A function f with domain X and range a subset of Y is written $f:X \longrightarrow Y$, following custom. If $Z \subseteq X$, then the *restriction* of f to Z is written $f|Z$.

$f''Z$ is the *image* of Z under f: $f''Z = \{f(x): x \in Z\}$. The axiom schema of replacement guarantees that $f''Z$ is a set. If $S \subseteq Y$, then $f^{-1}S$ is the *inverse image* of S under f, i.e. $f^{-1}S = \{x \in X: f(x) \in S\}$. If X and Y are sets, then the set of all functions with domain X and range Y is written ${}^{X}Y$.

Next we introduce some notation that is useful for doing cardinal arithmetic:
For any family $\{\kappa_i: i \in I\}$ of cardinals,

$$\sum_{i \in I} \kappa_i$$

is the cardinality of the disjoint union of the κ_i, and

$$\prod_{i \in I} \kappa_i$$

is the cardinality of the Cartesian product of the κ_i.
κ^λ is just $|{}^{\lambda}\kappa|$; there is a bijection between ${}^{\lambda}\kappa$ and $\prod_{\xi<\lambda} \kappa$.
If X is a set, then there is a bijection between the power set $\mathcal{P}(X)$ of X and the set of all maps with domain X and range a subset of $2 = \{0,1\}$, i.e. $|\mathcal{P}(X)| = |{}^{X}2|$. Hence the (class) function $P: \aleph \longrightarrow \aleph$ given by $P(\kappa) = 2^\kappa$ is called the *power function.*
If κ, λ are cardinals, then $\lambda^{<\kappa} = \Sigma\{\lambda^\alpha: \alpha \text{ a cardinal} < \kappa\}$.
The least ordinal λ such that there exists a family $\{\kappa_\xi: \xi < \lambda\}$ of cardinals $\kappa_\xi < \kappa$ with $\kappa = \sum_{\xi<\lambda} \kappa_\xi$, is called the *cofinality* of κ, and denoted $cf(\kappa)$. If X is a set of ordinals, then sup(X) is the least ordinal which is an upper bound for X, and inf(X) is the greatest ordinal which is a lower bound for X. Because the ordinals are well–ordered we always have $\inf(X) \in X$. A cardinal κ is *singular* if $cf(\kappa) < \kappa$, and *regular* otherwise.
An increasing sequence $(\kappa_\xi: \xi < \lambda)$ of cardinals $< \kappa$ is said to be *cofinal* in κ provided $\kappa = \sum_{\xi<\lambda} \kappa_\xi$. A cofinal sequence is said to be *continuous* provided $\kappa_\alpha = \sum_{\xi<\alpha} \kappa_\xi$ for every limit ordinal $\alpha < \lambda$. If λ is any ordinal, then λ^+ denotes its successor cardinal (i.e. the least cardinal $> \lambda$). In general, if μ is an ordinal, we denote the μ^{th} successor of λ by $\lambda^{+\mu}$ or $\lambda(+\mu)$, where:

$$\lambda(+0) = \lambda$$

$$\lambda(+\mu+1) = \lambda(+\mu)^+, \text{ and}$$

$$\lambda(+\delta) = \sup\{\lambda(+\mu) : \mu < \delta\} \text{ if } \delta \text{ is a limit ordinal.}$$

In addition, $\lambda(+2)$ and $\lambda(+3)$ may be written λ^{++} and λ^{+++} respectively.
The set of natural numbers will be denoted by the symbol ω. We shall write $\aleph_\alpha$ for $\omega(+\alpha)$.

A cardinal κ is said to be *limit* if it is not λ^+ for any cardinal λ, and a *successor* cardinal otherwise. A limit cardinal κ is said to be *strong limit* provided that whenever $\lambda < \kappa$, then also $2^\lambda < \kappa$.

Finally we introduce some notation from model theory:

$\vdash$ will denote the *satisfaction relation.*

$\cong$ will denote the *isomorphism relation.*

$\equiv$ will denote *elementary equivalence.*

$\leq_e$ will stand for *elementary submodel.*

$\Vdash$ will denote the *forcing relation.*

$\square$ marks the end of a proof.

§ 1.2 Elementary Results on Cardinal Arithmetic.

In this section we shall review some of the well–known properties of cardinal exponentiation. We will introduce the *gimel* function and indicate how it can be used to compute the power function. Two hypotheses on the behaviour of the power function, namely the Generalized Continuum Hypothesis and the Singular Cardinals Hypothesis will also be discussed.

Cantor first introduced the notion of cardinality using one–to–one functions and proved that the cardinality of the set of real numbers is strictly greater than the cardinality of the set of natural numbers using a diagonal argument. He also proved that the set of real numbers has the same cardinality as the power set of ω (where ω is the set of natural numbers), and proved generally that for any set X we have $2^{|X|} = |\mathcal{P}(X)| > |X|$. The set of real numbers therefore has cardinality 2^ω. Merely knowing that $2^\omega > \omega$ does not say very much about the size of 2^ω, however, and Cantor first made the conjecture that $2^\omega = \omega^+$ in 1878. This conjecture is known as the Continuum Hypothesis. The *Generalized Continuum Hypothesis* (GCH) was easily abstracted from this:

The Generalized Continuum Hypothesis:

For all infinite cardinals κ, we have $2^\kappa = \kappa^+$.

This hypothesis is equivalent to the following statement: The power function is *strictly* increasing and its range is the class of all successor cardinals.
In 1905 Julius König proved the following well–known lemma about cardinal arithmetic:

Lemma 1.2.1 [König 1905]:
If $\kappa_i < \lambda_i$ *for every* $i \in I$ *then* $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.

Proof: It is not hard to see that $\sum_{i\in I} \kappa_i \leq \prod_{i\in I} \kappa_i$. Thus it is enough to show that if $A_i \subseteq \prod_{i\in I} \lambda_i$ for $i \in I$ such that $|A_i| \leq \kappa_i$ then $\bigcup_{i\in I} A_i \neq \prod_{i\in I} \lambda_i$. Let $B_i = \{f(i): f \in A_i\}$. Then $B_i \subseteq \lambda_i$ and $|B_i| \leq \kappa_i < \lambda_i$ so we may choose $a_i \in \lambda_i - B_i$ for each $i \in I$. Let $f \in \prod_{i\in I} \lambda_i$ such that $f(i) = a_i$ for all $i \in I$. Then $f \notin \bigcup_{i\in I} A_i$ as required.

□

The following important corollaries follow easily from the above theorem. They will often be used without explicit mention.

Corollary 1.2.2:

(1) [Cantor 1878] $\kappa < 2^\kappa$ *for all cardinals* κ.

(2) $cf(2^\kappa) > \kappa$.

(3) $cf(\lambda^\kappa) > \kappa$ *for all cardinals* λ,κ.

(4) $\kappa^{cf\kappa} > \kappa$.

Proof: (1) $\kappa = \sum_\kappa 1 < \prod_\kappa 2 = 2^\kappa$.
(2) Let $(\gamma_\alpha: \alpha < \kappa)$ be a sequence of cardinals below 2^κ. Then $\sum_{\alpha<\kappa} \gamma_\alpha < \prod_{\alpha<\kappa} 2^\kappa = 2^\kappa$.
Hence $(\gamma_\alpha: \alpha < \kappa)$ cannot be cofinal in 2^κ.
(3) Similar to (2).
(4) Let $(\kappa_\xi: \xi < cf\kappa)$ be cofinal below κ. Then $\kappa = \sum_{\xi<cf\kappa} \kappa_\xi < \prod_{\xi<cf\kappa} \kappa = \kappa^{cf\kappa}$.

□

We have shown that the power function $P(\kappa) = 2^{\kappa}$ satisfies the following properties:

(1) If $\kappa < \lambda$ then $P(\kappa) \leq P(\lambda)$.

(2) $P(\kappa) > \kappa$

(3) $cf(P(\kappa)) > \kappa$

It is clear that (2) follows from (3). These conditions seem very weak, yet for regular cardinals this is all that we can say about P, for a theorem of Easton (Thm. 2.3.1) states that, for regular cardinals (1),(2),(3) are the only restrictions on the power function that are provable in ZFC. For some time it was expected that the power function would prove to be just as free at singular cardinals and that one only had to find the right forcing conditions. Already in 1965, however, a theorem due to Bukovsky (Theorem 1.1.4) showed that under certain conditions the value of the power function at a singular cardinal is determined by the axioms of ZFC. We say that the power function is eventually constant below a limit cardinal κ provided that there exists a cardinal $\lambda_0 < \kappa$ such that whenever $\lambda_0 \leq \lambda < \kappa$, we have $2^{\lambda} = 2^{\lambda_0}$. Note that in that case $2^{<\kappa} = 2^{\lambda_0}$. In order to prove Theorem 1.1.4, we need the following lemma.

Lemma 1.2.3 [Bukovsky 1965]:
For any cardinal κ, $2^{\kappa} = (2^{<\kappa})^{cf\kappa}$.

Proof: If κ is regular, then $cf\kappa = \kappa$, and so $2^{\kappa} \leq (2^{<\kappa})^{cf\kappa} \leq (2^{\kappa})^{\kappa} = 2^{\kappa}$.
Suppose now that κ is singular, and let $(\kappa_{\xi}: \xi < cf\kappa)$ be a continuous cofinal sequence below κ. Then $2^{\kappa} = 2^{\Sigma \kappa_{\xi}} = \Pi\, 2^{\kappa_{\xi}} \leq \Pi\, 2^{<\kappa} = (2^{<\kappa})^{cf\kappa} \leq (2^{\kappa})^{\kappa} = 2^{\kappa}$.

□

Theorem 1.2.4 [Bukovsky 1965]:
Let κ be a singular cardinal such that the power function below κ is eventually constant. Then $2^{\kappa} = 2^{<\kappa} = 2^{\lambda_0}$.

Proof: First note that $2^{\kappa} = (2^{<\kappa})^{cf\kappa}$ by Lemma 1.2.3. Now if the conditions of the theorem hold, then $2^{<\kappa} = 2^{\lambda_0}$, so since $\lambda_0 \leq \lambda_0 \cdot cf\kappa < \kappa$ we have $2^{\kappa} = 2^{\lambda_0 \cdot cf\kappa} = 2^{\lambda_0}$.

□

Note conversely that *if κ is a limit cardinal for which $2^{<\kappa} = 2^{\kappa}$, then the power function is eventually constant below κ*: For if the power function is not eventually constant below κ, then $\text{cf}(2^{<\kappa}) = \text{cf}(\kappa) \leq \kappa$, but $\text{cf}(2^{\kappa}) > \kappa$ by Corollary 1.2.2. Hence $2^{<\kappa} < 2^{\kappa}$.

The *gimel* function G: $\aleph \longrightarrow \aleph$ is the function $\kappa \longrightarrow \kappa^{\text{cf}(\kappa)}$. From Corollary 1.2.2 it follows that $G(\kappa) > \kappa$ and that $\text{cf}(G(\kappa)) > \text{cf}\kappa$. Note that the gimel function coincides with the power function on regular cardinals, and on strong limit cardinals: If κ is regular, then $\text{cf}\kappa = \kappa$, and so $G(\kappa) = \kappa^{\kappa} = 2^{\kappa}$. If κ is strong limit, then $2^{<\kappa} = \kappa$, and so $2^{\kappa} = (2^{<\kappa})^{\text{cf}\kappa} = \kappa^{\text{cf}\kappa}$. Hence the power function and the gimel function differ only on singular cardinals which are not strong limit. It turns out that the power function can be computed inductively in terms of the gimel function:

Theorem 1.2.5 [Bukovsky 1965]:

(1) *If κ is successor, then $2^{\kappa} = G(\kappa)$*

(2) *If κ is limit and the power function is eventually constant below κ, then $2^{\kappa} = 2^{<\kappa}.G(\kappa)$.*

(3) *If κ is limit and the power function is not eventually constant below κ, then $2^{\kappa} = G(2^{<\kappa})$.*

Proof: (1) If κ is successor, then κ is regular.

(2) If κ is singular, then $2^{\kappa} = 2^{<\kappa}$ and $2^{\kappa} \geq G(\kappa)$, whereas if κ is regular, then $2^{\kappa} = G(\kappa)$ and $2^{\kappa} \geq 2^{<\kappa}$.

(3) If the power function is not eventually constant below κ, then $\text{cf}(2^{<\kappa}) = \text{cf}\kappa$. Hence $G(2^{<\kappa}) = 2^{\kappa}$ by Lemma 1.2.3.

□

Thus the power function is completely determined by the gimel function, meaning that two models of set theory with the same gimel function must necessarily have the same power function as well. The reverse is not true, however, because with the aid of some large cardinal axioms one can construct transitive models of ZFC having the same cardinalities and cofinalities, the same power function, but different gimel functions. (This result follows from theorems due to Prikry and Silver. The argument is outlined in Section 3.5)

We shall now formulate a principle, called the *Singular Cardinals Hypothesis* (SCH), which will ensure that for singular cardinals κ, 2^κ is the least value that is consistent with the restrictions on the power function, namely that (1) the power function is non–decreasing, and (2) $\mathrm{cf}(2^{<\kappa}) > \kappa$.

The Singular Cardinals Hypothesis:

Let κ be a singular cardinal: If $2^{\mathrm{cf}\kappa} < \kappa$, then $\kappa^{\mathrm{cf}\kappa} = \kappa^+$.

Lemma 1.2.6:

If κ is singular and the SCH *holds, then*

$$2^\kappa = \begin{cases} 2^{<\kappa} & \text{if the power function is eventually constant below } \kappa \\ (2^{<\kappa})^+ & \text{otherwise} \end{cases}$$

Proof: Let $\lambda = 2^{<\kappa}$. Since κ is singular, it is limit so that $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = \lambda^{\mathrm{cf}(\kappa)}$. If the power function is eventually constant below κ, then by Theorem 1.2.4 we have $2^{<\kappa} = 2^\kappa$. Otherwise, $\mathrm{cf}\lambda = \mathrm{cf}\kappa$ and so $2^{\mathrm{cf}\lambda} = 2^{\mathrm{cf}\kappa} < 2^{<\kappa} = \lambda$ implies $\lambda^{\mathrm{cf}\lambda} = \lambda^+$. But $\lambda^{\mathrm{cf}\lambda} = (2^{<\kappa})^{\mathrm{cf}\kappa} = 2^\kappa$, and thus $2^\kappa = \lambda^+ = (2^{<\kappa})^+$, as is required.

□

Note that the singular cardinals hypothesis is clearly consistent with ZFC, as it is a consequence of the GCH. In Section 2.3 we shall see that it is also consistent with ¬GCH using forcing. However, if the SCH holds, then the GCH cannot fail everywhere:

Lemma 1.2.7:

If the SCH holds, then there exist arbitrarily large cardinals κ such that $2^\kappa = \kappa^+$.

Proof: Given any cardinal λ, define a sequence of cardinals $(\lambda_n)_{n\in\omega}$ as follows:

$$\lambda_0 = \lambda$$

$$\lambda_{n+1} = 2^{\lambda}.$$

Finally, let $\kappa = \sup\{\lambda_n : n \in \omega\}$.

Then $2^{<\kappa} = \sum_{n<\omega} 2^{\lambda_n} = \sum_{n<\omega} \lambda_{n+1} = \kappa$. Since κ is singular, we thus have $2^{\kappa} = (2^{<\kappa})^+ = \kappa^+$.

□

Theorem 2.3.1 will conclusively establish the possible values of the power function at regular cardinals, so our main effort will go into collecting a bag of theorems which deal similarly with the power function at singular cardinals. This is the so–called *singular cardinals problem.* We shall see in Chapter 3 that the existence of large cardinals may have a profound effect on cardinal arithmetic.In particular, a theorem of Solovay states that if there exists a compact cardinal κ in the universe, then the SCH holds for all cardinals above κ (Theorem 3.3.1). Compact cardinals are defined in Appendix 3.3.

The singular cardinals problem is deeply related to the existence of large cardinals in the following sense:

If no large cardinals exist, then there is no singular cardinals problem.

To be more precise, Jensen has proved that if a certain set of natural numbers called $0^{\#}$ does not exist, then the singular cardinals hypothesis holds (see [Devlin–Jensen 1975]; $0^{\#}$ is discussed in Appendix 1.2). In [Dodd–Jensen 1982] it is proved that the failure of the SCH implies the consistency measurable cardinals (For information on measurable cardinals, refer to Appendix 3.1) These results will be discussed in greater detail in Section 4.4.

§ 1.3 Inequalities for Cardinal Powers

In this section we present three theorems concerning the behaviour of the power function at singular cardinals of uncountable cofinality. We shall show that if κ is singular and $cf\kappa > \omega$, then 2^{κ} depends strongly on the range of the power function restricted to κ. In particular, it will emerge that the GCH cannot fail for the first time at such a singular κ. This is in contrast to Theorem 2.3.1, which allows one to "construct" models where the GCH first fails at any given *regular* λ.

We begin with a quick review of some elementary concepts.

Let κ be any cardinal, and let $C \subseteq \kappa$. We say that C is *closed unbounded* (club) provided the following conditions hold:

(1) C is closed: Whenever $A \subseteq C$ with $\sup(A) < \kappa$, then $\sup(A) \in C$.

(2) C is unbounded in κ: For any $\lambda < \kappa$ there is $\gamma \in C$ such that $\lambda < \gamma$).

If κ is regular, then the club subsets of κ are the natural copies of the order type of κ. The intersection of any two club subsets of κ is again club.

A subset $A \subseteq \kappa$ is said to be *stationary* if $A \cap C \neq \emptyset$ for any club $C \subseteq \kappa$. A subset $T \subseteq \kappa$ is said to be *thin* if it is not stationary. The *club–filter* over κ is the filter generated by all club subsets of κ. Dually, the family of thin subsets of κ forms an ideal, the *ideal of thin sets*.

Definition 1.3.1: Let κ be a regular cardinal.

(1) A filter $\mathcal{F}$ is said to be *κ–complete* if for any $X \subseteq \mathcal{F}$ such that $|X| < \kappa$, we have $\cap X \in \mathcal{F}$.

(2) A filter $\mathcal{F}$ over κ is said to be *normal* provided it is closed under *diagonal intersections*:
Whenever $\{x_\alpha : \alpha < \kappa\} \subseteq \mathcal{F}$, then the diagonal intersection $\Delta_{\alpha<\kappa} x_\alpha = \{\beta : \beta \in \bigcap_{\alpha<\beta} x_\beta\}$ $\in \mathcal{F}$.

These notions are easily dualised for ideals:

(1') An ideal $\mathcal{I}$ is said to be κ–complete if it is closed under unions of families of fewer than κ elements.

(2') An ideal $\mathcal{I}$ over κ is normal if $\{\beta : \beta \in \bigcup_{\alpha<\beta} x_\alpha\} \in \mathcal{I}$ whenever $\{x_\alpha : \alpha < \kappa\} \subseteq \mathcal{I}$, i.e. if $\mathcal{I}$ is closed under *diagonal unions*.

More information about filters and ideals may be found in the first section of Appendix 4. The proofs of the following facts are found in almost any standard textbook on set theory (e.g. [Jech 1978] or [Kunen 1980]).

Lemma 1.3.2:

(1) *The club filter over κ is κ–complete. Dually, the ideal of thin sets is κ–complete.*

(2) *The club filter over κ is normal. Moreover, the club filter is contained in any κ–complete normal filter over κ. Dually the ideal of thin sets is normal.*

(3) **Fodor's Theorem** [Fodor 1956]: *If F is a regressive function on a stationary $S \subseteq \kappa$, (i.e. $F(\alpha) < \alpha$ for any $\alpha \in S$ such that $\alpha \neq 0$), then there is a stationary $S_0 \subseteq S$ such that F is constant on S_0.*

(4) *One may partition κ into κ–many disjoint stationary sets (This is due to* [Solovay 1974]).

The club sets play an important role in the proof of Theorems 1.3.3, 1.3.4 and 1.3.5, which we state below for convenience, although their respective proofs will occupy us for the remainder of this section. The only reason why the proof does not work for singular cardinals of countable cofinality seems to be that ω does not have a club filter.

Theorem 1.3.3 [Silver 1974]:
Suppose that κ is a singular of uncountable cofinality, and let $\mu < \mathrm{cf}(\kappa)$. If $\{\lambda < \kappa : 2^\lambda \leq \lambda(+\mu)\}$ is stationary in κ, then $2^\kappa \leq \kappa(+\mu)$.

Theorem 1.3.4 [Galvin–Hajnal 1975]:
Let κ be a strong limit singular cardinal of uncountable cofinality such that $\kappa < \aleph_\kappa$. Then $2^\kappa < \aleph_\kappa$. In fact if $\kappa = \aleph_\eta < \aleph_\kappa$, then $2^\kappa < \aleph_\gamma$, where $\gamma = (2^{|\eta|})^+$.

Theorem 1.3.4 is actually true for all strong limit cardinals, but the more general version will only be proved in Section 6.4 (Theorem 6.4.9), and requires the development of altogether different machinery.

The next result we want to consider is of a different nature entirely: It gives a bound for $2^{\aleph_{\omega_1}}$ assuming that the *Chang Conjecture* holds. This needs some explanation:
By $(\kappa,\lambda) \longrightarrow (\kappa',\lambda')$ is meant the statement that if (A,R ...) is a model of some countable language such that $|A| = \kappa$ and $R \subsetneq A$ has $|R| = \lambda$, then (A,R ...) has an *elementary submodel* (A',R' ...) with $|A'| = \kappa'$ and $|R'| = \lambda'$.
The *Chang Conjecture* is $(\omega_2,\omega_1) \longrightarrow (\omega_1,\omega)$.

Theorem 1.3.5 [Magidor 1977]:

Assume that the Chang Conjecture holds. If $\aleph_{\omega_1}$ *is strong limit, then* $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.

Theorem 1.3.3 was proved by Silver using *generic ultrapowers* [Silver 1974], and we shall present a proof similar to his in Section 5.1. An elementary proof was subsequently found by Baumgartner–Prikry[1976]. Galvin and Hajnal used Prikry's ideas to prove Theorem 1.3.4, as well as a variety of other results involving cardinal exponentiation. Magidor originally proved Theorem 1.3.5 using a generic extension based on Silver's ideas [Magidor 1977]. Subsequently, Galvin and Benda noticed that this theorem follows directly from the results in [Galvin–Hajnal 1975].

Definition 1.3.6:
Let κ be an uncountable cardinal, and let $(A_\alpha: \alpha < \kappa)$ be a sequence of sets. A family $\mathcal{F} \subsetneq \Pi(A_\alpha: \alpha < \kappa)$ is called an *almost disjoint transversal* (a.d.t.) for $(A_\alpha: \alpha < \kappa)$ if $|\{\alpha < \kappa: f(\alpha) = g(\alpha)\}| < \kappa$ for all distinct f, g $\in \mathcal{F}$.

Thus if $\varphi: \kappa \longrightarrow \mathrm{On}$, then an a.d.t. for φ is a family $\mathcal{F}$ of maps on κ such that $f(\alpha) < \varphi(\alpha)$ for all $\alpha < \kappa$ and such that if f,g are distinct elements of $\mathcal{F}$, then $\{\alpha < \kappa: f(\alpha) = g(\alpha)\}$ has cardinality $< \kappa$.

Lemma 1.3.7:
Let κ be an uncountable cardinal, and let $(\kappa_\alpha: \alpha < \kappa)$ be any sequence of cardinals. Let $A_\alpha = \prod_{\beta<\alpha} \kappa_\beta$, *and let* $A = (A_\alpha: \alpha < \kappa)$. *Then there is an a.d.t.* $\mathcal{F}$ *for* A *such that* $|\mathcal{F}| = \prod_{\alpha<\kappa} \kappa_\alpha$.

Proof: For each $h \in \prod_{\alpha<\kappa} \kappa_\alpha$ let $f_h(\alpha) = h|\alpha$, and set $\mathcal{F} = \{f_h: h \in \prod_{\alpha<\kappa} \kappa_\alpha\}$. Then $\mathcal{F}$ is an a.d.t. for A with the required properties.

□

Let $\kappa > \omega$ be a regular cardinal. For any sequence $A = (A_\alpha : \alpha < \kappa)$, let

$$T(A) = \sup\{|\mathcal{F}| : \mathcal{F} \text{ is an a.d.t. for } A\}$$

Note that if $A = (A_\alpha : \alpha < \kappa)$ and $B = (B_\alpha : \alpha < \kappa)$ such that $|A_\alpha| \leq |B_\alpha|$ for all $\alpha < \kappa$, then $T(A) \leq T(B)$. Note also that if $\varphi \in {}^\kappa On$, then $T(\varphi)$ is defined.

Definition 1.3.8: Let κ be an uncountable regular cardinal, and let X be a stationary subset of κ. If $\varphi, \psi \in {}^\kappa On$ put $\varphi <_X \psi \longleftrightarrow \{\alpha \in X: \varphi(\alpha) \geq \psi(\alpha)\}$ is a thin subset of κ.

Proposition 1.3.9:

$<_X$ *is a wellfounded partial ordering on* ${}^\kappa On$.

Proof: It is clear that $<_X$ is transitive, since the union of two thin sets is again thin. If $\varphi <_X \psi$, then $Y = \{\alpha \in X: \varphi(\alpha) \geq \psi(\alpha)\}$ is thin, and so $\{\alpha \in X: \psi(\alpha) \geq \varphi(\alpha)\} \supseteq X - Y$ is stationary. Hence $\varphi <_X \psi$ implies $\neg(\psi <_X \varphi)$. It remains to see that $<_X$ is wellfounded. Assume otherwise that $(\varphi_n : n < \omega)$ is a sequence of ${}^\kappa On$–elements such that $n < m$ implies $\varphi_m <_X \varphi_n$. Thus $\{\alpha \in X: \varphi_n(\alpha) \leq \varphi_{n+1}(\alpha)\}$ is thin for each $n < \omega$. Since the ideal of thin sets is κ–complete, it follows that the set $\{\alpha \in X: \exists n<\omega(\varphi_n(\alpha) \leq \varphi_{n+1}(\alpha))\}$ is thin, hence not equal to X. Thus there is $\alpha \in X$ such that

$$\varphi_0(\alpha) > \varphi_1(\alpha) > \ldots > \varphi_n(\alpha) > \ldots (n < \omega),$$

a contradiction, because there is no infinite descending sequence of ordinals.

□

Since $<_X$ is wellfounded for each stationary subset X of κ, we may define *rank functions* $\|\varphi\|_X$ for $\varphi \in {}^\kappa On$ by:

$$\|\varphi\|_X = \sup\{\|\psi\|_X + 1 : \psi <_X \varphi\}.$$

Put $\varphi \equiv_X \psi \longleftrightarrow \{\alpha \in X: \varphi(\alpha) \neq \psi(\alpha)\}$ is thin.

Suppose $\varphi \equiv_X \varphi'$. Then $\psi <_X \varphi$ if and only if $\psi <_X \varphi'$, and thus $\|\varphi\|_X = \|\varphi'\|_X$.

If $X = \kappa$, we shall omit it, i.e. $\|\varphi\| = \|\varphi\|_\kappa$ and $\varphi < \psi$ iff $\varphi <_\kappa \psi$.

Lemma 1.3.10 [Galvin–Hajnal 1975]:

Let $\varphi \in {}^{\kappa}\mathrm{On}$, and let $\mu < \kappa^+$. There is a function $\varphi_\mu \in {}^{\kappa}\kappa$ such that $\|\varphi_\mu\| = \mu$ and for all $\psi \in {}^{\kappa}\mathrm{On}$, $\|\psi\| > \mu$ if and only if $\psi > \varphi_\mu$. Moreover, should φ'_μ be another function with the same properties, then $\varphi_\mu \equiv \varphi'_\mu$.

Proof: The proof is by *induction* on μ. Put $\varphi_0(\alpha) = 0$ for all $\alpha < \kappa$. Now assume $0 < \mu < \kappa^+$ and that φ_ν has been defined for all $\nu < \mu$. Since $\mathrm{cf}(\mu) \leq \kappa$, there is a sequence $(\mu_\xi\colon \xi < \kappa)$ such that $\mu = \sup\{\mu_\xi + 1\colon \xi < \kappa\}$. Define $\varphi_\mu(\alpha) = \sup\{\varphi_{\mu_\xi}(\alpha) + 1\colon \xi < \alpha\}$. We shall show that φ_μ has the required properties.
Clearly, if $\nu < \mu$, then there is $\xi < \kappa$ such that $\varphi_\nu \leq \varphi_{\mu_\xi}$; it follows that $\varphi_\nu < \varphi_{\mu_\xi} + 1$, and thus $\{\alpha < \kappa\colon \varphi_\nu(\alpha) \geq \varphi_{\mu_\xi}(\alpha) + 1\}$ is thin. In particular, $\{\alpha < \kappa\colon \varphi_\nu(\alpha) \geq \varphi_\mu(\alpha)\}$ is thin, proving that $\varphi_\nu < \varphi_\mu$ for all $\nu < \mu$. This and the induction hypothesis are sufficient to imply that $\|\varphi_\mu\| \geq \mu$.
Suppose now that $\|\psi\| > \mu$. By induction on μ we shall show that $\psi > \varphi_\mu$. Suppose that $\neg(\psi > \varphi_\mu)$ holds. Since $\|\psi\| > \mu$, there is $\psi' < \psi$ such that $\|\psi'\| = \mu$. It is easy to see that $\{\alpha < \kappa\colon \varphi_\mu(\alpha) \geq \psi(\alpha)\}$ is a stationary subset of κ, and thus $\{\alpha < \kappa\colon \varphi_\mu(\alpha) > \psi'(\alpha)\}$ is stationary as well. For $\xi < \kappa$, let $X_\xi = \{\alpha < \kappa\colon \psi'(\alpha) \leq \varphi_{\mu_\xi}(\alpha)\}$. Since $\mu_\xi < \mu$, by induction hypothesis $\|\psi'\| = \mu$ implies $\psi' > \varphi_{\mu_\xi}$ for all $\xi < \kappa$. Thus X_ξ is thin for all $\xi < \kappa$. The ideal of thin subsets of κ is normal, and so $X = \{\xi < \kappa\colon \exists\zeta < \xi(\xi \in X_\zeta)\}$ is a thin subset of κ. Clearly, however, $X \supseteq \{\xi < \kappa\colon \varphi_\mu(\xi) > \psi'(\xi)\}$, and this latter set we have seen to be stationary, contradiction. Hence $\psi > \varphi_\mu$ as required. This immediately provides us with a proof that $\|\varphi_\mu\| = \mu$; we know that $\|\varphi_\mu\| \geq \mu$, but if $\|\varphi_\mu\| > \mu$, we would get the absurd conclusion that $\varphi_\mu > \varphi_\mu$.
Finally, suppose that φ'_μ is a function with the same properties as φ_μ. Then $\|\varphi'_\mu + 1\| > \|\varphi'_\mu\| = \mu$, and thus $\varphi'_\mu + 1 > \varphi_\mu$; similarly $\varphi_\mu + 1 > \varphi'_\mu$, and thus the sets $\{\alpha < \kappa\colon \varphi_\mu'(\alpha) < \varphi_\mu(\alpha)\}$ and $\{\alpha < \kappa\colon \varphi_\mu(\alpha) < \varphi'_\mu(\alpha)\}$ are both thin. Hence so is their union, $\{\alpha < \kappa\colon \varphi'_\mu(\alpha) \neq \varphi_\mu(\alpha)\}$.

□

Corollary 1.3.11 [Galvin–Hajnal 1975]:

(1) *For all $\mu < \kappa$, $\|\varphi\| \leq \mu$ iff $\{\alpha < \kappa\colon \varphi(\alpha) \leq \mu\}$ is stationary.*

(2) *$\|\varphi\| = \kappa$ iff $\{\alpha < \kappa\colon \varphi(\alpha) \leq \alpha\}$ is stationary.*

Proof: (1) We shall show by induction on $\mu < \kappa$ that we may assume that the φ_μ is the constant function with value μ. For $\mu = 0$, this is obvious by definition of φ_0. If $\mu > 0$, let $(\mu_\xi: \xi < \kappa)$ be a sequence defined by:

$$\mu_\xi = \begin{cases} \xi \text{ if } \xi < \mu \\ 0 \text{ otherwise} \end{cases}$$

By induction hypothesis, we may assume $\varphi_\xi \equiv \xi$ for all $\xi < \mu$. Hence $\varphi_\mu(\alpha) = \sup\{\varphi_{\mu_\xi}(\alpha) + 1: \xi < \alpha\} = \mu$ if $\kappa > \alpha \geq \mu$. Now changing φ_μ at fewer than κ–many places does not affect the claims made about φ_μ in Lemma 1.3.10, and so we may assume $\varphi_\mu = \mu$ for all $\mu < \kappa$.

If $\|\varphi\| > \mu$, then $\varphi > \varphi_\mu = \mu$, and thus $\{\alpha < \kappa: \varphi(\alpha) \leq \mu\}$ is thin. It follows that $\|\varphi\| \leq \mu$ iff $\{\alpha < \kappa: \varphi(\alpha) \leq \mu\}$ is stationary.

(2) Let $(\mu_\xi: \xi < \kappa)$ be defined by $\mu_\xi = \xi$ for all $\xi < \kappa$. Then:

$$\varphi_\kappa(\alpha) = \sup\{\varphi_\xi(\alpha) + 1: \xi < \alpha\},$$

and by (1) we may assume that $\varphi_\xi = \xi$ for all $\xi < \kappa$. Hence $\varphi_\kappa(\alpha) = \alpha$ for all $\alpha < \kappa$, ie. we may assume that φ_κ is the *diagonal map*. This completes the proof.

□

Proposition 1.3.12 [Galvin–Hajnal 1975]:

Let $\varphi \in {}^\kappa\text{On}$; *then* $\|\varphi\| < |\Pi\{\varphi(\alpha): \varphi(\alpha) \neq 0\}|^+$

Proof: If $\|\varphi\| = 0$, this is obvious. Suppose now that $\|\varphi\| > 0$. For each $\psi < \varphi$, there is $\psi' < \varphi$ such that $\psi \equiv \psi'$ and $\forall \alpha < \kappa\ (\psi'(\alpha) < \varphi(\alpha)$ whenever $\varphi(\alpha) \neq 0)$. Clearly there are at most $|\Pi\ \{\varphi(\alpha) : \varphi(\alpha) \neq 0\}|$ – many such ψ', and thus $\|\varphi\| < |\Pi\ \{\varphi(\alpha): \varphi(\alpha) \neq 0\}|^+$.

□

We will now introduce some more terminology. Let $\mathcal{I}$ be an ideal over κ. We say that a set $X \subseteq \kappa$ is $\mathcal{I}$–positive provided $X \notin \mathcal{I}$. Thus, for example, if $\mathcal{I}$ is the ideal of thin sets, the $\mathcal{I}$–positive sets are just the stationary subsets of κ.

Proposition 1.3.13 [Shelah 1980]:

Let $\varphi \in {}^\kappa\lambda$, *where* λ *is an uncountable regular cardinal. Suppose that* $\forall \tau < \lambda(\text{cf}(\tau) = \kappa \longrightarrow \tau^\kappa < \lambda)$. *Then* $\|\varphi\| < \lambda$.

Proof: Suppose that $\|\varphi\| = \lambda$ for some $\varphi \in {}^{\kappa}\lambda$. Let

$$\mathcal{J} = \{Y \subseteq \kappa\colon Y \text{ stationary implies } \|\varphi\|_Y > \lambda\}.$$

It is not hard to see that $\mathcal{J}$ contains all the thin subsets of κ, and that if $X \subseteq Y \in \mathcal{J}$, then $X \in \mathcal{J}$, because $X \subseteq Y$ implies $\|\varphi\|_X \geq \|\varphi\|_Y$. We shall show that $\mathcal{J}$ is a κ–complete ideal over κ. To this end it suffices to show that if $\{Y_\xi\colon \xi < \mu\}$ is a family of pairwise disjoint stationary sets in $\mathcal{J}$, where $\mu < \kappa$, then $Y = \bigcup_{\xi<\mu} Y_\xi \in \mathcal{J}$, i.e. $\|\psi\|_Y > \lambda$.
We shall attain this by proving inductively that if $\|\psi\|_{Y_\xi} \geq \alpha$ for all $\xi < \mu$, then $\|\psi\|_Y \geq \alpha$ as well. The induction ranges over α and is obvious if $\alpha = 0$. Suppose now that $\alpha > 0$. Then $\|\psi\|_{Y_\xi} \geq \alpha$ implies that for all $\beta < \alpha$, there is $\psi_\xi^\beta <_{Y_\xi} \psi$ such that $\|\psi_\xi^\beta\| \geq \beta$. Since the Y_ξ are pairwise disjoint, we may define ψ^β such that $\psi^\beta | Y_\xi = \psi_\xi^\beta$ for all $\xi < \mu$, and define ψ^β arbitrarily on $\kappa - Y$. Clearly $\psi^\beta <_Y \psi$, and by induction hypothesis $\|\psi^\beta\|_Y \geq \beta$. Thus $\|\psi\|_Y \geq \alpha$, which was to be proved, completing the induction.
It follows that $\|\varphi\|_Y > \lambda$ and thus that $\mathcal{J}$ is a κ–complete nontrivial ideal over κ which extends the ideal of thin sets.

<u>Claim</u>: *There is a $\mathcal{J}$–positive set $Y \subseteq \kappa$ and a family of maps $\mathcal{F} \subseteq {}^{\kappa}\lambda$ of cardinality $< \lambda$ such that for any $\psi \in {}^{\kappa}\mathrm{On}$ with $\psi <_Y \varphi$, there is $f \in \mathcal{F}$ such that $\psi <_Y f <_Y \varphi$ (i.e. $\mathcal{F}$ is $<_Y$–cofinal in φ).*
The proof of this claim devolves into two cases. Let h be defined on κ by $h(\xi) = \mathrm{cf}\ \varphi(\xi)$ for $\xi < \kappa$. Then either (1) There is a $\mathcal{J}$–positive $Y \subseteq \kappa$ such that h is constant on Y, or
(2) There is no such Y.
In Case (1) let $h(\xi) = \rho$ for all ξ in some $\mathcal{J}$–positive $Y \subseteq \kappa$. Clearly ρ is a regular cardinal $< \lambda$. For each $\xi \in Y$, let $(h_\alpha(\xi)\colon \alpha < \rho)$ be a continuous cofinal sequence with limit $\varphi(\xi)$, and let $h_\alpha(\xi) = 0$ if $\xi \notin Y$. Suppose now that $\rho > \kappa$. In that case let $\mathcal{F} = \{h_\alpha\colon \alpha < \rho\}$. Then if $\psi <_Y \varphi$, the set $Z = \{\xi \in Y\colon \psi(\xi) \geq \varphi(\xi)\}$ is thin. For all $\xi \in Y - Z$, choose $\alpha(\xi) < \rho$ such that $\psi(\xi) < h_{\alpha(\xi)}(\xi)$ and put $\alpha(\xi) = 0$ for $\xi \notin Y - Z$. Choose α such that $\rho > \alpha > \alpha(\xi)$ for all $\xi < \kappa$. Since ρ is regular $> \kappa$, this is possible. Then $\psi(\xi) < h_\alpha(\xi)$ for all $\xi \in Y - Z$, and thus the set $\{\xi \in Y\colon \psi(\xi) \geq h_\alpha(\xi)\}$ is thin, i.e. $\psi <_Y h_\alpha$. So $\mathcal{F}$ is indeed $<_Y$–cofinal below φ.
On the other hand, if $\rho \leq \kappa$, let $\mathcal{F} = \prod_{\xi<\kappa} \{h_\alpha(\xi)\colon \alpha < \rho\}$. Then $|\mathcal{F}| = \rho^\kappa = 2^\kappa < \lambda$ (since $2^\kappa = \kappa^\kappa$ and κ has cofinality κ). This concludes the proof of Case (1).

In Case (2) let $Y_\eta = \{\xi < \kappa: h(\xi) < \eta\}$, and choose η least such that $Y_\eta \notin \mathcal{J}$. Our aim is to show that $cf(\eta) = \kappa$. Certainly if $cf(\eta) < \kappa$, then since $\{\xi < \kappa: h(\xi) < \alpha\} \in \mathcal{J}$ by assumption for all $\alpha < \eta$, and because $\mathcal{J}$ is κ–complete, we must have
$Y_\eta = \{\xi < \kappa: h(\xi) < \eta\} \in \mathcal{J}$, a contradiction. On the other hand, if $cf(\eta) > \kappa$, then
$\eta' = \sup\{h(\xi): \xi \in Y_\eta\} < \eta$, and $Y_{\eta'} = Y_\eta$, contradicting our choice of η. It follows that $cf(\eta) = \kappa$.
Since each $h(\xi)$ is a regular cardinal, η is necessarily a cardinal $< \lambda$. Put $Y = Y_\eta$; for each $\xi \in Y$, let $F_\xi \subseteq \varphi(\xi)$ be a set of cardinality $h(\xi) < \eta$ which is cofinal in $\varphi(\xi)$, and put
$F_\xi = \{0\}$ for $\xi \in \kappa - Y$. Finally let $\mathcal{F} = \prod_{\xi<\kappa} F_\xi$. Then $|\mathcal{F}| \leq \eta^\kappa < \lambda$ (by hypothesis, since $cf(\eta) = \kappa$). To see that $\mathcal{F}$ is $<_Y$– cofinal below φ, suppose that $\psi <_Y \varphi$. Define $Z =$
$\{\xi \in Y: \psi(\xi) \geq \varphi(\xi)\}$. Note that Z is thin. For each $\xi \in Y - Z$, let $\alpha(\xi) \in F_\xi$ such that $\psi(\xi) < \alpha(\xi)$ and put $\alpha(\xi) = 0$ for $\xi \notin Y - Z$. Define $f \in \mathcal{F}$ by $f(\xi) = \alpha(\xi)$. Clearly $\psi <_Y f <_Y \varphi$, as required. This concludes the proof of Case (2) and thus also the proof of the Claim.
We shall complete the proof of this lemma by showing that the Claim yields a contradiction. Let Y, $\mathcal{F}$ satisfy the assertions in the claim. Then $\|\varphi\|_Y =$
$\sup\{ \|f\|_Y +1: f \in \mathcal{F}$ and $f <_Y \varphi\}$, and $\|\varphi\|_Y = \lambda$ (since we assumed $\|\varphi\| = \lambda$ and because $Y \notin \mathcal{J}$). Thus $\|f\|_Y < \lambda$ for all $f \in \mathcal{F}$, and since $|\mathcal{F}| < \lambda$ this contradicts the regularity of λ.
The assumption $\|\varphi\| \geq \lambda$ for some $\varphi \in {}^\kappa\lambda$ thus leads to a contradiction.

□

Proposition 1.3.13 is an improvement of the following result of [Galvin–Hajnal 1975], which we state as a corollary:

Corollary 1.3.14 [Galvin–Hajnal 1975]:

Let $\varphi \in {}^\kappa\lambda$, where λ is an uncountable regular cardinal, and suppose that $\forall\tau < \lambda(\tau^\kappa < \lambda)$. Then $\|\varphi\| < \lambda$.

Recall that if $\varphi:\kappa \longrightarrow On$, $T(\varphi) = \sup\{|\mathcal{F}|: \mathcal{F}$ is an a.d.t. for $\varphi\}$. Define $T(\kappa,\delta) = T(\varphi)$ where φ is the function on κ with constant value δ. Thus $T(\kappa,\delta)$ is the supremum of $\{|\mathcal{F}|: \mathcal{F} \subseteq {}^\kappa\delta$ is an a.d.t.$\}$ Further recall that $\kappa(+\mu)$ denotes the μ^{th} successor cardinal of κ. The following lemma is the main device from which Theorems 1.3.3 and 1.3.4 follow:

Lemma 1.3.15 [Galvin–Hajnal 1975]:
Let κ be an uncountable regular cardinal, and suppose that $(\delta_\alpha : \alpha < \kappa)$ is a continuous (not necessarily strictly) increasing sequence of infinite cardinals. Let $\varphi \in {}^\kappa\mathrm{On}$, and define $\psi(\alpha) = \delta_\alpha(+\varphi(\alpha))$ for $\alpha < \kappa$. Let $\Delta = 2^\kappa \cdot \sum_{\alpha<\kappa} \mathrm{T}(\kappa,\delta_\alpha)$. Then $\mathrm{T}(\psi) \leq \Delta(+\|\varphi\|)$.

In order to prove Lemma 1.3.15 (which follows after Corollary 1.3.18) we shall need two results from combinatorial set theory. If X is a set, then $[X]^2$ is the set of all (unordered) pairs from $\{\alpha,\beta\} \subseteq X$. The first result we shall need states that if we partition $[(2^\kappa)^+]^2$ into $\leq\kappa$–many blocks, there is a set Y of cardinality κ^+ such that all of $[Y]^2$ is contained in one block. This is usually written

$$(2^\kappa)^+ \longrightarrow (\kappa)^2_\kappa$$

This result is an instance of the well–known *Erdös–Rado Partition Theorem* ([Erdös–Rado 1956]), and may also be stated as follows: For every $F:[(2^\kappa)^+]^2 \longrightarrow \kappa$, there is a set $Y \subseteq (2^\kappa)^+$ of cardinality κ^+ which is *homogeneous* for F, i.e. $|F''[Y]^2| = 1$.
For the sake of completeness, we include a proof of this theorem:

Theorem 1.3.16 [Erdös–Rado 1956]:
For any cardinal κ, $(2^\kappa)^+ \longrightarrow (\kappa^+)^2_\kappa$.

Proof: Let $\lambda = (2^\kappa)^+$, and let $F: [\lambda]^2 \longrightarrow \kappa$. For each $\xi < \lambda$, let $F_\xi : \lambda - \{\xi\} \longrightarrow \kappa$ be given by: $F_\xi(\zeta) = F(\{\xi,\zeta\})$. We construct a κ^+– sequence of subsets of λ as follows.
$X_0 \subseteq \lambda$ is arbitrary of cardinality 2^κ. For limit $\alpha < \kappa^+$, put $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$. Finally, suppose $X_\alpha \subseteq \lambda$ has cardinality 2^κ, and let $C \subseteq X_\alpha$ be of cardinality κ. There are only 2^κ–many maps from C to κ, and for each $\xi \in \lambda - C$, we have $F_\xi | C: C \longrightarrow \kappa$. Define an equivalence relation $\equiv_C$ on $\lambda - C$ as follows: $\xi \equiv_C \zeta$ if and only if $F_\xi | C = F_\zeta | C$. Now for each $C \subseteq X_\alpha$ of cardinality κ pick a representative from each $\equiv_C$–equivalence class. Let $X_{\alpha+1} \supseteq X_\alpha$ contain these representatives. We may assume that $|X_{\alpha+1}| = 2^\kappa$, because there are only 2^κ–many $C \subseteq X_\alpha$ of cardinality κ, and for every such C only 2^κ–many maps from C to κ. This completes the definition of the sequence $(X_\alpha : \alpha < \kappa^+)$.

Let $X = \bigcup_{\alpha<\kappa^+} X_\alpha$. Clearly $|X| = 2^\kappa$. X can also be seen to have the following property: Whenever $C \subseteq X$ is of cardinality κ, and whenever $\xi \in \lambda - C$, there is $\zeta \in X - C$ such that $F_\xi|C = F_\zeta|C$.

Choose $\xi \in \lambda - X$, and construct $\{y_\alpha : \alpha < \kappa^+\} \subseteq X$ as follows: $y_0 \in X$ is arbitrary.
Given $\{y_\alpha : \alpha < \beta\} = C$, let y_β be some $\zeta \in X - C$ such that $F_\zeta|C = F_\xi|C$.
Let $Y = \{y_\alpha : \alpha < \kappa^+\}$, and define $G: Y \longrightarrow \kappa$ by $G(y) = F_\xi(y)$. If $\alpha < \beta$, then $F(\{y_\alpha, y_\beta\}) = F_{y_\beta}(y_\alpha) = F_\xi(y_\alpha) = G(y_\alpha)$. Choose $Z \subseteq Y$ of cardinality κ^+ so that $G|Z$ is constant. Z exists because $G:Y \longrightarrow \kappa$ and $|Y| = \kappa^+$. It is clear that Z is homogeneous for F.

□

The second combinatorial result that is needed is due to [Hajnal 1961]: Suppose that X is a set. A *set mapping* on X is a map $f:X \longrightarrow \mathcal{P}(X)$ such that for all $x \in X$, $x \notin f(x)$. A set $F \subseteq X$ is said to be *free* with respect to a set mapping f provided $F \cap f(x) = \emptyset$ for all $x \in F$.

Lemma 1.3.17 [Hajnal 1961]:
Suppose that κ,λ are cardinals, and let f be a set mapping on κ such that $|f(\xi)| < \lambda$ for all $\xi < \kappa$. Then κ is the union of λ-many sets that are free with respect to f.

Proof: We shall construct a sequence $(x_{\alpha\beta} : \alpha < \mu \wedge \beta < \lambda)$ such that for each $\beta < \lambda$ the set $\{x_{\alpha\beta} : \alpha < \mu)$ is free with repect to f, and such that $\kappa = \{x_{\alpha\beta} : \alpha < \mu \wedge \beta < \lambda\}$.
Put $E_\xi = \{x_{\alpha\beta} : \alpha < \xi \wedge \beta < \lambda\}$, and assume that all the elements in E_ξ have been defined such that E_ξ is closed under f (i.e. $f(x) \subseteq E_\xi$ for all $x \in E_\xi$). If $E_\xi = \kappa$, put $\mu = \xi$, and we are done. Otherwise pick a set $F_\xi \subseteq \kappa - E_\xi$ of cardinality $\leq \lambda$ such that $E_\xi \cup F_\xi$ is closed under f. Write $F_\xi = \{y_{\xi\gamma} : \gamma < \lambda\}$ and define $x_{\xi\beta}$ to be $y_{\xi\gamma}$ where γ is the least ordinal such that $y_{\xi\gamma} \in F_\xi - \{x_{\xi\delta} : \delta < \beta\}$, and such that $\{y_{\xi\gamma}\} \cup \{x_{\alpha\beta}: \alpha < \xi\}$ is free with respect to f, provided such $y_{\xi\gamma}$ exists; otherwise put $x_{\xi\beta} = x_{0\beta}$.
Suppose that F_ξ is not a subset of $\{x_{\xi\beta}: \beta < \lambda\}$. We shall obtain a contradiction. Pick γ least such that $y_{\xi\gamma} \notin \{x_{\xi\beta} : \beta < \lambda\}$. Then for every $\beta < \lambda$ we must have either:
(1) $y_{\xi\gamma} = x_{\xi\delta}$ for some $\delta < \gamma$, or
(2) $\{y_{\xi\gamma}\} \cup \{x_{\alpha\beta} : \beta < \lambda\}$ is not free with respect to f.
Clearly (1) can occur for $< \lambda$–many β only. We shall show that the same is true for (2):

Since E_ξ is closed under f, necessarily $y_{\xi\gamma} \notin f(x_{\alpha\beta})$ for any $\alpha < \xi$ and $\beta < \lambda$. On the other hand, $x_{\alpha\beta} \in f(y_{\xi\gamma})$ can occur for only $<\lambda$–many β, since $|f(y_{\xi\gamma})| < \lambda$, and the family $\{\{x_{\alpha\beta} : \alpha < \xi\}: \beta < \lambda\}$ is pairwise disjoint. Thus both (1) and (2) can hold for only $<\lambda$–many $\beta < \lambda$, contradicting the fact that either (1) or (2) should hold for every $\beta < \lambda$. Hence $F_\xi \subseteq \{x_{\xi\beta} : \beta < \lambda\}$. By choice of F_ξ therefore, the set $E_{\xi+1} = \{x_{\alpha\beta} : \alpha \leq \xi \wedge \beta < \lambda\}$ is closed under f as well. Moreover the sets $\{x_{\alpha\beta} : \alpha \leq \xi\}$ (for $\beta < \lambda$) are obviously mutually disjoint. Continuing in this vein, we shall eventually find a μ such that $\kappa = \{x_{\alpha\beta} : \alpha < \mu \wedge \beta < \lambda\}$.

□

Corollary 1.3.18:

Let $\kappa > \lambda \geq \omega$ be cardinals such that κ is regular. Let f *be a set mapping on κ such that $|f(x)| < \lambda$ for all $x \in \kappa$. Then there is a set $F \subseteq \kappa$ of cardinality κ which is free with respect to* F.

The above corollary is true for singular κ as well, but we shall not need this fact.

Proof of Lemma 1.3.15:

By induction on $\|\varphi\|$: First suppose that $\|\varphi\| = 0$, and let $\mathcal{F}$ be an a.d.t. for ψ. Since $\{\alpha: \varphi(\alpha) = 0\}$ is stationary in κ, so is $X_0 = \{\alpha: \varphi(\alpha) = 0 \text{ and } \alpha \text{ is limit}\}$. By continuity of the $(\delta_\alpha: \alpha < \kappa)$–sequence, there is for each $f \in \mathcal{F}$ and each $\alpha \in X_0$ an ordinal $\beta(f,\alpha) < \alpha$ such that $f(\alpha) < \delta_{\beta(f,\alpha)}$. By Fodor's theorem (Lemma 1.3.2(3)) there exists an ordinal $\beta(f) < \kappa$ such that $f(\alpha) < \delta_{\beta(f)}$ for all α in some stationary $X_f \subseteq X_0$.
Let, for $X \subseteq X_0$ stationary and $\beta < \kappa$, $\mathcal{F}_{X,\beta} = \{f \in \mathcal{F}: X_f = X, \beta(f) = \beta\}$. Then $|\mathcal{F}_{X,\beta}| \leq T(\kappa,\delta_\beta) \leq \Delta$. Since there are at most 2^κ–many such pairs (X,β), it follows that $|\mathcal{F}| \leq 2^\kappa \cdot \Delta \leq \Delta$, as required. This completes the proof if $\|\varphi\| = 0$.
Next suppose that $\|\varphi\| = \nu > 0$, and that $\mathcal{F}$ is an a.d.t for ψ. If $f \in \mathcal{F}$, define

$$\varphi_f(\alpha) = \min\{\beta: |f(\alpha)| \leq \delta_\alpha(+\beta)\}$$

Since for all α, $|f(\alpha)| < \delta_\alpha(+\varphi(\alpha))$, we have: $\varphi_f(\alpha) \geq \varphi(\alpha)$ if and only if $\varphi(\alpha) = \varphi_f(\alpha) = 0$. Since $\|\varphi\| > 0$, it follows that $\{\alpha: \varphi(\alpha) = 0\}$ is not stationary in κ, and so $\{\alpha: \varphi_f(\alpha) \geq \varphi(\alpha)\}$ is not stationary in κ. Hence $\varphi_f < \varphi$ for all $f \in \mathcal{F}$ and so $\|\varphi_f\| < \nu$ for all f.
Define ψ_f by: $\psi_f(\alpha) = \delta_\alpha(+\varphi_f(\alpha))$.
Let $\mathcal{F}_\mu = \{f \in \mathcal{F}: \|\varphi_f\| = \mu\}$. Thus $\mathcal{F} = \bigcup_{\mu<\nu} \mathcal{F}_\mu$. Define a set mapping H on $\mathcal{F}_\mu$ by:

$$H(f) = \{g \in \mathcal{F}_\mu: \forall\alpha(g(\alpha) \leq f(\alpha))\} - \{f\}.$$

H(f) is clearly an a.d.t. for f+1, and since $|f(\alpha)+1| \leq \delta_\alpha(+\varphi_f(\alpha)) = \psi_f(\alpha)$, it follows that $|H(f)| \leq T(\psi_f)$. Also, $\psi_f(\alpha) = \delta_\alpha(+\varphi_f(\alpha))$ and $\|\varphi_f\| = \mu < \nu$ together imply that $T(\psi_f) \leq \Delta(+\mu)$ (by induction hypothesis). It follows that $|H(f)| \leq \Delta(+\mu)$ for all $f \in \mathcal{F}_\mu$. We will use this fact and Corollary 1.3.18 to show that $|\mathcal{F}_\mu| \leq \Delta(+\mu+1)$.

Suppose therefore that this condition fails, i.e. that $\mathcal{F}_\mu \geq \Delta(\mu+2)$. By Corollary 1.3.18 there is a set $\mathcal{G}_\mu \subseteq \mathcal{F}_\mu$ of cardinality $\Delta(+\mu+2)$ which is free for H. In particular, since $\Delta(+\mu+2) > 2^\kappa$, there exists a sequence $(f_\xi: \xi < (2^\kappa)^+)$ such that each $f_\xi \in \mathcal{F}_\mu$ and such that

$$\xi < \eta < (2^\kappa)^+ \text{ implies } f_\xi \notin H(f_\eta).$$

Now $f_\xi \notin H(f_\eta)$ implies that there is an ordinal $\alpha < \kappa$ such that $f_\xi(\alpha) > f_\eta(\alpha)$. Let $F(\xi,\eta)$ denote such an α for $\xi < \eta < (2^\kappa)^+$. Then $F:[(2^\kappa)^+]^2 \longrightarrow \kappa$, and so by the Erdös–Rado partition theorem, it follows that there is an infinite set $Y \subseteq (2^\kappa)^+$ which is homogeneous for F, [i.e. for any $\xi < \gamma < \eta$ in Y we have $F(\xi,\gamma) = F(\xi,\eta) = F(\gamma,\eta)$].

If $\xi_0 < \xi_1 < \dots < \xi_n < \dots$ $(n < \omega)$ is an ascending sequence in Y, and $F(\xi_n,\xi_m) = \alpha$ for any $n < m < \omega$, then

$$f_{\xi_0}(\alpha) > f_{\xi_1}(\alpha) > \dots > f_{\xi_n}(\alpha) > \dots \quad (n < \omega),$$

a contradiction because there is no infinite descending sequence of ordinals.

It follows that we must have $|\mathcal{F}_\mu| \leq \Delta(+\mu+1)$ for all $\mu < \nu = \|\varphi\|$, and so $|\mathcal{F}| = |\bigcup_{\mu<\nu} \mathcal{F}_\mu| \leq \sum_{\mu<\nu} \Delta(+\mu+1) \leq \nu \cdot \Delta(+\nu) = \Delta(+\|\varphi\|)$ as required.

□

Corollary 1.3.19 [Galvin–Hajnal 1975]:

Suppose that κ is a singular cardinal of cofinality $\lambda > \omega$. Suppose that $(\kappa_\xi: \xi < \lambda)$ is a non–decreasing continuous sequence of infinite cardinals $< \kappa$ such that $\kappa_\xi^\lambda \leq \kappa$ for all $\xi < \lambda$. Let $\varphi \in {}^\lambda On$ and let $\psi(\xi) = \kappa_\xi(+\varphi(\xi))$ for all $\xi < \lambda$. Then $T(\psi) \leq \kappa(+\|\varphi\|)$.

Proof: By Lemma 1.3.15, $T(\psi) \leq \Delta(+\|\varphi\|)$, where $\Delta = 2^\lambda \cdot \sum_{\xi<\lambda} T(\lambda,\kappa_\xi)$. By assumption, $2^\lambda < \kappa$. Also $T(\lambda,\kappa_\xi) \leq \kappa_\xi^\lambda < \kappa$ for each $\xi < \lambda$. Thus $\Delta \leq \kappa$, so $T(\psi) \leq \kappa(+\|\varphi\|)$.

□

Corollary 1.3.19 is all that is needed to prove Theorems 1.3.3 and 1.3.4:

Proof of Theorem 1.3.3: Suppose that κ is a singular cardinal of uncountable cofinality λ, and suppose that $\mu < \lambda$. Let $(\kappa_\xi: \xi < \lambda)$ be a continuous cofinal sequence in κ, and further suppose that $S = \{\xi < \lambda: 2^{\kappa_\xi} \leq \kappa_\xi(+\mu)\}$ is stationary in λ. Define $\varphi \in {}^\lambda On$ by :

$$\varphi(\xi) = \text{least ordinal } \gamma \text{ such that } 2^{\kappa_\xi} \leq \kappa_\xi(+\gamma).$$

Then $\|\varphi\| \leq \mu$. For every $X \subseteq \kappa$, define $f_X \in \prod_{\xi<\lambda} \mathcal{P}(\kappa_\xi)$ by: $f_X = (X \cap \kappa_\xi: \xi < \lambda)$, and let $\mathcal{F} = \{f_X: X \subseteq \kappa\}$. It is clear that $\mathcal{F}$ is an a.d.t. for $(\mathcal{P}(\kappa_\xi): \xi < \lambda)$ and that $|\mathcal{P}(\kappa_\xi)| \leq \kappa_\xi(+\varphi(\xi))$. Thus by Corollary 1.3.19, we have $|\mathcal{F}| \leq \kappa(+\|\varphi\|) \leq \kappa(+\mu)$, and since $|\mathcal{F}| = 2^\kappa$, the proof is complete.

□

Proof of Theorem 1.3.4: Assume that κ is a singular strong limit cardinal of cofinality $\lambda > \omega$ such that $\kappa < \aleph_\kappa$. We proceed exactly as in the proof Theorem 1.3.3: Let $(\kappa_\xi: \xi < \lambda)$ be a continuous cofinal sequence below κ. For every $X \subseteq \kappa$, define $f_X = (X \cap \kappa_\xi: \xi < \lambda) \in \prod_{\xi<\lambda} \mathcal{P}(\kappa_\xi)$, and let $\mathcal{F} = \{f_X: X \subseteq \kappa\}$. Since κ is strong limit, $|\mathcal{P}(\kappa_\xi)| = 2^{\kappa_\xi} < \kappa$ for all $\xi < \lambda$. Choose $\eta < \kappa$ such that $\kappa = \aleph_\eta$, and let $\varphi \in {}^\lambda\eta$ be a map such that $2^{\kappa_\xi} \leq \kappa_\xi(+\varphi(\xi))$. Then $\|\varphi\| < (|\eta|^\lambda)^+$; also $\eta > \lambda$, so $2^{|\eta|} \geq |\eta|^\lambda$, and thus $\|\varphi\| \leq (2^{|\eta|})^+$. By Corollary 1.3.19, it follows that $2^\kappa = |\mathcal{F}| \leq \aleph_{\eta+\|\varphi\|} < \aleph_\gamma$, where $\gamma = (2^{|\eta|})^+ < \kappa$ (since κ is strong limit). Hence $2^\kappa < \aleph_\kappa$.

□

Theorem 1.3.4 does not give us any information in case $\kappa = \aleph_\kappa$. We shall return to this question later in Chapter 5 when we consider the theory of ideals over uncountable sets, which is the natural generalization of the theory presented in this chapter.

We will now present some corollaries to Lemma 1.3.15, all involving products and powers of cardinals:

Corollary 1.3.20 [Galvin–Hajnal 1975]:

Suppose κ, $(\delta_\alpha: \alpha < \kappa)$ *and* Δ *are as in the statement of Lemma* 1.3.15. *Let* $\varphi \in {}^\kappa\mathrm{On}$, *and suppose that* $(\kappa_\alpha: \alpha < \kappa)$ *is a sequence of cardinals such that*

$$\prod_{\beta<\alpha} \kappa_\beta \leq \delta_\alpha(+\varphi(\alpha))$$

holds for all $\alpha < \kappa$. *Then*:

$$\prod_{\alpha<\kappa} \kappa_\alpha \leq \Delta(+\|\varphi\|).$$

Proof: Put $\psi(\alpha) = \delta_\alpha(+\varphi(\alpha))$ for all $\alpha < \kappa$. Let $A_\alpha = \prod_{\beta<\alpha} \kappa_\beta$, and let $A = (A_\alpha: \alpha < \kappa)$. By Lemma 1.3.7, there is an a.d.t. $\mathcal{T}$ for A such that $|\mathcal{T}| = \prod_{\alpha<\kappa} \kappa_\alpha$. Since $|A_\alpha| \leq \psi(\alpha)$ for all $\alpha \in \kappa$, it follows that $|\mathcal{T}| \leq T(\psi) \leq \Delta(+\|\varphi\|)$, as required.

□

Corollary 1.3.21 [Galvin–Hajnal 1975]:

Suppose κ, $(\delta_\alpha: \alpha < \kappa)$ *and* Δ *are as in the statement of Lemma* 1.3.15

(a) *If* $(\tau_\alpha: \alpha < \kappa)$ *is an increasing sequence of non-zero cardinals, and if* $\forall\alpha<\kappa(\prod_{\beta<\alpha} \tau_\beta \leq \delta_\alpha(+\varphi(\alpha)))$, *then* $(\sum_{\alpha<\kappa} \tau_\alpha)^\kappa \leq \Delta(+\|\varphi\|)$.

(b) *If* $(\lambda_\alpha: \alpha < \kappa)$ *is a strictly increasing sequence of infinite cardinals, if* $\lambda = \sum_{\alpha<\kappa} \lambda_\alpha$, *and if* ρ *is any cardinal, then*

$\forall\alpha<\kappa(\rho^{\lambda_\alpha} \leq \delta_\alpha(+\varphi(\alpha)))$ *implies* $\rho^\lambda \leq \Delta(+\|\varphi\|)$

Proof: (a) If we can show that $(\sum_{\alpha<\kappa} \tau_\alpha)^\kappa \leq 2^\kappa \cdot \prod_{\alpha<\kappa} \tau_\alpha$, then the result follows by Corollary 1.3.20 Let $\tau = \sum_{\alpha<\kappa} \tau_\alpha$. We distinguish two cases:

<u>Case 1</u>: There is $\beta < \kappa$ such that $\tau_\beta^\kappa \geq \tau$. In that case $\tau_\beta^\kappa = \tau^\kappa$, and $\tau_\beta^\kappa \leq \prod_{\alpha<\kappa} \tau_\alpha$.

<u>Case 2</u>: $\forall\alpha<\kappa(\tau_\alpha^\kappa < \tau)$. In that case $\tau^\kappa = (\sum_{\alpha<\kappa} \tau_\alpha)^\kappa \leq (\prod_{\alpha<\kappa} 2\cdot\tau_\alpha)^\kappa = 2^\kappa \cdot \prod_{\alpha<\kappa} \tau_\alpha^\kappa$.

But $\tau_\alpha^\kappa \leq \tau_{\beta(\alpha)}$ for some $\beta(\alpha) < \kappa$, and thus $\prod_{\alpha<\kappa} \tau_\alpha^\kappa \leq \prod_{\alpha<\kappa} \tau_{\beta(\alpha)} \leq \prod_{\alpha<\kappa} \tau_\alpha$.

(b) Set $\kappa_\alpha = \rho^{\lambda_\alpha}$; Then $\prod_{\beta<\alpha} \kappa_\beta \leq \rho^{\lambda_\alpha} \leq \delta_\alpha(+\varphi(\alpha))$. Hence by Corollary 1.3.20, $\rho^\lambda = \prod_{\alpha<\kappa} \rho^{\lambda_\alpha} = \prod_{\alpha<\kappa} \kappa_\alpha \leq \Delta(+\|\varphi\|)$.

□

Corollary 1.3.22 [Galvin–Hajnal 1975], [Shelah 1980]:

Let $\lambda > \kappa$ be uncountable regular cardinals, and suppose that $\tau^\kappa < \lambda$ for all $\tau < \lambda$ with $\mathrm{cf}(\tau) = \kappa$. *Suppose also that σ is an infinite cardinal such that $\sigma^\kappa < \sigma(+\lambda)$.*

(a) *Suppose that $(\kappa_\alpha: \alpha < \kappa)$ is a sequence of cardinals such that $\prod_{\beta<\alpha} \kappa_\beta < \sigma(+\lambda)$ for all $\alpha < \kappa$. Then $\prod_{\alpha<\kappa} \kappa_\alpha < \sigma(+\lambda)$.*

(b) *If $\rho^\tau < \sigma(+\lambda)$ for all $\tau < \kappa$, then $\rho^\kappa < \sigma(+\lambda)$.*

(c) *If ρ is a cardinal of cofinality κ, and if $2^\tau < \sigma(+\lambda)$ for all $\tau < \rho$, then $2^\rho < \sigma(+\lambda)$.*

Proof: (a) We will use Corollary 1.3.19. Put $\delta_\alpha = \sigma$ for all $\alpha < \kappa$, and define $\varphi(\alpha)$ to be the least ordinal γ such that $\prod_{\beta<\alpha} \kappa_\beta \leq \sigma(+\gamma)$. Then $\varphi \in {}^\kappa\lambda$, and thus by Lemma 1.3.13, $\|\varphi\| < \lambda$. By Corollary 1.3.9, $\prod_{\alpha<\kappa} \kappa_\alpha \leq \Delta(+\|\varphi\|)$. It is easy to see that $\Delta < \sigma(+\lambda)$, and so $\Delta(+\|\varphi\|) < \sigma(+\lambda)$ as well.

(b) Put $\kappa_\alpha = \rho$ for all $\alpha < \kappa$, and use (a).

(c) Since $\mathrm{cf}(\rho) = \kappa$, we may write $\rho = \sum_{\alpha<\kappa} \rho_\alpha$ for some increasing continuous sequence of ordinals $\rho_\alpha < \rho$. Define $\tau_\alpha = \sum_{\beta<\alpha} \rho_\beta$, and let $\kappa_\alpha = 2^{\rho_\alpha}$. Then $\prod_{\beta<\alpha} \kappa_\beta = 2^{\tau_\alpha} < \sigma(+\lambda)$ by hypothesis. Hence $2^\rho = \prod_{\alpha<\kappa} \kappa_\alpha < \sigma(+\lambda)$ by (a).

□

With the aid of Corollary 1.3.22(c), we can deduce things like: If $2^{\omega_1} = \omega_2$ and if $2^{\aleph_\alpha} < \aleph_{\omega_3}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_3}$. To see this, just put $\sigma = \omega$, $\kappa = \omega_1$, $\lambda = \omega_3$, and let $\rho = \aleph_{\omega_1}$.

Similarly, suppose that $2^{\aleph_4} = \aleph_{10}$ and further suppose that $2^{\tau} < \aleph_{\omega_{11}}$ for all $\tau < \aleph_{\omega_4}$. Then also $2^{\aleph_{\omega_4}} < \aleph_{\omega_{11}}$: Let $\kappa = \omega_4$, $\lambda = \omega_{11}$ and $\sigma = \omega$.

Assuming the Chang Conjecture, Magidor showed that we can strengthen the above: If $2^{\aleph_\alpha} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$. Galvin and Benda, amongst others, proved that the Chang Conjecture implies that $\|\varphi\| < \omega_2$ for all $\varphi:\omega_1 \longrightarrow \omega_1$. We shall prove this fact:

Lemma 1.3.23 :
Assume the Chang Conjecture holds. If $\varphi:\omega_1 \longrightarrow \omega_1$, then $\|\varphi\| < \omega_2$.

Proof: Suppose that $\varphi \in {}^{\omega_1}\mathrm{On}$ is a map such that $\|\varphi\| \geq \omega_2$. It suffices to show that there is $\xi < \omega_1$ such that $\varphi(\xi) \geq \omega_1$. By Lemma 1.3.10, for each $\mu < \omega_2$, there is a map $\varphi_\mu:\omega_1 \longrightarrow \omega_1$ such that $\|\varphi_\mu\| = \mu$, and such that if $\|\psi\| > \mu$, then $\psi > \varphi_\mu$. Thus if $\mu < \nu < \omega_2$, then $\varphi_\mu < \varphi_\nu < \varphi$, and thus there is a club set $C_{\mu\nu}$ such that $\varphi_\mu(\xi) < \varphi_\nu(\xi) < \varphi(\xi)$ for all $\xi \in C_{\mu\nu}$. Define a map h on $[\omega_2]^{<\omega}$ as follows: $h(X) = \inf \bigcap\{C_{\mu\nu}: \mu < \nu \text{ are in } X\}$ for every finite $X \subseteq \omega_2$. Note that $\mathrm{ran}(h) \subseteq \omega_1$, and consider the model (ω_2,ω_1,h). By the Chang Conjecture, there are $A \subseteq \omega_2$ and $R \subseteq \omega_1$ such that $|A| = \omega_1$, $|R| = \omega$, and $(A,R,h|A)$ is an elementary submodel of (ω_2,ω_1,h). Note that $\gamma = \sup(R) < \omega_1$, and that $h(X) \in R$ for all $X \in [A]^{<\omega}$. Now suppose that $\mu < \nu$ are in A. We claim that $\gamma \in C_{\mu\nu}$. If not, then $\gamma' = \sup\{\xi \in C_{\mu\nu}: \xi < \gamma\} < \gamma$, because $C_{\mu\nu}$ is closed. Choose $X \in [A]^{<\omega}$ such that $h(X) > \gamma'$. Then $h(X \cup \{\mu,\nu\}) > \gamma$ by definition of h, but this is a contradiction, since $h(X \cup \{\mu,\nu\}) \in R$. It follows that $\gamma \in C_{\mu\nu}$ whenever $\mu < \nu$ are in A. Thus $\varphi_\mu(\gamma) < \varphi_\nu(\gamma) < \varphi(\gamma)$ whenever $\mu < \nu$ in A, and since $|A| = \omega_1$, we must have $\varphi(\gamma) \geq \omega_1$, as required.

□

We will use the above lemma to prove Magidor's result:

Proof of Theorem 1.3.5: Let $\kappa = \omega_1$, $\delta_\alpha = \omega$ for all $\alpha < \omega_1$, and let $\kappa_\alpha = \aleph_\alpha$ for all $\alpha < \omega_1$ [illegible] $\varphi(\alpha)$ be such that $\prod_{\beta<\alpha} \aleph_\beta = \aleph_{\varphi(\alpha)}$; then since $\prod_{\beta<\alpha} \aleph_\beta \leq 2^{\aleph_\alpha} < \aleph_{\omega_1}$, it follows that $\varphi:\omega_1 \longrightarrow \omega_1$. Thus $\|\varphi\| < \omega_2$ by Lemma 1.3.23. If Δ is defined as usual, then

$\Delta = 2^{\omega_1} < \aleph_{\omega_1}$, and so $\Delta(+\|\varphi\|) < \aleph_{\omega_1}(+\omega_2) = \aleph_{\omega_2}$. It follows by Corollary 1.3.20 that $\prod_{\alpha<\omega_1} \aleph_\alpha < \aleph_{\omega_2}$. However, $\aleph_{\omega_1}$ is strong limit, and thus $\prod_{\alpha<\omega_1} \aleph_\alpha = \prod_{\alpha<\omega_1} 2^{\aleph_\alpha} = 2^{\aleph_{\omega_1}}$, proving the theorem.

□

Chapter 1 was concerned with showing what restrictions on the power function are provable in ZFC–set theory. We worked inside the universe of all sets to obtain our results, and thus the methods used might be described as *internal*. This chapter is occupied with showing what restrictions on the power function are *not* provable. In Section 2.1 we shall show that the GCH is consistent with ZFC set theory. In Section 2.2 we will take a model of set theory and generate an appropriate extension to which we "add" many subsets of ω, so that the GCH fails in the extension thus obtained. The material of Section 2.1 together with that of Section 2.2. combined proves the independence of the GCH. In Section 2.3 we discuss how to "add" subsets of many (even class–many) regular cardinals simultaneously, so that the power function is shown to be very incompletely described by the axioms of ZFC. All these subsets are added *generically*, from "outside" the universe as it were, and the methods used in Sections 2.2 and 2.3 are therefore best described as *external*. In Section 2.3. we also show that the SCH is consistent with $\neg$GCH (Theorem 2.3.6). The main results of this chapter are Theorems 2.1.1, 2.2.6 and 2.3.1.

§ 2.1 The Consistency of the GCH.

In this section we prove that the Generalized Continuum Hypothesis is consistent with the axioms of ZFC by exhibiting an inner model of ZFC + GCH, namely the *Constructible Universe* L. This was first done by Kurt Gödel ([Gödel 1938]). The background on L required for this section is given in Appendix 1.1.

Recall that L is built up as an ordinal–indexed hierarchy of transitive sets J_α ($\alpha \in \mathrm{On}$), each of which is closed under basic set operations. If $\alpha < \beta$, then $J_\alpha \in J_\beta$ (and so $J_\alpha \subsetneq J_\beta$ because J_β is transitive.) Each J_α has a canonical definable well–ordering $\leq_\alpha$, with the property that if $\alpha < \beta$, then $\leq_\beta$ is an end–extension of $\leq_\alpha$. Also, for any ordinal α, $|J_\alpha| = |\alpha|$. Further recall the *Condensation Lemma* for L (Lemma A.1.8): If $M \leq_e J_\alpha$, then there is an ordinal $\beta \leq \alpha$ and a *collapsing isomorphism* π: $M \longrightarrow J_\beta$ such that:

(1) π is the identity when restricted to transitive sets.

(2) $\pi(\alpha) \leq \alpha$ for all $\alpha \in \mathrm{On} \cap M$.

We shall use the Condensation Lemma to prove that $L \vdash$ GCH.

Theorem 2.1.1 [Gödel 1938]:
L $\vdash$ GCH. *Hence* GCH *is consistent with the axioms of* ZFC.

Proof: Let κ be an infinite cardinal in L, and suppose that $X \in \mathcal{P}(\kappa) \cap L$. We shall show that $X \in J_{\kappa^+}$. Let $\alpha \geq \kappa$ be the least ordinal such that $X \in J_\alpha$, and let M be an elementary submodel of J_α with the properties that:

(1) $X \in M$

(2) $\kappa \subseteq M$

(3) $|M| = \kappa$.

Such an M can be obtained via a suitable Skolemization using the canonical well–ordering of L. Let $\beta \leq \alpha$ and π: $M \longrightarrow J_\beta$ be the ordinal and collapsing isomorphism whose existence is guaranteed by the Condensation Lemma (Lemma A.1.8). Since $|M| = \kappa$, we must have $|\beta| = |J_\beta| = \kappa$, and thus $\beta < \kappa^+$. Moreover $\kappa \subseteq M$ is transitive, and thus $\pi|\kappa = \mathrm{id}|\kappa$. Since $X \subseteq \kappa$ we must have $\pi(X) = X$. Hence $X \in J_\beta \subseteq J_{\kappa^+}$.
$X \in \mathcal{P}^L(\kappa)$ was chosen arbitrarily, so it follows that $\mathcal{P}^L(\kappa) \subseteq J_{\kappa^+}$, and thus that $|\mathcal{P}^L(\kappa)| \leq |J_{\kappa^+}| = \kappa^+$. Hence L $\vdash 2^\kappa \leq \kappa^+$ for all infinite cardinals κ.

□

We shall encounter the above argument again in a slightly different setting when we prove that GCH is consistent with the existence of a measurable cardinal in Section 3.1.
It remains to show that the GCH cannot be disproved using the axioms of ZFC. This we tackle in the next section.

§ 2.2. Forcing and the Power Function

In this section we show that the GCH is independent of the axioms of ZFC. This was first done by Paul Cohen ([Cohen 1963–1964]), using the method of forcing. Cohen also proved the independence of the Axiom of Choice with the axioms of ZF at more or less at the same time. The forcing terminology and theory required as background for this section are contained in Appendix 2.1.

For the moment we shall be working in a transitive model V of ZFC with a forcing partial order $\mathbb{P} \in V$. Let $G \subseteq \mathbb{P}$ be generic over V, and let V[G] be the corresponding generic extension. In Appendix 2 we indicate how certain combinatorial properties on $\mathbb{P}$ govern the cardinal arithmetic in V[G]. Important are particularly Corollary A.2.6 and Theorem A.2.8: If $\mathbb{P}$ is $<\kappa$–distributive (see Definition A.2.4), then all cardinalities and cofinalities $\leq \kappa$ are preserved in the generic extension, and if $\mathbb{P}$ satisfies the κ–c.c (where κ is a regular cardinal; see Definition A.2.7), then all cardinalities and cofinalities $\geq \kappa$ are preserved. Chain conditions also govern how many new subsets of a cardinal κ may be found in the generic extension, and thus what the value of the power function at κ will be in V[G]. To illustrate this, we need the following concept.

Definition 2.2.1: Let $\dot{x}$ be a $\mathbb{P}$–name. A *nice name* for a subset of $\dot{x}$ is a name $\dot{y}$ of the form

$$y = \cup \{\{\dot{z}\} \times A_{\dot{z}} : \dot{z} \in \text{dom}(\dot{x}) \}$$

where each $A_{\dot{z}}$ is an antichain in $\mathbb{P}$.

Lemma 2.2.2 (Nice Name Lemma):

Every subset of a name $\dot{x}$ has a nice name. More precisely, if $\dot{y}$ is a name such that $p \Vdash \dot{y} \subseteq \dot{x}$ then there is a nice name $\dot{z}$ such that $p \Vdash \dot{y} = \dot{z}$.

Proof: Let $\dot{x}$, $\dot{y}$ be names and let $p \in \mathbb{P}$ such that $p \Vdash \dot{y} \subseteq \dot{x}$. For each $\dot{r} \in \text{dom}(\dot{x})$, let $A_{\dot{r}} \subseteq \mathbb{P}$ be such that

(a) $q \in A_{\dot{r}}$ implies $q \Vdash \dot{r} \in \dot{y}$
(b) $A_{\dot{r}}$ is an antichain in $\mathbb{P}$
(c) $A_{\dot{r}}$ is maximal with respect to (a) and (b).

Let $\dot{z} = \cup \{\{\dot{r}\} \times A_{\dot{r}} : \dot{r} \in \text{dom}(\dot{x}) \}$. Clearly $\dot{z}$ is a nice name. By Theorem A.2.2 one may then prove that $\dot{z}$ is a nice name for $\dot{y}$.

□

Nice names are important when discussing the power function because they can be used to give an upper bound on the number of subsets of a given set in the generic extension, as the next lemma demonstrates.

Lemma 2.2.3:

If $\mathbb{P}$ *has the* κ *– chain condition, then* $(2^\lambda)^{V[G]} \le [(|\mathbb{P}|^{<\kappa})^\lambda]^V$.

Proof: To every subset of λ in V[G] corresponds at least one nice name by Lemma 2.2.2 above. Now $\dot{x}$ is a nice name for a subset of ω provided

$$\dot{x} = \cup\{\{\check{\beta}\} \times A_\beta : \beta \in \lambda\}$$

for some antichains A_β. Now since every antichain has cardinality $< \kappa$, there are at most $|\mathbb{P}|^{<\kappa}$ – many antichains in $\mathbb{P}$. It follows That there are at most $(|\mathbb{P}|^{<\kappa})^\lambda$ – many nice names for subsets of λ in V, so 2^λ in V[G] is $\le (|\mathbb{P}|^{<\kappa})^\lambda$ in V.

□

Thus, for example, if $\mathbb{P}$ satisfies the countable chain condition, then $(2^{\aleph_0})^{V[G]} \le |\mathbb{P}|^\omega$ in V, and if $\mathbb{P}$ satisfies the κ^+ – c.c. then $(2^\lambda)^{V[G]} \le |\mathbb{P}|^{\kappa\lambda}$. Hence as a corollary, we cannot increase the power function at a cardinal λ by using a notion of forcing of cardinality $\le \lambda$, because such a notion of forcing must satisfy the λ^+–c.c. This is interesting as it implies that, starting from a model of ZFC + GCH, no set notion of forcing can affect the power function at arbitrarily large cardinals.

Definition 2.2.4: Let X and Y be sets and let λ be an infinite cardinal. We define $Fn(X,Y,\lambda)$ to be the set of all partial functions from X to Y of cardinality $< \lambda$, i.e.

$$Fn(X,Y,\lambda) = \{ p : p \text{ is a function, } dom(p) \subseteq X, ran(p) \subseteq Y, |p| < \lambda\}$$

A condition p is stronger that a condition q if and only if $p \supseteq q$.

Lemma 2.2.5:

Let X,Y *be sets in* V, *and let* λ *be an infinite cardinal in* V.

(1) $|Fn(X,Y,\lambda)| = |X \times Y|^{<\lambda}$

(2) *If* λ *is regular, then* $Fn(X,Y,\lambda)$ *is* $<\lambda$*–closed.*

(3) $Fn(X,Y,\lambda)$ *has the* $(|Y|^{<\lambda})^+$*–c.c.*

(4) *If* λ *is regular,* $|Y| \le \lambda$ *and* $2^{<\lambda} = \lambda$ *then* $Fn(X,Y,\lambda)$ *preserves all cardinalities and cofinalities.*

Proof : (1) is easily proved as $Fn(X,Y,\lambda) \subseteq \{p \subseteq X \times Y: |p| < \lambda\}$.

(2) Suppose that we are given a descending chain

$$p_0 \geq p_1 \geq \dots \geq p_\eta \geq \dots \quad \text{for } \eta < \xi < \lambda$$

Let p be the union of all the p_η. Then since λ is regular, $|p| < \lambda$, so $p \in Fn(X,Y,\lambda)$, and p is clearly stronger than every p_η ($\eta < \xi$). This proves the first assertion.

(3) Let $\theta = (|Y|^{<\lambda})^+$ and suppose that $\{p_\zeta : \zeta < \theta\}$ is an antichain. First suppose λ is regular, $(|Y|^{<\lambda})^{<\lambda} = |Y|^{<\lambda}$ and thus for every $\alpha < \theta$, also $\alpha^{<\lambda} < \theta$. By the Δ–System Lemma (Lemma A.2.10) it immediately follows that there is a set $I \subseteq \theta$ such that $\{dom(p_\zeta) : \zeta \in I\}$ forms a Δ–system with root $r \subseteq X$. There are $< \theta$ possibilities for $p_\zeta|r$, however, and thus we obtain a contradiction of the fact that the p_ζ are mutually incompatible. Next suppose λ is singular. Since $\theta > \lambda$ is regular there is a regular $\lambda' < \lambda$ such that $J = \{\zeta : |p_\zeta| < \lambda'\}$ is of cardinality θ. Then $\{p_\zeta : \zeta \in J\}$ contradicts the $(|Y|^{<\lambda'})^+$–c.c.

(4) By (2) and the Distributivity Theorem (Theorem A.2.5), $Fn(X,Y,\lambda)$ preserves cardinalities and cofinalities $\leq \lambda$. By (3) and the fact that $|Y|^{<\lambda} = \lambda$, it follows that $Fn(X,Y,\lambda)$ satisfies the λ^+–c.c. so that cardinalities and cofinalities $\geq \lambda^+$ are preserved.

□

Theorem 2.2.6 ([Cohen 1963–1964]):

Assume that, in the ground model V, $\lambda < \kappa$, λ *is regular*, $2^{<\lambda} = \lambda$, *and* $\kappa^\lambda = \kappa$. *If* $\mathbb{P} = Fn(\kappa \times \lambda, 2, \lambda)^V$, *then forcing with* $\mathbb{P}$ *preserves cardinalities and cofinalities, and if* $G \subseteq \mathbb{P}$ *is generic over* V, *then* $V[G] \vdash 2^\lambda = \kappa$.

Proof: That forcing with $\mathbb{P}$ preserves all cardinalities and cofinalities follows from Lemma 2.2.5 (4). By Lemma 2.2.5(3), $\mathbb{P}$ has the λ^+–c.c. Using Lemma 2.2.3 and Lemma 2.2.5(1), we see that $(2^\lambda)^{V[G]} \leq \kappa^\lambda = \kappa$, so that $V[G] \vdash 2^\lambda \leq \kappa$. It remains to prove the reverse inequality. For each $\alpha < \kappa$, let $X_\alpha = \{\beta < \lambda: \exists p \in G(p(\alpha,\beta) = 1)\}$. Then each X_α is a subset of λ, so we shall have proved the theorem if we can show that the X_α are all distinct. Given $\alpha \neq \alpha' < \kappa$, Let $\mathcal{D}_{\alpha\alpha'} = \{p \in \mathbb{P}: \exists\beta < \lambda(p(\alpha,\beta) \neq p(\alpha',\beta))\}$. It is not hard to see that $\mathcal{D}_{\alpha\alpha'}$ is dense in $\mathbb{P}$, and thus that $G \cap \mathcal{D}_{\alpha\alpha'} \neq \emptyset$. It follows that $X_\alpha \neq X_{\alpha'}$ whenever $\alpha \neq \alpha'$, so that $V[G] \vdash 2^\lambda \geq \kappa$ as well.

□

An iteration of Theorem 2.2.6 may be used to violate the GCH at any finite number of regular cardinals. An example will give the general idea.

Example 2.2.7: *If* ZFC *is consistent, so is* ZFC + $2^{\aleph_0} = \aleph_3 + 2^{\aleph_1} = \aleph_{100} + 2^{\aleph_{\omega+1}} = \aleph_{\omega_2}$.
Let V be the ground model. We will assume that $V \vdash$ GCH.
Let $\mathbb{P}_1 = \mathrm{Fn}(\omega_{\omega_2} \times \omega_{\omega+1}, 2, \omega_{\omega+1})^V$ and let $G_1 \subseteq \mathbb{P}_1$ be generic over V. Then by Theorem 2.2.6, $V[G_1] \vdash 2^{\omega_{\omega+1}} = \omega_{\omega_2}$. Now let $\mathbb{P}_2 = \mathrm{Fn}(\omega_{100} \times \omega_1, 2, \omega_1)^{V[G_1]}$,and let $G_2 \subseteq \mathbb{P}_2$ be generic over $V[G_1]$. Then $V[G_1][G_2] \vdash 2^{\omega_1} = \omega_{100}$. Moreover $V[G_1][G_2] \vdash 2^{\omega_{\omega+1}} = \omega_{\omega_2}$. To see this note that $\mathbb{P}_2$ has the ω_2–c.c. and that moreover $|\mathbb{P}_2|^{V[G_1]} = \omega_{100}{}^{\omega_1}$. Thus there are at most $(\omega_{100}{}^{\omega_1})^{\omega_{\omega+1}} = \omega_{\omega_2}$ – many nice names for subsets of $\omega_{\omega+1}$.
The third condition is obtained by forcing with $\mathbb{P}_3 = \mathrm{Fn}(\omega_3 \times \omega, 2, \omega)^{V[G_1][G_2]}$.
[Actually, since $\mathbb{P}_1$ is $<\omega_{\omega+1}$–closed and preserves cardinals, $\mathbb{P}_2 = \mathrm{Fn}(\omega_{100} \times \omega_1, 2, \omega_1)^V$ (by the Distributivity Theorem A.2.5), and similarly $\mathbb{P}_3 = \mathrm{Fn}(\omega_3 \times \omega, 2, \omega)^V$.]

§ 2.3 Easton's Theorem

In Section 2.2 we showed how to change the value of the power function at a regular cardinal λ. In particular, Theorem 2.2.6 shows that if κ is a cardinal $> \lambda$, if $2^{<\lambda} = \lambda$ and if $\kappa^\lambda = \kappa$, then the notion $\mathrm{Fn}(\kappa \times \lambda, 2, \lambda)$ forces $2^\lambda = \kappa$. Example 2.2.7 indicates how to change the power function at finitely many regular cardinals and can be seen to be a finite iteration. We will now show how to use product forcing to change the power function at infinitely many regular cardinals. The background material on product forcing needed for this section is given in Appendix 2.

The following facts about the power function were proved in Chapter 1:

(1) $\kappa \leq \lambda$ implies $2^\kappa \leq 2^\lambda$ (because $\mathcal{P}(\kappa) \subseteq \mathcal{P}(\lambda)$);

(2) $\mathrm{cf}(2^\kappa) > \kappa$ (Corollary 1.1.2).

The contents of the next theorem, due to Easton ([Easton 1970]), state that for regular cardinals this is *all* that can be proved in ZFC.

Theorem 2.3.1 [Easton 1970]:
Let V *be a transitive model of* ZFC + GCH *and let* F *be a function in* V *on a set of regular cardinals such that for any regular* $\kappa, \lambda \in \mathrm{dom}(F)$
we have:

(1) F *is increasing, i.e.* $\kappa \leq \lambda$ *implies* $F(\kappa) \leq F(\lambda)$.

(2) $F(\kappa) > \kappa$ *for all* κ *in the domain of* F.

Then there is a generic extension V[G] *of* V *having the same cardinalities and cofinalities* , *and for every regular* κ, $V[G] \vdash 2^\kappa = F(\kappa)$.

Thus $P^{V[G]} = F$ when restricted to regular cardinals (where P is the power function).
For every regular cardinal κ in dom(F) and let $\mathbb{P}_\kappa$ be the notion of forcing which adjoins $F(\kappa)$ – many subsets of κ, i.e. (see Theorem 2.2.6):

$$\mathbb{P}_\kappa = \mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa).$$

For the moment we assume that A = dom(F) is a set (as it would be if V were a set; for the full proof of Easton's theorem one needs to develop forcing with a *proper class* of conditions. This is discussed more fully later in this section.)

Definition 2.3.2:

(1) For $p \in \prod_{\kappa \in A} \mathbb{P}_\kappa$ we define the support of p by:

$$\mathrm{supp}(p) = \{\kappa \in A: p(\kappa) \neq \emptyset\}$$

(2) Let $\mathbb{P} \subseteq \prod_{\kappa \in A} \mathbb{P}_\kappa$ such that for every regular cardinal γ,

$$p \in \mathbb{P} \text{ implies } |\mathrm{supp}(p) \cap \gamma| < \gamma.$$

The ordering on $\mathbb{P}$ is coordinate–wise, i.e.

$$p \leq q \text{ in } \mathbb{P} \text{ iff } \forall \kappa \in A(p(\kappa) \leq q(\kappa) \text{ in } \mathbb{P}_\kappa)$$

$\mathbb{P}$ is called the *Easton product* of the $\mathbb{P}_\kappa$, and forcing with $\mathbb{P}$ is called Easton forcing, or forcing with Easton support. Note that this is a kind of product forcing, since all the $\mathbb{P}_\kappa$ are in the ground model.

We shall write $p(\kappa, \alpha, \beta)$ for $p(\kappa)(\alpha, \beta)$.

Clearly, therefore,every condition p may be regarded as a map with range $\subseteq \{0,1\}$ and domain a set of triples of the form (κ, α, β), where $\kappa \in A$, $\alpha < F(\kappa)$, $\beta < \kappa$ such that for every regular γ we have: $|\{(\kappa, \alpha, \beta) \in \text{dom}(p) : \kappa \leq \gamma\}| < \gamma$

(since γ is regular and there are $< \gamma$ – many $\kappa < \gamma$ such that $\text{dom}(p(\kappa)) \neq \emptyset$.)

Let G be V–generic over $\mathbb{P}$, and for every $\kappa \in A$, let G_κ be the projection of G to $\mathbb{P}_\kappa$, i.e. $G_\kappa = \{p(\kappa) : p \in G\}$. It is not hard to see that each G_κ is V–generic over $\mathbb{P}_\kappa$. For every $\alpha < F(\kappa)$ we shall denote the corresponding new subset of κ added by X^κ_α.

$$X^\kappa_\alpha = \{\beta < \kappa : \exists p \in G\ (p(\kappa, \alpha, \beta) = 1)\}.$$

Equivalently
$$X^\kappa_\alpha = \{\beta < \kappa : \exists p \in G_\kappa (p(\alpha, \beta) = 1)\}.$$

We shall now show that $\mathbb{P}$ preserves cardinalities and cofinalities. To this end, we need the following definition.

Definition 2.3.3: For every *regular* cardinal λ and every $p \in \mathbb{P}$,

$$p^{\leq\lambda} = p|\{(\kappa, \alpha, \beta) \in \text{dom}(p): \kappa \leq \lambda\}$$
$$p^{>\lambda} = p|\{(\kappa, \alpha, \beta) \in \text{dom}(p): \kappa > \lambda\}$$
$$\mathbb{P}^{\leq\lambda} = \{p^{\leq\lambda} : p \in \mathbb{P}\}$$
$$\mathbb{P}^{>\lambda} = \{p^{>\lambda} : p \in \mathbb{P}\}$$

Clearly $p = p^{\leq\lambda} \cup p^{>\lambda}$ for every $p \in \mathbb{P}$, and so $\mathbb{P} \cong \mathbb{P}^{\leq\lambda} \times \mathbb{P}^{>\lambda}$. Moreover $\mathbb{P}^{\leq\lambda}$ is the Easton product of $\mathbb{P}_\kappa$ for $\kappa \leq \lambda$, and $\mathbb{P}^{>\lambda}$ is the Easton product of $\mathbb{P}_\kappa$ for $\kappa > \lambda$.

Lemma 2.3.4 ([Easton 1970]):

(1) *Every* $\mathbb{P}^{>\lambda}$ *is* λ *– closed.*

(2) *Every* $\mathbb{P}^{\leq\lambda}$ *satisfies the* λ^+*– chain condition.*

Proof: (1) Let $(p^{>\lambda}_\xi : \xi < \lambda)$ be a descending sequence of $\mathbb{P}^{>\lambda}$– conditions, and let $p^{>\lambda}$ be the union of this sequence. It suffices to show that for every regular $\gamma > \lambda$ we have $|\text{supp}(p^{>\lambda}) \cap \gamma| < \gamma$. But $\text{supp}(p^{>\lambda}) = \bigcup \{\text{supp}(p^{>\lambda}_\xi) : \xi < \lambda\}$, so that

$$|\text{supp}(p^{>\lambda}) \cap \gamma| < \lambda \cdot \gamma = \gamma.$$

(2) Let $W \subseteq \mathbb{P}^{\leq\lambda}$ be a maximal antichain. By simultaneous induction we will construct two ascending chains

$$A_0 \subseteq A_1 \subseteq \dots \subseteq A_\alpha \subseteq \dots \qquad (\alpha < \lambda)$$
$$W_0 \subseteq W_1 \subseteq \dots \subseteq W_\alpha \subseteq \dots \qquad (\alpha < \lambda)$$

At limit α let $A_\alpha = \bigcup_{\xi<\alpha} A_\xi$, $W_\alpha = \bigcup_{\xi<\alpha} W_\xi$. For the successor stage, given A_α, W_α, choose for every $p^{\leq\lambda} \in \mathbb{P}^{\leq\lambda}$ with $\mathrm{dom}(p^{\leq\lambda}) \subseteq A_\alpha$ a $q \in W$ such that $p^{\leq\lambda} = q|A_\alpha$, provided such a q exists. Now let

$$W_{\alpha+1} = W_\alpha \cup \{\text{ all chosen } q\}$$
$$A_{\alpha+1} = \bigcup \{\mathrm{dom}(q) : q \in A_\alpha\}$$

Let
$$A = \bigcup_{\alpha<\lambda} A_\alpha.$$

If $p^{\leq\lambda} \in W$, then $|\mathrm{dom}(p)| < \lambda$, so there is $\alpha < \lambda$ such that $\mathrm{dom}(p) \cap A = \mathrm{dom}(p) \cap A_\alpha$. Hence there is a $q \in W_{\alpha+1}$ such that $q|A_\alpha = p|A_\alpha$. Then $p = q$ since W is an antichain. Hence $p \in W$ implies $\exists\alpha(p \in W_\alpha)$, which proves that $W = \bigcup_{\alpha<\lambda} W_\alpha$.

Next we shall show that $|A_\alpha| \leq \lambda$, $|W_\alpha| \leq \lambda$ for each $\alpha < \lambda$. This is also done by induction and is easy to see at limit stages. We may suppose that we have found $|A_\alpha|$, $|W_\alpha| \leq \lambda$. $|W_{\alpha+1}| = |W_\alpha \cup \{\text{chosen } q\}|$. But $|\{\text{chosen } q\}| \leq |A_\alpha^{<\lambda}| = \lambda^{<\lambda} = \lambda$ since GCH holds in the ground model. Hence $|W_{\alpha+1}| \leq \lambda$, and thus also $|A_{\alpha+1}| \leq \lambda$. Since W is just the union of W_α it follows that $|W| \leq \lambda$, i.e. $\mathbb{P}^{\leq\lambda}$ has the λ^+–chain condition.

□

Lemma 2.3.5 [Easton 1970]:
Easton forcing preserves cardinalities and cofinalities.

Proof: Let $\mathbb{P}$ be the Easton product defined earlier (Definition 2.3.2). To see that $\mathbb{P}$ preserves cofinalities it suffices to prove that $\mathbb{P}$ preserves all regular cardinals. Let $G \subseteq \mathbb{P}$ be generic over V, and let κ be regular in V. Suppose $\gamma < \kappa$ and suppose $f:\gamma \longrightarrow \kappa$ is cofinal in κ, where $f \in V[G]$. Then $\mathbb{P} \cong \mathbb{P}^{>\gamma} \times \mathbb{P}^{\leq\gamma}$ and $G \cong G^{>\gamma} \times G^{\leq\gamma}$ so that by Lemma A.2.18 we see that $f \in V[G^{\leq\gamma}]$. But $\mathbb{P}^{\leq\gamma}$ has the γ^+–c.c., so preserves cofinalities $> \gamma$. Hence κ should have cofinality $\leq \gamma$ in V, a contradiction.

□

We are now ready to complete the proof of Easton's theorem, at least for set–many regular cardinals, by showing that if $G \subseteq \mathbb{P}$ is generic over V, then $V[G] \vdash 2^{\lambda} = F(\lambda)$ for every regular λ in the domain of F.

Proof of Theorem 2.3.1 for a set of regular cardinals:

It is clear that in V[G], $2^{\lambda} \geq F(\lambda)$, since for every $\alpha < F(\lambda)$ we add a new subset X^{λ}_{α} of λ, and the X^{λ}_{α} are easily proved to be distinct by a density argument (as in the proof of Theorem 2.2.6). By Lemma A.2.18, $(2^{\lambda})^{V[G]} = (2^{\lambda})^{V[G^{\leq\lambda}]}$, so we may apply Lemma 2.2.3 to give an upper bound for $(2^{\lambda})^{V[G]}$. In fact, $(2^{\lambda})^{V[G]} \leq (|\mathbb{P}^{\leq\lambda}|^{\lambda})^{V}$, and since $|\mathbb{P}^{\leq\lambda}| = F(\lambda)$ it follows that $(2^{\lambda})^{V[G]} \leq F(\lambda)$. This completes the proof.

□

We cannot, however, use Easton's construction in the hope that it will tell us someting about the power function at singular cardinals, because Easton's models satisfy the Singular Cardinals Hypothesis.

Theorem 2.3.6 [Easton 1970]: *The Singular Cardinals Hypothesis holds in Easton's model. Hence the* SCH *is consistent with* ZFC + ¬GCH.

Proof: Let κ be singular in V[G], where G is a generic filter on $\mathbb{P}$, Easton's notion of forcing. Since $\mathbb{P}$ preserves cofinalities, it follows that $cf(\kappa)^{V} = cf(\kappa)^{V[G]}$. Suppose therefore that $\gamma = cf(\kappa)^{V}$ and that $V[G] \vdash 2^{\gamma} < \kappa$. We must show that $V[G] \vdash \kappa^{\gamma} = \kappa^{+}$.
Let $M = V[G^{\leq\gamma}]$. Then $(\kappa^{\gamma})^{V[G]} = (\kappa^{\gamma})^{M} \leq (2^{\kappa})^{M} \leq (F(\gamma)^{\kappa})^{V}$ by Lemma 1.2.7. But $F(\gamma) = (2^{\gamma})^{V[G]} = (2^{\gamma})^{M} = (\gamma^{+})^{V}$ since $\mathbb{P}^{\leq\gamma}$ preserves cardinalities and cofinalities $> \gamma$, and GCH holds in V. Since $\gamma < \kappa$, again by the GCH in V we have

$$(F(\gamma)^{\kappa})^{V} \leq (2^{\kappa})^{V} = (\kappa^{+})^{V} = (\kappa^{+})^{V[G]}$$

because $\mathbb{P}$ preserves of cardinals. Hence $V[G] \vdash \kappa^{cf(\kappa)} \leq \kappa^{+}$, and since the reverse inequality is a theorem of ZFC, it follows that the singular cardinals hypothesis holds in V[G].

□

Above we have shown how to change the power function at "set–many" places. But no notion of forcing, however large, can affect the power function at arbitrarily large places while preserving all cardinals: If $\mathbb{P}$ is any notion of forcing which is a *set* and $|\mathbb{P}| = \kappa$, then $\mathbb{P}$ certainly has the κ^+–c.c, and so by Lemma 2.2.3, $(2^\lambda)^{V[G]} \leq (2^{\kappa\lambda})^V$. Hence if $\mathbb{P}$ preserves cardinals and $\lambda \geq |\mathbb{P}|$, then $(2^\lambda)^{V[G]} = (2^\lambda)^V$.

It is therefore clear that in order to affect the power function at arbitrarily large places we need a notion of forcing that is a *proper class.* If $\mathbb{P}$ is a proper class, we can still define a forcing relation and the concept of a generic filter, although these will be classes. $\mathbb{P}$–names are defined as before, and are sets, and thus we may define the generic extension M[G]. The forcing theorem will still be true, but if $\mathbb{P}$ is a proper class notion it does not follow that the generic extension is a model of ZFC: In fact, if $\mathbb{P} = \bigcup_{\alpha\in On} Fn(\omega,\aleph_\alpha,\omega)$ with the ordering by inclusion, then every cardinal is collapsed to ω in the generic extension. It turns out that in many cases this difficulty can be circumvented,and we shall indicate how to do this for Easton forcing:

Proof of Theorem 2.3.1 for a <u>class</u> of regular cardinals:
Let F be an Easton function on the class REG of regular cardinals. Let $\mathbb{P}$ be the partial order whose conditions are *sets* p with

$$dom(p) \subseteq REG,$$
$$p(\kappa) \in Fn(F(\kappa) \times \kappa, 2, \kappa) \text{ for } \kappa \in dom(p)$$

and whose ordering is coordinate wise. As before we may define $\mathbb{P}^{\leq\lambda}$ and $\mathbb{P}^{>\lambda}$ for all regular cardinals λ (see Definition 2.3.3). Note that $\mathbb{P}^{\leq\lambda}$ is a set and that every condition $p \in \mathbb{P}$ is already in some $\mathbb{P}^{\leq\lambda}$. Let $G \subseteq \mathbb{P}$ be generic over V. It is not hard to see that $V[G] = \bigcup\{V[G^{\leq\lambda}]: \lambda \in REG\}$: If $\dot{x}$ is a name for an element of V[G], we may choose λ such that all the p required for the definition of $\dot{x}$ are in $\mathbb{P}^{\leq\lambda}$, because $\dot{x}$ is a set. Hence $\dot{x}$ is a name for an element of $V[G^{\leq\lambda}]$. Now each $V[G^{\leq\lambda}]$ is a model of ZFC, because each $\mathbb{P}^{\leq\lambda}$ is a set. Now if $V[G] \models \varphi(x,y)$, then there is $p \in G$ such that $p \Vdash \varphi(\dot{x},\dot{y})$. If we choose λ large enough so that $p \in \mathbb{P}^{\leq\lambda}$ and $\dot{x},\dot{y}$ are $\mathbb{P}^{\leq\lambda}$–names, then $V[G^{\leq\lambda}] \vdash \varphi(x,y)$. It follows that $V[G] \vdash ZFC$ (because φ cannot be the negation of an axiom of ZFC). The proof now continues in exactly the same way as the proof of Theorem 2.3.1 for a set of regular cardinals.

□

Example 2.3.7: Define $F(\kappa) = \kappa^{+n}$ for all $\kappa \in REG$, where $n < \omega$. Then F is clearly an Easton function. Let $\mathbb{P}$ be the associated notion of forcing, and let G be V–generic over $\mathbb{P}$.

Then $V[G] \vdash \forall\kappa(\kappa \text{ regular} \longrightarrow 2^{\kappa} = \kappa^{+n} \ \wedge \ \kappa \text{ singular} \longrightarrow 2^{\kappa} = \kappa^{+})$.
[Recall that V[G] satisfies the Singular Cardinals Hypothesis.]

Similarly, if $F(\kappa) = \kappa^{+\omega+n}$, then in V[G]:

$$2^{\kappa} = \begin{cases} \kappa^{+\omega+n} & \text{if } \kappa \text{ is regular} \\ \kappa^{+} & \text{if } \kappa \text{ is singular and not an } \omega^{th}\text{-successor cardinal} \\ \kappa^{+\omega+1} & \text{if } \kappa \text{ is } \lambda(+\omega) \text{ for some } \lambda \geq \omega. \end{cases}$$

Thus using Easton's methods, we can never get a model of ZFC where the GCH fails everywhere. Foreman and Woodin ([Foreman–Woodin 1991]) have accomplished this assuming the existence of a supercompact cardinal with infinitely many inaccessibles above it. Woodin also has obtained a model of ZFC + $\forall\kappa(2^{\kappa} = \kappa^{++})$. We shall discuss models in which the SCH fails in Chapter 4.

Early research on large cardinals and the GCH was founded on the hope that strong axioms of infinity might decide the GCH. For intance, Dana Scott proved that if there is a measurable cardinal, then $V \neq L$, ([Scott 1961]) and L was created by Gödel to prove the consistency of the GCH. Thus it was hoped that the existence of a measurable cardinal κ would prove $\neg$GCH, perhaps by ensuring that κ has many subsets. In retrospect this has proven impossible, but the theory of large cardinals still plays an important role since large cardinal axioms are mostly combinatorial in nature, and yield notions of forcing with strong closure properties, as we shall see in Section 3.4. when we discuss reverse Easton forcing. Furthermore, large cardinals often have reflection properties which are interesting should the GCH fail at a large cardinal. For example, just as the GCH cannot fail for the first time at a singular cardinal of uncountable cardinality (Theorem 1.3.3), the GCH cannot fail for the first time at a measurable cardinal. On the other hand, if the GCH does fail at a measurable cardinal, then one can prove the consistency of measurable cardinals. This chapter is mainly concerned with bringing together a mixed bag of theorems with this flavour, the main results being Theorems 3.1.3, 3.2.5, 3.3.1 and 3.4.1. Appendix 3 contains the various results on large cardinals which are required in this chapter.

§ 3.1 Elementary Results on Measurable Cardinals and the GCH

Our first aim is to prove that GCH is consistent with the existence of a measurable cardinal. This was done by Silver ([Silver 1971]), who exhibited an inner model of ZFC in which there exists a measurable cardinal, and which satisfies GCH.

Suppose κ is a measurable cardinal, and let $\mathcal{U}$ be a normal measure over κ (See Appendix 3.1 for the relevant definitions). Let $L[\mathcal{U}]$ be the class of sets constructible relative to $\mathcal{U}$ (See Appendix 1.1). $L[\mathcal{U}]$ is an inner model of ZFC. Our aims are to prove that κ is a measurable cardinal in $L[\mathcal{U}]$, and that $L[\mathcal{U}] \models$ GCH.

Let $\overline{\mathcal{U}} = \mathcal{U} \cap L[\mathcal{U}]$. Then there is an ordinal ξ such that $\overline{\mathcal{U}} = \mathcal{U} \cap J_\xi^{\mathcal{U}}$, and so $\overline{\mathcal{U}} \in J_{\xi+1}^{\mathcal{U}} \subseteq L[\mathcal{U}]$. It is easy to see that $L[\overline{\mathcal{U}}] = L[\mathcal{U}]$.

Lemma 3.1.1 ([Silver 1971]):

$\overline{\mathcal{U}}$ is a normal measure over κ in $L[\mathcal{U}]$. *Hence* $L[\mathcal{U}] \vdash$ "κ is measurable".

Proof: This follows straightforwardly from the fact that $\mathcal{U}$ is a normal measure over κ. For example, to see that $\overline{\mathcal{U}}$ is normal in $L[\mathcal{U}]$, let $f \in L[\mathcal{U}]$ be a regressive function on κ. Then there is $\gamma < \kappa$ such that $X = \{\alpha < \kappa: f(\alpha) = \gamma\} \in \mathcal{U}$. But because $f \in L[\mathcal{U}]$, also $X \in L[\mathcal{U}]$.

□

Lemma 3.1.2:

If $V = L[\mathcal{U}]$ *then* $2^\lambda = \lambda^+$ *for all* $\lambda \geq \kappa$.

Proof: We argue in $L[\mathcal{U}]$: Let $X \subseteq \lambda$ and choose ξ such that $X \in J_\xi^{\mathcal{U}}$ and $\mathcal{U} \in J_\xi^{\mathcal{U}}$. Let $M \leq_e J_\xi^{\mathcal{U}}$ such that $\lambda \subseteq M$, $\mathcal{U} \in M$ and $|M| = \lambda$. By the Condensation Lemma (Lemma A.1.8) there is $\gamma \leq \xi$ and a collapsing isomorphism π such that π: $M \cong N = J_\gamma^{\mathcal{V}}$, where $\mathcal{V} = \pi''(\mathcal{U} \cap M)$. Since $X \subseteq \lambda$, $\pi(X) = X$. Moreover, if $u \in \mathcal{U}$, then $u \subseteq \kappa \subseteq \lambda$, so $\pi(u) = u$. Hence $\mathcal{V} = \mathcal{U} \cap N$, so $N = J_\gamma^{\mathcal{U}}$. Since $|N| = \lambda$, it follows that $\gamma < \lambda^+$. Hence for any $X \subseteq \lambda$, $X \in J_{\lambda^+}^{\mathcal{U}}$, proving $|\mathcal{P}(\lambda)| \leq \lambda^+$.

□

The above proof is a typical condensation argument that would work for any $\mathcal{U} \subseteq \mathcal{P}(\lambda)$, not just $\mathcal{U}$ which are normal measures. To prove that GCH holds below κ in $L[\mathcal{U}]$, Silver used an indiscernibility argument provided partly by Rowbottom's Lemma (Lemma A.3.5).

Theorem 3.1.3: ([Silver 1971]):

If κ is a measurable cardinal and $\mathcal{U}$ is a normal measure over κ, then:

$$L[\mathcal{U}] \vdash \text{GCH} + \kappa \textit{ is measurable}.$$

Hence Con(ZFC + $\exists\kappa$(κ is measurable)) *implies* Con(ZFC + GCH + $\exists\kappa$(κ is measurable)).

Proof: We have already seen that $\mathcal{U} \cap L[\mathcal{U}]$ is a normal measure over κ in $L[\mathcal{U}]$ and that the GCH holds for all $\lambda \geq \kappa$. It thus suffices to show that if $V = L[\mathcal{U}]$, and $\lambda < \kappa$, then $2^\lambda = \lambda^+$. We argue by contradiction (in $L[\mathcal{U}]$): Suppose $\lambda < \kappa$ has at least λ^{++}– many subsets. Then there is $Y \subseteq \lambda$ such that, in the canonical well–order of $L[\mathcal{U}]$, Y is the λ^{+th} subset of λ. Let ξ be the least ordinal such that $Y \in J_\xi^{\mathcal{U}}$, and note that all the subsets of λ preceding Y are in $J_\xi^{\mathcal{U}}$. Let $\eta \geq \xi$ be least such that $\mathcal{U} \in J_\eta^{\mathcal{U}}$.

We will now apply Rowbottom's Lemma (Lemma A.3.5) with $A = J_\eta^{\mathcal{U}}$, $P = \mathcal{P}(\lambda) \cap A$, and $X = \lambda \cup \{\mathcal{U}, Y, \xi\}$. Thus we obtain an elementary submodel $B \leq_e A$ such that $|B| = \kappa$, $B \cap \kappa \in \mathcal{U}$ and $|P \cap B| \leq \lambda$. By a condensation argument (refer to Lemma A.1.8) we know that there is a collapsing isomorphism π: $B \longrightarrow N = J_\gamma^{\pi(\mathcal{U})}$ for some γ. It takes a little argument to see that $\pi(\mathcal{U}) = \mathcal{U} \cap N$. First note that $\kappa \cap B \in \mathcal{U}$ implies $|\kappa \cap B| = \kappa$. Hence $\pi(\kappa) = \kappa$. Since $\pi(\alpha) \leq \alpha$ for all $\alpha < \kappa$, it follows by normality that there is $D \in \mathcal{U}$ such that $D = \{\alpha < \kappa : \pi(\alpha) = \alpha\}$. Let $E \in \mathcal{U} \cap B$. Then $\pi(E) \supseteq \pi(E \cap D) = E \cap D \in \mathcal{U}$, so $\pi(E) \in \mathcal{U}$. Hence $\pi(\mathcal{U}) = \pi''(\mathcal{U} \cap B) \subseteq \mathcal{U} \cap N$. Similarly, if $\pi(E) \in \mathcal{U}$ for some $E \in B$, then $E \in \mathcal{U}$ (because $E \supseteq E \cap D = \pi(E) \cap D \in \mathcal{U}$). Thus $\mathcal{U} \cap N \subseteq \pi''(\mathcal{U} \cap B) = \pi(\mathcal{U})$.

So $N = J_\gamma^{\mathcal{U}}$. Now by the usual arguments $X = \pi(X)$ for every $X \subseteq \lambda$ in B, so that $\mathcal{P}(\lambda) \cap M = \mathcal{P}(\lambda) \cap B$. Also $\pi(Y) = Y \in J_\gamma^{\mathcal{U}}$. By the minimality of ξ with respect to that property, it follows that $\xi \leq \gamma$. This provides the required contradiction, since

$$\lambda^+ = |\mathcal{P}(\lambda) \cap J_\xi^{\mathcal{U}}| \leq |\mathcal{P}(\lambda) \cap J_\gamma^{\mathcal{U}}| = |\mathcal{P}(\lambda) \cap B| \leq \lambda.$$

□

The next result resembles Theorem 1.3.3 of Chapter 1, and uses an ultrapower construction. Ultrapowers are also discussed in Appendix 3.1.

Theorem 3.1.4 [Hanf–Scott 1961]:

Let κ be a measurable cardinal, and let $\mathcal{U}$ be a normal measure over κ.

If $2^\kappa > \kappa^+$, then the set $\{\xi < \kappa$: ξ a cardinal and $2^\xi > \xi^+\} \in \mathcal{U}$.

Hence if $2^\gamma = \gamma^+$ for all $\gamma < \kappa$, then $2^\kappa = \kappa^+$.

Proof: Let M be (the transitive isomorph of) $V^\kappa/\mathcal{U}$, and let j: V $\longrightarrow$ M be the associated elementary embedding. By Lemma A.3.3, ${}^\kappa M \subseteq M$, and thus $\mathcal{P}(\kappa) \subseteq M$. Hence $\mathcal{P}^M(\kappa) = \mathcal{P}(\kappa)$. Suppose that $2^\kappa > \kappa^+$. Then

$$|(2^\kappa)^M| = 2^\kappa > \kappa^+ \geq |(\kappa^+)^M|$$

so that $M \models 2^\kappa > \kappa^+$. Now κ is represented in the ultrapower by the diagonal (identity) function d on κ, since $\mathcal{U}$ is normal (Lemma A.3.4). Moreover, by Theorem A.3.2, $[d]^+ = [d^+]$, where $d^+:\kappa \longrightarrow \kappa$ is defined by $d^+(\xi) = \xi^+$. Hence by Theorem A.3.2 again:

$$M \models 2^{[d]} > [d]^+ \longleftrightarrow \{\xi < \kappa: V \models 2^\xi > \xi^+\} \in \mathcal{U}$$

proving the required result.

□

§ 3.2 **The Failure of the GCH at a Measurable Cardinal**

For a short while in this section we turn away from the study of the power function in order to analyse the structure of the model $L[\mathcal{U}]$ obtained in Section 3.1. We need to do this mainly in order to be able to present Kunen's result ([Kunen 1971a]) that the failure of the GCH at a measurable cardinal is consistency–wise stronger than the existence of a measurable cardinal (The results of Section 3.1. show that GCH + $\exists\kappa(\kappa$ is measurable) is *not*, consistency–wise, stronger than the existence of a measurable cardinal.)
In order to further investigate the structure of $L[\mathcal{U}]$, it is necessary to develop the theory of *iterated ultrapowers*, on which this section depends heavily. Since this is not directly relevant to the study of the power function, this material has been included in the Appendix 3.2. Other good sources are [Jech 1978] and [Kanamori–Magidor 1978].
Basically, the model $L[\mathcal{U}]$ is to the theory "ZFC + κ is measurable" what L is to ZFC. In $L[\mathcal{U}]$ we obtain a universe which is rich in combinatorial possibilities, because of the existence of a normal measure, but which nevertheless is very uniform, because of the way it is built up, and it is therefore more easily manageable.

Definition 3.2.1 ([Kunen 1970]):

(1) A transitive class M is said to be a *κ–model* if and only if there is a normal measure $\mathcal{U}$ over κ in M such that $M \vdash V = L[\mathcal{U}]$.

In that case $\mathcal{U}$ is called the *constructing measure* for M.

(2) A filter $\mathcal{F}$ over a cardinal κ is said to be a *strong filter* provided $L[\mathcal{F}]$ is a κ–model with constructing measure $\mathcal{F} \cap L[\mathcal{F}]$.

Note that if $\mathcal{F}$ is a strong filter over κ, then $\mathcal{F}$ is not necessarily a measure over κ. We shall see that $\mathcal{F}$ may even be the club filter over κ.

Recall that if $\mathcal{U}$ is a normal measure over a cardinal κ in a model M, and if $N_\alpha = Ult_\alpha(M,\mathcal{U})$ is the α^{th} iterated ultrapower for each ordinal α (where we put $N_0 = M$), then there are elementary embeddings $i_{\alpha\beta}: N_\alpha \longrightarrow N_\beta$ whenever $\alpha \leq \beta$. In particular, $i_{0\alpha}(\mathcal{U})$ is in N_α a normal measure over $i_{0\alpha}(\kappa)$. The next lemma provides a nice internal description of the $i_{0\alpha}(\mathcal{U})$.

Lemma 3.2.2 ([Kunen 1970]):

Suppose that M *is a κ–model with constructing measure $\mathcal{U}$. Suppose also that α is an infinite limit ordinal, and that* $N_\alpha = Ult_\alpha(M,\mathcal{U})$ *is well–founded. Then*

$$X \in i_{0\alpha}(\mathcal{U}) \longleftrightarrow X \in \mathcal{P}(i_{0\alpha}(\kappa)) \cap N_\alpha \text{ and } \exists\beta<\alpha\, [\{i_{0\gamma}(\kappa): \beta \leq \gamma < \alpha\} \subseteq X]$$

Proof: If N_α is wellfounded, then so is $N_\gamma = Ult_\gamma(M,\mathcal{U})$ for all $\gamma \leq \alpha$, and $i_{0\gamma}(\mathcal{U})$ is a normal measure over $i_{0\gamma}(\kappa)$ in N_γ. Moreover, since α is limit, N_α is the direct limit of $(N_\gamma: \gamma < \alpha)$ (by Lemma A.3.14). Thus if $X \in i_{0\alpha}(\mathcal{U})$, then there exists a $\beta < \alpha$ and a $Y \in i_{0\beta}(\mathcal{U})$ such that $X = i_{\beta\alpha}(Y)$. Then $i_{\beta\gamma}(Y) \in i_{0\gamma}(\mathcal{U})$ for all $\beta \leq \gamma < \alpha$. But clearly

$$Z \in i_{0\gamma}(\mathcal{U}) \longleftrightarrow Z \in \mathcal{P}(i_{0\gamma}(\kappa)) \cap N_\gamma \text{ and } i_{0\gamma}(\kappa) \in i_{\gamma,\gamma+1}(Z).$$

and hence $i_{0\gamma}(\kappa) \in i_{\beta,\gamma+1}(Y)$. Now $i_{\gamma+1,\alpha}$ fixes every element of $i_{\beta,\gamma+1}(Y)$ (by Lemma A.3.17(a)), and thus $i_{\beta,\gamma+1}(Y) \subseteq X$. It follows that if $X \in i_{0\alpha}(\mathcal{U})$, then $\{i_{0\gamma}(\kappa): \beta \leq \gamma < \alpha\} \subseteq X$, proving one direction. For the other direction, suppose that $X \in \mathcal{P}(i_{0\alpha}(\kappa)) \cap N_\alpha$ is such that $\{i_{0\gamma}(\kappa): \beta \leq \gamma < \alpha\} \subseteq X$ for some $\beta < \alpha$. Choose $Y \in N_\gamma$ for some $\gamma \geq \beta$ such that $i_{\gamma\alpha}(Y) = X$. Then $i_{0\gamma}(\kappa) \in i_{\gamma\alpha}(Y)$, and so $i_{0\gamma}(\kappa) \in i_{\gamma,\gamma+1}(Y)$ (because $i_{\gamma+1,\alpha}$ fixes $i_{0\gamma}(\kappa)$). It follows that $Y \in i_{0\gamma}(\mathcal{U})$, proving that $X \in i_{0\alpha}(\mathcal{U})$, as required.

□

Definition 3.2.3: If ρ is a limit cardinal, then the *cardinal filter* over ρ consists of all those $X \subseteq \rho$ which contain all cardinals below ρ from some point onwards, i.e. all those $X \subseteq \rho$ for which there is an $\xi < \rho$ such that "$\xi \leq \eta < \rho$ and η a cardinal" implies $\eta \in X$.

Note that if ρ is a limit cardinal of uncountable cofinality, then the club filter over ρ extends the cardinal filter over ρ.

Theorem 3.2.4 ([Kunen 1970]):
Suppose that M *is a* κ*-model (with constructing measure* $\mathcal{U}$*), and suppose that* ρ *is a cardinal in* M, $\rho > (\kappa^+)^M$. *If* $\mathcal{F}$ *is either the club filter over* ρ *(assuming the* real *cofinality of* ρ *is* $> \omega$*) or the cardinal filter over* ρ *(assuming that* ρ *is a limit cardinal in the* real *universe), then:*

(1) $\mathcal{F}$ *is a strong filter over* ρ;

(2) $L[\mathcal{F}] = \mathrm{Ult}_\rho(M,\mathcal{U})$, $i_{0\rho}(\kappa) = \rho$, *and* $i_{0\rho}(\mathcal{U}) = \mathcal{F} \cap L[\mathcal{F}]$.

Proof: Note that if M is a κ–model, then by Theorem 3.1.3 $M \vdash$ GCH, and thus $i_{0\rho}(\kappa) = \rho$ by Lemma A.3.19(2). Suppose that ρ is of uncountable cofinality, and let $X \in i_{0\rho}(\mathcal{U})$. Then by Lemma 3.2.2, there is a $\beta < \rho$ such that $\{i_{0\gamma}(\kappa): \beta \leq \gamma < \rho\} \subseteq X$. But this set is clearly club, and thus $X \in \mathcal{F}$ (where $\mathcal{F}$ is the club filter over ρ). Hence $i_{0\rho}(\mathcal{U}) \subseteq \mathcal{F} \cap \mathrm{Ult}_\rho(M,\mathcal{U})$, and thus $i_{0\rho}(\mathcal{U}) = \mathcal{F} \cap \mathrm{Ult}_\rho(M,\mathcal{U})$ (because $i_{0\rho}(\mathcal{U})$ is an ultrafilter). Since $M = L[\mathcal{U}]$, $\mathrm{Ult}_\rho(M,\mathcal{U}) = L[i_{0\rho}(\mathcal{U})] = L[\mathcal{F}]$. The same argument works if ρ is a limit cardinal and $\mathcal{F}$ is the cardinal filter over ρ, for in that case $\{i_{0\gamma}(\kappa): \beta \leq \gamma < \rho\}$ contains all cardinals below ρ from a certain point onwards (by Lemma A.3.19(2)).

□

We are now ready to state the promised theorem on the failure of the GCH at a measurable cardinal, due to Kunen. Recall that a filter is said to be *uniform* provided that all its members have the same cardinality.

Theorem 3.2.5 [Kunen 1971a]:

Suppose that there is a measurable cardinal κ such that at least one of the following holds:

(1) $2^\kappa > \kappa^+$;

(2) *Every κ–complete filter over κ can be extended to a κ–complete ultrafilter;*

(3) *There is a uniform κ–complete ultrafilter over κ^+.*

Then for all ordinals θ there is an inner model N *of* ZFC *such that*

N $\vdash$ (ZFC + $\exists$ θ–many measurable cardinals).

We shall only prove the result of Theorem 3.2.5 holds assuming (1), but will indicate why it is true assuming (2) or (3) towards the end of this section. The proof of Theorem 3.2.5 is quite complicated, and we shall need several seemingly unrelated propositions before we shall be able to tackle the proof (After Proposition 3.2.17). Because it is quite easy to get lost in all the details, we shall present a short proof of a weaker statement as an appetizer, which shows how we may use the machinery developed earlier in this section:

Theorem 3.2.6 ([Kunen 1971a]):

If there is a measurable cardinal κ so that $2^\kappa > \kappa^+$, then there is a transitive set model of "ZFC + $\exists\kappa$(κ is measurable)"

Proof: Let $\mathcal{U}$ be a normal ultrafilter over κ, let $N = L[\mathcal{U}]$, and let $\overline{\mathcal{U}} = \mathcal{U} \cap N$. For any ordinal α, let $N_\alpha = \text{Ult}_\alpha(N, \overline{\mathcal{U}}) = L[i_{0\alpha}(\overline{\mathcal{U}})]$, where $i_{0\alpha}: N \longrightarrow N_\alpha$ is the natural elementary embedding. Let $\mathcal{F} = \{X \subseteq i_{0\omega}(\kappa): \exists n \forall m(n < m < \omega \longrightarrow i_{0m}(\kappa) \in X\}$. By Lemma 3.2.2 $N_\omega = L[\mathcal{F}]$ is a $i_{0\omega}(\kappa)$–model with constructing measure $\mathcal{F} \cap L[\mathcal{F}]$.

Let M be the transitive isomorph of $V^\kappa/\mathcal{U}$, where V is the universe, and let $j: V \longrightarrow M$ be the natural embedding. By Lemma A.3.3, the sequence $(i_{0n}(\kappa): n < \omega)$ is an element of M, and since $\mathcal{F}$ can be defined in terms of this sequence, we must have $\mathcal{F} \cap M \in M$ as well. It follows that $L[\mathcal{F}] \subseteq M$. Since κ is measurable, it is strongly inaccessible, and thus $j(\kappa)$ is strongly inaccessible in M. It follows that $j(\kappa)$ is strongly inaccessible in $L[\mathcal{F}]$ as well (because $L[\mathcal{F}] \vdash$ GCH). By Lemma A.3.19(1), we have $i_{0\omega}(\kappa) < ((2^\kappa)^+)^N$. Since $N = L[\mathcal{U}]$ is a model of GCH, we have $((2^\kappa)^+)^N = (\kappa^{++})^N$, and by Lemma A.3.3 we have $2^\kappa < j(\kappa)$.

Also our assumption $2^\kappa > \kappa^+$ implies $\kappa^{++} \leq 2^\kappa$. Putting these inequalities together yields the following chain:

$$i_{o\omega}(\kappa) < ((2^\kappa)^+)^N = (\kappa^{++})^N \leq \kappa^{++} \leq 2^\kappa < j(\kappa).$$

Hence $i_{o\omega}(\kappa) < j(\kappa)$, and thus $i_{o\omega}(\kappa) \in J^{\mathcal{F}}_{j(\kappa)}$ (a member of the $L[\mathcal{F}]$ hierarchy; see Appendix 1.1). Moreover since $L[\mathcal{F}] \vdash$ "$j(\kappa)$ is inaccessible", we have $J^{\mathcal{F}}_{j(\kappa)} \vdash$ ZFC. Finally also $\mathcal{F} \cap L[\mathcal{F}] \subseteq \mathcal{P}(i_{o\omega}(\kappa))$, and so $\mathcal{F} \cap L[\mathcal{F}] \in J^{\mathcal{F}}_{j(\kappa)}$. Hence $J^{\mathcal{F}}_{j(\kappa)}$ is a model of ZFC in which there is a normal measure $\mathcal{F} \cap L[\mathcal{F}]$ over $i_{o\omega}(\kappa)$.

□

We now tackle the series of lemmas that culminate in the proof of Theorem 3.2.5, closely following the paper [Kunen 1971a]. Henceforth assume that κ is a measurable cardinal.

Proposition 3.2.7:

If $a \subseteq \kappa^+$, $\mathcal{S} \subseteq \mathcal{P}(\kappa)$ *and if* b *is a set of ordinals,* $|b| < \kappa$, *then*

$$|\mathcal{P}(\kappa) \cap L[a,b,\mathcal{S}]| \leq \kappa^+.$$

Proof: Via a condensation argument using Lemma A.1.8. Let $y \subseteq \mathcal{P}(\kappa) \cap L[a,b,\mathcal{S}]$ and let $|b| = \lambda < \kappa$. Choose γ sufficiently large that $y \in J_\gamma[a,b,\mathcal{S}]$, and let $M \leq_e J_\gamma[a,b,\mathcal{S}]$ such that $\{y\} \cup \kappa \cup b \subseteq M$ and $|M| = \kappa$. Let $j: M \longrightarrow N$ be the transitive collapsing isomorphism of M onto $N = J_\mu[j''a, j''b, j''\mathcal{S}]$ provided by Lemma A.1.8, where $\mu < \kappa^+$ since $|M| \leq \kappa$. Clearly $j(y) = y$, so $y \in N$. Let $x = j''b$; then $x \subseteq \kappa^+$ and $|x| = \lambda$. Also $j(z) = z$ for all $z \in \mathcal{S} \cap M$. It follows that $j''\mathcal{S} = \mathcal{S} \cap N$. Next let $\nu = \sup(On^M)$; then $a \cap M = a \cap \nu \subseteq \nu < \kappa^+$. Hence $j(\alpha) = \alpha$ for any $\alpha \in a \cap \nu$, proving that $j''a = a \cap \nu$. These facts add up to prove that:

$$y \in N = J_\mu[a \cap \nu, x, \mathcal{S}]$$

Since $y \in \mathcal{P}(\kappa) \cap L[a,b,\mathcal{S}]$ was arbitrary, we see that

$$\mathcal{P}(\kappa) \cap L[a,b,\mathcal{S}] \subseteq \bigcup \{J_\mu[a \cap \nu, x, \mathcal{S}] : \mu,\nu < \kappa^+ \text{ and } x \subseteq \kappa^+ \text{ such that } |x| = \lambda\}.$$

The latter set clearly has cardinality $\leq \kappa^+$ since κ is strongly inaccessible.

□

Proposition 3.2.8:

If $\{X_\alpha : \alpha < \lambda\}$ *is a set of subsets of a cardinal* λ, *then there is* $a \subseteq \lambda$ *such that each* $X_\alpha \in L[a]$.

Proof: Choose $f \in L$ such that $f: \lambda \times \lambda \longrightarrow \lambda$ is a bijection. Let $a = \bigcup \{f''X_\alpha \times \{\alpha\}: \alpha < \lambda\}$. Then $a \in L[a]$, $f \in L[a]$ and so $X_\alpha = \{\xi < \lambda: f(\xi,\alpha) \in a\} \in L[a]$.

□

Proposition 3.2.9 [Kunen 1971a]:

Suppose κ *is a measurable cardinal such that* $2^\kappa > \kappa^+$. *Assume further that* $b \subseteq On$ *is a set of cardinality* $< \kappa$, *and let* $\delta < \kappa^{++}$. *Let* $\mathcal{U}$ *be a* κ*–complete ultrafilter over* κ. *Then there is an inner model* M *of* ZFC *such that* $b \in M$, $\mathcal{U} \cap M \in M$ *and*

$$\delta < i_{01}^{\mathcal{U}\cap M}(\kappa) < \kappa^{++}$$

(*where* $i_{01}^{\mathcal{U}\cap M}: M \longrightarrow Ult_1(M, \mathcal{U} \cap M)$ *is the standard elementary embedding*).

Proof: From Lemma A.3.3 we know that $i_{01}^{\mathcal{U}}(\kappa) > 2^\kappa \geq \kappa^{++}$. Hence there exist functions $g_\alpha: \kappa \longrightarrow \kappa$ (for $\alpha \leq \delta$) representing ordinals $< i_{01}^{\mathcal{U}}(\kappa)$ in $Ult_1(V, \mathcal{U})$ such that $\{\xi < \kappa: g_\alpha(\xi) < g_\beta(\xi)\} \in \mathcal{U}$ whenever $\alpha < \beta \leq \delta$. Choose $a \subseteq \kappa^+$ such that all the g_α are in L[a] (by Lemma 3.2.8), and let $M = L[a,b,\mathcal{U}]$. Since each $g_\alpha \in M$, certainly $\delta < i_{01}^{\mathcal{U}\cap M}(\kappa)$. But by Proposition 3.2.7, there are at most κ^+–many maps $f:\kappa \longrightarrow \kappa$ in M, which implies that $i_{01}^{\mathcal{U}\cap M}(\kappa) < \kappa^{++}$.

□

Definition 3.2.10: For each ordinal θ, a θ*–set* is a set $b \subseteq On$ such that b has order type $\omega\theta$, and such that for each $\xi \in b$, $\xi > \sup(b \cap \xi)$.

Definition 3.2.11: If b is a θ–set, $\xi < \theta$ and $n < \omega$, then

(1) $\gamma^b_{\xi n}$ is the $(\omega\xi + n)^{th}$ ordinal in b;

(2) $A^b_\xi = \{\gamma^b_{\xi n} : n < \omega\}$;

(3) $\lambda^b_\xi = \sup(A^b_\xi)$;

(4) $\mathcal{F}^b_\xi = \{x \subseteq \lambda^b_\xi : \exists m\ \forall k \geq m\ (\gamma^b_{\xi k} \in x)\}$;

(5) $\vec{\mathcal{F}}^b = (\mathcal{F}^b_\xi : \xi < \theta)$;

(6) $\Psi(b,\xi)$ is the statement that $\mathcal{F}^b_\xi \cap L[\vec{\mathcal{F}}^b]$ is, in $L[\vec{\mathcal{F}}^b]$, a normal measure over λ^b_ξ.

Definition 3.2.12:

(1) $K_0 = \{\sigma : \sigma$ a strong limit cardinal $> 2^\kappa$ and $cf(\sigma) > \kappa\}$;

(2) $K_{\xi+1} = \{\sigma \in K_\xi : |\sigma \cap K_\xi| = \sigma\}$;

(3) $K_\beta = \bigcap \{K_\xi : \xi < \beta\}$ if β is a limit ordinal.

K_0 is choosen so that the elementary embeddings $i_{0\gamma}$ *preserve* $\sigma \in K_0$ for all $\gamma < \sigma$ (by Lemma A.3.19(4)). We shall see that for $\theta < \kappa$, if b is a θ–set $\subseteq K_{\omega\theta+1}$, then each λ^b_ξ is measurable in $L[\vec{\mathcal{F}}^b]$.

Proposition 3.2.13:
If M *is an inner model of* ZFC, $\mathcal{U}$ *a measure over* κ *such that* $\mathcal{U} \cap M \in M$, $\sigma \in K_0$ *and* $\mu < \sigma$, *then* $i^{\mathcal{U}\cap M}_{0\mu}(\sigma) = \sigma$.

Proof : In M, all the $\sigma \in K_0$ are still strong limit and larger than $(2^\kappa)^M$, and their cofinality is still larger than κ. Thus by Lemma A.3.19(4), $i^{\mathcal{U}\cap M}_{0\mu}(\sigma) = \sigma$ for all $\mu < \sigma$.

□

Proposition 3.2.14 [Kunen 1971a]:
If b *is a* θ*–set for some* $\theta < \kappa$ *and* $\xi < \theta$ *is such that*

$$\forall\eta<\xi\ (\lambda^b_\eta < \kappa^{++}) \quad \textit{and} \quad \forall n<\omega\ \forall\eta<\theta\ [\xi \leq \eta < \theta \longrightarrow \gamma^b_{\eta n} \in K_0]$$

then $\Psi(b,\xi)$.

Proof: Let $\delta = \sup\{\lambda^b_\eta : \eta < \xi\}$; then $\delta < \kappa^{++}$. We shall first assume that $\delta < \kappa$, and then prove the general case using an ultrapower argument. Thus assume $\delta < \kappa$, and let $\lambda = \lambda^b_\xi$. Then $\forall\eta<\lambda\ \forall n<\omega\ [i^{\mathcal{U}}_{0\lambda}(\gamma^b_{\eta n}) = \gamma^b_{\eta n}]$ since $\delta < \kappa$ and κ is critical. Now if $\xi \leq \eta < \theta$ and $n < \omega$, then $\gamma^b_{\eta n} \in K_0$ so also $i^{\mathcal{U}}_{0\lambda}(\gamma^b_{\eta n}) = \gamma^b_{\eta n}$. Hence $i^{\mathcal{U}}_{0\lambda}$ preserves all the $\gamma^b_{\eta n}$ for $\eta < \theta$ and $n < \omega$. Moreover, because $i^{\mathcal{U}}_{0\lambda}(\lambda^b_\eta) = \sup\{i^{\mathcal{U}}_{0\lambda}(\gamma_{\eta n}) : n < \omega\} = \lambda^b_\eta$, the λ^b_η are preserved as well for all $\eta < \theta$.

Now by definition of $\mathcal{F}^b_\eta$ and elementarity of $i^{\mathcal{U}}_{0\lambda}$ we have:

$$i^{\mathcal{U}}_{0\lambda}(\mathcal{F}^b_\eta) = \{x \subseteq i^{\mathcal{U}}_{0\lambda}(\lambda^b_\eta) : \exists m\ \forall k \geq m (i^{\mathcal{U}}_{0\lambda}(\gamma^b_{\eta k}) \in x\}$$

It follows that $i^{\mathcal{U}}_{0\lambda}(\mathcal{F}^b_\eta) = \mathcal{F}^b_\eta \cap \mathrm{Ult}_\lambda(V, \mathcal{U})$ for all $\eta < \theta$, and thus that

$$L[\vec{\mathcal{F}}^b] = L[(\mathcal{F}^b_\eta : \eta < \theta)] = L[(i^{\mathcal{U}}_{0\lambda}(\mathcal{F}^b_\eta) : \eta < \theta)]$$

Now $i^{\mathcal{U}}_{0\lambda}(\mathcal{U}) = \mathcal{F}^b_\xi \cap \mathrm{Ult}_\lambda(V, \mathcal{U})$ by Lemma 3.2.4(2), and so

$$\mathcal{F}^b_\xi \cap L[\vec{\mathcal{F}}^b] = i^{\mathcal{U}}_{0\lambda}(\mathcal{U}) \cap L[(i^{\mathcal{U}}_{0\lambda}(\mathcal{F}^b_\eta) : \eta < \theta)]$$

is a normal measure over λ in $L[(i^{\mathcal{U}}_{0\lambda}(\mathcal{F}^b_\eta) : \eta < \theta)]$. Hence $\mathcal{F}^b_\xi$ is a normal over λ in $L[\vec{\mathcal{F}}^b]$. This concludes the proof in case $\delta < \kappa$.

Assume now that $\delta < \kappa^{++}$ is arbitrary; by Proposition 3.2.9 there is an inner model M of ZFC such that $\mathcal{U} \cap M \in M$ and such that $\delta < i^{\mathcal{U}\cap M}_{01}(\kappa)$. We now reason as above inside $\mathrm{Ult}_1(M, \mathcal{U}\cap M)$ to prove $i^{\mathcal{U}\cap M}_{1\lambda}$ fixes all the $i^{\mathcal{U}\cap M}_{01}(\gamma^b_{\eta n})$.

□

Definition 3.2.15: Let $a \subseteq On$, $X \subseteq L[a]$. We define $\mathcal{H}(a,X)$ to be the smallest elementary submodel L[a] such that $X \cup a \cup \{a\} \subseteq \mathcal{H}(a,X)$, i.e. the class of all sets in L[a], definable from elements in $X \cup a \cup \{a\}$.

Proposition 3.2.16 [Kunen 1971a]:
Let $a \subseteq K_\alpha$ *be a set of order type* $\alpha < \kappa$. *Suppose that* $a = \{\sigma_\xi : \xi < \alpha\}$ *is in increasing order. Then for each* $\xi < \alpha$, *we have* $|\sigma_\xi \cap \mathcal{H}(a, K_{\xi+1} - \sigma_\xi)| \leq \kappa^+$.

Proof: By induction on $\xi < \alpha$; assume Proposition 3.2.16 holds for $\eta < \xi$, and let $\rho = \sup\{\sigma_\eta : \eta < \xi\}$. If $\rho = \sigma_\xi$, then we have

$$\sigma_\xi \cap \mathcal{H}(a, K_{\xi+1} - \sigma_\xi) \subseteq \cup \{\sigma_\eta \cap \mathcal{H}(a, K_{\xi+1} - \sigma_\xi) : \eta < \xi\}$$
$$\subseteq \cup \{\sigma_\eta \cap \mathcal{H}(a, K_{\eta+1} - \sigma_\eta) : \eta < \xi\}$$

and by induction hypothesis the latter set has cardinality $\leq \kappa^+$.

Thus we may assume that $\rho < \sigma_\xi$. Let $\mathcal{A} = \mathcal{H}(a, K_\xi - \rho) \supseteq \mathcal{H}(a, K_{\xi+1} - \sigma_\xi)$. Then

$$\rho \cap \mathcal{A} \subseteq \cup \{\sigma_\eta \cap \mathcal{A}: \eta < \xi\} \subseteq \cup \{\sigma_\eta \cap \mathcal{H}(a, K_{\eta+1} - \sigma_\eta): \eta < \xi\},$$

and the latter set has cardinality $\leq \kappa^+$. Thus $|\rho \cap \mathcal{A}| \leq \kappa^+$.

Let j: $\mathcal{A} \longrightarrow T$ be the transitive collapse of $\mathcal{A}$ (provided by Lemma A.1.8). If $\tau \in K_{\xi+1} - \sigma_\xi$, then $|\tau \cap K_\xi| = \tau$ by definition of $K_{\xi+1}$, and thus $j(\tau) = j''(\tau \cap \mathcal{A}) \geq \tau$. But $j(\tau) \leq \tau$ is a property of the collapsing isomorphism j, proving that $j(\tau) = \tau$. Similarly $j(\rho) = j''(\rho \cap \mathcal{A})$ and $|\rho \cap \mathcal{A}| \leq \kappa^+$ implies $j(\rho) < \kappa^{++}$. Let $\delta = j(\rho)$ and let $b = j(a)$. Then we may choose an inner model M of ZFC and a measure $\mathcal{U}$ over κ such that $b \in M$, $\mathcal{U} \cap M \in M$, and such that $\delta < i_{01}^{\mathcal{U}\cap M}(\kappa) < \kappa^{++}$ (by Proposition 3.2.9). Let $\pi \in \sigma_\xi \cap \mathcal{H}(a, K_{\xi+1} - \sigma_\xi)$. We shall show that $i_{1\pi}^{\mathcal{U}\cap M}(j(\pi)) = j(\pi)$. Choose finite sets $s \subseteq a \cap \sigma_\xi$, $t \subseteq K_{\xi+1} - \sigma_\xi$ such that π is definable in L[a] from $t \cup s \cup \{a\}$. Then $j(\pi)$ is definable in L[b] from $t \cup j(s) \cup \{b\}$ (since $j(\tau) = \tau$ for $\tau \in t \subseteq K_{\xi+1} - \sigma_\xi$).

Now if $x \in a \cap \sigma_\xi$, then $x \in a \cap \rho$, so $j(x) < j(\rho) < i_{01}^{\mathcal{U}\cap M}(\kappa)$. Hence $i_{1\pi}^{\mathcal{U}\cap M}(j(x)) = j(x)$.

Also if $\tau \in t$, then $\tau \geq \sigma_\xi > \pi$. Since $\tau \in K_0$, $i_{0\gamma}^{\mathcal{U}\cap M}(\tau) = \tau$ for all $\gamma < \tau$, so that

$$\tau = i_{0\pi}^{\mathcal{U}\cap M}(\tau) = i_{1\pi}^{\mathcal{U}\cap M} \cdot i_{01}^{\mathcal{U}\cap M}(\tau) = i_{1\pi}^{\mathcal{U}\cap M}(\tau).$$

Hence all the elements of $t \cup j(s) \cup \{b\}$ are preserved by $i_{1\pi}^{\mathcal{U}\cap M}$, and since $j(\pi)$ is definable from these elements in L[b], we may conclude that $i_{1\pi}^{\mathcal{U}\cap M}$ preserves $j(\pi)$ as well.

Now $j(\pi) \geq i_{01}^{\mathcal{U}\cap M}(\kappa)$ would imply also that $j(\pi) = i_{1\pi}^{\mathcal{U}\cap M}(j(\pi)) \geq i_{0\pi}^{\mathcal{U}\cap M}(\kappa) > \pi$, which contradicts $j(\pi) \leq \pi$; it follows that $j(\pi) < i_{01}^{\mathcal{U}\cap M}(\kappa)$.

Thus for all $\pi \in \sigma_\xi \cap \mathcal{H}(a, K_{\xi+1} - \sigma_\xi)$, we have $j(\pi) < i_{01}^{\mathcal{U}\cap M}(\kappa) < \kappa^{++}$. It follows that $|\sigma_\xi \cap \mathcal{H}(a, K_{\xi+1} - \sigma_\xi)| \leq \kappa^+$ as required.

□

We are now ready to prove Theorem 3.2.5. It is a direct consequence of the following proposition.

Proposition 3.2.17 [Kunen 1971a]:

If b $\subsetneq K_{\omega\theta+1}$ *is a* θ*–set, then* $\Psi(b,\xi)$ *holds for all* $\xi < \theta$. *Thus* $L[\vec{\mathcal{F}}^b]$ *is an inner model of* ZFC *in which* λ_ξ^b *is measurable for each* $\xi < \theta$.

Proof: First assume $\theta < \kappa$. Let $\mathcal{A} = \mathcal{H}(b, K_{\omega\theta} - \sup\{\lambda_\eta^b : \eta < \xi\})$, and let j be the transitive collapse of $\mathcal{A}$. For $\xi \leq \eta < \theta$ and $n < \omega$, $j(\gamma_{\eta n}^b) = \gamma_{\eta n}^b \in K_0$. For $\eta < \theta$ and $n < \omega$, $j(\gamma_{\eta n}^b) < \kappa^{++}$ by Proposition 3.2.16. Hence by Proposition 3.2.14, $\Psi(j(b), \xi)$ in $\mathcal{A}$. But $\mathcal{A} \leq_e$ L[b], and so we must have $\Psi(b,\xi)$ for all $\xi < \theta$.
To prove the proposition for all θ, carry out the entire proof in $\mathrm{Ult}_{\theta+1}(V, \mathcal{U})$, where $\theta < i_{0,\theta+1}^{\mathcal{U}}(\kappa)$.

□

Thus with the above hypotheses on θ and b, each λ_ξ^b ($\xi < \theta$) is a measurable cardinal in the inner model $L[\vec{\mathcal{F}}^b]$. We have now proved Theorem 3.2.5 from hypothesis (1) (in the statement of the theorem) only. However (3) implies either (1) or (2). The proof from (2) follows exactly as the proof from (1), since one may prove that for any $\delta < (2^\kappa)^+$ there is a κ–complete ultrafilter $\mathcal{U}$ over κ such that $\delta < i_{01}^{\mathcal{U}}(\kappa)$. Thus if $2^\kappa = \kappa^+$ and (2) holds, we may choose M = V in Proposition 3.2.9. Recall that κ is compact provided every κ–complete filter over a any cardinal $\lambda \geq \kappa$ can be extended to a κ–complete ultrafilter over λ. Thus if κ is a compact cardinal, then hypothesis (2) in the statement of Theorem 3.2.5 holds. It follows that compactness is, consistency–wise, stronger than measurability. On the other hand it is a theorem of Magidor ([Magidor 1976]) that the least measurable cardinal may be compact. (See also [Apter 1991] for an alternate proof.)

We shall see in Section 3.4 that if we assume the existence of a κ^{++}– supercompact cardinal κ (refer to Appendix 3.3 for the relevant definitions), we can find a notion of forcing $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} [\kappa \text{ is measurable and } 2^\kappa = \kappa^{++}]$$

Hence the consistency of the existence of a κ^{++}– supercompact cardinal already implies the consistency of the existence of arbitrarily many measurable cardinals.

§ 3.3 Compact Cardinals and the Power Function

In this section we want to discuss a theorem of Solovay ([Solovay 1974]) which further shows the deep connection between large cardinals and the power function: *If a compact cardinal κ exists, then the Singular Cardinals Hypothesis holds above κ.*
Hence if a compact cardinal exists, then the GCH cannot fail everywhere. This beautiful theorem shows how large cardinals may impose a structure on the universe which prevent it from being too chaotic. The notion of a compact cardinal is defined and discussed in Appendix 3.3. Briefly, if $\lambda \geq \kappa$ are cardinals, then κ is said to be λ–compact if there is a a *fine measure* over $[\lambda]^{<\kappa}$, i.e. a κ–complete ultrafilter $\mathcal{U}$ over $[\lambda]^{<\kappa}$ with the property that for any $P \in [\lambda]^{<\kappa}$, the set $\{Q \in [\lambda]^{<\kappa}: P \subseteq Q\} \in \mathcal{U}$. κ is said to be compact provided κ is λ–compact for all $\lambda \geq \kappa$.

Theorem 3.3.1 [Solovay 1974]:
Suppose that κ is a compact cardinal. Then the SCH holds above κ, i.e. whenever $\eta > \kappa$ is singular such that $2^{\mathrm{cf}(\eta)} < \eta$, then $\eta^{\mathrm{cf}(\eta)} = \eta^{+}$.

Solovay proved that if $\lambda \geq \kappa$ is regular and κ is λ–compact, then $\lambda^{<\kappa} = \lambda$ (see Appendix 3.1 for various equivalent formulations of the notion of λ–compactness). The proof of this bit of cardinal arithmetic is non–trivial, and we shall need several propositions before we obtain the result. In Section 5.3. we shall attain the same result from a weaker condition (see lemma 5.3.2), but as this necessitates the introduction of generic ultrapowers we have chosen not to present it in this chapter.
If $\mathcal{U}$ is an ultrafilter over some set S, then $\mathcal{U}$ is said to be *uniform* provided that $|X| = |S|$ for all $X \in \mathcal{U}$. We say that *almost all* $p \in S$ have a certain property φ (mod $\mathcal{U}$) if and only if

$$\{p \in S\colon p \text{ has property } \varphi\} \in \mathcal{U}.$$

Thus $\mathcal{U}$ is a fine measure over $[\lambda]^{<\kappa}$ provided that for any $P \in [\lambda]^{<\kappa}$, almost all $Q \in [\lambda]^{<\kappa}$ include P.

Proposition 3.3.2 [Solovay 1974]:
If κ is λ–compact and $\lambda > \kappa$ is regular, then there is a uniform κ–complete ultrafilter $\mathcal{U}$ over λ such that almost all (mod $\mathcal{U}$) *ordinals $< \lambda$ have cofinality $< \kappa$ (i.e. $\{\alpha < \lambda: \mathrm{cf}(\alpha) < \kappa\} \in \mathcal{U}$).*

Proof: Let $\mathcal{F}$ be a fine measure over $[\lambda]^{<\kappa}$, and consider the ultrapower Ult(V,$\mathcal{F}$). Let $j_{\mathcal{F}}$ be the associated elementary embedding of the universe and choose $f:[\lambda]^{<\kappa} \longrightarrow On$ such that $[f]_{\mathcal{F}} = \sup\{j_{\mathcal{F}}(\gamma) : \gamma < \lambda\}$. Define maps $g:[\lambda]^{<\kappa} \longrightarrow \lambda$ by $g(P) = \sup(P)$, and $h:[\lambda]^{<\kappa} \longrightarrow \lambda$ by $h(P) = \sup\{\alpha \in P: \alpha < f(P)\}$. Let $\gamma < \lambda$. Then $j_{\mathcal{F}}(\gamma) \leq [g]_{\mathcal{F}}$ since $\{P \in [\lambda]^{<\kappa}: \gamma \in P\} \in \mathcal{F}$ implies that $\{P \in [\lambda]^{<\kappa}: \gamma \leq g(P)\} \in \mathcal{F}$. By definition of f, it follows that $[f]_{\mathcal{F}} \leq [g]_{\mathcal{F}} \leq j_{\mathcal{F}}(\lambda)$.
Now define a κ–complete non–principal ultrafilter $\mathcal{U}$ over λ by:

$$X \in \mathcal{U} \quad \text{if and only if} \quad \{P \in [\lambda]^{<\kappa}: f(P) \in X\} \in \mathcal{F}.$$

Since $\{P \in [\lambda]^{<\kappa}: f(P) \geq \gamma\} \in \mathcal{F}$ for all $\gamma < \lambda$, it follows that $\lambda - \gamma \in \mathcal{U}$ for all $\gamma < \lambda$; thus $\mathcal{U}$ is non–principal and since moreover λ is regular, $\mathcal{U}$ is uniform as well.
Next note that since $\gamma \in P$ for almost all P (mod $\mathcal{F}$), and $f(P) \geq \gamma$ for almost all P (mod $\mathcal{F}$), we must have $\{P \in [\lambda]^{<\kappa}: h(P) \geq \gamma\} \in \mathcal{F}$ for all $\gamma < \lambda$. Hence by definition of f, $[f]_{\mathcal{F}} \leq [h]_{\mathcal{F}}$; however, $f(P) \geq h(P)$ for all $P \in [\lambda]^{<\kappa}$, so also $[f]_{\mathcal{F}} \geq [h]_{\mathcal{F}}$. Hence $f(P) = h(P)$ for almost all P (mod $\mathcal{F}$), but $\mathrm{cf}(h(P)) < \kappa$ for all P, so it follows that $\mathrm{cf}(f(P)) < \kappa$ for almost all P (mod $\mathcal{F}$). Hence $\{\alpha < \lambda: \mathrm{cf}(\alpha) < \kappa\} \in \mathcal{U}$, as required.

□

Proposition 3.3.3 [Solovay 1974]:
If κ is λ–compact, where $\lambda > \kappa$ is regular, then there is a uniform κ–complete ultrafilter $\mathcal{U}$ over λ and a set $\{M_\alpha: \alpha < \lambda\}$ such that

(1) $|M_\alpha| < \kappa$ *for all* $\alpha < \lambda$;
(2) *For all* $\gamma < \lambda$, $\{\alpha < \lambda: \gamma \in M_\alpha\} \in \mathcal{U}$.

Proof: Let $\mathcal{U}$ be the ultrafilter provided by Proposition 3.3.2 Then $X = \{\alpha < \lambda: \mathrm{cf}(\alpha) < \kappa\} \in \mathcal{U}$. For each $\alpha \in X$, choose A_α cofinal in α such that $|A_\alpha| < \kappa$, and choose $A_\alpha = \emptyset$ if $\alpha \notin X$. In the ultrapower Ult(V,$\mathcal{U}$), the sequence $(A_\alpha: \alpha < \lambda)$ represents a set A of ordinals which is cofinal in the ordinal [d] (where $d:\lambda \longrightarrow \lambda$ is the diagonal map, $d(\alpha) = \alpha$ for all $\alpha < \lambda$). Moreover, $[d] \geq j_{\mathcal{U}}(\gamma)$ for all $\gamma < \lambda$, and so we may inductively define a sequence

$(\eta_\gamma : \gamma < \lambda)$ as follows:

(i) $\eta_0 = 0$;

(ii) $\eta_{\gamma+1}$ = least ordinal η such that there is $\xi \in A$ with $j_{\mathcal{U}}(\eta_\gamma) \leq \xi < j_{\mathcal{U}}(\eta)$;

(iii) $\eta_\delta = \sup\{\eta_\gamma : \gamma < \delta\}$ if δ is limit.

Thus $A \cap \{\xi : j_{\mathcal{U}}(\eta_\gamma) \leq \xi < j_{\mathcal{U}}(\eta_{\gamma+1})\} \neq \emptyset$ for any $\gamma < \lambda$. Choose $I_\gamma = \{\xi : \eta_\gamma \leq \xi < \eta_{\gamma+1}\}$, and let $M_\alpha = \{\gamma < \lambda : I_\gamma \cap A_\alpha\} \neq \emptyset$. Since each A_α has cardinality $< \kappa$ and the I_γ are mutually disjoint, also $|M_\alpha| < \kappa$ for all $\alpha < \lambda$, proving (1). But $\{\alpha < \lambda : A_\alpha \cap I_\gamma \neq \emptyset\} \in \mathcal{U}$ for all $\gamma < \lambda$, so γ belongs to almost every M_α (mod $\mathcal{U}$), proving (2).

□

Proposition 3.3.4 [Solovay 1974]:

If κ is λ–compact, where $\lambda > \kappa$ is regular, then there is a subset $\{M_\alpha : \alpha < \lambda\}$ of $[\lambda]^{<\kappa}$ such that $[\lambda]^{<\kappa} = \bigcup \{\mathcal{P}(M_\alpha) : \alpha < \lambda\}$.

Proof: Let $\{M_\alpha : \alpha < \lambda\}$ be the set provided by Proposition 3.3.3. If $P \in [\lambda]^{<\kappa}$, then since each $\gamma < \lambda$ is in almost every M_α (mod $\mathcal{U}$), by κ–completeness $\{\alpha < \lambda : P \subseteq M_\alpha\} \in \mathcal{U}$. It follows that $P \in \mathcal{P}(M_\alpha)$ for almost all M_α (mod $\mathcal{U}$), so certainly $[\lambda]^{<\kappa} \subseteq \bigcup \{\mathcal{P}(M_\alpha) : \alpha < \lambda\}$. The reverse inclusion holds trivially because each $M_\alpha \in [\lambda]^{<\kappa}$.

□

Lemma 3.3.5 [Solovay 1974]: *If κ is λ–compact, where $\lambda \geq \kappa$ is regular, then $\lambda^{<\kappa} = \lambda$.*

Proof: Immediate from Proposition 3.3.4. If $\lambda = \kappa$, then κ is measurable, and hence strongly inaccessible. It follows that $\kappa^{<\kappa} = \kappa$. If $\lambda > \kappa$ then each $\mathcal{P}(M_\alpha)$ has cardinality $< \kappa$.

□

We can now get back to the problem on hand, which is to prove that the Singular Cardinals Hypothesis holds above a compact cardinal. We need the following lemma, the proof of which requires the method of almost disjoint transversals described in Section 1.3.

Lemma 3.3.6 [Silver 1974]:
If the Singular Cardinals Hypothesis holds for all cardinals of cofinality ω above some cardinal κ, then it holds for all singular cardinals above κ.

Proof: Let $\eta > \kappa$ be an arbitrary singular cardinal. By induction on the cofinality of η we prove that if η is singular such that $2^{\mathrm{cf}(\eta)} < \eta$, then $\eta^{\mathrm{cf}(\eta)} = \eta^+$. If $\mathrm{cf}(\eta) = \omega$, this is just our induction hypothesis. Thus assume that $\mathrm{cf}(\eta) > \omega$, and let $(\eta_\alpha : \alpha < \mathrm{cf}(\eta))$ be a cofinal sequence below η. For every $h:\mathrm{cf}(\eta) \longrightarrow \eta$ we may define a sequence $f_h = (h_\alpha : \alpha < \mathrm{cf}(\eta))$ of functions such that

$$h_\alpha(\xi) = \begin{cases} h(\xi) \text{ if } h(\xi) < \eta_\alpha \\ \emptyset \quad \text{otherwise} \end{cases}$$

The set $\mathcal{F} = \{f_h : h:\mathrm{cf}(\eta) \longrightarrow \eta\} \subseteq \Pi\{\eta_\alpha{}^{\mathrm{cf}(\eta)} : \alpha < \mathrm{cf}(\eta)\}$ is an almost disjoint transversal of cardinality $\eta^{\mathrm{cf}(\eta)}$.

Let $S = \{\alpha < \mathrm{cf}(\eta) : \mathrm{cf}(\eta_\alpha) = \omega, \text{ and } 2^\omega < \eta_\alpha\}$; then S is stationary in $\mathrm{cf}(\eta)$ and $\eta_\alpha{}^\omega = \eta_\alpha{}^+$ for $\alpha \in S$ by the induction hypothesis. Define $\varphi:\mathrm{cf}(\eta) \longrightarrow \eta$ by

$$\varphi(\xi) = \text{smallest } \gamma \text{ such that } \eta_\alpha{}^{\mathrm{cf}(\eta_\alpha)} = \eta_\alpha(+\gamma).$$

Then $\varphi = 1$ on a stationary set, so by Lemma 1.3.15 and Corollary 1.3.11, $\eta^{\mathrm{cf}(\eta)} = |\mathcal{F}| \leq \eta^+$.

□

We may now tackle the proof of Solovay's Theorem 3.3.1:

Proof of Thm 3.3.1: Let $\lambda \geq \kappa$ be an arbitrary cardinal. Then $\lambda^+ > \kappa$ is regular so by Theorem 3.3.5, $\lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+$. Hence in particular $\lambda^\omega \leq \lambda^+$ for all $\lambda \geq \kappa$. Thus if $\lambda > \kappa$ is singular of cofinality ω, then $\lambda^\omega = \lambda^+$, proving that the Singular Cardinals Hypothesis holds for all $\lambda > \kappa$ of cofinality ω. By Lemma 3.3.6 this is sufficient for it to hold everywhere above κ.

□

In Section 5.3 we shall discuss similar work (due to [Matsubara 1992]) which the SCH may be seen to hold over an interval of cardinals, given the existence an ideal with certain nice combinatorial properties. Another proof of Theorem 3.3.1 is also presented there.

§ 3.4 Supercompact Cardinals and the Power Function

We have seen that the failure of GCH at a measurable cardinal is stronger, consistency–wise, than the mere existence of a measurable cardinal (Section 3.2). Silver proved in unpublished work ([Menas 1976a] and [Baumgartner 1983] are good sources) that if there exists a supercompact cardinal κ in some transitive model V of ZFC, then there is a generic extension V[G] such that V[G] $\vdash$ κ is measurable $\wedge\ 2^{\kappa} > \kappa^{+}$. This is accomplished by the method of "Reverse Easton Forcing", which is a particular kind of iterated forcing. Combined with Prikry forcing (Section 4.1), this will enable us to obtain a model in which the Singular Cardinals Hypothesis fails. The required material on standard iterated forcing is presented in Appendix 2.3, and a discussion of supercompact cardinals may be found in Appendix 3.3. Briefly, a cardinal κ is λ–supercompact if and only if there exists a transitive class M an an elementary embedding j: V $\longrightarrow$ M of the universe with the property that κ is the critical point of j with $j(\kappa) > \lambda$, and such that M is closed under λ–sequences. An ultrafilter characterization of λ–supercompactness is also given in Appendix 3.3. Under the appropriate conditions we will be able to extend the elementary embedding j to a generic extension V[G] of V, and this will result in κ retaining some of its large cardinal character. The proof of main theorem of this section, Theorem 3.4.1, will follow after Lemma 3.4.6.

Theorem 3.4.1 [Silver]:
Suppose that κ is a κ^{+k}– supercompact cardinal in some transitive model V *of* ZFC + GCH, *where $k \in \omega$ and $k \geq 1$. Then there is a notion of forcing $\mathbb{P}$ such that*

$$\Vdash_{\mathbb{P}} \kappa \text{ is } \kappa^{+(k-1)}\text{–supercompact and } 2^{\kappa} = \kappa^{+k}.$$

We may as well assume that $k > 1$ in the statement of Theorem 3.4.1, for otherwise let $\mathbb{P}$ be the empty notion of forcing. Our aim is to add κ^{+k}– many subsets of κ in such a way that some degree of supercompactness of κ is preserved. Thus Reverse Easton Forcing (also called Silver Forcing) provides a method for manipulating the power function at a large cardinal κ while still retaining some of the large cardinal character of κ in the generic extension. For instance, κ will still be measurable in the generic extension, and thus the GCH can not fail for the first time at κ (Theorem 3.1.4). It is therefore advisable to add α^{+k}– many subsets to every cardinal $\alpha < \kappa$ as well. We shall obtain $\mathbb{P}$ as a κ+1–stage iteration as follows: Given $\mathbb{P}_\alpha$ let $\dot{\mu}_\alpha$ be a name in $\mathbb{P}_\alpha$–language such that

$$\Vdash_\alpha \text{"}\dot{\mu}_\alpha \text{ is the least regular cardinal} \geq \aleph^V_{\beta+k+1} \text{ for every } \beta < \alpha\text{"},$$

where $\Vdash_\alpha$ denotes the forcing relation with respect to $\mathbb{P}_\alpha$. Let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$–name such that

$$\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathrm{Add}(\dot{\mu}_\alpha, \dot{\mu}_\alpha^{+k})$$

where $\mathrm{Add}(\alpha,\beta) = \mathrm{Fn}(\beta,2,\alpha)$ is the standard notion of forcing that adds β–many subsets of α (see Definition 2.2.4 and Lemma 2.2.5): The conditions $p \in \mathrm{Add}(\alpha,\beta)$ are partial maps such that $\mathrm{dom}(p) \subseteq \beta$, $\mathrm{ran}(p) \subseteq 2$, and $|p| < \alpha$. Thus $\mathrm{Add}(\alpha,\beta)$ is α–directed closed and satisfies the β–chain condition. Now let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. If α is a limit ordinal which is not a strongly inaccessible cardinal, let $\mathbb{P}_\alpha$ be the *inverse limit* of $\mathbb{P}_\beta$ $(\beta < \alpha)$, and take *direct limits* at inaccessible cardinals. Finally, let $\mathbb{P} = \mathbb{P}_{\kappa+1}$. We shall see that $\mathbb{P}$ is the required notion of forcing, but before we can prove this we need to consider some structural propositions:

Proposition 3.4.2:

$\mathbb{P}_\kappa$ *satisfies the* κ*–chain condition,* $|\mathbb{P}_\kappa| = \kappa$*, and* $\Vdash_\kappa \dot{\mu}_\kappa = \check{\kappa}$.

Proof: Since κ is κ^{+k}–supercompact in the ground model V, it is Mahlo (for a definition of Mahlo cardinals refer to Appendix 3.3), hence strongly inaccessible, and a limit of strongly inaccessibles. By induction we shall show that for all $\alpha < \kappa$ we have:

(1) $|\mathbb{P}_\alpha| < \kappa$, and

(2) $\Vdash_\alpha \dot{\mu}_\alpha < \check{\kappa}$.

Given that (1) and (2) are true for α, note that $\Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| < \kappa$ (since κ remains regular when forcing with $\mathbb{P}_\alpha$), and thus it is easy to see that $|\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha| < \kappa$, i.e. $|\mathbb{P}_{\alpha+1}| < \kappa$. Moreover, since κ remains strongly inaccessible when forcing with $\mathbb{P}_{\alpha+1}$, it follows that $\Vdash_{\alpha+1} \dot{\mu}_{\alpha+1} < \check{\kappa}$. This completes the induction at successor ordinals, and the induction step for limit ordinals follows easily as well. Since κ is strongly inaccessible, $\mathbb{P}_\kappa$ is the direct limit of $\mathbb{P}_\alpha$ $(\alpha < \kappa)$, and thus $|\mathbb{P}_\kappa| = \kappa$. Moreover, the set

$$\{\alpha < \kappa \colon \mathbb{P}_\alpha \text{ is the direct limit of } \mathbb{P}_\beta\ (\beta < \alpha)\}$$

is stationary in κ since κ is Mahlo. It follows that $\mathbb{P}_\kappa$ satisfies the κ–chain condition (by Lemma A.2.23). Hence κ remains regular when forcing with $\mathbb{P}_\kappa$ and in particular,

$$\Vdash_\kappa \dot{\mu}_\kappa = \check{\kappa}.$$

□

Let j: V ⟶ M be an elementary embedding of the universe into a transitive class that witnesses the κ^{+k}–supercompactness of κ, i.e. ${}^{\kappa^{+k}}M \subseteq M$, κ is critical for j and $j(\kappa) > \kappa^{+k}$. Let G be V–generic over $\mathbb{P}$, and let G_α denote the projection of G to $\mathbb{P}_\alpha$. To show that κ retains a degree of supercompactness in V[G], we shall prove that we can extend j to V[G].

Note that $j(\mathbb{P})$ is (in M) a $j(\kappa)+1$–stage iteration; we shall denote the α^{th} stage of $j(\mathbb{P})$ by $\mathbb{P}^M_\alpha$ (and thus $j(\mathbb{P}) = \mathbb{P}^M_{j(\kappa)+1}$). Our first aim is to prove that the iteration $j(\mathbb{P})$ is an extension of the iteration $\mathbb{P}$:

Proposition 3.4.3:

For all $\alpha \leq \kappa+1$, $\mathbb{P}^M_\alpha = \mathbb{P}_\alpha$.

Proof: Since each $|\mathbb{P}_\alpha| < \kappa$ for $\alpha < \kappa$, it follows that we may assume that $\mathbb{P}_\alpha \in V_\kappa$ (by replacing $\mathbb{P}_\alpha$ with an isomorphic copy if necessary). Hence $j(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$. Clearly also $j(\mathbb{P}_\alpha) = \mathbb{P}^M_\alpha$, so $\mathbb{P}^M_\alpha = \mathbb{P}_\alpha$ for all $\alpha < \kappa$. Since κ remains strongly inaccessible in M, $\mathbb{P}^M_\kappa$ is the direct limit of $\mathbb{P}^M_\alpha$ $(\alpha < \kappa)$, and this is sufficient to ensure that $\mathbb{P}^M_\kappa = \mathbb{P}_\kappa$. Finally, to see that $\mathbb{P}^M_{\kappa+1} = \mathbb{P}_{\kappa+1}$ requires a little argument that actually involves the supercompactness property of κ: Suppose that $\Vdash_\kappa \dot{q} \in \dot{\mathbb{Q}}_\kappa$. Since κ^{+k} is preserved by $\mathbb{P}_\kappa$, we have

$$\Vdash_\kappa \operatorname{dom}(\dot{q}) \subseteq \kappa^{+\iota\tau} \wedge |\operatorname{dom}(\dot{q})| < \kappa.$$

For each $\alpha < \kappa^{+k}$, let A_α be a maximal antichain in $\{p \in \mathbb{P}_\kappa \colon p \Vdash_\kappa \dot{q}(\check{\alpha}) = \check{0}\}$, and let B_α be a maximal antichain in $\{p \in \mathbb{P}_\kappa \colon p \Vdash_\kappa \dot{q}(\check{\alpha}) = \check{1}\}$. Since $\mathbb{P}_\kappa$ satisfies the κ–chain condition and ${}^{\kappa^{+k}}M \subseteq M$ it follows that each A_α and B_α is in M, and thus $(A_\alpha: \alpha < \kappa)$, $(B_\alpha: \alpha < \kappa)$ $\in$ M. Clearly if two conditions $\dot{q}_1$, $\dot{q}_2$ give rise to the same sequences $(A_\alpha: \alpha < \kappa)$ and $(B_\alpha: \alpha < \kappa)$, then $\Vdash_\kappa \dot{q}_1 = \dot{q}_2$. Thus $\mathbb{P}_{\kappa+1}$ will be the same whether it is defined in V or in M, proving that $\mathbb{P}^M_{\kappa+1} = \mathbb{P}_{\kappa+1}$ as well.

□

In particular, if G is V–generic over $\mathbb{P}_{\kappa+1}$, then G is M–generic over $\mathbb{P}^M_{\kappa+1}$ (since any dense subset of $\mathbb{P}_{\kappa+1}$ in M is in V). The converse is not necessarily true. Also note that $M[G] \subseteq V[G]$ and that if M is a class of V, then M[G] is a class of V[G]

Proposition 3.4.4:

If $x \subseteq M[G]$, *where* $x \in V[G]$ *is such that* $V[G] \vDash |x| \leq \kappa^{+k}$, *then* $x \in M[G]$.

(*i.e.* $V[G] \vDash {}^{\kappa^{+k}}M[G] \subseteq M[G]$)

Proof: By the axiom of choice in V[G], we may as well assume that x is a set of ordinals in V[G]. Let $\dot{x}$ be a name for x, and let $D = \{\xi: \exists p \in \mathbb{P}(p \Vdash_{\mathbb{P}} \check{\xi} \in \dot{x})\}$. It is easy to see that $D \in V$, $|D| \leq \kappa^{+k}$ and $x \subseteq D$.

For each $\xi \in D$, let A_ξ be a maximal antichain in $\{p \in \mathbb{P}: p \Vdash_{\mathbb{P}} \check{\xi} \in \dot{x}\}$. Since ${}^{\kappa^{+k}}M \subseteq M$, it follows that $D \in M$, and thus that $(A_\xi: \xi \in D) \in M$. Clearly $x = \{\xi \in D: A_\xi \cap G\} \neq \emptyset$, so $x \in M[G]$ as required.

□

For now, define $\xi = \kappa + 1$. Recall (Appendix 2.3) that $\mathbb{P}^M_{\xi, j(\xi)} = \{p^\xi: p \in j(\mathbb{P})\}$ is a $(j(\xi) - \xi)$–iteration (Lemma A.2.24), where

$$p^\xi = p | \{\gamma: \xi \leq \gamma < j(\xi)\}.$$

Proposition 3.4.5:

$\mathbb{P}^M_{\xi, j(\xi)}$ *is* κ^{+k+1}– *directed closed in* V[G].

Proof: If A $\in$ V[G] is a directed subset of $\mathbb{P}^M_{\xi,j(\xi)}$ (i.e. any two members of A have a lower bound in A) of cardinality κ^{+k}, then A is in M[G] as well. Thus it suffices to show that $\mathbb{P}^M_{\xi,j(\xi)}$ is κ^{+k+1}–directed closed in M[G]. This is a direct consequence of Lemma A.2.25.

□

Our original aim is was to extend the elementary embedding j: V $\longrightarrow$ M to an elementary embedding h with domain V[G], such that h witnesses a certain degree of supercompact–ness. We shall show that under certain conditions, we may find a set H which is M[G]–generic over $\mathbb{P}^M_{\xi,j(\xi)}$ such that h: V[G] $\longrightarrow$ M[G][H].

Let $A = \{q^\xi: \exists p \in G(q = j(p))\}$. Clearly $A \subseteq \mathbb{P}^M_{\xi,j(\xi)}$. We shall show that A has a lower bound in $\mathbb{P}^M_{\xi,j(\xi)}$. Note that $|\mathbb{P}| = \kappa^{+k}$, and thus $|A| \leq \kappa^{+k}$. By Proposition 3.4.5 it therefore suffices to show that A is directed. Let $q_1^\xi, q_2^\xi \in A$, where $q_i = j(p_i)$ for some $p_i \in G$ (i = 1,2). Choose p $\in$ G such that $p \leq p_1, p_2$ and let q = j(p). Since $\mathbb{P}_\kappa$ is a direct limit, it follows that there is a bounded set $C \subseteq \kappa$ such that $\text{supp}(p) \subseteq C \cup \{\kappa\}$. Since j is an elementary embedding with critical point κ, it follows that $\text{supp}(q) \subseteq C \cup \{j(\kappa)\}$. Also, $p|\kappa = q|\kappa$ and $q(\kappa) = 1$ imply that $p \cup q^\xi \leq q$ in $\mathbb{P}^M_{j(\xi)} = j(\mathbb{P})$. Since $q \leq q_1, q_2$, we must have $p \cup q^\xi \leq p \cup q_1^\xi$ and $p \cup q_2^\xi$, and since p $\in$ G, this means that $q^\xi \leq q_1^\xi, q_2^\xi$ in A, proving that A is indeed directed. Thus we may fix a $q_0 \in \mathbb{P}^M_{\xi,j(\xi)}$ which is a lower bound for A. Choose H to be V[G]–generic over $\mathbb{P}^M_{\xi,j(\xi)}$ such that $q_0 \in$ H.

The following lemma is fundamental:

Lemma 3.4.6 [Silver]:
In V[G][H] *we may extend the elementary embedding* j: V $\longrightarrow$ M *to an elementary embedding* h: V[G] $\longrightarrow$ M[G][H].

Proof: Let G' = G * H. If p $\in$ G and q = j(p), then as above we have $p|\kappa = q|\kappa$ and $q(\kappa) = 1$. Hence $p \leq q|\kappa+1$ in $\mathbb{P}$. Moreover, since $q_0 \in$ H, it follows that $q^\xi \in$ H, and thus that q $\in$ G'. Hence p $\in$ G implies j(p) $\in$ G'. Define h: V[G] $\longrightarrow$ M[G][H] as follows:

If x $\in$ V[G], let $\dot{x}$ be a name for x; now define h(x) to be the realization of $j(\dot{x})$ in M[G][H], i.e. $h(x) = j(\dot{x})[G']$. Now if $V[G] \models \varphi(x,y)$, then there are names $\dot{x}$, $\dot{y}$ for x, y and $p \in G$ such that $p \Vdash_{\mathbb{P}} \varphi(\dot{x}, \dot{y})$. Since j is elementary, $j(p) \Vdash_{j(\mathbb{P})} \varphi(j(\dot{x}), j(\dot{y}))$. Also $p \in G$ implies $j(p) \in G'$, so that $M[G'] \models \varphi(h(x), h(y))$, as required. This argument immediately shows that h is an elementary embedding which is well defined (for if $p \Vdash_{\mathbb{P}} \dot{x}_1 = \dot{x}_2$ then $j(p) \Vdash_{j(\mathbb{P})} j(\dot{x}_1) = j(\dot{x}_2)$).

□

Proof of Theorem 3.4.1:

We must prove that:

(1) κ is $\kappa^{+(k-1)}$–supercompact in V[G]

(2) $V[G] \models 2^{\kappa} = \kappa^{+k}$.

We shall tackle the first of these now. In V[G][H], define $\mathcal{U}$ as follows:

$$X \in \mathcal{U} \longleftrightarrow X \subseteq [\kappa^{+k-1}]^{<\kappa}, X \in V[G] \text{ and } j''\kappa^{+k-1} \in h(X),$$

where h is the elementary embedding defined in the proof of Lemma 3.4.6. We now show that $\mathcal{U} \in V[G]$. Note that $|[\kappa^{+k-1}]^{<\kappa}| = \kappa^{+k-1}$. Thus $\mathcal{U} \subseteq \mathcal{P}([\kappa^{+k-1}]^{<\kappa})$ has cardinality κ^{+k} because $V \models GCH$. Since $\mathcal{U} \in V[G][H]$ and $\mathbb{P}^{M}_{\xi, j(\xi)}$ is κ^{+k+1}–directed closed in V[G] (by Proposition 3.4.5), it follows that $\mathcal{U} \in V[G]$, as required.

Next we shall show that in V[G], $\mathcal{U}$ is a normal fine measure over κ^{+k-1}. That $\mathcal{U}$ is a κ–complete ultrafilter over $[\kappa^{+k-1}]^{<\kappa}$ is easy to see. To see that $\mathcal{U}$ is a fine measure, we must show that for every $\alpha < \kappa^{+k-1}$, the set $\{P \in [\kappa^{+k-1}]^{<\kappa}: \alpha \in P\} \in \mathcal{U}$. Now $h(\{P: \alpha \in P\}) = \{P: j(\alpha) \in P\}$, so $j''\kappa^{+k-1} \in h(\{P: \alpha \in P\})$ for every $\alpha < \kappa^{+k-1}$, and thus $\{P: \alpha \in P\} \in \mathcal{U}$ for all $\alpha < \kappa^{+k-1}$ by definition of $\mathcal{U}$. To see that $\mathcal{U}$ is normal, let $f:[\kappa^{+k-1}]^{<\kappa} \longrightarrow \kappa^{+k-1}$ be a map in V[G] such that for all $P \in [\kappa^{+k-1}]^{<\kappa}$ we have $f(P) \in P$. Then h(f) is a map on $[j(\kappa^{+k-1})]^{<j(\kappa)} \cap M[G']$ such that $h(f)(P) \in P$ for all such P. In particular, $h(f)(j''\kappa^{+k-1}) \in j''\kappa^{+k-1}$, and so there is $\alpha < \kappa^{+k-1}$ such that $h(f)(j''\kappa^{+k-1}) = j(\alpha)$. Clearly we then have $j''\kappa^{+k-1} \in h(\{P: f(P) = \alpha\})$, so there is $\alpha < \kappa^{+k-1}$ such that $\{P: f(P) = \alpha\} \in \mathcal{U}$. This concludes the proof that κ is κ^{+k-1}–supercompact in V[G].

Next we turn our attention to (2). $\mathbb{P}_{\kappa}$ has the κ–c.c. and $|\mathbb{P}_{\kappa}| = \kappa$ by Proposition 3.4.2, and thus by Lemma 2.2.3 and the fact that κ is strongly inaccessible, we must have $(2^\kappa)^{V[G_\kappa]} = 2^\kappa$. Since $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathrm{Add}}(\kappa, \kappa^{+k})$ and $(\kappa^{+k})^\kappa = \kappa^{+k}$ (because V ⊩ GCH), we have $2^\kappa = \kappa^{+k}$ in V[G] by Lemma 2.2.6. This completes the proof of Theorem 3.4.1.

□

Remark 3.4.7: An investigation of the proof of Theorem 3.4.1 will show that the only reasons why we need to assume that V ⊩ GCH are:

(a) So that $|\mathcal{P}([\kappa^{+k-1}]^{<\kappa})| = \kappa^{+k}$

(b) So that $(\kappa^{+k})^\kappa = \kappa^{+k}$

Both of these follow if $2^{\kappa^{+k-2}} = \kappa^{+k-1}$ and $2^{\kappa^{+k-1}} = \kappa^{+k}$.

The next result should be read in conjunction with Theorem 3.2.5

Corollary 3.4.8 [Silver]:

Con(ZFC + $\exists\kappa^{++}$*–supercompact cardinal* κ) *implies* Con(ZFC + GCH *fails at a measurable cardinal*).

Two important general theorems about the power function are theorems 2.3.1 (due to Easton) and Theorem 1.3.3 (due to Silver). Theorem 2.3.1 says that for regular κ, the cardinality of $\mathcal{P}(\kappa)$ can be anything desired, within the weak constraints:

(i) $\mathrm{cf}(2^{\kappa}) > \kappa$

(ii) $\kappa_1 \leq \kappa_2$ implies $2^{\kappa_1} \leq 2^{\kappa_2}$

In particular, the GCH can fail for the first time at any regular cardinal.

Theorem 1.3.3, on the other hand, takes away this indeterminism for singular cardinals: If κ is singular of *uncountable* cofinality and the GCH fails at κ, then it fails already on a stationary subset of κ. As a corollary we see that if the GCH can fail for the first time at a singular cardinal, then the cofinality of that cardinal must be ω.

An adaptation of Lemma 3.3.6, also due to Silver, shows that if the SCH holds for all singular cardinals $\alpha < \kappa$ of countable cofinality, then it holds for all singular cardinals below κ. Hence if the SCH fails somewhere, it first fails at a singular cardinal of cofinality ω. In Section 4.1 we shall exhibit a model in which the SCH fails at a cardinal of cofinality ω, but this cardinal is quite large. Two natural questions that arise from the above considerations are:

(1) *Can the SCH fail at the smallest singular cardinal (i.e.* $\aleph_\omega$*)* ?

(2) *Can the GCH fail for the first time at* $\aleph_\omega$*?*

This chapter is mostly concerned with various constructions that answer these questions.In Section 4.2 we shall present a detailed exposition of Magidor's result ([Magidor 1977b]) that if there exists a cardinal κ is which is κ^{++}–supercompact, then there is a model in which $\aleph_\omega$ is strong limit, and in which $2^{\aleph_\omega} = \aleph_{\omega+2}$ (Theorem 4.2.1).

In Section 4.3 we shall consider, in less detail, several generalizations building on Magidor's result. In particular, we shall discuss another result by Magidor ([Magidor 1977c]) that the GCH *can* fail for the first time at $\aleph_\omega$ (assuming the existence of a very large cardinal). We shall also consider work by Apter ([Apter 1984]) in which the SCH fails at $\kappa(+\omega)$ for any cardinal κ, and a theorem of Shelah ([Shelah 1983]) which generalizes Theorem 4.2.1.

Section 4.4 will show that the failure of the SCH may be regarded as a large cardinal axiom. For instance, it implies the consistency of the existence of a measurable cardinal. The exact strength of the failure of the SCH has been determined by Gitik and Magidor

([Gitik 1992]). Sections 4.3 and 4.4 will not offer much in the way of proof (primarily to avoid this dissertation from assuming elephantine proportions, as the constructions involved are tremendously complicated). This chapter depends heavily on the material in Chapter 3 as well as the forcing methods developed in Appendix 2.

§ 4.1 The Consistency of ¬SCH

By Theorem 2.3.6 we know that the Singular Cardinals Hypothesis holds in Easton's models, and thus that SCH is consistent with ZFC + ¬GCH. We shall now show, assuming the existence of a cardinal with a certain degree of supercompactness, that ¬SCH is consistent with ZFC as well. In order to do this, we shall give a brief description of Prikry forcing, first introduced in [Prikry 1970].

Prikry forcing changes the cofinality of a measurable cardinal to ω, but preserves all cardinals. For background on forcing, refer to Appendix 2, and for information about measurable cardinals, refer to Appendix 3. Suppose that κ is a measurable cardinal, and let $\mathcal{U}$ be a *normal measure* over κ. Prikry's notion of forcing $\mathbf{P}_{\mathcal{U}}$ has as conditions ordered pairs of the form (p,X), where $p \in [\kappa]^{<\omega}$ is an ascending finite sequence of ordinals $< \kappa$, and $X \in \mathcal{U}$ is such that $\sup(p) < \inf(X)$. The idea is that the p–part is an approximation of (what will be) a cofinal ω–sequence with limit κ, and that the X–part consists of elements by which p can be extended. Demanding that $X \in \mathcal{U}$ ensures that the set of possible extensions of p is "large". Hence we define the order on $\mathbf{P}_{\mathcal{U}}$ as follows:

$$(p,X) \leq (q,Y) \longleftrightarrow q \text{ is an initial segment of } p,\ X \subseteq Y, \text{ and } p-q \subseteq Y.$$

Note that any two conditions (p,X) and (p,Y) are compatible, as $(p,X\cap Y)$ is a common extension. It follows that $\mathbf{P}_{\mathcal{U}}$ has the κ^+–chain condition, so that all cardinalities and cofinalities $> \kappa$ are preserved.

The fundamental lemma of Prikry forcing is:

Lemma 4.1.1 [Prikry 1970]:
Let φ be a sentence of the forcing language associated with $\mathbf{P}_{\mathcal{U}}$, and suppose that (p,X) *is a condition. Then there is a* $Y \in \mathcal{U}$ *such that* $Y \subseteq X$ *and* (p,Y) *decides* φ.

The proof of the above lemma requires the following lemma due to [Rowbottom 1971] which we state without proof. A proof may be found in [Jech 1978].

Lemma 4.1.2 [Rowbotttom 1971]: *Suppose that* $\mathcal{U}$ *is a normal measure over some cardinal* κ. *Suppose that* $\lambda < \kappa$ *and let* $X \in \mathcal{U}$. *If* $F:[X]^{<\omega} \longrightarrow \lambda$, *then there is a* $Y \in \mathcal{U}$ *such that* Y *is homogeneous for* F. (*i.e.* $|F"[Y]^n| = 1$ *for all* $n < \omega$).

Proof of Lemma 4.1.1: Define $F:[X]^{<\omega} \longrightarrow 3$ as follows:

$$F(q) = \begin{cases} 0 \text{ if } (p \cup q, Z) \Vdash \varphi \text{ for some } Z. \\ 1 \text{ if } (p \cup q, Z) \Vdash \neg\varphi \text{ for some } Z. \\ 2 \text{ otherwise.} \end{cases}$$

By Lemma 4.1.2 above, there is $Y \in \mathcal{U}$ homogeneous for F. We claim that (p,Y) decides φ.

Otherwise we can find $q,\bar{q}$ and $Z,\bar{Z}$ such that $(p \cup q, Z)$, $(p \cup \bar{q}, \bar{Z}) \leq (p,Y)$ and

$$(p \cup q, Z) \Vdash \varphi \quad \text{and} \quad (p \cup \bar{q}, \bar{Z}) \Vdash \neg\varphi$$

By extending one of $q,\bar{q}$ if necessary, we may assume that $q,\bar{q} \in [Y]^n$ for some $n \in \omega$. Since Y is homogeneous for F, this yields a contradiction. Hence either $(p,Y) \Vdash \varphi$ or $(p,Y) \Vdash \neg\varphi$.

□

Corollary 4.1.3 [Prikry 1970]:
Suppose that $G \subseteq \mathbb{P}_{\mathcal{U}}$ *is generic over* V *and suppose that* $y \in V$ *is a set of cardinality* $< \kappa$. *If* $x \in V[G]$ *such that* $x \subseteq y$, *then* $x \in V$.

Proof: Let $\dot{x}$ be a name for x and let $(p,X) \in G$ be a condition such that $(p,X) \Vdash \dot{x} \subseteq \check{y}$. For each $a \in y$, let $Y_a \subseteq X$ be such that $(p,Y_a) \Vdash \check{a} \in \dot{x}$ or $(p,Y_a) \Vdash \check{a} \notin \dot{x}$ (The Y_a exist by Lemma 4.1.1). Let $Y = \bigcap_{a \in y} Y_a$; then $Y \in \mathcal{U}$, and (p,Y) is a condition which extends all the (p,Y_a). Let $z = \{a \in y \colon (p,Y) \Vdash \check{a} \in \dot{x}\}$. Then $z \in V$, and $(p,Y) \Vdash \check{z} = \dot{x}$.

□

Theorem 4.1.4 [Prikry 1970]:
Let $\mathcal{U}$ be a normal measure over some measurable cardinal κ in the ground model V, *and let* $\mathbb{P}_{\mathcal{U}}$ *be the associated notion of forcing. Let* G *be* V*–generic over* $\mathbb{P}_{\mathcal{U}}$. *Then forcing with* $\mathbb{P}_{\mathcal{U}}$ *preserves all cardinals, and* $V[G] \vdash cf(\kappa) = \omega$. *Moreover, the power function in* V[G] *is identical to the power function in* V.

Proof: Standard forcing arguments show that $\bigcup \{p: \exists X((p,X) \in G)\}$ is a countable cofinal sequence with limit κ in V[G], so that $cf^{V[G]}(\kappa) = \omega$. By the κ^+–c.c., all cardinals $> \kappa$ are preserved, and by Corollary 4.1.3 all cardinals $< \kappa$ are preserved. Also κ, being a limit cardinal, is also preserved, proving that all cardinals are preserved. It remains to prove that the power function remains unchanged in the generic extension. By Corollary 4.1.3 and the fact that all cardinals are preserved, it follows that the power function is unchanged below κ. By Lemma 2.2.3 and the fact that $\mathbb{P}_{\mathcal{U}}$ has the κ^+–c.c, for $\lambda \geq \kappa$ we have $(2^\lambda)^{V[G]} \leq (2^\lambda)^V$, and preservation of cardinals thus guarantees that the power function remains unchanged on cardinals $\geq \kappa$ as well.

□

We may now construct a model in which the SCH fails in two stages, using reverse Easton forcing (described in Section 3.4) followed by a Prikry extension: Suppose that in the ground model V we have a cardinal κ which is κ^{++}– supercompact (see Appendix 3.3 for a definition). By Theorem 3.4.1 there is a generic extension V[G] of V such that

$$V[G] \vdash 2^\kappa = \kappa^{++} \text{ and } \kappa \text{ is measurable .}$$

We may then apply Prikry forcing *inside* V[G] to obtain a generic extension V[G][H] in which κ has cofinality ω, is strong limit, and satisfies $2^\kappa \geq \kappa^{++}$. Thus the SCH fails at κ in this model. This proves:

Theorem 4.1.5 [Silver]:
$ZFC + \exists\kappa(\kappa \textit{ is } \kappa^{++}\textit{– supercompact}) \vdash Con(ZFC + \neg SCH)$

We needed a large cardinal axiom to prove this result. Jensen, primarily, has been responsible for proving that we cannot do without a large cardinal axiom to prove the consistency of ¬SCH, using the set $0^{\#}$ isolated by Silver ([Silver 1971]). A definition of $0^{\#}$

is given in Appendix 1. Jensen proved

(1) If the SCH fails, then $0^{\#}$ exists [Devlin–Jensen 1975].

(2) If SCH fails, then there is an inner model with a measurable cardinal [Dodd–Jensen 1981].

Remembering that κ is measurable if and only if it is κ–supercompact, we see that in ZFC:

$$\mathrm{Con}(\exists\kappa(\kappa \text{ is } \kappa^{++}-\text{supercompact})) \longrightarrow \mathrm{Con}(\neg\mathrm{SCH}) \longrightarrow \mathrm{Con}(\exists\kappa(\kappa \text{ is } \kappa-\text{supercompact}))$$

This shows that we may think of $\neg$SCH as a *large cardinal axiom*, neatly wedged between two notions of supercompactness. We shall return to these matters in Section 4.4.

For now we want to give one more application of the argument used to prove Theorem 4.1.5. In Chapter 1, it was proved that the power function is determined by the gimel function, in that two models with the same gimel functions must have identical power functions as well. We shall now prove that the converse is not necessarily true, i.e. we shall construct two models which have the same power functions, but different gimel functions.

Theorem 4.1.6 [Silver]:

If there exists a κ^{++} – supercompact cardinal in a transitive model M *of* ZFC + GCH, *then we can construct two transitive models of* ZFC *with identical power functions, but different gimel functions. Hence the power function does not determine all cardinal arithmetic.*

Proof: (In outline) Let M be a transitive model of ZFC + GCH in which there is a κ^{++}–supercompact cardinal κ. Using a reverse Easton extension, we can obtain a model M[G] of ZFC in which κ remains measurable, but in which $2^{\kappa} = \kappa^{++}$ (Theorem 3.4.1). The notion of forcing used to accomplish this has cardinality κ^{++}, because GCH holds in the ground model. It also satisfies the κ^{++}–c.c. By Lemma 2.2.3, for each $\lambda \geq \kappa^{+}$, we have

$$(2^{\lambda})^{M[G]} \leq ((\kappa^{++})^{\lambda})^{M}$$

and thus, because GCH holds M, we must have M[G] $\models (2^{\lambda} = \lambda^{+}$ for all $\lambda \geq \kappa^{+})$. We can now collapse $2^{\aleph_0}$ to $\aleph_1$ using a notion of forcing of cardinality $< \kappa$. This notion of forcing will not destroy the measurability of κ, so we have now at our disposal a transitive model

N of ZFC with the following properties:

(1) $N \models \kappa$ is measurable and $2^{\kappa} = \kappa^{++}$

(2) In N, the GCH holds above κ.

(3) $N \models 2^{\aleph_0} = \aleph_1$.

Using Prikry forcing, we now obtain an extension N_1 of N in which all cardinals and all cofinalities are preserved, except that $cf(\kappa) = \omega$, and in which the power function is identical to the power function in N (Theorem 4.1.4). In particular, $2^{\kappa} = \kappa^{++}$ in N_1 and κ is strong limit in N_1, so (because the gimel– and the power functions coincide on strong limit cardinals), we have $N_1 \models \kappa^{\omega} = 2^{\kappa} = \kappa^{++}$.

Let N_3 be a generic extension of N_2 with the same cardinals and cofinalities obtained by adding κ^{++}– many subsets of $\aleph_1$ (Theorem 2.2.6). The notion of forcing which does this is $<\aleph_1$–closed, so there are no new ω–sequences in N_3. Hence in N_3 we have

$$\kappa^{\omega} = \kappa^{++}, 2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \kappa^{++} \text{ and GCH holds above } \kappa.$$

Next we go back to the original ground model M. Let M_2 be an extension of M using Prikry forcing to turn κ into a singular cardinal of cofinality ω. Because $M \models$ GCH and because Prikry forcing does not change the power function, we have $M_2 \models$ GCH. Let M_3 be a generic extension of M_2 with the same cardinals and cofinalities obtained by adding κ^{++}– many subsets of $\aleph_1$. Again, in M_3 there are no new ω–sequences, and thus in M_3

$$\kappa^{\omega} = \kappa^{+}, 2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \kappa^{++} \text{ and GCH holds above } \kappa.$$

Hence M_3 and N_3 have the same power functions, and $cf(\kappa) = \omega$ in both models, but

$$N_3 \models \kappa^{cf(\kappa)} = \kappa^{++} \quad \text{and} \quad M_3 \models \kappa^{cf(\kappa)} = \kappa^{+},$$

so their gimel functions do not coincide. It follows that the gimel function, and thus cardinal arithmetic, is not determined by the power function.

□

The argument given in the above proof was taken from [Jech 1973].

§ 4.2 Failure of the SCH at $\aleph_\omega$.

The first significant results in the direction of the singular cardinals problem were obtained by Magidor ([Magidor 1977b,c]). Starting with a model V of ZFC + GCH in which there is a $\kappa^{++}-$ supercompact cardinal, Magidor[1977b] obtained an extension in which $\aleph_\omega$ is strong limit such that $2^{\aleph_\omega} = \aleph_{\omega+2}$. This settles the question as to whether the SCH can fail at the first singular cardinal positively.

In this section we shall examine this work in detail. The main theorem of this section is:

Theorem 4.2.1 [Magidor 1977b]:
Suppose that $k \in \omega$ and that V *is a transitive model of* ZFC *in which there is a cardinal κ such that:*

$$(1)\ V \models 2^\kappa = \kappa^{+k}$$

$$(2)\ V \models \kappa \textit{ is } \kappa^{+k-1}-\textit{supercompact}$$

Then there is an extension V' *of* V *such that* $V' \models \kappa = \aleph_\omega$ *is strong limit and* $2^{\aleph_\omega} = \aleph_{\omega+k}$.

Note that in order to obtain a model in which conditions (1) and (2) hold it suffices to start with a model of ZFC in which there is a cardinal which is $\kappa^{+k}-$ supercompact, for we can then obtain a model in which (1) and (2) hold by a reverse Easton extension (Thm 3.4.1). Magidor's notion of forcing is a generalization of Prikry forcing (described in Section 4.1) in that it changes the cofinalities of each of κ, κ^{+1}, ... κ^{+k-1} to ω. Thus κ^{+1}, ... κ^{+k-1} will be collapsed to κ. We will show on the other hand that κ^{+k} is preserved. Since κ will become $\aleph_\omega$, and $2^\kappa = \kappa^{+k}$ the resulting generic extension will actually be a model of $2^{\aleph_\omega} = \aleph_{\omega+1}$, which is precisely what we don't want! However, a certain submodel of the generic extension will be seen to satisfy the required conclusions.

Suppose that $\kappa \leq \lambda$. Recall that $[\lambda]^{<\kappa}$ is the set of all subsets of λ whose cardinality is smaller than κ, and that if p is a set of ordinals, then otp(p) denotes the order type of p.

Definition 4.2.2:

(1) If p, q $\in [\lambda]^{<\kappa}$, then we shall define: p $\underset{\sim}{\subset}$ q $\longmapsto$ p $\subseteq$ q and otp(p) < otp(q $\cap$ κ) and say that p is *strongly included* in q.

(2) For any ordinal α, we will denote by α(p) the order type of p$\cap\alpha$.

We will need three lemmas concerning *normal fine measures* (see Appendix 3.3 for a definition).

Lemma 4.2.3:

Suppose that $\mathcal{U}$ is a normal fine measure over $[\lambda]^{<\kappa}$.

(1) *If* $F:[\lambda]^{<\kappa} \longrightarrow [\lambda]^{<\kappa}$ *is such that* F(p) $\underset{\sim}{\subset}$ p *for all* p $\neq \emptyset$, *then there is a* U $\in \mathcal{U}$ *such that* F *is constant on* U.

(2) *If* $\{U_p : p \in [\lambda]^{<\kappa}\} \subseteq \mathcal{U}$, *then the set* $\{q \in [\lambda]^{<\kappa} : q \in \cap\{U_p : p \underset{\sim}{\subset} q\} \in \mathcal{U}$ *as well. This set is called the diagonal intersection of* $\{U_p : p \in [\lambda]^{<\kappa}\}$.

Menas has shown that

Lemma 4.2.4 [Menas 1976b]:

If a cardinal κ is λ–supercompact, then there exists a normal fine measure $\mathcal{U}$ over $[\lambda]^{<\kappa}$ such that if $F:[[\lambda]^{<\kappa}]^{<\omega} \longrightarrow \xi$ *for some* $\xi < \kappa$, *then there is a* U $\in \mathcal{U}$ *which is homogeneous for* F (*i.e.* $|F''[U]^n| = 1$ *for all* $n < \omega$).

Lemma 4.2.5:

Suppose that $\mathcal{U}$ is a normal fine measure over $[\lambda]^{<\kappa}$ for some $\lambda \geq \kappa$. Then

(1) $\{p \in [\lambda]^{<\kappa} : \kappa(p) = \kappa\cap p$ is strongly inaccessible$\} \in \mathcal{U}$.

(2) *If* $\alpha \leq \lambda$ *is a (regular) cardinal, then* $\{p \in [\lambda]^{<\kappa} : \alpha(p)$ is a (regular) cardinal$\} \in \mathcal{U}$.

(3) *If* $\gamma < \kappa$ *and if* $\alpha,\beta \leq \lambda$ *such that* $\alpha^{+\gamma} = \beta$, *then* $\{p \in [\lambda]^{<\kappa} : \alpha(p)^{+\gamma} = \beta(p)\} \in \mathcal{U}$

(4) *If* $\alpha,\beta \leq \lambda$ *then*
(i) *if* $2^\alpha = \beta$, *then* $\{p \in [\lambda]^{<\kappa} : 2^{\alpha(p)} = \beta(p)\} \in \mathcal{U}$
(ii) *if* $2^\alpha > \beta$, *then* $\{p \in [\lambda]^{<\kappa} : 2^{\alpha(p)} > \beta(p)\} \in \mathcal{U}$
(iii) *if* $2^\alpha < \beta$, *then* $\{p \in [\lambda]^{<\kappa} : 2^{\alpha(p)} < \beta(p)\} \in \mathcal{U}$.

The Lemma 4.2.5 is easily proved using an ultrapower argument using the fact that for $\alpha \leq \lambda$ the ordinal α is represented in the ultrapower $(V^{[\lambda]^{<\kappa}}/\mathcal{U})$ by the map $\alpha(p)$.

Let $\mathrm{Coll}(\alpha,<\beta)$ is the (Lévy) notion of forcing which collapses an inaccessible cardinal β to the successor cardinal of α ([Lévy 1970]). The conditions of $\mathrm{Coll}(\alpha,<\beta)$ are partial maps f such that $\mathrm{dom}(f) \subseteq \alpha \times \beta$, $\mathrm{ran}(f) \subseteq \beta$, $|f| < \alpha$ and such that

$$(\gamma,\delta) \in \mathrm{dom}(f) \text{ implies } f(\gamma) < \delta.$$

The ordering on $\mathrm{Coll}(\alpha,<\beta)$ is by *reverse inclusion*. Thus a generic filter over $\mathrm{Coll}(\alpha,<\beta)$ adds for each $\delta < \beta$ a map of α onto δ. It is clear that $\mathrm{Coll}(\alpha,<\beta)$ is α–closed, and a Δ–system argument (Lemma A.2.10) shows that $\mathrm{Coll}(\alpha,<\beta)$ satisfies the β–chain condition.

Let k be a natural number, and let V be a model of ZFC in which there is a cardinal κ which is κ^{+k-1}– supercompact in V, and such that $2^\kappa = \kappa^{+k}$. Let $\mathcal{U}$ be a normal fine measure over $[\kappa^{+k-1}]^{<\kappa}$ which satisfies property of Lemma 4.2.4 above. By Lemma 4.2.5 the set:

$D = \{p \in [\kappa^{+k-1}]^{<\kappa}\colon p\cap\kappa \text{ is inaccessible and } (\kappa^{+i}(p))^+ = \kappa^{+i+1}(p) \text{ for } 0 \leq i \leq k-2\}$

is an element of $\mathcal{U}$.

We are now ready to define Magidor's notion of forcing $\mathbb{P}$:

Definition 4.2.6 [Magidor 1977b]: The conditions in $\mathbb{P}$ are of the form

$$\pi = (p_1,\dots,p_n, f_0,\dots,f_n, A, G), \quad \text{where}$$

(1) For $1 \leq i \leq n$, $p_i \in D$, and for $1 \leq i \leq n-1$, $p_i \subseteq p_{i+1}$.

(2) Let $\kappa_i = \kappa(p_i)$. Then

$$f_0 \in \mathrm{Coll}(\omega_1, <\kappa_1),$$
$$f_i \in \mathrm{Coll}(\kappa_i^{+k}, <\kappa_{i+1}) \text{ for } 1 \leq i \leq n-1, \text{ and}$$
$$f_n \in \mathrm{Coll}(\kappa_n^{+k}, <\kappa).$$

(3) $A \subseteq D$, $A \in \mathcal{U}$ and for every $q \in A$, $p_n \subseteq q$ and $f_n \in \mathrm{Coll}(\kappa_n^{+k}, <\kappa(q))$.

(4) G is a function defined on A such that for $q \in A$, $G(q) \in \mathrm{Coll}(\kappa(q)^{+k}, <\kappa)$. Moreover if $p \in A$ and $p \subseteq q$, then $G(p) \in \mathrm{Coll}(\kappa(p)^{+k}, \kappa(q))$.

If $\pi = (p_1 \dots p_n, f_0 \dots f_n, A, G)$, then n is said to be the *length* of π, and may be denoted by

n_π. $(p_1 \dots p_n)$ is called the p–*part* of π and may be denoted $(p^\pi_1 \dots p^\pi_{n_\pi})$ and similarly $(f_0 \dots f_n)$ is called the f–*part* of π and denoted $(f^\pi_0 \dots f^\pi_{n_\pi})$. Also $A^\pi = A$, $G^\pi = G$ and $\kappa^\pi_i = \kappa(p^\pi_i)$ for $1 \leq i \leq n_\pi$. Thus any condition π may be written as

$$\pi = (p^\pi_1 \dots p^\pi_{n_\pi}, f^\pi_0 \dots f^\pi_{n_\pi}, A^\pi, G^\pi)$$

but as this notation is very cumbersome, we will avoid doing so generally.

Thus far we have only defined what the forcing conditions are, but not yet how they are ordered.

Definition 4.2.7 [Magidor 1977b]: Suppose π and σ are two conditions,

$$\pi = (p_1 \dots p_n, f_0 \dots f_n, A, G) \text{ and } \sigma = (q_1 \dots q_m, g_0 \dots g_m, B, H).$$

Then $\pi \leq \sigma$ provided:

(a) $m \leq n$	(i.e. $n_\sigma \leq n_\pi$)
(b) $g_i \subseteq f_i$ for $0 \leq i \leq m$	(i.e. $f^\sigma_i \subseteq f^\pi_i$ for $0 \leq i \leq n_\sigma$)
(c) $p_i \in B$ for $m < i \leq n$	(i.e. $p^\pi_i \in A^\sigma$ for $n_\sigma < i \leq n_\pi$)
(d) $H(p_i) \subseteq f_i$ for $m < i \leq n$	(i.e. $G^\sigma(p^\pi_i) \subseteq f^\pi_i$ for $n_\sigma < i \leq n_\pi$)
(e) $B \supseteq A$	(i.e. $A^\sigma \supseteq A^\pi$)
(f) For all $p \in A$, $H(p) \subseteq G(p)$	(i.e. $G^\sigma(p) \subseteq G^\pi(p)$)

Note that if π and σ are compatible, then the p–part of σ is an initial segment of the p–part of π or vice versa; hence $\kappa^\sigma_i = \kappa^\pi_i$ for $1 \leq i \leq \min(n_\sigma, n_\pi)$. In this context we shall often just omit the π and σ and write κ_i. In particular, if G is a generic filter over $\mathbb{P}$ and $\pi, \sigma \in G$, then $\kappa^\pi_i = \kappa^\sigma_i$ ($= \kappa_i$ when the filter is understood from context).

The idea behind the above definitions is that each of the κ^{+i} must have its cofinality changed to ω, and given a condition $\pi \in \mathbb{P}$, the finite sequence whose j^{th} element is $\sup(p^\pi_j \cap \kappa^{+i})$ is an approximation of the cofinal ω–sequence below κ^{+i}. The set A^π is a "book–keeping set" from which possible p extending the p–part of π are chosen, i.e. if

$p = p^{\sigma}_{n_\pi+1}$ for some $\sigma \leq \pi$, then $p \in A^{\pi}$. Just as in Prikry forcing, we ensure that this set is large by demanding it to be an element of $\mathcal{U}$.

f^{π}_0 contains partial information about the collapse of all cardinals strictly between ω_1 and κ^{π}_1 to ω_1. f^{π}_1 contains information about the collapse of all cardinals between $(\kappa^{\pi}_1)^{+k}$ and κ^{π}_2 to $(\kappa^{\pi}_1)^{+k}$ and similar statements hold for the f^{π}_i $(1 \leq i < n_\pi)$.

$f^{\pi}_{n_\pi}$ should ideally contain information about the collapse of all cardinals between $(\kappa^{\pi}_{n_\pi})^{+k}$ and $\kappa^{\pi}_{n_\pi+1}$, but there is no $\kappa^{\pi}_{n_\pi+1}$. We therefore assume that $f^{\pi}_{n_\pi} \in \mathrm{Coll}((\kappa^{\pi}_{n_\pi})^{+k}, < \kappa)$, and if $p \in A$ is a possible extender of the p–part of π (i.e. a "possible" $p^{\pi}_{n_\pi+1}$), then

$f^{\pi}_{n_\pi} \in \mathrm{Coll}((\kappa^{\pi}_{n_\pi})^{+k}, < \kappa(p))$.

Finally the function G^{π} ensures that each possible extender of the f–part of π contains a certain amount of information, i.e. if $p = p^{\sigma}_k$ for some $\sigma \leq \pi$ and $k > n_\pi$, then $G^{\pi}(p) \subseteq f^{\sigma}_k$.

Definition 4.2.8: If $\pi \leq \sigma$, then for $0 \leq j \leq n_\sigma$, π is said to be a *j–direct extension* of σ provided:

(1) $f^{\pi}_i = f^{\sigma}_i$ for $j \leq i \leq n_\sigma$;

(2) $G^{\sigma}(p^{\pi}_i) = f^{\pi}_i$ for $n_\sigma < i \leq n_\pi$;

(3) $A^{\pi} = \{p \in A^{\sigma} : p^{\pi}_{n_\pi} \subseteq p\}$

(4) $G^{\pi} = G^{\sigma} | A^{\pi}$, i.e. if $p \in A^{\pi}$, then $G^{\pi}(p) = G^{\sigma}(p)$.

Note that if π is a j–direct extension of σ and χ is a j–direct extension of π, then χ is a j–direct extension of σ as well. Furthermore, if π is a j–direct extension of σ, then π is completely determined by the choice of $f^{\pi}_0 \ldots f^{\pi}_{j-1}$ and $p^{\pi}_{n_\sigma+1} \cdots p^{\pi}_{n_\pi}$.

Definition 4.2.9: If $\pi \leq \sigma$ and $0 \leq j \leq n_\sigma$, then π is said to be a *j–length preserving extension* of σ provided:

(1) $n_\pi = n_\sigma$ ie. π and σ have the same length;

(2) for $0 \leq i < j$, $f^{\pi}_i = f^{\sigma}_i$.

Clearly if π is a j–length preserving extension of σ and χ is a j–length preserving extension of π, then χ is a j–length preserving extension of σ.

Lemma 4.2.10:
Given two conditions $\pi \leq \sigma$, there is a unique χ such that
(1) $\pi \leq \chi \leq \sigma$;
(2) χ *is a* j*–direct extension of* σ;
(3) π *is a* j*–length preserving extension of* χ.

Proof: Clearly we must have:

$n_\chi = n_\pi$
$f_i^\chi = f_i^\pi$ for $0 \leq i < j$
$f_i^\chi = G^\sigma(p_i^\pi)$ for $n_\sigma < i \leq n_\pi$
$p_i^\chi = p_i^\pi$ for $1 \leq i \leq n_\pi$;
$f_i^\chi = f_i^\sigma$ for $j \leq i \leq n_\sigma$;
$A^\chi = \{p \in A^\sigma : p_{n_\pi}^\pi \subseteq p\}$;

and $G^\chi = G^\sigma | A^\chi$.

□

For $\pi \leq \sigma$ and $0 \leq j \leq n_\sigma$, the unique χ satisfying (1),(2),(3) in Lemma 4.2.10 above is called the j–*interpolant* of π and σ and is denoted j–int(π,σ). Note that if $\pi \leq \pi' \leq \pi''$, then j–int$(\pi,\pi') \leq$ j–int(π,π''). Moreover, if $\pi' =$ j–int(π,π''), then j–int$(\pi,\pi') = \pi'$.
Given a condition π and a natural number j such that $0 \leq j \leq n_\pi$,we shall call the sequence $(p_1^\pi \dots p_j^\pi, f_0^\pi \dots f_{j-1}^\pi)$ the *restriction* of π to j, and denote it $\pi | j$.

Lemma 4.2.11: *Let $\pi \in \mathbb{P}$ be a condition, and let $0 \leq j \leq n_\pi$. Suppose that $(\pi_\gamma : \gamma < \lambda)$ is a descending sequence of j–length preserving extensions of π, where $\lambda \leq \kappa_j^{+(k-1)}$ if j > 0, and $\lambda \leq \omega$ otherwise Then there is a condition π' which is a j–length preserving extension of each π_γ $(\gamma < \lambda)$ and of π.*

Proof: Let $\pi = (p_1 \dots p_n, f_0 \dots f_n, A, G)$. Since the π_γ $(\gamma < \lambda)$ are all j–length preserving extensions of π, each $\pi_\gamma = (p_1 \dots p_n, f_0 \dots f_{j-1}, f_j^\gamma \dots f_n^\gamma, A^\gamma, G^\gamma)$ for some $f_j^\gamma \dots f_n^\gamma$, A^γ and G^γ. Let $g_i = \bigcup \{f_i^\gamma : \gamma < \lambda\}$ for $j \leq i \leq n$. Since Coll$(\kappa_i^{+k}, < \kappa_{i+1})$ is κ_i^{+k}– closed, $g_i \in$ Coll$(\kappa_i^{+k}, < \kappa_{i+1})$, and g_i contains each f_i^γ $(\gamma < \lambda, j \leq i \leq n)$. Let $B = \bigcap \{A^\gamma : \gamma < \lambda\}$.

Since $\mathcal{U}$ is κ–closed, $B \in \mathcal{U}$, and B is contained in each A^{γ} ($\gamma < \lambda$). Define a function H on B by $H(p) = \cup \{G^{\gamma}(p) : \gamma < \lambda\}$ for $p \in B$. Now let $\pi' = (p_1 \ldots p_n, f_0 \ldots f_{j-1}, g_j \ldots g_n, B, H)$. Clearly π' is the required condition.

□

Next we will prove what Magidor calls his "main technical device." (Lemma 4.2.14), which is analogous to Lemma 4.1.1 for Prikry forcing. The proof is long, and we will need two preliminary lemmas before we are able to tackle it. The wording of the next lemma hardly but transparent, and some time should be spent ensuring that it is properly understood. Recall that a condition π *decides* a statement φ of the forcing language provided that either $\pi \Vdash \varphi$ or $\pi \Vdash \neg\varphi$.

Lemma 4.2.12:

Let φ be a statement of the forcing language of $\mathbb{P}$, and let π be a condition of length n. *Let* $j \leq n$ *be a natural number and suppose that η is the restriction to* j *of some extension of π. Let l be a natural number. Then there is a* j*–length preserving extension π' of π such that whenever $\pi'' \leq \pi$ has length* $n+l$ *and* $\pi''|j = \eta$, *then if π'' decides φ so does* j–int(π'', π').

Proof: The proof is by induction on l. Let $\pi = (p_1 \ldots p_n, f_0 \ldots f_n, A, G)$. To prove the lemma for $l = 0$, we distinguish two cases:

<u>Case 1</u>: There is $\sigma \leq \pi$ of length $n = n + 0$ such that $\sigma|j = \eta$ and σ decides φ. Suppose that $\sigma = (p_1 \ldots p_n, g_0 \ldots g_n, B, H)$. Let $\pi' = (p_1 \ldots p_n, f_0 \ldots f_{j-1}, g_j \ldots g_n, B, H)$. Then π' is a j–length preserving extension of π and if $\pi'' \leq \pi'$ satisfies $\pi''|j = \eta$, then $\pi'' \leq \sigma$ as well.

<u>Case 2</u>: Assume that Case 1 fails. Then there is nothing to prove since there is no extension of π, of the same length whose restriction to j is η, that decides φ.

We now commence with the induction step. As induction hypothesis we assume that the lemma has been proven for some fixed $l \geq 0$ i.e. for any π of any length n there is $\pi' \leq \pi$ such that if $\pi'' \leq \pi'$, $\pi''|j = \eta$ and π'' has length $n+l$, then if π'' decides φ so does the j–interpolant of π'' and π'. We now prove that the lemma holds for $l+1$ as well. The guiding idea is that a condition of length $n+(l+1)$ is a condition of length $(n+1)+l$, and we may then apply the induction hypothesis.

Let $\pi = (p_1 \ldots p_n, f_0 \ldots f_n, A, G)$. The ordering $\subseteq$ on D is well–founded, and thus may be extended to some well–ordering $\leq$ of D. By induction we will define a sequence $(\pi_q : q \in A)$ such that the following three properties hold:

(1) Each π_q has the form $\pi_q = (p_1 \ldots p_n, q, f_0 \ldots f_{j-1}, f_j^q \ldots f_n^q, f^q, B^q, H^q)$;

(2) Each $\pi_q \leq \pi$;

(3) If $t,q,s \in A$ such that $t \subseteq s$, $q \subseteq s$ and $s \in B^t \cap B^q$, then $H^t(s)$ and $H^q(s)$ are compatible.

The above sequence is chosen by induction on $\leq$: Thus suppose that we have defined π_q for all $q \leq p$ (and therefore, in particular, for all $q \subseteq p$), such that (1),(2),(3) hold for all $q,t \leq p$. Let $\chi_p = (p_1 \ldots p_n, p, f_0 \ldots f_n, g^p, A^p, G^p)$, where

$$g^p = \cup \{H^q(p) : q \in A, q \subseteq p\}, \quad (\text{if } p \notin B^q, \text{ assume } H^q(p) \text{ to mean } G(p)),$$

and $$A^p = A \cap \{t : p \subseteq t \wedge t \in \cap \{B^q : q \subseteq t, q \leq p\}\},$$

and G^p is defined on A^p by $G^p(t) = \cup \{H^q(t) : q \subseteq t, q \leq p\}$.

It is not hard to see that χ_p is a condition in $\mathbb{P}$: Each $H^q(p) \in \mathrm{Coll}(\kappa(p)^{+k}, < \kappa)$, and by property (3), any two of them are compatible. Moreover, the cardinality of the set $\{H^q(p) : q \subseteq p, q \in A\}$ is at most that of $\{q : q \subseteq p, q \in A\}$, and the latter set clearly has cardinality $|p|^{<|p \cap \kappa|}$. Since $p \in D$, and $p \subseteq \kappa^{+(k-1)}$, the properties of D ensure that $|p| = |p \cap \kappa|^{+(k-1)}$. Since $p \cap \kappa = \kappa(p)$ is inaccessible, it follows that $|p|^{<|p \cap \kappa|} = \kappa(p)^{+(k-1)}$. But $\mathrm{Coll}(\kappa(p)^{+k}, < \kappa)$ is $\kappa(p)^{+}$–closed, proving that $g^p \in \mathrm{Coll}(\kappa(p)^{+k}, < \kappa)$. Similar arguments prove that, $G^p(t) \in \mathrm{Coll}(\kappa(t)^{+k}, < \kappa)$. Finally, $A^p \in \mathcal{U}$, by normality of $\mathcal{U}$. Hence χ_p is a condition of length n+1 which extends π. By induction hypothesis, there is a j–length preserving extension π' of χ_p for which the lemma holds. Choose π_p to be this π'. Thus if π'' is a condition of length $(n+1)+l$ which extends π_p and such that $\pi''|j = \eta$, then if π'' decides φ, so does j–int(π'', π_p).

To complete the induction we still have to show that properties (1),(2),(3) hold for the extended sequence:

(1) holds because π_p is a j–length preserving extension of χ_p;

(2) holds because $\chi_p \leq \pi$, so $\pi_p \leq \pi$ as well;

(3) holds because if $p \subseteq s$, $q \subseteq s$ and $s \in B^p \cap B^q$, then $G^p(s) \subseteq H^p(s)$, so by the way G^p was defined, $H^p(s)$ and $H^q(s)$ must be compatible.

We have now completed the inductive definition of the sequence $(\pi_p : p \in A)$ satisfying (1),(2),(3). It remains to find a j–length preserving extension π' of π such that the lemma will hold for π, l+1 and n

<u>Claim</u>: There is $B \in \mathcal{U}$ such that $B \subseteq A$ and the sequence $(f_j^p \dots f_n^p)$ is constant for all $p \in B$. For $j \le i < n$, each $f_i^p \in \mathrm{Coll}(\kappa_i^{+k}, < \kappa_{i+1})$, and $\mathrm{Coll}(\kappa_i^{+k}, < \kappa_{i+1})$ has cardinality $< \kappa$. Thus since $\mathcal{U}$ is κ–complete, we can find $B_1 \subseteq A$ for such that the sequence $(f_j^p \dots f_{n-1}^p)$ is constant for $p \in B_1$. For all $p \in A$, $f_n^p \in \mathrm{Coll}(\kappa_n^{+k}, < \kappa(p))$, and since $\kappa(p)$ is inaccessible, there is $\alpha_p \in \kappa(p)$ such that $f_n^p \in \mathrm{Coll}(\kappa_n^{+k}, < \alpha_p)$. By normality of $\mathcal{U}$, there is $B_2 \in \mathcal{U}$ such that $B_2 \subseteq B_1$ and α_p is constant for $p \in B_2$. Call this constant α. Then $\mathrm{Coll}(\kappa_n^{+k}, < \alpha)$ has cardinality $< \kappa$, so by κ–completeness of $\mathcal{U}$ again, there is $B \in \mathcal{U}$ such that $B \subseteq B_2$ and such that for $p,q \in B$ we have $f_n^p = f_n^q$. B is clearly the required set.

Suppose that $g_j \dots g_n$ are the constant values of the $f_j^p \dots f_n^p$ for $p \in B$. Define π' as follows:

$$\pi' = (p_1 \dots p_n, f_0 \dots f_{j-1}, g_j \dots g_n, C, H),$$

where $$C = B \cap \{p : p \in \cap \{B^q : q \in A, q \subseteq p\}\}$$

and $$H(p) = f^p \text{ for } p \in C \text{ (where } f^p = f_{n+1}^{\pi_p})$$

By definition of f^p, $\cup \{H^q(p) : Q \in A, q \subseteq p\} \subseteq f^p$ and $G^p \subseteq f^p$.

We must now show that π' satisfies the lemma for $l+1$. Clearly π' is a j–length preserving extension of π (since for $p \in A$, $f_i^p \supseteq f_i$, and thus $g_i \supseteq f_i$). Suppose that $\pi'' \le \pi'$ is a condition of length $n+(l+1)$ such that $\pi''|j = \eta$ and π'' decides φ. Let χ = j–int(π'',π'). χ is determined by some $(p,q_1 \dots q_l)$, where $p,q_1 \dots q_n \in C$, and some functions $(h_0 \dots h_{j-1})$. Since $\pi''|j = \eta$, $\eta = (p_1 \dots p_j, h_0 \dots h_{j-1})$. Clearly $\pi'' \le \chi \le \pi'$; also, since $p,q_1 \dots q_n \in C$, we have $\chi \le \pi_p$. Thus $\pi'' \le \pi_p$, $\pi''|j = \eta$, π'' decides φ, and π'' has length $n+(l+1) = (n+1)+l$. By induction hypothesis, if χ' = j–int(π'',π_p), then χ' decides φ. But χ = j–int$(\pi'',\chi) \le$ j–int$(\pi'',\pi_p) = \chi'$, so χ decides φ, as required.

□

The above lemma may be used to prove the slightly more general

Lemma 4.2.13:

Let φ, π, j, η be as in the statement of Lemma 4.2.6. Then there is a j–length preserving extension π' of π such that whenever $\pi'' \leq \pi'$, $\pi''|j = \eta$ and π'' decides φ, then j–int(π'',π') decides φ as well.

Proof: We use induction on $l < \omega$ to construct a descending sequence $(\pi_l: l < \omega)$ of j–length preserving extensions of π: Let $\pi_0 = \pi$, and given π_l, let π_{l+1} be a j–length preserving extension of π_l which satisfies the conclusions of Lemma 4.2.12, i.e. if π'' is an extension of π_{l+1} of length n+l such that $\pi''|j = \eta$ and π'' decides φ, then j–int(π'',π_{l+1}) decides φ. Let π' be a j–length preserving extension of all the π_l ($l < \omega$), as given by Lemma 4.2.11. It is claimed that π' is the required condition: Let $\pi'' \leq \pi'$ such that $\pi''|j = \eta$, and π'' decides φ. Suppose the length of π'' is n+l for some $l \in \omega$. Since $\pi'' \leq \pi' \leq \pi_{l+1}$, it follows by definition of π_{l+1} that j–int(π'',π_{l+1}) decides φ. But clearly j–int(π'',π') $\leq$ j–int(π'',π_{l+1}).

□

Finally we are able to state and prove Magidor's "main technical device":

Lemma 4.2.14 [Magidor 1977b]:

Let φ be a statement of the forcing language of **P**, *and let π be a condition of length* n. *Suppose further that* j *is a natural number,* j $\leq$ n. *Then there is a j–length preserving extension π' of π which decides φ up to j–direct extensions, i.e. whenever $\pi'' \leq \pi'$ is a condition which decides φ, then* j–int(π'',π') *decides φ.*

Proof: It is not hard to compute that the set $\{\eta: \eta = \sigma|j$ for some $\sigma \leq \pi\}$ has cardinality κ_j because all the κ_j are inaccessible. Let $\{\eta_\gamma: \gamma < \kappa_j\}$ be an enumeration of this set, and by induction, construct a descending sequence $(\pi_\gamma: \gamma < \kappa_j)$ such that each π_γ is a j–length preserving extension of π: Let $\pi_0 = \pi$. If $\lambda < \kappa_j$ is limit, let π_λ be a condition which is a j–length preserving extension of all the π_γ for $\gamma < \lambda$ (such a sequence is provided by Lemma 4.2.11). Given π_γ, let $\pi_{\gamma+1}$ be any j–length preserving extension of π_γ which satisfies the conclusions of Lemma 4.2.13 with η replaced by η_γ (i.e. if $\pi'' \leq \pi_{\gamma+1}$, $\pi''|j = \eta_\gamma$ and π'' decides φ, so does j–int(π'',$\pi_{\gamma+1}$)).

Let π' be a j–length preserving extension of all the π_γ ($\gamma < \kappa_j$). Then π' is the required condition, because if $\pi'' \leq \pi'$ such that π'' decides φ, then $\pi''|j = \eta_\gamma$ for some $\gamma < \kappa_j$. Hence

$\pi'' \leq \pi_{\gamma+1}$, and by definition of $\pi_{\gamma+1}$ j–int$(\pi'',\pi_{\gamma+1})$ decides φ. But j–int$(\pi'',\pi') \leq$ j–int$(\pi'',\pi_{\gamma+1})$, proving the lemma.

□

Let $\mathcal{G} \subseteq \mathbb{P}$ be generic over V. (We use a script $\mathcal{G}$ to distinguish it from the G in the conditions of $\mathbb{P}$). Clearly if $\pi,\sigma \in \mathcal{G}$ such that $n_\pi \leq n_\sigma$, the p–part of π is an initial segment of the p–part of σ. Moreover, by simple arguments involving dense sets, for every $n < \omega$, there is $\pi \in \mathcal{G}$ such that $n_\pi \geq n$. Thus we may define $P_n = p_n^\pi$ for some (hence all) $\pi \in \mathcal{G}$ with length $\geq n$. Call $(P_n: 1 \leq n < \omega)$ the p–part of $\mathcal{G}$. Let $\kappa_n = \kappa \cap P_n$. Clearly if $\pi \in \mathcal{G}$ is of length $\geq n$, then $\kappa_n = \kappa_n^\pi$. Let $F_n = \cup \{f_n^\pi: \pi \in \mathcal{G}$ is of length $\geq n\}$, and call $(f_n: n < \omega)$ the f–part of $\mathcal{G}$. Again by simple arguments involving dense subsets of $\mathbb{P}$ it follows that each F_n is a map with domain $\kappa_n^{+k} \times \kappa_{n+1}$ and range κ_{n+1} and that for all $\kappa_n^{+k} < \gamma < \kappa_{n+1}$, the map $F_n(\_,\gamma)$ is a map with domain κ_n^{+k} and range γ. Thus each cardinal γ strictly between κ_n^{+k} and κ_{n+1} is collapsed to an ordinal of size κ_n^{+k}. We shall soon see that κ_n^{+k} and κ_{n+1} are preserved in V[$\mathcal{G}$], so that κ_{n+1} is collapsed to the successor of κ_n^{+k}.

Note 4.2.15: We shall temporarily accept the following convention: We shall let κ_n^{+r} denote the ordinal which is the r^{th} successor of κ_n in V; thus if we have a statement such as V[$\mathcal{G}$] $\models \varphi(\kappa_n^{+r})$, the κ_n^{+r} concerned is not the r^{th} successor of κ_n in V[$\mathcal{G}$], but the ordinal which is the r^{th} successor of κ_n in V. We have already seen, for instance, that κ_n^{+k+1} is not a cardinal in V[$\mathcal{G}$]. The successor of κ_n^{+k} in V[$\mathcal{G}$] will be denoted by $(\kappa_n^{+k})^+$.

Lemma 4.2.16:

Let $(P_n: 1 \leq n < \omega)$ *be the* p–*part of a* V–*generic filter over* $\mathbb{P}$, *and suppose that* $0 \leq i < k$. *Then* $(P_n \cap \kappa^{+i}: n < \omega)$ *is cofinal in* κ^{+i}. *In particular*, $(\kappa_n: n < \omega)$ *is cofinal in* κ.

Proof: Let $\alpha \in \kappa^{+i}$ and let $\pi \in \mathbb{P}$. If $\pi = (p_1 \dots p_n, f_0 \dots f_n, A, G)$, pick $p \in A$ such that $\alpha+1 \in p$ (which exists because $A \in \mathcal{U}$ and $\mathcal{U}$ is a fine measure over $[\kappa^{+k-1}]^{<\kappa}$. Let π' be the

0–direct extension of π determined by p. Then $\pi' \Vdash_{\mathbb{P}} \sup(P_{n+1} \cap \kappa^{+i}) > \alpha$. Hence the set of conditions in $\mathbb{P}$ which force that $(P_n \cap \kappa^{+i}: n < \omega)$ is cofinal in κ^{+i} is dense in $\mathbb{P}$.

□

So in $V[\mathcal{G}]$, each of the ordinals $\kappa, \kappa^{+1}, \ldots, \kappa^{+k-1}$ has cofinality ω, and thus none of the $\kappa^{+1}, \kappa^{+2}, \ldots, \kappa^{+k-1}$ remain cardinals in $V[\mathcal{G}]$.

Let $\mathbb{P}_{<j} = \mathrm{Coll}(\omega_1, < \kappa_1) \times \mathrm{Coll}(\kappa_1^{+k}, < \kappa_2) \times \ldots \times \mathrm{Coll}(\kappa_{j-1}^{+k}, < \kappa_j)$, and partially order $\mathbb{P}_{<j}$ component–wise. Let $\mathcal{G} \subseteq \mathbb{P}$ be generic over V, and let

$$\mathcal{G}_{<j} = \{(f_0 \ldots f_{j-1}) : \exists\, \pi \in \mathcal{G} \text{ such that } n_\pi \geq j-1 \text{ and } f_i^\pi = f_i \text{ for } 0 \leq i \leq j-1\}.$$

It is not hard to see that $\mathcal{G}_{<j}$ is V–generic over $\mathbb{P}_{<j}$, for if $\mathcal{D}$ is dense in $\mathbb{P}_{<j}$, then

$$\mathcal{D}' = \{\pi \in \mathbb{P}: \exists (f_0 \ldots f_{j-1}) \in \mathcal{D} \text{ such that } f_i^\pi = f_i \text{ for } 0 \leq i \leq j-1\}$$

is dense in $\mathbb{P}$.

The next theorem shows the relationship between sets in $V[\mathcal{G}_{<j}]$ and sets in $V[\mathcal{G}]$.

Lemma 4.2.17:

Let $\pi \in \mathbb{P}$, and let $\dot{x}$ be a name such that $\pi \Vdash_{\mathbb{P}} \dot{x} \subseteq \check{\alpha}$, where $\alpha \leq \kappa_j^{+k-1}$ for some j, and $n_\pi \leq j$. Suppose that $\mathcal{G}$ is V–generic over $\mathbb{P}$ such that $\pi \in \mathcal{G}$. Then $\dot{x}[\mathcal{G}] \in V[\mathcal{G}_{<j}]$.

Proof: We shall show that for each $\pi' \leq \pi$ there is $\chi < \pi'$ and a $\mathbb{P}_{<j}$–name $\dot{y}$, such that if $\bar{\mathcal{G}} = \mathcal{G}_{<j}$, then $\dot{y}[\bar{\mathcal{G}}] = \dot{x}[\mathcal{G}]$. Thus let $\pi' \leq \pi$. By Lemma 2.2.14, there is a j–length preserving extension χ' of π' such that whenever $\chi'' \leq \chi'$ is such that $\chi' \Vdash_{\mathbb{P}} \check{\lambda} \in \dot{x}$ (for $\lambda \in \alpha$), then also j–int$(\chi'',\chi') \Vdash_{\mathbb{P}} \check{\lambda} \in x$. Suppose $\chi' = (p_1 \ldots p_n, f_0 \ldots f_n, A, G)$, and let $\vec{f} = (f_0 \ldots f_{j-1})$. Define a map $F:[A]^{<\omega} \longrightarrow \mathcal{P}(\mathbb{P}_{<j} \times \alpha \times 3)$ as follows: If $q_1 \ldots q_l \in A$, then $F(\{q_1 \ldots q_l\})$ is the set of all 3–tuples $(\vec{g},\lambda,i) \in \mathbb{P}_{<j} \times \alpha \times 3$ such that $\vec{g} \leq \vec{f}$ in $\mathbb{P}_{<j}$, and if ρ is the j–direct extension of χ' determined by $\vec{g}$ and $(q_1 \ldots q_l)$, then:

(1) If $i = 0$, $\rho \Vdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$;

(2) If $i = 1$, $\rho \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$;

(3) If $i = 2$, $\rho \nVdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$, and $\rho \nVdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$.

Since the cardinality of $\mathcal{P}(\mathbf{P}_{<j} \times \alpha \times 3)$ is less than κ, $A \in \mathcal{U}$ and by choice of $\mathcal{U}$, there is $B \subseteq A$ such that $B \in \mathcal{U}$ and F is constant on each $[B]^l$. Call this constant c_l; thus for all $q_1 \dots q_l \in B$, $F(\{q_1 \dots q_l\}) = c_l$.

<u>Claim 1</u>: Suppose that $(\vec{g},\lambda,1) \in c_l$ and $(\vec{g}',\lambda,0) \in c_{l'}$ for some $\vec{g},\vec{g}' \in \mathbf{P}_{<j}$ and some $l,l' < \omega$. Then $\vec{g}$ and $\vec{g}'$ are incompatible in $\mathbf{P}_{<j}$.

Otherwise, let $\vec{g}''$ be an extension of $\vec{g}$ and $\vec{g}'$, let $l'' = \max(l,l')$ and choose $q_1 \dots q_l \in B$. Let ρ be the j–direct extension of χ' determined by $(q_1 \dots q_{l''})$ and $\vec{g}''$. Then ρ extends the j–direct extension of χ' determined by $(q_1 \dots q_l)$ and $\vec{g}$. Since $(\vec{g},\lambda,1) \in c_l = F(\{q_1 \dots q_l\})$, necessarily $\rho \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$. By a symmetric argument, because ρ extends the j–direct extension of χ' determined by $\vec{g}'$ and $(q_1 \dots q_{l'})$, also $\rho \Vdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$, a contradiction. This proves Claim 1.

Let $\mathcal{G}$ be V–generic over $\mathbf{P}_{<j}$ such that $\vec{f} \in \mathcal{G}$, and let $\dot{y}$ be a $\mathbf{P}_{<j}$–name such that

$\dot{y}[\mathcal{G}] = \{\lambda < \alpha: \exists l < \omega, \vec{g} \in \mathcal{G}$ such that $(\vec{g},\lambda,1) \in c_l\}$ for any such $\mathcal{G}$.

<u>Claim 2</u>: $\dot{y}[\mathcal{G}] = \alpha - \{\lambda < \alpha: \exists l < \omega, \vec{g} \in \mathcal{G}$ such that $(\vec{g},\lambda,0) \in c_l\}$

It suffices to show that either $(\vec{g},\lambda,1) \in c_l$ for some $\vec{g} \in \mathcal{G}$ and $l \in \omega$ or that $(\vec{g},\lambda,0) \in c_l$ for some $\vec{g} \in \mathcal{G}$ and $l \in \omega$. By Claim 1, both cannot hold simultaneously.

Let $(g_0 \dots g_{j-1}) = \vec{g} \le \vec{f} = (f_0 \dots f_{j-1})$ and consider the extension of χ' given by:

$$(p_1 \dots p_n, g_0 \dots g_{j-1}, f_j \dots f_n, B, G|B) \in \mathbf{P}.$$

Pick an extension χ'' of this condition such that either $\chi'' \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$, or $\chi'' \Vdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$. Since $\chi'' \le \chi'$, also either j–int$(\chi'',\chi') \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$, or j–int$(\chi'',\chi') \Vdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$. Now if $\chi'' = (p_1 \dots p_n, q_1 \dots q_l, h_0 \dots h_{n+l}, C, K)$, then j–int$(\chi'',\chi')$ is the j–direct extension of χ' determined by $(q_1 \dots q_l)$ and $\vec{h} = (h_0 \dots h_{j-1})$. Moreover $q_1 \dots q_l \in B$. Hence either $(\vec{h},\lambda,1) \in c_l$ (if $\chi'' \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$) or else $(\vec{h},\lambda,0) \in c_l$ (if $\chi'' \Vdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$).

The set $\{\vec{h}: \exists l \in \omega\ (\vec{h},\lambda,1) \in c_l$ or $(\vec{h},\lambda,0) \in c_l\}$ is therefore dense below $\vec{f}$. Since $\vec{f} \in \mathcal{G}$, there are $\vec{g} \in \bar{\mathcal{G}}$ and $l \in \omega$ such that either $(\vec{g},\lambda,1) \in c_l$ or $(\vec{g},\lambda,0) \in c_l$. This completes the proof of Claim 2.

Now let $\chi = (p_1 \dots p_n, f_0 \dots f_n, B, G|B)$. We claim that if $\mathcal{G} \subseteq \mathbf{P}$ is generic over V such that $\chi \in \mathcal{G}$, and if $\bar{\mathcal{G}} = \mathcal{G}_{<j}$, then $\dot{x}[\mathcal{G}] = \dot{y}[\bar{\mathcal{G}}]$. Note that if $\chi \in \mathcal{G}$, then $\vec{f} \in \bar{\mathcal{G}}$, so Claim 2 applies. Suppose that $\lambda \in \dot{x}[\mathcal{G}]$; then there is $\chi'' \leq \chi$ in $\mathcal{G}$ such that $\chi'' \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$. Since $\chi'' \leq \chi \leq \chi'$, also j–int$(\chi'',\chi') \Vdash_{\mathbf{P}} \check{\lambda} \in \dot{x}$, and the j–interpolant is the j–direct extension of χ' determined by some $\vec{g}$ and $(q_1 \dots q_l)$, where each $q_i \in B$. Since $\chi'' \in \mathcal{G}$, necessarily $\vec{g} \in \bar{\mathcal{G}}$, and thus $\lambda \in \dot{y}[\bar{\mathcal{G}}]$.

Conversely, if $\lambda \notin \dot{x}[\mathcal{G}]$, there is $\chi'' \leq \chi$ in $\mathcal{G}$ such that $\chi'' \Vdash_{\mathbf{P}} \check{\lambda} \notin \dot{x}$. By similar arguments, there is $\vec{g} \in \bar{\mathcal{G}}$ and $l < \omega$ such that $(\vec{g},\lambda,0) \in c_l$. The characterization of $\dot{y}[\bar{\mathcal{G}}]$ given by Claim 2 now ensures that $\lambda \notin \dot{y}[\bar{\mathcal{G}}]$. Hence $\dot{x}[\mathcal{G}] = \dot{y}[\bar{\mathcal{G}}]$, and thus $\dot{x}[\mathcal{G}] \in V[\mathcal{G}_{<j}]$.

□

Using Lemma 4.2.17, we see that if $\alpha \leq \kappa_j^{+k-1}$ is a cardinal in $V[\mathcal{G}_{<j}]$, it remains a cardinal in $V[\mathcal{G}]$, i.e. $V[\mathcal{G}]$ and $V[\mathcal{G}_{<j}]$ have the same cardinals $\leq \kappa_j^{+k-1}$.

Each $\mathrm{Coll}(\kappa_i^{+k}, < \kappa_{i+1})$ is κ_i^{+k}–closed and satisfies the κ_{i+1}–chain condition. Thus, as in Easton forcing (Section 2.3) one may show that $\mathbf{P}_{<j}$ preserves each of

$$\omega_1, \kappa_1, \kappa_1^{+1} \dots \kappa_1^{+k}, \kappa_2, \kappa_2^{+1} \quad \kappa_2^{+k}, \kappa_3 \dots \kappa_j$$

and that everything else below κ_j is collapsed. Thus $\kappa_1 = \omega_2$ in $V[\mathcal{G}_{<j}]$, $\kappa_1^{+1} = \omega_3$ etc, and the same holds in $V[\mathcal{G}]$. Clearly there are $2 + k(j-1)$ many cardinals below κ_j, so that in $V[\mathcal{G}]$, $\kappa_j = \aleph_{2+k(j-1)}$. Since $(\kappa_n : n < \omega)$ is cofinal in κ, it follows that κ is collapsed to $\aleph_\omega$ in $V[\mathcal{G}]$. Using Lemma 4.2.17 again, we may show that κ is strong limit in $V[\mathcal{G}]$: If $\alpha < \kappa$, then $\alpha < \kappa_j$ for some $j < \omega$. It follows that every subset of α in $V[\mathcal{G}]$ is already in $V[\mathcal{G}_{<j}]$. But κ remains inaccessible in $V[\mathcal{G}_{<j}]$, so $2^\alpha < \kappa$ in $V[\mathcal{G}_{<j}]$, and also in $V[\mathcal{G}]$. Hence in $V[\mathcal{G}]$, $\aleph_\omega$ is strong limit; however, since in V we started with $2^\kappa = \kappa^{+k}$ and $\kappa^{+1}, \dots, \kappa^{+k-1}$ were collapsed, it is possible that in $V[\mathcal{G}]$ we have $2^{\aleph_\omega} = \aleph_{\omega+1}$. We therefore consider a submodel of $V[\mathcal{G}]$ which includes all of the $V[\mathcal{G}_{<j}]$ in which we will prove that the cardinals $\kappa^{+1}, \dots, \kappa^{+k-1}$ are not collapsed. For $0 \leq i < k$, define

$$V_i = V[(P_j \cap \kappa^{+i}: j < \omega); (F_j: j < \omega)]$$

i.e. V_i is the smallest submodel of $V[\mathcal{G}]$ such that $V \subseteq V_i \subseteq V[\mathcal{G}]$ and the sequences $(P_j \cap \kappa^{+i}: j < \omega)$ and $(F_j: j < \omega)$ are in V_i. Note that $V_0 = V[(\kappa_j: j < \omega); (F_j: j < \omega)]$ and that $V_{k-1} = V[\mathcal{G}]$ (since each P_j is a subset of κ^{+k-1}). We also have the equation $P_j \cap \kappa^{+i} = (P_j \cap \kappa^{+i+1}) \cap \kappa^{+i}$, from which it is clear that $V \subseteq V_0 \subseteq V_1 \dots \subseteq V_{k-1} = V[\mathcal{G}]$. We shall prove that the model V_0 has the required properties.

First note that since $(\kappa_j: j < \omega) \in V_0$, we may define $\mathbb{P}_{<j}$ in V_0. Moreover, $\mathcal{G}_{<j} = \{\vec{g} \in \mathbb{P}_{<j}: g_0 \subseteq F_0 \wedge \dots \wedge g_{j-1} \subseteq F_{j-1}\}$; it follows that $\mathcal{G}_{<j} \in V_0$ for all $j < \omega$, and thus that $V \subseteq V[\mathcal{G}_{<j}] \subseteq V_0 \subseteq V[\mathcal{G}]$ for all $j < \omega$. It is clear, therefore, that V_0 and $V[\mathcal{G}]$ have the same bounded subsets of κ, and thus the same cardinals and cofinalities below κ. In particular, we have $V_0 \vDash$ "$\kappa = \aleph_\omega$ and $\aleph_\omega$ is strong limit". Since $V \subseteq V_0$ and $\kappa^{+k} \leq 2^\kappa$ in V, necessarily $V_0 \vDash \kappa^{+k} \leq 2^\kappa$ (where κ^{+k} is the k^{th} successor of κ in V). If we can show that the $\kappa^{+1}, \dots, \kappa^{+k-1}$ are not collapsed in V_0, then V_0 is a model of "$\aleph_\omega$ is strong limit and $2^{\aleph_\omega} \geq \aleph_{\omega+k}$". Hence the Singular Cardinals Hypothesis fails at $\aleph_\omega$ in V_0. In fact, we shall show that if $0 \leq i < k$, then $V_i \vDash$ "κ^{+i+1} is a cardinal". Before we can tackle this, however, we must make a small detour. We will return to the main argument after the proof of Lemma 4.2.20.

Definition 4.2.18: A partial map $\Gamma: \kappa^{+k-1} \rightarrow \kappa^{+k-1}$ is said to be *good* ([Magidor 1977b]) provided that whenever $\xi \in \kappa^{+j}$, then also $\Gamma\xi \in \kappa^{+j}$, and vice versa, whenever $\Gamma\xi \in \kappa^{+j}$ also $\xi \in \kappa^{+j}$ (for $0 \leq j < k$). Fix i such that $0 \leq i < k$, and let Aut(i) be the group of *good* automorphisms on κ^{+k-1} which, moreover, are the identity when restricted to κ^{+i}. Aut(i) acts on $\mathbb{P}$ as follows:

Let $\Gamma \in$ Aut(i):

(1) If $P \subseteq \kappa^{+k-1}$, define $\Gamma P = \{\Gamma\alpha: \alpha \in P\}$

(2) If $A \subseteq \mathcal{P}(\kappa^{+k-1})$, define $\Gamma A = \{\Gamma P: P \in A\}$

(3) If $\pi = (P_1 \dots P_n, f_0 \dots f_n, A\ G) \in \mathbb{P}$, define

$$\Gamma\pi = (\Gamma P_1 \dots \Gamma P_n, f_0 \dots f_n, \Gamma A, G\Gamma^{-1})$$

Lemma 4.2.19:

Each $\Gamma \in$ Aut(i) *induces an automorphism of* $\mathbb{P}$

Proof: Let $\Gamma \in \mathrm{Aut}(i)$, and let $\pi = (P_1 \dots P_n, f_o \dots f_n, A, G) \in \mathbb{P}$. We must first see that $\Gamma\pi \in \mathbb{P}$. Now Γ is the identity on κ, so $\Gamma P_j \cap \kappa = P_j \cap \kappa$ is an inaccessible cardinal, and since Γ is good, also $(\kappa^{+l}(\Gamma P_j))^+ = \kappa^{l+1}(\Gamma P_j)$. It follows that each $P_j \in D$ (Recall that D is defined just prior to Definition 4.2.6). Since $P_j \subseteq P_{j+1}$,

$$\mathrm{otp}(\Gamma P_j) = \mathrm{otp}(P_j) < \mathrm{otp}(P_{j+1} \cap \kappa) = \mathrm{otp}(\Gamma P_{j+1} \cap \kappa).$$

Hence $\Gamma P_j \subseteq \Gamma P_{j+1}$, and for similar reasons, the f_j are still in the correct $\mathrm{Coll}(\alpha, < \beta)$. If $A \in \mathcal{U}$ and $A \subseteq D$, we must still see that $\Gamma A \in \mathcal{U}$ and $\Gamma A \subseteq D$. We have shown that $P \in D$ implies $\Gamma P \in D$, so certainly $A \subseteq D$ implies $\Gamma A \subseteq D$. To see that $\Gamma A \in \mathcal{U}$, it suffices to show that $\{P: \Gamma P = P\} \in \mathcal{U}$, for then $\Gamma A \supseteq A \cap \{P: \Gamma P = P\} \in \mathcal{U}$. Suppose therefore that $B = \{P: \Gamma P \neq P\} \in \mathcal{U}$; then for each $P \in B$, there is $\beta_P \in P$ such that $\Gamma\beta_P \notin P$ or $\Gamma^{-1}\beta_P \notin P$. Since $\mathcal{U}$ is normal, there is β and $C \subseteq B$ in $\mathcal{U}$ such that for all $P \in C$, $\beta_P = \beta$. However, there is also $P \in C$ such that $\{\beta, \Gamma\beta, \Gamma^{-1}\beta\} \subseteq P$ (by normality again) which clearly contradicts $\beta = \beta_P$. Hence $\Gamma\pi \in \mathbb{P}$ whenever $\pi \in \mathbb{P}$, and Γ is easily seen to be an order preserving bijection, i.e. an automorphism of $\mathbb{P}$.

□

Next we prove a kind of "homogeneity" condition:

Lemma 4.2.20:
Let $\pi, \pi' \in \mathbb{P}$, *where* $\pi = (P_1 \dots P_n, f_o \dots f_n, A, G)$ *and* $\pi' = (Q_1 \dots Q_n, f_o \dots f_n, B, H)$ *such that*

(a) *For* $1 \leq j \leq n$, $Q_j \cap \kappa^{+i} = P_j \cap \kappa^{+i}$;

(b) *For* $P \in A \cap B$, $H(P) = G(P)$.

Then there is $\Gamma \in \mathrm{Aut}(i)$ *such that* $\Gamma\pi$ *is compatible with* π'.

Proof: If $i = k-1$, then $P_j = Q_j$ for all $1 \leq j \leq n$, and thus π, π' are compatible. We may therefore assume that $i < k-1$. The idea is to define Γ piecewise, but to do this we have to be sure that the pieces have the same cardinalities. Note that

$|P_j| = |P_j \cap \kappa^{+k-1}| = |P_j \cap \kappa^{+i}|^{+k-1-i} = |Q_j \cap \kappa^{+i}|^{+k-1-i} = |Q_j|$. Furthermore, $|P_1| < |P_2| < \dots < |P_n|$, and $|P_j| = |P_j \cap \kappa^{+k-1}| = |P_j \cap \kappa|^{+k-1} > |P_j \cap \kappa^{+i}|$. Hence for $1 \leq j < n$, $|P_{j+1} - P_j - \kappa^{+i}| = |P_{j+1} - P_j - (P_{j+1} - \kappa^{+i})| = |P_{j+1}|$, and similarly, $|Q_{j+1} - Q_j - \kappa^{+i}| = |Q_{j+1}| = |P_{j+1}|$. We are now ready to define Γ: Firstly, let Γ be the identity on κ^{+i}. Next pick any *good* 1–1 map from $(P_1 - \kappa^{+i})$ onto $(Q_1 - \kappa^{+i})$. and proceed by picking any good 1–1 map from $(P_{j+1} - P_j - \kappa^{+i})$ onto $(Q_{j+1} - Q_j - \kappa^{+i})$ for $1 \leq j < n$. Finally pick any good 1–1 map from $(\kappa^{+k-1} - P_n - \kappa^{+i})$ onto $(\kappa^{+k-1} - Q_n - \kappa^{+i})$, and glue all the pieces together to form Γ. It is not hard to see that $\Gamma \in \mathrm{Aut}(i)$ and that $\Gamma P_j = Q_j$ for all $1 \leq j \leq n$.

Thus $\Gamma\pi = (Q_1 \dots Q_n, f_0 \dots f_n, \Gamma A, G\Gamma^{-1})$. Let $C = \{P \in A \cap B: \Gamma P = P\}$. From the proof of Lemma 4.2.19 it is obvious that $C \in \mathcal{U}$; moreover if $P \in C$, then $H(P) = G(P) = G\Gamma^{-1}(P)$. It follows that $(Q_1 \dots Q_n, f_0 \dots f_n, C, H|C)$ extends both π' and $\Gamma\pi$, proving that these conditions are compatible.

□

This concludes the detour begun at Definition 4.2.18, and we now will return to the main argument. We have already seen that in V_0:

(1) κ is collapsed to $\aleph_\omega$,

(2) κ remains strong limit, and

(3) $V_0 \vdash \kappa^{+k} \leq 2^\kappa$, where κ^{+k} is the ordinal which is the k^{th} successor cardinal of κ in V.

If we can show that $\kappa^{+1} \dots \kappa^{+k}$ are cardinals in V_0, then $V_0 \vdash 2^{\aleph_\omega} \geq \aleph_{\omega+k}$. It is to this end that we now turn our attention.

Lemma 4.2.21: *For* $0 \leq i \leq k-1$, $V_i \vdash$ "κ^{+i+1} *is a cardinal*".

Proof: For $l \leq i$, $(P_j \cap \kappa^{+l}: j < \omega)$ is in V_i, and thus each of $\kappa^{+1} \dots \kappa^{+i}$ have cofinality ω in V_i. Thus each of these cardinals is collapsed. Assume that κ^{+i+1} is not a cardinal in V_i. Then it is singular in V_i and thus there is a regular cardinal $\mu < \kappa^{+i+1}$ which is the cofinality of κ^{+i+1} in V_i. Since κ is singular, we necessarily have $\mu < \kappa$. Let $\dot{s}$ be a name for a cofinal map from μ into κ^{+i+1} which lies in $V_i = V[(P_j \cap \kappa^{+i}: j < \omega); (F_j: j < \omega)]$.

We may assume that $\dot{s}$ is invariant under any automorphism of $\mathbb{P}$ which leaves some name for the pair $((P_j \cap \kappa^{+i}: j < \omega); (F_j: j < \omega))$ unchanged. If $\Gamma \in \mathrm{Aut}(i)$ and $P \in [\kappa^{+k-1}]^{<\kappa}$, then $P \cap \kappa^{+i} = \Gamma P \cap \kappa^{+i}$, and Γ applied to any $\pi \in \mathbb{P}$ does not change its f–part. Hence every $\Gamma \in \mathrm{Aut}(i)$ has the required property of leaving $\dot{s}$ invariant, i.e. $\Gamma b = b$ for all $\Gamma \in \mathrm{Aut}(i)$.

Let π be a condition such that $\pi \Vdash_{\mathbb{P}}$ "$\dot{s}:\check{\mu} \longrightarrow \check{\kappa}^{+i+1}$ is a cofinal map", and let $n = n_\pi$. We may choose π such that $\mu \leq \kappa_j$ for some $j \leq n$. By Lemma 4.2.14, there is a $\pi' \leq \pi$ such that $\pi' = (P_1 \dots P_n, f_0 \dots f_n, A, G)$ is a j–length preserving extension of π and if $\pi'' \leq \pi'$ and $\lambda \in \mu$ is such that $\pi'' \Vdash_{\mathbb{P}} \dot{s}(\check{\lambda}) = \check{\alpha}$ for some α, then the same is forced by j–int(π'',π').
For each $\lambda < \mu$ we may define a set $A_\lambda = \{\alpha : \exists \pi'' \leq \pi' \text{ such that } \pi'' \Vdash_{\mathbb{P}} \dot{s}(\check{\lambda}) = \check{\alpha}\}$.
If each $|A_\lambda| \leq \kappa^{+i}$, then since each $A_\lambda \in V$ and κ^{+i+1} is regular in V, there is $\delta < \kappa^{+i+1}$ such that $\cup \{A_\lambda: \lambda < \mu\} \subseteq \delta$. Clearly then $\pi' \Vdash_{\mathbb{P}} \mathrm{ran}(\dot{s}) \subseteq \check{\delta}$, which contradicts the fact that $\pi' \Vdash_{\mathbb{P}}$ "ran$(\dot{s})$ is cofinal in κ^{+i+1}". Hence to complete the proof of this lemma, it is only necessary to prove the following Claim.

<u>Claim</u>: For all $\lambda < \mu$, $|A_\lambda| \leq \kappa^{+i}$.

Fix $\lambda < \mu$, and for each $\alpha \in A_\lambda$ pick $\pi_\alpha \leq \pi'$ such that $\pi_\alpha \Vdash_{\mathbb{P}} \dot{s}(\check{\lambda}) = \check{\alpha}$. By the special property of π', we may assume that each π_α is an j–direct extension of π' (by replacing π_α with j–int(π_α, π') if necessary). Suppose that π_α is the j–direct extension of π' determined by $\vec{g} \in \mathbb{P}_{<j}$ and $(Q_1^\alpha \dots Q_{l_\alpha}^\alpha)$, and assume $|A_\lambda| = \kappa^{+i+1}$. Since κ is inaccessible in V, $\mathbb{P}_{<j}$ has cardinality $< \kappa$, and thus there is $B \subseteq A_\lambda$ (in V) of cardinality κ^{+i+1} and $\vec{g} \in \mathbb{P}_{<j}$, $l \in \omega$ such that for all $\alpha \in B$ we have $\vec{g}_\alpha = \vec{g}$, and $l_\alpha = l$. The number of possible sequences of the form $(Q_1^\alpha \cap \kappa^{+i}, \dots Q_l^\alpha \cap \kappa^{+i})$ is at most κ^{+i}, and by thinning out B if necessary, we may assume that for all $\alpha, \beta \in B$ and all $1 \leq j \leq l$ we have $Q_j^\alpha \cap \kappa^{+i} = Q_j^\beta \cap \kappa^{+i}$.
Since π_a is a j–direct extension of π' it must have the form

$$\pi_a = (p_1 \dots P_n, Q_1^\alpha \dots Q_l^\alpha, g_0 \dots g_{j-1}, f_j \dots f_n, G(Q_1^\alpha) \dots G(Q_l^\alpha), B_\alpha, G|B_\alpha)$$

for all $\alpha \in B$, where $(g_0 \dots g_{j-1}) = \vec{g}$. By similar arguments as above we may assume that there are functions $h_1 \dots h_l$ such that for all $\alpha \in B$ and all $1 \leq j \leq l$ we have $G(Q_j^\alpha) = h_j$ (by thinning out B again if necessary). Thus for $\alpha \in B$, π_α has the form

$$\pi_\alpha = (P_1 \ldots P_n, Q_1^\alpha \ldots Q_l^\alpha, g_0 \ldots g_{j-1}, f_j \ldots f_n, h_1 \ldots h_l, B_\alpha, G|B_\alpha).$$

The point of all this engineering is that we now have a set $B \subseteq A_\lambda$ of cardinality κ^{+i+1} for which whenever $\alpha, \beta \in B$, there is $\Gamma \in \mathrm{Aut}(i)$ such that $\Gamma\pi_\alpha$ and π_β are compatible (by Lemma 4.2.20). This is where we obtain a contradiction: By definition of π_α and π_β we have $\pi_\alpha \Vdash_{\mathbf{P}} \dot{s}(\check{X}) = \check{\alpha}$ and $\pi_\beta \Vdash_{\mathbf{P}} \dot{s}(\check{X}) = \check{\beta}$. Thus $\Gamma\pi_\alpha \Vdash_{\mathbf{P}} \dot{s}(\check{X}) = \check{\alpha}$ (since $\dot{s}$, $\check{X}$ and $\check{\alpha}$ are invariant under Γ). Hence $\Gamma\pi_\alpha$ and π_β cannot be compatible.

□

Proof of Theorem 4.2.1: By Lemma 4.2.21, each of $\kappa^{+1} \ldots \kappa^{+k}$ remain cardinals in V_0. Hence in V_0, $\kappa = \aleph_\omega$ is strong limit, and $2^{\aleph_\omega} \geq \aleph_{\omega+k}$. Forcing in V_0 with the Lévy partial order $\mathrm{Coll}(\kappa^{+k}, < (2^{\aleph_\omega})^+)$ will not destroy any cardinals below κ^{+k} nor will it change the power function below κ^{+k}, so that $\aleph_\omega$ will remain a strong limit cardinal in this extension.

□

A corollary of Theorem 4.2.1 is: If the notion of a κ^{++}– supercompact cardinal is consistent, then it is consistent for the Singular Cardinals Hypothesis to fail at the smallest singular cardinal. This is an improvement on Theorem 4.1.5, which states that it is consistent for the SCH to fail modulo the existence of a large cardinal. In the next section we shall discuss a theorem of Shelah ([Shelah 1983]) which shows how to make $2^{\aleph_\omega}$ even larger, while preserving its strong limit character. However, another theorem of Shelah ([Shelah 1992]) says (surprisingly!) that if $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_4}$. This last theorem is a statement true in all models of ZFC, independent of the possible existence of large cardinals. We shall turn to it in Chapter 6.

§ 4.3 Further Generalizations

In this section we shall present several results on the power function at cardinals of countable cofinality. Assuming various large cardinal axioms we shall see that:

(1) The GCH can fail for the first time at $\aleph_\omega$ (Theorem 4.3.1).

(2) It is consistent that the SCH fails at every ω^{th} successor cardinal (Theorem 4.3.14).

(3) It is consistent that, for any $\alpha < \omega_1$, $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\alpha+1}$ (Theorem 4.3.21).

These results are all generalizations of (and indeed were inspired by) the work by Magidor ([Magidor 1977b]) discussed in Section 4.2. In order to keep this dissertation within manageable bounds, we shall not endeavour to prove these theorems in detail, but content ourselves with mere descriptions of their proofs.

The first result that we want to present is again due to Magidor ([Magidor 1977c]). It supercedes Theorem 4.2.1, but requires stronger hypotheses. A cardinal κ is said to be *huge* provided that that there is a transitive class M and an elementary embedding j: $V \longrightarrow M$ of the universe with critical point κ such that M is closed under sequences of length $j(\kappa)$. More information about huge and supercompact cardinals may be found in Appendix 3.3.

Theorem 4.3.1 [Magidor 1977c]:
Suppose that V *is a transitive model of* ZFC *in which there is a huge cardinal with a supercompact cardinal below it. Then there is an extension* V' *of* V *such that*

$$V \models \forall n<\omega(2^{\aleph_n} = \aleph_{n+1}) \textit{ and } 2^{\aleph_\omega} = \aleph_{\omega+2}.$$

Next follow some definitions. Recall that $[\lambda]^\alpha$ is the set of all subsets of λ with order type exactly α if α is an ordinal, and that $[\lambda]^\alpha$ is the set of all subsets of λ with cardinality α if α is a cardinal. The assertion that κ is huge is equivalent to the assertion that there is a *normal fine measure* over $[\lambda]^\kappa$ for some $\lambda > \kappa$. An ultrafilter $\mathcal{U}$ over $[\lambda]^\kappa$ is *weakly normal* if for every choice function $F:U \longrightarrow \lambda$ on an element of $\mathcal{U}$ there is a $\beta < \lambda$ such that $\{p \in U: F(p) < \beta\} \in \mathcal{U}$. $\mathcal{U}$ is said to be (κ,λ)*–regular* provided that there is a λ–sequence of elements of $\mathcal{U}$ such that the intersection of any κ of them is empty.

Magidor proves the following lemma in [Magidor 1977c]

Lemma 4.3.2 [Magidor 1977c]:
Assume that ZFC + ($\exists$ *huge cardinal with a supercompact cardinal below it*) *is consistent. Then* ZFC *is consistent with the existence of a supercompact cardinal* κ *such that for some regular limit cardinal* λ *we have*

(1) $2^{\kappa} = \lambda^{+}$

(2) *For some ordinal* α *of cardinality* κ *there is a fine measure* $\mathcal{U}$ *over* $[\lambda]^{\alpha}$ *which is not* (κ,λ)*–regular.*

To describe the forcing conditions required in order to get a model in which the GCH fails for the first time at $\aleph_{\omega}$, we start in a model which satisfies the conditions described by Lemma 4.3.2. Fix κ, α, λ and $\mathcal{U}$ as in this lemma. For $p \in [\lambda]^{<\kappa}$ and $\alpha < \kappa$, let $\alpha(p) = \text{otp}(p \cap \alpha)$. The next lemma is an analogue of Lemma 4.2.5. (See [Magidor 1977c] for a proof):

Lemma 4.3.3:
There is a normal fine measure $\mathcal{V}$ *over* $[\lambda]^{<\kappa})$ *such that the set of all* $p \in [\lambda]^{<\kappa}$ *satisfying the following properties is in* $\mathcal{V}$.

(1) $\lambda(p)$ *is a regular cardinal.*

(2) $\kappa(p) = \kappa \cap p$ *is an inaccessible cardinal which is a limit of inaccessible cardinals.*

(3) $2^{\kappa(p)} = \lambda(p)^{+}$, *and* $\lambda(p)^{<\kappa(p)} = \lambda(p)$.

(4) $[\lambda(p)]^{\alpha(p)}$ *carries a* $\kappa(p)$*–complete fine measure which is not* $(\kappa(p), \lambda(p))$*–regular where* $|\alpha(p)| = \kappa(p)$.

Let $\mathcal{V}$ be fixed as the ultrafilter satisfying the conditions of Lemma 4.3.3, and let $D \in \mathcal{V}$ be the set satisfying properties (1) – (4). Thus for each $p \in D$, there is a $\kappa(p)$–complete fine measure over $[\lambda(p)]^{\alpha(p)}$ which is not $(\kappa(p), \lambda(p))$–regular. Since $\lambda(p) = \lambda(p)^{<\kappa(p)} = |[\lambda(p)]^{<\kappa(p)}|$ and because $|\alpha(p)| = \kappa(p)$, it follows that there is an ultrafilter with the same properties over $[[p]^{<\kappa(p)}]^{\kappa(p)}$. Denote such an ultrafilter by $\mathcal{U}(p)$ for each $p \in [\lambda]^{<\kappa}$.

We define $p \subseteq q$ as in Definition 4.2.2, and $\mathrm{Coll}(\alpha, < \beta)$ will denote the Lévy collapsing forcing. For $p \in [\lambda]^{<\kappa}$, define $\iota(p)$ to be the first strongly inaccessible cardinal above $\kappa(p)$. Note that if $p \subseteq q$, then $\mathrm{otp}(p) < \kappa(q)$, and thus $\iota(p) < \kappa(q)$ as well (because $\kappa(q)$ is an inaccessible limit of inaccessible cardinals).

Definition 4.3.4 [Magidor 1977c]:
The set of forcing conditions $\mathbb{P}$ is the set of all ordered tuples of the form:

$$\pi = (p_1 \dots p_n, f_o \dots f_n, g_1 \dots g_{n-1}, A, F, G, H)$$

where

(1) For $1 \le i \le n$, $p_i \in D$, and for $1 \le i < n$, $p_i \subseteq p_{i+1}$.

(2) $f_o \in \mathrm{Coll}(\omega_1, < \kappa(p_1))$, and
$f_i \in \mathrm{Coll}(\kappa^+(p_i), < \iota(p_i))$, for $1 \le i \le n$.

(3) $g_i \in \mathrm{Coll}(\iota(p_i)^+, < \kappa(p_{i+1})$ for $1 \le i \le n-1$.

(4) $A \in \mathcal{V}$, $A \subseteq D$, and for every $p \in A$, $p_n \subseteq p$.

(5) $\mathrm{Dom}(F) = \{(p,S)\colon p \in A,\ S \in [[p]^{<\kappa(p)}]^{\kappa(p)},\ p_i \in S \text{ for } 1 \le i \le n\}$
and if $(p,S) \in \mathrm{dom}(F)$, then $F(p,S) \in \mathrm{Coll}(\kappa(p)^+, < \iota(p))$.

(6) $\mathrm{Dom}(H) = \{(p,q)\colon p \in A \cup \{p_n\},\ q \in A, \text{ and } p \subseteq q\}$
and if $(p,q) \in \mathrm{dom}(H)$, then $H(p,q) \in \mathcal{U}(p)$, and for $S \in H(p,q)$,
$p \in S$ and $p_i \in S$ for $1 \le i \le n$.

(7) $\mathrm{Dom}(G) = \{(p,q,S)\colon (p,q) \in \mathrm{dom}(H) \text{ and } S \in H(p,q)\}$, and if
$(p,q,S) \in \mathrm{dom}(G)$, then $G(p,q,S) \in \mathrm{Coll}(\iota(p)^+, < \kappa(q))$

n is called the length of π, $(p_1 \dots p_n)$ the p–part of π, $(f_o \dots f_n)$ its f–part, and $(g_o \dots g_{n-1})$ its g–part. For $l \le n$, we define $\pi | l = (p_1 \dots p_l, f_o \dots f_l, g_1 \dots g_{l-1})$.

Definition 4.3.5 [Magidor 1977c]: We define the ordering on $\mathbb{P}$ as follows:

Let $\pi = (p_1 \dots p_n, f_o \dots f_n, g_1 \dots g_{n-1}, A,F,G,H)$

and $\bar{\pi} = (\bar{p}_1 \dots \bar{p}_l, \bar{f}_o \dots \bar{f}_l, \bar{g}_1 \dots \bar{g}_{l-1}, \bar{A},\bar{F},\bar{G},\bar{H})$

be conditions in $\mathbb{P}$. Then $\bar{\pi} \le \pi$ provided:

(1) $n \le l$. For $1 \le i \le n$, $p_i = \bar{p}_i$. For $n < i \le l$, $\bar{p}_i \in A$.

(2) For $0 \le i \le n$, $f_i \subseteq \bar{f}_i$, and for $0 \le i \le n-1$, $g_i \le \bar{g}_i$.

(3) $\bar{A} \subseteq A$.

(4) For $n < i \leq l$, there is $S_i \in H(\bar{p}_{i-1}, \bar{p}_i)$ such that $F(\bar{p}_i, S_i) \subseteq \bar{f}_i$, and if $n \leq i < l$, then $G(\bar{p}_i, \bar{p}_{i+1}, S_{i+1}) \subseteq \bar{g}_i$.

(5) If $(p,S) \in dom(\bar{F})$, then $F(p,S) \subseteq \bar{F}(p,S)$.

(6) If $(p,q) \in dom(H)$, then $\bar{H}(p,q) \subseteq H(p,q)$.

(7) If $(p,q,S) \in dom(\bar{G})$, then $G(p,q,S) \subseteq \bar{G}(p,q,S)$.

Note that this definition makes sense: (1) and (3) ensure that $dom(\bar{F}) \subseteq dom(F)$ and $dom(\bar{H}) \subseteq dom(H)$. Combined with (6), one now sees that $dom(\bar{G}) \subseteq dom(G)$ as well. We can now define the analogues to Definition 4.2.8 and Definition 4.2.9, and state a lemma analogous to Lemma 4.2.10.:

Definition 4.3.6: Let $\bar{\pi} \leq \pi$ be as in Definition 4.3.5., and let $j \leq n$. We say that $\bar{\pi}$ is a *j–direct extension* of π provided that:

(1) For $0 \leq i \leq j$ we have $\bar{f}_i = f_i$, and for $1 \leq i \leq j$, $\bar{g}_i = g_i$.

(2) $\bar{A} = \{q \in A: \bar{p}_l \subseteq q\}$.

(3) For $n < i \leq l$, the S_i provided by condition (4) of Definition 4.3.5 are such that $F(\bar{p}_i,S_i) = \bar{f}_i$, and similarly, for $n \leq i < l$, $G(\bar{p}_i,\bar{p}_{i+1},S_{i+1}) = \bar{g}_{i+1}$.

(4) $\bar{F} = F|dom(\bar{F})$ and $\bar{G} = G|dom(\bar{G})$.

(5) For $(p,q) \in dom(\bar{H})$, $\bar{H}(p,q) = \{S \in H(p,q)$: For $1 \leq i \leq l$, $\bar{p}_i \in S\}$.

If $j = 0$, we shall simply call $\bar{\pi}$ a direct extension of π.

If $\bar{\pi}$ is a j–direct extension of π, then $\bar{\pi}$ is determined by $(f_o \dots f_{j-1}, g_1 \dots g_{j-1})$, $(\bar{p}_{n+1} \dots \bar{p}_l)$ and $(S_{n+1} \dots S_l)$. However $(S_{n+1} \dots S_l)$ is not necessarily uniquely determined by π and $\bar{\pi}$. We shall say that $(f_o \dots f_{j-1}, g_1 \dots g_{j-1})$, $(\bar{p}_{n+1} \dots \bar{p}_l)$ and $(S_{n+1} \dots S_l)$ are *appropriate* for π provided that there is a j–direct extension of π determined by $(\bar{p}_{n+1} \dots \bar{p}_l)$, $(S_{n+1} \dots S_l)$, and $(\bar{f}_o \dots \bar{f}_{j-1}, \bar{g}_1 \dots \bar{g}_{j-1})$.

Definition 4.3.7: Let $\bar{\pi} \leq \pi$ be as in Definition 4.3.5. Then $\bar{\pi}$ is a *j–length preserving extension* of π provided that $n = l$ and for $0 \leq i < j$, $\bar{f}_i = f_i$ and $\bar{g}_i = g_i$. If $j = 0$, we shall simply call $\bar{\pi}$ a length preserving extension of π.

Lemma 4.3.8:

If $\bar{\pi} \leq \pi$ are as in Definition 4.3.5, and let $j \leq n$. There is a condition π' such that $\bar{\pi} \leq \pi' \leq \pi$ and with the property that $\bar{\pi}$ is a j–length preserving extension of π' and that π' is a j–direct extension of π.

The condition π' in Lemma 4.3.8 is known as the *j–interpolant* of $\bar{\pi}$ and π, and denoted j–int$(\bar{\pi},\pi)$. If $j = 0$, we shall simply write int$(\bar{\pi},\pi)$. Without proof, we will now state a technical lemma which is the analogue of Lemma 4.2.14:

Lemma 4.3.9 [Magidor 1977c]:

*Let π be a condition as in Definition 4.3.4, and let $\dot{\tau}$ be a name of the forcing language associated with **P** such that*

$$\pi \Vdash \text{"}\dot{\tau} \text{ is a map from } \check{\mu} \text{ into the ordinals"},$$

where $\mu \leq \kappa(p_j) = \kappa_j$ for some $j \leq n$, or where $\mu < \omega_1$ if $j = 0$. Then there is a j–length preserving extension $\bar{\pi}$ of π such that whenever $\pi' \leq \bar{\pi}$ and $\pi' \Vdash \dot{\tau}(\check{\alpha}) = \check{\delta}$ for some $\alpha < \mu$ and some ordinal δ, then also j–int$(\pi',\bar{\pi}) \Vdash \dot{\tau}(\check{\alpha}) = \check{\delta}$.

We need one more lemma which has no direct analogue in Section 4.2, and whose proof we shall also omit. If $\bar{\pi}$ is a j–direct extension of π, then $\bar{\pi}$ is determined by the sequences $(\bar{f}_0 \ldots \bar{f}_{j-1}, \bar{g}_1 \ldots \bar{g}_{j-1})$, $(q_1 \ldots q_l)$, and $(S_1 \ldots S_l)$. The next lemma claims that provided we know that the range of $\dot{\tau}$ is a subset of μ, then $(q_1 \ldots q_l)$, and $(S_1 \ldots S_l)$ are not relevant in determining the values of $\dot{\tau}$.

Lemma 4.3.10 [Magidor 1977c]:

Let π, $\dot{\tau}$, μ *and* j *be as in the statement of Lemma* 4.3.9, *and such that* $\pi \Vdash \mathrm{ran}(\dot{\tau}) \subseteq \check{\mu}$. *Then there is a* $\overline{\pi} \leq \pi$ *which satisfies: For all* $\delta, \alpha < \mu$, *the* j*–direct extension of* $\overline{\pi}$ *determined by* $(\bar{f}_0 \ldots \bar{f}_{j-1}, \bar{g}_1 \ldots \bar{g}_{j-1})$, $(q_1 \ldots q_l)$, *and* $(S_1 \ldots S_l)$ *forces* $\dot{\tau}(\check{\delta}) = \check{\alpha}$ *if and only if the* j*–direct extension of* $\overline{\pi}$ *determined by* $(\bar{f}_0 \ldots \bar{f}_{j-1}, \bar{g}_1 \ldots \bar{g}_{j-1})$, $(r_1 \ldots r_l)$ *and* $(T_1 \ldots T_l)$ *forces the same thing.*

Of course we are assuming that $(\bar{f}_0 \ldots \bar{f}_{j-1}, \bar{g}_1 \ldots \bar{g}_{j-1})$, $(q_1 \ldots q_l)$, $(S_1 \ldots S_l)$ and $(\bar{f}_0 \ldots \bar{f}_{j-1}, \bar{g}_1 \ldots \bar{g}_{j-1})$, $(r_1 \ldots r_l)$, $(T_1 \ldots T_l)$ in Lemma 4.3.10 are both apprpriate for $\overline{\pi}$.

The forcing conditions given in Definition 4.3.4 are quite a lot more complicated than the conditions given in Definition 4.2.6. Recall that λ is a regular limit cardinal with the property that $2^\kappa = \lambda^+$ and that for some ordinal α of cardinality κ there is fine measure on $[\lambda]^\alpha$ which is not (κ,λ)–regular (Lemma 4.3.2). Basically, we want to collapse κ to $\aleph_\omega$ in pretty much the same way as we did in Section 4.2, and we want to do this in such a way that the GCH holds below $\aleph_\omega$. We then want to ensure that κ, λ and λ^+ are not collapsed. For this we will need to look in a submodel of the generic extension obtained. Firstly, if $\mathcal{G} \subseteq \mathbb{P}$ is generic over V, then the p–part of any $\pi \in \mathcal{G}$ will be an initial segment of the p–part of any $\overline{\pi}$ stronger than π. Hence we may speak of the p–part of $\mathcal{G}$, which will be an ω–sequence $(p_n$: $1 \leq n < \omega)$ with the property that the p–part of any $\pi \in \mathcal{G}$ is an initial segment of this ω–sequence. It will turn out that $(\kappa(p_n)$: $1 \leq n < \omega)$ is a cofinal sequence in κ. How do we ensure that the GCH holds in the generic extension? f_0 ensures that $\kappa(p_1)$ is collapsed to ω_2. Then f_1 collapses $\iota(p_1)$ to ω_4, and every cardinal strictly between $\kappa(p_1)$ and $\iota(p_1)$ to ω_3. But $\iota(p_1)$ is an inaccessible cardinal $> \kappa(p_1)$, so it follows that $2^{\kappa(p_1)}$ is collapsed to ω_3 as well, so that $2^{\aleph_2} = \aleph_3$ in the generic extension. g_1 then ensures that $\kappa(p_2)$ is collapsed to $\iota(p_1)^+$,so that $\kappa(p_2)$ becomes ω_4 in the generic extension. f_2 now insures that $\iota(p_2)$ is collapsed to $\kappa^{++}(p_2) = \omega_5$, and that every cardinal between $\kappa^+(p_2)$ and $\iota(p_2)$ is collapsed to ω_4. In particular, $2^{\aleph_3} = \aleph_4$. The sequence of cardinals in V[$\mathcal{G}$] thus starts:

$$\omega, \omega_1, \kappa(p_1), \kappa^+(p_1), \iota(p_1), \iota(p_1)^+, \ldots, \kappa(p_l), \kappa^+(p_l), \iota(p_l), \iota(p_l)^+ \ldots$$

and since $(\kappa(p_l): 1 \leq l < \omega)$ is cofinal in κ, it is easy to see that κ is collapsed to $\aleph_\omega$ (provided that it is preserved) and that the GCH holds below κ in the generic extension.

We shall now present the arguments above in a little more detail. Let $\mathcal{G} \subseteq \mathbb{P}$ be generic over V, and let $(P_i: i < \omega)$ be the ω–sequence generated by p–parts of the conditions in $\mathcal{G}$. For each $i < \omega$, let $\kappa_i = \kappa(P_i) = \kappa \cap P_i$, let

$$[\mathcal{G}]_i = \{f: \exists \pi = (p_1 \ldots p_n, f_0 \ldots f_n, g_0 \ldots g_{n-1}, A,F,G,H) \in \mathcal{G} \text{ such that } f = f_i\}$$

$$[\mathcal{G}]^i = \{g: \exists \pi = (p_1 \ldots p_n, f_0 \ldots f_n, g_0 \ldots g_{n-1}, A,F,G,H) \in \mathcal{G} \text{ such that } g = g_i\}$$

Let $F_i = \bigcup [\mathcal{G}]_i$ and let $G_i = \bigcup [\mathcal{G}]^i$. It is not hard to see that $[\mathcal{G}]_i \subseteq \mathrm{Coll}(\kappa_i^+, < \iota(P_i))$ is generic over V if $0 < i$, and that $[\mathcal{G}]_0 \subseteq \mathrm{Coll}(\omega_1, < \kappa_1)$ is generic over V as well. Similarly, $[\mathcal{G}]^i \subseteq \mathrm{Coll}(\iota(P_i)^+, < \kappa_{i+1})$ is generic over V. Hence if $i > 0$, then F_i collapses all cardinals strictly between κ_i^+ and $\iota(P_i)$ to κ_i^+, and makes $\iota(P_i)$ into κ_i^{++}. F_0 collapses all cardinals between κ_1 and ω_1 to ω_1 and makes κ_1 into ω_2. Similarly G_i collapses all cardinals strictly between $\iota(P_i)^+$ and κ_{i+1} to $\iota(P_i)^+$, and makes κ_{i+1} into $\iota(P_i)^{++}$. $(\kappa_i: i < \omega)$ is a cofinal sequence with limit κ.

Let $\mu \leq \kappa_j$ for some $j < \omega$, and let $\dot{\tau}$ be a name for a subset of μ. It is not hard to see, using Lemma 4.3.10, that $\dot{\tau}[\mathcal{G}]$ is always a member of $V[[\mathcal{G}]_0, \ldots, [\mathcal{G}]_{j-1}, [\mathcal{G}]^1, \ldots [\mathcal{G}]^{j-1}] = V[F_0, \ldots, F_{j-1}, G_1 \ldots, G_{j-1}]$: Because if $\pi \in \mathcal{G}$ is a condition for which Theorem 4.3.10 holds and if For $\dot{\tau}$ is considered as a characteristic map from μ to 2, then it is clear that $\dot{\tau}(\check{\delta}) = \check{1}$ (i.e. $\delta \in \dot{\tau}[\mathcal{G}]$) if and only if there is a j–direct extension determined by $(q_1 \ldots q_k)$, $(S_1 \ldots S_k)$ and $(\bar{f}_0 \ldots \bar{f}_{j-1}, \bar{g}_1 \ldots \bar{g}_{j-1}) \in [\mathcal{G}]_0 \times \ldots \times [\mathcal{G}]_{j-1} \times [\mathcal{G}]^1 \times \ldots \times [\mathcal{G}]^{j-1}$ which also forces $\dot{\tau}(\check{\delta}) = \check{1}$. Let:

$$\mathbb{P}|j = \mathrm{Coll}(\omega_1, < \kappa_1) \times \mathrm{Coll}(\kappa_1^+, < \iota(P_1)) \times \ldots \times \mathrm{Coll}(\kappa_{j-1}^+, < \iota(P_{j-1})) \times \mathrm{Coll}(\iota(P_1)^+, < \kappa_2) \times \ldots \times \mathrm{Coll}(\iota(P_{j-1})^+, < \kappa_j)$$

and let

$$\mathcal{G}|j = [\mathcal{G}]_0 \times \ldots \times [\mathcal{G}]_{j-1} \times [\mathcal{G}]^1 \times \ldots \times [\mathcal{G}]^{j-1}.$$

Then $\mathcal{G}|j \subseteq \mathbb{P}|j$ is generic over V, and the structure of the cardinals and the power sets of cardinals $\leq \kappa_j$ is the same in $V[\mathcal{G}]$ and $V[\mathcal{G}|j]$. It is not hard to see (following the methods of section 2.3) that in $V[\mathcal{G}|j]$ the cardinals below κ_j are precisely:

$$\omega, \omega_1, \kappa_1, \kappa_1^+, \iota(P_1), \iota(P_1)^+, \kappa_2, \kappa_2^+, \iota(P_2), \dots \kappa_j,$$

i.e. all other cardinals are collapsed, and these cardinals are not collapsed. The same is therefore true in $V[\mathcal{G}]$, and since we can make j as large as we please, the sequence of cardinals in $V[\mathcal{G}]$ below κ must be given by:

$$\omega, \omega_1, \kappa_1, \kappa_1^+, \iota(P_1), \iota(P_1)^+, \dots, \kappa_j, \kappa_j^+, \iota(P_j), \iota(P_j)^+, \dots \quad (j < \omega)$$

Since κ is the limit of $(\kappa_j: j < \omega)$ and each of the κ_j is preserved, κ is itself a cardinal in $V[\mathcal{G}]$, and $V[\mathcal{G}] \vdash \kappa = \aleph_\omega$. Similarly, in $V[\mathcal{G}|j]$ it is not hard to see that

$$2^\omega = \omega_1, 2^{\omega_1} = \kappa_1, 2^{\kappa_1} = \kappa_1^+, 2^{\kappa_1^+} = \iota(P_1), 2^{\iota(P_1)} = \iota(P_1)^+, \dots 2^{\iota(P_{j-1})^+} = \kappa_j.$$

Hence the same is true in $V[\mathcal{G}]$, and so in $V[\mathcal{G}]$ the GCH holds below $\aleph_\omega$.

Unfortunately, λ is collapsed to κ (Recall that our ground model was chosen so that $2^\kappa = \lambda^+$):

Lemma 4.3.11:
For $\kappa \leq \beta \leq \lambda$, *if* $\mathrm{cf}(\beta) \geq \kappa$ *in* V, *then* $\mathrm{cf}(\beta) = \omega$ *in* $V[\mathcal{G}]$.

Proof: We shall show that $(\sup(P_i \cap \beta): \iota < \omega)$ is cofinal in β. If not, there is a $\pi \in \mathcal{G}$ and an ordinal $\alpha < \beta$ such that π forces $\sup(P_i \cap \beta) < \alpha$ for all $i < \omega$. Suppose that

$$\pi = (P_1 \dots P_n, f_0 \dots f_n, g_1 \dots g_n, A,F,G,H)$$

and pick $P \in A$ such that $P_n \subseteq p$ and $\alpha \in p$ (p exists because of the properties of the normal fine measure $\mathcal{V}$ given in Lemma 4.3.3). Pick $S \in H(P_n,p)$ such that $P_n \in S$ (S exists because $H(P_n,p) \in \mathcal{U}(p)$, and $\mathcal{U}(p)$ is a fine measure on $[[p]^{<\kappa(p)}]^{\kappa(p)}$. Let π' be a condition given by $\pi' = (P_1 \dots P_n, p, f_0 \dots f_n, F(p,S), g_1, \dots g_{n-1}, G(P_n,p,S), \dots)$. Then $\pi' \leq \pi$ and clearly $\pi' \Vdash \sup(\sup(P_i \cap \beta): i < \omega) > \alpha$, because $\alpha \in p = P_{n+1}$. This is a contradiction.

□

In particular, λ is collapsed to κ in $V[\mathcal{G}]$. However, we shall see that λ^+ is still a cardinal in $V[\mathcal{G}]$:

Lemma 4.3.12:

In V[$\mathcal{G}$], λ^+ *is a regular cardinal.*

Proof: If λ^+ is singular in V[$\mathcal{G}$], then by Lemma 4.3.11, $\mu = \mathrm{cf}(\lambda^+)^{V[\mathcal{G}]} < \kappa$. Hence there is a j < ω such that $\mu \leq \kappa_j$. Let $\dot{\tau}$ be a name, and let $\pi \in \mathcal{G}$ such that π forces that $\dot{\tau}$ is a cofinal map from μ to λ^+. We may assume that n = length of π is greater than j, by replacing π by a stronger condition if necessary. Let $\overline{\pi}$ be an extension of π for which Lemma 4.3.9 holds for $\dot{\tau}$ and j, and let $\delta < \mu$. Define $\mathrm{Val}_\delta = \{\alpha\colon \exists\pi' \leq \overline{\pi}(\pi' \Vdash \dot{\tau}(\check{\delta}) = \check{\alpha})\}$. If $\pi' \leq \overline{\pi}$ and π' forces $\dot{\tau}(\check{\delta}) = \check{\alpha}$ and , then j–int(π',$\overline{\pi}$) forces the same (by Lemma 4.3.9). Now j–int(π',$\overline{\pi}$) is a j–direct extension of $\overline{\pi}$, and thus

$$\mathrm{Val}_\delta = \{\alpha\colon \text{There is a j–direct extension of } \overline{\pi} \text{ which forces } \dot{\tau}(\check{\delta}) = \check{\alpha}\}$$

A j–direct extension of $\overline{\pi}$ is determined by some sequences $(q_1 \dots q_k)$, $(S_1 \dots S_k)$ and $(\bar{f}_0 \dots \bar{f}_{j-1}, \bar{g}_1 \dots \bar{g}_{j-1})$. The number of possible $(q_1 \dots q_k)$ is at most $|[\lambda]^{<\kappa}| = \lambda^{<\kappa} = \lambda$ (by Theorem 3.3.5, because λ is regular $\geq \kappa$ in the ground model). The number of $(S_1 \dots S_k)$ cannot exceed $|[\lambda]^{<\kappa}|^{<\kappa} = \lambda$, and the cardinality of the possible sequences $(\bar{f}_0 \dots \bar{f}_{j-1}, \bar{g}_1 \dots \bar{g}_{j-1})$ is at most κ. It follows from these simple cardinality considerations that $\overline{\pi}$ can have at most λ–many j–direct extensions, and thus Val_δ has cardinality $\leq \lambda$. Now Val_δ is in the ground model for each $\delta < \mu$, and λ^+ is regular in the ground model. It follows that $\bigcup_{\delta<\mu} \mathrm{Val}_\delta$ is bounded in λ^+. Choose $\rho < \lambda^+$ such that $\bigcup_{\delta<\mu} \mathrm{Val}_\delta \subseteq \rho$. Then $\overline{\pi} \Vdash$ "the range of $\dot{\tau}$ is contained in ρ" , which contradicts the fact that $\overline{\pi} \Vdash$ "the range of $\dot{\tau}$ is cofinal in λ^+".

□

In the model V[$\mathcal{G}$] it may be false that $2^{\aleph_\omega} > \aleph_{\omega+1}$ because λ is collapsed. We experienced this problem in Section 4.2 as well, and we deal with it in the same way, i.e. we shall work in a certain submodel V_0 of V[$\mathcal{G}$] in which λ is still a cardinal. We define V_0 by:

$$V_0 = V[(\kappa_i\colon i < \omega),(F_i\colon i < \omega), (G_i\colon 1 \leq i < \omega)]$$

Then it is not hard to see that the structure of the cardinals and power sets below κ in V_0 is the same as that in $V[\mathcal{G}]$, since for every $j < \omega$, we have $V[\mathcal{G}|j] \subseteq V_0$. It follows that in V_0 κ is still $\aleph_\omega$ and that the GCH holds in V_0 below $\aleph_\omega$.

Lemma 4.3.13:
$V_0 \vdash \lambda$ *is regular.*

Proof (in outline): We draw attention to the fact that we will assume, without proof, a combinatorial claim in the proof of this Lemma. The full proof may be found in [Magidor 1977c]
Assume that λ is not regular in V_0, and let $\beta < \lambda$ be the cofinality of λ. Let $\beta' = \max(\beta,\kappa)$, and let V_1 be the model defined by:

$$V_1 = V[(P_i \cap \beta': i < \omega), (F_i: i < \omega), (G_i: 1 \leq i < \omega)]$$

Clearly $V \subseteq V_0 \subseteq V_1 \subseteq V[\mathcal{G}]$. If $\beta' = \beta$, then $\kappa \leq \beta$, and all ordinals between κ and β are singular in V_1 (as in the proof of Lemma 4.3.11), so that λ has cofinality $< \kappa$ in V_1. On the other hand, if $\beta' = \kappa$, then, $\beta < \kappa$ (because κ is in V_0 a singular cardinal), and so λ has cofinality $< \kappa$ in that case as well. Let μ be the cofinality of λ in V_1, and choose $j < \omega$ such that $\mu \leq \kappa_j$. Let $\dot{\tau}$ be the name of a function from μ to λ which is cofinal in λ, and which moreover has the following property: If $\mathcal{G}$, $\mathcal{G}'$ are two generic filters generating sequences

$$(P_i: 1 \leq i < \omega), (F_i: i < \omega), (G_i: 1 \leq i < \omega) \quad \text{and}$$
$$(P_i': 1 \leq i < \omega), (F_i': i < \omega), (G_i': 1 \leq i \leq \omega)$$

respectively such that for all i, $P_i \cap \beta = P_i' \cap \beta$, $F_i = F_i'$, and $G_i = G_i'$, then

$$\dot{\tau}[\mathcal{G}] = \dot{\tau}[\mathcal{G}'].$$

$\dot{\tau}$ exists by definition of V_1. Let $\pi \in \mathcal{G}$ be such that $\pi \Vdash$ "$\dot{\tau}: \check{\mu} \longrightarrow \check{\lambda}$ cofinally" with length of $\pi = n \geq j$. By Theorem 4.3.9 there is an extension $\overline{\pi}$ of π such that whenever π' forces $\dot{\tau}(\check{\delta}) = \check{\alpha}$, then j–int$(\pi',\overline{\pi})$ forces the same thing. For $\delta < \mu$, let

$$\mathrm{Val}_\delta = \{\alpha: \exists\ \pi' \leq \overline{\pi} \text{ such that } \pi' \Vdash \dot{\tau}(\check{\delta}) = \check{\alpha}\}$$

Without proof we are going to assume the following claim.
Claim: $|A_\delta| < \lambda$ for all $\delta < \mu$.
(A proof of the Claim may be found in [Magidor 1977c].)
Since λ is regular in the ground model and each Val_δ is an element of the ground model, it

follows that there is an ordinal $\rho < \lambda$ such that $\bigcup_{\delta<\mu} \mathrm{Val}_\delta \subseteq \rho$. Then $\bar{\pi} \Vdash$ "ran($\dot{\tau}$) $\subseteq \check{\rho}$", which contradicts $\bar{\pi} \Vdash$ "ran($\dot{\tau}$) is cofinal in λ".

□

Proof of Theorem 4.3.1: V_0 has the same cardinal – and power set structure below κ as does $V[\mathcal{G}]$. In $V[\mathcal{G}]$, we have $\lambda^+ = \aleph_{\omega+1} = 2^\kappa \geq \lambda^+$ and in V_0, λ is still a cardinal, so that in V_0, λ^+ becomes $\aleph_{\omega+2}$. Hence

$$V_0 \Vdash \forall n<\omega(2^{\aleph_n} = \aleph_{n+1}) \text{ and } 2^{\aleph_\omega} \geq \aleph_{\omega+2}$$

so V_0 is the model which satisfies the assertion of Theorem 4.3.1.

□

Our next aim is to discuss a more recent development of Magidor's work, due to A. Apter ([Apter 1984]), who proved that it is consistent (modulo a certain degree of supercompactness) that every limit cardinal is strong limit and the GCH fails at every ω^{th} – successor cardinal (i.e. for any cardinal λ, $2^{\lambda(+\omega)} > \lambda(+\omega+1)$).
In order to obtain this result, we must ensure that $\aleph_\omega$ is strong limit, and that the GCH fails at $\aleph_\omega$. Then we must ensure that $\aleph_{\omega+\omega}$ is strong limit and that the GCH fails at $\aleph_{\omega+\omega}$ in such a way that these properties still hold at $\aleph_\omega$. Thus what we need is a kind of iteration of the notion of forcing defined in Section 4.2 (due to [Magidor 1977b]). Since the forcing must affect the power function at arbitrarily large cardinals, it is clear that we shall need a proper class forcing (as discussed in Section 2.3).

In order to obtain a model of "$\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$" Magidor required a κ^+– supercompact cardinal κ which moreover satisfies $2^\kappa = \kappa^{++}$. Thus to iterate Magidor's notion of forcing, we shall assume that we have a proper class $(\delta_\alpha\colon \alpha \in \mathrm{On})$ of cardinals such that each δ_α is δ_α^+– supercompact and such that $2^{\delta_\alpha} = \delta_\alpha^{++}$ for each $\alpha \in \mathrm{On}$. We shall also need to assume that $2^{\delta_\alpha^+} = \delta_\alpha^{++} = 2^{\delta_\alpha}$ for all $\alpha \in \mathrm{On}$. In many ways the proof we sketch resembles that of Magidor's Theorem 4.2.1. Apart from that, [Apter 1984] also borrows heavily from another paper by Magidor ([Magidor 1976]).

Theorem 4.3.4 [Apter 1984]:

Let M *be a model of* ZFC + $2^{\kappa} = \kappa^{+}$ *for all singular cardinals in which there is an unbounded sequence* $(\delta_\alpha : \alpha \in \mathrm{On})$ *of cardinals such that*

(1) δ_α *is* $\delta_\alpha{}^{+}$*– supercompact for all* $\alpha \in$ On.

(2) $2^{\delta_\alpha} = 2^{\delta_\alpha{}^{+}} = \delta_\alpha{}^{++}$ *for all* $\alpha \in$ on.

Then there is a model $\overline{\mathrm{M}}$ *of the theory*:

"ZFC + *every limit cardinal is strong limit* + $2^{\lambda(+\omega)} > \lambda(+\omega+1)$ *for all cardinals* λ."

Note that the existence of an unbounded sequence of cardinals such as required by the above theorem is a much weaker assumption then the existence of a supercompact cardinal. Suppose that κ is λ–supercompact for some strong limit cardinal λ, $\mathcal{U}_\lambda$ a normal fine measure over $[\lambda]^{<\kappa}$, and $\mathrm{N} = \mathrm{Ult}(\mathrm{V},\mathcal{U}_\lambda)$. If $\alpha < \lambda$, and $\mathcal{U}_\alpha$ is a normal fine measure over $[\alpha]^{<\kappa}$, then $\mathcal{U}_\alpha \in \mathrm{N}$ and $\mathcal{P}([\alpha]^{<\kappa}) = \mathcal{P}([\alpha]^{<\kappa}) \cap \mathrm{N}$ (because ${}^{\lambda}\mathrm{N} \subseteq \mathrm{N}$). It follows that $\mathrm{N} \vDash$ "κ is α–supercompact for all $\alpha < \kappa$". Recalling that in the ultrapower an ordinal $\xi \leq \lambda$ is represented by the map $f_\xi(p) = \mathrm{otp}(p \cap \xi)$ (see Appendix 3(§3)), it follows that the set

$$A = \{p \in [\lambda]^{<\kappa} : \kappa(p) = \kappa \cap p \text{ is } \kappa^{++}(p)\text{–supercompact}\} \in \mathcal{U}_\lambda.$$

Hence V_κ (the κ^{th} element of the cumulative hierachy of the universe) is a model of

"ZFC + $\exists$ unbounded sequence of ordinals δ (each δ is δ^{++}– supercompact)"

(V_κ is a model of ZFC because κ is strongly inaccessible). An iteration of reverse Easton extensions (refer to Section 3.4) with suitable modifications will yield a model of the required theory.

Let M be a transitive model of ZFC with a sequence $(\delta_\alpha : \alpha \in \mathrm{On})$ satisfying the above requirements. We shall henceforth reason in M. Let

$$A = \{\delta : \delta \text{ is } \delta^{+}\text{– supercompact and } 2^{\delta} = \delta^{++}\}.$$

Thus $\delta_\alpha \in A$ for all $\alpha \in On$, showing that A is a *proper* class. For each $\delta \in A$, let $\mathcal{U}^\delta$ be a normal fine measure over $[\delta^+]^{<\delta}$ satisfying the Lemma 4.2.4 and let

$$D^\delta = \{p \in [\delta^+]^{<\delta}: p\cap\delta \text{ is inaccessible and } \delta(p)^+ = \delta^+(p)\}.$$

(Recall that $\delta(p)$ is the order type of $p\cap\delta$; thus for $p \in D^\delta$ we have $p\cap\delta = \delta(p)$ is inaccessible). As in Magidor's proof, Lemma 4.2.5 implies that $D^\delta \in \mathcal{U}^\delta$. Note that we may assume that $\sup\{\alpha \in A: \alpha < \delta\} < \delta$, since otherwise the model $(V_{\delta_0})^M$ is a model of the required hypotheses (where δ_0 is the least ordinal such that $\sup\{\alpha \in A: \alpha < \delta_0\} = \delta_0$). We may now define Apter's notion of forcing: It is a sequence of "Magidor forcings".

Definition 4.3.14 [Apter 1984]: Let $\mathbb{P}$ to be the notion of forcing whose conditions are sequences of the form:

$$\pi = (p_1^\alpha \dots p_{l_\alpha}^\alpha, f_0^\alpha \dots f_{l_\alpha}^\alpha, A^\alpha, G^\alpha)_{\alpha \in A}$$

such that:

(1) $l_\alpha \in \omega$

(2) For $1 \leq i \leq l_\alpha$, $p_i^\alpha \in D^\alpha$, and for $1 \leq i < l_\alpha$, $p_i^\alpha \subseteq p_{i+1}^\alpha$.

(3) If we define $\delta_i^\alpha = \alpha(p_i^\alpha)$, then $f_0^\alpha o \in Coll(\omega_1, < \delta_1^\alpha o)$, where α_o is the least element of A. For $\alpha \in A$ such that $\alpha > \alpha_0$, we have

$f_0^\alpha \in Coll(\sup\{\beta \in A: \beta < \alpha\}^{+++}, < \delta_1^\alpha)$.

For $\alpha \in A$ and $1 \leq i < l_\alpha$, $f_i^\alpha \in Coll(\delta_i^{\alpha++}, < \delta_{i+1}^\alpha)$, and

$f_{l_\alpha}^\alpha \in Coll(\delta_{l_\alpha}^{\alpha++}, < \alpha)$.

(4) For all $\alpha \in A$, $A^\alpha \subseteq D^\alpha$ and $A^\alpha \in \mathcal{U}^\alpha$.

(5) For all $q \in A^\alpha$, $p_{l_\alpha}^\alpha \subseteq q$, and $f_{l_\alpha}^\alpha \in Coll(\delta_{l_\alpha}^{\alpha++}, < \alpha(q))$.

(6) For every $\alpha \in A$, G^α is a map on A^α such that for all $q \in A^\alpha$,

$G^\alpha(q) \in Coll(\alpha(q)^{++}, < \alpha)$ and if $p \in A^\alpha$ such that $q \subseteq p$, then

$G^\alpha(q) \in Coll(\alpha(q)^{++}, < \alpha(p))$.

(7) $l_\alpha \neq 1$ for at most finitely many $\alpha \in A$.

If $\pi \in \mathbb{P}$, $\pi = (p_1{}^\alpha \dots p_{l_\alpha}^\alpha, f_0{}^\alpha \dots f_{l_\alpha}^\alpha, A^\alpha, G^\alpha)_{\alpha \in A}$, we shall, for $\alpha \in A$, denote the "α^{th} coordinate" of π by π_α. Thus $\pi_\alpha = (p_1{}^\alpha \dots p_{l_\alpha}^\alpha, f_0{}^\alpha \dots f_{l_\alpha}^\alpha, A^\alpha, G^\alpha)$ is a kind of Magidor–condition (in the sense of Section 4.2).
$\pi | \alpha$ is defined to be the sequence $(\pi_\beta: \beta \in A, \beta < \alpha)$, and $\mathbb{P}_\alpha = \{\pi | \alpha : \pi \in \mathbb{P}\}$. The order on $\mathbb{P}$ (and on all the $\mathbb{P}_\alpha$) is defined coordinate–wise:

Definition 4.3.16 [Apter 1984]:

$$\pi \leq \sigma \text{ in } \mathbb{P} \text{ iff } \forall \alpha \in A(\pi_\alpha \leq \sigma_\alpha \text{ in Magidor's notion of forcing}).$$

More precisely, if $\pi = (p_1{}^\alpha \dots p_{l_\alpha}^\alpha, f_0{}^\alpha \dots f_{l_\alpha}^\alpha, A^\alpha, G^\alpha)_{\alpha \in A}$ and $\sigma = (q_1{}^\alpha \dots q_{n_\alpha}^\alpha, g_0{}^\alpha \dots g_{n_\alpha}^\alpha, B^\alpha, H^\alpha)_{\alpha \in A}$, then $\pi \leq \sigma$ iff for all $\alpha \in A$ we have:

(1) $l_\alpha \geq n_\alpha$.

(2) $p_i{}^\alpha = q_i{}^\alpha$ for $1 \leq i \leq n_\alpha$.

(3) $f_i{}^\alpha \supseteq g_i{}^\alpha$ for $1 \leq i \leq n_\alpha$.

(4) $p_i{}^\alpha \in B^\alpha$ and $H^\alpha(p_i{}^\alpha) \subseteq f_i{}^\alpha$ for $n_\alpha < i \leq l_\alpha$.

(5) $A^\alpha \subseteq B^\alpha$.

(6) For all $p \in A^\alpha$, $G^\alpha(p) \supseteq H^\alpha(p)$.

The idea behind Apter's notion of forcing $\mathbb{P}$ is that the least ordinal α_0 in A gets changed into $\aleph_\omega$ and that the $\delta_i{}^{\alpha_0}$ form a cofinal ω–sequence below α_0 in exactly the same way that Magidor accomplished this. Thus $f_0{}^{\alpha_0}$ contains partial information about the collapse of $\delta_1{}^\alpha$ to ω_2, and $f_1{}^{\alpha_0}$ contains partial information about the collapse of all cardinals strictly between $\delta_1{}^{\alpha_0{}++}$ and $\delta_2{}^{\alpha_0}$.
If $\alpha \in A$ is larger than α_0, then α gets changed into $\sup\{\beta \in A: \beta < \alpha\}^{+\omega}$ (see definition of $f_0{}^\alpha$ for $\alpha > \alpha_0$), and this is why we need the restriction that $\sup\{\beta \in A: \beta < \alpha\} < \alpha$ for all $\alpha \in A$.

Given two conditions π, $\pi' \in \mathbb{P}$ such that $\pi \leq \pi'$, define

$$|\pi_\alpha - \pi_\alpha'| = \text{length}(\pi_\alpha) - \text{length}(\pi'_\alpha)$$

Note that since length(π_α) = 1 for all but finitely many $\alpha \in A$, it follows that $|\pi_\alpha - \pi'_\alpha| = 0$ for all but finitely many $\alpha \in A$. We may also define the notions of j–direct and j–length preserving extensions in the same way as before (see Definitions 4.2.8.and 4.2.9). In what follows, however, we shall mainly be concerned with 0–interpolants (which we shall simply call *interpolants*). Thus if $\pi \leq \pi''$, then the interpolant of π_α and π''_α is a condition $\pi'_\alpha = \text{int}(\pi_\alpha, \pi''_\alpha)$ such that $\pi_\alpha \leq \pi'_\alpha \leq \pi''_\alpha$ and

(1) π'_α is a 0–direct extension of π''_α (determined by some $q_1 \dots q_n$).

(2) π_α is a 0–length preserving extension of π'_α.

Analogous to Lemma 4.2.14 we have:

Lemma 4.3.17 [Apter 1984]:

Let $\pi = (p_1{}^\alpha \dots p^\alpha_{l_\alpha}, f_0{}^\alpha \dots f^\alpha_{l_\alpha}, A^\alpha, G^\alpha)_{\alpha \in A}$, *and let* φ *be a formula of the forcing language associated with* $\mathbb{P}$. *Then there is a condition* $\pi' \in \mathbb{P}$ *such that*

(1) $|\pi'_\alpha - \pi_\alpha| = 0$ *for all* $\alpha \in A$.

(2) *If* $\pi'' \leq \pi'$ *and* π'' *decides* φ, *then so does* $((\pi'_\alpha\!: \alpha<\beta), (\text{int}(\pi''_\alpha,\pi'_\alpha)\!: \alpha \geq \beta)))$, *where* β *is the last coordinate such that* $|\pi''_\beta - \pi'_\beta| \neq 0$ (*and it decides* φ *in the same way*).

For a proof of Lemma 4.3.17, we refer the reader to [Apter 1984]. The conditions in $\mathbb{P}$ are more complicated than those of Section 4.2, but allow a simplification which is similar to Lemma 4.1.1 for Prikry forcing.

Lemma 4.3.18 [Apter 1984]:

Let $\pi \in \mathbb{P}$, *where* $\pi = (p^\alpha_1 \dots p^\alpha_{l_\alpha}, f^\alpha_0 \dots f^\alpha_{l_\alpha}, A^\alpha, G^\alpha)_{\alpha \in A}$, *and let* φ *be a formula of the forcing language associated with* $\mathbb{P}$. *Then there is* $\chi \leq \pi$ *such that:*

(1) *For all* $\alpha \in A$, $|\chi_\alpha - \pi_\alpha| = 0$;

(2) χ *decides* φ.

With the aid of the above technical machinery, we may sketch a proof of Theorem 4.3.14.

Recall that for $\alpha \in A$, $\mathbb{P}_\alpha = \{\pi | \alpha\!: \pi \in \mathbb{P}\}$, and that $\mathbb{P}^\alpha = \{\pi | A - \beta\!: \pi \in \mathbb{P}\}$. Clearly then $\mathbb{P} \cong \mathbb{P}_\alpha \times \mathbb{P}^\alpha$ for all $\alpha \in A$. Suppose that G is M–generic on $\mathbb{P}$; then for each $\alpha \in A$, G_α is M–generic on $\mathbb{P}_\alpha$, and $M[G] = \bigcup_{\alpha \in A} M[G_\alpha]$ (Note that $\mathbb{P}$, G are proper classes, so we have to take some care seeing that M[G] is a model of ZFC – see Section 2.3). Just as in the proof of Theorem 4.2.1, too many cardinals are collapsed in the model M[G], and we shall be interested in a certain submodel $\overline{M}$ of M[G]. For each $\alpha \in A$, the sequence $(\delta_n^{\alpha}: n < \omega)$ is a generic ω–sequence cofinal below α, and $(F_n^{\alpha}: n < \omega)$ is a generic sequence of Lévy collapsing functions, where:

(1) $F_0^{\alpha_0}$ is generic over $\mathrm{Coll}(\omega_1, < \delta_1^{\alpha_0})$, $\alpha_0 = \min(A)$.

(2) For $\alpha \in A$, $\alpha > \alpha_0$, F_0^{α} is generic over $\mathrm{Coll}(\sup(\{\beta \in A: \beta < \alpha\})^{+++}, < \delta_1^{\alpha})$.

(3) For $n \geq 1$, F_n^{α} is generic over $\mathrm{Coll}(\delta_n^{\alpha++}, < \delta_{n+1}^{\alpha})$.

Let $M^{\alpha} = M[((\delta_n^{\beta}: n < \omega), (F_n^{\beta}: n < \omega))_{\beta \in A \cap \alpha}]$ for each $\alpha \in A$, and let $\overline{M} = \bigcup_{\alpha \in A} M^{\alpha}$.

Lemma 4.3.19 [Apter 1984]:

In $\overline{M}$, *the* ω^{th} *successor of any cardinal is a strong limit cardinal that violates the* GCH.

Proof: Suppose that δ is a cardinal in $\overline{M}$. Let $\lambda = \sup\{\alpha \in A: \alpha \leq \delta\}$, and let β be the least member of A which is greater than δ. We shall show that

$$\overline{M} \vdash \delta^{+\omega} \text{ is strong limit and violates the GCH}$$

Let M_1 be the model $M[(\delta_n^{\beta}: n < \omega), (F_n^{\beta}: n < \omega)]$. As in the proof of Theorem 4.2.1, we see that $M_1 \vdash$ "$\beta = \lambda^{+\omega}$ is a strong limit cardinal that violates the GCH". Since $\lambda \leq \delta < \beta$, $M_1 \vdash$ "$\delta^{+\omega}$ is a strong limit cardinal that violates the GCH". We must show that this behaviour is preserved in $\overline{M}$. Let β' be the least element of A that is greater than β. Suppose that x is a subset of $(\beta^{++})^V$ in $V[G^{\beta'}]$ and let $\dot{x}$ be a name for x in the forcing language associated with $\mathbb{P}^{\beta'}$. Choose $\pi_0 \in \mathbb{P}^{\beta'}$ such that $\pi_0 \Vdash \dot{x} \subseteq \beta^{++}$, and inductively define a descending sequence $(\pi_{\alpha}: \alpha < \beta^{++})$ of $\mathbb{P}^{\beta'}$–conditions such that each π_{α} either forces "$\check{\alpha} \in \dot{x}$", or "$\check{\alpha} \notin \dot{x}$", and such that $|\pi_{\alpha+1} - \pi_{\alpha}| = 0$ for all $\alpha < \beta^{++}$. This is possible by Lemma 4.3.18. Let $\pi = \bigcup\{\pi_{\alpha}: \alpha < \beta^{++}\}$.

Because each $f_0^{\beta} \in \mathrm{Coll}(\sup(\{\alpha \in A: \alpha < \beta\})^{+++}, < \delta_1^{\beta})$, it follows that π is a condition in $\mathbb{P}^{\beta'}$. Thus $x = \{\alpha < \beta^{++}: \pi \Vdash \check{\alpha} \in \dot{x}\} \in M$. It follows that all the subsets of $(\beta^{++})^M$ in

$M[G^{\beta'}]$ are already in M, i.e. $\mathbb{P}^{\beta'}$ adds no new subsets of $(\beta^{++})^M$.
Consider now the model

$$M^1 = \bigcup_{\alpha \in A - \beta'} M[((\delta_n^{\gamma}: n < \omega), (F_n^{\gamma}: n < \omega))_{\beta' \leq \gamma < \alpha}].$$

Note that $M^1 \subsetneq M[G^{\beta'}]$, and thus that every subset of $(\beta^{++})^M$ in M^1 is already in M.
It follows that

$$M^1 \vdash \mathcal{U}^{\beta} \text{ is a normal fine measure over } [\beta^+]^{<\beta} \text{ and } 2^{\beta} = \beta^{++}$$

Thus because the sequences $(\delta_n^{\beta}: n < \omega)$, $(F_n^{\beta}: n < \omega)$ are also M^1–generic, it follows that the model $M^2 = M^1[((\delta_n^{\beta}: n < \omega),(F_n^{\beta}: n < \omega))]$ has

$$M^1 \vdash "\beta = \delta^{+\omega} = \lambda^{+\omega} \text{ is a strong limit cardinal that violates the GCH."}$$

[Note that M^2 includes both M_1 and M^1]. Now if we force with $\mathbb{P}_{\beta}$ over M^2, then since $|\mathbb{P}_{\beta}| \leq (\lambda^{++})^M$, we must have

$$M^2[G_{\beta}] \vdash "\delta^{+\omega} \text{ is a strong limit cardinal that violates the GCH."}$$

But $M^3 = M^2[((\delta_n^{\alpha}: n < \omega), (F_n^{\alpha}: n < \omega))_{\alpha \in A \cap \beta}] \subsetneq M^2[G_{\beta}]$, and so

$$M^3 \vdash "\delta^{+\omega} \text{ is a strong limit cardinal that violates GCH."}$$

If one considers the construction of M^3, it follows by a standard product forcing lemma that $M^3 = \overline{M}$, completing the proof.

□

In order to complete the proof of Theorem 4.3.14, we need the following well–known lemma, the proof of which may be found in [Jech 1978] on p.97. The basic rudimentary functions are defined in Appendix 1.1.

Lemma 4.3.20 [Hajnal 1956]:
Suppose that $\mathcal{M}$ is a transitive proper class which is closed under the basic rudimentary functions, and moreover has the following property:

Whenever $X \subsetneq \mathcal{M}$, there is a $Y \in \mathcal{M}$ such that $X \subsetneq Y$.

Then $\mathcal{M}$ is a model of ZF.

Proof of Theorem 4.3.14:

First note that every limit cardinal of $\overline{M}$ is strong limit: If $\lambda \in \overline{M}$ is a limit cardinal, and $\delta < \lambda$, then $\delta^{+\omega} \leq \lambda$ and $\delta^{+\omega}$ is strong limit by Lemma 4.3.19. It follows that $2^{\delta} < \delta^{+\omega} \leq \lambda$. Hence λ is strong limit as well. Lemma 4.3.19 also ensures that GCH fails at every ω^{th} successor cardinal. So it remains to show that $\overline{M}$ is a model of ZFC.

The only reason why this is not obvious is that $\mathbb{P}$ is a proper class notion of forcing. All axioms of ZFC excepting the Power Set Axiom and the Axiom of Replacement will hold in $\overline{M}$ (the proof is as for generic models using a set notion of forcing). In particular, $\overline{M}$ will be closed under the basic rudimentary functions and will satisfy the Axiom of Choice. Since $\overline{M}$ is a transitive proper class of M[G], it follows that we may apply Lemma 4.3.20. Now by definition, $\overline{M} = \bigcup_{\alpha \in A} M^{\alpha}$, where $M^{\alpha} = M[((\delta_n^{\beta}: n < \omega), (F_n^{\beta}: n < \omega))_{\beta \in A \cap \alpha}]$

Clearly $\alpha < \gamma$ implies $M^{\alpha} \subseteq M^{\gamma}$. Suppose now that $X \subseteq \overline{M}$. Since A is a proper class, there exists an $\alpha \in A$ such that $X \subseteq M^{\alpha}$. Since M^{α} is a proper class, there must be an ordinal β such that $X \subseteq (V_{\beta})^{M^{\alpha}}$ (i.e. $X \subseteq$ the β^{th} level of the cumulative hierarchy in M^{α}). Now $(V_{\beta})^{M^{\alpha}} \in (V_{\beta+1})^{M^{\alpha}} \subseteq M^{\alpha}$, and thus $X \subseteq (V_{\beta+1})^{M^{\alpha}} \in M^{\alpha} \subseteq \overline{M}$. By Lemma 4.3.20 and the fact that the Axiom of Choice holds in $\overline{M}$, $\overline{M} \models$ ZFC.

□

Shelah ([Shelah 1983]) generalized Magidor's work in another direction. In [Magidor1977b], Magidor proved that it is consistent that $\aleph_{\omega}$ is strong limit and that $2^{\aleph_{\omega}} = \aleph_{\omega+k}$ for any $k < \omega$ (This is Theorem 4.2.1). In the same paper, Magidor also indicated how to make $2^{\aleph_{\omega}}$ into $\aleph_{\omega+\omega+1}$, a little larger. The question thus arises: How large can $2^{\aleph_{\omega}}$ be, assuming that $\aleph_{\omega}$ is strong limit? It is this question which has been addressed by Shelah. He starts with a model V in which there is a cardinal κ which is λ_n–supercompact for some increasing sequence $(\lambda_n: n < \omega)$, and such that the λ_n–supercompactness of κ is preserved under any κ–directed closed notion of forcing. This is a much weaker assumption than the assumption of a supercompact cardinal. For suppose that κ is a supercompact cardinal. By a theorem of Laver ([Laver 1978]) there is a notion of forcing $\mathbb{P}$ of cardinality κ and satisfying the κ–chain condition such that forcing with $\mathbb{P}$ preserves the supercompactness of κ, and moreover has the following property: In the generic extension that results when forcing with $\mathbb{P}$, no subsequent forcing with a κ–directed closed notion can destroy the super – compactness of κ.

Theorem 4.3.21 [Shelah 1983]:
Assume that there is a transitive model V *of* ZFC *in which* κ *is* λ_n*–supercompact for* $n < \omega$, $\lambda_n < \lambda_{n+1}$, *and that moreover the* λ_n*–supercompactness of* κ *is preserved under any subsequent* κ*–directed closed forcing. Then the following statements are consistent:*

$$\aleph_\omega \text{ is strong limit and } 2^{\aleph_\omega} = \aleph_{\alpha+1}, \text{ for any } \alpha < \omega_1.$$

Shelah's notion of forcing also resembles the notion of forcing in Section 4.2. (due to Magidor) to a great extent and therefore we shall not discuss this work here. We will return to another aspect of Shelah's work on the possible size of $2^{\aleph_\omega}$ in Chapter 6.

There have recently been other approaches to the singular cardinals problem, using notions of forcing which do not resemble that of Magidor's. For example, [Foreman–Woodin 1991] have obtained a model in which $2^\kappa > \kappa^+$ for all cardinals κ, assuming the existence of a supercompact cardinal with infinitely many inaccessible cardinals above it. This is weaker, than, say, the existence of 2 supercompact cardinals.

Theorem 4.3.22 [Foreman–Woodin 1992]:
Let κ *be a supercompact cardinal with infinitely many inaccessible cardinals above* κ. *Then there is a notion of forcing* $\mathbb{P}$ *so that in the generic extension,*

$$V_\kappa \models \text{ZFC} \wedge \text{for all cardinals } \lambda,\ 2^\lambda > \lambda^+$$

In the same paper, it is asserted that Woodin has obtained the following improvement:

Theorem 4.3.23 (Woodin):
If there is a supercompact cardinal, then there is a model of ZFC *in which* $2^\kappa = \kappa^{++}$ *for all cardinals* κ.

In this model, the SCH fails at every singular cardinal, since every limit cardinal is strong limit. Another interesting result, which requires weaker large cardinal hypotheses, is the following theorem due to Cummings. A cardinal κ is $\mathcal{P}_3\kappa$–hypermeasurable if and only if there is a transitive model M such that $V_{\kappa+3} \subseteq M$ and an elementary embedding j: V ⟶ M of the universe with critical point κ.

Theorem 4.3.24 [Cummings 1992]:

If there exists a $\mathcal{P}_3\kappa$–hypermeasurable cardinal, then there is a model of ZFC *in which* $2^\lambda = \lambda^+$ *if* λ *is a successor cardinal, and in which* $2^\lambda = \lambda^{++}$ *if* λ *is a limit cardinal.*

Here, too, every limit cardinal is strong limit, and so the SCH fails at every singular cardinal, but yet the power function attains the smallest possible value at regular cardinals. Theorem 4.3.22 and 4.3.24 were both proved using the method of Radin forcing, which allows one to add a club set to a large ($\mathcal{P}_3\kappa$–hypermeasurable) cardinal generically in such a way that κ remains measurable in the generic extension ([Radin 1982]).

Various conjectures now spring to mind. It is known for instance, that if α is an ordinal and if $2^{\aleph_\gamma} = \aleph_{\gamma+\alpha}$ for all ordinals γ, then $\alpha < \omega$ (Choose β least such that $\beta + \alpha > \alpha$. Then $0 < \beta \leq \alpha$, and β is clearly a limit ordinal if α is infinite. Let $\kappa = \aleph_{\beta+\beta}$. Then κ is a singular cardinal of cofinality cf(β), and by hypothesis $2^\kappa = \aleph_{\beta+\beta+\alpha} > \aleph_{\beta+\alpha}$. Now for all $\xi < \beta$, we have $2^{\aleph_{\beta+\xi}} = \aleph_{\beta+\xi+\alpha} = \aleph_{\beta+\alpha}$ by choice of β. Thus the power function is eventually constant below $\kappa = \aleph_{\beta+\beta}$. By Theorem 1.2.4, therefore, we also have $2^\kappa = \aleph_{\beta+\alpha}$, a contradiction). The next question to ask is:

Can we have GCH at all successor cardinals, but $2^\lambda = \lambda(+\omega+1)$ at all limit cardinals λ? We don't know if this problem has been solved.

§ 4.4 The Strength of the Failure of the SCH

The purpose of this section is to discuss, largely without proof, the strength of the failure of the Singular Cardinals Hypothesis. We have already seen in Section 4.1 that $\neg$SCH is consistent with ZFC, assuming the existence of a cardinal with a certain degree of supercompactness. In this section we shall endeavour to show that $\neg$SCH may be regarded as a large cardinal axiom. In particular, we shall discuss work of Jensen ([Devlin–Jensen 1975]) that if the SCH fails, then $0^{\#}$ exists (see Appendix 1.2 for a definition of $0^{\#}$). It turns out that the the failure of the SCH implies the consistency of the existence of measurable cardinals (due to [Dodd–Jensen 1982]). This establishes the large cardinal character of $\neg$SCH. We shall then give a brief description of work by Magidor and Gitik ([Gitik 1989, 1991]) which determines the exact consistency strength of $\neg$SCH as a large cardinal axiom (see Theorem 4.4.8).

Definition 4.4.1: Suppose that M, N are two inner models of ZFC such that $M \subseteq N$. We shall say that N has the *covering property with respect to* M if and only if M is, in N, a definable class, and whenever $N \models X$ is an uncountable set of ordinals, then there is a $Y \in M$ such that $X \subseteq Y$ and $N \models |X| = |Y|$.

If N has the covering property with respect to $M \subseteq N$, then M and N have quite similar cardinal structures, and the power function in N will inherit some of the properties of the power function in M.

Lemma 4.4.2:
Assume that N *has the covering property with respect to* $M \subseteq N$. *Then any singular cardinal* κ *in* N *will be singular in* M *as well. Moreover,* $(\kappa^+)^N = (\kappa^+)^M$.

Proof: Suppose that κ is a singular cardinal in N, and let X be a cofinal subset of κ with the property that $|X| < \kappa$. Choose $Y \in M$ such that $N \models$ "$X \subseteq Y$ and $|X| = |Y|$". Then Y is a cofinal subset of κ with cardinality $< \kappa$ in M, proving that κ is singular in M as well.
In order to prove that $(\kappa^+)^N = (\kappa^+)^M$, it suffices to show that $(\kappa^+)^N \leq (\kappa^+)^M$ because $M \subseteq N$. Suppose therefore that $\alpha = (\kappa^+)^M < (\kappa^+)^N$. Then α is an ordinal of cardinality κ

in N, and thus has cofinality $< \kappa$. Reasoning as above, we see that α must have cofinality $< \kappa$ in M as well, contradicting that $\alpha = (\kappa^+)^M$ is a regular cardinal in M.

□

Lemma 4.4.3:
Suppose that N *has the covering property with respect to* $M \subseteq N$, *and that* $M \vdash$ GCH.
Then $N \vdash$ SCH.

Proof: Let κ be a singular cardinal in N such that $2^{cf(\kappa)} < \kappa$. We must show that $\kappa^{cf(\kappa)} = \kappa^+$ in N. We shall do this by proving that $|[\kappa]^{cf\kappa}| = \kappa^+$ in N, where $[\kappa]^{cf\kappa}$ is the set of all subsets of κ of cardinality cf(κ). Suppose that

$$N \vdash X \subseteq \kappa \text{ and } |X| = cf(\kappa).$$

Then there exists a $Y \in M$ such that $X \subseteq Y \subseteq \kappa$ and $N \vdash |Y| = cf(\kappa)$. Now M satisfies the GCH, and so there are exactly $(\kappa^{cf\kappa})^M = (\kappa^+)^M$ such Y. On the other hand for each Y there are at most $(2^{cf\kappa})^N$–many X in N such that X covered by Y. Thus

$$N \vdash |[\kappa]^{cf\kappa}| \leq 2^{cf\kappa} \cdot |(\kappa^+)^M|$$

By Lemma 4.3.2 and the fact that $2^{cf\kappa} < \kappa$, it follows that $N \vdash \kappa^{cf\kappa} = \kappa^+$.

□

We shall now give the statement of the Covering Lemma for L, due to Jensen. Recall that V denotes the universe and L the constructible universe.

Lemma 4.4.4 (The Covering Lemma for L) [Devlin–Jensen 1975]:
Suppose that $0^{\#}$ *does not exist. Then* V *has the covering property with respect to* L.

A proof of the Covering Lemma for L falls outside the scope of this dissertation. Jensen's original proof depended heavily on the *fine structure theory* for the constructible universe, which he himself developed. ([Jensen 1972]), although other proofs of a more elementary nature have been found since, e.g. by Magidor, using not much more than the notion of an elementary substructure, and another that uses the so–called Silver machines. Both are discussed in [Devlin 1984].

As an immediate consequence we have the following theorem (refer to [Devlin–Jensen 1975]):

Theorem 4.4.5 (Jensen):

If the Singular Cardinals Hypothesis fails, then $0^{\#}$ *exists.*

Proof: If $0^{\#}$ does not exist, then V has the covering property with respect to L by Lemma 4.4.5. Since L ⊢ GCH, it follows that V ⊢ SCH by Lemma 4.4.3.

□

Note that the Covering Lemma is fails badly if $0^{\#}$ does exist: In that case $\aleph_\omega$ is inaccessible in L (by Theorem A.1.15, because $\aleph_\omega$ is a Silver indiscernible), so the countable set $\{\aleph_n: n < \omega\}$ can only be covered by a constructible set of cardinality at least $\aleph_\omega$. One obtains the following nice characterization:

$$0^{\#} \text{ exists} \longleftrightarrow \aleph_\omega \text{ is regular in L}$$

Firstly, if $0^{\#}$ exists, then $\aleph_\omega$ is regular in L because it is a Silver indiscernible. On the other hand, if $\aleph_\omega$ is regular in L, then the covering property for V and L fails by Lemma 4.4.2. It follows by Lemma 4.4.4 that in that case $0^{\#}$ exists. Another characterization is that the existence of $0^{\#}$ is equivalent to the existence of non–trivial elementary embeddings of the constructible universe (Theorem A.1.15), which is already sufficient to give $0^{\#}$ a large cardinal character. [Dodd–Jensen 1981] have taken these results a step further by inventing the Core Model K. The core model is the most important inner model of set theory discovered since Gödel's constructible universe. It brings together the seemingly incompatible techniques of iterated ultrapowers and fine structure theory, and is essentially the largest inner model without any measurable cardinals. Suppose that $\mathcal{U}$ is a normal measure over a cardinal κ in some inner model of ZFC. One may take iterated ultrapowers of $L[\mathcal{U}]$ to obtain models $L[\mathcal{U}_\alpha]$, and it turns out that $K = \bigcap_{\alpha \in On} L[\mathcal{U}_\alpha]$. Dodd–Jensen define K in a way which does not presuppose the existence of measurable cardinals, however, by using iterations of the #– operator. One may define $x^{\#}$ for any set x (assuming sufficiently large cardinals) and then obtain elementary embeddings of L[x]. K is the smallest inner model of ZFC that is closed under the #–operator, i.e. if $x \in K$ and $x^{\#}$ exists, then $x^{\#} \in K$ as well. It follows immediately that, unlike L, K is not absolute for inner models of ZFC. For instance, if $0^{\#}$ does not exist, then $K = L$, whereas if $0^{\#}$ exists, but $0^{\#\#}$ does not, then $K = L[0^{\#}]$. K will always be a model of GCH, however, and have fairly uniform properties in that one may develop a fine structure theory for K as well (see [Dodd 1982] for a development thereof). One may then prove that if there are no inner models of ZFC with measurable cardinals, then V has the covering property with respect to K, so that in that case V satisfies the SCH. We will not get into the exact definition of K, which is rather complicated to say the least, but instead recommend the monograph by Dodd ([Dodd 1982]) for a comprehensive view. As a consequence of the above discussion and Lemma 4.4.3, we have:

Theorem 4.4.6 [Dodd–Jensen 1982]:
If the SCH *fails, then there exists an inner model of* ZFC + $\exists\kappa$(*κ is measurable*).

Theorem 4.1.5 and 4.4.6 together imply that ¬SCH has a large cardinal character, and place an upper and a lower bound on ¬SCH as a large cardinal axiom: The consistency of ¬SCH is weaker than the consistency of a supercompact cardinal, but stronger than the consistency of a measurable cardinal. The gap between a supercompact and a measurable cardinal is quite large, however, and a more precise formulation of the exact strength of the failure of the SCH has been given by Gitik ([Gitik 1991]), using techniques devised by Mitchell and Shelah.

Definition 4.4.7 [Mitchell 1974]: Suppose that $\mathcal{U}$ and $\mathcal{V}$ are normal measures over a cardinal κ. Define a partial ordering $<_M$ by: $\mathcal{U} <_M \mathcal{V} \longleftrightarrow \mathcal{U} \in \mathrm{Ult}(V,\mathcal{V})$.
$<_M$ is called the *Mitchell ordering.*

Alternatively, an equivalent definition is: $\mathcal{U} <_M \mathcal{V}$ if and only if there is a set $I \in \mathcal{V}$ and a sequence $(\mathcal{U}_\iota : \iota \in I)$ such that each $\mathcal{U}_\iota$ is a normal measure over ι, and for all $x \in \kappa$,

$$x \in \mathcal{U} \longleftrightarrow \{\iota \in I : x \cap \iota \in \mathcal{U}_\iota\} \in \mathcal{V}.$$

One may show that $\leq_M$ is a *well–founded* partial ordering ([Mitchell 1974]), and thus we may inductively define a rank function $o(\mathcal{U})$ on normal measures over κ by:

$$o(\mathcal{U}) = \sup\{o(\mathcal{V}) + 1 : \mathcal{V} <_M \mathcal{U}\}$$

If κ is a cardinal, we then define $o(\kappa) = \{o(\mathcal{U}) : \mathcal{U}$ a normal measure over $\kappa\}$.

One can easily see that for any cardinal κ, we have $o(\kappa) \leq (2^{\kappa})^{+}$: If $\mathcal{U}$ is any normal measure over κ, then $|\mathcal{P}(\mathcal{P}(\kappa)) \cap \mathrm{Ult}(V,\mathcal{U})| < (2^{\kappa})^{+}$ and so $o(\mathcal{U}) < (2^{\kappa})^{+}$. Hence indeed $o(\kappa) \leq (2^{\kappa})^{+}$. In models of ZFC + GCH, we thus have $o(\kappa) \leq \kappa^{++}$ always. Gitik proved the following theorem:

Theorem 4.4.8 [Gitik 1989, 1991]:

(a) Con(ZFC + $\exists\kappa(o(\kappa) = \kappa^{++})$) *implies* Con(ZFC + $\aleph_\omega$ *is strong limit* + $2^{\aleph_\omega} = \aleph_{\omega+2}$)

(b) *The strength of* $\neg$SCH *is at least* "$\exists\kappa(o(\kappa) = \kappa^{++})$".

(a) and (b) of Theorem 4.4.8 together provide the *exact* consistency strength of $\neg$SCH:

$$\mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{SCH}) \longleftrightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa(o(\kappa) = \kappa^{++}))$$

In this chapter we want to get some bounds on the power function using the technique of generic ultrapowers, which is decribed in Appendix 4. Essentially this involves generically adding an ultrafilter with which an ultrapower of the ground model may be taken in the generic extension. This was the method used by Silver ([Silver 1974]) to prove that if κ is a singular cardinal of uncountable cofinality and GCH holds below κ, then it holds at κ (Thm 1.3.3). Jensen, Baumgartner and Prikry later gave elementary proofs of this theorem using almost disjoint transversals (see [Baumgartner–Prikry 1976]), and it is this method which we used in Chapter 1. We shall present proof of Theorem 1.3.3 using generic ultrapowers in Section 1 (Thm. 5.1.1). There we shall also present some similar results due to [Jech–Prikry 1979], Theorem 5.1.3 being a good example. The notion of a "nice cardinal function" is introduced, and it allows us to prove a host of theorems about the power function (Thm. 5.1.15. and Thm. 5.1.16). In Section 2 we shall focus once again on the singular cardinals problem. A "defect" of Theorem 1.3.4 is that it gives no information about 2^{κ} if κ is a strong limit cardinal of uncountable cofinality with $\kappa = \aleph_{\kappa}$. This situation is remedied somewhat in Section 2, using the notion of nice cardinal functions: See Thm. 5.2.4 and the ensuing discussion. Most of the material presented here is owed to [Jech– Prikry 1979]. In Section 3 we shall discuss a result due to Matsubara ([Matsubara 1992]) which shows that the existence of certain ideals implies that the Singular Cardinals Hypothesis holds "locally" (i.e. on an interval of cardinals). This result will be used to obtain another proof of Solovay's Theorem 3.3.1.

§ 5.1 Saturated ideals and the GCH

The material in this section depends heavily on Appendix 4. Recall that if κ is a regular cardinal and $\mathcal{I}$ an ideal over κ, we may form the quotient Boolean algebra $\mathcal{P}(\kappa)/\mathcal{I}$ by identifying two sets if their symmetric difference is an element of $\mathcal{I}$. $\mathcal{I}$ is said to be λ–saturated if $\mathcal{P}(\kappa)/\mathcal{I}$ is λ–saturated (see Definition A.4.4). Using $\mathcal{P}(\kappa)/\mathcal{I}$ as a notion of forcing in the ground model V, we may adjoin a generic ultrafilter G on $\mathcal{P}(\kappa) \cap$ V. G is an M–κ–complete M–ultrafilter (see Definition A.4.5) which carries the same combinatorial properties as $\mathcal{I}$ (Lemma A.4.7). $\mathcal{I}$ is said to be precipitous if the ultrapower of V modulo G is well–founded. Jech and Prikry first arrived at the notion of precipitous ideals specifically when trying to extend Theorem 1.3.4 to cardinals of the type $\alpha = \aleph_\alpha$ ([Jech–Prikry 1976]) and proved that the existence of a precipitous ideal over $\aleph_1$ is equiconsistent to the existence of a measurable cardinal ([Jech–Magidor–Mitchell–Prikry 1980]). Thus the notion of precipitousness allows us to transport large cardinal properties to small cardinals (such as $\aleph_1$). In view of the fact that large cardinal properties often imply important consequences for the power function, it is not unexpected that the same turns out to be true for saturation– and precipitousness properties.

Recall that if $\mathcal{U}$ is an M–ultrafilter over κ (see Definition A.4.5), and if $x \in \mathrm{Ult}(M,\mathcal{U})$, then $\mathrm{ext}(x) = \{y \in \mathrm{Ult}(M,\mathcal{U}): y \in_{\mathcal{U}} x\}$, where $y \in_{\mathcal{U}} x$ if and only if $\{\alpha < \kappa: y(\alpha) \in x(\alpha)\} \in \mathcal{U}$. If j is the elementary embedding of M into $\mathrm{Ult}(M,\mathcal{U})$, then for any ordinal ν we have $|\mathrm{ext}(j(\nu)| \leq |\nu^\kappa|^M$ since every $x < j(\nu)$ in $\mathrm{Ult}(M,\mathcal{U})$ is represented by some $f:\kappa \longrightarrow \nu$ in M. Suppose that $f \in M$ is an ordinal valued function on κ. Let $f^+ \in M$ be the ordinal valued function on κ defined by $f^+(\alpha) = (f(\alpha))^+$ (where for any ordinal α, α^+ is the least cardinal larger than α). It is not hard to see that if f is an ordinal valued function on κ in M, then

$$|\mathrm{ext}[f^+]| \leq |\mathrm{ext}[f]|^+$$

for if x is such that $[x] \in_{\mathcal{U}} [f^+]$, then $\{\alpha < \kappa: x(\alpha) \in f^+(\alpha)\} \in \mathcal{U}$, and hence $\{\alpha < \kappa: \exists$ 1–1 map from $x(\alpha)$ to $f(\alpha)\} \in \mathcal{U}$. Thus in $\mathrm{Ult}(M,\mathcal{U})$, there is a 1–1 map from [x] into [f], which is easily seen to imply the existence of a 1–1 map from ext[x] into ext[f]. Hence $|\mathrm{ext}[x]| \leq |\mathrm{ext}[f]|$ for all x such that $[x] \in_{\mathcal{U}} [f^+]$, implying that $|\mathrm{ext}[f^+]| \leq |\mathrm{ext}[f]|^+$. This fact is very useful in proving the following theorem (a slightly weaker version of Theorem 1.3.3):

Theorem 5.1.1 [Silver 1974]:
Suppose that κ is a singular cardinal of uncountable cofinality such that for all cardinals $\alpha < \kappa$, $2^\alpha = \alpha^+$. Then $2^\kappa = \kappa^+$.

Proof: Suppose $\mathrm{cf}(\kappa) = \gamma \geq \omega_1$, and let $(\kappa_\alpha: \alpha < \gamma)$ be a continuous cofinal sequence converging to κ. Let M be the universe. Thus $M \vdash 2^{\kappa_\alpha} = \kappa_\alpha^+$ for all $\alpha < \gamma$. Let $\mathcal{I}$ be the ideal of thin subsets of γ in M, and let $e:\gamma \longrightarrow M$ be given by $e(\alpha) = \kappa_\alpha$ for all $\alpha < \gamma$. Let G be $\mathcal{I}$–generic, and in M[G], let N = Ult(M,G). By Łos' theorem, $N \vdash 2^{[e]} = [e^+]$.

Hence $|\text{ext } \mathcal{P}^N([e])| \le |\text{ext}[e^+]|$. Now $|\mathcal{P}^M(\kappa)| \le |\text{ext } \mathcal{P}^N([e])|$; to see this, let $f_X(\alpha) = X \cap \kappa_\alpha$ for each $X \in \mathcal{P}^M(\kappa)$. Then the map $\Psi:\mathcal{P}^M(\kappa) \longrightarrow \text{ext } \mathcal{P}^N([e])$ given by $\Psi(X) = [f_X]$ is clearly injective. If $[x] \in_G [e]$, then $Y_x = \{\alpha < \gamma: x(\alpha) \in \kappa_\alpha\} \in G$, and clearly also $Z = \{\alpha < \gamma: \kappa_\alpha \text{ is limit}\} \in G$. Thus $Y_x \cap Z \in G$, and for each $\alpha \in Y_x \cap Z$, there is $\gamma(\alpha) < \alpha$ such that $x(\alpha) < \kappa_{\gamma(\alpha)}$. By normality, there is an ordinal β such that $\gamma(\alpha) = \beta$ for all α in some set in G, and thus $[x] \in_G j(\kappa_\beta)$. Hence $[x] \in \text{ext } j(\kappa_\beta)$, and so $\text{ext}[e] \subseteq \bigcup_{\beta<\gamma} \text{ext } j(\kappa_\beta)$. Clearly, however, $|\text{ext } j(\kappa_\beta)| \le |(\kappa_\beta{}^\gamma)^M| < \kappa$, and therefore $|\text{ext}[e]| \le \kappa$.
Thus we have the following string of inequalities in M[G]

$$|(\kappa^+)^M| \le |\mathcal{P}^M(\kappa)| \le |\text{ext } \mathcal{P}^N([e])| = |\text{ext } 2^{[e]}| \le |\text{ext}[e^+]| \le |\text{ext}[e]|^+$$

where the latter inequality was proved earlier. It follows that $M[G] \vDash (2^\kappa)^M \le \kappa^+$. However, our notion of forcing (the set of all stationary subsets of γ with inclusion) has the $(2^\gamma)^+$–c.c. in M, and thus κ and κ^+ are preserved. Hopping back to the universe M, we see that $M \vDash 2^\kappa = \kappa^+$ as required.

□

Lemma 5.1.2 [Jech–Prikry 1979]:
Suppose that κ is a regular cardinal which carries a weakly normal λ–saturated ideal where $\lambda > \kappa$, and assume further that for all $\alpha < \kappa$, $2^\alpha = \alpha^+$.
Then $2^\kappa \le \lambda$.

Proof: Let $\mathcal{I}$ be the postulated weakly normal λ–saturated ideal over κ in the universe M, and let G be an $\mathcal{I}$–generic filter. Then G is weakly normal (by Lemma A.4.7). Let $N = \text{Ult}(M,G)$, and let $j:M \longrightarrow N$ be the canonical elementary embedding. Since G is weakly normal, $\kappa \in N$. Now $M \vDash \forall\, \alpha < \kappa(2^\alpha = \alpha^+)$ and thus $N \vDash \forall\, \alpha < j(\kappa)(2^\alpha = \alpha^+)$. In particular, $N \vDash 2^\kappa = \kappa^+$.
If $X \subseteq \kappa$ in M, then $j(X) \cap \kappa = X$, and so $\mathcal{P}^M(\kappa) \subseteq \mathcal{P}^N(\kappa)$, and thus in M[G] we have: $|(2^\kappa)^M| \le |\text{ext}(\kappa^+)^N| \le \kappa^+$, i.e. $|(2^\kappa)^M| \le (\kappa^+)^{M[G]}$. Since $\mathcal{I}$ is λ–saturated and $\lambda > \kappa$,

$(\kappa^+)^{M[G]} \leq \lambda$ and so $M[G] \vdash |(2^\kappa)^M| \leq \lambda$. Since all cardinals $\geq \lambda$ are preserved, the ordinal $(2^\kappa)^M$ cannot be larger than λ in M, and thus $M \vdash 2^\kappa \leq \lambda$ as required.

□

Since any κ^+–saturated ideal over κ is precipitous (By Lemma A.4.13), and thus weakly normal (by Lemma A.4.15), we immediately see:

Theorem 5.1.3 [Jech–Prikry 1979]:

If κ is a regular cardinal that carries a κ^+–saturated ideal, and if the GCH *holds below κ, then it does not fail at κ.*

Lemma 5.1.2 is easily generalized:

Lemma 5.1.4 [Jech–Prikry 1979]:

Assume that κ carries a weakly normal λ–saturated ideal $\mathcal{I}$ (where $\lambda > \kappa$), and that $2^\alpha \leq \alpha(+\beta)$ for all $\alpha < \kappa$, where $\beta < \kappa$. Then $2^\kappa \leq \lambda(+\beta)$. If $\beta < \omega$, then $2^\kappa < \lambda(+\beta)$.

Proof: As before, let G be $\mathcal{I}$–generic and let $j{:}M \longrightarrow N$ be the canonical elementary embedding of the universe M in the generic ultrapower. Since $M \vdash \forall\alpha < \kappa(2^\alpha \leq \alpha(+\beta))$, we have $N \vdash \forall\alpha < j(\kappa)\ (2^\alpha \leq \alpha(+\beta))$ (because $j(\beta) = \beta$), and since G is weakly normal, in particular we have $N \vdash 2^\kappa \leq \kappa(+\beta)$. Again $|(2^\kappa)^M| \leq |\mathrm{ext}(2^\kappa)^N|$ and so

$$M[G] \vdash |(2^\kappa)^M| \leq \kappa(+\beta).$$

Now $\kappa^+ \leq \lambda$, and thus $\kappa(+\beta) \leq \lambda(+\beta)$. (and if $\beta < \omega$, then $\kappa(+\beta) < \lambda(+\beta)$). Thus

$$M[G] \vdash |(2^\kappa)^M| \leq \lambda(+\beta)$$

(the inequality is strict if $\beta < \omega$) and thus by the λ–chain condition, it follows that $2^\kappa \leq \lambda(+\beta)$ in M (where the inequality is strict if $\beta < \omega$).

□

For the remainder of this section, we shall be concerned with $\kappa = \aleph_1$ only.

Lemma 5.1.6 [Jech–Prikry 1979]:

Let $\mathcal{U}$ *be a non–principal* M–κ–*complete* M–*ultrafilter over* $\kappa = \aleph_1^M$, *and let* j: M $\longrightarrow$ N *be the canonical elementary embedding of* M *in* N = Ult(M,$\mathcal{U}$). *Then in any model where* j *exists we have:*

(1) $|\mathrm{ext}(j(\kappa))| \leq \aleph_1$

(2) *If* $\delta > 1$, $|\mathrm{ext}\ j(\aleph_\delta^M)| \leq \aleph_\gamma$ *and* $\aleph_\delta^M \leq \xi < \aleph_{\delta+\alpha+1}^M$ *for some* $\alpha < \kappa$, *then* $|\mathrm{ext} j(\xi)| \leq \aleph_\gamma \cdot |\aleph_{\delta+\alpha}^M|$

Proof: (1) Let c_κ be the constant function with value κ. Thus $[c_\kappa] = j(\kappa)$. Clearly we have $[c_\kappa] = [c_\omega^+]$, and thus $|\mathrm{ext}\ j(\kappa)| = |\mathrm{ext}\ [c_\omega^+]| \leq |\mathrm{ext}\ [c_\omega]|^+ = \aleph_1$.

(2) By induction on ξ (for $\aleph_\delta^M \leq \xi < \aleph_{\delta+\kappa}^M$). If $\xi = \aleph_\delta^M$, then by hypothesis $|\mathrm{ext}\ j(\aleph_\delta^M)| \leq \aleph_\gamma$. Next suppose that $\xi < \aleph_{\delta+\kappa}^M$ and that ξ is not a cardinal in M. Thus there is $\alpha < \kappa$ such that $|\aleph_{\delta+\alpha}^M| = |\xi|^M < \xi$. Thus N $\models$ $|j(\xi)| = |j(\aleph_{\delta+\alpha}^M)|$. It follows that there is a 1–1 map from ext $j(\xi)$ into ext $j(\aleph_{\delta+\alpha}^M)$, and by induction hypothesis,

$$|\mathrm{ext}\ j(\aleph_{\delta+\alpha}^M)| \leq \aleph_\gamma \cdot |\aleph_{\delta+\alpha}^M|.$$

Thus also $|\mathrm{ext}\ j(\xi)| \leq \aleph_\gamma \cdot |\aleph_{\delta+\alpha}^M|$. Finally, suppose that ξ is a cardinal in M, i.e. assume $\xi = \aleph_{\delta+\alpha}^M$ for some $\alpha < \kappa$. We shall first show that $\mathrm{cf}(\xi) \neq \kappa$ in M: If α is a successor ordinal, then ξ is a successor cardinal in M, and thus regular. Since $\delta > 1$, we then necessarily have $\mathrm{cf}(\xi) > \kappa = \aleph_1^M$. If α is limit, then $\mathrm{cf}(\xi) = \mathrm{cf}(\alpha)$, and since $\alpha < \kappa$, we have $\mathrm{cf}(\xi) < \kappa$. Hence in either case, $\mathrm{cf}(\xi) \neq \kappa$. It follows that $j(\xi) = \lim_{\eta<\xi} j(\eta)$; ext $j(\xi)$ is therefore a linearly ordered set with a cofinal subset of cardinality $\aleph_{\delta+\alpha}^M$ and every $[f] \in \mathrm{ext}\ j(\xi)$ is already in some ext $j(\eta)$ $(\eta < \xi)$. By induction hypothesis, it follows that every initial segment of $j(\xi)$ has cardinality $\leq \aleph_\gamma \cdot |\aleph_{\delta+\beta}^M|$ for some $\beta < \alpha$, and thus $|\mathrm{ext}\ j(\xi)| \leq \aleph_\gamma \cdot |\aleph_{\delta+\alpha}^M|$ as required.

□

Part (2) of Lemma 5.1.6 looks very complicated. It is used primarily to find upper bounds for the cardinalities of the images under j of certain cardinals in M. Examples are:

Corollary 5.1.7 [Jech–Prikry 1979]:

(1) *If* $\delta > 1$, $|\text{ext } j(\aleph_\delta^M)| \le \aleph_\gamma$ *and* $\alpha < \kappa$, *then* $|\text{ext } j(\aleph_{\delta+\alpha}^M)| \le \aleph_\gamma \cdot |\aleph_{\delta+\alpha}^M|$.

(2) *If* $\alpha < \kappa$, *then* $|\text{ext } j(\aleph_\alpha^M)| \le \aleph_1 \cdot |\aleph_\alpha^M|$.

(3) *If* $\alpha < \kappa$, *then* $|\text{ext } j(\aleph_{\kappa+\alpha}^M)| \le \aleph_{\omega_1} \cdot |\aleph_{\kappa+\alpha}^M|$.

If the ultrapower is well–founded (e.g. If $\mathcal{U}$ is $\mathcal{I}$–generic for some precipitous ideal $\mathcal{I}$ over $\kappa = \aleph_1^M$ in M) we may identify ext(x) with x for each $x \in \text{Ult}(M,\mathcal{U})$. we then obtain the following:

Lemma 5.1.8 [Jech–Prikry 1979]:

If $\mathcal{U}$ *is an* M–κ–*complete* M–*ultrafilter over* $\kappa = \aleph_1^M$ *such that* $N = \text{Ult}(M,\mathcal{U})$ *is well–founded, then*:

(1) $j(\kappa) \le \aleph_1$;

(2) $j(\aleph_\kappa^M) \le \aleph_{\omega_1}$;

(3) *For any* $\alpha < \kappa$, $|j(\aleph_{\kappa+\alpha}^M)| \le \aleph_{\omega_1}$.

Proof: (1) $j(\kappa) = j(\aleph_1^M) = \aleph_1^N \le \aleph_1$.

(2) $j(\aleph_\kappa^M) = \aleph_{j(\kappa)}^N \le \aleph_{j(\kappa)} \le \aleph_{\omega_1}$ by (1) above.

(3) Since $j(\aleph_\kappa^M) \le \aleph_{\omega_1}$, it follows by Lemma 5.1.6(2) that $|j(\aleph_{\kappa+\alpha}^M)| \le \aleph_{\omega_1} \cdot |\aleph_{\kappa+\alpha}^M|$. Now $\kappa < j(\kappa) \le \aleph_1$, so κ is a countable ordinal. Thus also $|\aleph_{\kappa+\alpha}^M| < \aleph_{\omega_1}$.

□

With these combinatorial statements, we are going to prove that if $2^{\aleph_0} = \aleph_1$ and if $\aleph_1$ carries an $\aleph_2$–saturated ideal, then $2^{\aleph_1} = \aleph_2$. We need the following lemma to achieve this end.

Lemma 5.1.9 [Jech–Prikry 1979]:

Suppose that $\mathcal{I}$ is an ω_1–complete nonprincipal ideal over $\aleph_1$, let G *be $\mathcal{I}$–generic, and let* N = Ult(M,G), *where* M *is the universe. Then* $M[G] \vDash |(2^{\aleph_1})^M| \le |ext(2^{\aleph_0})^N|$.

Proof: Let d be the diagonal map on $\aleph_1$. Then $M \vDash \forall\alpha{<}\omega_1(d(\alpha)$ is countable), so [d] is a countable ordinal in N. The lemma will follow if we can demonstrate that there is an injective map of $\mathcal{P}(\aleph_1)$ into $\mathcal{P}^N([d])$. Given $X \subseteq \aleph_1$ in M, define f_X on ω_1 by:

$$f_X(\alpha) = X \cap \alpha$$

Then $M \vDash \forall\alpha{<}\omega_1(f_X(\alpha) \subseteq \alpha)$, so $N \vDash [f_X] \subseteq [d]$. The correspondence $X \longrightarrow [f_X]$ is necessarily injective, proving the lemma.

□

Theorem 5.1.10 [Jech–Prikry 1979]:

If $\aleph_1$ carries a λ–saturated ideal $\mathcal{I}$, where λ is a regular cardinal $> \aleph_1$, and if $2^{\aleph_0} < \aleph_{\omega_1}$, then $2^{\aleph_1} \le 2^{\aleph_0}\cdot\lambda$.

Proof: Choose $\alpha < \omega_1$ such that $2^{\aleph_0} = \aleph_\alpha$ in the universe M. Let G be $\mathcal{I}$–generic, and let N = Ult(M,G). Let $j{:}M \longrightarrow N$ be the canonical elementary embedding; then $M[G] \vDash |(2^{\aleph_1})^M| \le |ext(2^{\aleph_0})^N|$ by Lemma 5.1.9. Since $M \vDash 2^{\aleph_0} = \aleph_\alpha$, also $N \vDash 2^{\aleph_0} = \aleph_\alpha$, and thus $|(2^{\aleph_1})^M| \le |ext\ j(\aleph_\alpha{}^M)|$. It follows then by Lemma 5.1.6(2) that

$$M[G] \vDash |(2^{\aleph_1})^M| \le \aleph_1{}^{M[G]}\cdot|\aleph_\alpha{}^M|$$

By the λ–chain condition of our notion of forcing, $\lambda \ge \aleph_1{}^{M[G]}$, and so

$$M[G] \vDash |(2^{\aleph_1})^M| \le \lambda.|(2^{\aleph_0})^M|$$

Thus $M \vDash 2^{\aleph_1} \le 2^{\aleph_0}\cdot\lambda$ (since otherwise $2^{\aleph_1}$ remains a cardinal in M[G], and so we get a contradiction). This concludes the proof.

□

Corollary 5.2.11 [Jech–Prikry 1979]:

If $\aleph_1$ *carries an* $\aleph_2$*–saturated ideal and if* $2^{\aleph_0} = \aleph_1$, *then* $2^{\aleph_1} = \aleph_2$.

Thus the existence of an ω_2–saturated ideal over ω_1 is quite a strong assumption, giving much information about the power function. We will elaborate on this theme: Let $\mathcal{I}$ be an ω_2–saturated ideal over ω_1, let G be $\mathcal{I}$–generic , and let N = Ult(M,G), where M denotes the universe. Then ω_1^M is a countable ordinal in M[G], since

$$\omega_1^M < j(\omega_1^M) = \omega_1^N \leq \omega_1^{M[G]}.$$

By the ω_2–chain condition, all cardinals $\geq \omega_2^M$ in M are preserved. It follows that $\omega_2^M = \omega_1^{M[G]}$. More generally, for all $\alpha \geq \omega$ we have $\aleph_{1+\alpha}^M = \aleph_\alpha^{M[G]}$, and thus $\aleph_\alpha^M = \aleph_\alpha^{M[G]}$ for all infinite α. Suppose now that $2^{\aleph_0} = \aleph_\alpha$ in M. Then $(2^{\aleph_0})^N = j(\aleph_\alpha^M)$, and since $|\aleph_1^M| = |\aleph_0^N|$, we have:

$$M[G] \vdash (2^{\aleph_1})^M \leq j(|\aleph_\alpha^M|) = j((2^{\aleph_0})^M).$$

This explains why the size of $2^{\aleph_1}$ is dependent on the size of $2^{\aleph_0}$, as in Corollary 5.2.11. Now $(2^{\aleph_1})^M$ is (in M) a cardinal $\geq \aleph_2^M$, and such cardinals are preserved. Hence

$$M \vdash 2^{\aleph_1} \leq |j(2^{\aleph_0})|$$

so in order to find bounds for $2^{\aleph_1}$, it suffices to find bounds for $|j(2^{\aleph_0})|$ in the generic ultrapower.

As an example, suppose that $2^{\aleph_0} = \aleph_{\omega_1}$. By Lemma 5.2.8, $j(\aleph_{\omega_1}) \leq \aleph_\gamma$, where $\gamma = \omega_1^{M[G]}$. But $\omega_1^{M[G]} = \omega_2$, and so $2^{\aleph_1} \leq |j(\aleph_{\omega_1})| \leq \aleph_{\omega_2}$. Hence if ω_1 carries an ω_2–saturated ideal, and if $2^{\aleph_0} = \aleph_{\omega_1}$, then $2^{\aleph_1} \leq \aleph_{\omega_2}$. Similarly, one can show that if $2^{\aleph_0} = \aleph_{\omega_{\omega_1}}$, then $2^{\aleph_1} \leq \aleph_{\omega_{\omega_2}}$. We will now give a very general method for obtaining these results:

Definition 5.1.12: A *nice cardinal function* Ψ is an increasing enumeration of a class of cardinals which is definable *without parameters* such that:

(1) $\Psi^{M_1}(\alpha) \leq \Psi^{M_2}(\alpha)$ whenever $M_1 \subsetneq M_2$ are transitive models of ZFC.

(2) If M_1 is a transitive model of ZFC and if M_2 is a generic extension of M_1 via a λ–saturated notion of forcing, then $\text{ran}(\Psi^{M_1}) - \lambda = \text{ran}(\Psi^{M_2}) - \lambda$.

The following facts are immediately obvious:

Lemma 5.1.13 [Jech–Prikry 1979]:

(a) *If Ψ is a nice cardinal function, then so is the enumeration of all the limit points of Ψ.*

(b) *If Ψ is nice, then so is the enumeration of the class of fixed points of Ψ.*

(c) *If C is a class of cardinals which has a nice enumeration, then the class* $\{\alpha \in C: C \cap \alpha$ is stationary in $\alpha\}$ *has a nice enumeration.*

(d) *If Ψ is nice, then so are $\Psi(\alpha+\alpha)$ and $\Psi(\alpha\cdot\alpha)$.*

(e) *The composition of nice cardinal functions is nice.*

Lemma 5.1.14 [Jech–Prikry 1979]:

The following are examples of nice cardinal functions:

(1) $\Psi(\alpha) = \aleph_\alpha$

(2) $\Psi(\alpha) = \aleph_{\omega_\alpha}$

(3) $\Psi(\alpha) = \alpha^{\text{th}}$ *fixed point of the $\aleph$–function.*

(4) $\Psi(\alpha) = \alpha^{\text{th}}$ *weakly inaccessible cardinal.*

(5) $\Psi(\alpha) = \alpha^{\text{th}}$ *Mahlo cardinal.*

(6) $\Psi(\alpha) = \aleph_{1+\alpha}$.

Proof: (1) is clear. (2) follows from Lemma 5.1.13(e). (3) follows from Lemma 5.1.13(b). (4) holds because if $\alpha \geq \lambda$ is weakly inaccessible in M, it is weakly inaccessible in any generic extension of M via a λ–saturated notion of forcing. (5) holds because of (4) and Lemma 5.1.13(c). Finally, (6) follows from (1) and the fact that if $\nu = \aleph_{1+\alpha}^{M} \geq \lambda$, then ν remains a cardinal in M[G], and so $\nu \geq \aleph_1^{M[G]}$.

□

We can now state and prove the promised general result:

Theorem 5.1.15 [Jech–Prikry 1979]:
Suppose that there is a precipitous λ–saturated ideal I over $\aleph_1$. Suppose that Ψ is a nice cardinal function and that $\alpha < \omega_1$. Then

(a) *If* $2^{\aleph_0} < \Psi(\alpha)$, *then* $2^{\aleph_1} <$ *the* α^{th} *value of* Ψ *above* λ.

(b) *If* $2^{\aleph_0} = \Psi(\alpha)$, *then* $2^{\aleph_1} \leq$ *the* α^{th} *value of* Ψ *above* λ.

Proof: Let M be the universe and suppose that G is I–generic. Let N = Ult(M,G), and let j: M $\longrightarrow$ N be the associated elementary embedding. We have already shown in Lemma 5.1.9 that $(2^{\aleph_1})^M \leq j((2^{\aleph_0})^M) = (2^{\aleph_0})^N$. Now $(2^{\aleph_0})^N < j(\Psi^M(\alpha))$ in case (a), and $(2^{\aleph_0})^N = j(\Psi^M(\alpha))$ in case (b). But $j(\Psi^M(\alpha)) = \Psi^N(\alpha) \leq \Psi^{M[G]}(\alpha) \leq \alpha^{th}$ value of Ψ^M above λ.

□

Theorem 5.1.16 [Jech–Prikry 1979]:
Suppose that there is a precipitous λ–saturated ideal over $\aleph_1$, where $\lambda > \aleph_1$ is regular. Suppose further that Ψ is a nice cardinal function such that $2^{\aleph_0} \leq \Psi(\omega_1)$. Then:

(a) *If* $\lambda < \Psi(\lambda)$, *then* $2^{\aleph_1} \leq \Psi(\lambda)$;

(b) *If* $\lambda = \Psi(\lambda)$, *then* $2^{\aleph_1} \leq \Psi(\lambda+\lambda)$.

Proof: $(2^{\aleph_1})^M \leq j((2^{\aleph_0})^M) \leq j(\Psi^M(\omega_1^M)) = \Psi^N(\omega_1^N) \leq \Psi^N(\lambda) \leq \Psi^{M[G]}(\lambda) \leq \lambda^{th}$ value of Ψ^M above λ. Thus if $\lambda < \Psi(\lambda)$, then $\lambda \leq \Psi(\xi)$ for some $\xi < \lambda$, and thus $(2^{\aleph_1})^M \leq \Psi^M(\xi+\lambda) = \Psi^M(\lambda)$, whereas if $\lambda = \Psi(\lambda)$, then $(2^{\aleph_1})^M \leq \Psi^M(\lambda+\lambda)$.

□

As some examples of the applications of the above theorems we list the following:

Corollary 5.1.17 [Jech–Prikry 1979]:
Suppose that $\aleph_1$ carries an ω_2–saturated ideal. Then

(1) *If* $2^{\aleph_0} = \aleph_{\omega_1+\alpha}$ *for some* $\alpha < \omega_1$, *then* $2^{\aleph_1} \le \aleph_{\omega_2}$

(2) *If* $2^{\aleph_0} = \aleph_{\omega_1+\omega_1}$, *then* $2^{\aleph_1} \le \aleph_{\omega_2+\omega_2}$

(3) *If* $2^{\aleph_0} = \aleph_{\omega_{\omega_1}}$, *then* $2^{\aleph_1} \le \aleph_{\omega_{\omega_2}}$.

§ 5.2 Saturated Ideals and the Singular Cardinals Problem.

In Chapter 4 we examined the behaviour of the power function at strong limit cardinals of cofinality ω (in particular $\aleph_\omega$). In this section we shall to obtain bounds for the power function at singular κ of uncountable cofinalities, work first begun in Section 1.3. We shall assume that κ is a strong limit cardinal of cofinality ω_1 and that ω_1 carries a precipitous ideal. In Chapter 1 we proved a result that if $\kappa = \aleph_{\omega_1}$ is strong limit and if the Chang Conjecture holds, then $2^\kappa < \aleph_{\omega_2}$ (Theorem 1.3.5). We shall obtain the same result assuming instead that ω_1 carries an ω_2–saturated ideal.
Suppose that $\mathcal{I}$ is a precipitous ideal over $\aleph_1$ in the universe M, and let G be $\mathcal{I}$–generic. Let j: M $\longrightarrow$ N = Ult(M,G) be the canonical elementary embedding. Note that the notion of forcing (all $\mathcal{I}$–positive sets) has cardinality $2^{\aleph_1}$, and thus all cardinals $\ge (2^{\aleph_1})^+$ are preserved. In particular, all cardinals $\ge \kappa$ are preserved.

Note that $\kappa < j(\kappa)$, and that if $\alpha < \kappa$, then $j(\alpha) < \kappa$ as well: This follows because $j(\alpha) < (\alpha^{\omega_1})^+$ and $(\alpha^{\omega_1})^+ < \kappa$ (because κ is strong limit).
Thus κ is represented in the ultrapower by a map e: $\omega_1 \longrightarrow \kappa$, and for all $\alpha < \kappa$, the set $\{\gamma < \omega_1\colon e(\gamma) > \alpha\}$ is in G. If $X \subseteq \kappa$ in M, define a map t_X on ω_1 by: $t_X(\gamma) = X \cap e(\gamma)$. Each t_X represents a subset of κ in the ultrapower, and the correspondence $X \longmapsto [t_X]$ is injective. Thus $M[G] \models |(2^\kappa)^M| \le |(2^\kappa)^N|$. Also, $j(\kappa)$ is strong limit in N, because κ is strong limit in M. Thus $N \models 2^\kappa < j(\kappa)$ and so $M[G] \models |(2^\kappa)^M| < j(\kappa)$. Because cardinals > $j(\kappa)$ are preserved, we may conclude that:

Lemma 5.2.1 [Jech–Prikry 1979]:
Suppose that ω_1 carries a precipitous ideal and that κ is a singular strong limit cardinal of cofinality ω_1. If j: M $\longrightarrow$ Ult(M,G) *is the associated elementary embedding, then* $(2^\kappa)^M <$ $j(\kappa)$ *in* M.

This result will allow us to find bounds on 2^κ in M.

Theorem 5.2.2 [Jech–Prikry 1979]:
Suppose that $\aleph_1$ carries a λ–saturated precipitous ideal. Let Ψ be a nice cardinal function, and suppose that $\kappa = \Psi(\omega_1)$ is a strong limit cardinal of cofinality ω_1. Then $2^\kappa < \Psi(\lambda)$.

Proof: Let M,$\mathcal{I}$,G,N,j be as before. We have shown that $2^\kappa < j(\kappa)$ in M, and thus it suffices to prove that $j(\kappa) \leq \Psi^M(\lambda)$. Since $\kappa = \Psi(\omega_1)$, we immediately see that $j(\kappa) \leq \Psi^{M[G]}(\omega_1{}^{M[G]})$. The fact that $\mathcal{I}$ is λ–saturated implies that $\lambda \geq \aleph_1{}^{M[G]}$, and thus we may conclude that

$$
\begin{aligned}
& j(\kappa) \leq \Psi^{M[G]}(\lambda) \leq \lambda^{\text{th}} \text{ value of } \Psi^{M[G]} \text{ above } \kappa \\
& = \lambda^{\text{th}} \text{ value of } \Psi^{M} \text{ above } \kappa \ \ \text{(because } \Psi \text{ is nice)} \\
& = \Psi^M(\omega_1 + \lambda) \\
& = \Psi^M(\lambda).
\end{aligned}
$$

□

Corollary 5.2.3 [Jech–Prikry 1979]:

(a) *If $\aleph_1$ carries a precipitous ideal, and $\kappa = \Psi(\omega_1)$ is a singular strong limit cardinal of cofinality ω_1, then $2^\kappa < \Psi((2^{\aleph_1})^+)$*

(b) *If $\aleph_1$ carries an ω_2–saturated ideal, and $\kappa = \Psi(\omega_1)$ is a singular strong limit cardinal of cofinality ω_1, then $2^\kappa < \Psi(\omega_2)$.*

(c) *In particular, if $\aleph_{\omega_1}$ is strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_2}$.*

Note that If Ψ is the $\aleph$–function, and $\kappa = \aleph_{\omega_1}$ is strong limit, then $2^\kappa < \Psi((2^{\aleph_1})^+)$ by the theorem of Galvin–Hajnal (Theorem 1.3.4). Thus in this case we don't need the existence of a precipitous ideal. The next result gives another bound on 2^κ:

Theorem 5.2.4 [Jech–Prikry 1979]:
Suppose that $\aleph_1$ carries a precipitous ideal, and let Ψ be a nice cardinal function. If κ is a singular strong limit cardinal of cofinality ω_1 such that $\kappa \le \Psi(\alpha)$ for some $\alpha < \kappa$, then $2^\kappa <$ $\Psi(\kappa)$.

Proof: Let M,$\mathcal{I}$,G,N,j be as before. Since $\kappa \le \Psi^M(\alpha)$, $j(\kappa) \le \Psi^N(j(\alpha))$. Also, $j(\alpha) < \kappa$ for all $\alpha < \kappa$, and thus $j(\kappa) < \Psi^N(\kappa) \le \Psi^{M[G]}(\kappa)$. We may then argue as in theorem 3.2:

$$\Psi^{M[G]}(\kappa) = \kappa^{th} \text{ value of } \Psi^{M[G]} \text{ above } \kappa = \kappa^{th} \text{ value of } \Psi^M \text{ above } \kappa = \Psi^M(\kappa).$$

Thus $(2^\kappa)^M < j(\kappa) \le \Psi^M(\kappa)$, proving that $2^\kappa < \Psi(\kappa)$ in M.

□

Theorem 1.3.4 gives a bound on 2^κ if $\kappa < \aleph_\kappa$ is a singular strong limit cardinal of cofinality ω_1, namely $2^\kappa < \aleph_\kappa$ as well. We can use Theorem 5.2.4 to give bounds on 2^κ even if $\kappa = \aleph_\kappa$: For instance if $\Psi(\kappa) =$ the κ^{th} fixed point of $\aleph$, and if $\kappa = \aleph_\kappa < \Psi(\kappa)$ then $2^\kappa < \Psi(\kappa)$. Similarly, if $\theta(\kappa) = \kappa^{th}$ fixed point of Ψ and $\kappa = \aleph_\kappa = \Psi(\kappa) < \theta(\kappa)$, then $2^\kappa < \theta(\kappa)$.

A final theorem which gives bounds on 2^κ for particular κ is:

Theorem 5.2.5 [Jech–Prikry 1979]:
Suppose that ω_1 carries a precipitous ideal, and that Ψ is a nice cardinal function. Suppose further that κ is a strong limit cardinal of cofinality ω_1 such that the set of values of Ψ below κ is bounded below κ. Then

$$2^\kappa < \textit{least value of } \Psi \textit{ above } \kappa$$

Proof: Let $j{:}M \longrightarrow N = \mathrm{Ult}(M,G)$ be the associated elementary embedding, and let $\gamma = \sup(\mathrm{ran}(\Psi^M) \cap \kappa) < \kappa$. Then $\kappa \le$ least value of Ψ^M above γ, so that $j(\kappa) \le$ least value of

Ψ^N above $j(\gamma)$. Now $\gamma < \kappa$ implies $j(\gamma) < \kappa$, and so $j(\kappa) \leq$ least value of $\Psi^{M[G]}$ above κ = least value of Ψ^M above κ. Since $(2^\kappa)^M < j(\kappa)$, $M \vdash 2^\kappa <$ least value of Ψ above κ, as required.

□

Unfortunately, neither the methods used above, nor any other method can be used to find a bound on 2^κ for all singular strong limit cardinals of cofinality ω_1 (or ω):

Lemma 5.2.6:
Assume that V *is a model of* ZFC + GCH *in which there is a supercompact cardinal* κ, *and let* $\nu > \kappa$ *be arbitrary. Then there is a generic extension of* V *that preserves cardinals, in which* κ *is a strong limit singular cardinal of cofinality* ω_1 (or ω), *and such that* $2^\kappa > \nu$.

Proof (in outline): According to a theorem of Laver ([Laver 1978]) we may assume that the supercompactness of κ is not destroyed under κ–directed closed forcing. A standard Cohen extension then yields a model in which κ is supercompact, but $2^\kappa > \nu$. By a theorem of Magidor ([Magidor 1978]), there is a generic extension of this model which preserves all cardinals and in which κ is a singular strong limit cardinal of cofinality ω_1. We may use Prikry forcing (see Chapter 4.1) to change the cofinality of κ to ω for the other case.

□

§ 5.3 More about Saturated Ideals and the SCH

In Section 3.3 we proved that if a compact cardinal exists, then the SCH holds above it (Thm. 3.3.1). The proof of this fact hinged on the following result: If κ is λ–compact for some regular cardinal λ, then $\lambda^{<\kappa} = \lambda$ (Lemma 3.3.5). In this section we shall derive the same result from the existence of a λ–saturated precipitous ideal over $[\lambda]^{<\kappa}$ which satisfies one further property, namely that the ideal is bounding (see Definition 5.3.1). The main result of this section is Theorem 5.3.4, and is due to [Matsubara 1992]. We shall occasionally refer to Appendix 4, where many of the relevant definitions and standard results may be found.

Throughout this section, let κ be a strongly inaccessible cardinal and let λ be a cardinal $\geq \kappa$. If I is an ideal over $[\lambda]^{<\kappa}$, then $\mathcal{P}([\lambda]^{<\kappa})/I$ is the quotient Boolean algebra of the algebra of subsets $\mathcal{P}([\lambda]^{<\kappa})$, where X and Y are identified in $\mathcal{P}([\lambda]^{<\kappa})/I$ if their symmetric difference is an element of I. The quotient class of $X \subseteq [\lambda]^{<\kappa}$ in $\mathcal{P}([\lambda]^{<\kappa})/I$ will be denoted $[X]_I$. The notion of forcing associated with I, $\mathbb{P}_I$,is just the set of all I–positive sets ordered by inclusion. It may be identified with $\mathcal{P}([\lambda]^{<\kappa})/I - \{[\emptyset]_I\}$.

An ideal I over $[\lambda]^{<\kappa}$ will always be assumed to be κ–complete and fine (i.e. for each $\alpha < \lambda$, the set $\{x \in [\lambda]^{<\kappa}: \alpha \in x\}$ is in the dual filter of I).

An ideal I over $[\lambda]^{<\kappa}$ is said to satisfy the *disjointing property* provided that whenever $A \subseteq \mathcal{P}([\lambda]^{<\kappa})/I$ is a maximal antichain, then there is a pairwise disjoint family $\{X_a: \alpha \in A\}$ of subsets of $[\lambda]^{<\kappa}$ with the property that $[X_a]_I = a$ for each $a \in A$ (See Definition A.4.9). An ω_1–complete ideal I with the disjointing property is always precipitous, i.e. Ult(V,G) is always well–founded for any I–generic V–ultrafilter G (Lemma A.4.11).

Definition 5.3.1: Let I be an ideal over $[\lambda]^{<\kappa}$. I is a *bounding ideal* provided that

$$\Vdash_{\mathbb{P}_I} \exists\delta(\delta \text{ a regular cardinal} < \kappa) \longrightarrow \forall\alpha<\check{\lambda}(\mathrm{cf}\alpha = \delta \longrightarrow \mathrm{cf}^V\alpha < \check{\kappa})$$

Lemma 5.3.2:
Suppose that λ is a regular cardinal $\geq \kappa$, where κ is strongly inaccessible. If there is a λ–saturated bounding precipitous ideal on $[\lambda]^{<\kappa}$, then $\lambda^{<\kappa} = \lambda$.

Proof: Let I be an ideal with the properties cited in the statement of the Lemma, let $\mathbb{P}$ be the set of all I–positive sets ordered by inclusion, and let G be an I–generic V–ultrafilter. We shall first work in V[G]. Let M = Ult(V,G) be the generic ultrapower, and let $j_G: V \longrightarrow M$ be the corresponding elementary embedding. Also let id be the identity map on $([\lambda]^{<\kappa})^V$. Define $\gamma = \sup(j_G"\lambda)$ and define a filter $\mathcal{U}$ as follows:

$$X \in \mathcal{U} \longleftrightarrow X \in \mathcal{P}([\lambda]^{<\kappa}) \cap V \text{ and } [\mathrm{id}]_G \cap \gamma \in j_G(X)$$

We shall show that $\mathcal{U}$ is a weakly normal V–κ–complete V–ultrafilter (Recall that $\mathcal{U}$ is said to be weakly normal if and only if whenever f is a map such that $\{x \in [\lambda]^{<\kappa}: f(x) \in x\} \in \mathcal{U}$, then there is an ordinal $\alpha < \lambda$ such that $\{x: f(x) \leq \alpha\} \in \mathcal{U}$). If $X \in \mathcal{P}([\lambda]^{<\kappa}) \cap V$ and $X \notin \mathcal{U}$, then necessarily $[id]_G \cap \gamma \in j_G([\lambda]^{<\kappa} - X)$, proving that $\mathcal{U}$ is an ultrafilter over $\mathcal{P}([\lambda]^{<\kappa}) \cap V$. Since G is κ–complete and all cardinals $< \kappa$ are preserved by j_G it follows that $\mathcal{U}$ is κ–complete as well. Finally, suppose that f is a map with the property that $\{x: f(x) \in x\} \in \mathcal{U}$. Then $[id]_G \cap \gamma \in \{x \in [\gamma]^{<\kappa}: j_G f(x) \in x\}$. Hence $j_G([id]_G \cap \gamma) \leq j_G(\alpha)$ for some $\alpha < \lambda$, proving that $[id]_G \cap \gamma \in \{x: j_G f(x) \leq j_G(\alpha)\}$. It follows that $\{x: f(x) \leq \alpha\} \in \mathcal{U}$ for some $\alpha < \lambda$, and thus that $\mathcal{U}$ is weakly normal.

Let $j:V \longrightarrow N$ be the elementary embedding induced by $\mathcal{U}$. Since $\mathcal{U}$ is weakly normal, it follows that $\sup[id]_{\mathcal{U}} = \sup(j''\lambda)$, and since $\mathcal{I}$ is bounding, there is a $\delta < \kappa$ such that

$$V[G] \vdash \text{"}\delta \text{ is regular and } \forall\alpha<\lambda(cf(\alpha) = \delta \longrightarrow cf^V \alpha < \kappa).\text{"}$$

Let C be a subset of $\sup[id]_{\mathcal{U}}$ such that

$$N \vdash C \text{ is club in } \sup[id]_{\mathcal{U}}$$

Claim 1: In V[G], $j^{-1}(C \cap j''\lambda)$ is unbounded in λ and closed under suprema of δ–sequences.

First let $\alpha < \lambda$, and let $\alpha_0 = \alpha$. Inductively we define a sequence $(\alpha_\beta: \beta < \delta)$. Suppose we know α_β; since C is club in $\sup[id]_{\mathcal{U}} = \sup(j''\lambda)$, there is $\xi_\beta \in C$ such that $j(\alpha_\beta) \leq \xi_\beta$. Choose $\alpha_{\beta+1} < \lambda$ such that $\xi_\beta < j(\alpha_{\beta+1})$. If η is a limit ordinal, let α_η be the supremum of all the α_β that precede it. Finally, let $\overline{\alpha} = \sup(\alpha_\beta: \beta < \delta)$. Then $cf(\overline{\alpha}) = \delta$ in V[G], and thus by the bounding property, $cf^V(\overline{\alpha}) < \kappa$. It follows that

$$j(\overline{\alpha}) = \cup\{j(\gamma): \gamma < \overline{\alpha}\} = \sup\{\xi_\beta: \beta < \delta\} \in C.$$

Therefore $\overline{\alpha} \in j^{-1}(C \cap j''\lambda)$, and as $\overline{\alpha} > \alpha$, it follows that $j^{-1}(C \cap j''\lambda)$ is unbounded in λ. If $\{j(\beta_\alpha): \alpha < \delta\} \subseteq C \cap j''\lambda$. Then $\beta = \sup\{j(\beta_\alpha): \alpha < \delta\} \in C$ must have cofinality $< \kappa$ in V. As above it now follows that $\beta = j(\sup\{\beta_\alpha: \alpha < \delta\})$, and so $\sup\{\beta_\alpha: \alpha < \delta\} \in j^{-1}(C \cap j''\lambda)$, proving that this set is closed under suprema of δ–sequences as well. This completes the proof of Claim 1.

Claim 2: There is a club subset E of λ in V such that

$$E \cap \{\alpha < \lambda: cf^V(\alpha) = \delta\} \subseteq j^{-1}(C \cap j''\lambda)$$

Let D be the set of all limit points of $j^{-1}(C \cap j''\lambda)$. Then D is a club. subset of λ in V[G]. Since V[G] is obtained by a notion of forcing which satisfies the λ–chain condition, there is a club subset E of D such that $E \in V$. But $j^{-1}(C \cap j''\lambda)$ is closed under suprema of δ–

sequences (by Claim 1), and so $E \cap \{\alpha: cf^V(\alpha) = \delta\} \subseteq D \cap \{\alpha: cf^V(\alpha) = \delta\} \subseteq j^{-1}(C \cap j''\lambda)$. This completes the proof of Claim 2.

Next we shall work in V. Let $A = \{\alpha < \lambda: cf(\alpha) = \delta\}$. Then A is a stationary subset of λ and thus by Theorem 1.3.2 there is a partitioning $\{A_\alpha: \alpha < \lambda\}$ of A into λ–many mutually disjoint stationary subsets.

Claim 3: For each $\xi < \lambda$, the set $\{x \in [\lambda]^{<\kappa}: A_\xi \cap \sup(x)$ is stationary in $\sup(x)\} \in \mathcal{U}$.
It suffices to prove that

$$N \vdash j(A_\xi) \cap \sup[id]_{\mathcal{U}} \text{ is stationary in } \sup[id]_{\mathcal{U}}$$

If $N \vdash C$ is club in $\sup[id]_{\mathcal{U}}$, then since A_ξ is a stationary subset of $\{\alpha < \lambda: cf^V(\alpha) = \delta\}$, we have $\emptyset \neq E \cap A_\xi \subseteq E \cap A \subseteq j^{-1}(C \cap j''\lambda)$ by Claim 2. Hence $j(A_\xi) \cap C \neq \emptyset$, proving that (reasoning in N) $j(A_\xi) \cap \sup[id]_{\mathcal{U}}$ is stationary in $\sup[id]_{\mathcal{U}}$. This completes the proof of Claim 3.

Let Y be defined in V as follows:

$$Y = \{x \in [\lambda]^{<\kappa}: |\{\xi < \lambda: A_\xi \cap \sup(x) \text{ is stationary in } \sup(x)\}| < \kappa\}$$

Claim 4: $Y \in \mathcal{U}$.

Suppose $x \in [\lambda]^{<\kappa}$ such that $cf(\sup(x)) < \kappa$. Then there is a club subset C of $\sup(x)$ such that $|C| < \kappa$. Since the sets A_ξ are mutually disjoint, it follows that

$$|\{\xi < \lambda: A_\xi \cap \sup(x) \text{ is stationary in } \sup(x)\}| \leq |C| < \kappa$$

Hence $x \in Y$, i.e. $\{x \in [\lambda]^{<\kappa}: cf(\sup(x)) < \kappa\} \subseteq Y$. But $\{x \in [\lambda]^{<\kappa}: cf(\sup(x)) < \kappa\} \in \mathcal{U}$ because $\mathcal{U}$ is weakly normal, and thus $Y \in \mathcal{U}$ as well. This completes the proof of Claim 4.

We may now complete the proof of Lemma 5.3.2: Let $\beta < \lambda$ and, in V, define

$$X_\beta = \{\xi < \lambda: A_\xi \cap \beta \text{ is stationary in } \beta\}$$

For each $\xi < \lambda$, let $W_\xi = \{x \in [\lambda]^{<\kappa}: A_\xi \cap \sup(x)$ is stationary in $\sup(x)\}$. By Claim 3, each $W_\xi \in \mathcal{U}$. Let $x \in [\lambda]^{<\kappa}$. Since $\mathcal{U}$ is κ–complete and $Y \in \mathcal{U}$, we have $\bigcap_{\xi \in x} W_\xi \cap Y \in \mathcal{U}$ as well. Choose $s_x \in \bigcap_{\xi \in x} W_\xi \cap Y$, and let $\beta_x = \sup(s_x)$. For each $\xi \in x$, $A_\xi \cap \beta_x$ is stationary in β_x, so that $x \subseteq X_{\beta_x}$. Since moreover $s \in Y$, $|X_{\beta_x}| < \kappa$ as well. It follows that in V, $|[\lambda]^{<\kappa}| = \lambda$ (since each $x \in [\lambda]^{<\kappa}$ is a subset of some X_β, each X_β has cardinality $< \kappa$, and κ is strongly inaccessible).

□

In order to make use of Lemma 5.3.2, we need a way of obtaining some bounding ideals on $[\lambda]^{<\kappa}$. This is provided for by the following lemma.

Lemma 5.3.3 [Matsubara 1992]:

If I is an η–saturated normal ideal over $[\lambda]^{<\kappa}$, where κ is strongly inaccessible and $\eta < \lambda$, and $\eta < \kappa^{+\kappa}$, then I is bounding and precipitous.

Proof: By Lemma A.4.14, any λ^+–saturated normal ideal over $[\lambda]^{<\kappa}$ is precipitous, and so I is precipitous as well. Let G be I–generic and let M be the corresponding generic ultrapower. Let j: V ⟶ M be the associated elementary embedding. By Lemma A.4.12,

$$^{\lambda}M \cap V[G] \subseteq M$$

Since the notion of forcing with which V[G] was constructed is η–saturated, it follows that all cofinalities $\geq \eta$ are preserved in V[G]. Since j is the identity on ordinals $< \kappa$ and since $^{\lambda}M \subseteq M$ in V[G], it follows that cofinalities $< \kappa$ are preserved as well. Hence if $cf^V(\alpha) \neq cf^{V[G]}(\alpha)$, we must have $\kappa \leq cf^V(\alpha) < \eta$. Since $\eta < \kappa^{+\kappa}$ the set

$$\{cf^{V[G]}(\alpha): cf^V(\alpha) \neq cf^{V[G]}(\alpha)\}$$

must have cofinality $< \kappa$, and thus $\kappa - \{cf^{V[G]}(\alpha): cf^V(\alpha) \neq cf^{V[G]}(\alpha)\}$ is non–empty. If δ is a regular member of this set, then $cf^{V[G]}(\alpha) = \delta$ implies $cf^V(\alpha) = \delta$, and thus $cf^{V[G]}(\alpha) = \delta$ implies $cf^V(\alpha) < \kappa$.

□

Theorem 5.3.4 [Matsubara 1992]:

Suppose that κ is a strongly inaccessible cardinal with $\kappa < \eta < \kappa^{+\kappa} \leq \lambda$. If there exists a η–saturated normal ideal over $[\lambda]^{<\kappa}$, then the Singular Cardinals Hypothesis holds between η and λ.

Proof: By a trivial adaptation of Lemma 3.3.6, it suffices to prove that for all singular cardinals δ with $\eta \leq \delta \leq \lambda$ and $cf(\delta) = \omega$, we have $\delta^\omega = \delta^+$. If δ is such a cardinal, then $\delta^\omega \leq (\delta^+)^\omega \leq (\delta^+)^{<\kappa}$. Hence we shall be finished if we can prove that for every regular cardinal

γ with $\eta \leq \gamma \leq \lambda$, we have $\gamma^{<\kappa} = \gamma$. Let $\mathcal{I}$ be an ideal with the properties asserted by the statement of Theorem 5.3.4. For each regular γ between η and λ, define an ideal $\mathcal{J}$ over $[\gamma]^{<\kappa}$ by:

$$X \in \mathcal{J} \longleftrightarrow X \subseteq [\gamma]^{<\kappa} \text{ and } \{x \in [\lambda]^{<\kappa} : x \cap \gamma \in X\} \in \mathcal{I}$$

It is not hard to show that $\mathcal{J}$ is η–saturated normal ideal over $[\gamma]^{<\kappa}$, because $\mathcal{I}$ is an η–saturated normal ideal over $[\lambda]^{<\kappa}$. By Lemma 5.3.3, $\mathcal{J}$ is precipitous and bounding, and thus by Lemma 5.3.2 $\gamma^{<\kappa} = \gamma$, as required.

□

The methods of this section also allow us to present another proof of the fact that the SCH holds above a compact cardinal (Theorem 3.3.1 due to [Solovay 1974]): Suppose that κ is a compact cardinal and that λ is a regular cardinal $> \kappa$. Let $\mathcal{U}$ be a normal fine measure over $[\lambda]^{<\kappa}$, and let $\mathcal{I}$ be its dual ideal. Then since $\mathcal{I}$ is a maximal ideal, it is a bounding precipitous ideal which is 2–saturated. Hence $\lambda^{<\kappa} = \lambda$ by Lemma 5.3.2. It follows that $\lambda^{<\kappa} = \lambda$ for all regular $\lambda > \kappa$, and thus that the SCH holds above κ.

The material in this chapter is both basic (requiring not much more than the notions of club sets and reduced products) and surprising. In Chapter 1 we proved some theorems which give upper bounds for 2^{κ} when κ is a singular cardinal of uncountable cofinality, for example Theorems 1.3.3 and 1.3.4. Chapter 4 proved that these results can not be generalized in a straightforward way to singular cardinals of countable cofinality. In particular, Theorem 4.3.21 states that it is consistent for $\aleph_\omega$ to be a strong limit cardinal with $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for any $\alpha < \omega_1$. One might therefore expect to have considerable freedom for the value of the power function at $\aleph_\omega$ in the way that Theorem 2.3.1 (due to Easton) provides this freedom at regular cardinals. So it is quite surprising that the following theorem holds: *If* $\aleph_\omega$ *is strong limit, then* $2^{\aleph_\omega} < \aleph_{\omega_4}$. This result was proved by Saharon Shelah ([Shelah 1992]) using a new approach to cardinal arithmetic, namely *pcf theory*. If A is a set of cardinals and D an ultrafilter over A, then $\Pi A/D$ is a linearly ordered set, were for f,g $\in \Pi A$ we have $f \leq_D g$ iff $\{\alpha \in A: f(\alpha) \leq g(\alpha)\} \in D$. "pcf(A)" stands for the "possible cofinality" of ΠA modulo some ultrafilter D. Exact definitions will be given in Section 1. To investigate the potential cofinality rather than the true cofinality presents a shift of emphasis in cardinal arithmetic, and leads to many unexpected results. Though the material is basic (in that it does not require forcing or any other high powered machinery), the proofs are quite long and technical. We will therefore develop enough pcf theory in order to present a proof of Theorem 6.4.1, but will content ourselves with mere discussions of other results on the power function that have been obtained using pcf theory. Throughout we shall rely heavily on the introductory paper [Burke–Magidor 1990]. Sections 6.1, 6.2 and 6.3 are concerned with the development of pcf theory and as such have no direct relevance to the study of the power function. In Section 6.4 the material developed in Sections 6.1, 6.2 and 6.3 will be applied to give upper bounds for the values of the power function at singular strong limit cardinals. The important results of Section 6.4 are Theorems 6.4.9, 6.4.10 and 6.4.13.

§ 6.1 Pcf(A)

In this section A will always be an *infinite* set of *regular* cardinals with the property that

$$\min(A) > |A|^+.$$

Definition 6.1.1: Suppose that $\mathcal{I}$ is an ideal over A. If f, g $\in \Pi A$, we say that

$$f =_{\mathcal{I}} g \text{ if and only if } \{\alpha \in A: f(\alpha) \neq g(\alpha)\} \in \mathcal{I}$$

The product ΠA has a quasi orderering induced by $\mathcal{I}$ as follows:

$$f \leq_{\mathcal{I}} g \longmapsto \{\alpha \in A: f(\alpha) > g(\alpha)\} \in \mathcal{I}$$
$$f <_{\mathcal{I}} g \longmapsto \{\alpha \in A: f(\alpha) \geq g(\alpha)\} \in \mathcal{I}$$

Note that $\leq_{\mathcal{I}}$ is not necessarily antisymmetric, and thus not always a partial order. Note also that $f \leq_{\mathcal{I}} g$ and $f \neq g$ *does not* imply that $f <_{\mathcal{I}} g$. We shall also occasionally write "$f \leq g \pmod{\mathcal{I}}$" instead of "$f \leq_{\mathcal{I}} g$".

The *reduced product* of A by $\mathcal{I}$, denoted $\Pi A/\mathcal{I}$, is the set of all $=_{\mathcal{I}}$ – equivalence classes, defined in the same way as ultraproducts are (see Appendix 3.1). In fact if $\mathcal{I}$ is a maximal ideal, then $\Pi A/\mathcal{I}$ is just the ultraproduct of A modulo the dual filter of $\mathcal{I}$. $\leq_{\mathcal{I}}$ is a partial order on the reduced product, but even then we do not always have $f \leq_{\mathcal{I}} g$ and $f \neq_{\mathcal{I}} g$ implying $f <_{\mathcal{I}} g$.

Definition 6.1.2: The *cofinality* of a partially ordered set $(\mathbb{P}, \leq)$ is the least cardinal λ for which there exists a set $\{p_\alpha: \alpha < \lambda\}$ such that $(\forall p \in \mathbb{P})(\exists \alpha < \lambda)(p \leq p_\alpha)$. We shall let $cf(\mathbb{P})$ denote the cofinality of $\mathbb{P}$.

If there exists a *strictly increasing* $(p_\alpha: \alpha < \lambda)$ in $\mathbb{P}$ such that $(\forall p \in \mathbb{P})(\exists \alpha < \lambda)(p \leq p_\alpha)$, then $\mathbb{P}$ is said to have a *true cofinality.* We shall write $\lambda = tcf(\mathbb{P})$ if $\mathbb{P}$ has a true cofinality, and it is λ.

Note that whereas all partial orders have a cofinality, not all partial orders have a true cofinality (For example $\omega \times \omega_1$ with the induced (coordinate–wise) ordering is an example of a partially ordered set without a true cofinality). If $\mathbb{P}$ is a linear ordering, however, it does have a true cofinality, which coincides with its cofinality.

Definition 6.1.3: We shall say that λ is a *possible cofinality* of ΠA if there is an ultrafilter $\mathcal{F}$ over A such that $\lambda = cf(\Pi A/\mathcal{F})$. (Note that the ultraproduct $\Pi A/\mathcal{F}$ is linear and therefore does have a true cofinality). The set of all possible cofinalities of ΠA is denoted pcf(A).

If $B \subsetneq A$, then we shall say that "B forces $cf(\Pi A) < \lambda$" if and only if:

For any ultrafilter $\mathcal{F}$ over A, if $B \in \mathcal{F}$ then $cf(\Pi A/\mathcal{F}) < \lambda$.

It follows easily from the properties of an ultrafilter that the set

$$\mathcal{J}_{<\lambda}(A) = \{B \subsetneq A\text{: B forces } cf(\Pi A) < \lambda\}$$

is an ideal over A.

Remark 6.1.4: If $\mathcal{I}$ is an ideal over A and $\mathcal{F}$ is an ultrafilter over A such that $\mathcal{F} \cap \mathcal{I} = \emptyset$, then $f \leq_{\mathcal{I}} g$ implies $f \leq_{\mathcal{F}} g$. For if $f \leq_{\mathcal{I}} g$, then $\{\alpha \in A\text{: } f(\alpha) > g(\alpha)\}$ is in $\mathcal{I}$, and so it cannot be in $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, it follows that $\{\alpha \in A\text{: } f(\alpha) \leq g(\alpha)\} \in \mathcal{F}$, proving that $f \leq_{\mathcal{F}} g$ as well.

The main result about $\mathcal{J}_{<\lambda}(A)$ is:

Lemma 6.1.5 (Shelah):
If $\mathcal{F}$ is an ultrafilter over A *then* $cf(\Pi A/\mathcal{F}) < \lambda$ *if and only if there exists* $B \in \mathcal{F}$ *such that* B *forces* $cf(\Pi A) < \lambda$ *(i.e.* $cf(\Pi A/\mathcal{F}) < \lambda$ *iff* $\mathcal{F} \cap \mathcal{J}_{<\lambda}(A) \neq \emptyset$).

Lemma 6.1.5 follows directly from the following lemma, whose proof will be presented later.

Lemma 6.1.6:
The reduced product $\Pi A/\mathcal{J}_{<\lambda}(A)$ *is* λ*–directed, i.e. any* $B \subsetneq \Pi A/\mathcal{J}_{<\lambda}(A)$ *of cardinality* $< \lambda$ *has an upper bound in* $\Pi A/\mathcal{J}_{<\lambda}(A)$.

Proof of Lemma 6.1.5: Clearly if $\mathcal{F} \cap \mathcal{J}_{<\lambda}(A) \neq \emptyset$, then $cf(\Pi A/\mathcal{F}) < \lambda$. Now suppose that $\mathcal{F}$ is an ultrafilter over A such that $cf(\Pi A/\mathcal{F}) = \mu < \lambda$. Let $(g_\gamma/\mathcal{F}\text{: } \gamma < \mu)$ be a strictly increasing cofinal sequence in $\Pi A/\mathcal{F}$. By Lemma 6.1.6 there is a $g \in \Pi A$ which is an upper bound for $(g_\gamma\text{: } \gamma < \mu)$ in $\Pi A/\mathcal{J}_{<\lambda}(A)$. Suppose now that $\mathcal{F} \cap \mathcal{J}_{<\lambda}(A) = \emptyset$. Then:

$g_\gamma \leq g \pmod{\mathcal{J}_{<\lambda}(A)}$ implies $g_\gamma \leq g \pmod{\mathcal{F}}$ for all $\gamma < \mu$ (by Remark 6.1.4)

This contradicts the assertion that $(g_\gamma/\mathcal{F}\text{: } \gamma < \mu)$ is cofinal in $\Pi A/\mathcal{F}$.

□

Proof of Lemma 6.1.6: Let $\mathcal{J} = \mathcal{J}_{<\lambda}(A)$, and let $B \subseteq \Pi A/\mathcal{J}$ be of cardinality $< \lambda$. We shall show that B is bounded in $\Pi A/\mathcal{J}$. Without loss of generality it may be assumed that $\min(A) > |A|^+$. Suppose the lemma is false. Let $\mu < \lambda$ be the least cardinal for which there is a $B \subseteq \Pi A$ of cardinality μ which is unbounded in $\Pi A/\mathcal{J}$.

Note first that $\mu > |A|^+$. Otherwise since each $\alpha \in A$ is a *regular* cardinal $> |A|^+$, it follows that the function f defined by:

$$f(\alpha) = \sup\{g(\alpha): g \in B\}$$

is in ΠA. f dominates each $g \in B$ pointwise, and thus in particular $g \leq_{\mathcal{J}} f$ for all $g \in B$.

Thus $|B| = \mu > |A|^+$. We may as well assume that B is linearly ordered by $\leq_{\mathcal{J}}$ and that $B = \{g_\gamma: \gamma < \mu\}$ is an increasing enumeration (Otherwise we may replace each g_γ by an upper bound g'_γ for $\{g'_\delta: \delta < \gamma\} \cup \{g_\gamma\}$. Such a g'_γ exists by definition of μ). Note also that μ is regular (For if μ is singular, then B has a cofinal sequence of cardinality $< \mu$, and this sequence has an upper bound, by definition of μ, which will also be an upper bound for B).

Our aim is to inductively define a sequence $h_\beta \in \Pi A$ (for $\beta < |A|^+$). Let

$$b_\alpha^\beta = \{\gamma \in A: g_\alpha(\gamma) > h_\beta(\gamma)\}$$

The h_β will be defined in such a way that

(1) $\beta \leq \beta'$ implies $h_\beta \leq h_{\beta'}$ everywhere;

(2) $\forall \beta < |A|^+ \exists i_\beta \forall \alpha > i_\beta (b_\alpha^{\beta+1} \subset b_\alpha^\beta)$.

This will lead to a contradiction: Since $\mu > |A|^+$ is a regular cardinal, there is an $\alpha < \mu$ such that $i_\beta < \alpha$ for all $\beta < |A|^+$. Thus by (2) $(b_\alpha^\beta: \beta < |A|^+)$ is a *strictly* decreasing sequence of subsets of A, which is clearly impossible.

It remains to define the functions h_β such that (1) and (2) hold. Let $h_0 = g_0$, the least element of B. If $\beta < |A|^+$ is a limit cardinal, define $h_\beta(\alpha) = \sup\{h_\gamma(\alpha): \gamma < \beta\}$ for all $\alpha \in A$. Since each $\alpha \in A$ is regular $> |A|^+$, $h_\beta \in \Pi A$ as required.

Next we deal with successor ordinals: Suppose that h_β has been defined. By assumption, h_β is not an upper bound for B in $\Pi A/\mathcal{J}$, and hence for some α the set b_α^β is not an element of $\mathcal{I}$. Choose i_β to be the least such α. Because B is $\leq_{\mathcal{J}}$– increasing, it follows that $b_\alpha^\beta \notin \mathcal{J}$ for all $\alpha \geq i_\beta$. Since $b_{i_\beta}^\beta \notin \mathcal{J}$, there is an ultrafilter $\mathcal{F}$ over A such that $b_{i_\beta}^\beta \in \mathcal{F}$ and $cf(\Pi A/\mathcal{F}) \geq \lambda$. Thus there is $f \in \Pi A$ such that $f/\mathcal{F}$ is an upper bound for $\{g_\alpha/\mathcal{F}: \alpha < \mu\}$. Define

$$h_{\beta+1}(\gamma) = \max\{h_\beta(\gamma), f(\gamma)\} \text{ for all } \gamma \in A.$$

We must show that the sequence $(h_\beta : \beta < |A|^+)$ has the required properties (1) and (2).

Certainly (1) holds immediately. Also for any $\alpha < \mu$, $b_\alpha^{\beta+1} \subseteq b_\alpha^\beta$, since $g_\alpha(\gamma) > h_{\beta+1}(\gamma)$ implies $g_\alpha(\gamma) > h_\beta(\gamma)$. In order to prove (2) it thus suffices to show that for $\alpha \geq i_\beta$, $b_\alpha^{\beta+1} \neq b_\alpha^\beta$. We shall do so in two steps:

(i) $b_\alpha^\beta \in \mathcal{F}$ for all $\alpha \geq i_\beta$. Since $\mathrm{cf}(\Pi A/\mathcal{F}) \geq \lambda$, $\mathcal{F} \cap \mathcal{J} = \emptyset$. Now $b_{i_\beta}^\beta - b_\alpha^\beta \in \mathcal{J}$, and thus since $b_{i_\beta}^\beta \in \mathcal{F}$, also $b_\alpha^\beta \in \mathcal{F}$

(ii) $b_\alpha^{\beta+1} \notin \mathcal{F}$ for all $\alpha \geq i_\beta$. Since $f \leq h_{\beta+1}$ everywhere, and since $g_\alpha/\mathcal{F} \leq f/\mathcal{F}$ in $\Pi A/\mathcal{F}$, it follows that $b_\alpha^{\beta+1} \subseteq \{\gamma \in A: f(\gamma) < g_\alpha(\gamma)\} \notin \mathcal{F}$.

From (i) and (ii) we easily see that $b_\alpha^{\beta+1} \neq b_\alpha^\beta$ for all $\alpha \geq i_\beta$, proving (2) and thus the lemma.

□

Remark 6.1.7: (i) If $\kappa < \lambda$, then $\mathcal{J}_{<\kappa}(A) \subseteq \mathcal{J}_{<\lambda}(A)$.

(ii) If λ is a limit cardinal, then $\mathcal{J}_{<\lambda}(A) = \bigcup_{\kappa<\lambda} \mathcal{J}_{<\kappa}(A)$. By (i) we certainly have $\bigcup_{\kappa<\lambda} \mathcal{J}_{<\kappa}(A) \subseteq \mathcal{J}_{<\lambda}(A)$. On the other hand if $B \in \mathcal{J}_{<\lambda}(A) - \bigcup_{\kappa<\lambda} \mathcal{J}_{<\kappa}(A)$, we may choose an ultrafilter $\mathcal{F}$ over A such that $B \in \mathcal{F}$ and $\mathcal{F} \cap \mathcal{J}_{<\kappa}(A) = \emptyset$ for all $\kappa < \lambda$. Then by Lemma 6.1.5 we have $\mathrm{cf}(\Pi A/\mathcal{F}) < \lambda$ but $\geq \kappa$ for all $\kappa < \lambda$, a contradiction.

Lemma 6.1.8:
λ is a true cofinality of $\Pi A/\mathcal{I}$ for some ideal $\mathcal{I}$ over A if and only if $\mathcal{J}_{<\lambda}(A) \neq \mathcal{J}_{<\lambda^+}(A)$.

Proof: If $\mathcal{J}_{<\lambda}(A) \neq \mathcal{J}_{<\lambda^+}(A)$, there is a $B \in \mathcal{J}_{<\lambda^+}(A) - \mathcal{J}_{<\lambda}(A)$. Let $\mathcal{F}$ be an ultrafilter such that $B \in \mathcal{F}$ and $\mathrm{cf}(\Pi A/\mathcal{F}) \geq \lambda$. Then $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$, since $B \in \mathcal{J}_{<\lambda^+}(A)$, so the ideal $\mathcal{I}$ dual to $\mathcal{F}$ has the property that $\mathrm{tcf}(\Pi A/\mathcal{I}) = \lambda$.

Conversely, suppose that $\lambda = \mathrm{tcf}(\Pi A/\mathcal{I})$ for some ideal $\mathcal{I}$ over A. Extend the dual filter of $\mathcal{I}$ to an ultrafilter $\mathcal{F}$. Then $\mathcal{F} \cap \mathcal{I} = \emptyset$, and so $f \leq_\mathcal{I} g$ implies $f \leq_\mathcal{F} g$ for any $f, g \in \Pi A$. It follows that any increasing cofinal sequence in $\Pi A/\mathcal{I}$ is also increasing cofinal in $\Pi A/\mathcal{F}$, and thus $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$. It follows that there is $B \in \mathcal{F} \cap \mathcal{J}_{<\lambda^+}(A) - \mathcal{J}_{<\lambda}(A)$.

□

We shall now look a little more closely at the properties of the set

$$\mathrm{pcf}(A) = \{\lambda : \exists \text{ ultrafilter } \mathcal{F} \text{ over } A \text{ such that } \lambda = \mathrm{cf}(\Pi A/\mathcal{F})\}.$$

Lemma 6.1.9:

(a) $|\mathrm{pcf}(A)| \leq 2^{|A|}$.

(b) pcf(A) *has a maximal element.*

(c) $A \subseteq \mathrm{pcf}(A)$

(d) *If* $\min(A) > |\mathrm{pcf}(A)|$, *then* $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$

(e) $B \subseteq A$ *implies* $\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A)$

Proof: (a) $(\mathcal{J}_{<\lambda}(A)$: λ a cardinal) is an increasing sequence of subsets of $\mathcal{P}(A)$. Moreover, if $\lambda \in \mathrm{pcf}(A)$, then by Lemma 6.1.8, $\mathcal{J}_{<\lambda}(A) \subset \mathcal{J}_{<\lambda^+}(A)$. This is possible only if there are at most $|\mathcal{P}(A)|$– many $\lambda \in \mathrm{pcf}(A)$.

(b) First note that $A \in \mathcal{J}_{<\lambda}(A)$ for some cardinal λ. Otherwise $\bigcup \{\mathcal{J}_{<\lambda}(A): \lambda \text{ a cardinal}\}$ is a proper ideal over A. Let $\mathcal{F}$ be an ultrafilter extending its dual. Then $\mathcal{F} \cap \mathcal{J}_{<\lambda}(A) = \emptyset$ for all cardinals λ, but if $\mu = \mathrm{cf}(\Pi A/\mathcal{F})$, then $\mathcal{F} \cap \mathcal{J}_{<\mu^+}(A) \neq \emptyset$ by Theorem 6.1.5, a contradiction. We may therefore choose λ to be the least cardinal such that $A \in \mathcal{J}_{<\lambda}(A)$. Because $\mathcal{J}_{<\kappa}(A) = \bigcup_{\mu<\kappa} \mathcal{J}_{<\mu}(A)$, if κ is a limit cardinal, it follows by definition of λ that λ is successor, $\lambda = \kappa^+$ for some κ. Moreover $A \notin \mathcal{J}_{<\kappa}(A)$, again by definition of $\lambda = \kappa^+$ and thus it will follow from Lemma 6.1.8 that $\kappa \in \mathrm{pcf}(A)$ provided we can prove that κ is regular. However, if κ is singular, then $\mathcal{J}_{<\kappa^+}(A) = \mathcal{J}_{<\kappa}(A)$, a contradiction. Hence $\kappa \in \mathrm{pcf}(A)$, and κ is clearly an upper bound for pcf(A).

(c) Suppose $\alpha \in A$. Then α is a regular cardinal $> |A|^+$. Let $\mathcal{F}$ be the principal ultrafilter generated by α. It is clear that $\mathrm{cf}(\Pi A/\mathcal{F}) = \alpha$, and thus that $\alpha \in \mathrm{pcf}(A)$.

(d) Let $B = \mathrm{pcf}(A)$. By (c), $B \subseteq \mathrm{pcf}(B)$ and thus $\mathrm{pcf}(A) \subseteq \mathrm{pcf}(\mathrm{pcf}(A))$. Hence it suffices to show that $\mathrm{pcf}(B) \subseteq \mathrm{pcf}(A)$. Let $\lambda \in \mathrm{pcf}(A)$, and let $\mathcal{F}$ be an ultrafilter over B such that $\lambda = \mathrm{cf}(\Pi B/\mathcal{F})$. Similarly, for each $\beta \in B$, let $\mathcal{F}_\beta$ be an ultrafilter over A such that $\beta = \mathrm{cf}(\Pi A/\mathcal{F}_\beta)$. We may define an ultrafilter $\mathcal{F}^*$ over A by:

$$X \in \mathcal{F}^* \longleftrightarrow \{\beta \in B: X \in \mathcal{F}_\beta\} \in \mathcal{F}$$

For each $\beta \in B$, let $(f_\delta^\beta/\mathcal{F}_\beta: \delta < \beta)$ be an increasing cofinal sequence in $\Pi A/\mathcal{F}_\beta$, and let $(g_\delta/\mathcal{F}: \delta < \lambda)$ be cofinal in $\Pi B/\mathcal{F}$. Define, for $\delta < \lambda$, functions $h_\delta \in \Pi A$ as follows:

$$h_\delta(\alpha) = \sup\{f^\beta_{g_\delta(\beta)}(\alpha): \beta \in B\}$$

Then h_δ is indeed a member of A since $\min(A) > |pcf(A)| = |B|$. We intend to prove that:

$$\forall h \in \Pi A\ \exists \delta_1 < \lambda\ \forall \delta[\delta_1 \le \delta < \lambda \longrightarrow h/\mathcal{F}^* \le h_\delta/\mathcal{F}^*] \quad (\dagger)$$

Let $h \in A$, and for each $\beta \in B$ choose δ_β such that $h/\mathcal{F}_\beta \le f^\beta_{\delta_\beta}/\mathcal{F}_\beta$. Now choose $\delta_1 < \lambda$ such that $(\delta_\beta: \beta \in B)/\mathcal{F} \le g_{\delta_1}/\mathcal{F}$. Clearly if $\delta \ge \delta_1$, there is an $X \in \mathcal{F}$ such that for all $\beta \in X$, $\delta_\beta \le g_\delta(\beta)$. Let $Y = \{\alpha \in A: h(\alpha) \le h_\delta(\alpha)\}$ and let $\beta \in X$. There is $Y_\beta \in \mathcal{F}_\beta$ such that for all $\alpha \in Y_\beta$ we have $h(\alpha) \le f^\beta_{\delta_\beta}(\alpha) \le f^\beta_{g_\delta(\beta)}(\alpha) \le h_\delta(\alpha)$. It follows that $Y_\beta \subseteq Y$ for all $\beta \in X$ and thus that $Y \in \mathcal{F}^*$. Hence $(\dagger)$ holds. But having achieved $(\dagger)$, one easily obtains a subsequence of $(h_\delta/\mathcal{F}^*: \delta < \lambda)$ which is increasing cofinal in $\Pi A/\mathcal{F}^*$. It follows that $\lambda \in pcf(A)$, and thus that $pcf(B) \subseteq pcf(A)$ under the conditions stated.

(e) Suppose that $\mathcal{F}$ is an ultrafilter over A, $B \subseteq A$ and $B \in \mathcal{F}$. Let $\mathcal{F}|B = \{X \cap B: X \in \mathcal{F}\}$. It is easy to see that $\mathcal{F}|B$ is an ultrafilter over B, and that $\mathcal{F}|B \subseteq \mathcal{F}$. It is also not hard to show that $cf(\Pi A/\mathcal{F}) = cf(\Pi B/\mathcal{F}|B)$. With the aid of this fact we can prove (e): Let $B \subseteq A$, and let $\lambda \in pcf(B)$. Let $\mathcal{D}$ be an ultrafilter over B such that $\lambda = cf(\Pi B/\mathcal{D})$, and choose $\mathcal{D}^* \supseteq \mathcal{D}$ such that $\mathcal{D}^*$ is an ultrafilter over A. Then $B \in \mathcal{D}^*$ and $\mathcal{D}^*|B = \mathcal{D}$. It then follows by that $\lambda = cf(\Pi A/\mathcal{D}^*)$ and thus that $\lambda \in pcf(A)$. Hence $B \subseteq A$ implies $pcf(B) \subseteq pcf(A)$.

□

It follows from Lemma 6.1.9 that pcf(_) behaves very much like a closure operator. In particular if $\min(A) > 2^{|A|}$, then by (a) we have $\min(A) > |pcf(A)|$, and thus $pcf(pcf(A)) = pcf(A)$. Moreover if $B \subseteq A$, then $\min(B) \ge \min(A) > 2^{|A|} \ge 2^{|B|}$, so that we then have $pcf(pcf(B)) = pcf(B)$ for all $B \subseteq A$. Thus given that $A = pcf(A)$ and $\min(A) > 2^{|A|}$, pcf is exactly a closure operator on the subsets of A.

§ 6.2 A Technical Lemma

In this section we shall prove that if A is an interval of regular cardinals, then pcf(A) has some nice closure properties: Specifically, if $A = (\beta,\mu)$ is an interval of regular cardinals and $\lambda \in pcf(A)$, then any regular cardinal strictly between μ and λ is also in pcf(A)

(Lemma 6.2.2). In order to prove this, however, we shall need to prove the rather technical Lemma 6.2.3.

Definition 6.2.1: Suppose that A is a set of ordinals and $\mathcal{F}$ is an ultrafilter over A. We shall say $\mu = \lim_{\mathcal{F}}(A)$ if and only if for each $\beta < \mu$ the set $\{\alpha \in A: \beta < \alpha \leq \mu\}$ is in $\mathcal{F}$.

It is not hard to see that for each ultrafilter $\mathcal{F}$ there is a unique μ such that $\mu = \lim_{\mathcal{F}}(A)$: Let $\mu_0 = \sup(A)$. Given μ_n, define $\mu_{n+1} < \mu_n$ to be such that $\{\alpha \in A: \mu_{n+1} < \alpha \leq \mu_n\} \notin \mathcal{F}$, provided such a μ_{n+1} exists. This yields a descending and thus finite chain of ordinals. Hence for some $n < \omega$ there is no μ_{n+1}, i.e. for each $\beta < \mu_n$ the set $\{\alpha \in A: \beta < \alpha \leq \mu_n\} \in \mathcal{F}$. As in Section 6.1 we shall assume that A is an infinite set of regular cardinals with the property that $\min(A) > |A|^+$. $\mathcal{F}$ will be an ultrafilter over A.

Lemma 6.2.2:
If $A = (\beta,\mu)$ *is an infinite interval of regular cardinals,* $\lambda \in \text{pcf}(A)$ *and* λ' *is a regular cardinal such that* $\mu < \lambda' < \lambda$, *then* $\lambda \in \text{pcf}(A)$.

The proof of Lemma 6.2.2 is not hard once we know the statement of Lemma 6.2.3. The proof of Lemma 6.2.3 will be our main task in this section, and we shall have a complete proof only after Lemma 6.2.6.

Lemma 6.2.3:
Suppose $\mu = \lim_{\mathcal{F}}(A)$ *and* $\lambda = \text{cf}(\Pi A/\mathcal{F})$. *If* λ' *is regular and* $\mu < \lambda' < \lambda$, *then there exist a set* A' *of regular cardinals of cardinality* $\leq |A|$ *and an ultrafilter* $\mathcal{F}'$ *over* A' *such that* $\lim_{\mathcal{F}'}(A') = \mu$ *and* $\text{cf}(\Pi A'/\mathcal{F}') = \lambda'$.

Proof of Lemma 6.2.2: Suppose that $\mu < \lambda' < \lambda$, where $\lambda \in \text{pcf}(A)$. Choose an ultrafilter $\mathcal{F}$ over A such that $\lambda = \text{cf}(\Pi A/\mathcal{F})$, and let $\mu' = \lim_{\mathcal{F}}(A)$. Clearly $\mu' \leq \mu$, and thus $\mu' < \lambda' < \lambda$. By Lemma 6.2.3 there is a set A' of regular cardinals such that $|A'| \leq |A|$ and an ultrafilter $\mathcal{F}'$ over A' such that $\lim_{\mathcal{F}'}(A') = \mu'$ and $\text{cf}(\Pi A'/\mathcal{F}') = \lambda'$. In particular

$$B' = \{\alpha \in A': \beta < \alpha \leq \mu'\} \in \mathcal{F}'.$$

Let $\mathcal{D}' = \mathcal{F}'|B' = \{X \cap B': X \in \mathcal{F}'\}$. Then as in the proof of Lemma 6.1.9(e) we have $\text{cf}(\Pi B'/\mathcal{D}') = \text{cf}(\Pi A'/\mathcal{F}') = \lambda'$. Since $B' \subseteq (\beta,\mu)$ is a set of regular cardinals, we have $B' \subseteq A$. Thus $\lambda' \in \text{pcf}(B') \subseteq \text{pcf}(A)$, proving that $\lambda' \in \text{pcf}(A)$ as required.

□

Note that pcf(A) is not always an interval. For instance, if $A = \{\aleph_{2n+2}: n < \omega\}$, then $\aleph_{2n+3} \notin$ pcf(A) for any $n < \omega$. This is because if $\mathcal{F}$ is an ultrafilter over ω, and if $(f_\alpha/\mathcal{F}: \alpha < \aleph_{2n+3})$ is cofinal in $\Pi A/\mathcal{F}$, then g defined by

$$g(m) = \begin{cases} \sup\{f_\alpha(m): \alpha < \aleph_{2n+3}\} & \text{if } m > n \\ \emptyset & \text{otherwise} \end{cases}$$

clearly is an upper bound for $(f_\alpha: \alpha < \aleph_{2n+3})$ in ΠA modulo $\mathcal{F}$.

It is now time to turn to the proof of Lemma 6.2.3. We begin with some definitions:

Definition 6.2.4: Suppose that $h \in On^\kappa$, and let $\mathcal{F}$ be an ultrafilter over κ. Suppose that $(f_\alpha/\mathcal{F}: \alpha < \lambda)$ is an increasing sequence in $On^\kappa/\mathcal{F}$. We shall say that $h/\mathcal{F}$ *cuts* $(f_\alpha/\mathcal{F}: \alpha < \lambda)$ provided that there are ordinals $\alpha < \beta < \lambda$ such that $f_\alpha/\mathcal{F} < h/\mathcal{F} < f_\beta/\mathcal{F}$. Similarly, if $\mathcal{A} \subseteq On^\kappa$, we shall say that $\mathcal{A}$ *cofinally cuts* $(f_\alpha/\mathcal{F}: \alpha < \lambda)$ if and only if for each $\gamma < \lambda$ there is an $h/\mathcal{F} \in \mathcal{A}$ such that $h/\mathcal{F}$ cuts $(f_\alpha/\mathcal{F}: \alpha < \lambda)$ and $f_\gamma/\mathcal{F} < h/\mathcal{F}$.

Lemma 6.2.5:

Let $\mathcal{F}$ be an ultrafilter over an infinite cardinal κ, $\lambda > \kappa^+$ a regular cardinal and $(f_\alpha/\mathcal{F}: \alpha < \lambda)$ an increasing sequence in $On^\kappa/\mathcal{F}$. Then either

(1) *$(f_\alpha/\mathcal{F}: \alpha < \lambda)$ has a least upper bound in $On^\kappa/\mathcal{F}$, or else*

(2) *There are sets of ordinals S_α (for $\alpha < \kappa$) such that $|S_\alpha| \leq \kappa$ and such that $(\Pi_\alpha S_\alpha)/\mathcal{F}$ cofinally cuts $(f_\alpha/\mathcal{F}: \alpha < \lambda)$.*

Proof: By induction on $\beta < \kappa^+$ we shall attempt to define a decreasing sequence $(h_\beta/\mathcal{F}: \beta < \kappa^+)$ such that each $h_\beta/\mathcal{F}$ is an upper bound for $(f_\alpha/\mathcal{F}: \alpha < \lambda)$. For h_0 we take any upper bound with the property that for all $\delta < \kappa$ and all $\alpha < \lambda$ we have $f_\alpha(\delta) < h_0(\delta)$. Suppose that h_β has already been defined. If $h_\beta/\mathcal{F}$ is the least upper bound of $(f_\alpha/\mathcal{F}: \alpha < \lambda)$, then we are done and $h_{\beta+1}$ does not exist; otherwise choose $h_{\beta+1}$ such that $h_{\beta+1}/\mathcal{F} < h_\beta/\mathcal{F}$ is an upper bound for $(f_\alpha/\mathcal{F}: \alpha < \lambda)$.

It now remains to define h_β in case β is a limit ordinal, assuming that h_γ has been defined for all $\gamma < \beta$. For each $\delta < \kappa$, set $S_\delta = \{h_\gamma(\delta): \gamma < \beta\}$; since $\beta < \kappa^+$, $|S_\delta| \leq \kappa$ for $\delta < \kappa$.

Define g_α (for $\alpha < \lambda$) by:

$$g_\alpha(\delta) = \begin{cases} \text{minimum element of } S_\delta \text{ which is greater than } f_\alpha(\delta), \text{ where this exists} \\ \emptyset \text{ elsewhere} \end{cases}$$

Since each $h_\gamma/\mathcal{F}$ is an upper bound for $(f_\alpha/\mathcal{F}: \alpha < \lambda)$, it follows that $g_\alpha/\mathcal{F} \leq h_\gamma/\mathcal{F}$ for all $\gamma < \beta$. Also, if $\alpha < \alpha'$, then $f_\alpha/\mathcal{F} \leq f_{\alpha'}/\mathcal{F}$, and thus $g_\alpha/\mathcal{F} \leq g_{\alpha'}/\mathcal{F}$.

There are now two possibilities for the sequence $(g_\alpha/\mathcal{F}: \alpha < \lambda)$:

__Case 1__: $(g_\alpha/\mathcal{F}: \alpha < \lambda)$ is not eventually constant. In that case, if $\alpha < \alpha'$ and $g_\alpha/\mathcal{F} < g_{\alpha'}/\mathcal{F}$ then $g_\alpha/\mathcal{F} \leq f_{\alpha'}/\mathcal{F}$. Now each $g_\alpha \in \Pi_\delta S_\delta$ and $|S_\delta| \leq \kappa$. Hence $(\Pi_\delta S_\delta)/\mathcal{F}$ cofinally cuts $(f_\alpha/\mathcal{F}: \alpha < \lambda)$, which is the 2nd possibility given by the Lemma. So h_β does not exist.

__Case 2__: $(g_\alpha/\mathcal{F}: \alpha < \lambda)$ is eventually constant. Define $h_\beta/\mathcal{F}$ to be the constant value of this sequence. $h_\beta/\mathcal{F}$ is an upper bound for the sequence $(f_\alpha/\mathcal{F}: \alpha < \lambda)$; moreover, since $g_\alpha/\mathcal{F} \leq h_\gamma/\mathcal{F}$ for all $\alpha < \lambda$ and all $\gamma < \beta$, we also have $h_\beta/\mathcal{F} \leq h_\gamma/\mathcal{F}$ for all $\gamma < \beta$.

In this case we can define h_β.

Suppose we can define h_β for all $\beta < \kappa^+$ (i.e. no $h_\beta/\mathcal{F}$ is the least upper bound for $(f_\alpha/\mathcal{F}: \alpha < \lambda)$, and Case 1 never holds at limit stages). We shall obtain a ontradiction. Proceed in very much the same way as before. For $\delta < \kappa$, define

$$\overline{S}_\delta = \{h_\beta(\delta): \beta < \kappa^+\}$$

and similarly define $\overline{g}_\alpha$ for $\alpha < \lambda$ by:

$$\overline{g}_\alpha(\delta) = \begin{cases} \text{least element of } \overline{S}_\delta \text{ which is greater than } f_\alpha(\delta), \text{ where this exists} \\ \emptyset \text{ elsewhere} \end{cases}$$

Thus $\overline{g}_\alpha/\mathcal{F} \leq h_\beta/\mathcal{F}$ for all $\alpha < \lambda$ and all $\beta < \kappa^+$. For each $\alpha < \lambda$ we may choose a limit ordinal $\beta_\alpha < \kappa^+$ such that $\forall\delta{<}\kappa\exists\beta'{<}\beta_\alpha(\overline{g}_\alpha(\delta) < h_{\beta'}(\delta))$. Now λ is regular $> \kappa$ and hence there is an unbounded subset $C \subseteq \lambda$, such that for all $\alpha \in C$ the β_α take on the same value β. It follows that $[\alpha \in C \longrightarrow (\forall\delta < \kappa)(\exists\beta' < \beta)(\overline{g}_\alpha(\delta) < h_{\beta'}(\delta))]$.

At stage β, the g_α's defined would exactly coincide with the $\overline{g}_\alpha$'s for $\alpha \in C$. Since the induction proceeded, $h_\beta/\mathcal{F}$ is the constant value of the eventually constant sequence $(g_\alpha: \alpha < \lambda)$. Choose $\alpha \in C$ such that $\beta_\alpha = \beta$ and $g_\alpha/\mathcal{F} = h_\beta/\mathcal{F}$. Then

$\overline{g}_\alpha/\mathcal{F} = g_\alpha/\mathcal{F} = h_\beta/\mathcal{F}$. Now $h_\beta/\mathcal{F} < h_{\beta+1}/\mathcal{F}$, but $\overline{g}_\alpha/\mathcal{F} \leq h_{\beta+1}/\mathcal{F}$, a contradiction. Hence the induction must stop at some $\beta < \kappa^+$. If β is a successor ordinal, then we have arranged the first alternative in the statement of the Lemma, and if β is a limit ordinal, the 2^{nd} alternative holds.

□

In order to prove Lemma 6.2.3 we shall define an increasing sequence $(f_\alpha/\mathcal{F}: \alpha < \lambda)$ of members of $\Pi A/\mathcal{F}$ such that the second alternative of Lemma 6.2.5 is ruled out, i.e. there are no sets of ordinals S_δ $(\delta < \kappa)$ with $|S_\delta| \leq \kappa$ such that $(\Pi_\delta S_\delta)/\mathcal{F}$ cofinally cuts $(f_\alpha/\mathcal{F}: \alpha < \lambda)$. For each $\alpha < \lambda'$, choose a club subset $C_\alpha \subsetneq \alpha$ of order type $cf(\alpha)$. Further define $\mathcal{C}_\alpha = \{C_\beta \cap \alpha: \beta < \lambda'\}$ for each $\alpha < \lambda'$. The sequence $(\mathcal{C}_\alpha: \alpha < \lambda')$ has the following properties:

(1) $\mathcal{C}_\alpha \subsetneq \mathcal{P}(\alpha)$ for $\alpha < \lambda'$.

(2) $|\mathcal{C}_\alpha| \leq \lambda'$.

(3) There is a set $E \in \mathcal{C}_\alpha$ such that E is club in α and $otp(E) = cf(\alpha)$.

(4) If $\beta < \alpha$ and $X \in \mathcal{C}_\alpha$, then $X \cap \beta \in \mathcal{C}_\beta$.

[Burke–Magidor 1990] calls a family of sets $(\mathcal{C}_\alpha: \alpha < \lambda')$ with properties (1) – (4) a "silly square sequence", and any such sequence will do for our purposes.

Let $f_0/\mathcal{F} \in \Pi A/\mathcal{F}$ be arbitrary. Next suppose we have already defined $f_\gamma/\mathcal{F}$ for all $\gamma < \beta$, where $\beta < \lambda'$. We must define $f_\beta \in \Pi A$. Since $\beta < \lambda' < \lambda$, there is $h_\beta/\mathcal{F} \in \Pi A/\mathcal{F}$ such that $f_\gamma/\mathcal{F} < h_\beta/\mathcal{F}$ for all $\gamma < \beta$ (Recall that $cf(\Pi A/\mathcal{F}) = \lambda$). If $E \in \mathcal{C}_\beta$, define $g_E{}^\beta \in \Pi A$ by:

$$g_E{}^\beta(\alpha) = \max(h_\beta(\alpha), \sup\{f_\gamma(\alpha): \gamma \in E, \alpha > otp(E)\})$$

Note that if $\alpha > otp(E)$, then $g_E{}^\beta(\alpha) \in \alpha$, so $g_E{}^\beta \in \Pi A$.

Now define $f_\beta/\mathcal{F}$ to be a strict upper bound for $\{g_E{}^\beta/\mathcal{F}: E \in \mathcal{C}_\beta\}$. This completes the inductive definition of $(f_\alpha/\mathcal{F}: \alpha < \lambda')$.

Lemma 6.2.6:

The second alternative of Lemma 6.2.5 *fails: There do not exist* $\mu' < \mu$ *and* $S_\alpha \subsetneq \alpha$ (*for* $\alpha \in A$) *such that* $|S_\alpha| \leq \mu'$ *and* $(\Pi_\alpha S_\alpha)/\mathcal{F}$ *cofinally cuts* $(f_\alpha/\mathcal{F}: \alpha < \lambda')$.

Proof: Suppose that some such $\mu' < \mu$ and $S_\alpha \subsetneq \alpha$ with the above properties do exist. We shall arrive at a contradiction. It is clear that we may assume $\mu' > |A|$, since

$|A|^+ < \min(A) < \mu$ and $\mu < \lambda' < \lambda$. Let $B \subsetneq \lambda'$ be closed unbounded such that if $\gamma, \gamma' \in B$ and $\gamma < \gamma'$, then there exists $k \in \Pi_\alpha S_\alpha$ for which we have $f_\gamma/\mathcal{F} \leq k/\mathcal{F} \leq f_{\gamma'}/\mathcal{F}$. Such a B exists because the set $(\Pi_\alpha S_\alpha)/\mathcal{F}$ is assumed to cofinally cut $(f_\alpha/\mathcal{F}\colon \alpha < \lambda)$. Let β be a singular limit point of B such that $\mathrm{otp}(\beta) = (\mu')^+ \leq \mu$, and let $E \in \mathcal{C}_\beta$ be club in β such that $\mathrm{otp}(E) = \mathrm{cf}(\beta)$. Because β is a limit point of B, it follows that $E \cap B$ is club in β.

Let $E \cap B = \{\gamma_i\colon i < \mathrm{cf}(\beta)\}$ be an increasing enumeration, and for $i < \mathrm{cf}(\beta)$, choose $k_i \in \Pi_\delta S_\delta$ such that $f_{\gamma_i}/\mathcal{F} \leq k_i/\mathcal{F} \leq f_{\gamma_{i+1}}/\mathcal{F}$. Of course k_i exists by definition of B.

By the properties of the sequence $(\mathcal{C}_\alpha\colon \alpha < \lambda')$, $E \cap \gamma_i$ is an element of $\mathcal{C}_{\gamma_i}$ for all $i < \mathrm{cf}(\beta)$, and thus $f_{\gamma_i}/\mathcal{F} > g^{\gamma_i}_{E\cap\gamma_i}/\mathcal{F}$ for all $i < \mathrm{cf}(\beta)$. Also if $\alpha > \mathrm{otp}(E \cap \gamma_i)$, then $g^{\gamma_i}_{E\cap\gamma_i}(\alpha) \geq f_{\gamma_j}(\alpha)$ for all $j < i$ (by definition of $g^{\gamma_i}_{E\cap\gamma_i}$). This allows us to pick, for each $i < \mathrm{cf}(\beta)$, an $\alpha_i > \mathrm{otp}(E)$ such that:

(1) $f_{\gamma_i}(\alpha_i) \leq k_i(\alpha_i) \leq f_{\gamma_{i+1}}(\alpha_i)$, and

(2) $f_{\gamma_i}(\alpha_i) > g^{\gamma_i}_{E\cap\gamma_i}(\alpha_i)$.

Since $\mathrm{cf}(\beta) = (\mu')^+$ and $|A| < \mu'$, there are an ordinal α and a set $I \subsetneq \mathrm{cf}(\beta)$ of limit ordinals such that $|I| = \mathrm{cf}(\beta)$ and α_i takes on the constant value α for all $i \in I$.

If $i, j \in I$ and $i < j$, we see that $k_i(\alpha) \leq f_{\gamma_{i+1}}(\alpha) \leq g^{\gamma_j}_{E\cap\gamma_j}(\alpha) < f_{\gamma_j}(\alpha) \leq k_j(\alpha)$, and thus that the sequence $(k_i(\alpha)\colon i \in I)$ is strictly increasing. But $|S_\alpha| \leq \mu' < (\mu')^+ = \mathrm{cf}(\beta)$, so this is impossible.

□

Proof of Lemma 6.2.3: We have defined a sequence $(f_\alpha/\mathcal{F}\colon \alpha < \lambda')$ for which the second alternative of Lemma 6.2.5 fails, and thus the first alternative must hold, i.e.

$(f_\alpha/\mathcal{F}\colon \alpha < \lambda')$ must have a least upper bound in $\mathrm{On}^\kappa/\mathcal{F}$. Let this least upper bound be $g/\mathcal{F}$. We may make various assumptions about the function g. Firstly, because $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda > \lambda'$, we may assume that $g(\alpha) < \alpha$ for all $\alpha \in A$. Secondly, because $(f_\alpha/\mathcal{F}\colon \alpha < \lambda')$ is strictly increasing with least upper bound $g/\mathcal{F}$, we may assume that $g(\alpha)$ is a limit ordinal for each $\alpha \in A$ (since $\{\alpha \in A\colon g(\alpha) \text{ is limit}\} \in \mathcal{F}$ anyway).

For each $\alpha \in A$, fix a club $S_\alpha \subseteq g(\alpha)$ of order type $cf(g(\alpha))$, and let

$$S_\alpha = \{S_\alpha(i): i < cf(g(A))\}$$

be an increasing enumeration of S_α. In order to complete the proof of Lemma 6.2.3 we shall need the following four claims.

<u>Claim 1</u>: $\mathrm{Lim}_{\mathcal{F}}\{cf(g(\alpha)) : \alpha \in A\} = \mu$.

Certainly because $\lim_{\mathcal{F}} A = \mu$, we cannot have $\lim_{\mathcal{F}}\{cf(g(\alpha)): \alpha \in A\} > \mu$. Suppose therefore that $\lim_{\mathcal{F}}\{cf(g(\alpha)): \alpha \in A\} = \mu' < \mu$. Then there is a set $B \in \mathcal{F}$ such that $cf(g(\alpha)) \leq \mu'$ for each $\alpha \in B$. Each $g(\alpha)$ is a limit ordinal, so for any $\beta_0 < \lambda'$ there is a $g' \in \Pi_\alpha S_\alpha$ such that $f_{\beta_0}/\mathcal{F} < g'/\mathcal{F} < g/\mathcal{F}$. However, because $g/\mathcal{F}$ is the least upper bound of $(f_\alpha/\mathcal{F}: \alpha < \lambda')$, there is a $\beta_1 > \beta_0$ such that $g'/\mathcal{F} < f_{\beta_1}/\mathcal{F}$. It follows that $\Pi_\alpha S_\alpha/\mathcal{F}$ cofinally cuts $(f_\alpha/\mathcal{F}: \alpha < \lambda')$, a contradiction of Lemma 6.2.6. This proves Claim 1.

For $\beta < \lambda'$, define $\bar{f}_\beta(\alpha) = S_\alpha(i)$ for the least i such that $f_\beta(\alpha) \leq S_\alpha(i)$, provided that such i exists.

<u>Claim 2</u>: $(\bar{f}_\beta/\mathcal{F}: \beta < \lambda')$ is cofinal in $(\Pi_\alpha S_\alpha)/\mathcal{F}$.

Suppose that $f \in \Pi_\alpha S_\alpha$. Then $f/\mathcal{F} < g/\mathcal{F}$, and thus there is a $\beta < \lambda'$ such that $f/\mathcal{F} < f_\beta/\mathcal{F}$.

Since we clearly also have $f_\beta/\mathcal{F} \leq \bar{f}_\beta/\mathcal{F} \in (\Pi_\alpha S_\alpha)/\mathcal{F}$, the result follows.

<u>Claim 3</u>: $cf((\Pi_\alpha S_\alpha)/\mathcal{F}) = \lambda'$.

For suppose that $\mathcal{E}$ is a subset of $(\Pi_\alpha S_\alpha)/\mathcal{F}$ of cardinality $< \lambda'$. Choose $\beta_0 < \lambda'$ such that each member of $\mathcal{E}$ has an upper bound in $\{\bar{f}_\beta/\mathcal{F}: \beta < \beta_0\}$. Now each $\bar{f}_\beta/\mathcal{F} < g/\mathcal{F}$, so there is for each $\beta < \beta_0$ an ordinal $\xi(\beta)$ such that $\bar{f}_\beta/\mathcal{F} < f_{\xi(\beta)}/\mathcal{F}$. Let $\beta_1 < \lambda$ be an upper bound for $\{\xi(\beta): \beta < \lambda'\}$. Then clearly $\bar{f}_\beta/\mathcal{F} < f_{\beta_1}/\mathcal{F} \leq \bar{f}_{\beta_1}/\mathcal{F}$ for all $\beta < \beta_0$. Thus in particular $f/\mathcal{F} < \bar{f}_{\beta_1}/\mathcal{F}$ for all $f/\mathcal{F} \in \mathcal{E}$. Thus $\mathcal{E}$ is not cofinal in $(\Pi_\alpha S_\alpha)/\mathcal{F}$. In conjunction with Claim 2 we see that $cf((\Pi_\alpha S_\alpha)/\mathcal{F}) = \lambda'$ as required.

Let $A' = \{cf(g(\alpha)): \alpha \in A\}$. Define $\mathcal{F}'$ over A' as follows:

$$X \in \mathcal{F}' \longleftrightarrow \{\alpha \in A: cf(g(\alpha)) \in X\} \in \mathcal{F}.$$

Then $\mathcal{F}'$ is an ultrafilter over A'. Moreover $\lim_{\mathcal{F}'} A' = \mu$ by Claim 1 (because $\{\alpha \in A: cf(g(\alpha)) \geq \mu'\} \in \mathcal{F}$ for any $\mu' < \mu$). Thus to complete the proof of Lemma 6.2.3 it suffices to prove the following claim:

<u>Claim 4</u>: $cf(\Pi A'/\mathcal{F}') = \lambda'$.

For $\beta < \lambda'$, define T'_β by:

$$T'_\beta(\mathrm{cf}(g(\alpha)) = \sup\{i < \mathrm{cf}(g(\alpha)) : \exists\gamma \in A[\mathrm{cf}(g(\gamma)) = \mathrm{cf}(g(\alpha)) \text{ and } T_\beta(\gamma) = S_\gamma(i)]\}.$$

It is easy to see that $T'_\beta(\mathrm{cf}(g(\alpha)) < \mathrm{cf}(g(\alpha))$ for almost all $\alpha \in A$ (mod $\mathcal{F}$) and thus

$T'_\beta/\mathcal{F}' \in \Pi A'/\mathcal{F}'$. Suppose that $f' \in \Pi A'$; define $f \in \Pi_\alpha S_\alpha$ by:

$$f(\alpha) = S_\alpha(f'(\mathrm{cf}(g(\alpha)))).$$

Then for some $\beta < \lambda'$ we have $f/\mathcal{F} < T_\beta/\mathcal{F}$, and hence $f'/\mathcal{F}' < T'_\beta/\mathcal{F}'$. It follows that $\mathrm{cf}(\Pi A'/\mathcal{F}) \le \lambda'$. That actual equality holds follows as for Claim 3. This completes the proof of Claim 4 and thus Lemma 6.2.3.

□

§ 6.3 True Cofinalities and Ideals

In this section we examine the cofinality of $\Pi A/\mathcal{I}$ if $\mathcal{I}$ is not a maximal ideal. The usual hypotheses on A are assumed to hold throughout this section, i.e. each $\alpha \in A$ is a regular cardinal such that $\alpha > |A|$.

Lemma 6.3.1:
Let $\mathcal{I}$ be any ideal over A, λ a regular cardinal and $(f_\alpha : \alpha < \lambda)$ a sequence of members of ΠA. *If $(f_\alpha/\mathcal{I} : \alpha < \lambda)$ is increasing and unbounded in $\Pi A/\mathcal{I}$, then there is a sequence $(B_\gamma : \gamma < \lambda)$ of subsets of* A *such that for $\gamma, \gamma_1, \gamma_2 < \lambda$ we have:*

(a) $B_0 \notin \mathcal{I}$.

(b) $\gamma_1 < \gamma_2 \rightarrow B_{\gamma_1} \subseteq_{\mathcal{I}} B_{\gamma_2}$.

(c) $((f_\rho | B_\gamma)/\mathcal{I} : \rho < \lambda)$ *is cofinal in* $\Pi B_\gamma/\mathcal{I}$.

Moreover there is a map $g \in \Pi A$ *which is an upper bound for $(f_\gamma : \gamma < \lambda)$ modulo the ideal generated by $\{B_\gamma : \gamma < \lambda\} \cup \mathcal{I}$.*

Proof: We may assume without loss of generality that each $\alpha \in A$ satisfies $\alpha > |A|^+$. The sequence $(f_\gamma/\mathcal{I} : \gamma < \lambda)$ is unbounded in $\Pi A/\mathcal{I}$, and thus it follows that $\lambda > |A|^+$. Assume that the statement of Lemma 6.3.1 is false. Inductively we will define a sequence of maps

$g_\alpha \in \Pi A$ for each $\alpha < |A|^+$ such that:

$$\alpha < \beta < |A|^+ \longrightarrow \forall\delta \in A[g_\alpha(\delta) \leq g_\beta(\delta)]$$

We may then define subsets of A:

$$B_\gamma^{\ \alpha} = \{\delta \in A: g_\alpha(\delta) < f_\gamma(\delta)\} \quad (\alpha < |A|^+, \gamma < \lambda)$$

It will turn out that for all α there is a $\gamma(\alpha) < \lambda$ such that for all $\gamma \geq \gamma(\alpha)$, we have $B_\gamma^{\ \alpha+1} \subset B_\gamma^{\ \alpha}$ (i.e. a *proper* subset). Then since $\lambda > |A|^+$, there must be a $\gamma_0 < \lambda$ such that $\gamma(\alpha) \leq \gamma_0$ for all $\alpha < |A|^+$. Hence for all $\gamma \geq \gamma_0$ and all $\alpha < |A|^+$ we have $B_\gamma^{\ \alpha+1} \subset B_\gamma^{\ \alpha}$, thus yielding a strictly descending sequence $(B_\gamma^{\ \alpha}: \gamma_0 \leq \gamma < \lambda)$ of subsets of A whose length is $\lambda > |A|^+$, clearly a contradiction.

First note some basic facts concerning the $B_\gamma^{\ \alpha}$:

(1) Each g_α is an upper bound for the sequence $(f_\gamma: \gamma < \lambda)$ modulo the ideal generated by $I \cup \{B_\gamma^{\ \alpha}: \gamma < \lambda\}$; this follows directly from the definition of the $B_\gamma^{\ \alpha}$.

(2) If $\alpha < |A|^+$ and $\gamma_1 < \gamma_2 < \lambda$, then $B_{\gamma_1}^{\ \alpha} \subseteq_I B_{\gamma_2}^{\ \alpha}$ because $(f_\gamma/I: \gamma < \lambda)$ is increasing.

(3) For each $\alpha < |A|^+$ there is an $\xi(\alpha) < \lambda$ such that $\gamma \geq \xi(\alpha)$ implies that $B_\gamma^{\ \alpha} \notin I$. This holds because the sequence $(f_\gamma/I: \gamma < \lambda)$ is increasing and unbounded, so g_α cannot be an upper bound.

(4) If $\alpha < \beta$, then $B_\gamma^{\ \alpha} \supseteq B_\gamma^{\ \beta}$. In particular, $B_\gamma^{\ \alpha+1} \subseteq B_\gamma^{\ \alpha}$ for all $\alpha < |A|^+$. We aim to ensure that the latter inclusion is a proper inclusion.

To start off the inductive definition of the g_α, let g_0 be an arbitrary member of ΠA.

At limit α < ?A?$^+$, define $g_\alpha(\delta) = \sup\{g_\beta(\delta): \beta < \alpha\}$. Since $\alpha < \delta$ and $\delta \in A$ is regular, $g_\alpha \in \Pi A$.

Suppose now that we know g_α, and that $\xi(\alpha)$ is an ordinal such that $\gamma \geq \xi(\alpha)$ implies $B_\gamma^{\ \alpha} \notin I$. There is a $\gamma(\alpha) \geq \xi(\alpha)$ such that $((f_\rho|B_{\gamma(\alpha)}^{\ \alpha})/I: \rho < \lambda)$ is not cofinal in $(\Pi B_{\gamma(\alpha)}^{\ \alpha})/I$. The reason for this is quite tricky: We are assuming that Lemma 6.3.1 fails. Now if we define $B_\gamma = B_\gamma^{\ \alpha}$ for $\gamma \geq \xi(\alpha)$ and $B_\gamma = B_{\xi(\alpha)}^{\ \alpha}$ for $\gamma < \xi(\alpha)$, then (a),(b) of the Lemma hold automatically. Hence (c) must fail, yielding the required $\gamma(\alpha)$.

Choose $h \in \Pi A$ such that $(h|B_{\gamma(\alpha)}^{\ \alpha})/I$ is not bounded by $(f_\rho|B_{\gamma(\alpha)}^{\ \alpha})/I$ for any $\rho < \lambda$., and define $g_{\alpha+1}(\delta) = \max(g_\alpha(\delta), h(\delta))$ for $\delta \in A$. This completes the inductive definition of all the g_α. Clearly we have $\alpha < \beta < |A|^+ \longrightarrow \forall\delta \in A[g_\alpha(\delta) \leq g_\beta(\delta)]$, so it remains to prove that indeed $B_\gamma^{\alpha+1} \subset B_\gamma^\alpha$ for all $\gamma \geq \gamma(\alpha)$: Let $\gamma \geq \gamma(\alpha)$. Note that then $(h|B_\gamma^\alpha)/\mathcal{I}$ is not bounded by $(f_\rho|B_\gamma^\alpha)/\mathcal{I}$ for any $\rho < \lambda$. So we may choose $\delta \in B_\gamma^\alpha$ such that $f_\gamma(\delta) < h(\delta)$. Then $g_{\alpha+1}(\delta) > f_\gamma(\delta)$, so $\delta \notin B_\gamma^{\alpha+1}$.

□

Recall that a partially ordered set $\mathbb{P}$ has a *true* cofinality if and only if there is an *increasing* sequence of members of $\mathbb{P}$ such that every element of $\mathbb{P}$ is bounded above by some member of the sequence. The true cofinality of $\mathbb{P}$ is then the smallest cardinal κ for which there exists such a sequence of length κ. The true cofinality of $\mathbb{P}$, if it exists, is denoted by $\mathrm{tcf}(\mathbb{P})$.

Corollary 6.3.2:
If $\mathcal{I}$ is an ideal over A, $\Pi A/\mathcal{I}$ *is λ-directed, $\mathcal{F}$ is an ultrafilter over* A *disjoint from $\mathcal{I}$ and* $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$, *then there is a* $B \in \mathcal{F}$ *such that* $\mathrm{tcf}(\Pi B/\mathcal{I}) = \lambda$.

Proof: Let $(f_\rho: \rho < \lambda)$ be an increasing cofinal sequence in $\Pi A/\mathcal{F}$. Since $\Pi A/\mathcal{I}$ is λ–directed, we may assume that $(f_\rho/\mathcal{I}: \rho < \lambda)$ is increasing in $\Pi A/\mathcal{I}$ (by repacing each $f_\rho/\mathcal{I}$ by an upper bound $f_\rho'/\mathcal{I}$ for $\{f_\beta'/\mathcal{I}: \beta < \rho\} \cup \{f_\rho/\mathcal{I}\}$ if need be.) Because $(f_\rho/\mathcal{F}: \rho < \lambda)$ is unbounded in $\Pi A/\mathcal{F}$ and $\mathcal{F} \cap \mathcal{I} = \emptyset$, $(f_\rho/\mathcal{I}: \rho < \lambda)$ is unbounded in $\Pi A/\mathcal{I}$. Thus by Lemma 6.3.1 there is a sequence $(B_\gamma: \gamma < \lambda)$ of subsets of A such that $((f_\rho|B_\gamma)/\mathcal{I}: \rho < \lambda)$ is cofinal in $\Pi B_\gamma/\mathcal{I}$ for each $\gamma < \lambda$. Hence $\mathrm{tcf}(\Pi B_\gamma/\mathcal{I}) = \lambda$, and $(f_\rho: \rho < \lambda)$ is bounded in ΠA modulo the ideal $\mathcal{J}$ generated by $\mathcal{I} \cup \{B_\gamma: \gamma < \lambda\}$. Now $\mathcal{F} \cap \mathcal{J} \neq \emptyset$, because otherwise $(f_\rho/\mathcal{F}: \rho < \lambda)$ is bounded in $\Pi A/\mathcal{F}$, a contradiction. Hence for some $\gamma < \lambda$, $B_\gamma \in \mathcal{F}$. This completes the proof.

□

The next Corollary gives a sufficient condition for $\Pi A/\mathcal{I}$ to have a *true cofinality*.

Corollary 6.3.3:
Let $\mathcal{I}$ be an ideal over A *such that for each ultrafilter $\mathcal{F}$ over* A *disjoint from $\mathcal{I}$, we have* $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$. *Then* $\mathrm{tcf}(\Pi A/\mathcal{I}) = \lambda$.

Proof: First note that $\mathcal{J}_{<\lambda}(A) \subseteq \mathcal{I}$: If $B \in \mathcal{J}_{<\lambda}(A) - \mathcal{I}$, there is an ultrafilter $\mathcal{F}$ over A such that $B \in \mathcal{F}$ and $\mathcal{F} \cap \mathcal{I} = \emptyset$. It follows that $\mathrm{cf}(\Pi A/\mathcal{F}) < \lambda$, a contradiction. Hence by Lemma

6.1.6, $\Pi A/\mathcal{I}$ is λ–directed. Consider the ideal

$$\mathcal{I}^* = \{B \subseteq A: B \in \mathcal{I} \vee \mathrm{tcf}(\Pi B/\mathcal{I}) = \lambda\}$$

We shall show that $\mathcal{I}^*$ is *improper*, i.e. $A \in \mathcal{I}^*$. If not, there is an ultrafilter $\mathcal{F}$ disjoint from $\mathcal{I}^*$. Then $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$, so by Corollary 6.3.2, there is a $B \in \mathcal{F}$ such that $\mathrm{tcf}(\Pi B/\mathcal{I}) = \lambda$. Then $\mathcal{F}$ cannot be disjoint from $\mathcal{I}^*$, so $A \in \mathcal{I}^*$ as required.

□

Corollary 6.3.4:
If $B \in \mathcal{J}_{<\lambda^+}(A) - \mathcal{J}_{<\lambda}(A)$, *then* $\mathrm{tcf}(\Pi B/\mathcal{J}_{<\lambda}(A)) = \lambda$.

Proof: Let $\mathcal{I}$ be the ideal generated by $\mathcal{J}_{<\lambda}(A) \cup \{A-B\}$, and let $\mathcal{F}$ be an ultrafilter over A which is disjoint from $\mathcal{I}$. Then $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$: $\mathrm{cf}(\Pi A/\mathcal{F}) \geq \lambda$ because $\mathcal{F}$ is disjoint from $\mathcal{J}_{<\lambda}(A)$, and $\mathrm{cf}(\Pi A/\mathcal{F}) < \lambda^+$ because $A - B \in \mathcal{I}$ implies $B \in \mathcal{F}$. Thus by Corollary 6.3.3, $\mathrm{tcf}(\Pi A/\mathcal{I}) = \lambda$, and since $A - B \in \mathcal{I}$, this implies that $\mathrm{tcf}(\Pi B/\mathcal{J}_{<\lambda}(A)) = \lambda$ as well.

□

The last Lemma of this section will be useful in Section 4:

Lemma 6.3.5:
If $\min(A) > 2^{|A|}$, *then there is a* $C \subseteq A$ *such that* $\mathcal{J}_{<\lambda^+}(A)$ *is the ideal generated by* $\mathcal{J}_{<\lambda}(A) \cup \{C\}$.

Proof: Let $\mathcal{I} = \mathcal{J}_{<\lambda}(A)$ and let $\mathcal{J} = \mathcal{J}_{<\lambda^+}(A)$. Also let $\mu < \lambda$, and let $B_\alpha \in \mathcal{J}$. We shall show that:

Claim: There is a $B \in \mathcal{J}$ such that $B_\alpha - B \in \mathcal{I}$ for all $\alpha < \mu$.

Assume without loss of generality that each B_α is in $\mathcal{J} - \mathcal{I}$, so that $\mathrm{tcf}(\Pi B_\alpha/\mathcal{I}) = \lambda$ for all $\alpha < \mu$ (by Corollary 6.3.4). Choose functions $f_\rho^\alpha \in \Pi A$ for $\rho < \lambda$ and $\alpha < \mu$ such that $((f_\rho^\alpha | B_\alpha)/\mathcal{I}: \rho < \lambda)$ is increasing and cofinal in $\Pi B_\alpha/\mathcal{I}$. By Lemma 6.1.6 $\Pi A/\mathcal{I}$ is λ – directed. Thus we may inductively define maps $f_\rho^* \in \Pi A$ such that f_ρ^* is an upper bound for the functions in $\{f_\rho^\alpha: \alpha < \mu\} \cup \{f_\gamma^*: \gamma < \rho\}$ modulo the ideal $\mathcal{I}$. By Lemma 6.3.1 there is a $g \in \Pi A$ and a $\subseteq_{\mathcal{I}}$–increasing sequence $(C_\alpha : \alpha < \lambda)$ of subsets of A such that

(1) $((f_\rho^* | C_\alpha)/\mathcal{I} : \rho < \lambda)$ is cofinal in $\Pi C_\alpha/\mathcal{I}$;

(2) g is a bound for $\{f_\rho^* : \rho < \lambda\}$ modulo the ideal generated by $\mathcal{I} \cup \{C_\alpha : \alpha < \lambda\}$.

Suppose now that there is an $\alpha < \mu$ such that for every $\gamma < \lambda$ we have: $B_\alpha - C_\gamma \notin \mathcal{I}$.
We shall obtain a contradiction: There is an ultrafilter $\mathcal{F}$ disjoint from $\mathcal{I}$ such that $B_\alpha - C_\gamma \in \mathcal{F}$ for all $\gamma < \lambda$. By definition, $((f_\rho^* | B_\alpha)/\mathcal{I} : \rho < \lambda)$ is cofinal in $\Pi B_\alpha/\mathcal{I}$. Since $B_\alpha \in \mathcal{F}$, we must also have $(f_\rho^*/\mathcal{F} : \rho < \lambda)$ cofinal in $\Pi A/\mathcal{F}$. This yields the required contradiction, as the latter sequence is bounded by $g/\mathcal{F}$ (by definition of g). Thus we have shown that for all $\alpha < \mu$ there is a $\gamma(\alpha) < \lambda$ such that $B_\alpha - C_{\gamma(\alpha)} \in \mathcal{I}$.
Let $\gamma^* = \sup\{\gamma(\alpha) : \alpha < \mu\} < \lambda$, and let $B = C_{\gamma^*}$. Then for all $\alpha < \mu$, necessarily $B_\alpha - B \in \mathcal{I}$ because the sequence $(C_\gamma : \gamma < \lambda)$ was taken to be $\subseteq_{\mathcal{I}}$–increasing. To complete the proof of the Claim, we need only show that $B \in \mathcal{J}$. Now if $\mathcal{G}$ is any ultrafilter over A such that $B \in \mathcal{G}$, then either $\mathcal{G} \cap \mathcal{I} \neq \emptyset$, in which case $\mathrm{cf}(\Pi A/\mathcal{G}) < \lambda < \lambda^+$, or else $\mathcal{G} \cap \mathcal{I} = \emptyset$, in which case $((f_\rho^* | B)/\mathcal{G} : \rho < \lambda)$ is cofinal in $\Pi B/\mathcal{G}$. Since $B \in \mathcal{G}$, also $\mathrm{cf}(\Pi A/\mathcal{G}) < \lambda^+$ in this case, proving that $B \in \mathcal{J}_{<\lambda^+}(A) = \mathcal{J}$. This proves the Claim.
It is now a simple matter to prove the Lemma: If $\mathcal{J} - \mathcal{I} \neq \emptyset$, then there is an ultrafilter $\mathcal{F}$ such that $\mathrm{cf}(\Pi A/\mathcal{F}) = \lambda$. Hence $\lambda \geq \min(A) > 2^{|A|} \geq |\mathcal{J}|$. By the Claim there is a $C \in \mathcal{J}$ such that for all $B \in \mathcal{J}$, $B - C \in \mathcal{I}$. Hence $\mathcal{I} \cup \{C\}$ generates $\mathcal{J}$ as required.

□

§ 6.4 A Bound for $2^{\aleph_\omega}$ when $\aleph_\omega$ is Strong Limit.

In this section we will apply pcf theory to cardinal arithmetic by proving that $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$. As a corollary we have the following remarkable result (in the light of previous work on the power function at $\aleph_\omega$): If $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$ (Since $2^{\aleph_\omega} = (2^{<\aleph_\omega})^\omega \leq \aleph_\omega^\omega \leq 2^{\aleph_\omega}$). In particular:

If GCH holds below $\aleph_\omega$, then $2^{\aleph_\omega} < \aleph_{\omega_2}$ (†).

This might well be the best result obtainable. According to Theorem 4.3.21 it is consistent, modulo some degree of supercompactness, that

$$2^{\aleph_n} = \aleph_{n+1} \text{ for } n < \omega \text{ and } 2^{\aleph_\omega} = \aleph_{\alpha+1} \text{ for any } \alpha < \omega_1.$$

It may therefore be very hard to improve on (†), and that (†) is attainable at all is quite remarkable. The main lemma which we need in order to apply pcf theory to cardinal arithmetic is Lemma 6.4.1 stated below. The proof is technical and will only be completed after Proposition 6.4.8. The main theorems of this section (indeed, of this chapter) are Theorems 6.4.9, 6.4.10 and 6.4.13.

Lemma 6.4.1:

Suppose that A *is an interval of regular cardinals,*

$A = [\min(A), \sup(A))$, *such that* $\min(A)^{|A|} < \sup(A)$. *Then* $\max(\mathrm{pcf}(A)) = |\Pi A|$.

Of course we always have $\max(\mathrm{pcf}(A)) \leq |\Pi A|$ because for any ultrafilter $\mathcal{F}$ over A, $\mathrm{cf}(\Pi A/\mathcal{F}) \leq |\Pi A/\mathcal{F}| \leq |\Pi A|$. We begin the proof of Lemma 6.4.1 by showing that there is less to prove than one might originally think.

Proposition 6.4.2:

Suppose that Lemma 4.1 *holds for all intervals* A *of regular cardinals such that* $2^{|A|} <$ $\min(A)$ *and* $\min(A)^{|A|} < \sup(A)$. *Then Lemma* 4.1 *holds.*

Proof: Let A be any interval of regular cardinals, $A = [\min(A), \sup(A))$, such that $\min(A)^{|A|} < \sup(A)$. Define $A_0 = A \cap [0, \min(A)^{|A|}]$, $A_1 = A \cap (\min(A)^{|A|}, \sup(A))$. Clearly $|\Pi A_0| \leq \min(A)^{|A|} < \min(A_1)$. Also $2^{|A_1|} \leq 2^{|A|} \leq \min(A)^{|A|} < \min(A_1)$, and $\min(A_1)^{|A_1|} = \min(A_1)\cdot(\min(A)^{|A|})^{|A_1|} = \min(A_1)$ because A is an *interval*, and $\min(A_1) < \sup(A_1)$, so that $\min(A_1)^{|A_1|} < \sup(A_1)$ as well. By hypothesis, therefore, we must have $\max(\mathrm{pcf}(A_1)) = |\Pi A_1|$. Finally note that $\max(\mathrm{pcf}(A)) \leq |\Pi A| = |\Pi A_1| = \max(\mathrm{pcf}(A_1)) \leq \max(\mathrm{pcf}(A))$, proving that Lemma 6.4.1 holds for arbitrary A.

□

For the rest of this section, let A be an interval of regular cardinals such that $\min(A)^{|A|} < \sup(A)$ and $2^{|A|} < \min(A)$. The reason why we need the latter assumption is to employ Lemma 6.3.5. Let $\kappa = \max(pcf(A))$. As noted before, $|\Pi A| \geq \kappa$, and so we need to prove that $|\Pi A| \leq \kappa$. If θ is a regular cardinal, then $\mathcal{H}(\theta)$ will be the set of *hereditarily of cardinality* $< \theta$ sets, i.e. $\mathcal{H}(\theta) = \{x: |TC(x)| < \theta\}$. We will assume θ to be "sufficiently large" in which to do our work. How large exactly this is, will be clear after the proof of Lemma 6.4.1 has been completed.

Let $<^*$ be any arbitrary well–ordering of $\mathcal{H}(\theta)$. We shall identify $\mathcal{H}(\theta)$ with the structure $(\mathcal{H}(\theta), \in, <^*)$. For each cardinal $\lambda \leq \kappa$ there is, by Lemma 6.3.5, a set $B_\lambda \subseteq A$ such that $\mathcal{J}_{<\lambda^+}(A)$ is generated by $\mathcal{J}_{<\lambda}(A) \cup \{B_\lambda\}$. We may take $(B_\lambda: \lambda \leq \kappa)$ to be the $<^*$–least sequence of such sets with the property that $B_\kappa = A$ (Of course λ ranges over cardinals in this sequence).

Definition 6.4.3: If N is an elementary substructure of $\mathcal{H}(\theta)$, we wil say that N is *nice* provided:

(a) $|N| = \min(A)$

(b) There is an increasing elementary chain $(N_\gamma: \gamma < |A|^+)$ which is *continuous*, (i.e. if $\lambda < |A|^+$ is limit, then $N_\lambda = \cup\{N_\gamma: \gamma < \lambda\}$, and such that $N = \cup\{N_\gamma: \gamma < |A|^+\}$).

Moreover, $(N_\gamma: \gamma < \delta) \in N$ for any $\delta < |A|^+$.

(c) $A \in N$ and $\min(A) \subseteq N$.

By choosing θ large enough, $\theta > |TC(pcf(A))|$ for instance, we can assure $pcf(A) \in \mathcal{H}(\theta)$. Then if N is nice, we have $A \in N$, so that by elementarity and the fact that $\mathcal{H}(\theta)$ has a well–ordering we must also have $pcf(A) \in N$. Similarly $\min(A) \in N$. Since also $\min(A) \subseteq A$, and since there is in N a map from min(A) onto pcf(A) (because $\min(A) > 2^{|A|} \geq |pcf(A)|$ and again by elementarity), we also have $pcf(A) \subseteq N$. In particular, because $A \subseteq pcf(A)$, also $A \subseteq N$.
If $\lambda \in pcf(A)$, then by definition of $(B_\lambda: \lambda \leq \kappa)$, we also have $B_\lambda \in N$ and $\mathcal{J}_{<\lambda}(A) \in N$.
Finally, it is not hard to see that for all $x \in \mathcal{H}(\theta)$ there is a nice N such that $x \in N$: One can build such an N by stages.

Definition 6.4.4: $\chi_N(\eta) = \sup(N \cap \eta)$ for $\eta \in A$.

Proposition 6.4.5:
For nice models N, M, *if* $\chi_N = \chi_M$, *then* $N \cap \eta = M \cap \eta$ *for all cardinals* η *such that* $\min(A) \leq \eta \leq \sup(A)$. *Otherwise stated:* $N \cap \eta$ *is determined by* χ_N *for nice* N.
In particular, $N \cap \sup(A)$ *is determined by* χ_N *for nice* N.

Proof: By induction on cardinals η such that $\min(A) \leq \eta \leq \sup(A)$ we shall show that $N \cap \eta$ is determined by χ_N.
For any nice N, $\min(A) \subseteq N$ and thus $N \cap \min(A) = \min(A)$. Similarly if η is a limit cardinal in the relevant interval, then $N \cap \eta = \bigcup\{N \cap \gamma: \min(A) \leq \gamma < \eta, \gamma \text{ a cardinal}\}$, and thus $N \cap \eta$ is determined by χ_N provided $N \cap \gamma$ is determined by χ_N for all such γ.
It therefore remains to prove that if $N \cap \eta$ is determined by χ_N for some $\eta < \sup(A)$, then also $N \cap \eta^+$ is determined by χ_N. Now $\eta^+ \in A$ because η^+ is regular and A is an interval of regular cardinals. Let $E = \{\sup(N_\delta \cap \eta^+): \delta < |A|^+\}$, where $(N_\delta: \delta < |A|^+)$ is the increasing continuous elementary chain converging to N given by the *nicety* of N. It is clear that E is closed unbounded in $\sup(N \cap \eta^+) = \chi_N(\eta^+)$, and that $\mathrm{otp}(E) = |A|^+$. Also we have $E \subseteq N$: For each $\delta < |A|^+$, $N_\delta \in N$, and because $\eta^+ \in A \subseteq N$, also $\sup(N_\delta \cap \eta^+) \in N$. Hence there is a club $E \subseteq \chi_N(\eta^+)$ such that $E \subseteq N$ and $\mathrm{otp}(E) = |A|^+$. It follows that $\mathrm{cf}(\sup(N \cap \eta^+)) = |A|^+$.
Suppose now that M is another nice elementary submodel of $\mathcal{H}(\theta)$ such that $\chi_N = \chi_M$. We shall show that $N \cap \eta^+ = M \cap \eta^+$. By hypothesis we already have $N \cap \eta = M \cap \eta$. Also $\sup(N \cap \eta^+) = \chi_N(\eta^+) = \chi_M(\eta^+) = \sup(M \cap \eta^+)$. By the foregoing, it is clear that there is a club $F \subseteq \chi_N(\eta^+) = \chi_M(\eta^+)$ such that $F \subseteq N \cap M$ (Just take the intersections of the E's belonging to N, M respectively). Clearly then $\sup(N \cap \eta^+) = \sup(N \cap M \cap \eta^+) = \sup(M \cap \eta^+)$. Now choose $\alpha \in N \cap M \cap \eta^+$ arbitrarily, and let f be the $<^*$–least bijection $f: \eta \longrightarrow \alpha$. By elementarity, $f \in N \cap M$. Since also $N \cap \eta = M \cap \eta$, we have:

$$N \cap \alpha = f''(N \cap \eta) = f''(M \cap \eta) = M \cap \alpha$$

and thus since $N \cap M \cap \eta^+$ is cofinal in both $N \cap \eta^+$ and $M \cap \eta^+$, it follows that

$N \cap \alpha = M \cap \alpha$. Hence $N \cap \eta^+ = M \cap \eta^+$, completing the induction and proving the proposition.

□

Recall that $(B_\lambda : \lambda \leq \kappa)$ is the $<^*$–least sequence of sets such that $B_\kappa = A$ and such that $\mathcal{J}_{<\lambda^+}(A)$ is generated by $\mathcal{J}_{<\lambda}(A) \cup \{B_\lambda\}$. In particular, if $\lambda \in \mathrm{pcf}(A)$, then $B_\lambda \in \mathcal{J}_{<\lambda^+}(A) - \mathcal{J}_{<\lambda}(A)$ and so $\mathrm{tcf}(\Pi B_\lambda / \mathcal{J}_{<\lambda}(A)) = \lambda$ by Corollary 6.3.4.

Let $(g_\delta^\lambda : \delta < \lambda)$ be a sequence in ΠA such that $(g_\delta^\lambda | B_\lambda : \delta < \lambda)$ is an increasing cofinal sequence in ΠB_λ modulo $\mathcal{J}_{<\lambda}(A)$. Define $(f_\delta^\lambda : \delta < \lambda)$ in ΠA inductively as follows:

(a) If $\mathrm{cf}(\delta) \neq |A|^+$, then $f_\delta^\lambda | B_\lambda$ is strictly $\mathcal{J}_{<\lambda}(A)$–greater than $f_{\delta'}^\lambda | B_\lambda$ for any $\delta' < \delta$. Moreover $f_\delta^\lambda(\beta) \geq g_\delta^\lambda(\beta)$ for any $\beta \in A$

(b) If $\mathrm{cf}(\delta) = |A|^+$, then for any $\beta \in A$ we have
$f_\delta^\lambda(\beta) = \min\{\sup\{f_\gamma^\lambda(\beta) : \gamma \in C\} : C \text{ a club subset of } \delta\}$

We may arrange this in such a fashion that the resulting sequence $(f_\delta^\lambda : \delta < \lambda)$ is the $<^*$–least sequence with properties (a) and (b).

Remark 6.4.6: Suppose that $\delta < \lambda$ and $\mathrm{cf}(\delta) = |A|^+$; then we have:

(1) $f_\delta^\lambda(\beta) < \beta$: Let $C \subseteq \delta$ be a club subset of order type $|A|^+$. Because β is regular $> |A|^+$, it follows that $\sup\{f_\gamma^\beta(\beta) : \gamma \in C\} < \beta$.

(2) $f_\delta^\lambda | B_\lambda$ is $\mathcal{J}_{<\lambda}(A)$–greater than $f_\gamma^\lambda | B_\lambda$ for any $\gamma < \delta$: Choose $C_\beta \subseteq \delta$ to be closed unbounded such that $f_\delta^\lambda(\beta) = \sup\{f_\xi^\lambda(\beta) : \xi \in C_\beta\}$, and let $C = \bigcap\{C_\beta : \beta \in A\}$. Then $C \subseteq \delta$ is still closed unbounded, and by minimality considerations we have $f_\delta^\lambda(\beta) = \sup\{f_\xi^\lambda(\beta) : \xi \in C\}$ for each $\beta \in A$. Since C is cofinal in δ, it easily follows that $f_\delta^\lambda | B_\lambda \geq f_\gamma^\lambda | B_\lambda$ modulo $\mathcal{J}_{<\lambda}(A)$.

Proposition 6.4.7:

Suppose that N *is nice,* $\lambda \in \mathrm{pcf}(A)$ *and that* $\rho = \sup(N \cap \lambda)$. *Then* $f_\rho^\lambda \leq \chi_N$ *everywhere, and* $f_\rho^\lambda | B_\lambda = \chi_N | B_\lambda$ *modulo* $\mathcal{J}_{<\lambda}(A)$.

Proof: Note that by definition $\rho = \chi_N(\lambda)$. The nicety of N yields an increasing continuous elementary chain $(N_\delta: \delta < |A|^+)$ converging to N, and the set

$$E = \{\sup(N_\delta \cap \lambda): \delta < |A|^+\}$$

is a club subset of λ whose order type is $|A|^+$. It follows that $cf(\lambda) = |A|^+$. Moreover $E \subseteq N$. By Remark 6.4.6(2), there is a club $C \subseteq \rho$ such that for all $\beta \in A$ we have $f_\rho^\lambda(\beta) = \sup\{f_\gamma^\lambda(\beta): \gamma \in C\}$. We may therefore assume that $C \subseteq E \subseteq N$.

Hence $\gamma \in C$ implies $f_\gamma^\lambda(\beta) \in N$ (because $\gamma \in N$ and by $<^*$ – minimality of $(f_\delta^\lambda: \delta < |A|^+)$) and thus $f_\rho^\lambda(\beta)$ is the supremum of a subset of $N \cap \beta$, implying that $f_\rho^\lambda(\beta) \leq \chi_N(\beta)$ for all $\beta \in A$. It remains to show that $\chi_N | B_\lambda \leq f_\rho^\lambda | B_\lambda$ modulo $\mathcal{J}_{<\lambda}(A)$.

Let $X = \{\beta \in B_\lambda: \chi_N(\beta) > f_\rho^\lambda(\beta)\}$. If $\beta \in X$, let $\gamma(\beta) \in N \cap \beta$ be such that $\gamma(\beta) > f_\rho^\lambda(\beta)$, and choose $\delta < |A|^+$ such that $\{\gamma(\beta): \beta \in X\} \in N_\delta$. Clearly then $\gamma(\beta) < \chi_{N_\delta}(\beta)$ for each $\beta \in X$, and so $\chi_{N_\delta}(\beta) > f_\rho^\lambda(\beta)$ for all $\beta \in X$. By elementarity, and because $\chi_N \in N$, there must be a $\xi \in N$ with $\xi < \lambda$ such that $\chi_{N_\delta} | B_\lambda < f_\xi^\lambda | B_\lambda$ modulo $\mathcal{J}_{<\lambda}(A)$. Since $\xi \in N \cap \lambda$, we must have $\xi < \rho$ (by definition of ρ), and thus $\chi_{N_\delta} | B_\lambda < f_\xi^\lambda | B_\lambda \leq f_\rho^\lambda | B_\lambda$ modulo $\mathcal{J}_{<\lambda}(A)$. Hence $X = \{\beta \in B_\lambda: \chi_N(\beta) > f_\rho^\lambda(\beta)\} \subseteq \{\beta \in B_\lambda: \chi_{N_\delta}(\beta) \geq f_\rho^\lambda(\beta)\} \in \mathcal{J}_{<\lambda}(A)$.

It follows that $X \in \mathcal{J}_{<\lambda}(A)$ also, and thus that $\chi_N | B_\lambda \leq f_\rho^\lambda | B_\lambda$ modulo $\mathcal{J}_{<\lambda}(A)$.

□

All of this allows us to prove the following proposition:

Proposition 6.4.8:

(1) $|\{N \cap \sup(A): N \textit{ is nice}\}| \leq \kappa$

(2) $|\{\chi_N: N \textit{ is nice}\}| \leq \kappa$

Proof: By Proposition 6.4.5, (2) implies (1). So we prove (2): To begin we will define a finite sequence $<(\rho_m, \lambda_m, A_m): m \leq n>$ such that:

(a) $\kappa = \lambda_0 > \lambda_1 > \dots > \lambda_n$.

(b) $\forall m \leq n[\lambda_m \in pcf(A) \wedge \rho_m = \chi_N(\lambda_m)]$

(c) $\forall m \leq n[A_m \subseteq A]$

(d) $A_n = \emptyset$, $A_0 = \{\beta \in A: f_{\rho_0}^{\lambda_0}(\beta) < \chi_N(\beta)\}$

(e) $\forall m<n[A_m \in \mathcal{J}_{<\lambda^+_{m+1}}(A) - \mathcal{J}_{<\lambda_{m+1}}(A)]$

(f) $\forall m<n[A_{m+1} = \{\beta \in A_m: f_{\rho_{m+1}}^{\lambda_{m+1}}(\beta) < \chi_N(\beta)\}$

The definition is by induction: By Proposition 6.4.6, $A_0 \in \mathcal{J}_{<\lambda_0}(A)$. If $A_0 = \emptyset$, then $n = 0 < \lambda_0$, so we are done. Suppose therefore that $A_0 \neq \emptyset$. Let μ^+ be the least cardinal such that $A_0 \in \mathcal{J}_{<\mu^+}(A)$, and let $\lambda_1 = \mu$. Then $A_0 \in \mathcal{J}_{<\lambda_1^+}(A) - \mathcal{J}_{<\lambda_1}(A)$, so indeed $\lambda_1 \in pcf(A)$. The induction proceeds in the same fashion until $A_m = \emptyset$ for some m.

Now note that $\chi_N|(A - A_0) = f_{\rho_0}^{\lambda_0}|(A - A_0)$, and for all $m < n$, $\chi_N|(A_m - A_{M+1}) = f_{\rho_{m+1}}^{\lambda_{m+1}}|(A_m - A_{m+1})$. It follows that $\chi_N = \sup\{f_{\rho_0}^{\lambda_0}, \dots, f_{\rho_n}^{\lambda_n}\}$, and thus there are at most $|\{f_\rho^\lambda: \rho < \lambda \in pcf(A)\}|^{<\omega}$– many χ_N (where N ranges over nice elementary submodels of $\mathcal{H}(\theta)$). Hence there are at most $\max(pcf(A)) = \kappa$–many χ_N as required.

□

Proof of Lemma 6.4.1: We have to show that $\kappa \geq \Pi A$. Let $f \in \Pi A$, and choose N nice such that $f \in N$. Thus $f \subseteq N \cap (A \times \sup(A))$. Moreover since $A \subseteq N$, $f \subseteq A \times (N \cap \sup(A))$, i.e. $f \in (N \cap \sup(A))^A$. The number of $f \in \Pi A$ which belong to a nice M such that $M \cap \sup(A) = N \cap \sup(A)$ is therefore at most $|N \cap \sup(A)|^{|A|} \leq \min(A)^{|A|} < \sup(A) \leq \kappa$ (the first inequality holds because $|N| = \min(A)$, and the second is an assumption on A). Now by Proposition 6.4.7(1), $|\{N \cap \sup(A): N \text{ is nice}\}| \leq \kappa$, and so we have:

$|\Pi A| \leq |\{N \cap \sup(A): N \text{ is nice}\}| \cdot |N \cap \sup(A)|^{|A|} \leq \kappa \cdot \kappa = \kappa$.

□

Theorem 6.4.9 [Shelah 1992]:

(a) *If δ is a limit ordinal, then* $\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}$

(b) *If* $\aleph_\delta$ *is a strong limit cardinal, then* $2^{\aleph_\delta} < \aleph_{(2^{|\delta|})^+}$

(c) *If* $\aleph_\omega$ *is strong limit, then* $2^{\aleph_\omega} < \aleph_{(2^\omega)^+}$

Proof: (a) implies (b) implies (c) is immediately obvious, so we prove only (a). Let $A = [(2^{|\delta|})^+, \aleph_\delta)$ be an interval of regular cardinals, so that $\max(\mathrm{pcf}(A)) = |\Pi A| = \aleph_\delta^{|\delta|}$ by Lemma 6.4.1. We also know that $|\mathrm{pcf}(A)| \leq 2^{|A|} \leq 2^{|\delta|}$ from Lemma 6.1.9(a), and by Lemma 6.2.3 pcf(A) is an interval of regular cardinals. Thus

$$\aleph_\delta^{|\delta|} = \max(\mathrm{pcf}(A)) < \aleph_{|\mathrm{pcf}(A)|^+} \leq \aleph_{(2^{|\delta|})^+}$$

as required.

□

Note that conclusion (c) of Theorem 6.4.9 is quite weak: From (b) it actually follows that if $\alpha < \omega_1$ is an ordinal such that $\aleph_\alpha$ is strong limit, then $2^{\aleph_\alpha} < \aleph_{(2^\omega)^+}$. We have seen a weaker version of (b) before: Theorem 1.3.4 states that if $\aleph_\delta$ is a singular strong limit cardinal of uncountable cofinality with $\delta < \aleph_\delta$, then $2^{\aleph_\delta} < \aleph_{(2^{|\delta|})^+}$. The requirement of uncountable cofinality is therefore unnecessary in the light of Theorem 6.4.9.

Next we want to discuss various other (even stronger) theorems that have been proven using pcf theory, but in order to keep the length of this dissertation within manageable bounds we shall merely outline their proofs. The first result that we shall state was proved by Shelah ([Shelah 1982]), and is an improvement of Theorem 6.4.9(a). A proof may also be found in the paper [Burke–Magidor 1990].

Theorem 6.4.10 [Shelah 1982]:

If δ is a limit ordinal, then $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}$.

This theorem does give new information about the power function. For instance, if $\alpha = \omega_1 + \omega$ and $\aleph_\alpha$ is strong limit, then it follows from Theorem 6.4.9 that $2^{\aleph_\alpha} < \aleph_{(2^{\omega_1})^+}$, whereas Theorem 6.4.10 implies that $2^{\aleph_\alpha} < \aleph_{(\omega_1{}^{\omega})^+}$.

Let $\mu = \mathrm{cf}(\delta)$. We must prove that $\aleph_\delta^{\ \mu} < \aleph_{(|\delta|^\mu)^+}$. If $2^\mu > \aleph_\delta$, then obviously $\aleph_\delta^{\ \mu} < 2^\mu < \aleph_{(2^\mu)^+} \leq \aleph_{(|\delta|^\mu)^+}$, and thus Theorem 6.4.10 holds. Hence we may assume that $2^\mu < \aleph_\delta$. Let A be the interval of all regular cardinals strictly between 2^μ and $\aleph_\delta$, and let $\mathcal{S} = [A]^{\leq\mu} - \{\emptyset\}$ (where $[A]^{\leq\mu}$ is the set of all subsets of A of cardinality $\leq \mu$.) The cardinality of $\mathcal{S}$ is $\leq |\delta|^\mu$ because $|A| \leq |\delta|$. [Burke–Magidor 1990] start their proof of Theorem 6.4.10 by showing that if $B \in \mathcal{S}$ and λ is an infinite cardinal, then $B \in \mathcal{J}_{<\lambda}(A)$ if and only if there is a set $F \subseteq \Pi B$ of cardinality $< \lambda$ which is cofinal in ΠB (i.e. for all $f \in \Pi B$ there is a $g \in F$ such that $f \leq g$ everywhere). They then define

$$\mathrm{pcf}_\mu(A) = \{\mathrm{cf}(\Pi B/\mathcal{F})\colon B \in \mathcal{S}, \mathcal{F} \text{ is an ultrafilter over } B\}$$

and show that $\mathrm{pcf}_\mu(A)$ is an interval of regular cardinals using Lemma 6.2.2. Because $\aleph_\delta^{\ \mu} > \aleph_\delta$, it follows that there are cardinals $> \aleph_\delta$ in $\mathrm{pcf}_\mu(A)$. Let κ be the least regular cardinal such that $\mathcal{S} \subseteq \mathcal{J}_{<\kappa}(A)$. By Lemma 6.1.8 the sequence $(\mathcal{J}_{<\lambda}(A) \cap [A]^{\leq\mu}\colon \lambda \in \mathrm{pcf}_\mu(A))$ is a strictly increasing sequence of subsets of $[A]^{\leq\mu}$, and so $|\mathrm{pcf}_\mu(A)|$ cannot be greater than $|\delta|^\mu$. Since $\mathrm{pcf}_\mu(A)$ is an interval of regular cardinals, we have $\aleph_\delta < \kappa < \aleph_{(|\delta|^\mu)^+}$. Theorem 6.4.10 follows once the following Lemma is proved:

Lemma 6.4.11:

$\aleph_\delta^{\ \mu} \leq \kappa \cdot |\delta|^\mu$.

Assume that $\aleph_\delta^{\ \mu} > \kappa \cdot |\delta|^\mu$ and let us fix a sequence $(S_i\colon i < \kappa)$ of distinct subsets of $\aleph_\delta$, each of order type μ. Now $\aleph_\delta > |\delta|^\mu$ and δ is a limit cardinal, so if we define $\lambda = (|\delta|^\mu)^{++}$, then $\lambda < \aleph_\delta$ and $\lambda^\mu = \lambda$. For each $B \in \mathcal{S}$, let F_B be a cofinal subset of ΠB of cardinality $< \kappa$. Recall that for any set X, $[X]^\lambda$ denotes the set of all subsets of X of order type λ. One may then prove that there is a set $G_B \subseteq [F_B]^\lambda$ such that $|G_B| \leq |F_B|$ and such that for every $s \in [F_B]^\lambda$ there is a $t \in G_B$ such that $|s \cap t| = \lambda$. Let $\mathcal{S}^{>\lambda} = \{B \in \mathcal{S}: B \cap \lambda = \emptyset\}$. Let θ be a regular cardinal which is "sufficiently large" and let $\mathcal{H}(\theta) = \{x: |TC(x)| < \theta\}$. Let $<^*$ be an arbitrary well–ordering of $\mathcal{H}(\theta)$, and identify $\mathcal{H}(\theta)$ with the structure $(\mathcal{H}(\theta), \in, <^*)$. For each $i < \kappa$ and for each strictly increasing $\eta \in \lambda^{<\mu}$ one may define (by induction on the least α such that $\eta \in \alpha^{<\mu}$) a model M^i_η and maps f^i_α, $f^i_{\alpha,B}$ for $B \in \mathcal{S}^{>\lambda}$ such that the following conditions hold:

(1) M^i_η is an elementary submodel of $\mathcal{H}(\theta)$ of cardinality μ.

(2) $\{A,\lambda\} \cup S_i \subseteq M^i_\emptyset$ and $\mathrm{ran}(\eta) \subseteq M^i_\eta$.

(3) If α is a limit ordinal, then $M^i_\eta = \bigcup \{M^i_{\eta|\beta}: \beta < \mathrm{dom}(\eta)\}$.

(4) If α is a successor ordinal, let ξ be such that $\mathrm{dom}(\eta) = \xi + 1$. Then

  (i) $(M^i_{\eta|\gamma}: \gamma \leq \xi) \in M^i_\eta$

  (ii) $(f^i_\beta: \beta < \alpha) \in M^i_\eta$

  (iii) $(f^i_{\beta,B}: \beta < \alpha, B \in \mathcal{S}^{>\lambda}\} \in M^i_\eta$.

(5) $f^i_\alpha \in \Pi(A - \lambda)$ is given by $f^i_\alpha(\rho) = \sup(\bigcup\{M^i_\eta: \eta \in \alpha^{<\mu}\} \cap \rho)$ for $\rho \in A - \lambda$.

(6) For $B \in \mathcal{S}^{>\lambda}$, $f^i_\alpha|B < f^i_{\alpha,B} \in F_B$ and $f^i_{\beta,B} < f^i_{\alpha,B}$ for all $\beta < \alpha$ (where the inequality holds everywhere).

For $\eta \in \lambda^\mu$ one may define $M^i_\eta \in \bigcup_{\beta<\mu} M^i_{\eta|\beta}$. Since for $B \in \mathcal{S}^{>\lambda}$ we have $(f^i_{\alpha,B}: \alpha < \lambda)$ strictly increasing, it follows that there is $t^i_B \in G_B$ such that $t^i_B \cap \{f^i_{\alpha,B}: \alpha < \lambda\}$ is of cardinality λ. We may enumerate $t^i_B = \{g^i_{\alpha,B}: \alpha < \lambda\}$ in such a way that $t^i_B = t^j_B$ implies $g^i_{\alpha,B} = g^j_{\alpha,B}$ for all $\alpha < \lambda$. Next define $t^i_{\beta,B} = \{g^i_{\alpha,B}: \alpha < \beta\}$ and let $\mathcal{C}^i_B$ be the set of all $\beta < \lambda$ such that for all $\alpha < \beta$ we have:

(a) There are ξ, ζ such that $\alpha < \xi, \zeta < \beta$ and $g^i_{\zeta,B} = f^i_{\xi,B}$.

(b) If there is $\gamma < \lambda$ such that $g^i_{\alpha,B} < f^i_{\gamma,B}$, then there is a $\gamma < \beta$ with that property.

One may show that each $\mathcal{C}^i_B$ is a club subset of λ, and thus that $\mathcal{C}^i = \bigcap \{\mathcal{C}^i_B: B \in \mathcal{S}^{>\lambda}\}$ is club For each $i < \kappa$ choose $\beta(i) \in \mathcal{C}^i$ such that

$$\mathrm{cf}\beta(i) = \begin{cases} \mu & \text{if } \mu > \omega \\ (|\delta|^\mu)^+ & \text{if } \mu = \omega \end{cases}$$

There exists a set $I \subseteq \kappa$ and an ordinal $\beta < \lambda$ such that $|I| = \kappa$ and such that $\beta(i) = \beta$ for all $i \in I$. We distinguish two cases: Either $\mu > \omega$ (in which case $cf\beta = \mu$) or $\mu = \omega$ (in which case $cf\beta = (|\delta|^{\omega})^+$.

<u>Case 1</u>: Suppose that $\mu > \omega$. Let $\eta \in \beta^{\mu}$ be an increasing cofinal sequence in β. By thinning I somewhat if necessary, we may assume that there is a $B \in S^{>\lambda}$ such that for all $i \in I$ we have $M^i_\eta - \lambda = B$. [Burke–Magidor 1990] then prove the following claims in succession, thinning I somewhat along the way if necessary:

<u>Claim 1</u>: $f^i_\beta | B$ does not depend on $i \in I$.

<u>Claim 2</u>: For $\rho \in A$, $f^i_\beta(\rho) = \sup(M^i_\eta \cap \rho)$.

<u>Claim 3</u>: $M^i_\eta \cap \aleph_\delta$ does not depend on $i \in I$.

Let $S = M^i_\eta \cap \aleph_\delta$ for $i \in I$ and recall that $(S_i : i < \kappa)$ is a sequence of distinct elements of $[\aleph_\delta]^{\mu}$. Recall further that $S_i \subseteq M^i_\eta$ for all $i < \kappa$. Then for $i \in I$ we have $S_i \subseteq S$. Hence $\kappa \leq |\mathcal{P}(S)| = 2^{\mu} \leq |\delta|^{\mu} < \kappa$, a contradiction. Thus Lemma 6.4.11 holds for $\mu > \omega$.

<u>Case 2</u>: Suppose now that $\mu = \omega$. In that case $cf\beta = (|\delta|^{\omega})^+$. One may prove that for each $i \in I$ there is a $T^i \subseteq \beta^{<\omega}$ and a sequence $B_n \in S^{>\lambda}$ (for $n < \omega$) with the following properties:

(a) T^i is closed under initial segments.

(b) Every $\eta \in T^i$ is strictly increasing.

(c) For each $k < \omega$ and each $\eta \in T^i \cap \beta^k$ we have $(M^i_\eta \cap A) - \lambda = B_k$.

(d) For each $\eta \in T^i \cap \beta^k$, $\{\alpha < \beta : \eta^\frown(\alpha) \in T^i\}$ is stationary in β.

Let $B = \bigcup_{k<\omega} B_k$. By thinning I if necessary, one may show that $f^i_\beta | B$ does not depend on $i \in I$ (just as in Claim 1 of Case 1). Let $f = f^i_\beta | B$ for $i \in I$. One then shows that for each $\rho \in B$ and $i \in I$ we have $cf(f(\rho)) = cf(f^i_\beta(\rho)) = cf(\beta) = (|\delta|^{\omega})^+$, and so we may choose a club $C_\rho \subseteq f(\rho)$ of order type $cf(\beta)$. Let N be an elementary submodel of $\mathcal{H}(\theta)$ of cardinality λ such that

$$B \cup \lambda \cup \bigcup\{C_\rho : \rho \in B\} \subseteq N.$$

[Burke–Magidor 1990] then prove the following claim:

Claim 4: For each $\eta \in \beta^{<\omega}$ and each $\rho \in B_{dom(\eta)}$, there is an $\alpha < \beta$ and an ordinal γ such that $\gamma \in (\rho - \sup(M^i_\eta \cap \rho)) \cap M^i_{\eta^\frown(\alpha)} \cap N$.

Using Claim 4 and the fact that each $B_k \in [A]^\mu$ is countable, it follows that there is a sequence $(\eta_k: k < \omega)$ in $\beta^{<\omega}$ such that $dom(\eta_k) = k$ and such that if $k < l$, then $\eta_k \subsetneq \eta_l$. Moreover for all $k < \omega$ and all $\rho \in B_k$ there is $l > k$ such that

$$(\rho - \sup(M^i_{\eta_k} \cap \rho)) \cap M^i_{\eta_l} \cap N \neq \emptyset$$

Let $\eta = \bigcup_{k<\omega} \eta_k$ and let $M^i_\eta = \bigcup_{k<\omega} M_{\eta_k}$. Then $(M^i_\eta \cap A) - \lambda = B$ for all $i \in I$, and for each $\rho \in B$, the set $M^i_\eta \cap N \cap \rho$ is cofinal in $M^i_\eta \cap \rho$. By induction on cardinals $\rho \leq \aleph_\delta$ it is now easy to show that $M^i_\eta \cap \rho \subseteq N \cap \rho$ for all $i \in I$. In particular $S_i \subseteq M^i_\emptyset \cap \aleph_\delta \subseteq M^i_\eta \cap \aleph_\delta \subseteq N \cap \aleph_\delta$ for all $i \in I$, and so $S_i \in [N]^\mu$ for all $i \in I$. This leads to a contradiction, since $|I| = \kappa > \lambda = |N|$ and $\lambda^\mu = \lambda$. Hence if $\mu = \omega$, then Lemma 6.4.11 is true as well.

This completes the discussion of the proof of Theorem 6.4.10.

The next result, again due to Shelah ([Shelah 1992]) has some surprising consequences. We shall merely state it here and refer the reader to [Burke–Magidor 1990] for a proof.

Lemma 6.4.12:

Suppose that A *is an interval of regular cardinals such that* $\min(A) > 2^{|A|}$. *Then* $|pcf(A)| < |A|^{+3}$.

Theorem 6.4.13 [Shelah 1992]:

(a) *If* $2^{\aleph_0} < \aleph_\omega$, *then* $\aleph_\omega^{\ \omega} < \aleph_{\omega_4}$.

(b) *Hence if* $\aleph_\omega$ *is strong limit, then* $2^{\aleph_\omega} < \min(\aleph_{\omega_4}, \aleph_{(2^\omega)^+})$.

Proof: (a) Choose $m < \omega$ such that $2^{\aleph_0} < \aleph_m$, and put $A = [\aleph_m, \aleph_\omega)$. By Lemma 6.4.1 we have $\max(pcf(A)) = |\Pi A| = \aleph_\omega^{\ \omega}$, and by Lemma 6.4.12 we have $|pcf(A)| \leq |A|^{+3} = \omega_3$. Lemma 6.2.3 states that pcf(A) is an interval of regular cardinals with a maximal element.

It follows that $\aleph_\omega^{\ \omega} = \max(\mathrm{pcf}(A)) < \aleph_{|\mathrm{pcf}(A)|^+} = \aleph_{\omega_4}$.

(b) If $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} = \aleph_\omega^{\ \omega}$ and so by (a) we have $2^{\aleph_\omega} < \aleph_{\omega_4}$. On the other hand, by Theorem 6.4.10(c) we have $2^{\aleph_\omega} < \aleph_{(2^\omega)^+}$.

□

This concludes our discussion of the application of pcf–theory to cardinal arithmetic, and particularly to the problem of the power function at $\aleph_\omega$. Pcf–theory has also succesfully been applied to other branches of set theory and mathematical logic. For instance, there is a Jónsson algebra over $\aleph_{\omega+1}$ [i.e. $\aleph_{\omega+1}$ is not Jónsson]. Proof of this may be found in [Burke–Magidor 1990]. Possibly we will soon have an answer to the unsolved problem: Can $\aleph_\omega$ be Jónsson? Other applications include to set theoretical topology, partition calculus, the theory of Boolean algebras, and even free Abelian groups.

Epilogue: **A BRIEF HISTORY OF THE GCH**

The generalized continuum problem is probably the oldest problem in set theory, stemming directly from the subject's founder, George Cantor. In the 1870's, Cantor laid the foundations for basic cardinal arithmetic, and proved by means of a diagonal argument that the set of subsets of ω has cardinality $2^\omega > \omega$. But exactly how big was 2^ω? Since cardinal sums and products are essentially trivial, it must have been frustrating that not even the simplest cardinal power was amenable to calculation. Cantor hypothesized that $2^\omega = \omega_1$, and generally that $2^\kappa = \kappa^+$ for all cardinals κ. This is the celebrated Generalized Continuum Hypothesis, (abbreviated GCH), which Cantor first posited [Cantor 1878], but he could never prove nor disprove it, although he worked on the problem throughout his life. Various other results about cardinal arithmetic were proved in those early years of "naive set theory", the most important perhaps being the result of König [1905] which compares sums and products of cardinals, but no one came any closer to a resolution of the generalized continuum problem.

Since Cantor, set theory has been formalized, with various axiomatizations, due to Zermelo, Godel, Bernays, von Neumann, Morse and others. The most important axiomatization of set theory is ZFC (the axioms of ZFC may be found in Section 1.1). Axiomatization allowed the application of logical methods to set theory, on which further progress in set theory has heavily depended: The next important results concerning of the generalized continuum problem were obtained by Kurt Gödel in the 1938 using model theoretic methods. Gödel realized that the generalized continuum problem could not easily be resolved because the notion of power set is not adequately described by the axioms of ZFC. He proposed a hierarchical model of set theory, the constructible universe L, in which at any level of the hierarchy only subsets of already existing sets definable from already existing parameters in the previous levels are admitted (A description of L is given in Appendix 1). By the Axiom of Comprehension, all such subsets must exist, and thus the constructible universe is in some sense the minimal model of ZFC. Gödel then proved that $L \models$ GCH, and therefore the GCH cannot be disproved without introducing some new set theoretical principles. In addition, he resolved another of the important unsolved questions of his time, namely that the Axiom of Choice is consistent with the other axioms of ZFC [Gödel 1938].

It took decades before the next milestone in the resolution of the generalized continuum problem was reached. In 1963, Paul Cohen first applied forcing to set theory. Thus far forcing had been a curiosity in model theory, related to omitting types theorems, but when Cohen proved that negation of the GCH is consistent with ZFC, forcing had found its proper home. (A description of forcing is provided in Appendix 2). Cohen also managed simultaneously to deal with another problem, namely the consistency of the negation of the Axiom of Choice ([Cohen 1963–1964]). The next few years were perhaps the golden years of set theory, as one consistency result after the other was proved. Already in 1964, Easton solved the generalized continuum problem for regular cardinals, using an iteration of Cohen's method. The following statements are easily proved inside ZFC:

(1) For all cardinals κ, $\mathrm{cf}(2^{\kappa}) > \kappa$.

(2) For all cardinals $\kappa \leq \lambda$, $2^{\kappa} \leq 2^{\lambda}$.

Easton proved that if F is a (class) function in the universe V with domain the class of regular cardinals, and range the class of cardinals such that:

($\overline{1}$) For all regular κ, $\mathrm{cf}(F(\kappa)) > \kappa$

($\overline{2}$) For all regular $\kappa \leq \lambda$, $F(\kappa) \leq F(\lambda)$,

then there is a generic extension V[G] of V in which all cardinalities and cofinalities are preserved, and in which F is the power function (i.e. $V[G] \vdash 2^{\kappa} = F(\kappa)$ for all regular κ) ([Easton 1970])

Thus on regular cardinals, the power function can be almost anything, subject only to the meager restrictions imposed by ($\overline{1}$) and ($\overline{2}$). Easton's methods did not work for singular cardinals, however. It turned out that in Easton's models the singular cardinals hypothesis (SCH) held: If κ is a singular cardinal, then $2^{\kappa} = 2^{<\kappa}$ if $\mathrm{cf}(2^{<\kappa}) > \kappa$, and $(2^{<\kappa})^{+}$ otherwise, i.e. 2^{κ} takes on the smallest value permitted by the axioms of ZFC. Still, it was the general opinion of that time that one would have a similar freedom to choose the power function at singular cardinals, and that only the right forcing conditions would have to be discovered. There was a result due to Bukovsky [1965], however, which stated that if κ is singular and the power function is eventually constant below κ, then the power function takes on that constant value at κ as well. This theorem indicated that the behaviour of the power function might be a little more complicated at singular cardinals than at regular cardinals, but in the main the prevailing opinion was that the value of the power function at singular cardinals would also be largely indeterminate, as it is at regular cardinals.

This opinion was shown to be inconsistent with ZFC by Jack Silver in the early 1970's. Silver proved that, remarkably, the GCH cannot fail for the first time at singular cardinal of uncountable cofinality ([Silver 1974]). This result was similar to an earlier result by Hanf and Scott: The GCH cannot fail for the first time at a measurable cardinal (the first application of large cardinals to the GCH–problem). This similarity is no accident: [Hanf–Scott 1961] used an ultrapower argument to obtain this result, and Silver used a generic ultrapower via a generically added ultrafilter which extends the club filter (generic ultrapowers are discussed in Appendix 4, and measurable cardinals in Appendix 3). Elementary proofs of Silver's result, requiring little more than the notion of stationary sets, were soon exhibited by Baumgartner, Prikry and Jensen. These were developed further in the work of Galvin and Hajnal, who obtained upper bounds for the value of the power function at strong limit singular cardinals of uncountable cardinality ([Galvin–Hajnal 1975]), and also later by Shelah [1980].

We have already seen one application of large cardinal techniques to the elucidation of the power function. It was hoped that various large cardinal axioms might actually decide the GCH (although it was of course unlikely that anyone would revise his or her opinion about the behaviour of the power function solely on the basis of some weird large cardinal, whose existence would be extremely suspect anyway). Measurable cardinals were the most amenable to study and various interesting results about measurable cardinals and the GCH were proved. Silver exhibited an inner model of measurability, $L[\mathcal{U}]$, based on Gödel's constructible universe, and proved that here too the GCH holds. Thus GCH is consistent with the existence of measurable cardinals ([Silver 1971a]). Kunen then proved that failure of the GCH at a measurable cardinal had some strong consequences: If κ is measurable such that $2^\kappa > \kappa^+$, then for any ordinal θ, there is an inner model of set theory with at least θ–many measurable cardinals ([Kunen 1971a]). By Gödel's Incompleteness Theorems it therefore follows that the statement that the GCH fails at a measurable cardinal has is stronger consistency–wise than the statement that a measurable cardinal exists.

Thus far all the consistency results were obtained relative to ZFC. It soon turned out that large cardinal axioms, which generally yield elementary embeddings of the universe into an inner model with some strong closure properties, were an indispensable tool in the study of set theory in general, and the power function in particular. One could obtain a variety of independence results assuming the consistency of some large cardinal axioms. For example, assuming the existence of a cardinal κ which is κ^{++}–supercompact, Silver obtained a generic extension in which κ is still measurable but in which $2^{\kappa} = \kappa^{++}$. It follows that the negation of the GCH is consistent with the existence of a measurable cardinal too. Silver used "reverse Easton forcing" (described in Section 3.4) to obtain this result. This method allows one to manipulate the power function at a large cardinal without destroying its large cardinal character. Using Silver's method, it became clear that no large cardinal axiom is ever likely to decide GCH.

The techniques used for measurable cardinals were generalized, and various properties of ideals (saturated, precipitous etc. Refer to Appendix 4) led to further results concerning the power function. The work of Jech and Prikry was of particular importance in this regard. Some typical examples: (1) If $2^{\omega} = \omega_1$ and ω_1 carries an ω_2–saturated ideal, then $2^{\omega_1} = \omega_2$. (2) If ω_1 carries an ω_2–saturated ideal and $\aleph_{\omega_1}$ is strong limit, then $2^{\aleph_{\omega_1}} < \aleph_{\omega_{\omega_1}}$ ([Jech–Prikry 1979])

We have already mentioned the Singular Cardinals Hypothesis (SCH), which simply states that at singular cardinals the value of the power function is the smallest that is consistent. Jensen, in a series of three papers, proved the Covering Lemma for L: If $0^{\#}$ does not exist, then for any uncountable subset X of the ordinals, there is a constructible set Y of the same cardinality such that $X \subseteq Y$. ($0^{\#}$ is introduced in Appendix 1) It was immediately observed that if κ is singular, then κ is singular in L too. Another result of the Covering Lemma for L is that it implies that the SCH holds in the universe, i.e. if the SCH fails, then $0^{\#}$ exists. Further work by [Dodd–Jensen 1982] showed that the failure of the SCH has is stronger consistency–wise strength than the existence of a measurable cardinal.

Prikry [1970] had found a notion of forcing that preserved all cardinals but changed the cofinality of a given measurable cardinal to ω (see Section 4.1). Assuming the existence of a κ^{++}–supercompact κ, one can get, using a reverse Easton extension, a model in which κ is measurable and in which $2^{\kappa} = \kappa^{++}$. By following with a Prikry extension, it is then

possible to change the cofinality of κ to ω, making it singular strong limit, and yet have $2^\kappa = \kappa^{++}$. In this extension, therefore, the SCH fails.
The consistency strength of the failure of the SCH lies therefore somewhere between the consistency strength of the existence of a supercompact cardinal and that of a measurable cardinal. The exact strength was determined by Gitik and Magidor in the late 1980's: $o(\kappa) = \kappa^{++}$ (where $o(\kappa)$ is the Mitchell order of κ. This is explained in Section 4.4) [Gitik 1989, 1992]. Just as a further complication, Solovay [1974] had proved that the SCH holds above a compact cardinal. Hence it seems as though one needs a supercompact cardinal to show the consistency of $\neg$SCH, but if there is a supercompact cardinal, then the SCH holds almost everywhere!

We have mentioned how, using a κ^{++}–supercompact cardinal κ, there is a notion of forcing such that the generic extension is a model of $\neg$SCH. Such a κ would still be very large in the generic extension as no cardinals are collapsed. Magidor had also been thinking about the failure of the SCH and GCH at the smallest singular cardinal, $\aleph_\omega$. Of course, large cardinals would be required. In the mid 1970's, in two celebrated papers ([Magidor 1977b,c]) he obtained the following results relative to ZFC:

(1) Con(GCH + $\exists\kappa$(κ is κ^{++}–supercompact) $\longrightarrow$ Con($\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$)
(2) Con($\exists\kappa < \lambda$(κ is supercompact and λ is huge) $\longrightarrow$ Con(GCH fails first time at $\aleph_\omega$)
By (2), in particular, Silver's result concerning the power function at singular cardinals of uncountable cofinality cannot be generalized to all singular cardinals.
The work of Magidor was further generalized by Apter and Shelah. Apter [1984] proved it consistent (modulo some large cardinal hypotheses) that every limit cardinal is strong limit, but the GCH fails at every $\aleph_{\alpha+\omega}$. Similarly, Shelah [1983] proved that it is consistent for $\aleph_\omega$ to be strong limit, but $2^{\aleph_\omega} = \aleph_{\alpha+1}$ for any $\alpha < \omega_1$ (also modulo some degree of supercompactness). Woodin has shown how to get Magidor's result (1) from the existence of a $\mathcal{P}_2(\kappa)$–hypermeasurable cardinal.

There have also been several applications of another method of forcing, Radin forcing, which allows one to add club subsets to a large cardinal without destroying its large cardinal character (previous methods of shooting a club set through a large cardinal κ, such as those of [Prikry 1970] or [Magidor 1978], invariably turned κ into a singular cardinal).

Using Radin forcing, Cummings found a model in which GCH holds at successors but failed at limits, and Foreman and Woodin found a model in which GCH fails everywhere. Woodin also obtained a model of $2^{\kappa} = \kappa^{++}$ for all cardinals κ.

Shelah was still concerned with provable results on cardinal arithmetic, not just consistency results. For example, he showed that $\aleph_{\omega}$ being strong limit is consistent with $2^{\aleph_{\omega}}$ being any successor cardinal below $\aleph_{\omega_1}$ [Shelah 1983]. Exactly how big may $2^{\aleph_{\omega}}$ be under those conditions? In the late 1980's Shelah invented pcf theory (pcf stands for "possible cofinality") and proved some very powerful results concerning cardinal arithmetic inside ZFC which partially answer this question. For instance, if $\aleph_{\omega}$ is strong limit, then $2^{\aleph_{\omega}} < \min(\aleph_{(2^{\omega})^+}, \aleph_{\omega_4})$ ([Shelah 1992]). In particular, if GCH holds below $\aleph_{\omega}$, then $2^{\aleph_{\omega}} < \aleph_{\omega_2}$, so one cannot push up the bound for the possible values of $2^{\aleph_{\omega}}$ obtained in [Shelah 1983] too much. This is were matters stand today. Lest one becomes too discouraged, however, it should be noted that perhaps the most appealing aspect of pcf–theory is that it offers the possibility of explaining why some of the results in cardinal arithmetic seem so unexpected in the light of others.

§ A.1.1 Relative Constructibility

In this section we develop the background required for proving the consistency of the GCH (in Section 2.1) and the consistency of the GCH with the existence of a measurable cardinal (Section 3.1). This material is only presented here in order to make this dissertation fairly self-contained, and proofs will generally be omitted. A couple of good sources for the work discussed here are: [Jech 1978], [Devlin 1984] and [Dodd 1982].

Let $\mathcal{L}(A)$ be the language of set theory (cf. Section 1.1) augmented with a single unary predicate symbol A.

Definition A.1.1: The *basic rudimentary functions* (also called *Gödel functions*) are:

$$\mathcal{F}_0(x,y) = \{x,y\}$$
$$\mathcal{F}_1(x,y) = x - y$$
$$\mathcal{F}_2(x,y) = x \times y$$
$$\mathcal{F}_3(x,y) = \{(u,z,v): z \in x \wedge (u,v) \in y\}$$
$$\mathcal{F}_4(x,y) = \{(u,v,z): z \in x \wedge (u,v) \in y\}$$
$$\mathcal{F}_5(x,y) = \{(a,b) \in x \times y: a \in b\}$$
$$\mathcal{F}_6(x,y) = \bigcup x$$
$$\mathcal{F}_7(x,y) = \mathrm{dom}(x)$$
$$\mathcal{F}_8(x,y) = \{a: (z,a) \in x \wedge z \in y\}$$
$$\mathcal{F}_A(x,y) = x \cap A$$

Each basic rudimentary function is a total function which is Σ_0–definable in the language $\mathcal{L}(A)$, and only $\mathcal{F}_A$ mentions A. The functions $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_7$, and $\mathcal{F}_A$ all correspond to basic set operations. The other basic rudimentary functions enable one to shift variables so that the following theorem holds (see [Jech 1978]):

Lemma A.1.2:
If φ is a Σ_0–formula of $\mathcal{L}(\mathcal{U})$, then there is a composition of basic rudimentary functions $\mathcal{F}_\varphi$ such that $\mathcal{F}_\varphi(X_1, \dots X_n) = \{(x_1, \dots x_n) \in X_1 \times \dots \times X_n: \varphi(x_1, \dots, x_n)\}$.

We define a function $\mathcal{F}$ to be *rudimentary* provided that it is a composition of basic rudimentary functions.

Lemma A.1.3:

If $\mathcal{F}$ is a rudimentary n–ary function, then so is the function $\mathcal{G}$ given by

$$\mathcal{G}(x_1, \ldots, x_n) = \mathcal{F}''(x_1 \times \ldots \times x_n)$$

(See [Dodd 1982] for a proof of Lemma A.2.3)

Thus the function S^A defined by:

$$S^A(x) = x \cup \bigcup_{i=0}^{8} \mathcal{F}_i''(x \times x) \cup \mathcal{F}_A''(x \times x)$$

is also rudimentary.

With the aid of S^A we can inductively define an ordinal – indexed hierarchy of sets S_α^A as follows:

$$S_0^A = \emptyset$$
$$S_{\alpha+1}^A = S^A(S_\alpha^A \cup \{S_\alpha^A\})$$
$$S_\lambda^A = \bigcup_{\alpha<\lambda} S_\alpha^A \text{ if } \lambda \text{ is a limit ordinal.}$$

We are more interested in a sub–hierarchy of (S_α^A: $\alpha \in On$), however. Put $J_\alpha^A = S_{\omega\alpha}^A$ for each ordinal α. Then each J_α^A is closed under rudimentary functions by construction of the S_β^A – hierarchy. If $A = \emptyset$, we usually omit it, i.e. we write J_α instead of $J_\alpha^\emptyset$.

Lemma A.1.4:

For each ordinal α

(1) *Each J_α^A is a transitive set.*

(2) *If $\beta < \alpha$, then $J_\beta^A \subsetneq J_\alpha^A$.*

(3) *If $\beta < \alpha$, then $J_\beta^A \in J_\alpha^A$.*

(4) $|J_\alpha^A| = |\alpha|$.

Lemma A.1.5:

For $\alpha > 1$, *each* J_α^A *is a transitive model of the axioms*:

(1) $\forall x \exists y \forall z[z \in y \longleftrightarrow \exists w \in x(z \in w)]$ *Union*

(2) $\forall x \forall y \exists z \forall a[a \in z \longleftrightarrow a = x \vee a = y]$ *Pairing*

(3) $\exists x[x \text{ is an ordinal} \wedge \forall y \in x \exists z \in x(y \in z)]$ *Infinity*

(4) $\forall a \forall b \exists y[y = a \times b]$ *Cartesian Product*

$(5)_\varphi$ $\forall \vec{a} \forall x \exists y \forall x[z \in y \longleftrightarrow (z \in x \wedge \varphi(z,\vec{a}))]$ *for any* Σ_0*–formula* φ *of* $\mathcal{L}(A)$.

Σ_0*–Comprehension.*

Definition A.1.6: $L[A] = \bigcup_{\alpha \in On} J_\alpha^A$.

If $A = \emptyset$, we write L for $L[\emptyset]$.The class L is called the *constructible universe*, and was first introduced by Kurt Gödel to prove the consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the axioms of ZF ([Gödel 1938]). The hierarchy of sets that L is made up of, mamely the J_α ($\alpha \in On$), is called the Jensen hierarchy of the constructible universe.

The class L[A] is called the *universe of sets constructible relative to* A.

We shall be mainly interested in the cases $A = \emptyset$ and $A = \mathcal{U}$ is a normal measure over some measurable cardinal κ, because in both these cases the GCH holds.

Lemma A.1.7:

L[A] *is an inner model of* ZFC.

If $\leq_\alpha$ is a well–ordering of S_α^A, we may extend it to a well–ordering $\leq_{\alpha+1}$ of $S_{\alpha+1}^A$ in such a way that $\leq_{\alpha+1}$ is an end extension of $\leq_\alpha$. For expediency we momentarily let $\mathcal{F}_9 = \mathcal{F}_A$.

If $x,y \in S_{\alpha+1}^A$, put $x \leq_{\alpha+1} y$ if and only if:

(1) $x,y \in S_\alpha^A$ and $x \leq_\alpha y$.

(2) $x \in S_\alpha^A$ and $y \in S_{\alpha+1}^A - S_\alpha^A$.

(3) $x,y \in S_{\alpha+1}^A - S_\alpha^A$ and

(i) The least i such that $\exists u,v \in S_\alpha^A \cup \{S_\alpha^A\}[x = \mathcal{F}_i(u,v)]$ is smaller than the least j such that $\exists u,v \in S_\alpha^A \cup \{S_\alpha^A\}[y = \mathcal{F}_j(u,v)]$, OR

(ii) The least i equals the least j and the least u such that
$\exists v \in S_\alpha^A \cup \{S_\alpha^A\}[x = \mathcal{F}_i(u,v)]$ is $\leq_\alpha$ the least u' such that
$\exists v \in S_\alpha^A \cup \{S_\alpha^A\}[y = \mathcal{F}_i(u',v)]$, OR
(iii) The least i = the least j, the least u = the least u', and the least
$v \in S_\alpha^A \cup \{S_\alpha^A\}$ such that $x = \mathcal{F}_i(u,v)$ is $\leq_\alpha$ the least $v' \in S_\alpha^A \cup \{S_\alpha^A\}$ such that $y = \mathcal{F}_i(u,v')$.

Thus there is a definable well-ordering $\leq_{L[A]}$ of L[A], called the *canonical* well-ordering, given by repeatedly extending the ordering $\leq_0 = \emptyset$ on $S_0^A = \emptyset$ in the manner indicated above.

Recall that the symbol $\leq_e$ denotes "elementary submodel of". The next lemma is a modified version of a lemma Gödel used to prove that L ⊨ GCH ([Gödel 1938]).

Lemma A.1.8 (Condensation Lemma):
If $M \leq_e J_\alpha^A$, *then there is a collapsing isomorphism* $\pi : M \longrightarrow N = J_\beta^{\pi''(A \cap M)}$ *for some* $\beta \leq \alpha$. *Moreover,* π *is the identity when restricted to transitive subsets of* M, *and* $\pi(\alpha) \leq \alpha$ *for all ordinals* α. π *is given by* $\pi(u) = \pi''(u \cap M)$.

(This lemma can actually be strengthened: We need only assume that M is a Σ_1 – submodel of J_α^A for the conclusion to hold. The proof of Lemma A.1.8 may be found in [Devlin 1984], but it is true essentially because there is a Π_2– sentence φ of $\mathcal{L}(A)$ such that for any transitive set M, we have $M \models \varphi$ if and only if $M = J_\beta^B$ for some ordinal β and some set B.)
If $A = \emptyset$, then Lemma 2.1.8 states that any elementary (or even Σ_1) submodel of J_α is isomorphic to a J_β for some $\beta \leq \alpha$. The J_α are *absolute* for transitive models of ZFC, i.e. if M is a transitive model of ZFC and α is an ordinal in M, then $(J_\alpha)^M = J_\alpha$.
Finally, the *Axiom of Constructibility* (abbreviated V = L) is the assertion that all sets are constructible: $\forall x \exists \alpha \in On(x \in J_\alpha)$

§ A.1.2 $0^{\#}$

In this section we provide, without proofs, the background needed to understand the significance of Jensen's Covering Lemma to the singular cardinals problem (Section 4.4). Good references for the work presented here are [Jech 1978] and [Devlin 1984]. [Dodd 1982] also develops this work from a more general point of view, using the theory of mice.

The set $0^{\#}$ (which we shall define later) was isolated by Silver ([Silver 1971b]), building on indiscernibility methods in set theory which were developed by Rowbottom. Its existence is intimately connected with the structure of the (real) universe relative to the constructible universe. In L, there is no singular cardinals problem (because $L \vdash GCH$), and it transpires that if $0^{\#}$ does not exist, then the singular cardinals problem in V disappears as well.

Definition A.1.9: Suppose that $\mathcal{A}$ is a model of the language of set theory $\mathcal{L}$, such that $\kappa \subseteq \mathcal{A}$. A set $H \subseteq \kappa$ is a set of *indiscernibles* for $\mathcal{A}$ provided whenever φ is an $\mathcal{L}$–formula and $\xi_0 < \xi_1 < \dots < \xi_{n-1}$ and $\zeta_0 < \zeta_1 < \dots < \zeta_{n-1}$ are increasing sequences in H, then

$$\mathcal{A} \vDash \varphi(\xi_0, \dots, \xi_{n-1}) \text{ iff } \mathcal{A} \vDash \varphi(\zeta_0, \dots, \zeta_{n-1}).$$

Definition A.1.10: A set Σ of $\mathcal{L}$–formulas is called an *Ehrenfeucht–Mostowski* set (E–M set for short) if there is a model $\mathcal{A}$ of ZFC + V = L and an infinite set $H \subseteq On^{\mathcal{A}}$ which is indiscernible for $\mathcal{A}$ such that Σ is the set of all $\mathcal{L}$–formulas valid in $\mathcal{A}$ on increasing sequences from H.

Note that if $\mathcal{A}$ is a model of ZFC + V = L, and if $X \subseteq \mathcal{A}$, then

$$\mathcal{A}|X = \{t^{\mathcal{A}}(\vec{x}) \colon t \text{ is a term and } \vec{x} \in X\}$$

is an elementary submodel of $\mathcal{A}$ (by virtue of the definable well–ordering on $\mathcal{A}$).

Definition A.1.11: If α is an infinite ordinal and Σ an E–M set, then a (Σ,α)–model is a model $(\mathcal{A},H)$ such that:

(1) $\mathcal{A}$ is a model of ZFC + V = L.

(2) $H \subseteq On^{\mathcal{A}}$ is a set of indiscernibles for $\mathcal{A}$, and $otp(H) = \alpha$.

(3) $\mathcal{A} = \mathcal{A}|H = \{t^{\mathcal{A}}(\vec{x}) : \vec{x} \in H\}$.

(4) Σ is the set of all $\mathcal{L}$–formulas valid in $\mathcal{A}$ on increasing sequences from H.

For any E–M set Σ and any infinite ordinal α, it can be proved that there is a unique (up to isomorphism) (Σ,α)–model. We shall be concerned with E–M sets which have rather special properties:

Definition A.1.12:

(1) An E–M set Σ is said to be *cofinal* provided that

$$On(t(v_0 \dots v_{n-1})) \to t(v_0 \dots v_{n-1}) < v_n$$

is a formula in Σ for any term t of $\mathcal{L}$.

(2) An E–M set Σ is said to be *remarkable* provided that for every term t of $\mathcal{L}$, if

$$t(v_0 \dots v_{n-1}, v_n \dots v_{n+m}) < v_n$$

is a formula in Σ, then so is the formula

$$t(v_0 \dots v_{n-1}, v_n \dots v_{n+m}) = t(v_0 \dots v_{n-1}, v_{n+m+1} \dots v_{n+2m+1})$$

(3) An E–M set Σ is said to be *well–founded* provided that for any infinite ordinal α, the (Σ,α)–model is well–founded (and thus has a transitive isomorph which must be a J_κ by Lemma A.1.8). The transitive (Σ,α)–model is denoted by $M(\Sigma,\alpha)$.

Lemma A.1.13:

Let Σ *be a well–founded remarkable cofinal* E–M *set:*

(a) *Suppose that* α *is a limit ordinal, and* β *a limit ordinal* $< \alpha$. *Suppose also that* $(\mathcal{A},H)$ *is the* (Σ,α)*–model and that* $H = \{h_\gamma : \gamma < \alpha\}$ *is an increasing enumeration.* *Let* $K = \{h_\gamma : \gamma < \beta\}$ *and let* $\mathcal{B} = \mathcal{A}|K$. *Then* $(\mathcal{B},K)$ *is the* (Σ,β)*–model, and* $On^{\mathcal{B}} = \{x \in On^{\mathcal{A}} : x < h_\beta\}$.

(b) *If* λ *is a limit ordinal and if* $(\mathcal{A},H)$ *is the* (Σ,λ)*–model, then* H *is club in* $On^{\mathcal{A}}$.

(c) *If* κ *is an uncountable cardinal, and if* $(\mathcal{A},H) = M(\Sigma,\kappa)$, *then* $\mathcal{A} = J_\kappa$.

(d) *If* κ *is an uncountable cardinal,* (J_κ,H) *is the* (Σ,κ) *model, and* λ *is an uncountable cardinal* $< \kappa$, *then* $h_\lambda = \lambda$.

(e) *For any uncountable cardinal* κ, *let* H_κ *be the club subset of* κ *such that* (J_κ,H_κ) *is the* (Σ,κ)*–model. If* $\lambda < \kappa$ *are uncountable cardinals, then* $H_\lambda = H_\kappa \cap \lambda$, *and* $J_\lambda = J_\kappa | H_\lambda$.

It follows from Lemma A.1.13(d) that H_κ contains all uncountable cardinals below κ.
The point of the above is that if there exists a well–founded remarkable cofinal E–M set Σ, then it is unique: For $(J_{\omega_\omega}, H_{\omega_\omega})$ is the transitive (Σ,ω_ω)–model, and by Lemma A.1.13(e)

$\omega_n \in H_{\omega_\omega}$ for each $n < \omega$. It follows that for any $\mathcal{L}$–formula φ

$$\varphi(v_0 \ldots v_{n-1}) \in \Sigma \text{ if and only if } J_{\omega_\omega} \models \varphi(\omega_1 \ldots \omega_n).$$

Definition A.1.14: The *unique well–founded remarkable cofinal* E–M *set*, if it exists, is called $0^{\#}$.

If $0^{\#}$ exists, therefore, it is just the set of φ such that $J_{\omega_\omega} \models \varphi(\omega_1 \ldots \omega_n)$. The existence of $0^{\#}$ is intimately connected to the existence of non–trivial elementary embeddings of the constructible universe, as the following lemma asserts.

Lemma A.1.15: *The following statements are equivalent:*

(1) $0^{\#}$ *exists.*

(2) *There is a club class of ordinals* H, *containing all uncountable cardinals, such that for all uncountable cardinals* κ

(a) $\mathrm{otp}(H \cap \kappa) = \kappa$ *and* $H \cap \kappa$ *is club in* κ.

(b) $H \cap \kappa$ *is a set of indiscernibles for* J_κ.

(c) $J_\kappa = J_\kappa | (H \cap \kappa)$.

(3) *There exist limit ordinals* $\alpha < \beta$ *and an elementary embedding* $j: J_\alpha \longrightarrow J_\beta$ *such that* $j(\gamma) \neq \gamma$ *for some* $\gamma < |\alpha|$.

(4) *There is a non–trivial elementary embedding* $j: L \longrightarrow L$.

(5) *For any uncountable cardinal* κ, *there is a non–trivial elementary embedding* $j: J_\kappa \longrightarrow J_\kappa$.

The set H in part (2) of lemma A.1.15 is often called the set of *Silver indiscernibles* for **L**. Note that $0^{\#}$ may of course be regarded as a set of natural numbers, since formulas in the language of set theory may be regarded as natural numbers via some "Gödelization" process. If $0^{\#}$ exists, then it is absolute for transitive models of ZFC, because the formula $\Psi(x) \longleftrightarrow x = 0^{\#}$ is Π_1 (see [Devlin 1984]). Also note that the existence of $0^{\#}$ implies that $V \neq L$, because Kunen ([Kunen 1971b]) has proved that there is no non–trivial elementary embedding of the universe.

Appendix 2: FORCING

In order to make this dissertation as self–contained as possible, we provide here an overview of the method of forcing for obtaining independence proofs concerning the power function. In Section A.2.1 we introduce the notions and notation necessary for the application of forcing using a single generic extension. Section A.2.2 discusses the two–step iteration, and Section A.2.3 will discuss generalized iterated forcing, with particular emphasis on standard iterations along ordinals. Many proofs will be omitted, and good references for the material presented here are [Jech 1978], [Jech 1986], [Kunen 1980] and [Baumgartner 1983].

§ A.2.1 Forcing

Recall that the object of forcing is to extend a *transitive* model of ZFC–set theory, called the *ground model*, by adjoining a so–called *generic* set G. The model obtained, the *generic extension*, will also be a model of ZFC and will be the minimal model which contains the same ordinals as the ground model and has G as element. The elements of the generic extension will have *names* in the ground model, whose meaning is determined by the generic set.
Next follows a paragraph of important definitions.

Let V be a transitive model of ZFC and Let $(\mathbb{P},\leq)$ be a partially ordered set in V. $(\mathbb{P},\leq)$ is called a *notion of forcing*, and the elements of $\mathbb{P}$ are called the *forcing conditions.* We shall assume that $\mathbb{P}$ has a largest element, commonly denoted 1. If $p,q \in \mathbb{P}$ and $p \leq q$, we shall say that p is *stronger* than q.
A subset $D \subseteq \mathbb{P}$ is said to be *dense* in $\mathbb{P}$ provided that for every $p \in \mathbb{P}$ there is $q \in D$ such that q is stronger than p: $(\forall p \in \mathbb{P})(\exists q \in D)(q \leq p)$
A subset $D \subseteq \mathbb{P}$ is said to be *open dense* if D is dense and whenever $p \leq q$ and $q \in D$, also $p \in D$.
A subset $X \subseteq \mathbb{P}$ is said to be *directed* provided for all $x,y \in X$ there is $z \in X$ such that $z \leq x$ and $z \leq y$.
Two conditions p,q in $\mathbb{P}$ are *compatible* (abbreviated $p \mid q$) provided there is a third condition

r stronger than both, i.e. p¦q iff $\exists r \in \mathbb{P}(r \leq p \wedge r \leq q)$. Otherwise p,q are called *incompatible* (abbreviated $p \perp q$.)
A subset $A \subseteq \mathbb{P}$ is an *antichain* in $\mathbb{P}$ if any two of its elements are incompatible. A *partition* of $\mathbb{P}$ is a *maximal antichain* in $\mathbb{P}$. If $A \subseteq \mathbb{P}$ is a partition, then any element of $\mathbb{P}$ is compatible with some element of A.
Also, if A is a partition, then the set $D_A = \{p \in \mathbb{P}: \exists a \in A(p \leq a)\}$ is open dense in $\mathbb{P}$.
An antichain $A \subseteq \mathbb{P}$ is a *refinement* of an antichain $B \subseteq \mathbb{P}$ if for every $a \in A$ there is $b \in B$ such that $a \leq b$.
A subset $G \subseteq \mathbb{P}$ is a *filter* on $\mathbb{P}$ if and only if :

(1) G is a proper non–empty subset of $\mathbb{P}$.
(2) Any two elements of G are compatible.
(3) If $p \in G$ and $q \in \mathbb{P}$ is such that $p \leq q$, then $q \in G$.

A filter G on $\mathbb{P}$ is called *generic* over V iff one of the following equivalent conditions holds:

(i) For every dense $D \subseteq \mathbb{P}$, if $D \in V$ then $D \cap G \neq \emptyset$.
(ii) For every partition A of $\mathbb{P}$, $A \cap G$ has exactly one element.

The generic set G is not usually in the ground model V, since apart from trivial cases $\mathbb{P}\backslash G$ is dense in $\mathbb{P}$. We shall now indicate how to obtain the generic extension V[G] of V without going into details as it is unnecessary to be acquainted with the formal machinery of forcing to appreciate how it works. Roughly V[G] is the set of all sets constructed from G using set–theoretic processes definable in V. More precisely, each element of V[G] has a *name* in V which carries enough information to construct the element from G. The symbol $\dot{x}$ will denote a name in the ground model for a set $x \in V[G]$. However every element $y \in V$ has a *canonical* name, which will be denoted $\check{y}$, as does the generic filter, with name $\check{G}$.
The interpretation of a name $\dot{x}$ under G will be denoted $\dot{x}[G]$. Thus if $y \in V$, then $\check{y}[G] = y$, and $\check{G}[G] = G$. The generic extension is then defined as follows:

$$V[G] = \{\dot{x}[G] : \dot{x} \text{ is a name in } V\}$$

Thus V[G] can be described inside V in terms of the "unknown" set G. Since G has a canonical name, $G \in V[G]$, and since every element of V has a canonical name, $V \subseteq V[G]$.

The names are defined by transfinite recursion:

$\dot{x}$ is a name if it is of the form $\dot{x} = \{(\dot{y}_i, p_i) \mid \dot{y}_i \text{ is a name and } p_i \in \mathbb{P}, i \in I\}$

The interpretation of a name $\dot{x}$ under G is also defined inductively:

$$\dot{x}[G] = \{\dot{y}[G] : \exists p \in G((\dot{y},p) \in \dot{x})\}$$

If y ∈ V, the canonical name for y is defined by:

$$\check{y} = \{(\check{z},1) : z \in y\}$$

The canonical name for the generic filter is given by:

$$\dot{G} = \{(\check{p},p) : p \in \mathbb{P}\}$$

It is now not hard to verify the claims concerning names of the previous paragraph. Note that the notion of forcing $\mathbb{P}$ figures prominently in the definition of names, so that one should actually refer to names as "$\mathbb{P}$–names"; however, the notion of forcing is almost always understood from context.

We state the following well–known theorem without proof (See [Jech 1978] or [Kunen 1980]):

Generic Model Theorem A.2.1 [Cohen 1963–1964]:
Let V *be a transitive model of* ZFC *and let* $(\mathbb{P},\leq)$ *be a notion of forcing in* V. *If* $G \subseteq \mathbb{P}$ *is generic over* V, *then the generic extension* V[G] *satisfies the following*:

(a) V[G] *is a transitive model of* ZFC

(b) $V \subseteq V[G]$ *and* $G \in V[G]$

(c) V *and* V[G] *have the same ordinals.*

(d) V[G] *is universal with respect to* (a),(b),(c), *i.e. if* M *is another transitive model satisfying* (a),(b),(c), *then* $V[G] \subseteq M$.

We shall denote the relativization of a formula φ to a transitive model M by φ^M. Thus φ^M is true if and only if $M \models \varphi$. Similarly for a set X, $|X|^M$ will denote the cardinality of X in the model M, and $cf^M(\kappa)$ will denote the ordinal which is the cofinality of κ in M.

The *forcing relation* is a relation defined in the ground model between members of the forcing notion and formulas of the forcing language. The forcing language is just the usual language of set theory augmented with a constant symbol for every name. We shall write:

$$p \Vdash \varphi(\dot{x}_1 \ldots \dot{x}_n)$$

for p *forces* φ.

"$1 \Vdash \varphi$" is simply written "$\Vdash \varphi$" where 1 is the top element of $\mathbb{P}$.

The forcing relation is defined inductively on the complexity of a formula and we shall not go into details. The following is provable, however:

Forcing Theorem A.2.2(a) (First Version) [Cohen 1963–1964]:

Let $\mathbb{P}$ *be a notion of forcing in* V *and let* $G \subseteq \mathbb{P}$ *be generic over* V. *If* $\varphi(\dot{x}_1 \dots \dot{x}_n)$ *is a sentence of the forcing language and* $p \in \mathbb{P}$, *then:*

$$V[G] \models \varphi(\dot{x}_1[G] \dots \dot{x}_n[G]) \quad \textit{iff} \quad \exists p \in G \; p \Vdash \varphi(\dot{x}_1 \dots \dot{x}_n)$$

Forcing Theorem A.2.2(b) (Second Version) [Cohen 1963–1964]:

Let $\mathbb{P}$ *be a notion of forcing in* V *and let* $\varphi(\dot{x}_1 \dots \dot{x}_n)$ *be a sentence of the forcing language. Then:*

$$p \Vdash \varphi(\dot{x}_1 \dots \dot{x}_n) \;\; \textit{iff for all generic } G \subseteq \mathbb{P}, \textit{ if } p \in G \textit{ then } V[G] \models \varphi(\dot{x}_1[G] \dots \dot{x}_n[G]).$$

Version 2 is clearly equivalent to version 1 provided that for every $p \in \mathbb{P}$ there is a generic $G \subseteq \mathbb{P}$ such that $p \in G$. While one cannot always prove this (in fact generally one cannot even prove that a generic set exists) it is consistent to assume this, because it is true for countable transitive ground models : If V is a countable transitive model of ZFC, we may enumerate the dense subsets of $\mathbb{P}$ in V:

$$D_0, D_1, \dots D_n, \dots \qquad (n < \omega)$$

Then given $p \in \mathbb{P}$, we may choose $q_0 \in D_0$ such that $q_0 \leq p$ and in general given $q_n \in D_n$, we choose $q_{n+1} \in D_{n+1}$ such that $q_{n+1} \leq q_n$. If G is the filter generated by the q_n, then G is generic and $p \in G$.

Note A.2.3: For convenience we list the basic properties of the forcing relation:

(1) If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$
(2) $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$
(3) It is impossible that $p \Vdash \varphi$ and $p \Vdash \neg\varphi$
(4) $p \Vdash \forall x \varphi$ iff for all names $\dot{x}$ $p \Vdash \varphi(\dot{x})$
(5) $p \Vdash \neg\varphi$ iff for all $q \leq p$ $q \nVdash \varphi$
(6) $p \Vdash \varphi \vee \psi$ iff $\forall q{\leq}p\, \exists r{\leq}q\, (r \Vdash \varphi$ or $r \Vdash \psi)$
(7) $p \Vdash \exists x \varphi$ iff $\forall q{\leq}p\, \exists r{\leq}q\, \exists$ name $\dot{x}$ $(r \Vdash \varphi(\dot{x}))$
(8) Finally, for any $p \in \mathbb{P}$, and any sentence φ of the forcing language, there is $q \leq p$ such that q decides φ, ie. $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

Next we state certain combinatorial lemmas which describe how notions of forcing may affect cardinalities and cofinalities in the generic extension. Our primary aim is to describe notions of forcing which will change the power function in some desirable way. Thus, for example, in order to ensure that the Continuum Hypothesis fails, we must add at least $\aleph_2$– many new subsets of $\aleph_0$ to the generic extension. However, we must also make sure that $\aleph_1$ and $\aleph_2$ remain cardinals in the generic extension, for if either of them is collapsed, the generic extension will again be a model of CH. This is why we are often interested in notions of forcing which preserve cardinalities (i.e. leave the $\aleph$–function unchanged.)
For example, to ensure that a cardinal λ in the ground model remains one in the generic extension, we must ensure that there are no bijections from a $\kappa < \lambda$ to κ in the extension. One way to prevent the addition of unwanted bijections is simply to prevent the addition of new maps from κ to λ entirely!

Definition A.2.4: Let κ be an infinite cardinal, and let $(\mathbb{P},\leq)$ be a partially ordered set.

(1) If λ is another cardinal, then $\mathbb{P}$ is (κ, λ)*–distributive* if any $\leq \kappa$–many partitions of size $\leq \lambda$ have a common refinement.

(2) $\mathbb{P}$ is κ*–distributive* if any $\leq \kappa$–many partitions of $\mathbb{P}$ have a common refinement, i.e if $\mathbb{P}$ is (κ,λ)–distributive for all cardinals λ.

(3) $\mathbb{P}$ is κ*–closed* if for any descending chain of length $\leq \kappa$
$$p_0 \geq p_1 \geq \cdots \geq p_\xi \geq \cdots \quad (\xi < \lambda,\ \lambda \leq \kappa)$$
there is $p \in \mathbb{P}$ such that $p \leq p_\xi$ for every $\xi < \lambda$.

(4) $\mathbb{P}$ is κ*–directed closed* provided that whenever $X \subseteq \mathbb{P}$ is a directed set of cardinality $\leq \kappa$, there is a $p \in \mathbb{P}$ such that $\forall x \in X(p \leq x)$.

(5) We shall say that $\mathbb{P}$ is $<\kappa$–distributive (or $<\kappa$–closed) if $\mathbb{P}$ is ξ–distributuive (respectively: ξ–closed) for all cardinals $\xi < \kappa$.

If $\mathbb{P}$ is κ–directed closed, then $\mathbb{P}$ is κ–closed, and any κ–closed $\mathbb{P}$ is κ–distributive. However, it is usually easier to test for κ–(directed) closedness than for κ–distributivity.
An equivalent condition for κ–distributivity is the following: $\mathbb{P}$ is κ–distributive if and only if any intersection of $\leq$ k–many open dense subsets of $\mathbb{P}$ is open dense.
We shall be interested in the above notions relativized to the ground model.

Theorem A.2.5 (Distributivity Theorem):
$(\mathbb{P},\leq)$ is (κ,λ) – distributive in the ground model if and only if every map $f:\kappa \longrightarrow \lambda$ in the generic extension is already in the ground model.

Proof: Let $G \subseteq \mathbb{P}$ be generic over the ground model V, and suppose that $f:\kappa \longrightarrow \lambda$ is a map in V[G]. Choose $p_0 \in G$ and a name $\dot{f}$ for f such that $p_0 \Vdash \dot{f}:\check{\kappa} \longrightarrow \check{\lambda}$. For each $\gamma < \lambda$ and each $\xi < \kappa$, let $w_{\xi\gamma} \leq p_0$ be such that $w_{\xi\gamma} \Vdash \dot{f}(\check{\xi}) = \check{\gamma}$ (provided that such exists), and let $W_\xi = \{w_{\xi\gamma}: \gamma < \lambda\}$. Then there is a W which is a common refinement of all the W_ξ. Let $w \in W \cap G$, and define $g:\kappa \longrightarrow \lambda$ by : $g(\xi) = \gamma \longmapsto w \leq w_{\xi\gamma}$.

Clearly $g \in V$, and $w \Vdash \dot{f} = g$.
Conversely, suppose that any $f:\kappa \longrightarrow \lambda$ in V[G] necessarily belongs to V, and let $\{W_\xi: \xi < \kappa\}$ be a family of partitions each of cardinality $\leq \lambda$. Put $W_\xi = \{w_{\xi\gamma}: \gamma < \lambda\}$, and define $\dot{f} = \{((\check{\xi},\gamma), w_{\xi\gamma}): \xi < \kappa \text{ and } \gamma < \lambda\}$, where $(\check{\xi},\gamma)$ is the canonical name of the ordered pair (ξ,γ). Then $\dot{f}$ is a name for a map from κ to λ, and thus for any $p \in \mathbb{P}$ there are $w \leq p$ and $g \in V$ such that $w \Vdash \dot{f} = g$. Then $w \leq w_{\xi g(\xi)}$ for all $\xi < \kappa$, and so the W_ξ $(\xi < \kappa)$ have a common refinememt.

□

In particular, if $\mathbb{P}$ is κ–distributive in V then every map $\kappa \longrightarrow V$ in the generic extension is already in the ground model V. Clearly if $\mathbb{P}$ is (κ,λ)–distributive in V, then $|\lambda| \leq |\kappa|$ in the generic extension if and only if $\lambda \leq \kappa$ in the ground model. Hence the following corollary holds:

Corollary A.2.6:
Assume $\mathbb{P}$ is $<\kappa$–distributive in V*. Then $\mathbb{P}$ preserves cardinalities and cofinalities $\leq \kappa$, i.e. every ordinal which has cofinality (cardinality) $\leq \kappa$ in* V[G] *has the same cofinality (cardinality) in* V.

Another combinatorial property which governs cofinality–preservation in generic extensions is the the size of antichains in $\mathbb{P}$. Suppose that κ is a regular cardinal in the ground model, and suppose that $\lambda < \kappa$ is another cardinal. We want to ensure that there is no cofinal map from λ to κ in the generic extension. Let $f:\lambda \longrightarrow \kappa$ be a map in the generic extension, and let $\dot{f}$ be a name for f. For each $\alpha < \lambda$ and each pair $\beta_1, \beta_2 < \kappa$, if

$w_1 \Vdash \dot{f}(\check{\alpha}) = \check{\beta}_1$ and $w_2 \Vdash \dot{f}(\check{\alpha}) = \check{\beta}_2$, then clearly $w_1 \perp w_2$. Hence if every antichain in $\mathbb{P}$ has cardinality $< \kappa$, then the set of possible values of $\dot{f}(\check{\alpha})$ in the generic extension is bounded, below κ, and thus the possible range of $\dot{f}$ is bounded below κ (because $\lambda < \kappa$ and κ is regular). This motivates the following definition.

Definition A.2.7: Let κ be an infinite cardinal, and let $(\mathbb{P}, \leq)$ be a partially ordered set.

(a) $\mathbb{P}$ is said to satisfy the κ – *chain condition* (abbreviated κ–c.c.) provided that every antichain in $\mathbb{P}$ has cardinality $< \kappa$. In that case we also say that $\mathbb{P}$ is κ – *saturated.*

(b) Sat($\mathbb{P}$) is the least cardinal κ such that $\mathbb{P}$ is κ – saturated. Note that if $\mathbb{P}$ is κ – saturated and $\lambda > \kappa$, then $\mathbb{P}$ is λ – saturated as well. It is not hard to show that if Sat($\mathbb{P}$) is infinite, then it is a regular uncountable cardinal.

(c) We say that $\mathbb{P}$ has the *countable chain condition* (c.c.c) provided that $\mathbb{P}$ has the $\aleph_1$–c.c.

Theorem A.2.8 (Chain Condition Theorem):
Let κ be a cardinal in V *and suppose $\mathbb{P}$ is a notion of forcing which satisfies the κ–c.c.in* V. *Then $\mathbb{P}$ preserves cofinalities $\geq \kappa$. If κ is a regular cardinal in* V *then $\mathbb{P}$ preserves cardinalities $\geq \kappa$ as well.*

Proof: It suffices to prove that if κ is a regular cardinal in the ground model, the it is regular in the generic extension as well. Let $\lambda < \kappa$, let $\dot{f}$ be a $\mathbb{P}$–name, and let $p_0 \in \mathbb{P}$ such that $p_0 \Vdash \dot{f}:\check{\lambda} \longrightarrow \check{\kappa}$. For each $\alpha < \lambda$, the set $X_\alpha = \{\beta < \kappa: \exists w \leq p_0(w \Vdash \dot{f}(\check{\alpha}) = \check{\beta})\}$ has cardinality $< \kappa$, by the κ–c.c. Hence $\sup(\bigcup_{\alpha<\lambda} X_\alpha) < \kappa$.

It follows that $p_0 \Vdash$ "$\dot{f}$ is bounded below κ", and thus κ remains a regular cardinal in the generic extension.

□

Corollary A.2.9:
If $\mathbb{P}$ satisfies the c.c.c. it preserves all cardinalities and cofinalities.

One extremely useful technique for determining which chain conditions a partial ordering satisfies is the so–called Δ – *system lemma*: A family $\mathcal{A}$ of sets is called a Δ – *system* provided that there is a fixed set r, the *root* of the Δ – system, such that for any distinct a, b $\in \mathcal{A}$ we have $a \cap b = r$. Note that r may be the empty set.

Lemma A.2.10 (Δ– System Lemma):
Let κ be an infinite cardinal, and let θ be a regular cardinal $> \kappa$ such that for any $\alpha < \theta$, also $\alpha^{<\kappa} < \theta$. Assume $\mathcal{A}$ is a family of at least θ–many sets such that every element of $\mathcal{A}$ has cardinality $< \kappa$. Then there is a $B \subseteq \mathcal{A}$ of cardinality θ such that B forms a Δ–system.

Proof: We may first assume that $|\mathcal{A}| = \theta$, and then assume that $\bigcup\mathcal{A} \subseteq \theta$. A final assumption which we may make without loss of generality is that $\mathrm{otp}(x) = \rho$ for each $x \in \mathcal{A}$, where ρ is an ordinal $< \kappa$. Since $\mathcal{A}$ has cardinality θ and $\alpha^{<\kappa} < \theta$ for each $\alpha < \theta$, it follows that $\bigcup\mathcal{A}$ is unbounded in θ. Each $x \in \mathcal{A}$ has order type ρ, and so we may enumerate $x = \{x(\xi): \xi < \rho\}$ in ascending order. Now $\rho < \theta$ and θ is regular so there exists a least ξ_0 such that the set $\{x(\xi_0): x \in \mathcal{A}\}$ is unbounded in θ. Note that ξ_0 may be 0.
Let $\alpha_0 = \sup\{x(\eta) + 1: x \in \mathcal{A}$ and $\eta < \xi_0\}$. Then $\alpha_0 < \theta$ (by definition of ξ_0). We now pick a sequence x_μ from $\mathcal{A}$ by induction on $\mu < \theta$: x_0 is such that $x_0(\xi_0) > \alpha_0$.
Given $(x_\nu: \nu < \mu)$, pick $x_\mu \in \mathcal{A}$ such that $x_\mu(\xi_0) > \max(\alpha_0, \sup\{x_\nu(\eta): \eta < \rho$ and $\nu < \mu\})$.
Let $\mathcal{A}' = \{x_\mu: \mu < \theta\}$. It is not hard to see that $x \cap y \in \{z: z \subseteq \alpha_0$ and $|z| < \kappa\}$ for any distinct $x, y \in \mathcal{A}'$. Now $|\alpha_0{}^{<\kappa}| < \theta$ because $\alpha_0 < \theta$, and hence there is a set r and a set $B \subseteq \mathcal{A}'$ of cardinality θ such that $x \cap \alpha_0 = r$ for any $x \in B$. B is a Δ – system of the required size with root r.

□

§ A.2.2 The Two–step Iteration

In Section A.2.1 we showed how to use forcing to attain a single objective, but we will often interested in repeating forcing constructions many (even "class–many") times, in order to attain possibly infinitely many objectives. For example, it is easy to change the power function at one cardinal using a single notion of forcing (as is done in Section 2.2), but to change it at more than one place we may well need to force more than once, and this necessitates the development of multiple forcing techniques (set forth in this and the following section). We shall actually be concerned with only two types of multiple forcing, namely *product forcing* and *standard iterated forcing.* Product forcing is essentially the simultaneous addition of several generic objects to the ground model, whereas standard iterated forcing adds generic objects one by one:

The idea behind iterated forcing is to repeat the process of generating generic extensions ξ–many times, for some ordinal ξ. Thus one obtains an ascending chain

$$V = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq V_\eta \subseteq \dots \subseteq V_\xi$$

where each $V_{\eta+1}$ is $V_\eta[G_\eta]$ for some $G_\eta \subseteq \mathbb{P}_\eta \in V_\eta$ which is generic over V_η.

We will not endeavour to prove anything in this section; we present these results merely in an attempt to make this dissertation as self–contained as possible. Good references for the work discussed here are the books [Jech 1978], [Jech 1986] and the article [Baumgartner 1983].

Definition A.2.12: Let $\mathbb{P}$ be a notion of forcing in the ground model V, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$–name for a partial ordering, i.e. $\Vdash \dot{\mathbb{Q}}$ is a partially ordered set. $\mathbb{P} * \dot{\mathbb{Q}}$ is a notion of forcing in the ground model that amounts to first forcing with $\mathbb{P}$ in the ground model using a generic set G, and then forcing with $\dot{\mathbb{Q}}[G]$ in the generic extension obtained. We define:

$$\mathbb{P} * \dot{\mathbb{Q}} = \{(p, \dot{q}) : p \in \mathbb{P} \text{ and } \Vdash \dot{q} \in \dot{\mathbb{Q}}\}$$

$$(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \text{ if and only if } p_1 \leq p_2 \text{ and } p_1 \Vdash \dot{q}_1 \leq \dot{q}_2 .$$

This defines the two–step iteration of $\mathbb{P}$ and $\dot{\mathbb{Q}}$.

A special case of the above occurs if $\mathbb{P}$ and $\mathbb{Q}$ are both in the ground model. In that case we may define the product $\mathbb{P} \times \mathbb{Q}$ of $\mathbb{P}$ and $\mathbb{Q}$ with order defined coordinate–wise.

If both $\mathbb{P}$ and $\mathbb{Q}$ are in the ground model, and $\check{\mathbb{Q}}$ is the canonical name for $\mathbb{Q}$ in the forcing language associated with $\mathbb{P}$, then $\mathbb{P} \times \mathbb{Q}$ is isomorphic to a dense subset of $\mathbb{P} * \check{\mathbb{Q}}$ under the obvious inclusion: For suppose that $(p,\dot{q}) \in \mathbb{P} * \check{\mathbb{Q}}$. Then there is a $p' \leq p$ in $\mathbb{P}$ and a $q' \in \mathbb{Q}$ such that $p' \Vdash \dot{q} = \check{q}'$, and thus $(p',\check{q}') \leq (p,\dot{q})$. Hence $\mathbb{P} \times \mathbb{Q}$ is indeed dense in $\mathbb{P} * \check{\mathbb{Q}}$, and therefore forcing with $\mathbb{P} * \check{\mathbb{Q}}$ is the same as forcing with $\mathbb{P} \times \mathbb{Q}$. This is called product forcing. It differs from standard iterated forcing in that the order in which we force is not important (as one can similarly see that $\mathbb{Q} \times \mathbb{P}$ is also densely embeddable in $\mathbb{P} * \check{\mathbb{Q}}$.)

Let $\mathbb{P}$ be a notion of forcing in the ground model V, and let $\dot{\mathbb{Q}}$ be a name in the forcing language associated with $\mathbb{P}$ such that $\Vdash$ "$\dot{\mathbb{Q}}$ is a partially ordered set". Suppose that G is V–generic over $\mathbb{P}$ and that H is V[G]–generic over $\dot{\mathbb{Q}}[G]$. Then we can combine G and H to form a subset of $\mathbb{P} * \dot{\mathbb{Q}}$ as follows:

Define $$G * H = \{(p,\dot{q}): p \in G \text{ and } \dot{q}[G] \in H\}$$

Conversely, if G is V–generic over $\mathbb{P} * \dot{\mathbb{Q}}$, define projections

$$G_1 = \{p \in \mathbb{P}: \exists \dot{q}(p,\dot{q}) \in G\}$$
$$G_2 = \{\dot{q}[G_1]: \exists p(p,\dot{q}) \in G\}$$

That these definitions make sense is guaranteed by the following lemma:

Lemma A.2.13:

Let $\mathbb{P}$, $\dot{\mathbb{Q}}$ be as above. Suppose that G *is* V*–generic over* $\mathbb{P} * \dot{\mathbb{Q}}$. *Then* G_1 *is* V*–generic over* $\mathbb{P}$ *and* G_2 *is* $V[G_1]$*–generic over* $\dot{\mathbb{Q}}[G_1]$, *and* $G = G_1 * G_2$. *Moreover,* $V[G] = V[G_1][G_2]$. *Conversely, if* G_1 *is* V*–generic over* $\mathbb{P}$ *and* G_2 *is* $V[G_1]$*–generic over* $\dot{\mathbb{Q}}[G_1]$, *then* $G_1 * G_2$ *is* V*–generic over* $\mathbb{P} * \dot{\mathbb{Q}}$.

Suppose that $\mathbb{P}$, $\mathbb{Q}$ are both in the ground model V. If G is V–generic over $\mathbb{P} \times \mathbb{Q}$ we may define projections $G_1 = \{p \in \mathbb{P}: \exists q \in \mathbb{Q} \text{ s.t. } (p,q) \in G\}$, and G_2 similarly.
For product forcing, Lemma A.2.13 takes on the following guise:

Lemma A.2.14:
Suppose that $\mathbb{P}$ *and* $\mathbb{Q}$ *are two notions of forcing in the ground model* V. *Then* G_1 *is* V*–generic over* $\mathbb{P}$ *and* G_2 *is* $V[G_1]$*–generic over* $\mathbb{Q}$, *and* $G = G_1 \times G_2$. *Conversely if* G_1 *is* V*–generic over* $\mathbb{P}$ *and* G_2 *is* $V[G_1]$*–generic over* $\mathbb{Q}$, *then* $G_1 \times G_2$ *is* V*–generic over* $\mathbb{P} \times \mathbb{Q}$.

Corollary A.2.15:
If G *is* V*–generic over* $\mathbb{P} \times \mathbb{Q}$, *then* G_1 *is* $V[G_2]$*–generic over* $\mathbb{P}$, *and* G_2 *is* $V[G_1]$*–generic over* $\mathbb{Q}$. *Moreover* $V[G_1][G_2] = V[G_1 \times G_2] = V[G_2][G_1]$.

By Lemma A.2.13 two successive extensions of V by forcing can also be obtained as a single extension, and by Lemma A.2.14, if the two notions of forcing are both in the ground model, then the order in which we force is immaterial.

The two–step iteration preserves chain– and closure conditions:

Lemma A.2.16:

(a) *If κ is regular, $\mathbb{P}$ satisfies the κ – c.c. and $\Vdash$ "$\dot{\mathbb{Q}}$ satisfies the κ – c.c.", then $\mathbb{P} * \dot{\mathbb{Q}}$ also satisfies the κ – c.c.*

(b) *If $\mathbb{P}$ is κ–(directed) closed and $\Vdash$ "$\dot{\mathbb{Q}}$ is κ–(directed) closed", then $\mathbb{P} * \dot{\mathbb{Q}}$ is also κ–(directed) closed.*

(c) *If $\mathbb{P}$ is κ – distributive and $\Vdash$ "$\dot{\mathbb{Q}}$ is κ – distributive", then so is $\mathbb{P} * \dot{\mathbb{Q}}$.*

One may similarly prove that if both $\mathbb{P}$ and $\mathbb{Q}$ are in the ground model, and both $\mathbb{P}$, $\mathbb{Q}$ are κ – closed, then so is $\mathbb{P} \times \mathbb{Q}$. However, to say $\Vdash$ "$\check{\mathbb{Q}}$ satisfies the κ–c.c." is essentially stronger than saying "$\mathbb{Q}$ has the κ–c.c." as in V[G], $\mathbb{Q}$ might have more subsets, some of which may be antichains of cardinality $\geq \kappa$. It does not therefore follow that if both $\mathbb{P}$,$\mathbb{Q}$ have the κ–c.c. then $\mathbb{P} \times \mathbb{Q}$ has it too. The following lemma can be seen to be true, however:

Lemma A.2.17:
Let κ be a regular cardinal, and let $\mathbb{P}$, $\mathbb{Q}$ be two notions of forcing such that $\mathbb{Q}$ satisfies the κ–chain condition and $|\mathbb{P}| < \kappa$. Then $\mathbb{P} \times \mathbb{Q}$ satisfies the κ–chain condition.

We present one more lemma concerning product forcing. We present a proof in detail as this lemma is particularly important in the proof of Easton's Theorem (Theorem 2.3.1).

Lemma A.2.18:
Let λ be a regular cardinal and suppose that $\mathbb{P}$, $\mathbb{Q}$ are two notions of forcing such that $\mathbb{P}$ is λ–closed and $\mathbb{Q}$ satisfies the λ–chain condition. Suppose G $\times$ H *is* V*–generic over $\mathbb{P} \times \mathbb{Q}$, and let $\alpha < \lambda$; then any map* $f{:}\alpha \longrightarrow V$ *in* V[G $\times$ H] *is already in* V[H].

Proof: Let $f: \alpha \longrightarrow X$ be a function in $V[G \times H]$, where $X \in V$, and let $\dot{f}$ be a name for f such that $\Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{f}:\check{\alpha} \longrightarrow \check{X}$. For each $\xi < \alpha$, let $\mathcal{D}_\xi \subseteq \mathbb{P}$ be defined as follows:

$p \in \mathcal{D}_\xi$ iff there is a maximal antichain A in $\mathbb{Q}$ and a family $\{x^\xi_{p,q} : q \in A\}$

such that for each $q \in A$ $\quad (p,q) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\xi}) = \check{x}^\xi_{p,q}$.

We shall show that each $\mathcal{D}_\xi$ is open dense in $\mathbb{P}$: Given arbitrary $p_0 \in \mathbb{P}$ we shall find $p \in \mathcal{D}_\xi$ such that $p \leq p_0$. We can find $p_1 \leq p_0$, $q_1 \in \mathbb{Q}$ and $x_1 \in X$ such that $(p_1,q_1) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\xi}) = \check{x}_1$. By induction on $\gamma < \lambda$, we construct $p_\gamma \in \mathbb{P}$, $q_\gamma \in \mathbb{Q}$ and $x_\gamma \in X$ such that

$$p_0 \geq p_1 \geq \cdots \geq p_\gamma \geq \cdots,$$

such that the q_γ are mutually incompatible, and such that $(p_\gamma,q_\gamma) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\xi}) = \check{x}_\gamma$. Suppose therefore that we have p_ζ, q_ζ and x_ζ for all $\zeta < \gamma$ satisfying the above properties. If $\{q_\zeta: \zeta < \gamma\}$ is not a maximal antichain in $\mathbb{Q}$, choose $q' \in \mathbb{Q}$ incompatible with all the q_ζ. Since $\mathbb{P}$ is λ–closed, we may also find $p' \in \mathbb{P}$ smaller than all the p_ζ. Now choose $p_\gamma \leq p'$, $q_\gamma \leq q'$ and $x_\gamma \in X$ such that $(p_\gamma,q_\gamma) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\xi}) = \check{x}_\gamma$. This completes the induction. Since $\mathbb{Q}$ satisfies the λ–chain condition, there must be $\beta < \lambda$ at which this process comes to an end, i.e. when $A = \{q_\zeta: \zeta < \beta\}$ is a maximal antichain in $\mathbb{Q}$. Choose $p \in \mathbb{P}$ such that p is stronger than all the p_ζ (for $\zeta < \beta$). Then $p \in \mathcal{D}_\xi$, as witnessed by A and $\{x_\zeta: \zeta < \beta\}$, and $p \leq p_0$. This proves that each $\mathcal{D}_\xi$ is dense in $\mathbb{P}$. Finally each $\mathcal{D}_\xi$ is clearly open as well. Since $\mathbb{P}$ is λ–closed, it is λ–distributive, and thus $\bigcap\{\mathcal{D}_\xi: \xi < \alpha\}$ is open dense in $\mathbb{P}$. Hence there is $p \in G$ such that $p \in \mathcal{D}_\xi$ for all $\xi < \alpha$. Thus for each $\xi < \alpha$, we may choose (in V) a maximal antichain A_ξ in $\mathbb{Q}$ and a family $\{x^\xi_{p,q}: q \in A_\xi\}$ such that

$$(p,q) \Vdash_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\xi}) = \check{x}^\xi_{p,q}$$

holds for each $q \in A_\xi$. Since H is V–generic over $\mathbb{Q}$, there is a unique $q \in A_\xi \cap H$ for each $\xi < \alpha$. Thus $f(\xi) = x^\xi_{p,q}$, where q is the unique q in $A_\xi \cap H$. But then f is clearly definable in V[H].

□

§ A.2.3 Generalized Iterated Forcing

Let X be a partially ordered set. An *iteration along* X is a notion of forcing that adds to the ground model V a generic set $(G(x): x \in X)$ with the property that each $G(x)$ is $M[(G(y): y < x)]$–generic over some partial order $\mathbb{Q}_x$. The partial order X along which we force will be called the *support* partial order, and the partial orders with which we add generic objects will be called the *forcing* partial orders.

Let X be a support partial order, and let $\mathbb{P}$ be a forcing partial order whose elements are functions with domain X. If Y is a *downwards closed* subset of X, $x \in X$ and $p \in \mathbb{P}$, we may define the following:

$$\mathbb{P}_Y = \{p|Y: p \in \mathbb{P}\}$$
$$X|x = \{y \in X: y < x\} \text{ and } X(x) = \{y \in X: y \leq x\}$$
$$\mathbb{P}|x = \mathbb{P}_{X|x} \text{ and } \mathbb{P}(x) = \mathbb{P}_{X(x)}$$
$$p|x = p|\{y: y < x\}$$

Note that if $Y \subseteq X$ is downwards closed, then $\mathbb{P}_Y$ inherits a partial ordering from $\mathbb{P}$:

Put $p \leq q$ in $\mathbb{P}_Y$ iff $\exists \overline{p}, \overline{q} \in \mathbb{P}$ such that $p = \overline{p}|Y$, $q = \overline{q}|Y$, and $\overline{p} \leq \overline{q}$ in $\mathbb{P}$.
The forcing relation with respect to $\mathbb{P}|x$ will be denoted $\Vdash_x$.

Definition A.2.19 [Groszek–Jech 1991]: $\mathbb{P}$ is said to be an *iteration along* X provided:

(1) If $Y \subseteq X$ is downwards closed, $p \in \mathbb{P}$, $q \in \mathbb{P}_Y$ and $q \leq p|Y$ in $\mathbb{P}_Y$, then
$$\overline{p}(x) = \begin{cases} q(x) \text{ if } x \in Y \\ p(x) \text{ if } x \in X-Y \end{cases}$$
defines a condition in $\mathbb{P}$.

(2) There is a family of names $(\dot{\mathbb{Q}}_x: x \in X)$ such that for all $x \in X$, $\mathbb{P}(x)$ is a two–step iteration, $\mathbb{P}(x) = \mathbb{P}|x * \dot{\mathbb{Q}}_x$.

(3) $\mathbb{P}$ has a maximal element 1, where for all $x \in X$ we have
$\Vdash_x$ "$1(x) =$ maximal element 1 of $\dot{\mathbb{Q}}_x$"

(4) For any $p,q \in \mathbb{P}$, $p \leq q$ in $\mathbb{P}$ if and only if for every $x \in X$:
$p|x \leq q|x$ in $\mathbb{P}|x$ and $(p|x \Vdash_x p(x) \leq q(x))$.

Note that if $Y \subseteq X$ is downwards closed, then $\mathbb{P}_Y$ is an iteration along Y, and that $\mathbb{P} = \mathbb{P}_X$.
If $\mathbb{P}_X$ is an iteration along X, $Y \subseteq X$ is downwards closed, and G is V–generic over $\mathbb{P}_X$, then
$$G_Y = \{p|Y: p \in G\}$$

is V–generic over $\mathbb{P}_Y$. Each $\dot{\mathbb{Q}}_x$ is a name for a partial order in the forcing language associated with $\mathbb{P}|x$ in the model $V[G_{X|x}]$.

We are interested in only two types of support partial order. If X is a *trivial* partial order (i.e. all points are incomparable) then an iteration along X is called a *product* of the forcing partial orders ($\mathbb{Q}_x$: $x \in X$). All the $\mathbb{Q}_x$ are then elements of the ground model. If X is an *ordinal* α, then an iteration along X is called a *standard iteration of length* α. Note that each downwards closed subset of an ordinal is itself an ordinals.

We shall discuss standard iterations in greater detail. Thus from now on we shall be concerned with sequences $\mathbb{P}_\beta$ of partially ordered sets, each of which will give a different forcing relation. $\Vdash_\beta$ will denote the forcing relation associated with $\mathbb{P}_\beta$ and a β–name will be a name in the forcing language of $\mathbb{P}_\beta$.

Let $\alpha \geq 1$. Using Definition A.2.19 we see that a partial ordering $\mathbb{P}_\alpha$ is an α–stage iteration if and only if the following conditions are satisfied:

(1) $\mathbb{P}_\alpha$ is a set of sequences with domain α.

(2) There is a partial ordering $\mathbb{Q}_0$ such that $p \in \mathbb{P}_1$ iff $p(0) \in \mathbb{Q}_0$.

(3) If $\alpha = \beta + 1$, then $\mathbb{P}_\beta = \{p|\beta : p \in \mathbb{P}_\alpha\}$ is a β–stage iteration. Moreover, there is $\dot{\mathbb{Q}}_\beta$ such that $\Vdash_\beta$ "$\dot{\mathbb{Q}}_\beta$ is a partial ordering" and $p \in \mathbb{P}_\alpha$ if and only if $p|\beta \in \mathbb{P}_\beta$ and $\Vdash_\beta$ "$p(\beta) \in \dot{<}_\beta$"

(4) If α is limit, then for every $\beta < \alpha$, $\mathbb{P}_\beta$ is a β–stage iteration and

- (a) $\bar{1} \in \mathbb{P}_\alpha$, where $\bar{1}(\xi) = \dot{1}$ = greatest element of $\dot{\mathbb{Q}}_\xi$ for every $\xi < \alpha$.
- (b) If $\beta < \alpha$, $p \in \mathbb{P}_\alpha$, $q \in \mathbb{P}_\beta$ and $q = p|\beta$, then if r is such that $r|\beta = q$ and $r(\xi) = p(\xi)$ for every ξ in between β and α, then $r \in \mathbb{P}_\alpha$.
- (c) For all $p,q \in \mathbb{P}_\alpha$, $p \leq q$ if and only if $p|\beta \leq q|\beta$ for every $\beta < \alpha$.

By induction one trivially obtains the following facts:

If $p \in \mathbb{P}_\alpha$ then $p(0) \in \mathbb{Q}_0$ and for all $\beta < \alpha$, if $\beta \geq 1$, then $\Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$.
If p,q are in $\mathbb{P}_\alpha$ then $p \leq q$ if and only if $p(0) \leq q(0)$ and for all $\beta < \alpha$, if $\beta \geq 1$, then $p|\beta \Vdash_\beta p(\beta) \leq q(\beta)$.

It is clear that $\mathbb{P}_1$ is completely determined by $\dot{\mathbb{Q}}_0$, and that $\mathbb{P}_{\beta+1}$ is determined by $\mathbb{P}_\beta$ and $\dot{\mathbb{Q}}_\beta$. Usually, however, if β is a limit ordinal, then $\mathbb{P}_\beta$ is not completely determined by the $\mathbb{P}_\gamma$ for $\gamma < \beta$. This is simply because the conditions (a),(b),(c) of (4) in the definition of the α–stage iteration do not describe $\mathbb{P}_\beta$ completely, i.e. there may be several non–isomorphic partial orderings which are β–stage iterations of $(\mathbb{P}_\gamma : \gamma < \beta)$. The two most common are presented here and may be thought of as the extremes.

Definition A.2.20: Let α be a limit ordinal.

(1) We say that $\mathbb{P}_\alpha$ is a *direct limit* of $(\mathbb{P}_\beta : \beta < \alpha)$ if the following condition holds:
$p \in \mathbb{P}_\alpha$ if and only if $\exists \beta < \alpha\ (p|\beta \in \mathbb{P}_\beta$ and for all $\gamma, \beta \leq \gamma < \alpha$, we have $p(\gamma) = \dot{1})$

(2) We say that $\mathbb{P}_\alpha$ is an *inverse limit* of $(\mathbb{P}_\beta : \beta < \alpha)$ if the following condition holds:
$p \in \mathbb{P}_\alpha$ if and only if for all $\beta < \alpha$, $p|\beta \in \mathbb{P}_\beta$.

The direct limit of $(\mathbb{P}_\beta : \beta < \alpha)$ is essentially the smallest α–stage iteration of $(\mathbb{P}_\beta : \beta < \alpha)$ in the following sense: It is embeddable into any other α–stage iteration of $(\mathbb{P}_\beta : \beta < \alpha)$. Similarly, any α–stage iteration of $(\mathbb{P}_\beta : \beta < \alpha)$ is embeddable into the inverse limit of $(\mathbb{P}_\beta : \beta < \alpha)$, and thus the inverse limit may be thought of as essentially the greatest α–stage iteration of $(\mathbb{P}_\beta : \beta < \alpha)$.

If $p \in \mathbb{P}_\alpha$ then the *support* of p is defined by

$$\mathrm{supp}(p) = \{\beta < \alpha : p(\beta) \neq \dot{1}\}$$

If direct limits are taken at all limit stages, then supp(p) is clearly finite for every $p \in \mathbb{P}_\alpha$, so this kind of forcing is often called *finite support iteration.* The set of all finite subsets of a given set forms an *ideal* over that set, and thus one may generalize the notion of finite support iteration to arbitrary ideals.:

Definition A.2.21: Let $\mathcal{I}$ be an ideal over α. $\mathbb{P}_\alpha$ is an *α–stage iteration with supports in $\mathcal{I}$* if and only if $\mathbb{P}_\alpha$ is an α–stage iteration such that for any limit $\beta \leq \alpha$ we have:

$$p \in \mathbb{P}_\beta \longleftrightarrow \forall \xi < \beta(p|\xi \in \mathbb{P}_\xi \wedge \mathrm{supp}(p) \in \mathcal{I})$$

Thus if $\mathcal{I}$ is the ideal of countable subsets we obtain *countable support iteration,* whereas if $\mathcal{I}$ is the ideal of all subsets of α, we take inverse limits at all stages.

Next we present a sequence of lemmas which investigate the preservation of chain–, distributivity–, and closure conditions under iterations. Lemma A.2.16 shows that these conditions are preserved under two–step, and hence finite, iterations. However it is not true that they are preserved under all arbitrary iterations.

Lemma A.2.22:
Let κ be a regular cardinal and suppose that for all $\beta < \alpha$,

$$\Vdash_\beta \text{"}\dot{\mathbb{Q}}_\beta \text{ is } \kappa-\text{(directed) closed".}$$

Suppose furthermore that all limits are either inverse or direct and that if $\beta \leq \alpha$ is a limit ordinal of cofinality $< \kappa$, then $\mathbb{P}_\beta$ is the inverse limit of $(\mathbb{P}_\gamma : \gamma < \beta)$.
Then $\mathbb{P}_\alpha$ is $\kappa-$(directed) closed.

Next we examine the preservation of chain conditions:

Lemma A.2.23:
Let κ be a regular cardinal. Suppose $\mathbb{P}_\alpha$ is the direct limit of $(\mathbb{P}_\beta : \beta < \alpha)$ and suppose further that:

(a) *For every $\beta < \alpha$, $\mathbb{P}_\beta$ satisfies the κ–c.c.*
(b) *If $\text{cf}(\alpha) = \kappa$, then $\{\beta < \alpha$: $\mathbb{P}_\beta$ is the direct limit of $(\mathbb{P}_\gamma: \gamma < \beta)\}$ is stationary in α.*

Then $\mathbb{P}_\alpha$ has the κ–c.c.

As we have seen, the basic idea behind iterated forcing is that we force certain conditions one by one. Thus if $\mathbb{P}_\alpha$ is an α–stage iteration and $\beta < \alpha$, forcing with $\mathbb{P}_\alpha$ should amount to the same as first forcing with $\mathbb{P}_\beta$ and then forcing with a $(\alpha–\beta)$–stage iteration $\mathbb{P}_{\beta\alpha}$. These ideas are intuitively obvious, but some difficulty arises in making them precise because we have to skip between different forcing languages.

Let $\mathbb{P}_\alpha$ be an α–stage iteration (built up by terms $\dot{\mathbb{Q}}_\gamma$ for $\gamma < \alpha$), and let $\beta < \alpha$. If $p \in \mathbb{P}_\alpha$, define $p^\beta = p|\{\gamma\colon \beta \leq \gamma < \alpha\}$; clearly $p = p|\beta \cup p^\beta$ for any $\beta < \alpha$. Now define

$\mathbb{P}_{\beta\alpha} = \{p^\beta\colon p \in \mathbb{P}_\alpha\}$. If G_β is V–generic over $\mathbb{P}_\beta$ we can define a partial ordering on $\mathbb{P}_{\beta\alpha}$ by:

$$f \leq g \text{ iff } \exists p \in G_\beta(p \cup f \leq p \cup g \text{ in } \mathbb{P}_\alpha).$$

Since we have a canonical name for G_β, we can define $\mathbb{P}_{\beta\alpha}$ with its partial ordering in the language of $\mathbb{P}_\beta$. Let $\dot{\mathbb{P}}_{\beta\alpha}$ be a name for it in $\mathbb{P}_\beta$– language. Forcing with $\mathbb{P}_\alpha$ amounts to the same as forcing with $\mathbb{P}_\beta$ and then with $\dot{\mathbb{P}}_{\beta\alpha}$. This follows from the simple observation that the map $\varphi:\mathbb{P}_\alpha \longrightarrow \mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta\alpha}$ given by: $\varphi(p) = (p|\beta, \check{p}^\beta)$ is an order–preserving embedding whose range is dense in $\mathbb{P}_\beta * \dot{\mathbb{P}}_{\beta\alpha}$.

Lemma A.2.24:

$\Vdash_\beta$ "$\dot{\mathbb{P}}_{\beta\alpha}$ is an $(\alpha{-}\beta)$–stage iteration."

We thus see that every α–stage iteration may be regarded as a β–stage iteration followed by an $(\alpha{-}\beta)$–stage iteration, as expected. Another fairly technical lemma which we need for the discussion of reverse Easton forcing (section 3.5) is the following:

Lemma A.2.25:
Suppose that κ is a regular uncountable cardinal. Let $\beta < \alpha$ be an ordinal such that $\mathbb{P}_\beta$ has the κ–c.c. and such that if $\beta \leq \gamma < \alpha$, then $\Vdash_\gamma$"$\dot{\mathbb{Q}}_\gamma$ is κ–directed closed."
Suppose further that if $\beta < \gamma \leq \alpha$ and γ is limit, then $\mathbb{P}_\gamma$ is either the direct or inverse limit of the $\mathbb{P}_\delta$, $\delta < \gamma$, and that if cf$(\gamma) < \kappa$, *then $\mathbb{P}_\delta$ is the inverse limit of $\mathbb{P}_\delta$, $\delta < \gamma$.*

Then $\Vdash_\beta$ "$\dot{\mathbb{P}}_{\beta\alpha}$ is κ–directed closed."

This concludes our overview of the method of iterated forcing.

Appendix 3: LARGE CARDINALS

This appendix contains information on large cardinals that is required during the course of this dissertation, but is not directly relevant to the study of the power function. It is included solely in an attempt to make this dissertation self–contained, and proofs of theorems will frequently be omitted. Good sources for the work presented here are the books [Jech 1978] and [Drake 1974], and the articles [Kanamori–Magidor 1978] and [Solovay–Reinhardt–Kanamori 1978]. This appendix is particularly important for the study of Chapter 3. Section A.3.1 deals with measurable cardinals and ultrapowers, Section A.3.2 discusses iterated ultrapowers and Section A.3.3 introduces various other large cardinals, such as compact– and supercompact cardinals.

§ A.3.1 Measurable Cardinals

Recall that a proper filter $\mathcal{F}$ over a cardinal κ is:

(1) *λ–complete* (where λ is a cardinal) if whenever $X \subseteq \mathcal{F}$ with $|X| < \lambda$, then $\cap X \in \mathcal{F}$.

(2) *non–principal* if $\cap\mathcal{F} = \emptyset$.

(3) *normal* if $\mathcal{F}$ is closed under *diagonal intersections*: Whenever $\{F_\alpha : \alpha < \kappa\} \subseteq \mathcal{F}$, then $\Delta_{\alpha<\kappa} F_\alpha = \{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} F_\alpha\} \in \mathcal{F}$.

If κ is a regular cardinal, then the club–filter over κ is a normal κ–complete filter (Lemma 1.3.2). Recall that a function φ on a set of ordinals is said to be *regressive* provided that $\varphi(\alpha) < \alpha$ for all $\alpha \in \mathrm{dom}(\varphi) - \{0\}$. If $\mathcal{F}$ is a filter on a set X, we shall say that $Y \subseteq X$ is *$\mathcal{F}$–positive* provided that $\mathcal{F} \cup \{Y\}$ generates a proper filter. If $\mathcal{F}$ is the club–filter over κ, then the $\mathcal{F}$–positive sets are just the stationary subsets of κ.

An equivalent formulation of normality is given by the following: A filter $\mathcal{F}$ over a cardinal κ is normal provided that whenever F is $\mathcal{F}$–positive and $\varphi: F \longrightarrow \kappa$ is regressive, then there is $G \subseteq F$ such that G is $\mathcal{F}$–positive and $\varphi | G$ is a constant function.
If $\mathcal{F}$ is the club–filter over a regular cardinal κ, then this formulation is equivalent to Fodor's Theorem (Lemma 1.3.2).

Definition A.3.1 [Ulam 1930]: An uncountable cardinal κ is said to be *measurable* provided that there exists a *κ–complete non–principal* ultrafilter over κ. Such an ultrafilter is called a *measure* over κ.

A measurable cardinal is always strongly inaccessible (i.e. regular and strong limit.)

Our next aim is to review *ultrapowers* of models of ZFC and their relation to measurable cardinals. Let κ be a cardinal and let $\mathcal{U}$ be an ultrafilter over κ. Let M be a model of some countable language $\mathcal{L}$ ($\mathcal{L}$ will usually be the language of set theory, possibly augmented with some relation and constant symbols, and M will usually be a model of some sub– or supertheory of ZFC. M may be a proper class.) Define an equivalence relation $\sim_{\mathcal{U}}$ on ${}^{\kappa}M$ as follows:

$$f \sim_{\mathcal{U}} g \text{ if and only if } \{\alpha < \kappa\colon f(\alpha) = g(\alpha)\} \in \mathcal{U}.$$

Let $[f]_{\mathcal{U}}$ be the equivalence class of f, i.e. $[f]_{\mathcal{U}} = \{g \in {}^{\kappa}M\colon g \sim_{\mathcal{U}} f\}$.
(If M is a proper class, we employ *Scott's trick* ([Scott 1961]) to ensure that $[f]_{\mathcal{U}}$ is a set: We define $[f]_{\mathcal{U}} = \{g \in {}^{\kappa}M\colon g \sim_{\mathcal{U}} f \text{ and } g \text{ is of least rank}\}$.)
Put $M^{\kappa}/\mathcal{U} = \{[f]_{\mathcal{U}}\colon f \in {}^{\kappa}M\}$ and define a membership relation $E_{\mathcal{U}}$ on $M^{\kappa}/\mathcal{U}$ by:

$$[f]_{\mathcal{U}}\, E_{\mathcal{U}}\, [g]_{\mathcal{U}} \longleftrightarrow \{\alpha < \kappa\colon f(\alpha) \in g(\alpha)\} \in \mathcal{U}.$$

The model $(M^{\kappa}/\mathcal{U}, E_{\mathcal{U}})$ is known as the ultrapower of M modulo the ultrafilter $\mathcal{U}$, and may also be written $\mathrm{Ult}_{\mathcal{U}}(M)$.

The following theorem is a variant of the Fundamental Theorem of Ultraproducts (due to [Łos 1955]) and may be proved by induction on the complexity of $\mathcal{L}$–formulas φ using the properties of an ultrafilter. If $x \in M$, let $c_x \in {}^{\kappa}M$ be the function with constant value x.

Theorem A.3.2 [Łos 1955]:
For any formula φ of $\mathcal{L}$, $M^{\kappa}/\mathcal{U} \vdash \varphi[\vec{f}]$ if and only if

$$\{\alpha < \kappa\colon M \vdash \varphi(\vec{f}(\alpha))\} \in \mathcal{U}.$$

Thus the map $j_{\mathcal{U}}\colon M \longrightarrow M^{\kappa}/\mathcal{U}$ given by $j_{\mathcal{U}}(x) = [c_x]_{\mathcal{U}}$ is an elementary embedding.

$j_{\mathcal{U}}$ is called the canonical elementary embedding induced by $\mathcal{U}$.
Suppose now that V is the universe. If $\mathcal{U}$ is a measure over κ, then $\mathcal{U}$ is at least

ω_1–complete, and so $E_{\mathcal{U}}$ is a *wellfounded* relation over $V^\kappa/\mathcal{U}$. Thus by the Mostowski Collapsing Lemma ([Mostowski 1939]. See [Jech 1978] for a proof) there is a transitive model N and an isomorphism $\pi:(V^\kappa/\mathcal{U}, E_{\mathcal{U}}) \longrightarrow (N, \in)$. Hence there is an elementary embedding from V to N. We shall usually identify N and $V^\kappa/\mathcal{U}$ by identifying $[f]_{\mathcal{U}}$ and $\pi([f]_{\mathcal{U}})$.

Lemma A.3.3:

Let $j: V \longrightarrow N \cong V^\kappa/\mathcal{U}$ *be the canonical elementary embedding induced by a measure* $\mathcal{U}$ *over* κ, *where* N *is transitive. Then:*

(1) $j(\alpha) = \alpha$ *for all* $\alpha < \kappa$, *and* $j(\kappa) > \kappa$. *Hence* κ *is the least ordinal moved by* j.

(2) ${}^\kappa N \subseteq N$, *i.e.* N *is closed under* κ*–sequences.*

(3) $2^\kappa \leq (2^\kappa)^N < j(\kappa) < (2^\kappa)^+$

(4) $\mathcal{U} \notin N$.

If N is a transitive model of ZFC, and $j: V \longrightarrow N$ a non–trivial elementary embedding, then there exists a least ordinal κ such that $j(\kappa) > \kappa$, called the *critical point* of j. Thus if κ is measurable then there is an elementary embedding of the universe with critical point κ. Conversely, suppose that $j: V \longrightarrow N$ is an elementary embedding with critical point κ (where N is transitive). Define $X \in \mathcal{U}$ iff $X \subseteq \kappa$ and $\kappa \in j(X)$. It is not hard to show that $\mathcal{U}$ is a *normal* measure over κ and that thus κ is measurable. Hence measurable cardinals are very closely connected to elementary embeddings of the universe. the above argument also implies that if a cardinal is measurable, then there is a normal measure over κ.

Let $d: \kappa \longrightarrow \kappa$ be the identity map (also called the *diagonal map*). The following lemma is a useful characterization of normal measures.

Lemma A.3.4:

Let $\mathcal{U}$ *be a measure over a measurable cardinal* κ. *The following are equivalent:*

(1) $\mathcal{U}$ *is normal.*

(2) *In* $V^\kappa/\mathcal{U}$, *we have* $\kappa = [d]_{\mathcal{U}}$.

(3) *For all* $X \subseteq \kappa$, $X \in \mathcal{U}$ *iff* $\kappa \in j_{\mathcal{U}}(X)$

The following lemma is important for the proof of GCH in $L[\mathcal{U}]$ (Section 3.1).

Lemma A.3.5 ([Rowbottom 1971]):
Suppose κ is a measurable cardinal, $\mathcal{U}$ a normal measure over κ, and λ an infinite cardinal $< \kappa$. Suppose further that A *is a model of some language of cardinality $\leq \lambda$ such that* $A \supseteq \kappa$, $P \subseteq A$ *such that* $|P| < \kappa$ *and* $X \subseteq A$ *such that* $|X| \leq \lambda$.
Then A *has an elementary submodel* B *of cardinality κ such that* $X \subseteq B$, $B \cap \kappa \in \mathcal{U}$ *and* $|P \cap B| \leq \lambda$.

§ A.3.2 Iterated Ultrapowers.

In Section 3.1 we obtained an inner model of ZFC + GCH in which there exists a measurable cardinal, namely $L[\mathcal{U}]$. In order to further investigate the structure of $L[\mathcal{U}]$, it is necessary to develop the theory of iterated ultrapowers.
In Section A.3.1 we saw that if κ is measurable and if $\mathcal{U}$ is a measure over κ, then we may take an *ultrapower* of the universe denoted $V^\kappa/\mathcal{U}$ (identified with its transitive isomorph) associated with which is an elementary embedding $j_{\mathcal{U}} : V \longrightarrow V^\kappa/\mathcal{U}$. Then in $V^\kappa/\mathcal{U}$, $j_{\mathcal{U}}(\kappa)$ is a measurable cardinal, and $j_{\mathcal{U}}(\mathcal{U})$ is a measure over $j_{\mathcal{U}}(\kappa)$. So *inside* $V^\kappa/\mathcal{U}$ we can take an ultrapower. This is the basic idea behind iterated ultrapowers; Gaifman saw that one could iterate transfinitely by taking direct limits at limit stages ([Gaifman 1974]), and Kunen provided an internal description of the α^{th} iterated ultrapower inside the universe ([Kunen 1970]). Moreover, Kunen did not actually need the full strength of a normal measure to take ultrapowers of transitive (possibly proper class) models of set theory.
Our exposition will follow [Kunen 1970] rather closely, and the proofs omitted here may be found there. The article [Kanamori–Magidor 1977] also contains a good exposition of the material presented here.

Definition A.3.6: Suppose that M is a transitive model of ZFC, and κ is a cardinal in M. Then $\mathcal{U}$ is an M*–measure* over κ if and only if

(1) $\mathcal{U}$ is an ultrafilter on $\mathcal{P}(\kappa) \cap M$ and $\mathcal{U}$ contains no singletons.
(2) $\mathcal{U}$ is M–κ–complete, i.e. whenever $\eta < \kappa$ and $\{x_\xi : \xi < \eta\} \in \mathcal{P}(\mathcal{U}) \cap M$ then $\bigcap\{x_\xi : \xi < \eta\} \in \mathcal{U}$
(3) If $\langle x_\xi : \xi < \kappa\rangle$ is a sequence in M, then $\{\xi : x_\xi \in \mathcal{U}\} \in M$

If $\mathcal{U}$ is a V–measure over κ, where V is the universe, then $\mathcal{U}$ is a measure over κ and thus κ is a measurable cardinal. Similarly if $\mathcal{U}$ is an M–κ–complete ultrafilter over κ and $\mathcal{U} \in M$, then $M \vdash \kappa$ is measurable.

We do not need to assume that $\mathcal{U}$ is in M. However, should an M–measure $\mathcal{U}$ over κ exist, then $(\kappa \text{ is weakly compact})^M$. [Boos 1975] proves this using a ramification argument. One may also define the concept of an M–measure over some set I in M analogously.

Definition A.3.7:

(1) A function f with domain ${}^\alpha\kappa$ is said to have *finite support* y, provided y is a finite subset of α and for all s, t $\in {}^\alpha\kappa$, if $s|y = t|y$, then $f(s) = f(t)$.

(2) $X \subseteq {}^\alpha\kappa$ is said to have *finite support* $y \subseteq \alpha$ provided that y is a finite support for the characteristic function of X.

We will denote the set of finite support functions f with domain ${}^\alpha\kappa$ by $\mathcal{F}_\alpha(\kappa)$, and the set of finite support subsets $X \subseteq {}^\alpha\kappa$ by $\mathcal{P}_\alpha(\kappa)$. Note that $\mathcal{F}_\alpha(\kappa)$ and $\mathcal{P}_\alpha(\kappa)$ are defined in the *universe*, and that $\mathcal{F}_n(\kappa) = \{\text{all maps with domain } {}^n\kappa\}$ and $\mathcal{P}_n(\kappa) = \mathcal{P}({}^n\kappa)$.

Definition A.3.8: Let $j:\alpha \longrightarrow \beta$ be an injective order–preserving map. Then

(1) $j^*_\beta: {}^\beta\kappa \longrightarrow {}^\alpha\kappa$ is the map defined by: $(j^*_\beta(s))(\xi) = s(j(\xi))$.

(2) $j_{*\beta}: \mathcal{F}_\alpha(\kappa) \longrightarrow \mathcal{F}_\beta(\kappa)$ is the map defined by: $(j_{*\beta}(f))(s) = f(j^*_\beta(s))$

(3) $j_{*\beta}: \mathcal{P}_\alpha(\kappa) \longrightarrow \mathcal{P}_\beta(\kappa)$ is the map defined by: $s \in j_{*\beta}(x) \longleftrightarrow j^*_\beta(s) \in x$.

(4) For $\alpha \leq \beta$, $i_{\alpha\beta} = j_{*\beta}$ where $j:\alpha \longrightarrow \beta$ is the inclusion.

Definition A.3.9: Let M be a transitive model of ZFC.

(1) $\mathcal{F}_\alpha(M,\kappa)$ is the set of all $f \in \mathcal{F}_\alpha(\kappa)$ such that $f = j_{*\alpha}(g)$ for some order–preserving $j:n \longrightarrow \alpha$ and some $g \in \mathcal{F}_n(\kappa) \cap M$, where $n < \omega$.

(2) $\mathcal{P}_\alpha(M,\kappa)$ is the set of all $x \in \mathcal{P}_\alpha(\kappa)$ such that $x = j_{*\alpha}(y)$ for some order–preserving $j:n \longrightarrow \alpha$ and some $y \in \mathcal{P}_n(\kappa) \cap M$, where $n < \omega$.

Let $\mathcal{U}$ be an M–measure over some cardinal κ in M. We define a "quantifier" $\forall^*$ by:

$$\forall^* \alpha[\varphi(\alpha)] \longleftrightarrow \{\alpha: M \vdash \varphi(\alpha)\} \in \mathcal{U}$$

Definition A.3.10: By induction define *Rowbottom* M*–measures* $\mathcal{U}^n$ over ${}^n\kappa$ as follows:

$$\mathcal{U}^0 = \{\{0\}\}$$

$$\mathcal{U}^1 = \mathcal{U}$$

If $x \subseteq {}^{n+1}\kappa$, then $x \in \mathcal{U}^{n+1}$ iff $\forall^* \alpha(x_{(\alpha)} \in \mathcal{U}^n) \in \mathcal{U}$,

where $x_{(\alpha)} = \{(\alpha_1, ..., \alpha_n) : (\alpha, \alpha_1, ..., \alpha_n) \in x\}$

By unraveling the inductive definition of $\mathcal{U}^n$, it is not hard to see that

$$x \in \mathcal{U}^n \longleftrightarrow \forall^* \alpha_0 \forall^* \alpha_1 ... \forall^* \alpha_{n-1}[(\alpha_0, \alpha_1 ... \alpha_{n-1}) \in x]$$

Generally, if $s \in {}^\alpha\kappa$, and $x \in \mathcal{P}_{\alpha+\beta}(M,\kappa)$, we may define $x_{(s)} \in \mathcal{P}_\beta(M,\kappa)$ by

$$x_{(s)} = \{t \in {}^{m-n}\kappa \colon s\hat{\ }t \in x\}.$$

It follows then that $x \in \mathcal{U}^{m+n} \longleftrightarrow \{s \in {}^m\kappa \colon x_{(s)} \in \mathcal{U}^n\} \in \mathcal{U}^m$.

Next we define ultrafilters $\mathcal{U}_\alpha \subseteq \mathcal{P}_\alpha(M,\kappa)$:

Definition A.3.11: For all ordinals α,
If $x \in \mathcal{P}_\alpha(M,\kappa)$, then $x \in \mathcal{U}_\alpha \longleftrightarrow x = j*_\alpha(y)$ for some order–preserving $j{:}n \longrightarrow \alpha$ and some $y \in \mathcal{U}^n$.

In that case $\mathcal{U}_n$ coincides with $\mathcal{U}^n$ for $n < \omega$. The basic relation is given by the next lemma (See [Kunen 1970]):

Lemma A.3.12:

(1) *Let* $j{:}\alpha \longrightarrow \beta$ *be order–preserving, and let* $x \in \mathcal{P}_\alpha(M,\kappa)$. *Then*

$$x \in \mathcal{U}_\alpha \longleftrightarrow j*_\beta(x) \in \mathcal{U}_\beta.$$

(2) *Let* $x \in \mathcal{P}_{\alpha+\beta}(M,\kappa)$. *Then*

$$x \in \mathcal{U}_{\alpha+\beta} \longleftrightarrow \{s \in {}^\alpha\kappa \colon x_{(s)} \in \mathcal{U}_\beta\} \in \mathcal{U}_\alpha.$$

We may now define an equivalence relation $=_\alpha$ on $\mathcal{F}_\alpha(M,\kappa)$ by:

$$f =_\alpha g \text{ if and only if } \{s \in {}^\alpha\kappa : f(s) = g(s)\} \in \mathcal{U}_\alpha.$$

We define $[f]_\alpha$ to be the equivalence class of f, using Scott's trick (see Section A.3.1), i.e. $[f]_\alpha$ is the set of all $g =_\alpha f$ of minimal rank.

Let $N_\alpha = \{[f]_\alpha : f \in \mathcal{F}(\alpha,\kappa)\}$. On N_α we define a relation E_α by :

$$[f]_\alpha \, E_\alpha \, [g]_\alpha \text{ if and only if } \{s \in {}^\alpha\kappa : f(s) \in g(s)\} \in \mathcal{U}_\alpha.$$

Note that $N_0 = M$.

The α^{th} iterated ultrapower $\mathrm{Ult}_\alpha(M,\mathcal{U})$ is then *defined* to be the model (N_α, E_α), or its transitive isomorph, should that exist. Because for all $\vec{f} \in \mathcal{F}_\alpha(M,\kappa)$ and any formula φ we have $\{s \in {}^\alpha\kappa : M \models \varphi(\vec{f}(\alpha))\} \in \mathcal{P}_\alpha(M,\kappa)$, an appropriate version of Lemma A.3.2 holds.

If $j: \alpha \longrightarrow \beta$ is order–preserving, we may define an *elementary embedding*

$$j_{*\beta}: N_\alpha \longrightarrow N_\beta \text{ by } j_{*\beta}([f]_\alpha) = [j_{*\beta}(f)]_\beta.$$

If $\alpha \leq \beta$, then $i_{\alpha\beta}: N_\alpha \longrightarrow N_\beta = j_{*\beta}$, where $j:\alpha \longrightarrow \beta$ is the inclusion map.

In particular, $i_{0\alpha}: M \longrightarrow N_\alpha$ is an elementary embedding, and thus each N_α is a model of ZFC.

Note A.3.13:

(1) For any ordinal β, $i_{0\beta} : M \longrightarrow N_\beta$ is given by

$$i_{0\beta}(x) = [(x : s \in {}^\beta\kappa)].$$

(2) For ordinals $0 < \alpha \leq \beta$, $i_{\alpha\beta} : N_\alpha \longrightarrow N_\beta$ is given by

$$i_{\alpha\beta}([f]_\alpha) = [(f(s \mid \alpha) : s \in {}^\beta\kappa)].$$

Lemma A.3.14 ([Gaifman 1974]):

(1) *For all* $\alpha \leq \beta$, $i_{\alpha\beta}$ *is an elementary embedding.*

(2) *For* $0 \leq \alpha \leq \beta \leq \gamma$, $i_{\alpha\gamma} = i_{\beta\gamma} i_{\alpha\beta}$.

(3) *For limit* δ, N_δ *is isomorphic to the direct limit of the system* $(N_\alpha, i_{\alpha\beta})_{\alpha\leq\beta<\delta}$.

For (3), recall that N_δ is the direct limit of $(N_\alpha, i_{\alpha\beta})_{\alpha\leq\beta<\delta}$ provided that for all $x \in N_\delta$, there exists $\alpha < \delta$ and $y \in N_\alpha$ such that $x = i_{\alpha\gamma}(y)$.

So far we have *defined* $N_\alpha = \mathrm{Ult}_\alpha(M,\mathcal{U})$ to be the α^{th} iterated power of M modulo the M–ultrafilter $\mathcal{U}$. That its definition has anything to do with iterating the process of taking ultrapowers is not apparent. However, there is a natural way of defining an ultrafilter on $\mathcal{P}(i_{0\alpha}(\kappa)) \cap N_\alpha$:

$$\text{Put } \mathcal{U}^{(\alpha)} = \{[f]_\alpha \in \mathcal{P}^{(N_\alpha)}(i_{0\alpha}(\kappa)) : \{s \in {}^\alpha\kappa : f(s) \in \mathcal{U}\} \in \mathcal{U}_\alpha\}.$$

The next theorem connects the construction of the N_α to the original idea of iterating ultrapowers.

Theorem A.3.15 ([Gaifman 1974]):
If N_α *is well–founded, then for any* β *there is an isomorphism*

$$e_{\alpha\beta} : N_{\alpha+\beta} \longrightarrow \mathrm{Ult}_\beta(N_\alpha, \mathcal{U}^{(\alpha)})$$

such that the following diagram commutes:

$$\begin{array}{ccc} & i^{(\alpha)}_{0\beta} & \\ N_\alpha & \longrightarrow & \mathrm{Ult}_\beta(N, \mathcal{U}^{(\alpha)}) \\ i_{\alpha,\alpha+\beta} \searrow & & \uparrow \; e_\alpha \\ & & N_{\alpha+\beta} \end{array}$$

A great simplification occurs if we assume that the M–measure $\mathcal{U}$ over κ is an element of M. In that case κ is a measurable cardinal in M, and the construction of the α^{th} iterated ultrapowers can be done completely in M. In particular, $\mathcal{U}^{(\alpha)} = i_{0\alpha}(\mathcal{U})$, and thus by the Theorem A.3.15, $\mathrm{Ult}_{\alpha+\beta}(M, \mathcal{U}) \cong \mathrm{Ult}_\beta(\mathrm{Ult}_\alpha(M,\mathcal{U}), i_{0\alpha}(\mathcal{U}))$.
Henceforth we will assume that $\mathcal{U} \in M$.

Lemma A.3.16 ([Gaifman 1974]):
If in the inner model M, $\mathcal{U}$ *is a* ω_1– *complete ultrafilter over a cardinal* κ, *then every iterated ultrapower* (N_α, E_α) *is wellfounded and thus has a transitive isomorph.*

Lemma A.3.17 ([Kunen 1970]):
Let $\alpha < \beta$.

(a) $\xi < i_{0\alpha}(\kappa)$ implies $i_{\alpha\beta}(\xi) = \xi$

(b) $i_{0\beta}(\kappa) > i_{0\alpha}(\kappa)$

(c) $\mathcal{P}(i_{0\alpha}(\kappa)) \cap N_\alpha = \mathcal{P}(i_{0\alpha}(\kappa)) \cap N_\beta$

(d) $i_{0\alpha}(\mathcal{U}) \notin N_\alpha$

(e) $N_\alpha \supseteq N_\beta$, $N_\alpha \neq N_\beta$.

Lemma A.3.18 ([Kunen 1970]):
If $\delta > 0$ *is limit, then* $i_{0\delta}(\kappa) = \sup\{i_{0\alpha}(\kappa) \colon \alpha < \delta\}$

To conclude the outline of iterated ultrapowers we look at how ordinals are moved by the various elementary embeddings $i_{\alpha\beta}$:

Lemma A.3.19 ([Kunen 1970]):
Reasoning in M

(1) $i_{0\alpha}(\gamma) < (|\gamma|^{\kappa} \cdot |\alpha|)^{+}$

(2) *If* λ *is a cardinal* $> 2^{\kappa}$, *then* $i_{0\lambda}(\kappa) = \lambda$

(3) *If* δ *is limit so that* $\mathrm{cf}(\delta) > \kappa$, *then* $i_{0\gamma}(\delta) = \sup\{i_{0\gamma}(\xi) : \xi < \delta\}$

(4) *If* λ *is strong limit of cofinality* $> \kappa$, *then* $i_{0\gamma}(\lambda) = \lambda$ *for all* $\gamma < \lambda$.

§ A.3.3 Other Large Cardinals.

This section is little more than a sequence of definitions and equivalent formulations of large cardinal axioms that crop up in various places, particularly in Chapter 3. [Jech 1978] and [Kanamori–Magidor 1977] are excellent sources for the material described here.

We first introduce the notion of a *compact* cardinal (due to [Keisler–Tarski 1964]).
Recall that if S is a set and κ a cardinal, then $[S]^{<\kappa}$ is the set of all subsets of S whose cardinality is smaller than κ. If $|S| \geq \kappa$, then a *fine measure* $\mathcal{U}$ over $[S]^{<\kappa}$ via a κ–complete ultrafilter over $[S]^{<\kappa}$ with the property that for every $P \in [S]^{<\kappa}$ we have

$$\{Q \in [S]^{<\kappa} : Q \supseteq P\} \in \mathcal{U}$$

If κ and λ are cardinals, then the language $\mathcal{L}_{\kappa\lambda}$ is an infinitary language with κ–many variables, various relation, function and constant symbols, the usual logical connectives and also conjunctions and disjunctions of fewer than κ–many formulas and quantification over fewer than λ–many variables.

Definition A.3.20: A regular cardinal κ is *compact* if and only if one of the following three equivalent characterizations holds:

(1) For any set S, every κ–complete filter over S may be *extended* to a κ–complete ultrafilter over S.

(2) For any set S such that $|S| \geq \kappa$, there exists a *fine measure* over $[S]^{<\kappa}$.

(3) The infinitary language $\mathcal{L}_{\kappa\omega}$ satisfies the *strong compactness* property (i.e. whenever Σ is a set of $\mathcal{L}_{\kappa\omega}$– sentences such that every subset of Σ of cardinality $< \kappa$ has a model, then Σ has a model.)

Definition A.3.21: If $\kappa \leq \lambda$ are cardinals, then we say that κ is λ–*compact* provided any one of the following three equivalent statements holds.

(1) There is a fine measure over $[\lambda]^{<\kappa}$.

(2) There is a transitive class M and an elementary embedding j: V $\longrightarrow$ M of the universe with critical point κ such that: Whenever $X \subseteq M$ and $|X| \leq \lambda$, then there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

(3) If $\mathcal{F}$ is any κ–complete filter over a set S which is generated by $<\lambda$–many subsets of S, then $\mathcal{F}$ can be extended to a κ–complete ultrafilter over S.

Clearly κ is compact if and only if κ is λ–compact for each $\lambda \geq \kappa$. If κ is κ–compact, then κ is measurable.

Next we discuss a large cardinal notion which is stronger than of compactness, namely that of a *supercompact* cardinal. Supercompact cardinals were introduced by Solovay and Reinhardt ([Solovay–Reinhardt–Kanamori 1978]).

If S is a set, then a *normal fine measure* over $[S]^{<\kappa}$ is a fine measure over $[S]^{<\kappa}$ with an additional normality condition:

If $X \in \mathcal{U}$ and f is a choice function on X (i.e. $f(P) \in P$ for any $P \in X$)
then there is an $s \in S$ such that $\{P \in X: f(P) = s\} \in \mathcal{U}$.

Definition A.3.22: Suppose that $\kappa \leq \lambda$ are cardinals. We say that κ is λ–*supercompact* provided that any one of the following two equivalent conditions holds.

(1) There exists a transitive class M and an elementary embedding j: V $\longrightarrow$ M such that κ is the critical point of j, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.

(2) There is a normal fine measure over $[\lambda]^{<\kappa}$.

A cardinal κ is called *supercompact* if it is λ–supercompact for all $\lambda \geq \kappa$.

Clearly if $\kappa \leq \nu < \lambda$ and κ is λ–supercompact, then κ is also ν–supercompact.
κ is κ–supercompact precisely if κ is *measurable.* If κ is λ–supercompact, then κ is λ–compact. It is easily proven by an ultrapower argument that if κ is supercompact, then

there are at least κ–many measurable cardinals below it. Thus in this sense, the idea of supercompactness is really the natural successor of measurability. (Recall that it is consistent for the first measurable cardinal to be compact: Refer to [Magidor 1976] or [Apter 1991]).

Given a $j: V \longrightarrow M$ satisfying (1) of Definition A.3.22, we may define a normal fine measure $\mathcal{U}$ over $[\lambda]^{<\kappa}$ as follows:

$$X \in \mathcal{U} \longleftrightarrow X \subseteq [\lambda]^{<\kappa} \text{ and } j''\lambda \in j(X)$$

Let $\mathcal{U}$ be a normal fine measure over $[\lambda]^{<\kappa}$. We may define an ultrapower of the universe (modulo $\mathcal{U}$) in the usual way: let $M = \mathrm{Ult}(V,\mathcal{U})$ and let $j: V \longrightarrow \mathcal{U}$ be the canonical elementary embedding of the universe. Since $\mathcal{U}$ is κ–complete , we may assume that M is transitive. Note that if $d: [\lambda]^{<\kappa} \longrightarrow [\lambda]^{<\kappa}$ is the diagonal (identity) map, then

$$X \in \mathcal{U} \longleftrightarrow [d] \in j(X)$$

Moreover, $[d] = j''\lambda$, so we have: $X \in \mathcal{U} \longleftrightarrow j''\lambda \in j(X)$.

We now claim that if f is a map on $[\lambda]^{<\kappa}$, then $[f] = (jf)(j''\lambda)$: To see this, note that

$$\begin{aligned}[f] = [g] \quad &\longleftrightarrow \{p \in [\lambda]^{<\kappa}): f(p) = g(p)\} \in \mathcal{U} \\ &\longleftrightarrow j''\lambda \in j\{p \in [\lambda]^{<\kappa}: f(p) = g(p)\} \\ &\longleftrightarrow (jf)(j''\lambda) = (jg)(j''\lambda).\end{aligned}$$

Similarly, $[f] \in [g] \longleftrightarrow (jf)(j''\lambda) \in (jg)(j''\lambda)$, and so $[f] = (jf)(j''\lambda)$.

We next claim that if $\xi \leq \lambda$, then ξ is represented in M by the map $f_\xi(p) = \mathrm{otp}(p \cap \xi)$. This is because $[f_\xi] = (jf_\xi)(j''\lambda) = \mathrm{otp}(j''\lambda \cap j(\xi) = \mathrm{otp}\{j(\beta): \beta < \xi\} = \xi$.

Note also that if $p \in [\lambda]^{<\kappa}$, then $\mathrm{otp}(p \cap \lambda) < \kappa$; it follows that $\lambda < j(\kappa)$ in M.

Next suppose that $([f_\alpha]: \alpha < \lambda)$ is a sequence of M–elements. If we define $f: [\lambda]^{<\kappa} \longrightarrow V$ by $f(p) = \{f_\alpha(p): \alpha \in p\}$, then it is not hard to see that $[f] = \{[f_\alpha]: \alpha < \lambda\}$ and thus that M is closed under λ–sequences. Thus, given a normal fine measure $\mathcal{U}$ over $[\lambda]^{<\kappa}$, the elementary embedding j of the universe into the ultrapower satisfies condition (2) in Definition A.3.22. These considerations prove the equivalence of (1) and (2) of Definition A.3.22.

Next we give very brief definitions of various *large* large cardinals:

If $n < \omega$, then a cardinal κ is said to be $\mathcal{P}^n(\kappa)$–measurable, if there is an elementary embedding $j: V \longrightarrow M$ of the universe with critical point κ, satisfying the property that $\mathcal{P}^n(\kappa) \subseteq M$ (where $\mathcal{P}^n(\kappa) = \mathcal{P}(\mathcal{P}^{n-1}(\kappa))$.

A cardinal κ is said to be *huge* provided that there is a an elementary embedding j: $V \longrightarrow M$ of the universe with critical point κ, satisfying the property that ${}^{j(\kappa)}M \subseteq M$. A huge cardinal is clearly supercompact. The notion of hugeness is much stronger than the notion of supercompactness, however.

Next we introduce two *small* large cardinals. For this we need some notation (due to [Erdös–Rado 1956]):
If κ, λ are cardinals, α either an ordinal or a cardinal, and $n < \omega$, then

$$\kappa \longrightarrow (\alpha)^n_\lambda$$

abbreviates the statement: For any map $F:[\kappa]^n \longrightarrow \lambda$ there is a set $Y \subseteq \kappa$ of order type or cardinality α such that $|F''[Y]^n| = 1$. Such a set Y is said to be *homogeneous* for F.
Similarly, $\kappa \longrightarrow (\alpha)^{<\omega}_\lambda$ abbreviates the statement that for any $F:[\kappa]^{<\omega} \longrightarrow \lambda$ there is a set $Y \subseteq \kappa$ of order type or cardinality α such that for each $n < \omega$, Y is homogeneous for $F|[\kappa]^n$.

Definition A.3.23:

(1) A cardinal κ is said to be (*weakly*) *Mahlo* if κ is (weakly) inaccessible and the set of all regular cardinals below κ is stationary.

(2) An uncountable cardinal is said to be *weakly compact* If $\kappa \longrightarrow (\kappa)^2_2$.

(3) A cardinal κ is said to be *Ramsey* provided $\kappa \longrightarrow (\kappa)^{<\omega}_2$.

Any supercompact cardinal is compact is measurable is Ramsey is weakly compact is Mahlo is inaccessible.

Appendix 4: GENERIC ULTRAPOWERS

In this appendix we explore a method of generically adding an ultrafilter to the ground model V: If X is a set and $\mathcal{I}$ is an ideal over X, then $\mathcal{P}(X)/\mathcal{I}$ is the Boolean algebra obtained by identifying all those sets whose symmetric difference is in $\mathcal{I}$. Using $\mathcal{P}(X)/\mathcal{I}$ as a notion of forcing, the associated generic filter turns out to be an ultrafilter on $\mathcal{P}(X) \cap V$, and so in V[G] we may take an ultrapower of V modulo G. Hence we speak of "generic ultrapowers".

The material presented in this appendix is used mainly in Chapter 5, where bounds on 2^κ are obtained assuming the existence of ideals over κ with certain nice combinatorial properties. Good references for the results of this section are [Jech 1978] and [Jech–Prikry 1979], as well as [Foreman 1986].

Let Z be a set in the ground model V, and let $\mathcal{I}$ be an ideal over Z. Z will usually either be a regular cardinal κ or a $[\lambda]^{<\kappa}$ for some cardinal λ. An ideal over Z will always be assumed non–trivial and κ–complete. Sets in $\mathcal{I}$ will be understood to be "small" sets in some sense, analogous to the measure 0 sets in measure theory. Thus we will borrow the following terminology from measure theory.

Definition A.4.1: If $X \subseteq Z$, then:

(1) X has *$\mathcal{I}$–measure* 0 if $X \in \mathcal{I}$;

(2) X has *positive $\mathcal{I}$–measure* (or X is *$\mathcal{I}$–positive*) if $X \notin \mathcal{I}$;

(3) X has $\mathcal{I}$–measure 1 if $Z - X \in \mathcal{I}$

The sets of $\mathcal{I}$–measure 1 form a filter, called the *dual filter* of $\mathcal{I}$.

Definition A.4.2: An ideal $\mathcal{I}$ is *normal* provided that whenever $f:\mathcal{P}(Z) \longrightarrow V$ is a function such that $S = \{x: f(x) \in x\}$ is $\mathcal{I}$–positive, then there is a w and an $\mathcal{I}$–positive $T \subseteq S$ such that $T \subseteq \{x:f(x) = w\}$. If $Z = [\lambda]^{<\kappa}$, then an ideal $\mathcal{I}$ is *fine* provided that for any $\alpha < \lambda$ the set $\{x: \alpha \in x\}$ has $\mathcal{I}$–measure 1.

Definition A.4.3: Suppose that $Z = \kappa$ is a regular cardinal. An *$\mathcal{I}$–function* is a map f such that dom(f) is $\mathcal{I}$–positive. An $\mathcal{I}$–function $f:X \longrightarrow \kappa$ is said to be *unbounded* iff for all $\gamma < \kappa$, the set $\{\alpha \in X: f(\alpha) \leq \gamma\} \in \mathcal{I}$.

An unbounded $\mathcal{I}$–function f is *minimal unbounded* provided that there is no unbounded $\mathcal{I}$–function g such that $g(\alpha) < f(\alpha)$ for all α in some $\mathcal{I}$–positive set.
$\mathcal{I}$ is *weakly normal* if and only if for every $\mathcal{I}$–positive X there is a minimal unbounded $\mathcal{I}$–function f such that $\mathrm{dom}(f) \subseteq X$.

Clearly every normal ideal over κ is weakly normal, since the diagonal (identity) function d ($d(\alpha) = \alpha$) is a minimal unbounded $\mathcal{I}$–function uniformly.
The smallest ideal over κ is the set of all subsets of κ of cardinality $< \kappa$, whereas the smallest normal ideal over κ is the ideal of thin (i.e. non–stationary) sets.

Definition A.4.4: If $X \subseteq Z$ is $\mathcal{I}$–positive, then an *$\mathcal{I}$–partition* of X is a maximal family $\{X_\alpha : \alpha < \gamma\}$ of $\mathcal{I}$–positive subsets of X with the property that $X_\alpha \cap X_\beta \in \mathcal{I}$ for all $\alpha \neq \beta$.
If λ is a cardinal, then $\mathcal{I}$ is *λ–saturated* provided every $\mathcal{I}$–partition of Z has cardinality $< \lambda$. Thus $\mathcal{I}$ is λ–saturated iff the *quotient Boolean algebra* $\mathcal{P}(\kappa)/\mathcal{I}$ is λ–saturated (i.e. $\mathcal{P}(\kappa)/\mathcal{I}$ has the λ–chain condition). *Sat*($\mathcal{I}$) is the least cardinal λ such that $\mathcal{I}$ is λ–saturated. If $\mathrm{sat}(\mathcal{I})$ is infinite, it is regular and uncountable, and if $\mathcal{I}$ is a maximal ideal, then $\mathrm{sat}(\mathcal{I}) = 2$. Also, if $\mathcal{I}$ is an ideal over Z, we will always have $\mathrm{sat}(\mathcal{I}) \leq (2^{|Z|})^+$.

Next we will discuss some facts and definitions involving ultrafilters.

Definition A.4.5: Let V be a transitive model of ZFC, let $Z \in V$ and let κ be a cardinal in V. A filter $\mathcal{U}$ is said to be an *V–κ–complete* *V–ultrafilter* over Z provided that:

(1) $\mathcal{U}$ is a non–principal ultrafilter on $\mathcal{P}^V(Z)$;
(2) Whenever $\{X_\alpha : \alpha < \gamma\} \in V$ is a subset of $\mathcal{U}$ and $\gamma < \kappa$, then $\bigcap \{X_\alpha : \alpha < \gamma\} \in \mathcal{U}$.

Note that in our discussion of iterated ultrapowers in Appendix 3 we presented a different and much stronger definition of V–κ–complete V–ultrafilter due to [Kunen 1970]; Kunen's additional requirement was that for any family $\{X_\alpha : \alpha < \kappa\} \in V$, the set $\{\alpha < \kappa : X_\alpha \in \mathcal{U}\}$ is in V. Note also that here too $\mathcal{U}$ is not necessarily in V.

Definition A.4.6:

(1) A filter $\mathcal{U}$ over Z is said to be *normal* provided that whenever $f: \mathcal{P}(Z) \longrightarrow V$ is a function in V such that $\{x : f(x) \in x\} \in \mathcal{U}$, then there is w such that $\{x : f(x) = w\} \in \mathcal{U}$.

(2) If $\mathcal{U}$ is an ultrafilter over κ, then $\mathcal{U}$ is *weakly normal* provided that there is a function $f \in M$ such that for all $\gamma < \kappa$ we have $\{\alpha < \kappa: f(\alpha) \leq \gamma\} \notin \mathcal{U}$ and whenever g is such that $\{\alpha < \kappa: g(\alpha) < f(\alpha)\} \in \mathcal{U}$, then g is constant on some member of $\mathcal{U}$.

(3) If $\mathcal{U}$ is an ultrafilter over $[\lambda]^{<\kappa}$, then $\mathcal{U}$ is *fine* provided that for all $\alpha < \lambda$, the set $\{X \in [\lambda]^{<\kappa}: \alpha \in X\}$ belongs to $\mathcal{U}$.

As for ideals, the diagonal function proves that every normal V–ultrafilter is weakly normal. An ideal over $[\lambda]^{<\kappa}$ is said to be fine provided that the dual filter of $\mathcal{I}$ is fine.

Given a transitive model V of ZFC and a V–ultrafilter $\mathcal{U}$ over Z, it is possible to define an *ultrapower* of V as follows:
Let $f, g \in V$ be functions on Z. Put $f =_{\mathcal{U}} g$ iff $\{x \in Z: f(x) = g(x)\} \in \mathcal{U}$.
Let [f] denote the $=_{\mathcal{U}}$ – equivalence class of f, employing Scott's trick if necessary (see Appendix 1). Define $[f] \in_{\mathcal{U}} [g]$ iff $\{x \in Z: f(x) \in g(x)\} \in \mathcal{U}$. Let $\mathrm{Ult}(V,\mathcal{U})$ denote the class of all such equivalence classes [f] (where $f \in V$ is a function on Z), together with the $\in_{\mathcal{U}}$–relation. It is not hard to prove that Łos's Theorem holds, i.e.:

$$\mathrm{Ult}(V,\mathcal{U}) \vDash \varphi([f],[g]) \text{ iff } \{x \in Z: M \vDash \varphi(f(x),g(x))\} \in \mathcal{U}$$

for all functions $f,g \in V$ on Z and every formula φ of the language of set theory. Thus $\mathrm{Ult}(V,\mathcal{U})$ is a model of ZFC, although it is not necessarily well–founded.
There is a standard natural elementary embedding $j: V \longrightarrow \mathrm{Ult}(V,\mathcal{U})$ given by $j(x) = [c_x]$, where c_x is the function on κ with constant value x. The embedding j is such that:

(1) If $Z = \kappa$, then for all $\gamma < \kappa$, $j(\gamma) = \gamma$ and $j(\kappa) > \kappa$. If $Z = [\lambda]^{<\kappa}$, then for all $\gamma < \kappa$, $j(\gamma) = \gamma$ and $j(\kappa) > \lambda$.

(2) If $\mathcal{U}$ is a weakly normal V–ultrafilter over κ, then there is $f \in V$ such that $[f] = \kappa$ in $\mathrm{Ult}(V,\mathcal{U})$. If $\mathcal{U}$ is normal, then $[d] = \kappa$ in $\mathrm{Ult}(V,\mathcal{U})$, where d is the diagonal map.

(3) If $\mathcal{U}$ is a V–ultrafilter over κ and if ν is a limit ordinal, then
  (a) if $V \vDash \mathrm{cf}(\nu) \neq \kappa$, then $j(\nu) = \sup\{j(\xi): \xi < \nu\}$
  (b) if $V \vDash \mathrm{cf}(\nu) = \kappa$, then $j(\nu) > \sup\{j(\xi): \xi < \nu\}$

Suppose that $\mathcal{U}$ is a V–ultrafilter over κ. Given $x \in \mathrm{Ult}(V,\mathcal{U})$, let $\mathrm{ext}(x) = \{y \in \mathrm{Ult}(V,\mathcal{U}): y \in_{\mathcal{U}} x\}$ (the *extension* of x). It is easy to see that if ν is an ordinal, then $|\mathrm{ext}(j(\nu))| \leq |(\nu^{\kappa})^M|$, because every $x < j(\nu)$ in $\mathrm{Ult}(V,\mathcal{U})$ is represented by some $f{:}\kappa \longrightarrow \nu$

in V. Moreover, if the ultrapower is well–founded, we may identify x with ext(x), since in that case the ultrapower has a transitive isomorph.

We shall now show how we may add a V–ultrafilter using forcing: In a transitive model V of ZFC, let Z be a set, and let $\mathcal{I}$ be an ideal over Z.
Let $\mathbb{P} = \{X \subseteq Z : X$ has positive $\mathcal{I}$–measure$\}$, and put $X \leq Y$ iff $X \subseteq Y$. We claim that if G is V–generic over $\mathbb{P}$, then G is an V–ultrafilter. We shall call an V–generic filter over $\mathbb{P}$ simply *$\mathcal{I}$–generic.*

Lemma A.4.7:
Suppose that G *is $\mathcal{I}$–generic. Then*

(1) G *is an* V*–ultrafilter over* Z;

(2) G *is* V*–κ–complete and nonprincipal if $\mathcal{I}$ is (in* V*) κ–complete and non–trivial*;

(3) G *is normal if $\mathcal{I}$ is normal*;

(4) *If* $Z = \kappa$, *then* G *is weakly normal if $\mathcal{I}$ is weakly normal.*

(5) *If* $Z = [\lambda]^{<\kappa}$, *then* G *is fine if $\mathcal{I}$ is fine.*

Proof: (1) and (2): The set $\{P \in \mathbb{P}: P \subseteq X \ \vee \ P \subseteq Z - X\}$ is clearly dense in $\mathbb{P}$ for every $X \subseteq Z$. It follows that G is an V–ultrafilter. Also, $G \subseteq \mathbb{P}$ implies $G \cap \mathcal{I} = \emptyset$; thus if $\mathcal{I}$ is non–trivial, then every singleton subset of Z is in $\mathcal{I}$, so none are in G. This shows that G is non–principal. Suppose next that $\mathcal{I}$ is κ–complete in V, and let $\{X_\alpha: \alpha < \gamma\}$ be a partition of Z into $<\kappa$ – many blocks. The set $\{P \in \mathbb{P}: P \subseteq X_\alpha$ for some $\alpha < \gamma\}$ is then dense in $\mathbb{P}$, so there is $\alpha < \gamma$ such that $X_\alpha \in G$. This proves that G is V–κ–complete.
(3) Let $f: \mathcal{P}(Z) \longrightarrow V$ be a map in V such that $f(x) \in x$ for $x \in \mathcal{P}(Z)$. If $\mathcal{I}$ is normal, then the set

$$\{P \in \mathbb{P}: P \subseteq X \text{ and f is constant on P}\}$$

is dense below X, so f is constant on some $P \in G$. This proves that G is normal if $\mathcal{I}$ is normal.
(4) Next suppose that $\mathcal{I}$ is weakly normal over κ and let $\mathcal{F}$ be a maximal family of minimal unbounded $\mathcal{I}$–functions such that for $f \neq g$ in $\mathcal{F}$, $\mathrm{dom}(f) \cap \mathrm{dom}(g) \in \mathcal{I}$. By weak normality, the set $W_{\mathcal{F}} = \{\mathrm{dom}(f) : f \in \mathcal{F}\}$ is an $\mathcal{I}$–partition of κ, and thus there is a unique $h \in \mathcal{F}$ such that $\mathrm{dom}(h) \in G$. Because h is minimal unbounded, the set $\{\alpha \in \mathrm{dom}(h): h(\alpha) \leq \gamma\} \in \mathcal{I}$ for all $\gamma < \kappa$. Now suppose that $f \in M$ is a map with $\mathrm{dom}(f) \subseteq \mathrm{dom}(h)$, $\mathrm{dom}(f) \in G$, and such

that $f(\alpha) < h(\alpha)$ for all $\alpha \in dom(f)$. Since h is minimal unbounded, the set

$$\{P \in \mathbb{P}: P \subseteq dom(f) \text{ and } f \text{ is bounded on } P\}$$

is dense below $dom(f) \in G$, and thus there is $P \in G$ on which f is bounded. Since G is V–κ–complete, it follows that f is constant on some set in G.

(5) Suppose that $\mathcal{I}$ is fine over $[\lambda]^{<\kappa}$. Then for all $\alpha \in \lambda$, $\{X \in [\lambda]^{<\kappa}: \alpha \in X\}$ belongs to the dual filter of $\mathcal{I}$, and G contains the dual filter.

□

Thus in the generic extension V[G] of V we may take an ultrapower M = Ult(V,G) as described above. We want to develop some criteria which will ensure that this ultrapower is well–founded.

Definition A.4.8: We shall call an ideal $\mathcal{I}$ over Z *precipitous* provided that Ult(V,G) is well–founded for any $\mathcal{I}$–generic V–ultrafilter G.

The existence of a precipitous ideal has the same consistency strength as the existence of a measurable cardinal. This is maybe not unexpected, since it implies that if one wants wellfounded ultrapowers, one has to assume the existence of a measurable (or at least its consistency). More particularly, Jech et al have proved that the following are equiconsistent (see [Jech–Magidor–Mitchell–Prikry 1980]):

(1) ZFC + $\exists\kappa(\kappa$ is measurable)

(2) ZFC + $\exists$ precipitous ideal over $\aleph_1$

(3) ZFC + ideal of thin sets over $\aleph_1$ is precipitous.

Kunen has proved similar results in his paper on ultrapowers ([Kunen 1970]) assuming the existence of an $\aleph_2$–saturated ideal over $\aleph_1$. Such an ideal is necessarily precipitous, as we shall see in Lemma A.4.13.

Definition A.4.9: An ideal $\mathcal{I}$ over Z is said to satisfy the *disjointing property* provided that whenever $A \subseteq \mathcal{P}(Z)/\mathcal{I}$ is a maximal antichain, then there is a pairwise disjoint family $\{X_a: \alpha \in A\}$ of subsets of Z with the property that $[X_a]_{\mathcal{I}} = a$ for each $a \in A$.

Recall that the quotient Boolean algebra $\mathcal{P}(Z)/\mathcal{I}$ without its zero is, from a forcing point of view, equivalent to the notion of forcing $\mathbb{P}$ of all $\mathcal{I}$–positive sets.

Lemma A.4.10:

Suppose that I is an ideal over Z, *and that I has the disjointing property. Let* G *be I-generic over the ground model* V, *and let* M *be the generic ultrapower. Suppose that $\dot{\tau}$ is a* $\mathbb{P}$*-name for an element of* M. *There is a map* f: Z ⟶ V, f ∈ V *such that*

$$\Vdash_{\mathbb{P}} [f]_{\dot{G}} = \dot{\tau}.$$

Proof: By standard forcing arguments there is a maximal antichain A in $\mathbb{P}$ with the property that for each a ∈ A there is a map f_a:Z ⟶ V in V with the property that

$a \Vdash [f_a]_{\dot{G}} = \dot{\tau}$. Since I satisfies the disjointing property, we may assume that the a ∈ A are pairwise disjoint. Define f: Z ⟶ V by

$$f(x) = \begin{cases} f_a(x) \text{ where } x \in a \text{, provided such an a exists} \\ \text{arbitrary otherwise} \end{cases}$$

If G is I–generic, then there is a ∈ A ∩ G, and so $\dot{\tau}[G] = [f_a]_G = [f]_G$ (because f and f_a agree on an element of G).

□

Lemma A.4.11:

Suppose that I is an ω_1-complete ideal over Z. *If I has the disjointing property, then I is precipitous.*

Proof: Let G be I–generic, and suppose that I is not precipitous. Then the generic ultrapower M contains a descending chain of M–ordinals $(\alpha_n\colon n \in \omega)$. By Lemma A.4.10 there are maps f_n: Z ⟶ V in V such that $\Vdash [f_n]_{\dot{G}} = \alpha_n$. Since I is ω_1–complete, so is G (by Lemma A.4.7), and thus there is an X ∈ G such that for all x ∈ X we have

$$\forall n \in \omega(f_{n+1}(x) \in f_n(x))$$

Hence there is a descending ω–sequence of ordinals in V, a contradiction.

□

Lemma A.4.12:

Suppose that I is a normal fine ideal over $[\lambda]^{<\kappa}$, *where I has the disjointing property. Let* G *be I-generic, and let* M *be the associated generic ultrapower. Then* ${}^{\lambda}M \cap V[G] \subseteq M$.

Proof: Let $(\tau_\alpha: \alpha < \lambda)$ be a name for a λ-sequence in V[G] of elements of M. By Lemma A.4.10 there is in V a sequence $(f_\alpha: \alpha < \lambda)$ such that $\Vdash [f_\alpha]_{\dot{G}} = \tau_\alpha$. Define g on $[\lambda]^{<\kappa}$ by:

$$g(x) = \{f_\alpha(x): \alpha \in x\}$$

Since $\mathcal{I}$ is normal and fine, so is G (by Lemma A.4.7), and so $[g]_G = \{\tau_\alpha[G]: \alpha < \lambda\}$, proving that M is closed under λ-sequences in V[G].

□

Lemma A.4.13:

If $\mathcal{I}$ is a nontrivial κ-complete κ^+-saturated ideal over a regular cardinal κ, then $\mathcal{I}$ is precipitous.

Proof: Suppose that A is a maximal antichain in $\mathcal{P}(\kappa)/\mathcal{I}$. By κ^+-saturation, we may assume that $A = \{a_\xi: \xi < \kappa\}$. For each $a_\xi \in A$, let $X_\xi \subseteq \kappa$ be a set which represents a_ξ. We shall show that $\mathcal{I}$ has the disjointing property by exhibiting pairwise disjoint Y_ξ with the property that Y_ξ also represent a_ξ. Define :

$$Y_\xi = X_\xi - \bigcup_{\zeta<\xi} X_\zeta$$

The Y_ξ are clearly mutually disjoint. Now $X_\xi - X_\zeta \in \mathcal{I}$ for each $\zeta < \xi$, and thus since $\mathcal{I}$ is κ-complete, $X_\xi - Y_\xi \in \mathcal{I}$. It follows that Y_ξ also represents a_ξ in $\mathcal{P}(\kappa)/\mathcal{I}$, and thus that $\mathcal{I}$ has the disjointing property. Hence by Lemma A.4.11, $\mathcal{I}$ is precipitous.

□

Lemma A.4.14:

If $\mathcal{I}$ is a normal ω_1-complete fine λ^+-saturated ideal over $[\lambda]^{<\kappa}$, then $\mathcal{I}$ is precipitous.

Proof: We shall show that $\mathcal{I}$ has the disjointing property, and apply Lemma A.4.11. Let A be a maximal antichain in $\mathcal{P}([\lambda]^{<\kappa})/\mathcal{I}$. By λ-saturation and fineness it is permissible to assume that $A = \{[a_\alpha]_{\mathcal{I}}: \alpha < \lambda\}$, wheer a_α are representatives of the elements of A with the property that if $x \in a_\alpha$, then $\alpha \in x$. Let $\mathcal{F}$ be the dual ideal of $\mathcal{I}$. If $\alpha,\beta \in \lambda$, thenwe may pick $C_{\alpha\beta} \in \mathcal{F}$ such that $C_{\alpha\beta} \cap a_\alpha \cap a_\beta = \emptyset$. Since $\mathcal{F}$ is normal, it is closed under diagonal intersections and so $C = \{x: \forall\alpha,\beta \in x(x \in C_{\alpha\beta})\} \in \mathcal{F}$. We shall show that $C \cap a_\alpha \cap a_\beta = \emptyset$.

If not, pick $x \in C \cap a_\alpha \cap a_\beta$. Then $\alpha,\beta \in x$, and so $x \in C_{\alpha\beta}$, contradicting $C_{\alpha\beta} \cap a_\alpha \cap a_\beta = \emptyset$. Hence $C \in \mathcal{F}$. Now define $a'_\alpha = a_\alpha \cap C$. Then $[a'_\alpha]_{\mathcal{I}} = [a_\alpha]_{\mathcal{I}}$, and the a'_α are mutually disjoint. Hence $\mathcal{I}$ has the disjointing property.

□

Lemma A.4.15:

If $\mathcal{I}$ is a precipitous ideal over a regular cardinal κ, then $\mathcal{I}$ is weakly normal.

Proof: Suppose that there is an $\mathcal{I}$–positive set X such that no unbounded $\mathcal{I}$–function on X is minimal. We define a sequence F_n of families of $\mathcal{I}$–functions inductively as follows:

$F_0 = \{d\}$, where d is the diagonal (identity) map on X. Given F_n, pick for each $f \in F_n$ a family H_f of unbounded $\mathcal{I}$–functions which is maximal with respect to the following properties:

(1) $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ for all $g \in H_f$.

(2) $g(\alpha) < f(\alpha)$ for all $\alpha \in \mathrm{dom}(g)$

(3) $\mathrm{dom}(g_1) \cap \mathrm{dom}(g_2) \in \mathcal{I}$ for all distinct $g_1, g_2 \in H_f$.

Let $F_{n+1} = \bigcup \{H_f: f \in F_n\}$. This completes the inductive definition of the sequence F_n. The F_n have some interesting properties: Firstly if $f, g \in F_n$ and $f \neq g$, then $\mathrm{dom}(f) \cap \mathrm{dom}(g) \in \mathcal{I}$. Secondly, each $W_n = \{\mathrm{dom}(f): f \in F_n\}$ may be seen to be an $\mathcal{I}$–partition of X. Thirdly, if $f \in F_n$, $g \in F_{n+1}$ and $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$, then $g(\alpha) < f(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$. Let G be an $\mathcal{I}$–generic filter. In view of the three stated properties, we may define terms f_n of the forcing language associated with G such that for each $f \in F_n$, $\mathrm{dom}(f) \Vdash f_n = f$. It is then not hard to see that $\Vdash [f_{n+1}]_G \in [f_n]_G$, and thus that Ult(V,G) is not well–founded, contradicting the fact that $\mathcal{I}$ is precipitous.

□

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NOTATION AND ABBREVIATIONS

The symbols are shown here more or less in order of first appearance. The page number refers to the page on which the symbol is actually defined (not necessarily where it first appears).

$\wedge$	logical and 1
$\vee$	logical or 1
$\neg$	logical not 1
$\longrightarrow$	logical implication 1
$\forall$	universal quantifier 1
$\exists$	existential quantifier 1
$\in$	element relation 1
$\subseteq$	subset relation 1
$\emptyset$	empty set 1
$\aleph$	class of all cardinals 2
On	class of all ordinals 2
$\|X\|$	cardinality of X 2
$\cap$	set intersection 2
$\cup$	set union 2
$-$	set difference 2
$\times$	binary product 2
$\mathcal{P}(X)$	power set of X 2
$[X]^{\kappa}$	set of all subsets of cardinality or order type κ 2
$[X]^{<\kappa}$	set of all subsets of cardinality or order type $< \kappa$ 2
$[X]^{\leq\kappa}$	set of all subsets of cardinality or order type $\leq \kappa$ 2
otp(X)	order type of a set of ordinals 2
TC(X)	transitive closure of a set X
V	universe 2