

Hedging Interest-Rate Options Using Principal Components Analysis

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy to the University of Cape Town. It has not been submitted before for any degree or examination to any other university.

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Abstract

It is often a goal of the risk management of a portfolio of interest rate sensitive instruments to minimize the impact of movements in market rates on the value of the portfolio. This can be done by considering the sensitivity of the portfolio to each of the market rates that are used to bootstrap a yield curve. However, this is likely to lead to an excessive amount of trading due to an investment in a large number of hedging securities. As an alternative, we consider using principal components analysis (PCA) to condense most of the variability in the market rates into a much smaller number of risk factors, called the principal components. One can then construct a hedging portfolio so as to make the portfolio immune to shocks in these principal components, and hence to the most common movements in the yield curve. We compare the effectiveness of these two hedging strategies for hedging a portfolio of interest-rate options, both in the absence and presence of transaction costs. We also consider the additional feature of being able to update each hedging methodology on a daily basis and rebalance the hedge portfolios accordingly.

Key words: principal components analysis; hedging; portfolio management; European bond options; European swaptions.

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Chapter 1

Introduction

The values of multiple primary financial instruments such as deposits, forward rate agreements (FRAs), bonds and swaps are heavily linked to the level of market interest rates. The initial prices of such fixed income securities are obtained from the prevailing term structure of interest rates, after which the evolution of the interest rates from the present day through time leads to changes in the values of the securities. Fluctuations in the value of a portfolio of these securities caused by these term structure movements is known as interest-rate market risk.

This is a major risk for a trading desk with a large proportion of its portfolio invested in options written on interest rate sensitive instruments. Often the number of interest rate sensitive securities held in a portfolio is very large, with each underlying security potentially having multiple cash flows at various times in the future. This exposes the portfolio value to several market interest rates simultaneously. It is often a goal of the risk management of a portfolio of interest rate sensitive instruments to minimize the impact of movements in market rates on the value of the portfolio. This is the fundamental idea which led to the definition of the duration and convexity of a portfolio, and subsequently to the development of duration-convexity hedging (Fisher and Weil, 1971). This type of hedging involves measuring the sensitivity of the portfolio value to changes in its yield to maturity, which is a function of the prevailing term structure of interest rates at any given point in time.

However, the assumptions which duration-convexity hedging is based on are very restrictive, in that realistic shifts in the yield curve are not accounted for. The key assumption that is made in calculating the duration and convexity of a portfolio is that the yield curve shifts by a very small amount up or down in a perfectly parallel manner. Therefore, this does not allow for changes in the slope and curvature of the yield curve, which are features of term structure movements that are commonly observed in practice.

To relax these restrictive assumptions, a framework which allows for more flex-

ible shifts in the yield curve is required. To this end, it is possible to allow all the key rates along the yield curve to shift individually, and to measure the sensitivity of the portfolio value to these individual shifts in the rates. The key rates referred to here are the market observed rates that are used to bootstrap the yield or swap curve. However, this number of key rates used is likely to be very high, and this is problematic for hedging purposes because one would require as many instruments to construct the hedging portfolio as there are key rates.

Therefore, the challenge for a risk manager is to find a balance between capturing the most common movements in the yield curve while minimizing the cost of hedging by using as few hedging instruments as possible. This is where the technique of principal components analysis (PCA) can become useful. It is well-established that the risk factors that influence a term structure system, like yield and swap curves, are highly correlated ([Litterman and Scheinkman \(1991\)](#), [Falkenstein and Hanweck Jr \(1997\)](#)). By using PCA, it is possible to transform the set of highly correlated risk factors into an ordered set of orthogonal risk factors. It has often been noted in the literature, first by [Litterman and Scheinkman \(1991\)](#), that only the first three of these new risk factors are sufficient to explain as much as 97% of the variability of the term structure of interest rates. Once these factors are known, it is possible to invest in hedging instruments so as to offset the sensitivity of the portfolio value to variations in these factors. As will be seen, this will require a much smaller number of hedging instruments as compared to hedging against all the key rates along the yield curve.

A variety of approaches have been developed to use PCA in order to construct hedging portfolios. The first application of the technique on a term structure system of interest rates was by [Litterman and Scheinkman \(1991\)](#), in which the authors use PCA to obtain three orthogonal factors that can be used to explain the changes in the excess return over a risk-free rate of US Treasury bonds. The authors denote these first three principal components the level, steepness and curvature factors, respectively. [Litterman and Scheinkman \(1991\)](#) show that changes in the first principal component lead to an approximately constant shift in the yields across all maturities, which is a level shift in the yield curve. They highlight that hedging using only the first principal component is almost equivalent to duration hedging, and this connection is further explored by [Barber and Copper \(1996\)](#) and [Reitano \(1996\)](#). The second and third principal components are named the steepness and curvature factors respectively because of the effect that changes in these factors have on the shape of the yield curve. The authors use the first three principal components obtained from the historical yield curves to construct hedges for bond portfolios and compare the profit/loss (P&L) of these portfolios to those constructed using dura-

tion hedging, which naively assumes parallel shifts in the yield curve.

The drawbacks of duration hedging as compared to immunization using PCA are discussed by [Barber and Copper \(1996\)](#). In this article, the authors first discuss the methodology introduced by [Reitano \(1992\)](#), which allows immunization to be performed while allowing for non-parallel shifts in the yield curve. This methodology essentially explains the shift in a yield curve as a vector, θ , whose elements are the rate changes at different maturities. This θ -vector is the product of a scalar interest rate change, Δi , and a direction vector, η , i.e., $\theta = \Delta i \cdot \eta$. The magnitude and sign of the elements of η imply the size and direction of the shift of the yield curve at each maturity. However, the key problem with this methodology, as [Barber and Copper \(1996\)](#) point out, is that one has to choose the best single direction in which to anticipate a shift at each maturity. The authors therefore seek a more general model that allows more flexible changes in the shape of the yield curve. They suggest the use of PCA to decompose the single direction vector discussed above to a set of orthogonal vectors that represent a set of fundamental directions in which to anticipate spot rate changes. Further, they use the information provided by this set of independent direction vectors to immunize a liability stream.

[Soto \(2004\)](#) discusses a large number of strategies that portfolio managers could use to manage interest-rate risks, most of which focus on duration models. Since the first measure of duration introduced by [Macaulay \(1938\)](#), many authors have modified this measure to take into account more realistic changes in the level and shape of the yield curve, see, for example, [Bierwag \(1977\)](#), [Cox et al. \(1979\)](#) and [Chambers et al. \(1988\)](#). It is also possible to construct duration-type measures for a portfolio with respect to changes in the principal components that underly the variations in the yield curve, as is done by [Willner \(1996\)](#). Here, the author constructs principal component durations for the first three principal components by perturbing the values of the principal components by small amounts and measuring the changes in the portfolio value. This represents the sensitivity of the portfolio value to each of the three components.

In order to compare the hedging performance of each model, [Soto \(2004\)](#) compares the target return for a Spanish government bond portfolio to the effective portfolio return achieved. The author finds that there are significant differences between the models compared and that the model based on PCA with three principal components, which he denotes the common factor model, performs the best among all the strategies considered. For practical applications of PCA, the author suggests that the number of principal components used should be selected based on the criteria of meeting a minimum percentage of variability explained within the system. This threshold, in turn, would depend on the application to which the re-

sults of the PCA would be applied. Normally, for portfolio management purposes, the author finds that using the first three principal components provides the most parsimonious solution.

Clearly, there are a variety of approaches discussed in the literature involving different models and sensitivity measures which can be used in order to incorporate the method of PCA into a hedging strategy for a fixed income portfolio. However, an important question which seems to have been neglected in the literature is how to decide which instruments to use in constructing a hedge portfolio. The choice of the hedging instruments can have a significant impact in the performance of the hedging strategy. [Bagün *et al.* \(2000\)](#) perform an empirical study on using PCA to hedge a portfolio of coupon-paying bonds using zero-coupon bonds (ZCBs) and propose that the best hedging results are obtained when the instruments are well-spaced over the risk horizon under consideration. However, it could be of use to explore this further, particularly if the portfolio requiring hedging and the hedging instruments are more complex than that considered by [Bagün *et al.* \(2000\)](#).

The literature that has been reviewed here mainly focuses on constructing a hedge for a portfolio of fixed income securities, most often a portfolio of bonds. We find that most work in this area is of an empirical nature, with the mathematical development of the PCA technique as applied to hedging not well-formalized. Therefore, we begin by formalizing the mathematical foundations of constructing a hedging strategy based on PCA, after which we will consider performing an empirical study, as is mostly done in the work discussed above. However, instead of a portfolio of bonds, we consider constructing hedges for a portfolio of European options written on bonds and vanilla interest-rate swaps. We would like to make a comparison of the effectiveness of the key rate hedge and hedges based on PCA. This will be done by evaluating the hedging portfolios constructed using the two methods based on risk measures such as standard deviation, downside semi-variance (DSV) and Value at Risk (VaR). The two hedging frameworks will be compared using a buy-and-hold strategy as well as allowing for daily rebalancing of the hedge portfolio, both with and without accounting for transaction costs.

The rest of this dissertation is set out as follows. Chapter 2 develops the mathematical methodology upon which key rate hedging and hedging using PCA are based, and illustrates this methodology by constructing a hedge for a very simple portfolio of forward rate agreements. Chapters 3 and 4 extend the simple FRA example to hedge a portfolio of European bond options and European swaptions, using simulated and historical South African yield curves respectively. The effectiveness of the two hedging strategies will be compared in these two chapters. Chapter 5 concludes.

Chapter 2

Mathematical Specification of the Hedging Strategies

The aim of this chapter is to develop a mathematical procedure in order to construct a hedging portfolio for an existing portfolio of European options, written on fixed income instruments. The methods presented are, however, not particular to an option portfolio. They can be applied to hedge any portfolio of securities whose value is dependent on the term structure of interest rates. This existing portfolio of securities can simply be regarded as a sequence of cash flows at various points in time.

In order to construct a hedge for this stream of cash flows, it is important to isolate the effects of the various risk factors affecting the value of the portfolio. Separating the effects of the various risk factors allows one to identify the most important sources of risk, so as to take the necessary steps in order to mitigate the effects of these risk factors on the portfolio value. In many cases, the risk factors are highly correlated, which is particularly true for the rates along a yield curve. Matters are further complicated when one considers a large portfolio which is sensitive to many risk factors, for example, multiple interest rates along the yield curve. Due to this, it helps to transform the set of highly correlated risk factors into a set of uncorrelated risk factors. For instance, instead of using both the 3-month and the 2-year spot rates as risk factors, it is possible to use the 3-month rate and the 3-month to 2-year spread, where this spread is uncorrelated with the 3-month rate.

Constructing orthogonal risk factors manually using the spread between the original risk factors, as was done above, becomes infeasible as the number of risk factors increases. PCA provides a more efficient way to construct a set of uncorrelated risk factors. The technique also greatly reduces the dimension of the risk factor set because only a handful of the principal components are needed to explain almost all of the variability in any term structure system (Golub and Tilmán, 1997). Many risk managers often ignore the small percentage of variability that

is not captured by the first few principal components, considering the remaining variability to not have a material impact on the risk characteristics of the portfolio.

In this chapter, we will define PCA mathematically and describe how one goes about constructing a set of principal components from historical bond yield curve or swap curve data. We then discuss the measure of sensitivity that will be used to quantify how the values of the securities involved change when interest rates change. This will lead onto the mathematical definition of the key rate hedge and the PCA hedge using matrix algebra. We conclude the chapter by illustrating the described methods by constructing hedges for a simple portfolio consisting of forward rate agreements.

2.1 Mathematical Definition of PCA

Suppose we would like to construct the principal components of a system consisting of $T + 1$ observations of N different variables. For example, this could represent the observations of N market spot rates corresponding to N different maturity dates over a period of $T + 1$ days. From this data, we obtain the changes in the N spot rates, in basis points, from one day to the next. This differenced data set can be represented as a $T \times N$ matrix, $\mathbf{X} = (X_{ij})_{\{i=1,\dots,T;j=1,\dots,N\}}$, where each row represents the changes in the N rates over two consecutive days. To construct the principal components of this term structure system, it is necessary to use tools from matrix algebra.

2.1.1 Spectral Decomposition of a Matrix

The basis of the construction of the principal components of a system is the spectral decomposition of a positive, semi-definite matrix. From the data, \mathbf{X} , it is possible to obtain the $N \times N$ covariance matrix, Σ , defined as

$$\Sigma = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])^\top (\mathbf{X} - \mathbb{E}[\mathbf{X}])].$$

Here, the diagonal elements of Σ represent the variances of the daily changes in the N spot rates, whereas the off-diagonal elements are the covariances.

By definition, Σ is positive semi-definite, and so a spectral decomposition of this matrix can be performed. This involves expressing Σ as

$$\Sigma = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top,$$

where \mathbf{Q} is an $N \times N$ orthogonal matrix, i.e., $\mathbf{Q}^\top = \mathbf{Q}^{-1}$, with its columns the eigenvectors of Σ and $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_N)$ is a diagonal matrix where the entries are the

corresponding eigenvalues. We order the eigenvalues in descending order such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \dots, N - 1$. Correspondingly, the columns of \mathbf{Q} are ordered so that the position of the eigenvectors match the eigenvalues in \mathbf{D} . This ordering will be important while constructing the principal components of this system.

2.1.2 Constructing the Principal Components

The principal components of the system, \mathbf{X} , are defined to be linear combinations of the columns of \mathbf{X} , which are denoted by \mathbf{x}_i for $i = 1, \dots, N$. The weights in this linear combination are determined so as to ensure that the principal components are uncorrelated with each other, and that each subsequent component explains as much of the remaining variability in the system as possible. That is, the first principal component explains most of the variability in the term structure system, followed by the second, and then the third, and so on.

The matrix of principal components, \mathbf{P} , is a $T \times N$ matrix defined as the product of the original data matrix \mathbf{X} and the matrix of eigenvectors of Σ , that is, $\mathbf{P} = \mathbf{X}\mathbf{Q}$. It can be seen from this that the k^{th} principal component, \mathbf{p}_k , can be written as

$$\mathbf{p}_k = \sum_{j=1}^N q_{jk} \mathbf{x}_j,$$

where $[q_{1k}, \dots, q_{Nk}]^T$ is the eigenvector corresponding to the k^{th} largest eigenvalue λ_k .

We can check that the principal components are orthogonal by considering the covariance matrix of the components:

$$\Sigma_{\mathbf{P}} = \mathbb{E}[\mathbf{P}^T \mathbf{P}] = \mathbb{E}[\mathbf{Q}^T \mathbf{X}^T \mathbf{X} \mathbf{Q}] = \mathbf{Q}^T \Sigma \mathbf{Q} = \mathbf{D}.$$

Since the covariance matrix, $\Sigma_{\mathbf{P}}$, is the diagonal matrix \mathbf{D} , the principal components are uncorrelated with each other. Furthermore, the diagonal elements of \mathbf{D} represent the variances of each principal component, which are just the eigenvalues of Σ . The total variability in \mathbf{X} is given by the total variance of the principal components, $\sum_{j=1}^N \lambda_j$, and so, the proportion of the total variability explained by the k^{th} principal component is given by $\frac{\lambda_k}{\sum_{j=1}^N \lambda_j}$. Since λ_k decreases as k increases, this shows the diminishing explanatory power of each subsequent principal component, as described above.

In the above methodology, we have constructed a set of N principal components. However, it is often sufficient to construct only the first few principal components as the first six principal components explain on average 99% of the variability of a highly correlated system (Golub and Tilmann, 1997). The original data

can then be approximated using just the first l principal components

$$\mathbf{x}_j \approx \sum_{k=1}^l q_{jk} \mathbf{p}_k.$$

The above approximation demonstrates why PCA is a useful technique for reducing the dimensionality of the risk factor space. We have effectively suppressed majority of the variability into l risk factors, having originally started with N risk factors, where often $l \ll N$.

2.1.3 Performing PCA Using Correlation Matrices

The above methodology can also be applied in exactly the same way to the correlation matrix of the data set \mathbf{X} , rather than the covariance matrix. However, there is quite an important difference between the two approaches. If we perform PCA on the correlation matrix, then the principal components only depend on the correlations of the original risk factors, whereas if PCA is based on the covariance matrix, then the principal components are constructed based on both the correlations of the risk factors and the variability of each risk factor on its own. There is no general relationship between the principal components constructed using the covariance matrix and those constructed using the correlation matrix (Alexander, 2008a, Chap. 1.2.6).

For the application of PCA specifically for hedging a portfolio of interest rate sensitive instruments, it is important to consider both the correlations between the interest rates and also the volatility of each rate. Therefore, in this situation, it is preferable to use the covariance matrix to construct the principal components, rather than the correlation matrix.

2.2 Key Rate Hedging

We make the assumption that the zero-coupon yield curve is constructed from a total of N key market rates. In addition, it is assumed that there are liquid instruments that are associated with each of the tenors t_1, \dots, t_N that correspond to the input market rates. The aim of this section is to construct a portfolio consisting of these instruments in order to hedge the existing portfolio of interest rate sensitive instruments.

In order to do this, a measure of the sensitivity of the current portfolio value to changes in the market interest rates is required. One sensitivity measure which can be used is the present value of a basis point shift, which is also called PV01. This quantity is defined to be the change in the value of the portfolio if a particular rate

decreases by 1 basis point. We can therefore calculate a PV01 value for the existing portfolio with respect to each of the N rates, $PV01_j^P$, for $j = 1, \dots, N$. This vector of PV01's is denoted as $\mathbf{b} = [PV01_1^P, \dots, PV01_N^P]^\top$.

We can also construct another matrix of PV01's, where the entry in the i^{th} row and j^{th} column represents the sensitivity of the hedging instrument associated with the tenor t_j to the i^{th} market rate, denoted as $PV01_i^j$. This will be a $N \times N$ matrix,

$$\mathbf{A} = \begin{bmatrix} PV01_1^1 & \dots & PV01_1^N \\ PV01_2^1 & \dots & PV01_2^N \\ \vdots & \ddots & \vdots \\ PV01_N^1 & \dots & PV01_N^N \end{bmatrix}.$$

For example, if the hedging instrument associated with tenor t_j is a 5-year vanilla interest-rate swap, in which case $t_j = 5$, the j^{th} column of \mathbf{A} consists of the sensitivities of the value of the 5-year swap to each of the N rates under consideration.

The goal is to construct a hedging portfolio in such a way that the PV01 of the new combined portfolio is zero with respect to each of the N market rates. This involves solving for the holdings in each of the N instruments with tenors t_j , $\boldsymbol{\delta} = [\delta_1, \dots, \delta_N]^\top$, such that

$$\mathbf{A}\boldsymbol{\delta} = -\mathbf{b} \implies \boldsymbol{\delta} = -\mathbf{A}^{-1}\mathbf{b}.$$

This system amounts to solving for N unknowns, $[\delta_1, \dots, \delta_N]^\top$, using N equations. The new portfolio has been set up in such a way that it has zero sensitivity to each of the N key market rates.

2.3 Hedging Using PCA

In this section, we consider performing the same hedging exercise for a portfolio of interest rate sensitive instruments as done above, but this time using the principal components which have been constructed from the term structure system. As before, the existing portfolio of interest rate sensitive instruments is regarded simply as a stream of cash flows. It is also assumed that this stream of cash flows can be mapped to some of the tenors associated with the N key rates that have been used in order to construct the principal components.

2.3.1 Sensitivity of P&L to Principal Components

We will be considering the change in portfolio value that results from movements in the interest rates, which are also referred to as the risk factors. As discussed

before, the aim of PCA is to reduce the number of risk factors that need to be considered, so as to summarize most of the meaningful information into the first l principal components. To explain the P&L of a portfolio in terms of these principal components, a measure of the sensitivity of the P&L to changes in the values of the principal components is required.

The construction of a vector of PV01's, $\mathbf{b} = [\text{PV01}_1^P, \dots, \text{PV01}_N^P]^\top$, has already been discussed, where the j^{th} element represents the sensitivity of the portfolio value to the j^{th} market interest rate, which are the original risk factors. The change in the value of the portfolio at a particular time t , ΔP_t , can then be expressed as

$$\Delta P_t = - \sum_{j=1}^N \text{PV01}_j^P \Delta R_{j,t},$$

where $\Delta R_{j,t}$ is the change in the j^{th} rate at time t (in basis points).

On applying PCA to the covariance matrix, Σ , of the changes in the rates, $\Delta R_{j,t}$ can be written as

$$\Delta R_{j,t} \approx \sum_{k=1}^l q_{jk} p_{k,t},$$

where $p_{k,t}$ is the value of the k^{th} principal component at time t and q_{jk} is as before. We then have the following approximation for the change in the portfolio value in terms of the principal components,

$$\begin{aligned} \Delta P_t &\approx - \sum_{j=1}^N \text{PV01}_j^P \left(\sum_{k=1}^l q_{jk} p_{k,t} \right) \\ &= - \sum_{k=1}^l \left(\sum_{j=1}^N \text{PV01}_j^P q_{jk} \right) p_{k,t}. \end{aligned}$$

From the above, it can be seen that the sensitivity of the portfolio P&L to the k^{th} principal component is given by the dot-product of the PV01 vector \mathbf{b} and the k^{th} eigenvector of the covariance matrix Σ .

Having done this, we can now consider performing investments in some other securities which will add further cash flows into the portfolio such that the sensitivity of the new portfolio's value to the first l principal components is zero. If this is achieved, then the portfolio is immunized against the most common movements of the yield curve, as much of the variability in the yield curve is captured by the first l principal components that are used. It should be noted that performing the hedging in this way is not immunizing the portfolio against shifts in individual rates along a yield curve, like is done in the key rate hedge. Rather, we immunize the portfolio against shifts in the first l principal components, each of which have

an effect on all the rates along the yield curve in some way. It should also be noted that this requires an investment in l hedging instruments, instead of an investment in N instruments as before.

2.3.2 Constructing the Hedge Portfolio

The approach taken here is similar to that considered for the key rate hedging method discussed above. An important difference, however, is that now only l instruments are required in order to construct the hedge, rather than requiring the full set of N available instruments. Suppose we consider a subset of the full set of hedging instruments, choosing l of them in such a way that they are well-spaced over the risk horizon, based on the findings of [Bagün *et al.* \(2000\)](#).

We already have the PV01 vectors of these l instruments with respect to each of the N market interest rates. It is now necessary to find the sensitivities of all l instruments with respect to each of the l principal components. These sensitivities can be represented in a $l \times l$ matrix, \mathbf{S} , where the entry in the i^{th} row and the j^{th} column, s_{ij} , is the sensitivity of the j^{th} hedging instrument with respect to the i^{th} principal component. s_{ij} is given by the dot product of the PV01 vector for the j^{th} hedging instrument and the i^{th} eigenvector of Σ . We also need the sensitivity of the existing portfolio with respect to each principal component. This is calculated as the dot product of the vector of portfolio PV01's, with respect to the N rates, and the first l eigenvectors of the covariance matrix Σ . These sensitivities are stored in the column vector \mathbf{c} . \mathbf{S} and \mathbf{c} are therefore given as

$$\mathbf{S} = \begin{bmatrix} \sum_{j=1}^N \text{PV01}_j^1 q_{j1} & \cdots & \sum_{j=1}^N \text{PV01}_j^l q_{j1} \\ \sum_{j=1}^N \text{PV01}_j^1 q_{j2} & \cdots & \sum_{j=1}^N \text{PV01}_j^l q_{j2} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^N \text{PV01}_j^1 q_{jl} & \cdots & \sum_{j=1}^N \text{PV01}_j^l q_{jl} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \sum_{j=1}^N \text{PV01}_j^P q_{j1} \\ \sum_{j=1}^N \text{PV01}_j^P q_{j2} \\ \vdots \\ \sum_{j=1}^N \text{PV01}_j^P q_{jl} \end{bmatrix}.$$

Constructing the hedge portfolio then amounts to solving for l unknowns, $\delta^{\text{PC}} = [\delta_1^{\text{PC}}, \dots, \delta_l^{\text{PC}}]^{\text{T}}$, using the following system of l linear equations,

$$\mathbf{S}\delta^{\text{PC}} = -\mathbf{c} \implies \delta^{\text{PC}} = -\mathbf{S}^{-1}\mathbf{c}.$$

2.4 Adding a Self-Financing Condition

It is possible to make the key rate hedge a self-financing hedging strategy, which would make the initial value of the new combined portfolio equal to zero. This can be achieved by adding an extra instrument into the key rate hedging methodology,

the simplest of which is the overnight call account. This is exactly what is done to construct the classic, self-financing, delta-hedged position for a simple equity option. Therefore, to achieve the self-financing condition for the key rate hedge, we consider depositing or borrowing money from a bank account which earns the overnight rate of interest.

To state this more formally, we define V_0 and V_0^H to be the current values of the existing portfolio of instruments that is being hedged and the key rate hedging portfolio, respectively. If $V_0 + V_0^H > 0$, then there is a shortage of funds and so it is necessary to borrow this amount at the overnight rate. On the other hand, if $V_0 + V_0^H < 0$, then there are surplus funds of $|V_0 + V_0^H|$, which are deposited in the bank account, and subsequently accumulate interest at the overnight rate. If the portfolio is rebalanced at any point in time, it may be necessary to borrow or deposit further funds from this bank account.

It is desirable that the PCA hedges constructed are also self-financing hedging portfolios, as suggested by Alexander (2008b, Chap. 2.2.4) and Bagün *et al.* (2000). Again, this would make the initial value of the new combined portfolio equal to zero. Mathematically, this amounts to adding a further constraint into the linear system of equations $\mathbf{S}\delta^{\text{PC}} = -\mathbf{c}$, and so now $l + 1$ instruments are required to construct the hedging portfolio, instead of l . That is, δ^{PC} is now a vector of length $l + 1$.

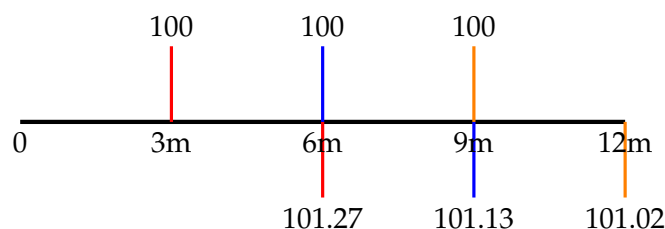
Incorporating the self-financing condition adds an extra row to \mathbf{S} and \mathbf{c} above. To ensure that \mathbf{S} remains a square matrix, it is also necessary to add an extra column to it. The first l entries of this additional column are similar to those shown in the last column in \mathbf{S} above, differing only in the use of the PV01 vector corresponding to the $(l + 1)^{\text{th}}$ hedging instrument, instead of the l^{th} . The particular form of the row added to \mathbf{S} depends on the kind of hedging instruments that are used to construct the hedging portfolio. For example, if we consider the case where ZCBs are used in order to construct the hedge, the row added to \mathbf{S} is $[P(0, T_1), \dots, P(0, T_{l+1})]$, where $P(0, T_i)$ is the time-0 price of a ZCB with unit face value and maturity date T_i . The single element added to \mathbf{c} is always equal to V_0 . This will ensure that the value of the hedging portfolio is equal to the negative of the current portfolio value, so the net value of the combined portfolio is zero. See Appendix A for more details.

2.5 Example: Hedging a Portfolio of FRAs

Having developed the general methodology to perform a key rate hedge and PCA hedge for any portfolio of interest rate sensitive instruments, we would like to now perform some implementations of the method and empirically analyze the relative

performances of the two hedging strategies. We begin with a very simple example in which we attempt to hedge a portfolio of three forward rate agreements.

Suppose, on 31 December 2007, a long position is taken in the following three GBP 100 million nominal FRAs: 3x6, 6x9 and 9x12. It is assumed that the FRAs are traded at the fair FRA rates prevailing on that date. Each FRA consists of two cash flows, a fixed payment $F(1 + K\Delta t)$ and a floating payment $F(1 + R(t, t + \Delta t)\Delta t)$. Here, F is the face value of the FRA contract, K is the fixed simple rate payable, $R(t, t + \Delta t)$ is the simple spot rate of interest which is realized on the reset date t and Δt is the tenor of the FRA. The floating payment which occurs on the maturity date of each FRA has an equivalent value equal to the face value of the FRA on the reset date. Therefore, the cash flows of this portfolio have the following fixed cash flow representation (in millions):



We consider comparing our two hedging strategies:

1. Key rate hedge – this involves hedging the above portfolio using four instruments because the cash flows of the portfolio are sensitive to a total of four spot interest rates.
2. PCA hedge – this involves hedging the portfolio using the PCA methodology developed above. We consider performing the hedging exercise using two principal components, where these have been constructed using UK spot interest rate data from January 2005 to December 2007¹. It is observed that the first two principal components explain 98.46% of the variability in the yield curve over that period.

Suppose we use ZCBs of maturities $\{3, 6, 9, 12\}$ months to construct the hedging portfolios. We also consider adding the self-financing condition to the hedging strategies, which was discussed in §2.4. By solving the relevant systems of linear equations, as described in §2.2 and §2.3.2, we obtain the amounts that need to be invested in each hedging instrument in order to construct each type of hedge. For the two component PCA hedge (without self-financing), we choose to use the 3- and 12-month ZCBs as the hedging instruments. Table 2.1 shows the positions

¹ Data can be obtained at <http://www.bankofengland.co.uk/statistics/yieldcurve/index.htm>

taken in the various hedging instruments, with all amounts being stated in millions.

Tab. 2.1: Positions taken in ZCBs to construct the hedges for a portfolio of FRAs

Hedging instrument	3m	6m	9m	12m
Key rate hedge	-100	1.27	1.13	101.02
PCA hedge (2 components)	-95.64	-	-	101.72
PCA hedge (2 components + self-financing)	-100.45	2.50	-	101.38

The key rate hedge involves making an investment in each of the four ZCBs in such a way that the net cash flows at each point in time are canceled out. This automatically makes the key rate hedge portfolio self-financing. On the other hand, the two component PCA strategy involves taking a short position of 95.64 million in the 3-month ZCB and a long position of 101.72 million in the 12-month ZCB. Adding the self-financing condition to the PCA hedge requires an investment in a further instrument, for which we choose the 6-month ZCB.

Suppose a buy-and-hold strategy is implemented, where each hedging portfolio is constructed and held for a period of time. We would like to assess the performance of each of the hedges. The key rate hedge is a perfect hedge, as the net cash flows at all points in time are zero. On the other hand, the two PCA hedges are imperfect hedges, but involve investing in two and three instruments respectively, rather than investing in all four ZCBs. To assess the performance of the PCA hedges, it is necessary to project the P&L of these hedged portfolios over a certain time horizon. Using a horizon of one day, we simulate scenarios of how the yield curve may look like in one day's time using the observed daily changes in the yield curve over the three year period 2005–2007. The P&Ls that arise from each of these scenarios are shown in Fig. 2.1 below. Table 2.2 provides some summary statistics of the observed P&L distributions (amounts in GBP, not millions).

Tab. 2.2: Summary statistics of the P&L distribution of the PCA hedge portfolios over one day for a portfolio of FRAs

	Median	Mean	Std Dev	5% VaR
PCA hedge (2 comp.)	-0.11	-0.40	8.61	11.58
PCA hedge (2 comp. + self-financing)	-0.01	-0.05	1.32	1.46

It can be seen that even though the PCA hedges are not perfect hedges, they still perform well in protecting the value of the portfolio against changes in interest rates, as the P&L that arises is of the order 10 even though the notional amounts of the FRAs being hedged is GBP 100 million. It is expected that if the cost of

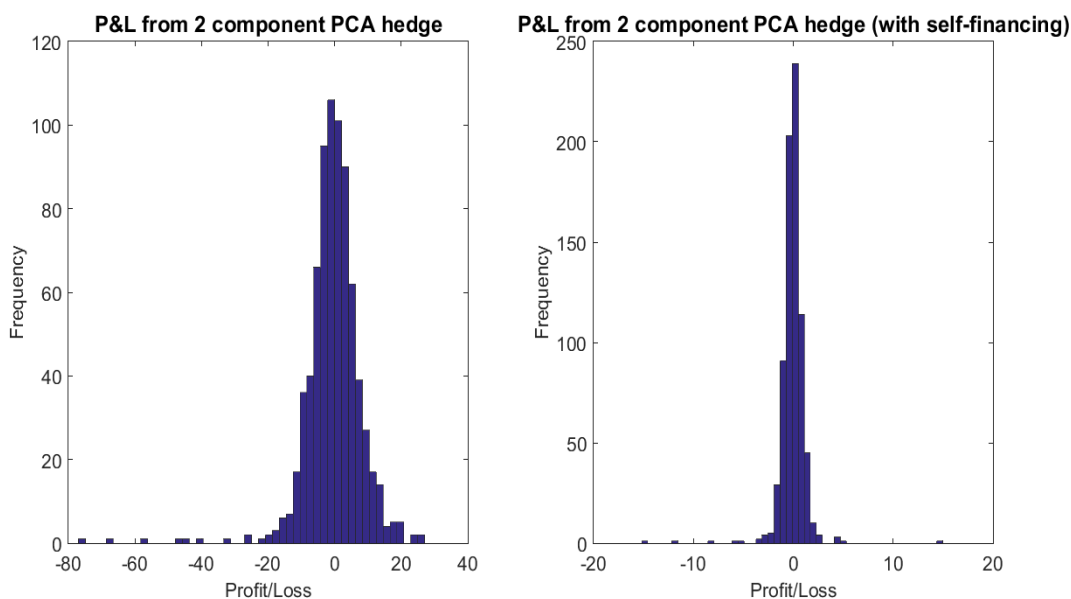


Fig. 2.1: P&L distribution of the PCA hedge portfolios over one day for a portfolio of FRAs

constructing the hedging portfolios is taken into consideration, the PCA hedges would perform better than the key rate hedge because they require an investment in fewer securities.

It should also be noted how much better the PCA hedge performs upon incorporating the self-financing condition of §2.4. This is a common feature that was identified throughout all the simulations that were carried out in this dissertation. Therefore, from here onwards, unless otherwise mentioned, we will show the results of constructing the PCA hedges along with incorporating the additional self-financing constraint. Also, to ensure a fair comparison, all key rate hedges will also be constructed in conjunction with the overnight call account, so that these hedging portfolios are self-financing as well.

Chapter 3

Implementation Using Simulated Rates

We now extend the implementation of the described methodology in two stages. The first involves performing the methods using model-generated yield curves, which we have chosen to model using simple, time-homogeneous short rate models. Of course, one can extend these models to be more complex in order to capture more realistic shapes and shifts in the yield curve. The second stage of our implementation involves performing the PCA and hedging exercise using actually observed yield curves from the South African market.

This chapter will deal with the first stage of the implementation, whereas the second stage will be covered in the subsequent chapter. In both chapters, we describe the results of performing PCA on the yield curves and present the empirical results of performing the hedging for a portfolio of interest-rate options – European zero-coupon bond options in this chapter and European swaptions in the next.

The most basic short rate model that is considered here is that of [Vasicek \(1977\)](#), in which the dynamics of the short rate $r(t)$ are given as

$$dr(t) = \alpha(\theta - r(t)) dt + \sigma dW_t,$$

where α, θ and σ are constants and W_t is a standard Brownian motion. The principle of risk-neutral valuation gives closed-form ZCB prices under this model. As before, if we define $P(t, T)$ to be the time- t price of a ZCB paying a unit amount at time T , then

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)}$$

where,

$$B(t, T) = \frac{1}{\alpha}(1 - e^{-\alpha(T-t)}),$$
$$A(t, T) = \exp \left\{ \left(\theta - \frac{\sigma^2}{2\alpha^2} \right) [B(t, T) - T + t] - \frac{\sigma^2}{4\alpha} B(t, T)^2 \right\}.$$

We simulate the short rate daily over a period of time, and use the above ZCB pricing formula to obtain a term structure of thirty-six ZCB prices over the simulated period. The ZCBs considered each have a different tenor, namely, eight short-term ZCBs of tenor increasing quarterly from 3 months out to 2 years, and twenty-eight longer term ZCBs of tenor 3 – 30 years. These tenors will be considered as the key tenors throughout this chapter and the next. The sequence of ZCB prices associated with these tenors gives a sequence of yield curves that evolve daily over the simulated period. The yield curves constructed are therefore assumed to consist of thirty-six key rates that correspond to each of the ZCBs considered.

Performing a principal components analysis on the changes in these key rates over time shows that a single principal component explains 100% of the variability of the system, as the covariance matrix of the changes in rates has a single non-zero eigenvalue. This is a feature of all one-factor short rate models. Therefore, based on the methodology presented, if one was to hedge a portfolio of interest-rate options, one would only require a single instrument to construct the PCA hedge. Of course, adding the self-financing condition would mean that two hedging instruments will be required.

To implement this in practice, however, it would be an implicit assumption that the yield curves actually observed are realizations of this single factor short rate model with suitably calibrated parameters. It is, however, well-known that a single factor model is not able to capture certain features of actually observed yield curves. It is for this reason that we extend our analysis to a two-factor short rate model, in which we combine two simpler models with Vasicek-type dynamics. This two-factor model does a slightly better job at simulating more reasonable shapes for yield curves and is the topic of discussion of the next section.

3.1 The G2++ Short Rate Model

In the [Vasicek \(1977\)](#) model, there is a single source of randomness that drives the short rate process, and this is what leads to the first principal component explaining all of the variation in the spot zero rates over time. We now consider a short rate model that has two sources of noise, which allows more flexible shifts in the yield curves while still maintaining analytical tractability.

The two-factor short rate model that is considered here is that proposed by [Brigo and Mercurio \(2001\)](#), which the authors refer to as the G2++ model. The model consists of three parts, namely, $x(t)$, $y(t)$ and $\varphi(t)$, and stipulates the following for the short rate $r(t)$:

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0,$$

where the processes $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ each satisfy Vasicek-type dynamics

$$\begin{aligned} dx(t) &= -\alpha x(t) dt + \sigma dW_1(t), & x(0) &= 0, \\ dy(t) &= -\beta y(t) dt + \nu dW_2(t), & y(0) &= 0. \end{aligned}$$

In the above, α, β, σ and ν are positive constants and $W_1(t)$ and $W_2(t)$ are correlated Brownian motions with correlation ρ , i.e., $dW_1(t) dW_2(t) = \rho dt$. [Brigo and Mercurio \(2006, Chap. 4.2.5\)](#) show that this model is equivalent to the two-factor Hull-White model ([Hull and White, 1994](#)). The above pair of stochastic differential equations can also be represented in terms of independent standard Brownian motions $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$, by using the Cholesky-decomposition of the correlation matrix of $(W_1(t), W_2(t))$,

$$\begin{aligned} dx(t) &= -\alpha x(t) dt + \sigma d\tilde{W}_1(t), & x(0) &= 0, \\ dy(t) &= -\beta y(t) dt + \nu \rho d\tilde{W}_1(t) + \nu \sqrt{1 - \rho^2} d\tilde{W}_2(t), & y(0) &= 0, \end{aligned}$$

where

$$dW_1(t) = d\tilde{W}_1(t), \quad dW_2(t) = \rho d\tilde{W}_1(t) + \sqrt{1 - \rho^2} d\tilde{W}_2(t).$$

The $\varphi(t)$ function is a deterministic-shift function that is used to calibrate the model to the term structure of rates initially observed at any point in time ([Brigo and Mercurio, 2001](#)). [Brigo and Mercurio \(2006, Chap. 4.2.2\)](#) show that to reproduce the observed yield curve, this function has to take the form

$$\varphi(T) = f^M(0, T) + \frac{\sigma^2}{2\alpha^2} (1 - e^{-\alpha T})^2 + \frac{\nu^2}{2\beta^2} (1 - e^{-\beta T})^2 + \rho \frac{\sigma\nu}{\alpha\beta} (1 - e^{-\alpha T})(1 - e^{-\beta T}),$$

where $f^M(0, T)$ is the market T -forward rate observed at the present moment in time. Similarly, we define $P^M(0, T)$ to be the current market price of a ZCB of maturity T . In order to simulate yield curves, we use the following closed-form ZCB pricing formula, which is again given by the principle of risk-neutral valuation ([Brigo and Mercurio, 2006, Chap. 4.2.2](#)),

$$P(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp(A(t, T)),$$

$$A(t, T) = \frac{1}{2} [V(t, T) - V(0, T) + V(0, t)] - \frac{1 - e^{-\alpha(T-t)}}{\alpha} x(t) - \frac{1 - e^{-\beta(T-t)}}{\beta} y(t),$$

where

$$\begin{aligned} V(t, T) &= \frac{\sigma^2}{\alpha^2} \left[T - t + \frac{2}{\alpha} e^{-\alpha(T-t)} - \frac{1}{2\alpha} e^{-2\alpha(T-t)} - \frac{3}{2\alpha} \right] + \\ &\quad \frac{\nu^2}{\beta^2} \left[T - t + \frac{2}{\beta} e^{-\beta(T-t)} - \frac{1}{2\beta} e^{-2\beta(T-t)} - \frac{3}{2\beta} \right] + \\ &\quad 2\rho \frac{\sigma\nu}{\alpha\beta} \left[T - t + \frac{e^{-\alpha(T-t)} - 1}{\alpha} + \frac{e^{-\beta(T-t)} - 1}{\beta} - \frac{e^{-(\alpha+\beta)(T-t)} - 1}{\alpha + \beta} \right]. \end{aligned}$$

We use the following parameter values to simulate the short rate over a ten year period – $r_0 = 0.07, \alpha = 0.10, \sigma = 0.01, \beta = 0.01, \nu = 0.025, \rho = 0.99$. The choice of these parameter values is motivated by the results obtained by [Heitmann and Trautmann \(1995\)](#). Furthermore, it is assumed that initially, the yield curve is flat at r_0 , after which the shape is driven by the simulated Brownian motions. It is then possible to obtain the zero rates for the thirty-six key tenors, for the full ten year period, which can be used to construct the principal components of this system.

Tab. 3.1: PCA results for yield curves simulated from the G2++ short rate model over a 10 year period

	PC1	PC2	PC3
Eigenvalue	8711	4.687	1.196e-06
% variance explained	99.95	0.05	≈ 0
Cumulative % variance explained	99.95	100.00	100.00

Performing the PCA leads to the results shown in [Table 3.1](#). It can be seen that the first two principal components explain all of the variability in the zero curves over the ten year period. In addition, the first principal component itself accounts for 99.95% of the variability in the rates. This is because of the high correlation ($\rho = 0.99$) between the two Brownian motions that drive the short rate process. The implication that this has on the hedging exercise is that theoretically we expect that a single component PCA hedge would perform almost as well as a two component PCA hedge. This will be discussed further later in this chapter.

3.2 Hedging a Portfolio of Bond Options

We now consider the problem of constructing a hedging portfolio for an existing portfolio of zero-coupon bond options. That is, there exists an investment in some call and put options written on zero-coupon bonds, and it is required to make further investments in a certain class of hedging instruments so as to protect the value of the full portfolio against movements in the bond yield curve. We investigate the performance of the key rate hedge relative to PCA hedges constructed using one and two principal components. Throughout this section, it will be assumed that the hedging portfolio will consist of investments in ZCBs of various terms, however, this can easily be extended to hedging using other, more complicated instruments.

Under the G2++ short rate model, there exist closed-form formulae for the prices of calls and puts on ZCBs. To define these, we require a function $\Sigma(t, T, S)$

(Brigo and Mercurio, 2006, Chap. 4.2.4), which is given as

$$\begin{aligned}\Sigma(t, T, S)^2 &= \frac{\sigma^2}{2\alpha^3} \left[1 - e^{-\alpha(S-T)}\right]^2 \left[1 - e^{-2\alpha(T-t)}\right] + \\ &\quad \frac{\nu^2}{2\beta^3} \left[1 - e^{-\beta(S-T)}\right]^2 \left[1 - e^{-2\beta(T-t)}\right] + \\ &\quad 2\rho \frac{\sigma\nu}{\alpha\beta(\alpha + \beta)} \left[1 - e^{-\alpha(S-T)}\right] \left[1 - e^{-\beta(S-T)}\right] \left[1 - e^{-(\alpha+\beta)(T-t)}\right].\end{aligned}$$

Proposition 3.1. Under the dynamics of the G2++ model, the time- t price of a call option C_t with strike K and expiring at T , on a ZCB of maturity S and of face value F is given as

$$C_t = FP(t, S) \Phi\left(\frac{\ln \frac{FP(t, S)}{KP(t, T)}}{\Sigma(t, T, S)} + \frac{1}{2}\Sigma(t, T, S)\right) - KP(t, T) \Phi\left(\frac{\ln \frac{FP(t, S)}{KP(t, T)}}{\Sigma(t, T, S)} - \frac{1}{2}\Sigma(t, T, S)\right),$$

where $\Phi(\cdot)$ represents the standard normal cumulative distribution function.

Proof. See Brigo and Mercurio (2006, App. C, Chap. 4.2). \square

Proposition 3.2. Under the dynamics of the G2++ model, the time- t price of a put option P_t with strike K and expiring at T , on a ZCB of maturity S and of face value F is given as

$$P_t = KP(t, T) \Phi\left(\frac{\ln \frac{KP(t, T)}{FP(t, S)}}{\Sigma(t, T, S)} + \frac{1}{2}\Sigma(t, T, S)\right) - FP(t, S) \Phi\left(\frac{\ln \frac{KP(t, T)}{FP(t, S)}}{\Sigma(t, T, S)} - \frac{1}{2}\Sigma(t, T, S)\right),$$

where $\Phi(\cdot)$ represents the standard normal cumulative distribution function.

Proof. See Brigo and Mercurio (2006, App. C, Chap. 4.2). \square

We construct a hedging portfolio for the following portfolio of three bond options, which are assumed to be held on 31 January 2018:

1. A call option maturing in 10 years written on a 20 year ZCB, with $K = 0.20$.
2. A put option maturing in 7 years written on a 15 year ZCB, with $K = 0.80$.
3. A put option maturing in 2 years written on a 8 year ZCB, with $K = 1.00$.

The face value F of all the ZCBs underlying this portfolio of options is R1 million. As mentioned before, ZCBs of various terms are used as the hedging instruments. Again, following the findings of Bagün *et al.* (2000), we choose the tenors for the ZCBs used to hedge to be as widely spaced over the risk horizon as possible. Therefore, for the single component PCA hedge, ZCBs of maturity $\{2, 20\}$ years are used and for the two component PCA hedge, we use the terms $\{2, 9, 20\}$ years. Tables 3.2 and 3.3 show the initial investments (to the nearest Rand) that have to be made in each security in order to construct the key rate hedge and the PCA hedges

respectively. It should be noted that in this case, the key rate hedge involves an investment in only six hedging instruments, out of the available thirty-six.

Tab. 3.2: Positions taken in ZCBs to construct the key rate hedge for a portfolio of bond options

Hedging instrument	2y	7y	8y	10y	15y	20y
Key rate hedge	-99,996	-79,550	99,976	3,297	95,973	-52,025

Tab. 3.3: Positions taken in ZCBs to construct PCA hedges for a portfolio of bond options

Hedging instrument	2y	9y	20y
PCA hedge (1 component)	-97,512	-	239,325
PCA hedge (2 components)	-104,069	36,142	102,946

We would now like to see how the PCA hedges perform, relative to each other and relative to the key rate hedge. This will be done by analyzing the P&L distribution that arises from these hedged portfolios over one day and over five days into the future, i.e., on 1 February 2018 and 5 February 2018 respectively. The same G2++ model has been used to simulate 10,000 scenarios of the yield curves on these future dates. The comparison of the strategies is initially done assuming there are no transaction costs, after which we relax the assumption of frictionless markets.

3.2.1 Assuming No Transaction Costs

Fig. 3.1 and Table 3.4 show the P&L profile and summary statistics of the P&L that arises from each hedged portfolio over one day. Clearly, the single component PCA hedge performs the worst, since it has the highest values of the three risk measures, namely, standard deviation, downside semivariance (DSV) and 5% Value at Risk (VaR). The two component PCA hedge and the key rate hedge are relatively similar, with the key rate hedge being only slightly better than the PCA hedge at reducing the risk of the portfolio.

Tab. 3.4: Summary statistics of the P&L distribution of the hedge portfolios over one day for a portfolio of bond options, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-0.03	0.07	6.47	20.55	10.45
PCA hedge (1 comp.)	-0.57	0.14	12.33	69.80	19.03
PCA hedge (2 comp.)	-0.05	0.08	7.19	25.30	11.59

As mentioned before, we expected the single component PCA hedge to perform

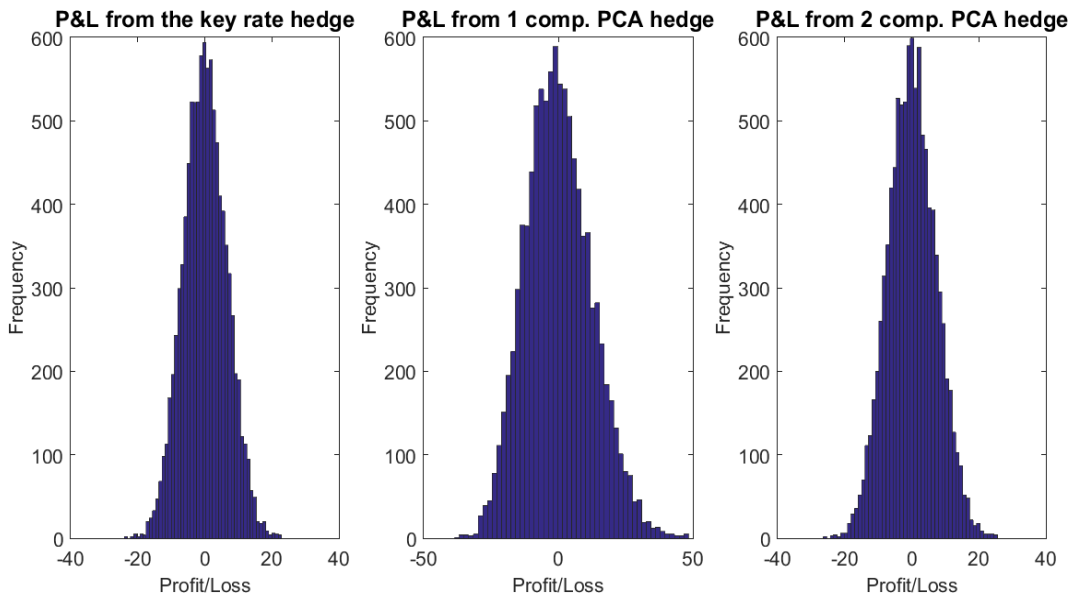


Fig. 3.1: P&L distribution of the hedge portfolios over one day for a portfolio of bond options, without transaction costs

just as well as the two component hedge, but this is not the case, in this example and in all examples that follow. This is also observed when the value of ρ is altered to 0 and -0.99 . This could be an indication that limiting the hedging strategy to consider just two simple hedging instruments restricts the ability of the strategy to provide a good hedge, even though most of the historical variability in the rates is captured by the first principal component.

We will now consider the accumulated P&L that arises from each portfolio over the course of five days. Two portfolio management techniques are considered:

- A buy-and-hold strategy, in which the amounts held in each hedging instrument are kept constant at the values shown in Tables 3.2 and 3.3 for each of the five days.
- A strategy in which the hedging portfolio is rebalanced every day, by retraining the hedging methodology using the newly observed yield curves on each day.

Fig. 3.2 and Table 3.5 show the results obtained for the buy-and-hold strategy, while Fig. 3.3 and Table 3.6 show the results obtained for the daily rebalancing strategy. The conclusions from both the buy-and-hold and daily rebalancing strategies are similar to those obtained from the results for the single day P&L results – the single component PCA hedge performs the worst whereas the key rate hedge is only marginally better than the two component PCA hedge. It should also be

noted how much more effective the daily rebalancing technique is at reducing risk as compared to the buy-and-hold portfolio management technique. The values of the risk measures over the five day period while allowing for daily rebalancing are comparable to the corresponding values in the one-day results, whereas those for the buy-and-hold strategy are much higher. The same conclusions are valid when considering the hedge performance over a thirty day period, rather than five days as considered above. The results for these simulations can be found in Appendix B.

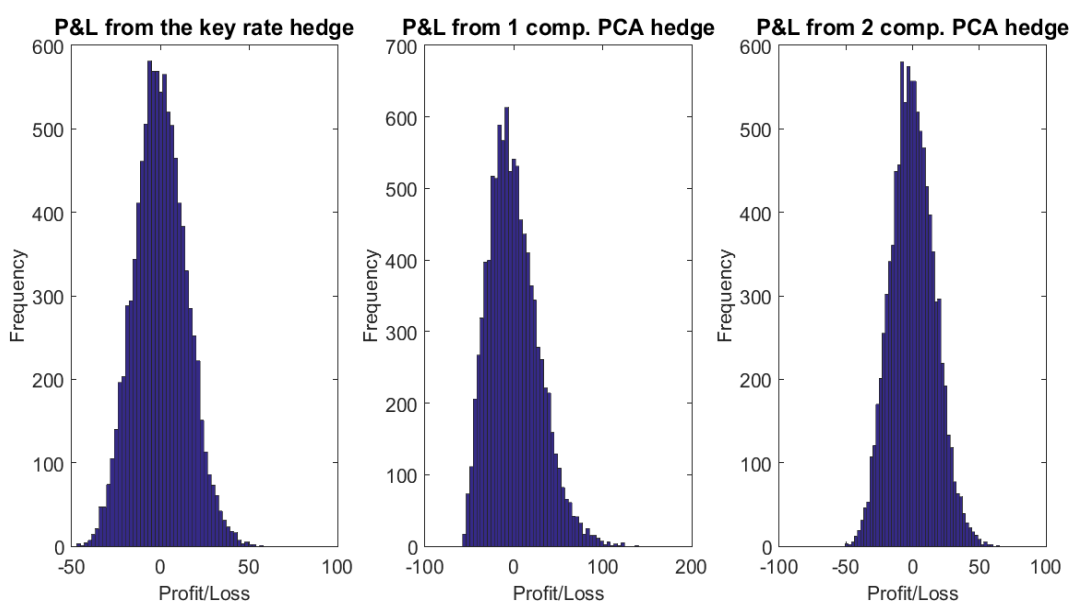


Fig. 3.2: Five day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, without transaction costs

Tab. 3.5: Summary statistics of the five day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-0.33	0.07	14.58	102.48	23.49
PCA hedge (1 comp.)	-3.14	0.17	28.14	326.79	40.38
PCA hedge (2 comp.)	-0.44	0.08	16.22	125.84	25.99

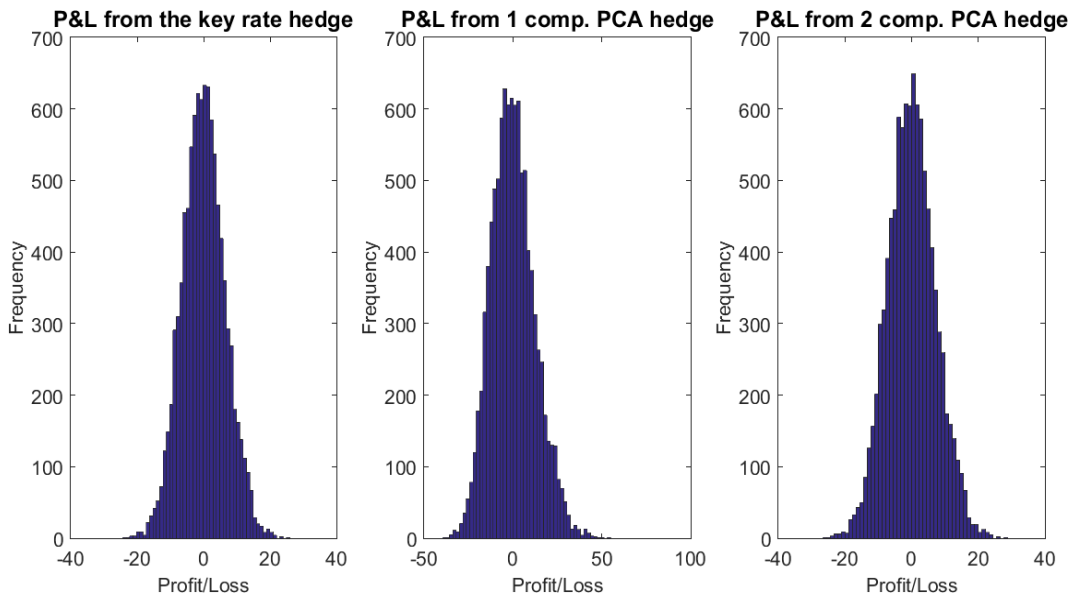


Fig. 3.3: Five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, without transaction costs

Tab. 3.6: Summary statistics of the five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	0.18	0.06	6.70	22.83	11.53
PCA hedge (1 comp.)	-0.22	0.13	12.75	77.34	20.61
PCA hedge (2 comp.)	0.19	0.06	7.45	28.09	12.82

3.2.2 Accounting for Transaction Costs

We now relax the assumption of frictionless markets, and consider a simplified regime of transaction costs. It is assumed that there is a transaction cost of 2 percent of the value of every trade that is made. The magnitude of this proportional transaction fee was decided upon based on the general consensus of a reasonable transaction cost that is incurred while hedging options (Leland (1985) and Clewlow and Hodges (1997)).

Fig. 3.4 and Table 3.7 show the P&L profile and summary statistics of the P&L that arises from each hedged portfolio over one day. The relative magnitudes of the mean, median and 5% VaR for the key rate hedge portfolio versus the PCA hedge portfolios show that the key rate hedge portfolio costs significantly more to set up. This is because the key rate hedge portfolio involves investing in a larger number

of hedging instruments, as compared to the PCA hedges. This leads to the key rate hedge portfolio performing the worst among the three, in terms of the sizes of the losses incurred.

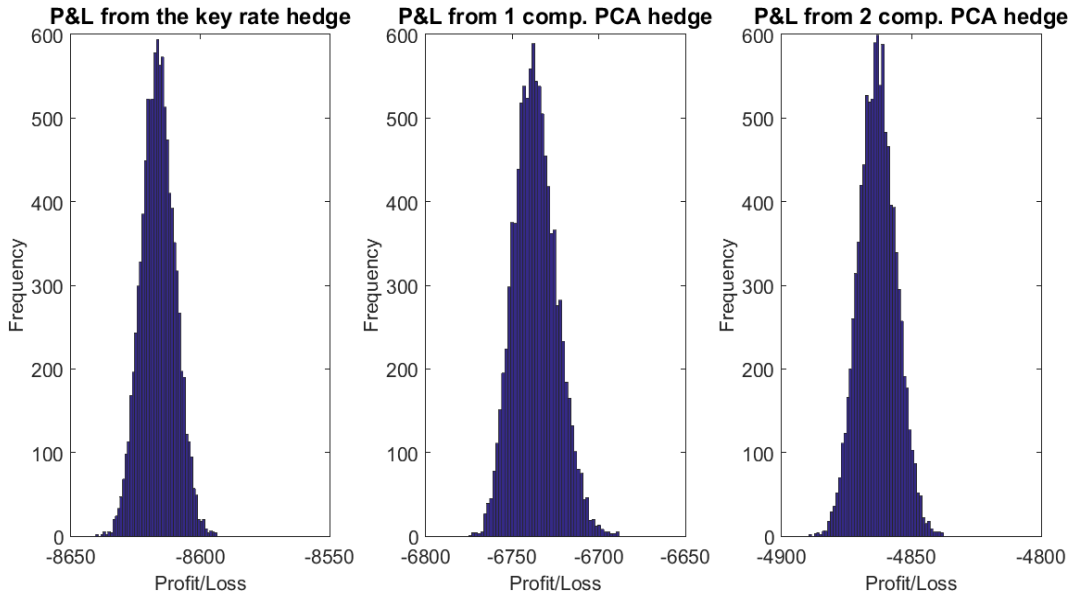


Fig. 3.4: P&L distribution of the hedge portfolios over one day for a portfolio of bond options, with transaction costs

Tab. 3.7: Summary statistics of the P&L distribution of the hedge portfolios over one day for a portfolio of bond options, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-8,616.40	-8,616.29	6.47	20.55	8,626.82
PCA hedge (1 comp.)	-6,737.32	-6,736.61	12.33	69.80	6,755.78
PCA hedge (2 comp.)	-4,863.21	-4,863.07	7.19	25.30	4,874.75

In addition, the two component PCA hedge performs the best; it even outperforms the single component PCA hedge, even though it trades one more instrument than the single component hedge. Similar conclusions are drawn from the accumulated P&L results over a five day period using a buy-and-hold strategy, which are shown in Fig. 3.5 and Table 3.8. This occurs for two reasons. Firstly, as in the case where we ignored transaction costs, we saw that the two component PCA hedge outperformed the single component hedge. Secondly, the sizes of the investments required to set up the single component PCA hedge are much bigger than those in the two component hedge, which leads to a larger amount of market friction costs being incurred when the single component PCA hedge is initially set up.

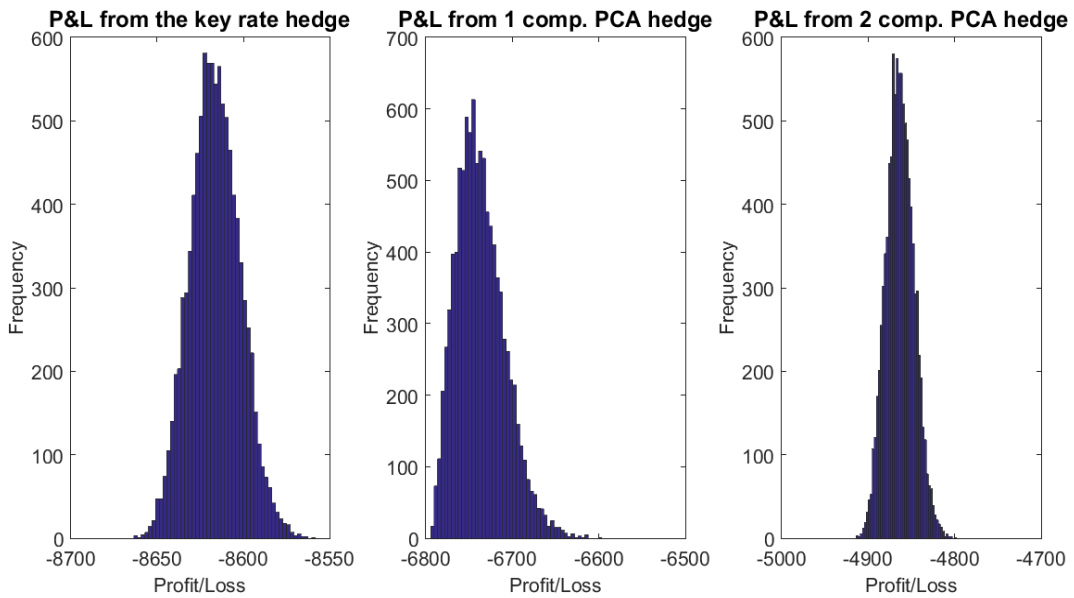


Fig. 3.5: Five day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, with transaction costs

Tab. 3.8: Summary statistics of the five day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-8,616.69	-8,616.29	14.58	102.48	8,639.85
PCA hedge (1 comp.)	-6,739.89	-6,736.58	28.14	326.79	6,777.13
PCA hedge (2 comp.)	-4,863.59	-4,863.07	16.22	125.84	4,889.14

Lastly, the accumulated P&L results over a five day period, allowing for daily rebalancing of the hedging portfolios, are shown in Fig. 3.6 and Table 3.9.

Tab. 3.9: Summary statistics of the five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-8,681.71	-8,683.87	23.61	317.64	8,726.36
PCA hedge (1 comp.)	-6,954.63	-6,960.70	76.29	3,281.26	7,096.92
PCA hedge (2 comp.)	-5,050.90	-5,056.35	65.74	2,433.41	5,172.94

In all scenarios we have considered in this subsection, and in particular that with the daily rebalanced hedging portfolio over a five day period, the two com-

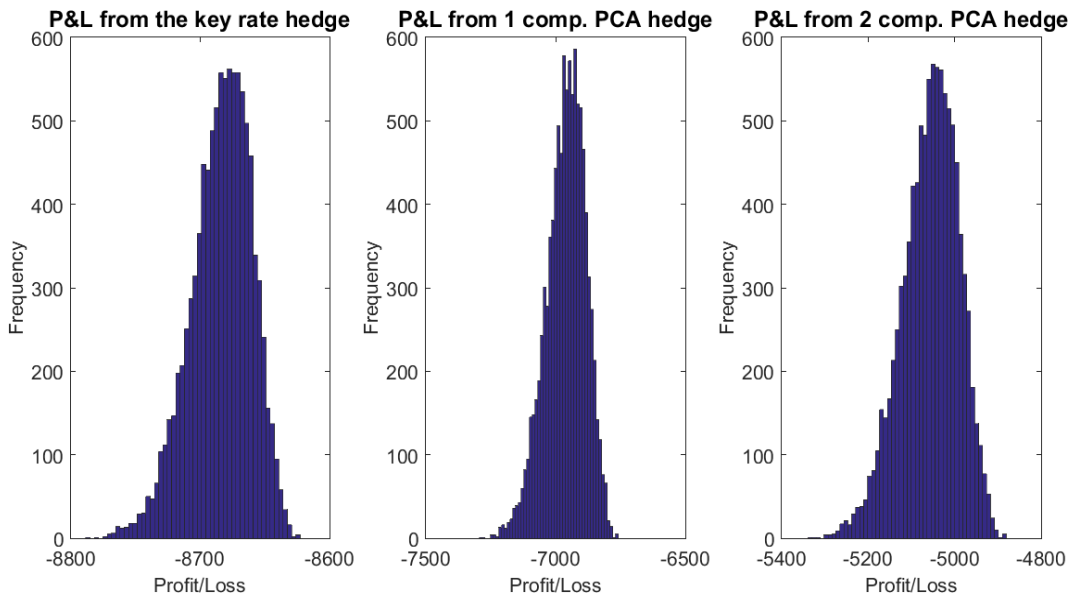


Fig. 3.6: Five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, with transaction costs

ponent PCA hedge outperforms the key rate hedge. This occurs because the two component PCA hedge involves significantly less trading and investing in fewer hedging instruments, which saves on trading costs. It is also interesting to note that when transaction costs are introduced, the buy-and-hold technique dominates the daily rebalancing technique, for all three hedging strategies. This is also true if one considers the P&L that arises over a thirty day horizon, rather than a five day horizon, as shown in Tables 3.10 and 3.11. It should be noted that the results that have been obtained are influenced by the type and size of transaction costs that are taken into consideration, which here have simply been taken to be a flat fee of 2 percent of the value of every trade. The conclusions that can be drawn may very well change if a different, more realistic regime of market friction costs are introduced.

Tab. 3.10: Summary statistics of the thirty day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-8,618.55	-8,616.16	35.78	577.59	8,671.23
PCA hedge (1 comp.)	-6,756.01	-6,736.65	74.33	1,786.74	6,816.79
PCA hedge (2 comp.)	-4,866.03	-4,862.95	40.01	712.59	4,923.89

Tab. 3.11: Summary statistics of the thirty day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-9,019.73	-9,021.84	57.40	1,737.71	9,119.30
PCA hedge (1 comp.)	-8,080.13	-8,084.97	187.86	18,361.80	8,398.03
PCA hedge (2 comp.)	-6,023.44	-6,028.57	163.70	13,988.63	6,301.71

We would now like to investigate whether all the above conclusions are still valid if the same hedging strategies and portfolio management techniques are compared using real world yield and swap curves.

Chapter 4

Implementation Using Market Rates

Having implemented the methods presented using model-generated yield curves, we now move on to the second stage of our analysis. This involves performing the principal components analysis and hedging exercise using historically observed interest rates from the South African (ZAR) market. In this chapter, hedging portfolios are constructed for an existing portfolio of European swaptions written on vanilla interest-rate swaps with quarterly payments.

4.1 Performing PCA on Market Swap Curves

The portfolio that requires hedging consists of swaptions, therefore it is necessary to consider the rates obtained from historical ZAR swap curves. The data that is used here is from the 10-year period 1 January 2005 to 31 December 2014. The results of performing a principal components analysis on the basis point shifts in the rates associated with the thirty-six key tenors are shown in Table 4.1.

Tab. 4.1: PCA results for ZAR swap curves observed from 1 January 2005 to 31 December 2014

	PC1	PC2	PC3	PC4	PC5	PC6
Eigenvalue	1601	110.2	32.69	17.89	10.80	6.661
% variance explained	89.30	6.15	1.82	1.00	0.60	0.37
Cumulative % variance explained	89.30	95.45	97.27	98.27	98.87	99.25

It can be seen that the first three principal components capture 97.27% of the historical variability in the rates, whereas six principal components capture over 99% of the variability.

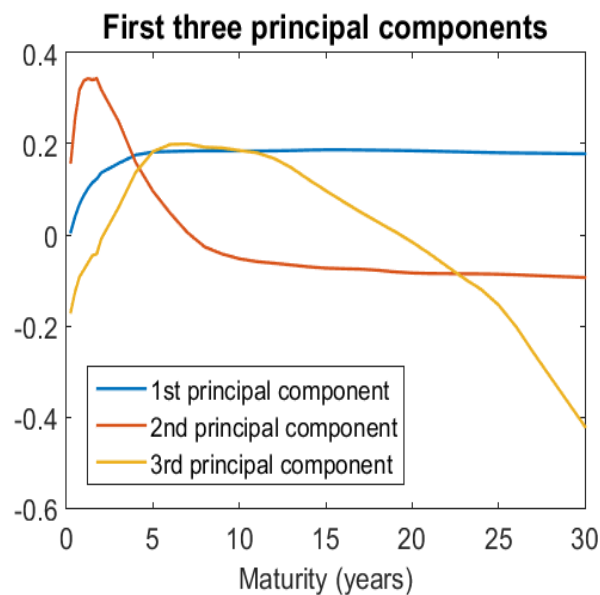


Fig. 4.1: First three principal components constructed from ZAR swap curve data

The shapes of the curves observed in Fig. 4.1 are typical of the first three principal components obtained when PCA is performed on a term structure of interest rates. The figure shows the reasoning behind [Litterman and Scheinkman \(1991\)](#) naming the first three components as the level, slope and curvature factors respectively. The flattening of the first principal component from a maturity of 5 years onwards means that changes in the first component leads to approximately parallel movements in the spot curve from 5 years to 30 years. Also, an increase in the second component has the effect of increasing rates at the short end of the curve while decreasing rates at the long end; therefore, a change in this component affects the overall slope of the yield curve. Finally, the humped shape of the third principal component causes rates in the very short end and the rates for maturities of 20 years and larger to move in the same direction, whereas the rates in between move in the opposite direction. This has the effect of changing the convexity or concavity of the yield curve.

Having constructed the principal components from this historical 10 year period, we now consider using these to construct a hedging portfolio for a portfolio of three European swaptions.

4.2 Hedging a Portfolio of Swaptions

The portfolio for which a hedge will be constructed consists of the following European swaptions, which are assumed to be held on 31 December 2014:

1. A payer swaption maturing in 1 year, written on a 10 year swap and struck at a fixed rate of 10%.
2. A payer swaption maturing in 3 years, written on a 15 year swap and struck at a fixed rate of 8%.
3. A receiver swaption maturing in 5 years, written on a 25 year swap and struck at a fixed rate of 20% .

The swaps that underly each of the swaptions have quarterly future payments and have a face value, F , of R1 million. As before, ZCBs of various terms will be used as the hedging instruments.

In order to carry out the methodology as described in Chapter 2, one needs to be able to price these swaptions given a term structure of interest rates. For this, a pricing model is required. We choose to use the G2++ model as the pricing model, as done in the previous chapter, as we are able to calibrate this model to the current yield curve and the current implied volatility surface prevailing in the market. Also, importantly, this model provides a semi-analytical solution for European swaption prices.

The price of a European swaption, written on a vanilla interest-rate swap with payment dates $\{t_1, \dots, t_n\}$, with face value F , strike rate X and maturing at time T , is given by the integral expression

$$F\omega P(0, T) \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{2\pi}} \left[\Phi(-\omega h_1(x)) - \sum_{i=1}^n \lambda_i(x) e^{\kappa_i(x)} \Phi(-\omega h_2(x)) \right] dx,$$

with $\omega = 1$ for a payer swaption, and $\omega = -1$ for a receiver swaption. See Appendix C for a more complete description of the pricing formula and the definitions of the various functions and variables involved in the above integral expression. This one-dimensional integral can be solved by using simple quadrature techniques.

In order to price the swaptions correctly, the G2++ model needs to be calibrated to the market by using the volatility surface implied by the prevailing market prices for caps of various tenors and terms. The data from 31 December 2014 that was used to perform the calibration, along with more details about the calibration process, can be found in Appendix D. The calibrated parameters of the G2++ model are the following:

$$\alpha = 0.305, \beta = 0.944, \sigma = 0.00958, \nu = 0.00897, \rho = -0.721.$$

We now proceed with the implementation of the hedging strategies. Again, three hedging strategies are compared. However, this time, the comparison is between the key rate hedge and PCA hedges constructed using two and three principal components respectively, as Table 4.1 shows that 97% of the variability in the

swap curve is captured by the first three components. ZCBs of maturity {1y, 15y, 30y} are used as the hedging instruments for the two component PCA hedge while the three component hedge is constructed using ZCBs of maturity {1y, 10y, 20y, 30y}. The initial investments in these (to the nearest Rand) are shown in Table 4.2.

Tab. 4.2: Positions taken in ZCBs to hedge a portfolio of swaptions

Hedging instrument	1y	10y	15y	20y	30y
PCA hedge(2 comp.)	-1,333,855	-	3,136,275	-	-4,988,870
PCA hedge(3 comp.)	-1,811,532	4,105,616	-	-6,341,041	1,619,088

Again, we will first consider the hedging performance of the various combinations of strategies and portfolio management techniques without transaction costs, after which these will be introduced. One-day ahead and five-day ahead scenarios of the swap curve are generated using the historical data and these are used to analyze the accumulated P&L that arises from each of the hedged portfolios over these time frames. The scenarios of the swap curve are generated by fitting a multivariate normal distribution to the historical ratios of the thirty-six key rates observed from one day to the next, and then simulating a random sample of size 1,000 from this distribution to propagate the current rates over each day. We restrict the number of swap curve scenarios generated to 1,000 because pricing the swaptions by using simple quadrature to numerically solve the integral above is computationally expensive.

4.2.1 Assuming No Transaction Costs

Fig. 4.2 and Table 4.3 show the P&L profile and summary statistics of the P&L that arises from each hedged portfolio over one day.

Tab. 4.3: Summary statistics of the P&L distribution of the hedge portfolios over one day for a portfolio of swaptions, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	98.81	76.99	8,009.93	3.27e+07	13,365.20
PCA hedge (2 comp.)	345.39	44.49	9,241.37	4.39e+07	15,485.77
PCA hedge (3 comp.)	-38.96	-9.59	8,224.39	3.33e+07	13,441.53

It turns out that the PCA hedge that uses two components actually has the highest median P&L, but the other realizations of the P&L from the strategy are widely dispersed around this value, so this strategy is the worst at reducing risk. On the other hand, the effectiveness of the key rate hedge and the three component PCA

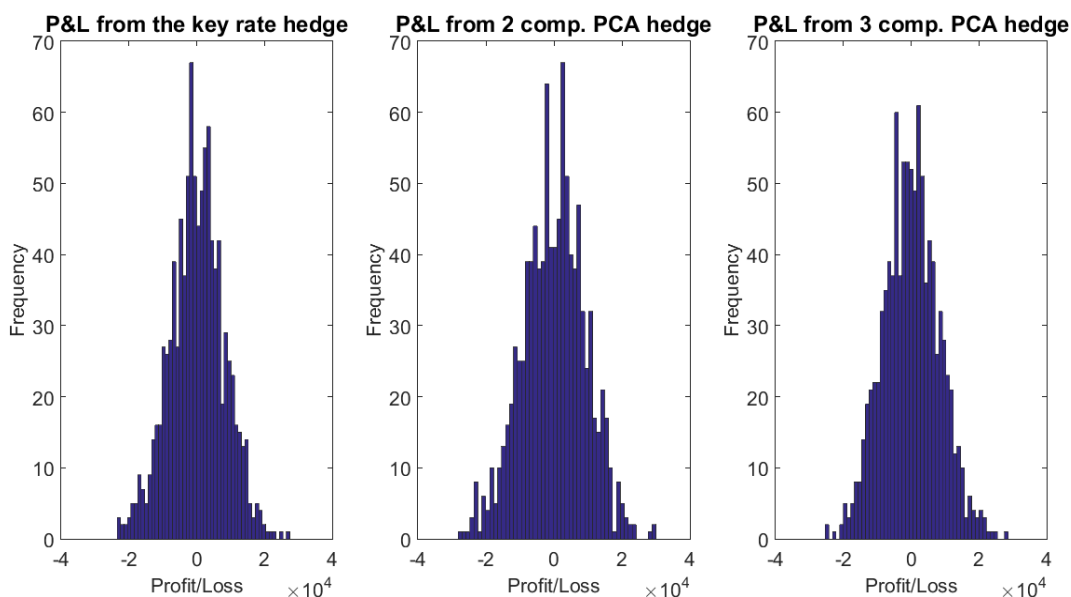


Fig. 4.2: P&L distribution of the hedge portfolios over one day for a portfolio of swaptions, without transaction costs

hedge in reducing risk is fairly similar. The key rate hedge does perform slightly better, just like we observed in the previous chapter. We also find that adding further principal components would improve the PCA hedge further still, however, at the added cost of trading in more instruments.

We will now consider the accumulated P&L that arises from each portfolio over the course of five days. Fig. 4.3 and Table 4.4 show the results obtained for the buy-and-hold strategy, while Fig. 4.4 and Table 4.5 show the results obtained for the hedge portfolio constructed with daily rebalancing.

Tab. 4.4: Summary statistics of the five day accumulated P&L distribution of the static hedge portfolios for a portfolio of swaptions, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-679.00	-476.32	17,117.34	1.44e+08	27,517.24
PCA hedge (2 comp.)	-378.71	-104.23	20,339.61	2.03e+08	34,280.23
PCA hedge (3 comp.)	445.10	394.05	17,408.95	1.54e+08	30,336.58
PCA hedge (4 comp.)	-180.89	96.75	17,258.17	1.52e+08	29,021.77

For the buy-and-hold technique, the results here are again consistent with the single day P&L results – the key rate hedge does the best job in reducing the riskiness of the portfolio, which is closely followed by the three component PCA hedge.

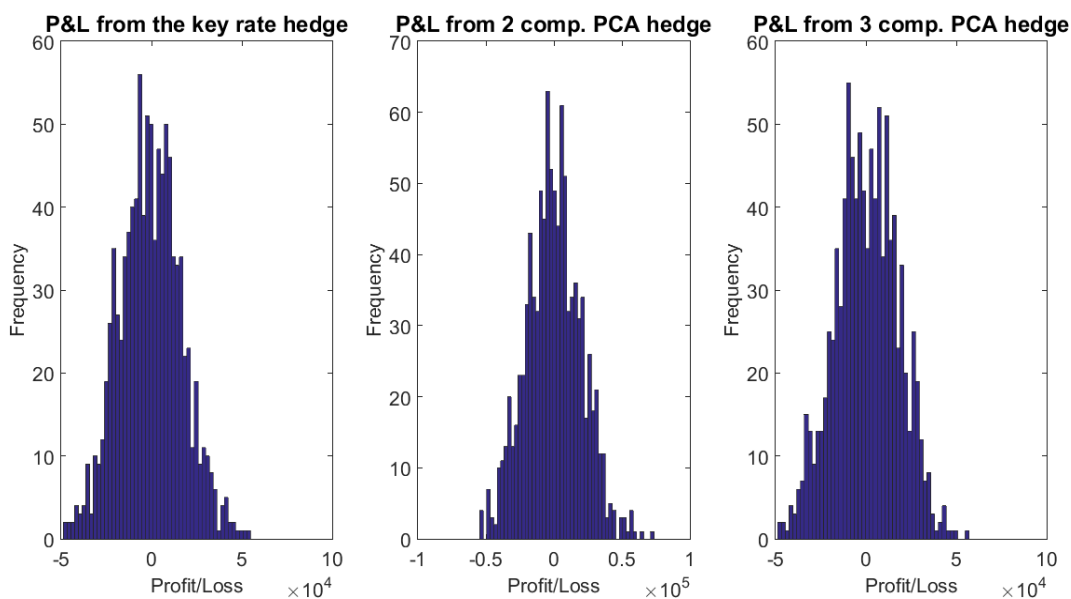


Fig. 4.3: Five day accumulated P&L distribution of the static hedge portfolios for a portfolio of swaptions, without transaction costs

Tab. 4.5: Summary statistics of the five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of swaptions, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-563.25	-321.66	7,142.40	2.45e+07	12,032.23
PCA hedge (2 comp.)	-462.61	-341.79	11,298.70	6.17e+07	18,782.68
PCA hedge (3 comp.)	-441.36	-273.31	7,241.89	2.52e+07	11,901.72
PCA hedge (4 comp.)	-439.48	-280.71	7,203.98	2.49e+07	11,853.16

On the other hand, for the daily rebalanced portfolio, even though the key rate hedge has a better performance in terms of lower standard deviation and DSV, the 5% VaR of the three component PCA hedge portfolio is slightly lower than that of the key rate hedge. These two strategies therefore seem to perform equally well in reducing the market risk. Also, as was observed in the previous chapter, it is clear that allowing for daily rebalancing reduces the riskiness of the portfolio a lot more as compared to maintaining a static hedged position over the five day period. Lastly, it can be seen from the last row of Tables 4.4 and 4.5 that adding a fourth component into the PCA hedge improves the hedging performance as compared to the three component hedge, as the values of the three risk measures decrease.

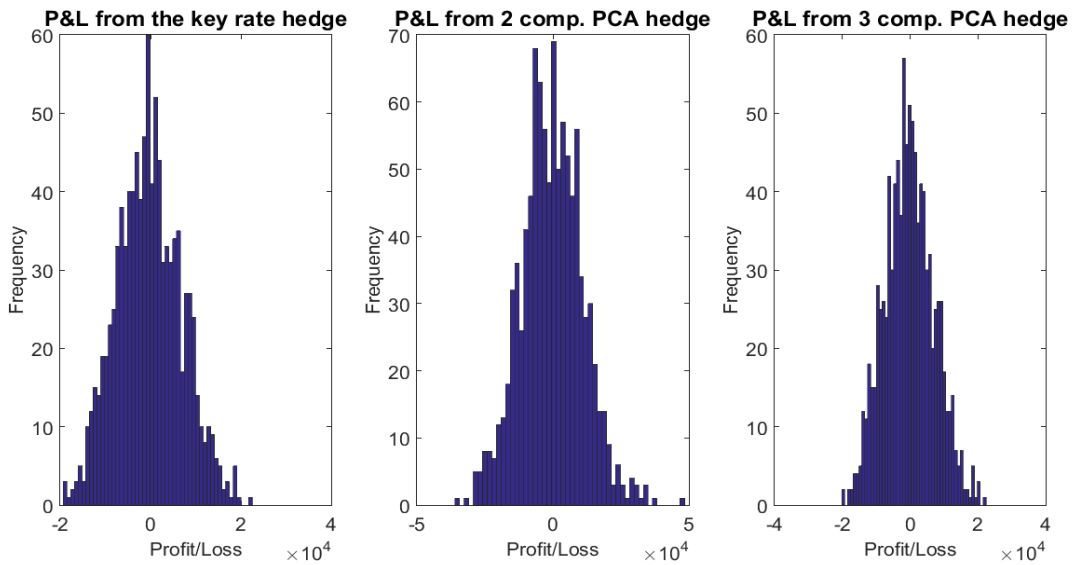


Fig. 4.4: Five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of swaptions, without transaction costs

4.2.2 Accounting for Transaction Costs

We now consider the relative performance of the three strategies with the added feature of accounting for a proportional transaction cost of 2 percent of the value of every trade made. Fig. 4.5 and Table 4.6 show the P&L profile and summary statistics of the P&L that arises from each hedged portfolio over one day.

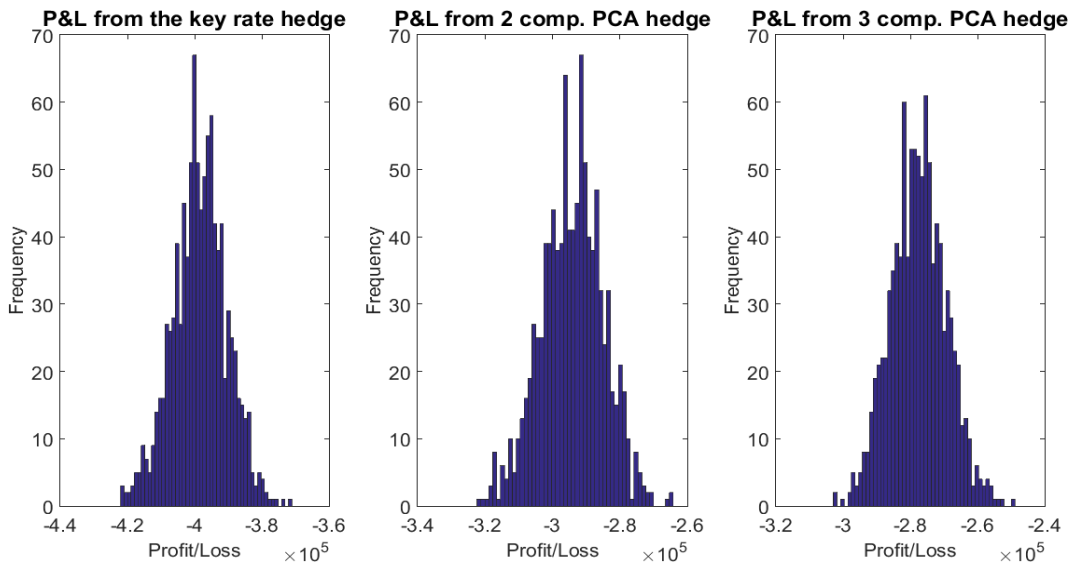


Fig. 4.5: P&L distribution of the hedge portfolios over one day for a portfolio of swaptions, with transaction costs

Tab. 4.6: Summary statistics of the P&L distribution of the hedge portfolios over one day for a portfolio of swaptions, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-398,418	-398,440	8,009.93	3.27e+07	411,882
PCA hedge (2 comp.)	-293,835	-294,136	9,241.37	4.39e+07	309,666
PCA hedge (3 comp.)	-277,585	-277,555	8,224.39	3.33e+07	290,987

The key rate hedge is again the best of the three in reducing the variability of the P&L that arises over a single day, as shown by the relative sizes of the standard deviation and the downside semivariance. However, due to the fact that the key rate hedge involves an investment in a much larger number of instruments, thirty-six instruments in this case, it incurs a large amount of market friction cost. This leads to a large decrease in the absolute size of the P&L, as shown by the median, mean and 5% VaR values. Consequently, the three component PCA hedge has the best performance in this scenario, because it reduces the variability of the P&L that arises and at the same time does not incur as much in transaction costs.

Analyzing the performance of the hedging strategies over a five day period leads to similar conclusions. Fig. 4.6 and Table 4.7 show the results obtained for the buy-and-hold strategy, while Fig. 4.7 and Table 4.8 show the results obtained for the daily rebalancing strategy.

Tab. 4.7: Summary statistics of the five day accumulated P&L distribution of the static hedge portfolios for a portfolio of swaptions, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-399,196	-398,993	17,117.34	1.44e+08	426,034
PCA hedge (2 comp.)	-294,559	-294,284	20,339.61	2.03e+08	328,460
PCA hedge (3 comp.)	-277,100	-277,152	17,408.95	1.54e+08	307,882
PCA hedge (4 comp.)	-294,603	-294,298	17,258.17	1.52e+08	325,648

Tab. 4.8: Summary statistics of the five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of swaptions, with transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-437,711	-437,807	7,212.25	2.65e+07	450,004
PCA hedge (2 comp.)	-350,256	-350,275	8,403.30	3.48e+07	363,716
PCA hedge (3 comp.)	-304,473	-304,647	7,269.06	2.72e+07	316,899
PCA hedge (4 comp.)	-312,657	-313,248	7,248.19	2.69e+07	326,883

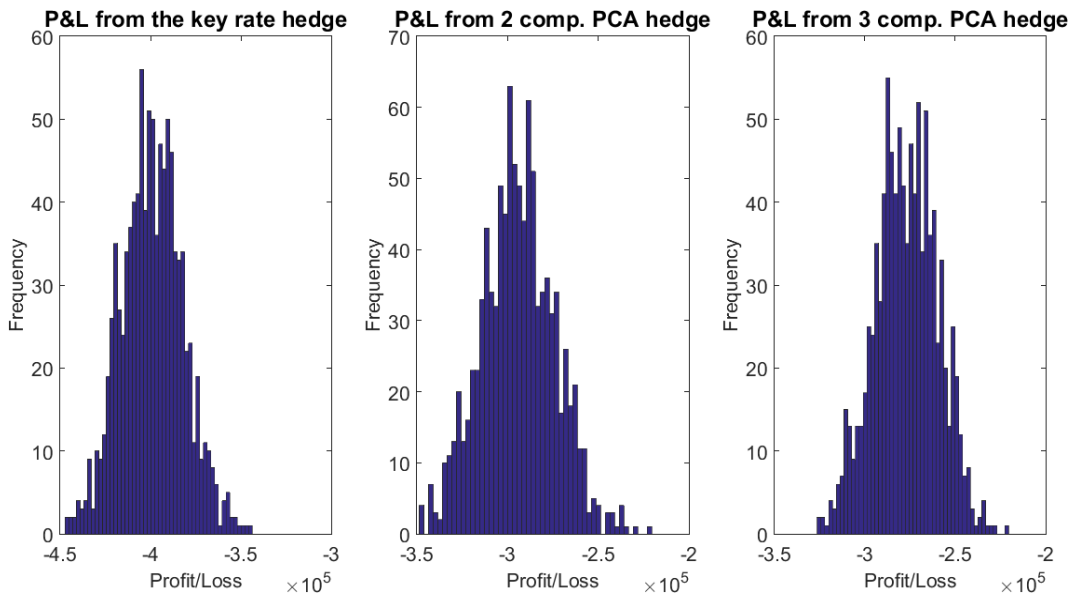


Fig. 4.6: Five day accumulated P&L distribution of the static hedge portfolios for a portfolio of swaptions, with transaction costs

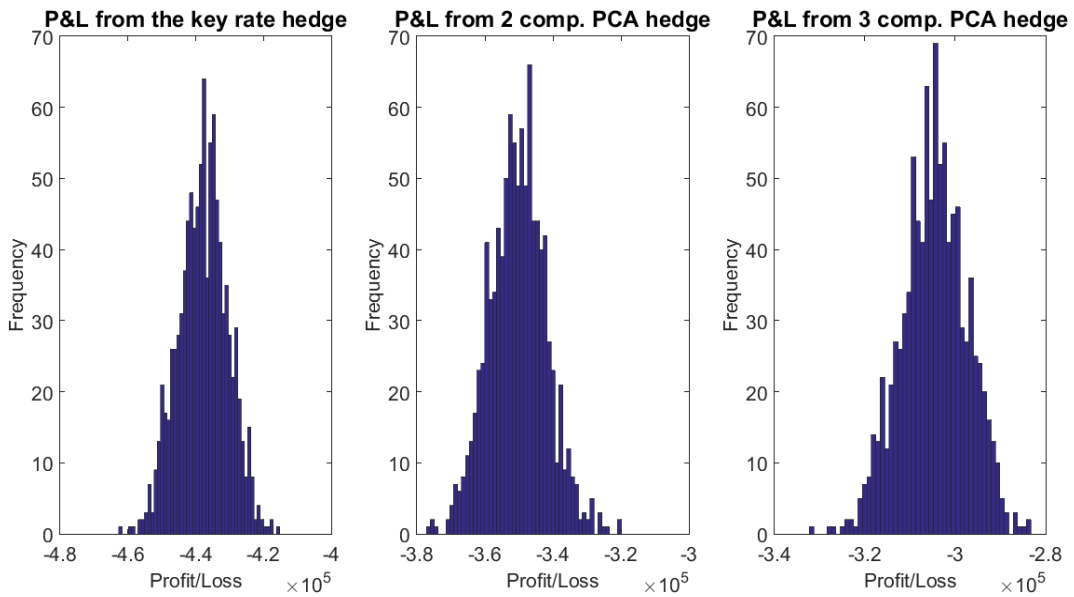


Fig. 4.7: Five day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of swaptions, with transaction costs

It is clear that the key rate hedge incurs a relatively large set up cost and also involves a large amount of trading when rebalancing, which leads to a deterioration in its hedging performance when transaction costs are accounted for. Also, it would not have been surprising if the two component PCA hedge (slightly) out-

performed the three component hedge after accounting for the market friction, due to the difference in the number of instruments each strategy trades in. However, this is not the case here - the three component hedge still outperforms the two component hedge. On the contrary, even though adding a fourth component into the PCA hedge reduces the risk in the portfolio, the median, mean and 5% VaR values in the last row of Tables 4.7 and 4.8 show that the transaction costs incurred in trading the extra instrument makes it expensive to do so. Lastly, we also observe once again that even though rebalancing the hedging portfolio daily reduces the variability of the P&L that arises quite significantly, the daily trading costs incurred has a detrimental effect on the P&L.

So we see that even though the key rate hedge is more often than not superior to the hedges constructed using PCA, this superiority is negated by the costs that are incurred to set up and maintain the portfolio with a large number of instruments. It seems that, in this case, the best balance between reducing the variability of the P&L and avoiding excessive transaction costs is obtained by using a three component PCA hedge. It should again be noted that these results are influenced by the assumptions of the type and size of transaction costs that are incurred. These conclusions may not be valid under a different regime of transaction costs.

Chapter 5

Conclusion

5.1 Concluding Remarks

We have made a comparison between two different hedging strategies that could be applied to hedge a portfolio of interest-rate options. The two strategies differ fundamentally by considering different risk factors which need to be hedged against. The key rate hedge considers the risk factors to be individual rates along a yield curve from which the entire curve is bootstrapped, whereas the PCA hedge takes the risk factors to be the principal components that drive the most common movements in the curve. The advantage of the latter strategy over the former is that it reduces the dimension of the hedging problem from considering a large number of key rates to a much smaller number of principal components.

It has been observed that the performance of the key rate hedge in reducing risk is consistently better than the PCA hedges constructed. However, after accounting for transaction costs, the key rate hedge is inferior. The number of instruments that have to be considered and the amount of trading that this strategy involves makes this strategy less attractive. In contrast, the PCA hedges considered do not perform as well as the key rate hedge at reducing the riskiness of the portfolio, but these do not incur as much in hedging costs, and so the overall performance of the PCA hedges is better. Under the transaction cost regime considered here, which is a proportional fee of 2 percent, it appears that a PCA hedge considering three principal components tends to perform well using historical yield curve data. This is because these three components capture most of the variability in the historical rates while utilizing only four instruments to construct the hedge.

In comparing two portfolio management techniques, it was observed that a daily rebalancing hedging strategy outperforms a buy-and-hold strategy, as one would expect. However, the opposite is true once transaction costs are introduced. Perhaps this is an indication that it may not be ideal to rebalance the hedge portfolio daily. A less frequent rebalancing could be more optimal, especially for a portfolio

of long-term options.

5.2 Avenues for Further Research

Throughout this dissertation, zero-coupon bonds have been used as the hedging instruments. It would be interesting to investigate the performance of other kinds of hedging instruments with each strategy, for example coupon-paying bonds, swaps, or perhaps even other interest-rate options such as caps and floors. The performance of a hedging portfolio that combines multiple kinds of instruments could also be considered.

Also, to reiterate an issue that was alluded to earlier, there is no set rule as to which of all available instruments to choose to perform the hedging. We have chosen to spread out our hedging instruments over the risk horizon under consideration, based on the findings of [Bagün *et al.* \(2000\)](#). It would be interesting to investigate how different strategies of selection affect the relative performance of the key rate and PCA hedges. Perhaps the best strategy of selection would depend on the constituents of the portfolio that needs to be hedged.

Lastly, [Perignon and Villa \(2006\)](#) suggest that the results of performing PCA on the yield curve depend heavily on the monetary policy regime that is currently active in the economy. So this raises the question as to how many years of historical data should be used to construct the principal components for the purposes of hedging. In this work, the principal components have been constructed using historical data over the most recent 10 year period. The problem with using data over so many years is that there may have been multiple changes in the economic climate and monetary policy regime active over that period of time. Perhaps it would be better to use data from a shorter time period in the recent past, one that matches the monetary policy regime that more suits the current regime that is under operation. This could have a material effect on the performance of the PCA hedges considered.

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Appendix A

Adding a Self-Financing Condition

If the additional self-financing condition is incorporated into the PCA hedging methodology, \mathbf{S} is now a $(l + 1) \times (l + 1)$ matrix, and \mathbf{c} is a $(l + 1) \times 1$ column vector. The additional row added to \mathbf{S} depends on the types of hedging instruments that are used in order to construct the hedging portfolio. If we use ZCBs in order to perform the hedging, then

$$\mathbf{S} = \begin{bmatrix} \sum_{j=1}^N \text{PV01}_j^1 q_{j1} & \dots & \sum_{j=1}^N \text{PV01}_j^{l+1} q_{j1} \\ \sum_{j=1}^N \text{PV01}_j^1 q_{j2} & \dots & \sum_{j=1}^N \text{PV01}_j^{l+1} q_{j2} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^N \text{PV01}_j^1 q_{jl} & \dots & \sum_{j=1}^N \text{PV01}_j^{l+1} q_{jl} \\ P(0, T_1) & \dots & P(0, T_{l+1}) \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \sum_{j=1}^N \text{PV01}_j^P q_{j1} \\ \sum_{j=1}^N \text{PV01}_j^P q_{j2} \\ \vdots \\ \sum_{j=1}^N \text{PV01}_j^P q_{jl} \\ V_0 \end{bmatrix},$$

where V_0 is the current value of the portfolio of options that is being hedged.

If some other kind of hedging instruments are used in order to construct the hedge, then the last row of \mathbf{S} would have to be changed. Essentially, we require that the dot product of the last row of \mathbf{S} with the vector of amounts invested in each hedging instrument, $\delta^{\text{PC}} = [\delta_1^{\text{PC}}, \dots, \delta_{l+1}^{\text{PC}}]^\top$, is equal to the present value of this hedging portfolio. This present value is set to be equal to $-V_0$, so that the net value of the combined portfolio is zero.

Therefore, the $(l + 1)^{\text{th}}$ row of both \mathbf{S} and \mathbf{c} serve to satisfy the desirable self-financing condition. In addition, the first l rows of \mathbf{S} and \mathbf{c} ensure that the combined portfolio is insensitive to shocks in the first l principal components, which are the key drivers of the yield or swap curve.

Appendix B

P&L Results from Bond Option Portfolio Over Thirty Day Horizon

As can be seen from the tables and figures below, the same conclusions can be drawn from the thirty-day P&L results that were drawn from the projected five-day results. That is, the single component PCA hedge performs the worst at reducing risk, whereas the key rate hedge slightly outperforms the two component PCA hedge. Maintaining a static hedge for thirty days is even more risky than doing so for a five day period, while allowing for daily rebalancing reduces risk quite significantly.

Tab. B.1: Summary statistics of the thirty day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-2.19	0.20	35.78	577.59	54.87
PCA hedge (1 comp.)	-19.26	0.10	74.33	1,786.74	80.04
PCA hedge (2 comp.)	-2.88	0.21	40.01	712.59	60.74

Tab. B.2: Summary statistics of the thirty day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, without transaction costs

	Median	Mean	Std Dev	DSV	5% VaR
Key rate hedge	-0.09	0.02	6.59	20.94	10.50
PCA hedge (1 comp.)	-0.65	0.03	12.59	71.41	19.16
PCA hedge (2 comp.)	-0.12	0.02	7.31	25.75	11.62

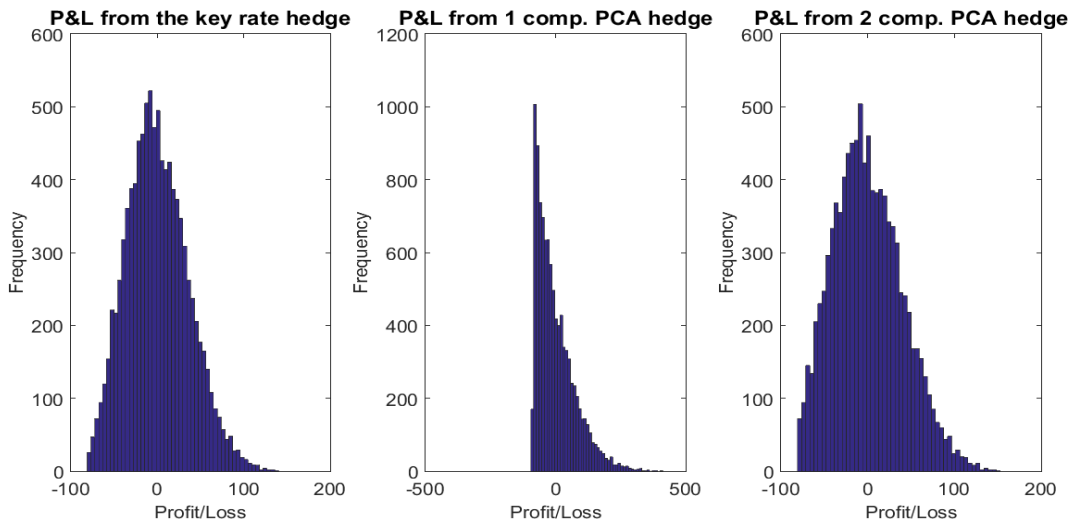


Fig. B.1: Thirty day accumulated P&L distribution of the static hedge portfolios for a portfolio of bond options, without transaction costs

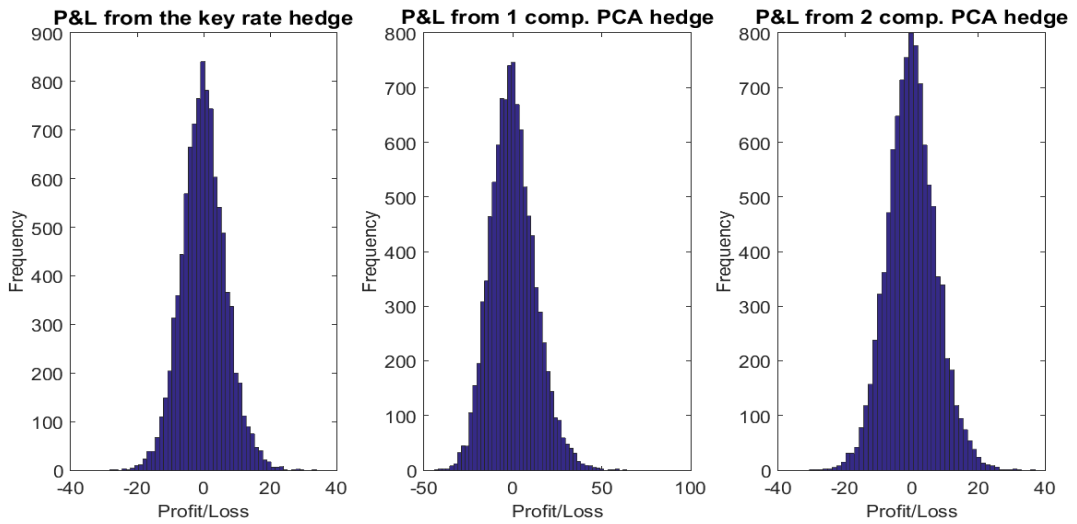


Fig. B.2: Thirty day accumulated P&L distribution of the dynamic hedge portfolios for a portfolio of bond options, without transaction costs

Appendix C

Pricing European Swaptions Under the G2++ Model

We need the following expressions to be able to price a European swaption under the G2++ model:

$$M_x^T(s, t) = \left(\frac{\sigma^2}{\alpha^2} + \rho \frac{\sigma\nu}{\alpha\beta} \right) \left[1 - e^{-\alpha(t-s)} \right] - \frac{\sigma^2}{2\alpha^2} \left[e^{-\alpha(T-t)} - e^{-\alpha(T+t-2s)} \right] - \rho \frac{\sigma\nu}{\beta(\alpha + \beta)} \left[e^{-\beta(T-t)} - e^{-\beta T - \alpha t + (\alpha + \beta)s} \right],$$

$$M_y^T(s, t) = \left(\frac{\nu^2}{\beta^2} + \rho \frac{\sigma\nu}{\alpha\beta} \right) \left[1 - e^{-\beta(t-s)} \right] - \frac{\nu^2}{2\beta^2} \left[e^{-\beta(T-t)} - e^{-\beta(T+t-2s)} \right] - \rho \frac{\sigma\nu}{\alpha(\alpha + \beta)} \left[e^{-\alpha(T-t)} - e^{-\alpha T - \beta t + (\alpha + \beta)s} \right],$$

$$C(t, T) = \frac{P^M(0, T)}{P^M(0, t)} \exp \left\{ \frac{1}{2} [V(t, T) - V(0, T) + V(0, t)] \right\},$$

$$B(z, t, T) = \frac{1 - e^{-z(T-t)}}{z},$$

$$\mu_x = -M_x^T(0, T), \quad \mu_y = -M_y^T(0, T),$$

$$\sigma_x = \sigma \sqrt{\frac{1 - e^{-2\alpha T}}{2\alpha}}, \quad \sigma_y = \nu \sqrt{\frac{1 - e^{-2\beta T}}{2\beta}},$$

$$\rho_{xy} = \frac{\rho\sigma\nu}{(\alpha + \beta)\sigma_x\sigma_y} \left[1 - e^{-(\alpha + \beta)T} \right].$$

The price of a European swaption, written on a vanilla interest-rate swap with payment dates $\{t_1, \dots, t_n\}$, with face value F , strike rate X and maturing at time $t_0 = T$, is given by the integral expression

$$F\omega P(0, T) \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} \left(\frac{x - \mu_x}{\sigma_x} \right)^2}}{\sigma_x \sqrt{2\pi}} \left[\Phi(-\omega h_1(x)) - \sum_{i=1}^n \lambda_i(x) e^{\kappa_i(x)} \Phi(-\omega h_2(x)) \right] dx,$$

with $\omega = 1$ for a payer swaption, and $\omega = -1$ for a receiver swaption, where

$$h_1(x) = \frac{\bar{y} - \mu_y}{\sigma_y \sqrt{1 - \rho_{xy}^2}} - \frac{\rho_{xy}(x - \mu_x)}{\sigma_x \sqrt{1 - \rho_{xy}^2}},$$

$$\begin{aligned}
h_2(x) &= h_1(x) + B(\beta, T, t_i)\sigma_y\sqrt{1 - \rho_{xy}^2}, \\
\lambda_i(x) &= c_i C(T, t_i)e^{-B(\alpha, T, t_i)x}, \\
\kappa_i(x) &= -B(\beta, T, t_i)\left[\mu_y - \frac{1}{2}(1 - \rho_{xy}^2)\sigma_y^2 B(\beta, T, t_i) + \rho_{xy}\sigma_y\frac{x - \mu_x}{\sigma_x}\right].
\end{aligned}$$

In the above expression for $\lambda_i(x)$, $c_i = X(t_i - t_{i-1})$ for $i = 1, \dots, n - 1$, and $c_n = 1 + X(t_n - t_{n-1})$. Finally, in the expression for $h_1(x)$, $\bar{y} = \bar{y}(x)$ is the solution to the equation

$$\sum_{i=1}^n c_i C(T, t_i)e^{-B(\alpha, T, t_i)x - B(\beta, T, t_i)\bar{y}} - 1 = 0.$$

Appendix D

Calibration of the G2++ Model

Table D.1 shows the implied volatility surface obtained from the market prices of caps on 31 December 2014. This data is used to calibrate the parameters of the G2++ model to the market. The cost function that was minimized during the calibration performed here was the sum of squared errors between observed market prices and model generated prices.

In order to perform the calibration, one would need the Black (1976) and G2++ cap pricing formulae, which are given below.

The Black (1976) time-0 price of a cap with face value F , maturity T_0 , payment dates $\{T_1, \dots, T_n\}$, strike K and constant volatility σ_B is given as

$$F \sum_{i=1}^n P(0, T_i) \tau_i D(K, L(0, T_{i-1}, T_i), v_i, 1),$$

where

$$D(K, L, v, \omega) = L\omega\Phi(\omega d_1(K, L, v)) - K\omega\Phi(\omega d_2(K, L, v)),$$

$$d_1(K, L, v) = \frac{\ln(L/K) + \frac{1}{2}v^2}{v},$$

$$d_2(K, L, v) = \frac{\ln(L/K) - \frac{1}{2}v^2}{v},$$

$$v_i = \sigma_B \sqrt{T_{i-1}},$$

$$\tau_i = T_i - T_{i-1}.$$

The G2++ model time-0 price of a cap with face value F , maturity T_0 , payment dates $\{T_1, \dots, T_n\}$ and strike K is given as

$$\sum_{i=1}^n \left[-F(1 + K\tau_i)P(0, T_i)\Phi\left(\frac{\ln \frac{P(0, T_{i-1})}{(1+K\tau_i)P(0, T_i)}}{\Sigma(0, T_{i-1}, T_i)} - \frac{1}{2}\Sigma(0, T_{i-1}, T_i)\right) + \right. \\ \left. FP(0, T_{i-1})\Phi\left(\frac{\ln \frac{P(0, T_{i-1})}{(1+K\tau_i)P(0, T_i)}}{\Sigma(0, T_{i-1}, T_i)} + \frac{1}{2}\Sigma(0, T_{i-1}, T_i)\right) \right],$$

where $\Sigma(t, T_{i-1}, T_i)$ is as defined in §3.2.

Tab. D.1: Volatility surface implied by market prices for caps on 31 December 2014

		Tenor (years)											
		0,25	0,5	1	2	3	4	5	6	7	8	9	10
Maturity (years)	0,25	8,67%	13,12%	13,31%	13,86%	14,19%	14,60%	15,04%	14,78%	14,54%	14,32%	14,12%	13,93%
	0,5	11,52%	15,00%	14,86%	14,85%	15,27%	15,76%	16,27%	16,08%	15,90%	15,75%	15,60%	15,47%
	1	11,28%	18,53%	18,31%	18,12%	17,63%	17,19%	16,79%	16,47%	16,16%	15,87%	15,59%	15,32%
	2	20,21%	22,01%	21,65%	21,23%	20,64%	20,11%	19,58%	19,25%	18,93%	18,64%	18,35%	18,06%
	3	22,40%	21,56%	20,91%	19,76%	19,62%	19,46%	19,33%	19,30%	19,28%	19,26%	19,23%	19,28%
	4	23,05%	20,89%	20,09%	18,64%	18,73%	18,84%	18,95%	19,15%	19,33%	19,50%	19,75%	20,00%
	5	23,20%	22,03%	21,03%	19,11%	18,96%	18,84%	18,75%	18,92%	19,09%	19,35%	19,61%	19,87%
	6	22,86%	21,72%	20,56%	18,31%	18,24%	18,21%	18,14%	18,36%	18,68%	19,00%	19,32%	19,80%
	7	22,41%	21,52%	20,17%	17,59%	17,54%	17,47%	17,40%	17,75%	18,09%	18,43%	18,95%	19,47%
	8	21,89%	19,98%	18,54%	15,80%	15,96%	16,14%	16,46%	16,84%	17,21%	17,78%	18,35%	18,90%
	9	21,45%	18,51%	17,03%	14,13%	14,52%	15,09%	15,63%	16,02%	16,63%	17,23%	17,82%	18,39%
10	20,70%	16,90%	15,40%	12,47%	13,28%	14,06%	14,81%	15,47%	16,10%	16,71%	17,32%	17,91%	