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Syntopogenous Structures and Realcompactness

by

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INTRODUCTION

The syntopogenous structures were introduced by Á. Császár. These are generalisations of classical continuity structures such as topologies, proximities and uniformities.

In his book, Foundations of General Topology (1963) [preceded by a French (1960) and a German (1963) edition], Császár treated many properties of syntopgenous structures. Among these properties were completeness and compactness, but not realcompactness.

Our purpose was to extend the definition of realcompactness from uniformisable topologies to arbitrary syntopogenous structures and to produce a realcompact reflection for arbitrary syntopogenous structures.

We did not fully accomplish this purpose. We have, in fact, first defined a notion of quasirealcompactness for arbitrary syntopogenous structures. For uniformisable Hausdorff topologies, realcompactness implies quasirealcompactness; we could not prove or disprove the converse implication. Nevertheless, we were able to give a characterisation of realcompactness for a uniformisable Hausdorff topology in terms of quasirealcompactness of a certain induced proximity; moreover, we produced a double quasirealcompact reflection in the category of separated syntopogenous structures, and from this retrieved the classical Hewitt realcompact reflection.

A syntopogenous structure is quasirealcompact iff each compressed filter base with the countable intersection property converges. A filter base, G , is compressed in a syntopogenous structure iff given a set A and a 'neighbourhood' B of A , there exists $R \in G$ such that $R \cap A = \emptyset$ or $R \subseteq B$. A syntopogenous structure is doubly quasirealcompact iff the symmetrisation of the structure is quasirealcompact.

A crucial role in the proof of the above characterisation of realcompactness was played by our theorem that a filter base with the countable intersection property is compressed in a proximity iff it is Cauchy in the uniformity induced by real-valued continuous functions.

The motivation for this procedure was the well-known theorem that a uniformisable Hausdorff space is realcompact if and only if it is complete in the weak uniformity induced by continuous real-valued functions. [Gillman and Jerison: Rings of Continuous Functions, pp. 225-226] .

A brief summary follows. Chapters 1, 2 and 3 contain a self-contained exposition of the basic theory of syntopogenous structures. We exhibit certain category-theoretical aspects of this theory, e.g. we show in Chapter 2 that the 'ordinary operations' of Császár are coreflections, and we show in Chapter 3 that the categories of the classical continuity structures are equivalent to certain subcategories of Syn , the category of syntopogenous structures. In Chapters 4 to 7 Császár's theory of compactness, double compactness and completeness with the appropriate reflections is expounded. Concurrently we introduce in Chapter 4 our notion of quasirealcompactness, from which in Chapter 6 we derive a characterisation of realcompactness of classical Tychonoff spaces. In Chapter 7, parallel to the exposition of the double compact and the complete reflections, we construct the double quasirealcompact reflection and from it retrieve the Hewitt realcompact reflection of Tychnoff spaces.

We attribute propositions and definitions by quoting references in brackets, thus '1.1 Definition [C] ' attributes item 1.1 to item [C] of the references.

The terminology used on categories coincides with that of Mitchell [M] , except that on reflections we use the standard terminology, which is dual to that of Mitchell.

As collateral material we have included our paper, On Alexandroff and Urysohn's General Metrisation Criterion, [F] .

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CHAPTER 1

THE CATEGORY Syn

1.0 This Chapter provides the definitions of the basic notions used. We begin by defining the relations which will be used in the definition of syntopogenous structures. We recall that a relation on a set E is just a subset of $E \times E$. A transitive relation on E is called an order.

1.1 Definition C

Let E be a set. The relation $<$ defined on the set of all subsets of E is called a topogenous order on E if it satisfies the following conditions:

(a) $\emptyset < \emptyset$ and $E < E$

(b) $A < B$ implies $A \subseteq B$

(c) $A \subseteq A' < B' \subseteq B$ implies $A < B$

(d) $A < B$ and $A' < B'$ imply $(A \cap A') < (B \cap B')$ and $(A \cup A') < (B \cup B')$

Note that on account of (b) and (c) a topogenous order is in fact a transitive relation.

A much used and important example of a topogenous order is given as follows.

1.2 Proposition [C]

Let E denote the real numbers and ε any positive number. The order $<_\varepsilon$ defined on the set of subsets of E by

$$A <_\varepsilon B \quad \text{iff} \quad \sup A + \varepsilon \leq \inf (E-B)$$

is a topogenous order.

PROOF: Taking $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$ we have that $\emptyset <_\varepsilon \emptyset$ and $E <_\varepsilon E$, verifying 1.1 (a). If $A <_\varepsilon B$ then it follows easily from the definition of $<_\varepsilon$ that $A \subseteq B$, giving 1.1 (b). To verify 1.1 (c) one need only notice that $A \subseteq A'$ implies $\sup A + \varepsilon \leq \sup A' + \varepsilon$ and that $B' \subseteq B$ implies $\inf (E-B') \leq \inf (E-B)$.

Finally, to verify 1.1 (d) let $A <_\varepsilon B$ and $A' <_\varepsilon B'$. One then has the relations

$$\begin{aligned}\sup (A \cap A') + \varepsilon &\leq \min (\sup A, \sup A') + \varepsilon \\ &\leq \min (\inf(E-B), \inf(E-B')) \\ &= \inf ((E-B) \cup (E-B')) = \inf (E-(B \cap B')),\end{aligned}$$

and

$$\begin{aligned}\sup (A \cup A') + \varepsilon &= \max (\sup A, \sup A') + \varepsilon \\ &\leq \max (\inf(E-B), \inf(E-B')) \\ &\leq \inf ((E-B) \cap (E-B')) = \inf (E-(B \cup B')).\end{aligned}$$

Thus 1.1 (d) is proved and it follows that $<_\varepsilon$ is a topogenous order on E.

□

We need the notion of 'finer than' for relations.

1.3 Definition [C]

If $<$ and $<'$ are two relations on the set E then $<'$ is finer than $<$ (or $<$ is coarser than $<'$) iff $< \subseteq <'$ where $\subseteq$ is the usual set inclusion. Thus $< \subseteq <'$. iff $A < B$ implies $A <' B$.

Two further examples of topogenous orders which are of basic importance in the development of the theory are provided by defining the following operations in topogenous orders.

1.4 The Operator c

Let $<$ be a topogenous order on the set E .
Define $c(<)$ as follows:

$$A \; c(<) \; B \quad \text{iff} \quad E - B < E - A .$$

The order $c(<)$ is a topogenous order on E . In fact, we verify that 1.1 (d) is satisfied. Suppose that $A \; c(<) \; B$ and $A' \; c(<) \; B'$, then $E - B < E - A$ and $E - B' < E - A'$.
Thus

$$(E-B) \cap (E-B') < (E-A) \cap (E-A')$$

giving

$$E - (B \cup B') < E - (A \cup A')$$

and so $A \cup A' \; c(<) \; B \cup B'$. In a similar way $A \cap A' \; c(<) \; B \cap B'$.

The other properties are easily verified.

1.4.1 Definition [C]

A topogenous order $<$ is symmetric iff $< \; = c(<)$.

We motivate the definition of the operator q below as follows: whereas it is easily verified that any intersection of topogenous orders is again a topogenous order it is not necessarily true that an arbitrary union of topogenous orders is a topogenous order, as the following example (cf. [C]) indicates.

Consider the following union of two topogenous orders on the real numbers where $\varepsilon > 0$: $< \; = \; <_\varepsilon \cup c(<_\varepsilon)$.

One has

$$(0,1) <_\varepsilon (-\infty, 1+\varepsilon) \quad \& \quad (0,1) \; c(<_\varepsilon) \; (-\varepsilon, +\infty)$$

and so

$$(0,1) < (-\infty, 1+\varepsilon) \quad \& \quad (0,1) < (-\varepsilon, +\infty) .$$

But since neither of the following two relations hold

$$(0,1) <_\varepsilon (-\varepsilon, 1+\varepsilon) \quad \& \quad (0,1) \; c(<_\varepsilon) \; (-\varepsilon, 1+\varepsilon),$$

it is untrue that the following holds.

$$(0,1) \cap (0,1) = (0,1) < (-\varepsilon, 1+\varepsilon) = (-\infty, 1+\varepsilon) \cap (-\varepsilon, +\infty) .$$

Thus $<$ is not a topogenous order.

However, as we shall see, one can generate a topogenous order from an arbitary union of topogenous orders on the same set by applying the operator q .

1.5 The Operator q

1.5.1 Definition [C]

If $<$ is an order on E satisfying only 1.1 (a), 1.1 (b) and 1.1 (c), then it is called a semitopogenous order on E .

1.5.2 Proposition [C]

If $<$ is a semitopogenous order on the set E then there is a topogenous order on E , denoted by $q(<)$, which is finer than $<$ and coarser than any topogenous order on E finer than $<$.

PROOF: Define $q(<)$ as follows.

1.5.3 $A \; q(<) \; B$ iff there exist positive integers m and n and sets A_i , B_j with $i \in I = \{1,2,\ldots,m\}$ and $j \in J = \{1,2,\ldots,n\}$ such that

$$A = \cup\{A_i : i \in I\}, \quad B = \cap\{B_j : j \in J\}$$

and

$$A_i < B_j \quad \text{for} \quad (i,j) \in I \times J .$$

It must be shown that $q(<)$ is a topogenous order. Now, 1.1 (a) is easily proved. Also, if $A \; q(<) \; B$ then $A_i \subseteq B_j$ and so $A \subseteq B_j$ giving $A \subseteq B$ and 1.1 (b).

Next, suppose that

$$A' \subseteq A \; q(<) \; B \subseteq B'$$

then 1.5.3 holds. And let

$$A'_i = A' \cap A_i \quad \text{and} \quad B'_j = B' \cup B_j .$$

Then

$$A' = \bigcup A'_i , \quad B' = \bigcap B'_j \quad \text{and}$$

$$A'_i \subseteq A_i < B_j \subseteq B'_j ,$$

giving

$$A'_i < B'_j \quad \text{by 1.1 (c).}$$

Hence $A' \; q(<) \; B'$, so that 1.1 (c) holds for $q(<)$.

Finally, 1.1 (d) is true: let $A \; q(<) \; B$ and $A' \; q(<) \; B'$. Then $A = \bigcup A_i$, $B = \bigcap B_j$, $A_i < B_j$ for $(i,j) \in I \times J$ and $A' = \bigcup A'_p$, $B' = \bigcap B'_r$, $A_p < B'_r$ for $(p,r) \in I' \times J'$, where I , I' , J and J' are finite. Thus

$$A \cap A' = \bigcup \{ A_i \cap A'_p : i \in I, \; p \in I' \}$$

and

$$B \cap B' = \bigcap \{ B_j \cap B'_r : j \in J, \; r \in J' \} .$$

Now

$$A_i \cap A'_p \subseteq A_i < B_j , \quad A_i \cap A'_p \subseteq A'_p < B'_r ,$$

thus

$$A_i \cap A'_p < B_j \cap B'_r$$

and hence

$$A \cap A' \; q(<) \; B \cap B' .$$

In a similar way it can be seen that $A \cup A' \; q(<) \; B \cup B'$.

It remains to show that if $<'$ is a topogenous order finer than $<$ then $q(<) \subseteq <'$. Suppose that $A \ q(<) \ B$. Then by 1.5.3 $A_i < B_j$ and therefore $A_i <' B_j$ and so $A <' B_j$ and $A <' B$ by 1.1 (d), since $<'$ is a topogenous order.

$\square$

1.5.4 Now, an arbitrary union $\cup \{<_a : a \in A\}$ of topogenous orders on the same set E is easily seen to be a semitopogenous order and so $q(\cup_\alpha <_\alpha)$ is a topogenous order on E.

1.5.5 Proposition [C]

The semitopogenous order $<$ is a topogenous order iff $< \; = \; q(<)$.

PROOF: Sufficiency is trivial since $q(<)$ is a topogenous order. Necessity follows by examining the definition of q and noting that $< \; \subseteq \; q(<)$.

We now prove some technical results about q which will be used later.

1.5.6 Proposition [C]

Let $<$ and $<'$ be semitopogenous orders on the set E. The following statements are true.

(a) $qq(<) = q(<)$

(b) $< \; \subseteq \; <'$ implies $q(<) \subseteq q(<')$.

(c) $cq(<) = qc(<)$

(d) $q(\bigcup\{<_i : i \in I\}) = q(\bigcup\{q(<_i) : i \in I\})$

where $<_i$ are semitopogenous orders.

(e) $q(<^2) \subseteq (q(<))^2$, where $<^2$ means $< \circ <$, which is the usual composition of relations.

PROOF:

(a) By 1.5.5, since $q(<)$ is a topogenous order, $q(<) = qq(<)$.

(b) By 1.5.2, $< \subseteq <' \subseteq q(<')$. Thus $q(<')$ is a topogenous order finer than $<$, and so by 1.5.2 $q(<) \subseteq q(<')$.

(c) Suppose that $A \; cq(<) \; B$, then $E - B \; q(<) \; E - A$ and so $E - B = \bigcup\{B_i : i \in I\}$, $E - A = \bigcap\{A_j : j \in J\}$ and $B_i < A_j$ for $(i,j) \in I \times J$, where I and J are finite (cf. 1.5.2). Now $E - A_j \; c(<) \; E - B_j$, thus

$$A = E - \bigcap A_j = \bigcup (E - A_j) \; qc(<) \; \bigcap (E - B_j)$$
$$= E - \bigcup B_j = B .$$

That is $A \; qc(<) \; B$. The argument is reversible and (c) is proved.

(d) By 1.5.2 $<_i \subseteq q(<_i) \subseteq q(\bigcup(q(<_i)))$ and therefore

$$q(\bigcup(<_i)) \subseteq q(\bigcup(q(<_i))) .$$

Conversely $q(<_i) \subseteq q(\cup(<_i))$ giving

$$\cup(q(<_i)) \subseteq q(\cup(<_i)) \quad \text{and} \quad q(\cup(q(<_i))) \subseteq q(\cup(<_i)) .$$

(e) Suppose that $A \; q(<^2) \; B$, then there are finite sets I and J and sets A_i, B_j such that $A = \cup\{A_i : i \in I\}$, $B = \cap\{B_j : j \in J\}$ and $A_i <^2 B_j$ for $(i,j) \in I \times J$. There are thus sets C_{ij} with $A_i < C_{ij} < B_j$. Put $C = \cup_i \cap_j C_{ij}$, then $\cap_j C_{ij} \subseteq C_{ij} < B_j$ and so $C \; q(<) \; B$. On the other hand, $C = \cap_j \cup_i C_{ij}$ and so $A_i < C_{ij} < \cup_i C_{ij}$ giving $A \; q(<) \; C$. Thus $A \; (q(<))^2 \; B$, as required.

$\square$

We now define the notion of a syntopogenous structure.

1.6 Definition [C]

Let S be a family of topogenous orders on the set E . S is a <u>syntopogenous structure</u> on E iff the following two conditions hold for E .

(a) If $<, <' \in S$ then there is $<'' \in S$ finer than both $<$ and $<'$.

(b) If $<' \in S$ then there is $< \in S$ such that $A <' B$ implies the existence of a set D satisfying $A < D < B$.

Bearing in mind the usual conventions for sets and relations, 1.6 (a) and (b) can respectively be rewritten as

$<, <' \in S$ implies there is $<'' \in S$ with $< \cup <' \subseteq <''$,

$<' \in S$ implies there is $< \in S$ such that $<' \subseteq <^2$.

1.7 Let E be the set of real numbers and for $\varepsilon > 0$ let $<_\varepsilon$ be the topogenous order defined in 1.2. Then the set $T = \{<_\varepsilon : \varepsilon > 0\}$ is a syntopogenous structure on E (cf. [C]). In fact, if $<_\varepsilon, <_{\varepsilon'} \in T$ and $\varepsilon'' = \max(\varepsilon, \varepsilon')$ then

$$<_\varepsilon \cup <_{\varepsilon'} \subseteq <_{\varepsilon''}$$

and also if $A <_\varepsilon B$ then, taking $D = (-\infty, \sup A + \varepsilon/2)$, one has that $A <_{\varepsilon/2} D <_{\varepsilon/2} B$.

☐

We wish to define the category Syn with the syntopogenous structures as objects. The morphisms in Syn are to be continuous functions; to describe them we will have to consider inverse images of syntopogenous structures. We proceed with this in mind.

1.8 Proposition [C]

Let f be a function on E into E' and let $<'$ be a topogenous order on E'. The order $<$ defined on the set of subsets of E by the condition

1.9 $A < B$ iff $f(A) <' E' - f(E-B)$

is a topogenous order on E.

PROOF: On account of 1.1 (a) and 1.1 (c) holding for $<'$,

$$f(\emptyset) = \emptyset <' \emptyset \quad E' - f(E-\emptyset)$$

and so $\emptyset \leq \emptyset$. Also

$$f(E) \subseteq E' <' E' = E' - f(E-E) ,$$

and thus $E < E$.

1.1 (b): On account of 1.1 (b) holding for $<'$,

$A < B$ implies $f(A) \subseteq E' - f(E-B)$ which implies $f(A) \cap f(E-B) = \emptyset$. It follows that $A \cap (E-B) = \emptyset$ and so $A \subseteq B$.

1.1 (c): On account of 1.1 (c) holding for $<'$,

$$f(A) \subseteq f(A') <' E' - f(E-B') \subseteq E' - f(E-B) ,$$

which implies $f(A) <' E' - f(E-B)$.

1.1 (d): Suppose $A < B$ and $A' < B'$. On account of 1.1 (d) holding for $<'$,

$$f(A) <' E' - f(E-B) \quad \& \quad f(A) <' E' - f(E-B')$$

imply that

$$f(A \cap A') \subseteq f(A) \cap f(A') <' E' - (f(E-B) \cup f(E-B')) \subseteq$$
$$E' - f((E-B) \cup (E-B')) = E' - f(E-(B \cap B')) ,$$

and thus $A \cap A' < B \cap B'$. The result for unions is also easily proved.

□

1.10 Definition [C]

If f is a function on E into E' and if $<'$ is a topogenous order on E' then the topogenous order $<$ defined on E by 1.9 is called the inverse image of $<'$ by f and is denoted $f^{-1}(<')$.

1.11 Definition [C]

Let S' be a syntopogenous structure on E' and let f be a function on E into E'. We define the inverse image of S' to be $\{f^{-1}(<') : <' \in S'\}$ and denote it by $f^{-1}(S')$.

If S' is a syntopogenous structure then so is $f^{-1}(S')$. A proof of this is accomplished in the next two results.

1.12 Proposition [C]

Let f be a function on E into E', let S' be a syntopogenous structure on E', let $<, <' \in S'$ and let I be arbitrary. The following statements are true.

(a) $< \subseteq <'$ implies $f^{-1}(<) \subseteq f^{-1}(<')$.

(b) $f^{-1}(\bigcup\{<_i : i \in I\}) = \bigcup\{f^{-1}(<_i) : i \in I\}$.

(c) $f^{-1}(<^2) \subseteq (f^{-1}(<))^2$.

(d) Let $<''$ be a semitopogenous order on E', then $f^{-1}(q(<'')) = q(f^{-1}(<''))$.

PROOF:

(a) $A \; f^{-1}(<) \; B$ iff $f(A) < E' - f(E-B)$ which implies $f(A) <' E' - f(E-B)$, and so $A \; f^{-1}(<') \; B$.

(b) $A \; f^{-1}(\cup <_i) \; B$ iff $f(A) \; (\cup <_i) \; E' - f(E-B)$
iff there is j with $f(A) <_j E' - f(E-B)$
iff $A \; \cup f^{-1}(<_i) \; B$.

(c) $A \; f^{-1}(<^2) \; B$ iff $f(A) <^2 E' - f(E-B)$
iff there is D' such that $f(A) < D' < E' - f(E-B)$.
Letting $D = f^{-1}(D')$ one has $D' = E' - f(E-D)$ and the result follows easily.

(d) Let $A \; f^{-1}(q(<'')) \; B$, then $f(A) \; q(<'') \; E' - f(E-B)$.
Thus there are sets A_i, B_j and finite sets I and J such that $f(A) = \cup\{A_i : i \in I\}$, $E' - f(E-B) = \cap\{B_j : j \in J\}$ and $A_i <'' B_j$ for $(i,j) \in I \times J$.
Put $A_i' = f^{-1}(A_i)$ and $B_j' = f^{-1}(E'-B_j)$, then $f(\cup A_i') = f(A)$ and $f(\cup B_j') = ff^{-1}(\cup(E'-B_j))$
$= E' - \cap B_j = f(E-B)$.

Next, it is easily seen that $f(A_i') <'' E' - f(B_j')$ giving $A_i' \; f^{-1}(<) \; E - B_j'$, from which it follows that $A \subseteq f^{-1}f(A)$
$= \cup A_i' \; q(f^{-1}(<'')) \; \cap (E-B_j') = E - \cup(f^{-1}(E'-B_j))$
$= E - f^{-1}f(E-B) \subseteq B$. We have shown that $f^{-1}(q(<'')) \subseteq q(f^{-1}(<''))$.

Conversely, $<'' \subseteq q(<'')$ and therefore $f^{-1}(<'') \subseteq f^{-1}(q(<''))$ by 1.12 (a). Since, by 1.8, $f^{-1}(q(<''))$ is a topogenous order, we have according to 1.5.2 that $q(f^{-1}(<'')) \subseteq f^{-1}(q(<''))$, as required.

□

1.13 Corollary [C]

If f and S' are as in 1.12 then $f^{-1}(S')$ is a syntopogenous structure on E .

PROOF: Let $f^{-1}(<)$, $f^{-1}(<') \in f^{-1}(S')$ and let $<'' \in S$ be such that $< \cup <' \subseteq <''$. Using 1.12,

$$f^{-1}(<) \cup f^{-1}(<') = f^{-1}(< \cup <') \subseteq f^{-1}(<'') \in f^{-1}(S') .$$

Also, let $f^{-1}(<') \in f^{-1}(S')$ and let $< \in S'$ be such that $<' \subseteq <^2$. Then using 1.13 ,

$$f^{-1}(<') \subseteq f^{-1}(<^2) \subseteq (f^{-1}(<))^2 \quad \& \quad f^{-1}(<) \in f^{-1}(S') .$$

□

The definition of continuity is now just a matter of arranging notation.

1.14 Definition

Let S and S' be syntopogenous structures on the set E . Then $S \leq S'$ iff given $< \in S$ there is $<' \in S'$ such that $< \subseteq <'$. In this case S is said to be coarser than S' .

$S \sim S'$ (S is equivalent to S') iff $S \lesssim S'$ and $S' \lesssim S$.

1.15 Definition [C]

Let f be a function on E into E' and let S and S' be syntopogenous structures on E and E' respectively, then f is (S,S')-continuous iff $f^{-1}(S') \lesssim S$.

We now define the basic category of the theory.

1.16 Definition

The category Syn is defined as follows: S is an object of Syn iff S is a syntopogenous structure defined on a set. If S and S' are objects of Syn the set of morphisms from S to S' is the set of all (S,S')-continuous functions.

We must verify that Syn is indeed a category.

1.17 Proposition

The compostion of two continuous functions is continuous.

PROOF: Let $g : (E,S) \rightarrow (E',S')$ and $f : (E',S') \rightarrow (E'',S'')$ be continuous functions.

We first show that $(fg)^{-1}(<'') = g^{-1}f^{-1}(<'')$ for $<'' \in S''$. In fact,

$$A \quad (fg)^{-1}(\lessdot'') \quad B \quad \text{iff}$$

$$fg(A) \lessdot'' \; E'' - fg(E-B) \;=\; E'' - f(E'-(E'-g(E-B))) \quad \text{iff}$$

$$g(A) \quad f^{-1}(\lessdot'') \quad (E'-g(E-B)) \quad \text{iff}$$

$$A \quad g^{-1}f^{-1}(\lessdot'') \quad B \,,$$

which furnishes the required equality.

Next, by hypothesis, $f^{-1}(S'') \leqslant S'$ and therefore, by 1.12 (a), $g^{-1}f^{-1}(S'') \leqslant g^{-1}(S')$ and, again by hypothesis, $g^{-1}(S') \leqslant S$ giving $g^{-1}f^{-1}(S'') \leqslant S$ as required. $\square$

The associativity axiom for categories is obviously satisfied. For identities on objects take the identity function, which is easily seen to be continuous. Syn is thus a category.

CHAPTER 2

SUBCATEGORIES OF Syn

2.0 The results of this chapter depend heavily on the methods and propositions in [C] . The classical structures such as uniformities, proximities and topologies will be characterized as equivalences of certain subcategories of Syn . The objects of these subcategories will be those which remain invariant under the action of appropriately chosen operators from Syn into Syn . The operators involved will all be examples of the so-called ordinary operations. These ordinary operations are actually retractions onto coreflective subcategories of Syn , as we show in our proposition 2.2.6.

In this chapter the ordinary operations are studied, while the structures themselves will be studied in Chapter 3.

As a preliminary to defining the ordinary operations we define

2.1 The operator g

If S is a family of topogenous orders defined on E then S is called directed iff it satisfies axiom 1.6 (a):

If $<, <' \in S$ then there is $<'' \in S$ finer than both $<$ and $<'$.

2.1.1. Let $S = \{<_i : i \in I\}$ be an arbitrary family of topogenous orders on the set E, F the set of finite subsets of I and $<_{I'} = q(\cup\{<_i : i \in I'\})$ where $\emptyset \neq I' \in F$.

Definition [C]

$$g(S) = \{<_{I'} : I' \in F \ \& \ I' \neq \emptyset\} .$$

2.1.2 Proposition [C]

Let S be a family of topogenous orders defined on the set E, then $g(S)$ is a directed family of topogenous orders on E which is finer than S but coarser than any other directed family finer than S.

PROOF: Let $I_1, I_2 \in F$ where S, I and F are as in 2.1.1 above. Then $<_{I_1} \cup <_{I_2} \subseteq <_{I_1 \cup I_2}$.

Next, it is easily seen by taking $I' = \{a\}$, say, for each $a \in I$, that $S \leq g(S)$.

Finally, suppose S' is another directed family on E finer than S. Let $<_{I'} = q(\cup\{<_i : i \in I'\}) \in g(S)$, and for each $i \in I'$ take $<'_i \in S'$ so that $<_i \subseteq <'_i$. It follows that $\cup\{<_i : i \in I'\} \subseteq \cup\{<'_i : i \in I'\}$, giving $<_{I'} \subseteq q(\cup\{<'_i : i \in I'\})$. Now let $<' \in S'$ be such that $\cup\{<'_i : i \in I'\} \subseteq <'$, then $<_{I'} \subseteq q(<') = <'$.

This shows that $g(S) \leqslant S'$.

□

2.1.3 Proposition [C]

Let S be a family of topogenous orders on the set E . S is a sytopogenous structure iff

$$g(S) \leqslant S \leqslant S^2 ,$$

where $S^2 = \{ <^2 : < \in S \}$.

PROOF: If S is a syntopogenous structure then by definition 1.6 both inequalities hold.

On the other hand if the above inequalities hold, then for $<_1, <_2 \in S$ it follows that $<_1 \cup <_2 \subseteq q(<_1 \cup <_2) \subseteq <_3$ for some $<_3 \in S$.

□

2.1.4 Proposition [C]

The operator g has the following properties:

(a) $S \leqslant g(S)$

(b) $gg(S) = g(S)$

(c) $S' \leqslant S$ implies $g(S') \leqslant g(S)$

(d) $g(S^2) \leqslant (g(S))^2$

(e) $f^{-1}(g(S)) = g(f^{-1}(S))$

PROOF:

(a) Let $<_i \in S$, then

$$<_i = q(<_i) = <_{\{i\}} \in g(S)$$

(b) Consider the member $q(\bigcup\{<_{I_i} : i = 1,2,\dots,n\})$ of $gg(S)$, where $S = \{<_i \,;\, i \in I\}$ and each $I_i \subseteq I$ is finite.

Now, given $1 \leqslant i \leqslant n$ there is n_i such that

$$q(\bigcup <_{I_i}) = q(\bigcup q(\bigcup\{<_{j,i} : j = 1,\dots,n_i)\})$$

where $<_{j,i} \in I_i$. By 1.5.6 (d)

$$q(\bigcup\{q(\bigcup\{<_{j,i} : j = 1,\dots,n_i\}) : i = 1,2,\dots,n\})$$
$$= q(\bigcup\{<_{j,i} : i = 1,\dots,n \;\&\; j = 1,\dots,n_i\})$$
$$\in g(S) .$$

Thus $gg(S) \subseteq g(S)$, while from the proof of (a) $S \subseteq g(S)$.

(c) In view of 1.5.6 (b) the proof of (c) is easy.

(d) We have $\bigcup\{<_i^2 : i = 1,\dots,n\} \subseteq (\bigcup\{<_i : i = 1,\dots,n\})^2$ and so $q(\bigcup <_i^2) \subseteq q((\bigcup <_i)^2) \subseteq (q(\bigcup <_i))^2$ by 1.5.6 (e).

(e) $f^{-1}(q(\bigcup\{<_i : i = 1,\dots,n\}))$

$= q(f^{-1}(\bigcup <_i))$ by 1.12 (d)

$= q(\bigcup f^{-1}(<_i))$ by 1.12 (b)

$\in g(f^{-1}(S))$.

$\square$

The properties 2.1.4 (a) to (e) are characteristic of the ordinary operations, which we now define.

2.2 Ordinary Operations

2.2.1 Definition [C]

An order family is a set of topogenous orders defined on the same set. Let k be a function whose domain is the set of all order families and such that $k(S)$ and S are always defined on the same set. The function k is called an ordinary operation iff it satisfies the following six conditions for arbitrary order families S, S' defined on the same set, and functions f whose codomain is this set.

(a) $S \leqslant k(S)$

(b) $kk(S) = k(S)$

(c) $S' \leqslant S$ implies $k(S') \leqslant k(S)$

(d) $k(S^2) \leqslant (k(S))^2$

(e) $g(k(S)) \leqslant k(g(S))$

(f) $f^{-1}(k(S)) = k(f^{-1}(S))$

It follows from 2.1.4 that g is an ordinary operation.

2.2.2 Proposition [C]

Let j and k be ordinary operations, then jk is also an ordinary operation if $jkjk(A) = jk(A)$ for all order families A.

$\square$

2.2.3 Proposition [C]

If k is an ordinary operation and $S \in |Syn|$, then $k(S) \in |Syn|$.

PROOF: According to 2.1.3 we need only show that $g(k(S)) \leqslant k(S) \leqslant (k(S))^2$. But by 2.2.1 (e), 2.1.3 and 2.2.1 (c) $g(k(S)) \leqslant k(g(S)) \leqslant k(S)$, and by 2.1.3, 2.2.1 (c), (d) $k(S) \leqslant k(S^2) \leqslant (k(S))^2$.

□

2.2.4 The following proposition is important because it shows that ordinary operations are functorial (cf. 2.2.5).

Proposition [C]

Let $f : (E,S) \rightarrow (E',S')$ be continuous, then f is $(k(S),k(S'))$-continuous, for any ordinary operation k, and also $(c(S),c(S'))$-continuous.

PROOF: By 2.2.1 (f), (c), (f), $f^{-1}(k(S')) = k(f^{-1}(S')) \leqslant k(f^{-1}(S)) = f^{-1}(k(S))$, and the first part of the statement is proved.

It is easily seen that 2.2.1 (c) holds with c replacing k, and so in view of the first paragraph above, we need only show that 2.2.1 (f) holds with c replacing k in order to prove the last part of the statement of the proposition. Accordingly, let $A \; f^{-1}(c(<)) \; B$ for $< \in S$. Now, $f(A) \; c(<) \; E' - f(E-B)$ iff $f(E-B) < E' - f(A)$ iff $E - B \; f^{-1}(<) \; E - A$ iff $A \; c(f^{-1}(<)) \; B$.

□

Several examples of ordinary operations will be given, but before becoming involved in those details we establish the fact that ordinary operations determine coreflections in Syn.

2.2.5 Definition

Let k be an ordinary operation. Define the full subcategory kSyn of Syn as follows: the objects of kSyn are all the objects of Syn satisfying $k(S) = S$.

2.2.6 Proposition

If k is an ordinary operation, then k retracts Syn onto its full, coreflective subcategory kSyn .

PROOF: For a morphism of $f \in$ Syn define $k(f) = f$, then k is a functor (2.2.4) .

To show that kSyn is coreflective, let S be a syntopogenous structure on the set E . By 2.2.1 (a) $S \leq k(S)$ and so the identity map $i : E \rightarrow E$ is $(k(S),S)$-continuous . Next let $S' \in$ kSyn and let $f : S' \rightarrow S$ be continuous in Syn . By 2.2.2 f is $(S',k(S))$-continuous and so the following diagram is commutative.

$$\begin{tikzcd} S & & k(S) \arrow[ll, "i"'] \\ & S' \arrow[ul, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

If $f' : S' \rightarrow k(S)$ makes the diagram commute then $f' = if' = f$, showing that f is unique and $k(S)$ is a coreflection. Finally, since k leaves morphisms unchanged and 2.2.1 (b) holds we have $kk = k$ and therefore k is a retraction onto kSyn .

$\square$

2.2.7 The coreflection mapping i is actually a monomorphism and so kSyn is a monocoreflection, and it is known that epireflections in classical topology have nice and interesting properties (cf. [HER]).

2.3. Several examples of ordinary operations fall into a subclass called the elementary operations. In order to avoid repetition we define elementary operations and show that they induce ordinary operations.

2.3.1 Definition [C]

Let e be a function which assigns to an arbitrary semitopogenous order $<$ (cf. 1.5.1) on an arbitrary set E another semitopogenous order $e(<)$ on the same set E . Then e is called an elementary operation if it satisfies the following conditions analogous to those of 2.2.1, for arbitrary semitopogenous orders $<, <'$ defined on E , and functions f whose codomains are E .

(a) $< \subseteq e(<)$

(b) $ee(<) = e(<)$

(c) $< \subseteq <'$ implies $e(<) \subseteq e(<')$

(d) $e(<^2) \subseteq (e(<))^2$

(e) $q(e(<)) \subseteq e(q(<))$

(f) $f^{-1}(e(<)) = e(f^{-1}(<))$

2.3.2 We adopt the convention that if e is an elementary operation and S a set of semitopogenous orders defined on an arbitrary set E , then $\{ e(<) : < \in S\}$ is denoted $e(S)$. Thus e induces an operation on order families by using this convention.

Proposition [C]

If e is an elementary operation then it is an ordinary operation.

PROOF: We prove that e satisfies 2.2.1 (e). Let S be a family of topogenous orders on E , then $e(g(S))$ is a directed family. In fact we have for arbitrary semitopogenous orders $<_1, <_2$ that $e(e(<_1) \cup e(<_2)) = e(<_1 \cup <_2)$. This can be seen as follows: $<_i \subseteq e(<_i)$ (by 2.3.1 (a) for i = 1,2) therefore

$<_1 \cup <_2 \subseteq e(<_1) \cup e(<_2)$ and so $e(<_1 \cup <_2) \subseteq e(e(<_1) \cup e(<_2))$

(by 2.3.1 (c). On the other hand for i = 1,2 $<_i \subseteq <_1 \cup <_2$ giving $e(<_i) \subseteq e(<_1 \cup <_2)$ (by 2.3.1 (c)) and

$e(<_1) \cup e(<_2) \subseteq e(<_1 \cup <_2)$ thus

$e(e(<_1) \cup e(<_2)) \subseteq ee(<_1 \cup <_2) = e(<_1 \cup <_2)$ (by 2.3.1 (c), (b))

as required.

Now let $<_1, <_2 \in g(S)$, which is directed. There is $< \in g(S)$ with $<_1 \cup <_2 \subseteq <$, thus $e(<_1) \cup e(<_2) \subseteq e(<_1 \cup <_2) \subseteq e(<)$ showing that $e(g(S))$ is directed. According to 2.1.2 , $g(e(S)) \leqslant e(g(S))$ because $e(S) \leqslant e(g(S))$ an account of $S \leqslant g(S)$ and 2.2.1 (c) (which follows independently of 2.2.1 (e)).

The other properties are easily verified.

□

2.3.3 Proposition [C]

The operator q is an elementary operation.

PROOF: 1.5.2; 1.5.6 (a), (b), (e) and 1.12 (d). $\square$

2.4 The Operator s

Let $<$ be a semitopogenous order on the set E.

2.4.1 Definition [C]

(a) Define $s(<) = q(< \cup c(<))$

(b) Define a semi-topogenous order $<$ to be symmetrical iff $< = c(<)$

2.4.2 Proposition [C]

(a) $qs = sq = s$

(b) $cs = sc = s$, and so $s(<)$ is symmetrical

(c) The order $q(<)$ is symmetrical iff $s(<) = q(<)$

PROOF:

(a) $qs(<) = q(q(< \cup c(<))) = q(< \cup c(<))$, by 1.5.6 (a).

Next, $sq(<) = q(qq(<) \cup qc(<)) = q(q(<) \cup qc(<))$

$= q(< \cup c(<))$, by 1.5.6 (a), (d).

(b) $cs(<) = cq(< \cup c(<)) = qc(< \cup c(<)) = q(c(<) \cup cc(<))$ $= s(<)$, using 1.5.6 (c). Also, $sc(<) = q(c(<) \cup cc(<))$ $= s(<)$.

(c) If $q(<) = cq(<)$, then using 1.5.6 (d) and (c)

$$s(<) = q(< \cup c(<)) = q(q(<) \cup qc(<)) = q(q(<) \cup cq(<))$$
$$= q(<) .$$

If $s(<) = q(<)$ then using 2.4.2 (b),
$q(<) = s(<) = cs(<) = cq(<)$.

□

2.4.3 Proposition [C]

Let $<$ be a topogenous order. The topogenous order $s(<)$ is coarser than any symmetrical topogenous order finer than $<$.

PROOF: Let $<'$ be a symmetrical topogenous order finer than $<$. It is easily seen that $c(<) \subseteq c(<')$. Next, using 1.5.6 (b),

$$s(<) = q(< \cup c(<)) \subseteq q(<' \cup c(<')) = <' .$$

□

2.4.4 Proposition [C]

The operator s is an elementary operation.

PROOF:

2.3.1 (a): $< \;\subseteq\; q(< \cup c(<)) \;=\; s(<)$

2.3.1 (b): We have $ss(<) = q(s(<) \cup cs(<)) = qs(<) = s(<)$, using 2.4.2.

2.3.1 (c): If $<_1 \subseteq <_1'$, then $q(<_1) \subseteq q(<_1')$, by using 1.5.6(b). Next, if $< \subseteq <'$ then $s(<) = q(< \cup c(<)) \subseteq q(<' \cup c(<'))$ $= s(<')$.

2.3.1 (d): First, $c(<^2) = (c(<))^2$. In fact, $A \; c(<^2) \; B$ iff $E - B <^2 E - A$, hence there is D such that $E - B < D < E - A$ and $A \; c(<) \; E - D \; c(<) \; B$, which shows that $A \; (c(<))^2 \; B$. The argument is reversible and so the reverse inclusion also holds. Next, $s(<^2) = q(<^2 \cup c(<^2)) = q(<^2 \cup (c(<))^2)$ $\subseteq q((< \cup c(<))^2)$. Finally, we need only observe that $q((< \cup c(<))^2) \subseteq (q(< \cup c(<)))^2$ on account of 1.5.6 (e).

2.3.1 (e): 2.4.2 (a).

2.3.1 (f): $f^{-1}(s(<)) = s(f^{-1}(<))$ will follow from the facts:

$$f^{-1}(q(<)) = q(f^{-1}(<)),$$
$$f^{-1}(\cup\{<_i : i \in I\}) = \cup\{f^{-1}(<_i) : i \in I\},$$
$$f^{-1}(c(<)) = c(f^{-1}(<)).$$

The last two facts have straightforward proofs and the first is 1.12 (d).

□

2.5 The Operator p

2.5.1 Definition [C]

The order $<$ is perfect iff for an arbitrary set I one has that $A_i < B_i$ for each $i \in I$ implies $\bigcup\{A_i : i \in I\} < \bigcup\{B_i : i \in I\}$.

2.5.2 Proposition [C]

The semitopogenous order $<$ is perfect iff $x < B$ for each $x \in A$ implies $A < B$.

PROOF: Suppose $<$ is perfect, then take $I = A$, $A_i = \{i\}$ for $i \in I$ and $B_i = B$; it follows that $A < B$.

Conversely, suppose the condition in the proposition is true; take $A = \bigcup\{A_i : i \in I\}$ and $B = \bigcup\{B_i : i \in I\}$, then the result follows on account of $a \in A_n < B_n \subseteq \bigcup\{B_i : i \in I\}$ for $a \in A$ and some index n.

$\square$

2.5.3 Proposition [C]

If $<$ is a semitopogenous order on the set E, there exists a perfect semitopogenous order, $p(<)$, finer than $<$ and coarser than all other perfect semitopogenous orders finer than $<$.

PROOF: In fact define $p(<)$ by

2.5.4 $A \ p(<) \ B$ iff there are sets I and A_i with $i \in I$ such that $A = \bigcup\{A_i : i \in I\}$ and for each $i \in I$ $A_i < B$.

Of the conditions required in 1.1 we verify only 1.1 (c). Accordingly suppose that $A \subseteq A' \ p(<) \ B' \subseteq B$, $A' = \bigcup A'_i$ and $A'_i < B$ for each $i \in I$, say. Let $A_i = A \cap A'_i$, then $A_i \subseteq A'_i < B$ and thus $A \ p(<) \ B$.

The order $p(<)$ is perfect. In fact, suppose for each $x \in A$, $x \ p(<) \ B$, then by 2.5.4 $x < B$. Again by 2.5.4, $A = \bigcup\{x : x \in A\} \ p(<) \ B$, and so according to 2.5.2 $p(<)$ is perfect.

It follows immediately from definition 2.5.4 that $p(<)$ is finer than $<$.

Now suppose that $<'$ is perfect and finer than $<$. If $A \ p(<) \ B$ then $A = \bigcup A_i$ with $A_i < B$ for i in some index set I. It follows that $A_i <' B$ and since $<'$ is perfect $A <' B$, showing that $p(<)$ is coarser than $<'$.

□

2.5.5 Proposition [C]

If $<$ is a semitopogenous order, $A \ p(<) \ B$ iff $a < B$ for each $a \in A$.

PROOF: If $a < B$ for each $a \in A$, then from 2.5.4 $A \ p(<) \ B$. If $A \ p(<) \ B$ and $a \in A$, then from 2.5.4 there is n such that $a \in A_n < B$ and so $a < B$.

$\square$

2.5.6 Proposition [C]

The semitopogenous order $<$ is perfect iff $p(<) = <$.

PROOF: Suppose $<$ is perfect, then $A \ p(<) \ B$ implies $a < B$ for $a \in A$, and therefore $A < B$ by definition 2.5.1. Also, by 2.5.3 $< \subseteq p(<)$ and thus $p(<) = <$.

Conversely, suppose $p(<) = <$ and let $a < B$ for each $a \in A$. Then $A \ p(<) \ B$ by 2.5.5 and so $A < B$, whence $<$ is perfect by 2.5.2.

$\square$

2.5.7 Proposition [C]

The operator p is an elementary operation.

PROOF:

2.3.1 (a): This has been shown in 2.5.3.

2.3.1 (b): We have $p(<) \subseteq pp(<)$ (2.5.3). On the other hand, if $A \ pp(<) \ B$ then for each $a \in A$, $a \ p(<) \ B$. But since $p(<)$ is perfect, $A \ p(<) \ B$, that is $pp(<) \subseteq p(<)$. It follows that $pp(<) = p(<)$, as required.

2.3.1 (c): Let $< \subseteq <'$. $A \; p(<) \; B$ implies $a < B$ for $a \in A$ and so $a <' B$ for $a \in A$, giving $A \; p(<') \; B$.

2.3.1 (d): $A \; p(<^2) \; B$ implies $a <^2 B$ for $a \in A$, and $a < C_a < B$ for some set C_a. On putting $D = \bigcup \{ C_a : a \in A \}$ it follows from 2.5.1 that $A \; p(<) \; D$ and $D \; p(<) \; B$, that is $A \; (p(<))^2 \; B$.

2.3.1 (e): Let $A \; qp(<) \; B$, then $A = \bigcup A_i$, $B = \bigcap B_j$ and $A_i \; p(<) \; B_j$ for i,j members of suitable finite sets I and J respectively. Thus for $x \in A$ we have that $x < B_j$ for $j \in J$ (2.5.5), and so for $x \in A$, $x \; q(<) \; B$. Again from 2.5.5, $A \; pq(<) \; B$.

2.3.1 (f): $A \; f^{-1}(p(<)) \; B$

iff $f(a) \; p(<) \; E' - f(E-B)$

iff for each $a \in A$, $f(a) < E' - f(E-B)$ (by 2.5.5)

iff for each $a \in A$, $a \; f^{-1}(<) \; B$

iff $A \; pf^{-1}(<) \; B$ (by 2.5.5).

□

2.5.8 Corollary [C]

If $<$ is a topogenous order, then so is $p(<)$.

PROOF: 2.5.7, 2.3.1 (e) and 1.5.5.

□

2.6 The Operator b

Let $<$ be a semitopogenous order on the set E .

2.6.1 Definition [C]

The order $<$ is biperfect iff given sets I , A_i and B_i with $i \in I$, the following condition is satisfied:

$$A_i < B_i \text{ for all } i \in I \text{ implies}$$
$$\bigcup\{A_i : i \in I\} < \bigcup\{B_i : i \in I\} \text{ and}$$
$$\bigcap\{A_i : i \in I\} < \bigcap\{B_i : i \in I\} .$$

Clearly, any biperfect semitopogenous order is perfect.

2.6.2 Example [C]

For each real number $\varepsilon > 0$ the topogenous order $<_\varepsilon$ defined by 1.2 is biperfect.

2.6.3 Proposition [C]

The semitopogenous order $<$ on E is biperfect iff $A < B$ whenever $x < E - y$ for every pair of points $x \in A$ and $y \in E - B$.

PROOF: Suppose $<$ is biperfect, $x \in A$, $y \in E - B$ and $x < E - y$, then $A = \bigcup\{a : a \in A\} < E - y$ and

$A < \bigcap\{E-b : b \in E-B\} = E - \bigcup\{b : b \in E-B\} = B$.

Conversely, if the condition in the statement is true and for $i \in I$ $A_i < B_i$, then for $a \in \bigcup A_i$ and $b \in E - \bigcup B_i = \bigcap E-B_i$ we have $a \in A_n$, say, and $b \in E - B_n$, that is $B_n \subseteq E - b$. Thus $a < E - b$ on account of $A_n < B_n$ and so $\bigcup A_i < \bigcup B_i$.

Next, if $a \in \bigcap A_i$ and $b \in E - \bigcap B_i$ then for some index n $b \in E - B_n$, $B_n \subseteq E - b$ and $a \in A_n$. Thus $a \in A_n < B_n \subseteq E - b$, giving $\bigcap A_i < \bigcap B_i$.

□

We now produce the operator b .

2.6.4 Proposition [C]

If $<$ is a semitopogenous order on the set E , then there is a biperfect topogenous order, $b(<)$, finer than $<$ and coarser than any other biperfect topogenous order finer than $<$.

2.6.5 The order $b(<)$ is defined as follows:

$A \; b(<) \; B$ iff there are sets I , J , A_i $(i \in I)$, B_j $(j \in J)$ with $A = \bigcup\{A_i : i \in I\}$, $B = \bigcap\{B_j : j \in J\}$ and $A_i < B_j$ for $i \in I$ and $j \in J$.

PROOF: We verify 1.1 (c). Suppose $A \subseteq A' \; b(<) \; B' \subseteq B$, then $A' = \bigcup A'_i$, $B' = \bigcap B'_j$ and $A'_i < B'_j$.
Let $A_i = A \cap A'_i$, $B_j = B \cup B'_j$, then $\bigcup A_i = A$, $\bigcap B_j = B$

and $A_i \subseteq A_i' < B_j' \subseteq B_j$, that is $A_i < B_j$ and so $A \; b(<) \; B$.

The order $b(<)$ is biperfect. Indeed, suppose, according to 2.6.3, that $x \in A$, $y \in E - B$ and $x \; b(<) \; E - y$. It follows from definition 2.6.5 that $x < E - y$. Again from 2.6.5 it follows that $A = \bigcup\{x : x \in A\} \; b(<) \; \bigcap\{E-y : y \in E-B\} = B$, and so 2.6.3 shows that $b(<)$ is biperfect.

The order $b(<)$ is obviously finer than $<$.

Finally, if $<'$ is a biperfect topogenous order finer than $<$, let $A \; b(<) \; B$. Then $A = \bigcup A_i$, $B = \bigcap B_j$ and $A_i < B_j$ thus $A_i <' B_j$ giving $A <' B$ since $<'$ is biperfect.

□

2.6.6 <u>Proposition [C]</u>

$A \; b(<) \; B$ iff $x < E - y$ for $x \in A$ and $y \in E - B$.

<u>PROOF</u>: Suppose $A \; b(<) \; B$, $x \in A$ and $y \in E - B$. By 2.6.5 there are A_i and B_j with $A = \bigcup A_i$, $B = \bigcap B_j$ and $A_i < B_j$. For suitable n and m $a \in A_n$ and $y \in E - B_m$, thus we have $a \in A_n < B_m \subseteq E - y$ giving $a < E - y$.

Conversely, suppose $x < E - y$ for $x \in A$ and $y \in E - B$, then for $x \in A$ and $y \in E - B$ $x \; b(<) \; E - y$ since $< \subseteq b(<)$. On account of 2.6.3 $A \; b(<) \; B$.

□

2.6.7 Proposition [C]

The semitopogenous order $<$ is biperfect iff $< \; = b(<)$.

PROOF: Suppose $<$ is biperfect. If $A \; b(<) \; B$ then for $a \in A$ and $b \in E - B$ we have $a < E - b$, thus $A < B$ by 2.6.3. That is $b(<) \subseteq \; <$. On the other hand $< \; \subseteq b(<)$ (2.6.4) giving $< \; = b(<)$.

The converse follows immediately from the fact that $b(<)$ is biperfect (2.6.4).

□

2.6.8 Proposition [C]

The operator b is an elementary operation.

PROOF:

2.3.1 (a): This has been shown in 2.6.4.

2.3.1 (b): We have that $b(<) \subseteq bb(<)$ (2.6.4). If $A \; bb(<) \; B$ then for each $a \in A$, $y \in E - b$ $a \; b(<) \; E - y$. But $b(<)$ is biperfect, hence $A \; b(<) \; B$, that is $bb(<) \subseteq b(<)$, giving $b(<) = bb(<)$.

2.3.1 (c): Let $< \; \subseteq \; <'$. $A \; b(<) \; B$ implies $a < E - y$ for $a \in A$, $y \in E - B$, and so $a <' E - y$. This means that $A \; b(<') \; B$.

2.3.1 (d): A $b(<^2)$ B implies $a <^2 E - y$ for $a \in A$, $y \in E - B$. That is, there is a set $C_{a,y}$ with $a < C_{a,y} < E - y$. Now, by 2.6.5,

$$a \;\; b(<) \;\; \bigcap \{C_{a,y} : y \in E\text{-}B\}$$

and therefore

$$A = \bigcup\{a : a \in A\} \;\; b(<) \;\; \bigcup\{\bigcap\{C_{a,y} : y \in E\text{-}B\} : a \in A\}$$

since $b(<)$ is biperfect (2.6.1). Next, $\bigcup_a \bigcap_y C_{a,y} \subseteq \bigcap_y \bigcup_a C_{a,y}$. However $\bigcup_a C_{a,y} \;\; b(<) \;\; E - y$ (2.6.5) and therefore, since $b(<)$ is biperfect,

$$\bigcap\{\bigcup_a C_{a,y} : y \in E\text{-}B\} \;\; b(<) \;\; \bigcap\{E\text{-}y : y \in E\text{-}B\}$$
$$= E - (E\text{-}B) = B .$$

This shows that

$$A \;\; b(<) \;\; \bigcup_a \bigcap_y C_{a,y} \subseteq \bigcap_y \bigcup_a C_{a,y} \;\; b(<) \;\; B$$

and so $A \;\; (b(<))^2 \;\; B$ as required.

2.3.1 (e): In fact we prove that $qb(<) = bq(<) = b(<)$. We have $q(<) \subseteq b(<)$ since $b(<)$ is a topogenous order finer than $<$, and so $bq(<) \subseteq bb(<) = b(<)$. Also, $< \subseteq q(<)$ implies $b(<) \subseteq bq(<)$ whence $bq(<) = b(<)$. Next, since $b(<)$ is a topogenous order, $qb(<) = b(<)$ (1.5.5).

2.3.1 (f): $A \; f^{-1}(b(<)) \; B$ iff $f(A) \; b(<) \; E' - f(E-B)$

iff for $a \in A$ and $y \in E - B$, $f(a) < E' - f(y)$ (2.6.6)

iff for $a \in A$ and $y \in E - B$, $f(a) < E' - f(E-(E-y))$

iff for $a \in A$ and $y \in E - B$, $a \; f^{-1}(<) \; E - y$

iff $A \; b(f^{-1}(<)) \; B$.

□

2.6.9 Proposition [C]

For any family A of semitopogenous orders on a set E

(a) $pb(A) = bp(A) = b(A)$

(b) $qb(A) = bq(A) = b(A)$.

PROOF:

(a): Let $< \in A$. The order $b(<)$ is a perfect topogenous order finer than $<$ and so $p(<) \subseteq b(<)$ (2.5.3), thus $bp(<) \subseteq bb(<) = b(<)$. On the other hand $< \subseteq p(<)$ and therefore $b(<) \subseteq bp(<)$ giving $b(<) = bp(<)$. This establishes that $bp(A) = b(A)$. Finally, since $b(A)$ is perfect, we have $pb(A) = b(A)$.

(b): See the proof of 2.3.1 (e) in proposition 2.6.8.

□

2.7 The Operator t

We now consider an ordinary operation which reduces an arbitrary family of topogenous orders to one consisting of a single topogenous order.

2.7.1 Definition [C]

A syntopogenous structure consisting of a single topogenous order is called a simple syntopogenous structure.

2.7.2 Definition [C]

Let $S = \{<_i : i \in I \neq \emptyset\}$ be a family of topogenous orders on a set E. Define $t(S)$ to be $\{<\}$ where $< \; = q(\bigcup\{<_i : i \in I\})$.

2.7.3 Proposition [C]

If $S = \{<_i : i \in I\}$ is a directed family of topogenous orders then $t(S) = \{<_i : i \quad I\}$.

PROOF: We must show that $q(\bigcup <_i) = \bigcup <_i$, that is we must show that $\bigcup <_i$ is a topogenous order (1.5.5) (a union of semitopogenous orders is easily seen to be a semitopogenous order). Since $\bigcup <_i$ is a semitopogenous order, we need only check that the condition

$$A \bigcup <_i B \text{ and } A' \bigcup <_i B' \text{ implies that}$$
$$A \cup A' \bigcup <_i B \cup B' \text{ and } A \cap A' \bigcup <_i B \cap B' .$$

In fact, assuming the conditions above we have for certain $<_1$ and $<_2 \in S$ that $A <_1 B$ and $A' <_2 B'$. As S is directed there is a topogenous order $< \in S$ finer than $<_1 \cup <_2$, whence $A < B$, $A' < B'$, giving $A \cup A' < B \cup B'$ and $A \cap A' < B \cap B'$. But $< \subseteq \bigcup <_i$.

□

2.7.4 Proposition [C]

The operator t is an ordinary operation.

PROOF: Suppose that $S = \{<_i : i \in I\}$ is a family of topogenous orders on the set E and let $t(S) = \{<\}$.

2.2.1 (a): Let $<_i \in S$, then $<_i \subseteq <$ (1.5.2)

2.2.1 (b): Easy.

2.2.1 (c): Since $< \subseteq <'$ implies $q(<) \subseteq q(<')$, 2.2.1 (c) follows easily.

2.2.1 (d): First, it is easily seen that $\bigcup\{<_i^2 : i \in I\} \subseteq (\bigcup\{<_i : i \in I\})^2$. Next, according to 1.5.6 (e), $q(<^2) \subseteq (q(<))^2$ and therefore $q(\bigcup_i <_i^2) \subseteq q((\bigcup_i <_i)^2) \subseteq (q(\bigcup_i <_i))^2$.

2.2.1 (e): Since $\{<\}$ is obviously directed, $g(\{<\}) \leqslant \{<\}$ by 2.1.2, that is $gt(S) \leqslant t(S)$. Next, $S \leqslant g(S)$ by 2.1.2, and so $t(S) \leqslant tg(S)$ (using 2.2.1 (c), proved above for t).

2.2.1 (f): $f^{-1}(t(S)) = f^{-1}(q(\bigcup_i <_i)) = qf^{-1}(\bigcup_i <_i)$, by 1.12 (d). Now let $A \; f^{-1}(\bigcup_i <_i) \; B$, then $f(A) \; (\bigcup_i <_i) \; E' - f(E-B)$. That is, for some index p, $f(A) <_p E' - f(E-B)$ giving $A \; f^{-1}(<_p) \; B$, which implies that $A \; (\bigcup_i f^{-1}(<_i)) \; B$.

The reverse implication also holds and so $f^{-1}(\cup_i <_i) = \cup_i f^{-1}(<_i)$. Therefore $q(f^{-1}(\cup_i <_i)) = q(\cup_i f^{-1}(<_i)) = tf^{-1}(S)$.

$\square$

2.7.5 Proposition [C]

A syntopogenous structure S is simple iff $S = t(S)$, and thus $t(S)$ is the coarsest simple syntopogenous structure finer than S .

$\square$

2.8 We now describe an important construction which enables us to define products in Syn .

2.8.1 Proposition [C]

Let $\{S_i : i \in I\}$ be a family of syntopogenous structures on the set E . There is a syntopogenous structure on E which is the coarsest syntopogenous structure finer than each of the S_i .

PROOF: Consider $S = g(\cup\{S_i : i \in I\})$. S is a directed family by 2.1.2. We have that $S_i \leqslant (S_i)^2$ and thus $\cup_i S_i \leqslant (\cup_i S_i)^2$ so that, since g is an ordinary operation, $g(\cup_i S_i) \leqslant g((\cup_i S_i)^2) \leqslant (g(\cup_i S_i))^2$. We have shown that S is a syntopogenous structure. Also $S_i \leqslant \cup_i S_i \leqslant g(\cup_i S_i)$. Finally, if S' is a syntopogenous structure on E such that $S_i \leqslant S'$ for each $i \in I$, then $\cup_i S_i \leqslant S'$ and since S' is directed $S = g(\cup_i S_i) \leqslant S'$.

$\square$

2.8.2 Definition [C]

Let $\{S_i : i \in I\}$ be a family of syntopogenous structures on the set E. The coarsest syntopogenous structure on E finer than each S_i (whose existence is given by 2.8.1) is denoted by $\bigvee\{S_i : i \in I\}$.

2.8.3 Proposition [C]

Let $\{S_i : i \in E\}$ be a family of syntopogenous structures on the set E and let $f : E' \to E$ be a function, then $f^{-1}(\bigvee(S_i)) = \bigvee(f^{-1}(S_i))$.

PROOF: $f^{-1}(\bigvee(S_i)) = f^{-1}(g(\bigcup(S_i))) = g(f^{-1}(\bigcup(S_i)))$, since g is an ordinary operation. But using 1.12 (b), $g(f^{-1}(\bigcup(S_i))) = g(\bigcup(f^{-1}(S_i))) = \bigvee(f^{-1}(S_i))$.

$\square$

We now produce products in Syn.

2.8.4 Proposition [C]

For each $i \in I$ let S_i be a syntopogenous structure on the set E_i, let the set E be the product of the sets E_i and let $p_i : E \to E_i$ be the projection functions. Then the syntopogenous structure $S = \bigvee\{p_i^{-1}(S_i) : i \in I\}$ on E is the category product of the S_i in Syn and $p_i : S \to S_i$ are the projection morphisms in Syn.

PROOF: We need only show that for each $i \in I$, $p_i : S \to S_i$ is continous and that if $S' \in \mathrm{Syn}$ and $q_i : S' \to S_i$, for $i \in I$, are continuous then there is a unique morphism $p : S' \to S$ with $q_i = p_i p$.

In fact, $p_i^{-1}(S_i) \leqslant S$ by 2.8.1 and so p_i is continuous for each $i \in I$.

Next, suppose that there is a syntopogenous structure S' on E' and $q_i : S' \to S_i$ continuous for each $i \in I$. Because E is a product in the category of sets and functions it follows that there is a unique function $p : E' \to E$ with the property that for each $i \in I$ $q = p_i p$. We show that $p : S' \to S$ is continuous :

$$p^{-1}(S) = p^{-1} \bigvee p_i^{-1}(S_i) = \bigvee p^{-1} p^{-1}_i(S_i) = \bigvee q_i^{-1}(S_i) \leqslant S'$$

using 2.8.3. The fact that p is unique as a function implies that p is unique as a morphism in Syn. □

2.8.5 Our next proposition modifies the proposition in [C, p.156].

For each $i \in I$ let $E_i = E$ and let S_i be a syntopogenous structure on E_i. Let the set P be the product of the sets $\{E_i : i \in I\}$ with projections $p_i : P \to E_i$, let the syntopogenous structure S defined on P be the product of the structures $\{S_i : i \in I\}$ and let $D : E \to P$ be the unique function induced by the identities $1 = 1_i : E_i \to E_i$. Since P is a product with projections p_i it follows that $1 = p_i D$ for each $i \in I$.

Proposition

$$D^{-1}(S) = \bigvee \{ S_i : i \in I \}$$

PROOF: $D^{-1}(S) = D^{-1}(\bigvee\{p_i^{-1}(S_i) : i \in I\})$

$= \bigvee D^{-1}p_i^{-1}(S_i) = \bigvee 1^{-1}(S_i) = \bigvee S_i$, using 2.8.3.

$\square$

We end the chapter by defining subspaces.

2.8.6 Definition [C]

Let S' be a syntopogenous structure on the set E', let E be a subset of E' and let $i : E \to E'$ be the inclusion function. The syntopogenous structure $i^{-1}(S')$ is called the subspace of S' induced by i and is denoted $S'|E$, its members are denoted $<|E$ where $< \in S'$. The order $<|E$ is also called 'the restriction of $<$ to E'.

CHAPTER 3

UNIFORMITIES, PROXIMITIES AND TOPOLOGIES

We proceed without further ado to embed the categories of the classical continuity structures in Syn .

The results of this chapter depend heavily on the methods of [C] , although we have reorganized the material somewhat and our proofs are not identical with those of [C] .

3.1 Definition [C]

Let S be an object of Syn .

3.1.1 S is a topology iff it is simple and perfect.

3.1.2 S is a proximity iff it is simple and symmetric.

3.1.3 S is a uniformity iff it is symmetric and biperfect.

3.2 Proposition [C]

Let S be an object of Syn .

3.2.1 S is a topology iff $pt(S) = S$.

3.2.2 S is a proximity iff $ts(S) = S$.

3.2.3 S is a uniformity iff $bs(S) = S$.

To prove 3.2 we need to be able to calculate with the operators t , s , b and p ; to this end we prove the preliminary results 3.3 to 3.7 and then return to the proof of 3.2 in item 3.8.

3.3 Proposition [C]

For any family A of topogenous orders on a set E and any elementary operation a , $atat(A) = tat(A) = ata(A) = at(A)$.

PROOF: First, $tat(A) = at(A)$ since $at(A)$ consists of a single topogenous order, and so $atat(A) = aat(A) = at(A)$. Next, using 2.2.1 (a), (c) one has $at(A) \leq ata(A) \leq atat(A)$ and $at(A) \leq tat(A) \leq atat(A)$. Thus $at(A) = atat(A) \sim ata(A) \sim tat(A)$.

However it is easy to see that if $<$ and $<'$ are topogenous orders, then $\{<\} \sim \{<'\}$ iff $\{<\} = \{<'\}$. Therefore $ata(A)$ and $tat(A)$ being simple, we have the result.

$\square$

3.4 Proposition [C]

For any family A of topogenous orders on the set E we have $ts(A) = st(A)$.

PROOF: First, $ts(A)$ is symmetrical. This can be seen as follows. Let $A = \{<_i : i \in I\}$. Now, $q(\bigcup_i s(<_i)) = ts(A)$. Consider $cts(A) = cq(\bigcup_i s(<_i)) = qc(\bigcup_i s(<_i)) = q(\bigcup_i cs(<_i))$

$= q(\bigcup_i s(<_i)) = ts(A)$, using 1.5.6 (c) and 2.4.2 (b). This shows that $ts(A)$ is symmetrical. Next, since $ts(A)$ is symmetrical one has $ts(A) = sts(A) = st(A)$ by 3.3.

□

The next two propositions are the technical ingredients of 3.7.

3.5 Proposition [C]

(a) $cb(<) = bc(<)$ for an arbitrary semitopogenous order $<$.

(b) Let $\{<_i : i \in I\}$ be a family of semitopogenous orders on E, then $b(\bigcup_i <_i) = b(\bigcup_i b(<_i))$.

PROOF:

(a): $A\ cb(<)\ B$ iff $E - B\ b(<)\ E - A$

iff for $y \in E - B$ and $x \in A$, $y < E - x$

iff for $x \in A$ and $y \in E - B$, $x\ c(<)\ E - y$

iff $A\ bc(<)\ B$.

(b): First, for $i \in I$ $<_i \subseteq b(<_i)$, $\bigcup_i <_i \subseteq \bigcup_i b(<_i)$ and $b(\bigcup_i <_i) \subseteq b(\bigcup_i b(<_i))$. Next, $<_i \subseteq b(\bigcup_i <_i)$, $b(<_i) \subseteq bb(\bigcup_i <_i)$, $\bigcup_i b(<_i) \subseteq b(\bigcup_i <_i)$, $b(\bigcup_i b(<_i)) \subseteq b(\bigcup_i <_i)$.

□

Our proof of the next proposition is simpler and more direct than that in [C, prop. 5.29] because we bypass the complicated result [C, 5.23].

3.6 Proposition [C]

For an arbitrary semitopogenous order, $<$, we have that $bs(<) = psb(<)$.

PROOF: By 2.6.9 (b) and 3.5 (b), $bs(<) = bq(< \cup c(<)) = b(< \cup c(<)) = b(b(<) \cup bc(<))$. But $b(b(<) \cup bc(<)) = pq(b(<) \cup bc(<))$. Indeed, let $A \; b(b(<) \cup bc(<)) \; B$, then for $x \in A$ and $y \in E - B$ $x \; (b(<) \cup bc(<)) \; E - y$ and so $x \; b(<) \; E - y$ or $x \; bc(<) \; E - y$. Let $E - B_1 = \{y \in E-B : x \; b(<) \; E-y\}$, let $E - B_2 = \{y \in E-B : x \; bc(<) \; E-y\}$ and suppose that neither $E - B_1 = \emptyset$ nor $E - B_2 = \emptyset$. It follows on account of 2.6.6 that $x \; b(<) \; B_1$ and $x \; bc(<) \; B_2$, giving $x \; (b(<) \cup bc(<)) \; B_1$ and $x \; (b(<) \cup bc(<)) \; B_2$. Therefore by 1.5.2 $x \; q(b(<) \cup bc(<)) \; B_1 \cap B_2 = B$. This holds for each $x \in A$ and so $A \; pq(b(<) \cup bc(<)) \; B$ (by 2.5.5). Thus $b(b(<) \cup bc(<)) \subseteq pq(b(<) \cup bc(<))$. The reverse inequality is easily seen to hold and the required equality follows.

Using 3.5 (a), the fact that $b(<)$ is a topogenous order and what has been proved above, $bs(<) = pq(b(<) \cup bc(<)) = pq(b(<) \cup cb(<)) = pqsb(<) = psb(<)$.

□

3.7 Proposition [C]

For any family A of topogenous orders on a set E , $bs(A) = psb(A) = sbs(A)$.

<u>PROOF</u>: That $bs(A) = psb(A)$ follows from 3.6. That $bs(A) = sbs(A)$ is seen as follows. Let $< \in A$, then by 3.5 (a) and the symmetry of $s(<)$, $cbs(<) = bcs(<) = bs(<)$ showing that $bs(<)$ is symmetrical. The result now follows at once from 2.4.2.

□

3.8 The proof of 3.2

<u>3.2.1</u>: If S is a topology then S is simple and perfect and so by 2.7.5 and 2.5.6 $pt(S) = p(S) = S$. On the other hand, if $pt(S) = S$ then $p(S) = ppt(S) = pt(S) = S$ and $t(S) = tpt(S) = pt(S)$, by 3.3, showing that S is perfect and simple.

<u>3.2.2</u>: It is easily seen that if S is a proximity then $ts(S) = S$. Conversely, if $ts(S) = S$ then $t(S) = tts(S)$ $= ts(S) = S$ and $s(S) = sts(S) = tss(S) = ts(S) = S$, using 3.4. Thus S is a proximity.

<u>3.2.3</u>: Suppose $bs(S) = S$. We have $b(S) = bbs(S) = bs(S) = S$. Also $s(S) = sbs(S) = bs(S)$.

□

Notice that if k is any one of pt, ts or bs then it is an ordinary operation. This is true, taking 2.2.2 into account, because k is idempotent: $ptpt = ptt = pt$ (3.3), $tsts = ttss = ts$ (3.4) and $bsbs = bbs = bs$ (3.7). We can conclude (2.2.6) that kSyn is a full, coreflective subcategory of Syn.

We must justify the terminology introduced in definition 3.1.

The basic ideas involved in the proofs of 3.9, 3.14 and 3.20 are due to [C] .

3.9 Proposition

Let k be the ordinary operation pt . The subcategory kSyn of Syn is isomorphic to the classical category of topological spaces and continuous maps.

PROOF: Let Top be the classical category of topological spaces and continuous maps. We must define an isomorphism, say, $T : \text{kSyn} \to \text{Top}$. Accordingly let S be an object of kSyn , $S = \{<\}$ and let f be a morphism of kSyn . Define T as follows.

3.10 $T(S) = S' = \{G \subseteq E : G < G\}$ and $T(f) = f$.

S' is a classical topology on E . In fact $\emptyset$ and $E \in S'$ since $\emptyset < \emptyset$ and $E < E$. S' is closed under finite intersections because $<$ is a topogenous order, and S' is closed under arbitrary unions because $<$ is perfect.

Next, T(f) is classically continuous. To see this let $f : V \to W$ in kSyn where $V = \{<_1\}$ and $W = \{<_2\}$ are defined on E and E' respectively. Denote T(V) and T(W) by V' and W' respectively and let H be open in W' . We must show that $f^{-1}(H)$ is open in V' . Now, $H <_2 H$ and $f^{-1}(<_2) \subseteq <_1$

giving $H <_2 H \subseteq E' - f(E-f^{-1}(H))$ since $H \cap f(E-f^{-1}(H)) = \emptyset$. However this implies that $f^{-1}(H) \; f^{-1}(<_2) \; f^{-1}(H)$ and therefore $f^{-1}(H) <_1 f^{-1}(H)$ meaning that $f^{-1}(H)$ is open in V'.

T is obviously a faithful covariant functor. To show that T is full we take a classically continuous function $f : V' \longrightarrow W'$ and show that it is continuous $V \rightarrow W$ where $V = \{<_1\}$ and $W = \{<_2\}$, as in the notation above. Accordingly let $A \; f^{-1}(<_2) \; B$, then $f(A) <_2 E' - f(E-B)$. Consider $G = \bigcup\{H : H <_2 E'-f(E-B)\}$. As $<_2$ is perfect, $G <_2 E' - f(E-B)$, and as $\{<_2\}$ is a syntopogenous structure, there is a set D satisfying $G <_2 D <_2 E' - f(E-B)$. However by the definition of G, $D \subseteq G$, so that $G = D$, $G <_2 G$ and G is open in W'. By the classical continuity of f, $f^{-1}(G)$ is open in V' which means, by definition of V', $f^{-1}(G) <_1 f^{-1}(G)$. We thus have that $A \subseteq f^{-1}f(A) \subseteq f^{-1}(G) <_1 f^{-1}(G) \subseteq f^{-1}(E'-f(E-B)) \subseteq B$, that is $A <_1 B$ as required.

To complete the proof we must show that T is one-to-one and onto for objects. We need the following construction. Given a classical topology S' on the set E, define a topology S in kSyn as follows.

3.11 $S = \{<\}$ where $A < B$ iff there is an open set $G \in S'$ satisfying $A \subseteq G \subseteq B$.

We verify that S is a topology. It is easily checked that $<$ is a perfect topogenous order. Next, let $A < B$ then there is an open set G satisfying $A \subseteq G \subseteq B$. According to 3.11 $A < G < B$ and S is a syntopogenous structure, as required.

To show that T is onto let S' be a classical topology and construct S by 3.11. Consider $T(S)$. $H \in T(S)$ implies $H < H$ and so there is $G \in S'$ with $H \subseteq G \subseteq H$ giving $H = G \in S'$, thus $T(S) \subseteq S'$. Also $G \in S'$ implies $G < G$ giving $G \in T(S)$ and so $S' \subseteq T(S)$ showing that $T(S) = S'$.

To show that T is one-to-one let $S_1 = \{<_1\}$ and $S_2 = \{<_2\}$ be topologies with $T(S_1) = T(S_2)$. We must show that $S_1 = S_2$. Let $A <_1 B$. The set $G = \bigcup\{H : H <_1 B\}$ satisfies $A \subseteq G <_1 G \subseteq B$. By 3.10 G is open in $T(S_1)$ and therefore also in $T(S_2)$. Using 3.10 again, it follows that $G <_2 G$ and thus $A <_2 B$. The argument is reversible and so $S_1 = S_2$.

□

We now consider proximity spaces. For ease of reference we recall the definition of a classical proximity space (cf. [C], [T]).

3.12 Definition

A relation R between the subsets of a set E is called a proximity structure on E iff

3.12.1 $A\ R\ B$ iff $B\ R\ A$.

3.12.2 $(A \cup B)\ R\ C$ iff at least one of $A\ R\ C$ or $B\ R\ C$ holds.

3.12.3 $x\ R\ x$ for all $x \in E$.

3.12.4 For any $A \subseteq E$, $A\ R\ \emptyset$ is false.

3.12.5 If $A\ R\ B$ is false then there are two sets P, Q such that $A\ R\ (E-P)$ is false, $B\ R\ (E-Q)$ is false and $P \cap Q = \emptyset$

3.13 Definition

Let (E,R) and (E',R') be classical proximity spaces. A function $f : E \to E'$ is p-continuous iff $A\ R\ B$ implies $f(A)\ R'\ f(B)$.

We are now ready to construct the equivalence between proximities and classical proximities.

3.14 Proposition

Let k be the ordinary operation ts. The subcategory kSyn of Syn is isomorphic to the classical category of proximity spaces and p-continuous maps.

PROOF: Let Prox be the classical category of proximity spaces and p-continuous maps. We must define an isomorphism $T : \text{kSyn} \to \text{Prox}$. To do this let S be a proximity on E with $S = \{<\}$ and let f be a morphism of kSyn. Define T by

3.15 $T(S) = R = \{(A,B) \subseteq E \times E : A \not< E-B$ and $T(f) = f$.

R is a classical proximity. In fact we prove that R satisfies 3.12.1 to 3.12.5.

3.12.1: $<$ is symmetrical.

3.12.2: We have $A \, R \, C$ false and $B \, R \, C$ false imply $A \cup B < E - C$ and so $(A \cup B) \; R \; C$ implies $A \, R \, C$ or $B \, R \, C$. On the other hand $A \, R \, C$ or $B \, R \, C$ implies $A \not< E - C$ or $B \not< E - C$, which implies $(A \cup B) \not< E - C$ and so $(A \cup B) \; R \; C$.

3.12.3: $x \qquad E - x$ is false.

3.12.4: $A < E = E - \emptyset$

3.12.5: If $A \, R \, B$ is false then $A < E - B$. Since $\{<\}$ is a syntopogenous structure there is P with $A < P < E - B$ and so $A \; R \; E - P$ is false. Also since $<$ is symmetric $B < E - P$. Putting $Q = E - P$ we have that $B \; R \; E - Q$ is false and clearly $P \cap Q = \emptyset$. This shows that R is a proximity.

Next, let $f : (E,S) \rightarrow (E',S')$ and put $S = \{<\}$, $S' = \{<'\}$, $T(S) = R$ and $T(S') = R'$. We must show that $T(f) = f$ is clasically p-continuous. In fact, if $f(A) \; R' \; f(B)$ is false then $f(A) <' E' - f(B)$ giving $A \; f^{-1}(<') \; E - B$, and so $A < E - B$ by continuity of f in kSyn. This means that $A \, R \, B$ is false, as required.

Clearly T is a faithful, covariant functor. T is also full: let $f : (E,R) \rightarrow (E,R')$ be classically p-continuous; we must show that f is continuous in kSyn . Let $A \ f^{-1}(<') \ B$, then $f(A) <' E' - f(E-B)$ and so by 3.15 $f(A) \ R' \ f(E-B)$ is false. By p-continuity of f , $A \ R \ E - B$ is false, and again by 3.15 $A < B$, as required.

To show that T is onto for object we first give a method of constructing a proximity from a classical proximity. Let R be a classical proximity on E . Define $\{<\}$ on E by

3.16 $\quad A < B$ iff $A \ R \ E - B$ is false.

We show that $<$ is a symmetrical topogenous order on E . In view of 3.12.4 $\emptyset < \emptyset$ and $E < E$. Next, $<$ is symmetrical since if $A < B$ then $A \ R \ E - B$ is false and so according to 3.12.1 $(E-B) \ R \ A$ is false showing that $E - B < E - A$. Next, if $A \subseteq A' < B' \subseteq B$ then $A' \ R \ (E-B')$ is false. Now $A \cup A' = A'$ thus in view of 3.12.2 $A \ R \ (E-B')$ is false and in view of 3.12.1 $(E-B') \ R \ A$ is false. But $E - B' = (E-B') \cup (E-B)$ and so again $(E-B) \ R \ A$ is false, that is $A \ R \ (E-B)$ is false and $A < B$ as required. To show that $A < B$ implies $A \subseteq B$, suppose on the contrary $x \in A \cap (E-B)$. Now, $\{x\} \subseteq A < B \subseteq E - \{x\}$ so that by what has just been proved $x < E - x$ which means that $x \ R \ x$ is false, contradicting 3.12.3.

Now suppose $A < B$ and $A' < B'$ then $A < B \subseteq B \cup B'$ giving $A < B \cup B'$ and similarly $A' < B \cup B'$. Thus $A \ R \ (E-(B \cup B'))$ is false and $A' \ R \ (E-(B \cup B'))$ is false which by 3.12.2 means that $(A \cup A') \; R \; (E-(B \cup B'))$ is false and so $A \cup A' < B \cup B'$. Next, since $<$ is symmetrical, $E - B < E - A$ and $E - B' < E - A'$, therefore $E - (B \cap B') = (E-B) \cup (E-B') < (E-A) \cup (E-A') = E - (A \cap A')$ and again by symmetry of $<$, $A \cap A' < B \cap B'$. We have shown that $<$ is a symmetrical topogenous order.

To show that $\{<\} = \{<^2\}$, let $A < B$. By 3.12.5 there are P, Q with $A \; R \; E - P$ false and $E - B \; R \; E - Q$ false and $P \cap Q = \emptyset$. Thus $A < P$ and $E - B < Q$, that is $A < P \subseteq E - Q < B$ or $A < P < B$ as required.

T is onto. Indeed, let R be a classical proximity on E. Let $<$ be defined by 3.16 then $T\{<\} = R$.

To complete the proof we must show that T is one-to-one on objects. Let $S_1 = \{<_1\}$ and $S_2 = \{<_2\}$ be proximities with $T(S_1) = T(S_2)$. We must show that $S_1 = S_2$. Let $A <_1 B$. If $A = \emptyset$, then trivially $A <_1 B$ iff $A <_2 B$. If $A \neq \emptyset$ then $A <_1 E - B$ is false, and so $(A,B) \in T(S_1) = T(S_2) = R$, say. Let $<$ be defined by 3.16. As has been shown above $T\{<\} = R$. But $A <_1 B$ iff $A < B$ iff $A <_2 B$, and thus $S_1 = S_2$.

□

Before explicating the equivalence between uniformities and classical uniformities we examine the nature of reflexive relations. The next result is basic and interesting in that it describes reflexive relations in terms of biperfect topogenous orders.

3.16 **Proposition [C]**

(a) Let $<$ be a biperfect topogenous order on the set E and define a relation R on E by

3.17 $x \, R \, y$ iff $x < E - y$ is false

then R is a reflexive relation and the following equivalence holds

3.18 $A < B$ iff whenever $x \in A$ and $x \, R \, y$, then $y \in B$.

(b) Let R be a reflexive relation on E and define $<$ by 3.18. Then $<$ is a biperfect topogenous order on E and 3.17 holds.

PROOF:

(a): If $<$ is a biperfect topogenous order on E then R defined by 3.17 is reflexive since $x < E - x$ is false. To show that 3.18 is true, suppose $A < B$, $x \in A$ and $x \, R \, y$, then $x \not< E - y$ and so $y \notin E - B$ by the biperfectness of $<$. Conversely, suppose $x \in A$, $x \, R \, y$ imply $y \in B$ and let $y \in E - B$. If $x < E - y$ is false then $x \, R \, y$ giving $y \in B$ which is a contradiction, therefore $x < E - y$ and biperfectness of $<$ give $A < B$.

(b): Suppose R is a reflexive relation. Let $<$ be defined by 3.18 then $<$ is a topogenous order. In fact $\emptyset < \emptyset$ by default and clearly $E < E$. Next, suppose $A < B$.

Since R is reflexive $x R x$ and so $A \subseteq B$. If $A \subseteq A' < B' \subseteq B$, $x \in A$ and $x R y$ then $y \in B'$ and therefore $y \in B$, whence $A < B$. It is easily seen that $A < B$ and $A' < B'$ imply $A \cap A' < B \cap B'$ and $A \cup A' < B \cup B'$ and so $<$ is a topogenous order.

To show that $<$ is biperfect let $x < E - y$ for $x \in A$ and $y \in E - B$. We must show that $A < B$. However since $x < E - y$, 3.18 implies that $x R y$ is false. Thus supposing $x \in A$ and $x R y$ it follows that $y \in B$ showing that $A < B$ as required. It is easily seen that 3.17 holds.

$\square$

The next two propositions describe the link between biperfect syntopogenous structures and uniformities and provide their equivalence.

3.19 Proposition [C]

(a) Let S be a biperfect syntopogenous structure on the set E, let $< \in S$, denote the reflexive relation corresponding to $<$ by $U_<$ (cf. 3.16(a)) and let $\mu = \{ U_< : < \in S \}$, then μ satisfies the conditions 3.19.1 to 3.19.3 below.

3.19.1 $U \in \mu$ implies U is reflexive.

3.19.2 $G, H \in \mu$ imply that there exists $U \in \mu$ with $U \subseteq G \cap H$

3.19.3 Given $H \in \mu$ there is $U \in \mu$ with $U^2 \subseteq H$.

The filter generated by μ is called a <u>quasi-uniformity</u> on E, and μ is called the <u>base</u> for this quasi-uniformity.

(b) Let μ be a quasi-uniform base on E, let $U \in \mu$, denote the biperfect topogenous order corresponding to U by $<_U$ (cf. 3.16 (b)) and let $S = \{<_U : U \in \mu\}$. Then S is a biperfect syntopogenous structure on E.

(c) Let the biperfect topogenous order $<_U$ and the reflexive relation U be related as in 3.17 and 3.18. Then $<_U$ is symmetrical iff U is symmetrical.

<u>PROOF</u>:

(a)

<u>3.19.1</u>: By 3.16 (a), if $U \in \mu$ then U is reflexive.

<u>3.19.2</u>: First we show that

3.19.4 $<_1 \subseteq <_2$ implies $U_{<_2} \subseteq U_{<_1}$.

Indeed if $x \; U_{<_2} \; y$ then $x \not<_2 E - y$ giving $x \not<_1 E - y$ and so $x \; U_{<_1} \; y$, as required. Next, suppose $G, H \in \mu$, $G = U_<$ and $H = U_{<'}$. Let $<'' \in S$ be such that $< \cup <' \subseteq <''$ and put $U = U_{<''}$. By 3.19.4 $U \subseteq G$ and $U \subseteq H$ giving $U \subseteq G \cap H$.

3.19.3: Let $H \in \mu$ with $H = U_{<'}$. There is $< \in S$ with $<' \subseteq <^2$. Let $U = U_<$. We show that $(U_<)^2 = U_{<}2$. Indeed, if $x \ (U_<)^2 \ y$ then there is a z such that $x \ U_< \ z$ and $z \ U_< \ y$, which means that $x \not< E - z$ and $z \not< E - y$. If $x \ U_{<}2 \ y$ is false then $x <^2 E - y$, therefore there exists D with $x < D < E - y$. Now if $z \in D$ then $z < E - y$ which is impossible, and if $z \notin D$ then $x < D \subseteq E - z$ which means that $x < E - z$, also impossible, thus $x \ U_{<}2 \ y$ is true and $(U_<)^2 \subseteq U_{<}2$. On the other hand suppose $x \ U_{<}2 \ y$, then $x <^2 E - y$ is false by 3.17. Thus no set D satisfies the condition $x < D < E - y$. Let $D = \{z : z < E-y\}$. Since $<$ is biperfect $D < E - y$ and therefore $x \not< D$. Again since $<$ is biperfect and $x \not< D$ there is $z \in E - D$ with $x \not< E - z$. According to 3.17 $x \ U_< \ z$ and $z \ U_< \ y$, which shows that $x \ (U_<)^2 \ y$. We have proved that $(U_<)^2 = U_{<}2$.

On account of 3.19.4 it follows that $(U_<)^2 = U_{<}2 \subseteq U_{<'}$, as required.

(b): By 3.16 all the members of S are biperfect topogenous orders on E. To show that S is directed consider $<_G, <_H \in S$ and let $U \in \mu$ be such that $U \subseteq G \cap H$, then $<_G \subseteq <_U$ and $<_H \subseteq <_U$, as required.

Next, given $<_U \in S$ let $G \in \mu$ be such that $G^2 \subseteq H$. We have that $<_U \subseteq <_{G}2$.

We show that $<_{G^2} = (<_G)^2$ and then it will follow that $<_U \subseteq (<_G)^2$ and thus S will be a syntopogenous structure, as required. In fact, we first prove 3.19.5 and 3.19.6 below.

Let G be a given reflexive relation, let $<$ be defined by using G in 3.18 and let $U_<$ be defined by using $<$ in 3.17. Then

3.19.5 $U_< = G$

This can be seen as follows: $x \; U_< \; y$ iff $x \not< E - y$ by 3.17. By 3.18 $x \not< E - y$ iff $x \, G \, z$ and $z \notin E - y$, that is $x \, G \, z$ and $z = y$. So $x \; U_< \; y$ iff $x \, G \, y$.

Next, let $<'$ be a given biperfect topogenous order, let R be defined by using $<'$ in 3.17 and let $<_R$ be defined by using R in 3.18, then

3.19.6 $<_R \; = \; <'$

This is so because by 3.18 $A <_R B$ iff $x \in A$, $x R y$ imply $y \in B$. By 3.17 this is equivalent to: $y \in B$ whenever $x \in A$ and $x \not<' E - y$. This is the case iff $x \in A$, $y \in E - B$ imply $x <' E - y$. Since $<'$ is biperfect it follows that $A <_R B$ iff $A <' B$.

Finally, set $<_G = <$ and $U_{<2} = R$. It was shown in the proof of 3.19.3 that $U_{<2} = (U_<)^2$ and so by 3.19.5 we have that $R = U_{<2} = (U_<)^2 = G^2$, and by 3.19.6 we have that $(<_G)^2 = <^2 = <_R = <_{G^2}$.

(c) Supposing U symmetrical, $y < E - x$ is false iff $y\, U\, x$ iff $x\, U\, y$ iff $x < E - y$ is false iff $y\ c(<)\ E - x$ is false. The proof of the converse is similar.

□

3.20 Proposition

Let k be the elementary operation bs. The subcategory kSyn of Syn is equivalent to the classical category of uniform spaces and uniformly continuous maps.

PROOF: Denote the category of uniform spaces and uniformly continuous maps by Unif. Define a covariant functor $T : \text{kSyn} \rightarrow \text{Unif}$ by taking

$T(S)$ to be the filter generated by the filter base $\{U_< : < \in S\}$, where $U_<$ is given by 3.17, and if $f : (E,S) \rightarrow (E',S')$ is a morphism in kSyn with $T(S) = U$ and $T(S') = U'$, define $T(f) = f$.

By 3.19 (a) and (c) $T(S)$ is a uniformity on E. T is obviously faithful. To show that T is full let $f : U \rightarrow U'$ be clasically uniformly continuous and let $<' \in S'$. We must produce $< \in S$ with $f^{-1}(<') \subseteq <$. Since $<'$, $<$, and hence $f^{-1}(<')$, are biperfect (using 2.2.1 (f)), it is enough to show that there is $< \in S$ with $x \; f^{-1}(<') \; E - y$ implying $x < E - y$. Accordingly let $x \; f^{-1}(<') \; E - y$ then $f(x) <' E' - f(y)$, so that by 3.17 $f(x) \; U_{<'} \; f(y)$ is false. Because f is classically uniformly continuous there is a base member $U_< \in U$ with $< \in S$ satisfying $(f \times f)(U_<) \subseteq U_{<'}$. This implies that $x \; U_< \; y$ is false and $x < E - y$ follows from 3.17 as required.

T is representative: let W be a member of a uniformity and let $< \; = <_W$ then by 3.19.5 $U_< = W$, and 3.19 (b), (c) complete the proof.

□

The operation pt defines a functor, denoted again by pt : Syn $\rightarrow$ Syn, which corresponds to the forgetful functors to <u>Top</u>, in the following sense.

3.21 <u>Proposition</u>

Let C be one of the classical categories <u>Quasiunif</u>, <u>Prox</u>, F : C $\rightarrow$ <u>Top</u> the usual forgetful functor, and G : C $\rightarrow$ Syn, H : <u>Top</u> $\rightarrow$ Syn the embedding functors established above. Then the diagram

$$\begin{tikzcd}
C \arrow[r, "G"] \arrow[d, "F"'] & \mathrm{Syn} \arrow[d, "pt"] \\
\underline{\mathrm{Top}} \arrow[r, "H"'] & \mathrm{Syn}
\end{tikzcd}$$

commutes.

PROOF: If C is Quasiunif , let μ be a quasiuniformity on the set E . Let $F :$ Quasiunif $\rightarrow$ Top be the forgetful functor defined by taking basic neighbourhoods of points as follows: for each $x \in E$ and each $U \in \mu$, $U(x) = \{y : x\, U\, y\}$ is a basic neighbourhood of x in $F(\mu)$.

Use 3.19 (b) to define G and 3.11 to define H .

If C is Prox , let R be a proximity on the set E . Let $F :$ Prox $\rightarrow$ Top be the forgetful functor defined by taking basic neighbourhoods of points as follows: for each $x \in E$, W is a basic neighbourhood of x in $F(R)$ iff $x\ R\ E - W$ is false. Use 3.16 to define G and 3.11 to define H .

□

CHAPTER 4

COMPLETENESS, COMPACTNESS AND QUASIREALCOMPACTNESS

This chapter will provide the definitions of completeness and compactness for arbitrary objects of Syn . The terminology agrees with classical terminology on subcategories of Syn respectively equivalent to the category of topological spaces and the category of uniformities, as will be shown in Chapter 6. We have here independently introduced the notion of quasirealcompactness for arbitrary objects in Syn . In Chapter 6 we use quasirealcompactness to define realcompactness. Various reflections will be produced in Chapter 7.

Convergence will be discussed through the medium of grills, which are just filter bases.

4.1 Definition [C]

Let E be a set. A family G of subsets of E is called a grill on E if it is not empty and satisfies

4.1.1 $R \in G$ implies $R \neq \emptyset$

4.1.2 $R_1 , R_2 \in G$ imply that there is $R \in G$ such that $R \subseteq R_1 \cap R_2$.

If G and G' are grills on E then we define G' to be <u>finer</u> than G iff each set $R \in G$ contains a member of G'.

Let G be a non-void class of subsets of E with the finite intersection property. The grill generated by G is defined to be the class of all finite intersections of members of G.

We need the following two well-known results on grills.

4.2 Proposition

Let G be a maximal class of subsets of the set E with the finite intersection property. Then

(a) G is a grill

(b) If A is a set which meets every member of G, then $A \in G$. $\square$

4.3 Proposition

If G is a family of subsets of E with the finite intersection property then there is a maximal family with the finite intersection property containing G. $\square$

4.4 Definition [C]

Let S be a syntopogenous structure on the set E and let $x \in E$.

(a) A subset V of E is a neighbourhood of x iff there is $< \in S$ such that $x < V$.

(b) The grill G on E converges to x in S, in symbols $G \to x(S)$, iff each neighbourhood of x contains a member of G .

4.5 Proposition [C]

Let $x \in E$ and let S be a syntopogenous structure on E . The set of all neighbourhoods of x is a grill.

PROOF: Let $x <_1 V_1$, $x <_2 V_2$ and $<_1 \cup <_2 \subseteq <$, then $x < V_1 \cap V_2$.

$\square$

4.6.1 Proposition [C]

(a) The neighbourhoods of a point x of the set E are the same in each of the syntopogenous structures S , $t(S)$, $p(S)$ and $pt(S)$.

(b) The following formulae are equivalent : $G \to x(S)$, $G \to x(t(S))$, $G \to x(p(S))$ and $G \to x(pt(S))$.

(c) If $S \leq S'$, then $G \to x(S')$ implies $G \to x(S)$.

$\square$

4.6.2 Proposition

Classical convergence in topological, proximity and quasiuniform spaces coincides with convergence in the equivalent syntopogenous structures.

PROOF: 3.21. $\square$

4.7 Definition [C]

Let $E \subseteq E'$ and let S' be a syntopogenous structure on E'. Then E is <u>dense</u> in E' iff $x \in E'$ implies that there is a grill G in E converging to x in S'.

We now define the notion of a Cauchy grill and show that it coincides with the classical notion for uniform spaces.

4.8 Definition [C]

Let S be a syntopogenous structure on E and let G be a grill on E, then G is <u>Cauchy</u> in S iff it satisfies:

4.8.1 given $< \in S$, there is $R \in G$ such that for all $A, B \subseteq E$, if $A < B$ then $A \cap R \neq \emptyset$ and $(E-B) \cap R \neq \emptyset$ cannot both hold.

An equivalent formulation is that

4.8.2 given $< \in S$, there is $R \in G$ such that for all $A, B \subseteq E$, $A < B$ and $A \cap R \neq \emptyset$ imply $R \subseteq B$.

The next proposition shows that the two terminologies coincide.

4.9 Proposition [C]

Let S be a uniformity on E. The grill G is a Cauchy grill in S iff for any entourage U in the classical uniformity associated with S (cf. 3.20) there is $R \in G$ satisfying: $x, y \in R$ implies $x\,U\,y$ (i.e., G is Cauchy in the classical sense).

PROOF: Suppose $x, y \in R$ imply $x\,U\,y$ and let $<$ be associated with U according to 3.17. If $A < B$ and $x \in A$, $y \in E - B$ then $x < E - y$ since S is biperfect, and so $x\,U\,y$ is false. Thus R cannot meet both A and $E - B$.

On the other hand suppose G is Cauchy and suppose $x, y \in R$ then since $<$ is biperfect we have that $x \not< E - y$. Thus $x\,U\,y$ and the proof is complete.

$\square$

We now prove that, in fact, a convergent grill in a uniformity is necessarily Cauchy.

4.10 Proposition [C]

Let S be a uniformity on the set E, let μ be the classical uniformity associated with S. If a grill G converges in S then it is Cauchy in S.

PROOF: Suppose $G \rightarrow x(S)$. Let $U \in \mu$. There is a member U' of μ with $(U')^2 \subseteq U$. Let $<'$ be the topogenous order corresponding to U' according to 3.18. Consider $V = \{y : x\,U'\,y\}$.

According to the definition of V, $x\ U'\ y$ implies $y \in V$ and so by 3.18 $x <' V$. V contains a set $R \in G$. But $r,s \in R$ implies $r,s \in V$ and hence $x\ U'\ r$ and $x\ U'\ s$. By symmetry of U', $r\ (U')^2\ s$ whence $r\ U\ s$. Then by 4.9 G is Cauchy.

□

Our proof of the next proposition differs from that of [C].

4.11 Proposition [C]

The Cauchy grills in the syntopogenous structures S, $bs(S)$, $c(S)$, $s(S)$, $p(S)$ and $b(S)$ are all the same.

PROOF: Suppose G is a Cauchy grill in $bs(S)$, then $A < B$ and $< \in S$ imply that $A\ bs(<)\ B$ and hence G is a Cauchy grill in S.

On the other hand suppose G is Cauchy in S, $< \in S$ and R satisfies 4.8.1. Now since $bs(<)$ is biperfect $A\ bs(<)\ B$ and $A \cap R \neq \emptyset \neq (E-B) \cap R$ imply for $x \in A \cap R$, $y \in (E-B) \cap R$ that $x\ s(<)\ E-y$. Thus $x\ q(< \cup c(<))\ E-y$. It is easily seen using 1.5.3 that then $x\ (< \cup c(<))\ E-y$.
If $x < E-y$ then by 4.8.2 $R \subseteq E-y$ which is a contradiction.
If $x\ c(<)\ E-y$ then $y < E-x$ and by 4.8.2 $R \subseteq E-x$ which is a contradiction. It follows that G is Cauchy in $bs(S)$.

The proof is completed, on account of 2.4.2 (b), 3.7 and 2.6.9 (a), by the following equalities: $bs(cS) = bs(S)$, $bs(sS) = bs(S)$, $bs(pS) = psbp(S) = psb(S) = bs(S)$, $bs(bS) = psbb(S) = psb(S) = bs(S)$.

□

The following proposition complements 4.6 (b).

4.12 Proposition [C]

If G is a Cauchy grill in the syntopogenous structure S, then $G \to x(S)$ iff $G \to x(b(S))$.

PROOF: Suppose $G \to x(b(S))$, then $G \to x(S)$ by 4.6 (c).

Conversely, suppose $G \to x(S)$ and suppose $x \; b(<') \; V$ for $<' \in S$ and let $<' \subseteq <^2$ where $< \in S$. Also, let $R \in G$ satisfy the Cauchy condition 4.8.2 for $<$. Now if $y \in R - V$ then $V \subseteq E - y$ and $x \; b(<') \; E - y$ which implies that $x <' E - y$ and so $x < D < E - y$ for a suitable set D. But this implies that $D \cap R = \emptyset$, which is impossible if $G \to x$. Thus $R \subseteq V$.

□

4.13 Definition [C]

A syntopogenous structure S is complete iff every Cauchy grill in S converges in S.

4.14 Proposition [C]

Let S be a syntopogenous structure.

(a) If any of $s(S)$, $p(S)$ or $b(S)$ is complete then so is S.

(b) If S is complete then so are $p(S)$ and $b(S)$.

(c) A simple syntopogenous structure is always complete.

4.17 Any Cauchy grill is compressed.

$\square$

Our proof of the next proposition is different from that in [C].

4.18 Proposition [C]

Let S be a syntopogenous structure. The compressed grills of the structures S , ts(S) , c(S) , s(S) and t(S) are all the same.

PROOF: Suppose G is a compressed grill in $ts(S) = \{<'\}$, say. If $A < B$ for each $< \in S$, then $A <' B$ and therefore G is a compressed grill in S .

On the other hand suppose G is a compressed grill in S . If $A <' B$ then there is $< \in S$ such that $A \ s(<) \ B$, that is $A \ q(< \cup c(<)) \ B$. Thus there are finite index sets I , J such that $A = \bigcup \{A_i : i \in I\}$, $B = \bigcap \{B_j : j \in J\}$ and $A_i \ (< \cup c(<)) \ B_j$. Now either $A_i < B_j$ or $E - B_j < E - A_i$. In either case, since G is compressed in S , there is $R_{i,j} \in G$ such that $R_{i,j}$ meets at most one of A_i , $E - B_j$. Let $R \in G$ be such that $R \subseteq \bigcap \{R_{i,j} : (i,j) \in I \times J\}$. If $R \cap A \neq \emptyset$ then for an index $r \in I$ $R \cap A_r \neq \emptyset$ and so for each $j \in J$ $R \subseteq B_j$ giving that $R \subseteq \bigcap B_j = B$. Thus G is compressed in ts(S) .

The remainder of the proposition follows from the equalities $ts(cS) = ts(S)$, $ts(sS) = ts(S)$, $ts(tS) = tts(S) = ts(S)$ (cf. 2.4.2 (b), 3.3 and 3.4).

$\square$

We now define compactness and quasirealcompactness.

4.19 <u>Definition</u>

Let S be a syntopogenous structure on the set E .

(a) [C] . S is <u>compact</u> iff each compressed grill in S converges in S .

(b) S is <u>quasirealcompact</u> iff each compressed grill in S with the countable intersection property converges in S .

The terminology will be 'justified' in Chapter 6.

The following proposition is important; it has been proved in [C] for compact structures.

4.20 <u>Proposition</u>

S is (quasireal)compact iff t(S) is (quasireal)compact.

<u>PROOF</u>: By 4.18 the compressed grills of S and t(S) are the same, and by 4.6 (b) a grill converges in S iff it converges in t(S) .

□

The following proposition has been proved in [C] for compact structures.

4.21 Proposition

If the syntopogenous structure S is (quasireal)compact then $p(S)$ and $pt(S)$ are also (quasireal)compact.

PROOF: Since $S \leqslant p(S)$ a compressed grill in $p(S)$ is also compressed in S, and by 4.6 (b) a grill converges in S iff it converges in $p(S)$. Thus if S is (quasireal)compact, so is $p(S)$. Also, if S is (quasireal)compact, so is $t(S)$ (4.20) and thus by what has just been proved, $pt(S)$ is (quasireal)compact.

□

For compact structures the converse to the above proposition is also true (see 6.4).

The following definition has been given in [C] for doubly complete and for doubly compact structures.

4.22 Definition

Let S be a syntopogenous structure. S is doubly complete, doubly compact or doubly quasirealcompact iff $s(S)$ is respectively complete, compact or quasirealcompact.

4.23 Proposition [C]

(a) If S is doubly complete then it is complete.

(b) S is doubly complete iff $bs(S)$ is complete.

(c) If one of the structures S, $c(S)$, $s(S)$, $p(S)$, $b(S)$ or $bs(S)$ is doubly complete then so are all the others.

PROOF: (a) and (b) follow from 4.14 (a) and (b). (c) follows from (b) and the equalities $bsc(S) = bs(S)$, $bss(S) = bs(S)$, $bsp(S) = psbp(S) = psb(S) = bs(S)$, $bsb(S) = psbb(S) = psb(S) = bs(S)$, $bsbs(S) = bbs(S) = bs(S)$.

□

The following proposition has been proved for compact structures in [C] .

4.24 Proposition

(a) If S is doubly (quasireal)compact then it is (quasireal)compact.

(b) S is doubly (quasireal)compact iff $ts(S)$ is (quasireal)compact.

(c) If one of the structures S , $c(S)$, $s(S)$, $t(S)$ or $ts(S)$ is doubly (quasireal)compact then so are the others.

PROOF: (a) and (b) follow from 4.18 and 4.6 (b) and (c). (c) follows from (b) and the equalities $ts(cS) = ts(S)$, $ts(sS) = ts(S)$ $ts(tS) = tts(S) = ts(S)$, $ts(tsS) = stts(S) = sts(S) = st(S) = ts(S)$.

□

CHAPTER 5

STRUCTURES ON THE REAL NUMBERS AND REAL-VALUED CONTINUOUS FUNCTIONS

in this chapter we examine certain syntopogenous structures defined on the real numbers. Then we study certain initial, i.e. weak, structures induced by real valued continuous functions. We allow arbitrary functions in these initial structures (cf. 5.2.5). This is an important departure from the approach in [C] , which allows only bounded functions, because it allows us to develop the theory of realcompactness in the sequel. Also of special interest is the analogue of Urysohn's lemma, proposition 5.2.10. The chapter ends with the definition of the uniformities induced by quasipseudometrics or quasiecarts as we shall call them following the usage in [C] .

5.1 Syntopogenous structures on the real numbers

Let E denote the set of real numbers.

5.1.1 The structure T

The orders $<_\varepsilon$ were defined in 1.2, and shown in 2.6.2 to be biperfect. Let T = $\{<_\varepsilon : \varepsilon > 0\}$.

One readily sees that T is a biperfect syntopogenous structure. Hence T is a quasi-uniformity on the real numbers (3.19 (a)). The nature of the basic entourages in the associated classical quasi-uniformity can be seen as follows: let U_ε correspond to $<_\varepsilon$ according to 3.17, then $x \, U_\varepsilon \, y$ iff $x \not<_\varepsilon E - y$. This is the case iff $\sup\{x\} + \varepsilon > \inf\{E-(E-y)\} = y$. That is

$$U_\varepsilon = \{(x,y) : y-x < \varepsilon\}$$

5.1.2 The topology pt(T)

First, by 2.7.3, $A \; t(T) \; B$ iff $\sup A < \inf (E-B)$. Next, $A \; pt(T) \; B$ iff $x \in A$ implies $x < \inf (E-B)$. The set G is open in this topology iff $G \; pt(T) \; G$, that is iff $x \in G$ implies $x < \inf (E-G)$, thus

G is open iff $G = (-\infty, p)$ for some real number p.

5.1.3 The uniformity $H = bs(T)$

H is equivalent to the natural uniformity on the real numbers.

Consider first the order $s(<_\varepsilon) \in s(T)$ for some positive real number ε. Our proof of the next proposition differs from that in [C].

Proposition [C]

$A \; s(<_\varepsilon) \; B$ iff there is a finite decomposition of A consisting of nonempty sets A_i $(i \in I)$ with $A = \bigcup\{A_i : i \in I\}$ and for each $i \in I$ $(\inf A_i - \varepsilon, \sup A_i + \varepsilon) \subseteq B$.

PROOF: Suppose $A \; s(<_\varepsilon) \; B$, then there are finite index sets I, J and sets A_i, B_j such that $A = \bigcup\{A_i : i \in I\}$, $B = \bigcap\{B_j : j \in J\}$ and for $(i,j) \in I \times J$ $A_i \; q(<_\varepsilon \cup c(<_\varepsilon)) \; B_j$. Now either $A_i <_\varepsilon B_j$ or $E - B_j <_\varepsilon E - A_i$. In the first case $\sup A_i + \varepsilon \leqslant \inf (E-B_j)$ and in the second case $\sup (E-B_j) + \varepsilon \leqslant \inf A_i$. This holds for each $j \in J$ and so $(\inf A_i - \varepsilon, \sup A_i + \varepsilon) \subseteq B_j$ for each $j \in J$; but $\bigcap B_j = B$.

Conversely, suppose that $A = \bigcup A_i$ and for each $i \in I$ $(\inf A_i - \varepsilon, \sup A_i + \varepsilon) \subseteq B$. For each $i \in I$ set $B_i = B \cup (-\infty, \sup A_i + \varepsilon)$ and $B'_i = B \cup (\inf A_i - \varepsilon, +\infty)$.

Now, $A = \bigcup A_i$, $B = \bigcap\{B_i \cap B'_i : i \in I\}$ and $A_i <_\varepsilon B_j$, $E - B'_j <_\varepsilon E - A_i$ for $(i,j) \in I \times I$. Thus $A_i \; (<_\varepsilon \cup c(<_\varepsilon)) \; B_j$ and $A_i \; (<_\varepsilon \cup c(<_\varepsilon)) \; B'_j$ so that $A_q \; (<_\varepsilon \cup c(<_\varepsilon)) \; B$, as required. □

Now we can say that $A \; bs(<_\varepsilon) \; B$ iff $x \in A$, $y \in E - B$ imply $x \; s(<_\varepsilon) \; E - y$. This is the case in view of the above theorem, iff $(x - \varepsilon, x + \varepsilon) \subseteq E - y$, which is equivalent to $|y-x| \geqslant \varepsilon$.

If for subsets A, B of E $d(A,B)$ is defined by $d(A,B) = \inf\{|y-x| : x \in A, y \in B\}$ then $A \; bs(<_\varepsilon) \; B$ iff $d(A,E-B) \geqslant \varepsilon$.

If U_ε is the basic entourage in the classical uniformity associated with $bs(<_\varepsilon)$ then $x \; U_\varepsilon \; y$ iff $x \; bs(<_\varepsilon) \; E - y$

is false, which means that

$$U_{\varepsilon} = \{(x,y) : |y-x| < \varepsilon\}.$$

5.1.4 The proximity $t(H) = tbs(T)$

According to 2.7.3 $t(H) = \{\cup\{bs(<_{\varepsilon}) : \varepsilon > 0\}\}$. Thus if d is defined as in 5.1.3 then $A\ t(H)\ B$ iff $d(A,E-B) > 0$.

If P is the classical proximity associated with $t(H)$ according to 3.15 then $A\ P\ B$ iff $A\ t(H)\ E-B$ is false, which is equivalent to

$$A\ P\ B \quad \text{iff} \quad d(A,B) = 0 .$$

5.1.5 The topology $pt(H) = ptbs(T)$

The topology $ptbs(T)$ is equivalent to the natural topology on the real numbers.

According to 6.1.4 $A\ pt(H)\ B$ iff $x \in A$ implies $d(x,E-B) > 0$.

G is open in the classical topology associated with $pt(H)$ iff $G\ pt(H)\ G$, that is iff $x \in G$ implies $d(x,E-G) > 0$. Thus G is open iff $x \in G$ implies there is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq G$.

5.2 Structures defined in terms of real-valued continuous functions

From now on let T denote the structure $\{<_\varepsilon : \varepsilon > 0\}$ mentioned in 5.1.1 and let R denote the set of real numbers.

The next two propositions are important.

5.2.1 Proposition [C]

Let $f : E \to R$ be a function, then $A \; f^{-1}(<_\varepsilon) \; B$ iff there is a number p, $-\infty \leqslant p \leqslant +\infty$, satisfying $\sup f(A) \leq p$ and $\inf f(E-b) \geqslant p + \varepsilon$.

PROOF: $A \; f^{-1}(<_\varepsilon) \; B$ iff $f(A) <_\varepsilon R - f(E-B)$, which is the case iff $\sup f(A) + \varepsilon \leqslant \inf f(E-B)$. Set $p = \sup f(A)$. Then there is p such that $f(A) \subseteq (-\infty, p)$ and $(-\infty, p+\varepsilon) \subseteq R - f(E-B)$.

On the other hand an easy computation shows that if there is p such that $f(A) \subseteq (-\infty, p)$ and $(-\infty, p+\varepsilon) \subseteq R - f(E-B)$, then $\sup f(A) + \varepsilon \leqslant \inf f(E-B)$. Thus $\sup f(A) + \varepsilon \leqslant \inf f(E-B)$ iff there is p such that $f(A) \subseteq (-\infty, p)$ and $(-\infty, p+\varepsilon) \subseteq R - f(E-B)$.

$\square$

5.2.2 Proposition [C]

Let S be a syntopogenous structure on the set E and let $f : E \to R$ be a function, then f is (S,T)-continuous iff for each real number p and each $\varepsilon > 0$ there is an order $< \in S$ such that $f^{-1}(-\infty,p] < f^{-1}(-\infty,p+\varepsilon)$.

PROOF: Suppose f is continuous, let p be a real number and let $\varepsilon > 0$ be given. According to 5.2.1 the sets $A = f^{-1}(-\infty,p]$ and $B = f^{-1}(-\infty,p+\varepsilon)$ satisfy $A \; f^{-1}(<_{\varepsilon}) \; B$. By continuity of f there is $< \in S$ such that $A < B$, as required.

Conversely, suppose that for all real numbers r , given $\varepsilon > 0$ there is $< \in S$ such that $f^{-1}(-\infty,p] < f^{-1}(-\infty,p+\varepsilon)$. Let $A \; f^{-1}(<_{\varepsilon}) \; B$, then there is a real number p satisfying $f(A) \leq p$ and $f(E-B) \geqslant p+\varepsilon$. Now using 5.2.1

$$A \subseteq f^{-1}f(A) \subseteq f^{-1}(-\infty,p] < f^{-1}(-\infty,p+\varepsilon) \subseteq f^{-1}(R-f(E-B)) \subseteq B .$$

□

The following is a useful result on real-valued continuous functions.

5.2.3 Proposition [C]

Let S be a syntopogenous structure and suppose that $f : S \rightarrow T$ is continuous, then

(a) If $r > 0$ then the function $rf : S \rightarrow T$ is continuous.

(b) If $r < 0$ then the function $rf : S \rightarrow c(T)$ is continuous.

(c) If $f : S \rightarrow s(T)$ is continuous then $rf : S \rightarrow s(T)$ is continous for any real number r .

PROOF:

(a): $A\ (rf)^{-1}(<_{\varepsilon})\ B$ implies by 5.2.1 that there is a number p such that $rf(A) \leq p$ and $rf(E-B) \geq p + \varepsilon$. It follows that $f(A) \leq p/r$ and $f(E-B) \geq p/r + \varepsilon/r$ whence, according to 5.2.1, $A\ f^{-1}(<_{\delta})\ B$ where $\delta = \varepsilon/r$. Continuity of f completes the proof of (a).

(b): It is easily seen for an arbitrary function g and order $<$ that $g^{-1}(c(<)) = cg^{-1}(<)$, hence it follows that $A\ (rf)^{-1}(c(<_{\varepsilon}))\ B$ implies $A\ c(rf)^{-1}(<_{\varepsilon})\ B$. Thus $E - B\ (rf)^{-1}(<_{\varepsilon})\ E - A$ and so according to 5.2.1 there is a number p with $rf(E-B) \leq p$ and $rf(A) \geq p + \varepsilon$. Since $r < 0$ we have $f(E-B) \geq p/r$ and $f(A) \leq p/r + \varepsilon/r$. Again by 5.2.1, $A\ f^{-1}(<_{\delta})\ B$ where $\delta = -\varepsilon/r$. Continuity of f completes the proof of (b).

(c): The function $f : S \rightarrow s(T)$ is continuous and therefore both (S,T)- and $(S,c(T))$-continuous. If $r > 0$ then rf is (S,T)-continuous. It is also $(S,c(T))$-continuous since f is $(c(S),T)$-continuous by 2.2.4 and so by (a) rf is $(c(S),T)$-continuous and by 2.2.4, $(S,c(T))$-continuous.

If $r < 0$ then rf is $(S,c(T))$-continuous. Now, since f is $(c(S),T)$-continuous by 2.2.4, rf is $(c(S),c(T))$-continuous by (b), and therefore (S,T)-continuous by 2.2.4.

We have shown that for any real number r , rf is both (S,T)- and $(S,c(T))$-continuous. This implies that rf is $(S,s(T))$-continuous. Indeed, let $< \in T$, then $(rf)^{-1}(s(<)) = (rf)^{-1}(q(< \cup c(<)))$
$= q((rf)^{-1}(<) \cup (rf^{-1})(c(<)))$ by 1.12 (d) and 1.12 (b) .
Letting $<' \in S$ be such that $(rf^{-1})(<) \cup (rf)^{-1}(c(<)) \subseteq <'$ (cf. 1.6 (a)), we have the result since $q(<') = <'$.

□

The next proposition motivates the definition, 5.2.5, of an ordering family.

5.2.4 Proposition

Let f and g be functions defined on E and mapping into R . The following statements are true.

(a) If r is a real number, then the function k defined by $k(s) = r$, for $x \in E$, is continuous.

(b) If f is (S,T)-continuous and r a real number, then $f + r$ is (S,T)-continuous.

(c) If f and g are (S,T)-continuous then so are $h = \min(f,g)$ and $m = \max(f,g)$.

PROOF:

(a) If $r \leq p$ then $f^{-1}(-\infty,p] = E = f^{-1}(-\infty,p+\varepsilon)$ and if $r > p$ then $f^{-1}(-\infty,p] = 0 = f^{-1}(-\infty,p+\varepsilon)$.
Proposition 5.2.2 furnishes the result.

(b): We have that $(f+r)^{-1}(-\infty,p] < (f+r)^{-1}(-\infty,p+\varepsilon)$ iff $f^{-1}(-\infty,p-r] < f^{-1}(-\infty,p-r+\varepsilon)$. Now since f is continuous, 5.2.2 implies that $f + r$ is also continuous.

(c) Consider $m^{-1}(<_{\varepsilon})$. By 5.2.1 $A \; m^{-1}(<_{\varepsilon}) \; B$ implies that there is a number p such that $m(x) \leq p$ for $x \in A$ and $m(x) \geq p + \varepsilon$ for $x \in E - B$.
Thus for $x \in A$ $f(x) \leq p$, $g(x) \leq p$ and there are F, G such that $E - B = F \cup G$, $f(x) \geq p + \varepsilon$ for $x \in F$ and $g(x) \geq p + \varepsilon$ for $x \in G$.
Again by 5.2.1 $A \; f^{-1}(<_{\varepsilon}) \; E - F$ and $A \; g^{-1}(<_{\varepsilon}) \; E - B$.
By continuity of f and g there are $<_1, <_2 \in S$ such that $A <_1 E - F$ and $A <_2 E - G$. Let $< \in S$ be such that $<_1 \cup <_2 \subseteq <$, then $A < E - F$, $A < E - G$ and so $A < (E-F) \cap (E-G) = B$.
Therefore $m^{-1}(<_{\varepsilon}) \subseteq <$ as required.

Now consider $h^{-1}(<_{\varepsilon})$. $A \; h^{-1}(<_{\varepsilon}) \; B$ implies that there is a number p such that $h(A) \leq p$ and $h(E-B) \geq p + \varepsilon$, thus $f(E-B) \geq p + \varepsilon$ and $g(E-B) \geq p + \varepsilon$. There are sets F, G such that $A = F \cup G$ and $f(F) \leq p$, $g(G) \leq p$. This implies that $F \; f^{-1}(<_{\varepsilon}) \; B$ and $G \; g^{-1}(<_{\varepsilon}) \; B$. By an argument similar to that for m, there is $< \in S$ such that $F < B$, $G < B$ and $A = F \cup G < B$, implying that $h^{-1}(<_{\varepsilon}) \subseteq <$.

□

In [C] the following definition is made only for bounded families of real valued functions. However our definition allows unbounded functions, and this is one of the crucial steps which enable us to develop the theory of realcompactness in the sequel.

5.2.5 Definition

An ordering family (OF) on a set E is a non-empty set F of real-valued functions defined on E and satisfying

(a) For $-\infty < r < +\infty$ the function $f(x) = r$ $(x \in E)$ is a member of F.

(b) If $f \in F$ then $f + r \in F$ for $-\infty < r < +\infty$.

(c) $f,g \in F$ imply $\max(f,g) \in F$ and $\min(f,g) \in F$.

The OF, F, is called symmetrical iff

(d) $f \in F$ implies $-f \in F$.

The OF, F, is called simple iff

(e) $f \in F$ implies $rf \in F$ for $0 < r < +\infty$.

5.2.6 Proposition 5.2.4 shows that if S is a syntopogenous structure on E then the set of (S,T)-continuous real-valued functions is an OF on E.

5.2.8 Proposition

If F is an OF then $<_{F,\varepsilon} = \bigcup\{f^{-1}(<_{\varepsilon}) : f \in F\}$ and $A <_{F,\varepsilon} B$ iff there is a function $f \in F$ satisfying $f(x) \leq 0$ for $x \in A$ and $f(x) \geq \varepsilon$ for $x \in E - B$.

PROOF: Denote $\bigcup\{f^{-1}(<_{\varepsilon}) : f \in F\}$ by $<$. We show that $<$ is a topogenous order, so that $< \; = q(<) = \; <_{F,\varepsilon}$. Indeed, $A < B$ iff there is $f \in F$ such that $A \quad f^{-1}(<_{\varepsilon}) \quad B$. This is equivalent, by 5.2.1, to the existence of a number p such that $f(x) \leq p$ for $x \in A$ and $f(x) \geq p + \varepsilon$ for $x \in E - B$. By 5.2.5 (b) we may suppose $p = 0$. Thus $A < B$ iff $f(x) \leq 0$ for $x \in A$ and $f(x) \geq \varepsilon$ for $x \in E - B$. A similar condition holds if $A' < B'$ where g, say, replaces f. Let $h = \max(f,g)$ then $h(x) \leq 0$ for $x \in A \cap A'$ and $h(x) \geq \varepsilon$ for $x \in E - (B \cap B')$. Letting $k = \min(f,g)$, we have $k(A \cup A') \leq 0$ and $k(E-(B \quad B')) \geq \varepsilon$. Thus $A \cap A' < B \cap B'$ and $A \cup A' < B \cup B'$; as required.

□

Let S be a syntopogenous structure and denote the OF of (S,T)-continuous real-valued functions by F. We now make comparisons between S and $S(F)$. The following result is straightforward, having been proved in [C] for bounded OF's.

5.2.9 Proposition

Let S be s syntopogenous structure on the set E and let F be the OF consisting of the (S,T)-continuous real-valued functions. We have $S(F) \leq S$.

PROOF: The results 5.2.1 and 5.2.2 show that if $A <_{F,\varepsilon} B$ then, bearing in mind the definition of F and 5.2.8, there is $< \in S$ such that $A < B$. □

To show the reverse inequality (cf. 5.2.12) we first prove a variant of Urysohn's lemma due to [C] in the next proposition.

5.2.10 Proposition [C]

Let $\{<_n : n = 0,1,2,\ldots\}$ be a sequence of topogenous orders on a set E such that $<_n \subseteq <_{n+1}^2$. If $A <_0 B$ then there is a function f of E into $[0,1]$ satisfying the following two conditions

(a) $f(A) = 0$, $f(E-B) = 1$

(b) If n is a positive integer and $\varepsilon > 0$, then $2^{-n} < \varepsilon$ implies $f^{-1}(<_\varepsilon) \subseteq <_{n+1}$.

PROOF: The proof proceeds by first reaching a definition of a set $A(t) \subseteq E$ for each real t, then using the sets $A(t)$ to define the function f and finally verifying that f has the required properties.

Put $A(0) = A$, $A(1) = B$. We proceed inductively. Suppose that for an integer $n \geqslant 0$ and $p = 0,1,\ldots,2^n$ the sets $A(p/2^n)$ satisfy

(c) $A(p.2^{-n}) <_n A((p+1).2^{-n})$ for $p = 0,1,\ldots,2^n-1$.

By our assumption on $\{<_n\}$ there is a set $A((2p+1).2^{-n-1})$, say, satisfying

(d) $A(p.2^{-n}) <_{n+1} A((2p+1).2^{-n-1}) <_{n+1} A((p+1).2^{-n})$.

Using (d) one sees that (c) holds with n replaced by $n + 1$. Thus for $n = 0,1,2,\ldots$ and $p = 0,1,\ldots,2^n$ the sets $A(p.2^{-n})$ are well defined.

If p and q are integers such that $0 \leqslant p < q \leqslant 2^n$ then since $p + 1 \leqslant q$, (c) shows that

(e) $A(p.2^{-n}) <_n A(q.2^{-n})$.

It follows from (e) that if r and r' are dyadic fractions satisfying $0 \leqslant r \leqslant r' \leqslant 1$ then

(f) $A(r) \subseteq A(r')$.

We now define $A(t)$ for all real numbers t :

(g) Put $A(t) = 0$ for $t < 0$,

$A(t) = E$ for $t > 1$,

$A(t) = \bigcup\{A(r) : 0 \leqslant r \leqslant t$ & r is a dyadic fraction$\}$

for $0 \leqslant t \leqslant 1$.

Then

(h) $A(s) <_{n+1} A(t)$ for $t - s \geqslant 2^{-n}$.

Indeed, if $s < 0$ then (h) follows from the first statement in (g). If $t > 1$ then (h) follows from the second statement in (g). Next, if $0 \leqslant s < s + 2^{-n} \leqslant t \leqslant 1$ then an integer p can be found to satisfy $0 \leqslant s \leqslant p.2^{-n-1} < (p+1).2^{-n-1} \leqslant t \leqslant 1$ by solving the inequalities $s \leqslant p.2^{-n-1}$, $(p+1).2^{-n-1} \leqslant t$ and remembering that $t - s \geqslant 2^{-n}$. Also, (f) and the last statement in (g) imply $A(s) \subseteq A(p.2^{-n-1})$ and $A((p+1).2^{-n-1}) \subseteq A(t)$. Thus (h) follows from (e).

The statement (h) implies

(i) $A(s) \subseteq A(t)$ for $s < t$.

We now define the function f by

(j) $f(x) = \inf\{t : x \in A(t)\}$.

It follows from (g) that $0 \leqslant f(x) \leqslant 1$ for $x \in E$. From this and the fact that $A(0) = A$ it follows that $f(A) = 0$, while from the fact that $A(1) = B$, (g) and (i) it follows that $f(E-B) = 1$, and (a) is proved.

We show that f satisfies (b). Choose n such that $0 < 2^{-n} < \varepsilon$. If $t < 0$ then $f^{-1}(-\infty, t] = 0$, if $t + \varepsilon > 1$ then $f^{-1}(-\infty, t+\varepsilon) = E$ and if $0 \leqslant t < u < u + 2^{-n} < t + \varepsilon \leqslant 1$ then $f^{-1}(-\infty, t] \subseteq A(u) <_{n+1} A(u+2^{-n}) \subseteq f^{-1}(-\infty, t+\varepsilon)$ by (j), (i) and (h). Therefore $G \; f^{-1}(<_\varepsilon) \; D$ implies $G <_{n+1} D$ (cf. 5.2.1), as required.

$\square$

5.2.11 Corollary

If S is a sytnopogenous structure on E , $< \in S$ and $A < B$ then there is a continuous function $f : S \to T$ mapping E into $[0,1]$ and satisfying $f(A) = 0$ and $f(E-B) = 1$.

$\square$

5.2.12 Let S be a syntopogenous structure on the set E and let F be the set of (S,T)-continuous real-valued functions.

Proposition

$S \leqslant S(F)$.

PROOF: 5.2.11 and 5.2.8 show that if $< \in S$ then $A < B$ implies $A <_{F,1} B$; that is $< \subseteq <_{F,1}$ and so indeed $S \leqslant S(F)$.

$\square$

5.2.13 According to propositions 5.2.9 and 5.2.12, if F is the set of real-valued (S,T)-continuous functions then $S \sim S(F)$.

5.3 Biperfect structures induced by quasiecarts

5.3.1 Definition [C]

A quasiecart, d , on the set E is a real-valued function defined on $E \times E$ and satisfying

(a) $d(x,x) = 0$ for $x \in E$

(b) $d(x,y) \geqslant 0$ for $x,y \in E$

(c) $d(x,z) \leqslant d(x,y) + d(y,z)$ for $x,y,z \in E$

If d is a quasiecart satisfying (d) below then it is called an <u>ecart</u>.

(d) $d(x,y) = d(y,x)$ for $x,y \in E$.

5.3.2 <u>Proposition [C]</u>

Let d be a quasiecart on E . Define a relation $U_{d,\varepsilon}$ for $\varepsilon > 0$ by

$$x \; U_{d,\varepsilon} \; y \quad \text{iff} \quad d(x,y) < \varepsilon .$$

The class $U_d = \{U_{d,\varepsilon} : \varepsilon > 0\}$ is a quasi-uniformity on E .

<u>PROOF</u>: Property 5.3.1 (a) implies that $U_{d,\varepsilon}$ is reflexive for $\varepsilon > 0$. If $0 < \varepsilon < \delta$ then $U_{d,\varepsilon} \cap U_{d,\delta} = U_{d,\varepsilon}$ since $U_{d,\varepsilon} \subseteq U_{d,\delta}$.

Let $\alpha = \varepsilon/2$. If $x \; U_{d,\alpha} \; y$ and $y \; U_{d,\alpha} \; z$ then $x \; U_{d,\varepsilon} \; z$ by 5.3.1 (c), showing that $U^2_{d,\alpha} \subseteq U_{d,\varepsilon}$. $\square$

5.3.3 <u>Proposition [C]</u>

Let d be a quasiecart on E and U_d the quasi-uniformity induced on E by d as in 5.3.2. The biperfect syntopogenous structure on E equivalent to U_d is the class

$S_d = \{<_{d,\varepsilon} : \varepsilon > 0\}$ where

$A <_{d,\varepsilon} B$ iff $x \in A$, $y \in E - B$ imply $d(x,y) \geq \varepsilon$.

PROOF: By 3.17, $A <_{d,\varepsilon} B$ iff $x \in A$ and $x \, U_{d,\varepsilon} \, y$ imply $y \in B$, which is the case iff $x \in A$, $y \in E - B$ imply the negation of $x \, U_{d,\varepsilon} \, y$. By 5.3.2 this is equivalent to $x \in A$ and $y \in E - B$, whence $d(x,y) \geqslant \varepsilon$.

□

5.3.4 U_d is a uniformity iff d is an ecart.

U_d is a uniformity iff S_d is a uniformity.

□

5.3.5 Definition [C]

A quasi-metric on a set E is a non-empty family of quasiecarts on E. A pseudo-metric on E is a non-empty family of ecarts on E.

We mention that the terminology is that of [C] which differs from classical usage.

5.3.6 Definition [C]

Let D be a quasi-metric on E. Define the biperfect syntopogenous structure $S(D)$ on E by $S(D) = b(\bigvee\{S_d : d \in D\})$.

5.3.7 Proposition [C]

Let f be a real-valued function on the set E. The function d_f defined by $d_f(x,y) = \max(f(y)-f(x),0)$ is a quasiecart on E.

The following proposition will also be needed.

5.3.8 Proposition

Let d_f be defined as in 5.3.7 and $<_{d_f,\varepsilon}$ as in 5.3.3, then $<_{d_f,\varepsilon} = f^{-1}(<_\varepsilon)$.

PROOF: Suppose that $A <_{d_f,\varepsilon} B$. Then for $x \in A$ and $y \in E - B$ we have that $d_f(x,y) \geq \varepsilon$.

which implies $f(y) - f(x) \geq \varepsilon$

which implies $f(x) \leq p$, $f(y) \geq p + \varepsilon$, where $p = f(x)$,

which implies $x \; f^{-1}(<_\varepsilon) \; E - y$, by 5.2.1,

which implies $A \; f^{-1}(<_\varepsilon) \; B$, since $f^{-1}(<_\varepsilon)$ is biperfect.

Conversely, if $A \; f^{-1}(<_\varepsilon) \; B$ then for $x \in A$, $y \in E - B$ we have $x \; f^{-1}(<_\varepsilon) \; E - y$

which implies $f(x) \leq p$, $f(y) \geq p + \varepsilon$, for some p by 5.2.1,

which implies $f(y) - f(x) \geq \varepsilon$

which implies $d_f(x,y) \geq \varepsilon$ and so $A <_{d_f,\varepsilon} B$ by 5.3.3.

□

CHAPTER 6

REALCOMPACTNESS, AND THE JUSTIFICATION OF THE DEFINITIONS OF COMPLETENESS, COMPACTNESS AND REALCOMPACTNESS

In this chapter we define realcompactness (see 6.7.8), which is a specialization of quasirealcompactness. We also show here that the definitions of completeness, compactness and realcompactness coincide with the classical definitions on suitably chosen objects of Syn .

6.1 Proposition

Let S be a uniformity on E and U an equivalent classical uniformity on E . S is complete iff U is classically complete.

PROOF: Proposition 4.9 shows that the Cauchy grills of S coincide with the classical Cauchy grills of U . By 3.21 the underlying topologies are the same.

□

We now consider compactness. We need a preliminary result.

6.2 Proposition [C, prop.15.71]

Let $S = \{<\}$ be a simple syntopogenous structure on E. S is compact iff the following statement is true

(a) If to each point $x \in E$ an S-neighbourhood V_x is assigned, then a finite number of these neighbourhoods cover E.

PROOF: Suppose that S is compact, then every compressed grill in S converges. Suppose that no finite subset of $\{V_x : x \in E\}$ covers E, then the family $G' = \{E-V_x : x \in E\}$ has the finite intersection property. Let G be a maximal grill generated by G'. G is compressed, since if not there would be sets A, B with $A < B$ and such that each set $R \in G$ meets A and $E - B$. This would mean that both A and $E - B$ belong to G (4.2 (b)) in contradiction to the fact that $A \cap (E-B) = \emptyset$.

By hypothesis there is a point $x \in E$ such that $G \rightarrow x(S)$. But this is impossible since the neighbourhood V_x of x does not meet $E - V_x \in G' \subseteq G$.

Conversely, suppose that (a) is true and that G is a non-convergent compressed grill in S. Then for each $x \in E$ there is an S-neighbourhood V_x of x which does not contain any member of G. For each x choose U_x to satisfy $x < U_x < V_x$. By hypothesis $E = \bigcup\{U_{x_i} : i = 1,\ldots,m\}$ for some finite set of points $\{x_i : i = 1,\ldots,m\} \subseteq E$. Now $U_{x_i} < V_{x_i}$, and since G is compressed, there is $R_i \in G$ satisfying $R_i \subseteq V_{x_i}$ whenever $R_i \cap U_{x_i} \neq \emptyset$. We can choose x in $\bigcap\{R_i : i = 1,\ldots,m\}$ and U_{x_i} such that $x \in U_{x_i}$. Then $R_i \cap U_{x_i} \neq \emptyset$ and so $R_i \subseteq V_{x_i}$. This contradicts the choice of V_{x_i} and therefore G converges in S.

□

6.3 Proposition [C, prop.15.79]

Let (E,S) be a compact syntopogenous space, let (E',S') be an arbitrary syntopogenous space and let f be an (S,S')-continuous function onto E', then S' is also compact.

PROOF: We use 6.2. Suppose that to each $x' \in E'$ a neighbourhood $V(x')$ is assigned. The sets $f^{-1}(V(x'))$ then cover E. Since $B' \subseteq E' - f(E-f^{-1}(B'))$, we see that $f^{-1}(V(x'))$ is a neighbourhood of x whenever $V(x')$ is a neighbourhood of $f(x)$. By 6.2 it follows that finitely many $f^{-1}(V(x'))$ cover E. Hence finitely many $V(x')$ cover E'.

□

6.4 Corollary

If (E,S) is a compact syntopogenous space then any syntopogenous structure on E coarser than S is compact.

PROOF: If $S' \leqslant S$ then the identity on E is (S,S')-continuous.

□

6.5 Proposition [C]

Let S be a syntopogenous structure. S is compact iff $pt(S)$ is compact.

PROOF: Suppose S is compact. If G is a compressed grill in $pt(S)$, then G is compressed in the coarser structure S and converges there. By 4.6.1 G converges in $pt(S)$.

Conversely if pt(S) is compact then so is S by 6.4.

□

6.6 Proposition

Let S be a toplogy and U the equivalent classical topology. S is compact iff U is clasically compact.

PROOF: According to 3.11 the neighbourhoods in S coincide with those in U , and proposition 6.2 furnishes the result.

□

6.7 Realcompactness

To motivate our definition of realcompactness (see 6.7.8) we consider the classical conditions for realcompactness. We collect together below in 6.7.1, without proof, a number of statements on classical realcompactness and uniform space theory.

6.7.1 Let U be a completely regular, Hausdorff topological space on the set E and let C be its ring of continuous real-valued functions.

(a) [GJ, p.226] . U is realcompact iff the uniformity on E induced by C is complete.

(b) [GJ, p.220] . A uniformity S is complete iff every Cauchy z-ultrafilter in S is fixed. The phrase 'is fixed' can be replaced by 'converges'.

(c) [GJ, p.225] . Every Cauchy z-ultrafilter in the uniformity induced by C is real, and so [GJ, p.77] has the countable intersection property.

We sketch our line of approach. Our definition of realcompactness must be applicable, at least, to uniformisable topologies in Syn . Accordingly, we first investigate in 6.7.4 under what conditions an arbitrary topology in Syn can be derived from a uniformity via the operator pt , that is under what conditions there is a uniformity compatible with the topology.

We then define, in 6.7.8, a uniformisable topology, U , in Syn to be realcompact iff the compatible proximity, induced by the real-valued continuous functions defined on U , is quasirealcompact. Classical realcompactness, however (cf. 6.7.1 (a)), requires that the classical uniformity induced by the real-valued continuous functions defined on U be complete. This necessitates the investigation of the correspondence between the compressed grills in a simple structure S , and the Cauchy grills in uniformities induced by real-valued continuous functions defined on S . It is shown in 6.7.11 that, if S is symmetrical, in particular a proximity, then the compressed grills in S with the countable intersection property coincide with the Cauchy grills in the uniformity induced by the continuous real-valued functions defined in S .

Next, we show, using 6.7.12 and 6.7.15, that if U is a given classical uniformisable topology and if $U' \in |\mathrm{Syn}|$ is equivalent to U , then C of 6.7.1 above is a family of functions that will induce a uniformity compatible with U' . This enables us to justify our definition of realcompactness in 6.7.17.

The next two propositions are required in the proof of 6.7.4. Our proof of 6.7.2 is more direct than that in [C] .

6.7.2 Proposition [C, prop.8.102]

Let $\{S_i : i \in I\}$ be a set of syntopogenous structures on the set E . Then $b(\bigvee\{S_i : i \in I\}) = p(\bigvee\{b(S_i) : i \in I\})$.

PROOF: According to 2.8.1 it must be shown that

(a) $bg(\bigcup\{S_i : i \in I\}) = pg(\bigcup\{b(S_i) : i \in I\})$.

Consider a member of the left hand side of (a). It has, according to 2.1.1, the form $bq(\bigcup\{<_i : i \in I'\})$, where I' is a finite subset of I and $<_i \in S_i$.

If $A \; bq(\bigcup\{<_i : i \in I'\}) \; B$, then

$$x \; q(\bigcup\{<_i : i \in I'\}) \; E - y$$

whenever $x \in A$ and $y \in E - B$. By 1.5.3 , this means that $E - y = \bigcap\{B_j : j \in J\}$, where J is finite, and for each $j \in J$

$$x \; (\bigcup\{<_i : i \in I'\}) \; B_j .$$

However, because $E - y = \bigcap_J B_j$ we have that $B_j = E - y$ for each $j \in J$. Thus

$$x \quad (\bigcup\{<_i : i \in I'\}) \quad E - y .$$

That is, there is $i \in I'$ such that $x <_i E - y$. Putting

$$Y_i = \{ y \in E-B : x <_i E-y \},$$

we have

$$x \quad b(<_i) \quad E - Y_i ,$$

and so

$$x \quad (\bigcup\{ b(<_i) : i \in I'\}) \quad E - Y_i$$

whence by 1.1 (d),

$$x \quad q(\bigcup\{b(<_i) : i \in I'\}) \quad \bigcap\{E-Y_i : i \in I'\} .$$

But $E - B = \bigcup\{Y_i : i \in I'\}$ since it is true for any $y \in E - B$ that

$$x \quad q(\bigcup\{<_i : i \in I'\}) \quad E - y$$

and thus $x <_r E - y$ for some $r \in I'$. It follows that

$$x \quad q(\bigcup\{b(<_i) : i \in I'\}) \quad B ,$$

and this holds for each $x \in A$, so that

$$A \quad pq(\bigcup\{b(<_i) : i \in I'\}) \quad B .$$

The order $pq(\bigcup\{b(<_i) : i \in I'\})$ is a member of the right hand side of (a). We have shown that

$$bq(\bigcup\{<_i : i \in I'\}) \subseteq pq(\bigcup\{b(<_i) : i \in I'\}) .$$

Conversely, consider the member of the right hand side of (a), $pq(\bigcup\{b(<_i) : i \in I'\})$. We have that

$$A \quad pq(\bigcup\{b(<_i) : i \in I'\}) \quad B$$

implies, for each $x \in A$,

$$x \quad q(\bigcup\{b(<_i) : i \quad I'\}) \quad B .$$

It follows that for $y \in E - B$,

$$x \quad q(\bigcup\{b(<_i) : i \in I'\}) \quad E - y$$

and so

$$x \quad (\bigcup\{b(<_i) : i \in I'\}) \quad E - y .$$

Hence $x <_r E - y$ for some $r \in I'$, and

$$x \quad bq(\bigcup\{<_i : i \in I') \quad E - y .$$

This holds for any $x \in A$, $y \in E - B$, therefore

$$A \quad bq(\bigcup\{<_i : i \in I'\}) \quad B .$$

We have shown that

$$pq(\bigcup\{b(<_i) : i \in I'\}) \subseteq bq(\bigcup\{<_i : i \quad I'\}) ,$$

and so they are equal.

$\square$

6.7.3 Proposition [C, prop.8.102]

Let $\{S_i : i \in I\}$ be a set of syntopogenous structures. Then $s(\bigvee\{S_i : i \in I\}) = \bigvee\{s(S_i) : i \in I\}$.

PROOF: The result follows easily from the following facts: $sq = s$ (2.4.2 (a)), $c(\bigcup_i <_i) = \bigcup_i c(<_i)$, and the equalities

$$\begin{aligned} s(\textstyle\bigcup_i <_i) &= q((\textstyle\bigcup_i <_i) \cup (c(\textstyle\bigcup_i <_i)) \\ &= q((\textstyle\bigcup_i <_i) \cup (\textstyle\bigcup_i c(<_i))) \\ &= q(\textstyle\bigcup_i (<_i \cup c(<_i))) \\ &= q(\textstyle\bigcup_i q(<_i \cup c(<_i))) \quad \text{(by 1.5.6 (d))} \\ &= q(\textstyle\bigcup_i s(<_i)) . \end{aligned}$$

$\square$

Although the following criterion for uniformisability is essentially the same (see 6.7.12) as that in [C, prop.12.62], our proof is different and more direct.

6.7.4 Proposition

Let $U = \{<\}$ be a topology on the set E. There is a uniformity W with $pt(W) = U$ iff the following condition is satisfied:

(a) If $x < B$ then there is a $(U,s(T))$-continuous function mapping into $[0,1]$ and satisfying $f(x) = 0$ and $f(E-B) = 1$.

PROOF: Suppose there is a uniformity W satisfying $pt(W) = U$ and let $x < B$. We have that $x \ t(W) \ B$ and hence there is $<' \in W$ with $x <' B$ (2.7.3). By 5.2.11 there is a (W,T)-continuous function f mapping into $[0,1]$ with $f(x) = 0$ and $f(E-B) = 1$. According to 2.2.4 f is $(W,s(T))$-continuous because W is symmetrical. Thus f is $(pt(W),s(T))$-continuous since $W \leq pt(W)$.

Conversely, suppose that (a) is satisfied. Define W by

6.7.5 $W = bs(\bigvee \{S_{d_f} : f \in K\})$ where $K = \{f : f$ is $(U,S(T))$-continuous$\}$, and d_f is defined by 5.3.7 and S_{d_f} by 5.3.3. That is d_f is given by

$$d_f(x,y) = \max\,(f(y)-f(x),0)\,,$$

and

$$S_{d_f} = \{<_{d_f,\varepsilon} : \varepsilon > 0\}$$

where

$<_{d_f,\varepsilon}$ is given by

$$A <_{d_f,\varepsilon} B \quad \text{iff} \quad d_f(x,y) \geq \varepsilon \quad \text{for all } x \in A,\ y \in E-B\,.$$

We shall use the abridged notation $S = bs(\bigvee S_{d_f})$.

The structure W is a uniformity on E . We show that pt(W) = U . First we prove

(b) $s(\bigvee s_{d_f}) \sim \bigvee s_{d_f}$ where $f \in K$.

Indeed, consider $s(S_{d_f})$ where $f \in K$. Now

$$A \; s(<_{d_f,\varepsilon}) \; B \quad \text{implies} \quad A \; q(<_{d_f,\varepsilon} \cup c(<_{d_f,\varepsilon})) \; B$$

and so by 1.5.3 $A = \bigcup_i A_i$, $B = \bigcap_j B_j$ and

$$A_i \; (<_{d_f,\varepsilon} \cup c(<_{d_f,\varepsilon})) \; B_j ,$$

where the i's and j's range over suitable finite sets. Either $A_i <_{d_f,\varepsilon} B_j$ or $E - B_j <_{d_f,\varepsilon} E - A_i$, which means that $A_i \; d_{-f,\varepsilon} \; B_j$ (cf. 5.3.3 and 5.3.7). But f is (U,s(T))-continuous and therefore by 5.2.3 (c) -f is (U,s(T))-continuous . Thus $q(<_{d_f,\varepsilon} \cup <_{d_{-f},\varepsilon})$ is a member of $\bigvee \{S_{d_f} : f \in K\}$. However,

$$A_i \; q(<_{d_f,\varepsilon} \cup <_{d_{-f},\varepsilon}) \; B_j$$

and so

$$A \; q(<_{d_f,\varepsilon} \cup <_{d_{-f},\varepsilon}) \; B .$$

It follows that

$$s(s_{d_f}) \leq \bigvee \{s_{d_f} : f \in K\} .$$

Using 6.7.3 we therefore have

$$s(\bigvee s_{d_f}) = \bigvee s(s_{d_f}) \leq s_{d_f}$$

The reverse inequality is obvious and so (b) is true.

Now we show that

(c) $$pt(W) = pt(\bigvee\{s_{d_f} : f \in K\}) .$$

In fact, using (b), 6.7.2, 3.3, the fact that s_{d_f} is biperfect (5.3.3) and the fact that a simple structure consists of a single order, we have that

$$pt(W) = ptbs(\bigvee s_{d_f}) \sim ptb(\bigvee s_{d_f})$$
$$= ptp(\bigvee b(s_{d_f})) = pt(\bigvee s_{d_f}) .$$

(d) $$U = pt(\bigvee\{s_{d_f} : f \in K\}) .$$

Indeed, $U \leq pt(\bigvee s_{d_f})$. To see this let $x < B$. By (a) there is a $(U,s(T)$-continuous f with $f(x) = 0$ and $f(E-B) = 1$. Thus for $y \in E - B$ $f(y) - f(x) \geqslant 1$ and so $d_f(x,y) \geqslant 1$, whence $x <_{d_f,1} B$ (5.3.3). Using 2.7.3 one then sees that

$$x \ t(\bigvee\{s_{d_f} : f \in K\}) \ B .$$

It follows from the perfectness of U that $U \leq pt(\bigvee s_{d_f})$. To show that $pt(\bigvee s_{d_f}) \leq U$ we first show that for each

$(U,s(T))$-continuous f , $S_{d_f} \leqslant U$, and then we operate on the left with $\bigvee$ and on both sides with pt . Accordingly let $A <_{d_f,\varepsilon} B$. Then by 5.3.8 $A \; f^{-1}(<_\varepsilon) \; B$. It follows that $A < B$ in view of the $(U,s(T))$-continuity of f and the inequalities $f^{-1}(T) \leqslant f^{-1}(s(T)) \leqslant U$. We have shown that (d) is true. It follows from (c) and (d) that $pt(W) = U$.

$\square$

6.7.6 Definition

Let K be an arbitrary set of real-valued functions defined on the set E . Then call the uniformity W defined by

$$W = bs(\bigvee\{S_{d_f} : f \in K\})$$

the uniformity induced by K . We accept circumlocutions for K as, for example, in the paragraph below.

If a topology U is uniformisable, then according to the proof of the previous proposition, the uniformity induced by the $(U,s(T))$-continuous functions is compatible with U .

6.7.7 Proposition

A topology on E is derivable from a uniformity on E iff it is derivable from a proximity on E .

PROOF: Suppose that the topology $U = \{<\}$ is derivable from the proximity S on E , that is $p(S) = U$. Let $S = \{<'\}$

and suppose that $x \ p(<') \ B$. Then $x <' B$ and so by 5.2.11 there is an (S,T)-continuous function f mapping into $[0,1]$ with $f(x) = 0$ and $f(E-B) = 1$. However by 2.2.4 f is $(s(S),s(T))$-continuous , so that

$$f^{-1}(s(T)) \leq s(S) = S \leq p(S) = U$$

whence 6.7.4 (a) is satisfied and one half of the statement is proved. The reverse implication is straightforward. □

6.7.8 Definition

Let U be a uniformisable topology on the set E , and let W be the uniformity induced by the $(U,s(T))$-continuous functions (cf. 6.7.6). We shall call U realcompact iff $t(W)$ is quasirealcompact.

We remark that $t(W)$ is the proximity induced by the uniformity W . In 6.7.17 we shall show the equivalence of the above definition with the classical definition of realcompactness for the case of Hausdorff uniformisable topologies.

In order to examine classical realcompactness we consider Cauchy grills in the uniformity induced by the continuous real-valued functions (cf. 6.7.1 (a)). Accordingly, we shall need a link between Cauchy grills in uniformities induced by continuous functions from a simple structure X into T , and the compressed grills in X . This is provided in 6.7.11. Propositions 6.7.9 and 6.7.10 are useful technical results.

Our proof of 6.7.9 differs from, and is more direct than that in [C] .

6.7.9 Proposition [C, prop.15.44]

Let D be a quasimetric on E and S(D) the quasiuniformity induced by D on E (cf. 5.3.5, 5.3.6). The following two statements are true.

(a) A grill $G \to x(S(D))$ iff for all $d \in D$ and all $\varepsilon > 0$, there is $R \in G$ such that $d(x,y) < \varepsilon$ for all $y \in R$.

(b) A grill G is Cauchy in S(D) iff for all $d \in D$ and all $\varepsilon > 0$, there is $R \in G$ such that $d(x,y) < \varepsilon$ for all $x,y \in R$.

PROOF:

(a): Suppose $G \to x(S(D))$. Let $d \in D$, $\varepsilon > 0$ and suppose that for each $R \in G$ there is $y_R \in R$ with $d(x,y_R) \geq \varepsilon$. Let $Y = \{y_R : R \in G\}$. By 5.3.3. $x <_{d,\varepsilon} E - Y$, whence by hypothesis there is $R \in G$ with $R \subseteq E - Y$, because $S_d \leq S(D)$ (cf. 5.3.6). But according to the definition of Y , $R \subseteq E - Y$ leads to a contradiction.

Conversely, suppose for all $d \in D$ and all $\varepsilon > 0$, there is $R \in G$ such that $d(x,y) < \varepsilon$ for $y \in R$. We must show that $G \rightarrow x(S(D))$. Accordingly let N be a neighbourhood of x in $S(D)$. There is a member of $S(D)$, $bq(\bigcup\{<_i : i \in I\})$, where I is a <u>finite</u> index set and $<_i = <_{d_i,\varepsilon_i}$ for $d_i \in D$ and $\varepsilon_i > 0$ (cf. 5.3.6, 2.8.1, 2.1.1), such that $x \; bq(\bigcup_i <_i) \; N$. According to 2.6.9 (b) and 2.6.6 this means that for $y \in E - N$,

$$x \; (\bigcup\{<_i : i \in I\}) \; E - y .$$

Thus for each $y \in E - n$ there is $d_y \in \{d_i : i \in I\}$ and $\varepsilon_y \in \{\varepsilon_i : i \in I\}$ such that $x <_{d_y,\varepsilon_y} E - y$. Now for each $i \in I$ let $R_i \in G$ be such that $d_i(x,w) < \varepsilon_i$ for $w \in R_i$, then for each $y \in E - N$ there is $R_y \in \{R_i : i \in I\}$ satisfying $d_y(x,w) < \varepsilon_y$ for $w \in R_y$. It follows that $R_y \subseteq E - y$ since if not then by the choice of d_y and ε_y a contradiction to the choice of R_y results.

The proof of (a) is completed by noting that $\bigcap\{R_y : y \in E-N\}$ is actually a finite intersection and so contains a member R of G. We thus have

$$R \subseteq \bigcap\{R_y : y \in E-N\} \subseteq \bigcap\{E-y : y \in E-N\} = N ,$$

as required.

The proof of (b) is similar.

$\square$

6.7.10 Proposition [C, prop.15.42]

Let $\{S_i : i \in I\}$ be a set of syntopogenous structures on the set E. If G is a Cauchy grill in S_i for each $i \in I$, then G is Cauchy in $\bigvee\{S_i : i \in I\}$.

PROOF: Let $q(\bigcup\{<_i : i \in I'\})$, where $I' \subseteq I$ is finite, be a member of $\bigvee\{S_i : i \in I\}$. Corresponding to each $<_i$ for $i \in I'$ there is a set $R_i \in G$ satisfying the Cauchy condition 4.8.2 in S_i.

Let $R \in G$ be such that $R \subseteq \bigcap\{R_i : i \in I'\}$ and suppose that

$$A \quad q(\bigcup\{<_i : i \in I'\}) \quad B$$

and that $A \cap R \neq \emptyset$. Then $A = \bigcup\{A_k : k \in K\}$, $B = \bigcap\{B_j : j \in J\}$ and $A_k \bigcup_i <_i B_j$, where J, K are finite and $(k,j) \in K \times J$. Since $A \cap R \neq \emptyset$ there is $k \in K$ such that $A_k \cap R \neq \emptyset$, whence $R \subseteq B_j$ for each $j \in J$. Thus $R \subseteq \bigcap_j B_j = B$.

□

Some remarks about proposition 6.7.11 are in order. Part (a) is due to [C] and is true for bounded continuous functions. For unbounded continuous functions we have one half of the equivalence in (a), namely the implication in (b). We obtain a partial converse to (b) in part (c), by requiring that the structure X be symmetrical and that the grill G have the countable intersection property.

Our proof of (a) is appreciably different from, and more direct than that in [C] .

6.7.11 Let $X = \{<\}$ be a simple syntopogenous structure on the set E and recall that T , of 5.1.1, is a quasi-uniformity on the real numbers. Let C_1 (resp. C_1^*) be the set of (resp. bounded) (X,T)-continuous real-valued functions and let $S(K)$, for $K = C_1$ (resp. $K = C_1^*$) , be the quasi-uniformity induced by K on E : $S(K) = b(\bigvee\{S_{d_f} : f \in K\})$ where $d_f(x,y) = \max (f(y)-f(x), 0)$ (cf. 5.3.3, 5.3.6, 5.3.7).

Proposition

(a) [C, prop.15.56] . G is a Cauchy grill in $S(C_1^*)$ iff G is compressed in X .

(b) If G is a Cauchy grill in $S(C_1)$, then G is compressed in X .

(c) Let G be a grill with the countable intersection property and suppose that X is symmetrical. If G is compressed in X then G is Cauchy in $S(C_1)$.

PROOF:

(a): Let G be a Cauchy grill in $S(C_1^*)$ and suppose that $A < B$. According to 5.2.11 there is an (X,T)-continuous function, f , mapping into $[0,1]$ with $f(A) = 0$ and $f(E-B) = 1$. Choose ε such that $0 < \varepsilon < 1$. There is, by 6.7.9 (b), a member R of G satisfying $d_f(x,y) < \varepsilon$ for all $x,y \in R$. This implies that $|f(y)-f(x)| < \varepsilon$ for $x,y \in R$ (cf. 5.3.3, 5.3.7).

Now suppose that $R \cap A \neq \emptyset$, then there is x with $x \in R \cap A$ and so for $y \in R$, $f(y) < f(x) + \varepsilon$. But $f(x) = 0$ since $x \in A$, therefore for $y \in R$ we have $f(y) < \varepsilon < 1$. This implies that $R \subseteq B$ because $f(E-B) = 1$, whence G is compressed in $X = \{<\}$.

Conversely, suppose G is a compressed grill in X, let f be bounded and (X,T)-continuous and let $\varepsilon > 0$. We shall show that G is Cauchy in S_{d_f}, and it will follow that G is Cauchy in $S(C_1^*) = b(\bigvee\{S_{d_f} : f \in C_1^*\})$ by 6.7.10 and 4.11.

Accordingly, if $A <_{d_f,\varepsilon} B$, then by 5.3.8 $A \; f^{-1}(<_\varepsilon) \; B$. There is thus a number p such that $f(A) \leqslant p$ and $f(E-B) \geqslant p + \varepsilon$ (cf. 5.2.1). Since f is (X,T)-continuous we have, by 5.3.2, that

$$f^{-1}(-\infty,p] < f^{-1}(-\infty,p+\varepsilon) .$$

Now let $m = \inf\{f(x) : x \in E\}$ and $M = \sup\{f(x) : x \in E\}$ We produce a member R of G depending only on f and ε, satisfying the Cauchy condition 4.8.2.

There is an integer n_0 such that $n_0 . \varepsilon/2 \leqslant m < (n_0 + 1). \varepsilon/2$ and an integer n_q, where $q = n_q - n_0$, such that $(n_q - 1). \varepsilon/2 \leqslant M < n_q . \varepsilon/2$. Set $n_i = n_0 + i$, then for $i = 0,1,2,\ldots,q$ we have that

$$f^{-1}(-\infty, n_i . \varepsilon/2] < f^{-1}(-\infty, n_i . \varepsilon/2 + \varepsilon/2)$$

by 5.3.2 since f is (X,T)-continuous. Because G is compressed in X there is for each $i = 0,1,2,\ldots,q$ a member R_i of G satisfying:

$$R_i \cap f^{-1}(-\infty, n_i . \varepsilon/2] \neq \emptyset \quad \text{implies}$$
$$R_i \subseteq f^{-1}(-\infty, n_i . \varepsilon/2 + \varepsilon/2) .$$

Let R be a member of G such that $R \subseteq \bigcap \{ R_i : i = 0,1,2,\ldots,q \}$. If $A \cap R \neq \emptyset$ then $R \cap f^{-1}(-\infty, p] \neq \emptyset$. Let $(n_r - 1).\varepsilon/2 < p \leq n_r . \varepsilon/2$. Then $R \cap f^{-1}(-\infty, n_r . \varepsilon/2] \neq \emptyset$ and so $R \subseteq f^{-1}(-\infty, n_r . \varepsilon/2 + \varepsilon/2)$, whence $R \subseteq f^{-1}(-\infty, p+\varepsilon)$ since $n_r . \varepsilon/2 + \varepsilon/2 < p + \varepsilon$. But $f^{-1}(-\infty, p+\varepsilon) \subseteq B$ and so $R \subseteq B$, as required.

(b): $C_1^* \subseteq C_1$ therefore if G is a Cauchy grill in $S(C_1)$ then it is Cauchy in $S(C_1^*)$ and (a) gives the result.

(c): Let G be a grill with the countable intersection property, compressed in X , and suppose that X is symmetrical . Let f be (X,T)-continuous and $\varepsilon > 0$. We shall show that G is Cauchy in S_{d_f} , and it will follow that G is Cauchy in $S(C_1) = b(\bigvee \{ S_{d_f} : f \in C_1 \})$ by 6.7.10 and 4.11.

We first show that there is a member of G on which f is bounded. Now, since f is (X,T)-continuous , $-f$ is

(X,T)-continuous because $X = c(X)$, X being symmetrical, and $cc(T) = T$ (cf. 5.2.3 (b), 2.2.4 and 2.4.2 (b)). According to 5.2.4 (c) $|f| = \max(f,-f)$ is also (X,T)-continuous. Let I be the set of nonnegative integers and define A_i for $i \in I$ by $A_i = |f|^{-1}(-\infty, i.\varepsilon]$. By 5.2.2, for $i \in I$,

$$A_i = |f|^{-1}(-\infty, i.\varepsilon] < |f|^{-1}(-\infty, i.\varepsilon+\varepsilon) \subseteq A_{i+1},$$

that is $A_i < A_{i+1}$ for $i \in I$.

Since G is compressed in X, there is for each $i \in I$ a set $R_{i+1} \in G$ such that $A_i \cap R_{i+1} \neq \emptyset$ implies $R_{i+1} \subseteq A_{i+1}$. G has the countable intersection property and so there is a point $x \in \bigcap\{R_i : i = 1,2,\ldots\}$.

However there is a number $p \in I$ such that $x \in A_p$ because $\bigcup\{A_i : i \in I\} = E$. This means that in particular $A_p \cap R_{p+1} \neq \emptyset$ and so

$$R_{p+1} \subseteq A_{p+1} = \{x \in E : |f|(x) \leq (p+1).\varepsilon\}.$$

This shows that f is bounded on the member R_{p+1} of G. Denote R_{p+1} by R.

Let $m = \inf\{f(x) : x \in R\}$, $M = \sup\{f(x) : x \in R\}$ and, for all x, $g(x) = \min(\max(f(x),m),M)$. The function g is bounded, (X,T)-continuous and agrees with f on R.

According to (a) and 6.7.9 (b) there is a member R' of G such that $d_g(x,y) < \varepsilon$ for $x,y \in R'$. Choosing $R'' \subseteq R \cap R'$ in G, we have $d_f(x,y) < \varepsilon$ for $x,y \in R''$. According to 6.7.9 (b) G is Cauchy in S_{d_f}.

□

We now characterize the classically continuous real-valued functions.

6.7.12 Proposition

Let U' be a classical completely regular topology on the set E and let $U = \{<\}$ be the corresponding topology in Syn. Then the function f is classically continuous from U' to the real numbers with their natural topology iff f is $(U,s(T))$-continuous.

PROOF: Suppose that f is a classically continuous function from U' to the real numbers with their natural topology. The natural topology on the real numbers corresponds to $\mathrm{ptbs}(T)$ (cf. 5.15) and so f is $(U,\mathrm{ptbs}(T))$-continuous and thus $(U,s(T))$-continuous, since $s(T) \leq \mathrm{ptbs}(T)$.

Conversely, suppose f is $(U,s(T))$-continuous and let W be the uniformity induced by the $(U,s(T))$-continuous functions. We have that $f^{-1}(<_\varepsilon) = <_{d_f,\varepsilon}$ by 5.3.8, thus $f^{-1}(T) = S_{d_f} \leq W$ and so f is (W,T)-continuous. According to 2.2.4 f is $(\mathrm{ptbs}(W),\mathrm{ptbs}(T))$-continuous. But $U = \mathrm{ptbs}(W)$ by 6.7.4 and its proof.

□

We present below two interesting examples.

6.7.13 Example

Let E be the real numbers endowed with their natural topology, U. Let C be the set of continuous functions $U \longrightarrow U$ and let W be the uniformity in Syn induced by C. We recall (cf. proof of 6.7.4) that $t(W)$ is a proximity such that $pt(W) = U$. We produce a grill, G, which is compressed in $t(W)$ but not in U.

Let r be a fixed real number and let G be the grill consisting of a countable set of nested intervals with midpoints r and lengths tending to zero.

G is compressed in $t(W)$. Indeed (cf. 6.7.11, 6.7.9), it must be checked for each $(t(W),s(T))$-continuous function f and the number $\varepsilon > 0$, that there is a member R of G such that $f(y) - f(x) < \varepsilon$ whenever $x,y \quad R$. But this condition is satisfied for all the $(U,S(T))$-continuous functions, because

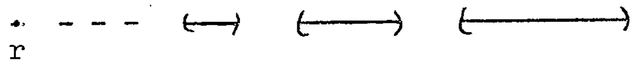
$$\{f : f \text{ is } (U,S(T)\text{-continuous}\} = \{f : f \text{ is } (U,U)\text{-continuous}\}.$$

It follows from the fact that

$$\{f : (t(W),s(T))\text{-continuous}\} \subseteq \{f : f \text{ is } (U,s(T))\text{-continuous}\},$$

that G is compressed in $t(W)$.

G is not compressed in U. This is seen as follows. Let X be the union of a set of disjoint open intervals lying to the right of r with lengths decreasing to zero and left hand end-points tending to r :



Setting $\{<\} = U$, we have that $X < X$ (cf. 3.11), each member of G meets X but no member of G is contained in X, whence G is not compressed in U.

□

6.7.14 Example

Let E be the set of all ordinals less that the first uncountable ordinal, endowed with the interval topology, U'. Denote by U the corresponding topology in Syn. U is uniformisable because U' is. Let $U = p(S)$ where $S = \{<\}$ is a prximity (cf. 6.7.7). We produce a grill G on E with the countable intersection property which is compressed in S but not in U.

Indeed, let $G = \{I(r) : r \in E\}$, where $I(r) = \{x \in E : x \geq r\}$.

The $(U,s(T))$-continuous functions are the same as the continuous functions from U' to the real numbers with their natural topology. Each of these functions is constant on a suitable set $I(r)$. But

$$\{f : f \text{ is } (S,s(T))\text{-continuous}\} \subseteq \{f : f \text{ is } (U,s(T))\text{-continuous}\} .$$

This means that each $(S,s(T))$-continuous function is constant on a suitable set $I(r)$, whence G is compressed in S (cf. 6.7.11, 6.7.9).

G is not compressed in U. Indeed, we produce a set A with $A \ p(<) \ A$, and such that each member of G intersects A but no member of G is contained in A. Let $A = \{x \in E : x$ is an isolated point of $U'\}$. A is open in U', and so $A \ p(<) \ A$ (cf. 3.11). But although A is cofinal in E is does not cover $I(r)$ for any $r \in E$.

□

We now retrieve the well-known fact that a topology which corresponds to a classical completely regular topology is uniformisable.

6.7.15 Proposition

Let U' be a classical completely regular topology on the set E, let C be the set of continuous functions from U' to the real numbers with their natural topology and let $U = \{<\}$ be the topology in Syn which corrresponds to U'. The following statements are true.

(a) If $x < B$ then there is a $(U,s(T))$-continuous function mapping into $[0,1]$ satisfying $f(x) = 0$ and $f(E-B) = 1$.

(b) $W = bs(\bigvee\{S_{d_f} : f \in C\})$ is the uniformity induced by the $(U,s(T))$-continuous functions and satisfies $pt(W) = U$.

PROOF:

(a): If $x \prec B$ then there is, by 3.11, an open set $G \in U'$ with $x \in G \subseteq B$. Thus $E - B$ is contained in $E - G$, which is closed in U'. Accordingly there is a function f in C mapping into $[0,1]$ such that $f(x) = 0$ and $f(E-B) = 1$.

(b): Propositions 6.7.4 and 6.7.12.

□

6.7.16 Corollary

Let C and W be as in the above proposition and let W' be the classical uniformity induced by C. Then the following statements are true.

(a) The grill G is Cauchy in W' iff it is Cauchy in W.

(b) The grill $G \rightarrow x(W')$ iff $G \rightarrow x(W)$.

PROOF: Propositions 6.7.15 and 6.7.9.

□

6.7.17 Let U' be a classical, completely regular, Hausdorff topological space on the set E, let U be the corresponding uniformisable topology on E (cf. 6.7.15 (b)) and let W be the uniformity induced by the $(U,s(T))$-continuous functions.

Proposition

(a) U' is classically realcompact iff U is realcompact.

(b) If U' is classically realcompact then U is quasirealcompact.

PROOF:

(a): We need only show that U' is classically realcompact iff $t(W)$ is quasirealcompact.

Suppose that $t(W)$ is quasirealcompact then, by 4.20, W is quasirealcompact. Let C be the set of classically continuous real-valued functions from U' to the reals with the natural topology and let W' be the classical uniformity induced on E by C with respect to the natural uniformity on the reals. We must show that every Cauchy z-ultrafilter in W' converges there (cf. 6.7.1 (b)).

Let G be a Cauchy z-ultrafilter in W', then according to 6.7.1 (c) G has the countable intersection property. G is Cauchy in W and so compressed there (cf. 4.17). It follows that G converges in W and therefore in W' (cf. 6.7.16 (b)). Thus W' is classically complete and U' is realcompact according to 6.7.1 (a).

Conversely suppose that U' is realcompact and let G be a compressed grill in $t(W)$ with the countable intersection property. Since $t(W)$ is symmetric G is Cauchy in the uniformity induced by the $(t(W),T)$-continuous functions. But this uniformity if equivalent to W. Indeed, f is $(t(W),T)$-continuous iff it is $(U,s(T))$-continuous. To see this let f be $(U,s(T))$-continuous, then $f^{-1}(T) = S_{d_f} \leq W \leq t(W)$ and so f is $(t(W),T)$-continuous (cf. 5.3.8, 5.3.3 and 6.7.5). The reverse implication is straightforward because $t(W) \preceq pt(W) = U$. Now,

$$\begin{aligned} W &= bs(\bigvee\{s_{d_f} : f \text{ is } (U,s(T))\text{-continuous}\}) \\ &\sim b(\bigvee\{s_{d_f} : f \text{ is } (U,s(T))\text{-continuous}\}) \\ &= b(\bigvee\{s_{d_f} : F \text{ is } (t(W),T)\text{-continuous}\}) , \end{aligned}$$

by 6.7.5 and (b) in 6.7.4. This proves the statement about uniformities above.

According to 6.7.11 (c) and what has just been proved, G is Cauchy in W , thus also in W' , and so converges in W , and therefore also in $t(W)$ (cf. 6.7.16 (a), (b) and 4.6 (b)).

(b): Let U' be classically realcompact, then by (a) $t(W)$ is quasirealcompact. But by 6.7.4 $pt(W) = U$, and so if G is a compressed grill in U with the countable intersection property then it is easily seen that G is compressed in $t(W)$, and therefore converges in $t(W)$. By 4.6 (b) G converges in U .

□

6.7.18 We have left open the question of whether there is a uniformisable topology which is quasirealcompact but not realcompact. One may still seek, however, sufficient conditions for quasirealcompactness to imply realcompactness. We mention, without goint into detail, two such conditions.

Let $U = \{<\}$ be a uniformisable topology on E and let W be the uniformity induced on E by the $(U,s(T))$-continuous functions.

(a) U is realcompact if U is quasirealcompact and the following condition, stronger than 6.7.4 (a), is satisfied:

If $A < B$, then there is a $(U,s(T))$-continuous function f with $f(A) = 0$ and $f(E-B) = 1$.

(b) U is realcompact if U is quasirealcompact and $t(W)$ is totally bounded (cf. [C, p.339, prop.19.1]).

PROOF: Both (a) and (b) force the condition $U = t(W)$. For the case of (b) see [C, prop.19.3].

☐

The above conditions are not very sensitive inasmuch as the equality $U = t(W)$ implies that U is a symmetrical topology. The symmetrical topologies correspond to classical topologies whose T_0-reflection is discrete. But a discrete topology is always quasirealcompact (its only compressed grills are the grills containing singletons as members).

6.7.19 After this work was done we found that our characterization of realcompactness for uniformisable Hausdorff spaces (6.7.8, 6.7.17 (a)) is essentially the same as one given by Hušek [HU, th.3]. This can be seen if one considers the fact that a grill is compressed in a proximity if and only if it is Cauchy in some compatible uniformity (cf. [C, pp.226-227]).

CHAPTER 7

COMPLETION AND COMPACTIFICATION

Our object is to show that the separated doubly compact, doubly quasirealcompact and doubly complete spaces form reflective subcategories of Syn . We accomplish this by producing appropriate extensions. If S is a syntopogenous structure on the set E , the idea is to adjoin extra points to E and extend the definition of S so that a selected class of grills in E will converge in this larger set. We then use the doubly compact and doubly quasirealcompact reflections to retrieve the well known compact and realcompact epireflections in uniformisable T_2 topologies.

Many of the results in this chapter are due to [CS] , where they are stated without proof. We mention that our universal double quasirealcompactification is new.

7.1 Extensions

In this section we deal with a method of extending the definition of a syntopogenous structure, S , on E to a structure S' on E' , where $E \subseteq E'$, $S'|E = S$ and S is appropriately dense in E' .

7.1.1 [CS] .

We recall that a grill is a <u>filter</u> iff $R \in G$ and $R \subseteq F$ imply $F \in G$.

Now let E be a set, and let $K = \{G(x) : x \in I\}$ be a set of filters on E, none of which is the filter generated by a singleton in E. We assume that the indexing of K is one-to-one and that I is disjoint from E. For $x \in E$, define $G(x)$ to be the filter generated by $\{x\}$, that is $G(x) = \{X \subseteq E : x \in X\}$. Set $E' = E \cup I$. Of course, if $x, y \in E'$, then $G(x) \neq G(y)$ whenever $x \neq y$.

We define the symbol $(\subseteq)$ and the set function h as follows:

If G is a grill on E and $X \subseteq E$, then $G \ (\subseteq)\ X$ iff there is a member $R \in G$ with $R \subseteq X$.

If $X \subseteq E$, then $h(X) = \{x \in E' : G(x)\ (\subseteq)\ X\}$.

We mention some properties of h.

7.1.2 Proposition

Suppose that h and K are as in 7.1.

(a) $A \subseteq B \subseteq E$ implies $h(A) \subseteq h(B)$.

(b) If $A \subseteq E$, then $A \subseteq h(A)$.

(c) If $A \subseteq E$, then $E \cap h(A) = A$.

PROOF:

(a): $x \in h(A)$ implies $G(x)\ (\subseteq)\ A \subseteq B$, whence $G(x)\ (\subseteq)\ B$ and so $x \in h(B)$.

(b): $x \in A$ implies $G(x)\ (\subseteq)\ A$ and so $x \in h(A)$.

(c): $x \in E \cap h(A)$ implies $G(x)\ (\subseteq)\ A$. But $x \in E$, whence $G(x) = \{Y \subseteq E : x \in Y\}$ and so $x \in A$. Conversely, if $x \in A$ then $G(x)\ (\subseteq)\ A$ and so $x \in E \cap h(A)$.

□

We now associate with each topogenous order on E a semi-topogenous order (cf. 1.5.1), $<'$, on E' as follows.

7.1.3 Definition [CS]

Let $<$ be a semi-topogenous order defined on E, and let K, E' and h be as in 7.1. Then we define the order $<'$, for subsets A' and B' of E', by

$A' <' B'$ iff there are sets $A, B \subseteq E$ satisfying $A < B$, $A' \subseteq h(A)$ and $h(B) \subseteq B'$.

We agree that the sentence

$<'$ extends $<$ from E to E' by K

will mean that $<$ is a semitopogenous order on E, that K, E'

and h are as in 7.1 and that $<'$ is defined as above. In the above sentence the phrases such as 'from E' and 'by K' will be omitted when the meaning is clear from the context.

7.1.4 Proposition [CS]

If $<'$ extends $<$ to E', then $<'$ is a semitopogenous order.

PROOF: It is easily seen that $\emptyset <' \emptyset$, while $E' <' E'$ follows from the fact that $E' = h(E)$.

Suppose that $A' <' B'$, then according to 7.3 $A' \subseteq h(A)$, $h(B) \subseteq B'$. But from 7.2 (a) $h(A) \subseteq h(B)$, whence $A' \subseteq B'$.

Finally let $A_1' \subseteq A' <' B' \subseteq B_1'$. It follows immediately from 7.3 that $A_1' <' B_1'$.

$\square$

We require the following proposition on restriction (cf. 2.8.6) to a subset.

7.1.5 Proposition [C, prop.6.21]

Suppose that $E \subseteq E'$ and that $<'$ is a semitopogenous order on E'. The following statements are true.

(a) For $A, B \subseteq E$, $A \; (<'|E) \; B$ iff $A <' B \cup (E'-E)$.

(b) $q(<'|E) = q(<')|E$.

PROOF: Let $i : E \to E'$ be the inclusion mapping.

(a): $A \; i^{-1}(<') \; B$ iff $i(A) <' E' - i(E-B)$

iff $A <' E' - (E-B) = B \cup (E'-E)$.

(b): $q(i^{-1}(<')) = i^{-1}(q(<'))$ (cf. 1.12 (d)).

$\square$

7.1.6 Proposition [CS]

Let $<'$ extend $<$ from E to E' . Then $<'|E = <$.

PROOF: Suppose $A', B' \subseteq E$ and $A' <'|E \; B'$. Then $A' <' B' \cup (E'-E)$ by 7.5 (a). According to 7.3 there are sets $A , B \subseteq E$ such that $A < B$, $A' \subseteq h(A)$, $h(B) \subseteq B' \cup (E'-E)$. Now $A' \subseteq E$, whence by 7.2 (c) $A' \subseteq A$. Also $B \subseteq h(B) \subseteq B' \cup (E'-E)$, but $E \cap (B' \cup (E'-E)) = B'$ and therefore, using 7.2 (c), we have $A' \subseteq A < B \subseteq B'$, so that $A' < B'$.

Conversely, suppose $A < B$ for $A , B \subseteq E$. Now by 7.2 (b) , $A \subseteq h(A)$. We show that $h(B) \subseteq B \cup (E'-E)$, so that according to 7.3 $A <'|E \; B$. Now, if $x \in h(B)$ then $G(x) \; (\subseteq) \; B$. For the case $x \in E$ this means that $x \in B$, while if $x \notin E$ then $x \in E' - E$ whence $h(B) \subseteq B \cup (E'-E)$, as required.

$\square$

We now turn to extensions of syntopogenous structures.

7.1.7 Proposition [CS]

Let S be a syntopogenous structure on the set E, and for each $< \in S$ let $<'$ extend $<$ to E'. Then $S' = \{q(<') : < \in S\}$ is a syntopogenous structure on E' satisfying $S'|E = S$.

PROOF: We first show that if $<_1, <_2 \in S$ are such that $<_1 \subseteq <_2$, then $<'_1 \subseteq <'_2$. Indeed, suppose that $A' <'_1 B'$ then according to 7.3 there are $A, B \subseteq E$ satisfying $A <_1 B$, $A' \subseteq h(A)$ and $h(B) \subseteq B'$. Because $A <_2 B$, it follows that $A' <'_2 B'$. We can conclude that S' is directed since $<'_1 \subseteq <'_2$ implies $q(<'_1) \subseteq q(<'_2)$.

Now suppose that $q(<'_1) \in S'$. Choose $<_2 \in S$ such that $<_1 \subseteq <_2^2$. We show that $<'_1 \subseteq (<'_2)^2$. Indeed, suppose $A' <'_1 B'$ and let $A, B \subseteq E'$ be such that $A <_1 B$, $A' \subseteq h(A)$ and $h(B) \subseteq B'$. There is $D \subseteq E$ such that $A <_2 D <_2 B$. According to 7.1.2 (a), $h(A) \subseteq h(D) \subseteq h(B)$. Set $D' = h(D)$, then it follows from the relations $A' \subseteq h(D)$, $h(D) \subseteq D'$, $D' \subseteq h(D)$ and $h(B) \subseteq B'$ that $A' <'_2 D' <'_2 B'$. Because q satisfies $q(<_2^2) \subseteq (q(<_2))^2$ (cf. 1.5.6 (e)), it follows that $S' \leqslant (S')^2$.

Finally, let $< \in S$. According to 7.1.6 $< \; = \; <'|E$ and so by 7.1.5 (b)

$$< \;=\; q(<) \;=\; q(<'|E) \;=\; q(<')|E \,.$$

But $\quad S'|E \;=\; \{q(<')|E : <' \in S'\}$.

$\square$

7.1.8 Let $E \subseteq E'$ and suppose S' is a syntopogenous structure on E' . We recall that E is <u>dense</u> in (E',S') iff each point of E' is the limit with respect to S' of a grill in E .

The next proposition provides a condition for our spaces to be dense in their extensions.

7.1.9 <u>Proposition [CS]</u>

Let S be a syntopogenous structure on the set E ; for each $< \in S$ let $<'$ extend $<$ to E' by K and let $S' = \{q(<') : < \in S\}$.

(a) If K consists of compressed filters in S , then $G(x) \rightarrow x(s(S'))$ for all $x \in E'$.

(b) If K consists of Cauchy filters in S , then $G(x) \rightarrow x(bs(S'))$ for all $x \in E'$.

<u>PROOF</u>:

(a): Let $x \in E'$ and consider a neighbourhood V of x in $(E',s(S'))$; thus $x \; sq(<') \; V$. On account of 2.4.2. (a) ,

$$sq(<') = s(<') = q(<' \cup c(<')) .$$

According to 1.5.3 there is a finite set J with $V = \bigcap\{V_j : j \in J\}$ and $x \ (<' \cup c(<')) \ V_j$ for $j \in J$. It follows that either $x <' V_j$ or $E' - V_j <' E' - x$.

If $x <' V_j$ there are by 7.1.3 $A_j, B_j \subseteq E$ with $A_j < B_j$ and $x \in h(A_j)$, $h(B_j) \subseteq V_j$. But from the definition of $h(A_j)$ (7.1.1), $x \in h(A_j)$ means that $G(x) \ (\subseteq) \ A_j$, and so there is $R_j \in G(x)$ with (cf. 7.1.2 (a), (b))

$$R_j \subseteq A_j \subseteq h(A_j) \subseteq h(B_j) \subseteq V_j .$$

If $E' - V_j <' E' - x$ there are by 7.1.3 $A_j, B_j \subseteq E$ with $A_j < B_j$, $E' - V_j \subseteq h(A_j)$ and $h(B_j) \subseteq E' - x$. Thus $x \in E' - h(B_j)$ and $E' - h(A_j) \subseteq V_j$. Since $G(x)$ is compressed in S, $R_j \in G(x)$ can be chosen such that the following statement is false (cf. 4.15.1).

$$R_j \cap A_j \neq \emptyset \neq R_j \cap (E-B_j) .$$

However, $x \in E' - h(B_j)$ implies that $R_j \cap (E-B_j) \neq \emptyset$ (cf. 7.1.1) and so $R_j \subseteq E - A_j \subseteq E \cap h(E-A_j)$.

We now show that $E \cap h(E-A_j) \subseteq E' - h(A_j)$, and it will follow that $R_j \subseteq E' - h(A_j) \subseteq V_j$ (see the first sentence in the paragraph above). Indeed, if $y \in E \cap h(E-A_j)$ then $y \in E - A_j$ by 7.1.2 (c), showing that no member of $G(y)$ is a subset of A_j and therefore $y \in E' - h(A_j)$, as required.

We have shown that for each $j \in J$ there is $R_j \in G(x)$ such that $R_j \subseteq V_j$. Taking $R \in G(x)$ with $R \subseteq \bigcap \{R_j : j \in J\}$, we have $R \subseteq \bigcap V_j = V$.

(b): It follows from (a) and 4.12 that $G(x) \rightarrow x(bs(S'))$ for $x \in E' - E$.

□

7.1.10 Proposition 7.1.9 shows that E is dense in $(E', s(S'))$.

7.2 Relative separation

This section deals with the problem of making the points of our extensions as separated as possible from each other and from the points of the space being extended.

7.2.1 Definition

(a) [C, p.251]. Let $E \subseteq E'$ and let S' be a syntopogenous structure on E'. We say that S' is relatively separated with respect to E iff given two points $x \neq y$ in E' there is an order $<' \in S'$ such that $x <' E' - y$ or $y <' E' - x$ unless both x and y belong to E.

(b) S' is <u>separated</u> iff given two points $x \neq y$ in E' there is an order $<' \in S'$ such that $x <' E' - y$ or $y <' E' - x$.

In the sequel we shall need to know under what conditions the structure $s(S)$ is relatively separated. This is examined in the next proposition.

7.2.2 Proposition [C, pp.197,198]

Let S' be a syntopogenous structure on the set E' and let $E \subseteq E'$. The following statements are true.

(a) For $x,y \in E'$, $x \ \mathrm{bts}(S') \ E - y$ iff there is $<' \in S'$ such that $x <' E' - y$ or $y <' E' - x$.

(b) If one of the structures S' or $s(S')$ is relatively separated with respect to E then so is the other.

PROOF:

(a): If $x <' E' - y$ or $y <' E' - x$ for $x,y \in E'$ and $<' \in S'$, then it follows that $x \ s(<') \ E - y$. But $s(S') \leq \mathrm{bts}(S')$.

Conversely, if $x \ \mathrm{bts}(S') \ E' - y$ then $x \ \mathrm{ts}(S') \ E' - y$ by 2.6.6. Next, $x \ s(<') \ E' - y$ for some $<' \in S'$ by 2.7.3, and so

$$x \ q(<' \cup c(<')) \ E' - y .$$

One sees, by applying 1.5.3 and examining the sets A_i , B_j that

$$x \ (\lessdot' \cup c(\lessdot')) \ E' - y .$$

Thus (a) follows.

(b): We have

$$\mathrm{bts}(s(S')) = \mathrm{btss}(S') = \mathrm{bts}(S') ,$$

and (b) follows easily from this.

$\square$

In our proof of 7.2.8 we shall need some purely technical auxiliary results. These are furnished by items 7.2.3 to 7.2.6.

7.2.3 Definition [CG, 16.51,16.2]

Let E , $<$, K , E' and h be as in 7.1.1. Define the set function g by

$$g(X) = \{x \in E' : \text{each member of } G(x) \text{ intersects } X\}$$

for each $X \subseteq E$.

Define the order $<^*$ on E' by

$A' <^* B'$ iff there are sets A , $B \subseteq E$ with $A < B$, $A' \subseteq h(A)$ and $g(B) \subseteq B'$.

By convention, we say that

$$<^* \text{ star-extends } < \text{ from } E \text{ to } E' \text{ by } K .$$

7.2.4 Proposition

Let E , K , E' and h be as in 7.1.1, and g as in 7.2.3. The following statements are true.

(a) If $A \subseteq B \subseteq E$, then $h(A) \subseteq g(A)$ and $g(A) \subseteq g(B)$.

(b) $\bigcup_j h(A_j) \subseteq h(\bigcup_j A_j)$.

(c) $g(\bigcap_j A_j) \subseteq \bigcap_j g(A_j)$.

(d) $E' - g(E-A) = h(A)$

(e) For finite J , $\bigcap_{j\in J} h(A_j) = h(\bigcap_{j\in J} A_j)$.

(f) For finite J , $\bigcup_{j\in J} g(A_j) = g(\bigcup_{j\in J} A_j)$.

PROOF:

(a): Follows at once from the definition.

(b): For each $j \in J$,

$$\begin{aligned} h(A_j) &= \{x \in E' : G(x) \ (\subseteq) \ A_j\} \\ &\subseteq \{x \in E' : G(x) \ (\subseteq) \ \textstyle\bigcup_j A_j\} \\ &= h(\textstyle\bigcup_j A_j) . \end{aligned}$$

(c): Follows easily, as (b) does, from the definition.

(d): If $x \in E' - g(E-A)$ then $G(x)$ has a member which is contained in A , so that $x \in h(A)$. Conversely if $x \in h(A)$, then $G(x) \ (\subseteq) \ A$ and so $x \in E' - g(E-A)$.

(e): If $x \in \bigcap h(A_j)$ then $G(x)$ $(\subseteq)$ A_j for each $j \in J$ and so, since J is finite, a suitable member of $G(x)$ is contained in $\bigcap A_j$. Thus $\bigcap h(A_j) \subseteq h(\bigcap A_j)$. The converse inclusion is easily proved.

(f): This follows from (e) and (d). □

7.2.5 Proposition [CG, items 16.61,16.56]

Let S be a syntopogenous structure on the set E. For each $< \in S$ let $<'$ extend $<$ to E' by K, let $<^*$ star-extend $<$ to E' by K, and let $S' = \{q(<') : < \in S\}$ and $S^* = \{q(<^*) : < \in S\}$. Suppose that K consists of compressed filters in S. The following statements are true.

(a) If $A < B$ and $< \in S$, then $g(A) \subseteq h(B)$.

(b) $s(S^*) \sim (s(S))^*$.

(c) $S' \sim S^*$.

PROOF:

(a): If $< \in S$, $A < B$ and $x \in g(A)$, then each member of $G(x)$ meets A. As $G(x)$ is compressed in S, there is a member $R \in G(x)$ satisfying 4.15.1 and $R \cap A \neq \emptyset$, and hence $R \subseteq B$. This implies that $G(x)$ $(\subseteq)$ B, so that $x \in h(B)$.

(b) Bearing in mind that $s = sq$ (2.4.2 (a)), suppose that A' $s(<^*)$ B', where $< \in S$. There are finite sets

I , J such that $A' = \bigcup\{A'_i : i \in I\}$, $B' = \bigcap\{B'_j : j \in J\}$ and $A'_i \; ({<^*} \cup c({<^*})) \; B'_j$ for each $i \in I$ and $j \in J$.

If $A'_i <^* B'_j$, then there are sets $A_i , B_j \subseteq E$ with $A_i < B_j$, $A'_i \subseteq h(A_i)$ and $g(B_j) \subseteq B'_j$. It follows that $A_i \; ({<} \cup c({<})) \; B_j$.

If $A'_i \; c({<^*}) \; B'_j$, then $E' - B'_j <^* E' - A'_i$ and there are sets $E - B_j , E - A_i \subseteq E$ with $E - B_j < E - A_i$, $E' - B'_j \subseteq h(E-B_j)$ and $g(E-A_i) \subseteq E' - A'_i$. It follows that $A_i \; ({<} \cup c({<})) \; B_j$, and $A'_i \subseteq h(A_i)$, $g(B_j) \subseteq B'_j$ on account of 7.2.4 (d).

We have shown that for $i \in I$, $j \in J$, $A'_i \; ({<^*} \cup c({<^*})) \; B'_j$ implies the existence of sets $A_i , B_j \subseteq E$ satisfying $A_i \; ({<} \cup c({<})) \; B_j$ and $A'_i \subseteq h(A_i)$, $g(B_j) \subseteq B'_j$.

By applying 7.2.4 (a), (b), (c) we have that $\bigcup A_i \; s({<}) \; \bigcap B_j$, $A' \subseteq h(\bigcup A_i)$ and $g(\bigcap B_j) \subseteq B'$, that is $A' \; (s({<}))^* \; B'$, and so $A' \; q((s({<}))^*) \; B'$.

Conversely, suppose that $A' \; (s({<}))^* \; B'$. There are sets $A , B \subseteq E$ with $A \; s({<}) \; B$, $A' \subseteq h(A)$ and $g(B) \subseteq B'$. Since $A \; s({<}) \; B$, there are finite sets I , J with $A = \bigcup\{A_i : i \in I$, $B = \bigcap\{B_j : j \in J\}$ and $A_i \; ({<} \cup c({<})) \; B_j$.

Let $< \subseteq <_1^3$, where $<_1 \in S$. If $A_i < B_j$ it follows that there are sets $D, F \subseteq E$ with $A_i <_1 D <_1 F <_1 B_j$ and $h(D) <_1^* g(F)$. By 7.2.5 (a) $g(A_i) \subseteq h(D) <_1^* g(F) \subseteq h(B_j)$. Thus $g(A_i)\ (<_1^* \cup c(<_1^*))\ h(B_j)$.

If $A_i\ c(<)\ B_j$ then there are sets $D_1, F_1 \subseteq E$ with $E - B_j <_1 D_1 <_1 F_1 <_1 E - A_i$ and $h(D_1) <_1^* g(F_1)$. By 7.2.5 (a) $g(E-B_j) \subseteq h(D_1) <_1^* g(F_1) \subseteq h(E-A_i)$. Thus $g(E-B_j)\ (<_1^* \cup c(<_1^*))\ h(E-A_i)$ and so by 7.2.4 (d) $E' - h(B_j)\ (<_1^* \cup c(<_1^*))\ E' - g(A_i)$ whence $g(A_i)\ (<_1^* \cup c(<_1^*))\ h(B_j)$.

We have shown that $A_i\ (< \cup c(<))\ B_j$ implies $g(A_i)\ (<_1^* \cup c(<_1^*))\ h(B_j)$. It follows from 7.2.4 (e), (f) that $g(A)\ q(<_1^* \cup c(<_1^*))\ h(B)$. Thus by 7.2.4 (a) $h(A)\ s(<_1^*)\ g(B)$ and it follows that $A'\ s(<_1^*)\ B'$. Thus $A'\ q((s(<))^*)\ B'$ implies $A'\ s(<_1^*)\ B'$.

(c): Let $A' <^* B'$. There are $A, B \subseteq E$ with $A < B$, $A' \subseteq h(A)$ and $g(B) \subseteq B'$. But $h(B) \subseteq g(B)$ and therefore $A' <' B'$.

On the other hand if $A' <' B'$, then there are $A, B \subseteq E$ with $A < B$, $A' \subseteq h(A)$ and $h(B) \subseteq B'$. Choosing $<_1 \in S$ such that $< \subseteq <_1^2$, we have $D \subseteq E$ satisfying $A <_1 D <_1 B$. By 7.2.5 (a) $g(D) \subseteq h(B) \subseteq B'$, so that $A' <_1^* B'$. It follows that $S' \sim S^*$.

$\square$

7.2.6 Corollary

With the data of 7.2.5, $s(S') \sim (s(S))'$.

PROOF: $s(S') \sim s(S^*) \sim (s(S))^* \sim (s(S))'$.

□

7.2.7 Definition [CG, p.240]

Let S be a syntopogenous structure on the set E and let G be a grill in E , then G is round in S iff for each $R \in G$ there are $< \in S$ and $R' \in G$ satisfying $R' < R$.

This definition clearly agrees with standard terminology for uniform and proximity spaces (cf. [T]) .

The following proposition is important.

7.2.8 Proposition [CS; CG, prop.16.48]

Let S be a syntopogenous structure on the set E and let K consist of filters in E which are compressed and round in $s(S)$ but not convergent in $s(S)$. For each $< \in S$ let $<'$ extend $<$ to E' by K , and let $S' = \{q(<') : < \in S\}$. Then the syntopogenous structure S' is relatively separated with respect to E .

PROOF: Suppose $x,y \in E'$. Let us say that $s(S')$ separates x and y iff there is $<' \in S'$ such that $x \; s(<') \; E' - y$ or $y \; s(<') \; E' - x$.

Let $N(x)$ denote the set of neighbourhoods of x in $s(S')$. We show that if $s(S')$ does not separate x and y then $N(x) = N(y)$. Indeed, suppose that for all $<' \in S'$ $x \; s(<') \; E' - y$ is false, then $y \; s(<') \; E'-x$ is false. Let $V' \in N(x)$. There is $<' \in S'$ such that $x \; s(<') \; V'$. There are $<_1' \in S'$ and V_1' such that

$$x \; s(<_1') \; V_1' \; s(<_1') \; V' .$$

If $y \in E' - V_1'$ then $s \; s(<_1') \; E' - y$, which is impossible, therefore $y \in V_1'$ and so $V' \in N(y)$. Thus $N(x) \subseteq N(y)$. Interchanging the roles of x and y, we see that $N(y) \subseteq N(x)$ and so $N(x) = N(y)$.

Next we show that for $x \in E' - E$, $G(x) = N(x)|E = \{N \cap E : N \in N(x)\}$. On the one hand, if $V \in N(x)$ then there is $R \in G(x)$ with $R \subseteq V$ because $G(x) \to x(s(S'))$ (cf. 7.1.9). It follows that $N(x)|E \subseteq G(x)$. On the other hand suppose that $R \in G(x)$. We show that $h(R) \in N(x)$, and it will follow that $G(x) \subseteq N(x)|E$ because by 7.1.2 (c) $h(R) \cap E \subseteq R$. In fact, as $G(x)$ is round in $s(S)$, there are $A \in G(x)$ and $< \in S$ such that $A \; s(<) \; R$. Since $G(x) \; (\subseteq) \; A$ we have that $x \in h(A)$. It follows from 7.1.3 that $s \; (s(<))' \; h(R)$, and from 7.2.6 that there is $<_1 \in S$ such that $x \; s(<_1') \; h(R)$. Hence $h(R) \in N(x)$ as required.

Now if $x \neq y$ with $x, y \in E' - E$ then $G(x) \neq G(y)$, implying that $N(x)|E \neq N(y)|E$ and so $N(x) \neq N(y)$. It follows from what has been proved above that $s(S')$ separates x and y.

If $x \in E$ and $y \in E' - E$ then $G(y)$ does not converge to x with respect to $s(S)$, so there is a neighbourhood $V \subseteq E$ of x with $< \in s(S)$ such that $x \; s(<) \; V$ and V contains no member of $G(y)$. Thus V contains no member of $N(y)|E$, that is $V \notin N(y)\ E$. However by 2.2.1 (f) and 7.1.6, $s(S) = s(S'|E) = s(S')|E$ and so $V \in N(x)|E$, whence $N(y)|E \neq N(x)|E$. We can conclude that $N(y) \neq N(x)$ and that $s(S')$ separates x and y. The result now follows from definition 7.2.1 and 7.2.2 (b).

□

7.3 Double reflections in Syn

7.3.1 Definition

Let $E \subseteq E'$ and let S and S' be syntopogenous structures on E and E' respectively such that $S'|E = S$.

(a) [CS]. (E', S') is a double compactification of (E, S) iff $s(S')$ is compact, E is dense in $s(S')$ and S' is relatively separated with respect to E.

(b) (E', S') is a double quasirealcompactification of (E, S) iff $s(S')$ is quasirealcompact, E is dense in $s(S')$ and S' is relatively separated with respect to E.

(c) [CS] . (E',S') is a <u>double completion</u> of (E,S) iff $s(S')$ is complete, E is dense in $bs(S')$ and S' is relatively separated with respect to E .

We mention that in [C, p.253] a double completion is called a completion.

Existence of double extensions is studied in items 7.3.2 to 7.3.4, and their uniqueness in 7.3.5 to 7.3.11. Reflectiveness is studied in items 7.3.12 to 7.3.16.

The next proposition produces double compactifications for arbitrary structures. Double compactifications are important, amongst other reasons, because they provide enough points with which to construct our other extensions. Indeed, we construct double quasirealcompactifications and double completions by suitably restricting double compactifications.

7.3.2 <u>Proposition [CS]</u>

Let (E,S) be a syntopogenous space. Then there is a double compactification (E',S') of (E,S) .

<u>PROOF</u>: Let $K = \{G(x) : x \in I\}$ be the set of all round, compressed filters in $s(S)$ which are not convergent in $s(S)$. We may suppose that the indexing of K is one-to-one. For each $< \in S$ let $<'$ extend $<$ to E' by K and let $S' = q(<') : < \in S\}$.

Proposition 7.2.8 shows that S' is relatively separated with respect to E, while according to 7.1.10 E is dense in $(E',s(S'))$.

We must show that $(E',s(S'))$ is compact. First, we show that any compressed grill in $s(S')$ generates a round, compressed filter in $s(S')$ and then that all the round, compressed filters in $s(S')$ converge. It follows that all the compressed grills in $s(S')$ converge.

Accordingly, let F be a compressed grill in $s(S')$. Define F' by $F' = \{R' : \text{there is } R \in F \text{ and there is } < \in s(S') \text{ such that } R < R'\}$. F' is easily seen to be a round filter in $s(S')$. F' is compressed in $s(S')$. Indeed, suppose that $A < B$ with $A, B \subseteq E'$ and $< \in s(S')$. There are $D, H \subseteq E'$ and $<_1 \in s(S')$ such that $A <_1 D <_1 H <_1 B$. Since F is compressed, we have $R \in F$ such that $R \cap D = \emptyset$ or $R \subseteq H$. If $R \cap D = \emptyset$ then $R \subseteq E' - D <_1 E' - A$, so that $E' - A \in F'$ and : $(E'-A) \cap (E'-B) = \emptyset$ or $A \cap (E'-A) = \emptyset$. If $R \cap D \neq \emptyset$ then $R \subseteq H <_1 B$, so that $B \in F'$ and : $B \cap A = \emptyset$ or $B \subseteq B$. Thus F' is compressed in $s(S')$. Clearly, if F' converges in $s(S')$ then so does F.

Now let G be a round, compressed filter in $s(S')$. Let $V \in G$. Then $V \cap E \neq \emptyset$. Indeed, there are, by roundness, $R \in G$ and $< \in s(S')$ such that $R < V$. Let $x \in R$; V is an $s(S')$-neighbourhood of x. Since $G(x)$ converges to x with respect to $s(S')$ there is $R' \in G(x)$ with $R' \subseteq V$ and it follows that $\emptyset \neq R' \subseteq V \cap E$.

Using what has just been proved one easily sees that $G|E = \{R \cap E : R \in G\}$ is a round grill in $s(S)$.

$G|E$ is compressed in $s(S)$. Indeed, suppose for $A, B \subseteq E$ that $A \; s(<) \; B$ with $< \in S$, then $A \; s(<') \; B \cup (E'-E)$ (cf. 7.1.5 (a)). Because G is compressed in $s(S')$, there is $R' \in G$ such that $R' \cap A = \emptyset$ or $R' \subseteq B \cup (E'-E)$. Letting $R = R' \cap E$ one sees that $R \in G|E$ and : $R \cap A = \emptyset$ or $R \subseteq B$.

Let G' be the filter in E generated by the compressed, round grill $G|E$. Then G' is compressed and round in $s(S)$. G' converges in $s(S')$ because it either converges in $s(S)$ or, by choice of K and 7.1.9, it converges in $s(S')$ so that $G|E$ converges in $s(S')$ to x, say. We show that G converges to x in $s(S')$. Indeed, let $x < D < V$ for suitable $< \in s(S')$ and $D, V \subseteq E'$. There is $R_1 \in G$ such that $R_1 \cap E \subseteq D$. Since G is compressed in $s(S')$ there is $R_2 \in G$ such that $R_2 \cap D = \emptyset$ or $R_2 \subseteq V$. However $R_2 \cap D \neq \emptyset$ because $(R_2 \cap E) \cap (R_1 \cap E) \neq \emptyset$, $G|E$ being a grill. It follows that $R_2 \subseteq V$.

□

7.3.3 Proposition

Let S be a syntopogenous structure on the set E and let (E', S') be a double compactification of (E, S). Then every grill which is compressed in S converges in $s(S')$.

PROOF: Let G be a compressed grill in S . G is compressed in S' . Indeed, let $A <' B$ where $A , B \subseteq E'$ and $<' \in S'$. If $A \cap E = \emptyset$ then any member R of G trivially satisfies the compression condition : $R \cap A \neq \emptyset$ implies $R \subseteq B$. If $A \quad E \neq \emptyset$ then by 7.1.5 (a)

$$A \cap E <'|E \; B \cap E .$$

Any member R of G satisfying the compression condition for $A \cap E$, $B \cap E$ and $<'|E$ will satisfy the compression condition for A , B and $<'$.

G, being compressed in S' , is compressed in $s(S')$ by 4.18 and so converges there.

$\square$

We now produce double quasirealcompactifications and double completions.

7.3.4 Proposition

Let S be a syntopogenous structure on the set E .

(a) There is a double quasirealcompactification, (E_1,S_1) , of (E,S) with the property that each point of E_1 is the limit in $s(S_1)$ of a grill in E with the countable intersection property.

(b) [C, p.253] . There is a double completion, (E_2,S_2) , of (E,S) .

PROOF: By 7.3.2 we have a double compactification (E',S') of (E,S) .

(a): Let X denote the set of all points in E' which are the limits in $s(S')$ of grills in E compressed in $s(S)$ with the countable intersection property, let $E_1 = E \cup X$ and $S_1 = S'|E_1$.

S_1 is relatively separated with respect to E . Indeed, suppose that x and y are distinct points of E_1 , not both in E . S' is relatively separated with respect to E , so suppose that $x <' E' - y$ where $<' \in S'$. It follows that $x <' (E_1-y)\cup(E'-E_1)$ and so $x \ (<'|E_1) \ E_1-y$. But $<'|E_1 \in S_1$, as required.

To see that E is dense in $(E_1,s(S_1))$, let $x \in E_1$. If $x \in E$ then x is the limit in $s(S')$ of the filter generated by $\{x\}$. If $x \in E_1 - E$ then x is the limit in $s(S')$ of a compressed grill in S with the countable intersection property. But if a grill in E converges in $s(S')$ then it converges in $s(S_1)$.

We must show that $(E_1,s(S_1))$ is quasirealcompact. Accordingly, let F be a grill in E_1 , compressed in $s(S_1)$ with the countable intersection property . We show that F generates a round grill G in $s(S_1)$ whose restriction to E is compressed in $s(S)$. Indeed, let $G = \{R' : \text{there is } R \in F \text{ and } < \in s(S_1) \text{ with } R < R'\}$. G is easily seen to be a round filter in $s(S_1)$ with the countable intersection property. G is compressed in $s(S_1)$, for suppose that $A < B$ where $< \in s(S_1)$. There are $D, H \subseteq E_1$ and $<_1 \in s(S_1)$ such that $A <_1 D <_1 H <_1 B$. Let $R \in F$ be such that $R \cap D = \emptyset$ or $R \subseteq H$. If $R \cap D = \emptyset$ then $R \subseteq E_1 - D <_1 E_1 - A$ and thus $E_1 - A$ is in G and 4.15.1 is satisfied. If $R \cap D \neq \emptyset$ then $R \subseteq H <_1 B$ and thus B is in G and 4.15.1 is satisfied. We have shown that G is compressed in $s(S_1)$.

Next, G has the property that if $V_i \in G$ for $i = 1,2,\ldots$, then $\bigcap\{V_i : i = 1,2,\ldots\} \cap E \neq \emptyset$. Indeed, there are by roundness $R_i \in G$ and $<_i \in s(S_1)$ such that $R_i <_i V_i$ for $i = 1,2,\ldots$. There is $x \in \bigcap\{R_i : i = 1,2,\ldots\}$, thus for each i V_i is an $s(S_1)$-neighbourhood of $x \in E_1$. By the definition of E_1 there is a grill G' in E with the countable intersection property converging to x in $s(S_1)$ and so there is for each i an $R'_i \in G'$ with $R'_i \subseteq V_i$. But $\emptyset \neq \bigcap R'_i \subseteq \bigcap V_i$. It follows that $G|E$ is a grill in E with the countable intersection property.

$G|E$ is compressed in $s(S)$. Indeed, suppose for $A, B \subseteq E$ that $A \; s(<) \; B$ with $< \in S$. Then $A \; s(<_1) \; B \cup (E_1 - E)$ where $<_1 = <'|E_1$. Because G is compressed in $s(S_1)$ there is $R' \in G$ such that $R' \cap A = \emptyset$ or $R' \subseteq B \cup (E_1 - E)$. Letting $R = R' \cap E$ it follows that $R \in G|E$ and $R \cap A = \emptyset$ or $R \subseteq B$.

By 7.3.3 and the definition of (E_1, S_1) $G|E$ converges in $s(S_1)$. It follows that G converges in $s(S_1)$. Indeed, suppose that $G|E \to x(s(S_1))$. Let $x < D < V$ where $< \in s(S_1)$. There is $R_1 \in G$ such that $R_1 \cap E \subseteq D$. Because G is compressed in $s(S_1)$ there is $R_2 \in G$ such that $R_2 \cap D = \emptyset$ or $R_2 \subseteq V$. However $R_2 \cap D \neq \emptyset$ since

$$(R_1 \cap E) \cap (R_2 \cap E) \neq \emptyset ,$$

$G|E$ being a grill. It follows that $R_2 \subseteq V$, and $G \to x(s(S_1))$.

Because G converges in $s(S_1)$ so does F.

(b): Let X denote the set of all points in E' which are the limits with respect to $s(S')$ of grills in E which are Cauchy in $s(S)$, let $E_1 = E \cup X$ and $S_1 = S'|E_1$.

Using the same arguments as in (a) one sees that S_1 is relatively separated with respect to E. Also by similar argument to that in (a) E is dense in $(E_1, s(S_1))$. Since the filters generated by $\{x\}$ for $x \in E$ and the grills which determine X are Cauchy, it follows from 4.12 that E is dense in $(E_1, bs(S_1))$.

We must show that $(E_1, s(S_1))$ is complete. Accordingly let F be a grill in E_1 which is Cauchy in $s(S_1)$.

We show that F generates a round grill in $bs(S_1)$ whose restriction to E is Cauchy in $bs(S)$. Indeed, according to 4.11 F is Cauchy in $bs(S_1)$. Let $G = \{R' \subseteq E_1 :$ there are $< \in bs(S_1)$ and $R \in F$ with $R < R'\}$. Let $R(<) = \bigcap \{R' : R < R'\}$ where $R \in F$ and $< \in bs(S_1)$. Then $R < R(<)$ because $<$ is biperfect. It follows that G is a round filter in $bs(S_1)$.

G is Cauchy in $bs(S_1)$. Indeed, suppose that $< \in bs(S_1)$, let $<_1 \in bs(S_1)$ be such that $< \subseteq <_1^4$ and choose $R \in F$ satisfying the Cauchy condition, 4.8.2, for $<_1$. If $A < B$ then there are sets D and H satisfying $A <_1 D <_1 H <_1 B$. If $R \cap D \neq \emptyset$ then $R \subseteq H$ and $R(<_1) \subseteq B$. If $R \cap D = \emptyset$ then $R \subseteq E_1 - D <_1 E_1 - A$ and $R(<_1) \subseteq E_1 - A$. Thus if $R(<_1)$ meets both A and $E_1 - B$ then $R \cap D \neq \emptyset$ and $R \cap D = \emptyset$, which is impossible. It has been shown that G is Cauchy in $bs(S_1)$.

We show that $G|E$ is a Cauchy grill in $bs(S)$.

First, for each $V \in G$, $V \cap E \neq \emptyset$. In fact there are, by roundness in $bs(S_1)$, $R \in G$ and $bs(<)$, where $< \in S_1$, such that $R \; bs(<) \; V$. Let $x \in R$. Then $x \; bs(<) \; V$. By the denseness of E in $(E_1, bs(S_1))$ there is a grill G' in E converging to x in $bs(S_1)$, and so there is $R' \in G'$ with $R' \subseteq V$. It follows that $V \cap E \neq \emptyset$. Thus $G|E$ is a grill in E.

Next, $G|E$ is Cauchy in $bs(S)$. Indeed, let $< \in S$ and let $R' \in G$ satisfy the Cauchy condition 4.8.2 for $<_1 \in S_1$, where $<_1 = <'|E_1$. Let $R = R' \cap E$ then $R \in G|E$. Suppose that $A \ bs(<) \ B$ and $R \cap A \neq \emptyset$, then $A \ bs(<_1) \ B \cup (E_1 - E)$ and $R' \cap A \neq \emptyset$, whence $R' \subseteq B \cup (E_1 - E)$. It follows that $R \subseteq B$.

By 7.3.3 and the definition of (E_1, S_1) $G|E$ converges in $s(S_1)$. It follows by a similar argument to that in (a) that G converges in $s(S_1)$ and therefore so does F.

□

We now deal with the uniqueness of the extensions produced so far. Some technical results are collected together in the next proposition.

7.3.5 Proposition

Let E and E' be sets, f a function from E into E' and S, S' syntopogenous structures on E and E' respectively. The following statements are true.

(a) If $B' \subseteq E'$ then $B' \subseteq E' - f(E - f^{-1}(B'))$.

(b) If G is a grill in E then $f(G)$ is a grill in E'.

(c) If G is a Cauchy (compressed) grill in S and f is (S,S')-continuous then $f(G)$ is Cauchy (compressed) in S'.

(d) If f is (S,S')-continuous and G is a grill in E, then $G \rightarrow x(S)$ implies $f(G) \rightarrow f(x)(S')$.

(e) Let $E \subseteq E'$. Then E is dense in (E',S') iff for each $x \in E'$, every neighbourhood of x in S' intersects E.

(f) Let the sets $E_0 \subseteq E_1 \subseteq E_2$ and let S_1, S_2 by syntopogenous structures on E_1 and E_2 respectively satisfying $S_2|E = S_1$. E_0 is dense in (E_2,S_2) whenever E_o is dense in (E_1,S_1) and E_1 is dense in (E_2,S_2).

(g) Let E_0, E_1, E_2, S_1 and S_2 be as in (f) above. Then S_2 is relatively separated with respect to E_o whenever S_1 is relatively separated with respect to E_o and S_2 is relatively separated with respect to E_1.

PROOF: We prove (c) and (d); the remaining statements are straightforward.

(c): Suppose that G is Cauchy in S. Let $<' \in S'$ be given and choose $< \in S$ such that $f^{-1}(<') \subseteq <$. Let $R \in G$ satisfy the Cauchy condition 4.8.2 for $<$. Now suppose that $A' <' B'$. From (a) it follows that $f^{-1}(A') < f^{-1}(B')$. If $f(R) \cap A' \neq \emptyset$ then $R \cap f^{-1}(A') \neq \emptyset$ and so $R \subseteq f^{-1}(B')$, whence $f(R) \subseteq B'$. We have shown that $f(R)$ satisfies the Cauchy condition 4.8.2 for $<'$.

The argument for compressed grills is similar.

(d): By continuity of f and by (a), given $<' \in S'$ one has $< \in S$ such that $f(x) <' V'$ implies $f^{-1}f(x) < f^{-1}(V')$, and so $x < f^{-1}(V')$. Now $G \rightarrow x(S)$, therefore there is $R \in G$ with $R \subseteq f^{-1}(V')$. It follows that $f(R) \subseteq V'$.

□

We need the following criterion for the continuity of a function.

7.3.6 Proposition [C, prop.16.33]

Let f be a function from E into E', S and S' syntopogenous structures on E and E' respectively, $E_0 \subseteq E$ with E_0 dense in $s(S)$, $f_0 = f|E$ an $(S|E_0,S')$-continuous mapping and suppose the following condition is satisfied:

for each grill G in E_0, $G \rightarrow x(s(S))$ implies $f_0(G) \rightarrow f(x)(s(S'))$.

Then it follows that f is (S,S')-continuous.

<u>PROOF</u>: We must show that $f^{-1}(S') \leq S$. Accordingly let $A \; f^{-1}(<') \; B$ for $A, B \subseteq E$ and $<' \in S'$. We shall produce an order $<_2 \in S$ satisfying : $f(A) <' E' - f(E-B)$ implies $A <_2 B$. It will then follow that f is continuous.

Indeed, suppose that $f(A) <' E' - f(E-B)$. Let $<'_1$ and $<'_2 \in S'$ be such that $<' \subseteq (<'_1)^2$ and $<'_1 \subseteq (<'_2)^2$.

By continuity of f_0 we can choose $<, <_1, <_2 \in S$ so that $f_0^{-1}(<_2') \subseteq <|E_0$, $< \subseteq <_1^2$ and $<_1 \subseteq <_2^2$. Now choose sets $C', D' \subseteq E'$ with

7.3.7 $\quad f(A) <_2' C' <_2' D' <_2' E' - f(E-B)$.

Next, $f_0^{-1}(C') \; f_0^{-1}(<_2') \; f_0^{-1}(D')$ because $C' <_2' D'$ and $D' \subseteq E' - f_0(E-f_0^{-1}(D'))$ (cf. 7.3.5 (a)). This means that $f_0^{-1}(C') < f_0^{-1}(D') \cup (E-E_0)$. Choose $C, D \subseteq E$ satisfying

7.3.8 $\quad f_0^{-1}(C') <_2 C <_2 D <_2 f_0^{-1}(D') \cup (E-E_0)$.

We show that $A \subseteq C$ and $D \subseteq B$. Indeed, let $x \in A$ then by denseness of E_0 in $s(S)$ there is a grill G in E_0 with $G \rightarrow x(s(S))$. Thus $f_0(G) \rightarrow f(x)(s(S'))$. Now $f(x) \in f(A) <_2' C'$ (7.3.7) and so there is a suitable member R of G with $f_0(R) \subseteq C'$ and $R \subseteq f_0^{-1}(C')$. If $x \in E - C$ then $x \; c(<_2) \; (E-f_0^{-1}(C'))$ (7.3.8) . Since $c(<_2) \subseteq s(<_2)$ this implies $x \; s(<_2) \; E - f_0^{-1}(C')$, whence $E - f_0^{-1}(C')$ contains a member of G , contradicting $R \subseteq f_0^{-1}(C')$. Thus $A \subseteq C$ as required. Now let $x \in E - B$. There is a grill G in E_0 converging to x in $s(S)$. Thus $f_0(g) \rightarrow f(x)(s(S'))$, $f(x) \in f(E-B)$ and so $f(x) \; c(<_2') \; E' - D'$ (7.3.7). This implies that $f(x) \; s(<_2') \; E' - D'$ and so there is $R \in G$ with $R \subseteq E_0$ and and $f_0(R) \subseteq E' - D'$, whence $R \subseteq f_0^{-1}(E' - D')$ and therefore

$$R \cap (f_0^{-1}(D') \cup (E-E_0)) = \emptyset .$$

If $x \in D$ then $f_0^{-1}(D') \cup (E-E_0)$ contains a member of G (7.3.8), which is impossible. Thus $x \in E - D$ and we have proved that $D \subseteq B$.

It follows immediately that $A <_2 B$.

□

The following result is standard.

7.3.9 Proposition

Suppose that (E_1,S_1) and (E_2,S_2) are symmetric syntopogenous spaces such that E is dense and relatively separated in both S_1 and S_2. Let $f_1 : S_1 \longrightarrow S_2$ and $f_2 : S_2 \longrightarrow S_1$ be continuous extensions of the inclusions $i_1 : E \subseteq E_1$ and $i_2 : E \subseteq E_2$ respectively. Then $f_2f_1 : E_1 \longrightarrow E_1$ as well as $f_1f_2 : E_2 \longrightarrow E_2$ are the identities. Thus f_1 and f_2 are isomorphisms inverse to each other.

PROOF: The composition f_2f_1 is the identity $i : E_1 \longrightarrow E_1$. Indeed, suppose on the contrary there is $x \in E_1 - E$ such that $f_2f_1(x) = y \neq x$. By denseness of E in (E_1,S_1) there is a grill G in E coverging to x in S_1. Now f_2f_1 is continuous in S_1 and so by 7.3.5 (d) $f_2f_1(G) \rightarrow f_2f_1(x) = y$. But $f_2f_1 = i$ on all members of G and so G converges to both x and y. Since $x \neq y$, a symmetric order $< \in S_1$ and a set $D \subseteq E_1$ can be chosen such that $x < D < E - y$, from which a contradiction can be derived, whence $x = y$.

In a similar way it can be seen that f_1f_2 is the identity on E_2.

$\square$

Now the uniqueness theorems can be proved.

7.3.10 Proposition

Let (E_0,S_0) be a syntopogenous space.

(a) [C, prop.16.49]. Let (E,S) and (E',S') be double compactifications of (E_0,S_0). Then the inclusion $i : E_0 \subseteq E'$ extends to a unique isomorphism of (E,S) onto (E',S').

(b) Let (E,S) and (E',S') be double quasirealcompactifications of (E_0,S_0) with the property that each point of E (resp. E') is the limit in $s(S)$ (resp. $s(S')$) of a grill in E_0 with the countable intersection property. Then the inclusion $i : E_0 \subseteq E'$ extends to a unique isomorphism of (E,S) onto (E',S').

(c) [C, prop.16.34]. Let (E,S) and (E',S') be double completions of (E_0,S_0). Then the inclusion $i : E_0 \subseteq E'$ extends to a unique isomorphism of (E,S) onto (E',S').

PROOF:

(a): We first provide a function $f : E-E_0 \rightarrow E'-E_0$. Indeed, let $x \in E - E_0$. There is a grill G in E_0 satisfying $G \rightarrow x(s(S))$. By 4.15 G is compressed in $s(S)$, therefore also in $S_0 = S|E_0$, and so there is $x' \in E'$ such that $G \rightarrow x'(s(S'))$.

We show that x' cannot be a member of E_0. Suppose on the contrary that $x' \in E_0$. First, $G \rightarrow x'(s(S))$. Indeed, if V is an $s(S)$-neighbourhood of x', then $(V \cap E_0) \cup (E'-E_0)$ is an $s(S')$-neighbourhood of x'. Since $G \rightarrow x'(s(S'))$, there is $R \in G$ such that $R \subseteq (V \cap E_0) \cup (E'-E_0)$. But $R \subseteq E_0$, whence $R \subseteq V$. Next, S and hence $s(S)$ (cf. 7.2.2) is relatively separated with respect to E_0, so there exists $< \in S$ such that $x' \; s(<) \; E - x$ and there exist $<_1 \in S$ and $D \subseteq E$ with $x' \; s(<_1) \; D \; s(<_1) \; E - x$. This leads to a contradiction because there are $R_1, R_2 \in G$ with $R_1 \subseteq D$ and $R_2 \subseteq E - D$ on account of G converging to both x' and x in $s(S)$.

We show that x', constructed as above, is uniquely defined by convergence. Indeed, suppose G_1 is a grill, different from G, in E_0 converging in $s(S)$ to x. Define the grill $G \; (\cup) \; G_1$ to be $\{R \cup R_1 : R \in G, R_1 \in G_1\}$ then $G \; (\cup) \; G_1$ converges to x in $s(S)$, is compressed in $s(S)$ by 4.15, and therefore also in $S_0 = S|E_0$, and thus converges in $s(S')$ to a point $x_1' \in E'$. It follows that $G \rightarrow x_1'(s(S'))$. As was shown above x_1' must belong to $E' - E_0$. Now suppose that $x' \neq x_1'$. S' and hence $s(S')$ (cf. 7.2.2) is relatively separated with respect to E_0. Thus there are suitable $<' \in S'$ and D such that $x' \; s(<') \; D \; s(<') \; E' - x_1'$. As above, this leads to a contradiction because G converges to both x' and x_1' in $s(S')$. Thus $x_1' = x'$ and f can be defined by

$$f(x) = x' \quad \text{for} \quad x \in E - E_0$$
$$f(x) = x \quad \text{for} \quad x \in E_0 .$$

We must show that f is (S,S')-continuous. But this follows easily from 7.3.6, using the construction of x' if $x \in E - E_0$ and if $x \in E_0$, from 7.3.5 (d).

Interchanging the roles of S and S' in the above argument provides an (S',S)-continuous extension, f_1, of the inclusion $i_1 : E_0 \subseteq E'$. Now, f is also $(s(S),s(S'))$-continuous, f_1 is $(s(S'),s(S))$-continuous (cf. 2.2.4), E_0 is dense in $s(S)$ and in $s(S')$, and $s(S)$ and $s(S')$ are both relatively separated with respect to E_0. It follows from 7.3.9 that $ff_1 : E' \to E'$ and $f_1f : E \to E$ are the identities. Thus f is an isomorphism onto E'.

If g is also an (S,S')-continuous extension of the inclusion $i : E_0 \subseteq E'$ then g is $(s(S),s(S'))$-continuous (cf. 2.2.4). It follows, as above from 7.3.9, that f_1g is the identity on E, whence $g = f$ and f is unique.

(b): Given the double quasirealcompactifications (E,S) and (E',S') of (E_0,S_0) with the property that each point of E (resp. E') is the limit in $s(S)$ (resp. $s(S')$) of a grill in E_0 with the countable intersection property, let (E_1,S_1) and (E_2,S_2) be double compactifications of (E,S) and (E',S') respectively. By 7.3.5 (f) and (g) (E_1,S_1)

and (E_2,S_2) are double compactifications of (E_0,S_0). According to 7.3.10 (a) there is a unique isomorphism $g : S_1 \rightarrow S_2$, onto E_2, which extends the inclusion $i : E_0 \subseteq E_2$.

We show that $g(E) \subseteq E'$. Indeed, if $x \in E_0$ then x is clearly in E'. If $x \in E - E_0$ then by hypothesis x is the limit in $s(S)$ of a grill G in E_0 with the countable intersection property. G is compressed in S_0, hence compressed in S' and so in $s(S')$, thus G converges in $s(S')$ to $x' \in E'$, say. It follows that G converges to x' in $s(S_2)$. According to 7.3.5 (d) $G = g(G)$ converges to $g(x)$ in $s(S_2)$. But $g(x)$ is uniquely defined by convergence (see (a)) and therefore $g(x) = x' \in E'$, as required.

Letting $f = g|E$, we have shown that f is a function from E into E'. Now if G is an arbitrary grill in E_0 converging to x in $s(S)$, then G converges to x in $s(S_1)$ and by 7.3.5 (d) $g(G) \rightarrow g(x)(s(S_2))$. It follows that $f(G) \rightarrow f(x)\ (s(S'))$. By 7.3.6 f is (S,S')-continous, whence by 2.2.4 f is $(s(S),s(S'))$-continuous.

In the same way $f^{-1} = g^{-1}|E$ is $(s(S'),s(S))$-continuous and it follows from 7.3.9 that f is an isomorphism onto E'.

If $w : S \rightarrow S'$ is another extension of the inclusion $E_0 \subseteq E'$ then $w : s(S) \rightarrow s(S')$ and from 7.3.9 we have that $w = f$.

(c): The pattern of the proof is the same as that of (b) .

□

7.3.11 Definition

Let (E,S) be a double quasirealcompactification of (E_0,S_0) with the property that each point of E is the limit in s(S) of a grill in E_0 with the countable intersection property. Then (E,S) is called a universal double quasirealcompactification of (E_0,S_0) .

Thus 7.3.10 (b) says that universal double quasirealcompactifications are unique up to isomorphism.

We remark that double compactifications and double completions are 'rigid'. That is, any two double compactifications or double completions of the same space are isomorphic. We do not know if double quasirealcompactifications are 'rigid'.

The next proposition is an essential step in proving that our extensions are reflections.

7.3.12 Proposition

(a) [C, prop.16.45] . Let (E_0,S_0) be a syntopogenous space, (E,S) a double compactification of (E_0,S_0) and (E',S') a separated doubly compact syntopogenous space. Let f_0 be (S_0,S')-continuous. Then f_0 extends to a unique (S,S')-continuous mapping.

(b) Let (E_0,S_0) be a syntopogenous space, (E,S) the universal double quasirealcompactification of (E_0,S_0) and (E',S') a separated doubly quasirealcompact syntopogenous space. Let f_0 be (S_0,S')-continuous. Then f_0 extends to a unique (S,S')-continuous mapping.

(c) [C, prop.16.30]. Let (E_0,S_0) be a syntopogenous space, (E,S) a double completion of (E_0,S_0) and (E',S') a separated doubly complete syntopogenous space. Let f_0 be (S_0,S')-continuous. Then f_0 extends to a unique (S,S')-continuous mapping.

PROOF: The proof proceeds along essentially the same lines as that of 7.3.10.

We first show that any convergent grill in a symmetric separated space (E_1,S_1) has a unique limit. Suppose, on the contrary, the grill G converges in S_1 to both x and y with $x \neq y$. There are $<_1 \in S_1$ and $D \subseteq E_1$ such that $x <_1 D <_1 E_1 - y$. Using the fact that the grill G converges and $<_1$ is symmetric, one chooses $R_1, R_2 \in G$ such that $R_1 \subseteq D$ and $R_2 \subseteq E_1 - D$, and derives a contradiction.

(a): With the data of (a), we produce a function f from E into E'. In fact, let $x \in E - E_0$, then there is a grill G in E_0 such that $G \rightarrow x(s(S))$. By 4.15 G is compressed in $s(S)$, therefore in $s(S_0)$. By 2.2.4 f_0 is $(s(S_0),s(S'))$-continuous and so by 7.3.5 (c) $f_0(G)$ is compressed in $s(S')$. Because $s(S')$ is compact there is $x' \in E'$ with $f_0(G) \longrightarrow x'(s(S'))$. Because $s(S')$ is separated and symmetric x' is unique.

It must be shown that x' is uniquely defined by convergence. Let G_1 be another grill in E_0 converging to x in $s(S)$. Let $G \ (\cup) \ G_1 = \{R \cup R_1 : R \in G \text{ and } R_1 \in G_1\}$. Now $G \ (\cup) \ G_1 \longrightarrow x(s(S))$, and therefore as above $f_0(G \ (\cup) \ G_1) \longrightarrow x''(s(S'))$ for some $x'' \in E'$. It follows that both $f_0(G)$ and $f_0(G_1)$ converge to x'' in $s(S')$. Since limits in $s(S')$ are unique $x' = x''$, as required. Accordingly, define the function f by

$$f(x) = x' \quad \text{for} \quad x \in E - E_0$$
$$f(x) = f_0(x) \quad \text{for} \quad x \in E_0 .$$

The function f is (S,S')-continuous. This follows from 7.3.5 (d), the definition of f and 7.3.6.

The uniqueness of f can be seen as follows. Let g be an (S,S')-continuous extension of f_0. If $G \longrightarrow x(s(S))$ for a grill G in E_0 then we have by 7.3.5 (d) that $g(G) \longrightarrow g(x)(s(S'))$. But $g = f_0$ on each member of G whence $g(x) = f(x)$.

(b): Given (E,S), a universal double quasirealcompactification of (E_0,S_0), (E',S'), a separated doubly quasirealcompact space and continuous $f_0 : S_0 \longrightarrow S'$, let (E_1,S_1) and (E_2,S_2) be double compactifications of (E,S) and (E',S')

respectively. Then by 7.3.5 (E_1,S_1) is a double compactification of (E_0,S_0) and it is easily seen that because S' is separated so is S_2. By (a) above there is a unique (S_1,S_2)-continuous extension f of f_0.

We need only show that f maps E into E'. Let $x \in E$, then there is a grill G in E_0 with the countable intersection property, compressed in $s(S)$, with $G \to x(s(S))$. By 7.3.5 the grill $f_0(G)$ is compressed in $s(S')$ and has the countable intersection property, therefore there is $x' \in E'$ such that $f_0(G) \to x'$ in $s(S')$ and thus in $s(S_2)$, and the limit is unique. By continuity of f, $f_0(G) \to f(x)$ whence $x' = f(x)$, as required.

By 7.3.6 $f|E$ is continuous while 7.3.5 (d) and the uniqueness of limits ensures the uniqueness of the extension of f_0 to (E,S).

(c): Similar to proof of (b).

□

7.3.13 Let (E_0,S_0) be a separated syntopogenous space. Proposition 7.3.12 shows that

(a) Any double compactification of (E_0,S_0) is a reflection in the category of separated doubly compact syntopogenous structures.

(b) The universal double quasirealcompactification of (E_0, S_0) is a reflection in the category of separated doubly quasirealcompact syntopogenous structures.

(c) Any double completion of (E_0, S_0) is a reflection in the category of separated doubly complete syntopogenous structures.

The reflection morphism is the inclusion map in each case.

Epireflections are more tractable than reflections (cf. [HER]). If a certain separation property stronger than separatedness is imposed then the reflection morphism of 7.3.13 is an epimorphism and the reflections of 7.3.13 are epireflections. Accordingly we define the separation property T_2.

7.3.14 <u>Definition [C, p.201]</u>

Let S be a syntopogenous structure on the set E. S is T_2 iff $x, y \in E$ with $x \neq y$ implies there are $< \in S$ and $D \subseteq E$ satisfying $x < D$ $c(<)$ $E - y$.

We observe that this property generalizes the classical Hausdorff separation property.

It is easily seen that a separated symmetric structure is T_2, and that limits in a T_2 structure are unique.

7.3.15 Proposition

Let (E_0,S_0) , (E,S) and (E',S') be syntopogenous spaces with $E_0 \subseteq E$ and $S_0 = S|E_0$. Let f , g be (S,S')-continuous mappings. If S' is T_2 , E_0 is dense in (E,S) and $fi = gi$, where $i : E_0 \subseteq E$ is the inclusion, then $f = g$.

PROOF: Suppose on the contrary that there is $x \in E$ with $f(x) \neq g(x)$. By denseness of E_0 in (E,S) there is a grill G in E_0 converging to x in S , and by continuity $f(G) \rightarrow f(x)(S')$, $g(G) \rightarrow g(x)(S')$. But also $fi(G) \rightarrow f(x)(S')$ and $gi(G) \rightarrow g(x)(S')$, so that $f(x) = g(x)$ as limits are unique in S' .

☐

7.3.16 Corollary

If in 7.3.13 'separated' is replaced by 'T_2' , then the reflections are epireflections.

☐

7.4 The compact and realcompact reflections for topological spaces

In this section we retrieve the well known fact that every T_2 uniformisable topology has a compact and a realcompact epireflection. To do this we prove lemma 7.4.3 and proposition 7.4.5 using the doubly compact and doubly quasirealcompact reflections produced in the preceding section.

7.4.1 Definition

Let (E,S) be a uniformisable topological space. Let (E',S') be a uniformisable syntopogenous space such that $E \subseteq E'$, E is dense in S', S' is relatively separated with respect to E and $S = S'|E$. Then

(a) [C, pp.270,271]. If (E',S') is compact it is called a compactification of (E,S).

(b) If (E',S') is realcompact it is called a realcompactification of (E,S).

7.4.2 Proposition

If (E,S) is a double compactification or a universal double quasirealcompactification of a symmetric, simple syntopogenous structure, then $S \sim s(S)$.

PROOF: Let the symmetric, simple syntopogenous structure be S_0, defined on E_0. The space $(E,s(S))$ is also a double compactification or a universal double quasirealcompactification of (E_0,S_0). According to 7.3.10 the inclusion $i_0 : (E_0,S_0) \subseteq (E,S)$ extends to a unique $(S,s(S))$-continuous isomorphism, f. The unique extension of $i_1 : (E_0,S_0) \subseteq (E,s(S))$ is the identity $i : (E,s(S)) \subseteq (E,S)$ because $S \leqslant s(S)$. However since f is also $(s(S),s(S))$-continuous (cf. 2.2.4), it follows from 7.3.9 that $fi = i$, whence $f = i$. Thus $S \sim s(S)$.

□

We retrieve two well known results on compactification and realcompactification in the following proposition.

7.4.3 Lemma

Let (E,U) be a uniformisable topological space. The following statements are true.

(a) [C, prop.16.52]. U has a compactification which is a topology.

(b) U has a realcompactification which is a topology.

PROOF: Let S be a proximity on E such that $p(S) = U$.

(a): Let (E',S') be a double compactification of (E,S). According to 7.4.2 S' may be taken to be symmetric. By 4.2.1 $pt(S')$ is compact and by 6.7.7 it is uniformisable. It is easily seen that E is dense in $pt(S')$. Because S' is relatively separated with respect to E, it is immediately seen that $pt(S')$ is relatively separated with respect to E.

(b): Let (E',S') be a universal double quasirealcompactification of (E,S). According to 7.4.2 S' may be taken to be symmetric. By 4.20 and 3.4 $t(S')$ is doubly quasirealcompact.

We must show that $pt(S')$ is realcompact. By 6.7.7 $pt(S')$ is uniformisable. Let W be the uniformity on E' induced by the $(pt(S'),s(T))$-continuous functions. According to definition 6.7.8 we must show that $t(W)$ is quasirealcompact.

Accordingly, let G be a grill in E' with the countable intersection property, compressed in $t(W)$. By 6.7.11 G is compressed in $t(W)$ iff G is Cauchy in the uniformity induced by the $(t(W),T)$-continuous functions. However this uniformity is equivalent to W (see the proof of 6.7.17 (a)). It follows that G is compressed in $t(W)$ iff it is Cauchy in W.

We shall show that since G is Cauchy in W, G is Cauchy in the uniformity induced by the (S',T)-continuous functions. It will then follow from 6.7.11 that G is compressed in S' and so will converge there, whence, via convergence in $pt(S')$, it will converge in $t(W)$ since $pt(W) = pt(S')$.

Indeed, $\{f : f \text{ is } (S',T)\text{-continuous}\} \subseteq \{f : f \text{ is } (S',s(T))\text{-continuous}\} \subseteq \{f : f \text{ is } (pt(S'),s(T))\text{-continuous}\}$. Thus, by 6.7.9 (b) with $d \equiv d_f$, G is Cauchy in the uniformity induced by the (S',T)-continuous functions.

We have shown that $t(W)$ is quasirealcompact.

It is easily seen that E is dense in $pt(S')$. Because S' is relatively separated with respect to E, it is immediately seen that $pt(S')$ is relatively separated with respect to E.

□

The functor pt (cf. 2.2.6) is forgetful from bsSyn to ptSyn. Proposition 6.7.4 provides the definition of a functor $F : \text{ptSyn} \to \text{bsSyn}$ on objects. The following proposition provides the definition of F on morphisms. We remark that the restriction of F to the category of uniformisable topological spaces is a right inverse of pt.

7.4.4 <u>Proposition [C, prop.12.64]</u>

Let (E,U) and (E',U') be uniformisable topological spaces. Let S and S' be the uniformities respectively induced by the $(U,s(T))$-continuous and $(U',s(T))$-continuous functions. If g is (U,U')-continuous then g is $(t(S),t(S'))$-continuous.

<u>PROOF</u>: According to 6.7.5

$$\begin{aligned} g^{-1}(t(S')) &= g^{-1}(t(bs(\bigvee\{S_{d_f} : f \text{ is } (U',s(T))\text{-continuous}\}))) \\ &= tbs(\bigvee\{g^{-1}S_{d_f} : f \text{ is } (U',s(T))\text{-continuous}\}) , \end{aligned}$$

by 2.2.1 (f) and 2.8.3.

Now for $<_{d_f,\varepsilon} \in S_{d_f}$ we have by 5.3.8

$$g^{-1}(<_{d_f,\varepsilon}) = g^{-1}f^{-1}(<_\varepsilon) = (fg)^{-1}(<_\varepsilon) = <_{d_{fg},\varepsilon} .$$

However, fg is $(U,s(T))$-continuous. This shows that $g^{-1}(S_{d_f}) \leqslant S_{d_{fg}}$. It follows that $g^{-1}(t(S')) \leqslant t(S)$.

□

We now retrieve the following two well known results.

7.4.5 Proposition

Let (E,U) be a T_2 uniformisable topological space. The following statements are true.

(a) [C, prop.16.55] . (E,U) has a compact epireflection in the category of T_2 uniformisable topological spaces.

(b) (E,U) has a realcompact epireflection in the category of T_2 uniformisable topological spaces.

PROOF:

(a): Let (E',U') be a T_2 uniformisable compact topological space, and let f_0 be a (U,U')-continuous function. According to 7.4.3 (a) there is a uniformisable topology (E_1,U_1) which is a compactification of (E,U) . Let S , S' and S_1 be the proximities respectively induced by the continuous functions from U , U' and U_1 into S(T) . Then p(S) = U , p(S') = U' and $p(S_1) = U_1$. It follows from 6.4 that S , S' and S_1 are compact and hence, being symmetrical, doubly compact. The function f_0 is (S,S')-continuous by 7.4.4. According to 7.3.12 f_0 extends to a unique (S_1,S')-continuous function, f . By 2.2.4 f is (U_1,U')-continuous , and so (E_1,U_1) is a reflection and the inclusion of E in E_1 is the reflection mapping. Further, it follows from 7.3.15 that the reflection mapping is an epimorphism.

It only remains to show that U_1 is T_2 . But this follows easily from the fact that S_1 , being symmetrical and separated, is T_2 .

(b): Let (E',U') be a T_2 , uniformisable realcompact topological space, and let f_0 be a (U,U')-continuous function. According to 7.4.3 (b) there is a uniformisable topology (E_1,U_1) which is a realcompactification of (E,U) . Let S , S' and S_1 be the proximities respectively induced by the continuous functions from U , U' and U_1 into $s(T)$. Then $p(S) = U$, $p(S') = U'$ and $p(S_1) = U_1$. By the same argument as was used in the proof of 7.4.3 (b) S , S' and S_1 are quasirealcompact and hence, being symmetrical, doubly quasirealcompact. The rest of the proof is the same as in (a) above.

□

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ON ALEXANDROFF AND URYSOHN'S GENERAL METRIZATION CRITERION

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OOR ALEXANDROFF EN URYSOHN SE ALGEMENE METRISERINGSKRITERIUM

Opsomming — Hierdie is 'n geskrif oor die kontemporêre belang van die Algemene Metriseringskriterium (1923) van Alexandroff en Urysohn. Enersyds word getoon hoe die Uniforme Metriseringstelling verkry kan word deur byvoeging aan 'n bewys van Dale Rolfsen vir die Algemene Metriseringskriterium. Andersyds word die Kriterium padlangs uit die Eenvormige Metriseringstelling herlei.

Summary — This is a note on the contemporary interest of the General Metrization Criterion (1923) of Alexandroff and Urysohn. On the one hand it is shown how the Uniform Metrization Theorem can be obtained by annexing to a proof given by Dale Rolfsen for the General Metrization Criterion. On the other hand the Criterion is deduced straightforwardly from the Uniform Metrization Theorem.

1. Introduction. Alexandroff and Urysohn's general criterion for the metrisability of a topological space is

(A). *A topological space is metrisable iff it is T_2 and admits a complete regular chain* [1]. (Definitions follow).

Since it is easily seen that a sequence of covers is a complete chain iff it is a development (see (1.1) below), the following is an equivalent statement:

(A′). *A topological space is metrisable iff it is T_2 and admits a regular development* [5, th. 2].

Rolfsen [5] shows how several topological metrisation theorems can be derived from (A′); moreover, (A) and (A′) give access to two profound classical results: (A) leads directly to the Niemytzki-Tychonoff theorem [4], while this note shows how (A′) and the methods used to prove (A′) lead to the uniform metrisation theorem:

(B). *A separated uniform space is uniformly metrisable iff its uniformity has a countable base.*

It is also shown how (A′) can be deduced from (B).

In spite of their utility (A) and (A′) are not treated in texts, although Thron [6, p. 220] and Kelley [3, p. 186] mention (A). Proposition (B) is mentioned in [2, p. 33].

Acquaintance with the notation, contents and definitions involved in [5, th. 2] and the basic definitions of uniform space theory [2, pp. 3-6] is assumed.

Define a *complete chain* $G = (G_n)$ on a topological space $(\chi, \mathscr{T})$ to be a sequence of open covers of χ satisfying for all $x \in \chi$, $x \in V_n \in G_n$ for each integer $n \geq O$ implies $\{V_n\}$ is a local base at x. A *regular sequence* is defined to be a sequence $G = (G_n)$ of covers of χ satisfying the condition that for each integer $n \geq O$, if two members of G_{n+1} intersect then their union is a subset of some member of G_n. For the definition of a *development* see [5]. A *normal development* for $(\chi, \mathscr{T})$ is a development for $(\chi, \mathscr{T})$ which is a normal sequence [2, p. 5].

Note that $(\chi, \mathscr{T})$ admits a regular development iff it admits a normal development. In fact, each normal sequence is clearly regular. On the other hand, supposing that $(\chi, \mathscr{T})$ admits the regular development G, one readily shows the sequence J defined by $J_n = \{\mathrm{star}(x,G_{2n}) : x \in \chi\}$ to be a normal development for $(\chi, \mathscr{T})$.

(1.1) Proposition. *A sequence of covers is a complete chain iff it is a development.*

Proof. Suppose that G is a complete chain on the topological space $(\chi, \mathscr{T})$ and that G is not a development. For each integer $n \geq O$ G_n is an open cover. Given open U and $x \in U$ there is for each integer $n \geq O$ a y_n with $y_n \in$ star (x, G_n) but $y_n \notin U$. For each $n \geq O$ choose $V_n \in G_n$ such that $x, y_n \in V_n$. The collection $\{V_n\}$ is clearly not a local base at x, which contradicts the hypothesis that G is a complete chain.

On the other hand if G is a development, if U is an open set containing x and if $x \in V_n \in G_n$ for each integer $n \geq O$ then there is an integer $m \geq O$ such that $\mathrm{star}(x,G_m) \subset U$. But $x \in V_m \subset \mathrm{star}(x,G_m)$ and therefore $\{V_n\}$ is a local base at x. The proof is complete.

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2. Some features of Rolfsen's proof of (A′) [5, th. 2] needed in the sequel are collected without proof below (statements 2.1-2.3). Let $(\chi, \mathscr{T})$ be a T_2 topological space.

(2.1). The metric provided by sufficiency in (A′) is defined in the following way. Let G be the regular development. Define a new development H on χ by letting $J_m = \cup\{G_i : i \geq m\}$ for $m \geq O$, $H_o = J_o \cup \{\chi\}$ and $H_n = J_{2n}$ if $n > O$. Call C a *chain* if it is a finite subcollection of H_o and if its distinct elements can be ordered $h_1, \ldots, h_k$ so that $h_i \cap h_{i+1} \ddagger \phi$ whenever $O \leq i < k$. Let C^* denote the union of the members of C. Next, for each member h of H_o define $\mu(h)$, the *size* of h, by $\mu(h) = O$ if $h \in H_n$ for all $n \geq O$, otherwise let m be the largest integer such that $h \in H_m$ and then define $\mu(h) = 2^{-m}$. Note that $\mu(h) \leq 2^{-n}$ if $h \in H_n$. Now define the *length* $\lambda(C)$ of a chain C by $\lambda(C) = \Sigma_{h \in c}\, \mu(h)$. The metric d is then given by $d(x,y) = \text{g.l.b.}\{\lambda(C) : C \text{ is a chain and } x,y \in C^*\}$ for $x,y \in \chi$.

(2.2). Lemma. A chain C has C^* a subset of some member of H_n if $\lambda(C) \leq 2^{-n}$ for some $n \geq 0$.

(2.3). Following from (2.2), $x \in S_d(x,2^{-n}) \subset \text{star}(x,H_n)$ for any integer $n \geq 0$.

Note that if G is a normal development then so are J and H.

3. The uniform metrisation theorem. The following result is required.

(3.1). Proposition. *In a preuniform space each preuniform cover is refined by an open preuniform cover.*

Proof. Suppose $\mathscr{U}$ is a preuniform cover of a preuniform space. Let int $\mathscr{U} = \{\text{int } U : U \in \mathscr{U}\}$ where int U is the interior of U (in the uniform topology). There is a $\mathscr{W}$ such that $\mathscr{W} <^* \mathscr{U}$; this implies $\mathscr{W} <$ int $\mathscr{U}$ and thus int $\mathscr{U}$ is a preuniform cover which is open and refines $\mathscr{U}$.

We prove that (A′) **implies** (B). Necessity in (B) is trivial.

Sufficiency: Let the T_2 space $(\chi, \mathscr{T})$ admit the separated uniformity ν with countable base $\{B_n\}$. ν has a base $\{G_n\}$ which is a normal development. In fact, define two sequences B′ and G inductively: start with $B_1' \in \{B_n\}$ such that $B_1' <^* B_1$ and, using (3.1), an open cover $G_1 \in \nu$ such that $G_1 < B'_1$. Then for each $n > 1$, choose B_n' and an open cover G_n satisfying $B'_n \in \{B_n\}$ and the following two conditions:

(3.2) $B'_n <^* G_{n-1}$, $B'_n <^* B_n$

(3.3) $G_n \in \nu$ with $G_n < B'_n$.

Using (3.2) and (3.3) one sees that the sequence G is a normal development.

Since G is a normal development for $(\chi, \mathscr{T})$, (A′) and (2.1) provide the metric d on χ. Define $\mathscr{B}_r = \{S_d(x,r) : x \in \chi\}$. G and $\mathscr{B}$ satisfy

(3.4) $G_{2n} < \mathscr{B}_{2^{-n}} < G_{2n-2}$ for $n \geq 1$.

This can be seen as follows. By (2.1), if $g \in G_{2n}$ then $g \in H_n$ and so $\lambda(g) = \mu(g) \leq 2^{-n}$. Therefore if x and y are in g then from the definition of d in (2.1), $d(x,y) \leq 2^{-n}$. Next, from (2.3), $S_d(x,2^{-n}) \subset \text{star}(\chi,H_n)$ which is a subset of some member of H_{n-1} since H is also a normal development. Now (3.4) follows from the relations $H_{n-1} = J_{2n-2} < G_{2n-2}$.

It follows immediately from (3.4) that $\{\mathscr{B}_r : r > 0\}$ is a base for ν and thus sufficiency in (B) is proved.

(B) **implies** (A′). Again, necessity in (A′) is trivial.

Sufficiency: Let the topological space be $(\chi, \mathscr{T})$. Suppose $(\chi, \mathscr{T})$ admits a regular development, then it admits a normal development which is a countable base for a separated uniformity compatible with $\mathscr{T}$. By (B) this uniformity is metrisable and therefore so is $\mathscr{T}$.

4. Remark. If the Hausdorff condition applying to topological and uniform spaces is dropped throughout then the results (A), (A′), (B) and the "equivalence" of (A′) with (B) still go through provided the following changes are made: "metric" is replaced by "pseudometric", "separated uniformity" is replaced by "preuniformity", lemma (2.2) is replaced by:

(2.2′). A chain C has C* a subset of some member of H_{n-1} if $\lambda(C) \leq 2^{-n}$ for some $n \geq 1$.

(The change in the subscript of H compensates for the fact that it is now possible that subsets of χ which are not singletons may also have zero size), (2.3) is replaced by:

(2.3′). Following from (2.2′), $x \in S_d(x,2^{-n}) \subset \text{star}(x,H_{n-1})$ for $n \geq 1$.

and (3.4) is replaced by:

(3.4′) $G_{2n} < \mathscr{B}_{2^{-n}} < G_{2n-4}$, for $n \geq 2$.

On the other hand, imposition of the T_0 separation axiom on the pseudometric spaces is enough to make the pseudometric into a metric. Therefore (A′) can be written:

A topological space (resp. T_0 *topological space) admits a regular development iff it is pseudometrisable (resp. metrisable).*

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