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GENERALIZED DF SPACES

BY

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0.0 INTRODUCTION

A DF space is a topological vector space sharing certain essential properties with the strong duals of Fréchet spaces. The class of DF spaces includes not only all such duals, but also every normed space and many other spaces besides. The definition of a DF space is due to Grothendieck [9], who derived almost all the important results concerning such spaces.

The "generalized DF spaces" in the title of this Thesis are locally convex topological vector spaces whose topologies are determined by their restrictions to an absorbent sequence of bounded sets. In the case when this sequence is a fundamental sequence of bounded sets, we obtain the gDF spaces. Many of the properties of DF spaces are shared by all gDF spaces.

The first Chapter of this Thesis comprises definitions and results that should be known to a reader familiar with the theory of locally convex spaces. Chapter 2 contains various results concerning DF spaces and the strong duals of Fréchet spaces. The emphasis here has been placed on properties peculiar to these spaces, as results common to all gDF spaces are discussed in more detail in subsequent Chapters. In Chapter 3 the basic features of gDF spaces are described.

A recurring theme throughout this Thesis is the interplay between precompactness, metrizability and equicontinuity. This starts almost imperceptibly in Chapter 1 with the basic Lemmas 1.3.6 and 1.3.7, but becomes progressively more dominant. Concepts closely connected to this theme are Schwartz spaces and quasinormability; these are discussed in Chapters 4 and 5. To make the connection more apparent, the idea of an M-quasinormable space is introduced (4.3.0): this includes Schwartz spaces and quasinormable spaces as special cases. A variation on the theme is provided by Theorem 2.4.7, a "locally convex" version of a more general result by Pfister [18], showing that in a DF space precompact sets are metrizable.

Chapter 6 is devoted to applications of the results of the preceding Chapters. Spaces of weakly compact and of compact linear maps are investigated, yielding generalizations of such well-known results as Schauder's Theorem (6.0.5) and the Banach-Dieudonné Theorem (6.1.8). Finally, an account is given of the duality between injective and projective tensor products of Fréchet spaces and of semiMontel gDF spaces.

The following results are original:

3.0.7, 3.2.2, 3.3.0,

4.3.0, 4.3.1, 4.3.2, 4.3.4, 4.4.2, 4.5.2,

5.3.4, 5.3.5,

6.0.3, 6.0.5, 6.0.7, 6.0.11.

1.0 PRELIMINARY DEFINITIONS AND RESULTS

0. This Chapter consists mainly of definitions. The reader should be familiar with most of these; the intention is to establish notation and terminology employed throughout the remainder of the thesis. Certain results are quoted without an explicit reference or a proof; any of the standard texts ([19], [17], [24], [13] or [26]) should be consulted.

1. All vector spaces will be over a fixed field $\mathbb{K}$, which can be either $\mathbb{R}$ or $\mathbb{C}$. By a *lcs space*, we mean a locally convex separated topological vector space. Some of our results generalize to a wider class of spaces; the interested reader is referred to the book [1] by Adasch, Ernst and Keim.

1.1 Discs and Mackey-convergent sequences.

0. If S is a subset of a lcs space E ,
ΓS denotes the *absolutely convex hull of* S *in* E ,
$\overline{\Gamma} S$ denotes the *closed absolutely convex hull of* S *in* E ,
span S denotes the *linear span of* S *in* E .

1. A *disc* is an absolutely convex set.
Associated with each disc A in E is a seminormed space E_A . The underlying space of E_A is span A

and its seminorm is the *gauge* p_A of A, defined by

$$p_A(x) = \sup\{\lambda > 0 \mid x \notin \lambda A\}$$

for all $x \in \operatorname{span} A$. The *kernel* of p_A is

$$N(A) = p_A^{\leftarrow}(0) = \{x \in E \mid (\forall\lambda > 0)\ x \in \lambda A\}\ .$$

The quotient space

$$E_{(A)} = E_A/N(A)$$

is normed. Let $\natural_A : E_A \to E_{(A)}$ be the canonical surjection. Taking the completion of $E_{(A)}$, we obtain a Banach space $\tilde{E}_{(A)}$.

2. Under certain conditions, such as when A is bounded in E, $N(A) = \{\underline{0}\}$. Then $E_A \cong E_{(A)}$, so E_A is a normed space. The norm topology on E_A is finer than the subspace topology inherited by $\operatorname{span} A$ from E.

3. If A is such that E_A is a Banach space, we say that A is a *Banach disc*. It can be shown that every closed and bounded sequentially complete disc in E is a Banach disc. In particular, compact discs in E are Banach discs.

4. A sequence (x_k) of points in E is said to be *Mackey-convergent* if there exists a bounded disc A in E such that A absorbs each x_k and (x_k)

converges in the normed space E_A . Every Mackey-convergent sequence converges in the usual sense; if E is such that every convergent sequence in E Mackey-converges, we say E satisfies the *Mackey-convergence condition.* This is the case with any metrizable lcs space ([17] §28.3(1)).

1.2 Linked topologies

0. Let τ_1 and τ_2 be linear topologies on a vector space E . We say τ_1 *is linked to* τ_2 if $\tau_2 \subseteq \tau_1$ and τ_1 has a local base consisting of τ_2-closed sets. This concept often proves useful. It can be shown ([29] 6-1-13) that if τ_1 is linked to τ_2 , then τ_2-complete sets are τ_1-complete. Furthermore ([17] §28.5(2)), τ_1 and τ_2 coincide on the τ_1-precompact subsets of E .

1.3 Topologies on spaces of linear maps

0. Let $L(E,F)$ denote the space of all linear maps $T: E \to F$, where E and F are lcs spaces. Suppose $\Lambda(E,F)$ is a linear subspace of $L(E,F)$ and M is a collection of closed and bounded discs in E satisfying

M1. $\cup\mathcal{M} = E$.

M2. $(\forall M_1, M_2 \in \mathcal{M})(\exists M_3 \in \mathcal{M}) \quad M_1 \cup M_2 \subseteq M_3$.

M3. $(\forall M \in \mathcal{M})(\forall \lambda > 0) \quad \lambda M \in \mathcal{M}$.

The *topology* $\tau_{\mathcal{M}}$ *on* $\Lambda(E,F)$ *of uniform convergence on* the members of $\mathcal{M}$ has as basic neighbourhoods of $\underline{0}$ all sets of the form

$$\Lambda[M,V] = \{T \in \Lambda(E,F) \mid TM \subseteq V\}$$

where $M \in \mathcal{M}$ and V is a closed absolutely convex neighbourhood of $\underline{0}$ in F . We shall abbreviate $(\Lambda(E,F),\tau_{\mathcal{M}})$ by $\Lambda_{\mathcal{M}}(E,F)$. Let $\mathcal{L}(E,F)$ be the space of all continuous linear maps $E \to F$. If $\Lambda(E,F) \subseteq \mathcal{L}(E,F)$ then $\Lambda_{\mathcal{M}}(E,F)$ is a lcs space.

1. Note that $\mathcal{L}(E,\mathbb{K}) = E'$, the *dual of* E , while $L(E,\mathbb{K}) = E^*$, the *algebraic dual of* E . If we set $V = \{\lambda \in \mathbb{K} \mid |\lambda| \leqslant 1\}$, then

$$\mathcal{L}[M,V] = M^0 ,$$

the *polar of* $M \in \mathcal{M}$ *in* E' . Thus the topology $\tau_{\mathcal{M}}$ on $E'_{\mathcal{M}} = \mathcal{L}_{\mathcal{M}}(E,\mathbb{K})$ is the usual $<E',E>$-polar topology induced by $\mathcal{M}$ on E' .

2. Possible choices of $\mathcal{M}$ include

$\mathcal{M}(s) = \{\overline{\Gamma}S \mid S$ is a finite subset of $E\}$,

$\mathcal{M}(k) = \{K \mid K$ is a $\sigma(E,E')$-compact disc in $E\}$,

$\mathcal{M}(b) = \{B \mid B$ is a closed and bounded disc in $E\}$,

$\mathcal{M}(p) = \{\overline{\Gamma}P \mid P$ is a precompact set in $E\}$,

$\mathcal{M}(ca) = \{K \mid K$ is a compact disc in $E\}$,

$\mathcal{M}(mc) = \{\overline{\Gamma}\{x_k\} \mid (x_k)$ Mackey-converges to $\underline{0}$ in $E\}$.

3. We shall write $\Lambda_s(E,F)$ for $\Lambda_{M(s)}(E,F)$, E'_k for $E'_{M(k)}$, etc. The topology τ_s is known as the topology of *simple convergence*, while τ_b is the topology of *bounded convergence*. It is traditional to use the notation $\sigma(E',E)$ for the *weak topology* on E'_s , $\mu(E',E)$ for the *Mackey topology* on E'_k and $\beta(E',E)$ for the *strong topology* on E'_b . We shall refer to E'_b as the *strong dual* of E .

4. If E is itself a space of linear maps defined on a lcs space G , we can also consider $M(e)$, the collection of all closed and bounded discs in E that are equicontinuous on G .

5. Associated with each $T \in L(E,F)$ is its adjoint $T' \in L(F'_s,E'_s)$, defined by

$$<x, T'g> = <Tx, g>$$

for all $x \in E$ and $g \in F'$.

6. **Lemma :**

Let E *and* F *be lcs spaces.*
On each equicontinuous subset H *of* $L(E,F)$ *the topologies of simple and of precompact convergence coincide.*

Proof:

Clearly $\tau_s \subseteq \tau_p$. Let $P \in M(p)$, and suppose V is a closed absolutely convex neighbourhood of $\underline{0}$ in F . If H is equicontinuous on E , there is a neighbourhood U of $\underline{0}$ in E with $TU \subseteq V$ for all $T \in H$. By the precompactness of P there is a finite $S \subseteq E$ with $P \subseteq S + \frac{1}{2}U$. Now $L[P,V]$ and $L[\overline{\Gamma}S,\frac{1}{2}V]$ are neighbourhoods of $\underline{0}$ in $L_p(E,F)$ and in $L_s(E,F)$ respectively. If $T \in H \cap L[\overline{\Gamma}S,\frac{1}{2}V]$, then

$$TP \subseteq T(S+\tfrac{1}{2}U) \subseteq T\overline{\Gamma}S + \tfrac{1}{2}TU \subseteq \tfrac{1}{2}V + \tfrac{1}{2}V = V .$$

Thus

$$H \cap L[\overline{\Gamma}S,\tfrac{1}{2}V] \subseteq L[P,V] . \qquad \square$$

7. As a simple consequence of the above Lemma, we obtain the following:

Lemma ([17] §42.1(8)):

Let $T \in L(E,F'_s)$, *where* E *and* F *are lcs spaces. Suppose* M *is a collection of closed and bounded discs in* E *as described in* 1.3.0, *and suppose* N *is a similar collection in* F . *The following statements are then equivalent :*

a. *For each* $M \in \mathcal{M}$ *,* TM *is precompact in* $F'_{\mathcal{N}}$.

b. *For each* $N \in \mathcal{N}$ *, the restriction of* T' *to* N *is a* $\sigma(F,F')$*-continuous map into* $E'_{\mathcal{M}}$.

c. *For each* $N \in \mathcal{N}$ *,* $T'N$ *is precompact in* $E'_{\mathcal{M}}$.

d. *For each* $M \in \mathcal{M}$ *, the restriction of* T *to* M *is a* $\sigma(E,E')$*-continuous map into* $F'_{\mathcal{N}}$

Proof:

a. ⇒ b.: Each $N \in \mathcal{N}$ is equicontinuous on $F'_{\mathcal{N}}$. By 1.3.6, the topology on F' of precompact convergence coincides with $\sigma(F',F)$ on N . Thus if TM is precompact, there exists a basic $\sigma(F',F)$-neighbourhood W of $\underline{0}$ in F' with $W \cap N \subseteq (TM)^0$. But then $T'(W \cap N) \subseteq M^0$, which expresses the desired continuity.

b. ⇒ c.: Each $N \in \mathcal{N}$ is bounded in F , hence $\sigma(F,F')$-precompact. Continuous maps preserve precompactness, so $T'N$ is precompact in $E'_{\mathcal{M}}$.

c. ⇒ d. and d. ⇒ a. follow by symmetry. □

8. In the case where the lcs spaces E and F form a dual pair and $T\colon E \to F'$ is the identity map, Lemma 1.3.7 reduces to the *Grothendieck Interchange Theorem :*

Each $M \in \mathcal{M}$ *is precompact in* $F'_{\mathcal{N}}$ *if and only if each* $N \in \mathcal{N}$ *is precompact in* $E'_{\mathcal{M}}$.

9. Note that if E and F are lcs spaces, with $\mathcal{M}$ as before, then the closure of a linear subspace $\Lambda(E,F)$ of $L(E,F)$ in $L_{\mathcal{M}}(E,F)$ is

$$\tilde{\Lambda}(E,F) = \cap \{\Lambda(E,F) + L[M,V] \mid M \in \mathcal{M} , V \in \mathcal{V}\} ,$$

where $\mathcal{V}$ is a local base for the topology on F . This yields a generalization of *Grothendieck's Completeness Theorems:* provided the lcs space F is complete, the completion of $\Lambda_{\mathcal{M}}(E,F)$ is $\tilde{\Lambda}_{\mathcal{M}}(E,F)$. In particular, $L_{\mathcal{M}}(E,F)$ is complete if and only if those $T \in L(E,F)$ that have continuous restrictions to each $M \in \mathcal{M}$ are continuous throughout E . (see [17] §39.6).

1.4 Compact linear maps

0. A linear map $T \in L(E,F)$ is said to be *of finite rank* if there exist continuous linear functionals $f_1, f_2, \dots f_n \in E'$ and points $y_1, y_2, \dots y_n \in F$ such that

$$Tx = \sum_{k=1}^{n} f_k(x) y_k$$

for all $x \in E$.

1. A linear map $T \in L(E,F)$ is *(weakly) compact* if T maps some neighbourhood of $\underline{0}$ in E onto a $(\sigma(F,F')-)$relatively compact set in F .

2. Define

$$\mathcal{F}(E,F) = \{T \in L(E,F) \mid T \text{ is of finite rank}\} ,$$
$$\mathcal{K}(E,F) = \{T \in L(E,F) \mid T \text{ is compact}\} ,$$
$$\mathcal{W}(E,F) = \{T \in L(E,F) \mid T \text{ is weakly compact}\} .$$

It is easy to show that

$$\mathcal{F}(E,F) \subseteq \mathcal{K}(E,F) \subseteq \mathcal{W}(E,F) \subseteq \mathcal{L}(E,F) \subseteq L(E,F) .$$

Interesting questions arise when one defines topologies on these spaces. Under what conditions is $\mathcal{K}(E,F)$ closed in $\mathcal{L}_b(E,F)$? When is $\mathcal{F}(E,F)$ dense in $\mathcal{L}_p(E,F)$? We shall return to some of these later.

3. From Lemma 1.3.7 we obtain

Schauder's Theorem :

If X *and* Y *are Banach spaces, then* $T \in \mathcal{K}(X,Y)$ *if and only if* $T' \in \mathcal{K}(Y',X')$.

1.5 Spaces of bilinear maps

0. Let E , F and G be lcs spaces. A function $h: E \times F \to G$ is said to be *bilinear* if for each $x \in E$ and $y \in F$ the induced maps $h(x,\_): F \to G$ and $h(\_,y): E \to G$ are linear.

Put $B(E \times F\ , G) = \{h: E \times F \to G \mid h \text{ is bilinear}\}$,

$$\mathcal{B}(E \times F\ , G) = \{h \in B(E \times F\ , G) \mid h \text{ is continuous}\}\ ,$$

$$B(E \times F) = B(E \times F\ , \mathbb{K})\ ,$$

$$\mathcal{B}(E \times F) = \mathcal{B}(E \times F\ , \mathbb{K})\ .$$

The elements of $B(E \times F)$ and of $\mathcal{B}(E \times F)$ are known as *bilinear forms* on $E \times F$.

1. Let $\mathcal{M}$ be a collection of closed and bounded discs in E as described in 1.3.0 and let $\mathcal{N}$ be a similar collection in F . The *topology on* $\mathcal{B}(E \times F)$ *of uniform convergence on* the members of $\mathcal{M}$ and $\mathcal{N}$ has as basic neighbourhoods of $\underline{0}$ all sets of the form

$$\mathcal{B}[M \times N] = \{h \in \mathcal{B}(E \times F) \mid (\forall x \in M,\ y \in N)\ |h(x,y)| \leqslant 1\}\ ,$$

where $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

2. Together with this topology, $\mathcal{B}(E \times F)$ is a lcs space, which we shall denote by $\mathcal{B}_{\mathcal{M}\mathcal{N}}(E \times F)$. If $\mathcal{M}$ and $\mathcal{N}$ are chosen from the lists given in 1.3.2 and 1.3.4, this is simplified to $\mathcal{B}_{ss}(E \times F)$, etc. The topology on $\mathcal{B}_{bb}(E \times F)$ is known as the topology of *bibounded convergence*, while that on $\mathcal{B}_{ee}(E' \times F')$ is known as the topology of *biequicontinuous convergence*.

1.6 Tensor products

0. Let E and F be lcs spaces. Each $(x,y) \in E \times F$ induces a linear functional $\chi(x,y)$ on $B(E \times F)$, given by

$$\langle h, \chi(x,y)\rangle = h(x,y)$$

for all $h \in B(E \times F)$. Thus we have a *canonical bilinear map* $\chi: E \times F \to (B(E \times F))$. The *tensor product* $E \otimes F$ of E and F is defined to be the linear span of $\chi(E \times F)$ in $(B(E \times F))^*$. It is customary to denote $\chi(x,y)$ by $x \otimes y$. Each $z \in E \otimes F$ is of the form $\sum_{i=1}^{n} x_i \otimes y_i$, where $(x_i,y_i) \in E \times F$.

1. Let $h \in B(E \times F, G)$, where G is a lcs space. Define $\dot{h} \in L(E \otimes F, G)$ by

$$\dot{h} \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} h(x_i,y_i) .$$

Then $\dot{h} \circ \chi = h$, so $\chi \in B(E \times F, E \otimes F)$ has the universal property that every bilinear map from $E \times F$ factors uniquely through χ. It follows that

$$L(E \otimes F, G) \cong B(E \times F, G) ;$$

in particular

$$(E \otimes F)^* \cong B(E \times F) .$$

(see [24] III §6.1).

2. Suppose $z = \sum_{i=1}^{n} x_i \otimes y_i$. We can identify $z \in E \otimes F$ with a linear map $T_z \in L(E',F)$ given by

$$T_z f = \sum_{i=1}^{n} f(x_i) y_i$$

where $f \in E'$. It follows that

$$E \otimes F \cong \mathcal{F}(E_s',F) .$$

Thus $E \otimes F$ can be thought of either as a space of linear functionals on $B(E \times F)$, or as a subspace of $\mathcal{L}(E_s',F)$. We can use either representation to define topologies on $E \otimes F$.

3. The *injective tensor product* $E \otimes_\varepsilon F$ is defined to be the lcs space $\mathcal{F}_e(E_s',F)$. Basic neighbourhoods of $\underline{0}$ in $E \otimes_\varepsilon F$ are of the form

$$\begin{aligned}\mathcal{F}[U^0,V] &= \{\sum_{i=1}^{n} x_i \otimes y_i \mid (\forall f \in U^0) \sum_{i=1}^{n} f(x_i) y_i \in V\} \\ &= \{\sum_{i=1}^{n} x_i \otimes y_i \mid (\forall f \in U^0, g \in V^0) \\ &\qquad\qquad |\sum_{i=1}^{n} f(x_i) g(y_i)| \leqslant 1\}\end{aligned}$$

where U and V are closed absolutely convex neighbourhoods of $\underline{0}$ in E and in F .

4. The *projective tensor product* $E \otimes_\pi F$ is defined to be $E \otimes F$ equipped with the finest locally convex topology making $\chi: E \times F \to E \otimes F$ continuous. Basic neighbourhoods of $\underline{0}$ are of

the form

$$\Gamma\chi(U \times V) = \{ \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \mid \sum_{i=1}^{n} |\lambda_i| \leq 1 , x_i \in U , y_i \in V\} .$$

5. It can be seen that $E \otimes_\varepsilon F \cong F \otimes_\varepsilon E$ and $E \otimes_\pi F \cong F \otimes_\pi E$. Because

$$\Gamma\chi(U \times V) \subseteq \mathcal{F}[U^0,V] ,$$

$E \otimes_\varepsilon F$ has a coarser topology than $E \otimes_\pi F$. The completion of $E \otimes_\varepsilon F$ is denoted by $E \tilde{\otimes}_\varepsilon F$ and that of $E \otimes_\pi F$ by $E \tilde{\otimes}_\pi F$.

6. It is easy to show that

$$(E \tilde{\otimes}_\pi F)' \cong \mathcal{B}(E \times F) .$$

We would like to know under which conditions this isomorphism becomes a topological one. In particular, when is

$$(E \tilde{\otimes}_\pi F)'_b \cong \mathcal{B}_{bb}(E \times F) \quad ?$$

1.7 Barrelled and bornological spaces

0. A *barrel* is a closed absorbent disc. If every barrel in E is a neighbourhood of $\underline{0}$, E is said to be *barrelled*. Barrels are the polars of $\sigma(E',E)$-bounded sets in E' , so an equivalent definition is that every $\sigma(E',E)$-bounded disc in E' is equicontinuous on E .

1. Every lcs space (E,τ) can be identified with a subspace of its *bidual* $E'' = (E_b')'$; in general $\tau \subseteq \beta(E'',E')|_E$. If $\tau = \beta(E'',E')|_E$, (E,τ) is said to be *quasibarrelled.* Such a space always has the Mackey topology $\mu(E,E')$. It can be seen that E is quasibarrelled if and only if every $\beta(E',E)$-bounded disc in E' is equicontinuous on E' .

2. A *bornivorous* set is one which absorbs all bounded sets. Thus E is quasibarrelled if and only if every bornivorous barrel in E is a neighbourhood of $\underline{0}$. If every bornivorous disc in E is a neighbourhood of $\underline{0}$, E is *bornological.*

3. (DeWilde [5] III)
An *ultrabornivorous* set absorbs every bounded Banach disc in E . It can be shown that every barrel is ultrabornivorous. If every ultrabornivorous disc in E is a neighbourhood of $\underline{0}$, E is *ultrabornological.* Ultrabornological spaces are clearly both barrelled and bornological.

4. In a sequentially complete lcs space, every closed and bounded disc is a Banach disc. Thus a sequentially complete bornological space is ultrabornological, and a sequentially complete quasibarrelled space is barrelled. Since a

metrizable lcs space is easily seen to be bornological, every Fréchet space is ultrabornological.

5. We have seen that barrelled spaces and quasi-barrelled spaces can be characterized in terms of their duals. Altering these characterizations slightly enables us to define a number of weaker forms of "barrelledness":

(a) (Husain)

A lcs space E is *countably (quasi)barrelled* if each $(\beta(E',E)-)$ $\sigma(E',E)$-bounded disc in E' that is the union of a sequence of equicontinuous subsets of E' is itself equicontinuous on E .

(b) (DeWilde and Houet [6], Saxon and Levin)

A lcs space E is σ-*(quasi)barrelled* if each $(\beta(E',E)-)$ $\sigma(E',E)$-bounded sequence in E' is equicontinuous on E .

(c) (Webb [28])

A lcs space E is *sequentially (quasi)barrelled* if every sequence in E' that $(\beta(E',E)-)\sigma(E',E)$-converges to $\underline{0}$ is equicontinuous on E .

1.8 Reflexivity

0. A lcs space (E,τ) is *semireflexive* if $E = E''$. This will occur if and only if every closed and bounded disc in (E,τ) is $\sigma(E,E')$-compact. If in addition $\tau = \beta(E'',E')$, (E,τ) is said to be *reflexive*. Thus E is reflexive if and only if E is both semireflexive and quasibarrelled.

1. If every closed and bounded disc in E is compact, E is *semi Montel*. A *Montel* space is both semi-Montel and quasibarrelled. Since a semiMontel space is semireflexive, a Montel space is reflexive.

2. It is easy to see that if E is reflexive, so is E'_b. Furthermore, every reflexive space is barrelled. Slightly more difficult to show is that if E is Montel, then so is E'_b ([24] IV §5.9).

3. If every closed and bounded disc in E is complete, E is *quasicomplete*. A semireflexive space E is quasicomplete: its closed and bounded discs are $\sigma(E,E')$-complete.

1.9 Fundamental sequences

0. Once again, let $\mathcal{M}$ be a collection of closed and bounded discs in E , as described in 1.3.0. A sequence $M_1, M_2, \ldots$ of members of $\mathcal{M}$ is *fundamental* if every $M \in \mathcal{M}$ is absorbed by some M_k . Given such a sequence, we can always replace it by one which satisfies

$$k\, M_k \subseteq M_{k+1}$$

for all $k \in \mathbb{N}$. Usually, we shall speak of a fundamental sequence of members of $\mathcal{M}(b)$ as being a *fundamental sequence of bounded sets in* E , while one of members of $\mathcal{M}(p)$ is known as a *fundamental sequence of precompact sets.*

1. If F is a metrizable lcs space, then its topology will have a local base consisting of a decreasing sequence of barrels $V_1, V_2, \ldots$. If $\mathcal{M}$ contains a fundamental sequence $M_1, M_2, \ldots$ and $\Lambda(E,F)$ is a subspace of $\mathcal{L}(E,F)$, then $\{\Lambda[M_k,V_k] \mid k \in \mathbb{N}\}$ is a countable local base for the topology on $\Lambda_{\mathcal{M}}(E,F)$, which is therefore metrizable.

2. On the other hand, $V_1^0, V_2^0, \ldots$ form a fundamental sequence of bounded sets in F_b' . It can be shown ([17] §29.1(2)) that any metrizable lcs space

containing a fundamental sequence of bounded sets must be normable, so in general the strong dual of a metrizable lcs space is not metrizable.

3. **Proposition:**

A fundamental sequence of members of $\mathcal{M}(\mathrm{mc})$ *is also a fundamental sequence of bounded sets.*

Proof:

Let $M_1, M_2, \ldots$ be a fundamental sequence of members of $\mathcal{M}(\mathrm{mc})$. Suppose there exists a bounded disc B in E not absorbed by any M_k. For each $k \in \mathbb{N}$, choose $x_k \in \frac{1}{k}B$ so that $x_k \notin M_k$. Then (x_k) Mackey-converges to $\underline{0}$ in E, but $\overline{\Gamma}\{x_k\}$ is not contained in any M_k, a contradiction. □

4. The above Proposition still holds if we replace $\mathcal{M}(\mathrm{mc})$ by $\mathcal{M}(\mathrm{p})$. Thus if E contains a fundamental sequence of precompact sets, then every bounded set in E is precompact. (For more results of this nature, see De Wilde [4]).

2.0 DF SPACES

0. In his classic paper [9], Grothendieck examined Fréchet spaces and their duals. He identified two characteristics of the strong dual of a Fréchet as being of particular importance, and made these the basis of his definition of a DF space. He was able to show that many of the properties of the strong dual of a Fréchet space are shared by all DF spaces.

1. Subsequent authors repeated this process, and used certain characteristics of a DF space to define new classes of spaces, such as the gDF spaces. Before we examine any of these generalizations, let us consider some of the basic results given in Grothendieck's paper. (Some of these also appear in [10], [17] and [14]).

2. **Definition:**

A DF *space is a countably quasibarrelled space containing a fundamental sequence of bounded sets.*

2.1 Strong duals of Fréchet spaces

0. Theorem:

If F *is a metrizable lcs space,* F'_b *is countably quasibarrelled.*

Proof:

Let M be the union of an increasing sequence of $\beta(F',F)$-equicontinuous discs $M_1, M_2, \ldots$ in F''. We shall show that if M is $\beta(F'',F')$-bounded in F'', then M is equicontinuous on F'_b.

As each M_n is equicontinuous on F'_b, for each $n \in \mathbb{N}$ there is a bounded disc B_n in F satisfying

$$B_n^0 \subseteq M_n^0 \quad .$$

Let $\{V_k \mid k \in \mathbb{N}\}$ be a local base for the topology on F. Taking polars first in F', then in F'', we see that each V_k^{00} is a $\beta(F'',F')$-neighbourhood of $\underline{0}$ in F''. Thus V_k^{00} absorbs M; let $\lambda_k > 0$ be such that

$$V_k^0 \subseteq \lambda_k M^0 \quad .$$

For each $n \in \mathbb{N}$, set

$$W_n = \bigcap_{k=1}^{n} \lambda_k V_k$$

and then define

$$A_n = W_n^0 + B_n^0 \quad .$$

Note that for $k = 1, 2, \ldots n$,

$$W_n^0 \subseteq (\lambda_k V_k)^0 = {}^1/_{\lambda_k} V_k^0 \subseteq M^0 .$$

Each W_n is a neighbourhood of $\underline{0}$ in F , so W_n^0 is $\sigma(F',F)$-compact in F' and A_n is a $\sigma(F',F)$-closed disc in F' . Put

$$A = \bigcap_{n=1}^{\infty} A_n .$$

Because A is a $\sigma(F',F)$-closed disc in F' , to show that $A^0 \cap F$ is bounded in F it is sufficient to prove that for each $m \in \mathbb{N}$, A absorbs V_m^0 . For $n = 1, 2, \ldots m$, V_m absorbs B_n , so there are $\alpha_n > 0$ with

$$\alpha_n B_n \subseteq V_m .$$

Then

$$\alpha_n V_m^0 \subseteq B_n^0 \subseteq A_n .$$

For $n = (m+1), (m+2), \ldots$, we have that

$$W_n \subseteq \lambda_m V_m ,$$

so

$$ {}^1/_{\lambda_m} V_m^0 \subseteq W_n^0 \subseteq A_n .$$

Thus if $\beta_m = \min\{\alpha_1, \alpha_2, \ldots \alpha_m, {}^1/_{\lambda_m}\}$, then

$$\beta_m V_m^0 \subseteq A_n$$

for all $n \in \mathbb{N}$, and so

$$\beta_m V_m^0 \subseteq A .$$

It follows that $A^0 \cap F$ is bounded in F , so A is a neighbourhood of $\underline{0}$ in F_b' . That M is equicontinuous on F_b' is a consequence of

$$A_n = W_n^0 + B_n^0 \subseteq M^0 + M_n^0 \subseteq M_n^0 + M_n^0 = 2M_n^0 .$$

□

1. **Corollary:**

The strong dual of a Fréchet space is a DF *space.*

2.2 Distinguished Fréchet spaces.

0. The strong duals of Fréchet spaces display certain properties that are not common to all DF spaces. A simple example is completeness: the strong dual of a Fréchet space (and indeed of any bornological space) is always complete, but there exist DF spaces which are not complete (such as any normed space which fails to be a Banach space).

1. A more interesting example of a property not shared by all DF spaces is the following:

Theorem ([5] III 3.12) :

If F *is a Fréchet space, the following are equivalent:*

a. F'_b *is ultrabornological.*
b. F'_b *is bornological.*
c. F'_b *is quasibarrelled.*
d. F'_b *is barrelled.*

The implications a. ⇒ b. and b. ⇒ c. are trivial, while c. ⇒ d. follows from the completeness of F'_b . To prove d. ⇒ a. we shall use the following:

2. **Definition:**

Let A be a disc in a vector space E . By the *algebraic closure of* A we mean the closure of A in its associated seminormed space E_A ; this is

$$A^a = \bigcap_{\varepsilon > 0} (1 + \varepsilon)A \, .$$

3. **Lemma:**

If a disc A *in the strong dual of a metrizable lcs space* F *can be expressed in the form*

$$A = \Gamma\left(\bigcup_{k=1}^{\infty} \lambda_k V_k^0\right) ,$$

where $\{V_k \mid k \in \mathbb{N}\}$ *is a local base for the neighbourhoods of* $\underline{0}$ *in* F *and each* $\lambda_k > 0$, *then*

$$A^a = \mathrm{cl}_{\beta(F',F)} A \, .$$

Proof:

From the definition of A^a we see that

$$A^a \subseteq \mathrm{cl}_{\beta(F',F)} A \, .$$

Given $f \in F' \setminus A^a$, we shall show

$$f \notin \mathrm{cl}_{\beta(F',F)} A \, .$$

For each $n \in \mathbb{N}$, put

$$A_n = \Gamma\left(\bigcup_{k=1}^{n} \lambda_k V_k^0\right),$$

so that

$$A = \bigcup_{n=1}^{\infty} A_n .$$

Being the absolutely convex hull of a $\sigma(F',F)$-compact set, each A_n is a $\sigma(F',F)$-compact disc in F'. It follows that

$$A_n = [\bigcap_{k=1}^{n} {}^{1}/_{\lambda_k} \; V_k]^0 .$$

Since $f \notin A^a$, $f \notin (1+\varepsilon)A$ for some $\varepsilon > 0$. Thus

$$f \notin (1+\varepsilon)A_n$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ choose

$$x_n \in \bigcap_{k=1}^{n} {}^{1}/_{\lambda_k} \; V_k$$

so that

$$|f(x_n)| > 1 + \varepsilon .$$

The sequence (x_n) is bounded in F, so its polar W is a neighbourhood of $\underline{0}$ in F'_b. We shall prove that $f \notin \mathrm{cl}_{\beta(F',F)}A$ by showing that

$$(f + \varepsilon W) \cap A = \emptyset .$$

Suppose $(f + \varepsilon g) \in A$ for some $g \in W$. Then $(f + \varepsilon g) \in A_N$ for some $N \in \mathbb{N}$, so

$$\begin{aligned} |f(x_N)| &\leqslant |(f + \varepsilon g)(x_N)| + \varepsilon \; |g(x_N)| \\ &\leqslant \quad 1 \quad + \varepsilon , \end{aligned}$$

contradicting the choice of x_N. $\square$

4. Proof of 2.1.1, d. ⇒ a. :

Let $\{V_k \mid k \in \mathbb{N}\}$ be a local base for the topology on F . Each V_k^0 is a bounded Banach disc in F_b' , so if U is an ultrabornivorous disc in F_b' then U absorbs V_k^0 . Thus there are $\lambda_k > 0$ such that

$$A = \Gamma\left(\bigcup_{k=1}^{\infty} \lambda_k V_k^0\right) \subseteq U .$$

By the Lemma, A^a is a closed disc in F_b' . Furthermore A^a is absorbent, so it is a barrel. Because

$$A^a \subseteq 2A \subseteq 2U$$

and F_b' is barrelled, U is a neighbourhood of $\underline{0}$ in F_b' . □

5. A lcs space E is *distinguished* if every $\sigma(E'',E')$-bounded set in E" lies in the $\sigma(E'',E')$-closure of a bounded disc in E . It is easy to see E is distinguished if and only if E_b' is barrelled, so we can express the above Theorem in the following form:

6. Corollary:

A Fréchet space F *is distinguished if and only if* F_b' *is bornological.*

□

7. Köthe ([17] §31.7) gives an example of a non-distinguished Fréchet space. Thus there exist Fréchet spaces whose strong duals are countably quasibarrelled, but not quasibarrelled.

8. For some time after the publication of Grothendieck's paper [9] it was conjectured that all quasibarrelled DF spaces are bornological. This was eventually proved false by Valdivia [27], who constructed a class of counterexamples.

2.3 Localization

0. The following Theorem describes what is probably the most important feature of a DF space: its topology is determined by its fundamental sequence of bounded sets.

1. **Theorem:**

Let $B_1, B_2, \ldots$ *be a fundamental sequence of bounded sets in a* DF *space* (E, τ) . *A disc* U *in* E *is a neighbourhood of* $\underline{0}$ *in* (E, τ) *if and only if* $U \cap B_n$ *is a* $\tau|_{B_n}$*-neighbourhood of* $\underline{0}$ *for each* $n \in \mathbb{N}$.

Proof:

$\Rightarrow$: Trivial.

$\Leftarrow$: For each $n \in \mathbb{N}$, let U_n be a closed absolutely convex neighbourhood of $\underline{0}$ in (E,τ) satisfying

$$U_n \cap B_n \subseteq U .$$

Put

$$W_n = cl_\tau[(U \cap B_n) + \tfrac{1}{2}U_n] \quad ;$$

W_n is then a closed absolutely convex neighbourhood of $\underline{0}$ in (E,τ) and satisfies

$$\begin{aligned} W_n \cap B_n &\subseteq [\{(U \cap B_n) + \tfrac{1}{2}U_n\} + \tfrac{1}{2}U_n] \cap B_n \\ &\subseteq [(U \cap B_n) + U_n] \cap B_n \\ &\subseteq (U \cap B_n) + (U_n \cap 2B_n) \\ &\subseteq (U \cap B_n) + 2(U_n \cap B_n) \\ &\subseteq 3(U \cap B_n) . \end{aligned}$$

We wish to show that $W = \bigcap_{n=1}^{\infty} W_n$ is a neighbourhood of $\underline{0}$ in (E,τ). To prove that W^0 is $\beta(E',E)$-bounded in E', we need only show that W absorbs each B_m. For $n = 1, 2, \ldots m$, there are $\alpha_n > 0$ such that

$$V_m \subseteq \alpha_n W_n \quad .$$

Furthermore, there is a $\lambda_m > 0$ such that

$$B_m \subseteq \lambda_m U_m \quad ,$$

so for $n = m + 1, m + 2, \ldots$,

$$\begin{aligned} B_m &\subseteq \lambda_m(U_m \cap B_m) \\ &\subseteq \lambda_m(U \cap B_m) \\ &\subseteq \lambda_m(U \cap B_n) \subseteq \lambda_m W_n \quad . \end{aligned}$$

Put $\beta_m = \max \{\alpha_1, \alpha_2, \ldots \alpha_m, \lambda_m\}$, then

$$B_m \subseteq \beta_m W_n$$

for all $n \in \mathbb{N}$, and so

$$B_m \subseteq \beta_m W \quad .$$

Thus W absorbs each B_m , and is therefore a neighbourhood of $\underline{0}$ in (E,τ) . Consider $x \in W$. Since $x \in B_N$ for some $N \in \mathbb{N}$,

$$x \in W_N \cap B_N \subseteq 3(U \cap B_N) \subseteq 3U \quad .$$

Thus $W \subseteq 3U$, which means that U is a neighbourhood of $\underline{0}$ in (E,τ) . □

2. In subsequent Chapters, we shall explore the consequences of this property in more detail.

2.4 Quasibarrelled DF spaces

0. In DF spaces, quasibarrelledness and separability are closely related. The following result shows why :

1. **Theorem** (De Wilde and Houet [6]):

On each separable subset A *of a* σ*-quasibarrelled space* (E,τ) *,* $\beta(E'',E')$ *and* τ *coincide.*

Proof:

Let N be a countable dense subset of A. Without any loss of generality, we can assume $A = E \cap N^{00}$. Since $\tau \subseteq \beta(E'',E')|_E$, we need only show $\beta(E'',E')|_A \subseteq \tau|_A$. Let B be a bounded disc in E'_b; $A \cap B^0$ is then a basic $\beta(E'',E')|_A$-neighbourhood of $\underline{0}$. The case when $A \subseteq B^0$ is trivial, so assume that $N \setminus B^0 \neq \emptyset$. For each $x \in N \setminus B^0$, choose $f_x \in B$ so that

$$|f_x(x)| > 1 ,$$

then set

$$M = \{f_x \mid x \in N \setminus B^0\} .$$

Because $M \subseteq B$, the elements of M form a bounded sequence in E'_b. As (E,τ) is σ-quasibarrelled, M is τ-equicontinuous. To show that $A \cap B^0$ is a $\tau|_A$-neighbourhood of $\underline{0}$, we shall prove that

$$A \cap \operatorname{int}_\tau(E \cap M^0) \subseteq B^0 .$$

Suppose not; then because N is dense in A, there will be an $x \in N \cap \operatorname{int}_\tau(E \cap M^0)$ with $x \notin B^0$. But then $x \in M^0$ and $f_x \in M$, so $|f_x(x)| \leqslant 1$. This contradicts the choice of f_x. □

2. Corollary:

A separable σ-quasibarrelled space is quasibarrelled.

□

3. Similarly, a separable σ-barrelled space is barrelled.

4. A DF space is by definition countably quasibarrelled, and hence σ-quasibarrelled. Thus separable DF spaces are quasibarrelled.

5. From Theorem 2.3.1 it follows that a DF space (E,τ) is quasibarrelled if and only if $\beta(E'',E')$ coincides with τ on each member of a fundamental sequence of bounded sets.

6. Corollary:

Let $B_1, B_2, \ldots$ *be a fundamental sequence of bounded sets in a* DF *space* (E,τ). *If* τ *is metrizable on each* B_n *, then* (E,τ) *is quasibarrelled.*

Proof:

The topology on a metric space is determined by its convergent sequences. Any sequence (x_n) is separable, so $\beta(E'',E')$ coincides with τ on $\{x_n \mid n \in \mathbb{N}\}$. Thus $\beta(E'',E')$ coincides with τ on each B_n.

□

7. **Theorem** (Pfister [18]) :

In a DF *space* (E,τ) *, precompact sets are metrizable.*

Proof:

Let K be a precompact disc in (E,τ) ; we shall use transfinite induction to construct a countable family of neighbourhoods of $\underline{0}$ in (E,τ) that will induce a metrizable topology on E coinciding with τ on K .

Let ω_1 denote the first uncountable ordinal. We start by choosing $U_0 = E$. Suppose for some countable ordinal β $(0 < \beta < \omega_1)$, we have chosen a family $\{U_\alpha \mid \alpha < \beta\}$ of closed absolutely convex neighbourhoods of $\underline{0}$ in (E,τ) . Let m_β be the topology on E that has this family as a local subbase. Clearly m_β is metrizable and $m_\beta \subseteq \tau$. If m_β coincides with τ on K , there is no need to carry on any further. So suppose $m_\beta|_K \neq \tau|_K$. We can then find a closed absolutely convex τ-neighbourhood of $\underline{0}$, U_β , such that $K \cap 2U_\beta$ is not an $m_\beta|_K$-neighbourhood of $\underline{0}$.

The problem is to show that there exists a countable $\beta < \omega_1$ such that

$$m_\beta|_K = \tau|_K \ .$$

Suppose not; we shall construct a sequence (x_n) in K and a neighbourhood U of $\underline{0}$ in (E,τ) such that

$$x_n - x_{n+r} \notin U$$

for all $n,r \in \mathbb{N}$. This will contradict the precompactness of K .

Let $B_1, B_2, \ldots$ be a fundamental sequence of bounded sets in (E,τ) . Using induction, we shall select a monotonically increasing sequence $\alpha(1), \alpha(2), \ldots$ of countable ordinals together with a sequence $p(1), p(2), \ldots$ of natural numbers such that

$$(\forall n,k \in \mathbb{N}) \quad n \leqslant k \Rightarrow B_n \subseteq p(n)\ U_{\alpha(k)} \ .$$

For each $p \in \mathbb{N}$, put

$$\Omega(1,p) = \{\alpha < \omega_1 \mid B_1 \subseteq pU_\alpha\} \ .$$

Because B_1 is absorbed by every U_α , $\bigcup_{p=1}^{\infty} \Omega(1,p)$ contains all countable ordinals, and is therefore uncountable. Thus there is a $p(1) \in \mathbb{N}$ such that $\Omega(1,p(1))$ is uncountable; choose $\alpha(1) \in \Omega(1,p(1))$. Now for each $n,p \in \mathbb{N}$, define

$$\Omega(n+1,p) = \{\alpha \in \Omega(n,p(n)) \mid B_{n+1} \subseteq pU_\alpha\} \ .$$

Again, because

$$\bigcup_{p=1}^{\infty} \Omega(n+1,p) = \Omega(n,p(n)) \ ,$$

there is a $p(n+1) \in \mathbb{N}$ such that $\Omega(n+1, p(n+1))$ is uncountable. Choose $\alpha(n+1) \in \Omega(n+1, p(n+1))$ so that

$$\alpha(n+1) > \alpha(n) \ .$$

Note that if $n \leqslant k$, then

$$\alpha(k) \in \Omega(k,p(k)) \subseteq \Omega(n,p(n)) \ ,$$

which means that

$$B_n \subseteq p(n)\ U_{\alpha(k)} \ .$$

Put

$$U = \bigcap_{k=1}^{\infty} U_{\alpha(k)} \ .$$

To show U is a neighbourhood of $\underline{0}$ in (E,τ) , we need only show U absorbs each B_n . For $k = 1, 2, \ldots n$, find $\lambda_k > 0$ so that

$$B_n \subseteq \lambda_k\ U_{\alpha(k)} \ .$$

For $k = n+1, n+2, \ldots$, we already have that

$$B_n \subseteq p(n)\ U_{\alpha(k)} \ .$$

Put $\mu = \max\ \{\lambda_1, \lambda_2, \ldots \lambda_n, p(n)\}$; then

$$B_n \subseteq \mu\ U \ ,$$

as required.

We can now construct the sequence (x_n) . For each $n \in \mathbb{N}$, $\bigcap_{k=1}^{n} U_{\alpha(k)}$ is an $m_{\alpha(n+1)}$-neighbourhood of $\underline{0}$. We chose $U_{\alpha(n+1)}$ so that $K \cap 2U_{\alpha(n+1)}$ would not be an $m_{\alpha(n+1)}|_K$-neighbourhood of $\underline{0}$, so

$$K \cap \bigcap_{k=1}^{n} U_{\alpha(k)} \not\subseteq 2U_{\alpha(n+1)} \ .$$

Choose $x_n \in K \cap \bigcap_{k=1}^{n} U_{\alpha(k)}$ so that $x_n \notin 2U_{\alpha(n+1)}$. We now check that if $n,r \in \mathbb{N}$, then

$$x_n - x_{n+r} \notin U .$$

Suppose $x_n \in x_{n+r} + U_{\alpha(n+1)}$. Then since

$$x_{n+r} \in \bigcap_{k=1}^{n+r} U_{\alpha(k)} \subseteq U_{\alpha(n+1)},$$

we have that

$$x_n \in x_{n+r} + U_{\alpha(n+1)} \subseteq 2U_{\alpha(n+1)} .$$

This contradicts the choice of x_n. □

8. Note that a precompact metric space is separable. Thus in a DF space E, precompact sets are separable, and therefore have the topology $\beta(E'',E')$.

9. **Corollaries:**

a. *If every bounded set in a* DF *space* E *is precompact, then* E *is quasibarrelled.*

b. *A semiMontel* DF *space is Montel.* □

10. A semireflexive DF space need not be reflexive, as the following example shows: Let X be a non-separable reflexive Banach space. On X', put the topology $\gamma(X',X)$ of uniform convergence on the bounded separable discs in X.

Then $\gamma(X',X) \neq \mu(X',X) = \beta(X',X)$, so $(X',\gamma(X',X))$ is not quasibarrelled. However, $(X', \gamma(X',X))$ is a semireflexive DF space.

3.0 gDF SPACES

0. In 2.3.0 we stated the most important feature of a DF space is that its topology is determined by its fundamental sequence of bounded sets. We will now examine this in more detail. We shall introduce a class of spaces, the gDF spaces, characterized by this property.

1. **Definitions:**

Let (E,τ) be a lcs space.
By an *absorbent sequence*, we mean an increasing sequence $A = (A_n)$ of discs in E whose union spans E. Let η_A denote the finest locally convex topology on E that agrees with τ on each A_n. If $\tau = \eta_A$, we say τ is *localizable on* A.

2. The idea of an absorbent sequence is due to De Wilde and Houet [6]. The topology η_A is an example of a *generalized inductive limit topology*, as introduced by Garling [7]. Subsequent authors have developed these concepts further; see [20] and [1].

3. We shall only consider absorbent sequences that satisfy

$$kA_k \subseteq A_{k+1}$$

for all $k \in \mathbb{N}$. If this holds, η_A has a local base consisting of all discs in E that intersect each A_n in a $\tau|_{A_n}$-neighbourhood of $\underline{0}$.

4. Let F be a lcs space. If $T \in L(E,F)$ has continuous restrictions to each A_n , T is η_A-continuous on E . Thus the topology on E is localizable on A if and only if for each lcs space F , every such T is continuous. We can sharpen this result slightly:

5. **Lemma:**

If $A = (A_n)$ *is an absorbent sequence in a lcs space* E *, then the following are equivalent:*

a. *The topology on* E *is localizable on* A .

b. *For each Banach space* Y *, those* $T \in L(E,Y)$ *that have continuous restrictions to each* A_n *are continuous on* E .

Proof:

We shall only show b. ⇒ a.

Let U be a disc in E intersecting each A_n in a $\tau|_{A_n}$-neighbourhood of $\underline{0}$. Then U spans E , so the canonical surjection $\natural_U : E_U \to E_{(U)}$ can be thought of as a mapping of E into the Banach space $\tilde{E}_{(U)}$. Let B be the closed unit ball in $\tilde{E}_{(U)}$; B is the closure of $\natural_U U$ in $\tilde{E}_{(U)}$. For each $n \in \mathbb{N}$

$$\natural_U(U \cap A_n) \subseteq \natural_U U \subseteq B \ ,$$

so $\natural_U$ is continuous on each A_n . Thus $\natural_U \in L(E, \tilde{E}_{(U)})$. Since

$$\natural_U^{\leftarrow}(B) = U \ ,$$

U is a neighbourhood of $\underline{0}$ in E . □

6. Again let M be a collection of closed and bounded discs in a lcs space E , as described in 1.3.0. Suppose F is a complete lcs space. By Grothendieck's Completeness Theorems (1.3.9), it follows from 3.0.4 that if the topology on E is localizable on a fundamental sequence of members of M , then $L_M(E,F)$ is complete. Similarly, Lemma 3.0.5 gives us the following:

7. **Theorem:**

Let E *be a lcs space and suppose* M *is as before. The following are equivalent:*

(a) *The topology on* E *is localizable on a fundamental sequence* $M_1, M_2, \ldots$ *of members of* M .

(b) *For each Fréchet space* F *,* $L_M(E,F)$ *is a Fréchet space.*

(c) *For each Banach space* Y *,* $L_M(E,Y)$ *is a Fréchet space.*

Proof:

(a) ⇒ (b):

Let $\{V_k \mid k \in \mathbb{N}\}$ be a local base for the topology on F . We can assume $V_1, V_2, \ldots$ is a decreasing sequence of barrels. Then

$$\{L[M_k,V_k] \mid k \in \mathbb{N}\}$$

forms a local base for the topology on $L_M(E,F)$. Completeness follows from 3.0.6.

(b) ⇒ (c): Trivial.

(c) ⇒ (a):

Choosing $Y = \mathbb{K}$, we see that E'_M has a countable local base for its topology; the polars of basic neighbourhoods of $\underline{0}$ in E'_M will be the desired fundamental sequence. That the topology on E is localizable on this sequence follows from the completeness of each $L_M(E,Y)$.

□

8. Theorem 2.3.1 says the topology of a DF space is localizable on its fundamental sequence of bounded sets.

9. **Definition:**

A gDF *space is a lcs space whose topology is localizable on a fundamental sequence of bounded sets.*

10. The term "gDF space" was invented by Ruess ([21], [22] and [23]) as an abbreviation for "generalized DF space" ; we shall use this name because much of what follows is based on his work. However, various other names have been used for the same concept, the most common being "σ-locally topological space" , which appears in [1] and [17].

11. From Theorem 3.0.7. we see that if E is a gDF space and F is a Fréchet space, then $L_b(E,F)$ is a Fréchet space. In particular, the strong dual of a gDF space is a Fréchet space. If E is a gDF space, $(E'_b)'_b$ is a DF space.

3.1 The completion of a gDF space

0. The following result can be proved by means of a construction similar to that used in the Dieudonné-Schwartz Theorem on bounded sets in countable inductive limits ([24] II 6.5).

Lemma:

Let $A = (A_n)$ *be an absorbent sequence in a lcs space* (E,τ) . *If* B *is a bounded disc in* (E,τ) , *then the following are equivalent:*

a. B *is bounded in* (E,η_A) .

b. For some $n \in \mathbb{N}$, B *is absorbed by* $\mathrm{cl}_\tau A_n$. □

1. **Lemma:**

Let E *be a dense linear subspace of a lcs space* F . *If* E *contains an absorbent sequence* $A = (A_n)$ *such that* $\overline{A} = (\mathrm{cl}_F A_n)$ *is an absorbent sequence in* F , *then*

$$\eta_A = \eta_{\overline{A}}|_E \quad .$$ □

2. **Proposition:**

The completion of a gDF *space is a* gDF *space.* □

3. **Corollary:**

Quasicomplete gDF *spaces are complete.* □

3.2 The countable neighbourhood condition

0. In some respects, this property is a weaker version of countable quasibarrelledness.

1. **Definition:**

A lcs space E satisfies the *countable neighbourhood condition* if for each sequence (U_n) of closed absolutely convex neighbourhoods of $\underline{0}$ in E there exist $\alpha_n > 0$ such that

$$U = \bigcap_{n=1}^{\infty} \alpha_n U_n$$

is again a neighbourhood of $\underline{0}$ in E.

2. **Proposition:**

Let E *be a lcs space, with* M *as before. If the topology on* E *is localizable on a fundamental sequence* $M_1, M_2, \ldots$ *of members of* M *, then* E *satisfies the countable neighbourhood condition.*

Proof:

Suppose we are given a sequence (U_n) of closed absolutely convex neighbourhoods of $\underline{0}$ in E. Because each M_k is bounded, there are $\lambda_{nk} > 0$ such that

$$M_k \subseteq \lambda_{nk} U_n$$

for all $n, k \in \mathbb{N}$. For each $n \in \mathbb{N}$, put

$$\alpha_n = \max\{\lambda_{n1}, \lambda_{n2}, \dots \lambda_{nn}\} .$$

Then for $k = 1, 2, \dots n$,

$$M_k \subseteq \lambda_{nk} U_k \subseteq \alpha_n U_n .$$

Put

$$U = \bigcap_{n=1}^{\infty} \alpha_n U_n .$$

Then for each $k \in \mathbb{N}$,

$$U \cap M_k = \bigcap_{n=1}^{k} \alpha_n U_n \cap \bigcap_{n=k+1}^{\infty} (\alpha_n U_n \cap M_k) ,$$

$$= \left(\bigcap_{n=1}^{k} \alpha_n U_n \right) \cap M_k$$

Since each $\left(\bigcap_{n=1}^{k} \alpha_n U_n \right)$ is a neighbourhood of $\underline{0}$ in E, so is U.

□

3. Corollary:

Every gDF *space satisfies the countable neighbourhood condition.*

□

4. Any linear subspace of a lcs space satisfying the countable neighbourhood condition inherits that property. This is a useful observation, for the property of being localizable on a fundamental sequence of members of a particular collection M need not be inherited by subspace topologies.

3.3 Sequential quasibarrelledness

0. Proposition:

Let E *and* F *be lcs spaces, with* M *as before. Suppose the topology on* E *is localizable on a fundamental sequence of members of* M *. Then every precompact subset* H *of* $L_M(E,F)$ *is equicontinuous on* E *.*

Proof:

Let V be a closed absolutely convex neighbourhood of $\underline{0}$ in F . For each $k \in \mathbb{N}$ there is a finite subset $\{T_{1k}, T_{2k}, \dots T_{nk}\}$ of H such that

$$H \subseteq \bigcup_{i=1}^{n} T_{ik} + \tfrac{1}{2} L[M_k, V] \quad .$$

Let U_k be a closed absolutely convex neighbourhood of $\underline{0}$ in E satisfying

$$\bigcup_{i=1}^{n} T_{ik} U_k \subseteq \tfrac{1}{2} V \ .$$

Consider $T \in H$. Since

$$T = T_{ik} + \frac{1}{2}S$$

for some $i \in \{1, 2, \ldots n\}$ and $S \in L[M_k, V]$,

$$T(U_k \cap M_k) \subseteq T_{ik}U_k + \frac{1}{2}SM_k \subseteq V .$$

Put

$$U = \Gamma \bigcup_{k=1}^{\infty} (U_k \cap M_k) ;$$

then U is a neighbourhood of $\underline{0}$ in E satisfying

$$TU \subseteq V$$

for all $T \in H$.

□

1. **Corollary:**

In the strong dual of a gDF *space, precompact sets are equicontinuous.*

□

2. A countably quasibarrelled gDF space is automatically a DF space. We shall see later there exist gDF spaces which are not DF spaces, and are therefore not countably quasibarrelled. However, every gDF space does have a certain element of barrelledness:

3. **Corollary:**

Every gDF *space is sequentially quasibarrelled.*

Proof:

If E is a lcs space, then any sequence that $\beta(E',E)$-converges to $\underline{0}$ in E'_b forms a precompact set in E'_b . □

4.0 SCHWARTZ SPACES AND QUASINORMABILITY

0. Apart from DF spaces, Grothendieck discussed two other classes of spaces in his paper [9]. In general, if H is a closed subspace of a lcs space E, not every bounded set in E/H need lie in the canonical image of a bounded disc in E. This can happen even if E is a Fréchet Montel space ([17] §31.5). Thus a separated quotient of a Fréchet Montel space need not be reflexive, let alone Montel. To avoid such problems, Grothendieck introduced the notion of a Schwartz space, a natural generalization of which is that of a quasinormable space. Interestingly enough, Grothendieck did not show that every DF space is quasinormable, it was left to Kats [16] to prove this is true.

1. Let U and V be discs in a lcs space E. If U absorbs V, there is a natural linear map $\Phi_{VU} \in L(E_{(V)}, E_{(U)})$ given by

$$\Phi_{VU}(x + N(V)) = x + N(U) .$$

This extends to a map $\tilde{\Phi}_{VU} \in L(\tilde{E}_{(V)}, \tilde{E}_{(U)})$; the following diagram commutes :

$$\begin{tikzcd}
E_V \arrow[r, hook] \arrow[d, two heads, "\natural_V"'] & E_U \arrow[d, two heads, "\natural_U"] & \text{(seminormed)} \\
E_{(V)} \arrow[r, "\Phi_{VU}"] \arrow[d, hook] & E_{(U)} \arrow[d, hook] & \text{(normed)} \\
\tilde{E}_{(V)} \arrow[r, "\tilde{\Phi}_{VU}"] & \tilde{E}_{(U)} & \text{(Banach)}
\end{tikzcd}$$

2. If U and V are both neighbourhoods of $\underline{0}$ in E, then $(\tilde{E}_{(U)})' \cong E'_{(U^0)} \cong E'_U$

and $(\tilde{E}_{(V)})' \cong E'_{(V^0)} \cong E'_V$.

It follows that the adjoint $(\tilde{\Phi}_{VU})'$ of $\tilde{\Phi}_{VU}$ can be identified with $\Phi_{U^0V^0}$ and hence with the inclusion of E'_{U^0} into E'_{V^0} :

$$\begin{tikzcd}
(\tilde{E}_{(U)})' \arrow[r, phantom, "\cong"] \arrow[d, "(\tilde{\Phi}_{VU})'"] & E'_{(U^0)} \arrow[r, phantom, "\cong"] \arrow[d, "\Phi_{U^0V^0}"] & E'_{U^0} \arrow[d, hook] \\
(E_{(V)})' \arrow[r, phantom, "\cong"] & E'_{(V^0)} \arrow[r, phantom, "\cong"] & E'_{V^0}
\end{tikzcd}$$.

3. **Definition** (Swart [25] 2.1) :

Let U be a disc in a vector space E . We say $V \subseteq E$ is *totally bounded relative to* U if $(\forall \varepsilon > 0)\ (\exists$ finite $S \subseteq V)\quad V \subseteq S + \varepsilon U$.

4. **Proposition** ([25] 2.6 and 2.8) :

If U *and* V *are closed absolutely convex neighbourhoods of* $\underline{0}$ *in a lcs space* E *, the following are equivalent :*

a. V *is totally bounded relative to* U .

b. U *absorbs* V *and* $\natural_U V$ *is precompact in* $E_{(U)}$.

c. U *absorbs* V *and* $\tilde{\Phi}_{VU} \in K(\tilde{E}_{(V)}, \tilde{E}_{(U)})$.

d. V^0 *absorbs* U^0 *and* U^0 *is relatively compact in* E'_{V^0} .

Proof:

a. ⇒ b.: Trivial.

b. ⇒ c.: Let B be the closure of $\natural_V V$ in $\tilde{E}_{(V)}$. Because $\Phi_{VU} \natural_V V = \natural_U V$ is precompact in $E_{(U)}$, $\tilde{\Phi}_{VU} B$ is relatively compact in $\tilde{E}_{(U)}$; B is the unit ball in $\tilde{E}_{(V)}$.

c. ⇒ d.: Since $U^0 = (\tilde{\Phi}_{VU})' U^0$, this follows from Schauder's Theorem.

d. ⇒ a.: Let $\varepsilon > 0$ be given. By the precompactness of U^0 in E'_{V^0} , there is a finite $C \subseteq U^0$ with

$$U^0 \subseteq C + {}^1/_3\ \varepsilon\ V^0 .$$

Thus if $f \in U^0$, then $f = g + {}^1/_3\, \varepsilon\, h$ for some

$g \in C$ and $h \in V^0$. Because V^0 absorbs U^0, U absorbs V and so $\natural_V C^0$ is a $\sigma(E_{(V)}, E'_{V^0})$-neighbourhood of $\underline{0}$ in $E_{(V)}$. Since $\natural_V V$ is contained in the unit ball of $E_{(V)}$, $\natural_V V$ is $\sigma(E_{(V)}, E'_{V^0})$-precompact. Thus there is a finite $S \subseteq V$ with

$$\natural_V V \subseteq \natural_V S + \tfrac{1}{6}\,\varepsilon\, \natural_V C^0 ,$$

i.e.

$$V \subseteq (S + \tfrac{1}{6}\,\varepsilon\, C^0) + N(V) .$$

Since U absorbs V and $C \subseteq U^0$,

$$N(V) \subseteq \tfrac{1}{6}\,\varepsilon\, U \subseteq \tfrac{1}{6}\,\varepsilon\, C^0 ,$$

which means that

$$V \subseteq S + \tfrac{1}{3}\,\varepsilon\, C^0 .$$

If $x \in V$, then for some $y \in S$,

$$x - y \in \tfrac{1}{3}\,\varepsilon\, C^0 .$$

Thus

$$|f(x-y)| \leqslant |g(x-y)| + \tfrac{1}{3}\,\varepsilon\, |h(x)| + \tfrac{1}{3}\,\varepsilon\, |g(x)| \leqslant \varepsilon ,$$

and so $x - y \in \varepsilon U^{00} = \varepsilon U$. We therefore have

$$V \subseteq S + \varepsilon U .$$

□

4.1 Schwartz topologies

0. Definition (Grothendieck [9]):

A *Schwartz space* is a lcs space E in which for each closed absolutely convex neighbourhood U of

$\underline{0}$ there exists a neighbourhood of $\underline{0}$ totally bounded relative to U .

1. According to the above Proposition, an equivalent requirement is that for each such U there must be an absolutely convex neighbourhood V of $\underline{0}$ such that U is relatively compact in the Banach space E'_{V^0} .

2. **Definition** ([25] 2.10) :

If (E,τ) is a Schwartz space, τ is referred to as a *Schwartz topology for* E .

3. If U is a $\sigma(E,E')$-neighbourhood of $\underline{0}$ in a lcs space E , then E'_{U^0} is finite-dimensional. Because U^0 is relatively compact in E'_{U^0} , $\sigma(E,E')$ is always a Schwartz topology on E .

4. Schwartz spaces have remarkable permanence properties. The class of all such spaces is closed under the taking of subspaces, products, projective limits, separated quotients, countable inductive limits and countable direct sums. (For a proof, see [13] 3, §15.)

5. (Swart [25] 3.1)

Let (E,τ) be a lcs space. The collection of all Schwartz topologies coarser than τ contains $\sigma(E,E')$. Taking a projective limit, we find there is a finest Schwartz topology τ^S coarser than τ . We shall refer to (E,τ^S) as the *Schwartz space associated with* (E,τ) .

6. It can be shown (Swart [25] 4.6) that each $T \in L(E,F)$ remains continuous if both E and F are replaced by their associated Schwartz spaces. In the language of category theory, this says the Schwartz spaces form a full epireflective subcategory of the category of all lcs spaces and their continuous linear maps.

4.2 Quasinormable spaces

0. Definition:

A lcs space E is *quasinormable* if for each closed absolutely convex neighbourhood U of $\underline{0}$ in E there exists a neighbourhood V of $\underline{0}$ satisfying

$$(\forall \varepsilon > 0)\ (\exists \text{ bounded disc } B \text{ in } E)\ V \subseteq B + \varepsilon U .$$

1. Obviously every Schwartz space is quasinormable. In fact, it is easy to prove the following:

Proposition:

The Schwartz spaces are exactly those quasinormable spaces in which every bounded disc is precompact. □

2. **Corollaries:**

a. Every quasicomplete Schwartz space is semiMontel.
b. Every Fréchet Schwartz space is Montel. □

3. It follows from Theorem 2.2.1 and Corollary 4.2.2b that the strong dual of a Fréchet Schwartz space is ultrabornological. (In fact, the strong dual of any complete Schwartz space is ultrabornological ([5] III 3.10)).

4. **Theorem:**

Every gDF *space* E *is quasinormable.*

Proof (Kats [16]):

Let U be a closed absolutely convex neighbourhood of $\underline{0}$ in E . Suppose $B_1, B_2, \ldots$ is a fundamental sequence of bounded sets in E . For each $k \in \mathbb{N}$, put

$$V_k = B_k + \tfrac{1}{k}U ,$$

then set

$$V = \bigcap_{k=1}^{\infty} V_k .$$

We must show V is a neighbourhood of $\underline{0}$ in E. Let $n \in \mathbb{N}$. For $k = 1, 2, \dots n$,

$$B_n \cap {}^1\!/_n U \subseteq {}^1\!/_k U \subseteq B_k + {}^1\!/_k U = V_k ,$$

so

$$B_n \cap {}^1\!/_n U \subseteq \bigcap_{k=1}^{n} V_k .$$

For $k = n + 1, n + 2, \dots$, $B_n \subseteq B_k \subseteq V_k$, so

$$B_n \cap {}^1\!/_n U \subseteq \bigcap_{k=n+1}^{\infty} V_k .$$

Thus

$$B_n \cap {}^1\!/_n U \subseteq V$$

which means that V is a neighbourhood of $\underline{0}$ in E. Now let $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ so that ${}^1\!/_k < \varepsilon$. Then

$$V \subseteq V_k = B_k + {}^1\!/_k U \subseteq B_k + \varepsilon U . \qquad \Box$$

5. From 1.9.4, 4.2.1 and 4.2.4, we see that a lcs space E is a Schwartz gDF-space if and only if its topology is localizable on a fundamental sequence of precompact sets. Thus every semiMontel gDF space is a Schwartz space. This has the following converse :

6. **Proposition:**

The completion of a Schwartz gDF *space* E *is semiMontel.*

Proof:

Proposition 3.1.2 says $\tilde{E}$ is a gDF space. If $P_1, P_2, \ldots$ form a fundamental sequence of precompact sets in E , then their completions $\tilde{P}_1, \tilde{P}_2, \ldots$ will be a fundamental sequence of bounded sets in $\tilde{E}$ consisting of compact discs. □

4.3 A generalization of quasinormability

0. Once again let M be a collection of closed and bounded discs in E, as described in 1.3.0. We shall refer to E as being *M-quasinormable* if for each absolutely convex neighbourhood U of $\underline{0}$ in E there exists a neighbourhood V of $\underline{0}$ satisfying

$$(\forall\varepsilon > 0)(\exists M \in M) \quad V \subseteq M + \varepsilon U .$$

1. Clearly in the case when $M = M(s)$, this is the definition of a Schwartz space given in 4.1.0, while if $M = M(b)$ we have the definition of quasinormability used in 4.2.0. It is easy to modify Theorem 4.2.4 so as to prove that whenever the topology on E is localizable on some fundamental sequence of members of M , then E is M-quasinormable.

2. Proposition:

Let E *and* F *be lcs spaces, with* $\mathcal{M}$ *as above. If* E *is* $\mathcal{M}$*-quasinormable and* F *is metrizable, then each equicontinuous subset* H *of* $L_{\mathcal{M}}(E,F)$ *is metrizable.*

Proof:

Let $W_1, W_2, \ldots$ be a decreasing sequence of barrels forming a local base for the topology on F . For each $k \in \mathbb{N}$, let U_k be a closed absolutely convex neighbourhood of $\underline{0}$ in E satisfying

$$\cup \{ TU_k \mid T \in H\} \subseteq W_k .$$

By the $\mathcal{M}$-quasinormability of E , there is a closed absolutely convex neighbourhood V_k of $\underline{0}$ in E satisfying

$$(\forall \varepsilon > 0)(\exists M \in \mathcal{M}) \quad V_k \subseteq M + \varepsilon U_k .$$

Now $\{L[V_k,W_k] \mid k \in \mathbb{N}\}$ is a local subbase for a pseudometrizable topology on $L(E,F)$. Since each V_k absorbs every $M \in \mathcal{M}$, this topology is finer than the one on $L_{\mathcal{M}}(E,F)$. To show the two topologies coincide on H , let $k \in \mathbb{N}$ be given. Choose $M \in \mathcal{M}$ so that

$$V_k \subseteq M + \tfrac{1}{2}U_k .$$

Suppose $T \in H \cap L[M, \frac{1}{2}W_k]$; then

$$TV_k \subseteq TM + \tfrac{1}{2}TU_k \subseteq W_k$$

and so

$$H \cap L[M, \tfrac{1}{2}W_k] \subseteq L[V_k, W_k] \ . \qquad \square$$

3. Corollaries:

a. If E *is a Schwartz space, then every equicontinuous subset of* E' *is* $\sigma(E',E)$*-metrizable.*

b. If E *is a quasinormable space, then every equicontinuous subset of* E' *is* $\beta(E',E)$*-metrizable.* ☐

4. Proposition:

Let E *be a lcs space, with* M *as before. The following statements are equivalent:*

a. E *is M-quasinormable.*

b. For each equicontinuous disc H *in* E' *, there exists an absolutely convex neighbourhood* V *of* $\underline{0}$ *in* E *such that on* H *the topologies of uniform convergence on the members of* M *and on* V *coincide.*

Proof:

a. ⇒ b.: This is a special case of Proposition 4.3.2.

b. ⇒ a.: If U is a closed absolutely convex neighbourhood of $\underline{0}$ in E, then $H = U^0$ is equicontinuous on E. Let V be the neighbourhood described in b. For each $\varepsilon > 0$ there is an $M \in \mathcal{M}$ with

$$H \cap M^0 \subseteq \varepsilon V^0 ,$$

so that

$$V \subseteq V^{00} \subseteq \varepsilon(H \cap M^0)^0 \subseteq \varepsilon(M \cup U)^{00} \subseteq \varepsilon(M + U) .$$

Since $\varepsilon M \in \mathcal{M}$, the result follows. □

5. Corollary:

A lcs space E *is quasinormable if and only if for each equicontinuous disc* H *in* E'*, there exists an absolutely convex neighbourhood* V *of* $\underline{0}$ *in* E *such that* $\beta(E',E)$ *coincides on* H *with the topology of uniform convergence on* V *.* □

6. The above characterization suggests why such spaces should be called "quasinormable". In fact, Grothendieck used this result as his original definition of quasinormability([9] III Dèfinition 4).

4.4 Equihypocontinuity

0. Definitions ([17] §40.2) :

Let E, F and G be lcs spaces. Suppose $\mathcal{R}$ and $\mathcal{S}$ are collections of subsets of E and of F respectively, and H is a subset of $B(E \times F, G)$. We say H is $\mathcal{R} \times \mathcal{S}$-*equihypocontinuous* if for each $R \in \mathcal{R}$ the set

$$H(R,\_) = \{h(x,\_) \mid h \in H, x \in R\}$$

is equicontinuous on F and for each $S \in \mathcal{S}$ the set

$$H(\_,S) = \{h(\_,y) \mid h \in H, y \in S\}$$

is equicontinuous on E . If $\mathcal{R}$ and $\mathcal{S}$ are the collections of all closed and bounded discs in E and in F respectively, we say H is *equihypocontinuous*, and refer to an $h \in H$ as being *hypocontinuous*.

1. If $\mathcal{R}$ and $\mathcal{S}$ consist of bounded sets in E and in F , then every equicontinuous subset of $\mathcal{B}(E \times F, G)$ is $\mathcal{R} \times \mathcal{S}$-equihypocontinuous.

2. Theorem:

Let E , F *and* G *be lcs spaces. Suppose* $\mathcal{M}$ *is a collection of closed and bounded discs in* E , *as described in* 1.3.0, *and* $\mathcal{A} = (A_n)$ *is an absorbent sequence in* F . *Provided*

a. E *is* $\mathcal{M}$*-quasinormable,*

b. E *satisfies the countable neighbourhood condition,*

c. the topology on F *is localizable on* $\mathcal{A}$,

every $\mathcal{M} \times \mathcal{A}$*-equihypocontinuous subset* H *of* $B(E \times F, G)$ *is equicontinuous.*

Proof: (based on [23], Proposition 1.7) :

Let W be a closed absolutely convex neighbourhood of $\underline{0}$ in G . For each $n \in \mathbb{N}$, $H(\_,A_n)$ is equicontinuous on E , so there is a neighbourhood W_n of $\underline{0}$ in E with

$$H(W_n,A_n) = \{h(x,y) \mid h \in H,\ x \in W_n,\ y \in A_n\} \subseteq \tfrac{1}{2}W .$$

By the $\mathcal{M}$-quasinormability of E , to each W_n there corresponds an absolutely convex neighbourhood U_n of $\underline{0}$ in E satisfying

$$(\forall\varepsilon > 0)(\exists M \in \mathcal{M})\ U_n \subseteq M + \varepsilon W_n .$$

Applying the countable neighbourhood condition, we obtain a sequence (ε_n) of positive numbers such that

$$U = \bigcap_{n=1}^{\infty} \frac{1}{\varepsilon_n} U_n$$

is again a neighbourhood of $\underline{0}$ in E. For each $n \in \mathbb{N}$, choose $M_n \in \mathcal{M}$ so that

$$U_n \subseteq M_n + \varepsilon_n W_n .$$

Because $H(M_n,\_)$ is equicontinuous on F, there is a neighbourhood V_n of $\underline{0}$ in F with

$$H(M_n,V_n) = \{h(x,y) \mid h \in H,\ x \in M_n,\ y \in V_n\} \subseteq \tfrac{1}{2}W .$$

Put

$$V = \bigcup_{n=1}^{\infty} (A_n \cap \varepsilon_n V_n) \quad ;$$

then V is a neighbourhood of $\underline{0}$ in F.

We wish to show that

$$H(U,V) = \{h(x,y) \mid h \in H,\ x \in U,\ y \in V\} \subseteq W .$$

Let $h \in H$, $x \in U$ and $y \in V$ be given. For some $n \in \mathbb{N}$,

$$y \in A_n \cap \varepsilon_n V_n .$$

Then

$$x \in U \subseteq \frac{1}{\varepsilon_n} U_n \subseteq \frac{1}{\varepsilon_n} M_n + W_n$$

so

$$h(x,y) \in H(\frac{1}{\varepsilon_n} M_n,\ \varepsilon_n V_n) + H(W_n,A_n) \subseteq W . \qquad \square$$

3. **Corollary:**

If E *and* F *are* gDF *spaces, then every equihypocontinuous subset of* B(E × F) *is equicontinuous.* □

4.5 The "Problême des Topologies"

0. At the end of his paper [9], Grothendieck listed a number of open problems. One he singled out as being of particular importance: Is $\mathcal{B}_{bb}(E \times F)$ a DF space whenever E and F are Fréchet spaces? An answer to this question would enable us to strengthen Corollary 2.1.1 considerably.

1. The difficulty lies in the fact that although $\mathcal{B}(E \times F)$ can always be identified with $(E \tilde{\otimes}_\pi F)'$, there are lcs spaces E and F for which the strong topology on $(E \tilde{\otimes}_\pi F)'_b$ is strictly finer than the topology on $\mathcal{B}_{bb}(E \times F)$ ([17] §41.6). We would like to know if there are Fréchet spaces with this property. The question as to whether
$$(E \tilde{\otimes}_\pi F)'_b \cong \mathcal{B}_{bb}(E \times F) \quad .$$
for all lcs spaces E and F belonging to a given class is referred to as the "Problème des Topologies" for that class. Grothendieck's original conjecture remains unanswered, but we are able to solve the "Problème des Topologies" for gDF spaces.

2. **Theorem:**

Let E *and* F *be lcs spaces, with* M *a collection of closed and bounded discs in* E *, as described in*

1.3.0, and $\mathcal{N}$ *a similar collection in* F *. If the topology on* E *is localizable on a fundamental sequence* $M_1, M_2, \ldots$ *of members of* $\mathcal{M}$ *, and that on* F *on a corresponding sequence* $N_1, N_2, \ldots$ *of members of* $\mathcal{N}$ *, then the projective tensor product topology on* $E \otimes_\pi F$ *is localizable on* $(\overline{\Gamma}\chi(M_n \times N_n))$.

Proof:

It is easy to check that $\mathcal{A} = (\overline{\Gamma}\chi(M_n \times N_n))$ is indeed an absorbent sequence in $E \otimes F$. To show that $\eta_{\mathcal{A}}$ coincides with the projective tensor product topology on $E \otimes_\pi F$, we need only prove that $\chi\colon E \times F \to E \otimes F$ is a continuous map into $(E \otimes F, \eta_{\mathcal{A}})$. In fact, it is sufficient to show that $\{\chi\}$ is $\mathcal{M} \times \mathcal{N}$-equihypocontinuous with respect to $\eta_{\mathcal{A}}$; because of the obvious symmetry, we shall only check that

$$\chi(M,\_) = \{\chi(x,\_) \mid x \in M\}$$

is equicontinuous on F for all $M \in \mathcal{M}$.

A basic $\eta_{\mathcal{A}}$-neighbourhood of $\underline{0}$ in $E \otimes F$ is a disc W such that for each $n \in \mathbb{N}$ there exist closed absolutely convex neighbourhoods of $\underline{0}$, U_n in E and V_n in F, satisfying

$$\Gamma\chi(U_n \times V_n) \cap \overline{\Gamma}\chi(M_n \times N_n) \subseteq W .$$

To show that $\chi(M,\_)$ is equicontinuous on F, we need only check that

$$V = \{y \in F \mid (\forall x \in M)\ \ x \otimes y \in W\}$$

is a neighbourhood of $\underline{0}$ in F. Since M is bounded in E, for each $n \in \mathbb{N}$ there is an $\alpha_n > 0$ such that $\alpha_n M \subseteq U_n$. For all $n \in \mathbb{N}$ sufficiently large, we have that $M \subseteq M_n$. For such n,

$$\chi(M \times (\alpha_n V_n \cap N_n)) \subseteq \chi(U_n \times V_n) \cap \chi(M_n \times N_n) \subseteq W .$$

This means that

$$\alpha_n V_n \cap N_n \subseteq V \ ;$$

since the topology on F is localizable on (N_n), it follows that V is a neighbourhood of $\underline{0}$ in F.

□

3. **Proposition** (Ruess [23] Proposition 1.8) :

Let E *and* F *be* gDF *spaces, with fundamental sequences of bounded sets* $A_1, A_2, \ldots$ *in* E *and* $B_1, B_2, \ldots$ *in* F. *Then* $E \otimes_\pi F$ *is a* gDF *space, with* $\overline{\Gamma}\chi(A_1 \times B_1)$, $\overline{\Gamma}\chi(A_2 \times B_2)$, ... *as a fundamental sequence of bounded sets.*

Proof:

In view of the above Theorem, we need only check $\overline{\Gamma}\chi(A_1 \times B_1)$, $\overline{\Gamma}\chi(A_2 \times B_2)$, ... is a fundamental sequence of bounded sets in $E \otimes_\pi F$. It is easy to see that each $\overline{\Gamma}\chi(A_n \times B_n)$ is bounded in $E \otimes_\pi F$. Because the topology on $E \otimes_\pi F$ is localizable on $(\overline{\Gamma}\chi(A_n \times B_n))$, it follows from Lemma 3.1.0 that every bounded disc in $E \otimes_\pi F$ is absorbed by some $\overline{\Gamma}\chi(A_n \times B_n)$. □

4. Corollaries:

If E *and* F *are* gDF *spaces, then*

a. $E \tilde{\otimes}_\pi F$ *is a* gDF *space.*

b. $(E \tilde{\otimes}_\pi F)'_b \cong \mathcal{B}_{bb}(E \times F)$.

c. $\mathcal{B}_{bb}(E \times F)$ *is a Fréchet space.*

Proof:

a. From Proposition 3.1.2.

b. We need only check that every basic neighbourhood of $\underline{0}$ in $(E \tilde{\otimes}_\pi F)'_b$ can be identified with a neighbourhood of $\underline{0}$ in $\mathcal{B}_{bb}(E \times F)$. If C is a bounded disc in $E \tilde{\otimes}_\pi F$, then by the Proposition there exist bounded discs, A in E and B in F, with

$$C \subseteq \overline{\Gamma}\chi(A \times B) .$$

Thus

$$\mathcal{B}[A \times B] \subseteq C^0 ,$$

as required.

c. By b. $\mathcal{B}_{bb}(E \times F)$ is the strong dual of a gDF space. □

5.0 SEMIMONTEL gDF SPACES

0. Having examined DF spaces in some detail, we now look at a class of gDF spaces which might be thought of as lying at the opposite end of the spectrum. Whereas DF spaces were originally modelled after the strong duals of Fréchet spaces, semiMontel gDF spaces are all of the form F'_p for some Fréchet space F. Since DF spaces have by definition a fairly strong barrelledness property, their topologies are comparatively fine. By contrast, the topology on a semiMontel gDF space is in a sense the coarsest topology localizable on the same fundamental sequence of bounded sets.

1. **Definitions** (Köthe [17] §23.9) :

We refer to E'_p as the *polar dual* of the lcs space E. If E can be identified algebraically with $(E'_p)'$, we say E is *polar semireflexive.* If in addition the topologies on E and on $(E'_p)'_p$ coincide, E is *polar reflexive.*

2. It is easy to see E is polar semireflexive if and only if each precompact disc in E is relatively compact. Thus a Schwartz space is polar semireflexive

if and only if it is semiMontel. Every quasicomplete space is polar semireflexive. (The converse is not true - for a counter-example see [11]).

3. **Proposition:**

Every semiMontel gDF *space* E *is polar reflexive.*

Proof:

Because E is quasicomplete, it is polar semireflexive. To see that the topology on E is coarser than that of $(E_p')_p'$, let U be a closed absolutely convex neighbourhood of $\underline{0}$ in E. By the Grothendieck Interchange Theorem, U^0 is precompact in E_p', so $U = U^{00}$ is a neighbourhood of $\underline{0}$ in $(E_p')_p'$.

Conversely, 4.2.5 says the topology on E is localizable on a fundamental sequence of precompact discs. By Proposition 3.3.0 this means each precompact disc in E_p' is equicontinuous on E, so the topology on $(E_p')_p'$ is coarser than that of E. □

4. **Corollary:**

Every semiMontel gDF *space* E *is the polar dual of a Fréchet space.*

Proof:

By Theorem 3.0.7, E'_p is a Fréchet space. □

5.1 The Banach-Dieudonné Theorem

0. We would like to show that the converse to Corollary 5.0.4 is also true.

1. Lemma:

The polar dual of a barrelled space F *is semiMontel.*

Proof:

Every bounded disc B in F'_p is equicontinuous on F ; that B is relatively compact in F'_p follows from the Grothendieck Interchange Theorem. □

2. If F is a lcs space, let $\nu(F',F)$ denote the topology on F'_{mc} of uniform convergence on Mackey-convergent sequences in F . (Note that this topology is a duality-invariant: it is determined by the dual pair $<F',F>$, not by the specific topology on F .)

3. Theorem:

If F *is a metrizable lcs space, then* $\nu(F',F)$ *is the finest topology on* F' *coinciding with* $\sigma(F',F)$ *on equicontinuous sets.*

Proof:

See [24] IV §6.3. □

4. This is one formulation of the classic Banach-Dieudonné Theorem; there are many others. It follows immediately that if F is a metrizable lcs space, then $\nu(F',F)$ is localizable on a fundamental sequence of equicontinuous discs in F'_s .

5. Corollaries:

If F *is a Fréchet space, then*

a. $F'_{mc} = F'_p$.

b. F'_p *is a semiMontel* gDF *space.*

c. F *is polar reflexive.*

Proof:

a. By Lemma 1.3.6 the topology on F'_p coincides with $\sigma(F',F)$ on the equicontinuous subsets of F' ; $\nu(F',F)$ is the finest topology with this property.

b. From 3.1.4 we have that F'_{mc} is a gDF space; Lemma 5.1.1 says F'_p is semiMontel.

c. By Proposition 5.0.3 F'_p is polar reflexive, hence so is $(F'_p)'_p = F$. □

6. Corollary:

Let D *be a dense linear subspace of a metrizable lcs space* F *. If* P *is a subset of* F *, the following are equivalent:*

a. P *is precompact in* F .

b. There exists a null sequence (x_k) *in* D *with* $P \subseteq \overline{\Gamma}\{x_k\}$.

Proof:

$$F'_p = F'_{mc} = D'_{mc} .$$

□

5.2 The duality between Fréchet spaces and semiMontel gDF spaces.

0. Together Corollaries 5.0.4 and 5.1.5b describe a one-to-one correspondence between Fréchet spaces and semiMontel gDF spaces. Applying Lemma 1.3.7, we see this correspondence extends to the continuous linear maps between such spaces: if E and F are either Fréchet spaces or semiMontel gDF spaces, then $T \in L(E,F) \Leftrightarrow T' \in L(F'_p,E'_p)$. This duality between the class of Fréchet spaces and the class of semiMontel gDF spaces was first discussed by Brauner [2].

1. Since every Montel space is barrelled, all Montel gDF spaces are DF spaces. By 1.8.2, a lcs space E is a Montel DF space if and only if $E'_b = E'_p$ is a Fréchet Montel space. Thus under the duality mentioned above, the Montel DF spaces are in correspondence with the Fréchet Montel spaces; again this correspondence can be extended to the continuous linear maps between these spaces.

2. By Corollary 2.4.9b, a semiMontel DF space must be Montel. However, since there exist Fréchet spaces which are not reflexive, there do exist semiMontel gDF spaces which are not Montel, and which are therefore not DF spaces.

3. Corollary 2.4.2 says every separable σ-quasibarrelled space is quasibarrelled. However, there do exist semiMontel gDF spaces which are separable but not quasibarrelled, and which are therefore sequentially quasibarrelled but not σ-quasibarrelled. For an example, let X be a separable non-reflexive Banach space, and consider $(X'_b)'_p$.

4. Theorem 2.4.7 says every precompact disc in a DF space is metrizable. In a semiMontel gDF space, precompact discs need not be metrizable; they need not even be separable. Let B be the unit ball in a non-separable reflexive Banach space X . By the Grothendieck Interchange Theorem, B is precompact in $E = (X'_b)'_p$, but if B were separable in E , it would have to be separable in X .

5. Proposition 4.2.6 says the completion of a Schwartz gDF space is semiMontel. We can use this to obtain the following characterization:

6. Theorem:

If E *is a lcs space, the following are equivalent:*

a. E *is a Schwartz* gDF *space.*

b. E *is dense in the polar dual of a Fréchet space.*

c. E is sequentially quasibarrelled and contains a fundamental sequence of precompact sets.

Proof:

a. ⇒ b.: By Corollary 5.0.4, $\tilde{E}$ is the polar dual of a Fréchet space.

b. ⇒ c.: Suppose E is dense in F'_p, where F is a Fréchet space. If $\{V_k \mid k \in \mathbb{N}\}$ is a local base for the topology on F, then $E \cap V_1^0$, $E \cap V_2^0$, ... form a fundamental sequence of precompact sets in E. Since F'_p is a gDF space, it is sequentially quasibarrelled, and hence so is E.

c. ⇒ a.: Because E contains a fundamental sequence of precompact sets, E'_p is metrizable. If U is a neighbourhood of $\underline{0}$ in E, then U^0 is precompact in E'_p, so by Corollary 5.1.6 there exists a null sequence (f_n) in E'_p such that $U^0 \subseteq \overline{\Gamma}\{f_n\}$. Thus the topology on E is coarser than $\nu((E'_p)',E')|_E$. Conversely, if (g_n) is a null sequence in $E'_p = E'_b$, it follows from the sequential quasibarrelledness of E that $\{g_n\}$ is equicontinuous on E; therefore the topology on E is $\nu((E'_p)',E')|_E$. By 5.1.4, E is a gDF space; by 4.2.5 it is a Schwartz space. □

7. The Theorem above should be compared with the following :

Theorem (Brudovskii [3]):

If E *is a lcs space, the following are equivalent:*

a. E *is a Schwartz* DF *space.*

b. E *is dense in the strong dual of a Fréchet Montel space.*

c. E *is countably quasibarrelled and contains a fundamental sequence of precompact sets.* □

8. **Theorem** (Garling [8]) :

If E *is a lcs space, the following are equivalent:*

a. E *is a reflexive* DF *space.*

b. E *is the strong dual of a reflexive Fréchet space.*

c. E *is barrelled and contains a fundamental sequence of* $\sigma(E,E')$*-compact discs.*

Proof:

We shall only show c. ⇒ a.:
Because E'_k is metrizable, it is quasibarrelled, and so each $\beta(E,E')$-bounded closed disc in E is

$\sigma(E,E')$-compact. But E is barrelled, so all bounded subsets of E are $\beta(E,E')$-bounded. It follows that the fundamental sequence of $\sigma(E,E')$-compact discs in E is also a fundamental sequence of bounded sets. As E is barrelled, this means E is a DF space. □

5.3 The associated Schwartz space

0. While proving 5.2.6, we noted that the topology on a Schwartz gDF space E is the topology $\nu((E'_p)',E')|_E$ of uniform convergence on the Mackey-convergent sequences in E'_p. Jarchow and Swart [15] have shown how to characterize the Schwartz topology associated with any lcs space E in terms of suitable sequences in E'. Using their construction, we shall investigate the Schwartz topology associated with a gDF space.

1. **Theorem:**

If $\mathcal{U}$ *is a local base for the topology* τ *on a lcs space* E *, then*

$$\mathcal{U}^s = \{\{f_n\}^0 \mid (\exists U \in \mathcal{U}) \ (f_n) \text{ converges to } \underline{0} \text{ in } E'_{U^0}\}$$

is a local base for the associated Schwartz topology τ^s *on* E .

Proof:

We first check that the topology that has $\mathcal{U}^s$ as its local base is a Schwartz topology on E. Let $\{f_n\}^0 \in \mathcal{U}^s$, then (f_n) converges to $\underline{0}$ in E'_{U^0} for some $U \in \mathcal{U}$. For each $n \in \mathbb{N}$, let $r_n = p_{U^0}(f_n)$ be the norm of f_n in E'_{U^0}. Then $(r_n^{-\frac{1}{2}} f_n)$ also converges to $\underline{0}$ in E'_{U^0}, so $V = \{r_n^{-\frac{1}{2}} f_n\}^0 \in \mathcal{U}^s$. Now (f_n) converges to $\underline{0}$ in E'_{V^0}, so $\Gamma\{f_n\}$ is precompact in E'_{V^0}. Since the norm topology on E'_{V^0} is linked to $\sigma(E',E)$, the closure of $\Gamma\{f_n\}$ in E'_{V^0} is $\{f_n\}^{00}$. Thus $\{f_n\}^{00}$ is compact in E'_{V^0}; according to Proposition 4.0.4 this means that $V \in \mathcal{U}^s$ is totally bounded relative to $\{f_n\}^0$. We therefore have that $\mathcal{U}^s$ is a local base for a Schwartz topology. To show that there is no finer Schwartz topology on E coarser than τ, we shall prove that whenever $V \in \mathcal{U}$ is totally bounded relative to $U \in \mathcal{U}$, there exists an $\{f_n\}^0 \in \mathcal{U}^s$ such that $\{f_n\}^0 \subseteq U$. Since U^0 is relatively compact in E'_{V^0} by Proposition 4.0.4, it follows from Corollary 5.1.6 that there exists a null sequence (f_n) in E'_{V^0} such that $U^0 \subseteq \{f_n\}^{00}$.

□

2. **Corollary:**

The Schwartz space associated with E'_b *is* E'_{mc} . □

3. In 3.0.11 we mentioned that if E is a gDF space, then $(E'', \beta(E'', E'))$ is a (complete) DF space. From Corollary 5.1.5b we have that $(E'', \nu(E'', E'))$ is a semiMontel gDF space; it follows from the above result that $\nu(E'', E')$ is the Schwartz topology associated with $\beta(E'', E')$.

4. **Proposition:**

If (E, τ) *is a* gDF *space, then so is its associated Schwartz space* (E, τ^s) .

Proof:

Because τ and τ^s share the same bounded sets, the fundamental sequence $B_1, B_2, \ldots$ of bounded sets in (E, τ) will also serve as a fundamental sequence of bounded sets in (E, τ) . In fact, since bounded sets in (E, τ^s) are precompact, $B_1, B_2, \ldots$ will be a fundamental sequence of precompact sets in (E, τ^s) . According to Theorem 5.2.6, to show that (E, τ^s) is a gDF space, we need only prove that (E, τ^s) is sequentially quasibarrelled. Let (f_n) be a null sequence in E'_b . Because (E,τ) is sequentially quasibarrelled, $\{f_n\}$ is equicontinuous on (E,τ) . Since (E,τ) is quasinormable, we have from Corollary 4.3.5 that $\beta(E',E)$ coincides on $\{f_n\}^{00}$ with the topology of uniform convergence on some neighbourhood U of $\underline{0}$ in (E,τ) . Thus (f_n) converges to $\underline{0}$ in E'_{U^0}. By Theorem 5.3.1, this means that $\{f_n\}$ is equicontinuous on (E,τ^S) , as required. □

5. Note that because τ^S is linked to $\sigma(E,E')$, on each bounded disc in (E,τ) the topologies τ^S and $\sigma(E,E')$ coincide. Thus if (E,τ) is a gDF space, its associated Schwartz topology τ^S may be thought of as the coarsest topology on E that is both finer than $\sigma(E,E')$ and localizable on the same fundamental sequence of bounded sets.

6.0 LINEAR MAPS BETWEEN gDF AND FRÉCHET SPACES

0. In the case when X and Y are Banach spaces, the spaces $K_b(X,Y)$, $W_b(X,Y)$ and $L_b(X,Y)$ have been the subject of much study. In this Chapter we turn our attention to $K(E,F)$, $W(E,F)$ and $L(E,F)$, where E and F are lcs spaces. We shall see many of the well-known results of the Banach space theory carry through provided we choose E to be a gDF space and F a Fréchet space.

1. In many ways, this is the central result:

Proposition:

If E *and* F *are lcs spaces, then*

a. $K(E,F) \cong B(E \times F'_{ca}) \cong (E \tilde{\otimes}_\pi F'_{ca})'$.

b. $W(E,F) \cong B(E \times F'_k) \cong (E \tilde{\otimes}_\pi F'_k)'$.

Proof:

We shall only prove a.

Since $B(E\times F) \cong (E \tilde{\otimes}_\pi F)'$ for any lcs spaces E and F, we need only show that $K(E,F) \cong B(E \times F'_{ca})$.

Given $T \in \mathcal{K}(E,F)$, define $\hat{T} \in B(E \times F')$ by

$$\hat{T}(x\,,g) = \langle Tx\,,g\rangle$$

for all $x \in E$ and $g \in F'$. Because T is a compact linear map, there exists a neighbourhood U of $\underline{0}$ in E together with a compact disc K in F such that

$$TU \subseteq K\,.$$

Since $U \times K^0$ is a neighbourhood of $\underline{0}$ in $E \times F'_{ca}$ and

$$|\hat{T}(x,g)| \leqslant 1$$

whenever $(x,g) \in U \times K^0$, $\hat{T}$ is continuous on $E \times F'_{ca}$. Given $h \in \mathcal{B}(E \times F'_{ca})$, define $\dot{h} \in L(E,F)$ by

$$\langle \dot{h}x, g\rangle = h(x,g)$$

for all $x \in E$ and $g \in F'$. A similar argument shows $\dot{h} \in \mathcal{K}(E,F)$; we therefore have an isomorphism between $\mathcal{K}(E,F)$ and $\mathcal{B}(E \times F'_{ca})$. □

2. Compact linear maps were first studied in the context of Banach spaces: there $T \in \mathcal{L}(X,Y)$ is compact if T maps the unit ball in X onto a relatively compact disc in Y. We have chosen to generalize this to lcs spaces by defining $T \in \mathcal{L}(E,F)$ to be compact if T maps some neighbourhood of $\underline{0}$ in E onto a relatively compact disc in F. However, in a Banach space

the unit ball is not only a neighbourhood of $\underline{0}$; it is also a fundamental bounded set. We could equally well have chosen to define $T \in L(E,F)$ to be compact if T maps the bounded discs in E onto relatively compact discs in F . In general, this definition results in a different class of linear maps; we shall now prove that in the case when E is a gDF space and F is a Fréchet space, the two definitions are equivalent.

3. **Proposition:**

Let $\mathcal{M}$ *be a collection of closed and bounded discs in a lcs space* E *, as described in 1.3.0, such that* E *is* $\mathcal{M}$*-quasinormable and satisfies the countable neighbourhood condition. If* H *is an equicontinuous subset of* $L(E,F)$ *, where* F *is a Fréchet space, then the following are equivalent:*

a. *There exists a neighbourhood* U *of* $\underline{0}$ *in* E *such that*

$$H(U) = \{Tx \mid T \in H , x \in U\}$$

is $(\sigma(F,F')-)$ *relatively compact in* F .

b. *For each* $M \in \mathcal{M}$

$$H(M) = \{Tx \mid T \in H , x \in M\}$$

is $(\sigma(F,F')-)$ *precompact in* F .

Proof (based on [23] Proposition 2.1) :

a. ⇒ b.: Each $M \in \mathcal{M}$ is bounded, hence absorbed by U .

b. ⇒ a.: We shall only consider the "compact" case.
If $\{V_k \mid k \in \mathbb{N}\}$ is a local base for the topology on F , then by Theorem 5.1.3 the topology on $F'_p = F'_{ca}$ is localizable on $\mathcal{A} = (V_n^0)$. Given $T \in H$, define $\hat{T} \in B(E \times F')$ by

$$\hat{T}(x,g) = \langle Tx, g\rangle$$

for all $x \in E$ and $g \in F'$. It follows from b. that

$$\hat{H} = \{\hat{T} \mid T \in H\}$$

is $\mathcal{M} \times \mathcal{A}$-equihypocontinuous on $E \times F'_{ca}$. By Theorem 4.4.2 $\hat{H}$ is an equicontinuous subset of $\mathcal{B}(E \times F'_{ca})$; so there exists a neighbourhood U of $\underline{0}$ in E together with a compact disc K in F such that

$$|\hat{T}(x,g)| \leqslant 1$$

for all $T \in H$, $x \in U$ and $g \in K^0$. This means

$$H(U) \subseteq K^{00} = K .$$

□

4. **Corollary:**

Let E *be a* gDF *space and* F *a Fréchet space. A linear map* $T \in L(E,F)$ *is* (weakly) *compact if and only if* T *maps bounded discs in* E *onto* $(\sigma(F,F')\text{-})$ *precompact discs in* F . □

5. Comparing these results with Lemma 1.3.7, we immediately obtain a generalization of Schauder's Theorem:

Proposition:

Let E *and* F *be lcs spaces, with* $\mathcal{M}$ *a collection of closed and bounded discs in* E , *as described in 1.3.0, and* $\mathcal{N}$ *a similar collection in* F . *Suppose the topologies on* E *and on* F *are localizable on fundamental sequences of members of* $\mathcal{M}$ *and of* $\mathcal{N}$. *Then*

a. $T \in K(E,F'_{\mathcal{N}})$ *if and only if* $T' \in K(F,E'_{\mathcal{M}})$.

b. $T \in W(E,F'_{\mathcal{N}})$ *if and only if* $T' \in W(F,E'_{\mathcal{M}})$. □

6. Since continuous linear maps preserve precompactness, the following can be derived immediately from Corollary 6.0.4 :

Corollaries:

Let F *be a Fréchet space.*

a. *If* E *is a Schwartz* gDF *space, then*

$$\mathcal{K}(E,F) = \mathcal{L}(E,F) .$$

b. *If* E *is a semireflexive* gDF *space, then*

$$\mathcal{W}(E,F) = \mathcal{L}(E,F) .$$ □

7. In the case when E is an $\mathcal{M}$-quasinormable space satisfying the countable neighbourhood condition and F is a Fréchet space, we can use Lemma 1.3.7 to identify the compact linear maps in $\mathcal{K}(E,F)$ as being exactly those $T \in \mathcal{L}(E,F)$ whose restrictions to each $M \in \mathcal{M}$ are $\sigma(E,E')$-continuous. In particular, when the topology on E is localizable on a fundamental sequence of members of $\mathcal{M}$, they are exactly the linear maps in $\mathcal{L}(E,F)$ that remain continuous when E is replaced by its associated Schwartz space.

8. In effect, Corollaries 6.0.6a and 6.0.6b give conditions that can be placed on a lcs space E to ensure that every continuous linear map from E into a Fréchet space F is either compact or weakly compact. We can get away with less stringent conditions on E provided we restrict our choice of F.

9. **Propositions** (Ruess [22] Theorem 2.6) :

Let E *be a lcs space satisfying the countable neighbourhood condition.*

a. *If* F *is a Fréchet Montel space, then*

$$K(E,F) = L(E,F) .$$

b. *If* F *is a reflexive Fréchet space, then*

$$W(E,F) = L(E,F) .$$

Proof: Straightforward. □

10. Even when X and Y are Banach spaces, it is possible for $K(X,Y)$, $W(X,Y)$ and $L(X,Y)$ to be distinct. However, $K_b(X,Y)$, $W_b(X,Y)$ and $L_b(X,Y)$ are Banach spaces; we now look for analogues of this result.

11. **Proposition:**

Let E *be a lcs space and* F *a Fréchet space, with* M *as before. If* E *is M-quasinormable and satisfies the countable neighbourhood condition, then both* $K(E,F)$ *and* $W(E,F)$ *are closed in* $L_M(E,F)$.

Proof (based on [23] Proposition 2.5) :

Consider a linear map T lying in the closure of $\mathcal{K}(E,F)$ in $\mathcal{L}_{\mathcal{M}}(E,F)$. In view of Proposition 6.0.3, to show that $T \in \mathcal{K}(E,F)$, we need only prove T maps each $M \in \mathcal{M}$ onto a precompact disc in F . Let V be a closed absolutely convex neighbourhood of $\underline{0}$ in F . Because $\mathcal{L}[M,V]$ is a neighbourhood of $\underline{0}$ in $\mathcal{L}_{\mathcal{M}}(E,F)$, there is an $S \in \mathcal{K}(E,F)$ with

$$(S - T) \in \mathcal{L}[M,V] \ .$$

This means

$$TM \subseteq SM + V \ .$$

Since SM is precompact in F , it follows that so is TM . A similar argument shows $\mathcal{W}(E,F)$ is closed in $\mathcal{L}_{\mathcal{M}}(E,F)$. □

12. **Corollary :**

If E *is a* gDF *space and* F *is a Fréchet space, then* $K(E,F)$ *and* $\mathcal{W}(E,F)$ *are closed in* $\mathcal{L}_b(E,F)$. □

13. Thus whenever the topology on E is localizable on a fundamental sequence of members of $\mathcal{M}$, and F is a Fréchet space, $K_{\mathcal{M}}(E,F)$ and $\mathcal{W}_{\mathcal{M}}(E,F)$ are Fréchet spaces.

6.1 A generalization of the Banach-Dieudonné Theorem

0. In the previous Chapter we discussed the duality between Fréchet spaces and semiMontel gDF spaces. If both E and F are Fréchet spaces, then so is $E \tilde{\otimes}_\pi F$; we would like to know if there is any direct relationship between the semiMontel gDF spaces E'_p, F'_p and $(E \tilde{\otimes}_\pi F)'_p$. It turns out there is; in fact if either E or F satisfies a certain extra condition, then

$$(E \tilde{\otimes}_\pi F)'_p \cong E'_p \tilde{\otimes}_\varepsilon F'_p .$$

What is particularly interesting is that there exists a symmetrical relationship between E'_p, F'_p and $(E \tilde{\otimes}_\pi F)'_p$ when E and F are semiMontel gDF spaces. This symmetry, in the case when E or F satisfies the extra condition, was first observed by Buchwalter; the more general case was established by Köthe ([17] §45.3) and Hollstein [12].

1. Corollary 5.1.6 says that in a Fréchet space F containing a dense linear subspace D a set R is precompact if and only if there exists a null sequence (x_k) in D such that each $x \in R$ can be expressed in the form

$$x = \sum_{k=1}^{\infty} \lambda_k x_k$$

for some $\lambda_k \in \mathbb{K}$ with $\sum_{k=1}^{\infty} |\lambda_k| \leqslant 1$. This can be used to prove the following:

2. **Lemma** ([19] VII §2) :

If E *and* F *are metrizable lcs spaces, the following are equivalent:*

a. R *is a precompact set in* $E \tilde{\otimes}_\pi F$.

b. *There exist precompact discs* P *in* E *and* Q *in* F *such that*

$$R \subseteq \overline{\Gamma}\chi(P \times Q) \ . \qquad \square$$

3. **Proposition:**

If E *and* F *are Fréchet spaces, then*

$$(E \tilde{\otimes}_\pi F)'_p \cong \mathcal{K}_p(E,F'_p) \ .$$

Proof:

$$(E \tilde{\otimes}_\pi F)' = (E \tilde{\otimes}_\pi (F'_p)'_p)' \quad \text{by polar reflexivity.}$$

$$= (E \tilde{\otimes}_\pi (F'_p)'_{ca})' \quad \text{since } F'_p \text{ is quasi-complete.}$$

$$\cong \mathcal{K}(E,F'_p) \quad \text{by Proposition 6.0.1b.}$$

Basic neighbourhoods of $\underline{0}$ in $(E \tilde{\otimes}_\pi F)'_p$ are of the form R^0 , where R is a precompact set in $E \tilde{\otimes}_\pi F$. By Lemma 6.1.2 there are precompact discs P in E and Q in F such that

$$R \subseteq \overline{\Gamma}\chi(P \times Q) \ .$$

Put

$$\mathcal{B}[P \times Q] = \{h \in \mathcal{B}(E \times F) \mid (\forall x \in P, y \in Q) \quad |h(x,y)| \leqslant 1\} \ ;$$

then

$$\mathcal{B}[P \times Q] \subseteq R^0 .$$

Basic neighbourhoods of $\underline{0}$ in $\mathcal{K}_p(E,F'_p)$ are of the form $\mathcal{K}[P,Q^0]$, where P and Q are precompact discs in E and in F . It is straightforward to check that under the isomorphism between $(E \tilde{\otimes}_\pi F)'$ and $\mathcal{K}(E,F'_p)$, $\mathcal{B}[P \times Q]$ corresponds to $\mathcal{K}[P,Q^0]$. □

4. **Lemma** ([22] Theorem 2.2) :

If E *and* F *are Fréchet spaces, then*

$$\mathcal{K}(E,F'_p) = \mathcal{L}(E,F'_p) .$$

Proof:

Let $\{U_j \mid j \in \mathbb{N}\}$ and $\{V_j \mid j \in \mathbb{N}\}$ be local bases for the topologies on E and on F . To show that $T \in \mathcal{L}(E,F'_p)$ is a compact linear map, it is sufficient to show that for some $n,j \in \mathbb{N}$,

$$TU_n \subseteq V_j^0 .$$

Suppose not, then for each $j \in \mathbb{N}$ there is an $x_j \in U_j$ with $Tx_j \notin V_j^0$. Since (x_j) is a null sequence in E , $\{Tx_j\}$ is precompact in F'_p , contradicting the fact that $V_1^0, V_2^0, \ldots$ form a fundamental sequence of precompact sets in F'_p . □

5. (The above Lemma should be compared with Corollary 6.0.6a, which says that if E and F are Fréchet spaces, then

$$K(E'_p, F) \cong L(E'_p, F)\ .)$$

6. **Theorem** (Hollstein [12] Satz 6) :

If E *and* F *are Fréchet spaces, then*

$$(E \mathbin{\tilde{\otimes}_\pi} F)'_p \cong L_p(E, F'_p)\ . \qquad \square$$

7. Thus if E and F are Fréchet spaces, then $L_p(E, F'_p)$ is a semiMontel gDF space. This extends Corollary 5.1.5b. In fact, this result can be thought of as a generalization of the Banach-Dieudonné Theorem; using Lemma 1.3.6 we immediately have the following :

8. **Corollary** (Brauner [2] Corollary 2.7) :

If E *and* F *are Fréchet spaces, then the topology on* $L_p(E, F'_p)$ *of precompact convergence is the finest topology on* $L(E, F'_p)$ *that coincides with the topology of simple convergence on equicontinuous sets.*

□

6.2 The ε-product

0. At this stage it is convenient to introduce a concept invented by L Schwartz:

Definition (Köthe [17] §43.3(3')):

If E *and* F *are lcs spaces, then*

$$E \,\varepsilon\, F = L_e(E'_{ca}, F) .$$

1. We can now express Theorem 6.1.6 in the following form :

Proposition ([17] §45.3(1)) :

If E *and* F *are Fréchet spaces, then*

$$(E \tilde{\otimes}_\pi F)'_p \cong E'_p \,\varepsilon\, F'_p .$$

Proof:

By polar reflexivity

$$L_p(E, F'_p) \cong L_e((E'_p)'_{ca}, F'_p) .$$

□

2. Exploiting the duality between Fréchet spaces and semiMontel gDF spaces, we obtain the following:

Corollary:

If E *and* F *are semiMontel* gDF *spaces, then*

$$(E'_p \tilde{\otimes}_\pi F'_p)'_p \cong E \varepsilon F .$$ □

3. Proposition ([17] §45.3(5)) :

If E *and* F *are Fréchet spaces, then*

$$E \varepsilon F \cong (E'_p \tilde{\otimes}_\pi F'_p)'_p .$$

Proof:

From Corollary 6.0.6a we have that

$$\begin{aligned} E \varepsilon F &= \mathcal{L}_e(E'_{ca},F) \\ &= \mathcal{L}_e(E'_p, F) \\ &= \mathcal{K}_e(E'_p,F) . \end{aligned}$$

It follows from Proposition 6.0.1a that

$$\begin{aligned} \mathcal{K}(E'_p,F) &\cong (E'_p \tilde{\otimes}_\pi F'_{ca}) \\ &\cong (E'_p \tilde{\otimes}_\pi F'_p). \end{aligned}$$

Since E'_p and F'_p are both gDF spaces, Corollary 4.5.4b says

$$\begin{aligned} (E'_p \tilde{\otimes}_\pi F'_p)'_b &\cong \mathcal{B}_{bb}(E'_p \times F'_p) \\ &\cong \mathcal{B}_{ee}(E'_p \times F'_p) . \end{aligned}$$

Let $\{U_j \mid j \in \mathbb{N}\}$ and $\{V_j \mid j \in \mathbb{N}\}$ be local bases for the topologies on E and on F. A routine calculation shows that under the isomorphism between $\mathcal{K}(E'_p,F)$ and $\mathcal{B}(E'_p \times F'_p)$,

$\mathcal{K}[U_j^0, V_j]$ corresponds to $\mathcal{B}[U_j^0 \times V_j^0]$.

Thus

$$\mathcal{K}_e(E'_p, F) \cong \mathcal{B}_{ee}(E'_p, F'_p)$$

and so

$$E \; \varepsilon \; F \cong (E'_p \tilde{\otimes}_\pi F'_p)'_b \; .$$

Furthermore, since $\overline{\Gamma}\chi(U_1^0 \times V_1^0)$, $\overline{\Gamma}\chi(U_2^0 \times V_2^0)$, ... form a fundamental sequence of bounded sets in $E'_p \tilde{\otimes}_\pi F'_p$, $E'_p \tilde{\otimes}_\pi F'_p$ is semiMontel. It follows that

$$(E'_p \tilde{\otimes}_\pi F'_p)'_b \cong (E'_p \tilde{\otimes}_\pi F'_p)'_p \; .$$

□

4. **Corollary:**

If E *and* F *are semiMontel* gDF *spaces, then*

$$E'_p \; \varepsilon \; F'_p \cong (E \tilde{\otimes}_\pi F)'_p \; .$$

□

5. To summarize :

Theorem (Hollstein [12] Satz 1) :

If E *and* F *are either both Fréchet spaces or both semiMontel* gDF *spaces, then*

a. $(E \tilde{\otimes}_\pi F)'_p \cong E'_p \; \varepsilon \; F'_p$.   a'. $E \tilde{\otimes}_\pi F \cong (E'_p \; \varepsilon \; F'_p)'_p$.

b. $(E'_p \tilde{\otimes}_\pi F'_p)'_p \cong E \; \varepsilon \; F$.   b'. $E'_p \tilde{\otimes}_\pi F'_p \cong (E \; \varepsilon \; F)'_p$.

□

6.3 The approximation property

0. If E and F are lcs spaces, then $E \otimes_\varepsilon F$ can be identified with $\mathcal{F}_e(E'_s, F)$, which in turn forms a subspace of $\mathcal{L}_e(E'_{ca}, F)$. Thus $E \otimes_\varepsilon F$ is a subspace of $E \varepsilon F$.

1. **Definition** ([17] §43.1) :

A lcs space E is said to have *the approximation property* if $\mathcal{F}(E,E)$ is dense in $\mathcal{L}_p(E,E)$.

2. **Proposition** ([17] §43.1(1)) :

Let E *and* F *be lcs spaces. If either* E *or* F *has the approximation property, then* $\mathcal{F}(E,F)$ *is dense in* $\mathcal{L}_p(E,F)$.

Proof:

Suppose $T \in \mathcal{L}(E,F)$, P is a precompact disc in E and V is a closed absolutely convex neighbourhood of $\underline{0}$ in F. Consider first the case when E has the approximation property. Let U be a neighbourhood of $\underline{0}$ in E such that $TU \subseteq V$. Since $\mathcal{F}(E,E)$ is dense in $\mathcal{L}_p(E,E)$, there is an $S \in \mathcal{F}(E,E)$ with

$$Sx - x \in U$$

for all $x \in P$. But then $TS \in \mathcal{F}(E,F)$ and

$$TS - T \in \mathcal{L}[P,V] ,$$

as required.

Now suppose F has the approximation property. Since TP is precompact in F and $\mathcal{F}(F,F)$ is dense in $\mathcal{L}_p(F,F)$, there exists an $R \in \mathcal{F}(F,F)$ with

$$Ry - y \in V$$

for all $y \in TP$. Then $RT \in \mathcal{F}(F,F)$ and

$$RT - T \in \mathcal{L}[P,V] ,$$

as required. □

3. **Lemma:**

A lcs space E has the approximation property if and only if E'_p has the approximation property.

Proof:

$$T \in \mathcal{L}(E,E) \Leftrightarrow T' \in \mathcal{L}(E'_p,E'_p)$$
$$S \in \mathcal{F}(E,E) \Leftrightarrow S' \in \mathcal{F}(E'_p,E'_p) .$$

□

4. **Proposition:**

Let E *and* F *be either Fréchet spaces or semi-Montel* gDF *spaces. If either* E *or* F *has the approximation property,* $E \otimes F$ *is dense in* $E \varepsilon F$.

Proof:

$$E \varepsilon F = L_e(E'_{ca},F) = L_p(E'_p,F) .$$
$$E \otimes F \cong F(E'_s,F) = F(E'_p,F) . \qquad \square$$

5. The following is a simple variation of Lemma 1.3.7; it proves useful in establishing some of the basic properties of $E \varepsilon F$:

6. **Lemma:**

Let E *and* F *be lcs spaces.*
If $T \in L(E',F)$ *is* $(\sigma(E',E), \sigma(F,F'))$*-continuous, then the following are equivalent:*

a. T *maps equicontinuous discs in* E' *onto relatively compact discs in* F .

b. $T' \in L(F'_{ca},E)$.

c. For each neighbourhood V *of* $\underline{0}$ *in* F *, the restriction of* T' *to* V^0 *is a* $\sigma(F',F)$*-continuous map into* E .

a'. T' *maps equicontinuous discs in* F' *onto relatively compact discs in* E .

b'. $T \in L(E'_{ca},F)$.

c'. *For each neighbourhood* U *of* $\underline{0}$ *in* E *, the restriction of* T *to* U^0 *is a* $\sigma(E',E)$*-continuous map into* F .

□

7. Note that $E \varepsilon F \cong F \varepsilon E$.

8. **Lemma:**

Let E *and* F *be lcs spaces. If either* E *or* F *is complete, then so is* $E \varepsilon F$.

Proof:

Suppose F is complete. By 1.3.9, to show that $L_e(E'_{ca},F)$ is complete, we need only prove that if a $(\sigma(E',E), \sigma(F,F'))$-continuous linear map $T \in L(E',F)$ is continuous on each equicontinuous disc in E'_{ca} , then T is continuous throughout E'_{ca} ; this follows immediately from Lemma 6.3.6.

□

9. **Proposition:**

Let E *and* F *be either Fréchet spaces or semi-Montel* gDF *spaces. If either* E *or* F *has the approximation property, then*

$$E \tilde{\otimes}_{\varepsilon} F = E \varepsilon F \,. \qquad \square$$

10. **Theorem** (Hollstein [12] Korollar 1) :

Let E *and* F *be either both Fréchet spaces or both semiMontel* gDF *spaces. If either* E *or* F *has the approximation property, then*

a. $(E \tilde{\otimes}_{\pi} F)'_p \cong E'_p \tilde{\otimes}_{\varepsilon} F'_p$.

b. $(E'_p \tilde{\otimes}_{\pi} F'_p)'_p \cong E \tilde{\otimes}_{\varepsilon} F$.

a'. $E \tilde{\otimes}_{\pi} F \cong (E'_p \tilde{\otimes}_{\varepsilon} F'_p)'_p$.

b'. $E'_p \tilde{\otimes}_{\pi} F'_p \cong (E \tilde{\otimes}_{\varepsilon} F)'_p$. $\square$

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