Implementing Short-rate Models With Jumps At Deterministic Times

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy at the University of Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

Macroeconomic announcements have a direct impact on short-term interest rates during a financial year. However, this is not directly reflected in the continuous-time interest rate models. In this paper, we work with short-rate models which include the possibility of jumps at deterministic times. An application of the finite-difference method enables the pricing of bonds and bond options in these short-rate models with different types of jump distributions. A closed-form solution for bond prices, when the jumps are normally distributed, is available in the literature, but not for other jump distributions. The Monte Carlo method is used to compare the finite-difference calculations for these cases. An illustration of varying important model parameters is provided in which we observe that an increase in option prices could result from an increase in the jump variances and/or volatility parameters.

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Chapter 1

Introduction

The decisions taken by investors and risk departments in the financial markets rely considerably on the movements and potential movements of interest rates. In this concern, short-rate models have proved to be very helpful to determine the evolution of interest rates over time. These continuous-time models have dynamics represented by an Itô stochastic differential equation. There are many different types of short-rate models which have been proposed to date, with each one having its own specificity. Models such as those presented by Vasicek (1977) and Cox *et al.* (1985) are convenient to use as they work under the assumption that the evolution of the interest rates depends on one stochastic factor only.

Johannes (2004) emphasised the importance of accounting for jumps in these types of models, explaining the crucial role they play in determining the dynamics of interest rate movements. Jumps can be classified as expected jumps, where the timing of the jump is known in advance, and unexpected jumps, where both the jump time and jump size are unknown (Backwell and Hayes, 2021). The occurrence of the deterministically-timed jumps during a financial year are usually due to prescheduled macroeconomic announcements. Backwell and Hayes (2021) worked closely with the SONIA data, which is the major overnight rate in the United Kingdom. They showed that jumps are in fact by far the most important dynamic of SONIA.

Benchmark reform—the transition from London Interbank Offered Rates (LIBORs) to a new set of reference interest rates—has created renewed interest in short-rate models. Gellert and Schlögl (2021) directed their research towards the Secured Overnight Funding Rate (SOFR), which is central to the benchmark reform proposals in the USA. They focused on expected spikes (jumps that quickly return to pre-jump level), which are an important feature of SOFR, and expected jumps. Andersen and Bang (2020) also worked on SOFR, considering short-rate models in connection with the benchmark reform in which they modelled for both expected and unexpected spikes.

Kim and Wright (2014) presented one of the few papers in the literature that focused on providing a term structure model for bond yields which caters for the occurrence of deterministically-timed jumps. They developed a closed-form representation for pricing bonds that follow a jump-diffusion, in which the jumps are normally distributed with a state-dependent mean. In addition to considering the pricing of bonds in this kind of framework, they applied their model by analysing the expected excess returns on jump days compared to non-jump days in which it was concluded that the bond risk premia were greater in absolute values when jumps occurred. The modelling of deterministically-timed interest rate jumps traces back to Piazzesi (2001).

This dissertation will focus on short-rate models which include the possibility of jumps at deterministic times. We price bonds and bond options in these models using finite-difference methods. We make use of the Kim and Wright (2014) closed-form representation to verify our model. The Monte Carlo method is used to verify the finite-difference calculations when pricing bond options or bonds under other jump distributions. Finally, an illustration of the effects of varying different model parameters is provided.

Following this introduction, the short-rate model generalised to include jumps is provided in Chapter 2. The finite-difference method, as applied to these short-rate models is presented in Chapter 3. Chapter 4 discusses the important results obtained from the implemented model, and Chapter 5 concludes on the work.

Chapter 2

Modelling Of Jumps

In this chapter we start by illustrating the general form of a short-rate model. The Vasicek model is then introduced. A change of measure is performed to move from the real-world dynamics to the risk-neutral dynamics. An overview for jump implementations in the model is provided by referring to the work conducted by Kim and Wright (2014). A closed-form formula for pricing a zero-coupon bond following a Vasicek stochastic differential equation (SDE) modified for normal jumps is then illustrated.

2.1 Continuous short-rate models

The dynamics of the short rate, r_t, are given by an Itô process as

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{2.1}$$

where $\mu(t, r_t)$ is the drift function, $\sigma(t, r_t)$ is the volatility function and W_t is a standard Brownian motion. While there are different models which have been proposed to date with each one having its own advantages and disadvantages, in this work we make use of the Vasicek model for its mean reversion features and also because there exists a closed-form solution which will allow us to verify our implementation. In general, one could allow for more than one stochastic process to feed into the drift and volatility functions. However, we will consider *one-factor* models only in this paper.

The Vasicek model, using the Ornstein-Uhlenbeck process, produces short rates with the following dynamics:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \tag{2.2}$$

where the drift component consists of κ which is the rate of mean reversion and θ, the mean reversion level. The drift term exhibits mean reversion, which means that the interest rates converge to a mean reversion level θ at a speed of κ. We can consider the parameter θ here as the long-term interest rate level. A drop of the short-rate below the long-term rate causes it to be pushed back up and the same applies for when there is a shift above the long-term rate (Burgess, 2014).

To ensure that discounted prices are martingales, we have to move from the real-world measure, $\mathbb{P}$, to a risk-neutral measure, $\mathbb{Q}$. In so doing, we ensure that there are no arbitrage opportunities when pricing a claim (Björk, 2009). The Vasicek SDE under this measure is then given as

$$dr_t = \kappa_{\mathbb{Q}}(\theta_{\mathbb{Q}} - r_t)dt + \sigma dW_t^{\mathbb{Q}}. \tag{2.3}$$

Now, by applying this martingale property at a time t, the price of a zero-coupon bond maturing at time T is obtained as follows:

$$P(t,T) = \mathbb{E}_t\left[\exp\Big(-\int_t^T r_s ds\Big)\right], \tag{2.4}$$

where $\mathbb{E}_t[.]$ denotes the time-t conditional expectation under $\mathbb{Q}$. The expression is key to the derivation of the analytical formula used for pricing bonds under the Vasicek SDE. The closed-form solution is given as

$$P(t,T) = \exp\Big(-A(t,T)r_t + B(t,T)\Big), \tag{2.5}$$

where

$$A(t,T) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}), \tag{2.6}$$

$$B(t,T) = \Big(\frac{\sigma^2}{2\kappa^2} - \theta\Big)[(T-t) - A(t,T)] - \frac{\sigma^2}{4\kappa}A^2(t,T). \tag{2.7}$$

While the bond considered in equation 2.4 has a basic payoff function, one can consider more complicated claims. For example, the price of a call option written on a zero-coupon bond is given as

$$V_t = \mathbb{E}_t\left[\exp\Big(-\int_t^S r_s ds\Big)(P(S,T) - K)^+\right], \tag{2.8}$$

where S is the option maturity, T is the bond maturity, with $T > S$, and K is the strike value.

2.2 Short-rate models with jumps

The Vasicek SDE can be modified to accommodate for the occurrence of jumps as follows:

$$dr_t = \kappa_{\mathbb{Q}}(\theta_{\mathbb{Q}} - r_t)dt + \sigma dW_t^{\mathbb{Q}} + \mathcal{J}_t dN_t, \tag{2.9}$$

where N_t is a counting process for a set of jumps with deterministic timing (i.e., dN_t is an indicator for a jump time, with $dN_t = 1$ if t is a jump time, and $dN_t = 0$ otherwise), and $\mathcal{J}_t$ gives the size of a jump if one occurs at time t. We do not need to specify the jump distribution at this stage; a simple discrete distribution could be specified, while continuous distributions such as the normal or t distribution could also be considered.

In the case of one-factor models, Kim and Wright (2014) follows the dynamics in equation 2.9 where the jumps are made to follow a normal distribution under $\mathbb{Q}$ as: $\mathcal{J}_t \sim \mathcal{N}(\mu(r_{T_{j-}}), s^2)$, where $r_{T_{j-}}$ refers to the short rate value immediately before a jump time T_j.

While the closed-form representation by Kim and Wright (2014) allows the mean of the jump to depend on the pre-jump short-rate, in our case we set the mean to zero. So, under the short-rate dynamics of equation 2.9, the expectation in 2.4 is given as

$$P(t,T) = \exp\Big(- \tilde{A}(t,T) r_t + \tilde{B}(t,T) \Big), \tag{2.10}$$

where

$$\tilde{A}(t,T) = A(t,T), \tag{2.11}$$

$$\tilde{B}(t,T) = B(t,T) + \sum_{t \le T_j \le T} \Big(\frac{s^2}{2\kappa^2} (1 - e^{-\kappa(T-T_j)})^2 \Big), \tag{2.12}$$

where A and B are given in equations 2.6 and 2.7; these are the coefficient functions for the Vasicek model (without jumps). A summation over all jump times, T_j, is made between t and T as shown in equation 2.12.

Chapter 3

Model Implementation

In this chapter, we start by providing a general overview of the finite-difference method. We then show a brief derivation of the Vasicek partial differential equation (PDE) using the Feynman-Kac theorem and assuming the Markov property. After that, we apply the finite-difference scheme on the Vasicek PDE in order to price a simple zero-coupon bond. The finite-difference scheme is then modified to accommodate for jumps in the model as introduced in Chapter 2. Lastly, a brief overview on the Monte Carlo approach is provided, which will be used to compare with the finite-difference model in Chapter 4 to verify the accuracy of the approach used when pricing a bond option.

3.1 An introduction to finite-difference methods

Finite-difference methods are devised to obtain accurate numerical solutions to partial differential equations. We provide a brief qualitative introduction to these methods, before explaining how they are applied in the context of short-rate models. Note that for a more detailed coverage of finite-difference methods in finance, two textbooks namely, Crepey (2013, Ch.8) and Zhu *et al.* (2004) can be consulted.

The principal idea of this method consists of replacing the partial derivatives occurring in the PDE by finite difference approximations, i.e., we need to represent the partial derivatives in terms of solution values at discrete points in the domain. The errors of these approximations can be studied with Taylor-series results. Now, derivatives with respect to time and also the short rate (referred to as the spatial derivative) are used in the PDE. This is done by dividing the axes into equally spaced nodes, a distance δt and δr for the time and spatial dimensions respectively. In addition to discretising the time and spatial dimensions, finite-difference methods require that we *truncate* the spatial dimensions to specify a minimum and maximum value which cover the two extremities in the range. In order to do this, boundary conditions have to be specified to supplement the application of the PDE at the boundaries. A more detailed explanation for choosing these boundary values is provided in section 3.4.

By applying the forward, backward and central difference approximations, we can set up an explicit scheme as well as an implicit scheme. It must imperatively be noted that the model we want to consider in this work must be able to solve for prices starting with a known terminal condition at maturity time, T. So, in the case of an explicit scheme, with u representing the variable being solved, its values at time t_j depends entirely and explicitly on the ones at time step t_{j+1}. In the implicit scheme, we have a system of equations which needs to be solved in an implicit manner to determine the values of u at time t_j. While the explicit scheme is faster and simpler, the implicit scheme addresses potential stability problems associated with the explicit scheme. So for this reason, in this work we make use of the implicit finite-difference scheme to price the different financial instruments.

3.2 Feynman-Kac approach

The Feynman-Kac theorem relates a stochastic differential equation (SDE) to a PDE, stating that the conditional expectation of some stochastic process can be obtained as a solution of its corresponding PDE (Crepey, 2013, Ch.3). Given a SDE as in equation 2.1, by using the Feynman-Kac theorem, the expectation

$$u(t, r_t) = \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\phi(r_T)\Big], \tag{3.1}$$

where $\phi(r_T)$ represents a payoff, is a solution to the PDE as shown:

$$u_t + \mu(t,r)u_r + \frac{1}{2}\sigma^2(t,r)u_{rr} - ru = 0, \tag{3.2}$$

where u_t is a partial derivative with respect to time, and u_r and u_{rr} are first and second-order partial derivatives with respect to the short rate, respectively. A terminal condition for this PDE is given by $u(T,r) = \phi(r)$. It should be noted that the SDE must follow the Markov property to allow the expectation to be expressed as a function of the current short rate, as is done in equation 3.1.

3.3 Implicit scheme applied to Vasicek PDE

Applying the Feynman-Kac formula to the Vasicek SDE as given by equation 2.2, we can express the PDE as

$$u_t + \kappa(\theta - r)u_r + \frac{1}{2}\sigma^2 u_{rr} - ru = 0. \tag{3.3}$$

The following approximations are made for the implicit finite-difference scheme:

$$\frac{\partial u}{\partial t}(i,j) \approx \frac{u_{i,j+1} - u_{i,j}}{\delta t},$$
$$\frac{\partial u}{\partial r}(i,j) \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\delta r},$$
$$\frac{\partial^2 u}{\partial r^2}(i,j) \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta r)^2},$$

where subscript i refers to the rate index ranging from the minimum rate to the maximum rate and subscript j refers to the time index going from time 0 to maturity. This means that $u_{0,j}$ represents the lower boundary of the spatial dimension, and $u_{M,j}$ represents the upper boundary, where we let M denote the number of spatial steps. Also, $u_{i,0}$ presents current prices for the claim being considered, while $u_{i,N}$ corresponds to the terminal time prices, and needs to be given by a suitable terminal payoff condition.

By substituting these approximations into the Vasicek PDE, we obtain

$$\frac{u_{i,j+1} - u_{i,j}}{\delta t} + \kappa(\theta - r)\frac{u_{i+1,j} - u_{i-1,j}}{2\delta r} + \frac{1}{2}\sigma^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta r)^2} - ru_{i,j} = 0. \quad (3.4)$$

Rearranging the terms as

$$u_{i,j+1} = u_{i-1,j}\left(\frac{\kappa(\theta - r)\delta t}{2\delta r} - \frac{\sigma^2 \delta t}{2(\delta r)^2}\right) + u_{i,j}\left(1 + r\delta t + \frac{\sigma^2 \delta t}{(\delta r)^2}\right) - \frac{1}{2}u_{i+1,j}\left(\frac{\kappa(\theta - r)\delta t}{\delta r} + \frac{\sigma^2 \delta t}{(\delta r)^2}\right), \quad (3.5)$$

and regrouping the parameters into

$$a = \frac{1}{2}\left(\frac{\kappa(\theta - r)\delta t}{\delta r} - \frac{\sigma^2 \delta t}{(\delta r)^2}\right),$$
$$b = \left(1 + r\delta t + \frac{\sigma^2 \delta t}{(\delta r)^2}\right),$$
$$c = -\frac{1}{2}\left(\frac{\kappa(\theta - r)\delta t}{\delta r} + \frac{\sigma^2 \delta t}{(\delta r)^2}\right),$$

we get

$$u_{i,j+1} = au_{i-1,j} + bu_{i,j} + cu_{i+1,j}. \quad (3.6)$$

Bonds and bond options can be priced under this approach by correctly choosing the boundary and terminal conditions respectively.

3.4 Implementing the implicit scheme for pricing

We now explain how the above implicit scheme can be implemented. By using the known terminal payoff at maturity time, T, we can perform an iteration in a backward loop to find the value of the claim at all discrete points in time. We start by re-writing equation 3.6, for all values of i (i.e., for all values of the short rate), in matrix form as

$$AU_j = U_{j+1}, \tag{3.7}$$

where

$$A = \begin{bmatrix} -1 & 2 & -1 & 0\dots & 0 \\ a_1 & b_1 & c_1 & 0\dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & \dots & -1 & 2 & -1 \end{bmatrix},$$

$$U_{j+1} = \begin{bmatrix} U_0 & \dots & U_i & \dots & U_M \end{bmatrix}^T.$$

The matrix A contains the boundary conditions in the first and last rows, which we shall now explain, while the parameters a, b and c run diagonally throughout the matrix, ensuring that equation 3.7 captures equation 3.6. The values that we choose at the boundary ensures that the slope which is next to the boundary is continued into the boundary. Given that U_0, U_1 and U_2 are the first three inputs of vector U_j, the first row of A multiplies into U_j to give

$$-U_0 + 2U_1 - U_2 = 0, \tag{3.8}$$

which can be re-expressed as

$$U_0 - U_1 = U_1 - U_2. \tag{3.9}$$

This particular condition guarantees that the finite-difference slope is continued at the boundary, in other words, the slope that is observed near the boundary (measured by $U_1 - U_2$) is specified to apply at the boundary (by setting $U_0 - U_1$ as above). We will see in Chapter 4 that this condition is indeed an effective way to specify the boundaries. Since A is a square matrix, we can find the prices at each point in time simply by taking its inverse as shown:

$$U_j = A^{-1}U_{j+1}. \tag{3.10}$$

Let us take the example of a zero-coupon bond as previously shown with equation 2.4. We know that this particular type of bond pays the nominal amount at terminal time, implying that we should set this as the terminal condition. By choosing an appropriate range for the short rates and setting the maturity time, T, we perform an iteration using equation 3.10. This leads us to obtain the initial price corresponding to the discrete r_0 values. The results can then be compared with the closed-form solution provided by equations 2.5, 2.6 and 2.7 so as to verify for the accuracy of the model.

3.5 Introducing jumps

The implicit finite-difference scheme needs to be modified to accommodate for the jumps. At a given jump time, T_j, a risk-neutral expectation of the price just after the jump is taken to find the price immediately before the jump. A paper written by Backwell and Hayes (2021) explains that since the jumps take place instantaneously in time, the discounted risk-neutral formula must therefore be applied over an instantaneous jump time as shown:

$$U_{T_{j-}} = \mathbb{E}_{T_{j-}}\left[U_{T_{j+}}\right]. \tag{3.11}$$

In order to implement the risk-neutral expectation in the finite-difference scheme, a step needs to be inserted in the backward iteration to implement equation 3.11, whenever a jump time is encountered. A matrix B is created, which contains probabilities for all jump possibilites, discretised as per the finite-difference scheme. Each row contains a discretised distribution for the jump, where the various rows correspond to the various discrete values that the short rate could take on before the jump. Intuitively a vector, U_{postjump}, is defined containing the prices after the jump. A product of the B matrix and the U_{postjump} vector results in prices corresponding to a time step just before the jump's occurrence as follows:

$$U_{\text{prejump}} = BU_{\text{postjump}}. \tag{3.12}$$

3.5.1 A simple discrete example

Consider a model where, at certain times, the short rate can jump either up or down by a given amount x. If the probability of the up-jump is denoted by p, then a typical row of the B matrix will look as follows:

$$\begin{bmatrix} \dots & 0 & p & \dots & 0 & \dots & (1-p) & 0 & \dots \end{bmatrix}.$$

These probabilities can then be multiplied with the values inside the U_{postjump} vector that correspond to the rates $r_i + x$ and $r_i - x$ respectively, which results in the prices instantaneously before the jump. Note that in the absence of jumps, the rows of the B matrix consists purely of $1s$ on its main diagonal.

3.5.2 Normally distributed jumps

As indicated in section 2.2, the jumps can follow a normal distribution. Consider normally distributed jumps with zero mean. The implementation of such jumps in the finite-difference scheme is done by modifying the B matrix so that there is a continuous probability distribution throughout each of its rows. With a mean of zero, the highest probability in each row sits on the diagonal of B. Let us provide an example where we have short rates ranging from 0% to 10%, with a jump distribution given by: $\mathcal{J}_t \sim \mathcal{N}(0, 1\%^2)$. Figure 3.1 plots the probability distribution along a row of the B matrix which corresponds to a rate of 5%, the middle row of B, against the short rates.

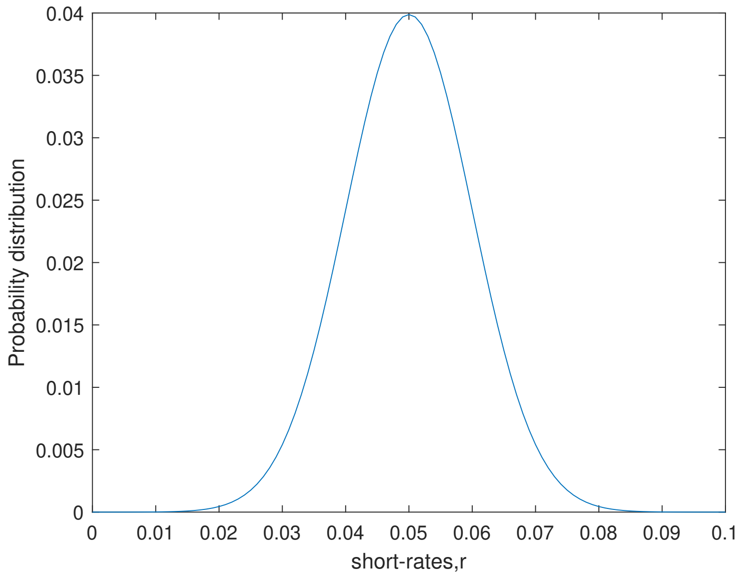


Fig. 3.1: Probability distribution along a row of B matrix corresponding to a short-rate of 5%.

In summary, if the short rate is currently 5%, then a set of probabilities (contained in a row of B as plotted in Figure 3.1) can be multiplied onto the post-jump values to give the pre-jump values that apply at a 5% rate.

3.5.3 Jumps outside of the finite-difference range

There are situations where jumps can go out of the truncated range. In the normal distribution, this is always the case, as the jumps can take on any value on the real line. However, as we move closer to the boundary, the probabilities associated with these jumps become more significant. In order to deal with this, we can simply truncate the jumps by putting the overflowing probability on the boundary node.

This will of course create some error. A more sophisticated way exists in which the post-jump prices can be extended using a linear extrapolation. The size of the B matrix must correspondingly be increased to handle a larger range that can then be multiplied with the extended post-jump prices. In this way, equation 3.12 is modified as follows:

$$U_{\text{prejump}} = B^* U^*_{\text{postjump}}, \tag{3.13}$$

where B^* is now a $(M+1)$ by $(M+2m+1)$ matrix with m additional columns extending the jump range on each side, and where U^*_{postjump} is a linearly extrapolated version of U_{postjump} with m additional values added to each side. We set $m = 40$ for the different simulations in Chapter 4.

3.6 The Monte Carlo method

Monte Carlo simulation is a technique which involves the use of computational algorithms relying on random sampling several times to provide approximate solutions that are very complex to obtain analytically. See Glasserman (2004) for a general treatment of Monte Carlo methods, and see Ayranci and Özgürel (2014) for an example of Monte Carlo simulation in the context of the Vasicek model. In this work, we make use of this method to check for the accuracy of the prices of bond options obtained when using the finite-difference method. It should be noted that the simulation takes place in increasing time steps, working forward from initial values, which is the opposite of the way that the finite-difference scheme was used.

Using the property that the increments of a standard Brownian motion is normally distributed, the short-rate realisations are generated under the Vasicek model as

$$r_j = r_{j-1} + \kappa(\theta - r_{j-1})\Delta t_j + \sigma\sqrt{\Delta t_j} Z_j + \mathcal{J}_j \Delta N_j, \tag{3.14}$$

where $\Delta t_j = t_j - t_{j-1}$ and each Z_j is an independent standard normal random variable. At a jump time, T_j , the counting indicator ΔN_j turns on as previously stated in section 2.2.

Now, in order to price a bond, we also have to find the realisations of the discount factor expressed as shown:

$$\frac{1}{\beta(t)} = \exp\Big(- Y(t) \Big), \tag{3.15}$$

where

$$Y(t) = \int_0^t r(u)du. \tag{3.16}$$

Once the rates up until maturity time, T_m, are found, $Y(t_m)$ can be calculated by making use of a trapezoidal quadrature as:

$$Y(t_m) \approx \sum_{j=1}^{m} \Big(r(t_{j-1}) + r(t_j) \Big) \frac{\Delta t_j}{2}. \tag{3.17}$$

In implementation, a large number of paths, each as per equation 3.14, are simulated. The respective discount factors are then calculated using equation 3.17. An expectation for the bond price is obtained by averaging through all the simulated paths. In case of a bond option, the payoff at the option maturity time has to be taken inside the risk-neutral expectation as shown earlier with the call option in equation 2.8.

Chapter 4

Results Analysis Of Implemented Models

Using the theory established on the addition of jumps in short-rate models in Chapters 2 and 3, we now analyse the effect of these jumps on the prices of bonds and bond options. In section 4.1 we first compare the accuracy of the bond prices obtained using the implicit finite-difference method to the closed-form solution provided by Kim and Wright (2014). A bond option is then priced and checked for consistency with the Monte Carlo method. In the same section, we analyse the pricing profiles of these financial instruments under normally distributed jumps. In section 4.2, a deeper analysis is conducted on options in which we study how the choice of the jump distribution affects the simulated prices. We finally conclude this chapter by investigating the effects of varying different parameters in the model.

4.1 Normal jumps implementation

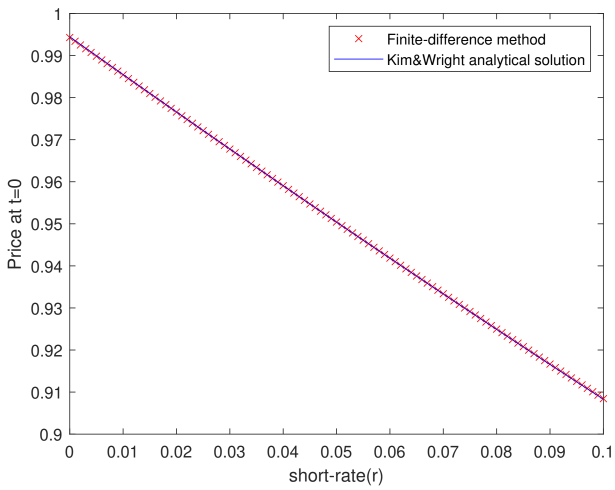
We implemented the normally distributed jumps in the finite-difference model as explained in section 3.5.2, with the jumps following a distribution $\mathcal{J}_t \sim \mathcal{N}(0, 1\%^2)$. Contracts with a maturity of one year were considered, in which four jump times were evenly spaced out. The specific parameters used for the analysis produced in sections 4.1.1 and 4.1.2 are illustrated in Table 4.1.

4.1.1 Zero-coupon bond

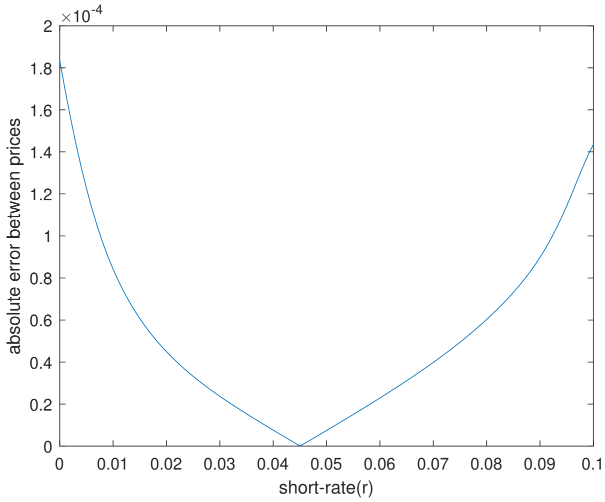
The bond prices under the Vasicek short-rate model with associated normal jumps were generated using the finite-difference method and compared to the closed-form formula from Kim and Wright (2014). The convergence between the prices, as shown in Figure 4.1, confirms the correct implementation of the implicit scheme.

Parameters	
$r_{t,min}$	0%
$r_{t,max}$	10%
δt	0.0125
δr	0.001
σ	1%
κ	20%
θ	0.06
T_j	0.2, 0.4, 0.6, 0.8

Tab. 4.1: Parameters used for normal jumps implementation in the Vasicek short-rate model.



(a) Pricing profiles



(b) Relative error

Fig. 4.1: Comparison between the finite-difference method and the Kim and Wright (2014) closed-form formula.

Starting with a constant terminal condition visible in Figure 4.2, the surface plot shows, as expected, a general decrease in the prices, as we move further from maturity time and as the rate increases. We note that these pricing results are similar to what would be obtained from the standard Vasicek model. This is explained by the fact that zero-coupon bonds have a roughly symmetrical dependence on the short rate, i.e., if the short rate jumps up, there is more discounting driving the bond prices down and on the other hand if the short rate jumps down, there is less discounting driving the bond prices up, and therefore by taking a risk-neutral expectation of these two scenarios, there is more or less a cancellation effect for the jumps' occurrence.

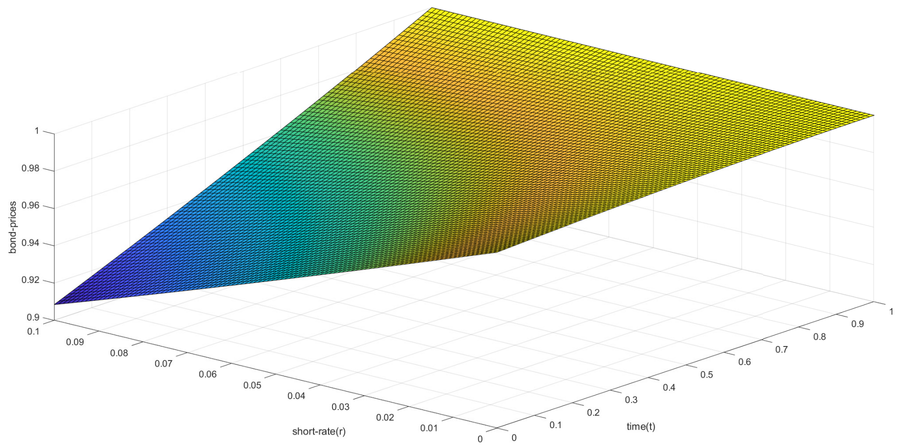
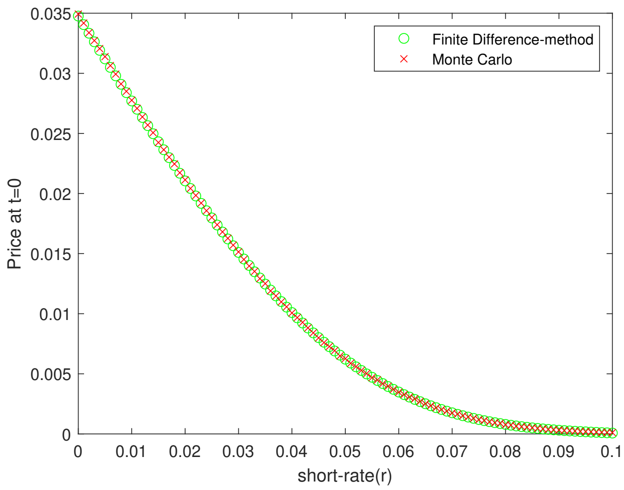


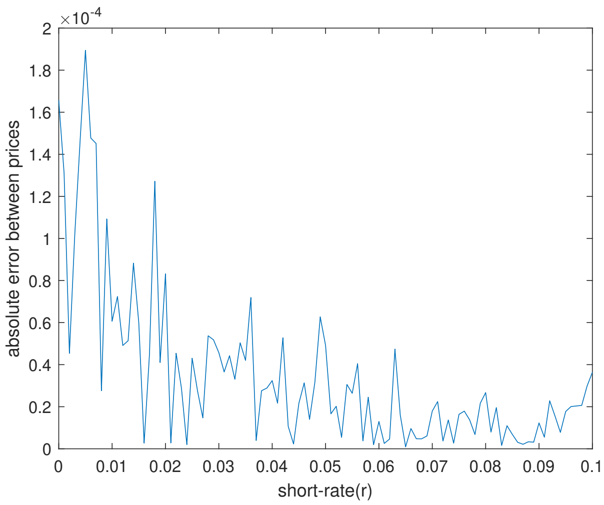
Fig. 4.2: Pricing surface of the zero-coupon bond.

4.1.2 Bond options

A zero-coupon bond option must be written on a bond which reaches maturity after the option expiry. Using the finite-difference scheme, it is only possible to price the option if the payoff at the time of the option expiry is known. An assumption was made whereby no jump times occurred after the option expiry; although one could simply apply the Kim and Wright (2014) closed-form representation to find the terminal conditions if jumps were to be considered. In our case, the closed-form Vasicek formula, as shown earlier in equations 2.5, 2.6 and 2.7, was used to find the payoff at the time of expiry. We priced a call option expiring in 1 year with a strike price of 0.95 on a zero-coupon bond having a time maturity of 2 years. The extension of the B matrix, as explained in section 3.5.3, was necessary in order to capture the cases where the short-rate jumped to values below the lower boundary, where the option was deeply in the money. Since the Kim and Wright (2014) formula cannot be used to price bond options, the Monte Carlo method was applied to provide a good approximate check for the simulated prices. In Figure 4.3, a very close match between the plots obtained using the two techniques can be observed. The results of Figure 4.1 and 4.3 also confirm the effectiveness of the approach, as previously explained in section 3.4, for choosing the boundary conditions.



(a) Pricing profiles



(b) Relative error

Fig. 4.3: Comparison between the finite-difference method and the Monte Carlo method.

In Figure 4.4, we can see that the terminal payoff has a hockey-stick profile indicating that the bond option goes out of the money beyond a certain rate. This hockey-stick profile becomes gradually smoother as we move away from maturity, with the option prices determined using the Vasicek PDE, as well as the adjustments made for jumps (recall equation 3.11) as each jump time is passed. The effect of jumps tends to be more significant for options than for bonds. The addition of jumps in the model can cause the option to jump into the money as well as out of the money, depending on the direction of these jumps. These two effects do not usually cancel when expectations are taken.

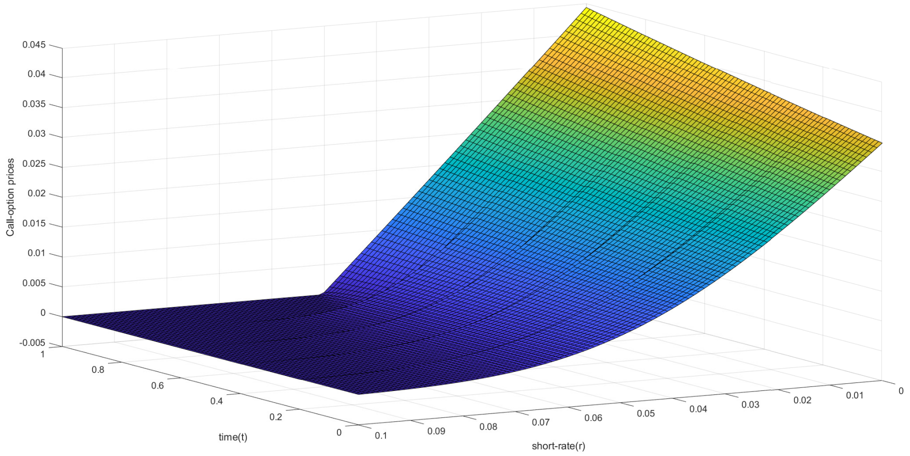


Fig. 4.4: Pricing surface of the callable bond option.

4.2 Option price analysis

It should be noted that finite-difference methods are very flexible in the sense that they can be used to accommodate for different types of jump distributions. Consider the discrete distribution from section 3.5.1. By setting probability parameter p to 0.5, we ensure that there is an equal possibility of coming across an up-jump or a down-jump at any jump time. We want to analyse how the choice of the jump distribution impact on the values of a claim. In Figure 4.5, we simulated the prices using both discrete and normally distributed jumps and also produced a plot for the option in the absence of jumps. Furthermore, we compared these options, all facing different jump distributions, against their intrinsic values, which is defined as the value of exercising the call option now (even though this is a European option, which cannot be exercised until the expiry time). We could, of course, consider other kinds of continuous distribution as well. For example, we could implement t-distributed jumps by changing the cumulative distribution function that is used to populate the entries of B. Such a modification would lead the probability distribution as previously illustrated in Figure 3.1 to have a higher kurtosis.

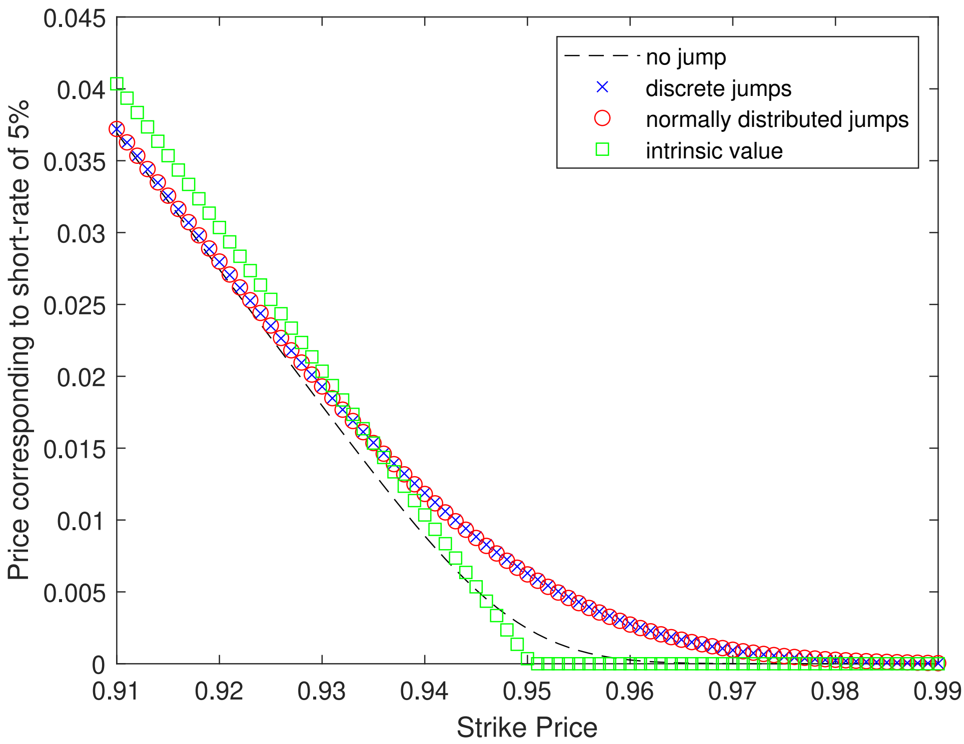


Fig. 4.5: Comparison of call option prices corresponding to an initial short rate of 5% using different jump distributions.

The discrete and normal jump distributions produced a similar profile with very small deviations which are not visible on the plot. Comparing these two distributions to the plot with no jump, we see that there is a small deviation while the option is in the money, and this deviation increases to a maximum when it is around the money. The presence of jumps in the model increases the price of the option for all strike values, with the no-jump profile reducing to negligible values at a considerably lower strike price. It was observed that the option prices lie below the intrinsic values in the region where strikes are low, corresponding to high interest rates; there is then more significant discounting of the payoff to get the current price which is therefore just a result of time value of money. Aside from this effect, the option values are above the intrinsic value, reflecting the additional "time value" that the options get from the remaining variation of the short rate.

While the pricing surface in Figure 4.4 shows how the option prices evolve over time and different short rate values, another kind of analysis can be made by varying different parameters in the model. It is well-known that an option holder benefits from a higher volatility. A high fluctuation in the rates would normally represent more uncertainty but by holding an option, the consequences on one side of the distribution are limited, as the option simply goes out of the money; at the same time, the option holder enjoys the possibility of making increasingly large profits if the short rate jumps further into the money. We quantified this by pricing call options under the dynamics of equation 2.9. We first analysed for different volatilities in Figure 4.6 by varying the σ parameters from 0% to 10%. For each σ value, the finite-difference algorithm was ran for several strike prices, and for each of them the option price corresponding to $r_0 = 5\%$ was taken. The observed

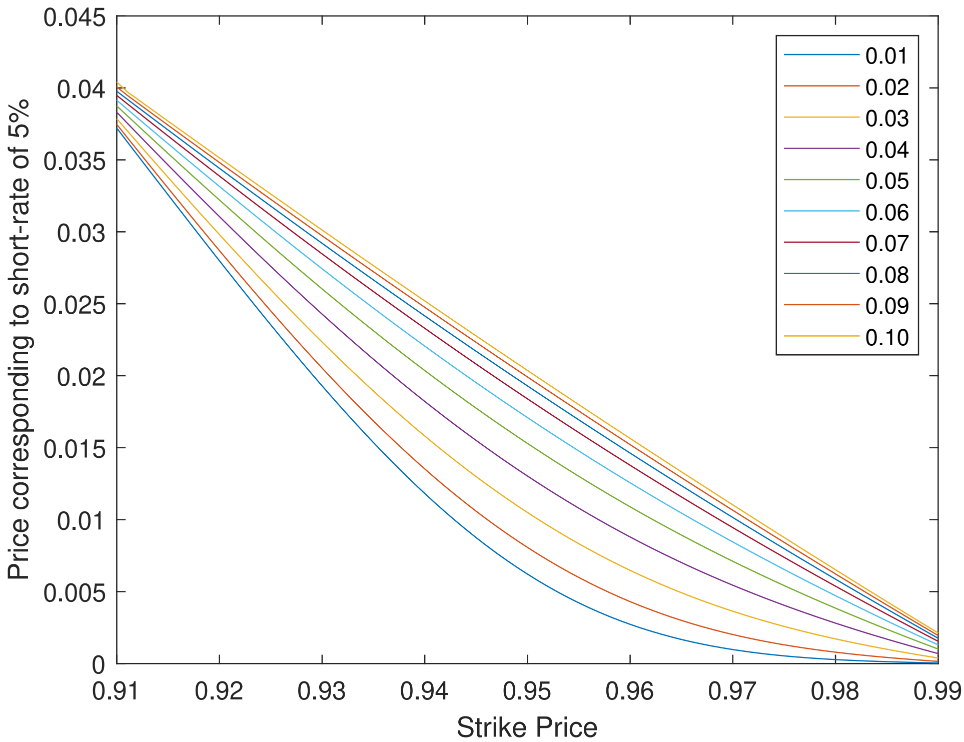


Fig. 4.6: Call option prices corresponding to an initial short rate of 5% for different volatilities against strike price.

behaviour is in line with our expectations, with a higher profile corresponding to an increase in volatility, which confirms the relationship between the option price and the Brownian motion driven variation. We further observed that around the money, the gap between each adjacent pricing profile increases. These gaps then decreases with increasing volatility. Therefore, we can deduce that an additional volatility is most beneficial around the money and its effect is more important for a certain range of volatility rates. In the second case, we analysed for different jump parameters by varying the standard deviations of the jump distribution from 1% to 3%. It was observed that a higher variance in the jump leads to a higher option price, as illustrated in Figure 4.7, with the impact of this variation being mostly observed around the money, in a similar way to the effect of the Vasicek Brownian motion parameters.

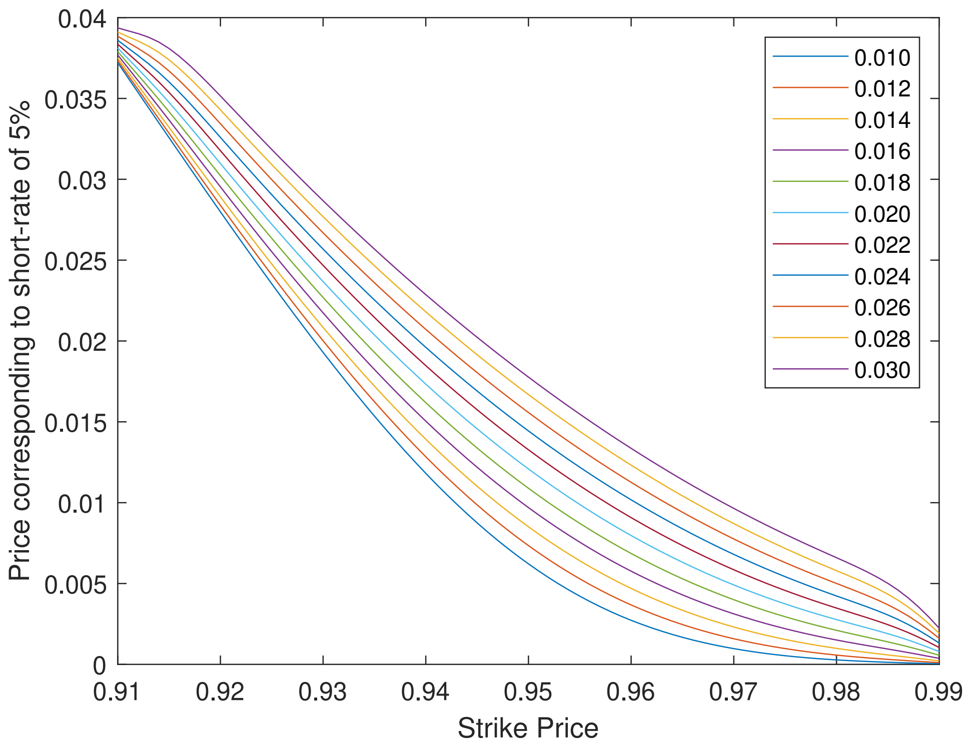


Fig. 4.7: Call option prices corresponding to an initial short rate of 5% for different jump standard deviations against strike price.

We have quantified the effect of short rate volatility from two different sources (the standard Brownian motion driven variation in the standard Vasicek model and also via the addition of jumps to the model). Both lead to increased option prices, especially around the money.

Finally, we return to a point made in Chapter 2, where we mentioned that Kim and Wright (2014) allow for the jump mean to depend on the pre-jump short rate. However, they cannot accommodate a variance that depends on the short rate, which is an aspect that the finite-difference methods allow us to undertake in a simple way. After fixing the Vasicek Brownian motion variation parameter, we let the jump standard deviation vary with the pre-jump short rate in a linear fashion as shown:

$$s = 0.005 + k\, r_{T_{j^-}}, \tag{4.1}$$

where k is a parameter which we varied from 0 to 0.4 as illustrated in Figure 4.8.

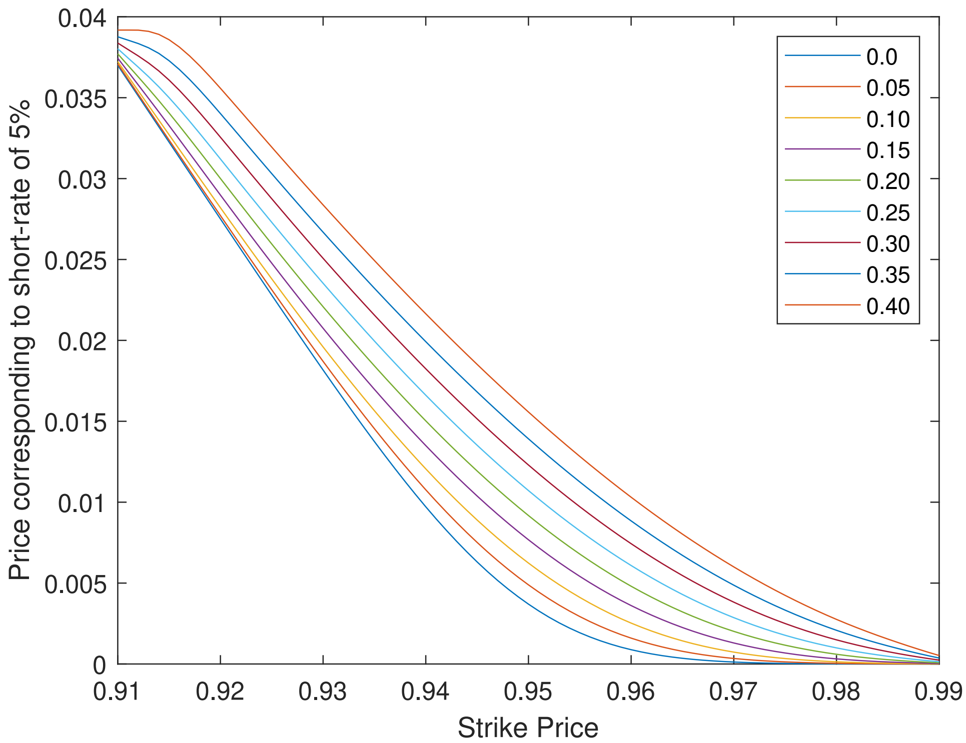


Fig. 4.8: Call option prices corresponding to an initial short rate of 5% with jump standard deviations varying with the pre-jump short rate against strike price.

We observed that an increase in k leads to an increase in the option prices, which is an expected result supported by the previous observations made in Figure 4.7. However, in comparison with the previous two plots, we observed that the effect of increasing volatility is prominent not just around the money, but at lower strike prices as well. These options depend on the larger side of the short rate distribution (unlike high-strike options, which are out-the-money if the short rate is large), and the volatility is especially large for non-zero k values.

Chapter 5

Conclusion

A strong advantage of the finite-difference methods lies in their flexibility. In the context of the short-rate models, they allow for the inclusion of different types of jump distributions in the model, giving the opportunity to price and analyse various instruments by a simple modification of the terminal conditions. The bond prices obtained in the Vasicek short-rate model modified for deterministically-timed jumps, based on an implicit finite-difference scheme, were successfully verified with the Kim and Wright (2014) closed-form solution. In order to compare with option prices, we applied the Monte Carlo method, where a very close match between the plots indicated an accurate implementation of the implicit scheme for this case as well.

We then went along to analyse the effects of including these jumps in the model. It was observed that zero-coupon bonds, having a roughly symmetrical dependence on the short-rate, were not very sensitive to the presence of jumps (for a symmetrical jump distribution). Bond options, on the other hand, clearly showed an increase in the prices as the option moved around the money, which is explained by their asymmetric nature. By varying the volatility parameter, we confirmed the well-known relationship stating that a higher volatility leads to a higher option price. A similar observation was made by increasing the jump variance in the short-rate model. Lastly, after establishing a linear relationship between the jump standard deviation and the pre-jump short rate, we could observe that options with lower strike prices highly depend on the larger side of the short rate distribution.

An extension to this paper could be made by calibrating the short-rate models with jumps to market option prices. However, this might be challenging when it comes to distinguishing the effects of the jump parameters from that of the Brownian motion parameters since, as we observed, higher option prices can result from an increase in the variation of jumps or the Brownian motion aspect of the model itself. Another idea for future research would consist of looking at option pricing on new benchmarks, by adjusting the terminal conditions.

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