Asymptotic Analysis of the Parametrically Driven Damped Nonlinear Evolution Equations.

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Submitted in fulfilment of

the M.Sc. Degree

in the Applied Mathematics Department

at the University of Cape Town.

September 1996

Abstract

Singular perturbation methods are used to obtain amplitude equations for the parametrically driven damped linear and nonlinear oscillator, the linear and nonlinear Klein-Gordon equations in the small-amplitude limit in various frequency regimes.

In the case of the parametrically driven linear oscillator, we apply the Lindstedt-Poincaré method and the multiple-scales technique to obtain the amplitude equation for the driving frequencies $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0$ and $(1/2)\omega_0$. The Lindstedt-Poincaré method is modified to cater for solutions with slowly varying amplitudes; its predictions coincide with those obtained by the multiple-scales technique. The scaling exponent for the damping coefficient and the correct time scale for the parametric resonance are obtained.

We further employ the multiple-scales technique to derive the amplitude equation for the parametrically driven pendulum for the driving frequencies $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0, (1/2)\omega_0$ and $4\omega_0$. We obtain the correct scaling exponent for the amplitude of the solution in each of these frequency regimes.

Proceeding to the damped linear Klein-Gordon equation, we identify the adequate set of "slow variables" induced by the parametric pumping. We obtain the amplitude equations and find the onset of the parametric instability in each of the four cases: $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0$ and $(1/2)\omega_0$. This onset is shown to coincide with the lower bound of the instability window of the linear oscillator.

The discussion of the parametrically driven damped sine-Gordon equation focuses on its breather and radiation wave solutions. We obtain the amplitude equations (which are a family of nonlinear Schrödinger equations) for both of these. The driving frequencies considered are $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0, (1/2)\omega_0$ and $4\omega_0$.

We found that the same driving frequency ω_{dr} can excite radiation waves with different carrier frequencies, $\Omega_N = (N/2)\omega_{dr}$, and wavenumbers: $k_N = \sqrt{\Omega_N^2 - \omega_0^2}$. Analysing existence and stability of solutions to the corresponding amplitude equations, we determine the correct scaling for the amplitude of radiation waves in each of these cases.

Contents

Chapter 1

Introduction

1.1 Nonlinear evolution equations and solitons

For a long time, it has been accepted that a wide variety of physical problems could be modelled by linear equations. Nonlinear effects were neglected with the assumption that only small variations to the solution could arise if nonlinearity is taken into consideration. This assumption holds for small amplitude solutions. When the amplitude is large, however, nonlinear effects cannot be ignored. Nonlinearity is a fascinating aspect of nature the importance of which has been appreciated when considering large- amplitude waves observed in various fields such as hydrodynamics, solid-state physics, nonlinear optics, and so on. This gave rise to the development of what is nowadays known as theory of nonlinear waves.

In the course of the development of nonlinear wave theory several nonlinear equations which model a wide range of physical phenomena were identified. Such equations include the Korteweg-de Vries, Boussinesq, Kadomtsev-Petviashvili, modified Korteweg-de-Vries, nonlinear Schrödinger and derivative nonlinear Schrödinger, sine- and sinh-Gordon equation, φ^4-theory and so on. Amongst these there are equations that are universal in that they may be encountered, just like the d'Alembert linear wave equation, in diverse problems. Examples of such equations are the Korteweg-de-Vries (KdV), the nonlinear Schrödinger (NLS), and the sine-Gordon (SG) equation.

This dissertation is concerned with various questions relating to the *parametrically driven, damped* SG equation. In order to place this work into perspective, we will begin by reviewing some topics pertaining to nonlinear partial differential equations. First, we will illustrate how the KdV, NLS and SG equations arise; next we will discuss features of solutions to these equations, in particular, soliton solutions, kinks and breathers, and the effects of dissipation and driving on these solutions. With this background in place, the aims of the dissertation can then be outlined in some detail. This will be done in Section 1.2.

1.1.1 Some nonlinear evolution equations of physical significance.

KdV Equation. The Korteweg-de-Vries equation,

$$u_t + u_{xxx} + uu_x = 0, \tag{1.1}$$

is typically a useful approximation for systems with weak nonlinearity and weak dispersion. It is encountered in the modelling of electrical transmission lines [25], anharmonic lattices [79], plasma (as a model equation for magnetohydrodynamic waves) [57], longitudinal dispersive waves in elastic rods [69], pressure waves in liquid-gas bubble mixtures [82], and so on.

Let us illustrate how this equation arises in the description of weakly nonlinear weakly dispersive waves. A broad class of wave processes is modelled by the d'Alembert equation,

$$\psi_{tt} - c_0^2\psi_{xx} = 0, \tag{1.2}$$

which describes the propagation of waves travelling with a constant velocity c_0. There are several assumptions that are made in deriving this equation: there is no dissipation, nonlinearity is negligible and there is no dispersion. Let us now introduce nonlinearity and dispersion. The general solution of equation (1.2) has the form of two waves travelling in opposite directions,

$$\psi = \psi_1(x - c_0t) + \psi_2(x + c_0t). \tag{1.3}$$

For very large times, these waves can be treated independently of each other even if small nonlinearity and dispersion are taken into consideration. (We assume that the dispersive broadening of these waves occurs slower than the rate at which the distance between them increases.) Each of these waves satisfies a first order equation. The wave $\psi(x - c_0t)$, in particular, satisfies

$$\psi_t + c_0\psi_x = 0. \tag{1.4}$$

Let us find corrections to this equation due to dispersion and nonlinearity.

We start with the dispersion law for linear waves:

$$\omega = kc(k).$$

As $k \to 0$ the phase velocity $c \to c_0$. In the absence of dissipation, $c(k)$ can be expanded as a power series in k^2. Thus ω can be represented as a function of k with real coefficients. For first corrections,

$$\omega = c_0k - \beta k^3,$$

where β is a constant. This dispersion relation implies that we have to add the third order derivative term to (1.4):

$$\psi_t + c_0\psi_x + \beta\psi_{xxx} = 0. \tag{1.5}$$

The correction due to nonlinearity is accounted for with the use of conservation laws. Let, for instance,

$$\psi_t + \eta_x = 0, \tag{1.6}$$

where ψ is the conserved density and η is the corresponding flux. From the comparison of (1.5) and (1.6), we have that in an approximation linear in ψ

$$\eta = c_0\psi + \beta\frac{\partial^2\psi}{\partial x^2}.$$

In the next approximation, we have to include a term containing the second power of ψ:

$$\eta = c_0\psi + \beta\frac{\partial^2\psi}{\partial x^2} + \frac{\alpha}{2}\psi^2, \tag{1.7}$$

where α =const. Substituting (1.7) in (1.6) we obtain

$$\psi_t + c_0\psi_x + \beta\psi_{xxx} + \alpha\psi\psi_x = 0. \tag{1.8}$$

By change of variables

$$\xi = x - c_0 t, \quad \psi = \frac{\beta}{\alpha}u,$$

the equation (1.8) becomes

$$u_t + u_{\xi\xi\xi} + uu_\xi = 0, \tag{1.9}$$

the KdV equation.

NLS Equation. The nonlinear Schrödinger equation,

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0, \tag{1.10}$$

describes the evolution of small-amplitude pulses in weakly nonlinear strongly dispersive systems. In contrast to the KdV, the NLS equation governs the evolution of an *envelope* of weakly nonlinear waves and admits standing wave solutions. It occurs in a large class of physical contexts including plasma physics [46], nonlinear optics [10], solid-state physics (where it describes the dynamics of quasi-one-dimensional ferromagnets with easy axis anisotropy [33], and that of strong phonon beams [77]), hydrodynamics [16], and in various other fields.

How does this equation arise in the wave theory? Consider a wave which consists of a sinusoidal carrier wave of a constant frequency ω_0 and a wavenumber k_0, modulated by a waveform ψ_0. We assume that this wave envelope varies slowly in time and space as compared to the variations of the carrier wave:

$$\psi = \psi_0(X,T)e^{i(\omega_0 t - k_0 x)}, \tag{1.11}$$

where $X = \varepsilon x$, $T = \varepsilon t$ with $\varepsilon \ll 1$ represent the "slow" space and time variables. As the envelope of the wavepacket is slowly varying, it contains a large number of crests of the carrier wave and the amplitude distribution is concentrated in wave numbers close to the value $k = k_0$. Under these conditions the dispersion law $\omega = \omega(k)$ can be expanded in a Taylor series about the wavenumber k_0 of the sinusoidal carrier wave:

$$\omega = \omega(k_0) + c_0(k - k_0) + \beta(k - k_0)^2.$$

Thus

$$i\psi_t = \omega_0\psi + c_0(\frac{1}{i}\frac{\partial}{\partial x} - k_0)\psi + \beta(\frac{1}{i}\frac{\partial}{\partial x} - k_0)^2\psi. \tag{1.12}$$

Substituting (1.11) in (1.12), then we have

$$i(\frac{\partial\psi_0}{\partial t} + c_0\frac{\partial\psi_0}{\partial x}) = -\beta\frac{\partial^2\psi_0}{\partial x^2}. \tag{1.13}$$

The nonlinearity is exhibited in the form of the dependence of ω on $|\psi_0|$. For first corrections we have

$$\omega = \omega_0 + \alpha|\psi_0|^2. \tag{1.14}$$

Combining (1.13) and (1.14) we obtain

$$i\left(\frac{\partial\psi_0}{\partial t} + c_0\frac{\partial\psi_0}{\partial x}\right) = -\beta\frac{\partial^2\psi_0}{\partial x^2} + \alpha|\psi_0|^2\psi_0. \tag{1.15}$$

This equation describes the propagation of a wave packet with the group velocity c_0, and is known as the nonlinear Schrödinger equation.

SG Equation. The sine-Gordon equation,

$$u_{xx} - u_{tt} = \sin u, \tag{1.16}$$

describes systems with structural periodicities, e.g. when u is an angular variable. It is encountered in the theory of Josephson junctions [60, 72], spin excitations in liquid ^{3}He, resonant optical pulses [31], chains of pendula connected by a spring [72], and so on. This equation also arises in the theory of charge-density-waves, and describes the dynamics of quasi-one-dimensional ferromagnets with easy-plane anisotropy and that of ferroelectric systems [25].

It is worth illustrating how this equation arises in the theory of Josephson junctions [72], for example. The Josephson-junction transmission line consists of two superconductor strips separated by an oxide layer which is thin enough to permit coupling of the superconducting wave functions. The two superconductors can be described by wave functions for the superconducting state of the form

$$\psi_1 = (\rho_1)^{1/2}e^{i\varphi_1} \quad \text{and} \quad \psi_2 = (\rho_2)^{1/2}e^{i\varphi_2},$$

where ρ_1 and ρ_2 are the electronic charge densities. The Josephson current is related to the difference $\varphi = \varphi_1 - \varphi_2$, between the two phases of the wave functions on the two sides of the oxide layer. In particular, the Josephson current per unit area is

$$I = I_0 \sin\varphi \tag{1.17}$$

where φ is associated with the applied voltage by

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar}v. \tag{1.18}$$

Note that equation (1.17) describes the internal periodic structure of the Josephson junction. If we take the magnetic flux to be

$$\Phi = \int v dt,$$

equation (1.17) can be rewritten as

$$I = I_0 \sin(2\pi\Phi/\Phi_0) \tag{1.19}$$

where $\Phi_0 = \hbar/2e$ is the flux quantum. From equations (1.17), (1.18) and (1.19) the partial differential equations for the shunt voltage, v, and the surface current, i, on the transmission line can be written, by applying Kirchhoff's laws, as:

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}, \tag{1.20}$$

$$\frac{\partial i}{\partial x} = -C\left(\frac{\partial v}{\partial t} - J_0 \sin\varphi\right), \tag{1.21}$$

and

$$\frac{\partial \varphi}{\partial t} = \frac{2e}{\hbar}v, \tag{1.22}$$

where L is the inductance, C the shunt capacitance, and J_0 the maximum Josephson current density per unit length. Combining equations (1.20), (1.21), and (1.22) we obtain a single partial differential equation for the phase angle difference φ:

$$\frac{\partial^2 \varphi}{\partial x^2} - LC\frac{\partial^2 \varphi}{\partial t^2} = \frac{2eJ_0L}{\hbar}\sin\varphi. \tag{1.23}$$

If we measure distance in units of the Josephson length

$$l = (\hbar/2eJ_0L)^{1/2},$$

and time in units of

$$\tau = (\hbar C/2eJ_0),$$

the sine-Gordon equation (1.23) reduces to the normalised form,

$$\varphi_{xx} - \varphi_{tt} = \sin\varphi.$$

The equations discussed above are of importance to mathematicians since their solutions can be found exactly by the inverse scattering method. Such equations are *completely integrable.* Through the inverse scattering method, it is possible to find all the remarkable properties possessed by these equations such as the existence of stable N-soliton solutions, infinite number of conservation laws, large time asymptotics, Lie-Bäcklund symmetries, and so on. If one wishes to find only some of these properties, this method can be laborious. There are, however, other methods that can be used in that regard such as the Bäcklund transformations, Hirota bilinear method and so on. There is also a wide variety of *non-integrable* equations that are of no lesser importance. They appear in many fields such as superfluids, solid state, high-energy and nuclear physics, nonlinear optics, and so on.

These equations include the so-called ϕ^4 theory; the double sine-Gordon and the driven sine-Gordon; Maxwell-Bloch equations, the higher order NLS equations,

$$i\psi_t + \psi_{xx} + |\psi|^{2n}\psi = 0, \quad n \geq 2,$$

to name but a few. Although they cannot be solved by the inverse scattering transform, large classes of solutions of these equations may be found through other methods, for instance perturbatively [30, 56] and numerically.

1.1.2 Behaviour of solutions.

In both integrable and nonintegrable systems, there may occur waves that are characterised by a balance between nonlinearity and dispersion. In general, as a wave propagates, nonlinearity tends to steepen and break it which often leads to formation of discontinuities and shock waves. On the other hand, dispersion spreads out the wave because different Fourier components of any initial condition will propagate at different velocities and hence any profile will spread. Zabusky and Kruskal [87] studied numerically the KdV equation and noted that initially the nonlinearity uu_x dominates over dispersion u_{xxx} thereby resulting in the steepening of the wave in the region where $u(x,t)$ has a negative slope. After $u(x,t)$ has steepened sufficiently, the dispersion u_{xxx} comes into play and prevents the formation of a discontinuity. Instead, short wavelength ripples develop on the front. The amplitudes of the ripple oscillations grow and finally each oscillation achieves an almost steady amplitude. In systems which are slightly nonlinear, the competition between these two effects of steepening and spreading may lead to a balance that gives rise to a steady-state pulse. This pulse is called a solitary wave.

Solitons. In completely integrable systems these solitary waves emerge from a collision without change of form or speed. (There is only a phase shift, or time delay). This particle-like behaviour prompted Zabusky and Kruskal [87] to coin the term *soliton* for a solitary wave which preserves its shape and speed in a collision with another solitary wave. Like the solitary wave of the KdV equation, the solitary waves of the SG and NLS equations are also solitons. On the contrary, in nonintegrable systems collisions are not elastic. Solitary waves may destroy each other, form bound states, or simply emit radiation in the form of small-amplitude dispersive linear waves.

The soliton solution of the KdV equation (1.1) has the form

$$u = 3v\text{sech}^2\left[\frac{\sqrt{v}}{2}(x - vt)\right].$$

It represents a disturbance that moves in the positive x direction at a constant velocity v. The velocity of this solitary wave is proportional to its amplitude and so larger-amplitude pulses travel faster than smaller ones.

The soliton solution of the NLS equation reads

$$u = A\text{sech}[A(x - vt)]\exp\left[i\frac{v}{2}(x - \frac{v}{2}t) + iA^2t\right].$$

In contrast to the KdV soliton, the velocity v of the NLS soliton is independent of the amplitude A. Any initial condition of the KdV and NLS equations evolves into a sequence of solitons and an oscillatory tail decaying as $1/\sqrt{t}$.

Kinks and Breathers. The sine-Gordon equation exhibits two fundamental types of soliton solutions, the kink and breather. The kink solution has the form

$$u_{\pm} = 4\tan^{-1}\left[\exp\left(\pm\frac{x - vt}{\sqrt{1 - v^2}}\right)\right],$$

where v is its velocity. If we choose the upper sign, u increases monotonically from 0 to 2π as x changes from $-\infty$ to $+\infty$. The negative sign corresponds to a negative sense of rotation and the pulse is referred to as an antikink. An important property of the kink is that it has nonzero topological charge. The topological charge is defined by

$$n = \frac{1}{2\pi}\int u_x dx.$$

The topological charge of an arbitrary configuration is therefore

$$n = \frac{1}{2\pi}[u(\infty) - u(-\infty)]$$

which is the difference between the number of kinks and the number of antikinks in this configuration. It is clearly integer-valued.

The energy of the kink is

$$E_K = 8/\sqrt{1 - v^2}.$$

Notice that E_K does not vanish when $v \to 0$. Consequently, a finite amount of energy is necessary to excite even a stationary kink.

If a kink and an antikink travelling with the same velocities undergo a collision, they may pass through each other as described by the doublet solution

$$u_D = 4\tan^{-1}\left[\frac{\sinh(vt/\sqrt{1 - v^2})}{v\cosh(x/\sqrt{1 - v^2})}\right], \tag{1.24}$$

where v is an arbitrary parameter. As time increases, the doublet solution separates into the antikink travelling to the left with velocity $(-v)$ and the kink travelling to the right with velocity $(+v)$. The doublet wave form resulting from the collision between two kinks (or two antikinks) travelling with equal velocities has the form

$$u_D = 4\tan^{-1}\left[\frac{v\sinh(x/\sqrt{1 - v^2})}{\cosh(vt/\sqrt{1 - v^2})}\right]. \tag{1.25}$$

The energy of each of the solutions (1.24) and (1.25) is the sum of energies of the two individual kinks comprising the doublets. In general, there exist N-kink solutions of the SG equation [1].

The other fundamental type of a soliton solution is the breather. It is localised in space and oscillates in time:

$$u_B = 4\tan^{-1}\left[\frac{\tan\nu\sin[(\cos\nu)t]}{\cosh[(\sin\nu)x]}\right].$$

Unlike the kink, it is a nontopological solution ($n = 0$) and its amplitude oscillates between 4ν and (-4ν). This solution may be considered as a bound state of a kink and antikink. However, the breather cannot dissociate into a kink and antikink as $t \longrightarrow \pm\infty$. Its rest energy is given by

$$E_B = 16\sin\nu$$

and varies from 16, the rest energy of two kinks, down to zero as ν tends to zero. Consequently, even small amounts of energy are sufficient to excite the breather. The breather frequency equals $\cos\nu$ and is always less than unity. When the frequency of the internal oscillations tends to 1 (the value of the natural frequency of the system), the breather's amplitude tends to zero and the breather approaches a small-amplitude linear wave.

Let us show that slowly varying amplitude of the small-amplitude breather of the sine-Gordon equation satisfies the NLS equation. In the small-amplitude limit the SG equation (1.16) can be approximated by

$$u_{xx} - u_{tt} + u - \frac{1}{6}u^3 = 0. \tag{1.26}$$

We then assume that

$$u = \varepsilon\sum_{i=0}^{\infty}\varepsilon^i u_i, \quad \varepsilon \ll 1$$

and that

$$u = \psi(X,T)e^{i\Omega t} + c.c. \tag{1.27}$$

where $X = \varepsilon x$, $T = \varepsilon t$ and c.c. denotes complex conjugate. The amplitude function $\psi(\varepsilon x, \varepsilon t)$ is constant with respect to rapid variations in x and t of the carrier wave. Inserting eq. (1.27) into (1.26) and neglecting the third harmonic, we obtain at a first approximation a nonlinear dispersion relation

$$\Omega^2 = 1 - \frac{1}{2}\varepsilon^2|\psi|^2,$$

and the NLS equation for the envelope ψ:

$$2i\psi_T - \psi_{XX} - \frac{1}{2}|\psi|^2\psi = 0.$$

The concept of solitary wave retains its relevance in nonintegrable systems. In contrast to completely integrable systems where solitons are always stable, nonintegrable systems admit solitary waves which can be both stable and unstable with respect to small perturbations. If the solitary wave is stable, small perturbations die off as $t \longrightarrow \infty$. If solitary waves are unstable their amplitudes may grow infinitely, the phenomenon called collapse. In other instances, unstable solitary waves may disperse into linear waves. As far as stable solitary waves are concerned, during collisions they may pass through or bounce off each other (with the emission of radiation), destroy each other [44], or form breather-like bound states [38, 39, 40].

The result of a collision may depend on the initial velocities of the stable solitary waves. If velocities of colliding waves are high, the waves typically pass through each other, or get reflected, without being destroyed. Their energies, amplitudes and velocities decrease after collision due to the emission radiation. When the velocities are small, the two solitary waves may be bound together in a breather-like state. Campbell *et al* [38] have shown that for particular nonintegrable systems such as the φ^4-theory, there can be however more than one range of initial velocities for which solitary waves form bound states. They have also observed that there is a sequence of regions of intermediate initial velocities in which a creation of bound-states and repulsion alternate – they referred to these regions as *reflection windows.* Due to inelastic collisions in nonintegrable systems solitary waves are not genuine solitons [85]. However, in recent literature, especially physics literature, no distinction is made between a solitary wave and a soliton. It has become common to use the term soliton for any solitary wave solution. Accordingly, we shall not make such a distinction in what follows.

Solitons are important structures in nonlinear evolution equations. In completely integrable systems, every localised initial condition breaks up asymptotically into a set of solitons, propagating with constant velocities, accompanied by the background radiation that typically dies off as $1/\sqrt{t}$. Since collisions of solitons of completely integrable systems are elastic, the number of solitons remains constant. Hence the evolution of any initial data in these systems reduces asymptotically to soliton dynamics. The existence of stable solitons in nonintegrable systems is even more significant. In this case collisions are no longer elastic but produce the background radiation. This radiation can, however, be absorbed by other solitons. This leads to a loss of "mass" by some solitons in the system. The implication of this behaviour for systems in a bounded region, where solitons are confined to keep on colliding, is that after some time only one stable soliton survives [44]. In this case one may say that stable solitons behave as statistical attractors. In general, solitons represent structures that serve as footholds for the nonlinear analysis even in the presence of chaotic phenomena that may be caused by some perturbations.

Effects of dissipation. Let us first consider a situation when dissipation appears as a perturbation to completely integrable system. If one wishes that nonlinear structures such as solitons exist over long periods of time (for example, if one wants to use solitons in optical transmission lines) it is necessary that dissipative factors be small. When the energy of the soliton is dissipated, the soliton's amplitude decreases and its width increases. Damping may appear in different forms depending on the underlying physics.

In the case of the sine-Gordon equation, the damping may occur as the first time derivative:

$$\varphi_{tt} - \varphi_{xx} + \omega_0^2 \sin\varphi = -R(\varphi)$$

where $R(\varphi) = \lambda\varphi_t$, or $\lambda\varphi_{xxt}$ $(\lambda > 0)$. In some instances, a combination of these may be encountered. For example, in the theory of Josephson junctions, the damping $\lambda\varphi_t$ accounts for the dissipation due to tunnelling of normal electrons *across* the dielectric barrier, while the damping $\lambda\varphi_{xxt}$ accounts for the losses due to the current *along* the barrier. Damping, in this case, slows down kinks and breathers. In the case of the breather, the internal breathing will tend to slow down and energy will be dissipated until the breather decays into linear waves. As for the kinks and antikinks, as the energy decreases the velocities also decrease. In this case the kink and antikink may be bound together into a breather.

Damping in the NLS equation may appear in different forms as well. Consider

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = -R(\psi). \tag{1.28}$$

Dissipation may arise, for example, in one of the following forms or a combination of these: $R(\psi) = i\gamma\psi$, $R = i\gamma\psi_{xxt}$, $R = i\gamma\psi|\psi|^2$, and so on. (Here γ is a damping coefficient.) Let us now consider the damped NLS equation with the dissipation term $R(\psi) = i\gamma\psi$. This type of damping is encountered in optics where the fiber loss acts as a dissipating factor. If γ is small, the one-soliton solution can be obtained through perturbation methods [30, 56]:

$$\psi(x,t) = A(t)\text{sech}[A(t)x]\exp\left[i\omega(t)\right] + O(\gamma), \tag{1.29}$$

where

$$A(t) = A_0 \exp\left(-2\gamma t\right) \tag{1.30}$$

and

$$\omega(t) = \omega_0[1 - \exp\left(-4\gamma t\right)]. \tag{1.31}$$

According to (1.30), the soliton's amplitude decreases as $\exp\left(-2\gamma t\right)$, while the width increases as $\exp\left(2\gamma t\right)$. (Consequently, the soliton propagates by retaining the property that the amplitude times width remains constant.) The energy of the soliton also decreases as $\exp\left(-2\gamma t\right)$. On the other hand, if γ is large, no soliton can be formed at all. The initial condition will rapidly decay to zero.

External and Parametric Pumping. In order to compensate for energy losses the energy should be pumped into the system. This can be done in different ways. The simplest example of a damped driven system is a playground swing. In this case there are two ways of pumping-in the energy. First, somebody from the outside may push the swing periodically. This situation is described by the equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 \sin x = f\cos\Omega t,$$

where γ is the damping coefficient and f the amplitude of the external force. The driving term appears as an inhomogeneity in the equation. This type of a driver is called *external* driver.

Alternatively, a person on the swing may amplify his swaying motion by lowering the centre of gravity of his body as the swing descends and raising it as it ascends. In this case the pumping appears as a time-dependent coefficient:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2(1 - \varepsilon\cos\Omega t)\sin x = 0.$$

This type of pumping is called *parametric* driving. When both dissipative and driving terms appear in a system, there is a competition between the pumping and dissipation. A steady-state or periodic solution may result if these two perturbing factors balance each other.

In both the external and parametric driving, resonance may occur between the natural frequency ω_0 of the system and the frequency Ω of the driving term. In the case of the *external* driving, the resonance occuring for $\Omega \sim \omega_0$ is called the principal resonance. When the system is nonlinear, a resonance may also occur between the main harmonic and higher harmonics of the nonlinear term; this is referred to as subharmonic resonance. *Parametric* resonance occurs when the frequency ω_0 is a rational multiple of Ω and is strongest when the frequency of the driver is close to twice the natural frequency of the system (principal resonance). In the absence of dissipative factors, the amplitude of oscillation will grow indefinitely as $t \to \infty$.

Damped Driven PDEs. In partial differential equations the situation is even more interesting. In this case there are four factors that interplay: dispersion, nonlinearity, damping and driving.

External Driving. Let us first consider the case of external excitation in soliton theory, and describe what effect does the resonance between the natural frequency of the system and the driving frequency have on the soliton solution. Consider for example, the externally driven, damped sine-Gordon equation:

$$u_{tt} - u_{xx} + \omega_0^2 \sin u = -\lambda u_t + F \cos \Omega t \tag{1.32}$$

where ω_0 is the natural frequency of the system, λ the damping coefficient, F the amplitude and Ω the frequency of the driver. The sine-Gordon equation exhibits two fundamental soliton solutions but it is relatively easier to excite the breather compared to the kink since the former does not have the excitation threshold. The energy of the quiescent breather is zero whereas the energy of the quiescent kink equals 8. For this reason, we shall only consider the excitation of the breathers here. In small-amplitude limit the breather of the externally driven SG equation has been shown to satisfy the NLS equation [74, 76] of the form

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = -ihe^{i\omega t} - i\gamma\psi. \tag{1.33}$$

This correspondence takes place for small λ and F; one also assumes that $\Omega \approx \omega_0$.

Kaup and Newell [56] have discovered an important property of breathers in perturbed systems. They have shown that the driving frequency can resonate with the frequency of the breather's oscillations and as a result of this the breather can phase-lock to the frequency of the driver. This phenomenon of phase locking may be observed provided the difference between the forcing frequency and the natural frequency of the breather is small [56]. The phase-locked breathers can occur in a broad variety of damped driven near-integrable systems [56].

The spatial and temporal structure of the phase-locked breathers and other solutions arising in damped driven partial differential equations strongly depend on the interplay between the driver's amplitude F, frequency Ω, and the damping coefficient λ. The sine-Gordon equation (1.32) has been discussed by Taki *et al* [76], Spatschek *et al* [74], Bishop *et al* [35], McLaughlin *et al* [78] and Abdullaev [28]. Taki *et al* [76] considered eq. (1.32) on a relatively large spatial interval, $L \equiv 80$, and for Ω very close to ω_0: $\Omega = 0.98$ (the nonlinear Schrödinger regime.) Keeping λ fixed ($\lambda = 0.004$), they varied F. For $F < 0.0032$, only the flat solution ($u_{xx} = 0$) exists. When $F = 0.0032$, the phase-locked breather appears. In the

Poincaré map defined by the period of the driver, this solution corresponds to a fixed point. When F is further increased, the fixed point becomes unstable and a limit cycle appears instead. On further increasing F, a sequence of period-doubling bifurcations occurs: the limit cycle bifurcates into a period-2 solution, then the period-2 solution becomes unstable and is replaced by a period-4 solution, and so on. This sequence terminates when a chaotic attractor appears. For larger values of F it was found that a chaotic attractor disappears (the so-called crisis) and the limit cycle reappears.

For a different range of control parameters in eq. (1.32) Bishop *et al* [35] reported the quasi-periodic transition to chaos. The difference has been attributed to the fact that the values of the driver's strength and dissipation coefficient examined in [76] and [35], differ by order of magnitude.

The period-doubling scenario occurs also in the case of externally driven NLS equation eq.(1.33). This has been discussed by Nozaki and Bekki [66, 67, 68] who used (1.33) as a model for a uniform plasma driven by an rf-field. They have demonstrated that for small damping and small driver's amplitude, only a flat solution exists. As the strength of the driver increases, the phase-locked soliton appears as a fixed point in the corresponding Poincaré map. When the strength increases further, a sequence of bifurcations is observed due to couplings of the soliton and the long-wavelength radiation and eventually the soliton becomes temporally chaotic [74].

Parametric Driving. Numerous experiments were conducted in systems which are parametrically driven (e.g. Ciliberto & Gollub [42], Ezerskiĭ *et al* [45], Funakoshi & Inoue [48], Guthart & Wu [49], Wang & Wei [80], Wu *et al* [83]). The basic theory of parametrically driven systems was developed by Miles [64, 65] and Larraza & Putterman [61]. Miles [65] investigated the resonance that may occur in the nonlinear Faraday system and Ciliberto and Gollub studied mode coupling in Faraday resonance. Wu *et al* [83] and Wang and Wei [80] were motivated by Miles' work and conducted experiments in a water tank. They observed a hydrodynamic soliton in a system driven at a frequency close to twice the natural frequency of the system. Miles [64] in turn, established that the soliton observed by Wu *et al* satisfies the parametrically driven nonlinear Schrödinger equation with weak damping

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = he^{2it}\bar{\psi} - i\gamma\psi. \tag{1.34}$$

The same equation was derived by Barashenkov *et al* [33] for the easy plane ferromagnet with an rf magnetic field in the easy plane. In the context of magnetic systems we should also mention the work of Bishop and Wysin [84] and Yamazaki and Mino [86] who performed numerical simulations of parametrically driven ferromagnets.

Barashenkov *et al* [33] considered the parametrically driven, damped SG equation

$$u_{tt} - u_{xx} + \tilde{\omega}_0^2(t)\sin u = -\lambda u_t, \tag{1.35}$$

where

$$\tilde{\omega}_0^2(t) = \omega_0^2(1 - F\cos\Omega t).$$

This equation describes, for example, a chain of pendula connected by springs and driven by vertical oscillations of the bar. Also, it arises in the theory of magnetism [63] and describes amplitude modulated Josephson junctions [73]. A number of parametrically driven systems that are encountered in other fields, are discussed by Kivshar and Malomed [29] and Abdullaev [28].

The authors of [33] assumed that (a) F and λ are small; (b) the driving frequency Ω is close to twice the natural frequency ω_0 of the system; and finally, (c) solution is small in amplitude. They demonstrated that under these conditions, eq. (1.35) can be reduced to the parametrically driven damped NLS equation (1.34) where h is the amplitude of the driver, and γ is the damping coefficient. In other words, similarly to to the externally driven case, the small amplitude breather of the parametrically driven SG equation corresponds to the NLS soliton.

In equation (1.34) the parameters h and γ are not necessarily small. Barashenkov *et al* [33] have found exact soliton solutions for arbitrary values of h and γ:

$$\psi_\pm(x,t) = A_\pm \exp\left[i(t-\theta_\pm)\right]\mathrm{sech}(A_\pm x), \tag{1.36}$$

where $A_\pm > 0$ and

$$\begin{aligned} A_+^2 &= 1 + (h^2-\gamma^2)^{1/2}, \quad 2\theta_+ = \arcsin(\gamma/h), \\ A_-^2 &= 1 - (h^2-\gamma^2)^{1/2}, \quad 2\theta_- = \pi - \arcsin(\gamma/h). \end{aligned} \tag{1.37}$$

As follows from (1.37) both ψ_+ and ψ_- exist only above the threshold $h=\gamma$. The soliton ψ_- was found to be unstable for any h and γ [33]. Consequently, the authors of [33] have focused their attention on the stable soliton, ψ_+.

Barashenkov *et al* have described the stability diagram of this stable soliton in the (h,γ)-plane (fig.1). As we already mentioned, the soliton ψ_+ exists above the line $h=\gamma$. Below this line no localised solutions exist and the only attractor is the trivial solution $\psi \equiv 0$ (*zero attractor*). Above the line $h > (1+\gamma^2)^{1/2}$, the zero solution and, therefore, soliton is unstable with respect to the excitations of continuous spectrum waves. Above the curve 1 the soliton ψ_+ is unstable with respect to a local mode.

Bondila *et al* [37] studied the parametrically driven NLS equation (1.34) using numerical methods. These authors analysed attractors in the (h,γ)-plane above the Hopf bifurcation curve (i.e. in the region where the soliton ψ_+ is unstable). They have uncovered the following attractors in that region (see the figure below). Above the line 1, in the domain marked by open circles, the nontrivial attractor is temporally periodic. Attractors resulting from the subsequent period-doubling (2-periodic, 4-periodic, 8- and higher periodic, and, finally, complex cycles and strange attractors) occupy a narrow band of boxes, diamonds and white blobs. Next, above the curve 2 only the zero attractor exists. The latter region is marked by empty triangles. Thus the final stage of the soliton's instability following period-doubling sequence is its decay to zero.

Crossing the bifurcation line 3, the limit cycle is replaced by the spatio-temporal chaos. As opposed to the period-doubling scenario, there are no intermediate attractors here. This transition to chaos is called *quasi-periodic* route The line 4 is the interface between the spatio-temporal chaos and the region of the zero attractor. The lines 2, 3 and 4 meet at a critical point $\gamma_c = 0.25$, $h_c = 0.81$ which separates the period-doubling and quasiperiodic routes. (see the figure 1 below[1]).

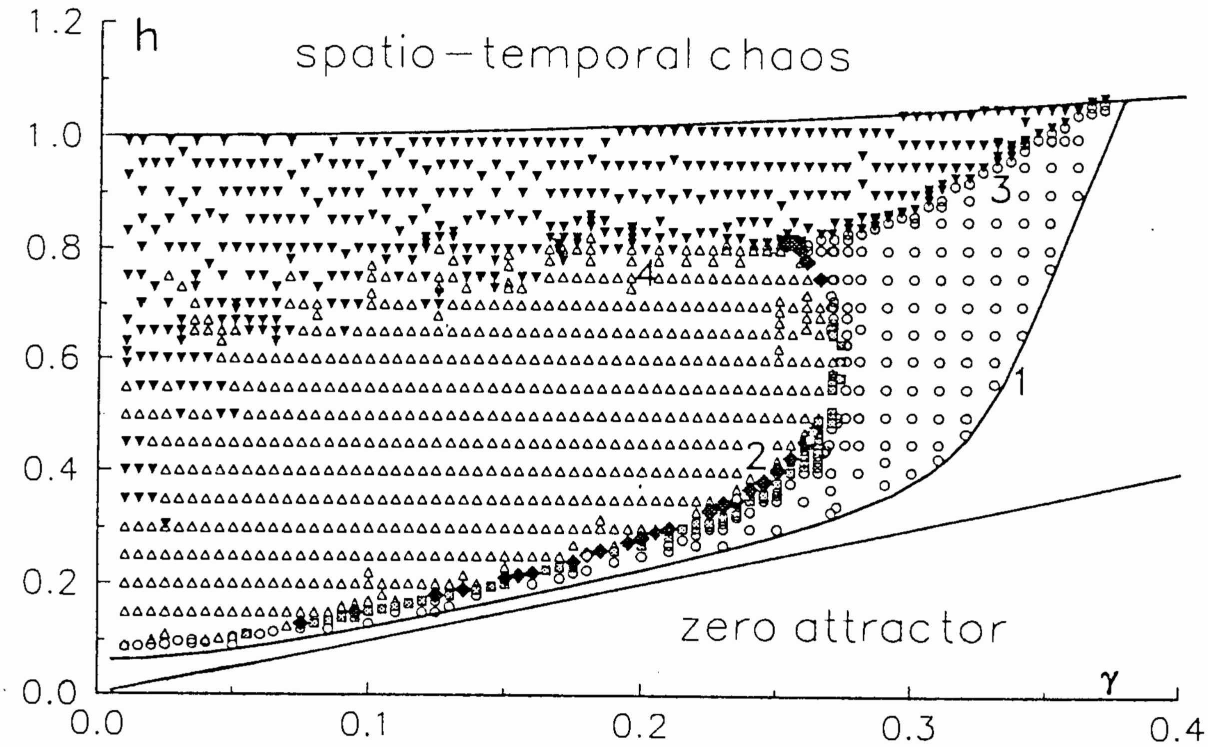

Fig. 1

We have already mentioned that there is a correspondence between the sine-Gordon and the NLS equations. In view of this it can be inferred that all the features observed for the parametrically driven damped NLS equation may be encountered in the analysis of the parametrically driven sine-Gordon equation.

1.2 Aims and scope of this work

Suppose we are considering one of the physical systems modelled by the damped sine-Gordon equation, for instance, a ferromagnet, or the long Josephson junction, or simply a chain of pendula connected by springs. Should we want to observe nondecaying spatio-temporal structures in this system, it would be necessary to compensate dissipative losses by pumping energy into the system by, for example, periodically changing the parameters of the system.

[1]This diagram is reproduced with permission from the authors of [37]

In the case of an easy-plane ferromagnet, this is accomplished by exposing the pattern to the microwave radiation at a properly chosen angle with respect to the crystal axes [33]. In the case of the chain of pendula one moves the bar periodically in the vertical direction. There are natural realisations of the parametric driving in the Josephson junction setting as well [73]. In all these cases the arising sine-Gordon equation looks the same:

$$\varphi_{tt} - \varphi_{xx} + \tilde{\omega_0}^2(t)\sin\varphi = -\lambda\varphi_t, \tag{1.38}$$

where

$$\tilde{\omega_0}^2(t) = \omega_0^2(1 - 2F\cos\Omega t),$$

ω_0 is a natural frequency of the system which is entirely determined by its physical characteristics, Ω is the frequency of the driver, $2F$ is the driver's strength (assumed small), and λ is the dissipation coefficient (also assumed small).

In this dissertation we address the following questions:

1. What kind of nonlinear localised structures should we expect to occur in this system?
2. What will be their characteristic amplitudes and, therefore, how difficult will be their detection?
3. What other kinds of structures will be generated in the system, and how will their magnitudes compare to those of localised objects?
4. How can all these structures be described quantitatively?

As we will see, the answers heavily depend on the interplay between ω_0, Ω and other parameters in equation (1.38). In the case $\Omega \approx 2\omega_0$ (principal resonance), Barashenkov *et al* [33] showed that the parametrically drive sine-Gordon equation admits a stable localised solution in the form of the breather with slowly varying amplitude, whose evolution is described by the NLS equation. The above authors also studied localised solutions to NLS and constructed the stability chart of the soliton. Subsequently, Bondila *et al* [37] have observed a variety of localised attractors in the instability region and described transitions to spatio-temporal chaos. However, what happens in other frequency regimes of the parametrically driven sine-Gordon system (for example, when $\Omega \approx (p/q)\omega_0$) remained an open question. In this thesis we attempt to answer this question.

The structure of the dissertation is as follows:

In Chapter 2 we review singular perturbation methods, for which some will be subsequently used to reduce the equations of the linear and nonlinear oscillator, the linear and nonlinear Klein-Gordon equations to the corresponding amplitude equations. The methods discussed include the Lindstedt method [11, 12, 21], the Krylov-Bogoliubov-Mitropolsky (KBM) approach [5, 11, 20], and multiple-scale formalism [11, 12, 21, 23].

In Chapter 3, we modify the Lindstedt-Poincaré method and apply it to the parametrically driven *linear* oscillator equation in the small amplitude limit for frequencies $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0$ and $(2/4)\omega_0$. We obtain the amplitude equation of the main harmonic of the parametrically driven oscillator equation and stability of the trivial solution analysed in each of these frequency regimes.

In Chapter 4 we apply the multiple-scales technique to the parametrically driven linear oscillator equation and compare results with those obtained through the Lindstedt-Poincaré method and in literature.

In Chapter 5 we continue with the multiple scales and apply it to the parametrically driven pendulum. The amplitude equation corresponding to the equation of the parametrically driven pendulum is obtained and the correct scaling for the solution obtained for the frequencies $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0, (2/4)\omega_0$ and $4\omega_0$.

In Chapter 6, the amplitude equation corresponding to the parametrically driven, damped *linear* Klein-Gordon equation in various frequency regimes is obtained for the cases $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0$ and $(2/4)\omega_0$. We identify the correct set of "slow variables" introduced by the multiple scales technique. The parametric instability windows are obtained in each of the frequency regimes.

In Chapter 7, we add sinusoidal nonlinearity to the parametrically driven, damped Klein-Gordon equation, so that we consider the sine-Gordon equation in the following frequency regimes: $\omega_{dr} \approx 2\omega_0, \omega_0, (2/3)\omega_0, (2/4)\omega_0$ and $4\omega_0$. The sine-Gordon equation is reduced to a family of NLS equations by means of the multiple-scales technique under the assumption that the amplitude of the breather solution of the SG equation is small, and the damping coefficient and the driver's amplitude are also small.

In Chapter 8 we investigate the radiation waves which may constitute the background to the breather solution of the SG equation, in the same frequency regimes as the breather. We obtain the amplitude equations corresponding to the parametrically driven sine-Gordon equation and determine the correct scaling of the amplitude of the solution. Linear stability of the trivial solution is analysed and the existence of nonlinear solutions investigated.

Chapter 9 summarises the results obtained and outlines further research directions.

Chapter 2

Singular Perturbation Methods

2.1 Introduction

In chapter 1 we assumed that the parametrically driven damped sine-Gordon equation (1.38) has small driver's amplitude and small damping coefficient. Denoting ε the smallness parameter, equation (1.38) may be rewritten in the form

$$\varphi_{tt} - \varphi_{xx} + \tilde{\omega}_0^2(t)\sin\varphi = -\varepsilon\lambda_0\varphi_t, \tag{2.1}$$

where

$$\tilde{\omega}_0^2(t) = \omega_0^2(1 - 2\varepsilon F_0 \cos\Omega t).$$

When ε=0, equation (2.1) has exact solution in the form of a kink or a breather. On the other hand, if ε is nonzero, eq. (2.1) does not have exact solutions and we have to resort to approximate or numerical methods.

In this chapter we introduce a collection of methods that yield approximate solutions for a large class of ordinary and partial differential equations. Generally speaking, these methods are used when a small parameter occurs in the equation. According to Poincaré [18, 19], if coefficients of an ordinary differential equation are analytic functions of ε, the solution will also be an analytic function of ε. Then the solution can be sought as a series in powers of ε. This technique generally referred to as the perturbation method, was originally developed for the theory of nonlinear oscillations and subsequently generalised for partial differential equations. Since the case of ordinary differential equations is conceptually simpler, we will illustrate perturbation methods by means of an example from the theory of oscillations. (In subsequent sections we will demonstrate how these methods can be generalised for partial differential equations.) Here we mainly follow Nayfeh [22], Jeffrey & Kawahara [11], and Jordan & Smith [12].

We consider the differential equation of the form

$$\ddot{u} + u + \varepsilon F(u, \dot{u}) = 0, \tag{2.2}$$

where ε is a small parameter and F is assumed to be an analytic nonlinear function of u and $\dot{u}$. When $\varepsilon = 0$, the solution is periodic. When $\varepsilon \neq 0$, we assume that periodic solution still exists and can be looked for as

$$u = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots + \varepsilon^n u_n(t) + \cdots \tag{2.3}$$

where the coefficients at powers of ε are functions of the independent variable t. Substituting the expansion (2.3) in (2.2) and equating coefficients at the same powers of ε, leads, in general, to an infinite system of inhomogeneous equations that can be solved by standard methods. If the series (2.3) approaches the solution of the original equation as $\varepsilon \to 0$, the approximation is said to be ***asymptotic.*** If the series converges, the perturbation method is called regular; otherwise it is said to be singular.

To illustrate this set of ideas and to motivate the necessity of singular perturbation methods, we consider a simple example, the undamped Duffing equation:

$$\ddot{u} + u + \varepsilon u^3 = 0, \tag{2.4}$$

where ε $(0 < \varepsilon << 1)$ is a dimensionless quantity which measures the strength of the nonlinearity, and dots represent differentiation with respect to time, t.

We look for a solution in the form of a power series in ε:

$$u(t;\varepsilon) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots. \tag{2.5}$$

Substituting this in equation (2.4) we obtain, by collecting terms of the same order of ε:

$$\varepsilon^0 : \quad \ddot{u_0} + u_0 = 0, \tag{2.6}$$

$$\varepsilon^1 : \quad \ddot{u_1} + u_1 = -u_0^3, \tag{2.7}$$

$$\varepsilon^2 : \quad \ddot{u_2} + u_2 = -3u_0^2 u_1, \tag{2.8}$$

and so on. The solution of eq. (2.6) is

$$u_0 = a_0 \cos(t + \beta_0), \tag{2.9}$$

where a_0 and β_0 are arbitrary constants to be determined by initial conditions. (Here we shall not impose any initial conditions since we are concerned only with general properties of solutions.) Equation (2.7) then becomes

$$\ddot{u_1} + u_1 = a_0^3 \cos^3(t + \beta_0). \tag{2.10}$$

The general solution of this inhomogeneous equation consists of the sum of a homogeneous solution and a particular nonhomogeneous solution. To determine a particular solution one expands the inhomogeneity in simple harmonics:

$$\ddot{u_1} + u_1 = -\frac{3}{4} a_0^3 \cos(t + \beta_0) - \frac{1}{4} a_0^3 \cos(3t + 3\beta_0). \tag{2.11}$$

The general solution is then found to be

$$u_1 = a_1 \cos(t + \beta_1) - \frac{3}{8} a_0^3 t \sin(t + \beta_0) + \frac{1}{32} a_0^3 \cos(3t + 3\beta_0). \tag{2.12}$$

Since the term $a_1 \cos(t + \beta_1)$ can be included eventually in the zeroth-order solution, we can put, without loss of generality, $a_1 = 0$. Alternatively, one can ignore the homogeneous solutions in all of u_n, for $n \geq 1$, until the last step. Then, considering the constants of integration in x_0 in powers of ε. The latter alternative is preferable since there is less algebraic manipulations and in many instances, we are only concerned with steady-state responses [23].

Thus, the zeroth-order approximation to the solution u is

$$u_0 = a_0 \cos(t + \beta_0),$$

and the first order correction is

$$u_1 = \varepsilon \left\{ -\frac{3}{8} a_0^3 t \sin(t + \beta_0) + \frac{1}{32} a_0^3 \cos(3t + 3\beta_0) \right\}.$$

We note that this correction is small, as it is supposed to be, only when $\varepsilon t \ll 1$. When $\varepsilon t \geq 1$, the "small correction" term becomes larger than the main term. Hence the straightforward expansion is only valid for times $t \ll \varepsilon^{-1}$. This means that such expansions are nonuniform and break down for large times. The reason for the breakdown is the presence of the unbounded term $t \sin(t + \beta_0)$ which causes the expansion (2.3) to be divergent. Such terms are usually referred to as *secular terms*. In order to obtain uniformly valid approximate solutions, the straightforward perturbation procedure should be replaced by more sophisticated techniques, the so-called singular perturbation methods.

We shall illustrate these methods by means of a differential equation containing a small parameter in front of nonlinear term, namely the Duffing equation (2.4). As we shall see, this example is in many respects, the simplest. We shall describe techniques such as the multiple-scale formalism [8, 9, 11, 12, 21, 23], the Lindstedt-Poincaré method [11, 12, 19, 21, 23], and the Krylov-Bogoliubov-Mitropolsky (KBM) approach [5, 11, 19, 20].

2.2 The Lindstedt-Poincaré technique

The breakdown of the straightforward expansion is, in particular, due to its failure to account for the fact that in nonlinear systems the frequency depends on the amplitude. To account for this, Lindstedt explicitly exhibited the frequency ω of the assumed oscillating solution in the equation. (This idea goes back to Stokes). Lindstedt introduced the transformation $\tau = \omega t$ where

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{2.13}$$

The Duffing equation (2.4) then becomes

$$\omega^2 u'' + u + \varepsilon u^3 = 0, \tag{2.14}$$

where the prime indicates the derivative with respect to τ. The approximate solution is still sought as a power series in ε:

$$u = u_0(\tau) + \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \cdots \tag{2.15}$$

Later, it was proved by Poincaré that this expansion is asymptotic and uniformly valid. The first term in the expansion for ω is the natural frequency ω_0 (unity in this case since the $\varepsilon = 0$ solution has frequency 1). The dependence of τ on ε has to be determined from the conditions for nonsecularity. Substituting the above two expansions into eq. (2.14) and equating coefficients of same powers of ε we obtain

$$\varepsilon^0: \quad u_0'' + u_0 = 0, \tag{2.16}$$

$$\varepsilon^1: \quad u_1'' + u_1 = -u_0^3 - 2\omega_1 u_0''. \tag{2.17}$$

The general solution of equation (2.16) is $u_0 = a\cos(\tau+\beta)$, and (2.17) becomes

$$u_1'' + u_1 = (2\omega_1 a - \frac{3}{4}a^3)\cos(\tau+\beta) - \frac{1}{4}a^3\cos[3(\tau+\beta)]. \tag{2.18}$$

The first term on the right-hand side leads to a secular term, and must be suppressed in order that the expansion be uniform.

In contrast to the straightforward expansion where the secular term cannot be removed unless $a = 0$ (i.e. trivial solution for u), in the case at hand we still have the freedom of choosing the parameter ω_1. We choose it in such a way that the secular term is absent:

$$2\omega_1 a - \frac{3}{4}a^3 = 0.$$

Disregarding the trivial case $a = 0$, we find that $\omega_1 = \frac{3}{8}a^2$ and so

$$u_1 = \frac{1}{32}a^3\cos(3\tau + 3\beta).$$

Finally, the solution of equation (2.14) is

$$u = a\cos\left[(1+\frac{3}{8}\varepsilon a^2)t + \beta\right] + \frac{1}{32}\varepsilon a^3\cos\left[3(1+\frac{3}{8}\varepsilon a^2)t + 3\beta\right] + O(\varepsilon^2). \tag{2.19}$$

This expansion is uniform to first order.

The main advantage of the Lindstedt method is that it is very simple and straightforward to apply. It allows one to obtain uniformly valid expansions for periodic solutions. (The higher-order approximations may be cumbersome, but in general these are not difficult to obtain.) The scope of applications of this method is, however, limited. Since we assumed in the derivation that each $u_i(t)$ is periodic, the Lindstedt method will only generate periodic solutions, and will not give us any information about how quickly the system settles down to these nonlinear oscillations neither will it be used to describe resonantly growing solutions.

2.3 The Krylov-Bogoliubov-Mitropolsky method

The solution of the unperturbed problem ($\varepsilon = 0$) in (2.4) is purely harmonic, eq. (2.9). However, in the case $\varepsilon \neq 0$, there occurs a systematic change in the amplitude of the oscillation owing to the nonlinear "perturbing force". When ε is small, the amplitude and phase can be assumed to be slowly varying functions of time. That is, $da/dt = O(\varepsilon)$ and $d\beta/dt = O(\varepsilon)$. The key idea of this method may then be formulated as the following systematic perturbation scheme.

The perturbed solution is sought as

$$u = a\cos\theta + \sum_{n=1}^{\infty} \varepsilon^n u_n(a,\theta), \quad \theta = t + \beta, \tag{2.20}$$

where $u_1, u_2, u_3, \ldots$, are assumed to be periodic functions of θ. Here slow variations of the amplitude and phase are taken into account by introducing the following differential equations:

$$\dot{a} = \sum_{n=1}^{\infty} \varepsilon^n A_n(a), \tag{2.21}$$

$$\dot{\theta} = 1 + \sum_{n=1}^{\infty} \varepsilon^n \Theta_n(a), \tag{2.22}$$

where overdot represents differentiation with respect to t. The problem is to determine the functions u_n, A_n and Θ_n in such a way that equation (2.4) is satisfied. By the chain rule we have

$$\frac{d}{dt} = \dot{a}\frac{\partial}{\partial a} + \dot{\theta}\frac{\partial}{\partial \theta} \tag{2.23}$$

and

$$\frac{d^2}{dt^2} = \dot{a}^2\frac{\partial^2}{\partial a^2} + 2\dot{a}\dot{\theta}\frac{\partial^2}{\partial a \partial \theta} + \dot{\theta}^2\frac{\partial^2}{\partial \theta^2} + \ddot{a}\frac{\partial}{\partial a} + \ddot{\theta}\frac{\partial}{\partial \theta}, \tag{2.24}$$

whence

$$\ddot{a} = \varepsilon^2 A_1 \frac{dA_1}{da} + O(\varepsilon^3) \tag{2.25}$$

and

$$\ddot{\theta} = \varepsilon^2 A_1 \frac{d\Theta}{da} + O(\varepsilon^3). \tag{2.26}$$

Thus the concept of a slow variation of both amplitude and phase is taken into consideration in a systematic way. Let $u_0 = a\cos\theta$. Then

$$\ddot{u}_0 = -a\cos\theta - 2\varepsilon(A_1\sin\theta + \Theta_1 a\cos\theta) - \varepsilon^2\Theta_1(2A_1\sin\theta + \Theta_1 a\cos\theta) + O(\varepsilon^3). \tag{2.27}$$

Also,

$$\ddot{u}_1 = \frac{\partial^2 u_1}{\partial \theta^2} + 2\varepsilon\Theta_1\frac{\partial^2 u_1}{\partial \theta^2} + \varepsilon^2\left[A_1\left(\frac{dA_1}{da}\frac{\partial u_1}{\partial a} + \frac{d\Theta}{da}\frac{\partial u_1}{\partial \theta}\right) + A_1^2\frac{\partial^2 u_1}{\partial a^2} + \Theta_1^2\frac{\partial^2 u_1}{\partial \theta^2}\right] + O(\varepsilon^3). \tag{2.28}$$

Substituting eqs. (2.20) to (2.28) into the Duffing equation (2.4) we obtain, at ε^1:

$$\frac{\partial^2 u_1}{\partial \theta^2} + u_1 = 2A_1\sin\theta + (2\Theta_1 a - \frac{3}{4}a^3)\cos\theta - \frac{1}{4}a^3\cos 3\theta. \tag{2.29}$$

This equation provides the nonsecular periodic solution for u_1,

$$u_1 = \frac{1}{32}a^3\cos 3\theta, \tag{2.30}$$

if the nonsecularity conditions are satisfied, namely, $A_1 = 0$ and $\Theta_1 = \frac{3}{8}a^2$. Here, as in the previous section, we have assumed that the first harmonic is not present in the expression for u_1. (In other words, we have selected the amplitude a of the lowest order solution as the full amplitude of the first fundamental harmonic mode of oscillation.)

Substituting the above nonsecularity conditions into equations (2.21) and (2.22) leads to

$$\dot{a} = 0 + O(\varepsilon^2) \tag{2.31}$$

and

$$\dot{\theta} = 1 + \frac{3}{8}\varepsilon a^2 + O(\varepsilon^2). \tag{2.32}$$

Equation (2.32) implies that

$$\theta = (1 + \frac{3}{8}\varepsilon a^2)t + \beta_0 + O(\varepsilon^2).$$

Since, on the other hand, $\theta = t + \beta$, we have

$$\beta = \frac{3}{8}\varepsilon a^2 t + \beta_0, \tag{2.33}$$

whence

$$u = a\cos\left[(1 + \frac{3}{8}\varepsilon a^2)t + \beta_0\right] + \frac{1}{32}\varepsilon a^3 \cos\left\{3\left[(1 + \frac{3}{8}\varepsilon a^2)t + \beta_0\right]\right\} + O(\varepsilon^2), \tag{2.34}$$

in the first order of approximation.

In this method conditions for nonsecularity give rise to small corrections to the amplitude and phase. If there are periodic solutions, the method allows us to determine them. In addition, it describes the transient regime as the system settles down to these periodic solutions through the variations in phase and amplitude. The disadvantage of this method is that higher-order calculations are extremely long and cumbersome, and, in general, the amount of effort required to calculate higher-order terms is not justified by the small amount of additional information obtained.

2.4 The multiple-scale formalism

If we consider solution (2.19) obtained by the Lindstedt technique,

$$u = a\cos(t + \beta + \frac{3}{8}\varepsilon a^2 t) + \frac{1}{32}\varepsilon a^3 \cos(3t + 3\beta + \frac{9}{8}\varepsilon a^2 t) + \cdots$$

we note that u contains two different time scales, t and εt. Consequently, when constructing solutions it is natural to introduce the dependence on slow variables $\varepsilon t, \varepsilon^2 t, \varepsilon^3 t, \cdots$ in addition to the dependence on t. We write

$$u(t;\varepsilon) = u(T_0, T_1, T_2, \ldots; \varepsilon)$$

where $T_n = \varepsilon^n t$. Thus instead of considering u as a function of t, we think of it as a function of $T_0, T_1, T_2, \ldots$. Using the chain rule, we have

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots,$$

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2 \left(2\frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2} \right) + \cdots$$

and so on. In terms of the new variables, the Duffing equation (2.4) becomes

$$\frac{\partial^2 u}{\partial T_0^2} + 2\varepsilon \frac{\partial^2 u}{\partial T_0 \partial T_1} + \varepsilon^2 \left(2\frac{\partial^2 u}{\partial T_0 \partial T_2} + \frac{\partial^2 u}{\partial T_1^2} \right) + \cdots + u + \varepsilon u^3 = 0. \tag{2.35}$$

Hence the original ordinary differential equation has been replaced by a partial differential equation.

We now seek a solution to eq. (2.35) in the form

$$u = u_0(T_0, T_1, T_2, \ldots) + \varepsilon u_1(T_0, T_1, T_2, \ldots) + \ldots. \tag{2.36}$$

Substituting (2.36) in equation (2.35) and comparing coefficients at same powers of ε we obtain

$$\varepsilon^0 : \quad \frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0, \tag{2.37}$$

$$\varepsilon^1 : \quad \frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} - u_0^3. \tag{2.38}$$

The general solution of equation (2.37) can be written as

$$u_0 = a(T_1, T_2, \ldots) \cos\left[T_0 + \beta(T_1, T_2, \ldots)\right]. \tag{2.39}$$

Similarly to the KBM method, a and β are no longer constants but functions of slow times $T_1, T_2, \ldots$. The dependence of a and β on $T_1, T_2, \ldots$, is not known at this level of approximation; it is determined at subsequent levels by eliminating secular terms. Substituting u_0 in (2.38) we obtain

$$\begin{aligned}
\frac{\partial^2 u_1}{\partial T_0^2} + u_1 &= -2\frac{\partial^2}{\partial T_0 \partial T_1}\left[a \cos(T_0 + \beta)\right] - a^3 \cos^3(T_0 + \beta) \\
&= 2\frac{\partial a}{\partial T_1} \sin(T_0 + \beta) + \left[2a\frac{\partial \beta}{\partial T_1} - \frac{3}{4}a^3\right] \cos(T_0 + \beta) - \\
&- \frac{1}{4}a^3 \cos(3T_0 + 3\beta).
\end{aligned} \tag{2.40}$$

In order to get a uniform expansion, the resonant terms must be eliminated. This is accomplished by setting each of the coefficients of $\sin(T_0 + \beta)$ and $\cos(T_0 + \beta)$ equal to zero:

$$\frac{\partial a}{\partial T_1} = 0, \quad \frac{\partial \beta}{\partial T_1} = \frac{3}{8}a^2,$$

that is, $a = a(T_2, T_3, \ldots)$ and $\beta = \frac{3}{8}a^2 T_1 + \beta_0(T_2, T_3, \ldots)$ where β_0 is a constant of integration. These results coincide with those obtained by the KBM method in section 2.3. A particular solution for u_1 is

$$u_1 = \frac{1}{32} a^3 \cos(3T_0 + 3\beta), \tag{2.41}$$

and so

$$u = a \cos\left[(1 + \frac{3}{8}\varepsilon a^2)t + \beta_0\right] + \frac{1}{32}\varepsilon a^3 \cos\left[3\left(1 + \frac{3}{8}\varepsilon a^2\right)t + 3\beta_0\right] + \cdots. \tag{2.42}$$

In the method of multiple-scales the conditions for nonsecularity are given by partial differential equations with respect to slow variables, whereas in the KBM method these conditions were given by ordinary differential equations with respect to t. Changing the original ODE to a system of PDE's allows enough generality in the form of the solution to obtain a uniform approximation, that is, to make (u_n/u_{n-1}) bounded for all $T_0, T_1, T_2, \ldots$. The disadvantage of this method is that even to obtain first-order results, one usually has to solve a number of PDE's. At higher-order approximations the computations can become quite cumbersome.

2.5 Discussion

The essence of the Lindstedt method is to transform the independent variable t to $\tau = \omega t$ and expand ω as a series in the small parameter ε: $\omega = \omega_0 + \varepsilon\omega_1 + \cdots$. This transformation allows us to avoid the occurence of secular terms by adjusting ω_i's. By expanding ω in terms of ε, the solution u will contain terms $\varepsilon t, \varepsilon^2 t, \ldots$. Thus the slow scales are introduced *implicitly*. Similarly to the Lindstedt method, the Krylov-Bogoliubov-Mitropolsky method introduces the slow scales implicitly through the ε-expansion of temporal derivatives of amplitude and phase. The multiple-scales technique, on the other hand, takes into account existence of processes with different time scales by *explicitly* introducing $T_0, T_1, T_2, \ldots$.

Whereas the Lindstedt method assumes that the solution is periodic by explicitly introducing the frequency of oscillations, the KBM method and the multiple-scales techniques do not assume any periodicity. The KBM approach and the multiple-scales techniques are therefore more general than the Lindstedt method.

We have demonstrated that for the first order approximation to the Duffing equation, the three methods give equivalent results. Amongst these methods, the Lindstedt method is the simplest and the most straightforward to apply. However, it is inapplicable to solutions other than periodic ones. On the contrary, the KBM method and the multiple-scales technique can describe resonant growth of amplitudes, various transient regimes and other nonstationary effects. The major drawback of these methods is that higher-order calculations are extremely lengthy and cumbersome and, in general, the amount of effort required to calculate higher order terms is not justified by the small amount of additional information obtained.

In this work we shall employ only two of these methods, the Lindstedt- Poincaré method and the multiple-scales technique in order to reduce the parametrically driven damped linear oscillator equations to the corresponding amplitude equations. In both cases we assume that the amplitude is slowly varying. Although as we have just mentioned, the Lindstedt-Poincaré method is inapplicable to variable amplitude systems, we will modify it by introducing a feature from the multiple-scales techniques: we will assume that the amplitude is a function of two different scales - fast scale t, and slow scale $T = \varepsilon t$. The reason why we opt to modify the Lindstedt method is that we wish to rederive the results by means of Lindstedt-Poincaré and compare to those obtained by the multiple-scales technique.

We will, however, mostly use the multiple-scales technique in order to reduce the parametrically driven damped linear and nonlinear oscillators, and the linear and nonlinear Klein-Gordon equations to the corresponding amplitude equations.

Chapter 3

The parametrically driven, damped linear oscillator: Lindstedt-Poincaré method.

In the previous chapter we presented different techniques of finding secular-free expansions of solutions of ordinary differential equations, exemplified by the Duffing equation. In this chapter, we study an equation of the linear oscillator perturbed by a small driving term which appears as a time-dependent coefficient. We expect that there will be a resonance between the natural frequency of the oscillator and the frequency of the driver. This means that the amplitude will vary with time. The aim of this chapter is to show how the Lindstedt-Poincaré method can lead us into determining the range of the driving frequencies for which the parametric resonance will occur. However, in the previous discussions we mentioned that the Lindstedt-Poincaré method is inadequate for the treatment of solutions with varying amplitude. We shall therefore show how this technique can be modified to meet the present needs. The results obtained using this method will later be compared to those obtained using the multiple-scales formalism.

The equation of the parametrically driven, damped linear oscillator reads:

$$\varphi_{tt} + 2\lambda\varphi_t + \omega_0^2\left[1 - 2F_0\varepsilon^2\cos\left(\frac{2}{N}\Omega t\right)\right]\varphi = 0. \tag{3.1}$$

Here $2\varepsilon^2 F_0$ and $\frac{2}{N}\Omega$ are, respectively, the driver's amplitude and frequency. N is a positive integer and ε^2 a small parameter ensuring that the driver's strength is small. (The reason for using ε^2 instead of ε will become clear in chapter 5.) Equation (3.1) is known as the damped Mathieu equation and has been extensively studied [9, 12, 21, 23, 89]. We, however, revisit it here to lay the foundation for the subsequent analysis of the *nonlinear* oscillator, and the sine-Gordon equation. Another reason for this revision is that in refs. [9, 12, 21, 23, 89], the driver's frequency is considered as fixed while the natural frequency is varied. On the contrary, our formulation of the problem is such that the natural frequency ω_0 is fixed but the driver is variable. (The results of the two approaches can, of course, be recalculated to one another.)

Since the driver causes the amplitude of the oscillations to vary, Lindstedt's method in its standard form is inapplicable in these cases. We shall therefore modify it by including one more time scale. We first rewrite (3.1) in the form

$$\varphi_{tt} + \Omega^2\varphi = (\Omega^2 - \omega_0^2)\varphi + 2\omega_0^2 F_0 \varepsilon^2 \cos\left(\frac{2}{N}\Omega t\right)\varphi - 2\lambda\varphi_t. \tag{3.2}$$

In so doing we ensure that the left-hand side is of the same form as the undriven, undamped equation, $\varphi_{tt} + \omega_0^2\varphi = 0$, with $\omega_0 \to \Omega$. This is necessary to determine the terms that may resonate with the solution of the undriven equation.

We then make a transformation $\tau = \Omega t$ so that eq. (3.2) becomes

$$\varphi_{\tau\tau} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + 2\frac{\omega_0^2}{\Omega^2} F_0 \varepsilon^2 \cos\left(\frac{2}{N}\tau\right) - 2\lambda\Omega\varphi_\tau. \tag{3.3}$$

Letting $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$, and expanding the detuning and the damping term in powers of μ:

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots, \tag{3.4}$$

$$\lambda\Omega = \mu^{2q}\lambda_0, \tag{3.5}$$

we arrive at the equation of the form

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \cdots)\varphi + 2\mu^2 F_0 \cos\left(\frac{2}{N}\tau\right)\varphi - 2\mu^{2q}\lambda_0\varphi_\tau. \tag{3.6}$$

We will look for solution φ of eq. (3.6) as a power series:

$$\varphi = \varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots, \tag{3.7}$$

Notice that we have expressed the detuning and the damping term in powers of μ^2 since the perturbing parameter enters the system as μ^2. It will become clear later that this choice leads to sensible conclusions. To simplify the calculations, we first consider the undamped case.

3.1 The driving frequency $\sim 2\omega_0$.

We start with the case of the principal resonance:

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots)\varphi + 2\mu^2 F_0 \cos(2\tau)\varphi. \tag{3.8}$$

In this case $N = 1$, that is, the driving frequency is close to twice the natural frequency of the system. Using the expansions (3.7) and comparing coefficients of the same power of μ, we obtain at μ^0 the equation

$$\frac{d^2\varphi_0}{d\tau^2} + \varphi_0 = 0,$$

whose general solution is

$$\varphi_0 = a_0 e^{i\tau} + \bar{a}_0 e^{-i\tau}.$$

Since equation (3.8) is driven, we expect its solution to grow. However, the method of Lindstedt does not apply when the amplitude is varying. Let us try to accomodate the variation of the amplitude by allowing a_0 to change slowly with time: $a_0 = a_0(T)$, where $T = \mu^2\tau$.

At the order μ^2 we have

$$\begin{aligned}\frac{d^2\varphi_0}{d\tau^2} + \varphi_0 &= -2i\frac{da_0}{dT}e^{i\tau} + \alpha_1 a_0 e^{i\tau} + F_0(e^{2i\tau} + e^{-2i\tau})\varphi_0 \\ &= \left(-2i\frac{da_0}{dT} + \alpha_1 a_0 + F_0\bar{a}_0\right)e^{i\tau} + F_0 a_0 e^{3i\tau} + c.c. \end{aligned} \tag{3.9}$$

For nonsecularity we require that the coefficient of $e^{i\tau}$ be equal to zero:

$$-2i\frac{da_0}{dT} + \alpha_1 a_0 + F_0\bar{a}_0 = 0. \tag{3.10}$$

This equation describes the evolution of the amplitude of the main harmonic φ_0. The leading approximation to the solution of (3.8) is thereby completely determined.

Stability analysis. We consider the equation of the main harmonic eq. (3.10)

$$-2i\frac{da_0}{dT} + \alpha_1 a_0 + F_0\bar{a}_0 = 0,$$

and differentiate it with respect to T to get

$$2i\frac{d^2a_0}{dT^2} = \alpha_1\frac{da_0}{dT} + F_0\frac{d\bar{a}_0}{dT}.$$

Using eq.(3.10) to eliminate $d\bar{a}_0/dT$, we arrive at

$$4\frac{d^2a_0}{dT^2} = (F_0^2 - \alpha_1^2)a_0,$$

whose solution is given by $a_0 = e^{sT}$. Parametric resonance occurs when $s^2 > 0$, i.e.,

$$|\alpha_1| < F_0,$$

or, equivalently,

$$-F_0 < \alpha_1 < F_0. \tag{3.11}$$

Recalling that $1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + O(\mu^4)$, and $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$, we have the condition for instability (parametric resonance) in the form

$$\omega_0^2(1 - \varepsilon^2 F_0) < \Omega^2 < \omega_0^2(1 + \varepsilon^2 F_0). \tag{3.12}$$

3.2 The driving frequency $\sim \omega_0$.

We proceed to equation (3.1) with $N = 2$:

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots)\varphi + 2\mu^2 F_0 \cos(\tau)\varphi, \tag{3.13}$$

and assume the expansion (3.7). Comparing equations at same power of μ we obtain at μ^0,

$$\frac{d^2\varphi_0}{d\tau^2} + \varphi_0 = 0,$$

whence

$$\varphi_0 = a_0 e^{i\tau} + \bar{a}_0 e^{-i\tau}.$$

Here a_0 is assumed to be a function of "the slow time" $T = \mu^r\tau$ where r is an even integer to be determined. Apriori it is not obvious what r should be equal to. We shall therefore examine several possibilities.

Let $r = 2$. At the order μ^2 we then obtain

$$\begin{aligned}\frac{d^2\varphi_0}{d\tau^2} + \varphi_0 &= -2i\frac{da_0}{dT}e^{i\tau} + \alpha_1\varphi_0 + F_0(e^{i\tau} + e^{-i\tau})\varphi_0 \\ &= (-2i\frac{da_0}{dT} + \alpha_1 a_0)e^{i\tau} + F_0(a_0 e^{2i\tau} + a_0) + c.c.\end{aligned} \tag{3.14}$$

The condition for the absence of secular terms is that

$$-2i\frac{da_0}{dT} + \alpha_1 a_0 = 0, \tag{3.15}$$

which is undriven. Hence the choice $r = 2$ does not lead to the parametric resonance.

Let $r = 4$. At the order μ^2 we get

$$\begin{aligned}\frac{d^2\varphi_1}{d\tau^2} + \varphi_1 &= \alpha_1\varphi_0 + F_0(e^{i\tau} + e^{-i\tau})\varphi_0 \\ &= \alpha_1 a_0 e^{i\tau} + F_0(a_0 e^{2i\tau} + a_0) + c.c.\end{aligned} \tag{3.16}$$

The nonsecularity condition is simply $\alpha_1 = 0$. The solution of (3.16) is then given by

$$\varphi_1 = a_1 e^{2i\tau} + b_1 + c.c.,$$

where

$$a_1 = -\frac{1}{3}F_0 a_0, \quad b_1 = F_0 a_0.$$

Next, at the order μ^4 we have

$$\begin{aligned}\frac{d^2\varphi_2}{d\tau^2} &= -2i\frac{da_0}{dT}e^{i\tau} + \alpha_2\varphi_0 + F_0(e^{i\tau} + e^{-i\tau})\varphi_1 \\ &= \left[-2i\frac{da_0}{dT} + \alpha_2 a_0 + F_0(a_1 + b_1 + \bar{b}_1)\right] e^{i\tau} + F_0 a_1 e^{3i\tau} + c.c.\end{aligned} \tag{3.17}$$

Setting to zero coefficients of secular terms and substituting a_1 and b_1 we obtain

$$-2i\frac{da_0}{dT} + \left(\alpha_2 + \frac{2}{3}F_0^2\right)a_0 + F_0^2\bar{a}_0 = 0. \tag{3.18}$$

This is a parametrically driven equation; we shall now show that it exhibits parametric resonance. Thus when the driving frequency is $\sim \omega_0$ (and not $\sim 2\omega_0$), the amplitude of the oscillation will grow much slower than in the case of the principal resonance: $T = \mu^4\tau$ (and not $\mu^2\tau$).

Stability. Proceeding as before we find by differentiating eq. (3.18), the equation

$$4\frac{d^2a_0}{dT^2} = F_0^4 a_0 - \left(\alpha_2 + \frac{2}{3}F_0^2\right)^2 a_0,$$

whose solution is given by $a_0 = e^{sT}$ where

$$4s^2 = F_0^4 - \left(\alpha_2 + \frac{2}{3}F_0^2\right)^2.$$

Resonance occurs when

$$\left|\alpha_2 + \frac{2}{3}F_0^2\right| < F_0^2.$$

Equivalently,

$$-\frac{5}{3}F_0^2 < \alpha_2 < \frac{1}{3}F_0^2. \tag{3.19}$$

Recalling that $1 - \frac{\omega_0^2}{\Omega^2} = \alpha_2\mu^4 + O(\mu^6)$ (since $\alpha_1 = 0$) and $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$, we have the region of instability as

$$\omega_0^2\left(1 - \frac{5}{3}\frac{\omega_0^2}{\Omega^2}\varepsilon^4 F_0^2\right) < \Omega^2 < \omega_0^2\left(1 + \frac{1}{3}\frac{\omega_0^2}{\Omega^2}\varepsilon^4 F_0^2\right).$$

Since $\omega_0^2/\Omega^2 = O(1)$, this can be simplified to

$$\omega_0^2\left(1 - \frac{5}{3}\varepsilon^4 F_0^2 + O(\varepsilon^6)\right) < \Omega^2 < \omega_0^2\left(1 + \frac{1}{3}\varepsilon^4 F_0^2 + O(\varepsilon^6)\right). \tag{3.20}$$

Outside this region there is no parametric resonance.

3.3 The driving frequency $\sim \frac{2}{3}\omega_0$.

In this section we shall determine the scaling exponent, r, for the "slow time" $T = \mu^r\tau$ when the system is driven at the frequency $\sim \frac{2}{N}\omega_0$. We shall first find the scaling for the driving frequency $\frac{2}{3}\omega_0$ and then generalise our conclusion to arbitrary N. In addition, we will analyse the behaviour of the solution of the system driven at a frequency $\sim \frac{2}{3}\omega_0$.

Thus, we consider the equation

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots)\varphi + 2\mu^2 F_0 \cos\left(\frac{2}{3}\tau\right)\varphi. \tag{3.21}$$

Substituting (3.7) into (3.21), we obtain at the order μ^0:

$$\frac{d^2\varphi_0}{d\tau^2} + \varphi_0 = 0,$$

whence

$$\varphi_0 = a_0 e^{i\tau} + \bar{a}_0 e^{-i\tau}$$

with $a_0 = a_0(T)$, where $T = \mu^r \tau$, and r is an even integer to be determined. In order that the equation of the main harmonic be driven, we must have the first derivative term appearing at the same order of μ as the complex conjugate term. (The latter is the driver.) Consequently, to determine r we should identify the order of μ at which the complex conjugate term appears.

At the order μ^2 we have

$$\begin{aligned}\frac{d^2\varphi_1}{d\tau^2} + \varphi_1 &= \alpha_1\varphi_0 + F_0(e^{(2/3)i\tau} + e^{-(2/3)i\tau})\varphi_0 \\ &= \alpha_1 a_0 e^{i\tau} + F_0(a_0 e^{i(1+2/3)\tau} + a_0 e^{i(1-2/3)\tau}) + c.c.,\end{aligned} \tag{3.22}$$

where $c.c$ denotes complex conjugate. The nonsecularity condition yields $\alpha_1 = 0$, and a particular solution is given by

$$\varphi_1 = a_1 e^{i(1+2/3)\tau} + b_1 e^{i(1-2/3)\tau} + c.c.,$$

where

$$a_1 = -\frac{9}{16}F_0 a_0, \quad b_1 = \frac{9}{8}F_0 a_0. \tag{3.23}$$

At the order μ^4 we have

$$\begin{aligned}\frac{d^2\varphi_2}{d\tau^2} + \varphi_2 &= \alpha_2\varphi_0 + F_0(e^{(2/3)i\tau} + e^{-(2/3)i\tau})\varphi_0 \\ &= [\alpha_2 a_0 + F_0(a_1 + b_1)]e^{i\tau} + F_0(a_1 e^{i(1+4/3)\tau} + b_1 e^{i(1-4/3)\tau}) + c.c.\end{aligned} \tag{3.24}$$

To suppress the secular terms we require

$$\alpha_2 a_0 + F_0(a_1 + b_1) = 0. \tag{3.25}$$

Invoking (3.23), this simply yields

$$\alpha_2 = -\frac{9}{16}F_0^2.$$

A particular solution of (3.24) is then given by

$$\varphi_2 = a_2 e^{i(1+4/3)\tau} + b_2 e^{i(1-4/3)\tau} + c.c.,$$

where

$$a_2 = \frac{9 \times 9}{40 \times 16}F_0^2 a_0, \quad b_2 = \left(\frac{9}{8}\right)^2 F_0^2 a_0. \tag{3.26}$$

Notice that we still have not obtained the equation for the amplitude of the main harmonic; thus we have to go to higher orders. At the order μ^6 we have

$$\begin{aligned}\frac{d^2\varphi_3}{d\tau^2} + \varphi_3 &= \alpha_2\varphi_1 + \alpha_3\varphi_0 + F_0(e^{(2/3)i\tau} + e^{-(2/3)i\tau})\varphi_2 \\ &= (\alpha_3 a_0 + F_0\bar{b}_2)e^{i\tau} + (\alpha_2 a_1 + F_0 a_2)e^{i(1+2/3)\tau} + \\ &+ (\alpha_2 b_1 + F_0 b_2)e^{i(1-2/3)\tau} + F_0 a_2 e^{i(1+6/3)\tau} + c.c. \end{aligned} \tag{3.27}$$

For nonsecularity we require that

$$\alpha_3 a_0 + F_0\bar{a}_2 = 0. \tag{3.28}$$

Substituting (3.26) into (3.28), we obtain

$$\alpha_3 a_0 + \left(\frac{9}{8}\right)^2 F_0^3\bar{a}_0 = 0. \tag{3.29}$$

Equation (3.29) includes the desired complex conjugate quantity which originates from the term $b_2 e^{i(1-6/3)\tau} = b_2 e^{-i\tau}$ of (3.27). That is,

$$b_2 \exp\left\{i\left(1 - \frac{2}{3} - \frac{2}{3} - \frac{2}{3}\right)\tau\right\} = b_2 \exp(-i\tau),$$

or $b_2 \exp\left\{i\left(1 - m\frac{2}{3}\right)\tau\right\} = b_2 \exp(-i\tau)$ where $m = 3$.

As we have already mentioned, equation (3.25) simply defines α_2. If we let $r = 4$ and added the term $-2i\dfrac{da_0}{dT}$ at the order μ^4, equation (3.25) would become

$$-2i\frac{da_0}{dT} + \alpha_2 a_0 + \frac{9}{16}F_0^2 a_0 = 0, \tag{3.30}$$

which is an undriven equation (no parametric resonance).

If we, instead, assume that $r = 6$, that is, $T = \mu^6\tau$, then the equation governing the main harmonic will be eq.(3.29):

$$-2i\frac{da_0}{dT} + \alpha_3 a_0 + \left(\frac{9}{8}\right)^2 F_0^3\bar{a}_0 = 0. \tag{3.31}$$

This equation is driven. Assuming that $a_0 = e^{sT}$ we get

$$s^2 = \frac{1}{4}\left[\left(\frac{9}{8}\right)^4 F_0^6 - \alpha_3\right], \tag{3.32}$$

which can be positive if α_3 is not too large (parametric resonance). Thus, the idea that we should choose r in such a way that the term $-2ida_0/dT$ appear at the same order of μ as $\bar{a}_0$, remains productive for general N. If we consider the driving frequency $\sim \frac{2}{N}\omega_0$ and write $r = 2m$, the complex conjugate term will appear when

$$\exp(i\tau) \times \exp\left\{i\left(-\frac{2}{N} - \frac{2}{N} - \frac{2}{N} - \cdots\right)\tau\right\} = \exp(-i\tau),$$

i.e.

$$1-\underbrace{\frac{2}{N}-\frac{2}{N}-\frac{2}{N}-\cdots}_{m \quad \text{times}}=-1.$$

That is, $1-m\frac{2}{N}=-1$, whence $m=N$. Thus the scaling law is $T=\mu^{2N}\tau$.

We close this section by writing out the condition for parametric resonance in (3.31). According to eq. (3.32), exponentially growing solutions exist if

$$-\left(\frac{9}{8}\right)^2 F_0^3<\alpha_3<\left(\frac{9}{8}\right)^2 F_0^3,$$

or equivalently

$$\omega_0^2\left[1-\frac{9}{16}\frac{\omega_0^2}{\Omega^2}\varepsilon^4F_0^2-\left(\frac{9}{8}\right)^2\frac{\omega_0^4}{\Omega^4}\varepsilon^6F_0^3\right]<\Omega^2<\omega_0^2\left[1-\frac{9}{16}\frac{\omega_0^2}{\Omega^2}\varepsilon^4F_0^2+\left(\frac{9}{8}\right)^2\frac{\omega_0^4}{\Omega^4}\varepsilon^6F_0^3\right].$$

Since $1-\dfrac{\omega_0^2}{\Omega^2}=O(\varepsilon^4)$ (for $\alpha_1=0$) and therefore $\omega_0^2/\Omega^2=1+O(\varepsilon^4)$, we have, finally:

$$\omega_0^2\left[1-\frac{9}{16}\varepsilon^4F_0^2-\left(\frac{9}{8}\right)^2\varepsilon^6F_0^3\right]<\Omega^2<\omega_0^2\left[1-\frac{9}{16}\varepsilon^4F_0^2+\left(\frac{9}{8}\right)^2\varepsilon^6F_0^3\right].$$

There is no parametric resonance outside this region.

3.4 The driving frequency $\sim \frac{2}{4}\omega_0$.

Finally we shall show how the general recipe designed in section (3.3) will work when applied to the case of the driving frequency $\sim \frac{2}{4}\omega_0$:

$$\varphi_{\tau\tau}+\varphi=(\alpha_1\mu^2+\alpha_2\mu^4+\alpha_3\mu^6+\cdots)\varphi+2\mu^2F_0\cos\left(\frac{2}{4}\tau\right)\varphi. \tag{3.33}$$

Proceeding as usual, we find that

$$\varphi_0=a_0e^{i\tau}+\bar{a}_0e^{-i\tau}.$$

According to the scaling law derived in section (3.3), we should assume $a_0=a_0(\mu^8\tau)$. At the order μ^2 we obtain

$$\begin{aligned}\frac{d^2\varphi_1}{d\tau^2}+\varphi_1 &= \alpha_1\varphi_0+F_0(e^{2/4i\tau}+e^{-2/4i\tau})\varphi_0\\ &= \alpha_1a_0e^{i\tau}+F_0(a_0e^{i(1+2/4)\tau}+a_0e^{i(1-2/4)\tau})+c.c.\end{aligned} \tag{3.34}$$

whose solution is given by

$$\varphi_1=a_1e^{i(1+2/4)\tau}+b_1e^{i(1-2/4)\tau}+c.c.,$$

where

$$a_1 = -\frac{4}{5}F_0 a_0, \quad b_1 = \frac{4}{3}F_0 a_0.$$

(The nonsecularity condition is $\alpha_1 = 0$.) At the order μ^4, we have

$$\begin{aligned} \frac{d^2\varphi_2}{d\tau^2} + \varphi_2 &= \alpha_2\varphi_0 + F_0(e^{(2/4)i\tau} + e^{-(2/4)i\tau})\varphi_0 \\ &= [\alpha_2 a_0 + F_0(a_1 + b_1)]e^{i\tau} + F_0(a_1 e^{i(1+4/4)\tau} + b_1 e^{i(1-2/4)\tau} + c.c. \end{aligned} \tag{3.35}$$

This time the condition for nonsecularity is

$$\alpha_2 = -\frac{8}{15}F_0^2 a_0, \tag{3.36}$$

and the solution reads

$$\varphi_2 = a_2 e^{i(1+4/4)\tau} + b_2 e^{i(1-4/4)\tau} + c.c.,$$

where

$$a_2 = \frac{4}{15}F_0^2 a_0, \quad b_2 = \frac{4}{3}F_0^2 a_0.$$

Next, at the order μ^6 the perturbation expansion yields:

$$\begin{aligned} \frac{d^2\varphi_3}{d\tau^2} + \varphi_3 &= \alpha_2\varphi_1 + \alpha_3\varphi_0 + F_0(e^{(2/4)i\tau} + e^{-(2/4)i\tau})\varphi_2 \\ &= \alpha_3 a_0 e^{i\tau} + (\alpha_2 a_1 + F_0 a_2)e^{i(1+2/4)\tau} + \\ &+ [\alpha_2 b_1 + F_0(b_2 + \bar{b}_2)]e^{i(1-2/3)\tau} + F_0 a_2 e^{i(1+6/4)\tau} + c.c., \end{aligned} \tag{3.37}$$

whose solution is

$$\varphi_3 = a_3 e^{i(1+6/4)\tau} + b_3 e^{i(1-6/4)\tau} + c_3 e^{i(1+2/4)\tau} + c.c.,$$

with

$$a_3 = -\frac{4\times 4}{21\times 15}F_0^3 a_0, \quad b_3 = \frac{112}{3\times 45}F_0^3\bar{a}_0 + \left(\frac{4}{3}\right)^2 F_0^3 a_0, \quad c_3 = -\frac{208}{15\times 25}F_0^3 a_0.$$

If we proceed to higher order terms, we find that at μ^8,

$$\begin{aligned} \frac{d^2\varphi_4}{dT^2} + \varphi_4 &= -2i\frac{da_0}{dT}e^{i\tau} + \alpha_4\varphi_0 + \alpha_2\varphi_2 + F_0(e^{(2/4)i\tau} + e^{-(2/4)i\tau})\varphi_3 \\ &= \left[-2i\frac{da_0}{dT} + \alpha_4 a_0 + F_0(\bar{b}_3 + c_3)\right]e^{i\tau} + [\alpha_2 a_2 + F_0(a_3 + c_3)] \times \\ &\times \; e^{i(1+4/4)\tau} + (\alpha_2 b_2 + F_0 b_3)e^{i(1-4/4)\tau} + F_0 a_3 e^{i(1+8/4)\tau} + c.c. \end{aligned} \tag{3.38}$$

This time, the suppression of secular terms yields a first order differential equation in a_0 which involves complex conjugates:

$$-2i\frac{da_0}{dT} + \alpha_4 a_0 + F_0(\bar{b}_3 + c_3) = 0. \tag{3.39}$$

Substituting for known variables $\bar{b}_3$ and c_3 results in

$$-2i\frac{da_0}{dT}+\alpha_4 a_0+\frac{928}{45\times 75}F_0^4 a_0+\left(\frac{4}{3}\right)^2 F_0^4\bar{a}_0=0. \tag{3.40}$$

This is a parametrically driven equation.

To determine the range of resonance, we proceed in the same way as in the previous section. Parametric resonance occurs if

$$\left|\alpha_4+\frac{928}{75\times 45}F_0^4\right|<\left(\frac{4}{3}\right)^2 F_0^4.$$

Equivalently,

$$\omega_0^2\left[1-\frac{8}{15}\frac{\omega_0^2}{\Omega^2}\varepsilon^4F_0^2-\frac{5968}{75\times 45}\frac{\omega_0^6}{\Omega^6}\varepsilon^8F_0^4\right]<\Omega^2<\omega_0^2\left[1-\frac{8}{15}\frac{\omega_0^2}{\Omega^2}\varepsilon^4F_0^2+\frac{6032}{75\times 45}\frac{\omega_0^2}{\Omega^2}\varepsilon^8F_0^4\right].$$

Recalling that

$$1-\frac{\omega_0^2}{\Omega^2}+\frac{8}{15}F_0^2\frac{\omega_0^4}{\Omega^4}\varepsilon^4=0,$$

we arrive at the resonance range

$$\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2-\frac{6928}{75\times 45}\varepsilon^8F_0^4\right]<\Omega^2<\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2+\frac{5072}{75\times 45}\varepsilon^8F_0^4\right].$$

3.5 Effects of damping.

We now consider the effects of damping by studying the equation

$$\begin{aligned}\varphi_{\tau\tau}+\varphi &= (\alpha_1\mu^2+\alpha_2\mu^4+\cdots)+2\mu^2F_0\cos\left(\frac{2}{N}\right)\tau\varphi- \\ &- 2\mu^{2q}\lambda_0\end{aligned} \tag{3.41}$$

We assume that the damping is weak enough to allow the parametric resonance to occur. We first wish to determine the value of q such that the dissipation features in the amplitude equation.

We shall show how weak damping affects the solution of the system by studying the case $N=2$ and deriving the scaling for the damping coefficient. Since we are interested in the effects of damping on the evolution of the amplitude we wish to find the damping term that will counteract the driver. Thus, the damping term must appear at the same order of approximation as the driver.

Let $q=1$. Comparing coefficients at like powers of μ, we obtain at μ^0

$$\frac{d^2\varphi_0}{d\tau^2}+\varphi_0=0,$$

whose solution is given by

$$\varphi_0=a_0e^{i\tau}+\bar{a}_0e^{-i\tau},$$

with $a_0 = \mu^4\tau$ as we have found in section (3.2). At the order μ^2 we obtain

$$\begin{aligned}\frac{d^2\varphi_1}{d\tau^2} + \varphi_1 &= \alpha_1\varphi_0 + F_0(e^{i\tau} + e^{-i\tau})\varphi_0 - 2i\lambda_0 a_0 e^{i\tau} \\ &= (\alpha_1 a_0 - 2i\lambda_0 a_0)e^{i\tau} + F_0(a_0 e^{2i\tau} + a_0) + c.c.\end{aligned} \tag{3.42}$$

The nonsecularity condition is

$$(\alpha_1 - 2i\lambda_0)a_0 = 0.$$

This implies that either $a_0 = 0$ or $\alpha_1 = 2i\lambda_0$, both of which are impossible since α_1 is a real constant and a_0 cannot be zero. Thus $q \neq 1$.

Let $q = 2$. At the order μ^2 we obtain

$$\begin{aligned}\frac{d^2\varphi_1}{d\tau^2} + \varphi_1 &= \alpha_1\varphi_0 + F_0(e^{i\tau} + e^{-i\tau})\varphi_0 \\ &= \alpha_1 a_0 e^{i\tau} + F_0(a_0 e^{2i\tau} + a_0) + c.c.\end{aligned} \tag{3.43}$$

The nonsecularity condition is simply $\alpha_1 = 0$ and a particular solution given by

$$\varphi_1 = a_1 e^{2i\tau} + b_1 + c.c.,$$

where

$$a_1 = -\frac{1}{3}F_0, \quad b_1 = F_0 a_0.$$

At the order μ^4 we arrive at

$$\begin{aligned}\frac{d^2\varphi_2}{d\tau^2} + \varphi_2 &= -2i\frac{da_0}{dT}e^{i\tau} + \alpha_2\varphi_0 + F_0(e^{i\tau} + e^{-i\tau})\varphi_1 - 2i\lambda_0 a_0 e^{i\tau} \\ &= \left[-2i\frac{da_0}{dT} + \alpha_2 a_0 + F_0(a_1 + b_1 + \bar{b}_1) - 2i\lambda_0 a_0\right] e^{i\tau} + F_0 a_1 e^{3i\tau} + c.c.\end{aligned}$$

Suppressing secular terms we arrive at

$$-2i\frac{da_0}{dT} + \left(\alpha_2 + \frac{2}{3}F_0^2\right) a_0 + F_0^2\bar{a}_0 - 2i\lambda_0 a_0 = 0, \tag{3.44}$$

where we have substituted for a_1 and b_1. Equation (3.44) is the parametrically driven, damped "Schrödinger" equation. Thus $q = 2$ (i.e. $\lambda = \mu^4\lambda_0$). Similarly it can be shown that $\lambda = \mu^{2N}\lambda_0$ for an arbitrary integer N.

Stability. Analysing the stability of the trivial solution of the damped driven amplitude equation (3.44) we proceed as follows. Let $a_0 = Ae^{-\lambda_0 T}$ so that

$$\frac{da_0}{dT} = \left(\frac{dA}{dT} - \lambda_0 A\right) e^{-\lambda_0 T}.$$

Substituting this expression in (3.44) we arrive at the equation in the form (3.18):

$$-2i\frac{dA}{dT} = \left(\alpha_2 + \frac{2}{3}F_0^2\right) A + F_0^2\bar{A} = 0. \tag{3.45}$$

So we can use the result for eq.(3.18). Equation (3.45) has solutions growing as e^{sT} where

$$4s^2 = F_0^4 - \left(\alpha_2 + \frac{2}{3}F_0^2\right)^2 .$$

Then $a_0 \sim e^{(s-\lambda_0)T}$ and instability occurs when $s > \lambda_0$. That is, when

$$-\frac{2}{3}F_0^2 - \sqrt{F_0^4 - 4\lambda_0^2} < \alpha_2 < -\frac{2}{3}F_0^2 + \sqrt{F_0^4 - 4\lambda_0^2}.$$

Recalling that $1 - \dfrac{\omega_0^2}{\Omega^2} = \alpha_2\mu^4 + \cdots$ we arrive finally at

$$\omega_0^2\left[1 - \frac{2}{3}\varepsilon^4 F_0^2 - \varepsilon^4\sqrt{F_0^4 - 4\lambda_0^2}\right] < \Omega^2 < \omega_0^2\left[1 - \frac{2}{3}\varepsilon^4 F_0^2 + \varepsilon^4\sqrt{F_0^4 - 4\lambda_0^2}\right].$$

Résumé. When the linear oscillator is parametrically driven, the scaling for the damping coefficient is given in the general case by $\lambda = \mu^{2N}\lambda_0$. It can be easily shown then that the instability windows for $N = 1, 2, 3$ and 4 are given by

- $N = 1:\quad \omega_0^2\left[1 - \varepsilon^2\sqrt{F_0^2 - 4\lambda_0^2}\right] < \Omega^2 < \omega_0^2\left[1 + \varepsilon^2\sqrt{F_0^2 - 4\lambda_0^2}\right],$

- $N = 2:\quad \omega_0^2\left[1 - \frac{2}{3}\varepsilon^4 F_0^2 - \varepsilon^4\sqrt{F_0^4 - 4\lambda_0^2}\right] < \Omega^2 < \omega_0^2\left[1 - \frac{2}{3}\varepsilon^4 F_0^2 + \varepsilon^4\sqrt{F_0^4 - 4\lambda_0^2}\right],$

- $N = 3:\quad \omega_0^2\left[1 - \frac{9}{16}\varepsilon^4 F_0^2 - \varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4 F_0^6 - 4\lambda_0^2}\right] < \Omega^2$

$$< \omega_0^2\left[1 - \frac{9}{16}\varepsilon^4 F_0^2 + \varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4 F_0^6 - 4\lambda_0^2}\right],$$

- $N = 4:\quad \omega_0^2\left[1 - \frac{8}{15}\varepsilon^4 F_0^2 + \frac{928}{75\times 45}\varepsilon^8 F_0^4 - \varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4 F_0^8 - 4\lambda_0^2}\right] < \Omega^2$

$$< \omega_0^2\left[1 - \frac{8}{15}\varepsilon^4 F_0^2 + \frac{928}{75\times 45}\varepsilon^8 F_0^4 + \varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4 F_0^8 - 4\lambda_0^2}\right].$$

3.6 Conclusions.

In this chapter we have studied the linear oscillator driven at a frequency $(2/N)\omega_0$ where

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \cdots,$$

and N is integer. Using the modified Lindstedt-Poincaré method we have shown that in order to obtain the damped driven differential equation describing slow variation of the amplitude of oscillations, the "slow time" T should scale as $T = \varepsilon^{2N} t$ and the damping coefficient as $\lambda = \mu^{2N}\lambda_0$.

Using this scale for T, we have found the following equations for the amplitude of the main harmonic:

- $N = 1: \quad -2i\dfrac{da_0}{dT} + \alpha_1 a_0 + F_0 \bar{a}_0 - 2i\lambda_0 a_0 = 0,$

- $N = 2: \quad -2i\dfrac{da_0}{dT} + \left(\alpha_2 + \dfrac{2}{3}F_0^2\right) a_0 + F_0^2 \bar{a}_0 - 2i\lambda_0 a_0 = 0,$

- $N = 3: \quad -2i\dfrac{da_0}{dT} + \alpha_2 a_0 + \left(\dfrac{9}{8}\right)^2 F_0^3 \bar{a}_0 - 2i\lambda_0 a_0 = 0,$

- $N = 4: \quad -2i\dfrac{da_0}{dT} + \left(\alpha_4 + \dfrac{928}{75 \times 45}F_0^4\right) a_0 + \left(\dfrac{4}{3}\right)^2 F_0^4 \bar{a}_0 - 2i\lambda_0 a_0 = 0.$

In the case of the *undamped* linear oscillator parametric resonance occurs in the following "windows" of the driving frequency Ω:

- $N = 1: \quad \omega_0^2(1 - \varepsilon^2 F_0) < \Omega^2 < \omega_0^2(1 + \varepsilon^2 F_0),$

- $N = 2: \quad \omega_0^2\left(1 - \dfrac{5}{3}\varepsilon^4 F_0^2\right) < \Omega^2 < \omega_0^2\left(1 + \dfrac{1}{3}\varepsilon^4 F_0^2\right),$

- $N = 3: \quad \omega_0^2\left[1 - \dfrac{9}{16}\varepsilon^4 F_0^2 - \left(\dfrac{9}{8}\right)^2 \varepsilon^6 F_0^3\right] < \Omega^2 < \omega_0^2\left[1 - \dfrac{9}{16}\varepsilon^4 F_0^2 + \left(\dfrac{9}{8}\right)^2 \varepsilon^6 F_0^3\right],$

- $N = 4: \quad \omega_0^2\left[1 - \dfrac{8}{15}\varepsilon^4 F_0^2 - \dfrac{6928}{75 \times 45}\varepsilon^8 F_0^4\right] < \Omega^2 < \omega_0^2\left[1 - \dfrac{8}{15}\varepsilon^4 F_0^2 + \dfrac{5072}{75 \times 45}\varepsilon^8 F_0^4\right],$

As N increases, the interval of resonant frequencies narrows. The width of the resonance window in the case of the driving frequency $\sim \frac{2}{N}\omega_0$ is of order ε^{2N}. (This fact has already appeared in literature). It is therefore much more difficult to achieve parametric resonance at low driving frequencies ($\frac{2}{N}\omega_0$ with N large) in the sense that one has to tune the frequency more accurately there.

When we pass to the *damped* linear oscillator, the above resonance windows are modified to

- $N = 1: \quad \omega_0^2\left[1 - \varepsilon^2\sqrt{F_0^2 - 4\lambda_0^2}\right] < \Omega^2 < \omega_0^2\left[1 + \varepsilon^2\sqrt{F_0^2 - 4\lambda_0^2}\right],$

- $N=2:\quad \omega_0^2\left[1-\frac{2}{3}\varepsilon^4F_0^2-\varepsilon^4\sqrt{F_0^4-4\lambda_0^2}\right]<\Omega^2<\omega_0^2\left[1-\frac{2}{3}\varepsilon^4F_0^2+\varepsilon^4\sqrt{F_0^4-4\lambda_0^2}\right],$

- $N=3:\quad \omega_0^2\left[1-\frac{9}{16}\varepsilon^4F_0^2-\varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4F_0^6-4\lambda_0^2}\right]<\Omega^2$

$$<\omega_0^2\left[1-\frac{9}{16}\varepsilon^4F_0^2+\varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4F_0^6-4\lambda_0^2}\right],$$

- $N=4:\quad \omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2+\frac{928}{75\times 45}\varepsilon^8F_0^4-\varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4F_0^8-4\lambda_0^2}\right]<\Omega^2$

$$<\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2+\frac{928}{75\times 45}\varepsilon^8F_0^4+\varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4F_0^8-4\lambda_0^2}\right].$$

In the next chapter we shall compare the above results with conclusions of the method of multiple time scales and with results available in literature.

Chapter 4

The parametrically driven, damped linear oscillator: the method of multiple scales.

In chapter 3 we appied the modified Lindstedt-Poincaré method to the parametrically driven, damped linear oscillator equation:

$$\varphi_{tt} + 2\lambda\varphi_t + \omega_0^2\left[1 - 2\varepsilon^2 F_0 \cos\left(\frac{2}{N}\right)\Omega t\right]\varphi = 0. \tag{4.1}$$

From the discussion in chapter 2, we know that the Lindstedt-Poincaré method gives limited results, that is, only periodic solutions although we modified it in the previous chapter to include not only periodic solutions. In this chapter we apply the multiple-scales technique to the linear oscillator and compare results with those obtained by means of the Lindstedt-Poincaré method. We shall follow the multiple- scales as used by Nayfeh [21, 22, 23] to find the secular free expansions of the solution of the parametrically driven linear oscillator.

A special case of the linear oscillator (the Mathieu equation for N integer), has been studied in literature (e.g. Arnold [2], Grimshaw [9], Nayfeh [21, 23] Jordan and Smith [12].) Here we shall revisit it to prepare for the study of the parametrically driven damped sine-Gordon equation.

We begin with the undamped case and proceed as follows: By first rewriting equation (4.1) in the form

$$\varphi_{tt} + \Omega^2\varphi = (\Omega^2 - \omega_0^2)\varphi + 2\omega_0^2 F_0\varepsilon^2 \cos\left(\frac{2}{N}\Omega t\right)\varphi \tag{4.2}$$

and making the transformation $\tau = \Omega t$, (4.1) becomes

$$\varphi_{\tau\tau} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + 2\frac{\omega_0^2}{\Omega^2}F_0\varepsilon^2 \cos\left(\frac{2}{N}\tau\right)\varphi. \tag{4.3}$$

We denote $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$ and assume that

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \cdots \tag{4.4}$$

so that (4.3) becomes

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \cdots)\varphi + 2\mu^2 F_0 \cos\left(\frac{2}{N}\tau\right)\varphi. \tag{4.5}$$

Solution of this equation can be looked for as a series expansion in powers of μ^2:

$$\varphi(\tau;\mu) = \varphi_0(\tau) + \mu^2\varphi_1(\tau) + \mu^4\varphi_2(\tau) + \cdots. \tag{4.6}$$

Recalling that the multiple-scales method takes into account the existence of processes with different time scales, we introduce multiple variables $T_n = \mu^n\tau$ so that $\varphi(\tau) = \varphi(T_0, T_1, T_2, \ldots)$. By the chain rule we have

$$\begin{aligned} & [D_0^2 + 2\mu D_0 D_1 + \mu^2(D_1^2 + 2D_0D_2) + \mu^3(2D_0D_3 + 2D_1D_2) + \mu^4(D_2^2 + \\ + \; & 2D_0D_4 + 2D_1D_3) + \mu^5(2D_0D_5 + 2D_1D_4 + 2D_2D_3) + \mu^6(D_3^2 + 2D_0D_6 + \\ + \; & 2D_1D_5 + 2D_2D_4) + \mu^7(D_4^2 + 2D_1D_6 + 2D_2D_5 + 2D_3D_4) + \mu^8(D_4^2 + \\ + \; & 2D_0D_8 + 2D_1D_7 + 2D_2D_6 + 2D_3D_5) + \cdots + 1](\varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots) = \\ = \; & (\alpha_1\mu^2 + \alpha_2\mu^4 + \cdots)(\varphi_0 + \mu^2\varphi_1 + \cdots) + 2\mu^2 F_0 \cos\left(\frac{2}{N}T_0\right)(\varphi_0 + \mu^2\varphi_1 + \cdots), \end{aligned} \tag{4.7}$$

where $D_n = \partial/\partial T_n$. We shall consider the system (4.5) in various frequency regimes and use the multiple-scales method in the formulation of Nayfeh and Mook [23].

4.1 The driving frequency $\sim 2\omega_0$.

We consider eq.(4.5) and first assume that $N = 1$, the case of the principal resonance:

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \cdots)\varphi + 2\mu^2 F_0 \cos(2\tau)\varphi. \tag{4.8}$$

Using eq.(4.7) and comparing coefficients at same powers of μ, we obtain at the zeroth order

$$D_0^2\varphi_0 + \varphi_0 = 0,$$

the general solution of which is given by

$$\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c., \tag{4.9}$$

where *c.c.* denotes complex conjugate of the preceding term. At the order μ^1 we have

$$2D_0D_1\varphi_0 = 0,$$

which implies that

$$D_1 a_0 = 0. \tag{4.10}$$

Thus $a_0 = a_0(T_2, T_3, \ldots)$. At μ^2 we obtain the equation

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= -(D_1^2 + 2D_0D_2)\varphi_0 + \alpha_1\varphi_0 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1 a_0 + F_0\bar{a}_0)e^{iT_0} + F_0 a_0 e^{3iT_0} + c.c., \end{aligned} \tag{4.11}$$

where we have made use of (4.9). To ensure that the solution remains bounded, we set to zero coefficients of the secular terms, that is, coefficients of e^{iT_0}:

$$-2iD_2a_0 + \alpha_1 a_0 + F_0\bar{a}_0 = 0. \tag{4.12}$$

Equation (4.12) describes the evolution of the amplitude of the main harmonic φ_0. In terms of the original variables it becomes

$$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^2 F_0\bar{a}_0 = 0. \tag{4.13}$$

Below we will show that it exhibits resonant growth for α_1 in a certain range. Therefore, we do not need to go to higher order terms in the perturbation expansion.

Stability. Differentiate (4.12) with respect to T_2 and obtain

$$2i\ddot{a}_0 = \alpha_1\dot{a}_0 + F_0\dot{\bar{a}}_0,$$

where the overdot represents differentiation with respect to T_2. Using (4.12) to eliminate $\dot{a}_0$ and $\dot{\bar{a}}_0$, we get

$$-4\ddot{a}_0 = (\alpha_1^2 - F_0^2)a_0.$$

Letting $a_0 = e^{sT_2}$, we find

$$4s^2 = F_0^2 - \alpha_1^2.$$

Instability occurs when $s^2 > 0$, that is, when

$$|\alpha_1| < F_0,$$

or equivalently,

$$-F_0 < \alpha_1 < F_0. \tag{4.14}$$

Recalling that $\mu^2(\alpha + O(\mu^2)) = 1 - \frac{\omega_0^2}{\Omega^2}$ and that $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$, we have

$$\alpha_1 = \varepsilon^{-2}\left(\frac{\Omega^2}{\omega_0^2} - 1\right) + O(\varepsilon^2).$$

Substituting for α_1 in the inequality (4.14), we obtain the instability boundaries:

$$\omega_0^2[1 - \varepsilon^2 F_0 + O(\varepsilon^4)] < \Omega^2 < \omega_0^2[1 + \varepsilon^2 F_0 + O(\varepsilon^4)]. \tag{4.15}$$

Outside this region there is no parametric resonance.

Résumé: When the system is driven at a frequency $\sim 2\omega_0$, the amplitude evolves on the "slow" scale T_2 and the detuning, eq. (4.4), is of order μ^2. The resulting equation for the amplitude of the main harmonic is

$$-2i\frac{\partial a_0}{\partial T_2} + \alpha_1 a_0 + F_0\bar{a}_0 = 0.$$

The resonant growth occurs in the region (4.15).

4.2 The driving frequency $\sim \omega_0$.

We proceed to equation (4.5) with $N = 2$:

$$\varphi_{\tau\tau} + \varphi = (\alpha_1\mu^2 + \alpha_2\mu^4 + \cdots)\varphi + 2F_0\mu^2\cos(\tau)\varphi, \tag{4.16}$$

and assume expansion (4.6). At μ^0 we obtain

$$D_0^2\varphi_0 + \varphi_0 = 0,$$

whence

$$\varphi_0 = a_0 e^{iT_0} + \bar{a}_0 e^{-iT_0}, \tag{4.17}$$

where a_0 is a function of "slow times" $T_n = \mu^n t$, $n \geq 1$. At the order μ^1 we then obtain

$$2D_0D_1\varphi_0 = 0.$$

Substituting eq.(4.17) for φ_0, we get

$$D_1 a_0 = 0 \tag{4.18}$$

whence $a_0 = a_0(T_2, T_3, \ldots)$. At the order μ^2, we have the equation

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= -(D_1^2 + 2D_0D_2)\varphi_0 + \alpha_1\varphi_0 + F_0(e^{iT_0} + e^{-iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1 a_0)e^{iT_0} + F_0(a_0e^{2iT_0} + a_0) + c.c. \end{aligned} \tag{4.19}$$

The condition for the absence of secular terms is

$$-2iD_2a_0 + \alpha_1 a_0 = 0. \tag{4.20}$$

This equation describes free (unforced) oscillations of the solution a_0. To detect the effect of parametric driving, we have to proceed to higher order terms. A particular solution of (4.19) is given by

$$\varphi_1 = a_1 e^{2iT_0} + b_1 + c.c., \tag{4.21}$$

where

$$a_1 = -\frac{1}{3}F_0a_0, \quad \text{and} \quad b_1 = F_0a_0. \tag{4.22}$$

Here we have ignored the homogeneous solution as explained in section 2.1.

Next, at order μ^3 we have

$$(2D_0D_3 + 2D_1D_2)\varphi_0 + 2D_0D_1\varphi_1 = 0.$$

If we substitute for φ_0 and φ_1 from (4.17) and (4.21), respectively, we find that

$$D_3a_0 = D_1a_1 = 0.$$

Thus $a_0 = a_0(T_2, T_4, T_5, \ldots)$ and $a_1 = a_1(T_2, T_3, \ldots)$. At order μ^4 we then get

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= -(D_2^2 + 2D_0D_4 + 2D_1D_3)\varphi_0 - (D_1^2 + 2D_0D_2)\varphi_1 + \alpha_1\varphi_1 + \\ &+ \alpha_2\varphi_0 + F_0(e^{iT_0} + e^{-iT_0})\varphi_1 \\ &= [-2iD_4a_0 - D_2^2a_0 + \alpha_2a_0 + F_0(a_1 + A + \bar{A})]e^{iT_0} + \\ &+ (\alpha_1a_1 - 4iD_2a_1)e^{2iT_0} + \alpha_1 A + F_0a_1e^{3iT_0} + c.c. \end{aligned} \tag{4.23}$$

Setting to zero coefficients of secular terms and substituting a_1 and b_1 from (4.22) we obtain

$$-2iD_4a_0 - D_2^2a_0 + \left(\alpha_2 + \frac{2}{3}F_0^2\right)a_0 + F_0^2\bar{a}_0 = 0 \tag{4.24}$$

By means of equation (4.20) we find that

$$D_2^2a_0 = -\frac{1}{4}\alpha_1^2a_0;$$

hence eq.(4.24) becomes

$$-2iD_4a_0 + \frac{1}{4}\alpha_1^2a_0 + \alpha_2a_0 + \frac{2}{3}F_0^2a_0 + F_0^2\bar{a}_0 = 0. \tag{4.25}$$

Here we notice that the amplitude does not depend on all "slow scales" but is in fact a function of T_2, T_4 and slower times: $a_0 = a_0(T_2, T_4, \ldots)$. Next, since the amplitude a_0 evolves on the scales T_2 and T_4, it becomes imperative that stability analysis should encompass the evolution on both time scales. It can be easily shown that eqs.(4.20) and (4.25) result from a multiple-scales expansion of

$$-2i\frac{\partial a_0}{\partial \tau} + \left(\mu^2\alpha_1 + \frac{1}{4}\mu^4\alpha_1^2 + \mu^4\alpha_2 + \frac{2}{3}F_0^2\mu^4\right)a_0 + F_0^2\mu^4\bar{a}_0 = 0. \tag{4.26}$$

(The idea of assembling amplitude equations pertaining to different time scales into one equation, belongs to Nayfeh and Mook [23].) In the next subsection we will find out when it exhibits parametric resonance.

Stability. Proceeding as in section (4.1), the solution of (4.26) is given by $a_0 = e^{s\tau}$ where

$$s^2 = -\frac{1}{4}\left[\left(\mu^2\alpha_1 + \frac{1}{4}\mu^4\alpha_1^2 + \mu^4\alpha_2 + \frac{2}{3}\mu^4F_0^2\right)^2 - \mu^8F_0^4\right]. \tag{4.27}$$

The condition for parametric resonance is $s^2 > 0$, that is,

$$\left|\mu^2\alpha_1 + \frac{1}{4}\mu^4\alpha_1^2 + \mu^4\alpha_2 + \frac{2}{3}\mu^4F_0^2\right| < \mu^4F_0^2.$$

Equivalently,

$$-\frac{5}{3}\mu^4F_0^2 < \mu^2\alpha_1 + \frac{1}{4}\mu^4\alpha_1^2 + \mu^4\alpha_2 < \frac{1}{3}\mu^4F_0^2,$$

so that when we compare coefficients of the same order of μ we obtain $\alpha_1 = 0$. Thus we are left with

$$-\frac{5}{3}F_0^2 < \alpha_2 < \frac{1}{3}F_0^2. \tag{4.28}$$

We recall that $\alpha_2 = \mu^{-4}\left(1 - \frac{\omega_0^2}{\Omega^2}\right)$ and $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$, and so the region of instability is given by (an implicit) inequality

$$\omega_0^2\left(1 - \frac{5}{3}\frac{\omega_0^2}{\Omega^2}\varepsilon^4F_0^2\right) < \Omega^2 < \omega_0^2\left(1 + \frac{1}{3}\frac{\omega_0^2}{\Omega^2}\varepsilon^4F_0^2\right). \tag{4.29}$$

Next, since $\omega_0^2/\Omega^2 = O(1)$, this becomes an implicit condition for Ω:

$$\omega_0^2\left[1-\frac{5}{3}\varepsilon^4F_0^2+O(\varepsilon^6)\right] < \Omega^2 < \omega_0^2\left[1+\frac{1}{3}\varepsilon^4F_0^2+O(\varepsilon^6)\right]. \tag{4.30}$$

There is no parametric resonance outside this region.

Discussion. When the system is driven at the frequency $\sim \omega_0$, the parametric resonance occurs when $\alpha_1 = 0$, which gives the detuning

$$1-\frac{\omega_0^2}{\Omega^2} = \alpha_2\mu^4+\cdots.$$

Consequently, the derivative with respect to T_2 vanishes (see eq.(4.20)). Thus the temporal dependence of a_0 is $a_0 = a_0(T_4,\ldots)$ and hence, the equation governing the evolution of the main harmonic can be rewritten in terms of T_4 as

$$-2i\frac{\partial a_0}{\partial T_4}+\left(\alpha_2+\frac{3}{2}F_0^2\right)a_0+F_0^2\bar{a}_0 = 0. \tag{4.31}$$

The instability region is given by (4.30).

4.3 The driving frequency $\sim (2/3)\omega_0$.

As before, we rewrite the equation (4.1) as

$$\varphi_{\tau\tau}+\varphi = \left(1-\frac{\Omega^2}{\omega_0^2}\right)\varphi+2\mu^2F_0\cos\left(\frac{2}{3}\tau\right)\varphi. \tag{4.32}$$

Substituting the expansions (4.4), (4.6) into (4.32) and using eq.(4.7) we obtain, at μ^0:

$$D_0^2\varphi_0+\varphi_0 = 0,$$

whose solution is given by

$$\varphi_0 = a_0e^{iT_0}+c.c.$$

with $a_0 = a_0(T_1, T_2,\ldots)$. In order that the equation of the main harmonic be driven, we must have the first derivative term appearing at the same order of μ as the complex conjugate term. (The latter acts as the driver.) Consequently, to determine the correct scaling of the detuning we should identify the order of μ at which the evolution (forced or unforced) takes place.

At the order μ^1 we have

$$2D_0D_1\varphi_0 = 0,$$

which gives us

$$D_1a_0 = 0. \tag{4.33}$$

At the order μ^2 we have

$$\begin{aligned} D_0^2\varphi_1+\varphi_1 &= -(D_1^2+2D_0D_2)\varphi_0+\alpha_1\varphi_0+F_0(e^{2/3iT_0}+e^{-2/3iT_0})\varphi_0 \\ &= [-2iD_2a_0+\alpha_1a_0]e^{iT_0}+F_0\left(a_0e^{i(1+2/3)T_0}+a_0e^{i(1-2/3)T_0}\right)+c.c. \end{aligned} \tag{4.34}$$

where $c.c.$ denotes complex conjugate of the preceeding term. The nonsecularity condition yields

$$-2iD_2a_0 + \alpha_1 a_0 = 0. \tag{4.35}$$

A particular solution of (4.34) is given by

$$\varphi_1 = a_1 e^{i(1+2/3)T_0} + b_1 e^{i(1-2/3)T_0} + c.c.,$$

where

$$a_1 = -\frac{9}{16}F_0 a_0, \quad b_1 = \frac{9}{8}F_0 a_0. \tag{4.36}$$

At the order μ^3 we obtain

$$2D_0D_1\varphi_1 + (2D_0D_3 + 2D_1D_2)\varphi_0 = 0,$$

whence

$$D_3 a_0 = 0. \tag{4.37}$$

Thus, $a_0 = a_0(T_2, T_4, T_5, \ldots)$.

At the order μ^4 we have

$$\begin{aligned}
D_0^2\varphi_2 + \varphi_2 &= \\
&= -(D_2^2 + 2D_0D_4)\varphi_0 - 2D_0D_2\varphi_1 + \alpha_1\varphi_1 + \alpha_2\varphi_0 + F_0(e^{2/3iT_0} + e^{-2/3iT_0})\varphi_1 \\
&= [-2iD_4a_0 - D_2^2a_0 + \alpha_2 a_0 + F_0(a_1 + b_1)]e^{iT_0} + \left(-\frac{10}{3}iD_2a_1 + \alpha_1 a_1\right)e^{i(1+2/3)T_0} + \\
&+ \left(-\frac{2}{3}iD_2b_1 + b_1\right)e^{i(1-2/3)T_0} + F_0(a_1e^{i(1+4/3)T_0} + b_1e^{i(1-4/3)T_0}) + c.c.
\end{aligned} \tag{4.38}$$

To suppress the secular terms we require

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2 a_0 + F_0(a_1 + b_1) = 0.$$

But $D_2^2 a_0 = -\frac{1}{4}\alpha_1^2 a_0$ and so invoking (4.36), this yields

$$-2iD_4a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{9}{16}F_0^2\right)a_0 = 0. \tag{4.39}$$

This equation tells us that the amplitude a_0 evolves with the characteristic time scale T_4. The evolution is undriven. A particular solution of (4.38) is then given by

$$\varphi_2 = a_2e^{i(1+4/3)T_0} + b_2e^{i(1+2/3)T_0} + c_2e^{i(1-2/3)T_0} + c.c.$$

where

$$a_2 = \frac{9\times 9}{40\times 16}F_0^2a_0, \quad b_2 = -\frac{9\times 3}{16\times 8}F_0\alpha_1 a_0, \quad c_2 = \frac{27}{32}F_0\alpha_1 a_0 + \left(\frac{9}{8}\right)^2 F_0^2\bar{a}_0. \tag{4.40}$$

Since we are interested in the resonant effects we must get the driven equation for the amplitude of the main harmonic. Thus we have to go to higher orders of μ.

At μ^5 we obtain

$$(2D_0D_5 + 2D_2D_3 + 2D_1D_4)\varphi_0 + (2D_0D_3 + 2D_1D_2)\varphi_1 + 2D_0D_1\varphi_2 = 0,$$

producing $D_5a_0 = 0$. Since $D_1a_0 = D_3a_0 = 0$ and a_1, b_1, a_2 and b_2 are proportional to a_0, their derivatives with respect to T_3 and T_1 vanish.

At the order μ^6 we have

$$\begin{aligned}
D_0^2\varphi_3 + \varphi_3 &= -(2D_0D_6 + 2D_2D_4)\varphi_0 - (D_2^2 + 2D_0D_4)\varphi_1 - 2D_0D_2\varphi_2 + \alpha_1\varphi_2 + \\
&+ \alpha_2\varphi_1 + \alpha_3\varphi_0 + F_0(e^{(2/3)iT_0} + e^{-(2/3)iT_0})\varphi_2 \\
&= [-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 + F_0(b_2 + c_2)]e^{iT_0} + \\
&+ \left[-D_2^2a_1 - \frac{10}{3}i(D_4a_1 + D_2b_2) + \alpha_1b_2 + \alpha_2a_1 + F_0a_2\right]e^{i(1+2/3)T_0} + \\
&+ \left[-D_2^2b_1 - \frac{2}{3}i(D_4b_1 + D_2c_2) + \alpha_2b_1 + \alpha_1c_2 + F_0\bar{c}_2\right]e^{i(1-2/3)T_0} + \\
&+ \left[-\frac{14}{2}iD_2a_2 + \alpha_1a_2 + F_0b_2\right]e^{i(1+4/3)T_0} + F_0a_2e^{i(1+6/3)T_0} + c.c. \qquad (4.41)
\end{aligned}$$

For nonsecularity we require that

$$-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 + F_0(b_2 + c_2) = 0. \qquad (4.42)$$

But

$$D_2D_4a_0 = -\frac{1}{4}\alpha_1\alpha_2a_0 - \frac{9}{64}F_0^2\alpha_1a_0 + \frac{1}{16}\alpha_1^3a_0;$$

substituting this equation and eqs.(4.40) into (4.42), we obtain

$$-2iD_6a_0 + \left(\frac{1}{2}\alpha_1\alpha_2 + \frac{117}{128}F_0^2\alpha_1 + \frac{1}{8}\alpha_1^3 + \alpha_3\right)a_0 + \left(\frac{9}{8}\right)^2 F_0^3\bar{a}_0 = 0. \qquad (4.43)$$

Eq.(4.43) includes the desired complex conjugate term which originates from the term $b_2\exp\left\{i\left(1-\frac{6}{3}\right)T_0\right\} = b_2\exp(-iT_0)$ of eq.(4.41). That is,

$$b_2\exp\left\{i\left(1 - \frac{2}{3} - \frac{2}{3} - \frac{2}{3}\right)T_0\right\} = b_2\exp(-iT_0)$$

or $b_2\exp\left\{i\left(1-m\frac{2}{3}\right)T_0\right\} = b_2\exp(-iT_0)$, where $m = 3$. We see that in order that the equation of the main harmonic be driven, the complex conjugate quantity must appear at the same order of μ as the time derivative. In the case at hand, this order is $\mu^{2m} = \mu^6 = \mu^{2N}$. This is the general rule. In the case of general N the complex conjugate term should appear at the order μ^{2N}. (In particular, when $N = 1$ and 2, the driving term is to appear at the order μ^2 and μ^4 respectively; in the previous sections we saw that this is indeed the case.) In the next section, we shall show that when $N = 4$ the driving term has to arise at the order $\mu^8 = \mu^{2N}$.

At this stage we realise that the amplitude a_0 evolves on three time scales T_2, T_4 and T_6. Although the driving is effective only on the scale T_6, taking into account the evolution with respect to T_2 and T_4, helps us to fix the coefficients α_1 and α_2.

Thus in order to derive the amplitude equation we need to make use of all the three evolution equations, (4.35), (4.39) and (4.43). It is straightforward to show that these three equations result from a multiple-scale expansion of

$$\begin{aligned}
&-2i\frac{\partial a_0}{\partial \tau}+\mu^2\alpha_1 a_0+\mu^4\left(\alpha_2+\frac{1}{4}\alpha_1^2+\frac{9}{16}F_0^2\right)a_0+\\
&+\ \mu^6\left(\frac{1}{2}\alpha_1\alpha_2+\frac{1}{8}\alpha_1^3+\alpha_3+\frac{117}{128}\alpha_1F_0^2\right)a_0+\left(\frac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0=0.
\end{aligned}\tag{4.44}$$

Stability. Proceeding in a standard way we obtain the solution of (4.44): $a_0 = \exp(s\tau)$, where

$$\begin{aligned}
-4s^2 &= \left[\alpha_1\mu^2+\mu^4\left(\alpha_2+\frac{1}{4}\alpha_1^2+\frac{9}{16}F_0^2\right)+\mu^6\left(\frac{1}{2}\alpha_1\alpha_2+\frac{1}{8}\alpha_1^3+\alpha_3+\frac{117}{128}\alpha_1F_0^2\right)\right]^2-\\
&- \left(\frac{9}{8}\right)^4F_0^6\mu^{12}.
\end{aligned}$$

Resonance occurs when $s^2>0$, that is when

$$\left|\alpha_1\mu^2+\mu^4\left(\alpha_2+\frac{1}{4}\alpha_1^2+\frac{9}{16}F_0^2\right)+\mu^6\left(\frac{1}{2}\alpha_1\alpha_2+\frac{1}{8}\alpha_1^3+\alpha_3+\frac{117}{128}\alpha_1F_0^2\right)\right|<\left(\frac{9}{8}\right)^2F_0^3\mu^6.$$

Equivalently,

$$\alpha_1\mu^2+\mu^4\left(\alpha_2+\frac{1}{4}\alpha_1^2\right)+\mu^6\left(\frac{1}{2}\alpha_1\alpha_2+\frac{1}{8}\alpha_1^3+\alpha_3+\frac{117}{128}\alpha_1F_0^2\right)>-\left(\frac{9}{8}\right)^2F_0^3a_0\mu^6-\frac{9}{16}F_0^2\mu^4,$$

and should hold together with

$$\alpha_1\mu^2+\mu^4\left(\alpha_2+\frac{1}{4}\alpha_1^2\right)+\mu^6\left(\frac{1}{2}\alpha_1\alpha_2+\frac{1}{8}\alpha_1^3+\alpha_3+\frac{117}{128}\alpha_1F_0^2\right)<\left(\frac{9}{8}\right)^2F_0^3\mu^6-\frac{9}{16}F_0^2\mu^4.$$

By comparing coefficients of same order of μ we find that $\alpha_1=0$; $\alpha_2=-\frac{9}{16}F_0^2$ and

$$-\left(\frac{9}{8}\right)^2F_0^3<\alpha_3<\left(\frac{9}{8}\right)^2F_0^3.\tag{4.45}$$

Thus by (4.35) and (4.39), $a_0=a_0(T_6,\ldots)$. Recalling that $1-\frac{\omega_0^2}{\Omega^2}=\alpha_1\mu^2+\alpha_2\mu^4+\alpha_3\mu^6\cdots$ we then have

$$\omega_0^2\left[1-\frac{9}{16}\varepsilon^4F_0^2\frac{\omega_0^2}{\Omega^2}-\left(\frac{9}{8}\right)^2\varepsilon^6F_0^3\frac{\omega_0^4}{\Omega^4}\right]<\Omega^2<\omega_0^2\left[1-\frac{9}{16}\varepsilon^4F_0^2\frac{\omega_0^2}{\Omega^2}+\left(\frac{9}{8}\right)^2\varepsilon^6F_0^3\frac{\omega_0^4}{\Omega^4}\right].$$

Since $\frac{\omega_0^2}{\Omega^2}=1+O(\mu^4)$, the resonance window is given by

$$1-\frac{9}{16}\varepsilon^4F_0^2-\left(\frac{9}{8}\right)^2\varepsilon^6F_0^3+O(\varepsilon^8)<\frac{\Omega^2}{\omega_0^2}<1-\frac{9}{16}\varepsilon^4F_0^2+\left(\frac{9}{8}\right)^2\varepsilon^6F_0^3+O(\varepsilon^8).\tag{4.46}$$

Notice that the term $-\frac{9}{16}F_0^2$ is important in providing the boundaries of the instability domain. This term results from the unforced evolution on the time scale T_4.

Résumé. When $N = 3$, the parametric resonance occurs for the detunings

$$1 - \frac{\omega_0^2}{\Omega^2} = -\frac{9}{16}F_0^2\mu^4 + \alpha_3\mu^6 + \cdots.$$

In the resonant case the equation of the main harmonic eq. (4.44) can be written in terms of T_6 as

$$-2i\frac{\partial a_0}{\partial T_6} + \alpha_3 a_0 + \left(\frac{9}{8}\right)^2 F_0^3\bar{a}_0 = 0, \tag{4.47}$$

where $a_0 = a_0(T_6, \ldots)$. The resonance window is given by eq.(4.46).

4.4 The driving frequency $\sim (2/4)\omega_0$.

In this section

$$\varphi_{\tau\tau} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + 2\mu^2 F_0 \cos\left(\frac{2}{4}\Omega t\right)\varphi. \tag{4.48}$$

We let

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots \tag{4.49}$$

and expand φ as in (4.6). In accordance with the result found in the previous section, we expect that the driver will appear at the order μ^8.

Using multiple scales and comparing coefficients from eq.(4.7) we obtain at $O(\mu^0)$,

$$\varphi_0 = a_0(T_1, T_2, T_3, \ldots)e^{iT_0} + c.c.$$

At order μ^1 we obtain

$$2D_0D_1\varphi_0 = 0,$$

which yields $D_1a_0 = 0$. At order μ^2, we have

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= -2D_0D_2\varphi_0 + \alpha_1\varphi_0 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1 a_0)e^{iT_0} + F_0(a_0e^{i(1+2/4)T_0} + a_0e^{i(1-2/4)T_0}) + c.c. \end{aligned} \tag{4.50}$$

For nonsecularity we require

$$-2iD_2a_0 + \alpha_1 a_0 = 0 \tag{4.51}$$

and a particular solution of eq.(4.50) is given by

$$\varphi_1 = a_1e^{i(1+2/4)T_0} + b_1e^{i(1-2/4)T_0} + c.c.$$

where

$$a_1 = -\frac{4}{5}F_0a_0, \quad b_1 = \frac{4}{3}F_0a_0. \tag{4.52}$$

Order μ^3 yields

$$(2D_0D_3 + D_1D_2)\varphi_0 + 2D_0D_1\varphi_1 = 0,$$

whence $D_3 a_0 = 0$. At the order μ^4 we get

$$\begin{aligned}
D_0^2\varphi_2 + \varphi_2 &= -(D_2^2 + 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + F_0(e^{i(2/4)T_0} + e^{i(-2/4)T_0})\varphi_1 \\
&= [-2iD_4a_0 - D_2^2a_0 + \alpha_2a_0 + F_0(a_1 + b_1)]e^{iT_0} + \\
&+ (-3iD_2a_1 + \alpha_1a_1)e^{i(1+2/4)T_0} + (-iD_2b_1 + \alpha_1b_1)e^{i(1-2/4)T_0} + \\
&+ F_0[a_1e^{i(1+4/4)T_0} + b_1e^{i(1-4/4)T_0}] + c.c.
\end{aligned}$$

whose nonsecularity condition is given by

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2a_0 + F_0(a_1 + b_1) = 0,$$

or by using (4.51) and (4.52),

$$-2iD_4a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right)a_0 = 0. \tag{4.53}$$

φ_2 is given by

$$\varphi_2 = a_2e^{i(1+4/4)T_0} + b_2e^{i(1-4/4)T_0} + c_2e^{i(1+2/4)T_0} + d_2e^{i(1-2/4)T_0} + c.c.,$$

where

$$a_2 = \frac{4}{15}F_0^2a_0, \quad b_2 = \frac{4}{3}F_0^2a_0, \quad c_2 = -\frac{8}{25}F_0\alpha_1a_0, \quad d_2 = \frac{8}{9}F_0\alpha_1a_0. \tag{4.54}$$

At order μ^5 the perturbation expansion gives

$$(2D_0D_5 + 2D_2D_3 + 2D_1D_4)\varphi_0 + (2D_0D_3 + 2D_1D_2)\varphi_1 + 2D_0D_1\varphi_2 = 0.$$

The result of this equation is that $D_5a_0 = 0$. Thus $a_0 = a_0(T_2, T_4, T_6, \ldots)$. Proceeding to higher order terms, we have at $O(\mu^6)$ the following:

$$\begin{aligned}
D_0^2\varphi_3 + \varphi_3 &= (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_0 + (-D_2^2 + 2D_0D_4 + \alpha_2)\varphi_1 + \\
&+ (-2D_0D_2 + \alpha_1)\varphi_2 + F_0(e^{2/4iT_0} + e^{-2/4iT_0})\varphi_2 \\
&= (-2iD_6a_0 - 2D_2D_4 + \alpha_3a_0)e^{iT_0} + (-D_2^2a_1 - 3iD_4a_1 + \alpha_1a_1 - \\
&- 3iD_2c_2 + \alpha_2a_1 + \alpha_1c_2 + F_0a_2)e^{i(1+2/4)T_0} + [-D_2^2\bar{b}_1 - iD_4\bar{b}_1 + \alpha_2\bar{b}_1 + \\
&+ \alpha_1\bar{d}_2 - iD_2\bar{d}_2 + F_0(b_2 + \bar{b}_2)]e^{i(1-6/4)T_0} + (-4iD_2a_2 + \alpha_1a_2 + F_0c_2) \times \\
&\times e^{i(1+4/4)T_0} + (\alpha_1b_2 + F_0d_2)e^{i(1-4/4)T_0} + F_0a_2e^{i(1+6/4)T_0} + c.c.
\end{aligned} \tag{4.55}$$

To suppress secular terms, the coefficient of the term proportional to e^{iT_0} must be set to zero. Thus

$$-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 + F_0(c_2 + d_2) = 0. \tag{4.56}$$

Using (4.51), (4.53) and (4.54) we obtain

$$-2iD_6a_0 + \left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \alpha_3 + \frac{188}{15\times 15}F_0^2\alpha_1\right)a_0 = 0. \tag{4.57}$$

A particular solution at order μ^6 is given by

$$\varphi_3 = a_3e^{i(1+6/4)T_0} + b_3e^{i(1-6/4)T_0} + c_3e^{i(1+4/4)T_0} + d_3e^{i(1-4/4)T_0} + e_3e^{i(1+2/4)T_0} + c.c.,$$

where

$$a_3 = -\frac{4 \times 4}{21 \times 15} F_0^3 a_0, \quad b_3 = \left(\frac{4}{3}\right)^2 F_0 \left[\frac{11}{24}\alpha_1^2 + \frac{1}{2}\alpha_2 + \frac{11}{15}F_0^2\right] \bar{a}_0 + \left(\frac{4}{3}\right)^2 F_0^3 a_0,$$

$$c_3 = \frac{4 \times 11}{15 \times 15} F_0^2 \alpha_1 a_0, \quad d_3 = \frac{4}{3} F_0^2 \alpha_1 a_0,$$

$$e_3 = -\left(\frac{4}{5}\right)^2 F_0 \left[\frac{13}{40}\alpha_1^2 - \frac{1}{2}\alpha_2 - \frac{17}{15}F_0^2\right] a_0.$$

At order μ^7 we obtain

$$\begin{aligned} & (2D_0D_7 + 2D_1D_6 + 2D_2D_5 + 2D_3D_4)\varphi_0 + (2D_0D_5 + 2D_2D_3 + 2D_1D_4)\varphi_1 + \\ + \ & (2D_0D_3 + 2D_1D_2)\varphi_2 + 2D_0D_1\varphi_3 = 0, \end{aligned}$$

which gives $D_7 a_0 = 0$. Notice that we have not yet found an equation that describes the forced evolution of the amplitude. Thus we should proceed further to higher order approximations. At order μ^8, we have

$$\begin{aligned} D_0^2\varphi_4 + \varphi_4 &= -(D_4^2 + 2D_0D_8 + 2D_2D_6)\varphi_0 - (2D_0D_6 + 2D_2D_4)\varphi_1 - (D_2^2 + 2D_0D_4)\varphi_2 - \\ &- 2D_0D_2\varphi_3 + \alpha_1\varphi_3 + \alpha_2\varphi_2 + \alpha_3\varphi_1 + \alpha_4\varphi_0 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_3 \\ &= [-D_4^2 a_0 - 2iD_8 a_0 - 2D_2D_6 + \alpha_4 a_0 + F_0(\bar{b}_3 + e_3)]e^{iT_0} + [-3iD_6 a_1 - \\ &- 2D_2D_4 a_1 + \alpha_3 a_1 - D_2^2 c_2 - 3iD_4 c_2 + \alpha_2 c_2 - 3iD_2 e_3 + \alpha_1 e_3 + F_0 c_3] \times \\ &\times e^{i(1+2/4)T_0} + [-iD_6 b_1 - 2D_2D_4 + \alpha_3 b_1 + 2iD_2\bar{b}_3 + \alpha_1\bar{b}_3 - D_2^2 d_2 - iD_4 d_2 + \\ &+ F_0(d_3 + \bar{d}_3)]e^{i(1-2/4)T_0} + [-D_2^2 a_2 - 4iD_4 a_2 + \alpha_2 a_2 + F_0(a_3 + e_3) - \\ &- 4iD_2 c_3 + \alpha_1 c_3]e^{i(1+4/4)T_0} + [-D_2^2 b_2 + \alpha_2 b_2 + \alpha_1 d_3 + F_0 b_3]e^{i(1-4/4)T_0} + \\ &+ [-5iD_2 a_3 + \alpha_1 a_3 + F_0 c_3]e^{i(1+6/4)T_0} + F_0 a_3 e^{i(1+8/4)T_0} + c.c. \end{aligned} \tag{4.58}$$

Putting coefficients of secular terms to zero yields

$$-2iD_8 a_0 - D_4^2 a_0 - 2D_2D_6 a_0 + \alpha_4 a_0 + F_0(\bar{b}_3 + e_3) = 0. \tag{4.59}$$

Using (4.51), (4.53) and (4.57) we obtain

$$\begin{aligned} & -2iD_8 a_0 + \left(\alpha_4 + \frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 - \frac{164}{45 \times 25}\alpha_1^2 F_0^2 + \frac{1}{4}\alpha_2^2 + \right. \\ + \ & \left. \frac{1}{2}\alpha_1\alpha_3 + \frac{188}{15 \times 15}\alpha_2 F_0^2 + \frac{2192}{25 \times 135}F_0^4\right) a_0 + \left(\frac{4}{3}\right)^2 F_0^4 \bar{a}_0 = 0, \end{aligned} \tag{4.60}$$

which is a driven equation that was required to determine the effect of the parametric driving on the solution a_0. As we have shown, the driver appears at the order $\mu^8 = \mu^{2N}$. We shall show how the coefficients α_i are fixed when we analyse stability boundaries. In order to understand the evolution of the amplitude at different time scales, all the equations (4.51), (4.53), (4.57) and (4.60) must be taken into consideration. (This is the key idea of the method of multiple scales in the formulation by Nayfeh and Mook [23]. It is straightforward to show that all these equations result from a multiple-scales expansion of the equation

$$\begin{aligned} -2i\frac{\partial a_0}{\partial\tau} &+ \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right)a_0 + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 - \frac{34}{75}F_0^2\alpha_1 + \right. \\ &+ \left.\alpha_3\right)a_0 + \mu^8\left(\frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{1}{4}\alpha_2^2 - \frac{164}{45\times 25}F_0^2\alpha_1^2 + \frac{188}{15\times 15}F_0^2\alpha_2 + \right. \\ &+ \left. + \frac{1}{2}\alpha_1\alpha_3 + \alpha_4 + \frac{2192}{25\times 135}F_0^4\right)a_0 + \left(\frac{4}{3}\right)^2 F_0^4\mu^8\bar{a}_0 = 0. \end{aligned} \tag{4.61}$$

Stability. To analyse stability of the solution of (4.61) we differentiate this equation with respect to τ to obtain

$$\begin{aligned} 2i\frac{\partial^2 a_0}{\partial\tau^2} &= \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right) + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \alpha_3\right) + \right. \\ &+ \mu^8\left(\frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{1}{4}\alpha_2^2 - \frac{164}{45\times 25}F_0^2\alpha_1^2 + \frac{188}{15\times 15}F_0^2\alpha_2 + \right. \\ &+ \left.\left. \frac{1}{2}\alpha_1\alpha_3 + \alpha_4 + \frac{2192}{25\times 135}F_0^4\right)\right]\frac{\partial a_0}{\partial\tau} + \left(\frac{4}{3}\right)^2\mu^8 F_0^4\frac{\partial\bar{a}_0}{\partial\tau}. \end{aligned} \tag{4.62}$$

Let $\mathcal{L}$ denote the coefficient of $\dot{a}_0$ in the above equation. Then, using (4.61) we obtain a second-order differential equation of the form

$$-4\ddot{a}_0 = \mathcal{L}^2 a_0 - \left(\frac{4}{3}\right)^4\mu^{16}F_0^8 a_0,$$

whose solution $a_0 = e^{s\tau}$ is unstable when $s^2 > 0$, that is when

$$|\mathcal{L}| < \left(\frac{4}{3}\right)^2\mu^8 F_0^4.$$

Equivalently,

$$\begin{aligned} -\left(\frac{4}{3}\right)^2\mu^8 F_0^4 &< \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right) + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \alpha_3 - \frac{34}{75}F_0^2\alpha_1\right)\right. \\ &+ \mu^8\left(\frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{1}{4}\alpha_2^2 - \frac{164}{45\times 25}\alpha_1^2F_0^2 + \frac{188}{15\times 15}F_0^2\alpha_2 + \right. \\ &+ \left.\left. \frac{1}{2}\alpha_1\alpha_3 + \alpha_4 + \frac{2192}{25\times 135}F_0^4\right)\right] \\ &< \left(\frac{4}{3}\right)^2\mu^8 F_0^4. \end{aligned} \tag{4.63}$$

Comparing coefficients of same powers of μ we obtain at μ^2 that $\alpha_1 = 0$. At the order μ^4 we then obtain

$$\alpha_2 = -\frac{8}{15}F_0^2, \tag{4.64}$$

and at the order μ^6 we have $\alpha_3 = 0$. Order μ^8 gives us the range of instability as

$$-\frac{6928}{75\times 45}F_0^4 < \alpha_4 < \frac{5072}{75\times 45}F_0^4, \tag{4.65}$$

where we have made use of eq.(4.64).

Recalling that

$$1-\frac{\omega_0^2}{\Omega^2}=-\frac{8}{15}F_0^2\mu^4+\alpha_4\mu^8, \tag{4.66}$$

we then have

$$\left[1+\frac{8}{15}F_0^2\mu^4-\frac{5072}{75\times 45}F_0^4\mu^8\right]<\frac{\omega_0^2}{\Omega^2}<\left[1+\frac{8}{15}\mu^4F_0^2+\frac{6928}{75\times 45}F_0^4\mu^8\right].$$

Using the binomial series and recalling that $\mu^2=\frac{\omega_0^2}{\Omega^2}\varepsilon^2$ we obtain the resonance window

$$\omega_0^2\left[1-\frac{8}{15}\frac{\omega_0^4}{\Omega^4}\varepsilon^4F_0^2-\frac{5968}{75\times 45}\frac{\omega_0^8}{\Omega^8}\varepsilon^8F_0^4\right]<\Omega^2<\omega_0^2\left[1-\frac{8}{15}\frac{\omega_0^4}{\Omega^4}\varepsilon^4F_0^2+\frac{6032}{75\times 45}\frac{\omega_0^8}{\Omega^8}\varepsilon^8F_0^4\right].$$

Since

$$\frac{\omega_0^2}{\Omega^2}=1-\frac{8}{15}F_0^2\mu^4+O(\mu^8),$$

we can rewrite the instability condition as

$$\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2\left(1+\frac{8}{15}\frac{\omega_0^4}{\Omega^4}\varepsilon^4\right)-\frac{5968}{75\times 45}F_0^4\varepsilon^8(1+O(\varepsilon^8))+\right]<$$

$$<\ \Omega^2<\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2\left(1+\frac{8}{15}F_0^2\varepsilon^4\frac{\omega_0^4}{\Omega^4}\right)+\frac{6032}{75\times 45}\varepsilon^8F_0^4(1+O(\varepsilon^4))\right].$$

Finally,

$$\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2-\frac{6928}{75\times 45}\varepsilon^8F_0^4+O(\varepsilon^{10})\right]<\Omega^2<\omega_0^2\left[1-\frac{8}{15}\varepsilon^4F_0^2+\frac{5072}{75\times 45}\varepsilon^8F_0^4+O(\varepsilon^{10})\right]. \tag{4.67}$$

Discussion. When the system is driven at the frequency $\sim(2/4)\omega_0$, the parametric resonance occurs when $\alpha_1=\alpha_3=0$ and the detuning

$$1-\frac{\omega_0^2}{\Omega^2}=-\frac{8}{15}F_0^2\mu^4+\alpha_4\mu^8+\cdots.$$

Thus, the detuning must be of order μ^4. However, if we limit ourselves to the order μ^4 in the expansion, the resulting equation of the amplitude of the main harmonic is unforced. The driven equation only arises at the order $\mu^8=\mu^{2N}$, in agreement with the general principle explained in the previous section.

Since the perturbation parameter μ enters the original equation (4.1) only as μ^2 we could have used only $T_n=\mu^n\tau$ with n even, so that the slow time scales are defined by powers of μ^2 instead of μ. (Indeed, we have seen that all derivatives with respect to "odd" times, i.e. $T_n=\mu^n\tau$ with n odd, vanish.)

If we substitute α_1,α_2 and α_3 in (4.51), (4.53), (4.57) and (4.60), we obtain $D_2a_0=D_4a_0=D_6a_0$, and the equation for the amplitude of the main harmonic is simplified to

$$-2i\frac{\partial a_0}{\partial T_8}+\left(\alpha_4+\frac{928}{25\times 135}F_0^4\right)a_0+\left(\frac{4}{3}\right)^2\mu^8F_0^4\bar{a}_0=0. \tag{4.68}$$

4.5 The effect of damping.

So far in this chapter we ignored dissipative effects. Let us now include damping so that the equation reads

$$\varphi_{tt} + 2\lambda\varphi_t + \omega_0^2\left[1 - 2F_0\varepsilon^2\cos\left(\frac{2}{N}\Omega t\right)\right]\varphi = 0. \tag{4.69}$$

We shall exemplify the effects of damping by treating the case $N = 2$. As before we make a transformation $\tau = \Omega t$ which yields

$$\varphi_{\tau\tau} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + 2\frac{\omega_0^2}{\Omega^2}F_0\varepsilon^2\cos(\tau)\varphi - 2\lambda\Omega\varphi_\tau, \tag{4.70}$$

and define $\mu = (\omega_0/\Omega)\varepsilon$, $1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \cdots$. Since we are interested in the solution that will exist for a long time we assume that the damping is small:

$$\lambda\Omega = \mu^2(\lambda_0 + \mu\lambda_1 + \mu^2\lambda_2 + \cdots).$$

By using multiple scales as before, and comparing coefficients at like powers of μ, we obtain at order μ^0 the equation

$$D_0^2\varphi_0 + \varphi_0 = 0,$$

whose solution is

$$\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$$

At order μ^1 we have

$$2D_0D_1\varphi_0 = 0.$$

Upon the substitution of φ_0, we then have $D_1a_0 = 0$. At the order μ^2 we get

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= -2D_0D_2\varphi_0 + \alpha_1\varphi_0 + F_0(e^{iT_0} + e^{-iT_0})\varphi_0 - 2\lambda_0D_0\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1a_0 - 2i\lambda_0a_0)e^{iT_0} + F_0(a_0e^{2iT_0} + a_0) + c.c. \end{aligned} \tag{4.71}$$

The nonsecularity condition gives

$$-2iD_2a_0 + \alpha_1a_0 - 2i\lambda_0a_0 = 0. \tag{4.72}$$

A particular solution of (4.71) is given by

$$\varphi_1 = a_1e^{2iT_0} + b_1 + c.c.,$$

where

$$a_1 = -\frac{1}{3}F_0a_0, \quad b_1 = F_0a_0. \tag{4.73}$$

The order μ^3 gives

$$2D_0D_3\varphi_0 + 2\lambda_1D_0\varphi_0 = 0,$$

whence

$$-2iD_3a_0 - 2i\lambda_1a_0 = 0. \tag{4.74}$$

At the order μ^4 we obtain

$$\begin{aligned}
D_0^2\varphi_2+\varphi_2 &= -(D_2^2+2D_0D_4)\varphi_0-2D_0D_2\varphi_1+\alpha_1\varphi_1+\alpha_2\varphi_0+F_0(e^{iT_0}+e^{-iT_0})\varphi_1- \\
&- 2\lambda_2D_0\varphi_0-2\lambda_0D_0\varphi_1 \\
&= [-D_2^2a_0-2iD_4a_0+\alpha_2a_0-2i\lambda_2a_0+F_0(a_1+b_1+\bar{b}_1)]e^{iT_0}+ \\
&+ (-4iD_2a_1+\alpha_1a_1-4i\lambda_0a_1)e^{2iT_0}+\alpha_1b_1+F_0a_1e^{3iT_0}+c.c. \qquad (4.75)
\end{aligned}$$

For nonsecularity, we require

$$-D_2^2a_0-2iD_4a_0+\alpha_2a_0-2i\lambda_2a_0+F_0(a_1+b_1+\bar{b}_1)=0,$$

which upon substitution of

$$D_2^2a_0=-\frac{1}{4}\alpha_1^2a_0+i\alpha_1\lambda_0a_0+\lambda_0^2a_0,$$

becomes

$$-2iD_4a_0+\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{2}{3}F_0^2\right)a_0-i\alpha_1\lambda_0a_0-\lambda_0^2a_0-2i\lambda_2a_0+F_0^2\bar{a}_0=0, \qquad (4.76)$$

where we have also used (4.73). This equation is damped and driven. Thus we do not need to go to higher order approximations.

Equations (4.72), (4.74) and (4.76) result from the multiple scale expansion of

$$\begin{aligned}
&-2i\frac{\partial a_0}{\partial\tau}+\left[\mu^2\alpha_1+\mu^4\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{2}{3}F_0^2-\lambda_0^2\right)\right]a_0- \\
&- 2i\left[\mu^2\lambda_0+\mu^3\lambda_1+\mu^4\left(\frac{1}{2}\alpha_1\lambda_0+\lambda_2\right)\right]+\mu^4F_0^2\bar{a}_0=0. \qquad (4.77)
\end{aligned}$$

Writing

$$a_0(\tau)=A(\tau)\exp\left(-\mu^2\lambda_0-\mu^3\lambda_1-\frac{1}{2}\mu^4\alpha_1\lambda_0-\mu^4\lambda_2\right)\tau,$$

and substituting for a_0 in (4.77), we obtain the undamped equation for A,

$$-2i\frac{\partial A}{\partial\tau}+\mu^2\alpha_1A+\mu^4\left(\frac{1}{4}\alpha_1^2A-\lambda_0^2A+\alpha_2A+\frac{3}{2}F_0^2A\right)+\mu^4F_0^2\bar{A}=0, \qquad (4.78)$$

whose solution is given by $A=e^{s\tau}$ where

$$4s^2=F_0^4\mu^8-\left[\mu^2\alpha_1+\mu^4\left(\frac{1}{4}\alpha_1^2-\lambda_0^2+\alpha_2+\frac{3}{2}F_0^2\right)\right]^2. \qquad (4.79)$$

Thus

$$a_0=\exp\left[s-\left(\mu^2\lambda_0+\mu^3\lambda_1+\frac{1}{2}\mu^4\alpha_1\lambda_0+\mu^4\lambda_2\right)\right]\tau.$$

For resonance, we need to have

$$s^2>\left(\mu^2\lambda_0+\mu^3\lambda_1+\frac{1}{2}\mu^4\alpha_1\lambda_0+\mu^4\lambda_2\right)^2,$$

that is

$$\mu^8 F_0^4 - \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 - \lambda_0^2 + \alpha_2 + \frac{3}{2}F_0^2\right)\right]^2 > 4\left(\mu^2\lambda_0 + \mu^3\lambda_1 + \frac{1}{2}\mu^4\alpha_1\lambda_0 + \mu^4\lambda_2\right)^2 . \tag{4.80}$$

By comparing terms of same power of μ we obtain at the order μ^4,

$$-\alpha_1^2 \geq 4\lambda_0^2,$$

which can only hold for $\alpha_1 = \lambda_0 = 0$. At the order μ^6 we have that $4\lambda_1^2 \leq 0$, a condition which is satisfied only when $\lambda_1 = 0$. The order μ^8 gives

$$F_0^4 - \left(\alpha_2 + \frac{3}{2}F_0^2\right)^2 > 4\lambda_2^2,$$

or equivalently,

$$\left|\alpha_2 + \frac{3}{2}F_0^2\right| < \sqrt{F_0^4 - 4\lambda_2^2},$$

i.e.,

$$-\frac{3}{2}F_0^2 - \sqrt{F_0^4 - 4\lambda_2^2} < \alpha_2 < -\frac{3}{2}F_0^2 + \sqrt{F_0^4 - 4\lambda_2^2}.$$

The necessary condition of resonance is $\lambda_2 < \frac{1}{2}F_0^2$, that is, the dissipation coefficient cannot be greater than the driver's strength.

Recalling that $\alpha_2 = \mu^{-4}(1 - \omega_0^2/\Omega^2)$, we have

$$\omega_0^2\left[1 - \frac{3}{2}F_0^2\mu^4 - \mu^4\sqrt{F_0^4 - 4\lambda_2^2}\right] < \Omega^2 < \omega_0^2\left[1 - \frac{3}{2}F_0^2\mu^4 + \mu^4\sqrt{F_0^4 - 4\lambda_2^2}\right].$$

Since $\mu^2 = (\omega_0^2/\Omega^2)\varepsilon^2$ and $\omega_0^2/\Omega^2 = 1 + O(\mu^4)$ we then have

$$\omega_0^2\left[1 - \frac{3}{2}F_0^2\varepsilon^4 - \varepsilon^4\sqrt{F_0^4 - 4\lambda_2^2}\right] < \Omega^2 < \omega_0^2\left[1 - \frac{3}{2}F_0^2\varepsilon^4 + \varepsilon^4\sqrt{F_0^4 - 4\lambda_2^2}\right]. \tag{4.81}$$

Résumé.

When the system is driven at the frequency $\sim \omega_0$, then for resonance, the damping coefficient must be $\mu^2(\mu^2\lambda) = \mu^4\lambda = \mu^{2N}\lambda$. In the same way it can be shown that for general N the damping should scale as μ^{2N}. With this scaling of the damping term, the resulting damped driven amplitude equation is

$$-2i\frac{\partial a_0}{\partial T_4} + \left(\alpha_2 + \frac{3}{2}F_0^2\right)a_0 - 2i\lambda_2 a_0 + F_0^2\bar{a}_0 = 0, \tag{4.82}$$

and the corresponding resonance window is given by (4.81). This equation correspond to the resonant driving frequencies. In the same way it can be shown that for general N the damping should scale as μ^{2N}.

Below we give the resonance windows for $N = 1, 2, 3$ and 4.

- $N = 1:\quad \omega_0^2\left[1 - \varepsilon^2\sqrt{F_0^2 - 4\lambda_0^2}\right] < \Omega^2 < \omega_0^2\left[1 + \varepsilon^2\sqrt{F_0^2 - 4\lambda_0^2}\right],$

- $N = 2:$ $\omega_0^2 \left[1 - \frac{2}{3}F_0^2\varepsilon^4 - \varepsilon^4\sqrt{F_0^4 - 4\lambda_2^2}\right] < \Omega^2 < \omega_0^2 \left[1 - \frac{2}{3}\varepsilon^4 F_0^2 + \varepsilon^4\sqrt{F_0^4 - 4\lambda_2^2}\right],$

- $N = 3:$ $\omega_0^2 \left[1 - \frac{9}{16}\varepsilon^4 F_0^2 - \varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4 F_0^6 - 4\lambda_4^2}\right] < \Omega^2 <$

  $< \omega_0^2 \left[1 - \frac{9}{16}\varepsilon^4 F_0^2 + \varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4 F_0^6 - 4\lambda_4^2}\right],$

- $N = 4:$ $\omega_0^2 \left[1 - \frac{8}{15}\varepsilon^4 F_0^2 + \frac{928}{75\times 45}\varepsilon^8 F_0^4 - \varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4 F_0^8 - 4\lambda_6^2}\right] < \Omega^2 <$

  $< \omega_0^2 \left[1 - \frac{8}{15}\varepsilon^4 F_0^2 + \frac{928}{75\times 45}\varepsilon^8 F_0^4 + \varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4 F_0^8 - 4\lambda_6^2}\right].$

4.6 Conclusions.

In this chapter we have considered the parametrically driven linear oscillator driven at a frequency $(2/N)\Omega$, where $\Omega \approx \omega_0$; more precisely,

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots.$$

Here α_i's are constants and $\mu = (\omega_0/\Omega)\varepsilon$ is a small parameter. We have shown that in order to have parametric resonance in various frequency regimes the driving frequency should be tuned into the following windows:

- $N = 1:$ $1 - \dfrac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \cdots$

- $N = 2:$ $1 - \dfrac{\omega_0^2}{\Omega^2} = \alpha_2\mu^4 + \cdots$

- $N = 3:$ $1 - \dfrac{\omega_0^2}{\Omega^2} = -\dfrac{9}{16}F_0^2\mu^4 + \alpha_3\mu^6 + \cdots$

- $N = 4:$ $1 - \dfrac{\omega_0^2}{\Omega^2} = -\dfrac{8}{15}F_0^2\mu^4 + \alpha_4\mu^8 + \cdots.$

We also found that the driver appears at the order μ^{2N}, in the form of a complex conjugate term arising from the term

$$\exp\left\{i\left(1 - \frac{2}{N} - \frac{2}{N} - \frac{2}{N} - \cdots\right)T_0\right\} = \exp\left(-iT_0\right).$$

This term arises when the system evolves on the "slow" scale $T = \mu^{2N}\tau$.

If the parametrically driven linear oscillator is damped, the scaling for the damping coefficient should be μ^{2N} since the damping term must appear at the same order as the driver.

With these scalings, the parametric resonance occurs and the main harmonic is governed by the following equations:

- $N=1: \quad -2i\frac{\partial a_0}{\partial T_2} + \alpha_1 a_0 + F_0\bar{a}_0 - 2i\lambda_0 a_0 = 0,$
- $N=2: \quad -2i\frac{\partial a_0}{\partial T_4} + \left(\alpha_2 + \frac{3}{2}F_0^2\right)a_0 + F_0^2\bar{a}_0 - 2i\lambda_2 a_0 = 0,$
- $N=3: \quad -2i\frac{\partial a_0}{\partial T_6} + \alpha_3 a_0 + \left(\frac{9}{8}\right)^2 F_0^3\bar{a}_0 - 2i\lambda_4 a_0 = 0,$
- $N=4: \quad -2i\frac{\partial a_0}{\partial T_8} + \left(\alpha_4 + \frac{928}{75\times 45}F_0^4\right)a_0 + \left(\frac{4}{3}\right)^2 F_0^4\bar{a}_0 - 2i\lambda_6 a_0 = 0.$

Analysing the stability of the *undamped* solution the following resonance windows were obtained:

- $N=1: \quad -F_0 < \alpha_1 < F_0,$
- $N=2: \quad \alpha_1 = 0, \quad -\frac{5}{3}F_0^2 < \alpha_2 < \frac{1}{3}F_0^2,$
- $N=3: \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{9}{16}F_0^2, \quad -\left(\frac{9}{8}\right)^2 F_0^3 < \alpha_3 < \left(\frac{9}{8}\right)^2 F_0^3,$
- $N=4: \quad \alpha_1 = \alpha_3 = 0, \quad \alpha_2 = -\frac{8}{15}F_0^2,$

$$-\frac{6928}{75\times 45}F_0^4 < \alpha_4 < \frac{5072}{75\times 45}F_0^4. \tag{4.83}$$

Expressing these regions of instability in terms of the driving frequency Ω and natural frequency ω_0 we have:

- $N=1: \quad \omega_0^2(1-\varepsilon^2F_0) < \Omega^2 < \omega_0^2(1+\varepsilon^2F_0),$
- $N=2: \quad \omega_0^2\left(1-\frac{5}{3}\varepsilon^4F_0^2\right) < \Omega^2 < \omega_0^2\left(1+\frac{1}{3}\varepsilon^4F_0^2\right),$
- $N=3: \quad \omega_0^2\left(1-\frac{9}{16}\varepsilon^4F_0^2 - \left(\frac{9}{8}\right)^2\varepsilon^6F_0^3\right) < \Omega^2 < \omega_0^2\left(1-\frac{9}{16}\varepsilon^4F_0^2 + \left(\frac{9}{8}\right)^2\varepsilon^6F_0^3\right),$
- $N=4: \quad \omega_0^2\left(1-\frac{8}{15}\varepsilon^4F_0^2 - \frac{6928}{75\times 45}\varepsilon^8F_0^4\right) < \Omega^2$

$$< \omega_0^2\left(1-\frac{8}{15}\varepsilon^4F_0^2 + \frac{5072}{75\times 45}\varepsilon^8F_0^4\right). \tag{4.84}$$

In the *damped* case the above resonance windows modify into:

- $N=1: \quad \omega_0^2\left(1-\varepsilon^2\sqrt{F_0^2-4\lambda_0^2}\right) < \Omega^2 < \omega_0^2\left(1+\varepsilon^2\sqrt{F_0^2-4\lambda_0^2}\right),$

- $N=2: \quad \omega_0^2\left(1-\frac{2}{3}F_0^2\varepsilon^4-\varepsilon^4\sqrt{F_0^4-4\lambda_2^2}\right)<\Omega^2<\omega_0^2\left(1-\frac{2}{3}\varepsilon^4F_0^2+\varepsilon^4\sqrt{F_0^4-4\lambda_2^2}\right),$
- $N=3: \quad \omega_0^2\left(1-\frac{9}{16}\varepsilon^4F_0^2-\varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4F_0^6-4\lambda_4^2}\right)<\Omega^2<$

$$< \ \omega_0^2\left(1-\frac{9}{16}\varepsilon^4F_0^2+\varepsilon^6\sqrt{\left(\frac{9}{8}\right)^4F_0^6-4\lambda_4^2}\right),$$

- $N=4: \quad \omega_0^2\left(1-\frac{8}{15}\varepsilon^4F_0^2+\frac{928}{75\times 45}\varepsilon^8F_0^4-\varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4F_0^8-4\lambda_6^2}\right)<\Omega^2<$

$$< \ \omega_0^2\left(1-\frac{8}{15}\varepsilon^4F_0^2+\frac{928}{75\times 45}\varepsilon^8F_0^4+\varepsilon^8\sqrt{\left(\frac{4}{3}\right)^4F_0^8-4\lambda_6^2}\right). \tag{4.85}$$

Let us now compare the results obtained by the Lindstedt-Poincaré method (Chapter 3) and the multiple scale technique by using the case $N=2$, for simplicity. Using the modified Lindstedt-Poincaré method we first assumed that $r=2$, where r is the scaling exponent for $T=\mu^r\tau$. We arrived at the equation

$$-2i\frac{da}{dT}+\alpha_1 a_0=0, \tag{4.86}$$

which is not driven, and therefore discarded the scaling $r=2$. We then assumed that $r=4$ and the resulting equation is

$$-2i\frac{da}{dT}+\left(\alpha_2+\frac{2}{3}F_0^2\right)a_0+F_0^2\bar{a}_0=0, \tag{4.87}$$

which is indeed, a driven equation.

On the other hand, by the multiple scale technique, at the order μ^2 we arrived at

$$-2iD_2a_0+\alpha_1a_0=0, \tag{4.88}$$

which is also undriven. However, we went further to higher orders until we found the driven equation,

$$-2iD_4a_0+\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{2}{3}F_0^2\right)a_0+F_0\bar{a}_0=0. \tag{4.89}$$

Equation (4.88) shows that the evolution of the amplitude is free on the scale T_2, whereas the evolution of (4.89) implies forced oscillations.

As we can see from the facts above, the modified Lindstedt method, discarded free oscillations on the time scale T_2, while the multiple-scales technique has taken into account both T_2 (free oscillations) and T_4 (forced oscillations). Thus the Lindstedt method gives limited results.

However, in determining the resonant detuning, that is the coefficients α_i for which resonance occurs, the Lindstedt method seems to be convenient as these coefficients are obtained from nonsecularity conditions. Using the multiple-scales technique, we are only able to obtain these variables when we analyse stability of the amplitude equation.

In general, these two methods yield the same results in the resonant case, but the multiple scales method takes into account both free and forced oscillations. In addition, using the multiple scales, we do not have to determine any scaling exponent, but the time scale comes out naturally. Although the modified Lindstedt method is straightforward to apply, it gives limited results and in what follows, we shall make use of the multiple-scales technique. The other reason for continuing with the multiple scales method is that the Lindstedt method is inapplicable to nonlinear equations with varying amplitude since the essence of this method is to introduce the frequency-amplitude dependence. In time varying amplitudes, this dependence is unacceptable since the frequency will be varying instead of being constant.

It can also be shown that there is an exact correspondence between our results and those obtained in literature for the Mathieu equation (see e.g. Abramowitz and Stegun [89]):

$$\varphi_{\theta\theta} + \{a - 2q\cos(2\theta)\}\varphi = 0. \tag{4.90}$$

In comparing these results we note that in the case of equation (4.90), the driving frequency is fixed equal to 2 and the natural frequency, a, is variable. On the other hand, in our equation

$$\varphi_{tt} + \omega_0^2\left[1 - 2F_0\varepsilon^2\cos\left(\frac{2}{N}\Omega t\right)\right]\varphi = 0, \tag{4.91}$$

we vary the driving frequency, $2\Omega/N$, rather than the natural frequency, ω_0. The transformation $\tau = \Omega t$ takes (4.91) to

$$\varphi_{\tau\tau} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi - 2F_0\mu^2\cos\left(\frac{2}{N}\tau\right)\varphi = 0, \tag{4.92}$$

where $\mu = (\omega_0/\Omega)\varepsilon$. This equation is brought to the form (4.90) by writing $\tau = N\theta$ and identifying

$$F_0\mu^2 = \frac{q}{N^2}, \quad \frac{\omega_0^2}{\Omega^2} = \frac{a}{N^2}. \tag{4.93}$$

The equations (4.93) establish the correspondence between our resonance windows and instability boundaries

$$b_N(q) < a < a_N(q)$$

for the Mathieu equation (4.90), available in the literature. Abramowitz and Stegun [89] give a_N and b_N for $N = 1, 2, \ldots, 6$; it is not difficult to verify that for $N = 1, 2, 3, 4$ their instability boundaries give rise to exactly equations (4.84).

Chapter 5

The parametrically driven pendulum.

In chapters 3 and 4, we considered the linear oscillator as an example of a parametrically driven system. In reality, however, most physical problems are modelled by nonlinear equations. In this chapter we shall add sinusoidal nonlinearity to the linear oscillator equation and consider the parametrically driven pendulum:

$$\varphi_{tt} + 2\lambda\varphi_t + \omega_0^2 \left[1 - 2F_0\varepsilon^2 \cos\left(\frac{2}{N}\Omega t\right)\right] \sin\varphi = 0. \tag{5.1}$$

We add nonlinearity of the form $\sin\varphi$ since our ultimate interest is in the parametrically driven sine-Gordon equation.

As in the previous chapters, we make use of the transformation $\tau = \Omega t$, define $\mu^2 = (\omega_0^2/\Omega^2)\varepsilon^2$ and expand

$$\sin\varphi = \varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 + \cdots.$$

Then eq.(5.1) becomes

$$\begin{aligned} \varphi_{\tau\tau} + \varphi &= (\alpha_1\mu^2 + \alpha_2\mu^4 + \cdots)\varphi + \frac{1}{3!}(1 - \alpha_1\mu^2 - \alpha_2\mu^4 - \cdots)\varphi^3 - \\ &+ \frac{1}{5!}(1 - \alpha_1\mu^2 - \alpha_2\mu^4 - \cdots)\varphi^5 + \cdots + \\ &+ 2\frac{\omega_0^2}{\Omega^2}F_0\varepsilon^2 \cos\left(\frac{2}{N}\tau\right)\left(\varphi - \frac{1}{3!}\varphi^3 + \cdots\right). \end{aligned} \tag{5.2}$$

Since we are interested in *weakly* nonlinear solutions, we write

$$\varphi = \mu^p(\varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots),$$

with $p > 0$. This ensures that the nonlinear term is small. Making use of the multiple-scales techniques, we compare coefficients at like powers of μ. We shall determine the correct scaling of the amplitude of the solution by examining different values of p.

5.1 The frequency $\sim 2\omega_0$.

$p = 1$. We first assume that $p = 1$, so that

$$\varphi = \mu\varphi_0 + \mu^3\varphi_1 + \mu^5\varphi_2 + \cdots,$$

where $\varphi_0, \varphi_1, \varphi_2, \ldots = O(\mu^0)$. Comparing coefficients at equal powers of μ, we obtain at μ^1:

$$D_0^2\varphi_0 + \varphi_0 = 0,$$

the solution of which is given by

$$\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$$

At the order μ^2 we obtain

$$2D_0D_1\varphi_0 = 0.$$

Upon the substitution of $\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$ we obtain $D_1a_0 = 0$. At the order μ^3 one arrives at

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + \frac{1}{3!}\varphi_0^3 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_0 \\ &= \left(-2iD_2a_0 + \alpha_1 a_0 + \frac{1}{2}|a_0|^2a_0 + F_0\bar{a}_0\right)e^{iT_0} + \left(\frac{1}{6}a_0^3 + F_0a_0\right)e^{3iT_0} + \\ &+ \ c.c. \end{aligned} \tag{5.3}$$

The nonsecularity condition is given by

$$-2iD_2a_0 + \alpha_1 a_0 + \frac{1}{2}|a_0|^2a_0 + F_0\bar{a}_0 = 0, \tag{5.4}$$

which is nonlinear and driven and therefore we do not need to proceed to higher order terms. We only notice that equation (5.4) results from a multiple scale expansion of

$$-2i\frac{\partial a_0}{\partial \tau} + \alpha_1\mu^2a_0 + \frac{1}{2}\mu^2|a_0|^2a_0 + \mu^2F_0\bar{a}_0 = 0. \tag{5.5}$$

Let $p = 2$, that is

$$\varphi = \mu^2\varphi_0 + \mu^4\varphi_1 + \mu^6\varphi_2 + \cdots$$

At the order μ^2 we have the solution $\varphi_0 = a_0(T_1, T_2, \cdots)e^{iT_0} + c.c.$ The order μ^3 gives

$$2D_0D_1\varphi_0 = 0,$$

which upon substitution of $\varphi_0 = a_0e^{iT_0} + c.c.$ yields $D_1a_0 = 0$. At the order μ^4 we obtain

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1a_0 + F_0\bar{a}_0)e^{iT_0} + F_0a_0e^{3iT_0} + c.c. \end{aligned} \tag{5.6}$$

For nonsecularity we require that

$$-2iD_2a_0 + \alpha_1a_0 + F_0\bar{a}_0 = 0. \tag{5.7}$$

Since this equation is linear, we have to go to higher order terms so that we may have a nonlinear equation. A solution φ_1 is given by

$$\varphi_1 = a_1 e^{3iT_0} + c.c.,$$

where $a_1 = -\frac{1}{8}F_0 a_0$. At the order μ^5 we have

$$2D_0 D_3 \varphi_0 = 0,$$

whence $D_3 a_0 = 0$. At the order μ^6 we obtain

$$\begin{aligned}
D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + \\
&+ \frac{1}{3!}\varphi_0^3 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_1 \\
&= \left(-D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0 a_1\right) e^{iT_0} + \\
&+ \left(-6iD_2 a_1 + \alpha_1 a_1 + \frac{1}{6}a_0^3\right) e^{3iT_0} + F_0 a_1 e^{5iT_0} + c.c.
\end{aligned} \tag{5.8}$$

Suppressing secular terms we arrive at a nonlinear equation,

$$- D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0 a_1 = 0, \tag{5.9}$$

where $a_1 = -\frac{1}{8}F_0 a_0$. Since, in view of (5.7)

$$D_2^2 a_0 = \left(-\frac{1}{4}\alpha_1^2 + \frac{1}{4}F_0^2\right) a_0,$$

eq. (5.9) can be rewritten as

$$- 2iD_4 a_0 + \left(\frac{1}{4}\alpha_1^2 - \frac{3}{8}F_0^2 + \alpha_2\right) a_0 + \frac{1}{2}|a_0|^2 a_0 = 0. \tag{5.10}$$

Equations (5.7) and (5.10) result from a multiple scale expansion of

$$- 2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 - \frac{3}{8}F_0^2 + \alpha_2\right) a_0 + \frac{1}{2}\mu^4|a_0|^2 a_0 + \mu^2 F_0 \bar{a}_0 = 0. \tag{5.11}$$

Discussion.

When the nonlinear oscillator is driven at the frequency $\sim 2\omega_0$, the amplitude of oscillations satisfies the following equations:

- $p = 1:$ $$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \frac{\mu^2}{2}|a_0|^2 a_0 + \mu^2 F_0 \bar{a}_0 = 0, \tag{5.12}$$

- $p = 2:$ $$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{\alpha_1^2}{2} - \frac{3}{8}F_0^2 + \alpha_2\right) a_0 + \frac{\mu^4}{2}|a_0|^2 a_0 + \mu^2 F_0 \bar{a}_0 = 0. \tag{5.13}$$

If a_0 is small, we can ignore the nonlinear terms. This assumption gives two linear equations which differ by terms of order μ^4. The linear limit of eq.(5.12) is precisely the equation that we derived in the *linear* oscillator case, eq.(4.13).

We recall that the instability window for the linear oscillator is given by

$$-F_0 < \alpha_1 < F_0.$$

Let us demonstrate how this instability saturates in the *nonlinear* case; this will give us a clue on the correct choice of p. For simplicity, we confine ourselves to stationary solutions. Putting $\dfrac{\partial a_0}{\partial \tau} = 0$, we obtain for the case $p = 1$ (see eq.(5.12)):

$$\mu^2 \alpha_1 a_0 + \frac{1}{2}\mu^2 |a_0|^2 a_0 + \mu^2 F_0 \bar{a}_0 = 0.$$

Letting $a_0 = Ae^{i\theta}$, we have

$$\alpha_1 + \frac{1}{2}A^2 + F_0 e^{-2i\theta} = 0.$$

By setting the imaginary part to zero, we end up with $\sin 2\theta = 0$ which gives $\theta = \dfrac{\pi}{2}n$, $n = 0, \pm 1, \pm 2, \ldots$. Hence $\cos 2\theta = \cos \pi n = (-1)^n$. The real part is then

$$\alpha_1 + \frac{1}{2}A^2 + (-1)^n F_0 = 0,$$

whence

$$A^2 = -2\left[\alpha_1 + (-1)^n F_0\right]. \tag{5.14}$$

Thus A is real only if $\alpha_1 < (-1)^{n+1} F_0$. Recalling that in the linear case the region of the instability of the trivial solution was given by $-F_0 < \alpha_1 < F_0$, we observe that this instability will be saturated by a solution (5.14) with n odd:

$$A = \sqrt{-2(\alpha_1 - F_0)}.$$

This fact is in agreement with the assumption that $a_0 = O(1)$.

In the case $p = 2$, setting $\dfrac{\partial a_0}{\partial \tau} = 0$ yields

$$\alpha_1 a_0 + \mu^2 \left(\frac{1}{2}\alpha_1^2 - \frac{3}{8}F_0^2 + \alpha_2\right) a_0 + \frac{1}{2}\mu^2 |a_0|^2 a_0 + F_0 \bar{a}_0 = 0.$$

By following the same procedure as above we end up with

$$\alpha_1 + \mu^2 \left(\frac{\alpha_1^2}{2} - \frac{3}{8}F_0^2 + \alpha_2\right) + \frac{1}{2}\mu^2 A^2 + (-1)^n F_0 = 0,$$

which gives

$$A^2 = -2\mu^{-2}\left[\alpha_1 + (-1)^n F_0 + \mu^2 \left(\frac{\alpha_1^2}{2} - \frac{3}{8}F_0^2 + \alpha_2\right)\right].$$

If $\alpha_1 \neq \pm F_0$, we have $a_0 = O(\mu^{-1})$ which contradicts the assumption that $\varphi_0, \varphi_1, \ldots$ are of order 1. The order-1 solution arises only if $\alpha_1 = \pm F_0$. In this case $A^2 = -2\left(\alpha_2 + \frac{1}{8}F_0^2\right)$ which yields real A provided $\alpha_2 < -\frac{1}{8}F_0^2$.

To conclude, the driving frequency with $-F_0 < \alpha_1 < F_0$ will sustain stationary solutions with amplitude of order μ^1. The driver with $\alpha_1 = \pm F_0$ and $\alpha_2 < -\frac{1}{8}F_0^2$ will support stationary solutions of order μ^2. We did not discuss frequencies outside the instability window $-F_0 \leq \alpha_1 \leq F_0$ although these, nonresonant frequencies can also support stationary nonlinear solutions. Outside the resonance window *small* oscillations will die off when we "switch on" the dissipation. This does not mean, however, that *nonlinear* solutions will be damped off. It would be interesting to study the effect of dissipation on stationary nonlinear solutions, as well as to study their stability. This problem lies beyond the scope of this work, however.

5.2 The driving frequency $\sim \omega_0$.

In this section we consider the case $N = 2$. As in the previous section, we vary p in order to find the correct scaling of the amplitude.

Let $p = 1$, that is, $\varphi = O(\mu)$. At the order μ we obtain

$$D_0^2\varphi_0 + \varphi_0 = 0$$

with a nonpropagating solution

$$\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$$

The order μ^2 yields

$$2D_0D_1\varphi_0 = 0,$$

and hence $D_1 a_0 = 0$. At the order μ^3 results

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + \frac{1}{3!}\varphi_0^3 + F_0(e^{iT_0} + e^{-iT_0}) \\ &= \left(-2iD_2a_0 + \alpha_1 a_0 + \frac{1}{2}|a_0|^2 a_0\right) e^{iT_0} + F_0(a_0e^{2iT_0} + a_0) + \\ &+ \frac{1}{6}a_0^3 e^{3iT_0} + c.c., \end{aligned} \tag{5.15}$$

and the nonsecularity condition is

$$-2iD_2a_0 + \alpha_1 a_0 + \frac{1}{2}|a_0|^2 a_0 = 0. \tag{5.16}$$

A particular solution φ_1 is given by

$$\varphi_1 = a_1e^{3iT_0} + b_1e^{2iT_0} + c_1 + c.c.,$$

where

$$a_1 = -\frac{1}{48}a_0^3, \quad b_1 = -\frac{1}{3}F_0 a_0, \quad c_1 = F_0 a_0. \tag{5.17}$$

At the order μ^4 we have

$$2D_0 D_3 \varphi_0 = 0$$

which upon substitution of $\varphi_0 = a_0 e^{iT_0} + c.c.$ yields $D_3 a_0 = 0$. The order μ^5 gives

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + \frac{1}{2}\varphi_0^2\varphi_1 - \\ &- \frac{1}{3!}\alpha_1\varphi_0^3 - \frac{1}{5!}\varphi_0^5 + F_0(e^{iT_0} + e^{-iT_0})\left(\varphi_0 - \frac{1}{3!}\varphi_0^3\right). \end{aligned} \tag{5.18}$$

Substituting for φ_0 and φ_1 and setting to zero coefficients of secular terms we obtain

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2 a_0 - \frac{1}{2}\alpha_1|a_0|^2a_0 + \frac{1}{2}\bar{a}_0^2 a_1 - \frac{1}{12}|a_0|^4a_0 + F_0(b_1 + c_1 + \bar{c}_1) = 0. \tag{5.19}$$

But since, in view of eq.(5.16)

$$D_2^2 a_0 = \left(-\frac{1}{4}\alpha_1^2 - \frac{1}{4}\alpha_1|a_0|^2 + \frac{3}{16}|a_0|^4\right)a_0,$$

eq.(5.19) becomes

$$-2iD_4a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{2}{3}F_0^2\right)a_0 - \frac{1}{4}\alpha_1|a_0|^2a_0 - \frac{9}{32}|a_0|^4a_0 + F_0^2\bar{a}_0 = 0. \tag{5.20}$$

Equations (5.16) and (5.20) result from a multiple scale expansion of

$$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{2}{3}F_0^2\right)\right]a_0 + \\ &+ \left(\frac{1}{2}\mu^2 - \frac{1}{4}\mu^4\alpha_1\right)|a_0|^2a_0 - \frac{9}{32}\mu^4|a_0|^4a_0 + \mu^4F_0^2\bar{a}_0 = 0. \end{aligned} \tag{5.21}$$

Let $p = 2$. In this case

$$\varphi = \mu^2\varphi_0 + \mu^4\varphi_1 + \mu^6\varphi_2 + \cdots$$

At the order μ^2 we obtain $\varphi_0 = a_0e^{iT_0} + c.c.$ and the order μ^3 yields $D_1a_0 = 0$. Now, at the order μ^4 we obtain

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2iD_0D_2 + \alpha_1)\varphi_0 + F_0(e^{iT_0} + e^{-iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1a_0)e^{iT_0} + F_0(a_0e^{2iT_0} + a_0) + c.c. \end{aligned} \tag{5.22}$$

For nonsecularity we require

$$-2iD_2a_0 + \alpha_1a_0 = 0, \tag{5.23}$$

thus, a particular solution is given by

$$\varphi_1 = a_1e^{2iT_0} + b_1 + c.c.,$$

with

$$a_1 = -\frac{1}{3}F_0 a_0, \quad b_1 = F_0 a_0.$$

At the order μ^5 we have

$$2D_0D_3\varphi_0 = 0$$

which yields $D_3a_0 = 0$. The order μ^6 gives

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (2D_0D_2 + \alpha_1)\varphi_1 + \frac{1}{3!}\varphi_0^3 + F_0(e^{iT_0} + e^{-iT_0})\varphi_1 \\ &= \left[-2D_2^2a_0 - 2iD_4a_0 + \alpha_2a_0 + \frac{1}{2}|a_0|^2a_0 + F_0(a_1 + b_1 + \bar{b}_1)\right]e^{iT_0} + \\ &+ (-4iD_2a_1 + \alpha_1a_1)e^{2iT_0} + \alpha_1b_1 + \left(\frac{1}{6}a_0^3 + F_0a_1\right)e^{3iT_0} + c.c. \end{aligned} \tag{5.24}$$

Suppressing secular terms we obtain

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2a_0 + \frac{1}{2}|a_0|^2a_0 + F_0(a_1 + b_1 + \bar{b}_1) = 0.$$

But $D_2^2a_0 = -\frac{1}{4}\alpha_1^2a_0$, such that the nonsecular condition is given by

$$-2iD_4a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{2}{3}F_0^2\right)a_0 + \frac{1}{2}|a_0|^2a_0 + F_0^2\bar{a}_0 = 0. \tag{5.25}$$

Equations (5.23) and (5.25) result from a multiple scale expansion of

$$-2i\frac{\partial a_0}{\partial\tau} + \mu^2\alpha_1a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2a_0 + \alpha_2a_0 + \frac{2}{3}F_0^2a_0\right) + \frac{1}{2}\mu^4|a_0|^2a_0 + \mu^4F_0^2\bar{a}_0 = 0. \tag{5.26}$$

Discussion.

To determine the correct order of magnitude of nonlinear solutions, we consider the resonant regime. When analysing linear stability of the trivial solution, the instability window was found to be

$$-\mu^4F_0^2 < \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{2}{3}F_0^2\right)\right] < \mu^4F_0^2,$$

which gave us $\alpha_1 = 0$ and $-\frac{5}{3}F_0^2 < \alpha_2 < \frac{1}{3}F_0^2$, see eq.(4.28). When $\alpha_1 = 0$, the nonlinear equations for the amplitude of the main harmonic, eqs.(5.21) and (5.26), reduce to

- $p = 1: \quad -2i\frac{\partial a_0}{\partial\tau} + \mu^4\left(\alpha_2 + \frac{2}{3}F_0^2\right)a_0 + \frac{1}{2}\mu^2|a_0|^2a_0 - \frac{9}{32}\mu^4|a_0|^4a_0 + \mu^4F_0^2\bar{a}_0 = 0,$

- $p = 2: \quad -2i\frac{\partial a_0}{\partial\tau} + \mu^4\left(\alpha_2 + \frac{2}{3}F_0^2\right)a_0 + \frac{1}{2}\mu^4|a_0|^2a_0 + \mu^4F_0^2\bar{a}_0 = 0.$

(Notice that the linear limits of these equations coincide and are the same as eq.(4.31) of the previous chapter.) To determine which p is acceptable we proceed in the same way as in the case $N = 1$ and examine stationary solutions.

When $p = 1$, we have

$$\frac{1}{2}|a_0|^2a_0 + \mu^2\left(\alpha_2 + \frac{2}{3}F_0^2\right)a_0 - \frac{9}{32}\mu^2|a_0|^4a_0 + \mu^2F_0^2\bar{a}_0 = 0.$$

Letting $a_0 = Ae^{i\theta}$ yields

$$-\frac{9}{32}\mu^2 A^4 + \frac{1}{2}A^2 + \mu^2\left[\alpha_2 + \frac{2}{3}F_0^2 + (-1)^n F_0^2\right] = 0.$$

Expanding $A = A_0 + \mu A_1 + \mu^2 A_2 + \ldots$ and comparing coefficients at like powers of μ, we find that, provided $\alpha_2 + \frac{2}{3}F_0^2 + (-1)^n F_0^2 \neq 0$, $A_0 = 0$, that is $a_0 = O(\mu)$ which contradicts the assumption that φ is of order 1. And if $\alpha_2 + \frac{2}{3}F_0^2 + (-1)^n F_0^2 = 0$, we have $A \sim \mu^{-1}$. This disqualifies $p = 1$.

For $p = 2$ we arrive at

$$\frac{1}{4}A^2 + \alpha_2 + \frac{2}{3}F_0^2 + F_0^2(-1)^n = 0,$$

or

$$A^2 = -4\left[\alpha_2 + \frac{2}{3}F_0^2 + (-1)^n F_0^2\right], \tag{5.27}$$

which gives real A if

$$\alpha_2 + \frac{2}{3}F_0^2 + (-1)^n F_0^2 < 0. \tag{5.28}$$

When n is odd, eq.(5.28) becomes $\alpha_2 < \frac{1}{3}F_0^2$ which coincides with the upper bound of the corresponding linear instability window $(-\frac{5}{3}F_0^2 < \alpha_2 < \frac{1}{3}F_0^2)$. On the contrary when n is even, eq.(5.28) becomes $\alpha_2 < -\frac{5}{3}F_0^2$ which is just outside the linear instability region. This means that we need to retain only n odd in eq.(5.27) and so the instability will saturate by the solution $a_0 = \pm A$, with

$$A = 2\sqrt{\frac{1}{3}F_0^2 - \alpha_2}, \tag{5.29}$$

which is of order 1, as it should be. Thus, $p = 2$ is an acceptable scaling exponent. If we had considered the cases $p = 3$ and higher, we would have found that stationary solutions of order μ^3 are also possible. They should arise at the edges of the instability window, i.e., for $\alpha_2 = -\frac{5}{3}F_0^2$ or $\frac{1}{3}F_0^2$. (However, we did not check this.)

Finally, we should mention that, as in the previous section, we did not consider stationary nonlinear solutions outside the resonance window. Although for these solutions $p = 1$ seems to be an acceptable scaling, it is not clear whether they will survive the effect of dissipation. The stability of these solutions is also questionable.

5.3 The frequency $\sim (2/3)\omega_0$.

We consider the case $N = 3$:

$$\varphi_{tt} + \omega_0^2\left[1 - 2F_0\varepsilon^2\cos\left(\frac{2}{3}\Omega t\right)\right]\sin\varphi = 0. \tag{5.30}$$

Proceeding in a standard way, we make a transformation $\tau = \Omega t$ and expand $\sin\varphi$ to obtain

$$\begin{aligned}\varphi_{\tau\tau} + \varphi &= \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + \frac{1}{3!}\frac{\omega_0^2}{\Omega^2}\varphi^3 - \frac{1}{5!}\frac{\omega_0^2}{\Omega^2}\varphi_0^5 + \cdots + \\ &+ 2\frac{\omega_0^2}{\Omega^2}F_0\varepsilon^2\cos\left(\frac{2}{3}\tau\right)\left(\varphi_0 - \frac{1}{3!}\varphi^3 + \cdots\right).\end{aligned} \tag{5.31}$$

Letting $\mu^2 = \frac{\omega_0^2}{\Omega^2}\varepsilon^2$, and

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots,$$

we define the slow scales as $T_n = \mu^n\tau$ and assume that

$$\varphi = \mu^p(\varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots),$$

where $\varphi_0, \varphi_1, \ldots = O(\mu^0)$. Our aim here is to find the correct scaling exponent p for this case, i.e. we try to determine how small the arising nonlinear solution will be.

Let $p = 1$. At the order μ^1 we obtain the equation

$$D_0^2\varphi_0 + \varphi_0 = 0,$$

whose stationary solution is given by

$$\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$$

At the order μ^2 we have

$$2D_0D_1\varphi_0 = 0$$

which upon substitution of φ_0 yields $D_1a_0 = 0$. At the order μ^3 we obtain the equation of the form

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + \frac{1}{3!}\varphi_0^3 + F_0\left(e^{(2/3)iT_0} + e^{-(2/3)iT_0}\right)\varphi_0 \\ &= \left[-2iD_2a_0 + \alpha_1a_0 + \frac{1}{2}|a_0|^2a_0\right]e^{iT_0} + \frac{1}{3!}a_0^3e^{3iT_0} + \\ &+ F_0\left(a_0e^{i(1+2/3)T_0} + a_0e^{i(1-2/3)T_0}\right) + c.c. \end{aligned} \tag{5.32}$$

The nonsecularity condition is

$$-2iD_2a_0 + \alpha_1a_0 + \frac{1}{2}|a_0|^2a_0 = 0 \tag{5.33}$$

and φ_1 is given by

$$\varphi_1 = a_1e^{i(1+2/3)T_0} + b_1e^{i(1-2/3)T_0} + c_1e^{3iT_0} + c.c.$$

where

$$a_1 = -\frac{9}{16}F_0a_0, \quad b_1 = \frac{9}{8}F_0a_0, \quad c_1 = -\frac{1}{48}a_0^3. \tag{5.34}$$

The order μ^4 yields

$$2D_0D_3\varphi_0 = 0$$

whence $D_3a_0 = 0$. At the fifth order of μ we have

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + \frac{1}{2}\varphi_0^2\varphi_1 - \\ &- \frac{1}{3!}\alpha_1\varphi_0^3 - \frac{1}{5!}\varphi_0^5 + F_0(e^{(2/3)iT_0} + e^{-(2/3)iT_0})\left(\varphi_1 - \frac{1}{3!}\varphi_0^3\right) \end{aligned} \tag{5.35}$$

Substituting for φ_0 and φ_1, the nonsecularity condition is given by

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2 a_0 - \frac{1}{2}\alpha_1|a_0|^2a_0 + \frac{1}{2}\bar{a}_0c_1 - \frac{1}{12}|a_0|^4a_0 + F_0(a_1+b_1) = 0, \tag{5.36}$$

which upon substituting for a_1, b_1 and using eq.(5.33) yields

$$-2iD_4a_0 + \left(\alpha_2 + \frac{1}{4}\alpha_1^2 + \frac{9}{16}F_0^2\right)a_0 - \frac{1}{4}\alpha_1|a_0|^2a_0 - \frac{1}{32}|a_0|^4a_0 = 0. \tag{5.37}$$

The fifth order correction is given by

$$\varphi_2 = a_2e^{i(1+8/3)T_0} + b_2e^{i(1-8/3)T_0} + c_2e^{i(1+4/3)T_0} + d_2e^{i(1-4/3)T_0} + e_2e^{5iT_0} + f_2e^{3iT_0} + c.c.,$$

where

$$a_2 = \frac{9\times 43}{112\times 96}F_0a_0^3, \quad b_2 = -\left(\frac{9}{16}\right)^2\times\frac{2}{3}F_0\alpha_1\bar{a}_0 + \frac{9}{16}\times\frac{1}{32}F_0|a_0|^2\bar{a}_0,$$

$$c_2 = -\frac{9\times 19}{40\times 32}F_0a_0^3 + \frac{9\times 9}{16\times 40}F_0^2a_0, \quad d_2 = \left(\frac{9}{8}\right)^2\times\frac{2}{3}F_0\alpha_1\bar{a}_0 + \frac{9\times 5}{8\times 32}F_0|a_0|^2\bar{a}_0 + \left(\frac{9}{8}\right)^2F_0^2a_0,$$

$$e_2 = \frac{1}{160\times 8}a_0^5, \quad f_2 = -\frac{1}{8\times 32}|a_0|^2a_0^3.$$

The order μ^6 yields

$$2D_0D_5\varphi_0 = 0$$

which gives $D_5a_0 = 0$.

At the order μ^7 we arrive at

$$\begin{aligned} D_0^2\varphi_3 + \varphi_3 &= (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_0 + (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_1 + \\ &+ (-2D_0D_2 + \alpha_1)\varphi_2 + \frac{1}{2}(\varphi_0\varphi_1^2 + \varphi_0\varphi_1^2) - \frac{1}{3!}\alpha_2\varphi_0^3 - \frac{1}{2}\alpha_1\varphi_0^2\varphi_1 - \\ &- \frac{1}{24}\varphi_0^4\varphi_1 - \frac{1}{5!}\alpha_1\varphi_0^5 + \frac{1}{7!}\varphi_0^7 + \\ &+ F_0(e^{(2/3)iT_0} + e^{-(2/3)iT_0})\left(\varphi_2 - \frac{1}{2}\varphi_0^2\varphi_1 + \frac{1}{5!}\varphi_0^5\right). \end{aligned} \tag{5.38}$$

Making relevant substitutions we arrive at

$$\begin{aligned} &-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 - \frac{1}{2}\alpha_2|a_0|^2a_0 - \frac{1}{2}\alpha_1\bar{a}_0^2c_1 + \frac{1}{2}\bar{a}_0^2f_2 + |a_1|^2a_0 + \\ &+ |b_1|^2a_0 + |c_1|^2a_0 + \bar{a}_0a_1b_1 - \frac{1}{24}(a_0^4\bar{c}_1 + 4|a_0|^2\bar{a}_0^2c_1) - \frac{1}{12}\alpha_1|a_0|^4a_0 + \\ &+ \frac{35}{7!}|a_0|^6a_0 + F_0\left(\bar{b}_2 + \bar{d}_2 - \frac{1}{2}a_0^2\bar{a}_1 - |a_0|^2b_1 - \frac{1}{2}a_0^2\bar{a}_1\right) = 0, \end{aligned} \tag{5.39}$$

which is the nonsecularity condition. Equivalently,

$$\begin{aligned} &-2iD_6a_0 + \left(\frac{1}{2}\alpha_1\alpha_2 + \frac{1}{8}\alpha_1^3 + \frac{117}{128}F_0^2\alpha_1 + \alpha_3\right)a_0 - \left(\frac{1}{16}\alpha_1^2 - \frac{1}{4}\alpha_2 + \right. \\ &+ \left.\frac{369}{512}F_0^2\right)|a_0|^2a_0 - \frac{53}{192}\alpha_1|a_0|^4a_0 + \frac{1}{128}|a_0|^6a_0 + \left(\frac{9}{8}\right)^2F_0^3\bar{a}_0 = 0 \end{aligned} \tag{5.40}$$

Equations (5.33), (5.37) and (5.40) are a result of a multiple scale expansion of

$$\begin{aligned} & -2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\alpha_2 + \frac{1}{4}\alpha_1^2 + \frac{9}{16}F_0^2\right)a_0 + \mu^6\left(\frac{1}{2}\alpha_1\alpha_2 + \frac{1}{8}\alpha_1^3 + \right.\\ + & \left.\frac{117}{128}F_0^2\alpha_1 + \alpha_3\right)a_0 + \left[\frac{1}{2}\mu^2 - \frac{1}{4}\mu^4\alpha_1 - \mu^6\left(\frac{1}{16}\alpha_1^2 + \frac{1}{4}\alpha_2 + \frac{369}{512}F_0^2\right)\right]|a_0|^2 a_0 - \\ - & \left(\frac{1}{32}\mu^4 + \frac{53}{192}\alpha_1\mu^6\right)|a_0|^4 a_0 + \frac{1}{128}\mu^6|a_0|^6 a_0 + \left(\frac{9}{8}\right)^2\mu^6 F_0^3\bar{a}_0 = 0. \end{aligned} \tag{5.41}$$

As we have shown in the previous chapter, the linear instability occurs for $\alpha_1 = 0$, $\alpha_2 = -\frac{9}{16}F_0^2$ and

$$-\left(\frac{9}{8}\right)^2 F_0^3 < \alpha_3 < \left(\frac{9}{8}\right)^2 F_0^3,$$

see eq.(4.45). For these values of α_1 and α_2 the equation governing the amplitude of the main harmonic is

$$\begin{aligned} & -2i\frac{\partial a_0}{\partial \tau} + \mu^6\alpha_3 a_0 + \left(\frac{1}{2}\mu^2 + \frac{9}{64}\mu^6 + \frac{369}{512}\mu^6 F_0^2\right)|a_0|^2 a_0 - \\ - & \frac{1}{32}\mu^4|a_0|^4 a_0 + \frac{1}{128}\mu^6|a_0|^6 a_0 + \left(\frac{9}{8}\right)^2\mu^6 F_0^3\bar{a}_0 = 0. \end{aligned} \tag{5.42}$$

Let $p = 2$.
In this case we assume that the solution

$$\varphi = \mu^2\varphi_0 + \mu^4\varphi_1 + \mu^6\varphi_2 + \cdots.$$

At μ^2 we have $\varphi_0 = a_0 e^{iT_0} + c.c.$ The order μ^3 gives $D_1 a_0 = 0$. At the order μ^4 we have

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0\left(e^{(2/3)iT_0} + e^{-(2/3)iT_0}\right)\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1 a_0)e^{iT_0} + F_0\left(a_0e^{i(1+2/3)T_0} + a_0e^{i(1-2/3)T_0}\right) + c.c. \end{aligned} \tag{5.43}$$

For nonsecularity we require

$$-2iD_2a_0 + \alpha_1 a_0 = 0, \tag{5.44}$$

thus,

$$\varphi_1 = a_1 e^{i(1+2/3)T_0} + b_1 e^{i(1-2/3)T_0} + c.c.,$$

where

$$a_1 = -\frac{9}{16}F_0 a_0, \quad b_1 = \frac{9}{8}F_0 a_0.$$

At μ^5 we find $D_3 a_0 = 0$. The order μ^6 gives us

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + \frac{1}{3!}\varphi_0^3 + \\ &+ F_0\left(e^{(2/3)iT_0} + e^{-(2/3)iT_0}\right)\varphi_1. \end{aligned} \tag{5.45}$$

Substituting for φ_0 and φ_1, the nonsecularity condition is given by

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0(a_1 + b_1) = 0, \tag{5.46}$$

or equivalently,

$$-2iD_4a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{9}{16}F_0^2\right)a_0 + \frac{1}{2}|a_0|^2a_0 = 0. \qquad (5.47)$$

A solution φ_2 is given by

$$\varphi_2 = a_2e^{i(1+4/3)T_0} + b_2e^{i(1-4/3)T_0} + c_2e^{i(1+2/3)T_0} + d_3e^{3iT_0} + c.c.,$$

where

$$a_2 = \frac{9\times 9}{40\times 16}F_0^2a_0, \quad b_2 = \left(\frac{9}{8}\right)^2 \times \frac{2}{3}F_0\alpha_1\bar{a}_0 + \left(\frac{9}{8}\right)^2 F_0^2a_0, \quad c_2 = -\left(\frac{9}{16}\right)^2 \times \frac{2}{3}F_0\alpha_1a_0,$$

$$d_3 = -\frac{1}{48}a_0^3.$$

At the order μ^7 we have $D_5a_0 = 0$. At the order μ^8 we obtain

$$\begin{aligned} D_0^2\varphi_3 + \varphi_3 &= (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_0 + (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_1 - \frac{1}{3!}\alpha_1\varphi_0^3 + \\ &+ (-2D_0D_2 + \alpha_1)\varphi_2 + \frac{1}{2}\varphi_0^2\varphi_1 + F_0\left(e^{(2/3)iT_0} + e^{-(2/3)iT_0}\right)\varphi_2. \qquad (5.48)\end{aligned}$$

At this order the nonsecularity condition is given by

$$-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 - \frac{1}{2}\alpha_1|a_0|^2a_0 + F_0(\bar{b}_2 + c_2) = 0, \qquad (5.49)$$

or equivalently,

$$-2iD_6a_0 + \left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \alpha_3 + \frac{117}{128}F_0^2\alpha_1\right)a_0 - \frac{1}{4}\alpha_1|a_0|^2a_0 + \left(\frac{9}{8}\right)^2 F_0^3\bar{a}_0 = 0. \quad (5.50)$$

Equations (5.44), (5.47), and (5.50) are a result of a multiple scale expansion of

$$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{9}{16}F_0^2\right) + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \right.\right. \\ &+ \left.\left.\alpha_3 + \frac{117}{128}F_0^2\alpha_1\right)\right]a_0 + \left(\frac{1}{2}\mu^4 - \frac{1}{4}\alpha_1\right)|a_0|^2a_0 + \left(\frac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0 = 0. \qquad (5.51)\end{aligned}$$

The instability arises for $\alpha_1 = 0$, $\alpha_2 = -\frac{9}{16}F_0^2$ and α_3 satisfying $-\left(\frac{9}{8}\right)^2F_0^3 < \alpha_3 < \left(\frac{9}{8}\right)^2F_0^3$. In this case the equation of the amplitude of the main harmonic reduces to

$$-2i\frac{\partial a_0}{\partial \tau} + \alpha_3\mu^6a_0 + \frac{1}{2}\mu^4|a_0|^2a_0 + \left(\frac{9}{8}\right)^2F_0^3\mu^6\bar{a}_0 = 0. \qquad (5.52)$$

Let $p = 3$.
At $O(\mu^3)$ we find the solution $\varphi_0 = a_0e^{iT_0} + c.c.$, and $O(\mu^4)$ gives $D_1a_0 = 0$. At the order μ^5 one finds

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0\left(e^{(2/3)iT_0} + e^{-(2/3)iT_0}\right)\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1a_0)e^{iT_0} + F_0\left(a_0e^{i(1+2/3)T_0} + a_0e^{i(1-2/3)T_0}\right) + c.c. \quad (5.53)\end{aligned}$$

For nonsecularity we require that

$$-2iD_2a_0+\alpha_1a_0=0 \tag{5.54}$$

and

$$\varphi_1=a_1e^{i(1+2/3)T_0}+b_1e^{i(1-2/3)T_0}+c.c.,$$

where

$$a_1=-\frac{9}{16}F_0a_0,\quad b_1=\frac{9}{8}F_0a_0.$$

The sixth order in the expansion gives $D_3a_0=0$. At the next order μ^7 we get

$$\begin{aligned}
&D_0^2\varphi_2+\varphi_2=\\
&=\ (-D_2^2-2D_0D_4+\alpha_2)\varphi_0+(-2D_0D_2+\alpha_1)\varphi_1+F_0(e^{(2/3)iT_0}+e^{-(2/3)iT_0})\varphi_1\\
&=\ [-D_2^2a_0-2iD_4a_0+\alpha_2a_0+F_0(a_1+b_1)]e^{iT_0}+\left(-\frac{1}{3}D_2a_1+\alpha_1a_1\right)e^{i(1+2/3)T_0}+\\
&+\ \left(-\frac{2}{3}iD_2b_1+\alpha_1b_1\right)e^{i(1-2/3)T_0}+F_0\left(a_1e^{i(1+4/3)T_0}+b_1e^{i(1-4/3)T_0}\right)+c.c.
\end{aligned} \tag{5.55}$$

The nonsecularity condition is given by

$$-2iD_4a_0-D_2^2a_0+\alpha_2a_0+F_0(a_1+b_1)=0,$$

which upon substitution of known variables yields

$$-2iD_4a_0+\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{9}{16}F_0^2\right)a_0=0 \tag{5.56}$$

and

$$\varphi_2=a_2e^{i(1+4/3)T_0}+b_2e^{i(1-4/3)T_0}+c_2e^{i(1+2/3)T_0}+c.c.$$

where

$$a_2=\frac{9\times9}{40\times16}F_0^2a_0,\quad b_2=\left(\frac{9}{8}\right)^2\times\frac{2}{3}F_0\alpha_1\bar{a}_0+\left(\frac{9}{8}\right)^2F_0^2a_0,\quad c_2=-\left(\frac{9}{16}\right)^2\times\frac{2}{3}F_0\alpha_1a_0.$$

At μ^8, we have $D_5a_0=0$. Moving further to higher order terms, we obtain at μ^9

$$\begin{aligned}
D_0^2\varphi_3+\varphi_3\ &=\ (-2D_0D_6-2D_2D_4+\alpha_3)\varphi_0+(-2D_2^2-2D_0D_4+\alpha_2)\varphi_1+\\
&+\ (-2D_0D_2+\alpha_1)\varphi_2+\frac{1}{3!}\varphi_0^3+F_0\left(e^{(2/3)iT_0}+e^{-(2/3)iT_0}\right)\varphi_2
\end{aligned} \tag{5.57}$$

and the condition for the absence of secular terms is

$$-2iD_6a_0-2D_2D_4a_0+\alpha_3a_0+\frac{1}{2}|a_0|^2a_0+F_0(\bar{b}_2+c_2)=0$$

or

$$-2iD_6a_0+\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2+\alpha_3+\frac{117}{128}F_0^2\alpha_1\right)a_0+\frac{1}{2}|a_0|^2a_0+\left(\frac{9}{8}\right)^2F_0^3\bar{a}_0=0. \tag{5.58}$$

Eqs.(5.54), (5.56), and (5.58) can be derived from a multiple-scale expansion of

$$-2i\frac{\partial a_0}{\partial \tau}+\mu^2\alpha_1 a_0+\mu^4\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{9}{16}F_0^2\right)a_0+\mu^6\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2+\right.$$
$$+\left.\quad\alpha_3+\frac{117}{128}F_0^2\alpha_1\right)a_0+\frac{1}{2}\mu^6|a_0|^2a_0+\left(\frac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0=0. \tag{5.59}$$

The instability arises if $\alpha_1=0$, $\alpha_2=-\frac{9}{16}F_0^2$, and $-\left(\frac{9}{8}\right)^2F_0^3<\alpha_3<\left(\frac{9}{8}\right)^2F_0^3$. Hence,

$$-2i\frac{\partial a_0}{\partial \tau}+\mu^6\alpha_3 a_0+\frac{1}{2}\mu^6|a_0|^2a_0+\left(\frac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0=0. \tag{5.60}$$

Discussion.

When the nonlinear oscillator is driven at the frequency $\sim(2/3)\omega_0$, the instability is saturated by solutions for the following nonlinear equations:

- $p=1:\quad -2i\dfrac{\partial a_0}{\partial \tau}+\mu^6\alpha_3 a_0+\left(\dfrac{1}{2}\mu^2+\dfrac{9}{64}\mu^6+\dfrac{369}{512}\mu^6F_0^2\right)|a_0|^2a_0-$

  $-\dfrac{1}{32}\mu^4|a_0|^4a_0+\dfrac{1}{128}\mu^6|a_0|^6a_0+\left(\dfrac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0=0,$

- $p=2:\quad -2i\dfrac{\partial a_0}{\partial \tau}+\mu^6\alpha_3 a_0+\dfrac{1}{2}\mu^4|a_0|^2a_0+\left(\dfrac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0=0,$

- $p=3:\quad -2i\dfrac{\partial a_0}{\partial \tau}+\mu^6\alpha_3 a_0+\dfrac{1}{2}\mu^6|a_0|^2a_0+\left(\dfrac{9}{8}\right)^2\mu^6F_0^3\bar{a}_0=0.$

We now let $\dfrac{\partial a_0}{\partial \tau}=0$ to determine which p to choose.

For $p=1$, this gives

$$\mu^4\alpha_3 a_0+\left(\frac{1}{2}+\frac{9}{64}\mu^4+\frac{369}{512}\mu^4F_0^2\right)|a_0|^2a_0-\frac{1}{32}\mu^2|a_0|^4a_0+\frac{1}{128}\mu^4|a_0|^6a_0+\left(\frac{9}{8}\right)^2\mu^4F_0^3\bar{a}_0=0.$$

Letting $a_0=Ae^{i\theta}$ we obtain

$$\mu^4\alpha_3+\frac{1}{2}A^2-\frac{1}{32}\mu^2A^4+\frac{1}{128}\mu^4A^6+\left(\frac{9}{8}\right)^2\mu^4F_0^3(-1)^n=0,$$

where we have neglected small corrections. Expanding A as

$$A=A_0+\mu^1A_1+\mu^2A_2+\cdots,$$

and comparing coefficients at like powers of μ, we find $A_0=A_1=0$ and

$$A_2^2=-2\left[\alpha_3+\left(\frac{9}{8}\right)^2F_0^3(-1)^n\right].$$

Thus $a_0=O(\mu^2)$, which contradicts the assumption that the amplitude a_0 is of order 1.

For $p = 2$ we arrive at the equation

$$\alpha_3 \mu^2 a_0 + \frac{1}{2}|a_0|^2 a_0 + \left(\frac{9}{8}\right)^2 \mu^2 F_0^3 a_0 = 0,$$

whence

$$A^2 = -2\mu^2 \left[\alpha_3 + \left(\frac{9}{8}\right)^2 F_0^3 (-1)^n\right].$$

This gives a_0 of order μ and so $p = 2$ is not the correct scaling either.

Finally, when $p = 3$, we have

$$\alpha_3 a_0 + \frac{1}{2}|a_0|^2 a_0 + \left(\frac{9}{8}\right)^2 F_0^3 \bar{a}_0 = 0,$$

whence

$$A^2 = -2\left[\alpha_3 + \left(\frac{9}{8}\right)^2 F_0^3 (-1)^n\right], \tag{5.61}$$

which yields $a_0 = O(1)$. This equation gives real A if

$$\alpha_3 < \left(\frac{9}{8}\right)^2 F_0^3 (-1)^{n+1}.$$

We recall that the linear instability window is given by $-\left(\frac{9}{8}\right)^2 F_0^3 < \alpha_3 < \left(\frac{9}{8}\right)^2 F_0^3$, eq.(4.45). This instability will be saturated by the solution (5.61) with n odd. (When n is even, $\alpha_3 < -\left(\frac{9}{8}\right)^2 F_0^3$ which is outside the linear instability region.) Thus for these α_3 the instability of the zero solution will saturate by a stationary solution $a_0 = \pm A$, with $A = 2\sqrt{\left(\frac{9}{8}\right)^2 F_0^3 - \alpha_3}$. Consequently, $p = 3$ is an acceptable scaling. If we had considered the cases $p = 4$ and higher, we would have found that stationary solutions of order μ^4 are also possible. They should arise at the edges of the instability window, that is, $\alpha_3 = -\left(\frac{9}{8}\right)^2 F_0^3$ or $\left(\frac{9}{8}\right)^2 F_0^3$.

As in the previous sections, we did not consider stationary solutions outside the instability region. This is beyond the scope of this work.

5.4 The frequency $\sim (2/4)\omega_0$.

In this case the equation is

$$\varphi_{tt} + \omega_0^2 \left[1 - 2F_0 \varepsilon^2 \cos\left(\frac{2}{4}\Omega t\right)\right] \sin\varphi = 0. \tag{5.62}$$

Proceeding as in the previous sections we have

$$\begin{aligned}
\varphi_{\tau\tau} + \varphi &= \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + \frac{1}{3!}\frac{\omega_0^2}{\Omega^2}\varphi^3 - \frac{1}{5!}\frac{\omega_0^2}{\Omega^2}\varphi^5 + \cdots + \\
&+ 2\mu^2 F_0 \cos\left(\frac{2}{4}\tau\right)\left(\varphi - \frac{1}{3!}\varphi^3 + \cdots\right)
\end{aligned}$$

we let

$$1-\frac{\omega_0^2}{\Omega^2}=\alpha_1\mu^2+\alpha_2\mu^4+\cdots$$

Assuming that the solution is small, we define

$$\varphi=\mu^p(\varphi_0+\mu^2\varphi_1+\mu^4\varphi_2+\cdots),$$

and determine the correct scaling exponent p.

Let $p=1$. Comparing terms at same order of μ, the multiple-scales expansion yields at the order μ^1,

$$D_0^2\varphi_0+\varphi_0=0,$$

whose nonpropagating solution is given by

$$\varphi_0=a_0(T_1,T_2,\ldots)e^{iT_0}+c.c.$$

At the order μ^2 we have $2D_0D_1\varphi_0=0$, from which we find $D_1a_0=0$. At the order μ^3 we obtain

$$\begin{aligned} D_0^2\varphi_1+\varphi_1 &= (-2D_0D_2+\alpha_1)\varphi_0+\frac{1}{3!}\varphi_0^3+F_0(e^{(2/4)iT_0}+e^{-(2/4)iT_0}) \\ &= (-2iD_2a_0+\alpha_1a_0+\frac{1}{2}|a_0|^2a_0)e^{iT_0}+\frac{1}{6}a_0^3e^{3iT_0}+ \\ &+ F_0\left(a_0e^{i(1+2/4)T_0}+a_0e^{i(1-2/4)T_0}\right)+c.c. \end{aligned} \tag{5.63}$$

For nonsecularity, we require that

$$-2iD_2a_0+\alpha_1a_0+\frac{1}{2}|a_0|^2a_0=0, \tag{5.64}$$

and the solution φ_1 is given by

$$\varphi_1=a_1e^{i(1+2/4)T_0}+b_1e^{i(1-2/4)T_0}+c_1e^{3iT_0}+c.c.,$$

where

$$a_1=-\frac{4}{5}F_0a_0,\quad b_1=\frac{4}{3}F_0a_0,\quad c_1=-\frac{1}{48}a_0^3.$$

At the order μ^4 we have $D_3a_0=0$. At the order μ^5 we arrive at

$$\begin{aligned} D_0^2\varphi_2+\varphi_2 &= (-D_2^2-2D_0D_4+\alpha_2)\varphi_0+(-2D_0D_2+\alpha_1)\varphi_1+\frac{1}{2}\varphi_0^2\varphi_1- \\ &- \frac{1}{3!}\alpha_1\varphi_0^3-\frac{1}{5!}\varphi_0^5+F_0\left(e^{(2/4)iT_0}+e^{-(2/4)iT_0}\right)\left(\varphi_1-\frac{1}{3!}\varphi_0^3\right). \end{aligned} \tag{5.65}$$

Substituting for φ_0 and φ_1 we obtain

$$-2iD_4a_0-D_2^2a_0+\alpha_2a_0-\frac{1}{2}\alpha_1|a_0|^2a_0+\frac{1}{2}\bar{a}_0c_1-\frac{1}{12}|a_0^4a_0+F_0(a_1+b_1)=0.$$

Using eq.(5.64) we find

$$D_2^2a_0=-\frac{1}{4}\alpha_1^2a_0-\frac{1}{4}\alpha_1|a_0|^2a_0-\frac{1}{16}|a_0|^4a_0,$$

hence the equation of the main harmonic is

$$-2iD_4a_0+\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{8}{15}F_0^2\right)a_0-\frac{1}{4}\alpha_1|a_0|^2a_0-\frac{1}{32}|a_0|^4a_0=0. \tag{5.66}$$

Subsequently, a solution φ_2 is given by

$$\begin{aligned}\varphi_2 &= a_2e^{i(1+4/4)T_0}+b_2e^{i(1-4/4)T_0}+c_2e^{i(1+2/4)T_0}+d_2e^{i(1-2/4)T_0}+e_2e^{i(1+10/4)T_0}+\\ &+ f_2e^{i(1+6/4)T_0}+g_2e^{5iT_0}+h_2e^{3iT_0}+c.c.,\end{aligned}$$

where

$$a_2=\frac{4}{15}F_0^2a_0,\quad b_2=\frac{4}{3}F_0^2a_0,\quad c_2=\frac{2}{75}F_0|a_0|^2a_0-\frac{8}{25}F_0\alpha_1a_0,$$
$$d_2=\frac{8}{9}F_0\alpha_1a_0+\frac{2}{15}F_0|a_0|^2a_0,\quad e_2=\frac{47}{45\times 20}F_0a_0^3,\quad f_2=-\frac{23}{21\times 12}F_0a_0^3,$$
$$g_2=\frac{1}{8\times 160}a_0^5,\quad h_2=-\frac{1}{24}|a_0|^2a_0^3.$$

At the order μ^6 we have $D_5a_0=0$. At the order μ^7 we obtain

$$\begin{aligned}D_0^2\varphi_3+\varphi_3 &= (-2D_0D_6-2D_2D_4+\alpha_3)\varphi_0+(-D_2^2-2D_0D_4+\alpha_2)\varphi_1+\\ &+ (-2D_0D_2+\alpha_1)\varphi_2+\frac{1}{2}\varphi_0^2\varphi_2+\frac{1}{2}\varphi_0\varphi_1^2-\frac{1}{24}\varphi_0^4\varphi_1+\\ &+ \frac{1}{7!}\varphi_0^7-\frac{1}{2}\alpha_1\varphi_0^2\varphi_1-\frac{1}{3!}\alpha_2\varphi_0^3+\frac{1}{5!}\alpha_1\varphi_0^5+\\ &+ F_0\left(e^{(2/4)iT_0}+e^{-(2/4)iT_0}\right)\left(\varphi_2-\frac{1}{2}\varphi_0^2\varphi_1+\frac{1}{5!}\varphi_0^5\right)\end{aligned} \tag{5.67}$$

Substituting for φ_0, φ_1, φ_2, the nonsecularity condition is given by

$$\begin{aligned}&-2iD_6a_0+\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2+\alpha_3+\frac{188}{15\times 15}F_0^2\alpha_1\right)a_0+\left(\frac{1}{4}\alpha_2-\frac{1}{16}\alpha_1^2+\right.\\ &+\left.\frac{38}{45}F_0^2\right)|a_0|^2a_0-\frac{5}{64}\alpha_1|a_0|^4a_0+\frac{1}{16\times 32}|a_0|^6a_0=0.\end{aligned} \tag{5.68}$$

A particular solution φ_3 is given by

$$\begin{aligned}\varphi_3 &= a_3e^{i(1+18/4)T_0}+b_3e^{i(1+14/4)T_0}+c_3e^{i(1+12/4)T_0}+d_3e^{i(1+10/4)T_0}+e_3e^{i(1+6/4)T_0}+\\ &+ f_3e^{i(1+4/4)T_0}+g_3e^{i(1-4/4)T_0}+h_3e^{i(1+2/4)T_0}+j_3e^{i(1-2/4)T_0}+k_3e^{7iT_0}+\\ &+ l_3e^{5iT_0}+m_3e^{3iT_0}+c.c.,\end{aligned}$$

where

$$a_3=-\frac{5509}{117\times 40\times 360}F_0a_0^5,\quad b_3=\frac{457}{77\times 48\times 12}F_0a_0^5,\quad c_3=-\frac{163}{225\times 12}F_0^2a_0^3,$$
$$d_3=\frac{1037}{45\times 450}F_0\alpha_1a_0^3-\frac{5621}{75\times 45\times 64}F_0|a_0|^2a_0^3,$$
$$e_3=-\frac{523}{21\times 126}F_0\alpha_1a_0^3-\frac{227}{315\times 21\times 21}F_0|a_0|^2a_0^3-\frac{16}{21\times 15}F_0^3a_0,$$

$$f_3 = \frac{44}{225}F_0^2\alpha_1 a_0 - \frac{2}{75}F_0^2|a_0|^2a_0 - \frac{2677}{45\times 210}F_0^2a_0^3,$$
$$g_3 = \frac{20}{9}F_0^2\alpha_1 a_0 - \frac{16}{15}F_0^2|a_0|^2\bar{a}_0 + \frac{58}{45}F_0^2|a_0|^2a_0,$$
$$h_3 = -\left(\frac{4}{5}\right)^2\left(\frac{13}{40}F_0\alpha_1^2 + \frac{1}{2}F_0\alpha_2 - \frac{17}{15}F_0^3\right)a_0 + \left(\frac{2}{225}F_0\alpha_1 - \frac{401}{75\times 120}F_0 + \frac{4}{5}F_0^2\right)|a_0|^2a_0,$$
$$j_3 = \left(\frac{4}{3}\right)^2 F_0\left(\frac{11}{24}\alpha_1^2 + \frac{11}{15}F_0^2 + \frac{1}{2}\alpha_2\right)a_0 + \left(\frac{4}{3}\right)^2 F_0^3\bar{a}_0 - \frac{542}{225}F_0\alpha_1|a_0|^2a_0 - \frac{6687}{360\times 105}F_0|a_0|^4a_0,$$
$$k_3 = \frac{69}{7!\times 32\times 16}a_0^7, \quad l_3 = \frac{1}{8\times 160}\alpha_1 a_0^5 + \frac{14903}{7!\times 24\times 106}|a_0|^2a_0^5,$$
$$m_3 = \frac{1}{8\times 32}\alpha_1|a_0|^2a_0^3 + \frac{1}{128\times 16}|a_0|^4a_0^3 + \frac{359}{96\times 21}F_0^2a_0^3.$$

At the order μ^8 we obtain $D_7a_0 = 0$. The ninth order of μ gives

$$\begin{aligned}
D_0^2\varphi_4 + \varphi_4 = &\\
& (-D_4^2 - 2D_0D_8 - 2D_2D_6 + \alpha_4)\varphi_0 + (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_1 + \\
+ \;& (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_2 + (-D_0D_2 + \alpha_1)\varphi_3 + \frac{1}{3!}(\varphi_1^3 + 3\varphi_0^2\varphi_3 + \\
+ \;& 6\varphi_0\varphi_1\varphi_2) - \frac{1}{5!}(5\varphi_0^4\varphi_2 + 10\varphi_0^3\varphi_1^2) + \frac{7}{7!}\varphi_0^6\varphi_1 - \frac{1}{9!}\varphi_0^9 - \frac{1}{2}\alpha_1\varphi_0^2\varphi_2 + \\
+ \;& \frac{1}{2}\alpha_1\varphi_0\varphi_1^2) - \frac{1}{2}\alpha_2\varphi_0^2\varphi_1 - \frac{1}{6}\alpha_3\varphi_0^3 + \frac{5}{5!}\alpha_1\varphi_0^4\varphi_1 + \frac{1}{5!}\alpha_2\varphi_0^5 - \frac{1}{7!}\alpha_1\varphi_0^7 + \\
+ \;& F_0\left(e^{(2/4)iT_0} + e^{-(2/4)iT_0}\right)\left(\varphi_3 - \frac{1}{2}\varphi_0^2\varphi_2 - \frac{1}{2}\varphi_0\varphi_1^2 + \frac{5}{5!}\varphi_0^4\varphi_1 - \frac{1}{7!}\varphi_0^7\right),
\end{aligned} \tag{5.69}$$

whose nonsecularity condition is given by

$$\begin{aligned}
& -2iD_8a_0 + \left(\alpha_4 + \frac{1}{2}\alpha_1\alpha_3 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{5}{64}\alpha_1^4 + \frac{2192}{75\times 45}F_0^4 + \frac{188}{15\times 15}F_0^2\alpha_2 + \right.\\
+ \;& \left.\frac{4223}{75\times 45}F_0^2\alpha_1^2\right)a_0 - \left(\frac{1}{32}\alpha_1^3 + \frac{11}{375}F_0^2\alpha_1 - \frac{1}{8}\alpha_1\alpha_2 - \frac{1}{64\times 64} + \frac{3}{4}\alpha_3\right)|a_0|^2a_0 - \\
- \;& \left(\frac{1}{120} - \frac{9}{64}\alpha_2 + \frac{3}{64}\alpha_1^2 + \frac{157339}{3150\times 96}F_0^2\right)|a_0|^4a_0 + \frac{237405}{7!\times 192}\alpha_1|a_0|^6a_0 + \\
+ \;& \frac{3087}{7!\times 512}|a_0|^8a_0 - \frac{8}{375}F_0^2|a_0|^2a_0^3 + \left(\frac{4}{3}\right)^2 F_0^4\bar{a}_0 = 0.
\end{aligned} \tag{5.70}$$

Equations (5.64), (5.66), (5.68) and (5.70) result from the multiple scale expansion of

$$\begin{aligned}
& -2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \frac{1}{2}\mu^2|a_0|^2a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right)a_0 - \frac{1}{4}\mu^4\alpha_1|a_0|^2a_0 - \\
- \;& \frac{1}{32}\mu^4|a_0|^4a_0 + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \frac{188}{15\times 15}F_0^2\alpha_1 + \alpha_3\right)a_0 - \frac{1}{16}\mu^6\alpha_1^2|a_0|^2a_0 - \\
- \;& \frac{5}{64}\mu^6\alpha_1|a_0|^4a_0 + \frac{38}{45}\mu^6F_0^2|a_0|^2a_0 + \frac{1}{4}\mu^6\alpha_2|a_0|^2a_0 + \frac{1}{16\times 32}\mu^6|a_0|^6a_0 + \\
+ \;& \mu^8\left(\frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \alpha_4 + \frac{1}{2}\alpha_1\alpha_3 + \frac{4223}{75\times 45}F_0^2\alpha_1^2 + \frac{188}{15\times 15}F_0^2\alpha_2 + \frac{2192}{75\times 45}F_0^4 + \right.
\end{aligned}$$

$$
\begin{aligned}
+ & \frac{1}{4}\alpha_2^2\Big) a_0 - \mu^8 \left(\frac{1}{32}\alpha_1^3 + \frac{11}{375}F_0^2\alpha_1 - \frac{1}{8}\alpha_1\alpha_2 - \frac{1}{64\times 64} + \frac{3}{4}\alpha_3\right) |a_0|^2 a_0 - \mu^8 \Big(\frac{1}{120} - \\
- & \frac{9}{64}\alpha_2 + \frac{3}{64}\alpha_1^2 + \frac{157339}{3150\times 96}F_0^2\Big) |a_0|^4 a_0 + \frac{237405}{7!\times 192}\alpha_1\mu^8 |a_0|^6 a_0 + \\
+ & \frac{3087}{7!\times 512}\mu^8 |a_0|^8 a_0 - \frac{8}{375}\mu^8 F_0^2 |a_0|^2 a_0^3 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0. \qquad (5.71)
\end{aligned}
$$

We recall that linear instability occurs when $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = -\frac{8}{15}F_0^2$, and α_4 satisfies

$$
-\frac{6928}{75\times 45}F_0^4 < \alpha_4 < \frac{5072}{75\times 45}F_0^4
$$

[see eq.(4.65)]. In this case the amplitude equation is

$$
\begin{aligned}
& -2i\frac{\partial a_0}{\partial \tau} + \mu^8\left(\alpha_4 + \frac{928}{75\times 45}F_0^4\right) a_0 + \left(\frac{1}{2}\mu^2 + \frac{2}{3}\mu^6 F_0^2 + \frac{1}{64\times 64}\mu^8\right) |a_0|^2 a_0 - \\
- & \left(\frac{1}{32}\mu^4 + \frac{134659}{3150\times 96}\mu^8 F_0^2 + \frac{1}{375}\mu^8\right) |a_0|^4 a_0 + \frac{1}{16\times 32}\mu^6 |a_0|^6 a_0 - \\
- & \frac{3087}{7!\times 512}\mu^8 |a_0|^8 a_0 - \frac{8}{375}\mu^8 F_0^2 |a_0|^2 a_0^3 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0. \qquad (5.72)
\end{aligned}
$$

Let $p = 2$.

At the order μ^2 we find

$$
D_0^2\varphi_0 + \varphi_0 = 0,
$$

whose solution is

$$
\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c
$$

At the order μ^3 we have $D_1 a_0 = 0$. The order μ^4 gives

$$
\begin{aligned}
D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_0 \\
&= (-2iD_4 a_0 + \alpha_1 a_0)e^{iT_0} + F_0(a_0 e^{i(1+2/4)T_0} + a_0 e^{i(1-2/4)T_0}) + c.c. \qquad (5.73)
\end{aligned}
$$

The nonsecularity condition is given by

$$
-2iD_2 a_0 + \alpha_1 a_0 = 0, \qquad (5.74)
$$

and a solution φ_1 is

$$
\varphi_1 = a_1 e^{i(1+2/4)T_0} + b_1 e^{i(1-2/4)T_0} + c.c.,
$$

where

$$
a_1 = -\frac{4}{5}F_0 a_0, \quad b_1 = \frac{4}{3}F_0 a_0.
$$

The order μ^5 yields $D_3 a_0$ and at $O(\mu^6)$ we arrive at

$$
\begin{aligned}
D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + \\
&+ \frac{1}{3!}\varphi_0^3 + F_0(e^{(2/4)iT_0} + e^{-(2/4)T_0})\varphi_1 \\
&= \left[-D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0(a_1 + b_1)\right] e^{iT_0} + \\
&+ (-3iD_2 a_1 + \alpha_1 a_1)e^{i(1+2/4)T_0} + (-iD_2 b_1 + \alpha_1 b_1)e^{i(1-2/4)T_0} + \\
&+ \frac{1}{6}a_0^3 e^{3iT_0} + F_0(a_1 e^{i(1+4/4)T_0} + b_1 e^{i(1-4/4)T_0}) + c.c. \qquad (5.75)
\end{aligned}
$$

For nonsecularity we require

$$-D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0(a_1 + b_1) = 0,$$

which upon substitution of (5.74), a_1 and b_1 gives

$$-2iD_4 a_0 + \left(\alpha_2 + \frac{1}{4}\alpha_1^2 + \frac{8}{15}F_0^2\right) a_0 + \frac{1}{2}|a_0|^2 a_0 = 0. \tag{5.76}$$

A particular solution φ_2 is then given by

$$\varphi_2 = a_2 e^{i(1+4/4)T_0} + b_2 e^{i(1-4/4)T_0} + c_2 e^{i(1+2/4)T_0} + d_2 e^{i(1-2/4)T_0} + e_2 e^{i(1+8/4)T_0} + c.c,$$

where

$$a_2 = \frac{4}{15}F_0^2 a_0, \quad b_2 = \frac{4}{3}F_0^2 a_0, \quad c_2 = -\frac{8}{25}F_0 \alpha_1 a_0,$$
$$d_2 = \frac{8}{9}F_0 \alpha_1 a_0, \quad e_2 = -\frac{1}{48}a_0^3.$$

At the order μ^7 we have $D_5 a_0 = 0$ whereas at the order μ^8 we arrive at

$$\begin{aligned} D_0^2\varphi_3 + \varphi_3 &= (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_0 + (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_1 - \frac{1}{3!}\alpha_1\varphi_0^3 + \\ &+ (-2D_0D_2 + \alpha_1)\varphi_2 + \frac{1}{2}\varphi_0\varphi_1 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_2. \end{aligned} \tag{5.77}$$

The nonsecularity condition is given by

$$-2iD_6 a_0 - 2D_2D_4 a_0 + \alpha_3 a_0 - \frac{1}{2}\alpha_1 |a_0|^2 a_0 + F_0(c_2 + d_2) = 0.$$

Making use of (5.74) and (5.76), and substitute for c_2 and d_2 we have the condition for the absence of secular terms given by

$$-2iD_6 a_0 + \left(\frac{1}{2}\alpha_1\alpha_2 + \frac{1}{8}\alpha_1^3 + \alpha_3 + \frac{188}{15\times 15}F_0^2\alpha_1\right) a_0 - \frac{1}{4}\alpha_1 |a_0|^2 a_0 = 0. \tag{5.78}$$

A particular solution φ_3 is

$$\begin{aligned} \varphi_3 &= a_3 e^{i(1+6/4)T_0} + b_3 e^{i(1-6/4)T_0} + c_3 e^{i(1+4/4)T_0} + d_3 e^{i(1-4/4)T_0} + e_3 e^{i(1+2/4)T_0} + \\ &+ f_3 e^{3iT_0} + g_3 e^{i(1+10/4)} + c.c \end{aligned}$$

where

$$a_3 = -\frac{31}{12\times 21}F_0 a_0^3 + \frac{4}{15}F_0^3 a_0,$$
$$b_3 = \left(\frac{4}{3}\right)^2 \left(\frac{11}{24}F_0\alpha_1^2 - \frac{1}{2}F_0\alpha_2 - \frac{11}{15}F_0^3\right)\bar{a}_0 + \frac{4}{5}F_0|a_0|^2\bar{a}_0 + \left(\frac{4}{3}\right)^2 F_0^3 a_0,$$
$$c_3 = \frac{4\times 11}{15\times 15}F_0^2\alpha_1 a_0, \quad d_3 = \frac{20}{9}F_0^2\alpha_1 a_0,$$

$$e_3 = -\left(\frac{4}{5}\right)^2\left(\frac{13}{40}F_0\alpha_1^2 - \frac{1}{2}F_0\alpha_2 - \frac{17}{15}F_0^3\right)a_0 - \frac{4}{15}F_0|a_0|^2a_0,$$

$$f_3 = 0, \quad g_3 = \frac{101}{60\times 45}F_0a_0^3.$$

At the order μ^9 we arrive at $D_7a_0 = 0$. The tenth order of μ yields

$$\begin{aligned} & D_0^2\varphi_4 + \varphi_4 = \\ = \;& (-D_4^2 - 2D_0D_8 - 2D_2D_6 + \alpha_4)\varphi_0 + (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_1 + \\ + \;& (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_2 + (-2D_0D_2 + \alpha_1)\varphi_3 + \frac{1}{2}(\varphi_0^2\varphi_2 + \varphi_0\varphi_1^2) - \\ - \;& \frac{1}{5!}\varphi_0^5 - \frac{1}{2}\alpha_1\varphi_0^2\varphi_1 - \frac{1}{2}\alpha_2\varphi_0^3 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\left(\varphi_3 - \frac{1}{2}\varphi_0^2\varphi_1\right). \end{aligned} \tag{5.79}$$

The nonsecularity condition is given by

$$-D_4^2a_0 - 2iD_8a_0 - 2D_2D_6a_0 + \alpha_4a_0 - \frac{1}{12}|a_0|^4a_0 - \frac{1}{2}\alpha_2|a_0|^2a_0 + F_0(\bar{b}_3 + e_3) = 0.$$

Equivalently,

$$\begin{aligned} & -2iD_8a_0 + \left(\frac{1}{4}\alpha_2^2 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{5}{64}\alpha_1^4 + \frac{188}{15\times 15}F_0^2\alpha_2 + \frac{4223}{75\times 45}F_0^2\alpha_1^2 + \frac{2192}{75\times 45}F_0^4 + \right. \\ + \;& \left.\frac{1}{2}\alpha_1\alpha_3 + \alpha_4\right)a_0 - \left(\frac{1}{16}\alpha_1^2 + \frac{1}{4}\alpha_2 - \frac{2}{3}F_0^2\right)|a_0|^2a_0 - \frac{1}{48}|a_0|^4a_0 + \left(\frac{4}{3}\right)^2F_0^4\bar{a}_0 = 0 \end{aligned} \tag{5.80}$$

The amplitude equation is given by

$$\begin{aligned} & -2i\frac{\partial a_0}{\partial\tau} + \left[\mu^2\alpha_1 + \mu^4\left(\alpha_2 + \frac{1}{4}\alpha_1^2 + \frac{8}{15}F_0^2\right) + \mu^6\left(\frac{1}{2}\alpha_1\alpha_2 + \frac{1}{8}\alpha_1^3 + \frac{188}{15\times 15}F_0^2\alpha_1 + \alpha_3\right)a_0 + \right. \\ + \;& \left.\mu^8\left(\frac{1}{4}\alpha_2^2 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{5}{64}\alpha_1^4 + \frac{188}{15\times 15}F_0^2\alpha_2 + \frac{4223}{75\times 45}F_0^2\alpha_1^2 + \frac{2192}{75\times 45}F_0^4 + \frac{1}{2}\alpha_1\alpha_3 + \alpha_4\right)\right]a_0 + \\ + \;& \left(\frac{1}{2}\mu^4 - \frac{1}{4}\mu^6\alpha_1 - \frac{1}{16}\mu^8\alpha_1^2 - \frac{1}{4}\mu^8\alpha_2 + \frac{3}{2}\mu^8F_0^2\right)|a_0|^2a_0 - \\ - \;& \frac{1}{48}\mu^8\alpha_2|a_0|^4a_0 + \left(\frac{4}{3}\right)^2\mu^8F_0^4\bar{a}_0 = 0. \end{aligned} \tag{5.81}$$

We recall that linear instability occurs when $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = -\frac{8}{15}F_0^2$ and α_4 satisfies

$$-\frac{6928}{75\times 45}F_0^4 < \alpha_4 < \frac{5072}{75\times 45}F_0^4,$$

see eq.(4.65). For these α_1, α_2 and α_3 eq. (5.81) becomes

$$\begin{aligned} & -2i\frac{\partial a_0}{\partial\tau} + \mu^8\left(\alpha_4 + \frac{928}{45\times 75}F_0^4\right)a_0 + \left(\frac{1}{2}\mu^4 + \frac{4}{5}\mu^8F_0^2\right)|a_0|^2a_0 - \\ - \;& \frac{1}{48}\mu^8|a_0|^4a_0 + \left(\frac{4}{3}\right)^2\mu^8F_0^4\bar{a}_0 = 0. \end{aligned} \tag{5.82}$$

Let $p = 3$.
At the lowest order we find $\varphi_0 = a_0 e^{iT_0} + c.c.$ and at μ^4 we have $D_1 a_0 = 0$. At the order μ^5 we arrive at

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1 a_0)e^{iT_0} + F_0(a_0e^{i(1+2/4)T_0} + a_0e^{i(1-2/4)T_0}) + c.c. \end{aligned} \tag{5.83}$$

whose nonsecularity condition is given by

$$-2iD_2a_0 + \alpha_1 a_0 = 0, \tag{5.84}$$

and particular solution is

$$\varphi_1 = a_1 e^{i(1+2/4)T_0} + b_1 e^{i(1-2/4)T_0} + c.c.,$$

where

$$a_1 = -\frac{4}{5}F_0a_0, \quad b_1 = \frac{4}{3}F_0a_0.$$

The order μ^6 yields $D_3a_0 = 0$ and order μ^7 gives

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_1 \\ &= [-D_2^2a_0 - 2iD_4a_0 + \alpha_2a_0 + F_0(a_1 + b_1)]e^{iT_0} + (-3iD_2a_0 + \alpha_1a_1) \times \\ &\times \ e^{i(1+2/4)T_0} + (-iD_2b_1 + \alpha_1b_1)e^{i(1-2/4)T_0} + F_0(a_1e^{i(1+4/4)T_0} + \\ &+ \ b_1e^{i(1-4/4)T_0}) + c.c. \end{aligned} \tag{5.85}$$

Suppressing secular terms we have

$$-2iD_4a_0 - D_2^2a_0 + \alpha_2a_0 + F_0(a_1 + b_1) = 0,$$

and by using (5.84) we have

$$-2iD_4a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right)a_0 = 0, \tag{5.86}$$

and a particular solution given by

$$\varphi_2 = a_2e^{i(1+4/4)T_0} + b_2e^{i(1-4/4)T_0} + c_2e^{i(1+2/4)T_0} + d_2e^{i(1-2/4)T_0} + c.c.,$$

where

$$a_2 = \frac{4}{15}F_0^2a_0, \quad b_2 = \frac{4}{3}F_0^2a_0, \quad c_2 = -\frac{8}{25}F_0\alpha_1a_0, \quad d_2 = \frac{8}{9}F_0\alpha_1a_0.$$

The order μ^8 gives $D_5a_0 = 0$ and at the order μ^9 we arrive at

$$\begin{aligned} D_0^2\varphi_3 + \varphi_3 &= (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_0 + (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_1 + \\ &+ \ (-2D_0D_2 + \alpha_1)\varphi_2 + \frac{1}{3!}\varphi_0^3 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_2 \end{aligned} \tag{5.87}$$

and the suppression of secular terms yields

$$-2iD_6 - 2D_2D_4a_0 + \alpha_3a_0 + \frac{1}{2}|a_0|^2a_0 + F_0(c_2 + d_2) = 0,$$

which upon substitution of eqs.(5.84) and (5.86) becomes

$$-2iD_6a_0+\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2+\alpha_3+\frac{188}{15\times 15}F_0^2\alpha_1\right)a_0+\frac{1}{2}|a_0|^2a_0=0. \tag{5.88}$$

A particular solution φ_3 is given by

$$\varphi_3=a_3e^{i(1+6/4)T_0}+b_3e^{i(1-6/4)T_0}+c_3e^{i(1+4/4)T_0}+d_3e^{i(1-4/4)T_0}+e_3e^{i(1+2/4)T_0}+f_3e^{i(1+8/4)T_0}+c.c.,$$

where

$$a_3=-\frac{4\times 4}{21\times 15}F_0^3a_0,\quad b_3=\left(\frac{4}{3}\right)^2\left(\frac{11}{24}F_0\alpha_1^2+\frac{1}{2}F_0\alpha_2+\frac{11}{15}F_0^3\right)\bar{a}_0+\left(\frac{4}{3}\right)^2F_0^3a_0,$$

$$c_3=\frac{44}{15\times 15}F_0^2\alpha_1a_0,\quad d_3=\frac{4}{3}F_0^2\alpha_1a_0,$$

$$e_3=\left(\frac{4}{5}\right)^2\left[\frac{13}{40}F_0\alpha_1^2-\frac{1}{2}F_0\alpha_2-\frac{17}{15}F_0^3\right]a_0,\quad f_3=-\frac{1}{48}a_0^3.$$

At the order μ^{10}, the perturbation expansion gives $D_7a_0=0$, and going further to higher order $O(\mu^8)$ we arrive at

$$\begin{aligned}D_0^2\varphi_4+\varphi_4 &= (-D_4^2-2D_0D_8-D_2D_6+\alpha_4)\varphi_0+(-2D_0D_6-2D_2D_4+\\&+\alpha_3)\varphi_1+(-D_2^2-2D_0D_4+\alpha_2)\varphi_2+(-2D_0D_2+\alpha_1)\varphi_3+\\&+\frac{1}{2}\varphi_0^2\varphi_1-\frac{1}{3!}\alpha_1\varphi_0^3+F_0(e^{(2/4)iT_0}+e^{-(2/4)iT_0})\varphi_3,\end{aligned} \tag{5.89}$$

and the nonsecularity condition is given by

$$-2iD_8a_0-D_4^2a_0-2D_2D_6a_0+\alpha_4a_0-\frac{1}{2}\alpha_1|a_0|^2a_0+F_0(\bar{b}_3+e_3)=0,$$

where we have substituted for φ_i's. Making use of (5.84), (5.86), and (5.88) we arrive at

$$\begin{aligned}&-2iD_8a_0+\left(\alpha_4+\frac{5}{64}\alpha_1^2+\frac{3}{8}\alpha_1^2\alpha_2+\frac{4223}{75\times 45}F_0^2\alpha_1^2+\frac{1}{4}\alpha_2^2+\frac{1}{2}\alpha_1\alpha_3+\right.\\&+\left.\frac{188}{15\times 15}F_0^2\alpha_2+\frac{2192}{75\times 45}F_0^4\right)a_0-\frac{1}{4}\alpha_1|a_0|^2a_0+\left(\frac{4}{3}\right)^2F_0^4\bar{a}_0=0.\end{aligned} \tag{5.90}$$

Equations (5.84), (5.86), (5.88) and (5.90) result from a multiple scale expansion of

$$\begin{aligned}&-2i\frac{\partial a_0}{\partial\tau}+\left[\mu^2\alpha_1+\mu^4\left(\frac{1}{4}\alpha_1^2+\alpha_2+\frac{8}{15}F_0^2\right)+\mu^6\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2+\right.\right.\\&+\left.\alpha_3+\frac{188}{15\times 15}F_0^2\alpha_1\right)+\mu^8\left(\alpha_4+\frac{5}{64}\alpha_1^4+\frac{3}{8}\alpha_1^2\alpha_2+\frac{4223}{75\times 45}\alpha_1^2F_0^2+\frac{1}{4}\alpha_2^2+\right.\\&+\left.\left.\frac{1}{2}\alpha_1\alpha_2+\frac{1}{4}\alpha_2^2+\frac{1}{2}\alpha_1\alpha_3+\frac{188}{15\times 15}F_0^2\alpha_2+\frac{2192}{75\times 45}F_0^4\right)\right]a_0+\left(\frac{1}{2}\mu^6-\frac{1}{4}\mu^8\alpha_1\right)|a_0|^2a_0-\\&-\left(\frac{4}{3}\right)^2\mu^8F_0^4\bar{a}_0=0.\end{aligned} \tag{5.91}$$

The linear instability occurs when $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = -\frac{8}{15}F_0^2$ and α_4 satisfying

$$-\frac{6928}{25 \times 135}F_0^4 < \alpha_4 < \frac{5072}{25 \times 135}F_0^4.$$

[see eq.(4.65).] In this case the amplitude equation is

$$-2i\frac{\partial a_0}{\partial \tau} + \mu^8 \left(\alpha_4 + \frac{928}{25 \times 135}F_0^4\right) a_0 + \frac{1}{2}\mu^6 |a_0|^2 a_0 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0. \qquad (5.92)$$

Let $p = 4$.

At the order μ^4 we have $D_0^2\varphi_0 + \varphi_0 = 0$ whose solution is $\varphi_0 = a_0 e^{iT_0} + c.c.$ At the order μ^5 we arrive at $D_1 a_0 = 0$ and order μ^6 gives

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1 a_0)e^{iT_0} + F_0(a_0 e^{i(1+2/4)T_0} + a_0 e^{i(1-2/4)iT_0}) + c.c. \end{aligned} \qquad (5.93)$$

whose solution is

$$\varphi_1 = a_1 e^{i(1+2/4)T_0} + b_1 e^{i(1-2/4)T_0} + c.c.,$$

where

$$a_1 = -\frac{4}{5}F_0 a_0, \quad b_1 = \frac{4}{3}F_0 a_0,$$

when

$$-2iD_2 a_0 + \alpha_1 a_0 = 0. \qquad (5.94)$$

At the order μ^7 we have $D_3 a_0 = 0$, whereas at the eighth order one obtains

$$\begin{aligned} & D_0^2\varphi_2 + \varphi_2 = \\ &= (-D_2^2 + 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_1 \\ &= [-2D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + F_0(a_1 + b_1)]e^{iT_0} + (-2iD_2 a_1 + \alpha_1 a_1)e^{i(1+2/4)T_0} + \\ &+ (-iD_2 b_1 + \alpha_1 b_1 + \alpha_1 b_1)e^{i(1-2/4)T_0} + F_0(a_1 e^{i(1+4/4)T_0} + b_1 e^{i(1-4/4)T_0}) + c.c. \end{aligned} \qquad (5.95)$$

Suppressing secular terms we have

$$-2iD_4 a_0 - D_2^2 a_0 + \alpha_2 a_0 + F_0(a_1 + b_1) = 0,$$

or equivalently,

$$-2iD_4 a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right) a_0 = 0. \qquad (5.96)$$

The solution φ_2 is given by

$$\varphi_2 = a_2 e^{i(1+4/4)T_0} + b_2 e^{i(1-4/4)T_0} + c_2 e^{i(1+2/4)T_0} + d_2 e^{i(1-2/4)T_0} + c.c.,$$

where

$$a_2 = \frac{4}{15}F_0^2 a_0, \quad b_2 = \frac{4}{3}F_0^2 a_0, \quad c_2 = -\frac{8}{25}F_0\alpha_1 a_0, \quad d_2 = \frac{8}{9}F_0\alpha_1 a_0.$$

The order μ^9 yields $D_5 a_0 = 0$. Going further to order μ^{10} we arrive at

$$
\begin{aligned}
D_0^2\varphi_3 + \varphi_3 &= (-2D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_0 + (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_1 + \\
&+ (-2D_0D_2 + \alpha_1)\varphi_2 + F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_2 \\
&= [-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 + F_0(c_2 + d_2)]e^{iT_0} + \\
&+ [-D_2^2a_1 - 3iD_4a_1 + \alpha_2a_1 - 3iD_2c_2 + \alpha_1c_2 + F_0a_2]e^{i(1+2/4)T_0} + \\
&+ [-D_2^2b_1 - iD_4b_1 + \alpha_2b_1 - iD_2d_2 + \alpha_1d_2 + F_0(b_2 + \bar{b}_2)]e^{i(1-2/4)T_0} + \\
&+ [-4iD_2^2 + \alpha_1a_2 + F_0c_2]e^{i(1+4/4)T_0} + [\alpha_1b_2 + F_0d_2]e^{i(1-4/4)T_0} + \\
&+ F_0a_2e^{i(1+6/4)T_0} + c.c
\end{aligned}
\tag{5.97}
$$

For nonsecularity we have

$$
-2iD_6a_0 - 2D_2D_4a_0 + \alpha_3a_0 + F_0(c_2 + d_2) = 0,
$$

and by making relevant substitutions we obtain

$$
-2iD_6a_0 + \left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \alpha_3 + \frac{188}{15\times15}F_0^2\alpha_1\right)a_0 = 0. \tag{5.98}
$$

A solution φ_3 is given by

$$
\varphi_3 = a_3e^{i(1+6/4)T_0} + b_3e^{i(1-6/4)T_0} + c_3e^{i(1+4/4)T_0} + d_3e^{i(1-4/4)T_0} + e_3e^{i(1+2/4)T_0} + c.c.,
$$

where

$$
a_3 = -\frac{16}{21\times15}F_0^3a_0, \quad b_3 = \left(\frac{4}{3}\right)^2\left[\frac{11}{24}F_0\alpha_1^2 + \frac{1}{2}F_0\alpha_2 + \frac{11}{15}F_0^3\right]\bar{a}_0 + \left(\frac{4}{3}\right)^2F_0^3a_0,
$$

$$
c_3 = \frac{44}{15\times15}F_0^2\alpha_1a_0, \quad d_3 = \frac{4}{3}F_0^2\alpha_1a_0,
$$

$$
e_3 = -\left(\frac{4}{5}\right)^2\left[\frac{13}{40}F_0\alpha_1^2 + \frac{1}{2}F_0\alpha_2 + \frac{17}{15}F_0^3\right]a_0.
$$

At the order μ^{11} we have $2D_0D_7a_0 = 0$ which yields $D_7a_0 = 0$. The order μ^{12} gives us

$$
\begin{aligned}
D_0^2\varphi_4 + \varphi_4 &= (-D_4^2 - 2D_0D_8 - 2D_2D_6 + \alpha_4)\varphi_0 + (-D_0D_6 - 2D_2D_4 + \alpha_3)\varphi_1 + \\
&+ (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_2 + (-2D_0D_2 + \alpha_1)\varphi_3 + \frac{1}{3!}\varphi_0^3 + \\
&+ F_0(e^{(2/4)iT_0} + e^{-(2/4)iT_0})\varphi_3 \\
&= [-D_4^2a_0 - 2iD_8a_0 - 2D_2D_6a_0 + \alpha_4a_0 + \frac{1}{2}|a_0|^2a_0 + F_0(\bar{b}_3 + e_3)]e^{iT_0} + \\
&+ [-3iD_6a_1 - 2D_2D_4a_1 + \alpha_3a_1 - D_2^2c_2 - 3iD_4c_2 + \alpha_2c_2 - 3iD_2e_3 + \alpha_1e_3 + \\
&+ F_0c_3 + [-iD_6b_1 - 2D_2D_4b_1 + \alpha_3b_1 - D_2^2d_2 - iD_4d_2 + \alpha_2d_2 - iD_2\bar{b}_3 + \\
&+ \alpha_1\bar{b}_3 + F_0(d_3 + \bar{d}_3)]e^{i(1-2/4)T_0} + [-D_2^2a_2 - 4iD_4a_2 + \alpha_2a_2 - 4iD_2c_3 + \alpha_1c_3 + \\
&+ F_0(a_3 + e_3)]e^{i(1+4/4)T_0} + [-D_2^2b_2 + \alpha_2b_2 + \alpha_1d_3 + F_0b_3]e^{i(1-4/4)T_0} + \\
&+ [-5iD_2a_3 + \alpha_1a_3 + F_0c_3]e^{i(1+6/4)T_0} + \left(\frac{1}{6}a_0^3 + F_0a_3\right)e^{i(1+8/4)T_0} + c.c.
\end{aligned}
\tag{5.99}
$$

The nonsecularity condition is given by

$$-2iD_8a_0 - D_4^2a_0 - 2D_2D_6a_0 + \alpha_4 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0(\bar{b}_3 + e_3) = 0, \tag{5.100}$$

for which upon substitution of (5.94), (5.96) and (5.98) we arrive at

$$\begin{aligned} &-2iD_8a_0 + \left(\alpha_4 + \frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{4223}{75\times 45}F_0^2\alpha_1^2 + \frac{1}{4}\alpha_2^2 + \frac{1}{2}\alpha_1\alpha_3 + \right.\\ &+ \left. \frac{188}{15\times 15}F_0^2\alpha_2 + \frac{2192}{25\times 135}F_0^4\right) a_0 + \frac{1}{2}|a_0|^2 a_0 + \left(\frac{4}{3}\right)^2 F_0^4 \bar{a}_0 = 0. \end{aligned} \tag{5.101}$$

Equations (5.94), (5.96), (5.98) and (5.101) result from a multiple scale expansion of

$$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right) + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \right.\right.\\ &+ \left. \alpha_3 + \frac{188}{15\times 15}F_0^2\alpha_1\right) + \mu^8\left(\alpha_4 + \frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{4223}{75\times 45}F_0^2\alpha_1^2 + \frac{1}{4}\alpha_2^2 + \right.\\ &+ \left.\left. \frac{1}{2}\alpha_1\alpha_3 + \frac{188}{15\times 15}F_0^2\alpha_2 + \frac{2192}{75\times 45}F_0^4\right)\right] a_0 + \frac{1}{2}\mu^8|a_0|^2 a_0 + \\ &+ \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0. \end{aligned} \tag{5.102}$$

Using eq.(4.65), $\alpha_1 = \alpha_3 = 0$, and $\alpha_2 = -\frac{8}{15}F_0^2$ the equation governing the main harmonic is then given by

$$-2i\frac{\partial a_0}{\partial \tau} + \mu^8\left(\alpha_4 + \frac{928}{75\times 45}F_0^4\right) a_0 + \frac{1}{2}\mu^8|a_0|^2 a_0 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0. \tag{5.103}$$

Discussion.

When driving the pendulum at a frequency one-half of ω_0, we find, for various p's, the following amplitude equations:

- $p = 1:$ $$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \mu^8\left(\alpha_4 + \frac{928}{75\times 45}F_0^4\right) + \frac{1}{2}\mu^2|a_0|^2 a_0 - \frac{1}{32}\mu^4|a_0|^4 a_0 + \frac{2}{3}\mu^6 F_0^2|a_0|^2 a_0 + \\ &+ \frac{1}{16\times 32}\mu^6|a_0|^6 a_0 + \frac{1}{64\times 64}\mu^8|a_0|^2 a_0 - \frac{1}{375}\mu^8|a_0|^4 a_0 - \frac{134659}{3150\times 96}\mu^8 F_0^2|a_0|^4 a_0 - \\ &- \frac{3087}{7!\times 512}\mu^8|a_0|^8 a_0 - \frac{8}{375}\mu^8 F_0^2|a_0|^2 a_0^3 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0, \end{aligned}$$
- $p = 2:$ $$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \mu^8\left(\alpha_4 + \frac{928}{75\times 45}F_0^4\right) a_0 + \frac{1}{2}\mu^4|a_0|^2 a_0 + \frac{4}{5}\mu^8 F_0^2|a_0|^2 a_0 - \\ &+ \frac{1}{48}\mu^8|a_0|^4 a_0 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0, \end{aligned}$$
- $p = 3:$ $$-2i\frac{\partial a_0}{\partial \tau} + \mu^8\left(\alpha_4 + \frac{928}{75\times 45}F_0^4\right) a_0 + \frac{1}{2}\mu^6|a_0|^2 a_0 + \left(\frac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0,$$

- $p=4$: $\quad -2i\dfrac{\partial a_0}{\partial\tau} + \mu^8\left(\alpha_4 + \dfrac{928}{75\times 45}F_0^4\right)a_0 + \dfrac{1}{2}\mu^8|a_0|^2 a_0 + \left(\dfrac{4}{3}\right)^2 \mu^8 F_0^4 \bar{a}_0 = 0.$

Similarly to the cases $N = 1, 2$ and 3, we wish to find what p is the correct scaling exponent for the amplitude of the nonlinear solution. As before, we do this by analysing static solutions. Writing $a_0 = Ae^{iT_0}$, we obtain for $p = 1$ the following equation:

$$\left(\frac{1}{2}+\frac{2}{3}\mu^4F_0^2+\frac{1}{64\times 64}\mu^6\right)A^2-\left(\frac{1}{32}\mu^2+\frac{1}{375}\mu^6+\frac{134659}{3150\times 96}F_0^2\mu^6-\frac{8}{375}\mu^6F_0^2\right)A^4+$$
$$+\quad \frac{1}{16\times 32}\mu^4A^6+\frac{3087}{7!\times 512}\mu^6A^8+\mu^6\left(\alpha_4+\frac{928}{75\times 45}F_0^4\right)+\left(\frac{4}{3}\right)^2\mu^6F_0^4(-1)^n=0.$$

Expanding $A = A_0 + \mu A_1 + \mu^2 A_2 + \cdots$ we find $A_0 = A_1 = A_2 = 0$ and

$$\frac{1}{2}A_3^2+\alpha_4+\frac{928}{75\times 45}F_0^4+\left(\frac{4}{3}\right)^2F_0^4(-1)^n=0,$$

which implies that $a_0 \sim O(\mu^3)$. This result contradicts the assumption that $\varphi_0, \varphi_1 \ldots = O(1)$. Thus, $p \neq 1$.

Proceeding in the same way for greater p's we find that $p = 2$ and $p = 3$ are not acceptable either, since they give $a_0 \sim O(\mu^2)$ and $a_0 \sim O(\mu)$, respectively. Lastly, the case $p = 4$ gives

$$A_0^2=-2\left(\alpha_4+\frac{928}{75\times 45}F_0^4+\left(\frac{4}{3}\right)^2F_0^4(-1)^n\right). \tag{5.104}$$

This gives A real only if

$$\alpha_4<\left(\frac{4}{3}\right)^2F_0^4(-1)^{n+1}-\frac{928}{75\times 45}F_0^4.$$

For n odd, this amounts to

$$\alpha_4<\frac{5072}{75\times 45}F_0^4 \tag{5.105}$$

which coincides with the upper bound of the instability window for the linear oscillator:

$$-\frac{6928}{75\times 45}F_0^4<\alpha_4<\frac{5072}{75\times 45}F_0^4$$

[see eq.(4.65)]. When n is even, we have

$$\alpha_4<-\frac{6928}{25\times 135}F_0^4,$$

which is just outside the linear instability region.

Thus, for α_4 eq.(5.105) the instability of the zero solution will saturate by a stationary solution $a_0 = \pm A$, with

$$A=-2\sqrt{\frac{5072}{75\times 45}F_0^4-\alpha_4}.$$

Thus $p = 4$ is the correct scaling exponent. In the same way as in the previous sections, we can show that the scaling $p = 5$ will support stationary solutions of order μ^5.

We did not discuss solutions outside and at the edge of the instability window. This is beyond the scope of this work.

5.5 The driving frequency $\sim 4\omega_0$.

In this section we consider the pendulum driven at the frequency $\sim 4\omega_0$:

$$\varphi_{tt} + \omega_0^2 \left[1 - 2F_0\varepsilon^2 \cos 4\Omega t\right] \sin\varphi = 0. \tag{5.106}$$

Proceeding as before we transform eq.(5.106) to

$$\begin{aligned} \varphi_{\tau\tau} + \varphi &= \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + \frac{1}{3!}\frac{\omega_0^2}{\Omega^2}\varphi^3 - \frac{1}{5!}\frac{\omega_0^2}{\Omega^2}\varphi^5 + \cdots + \\ &+ 2\mu^2 F_0 \cos(4\tau)\left(\varphi - \frac{1}{3!}\varphi^3 + \frac{1}{5!}\varphi^5 + \cdots\right) \end{aligned} \tag{5.107}$$

and let

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots$$

Assuming that

$$\varphi = \mu^p(\varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots),$$

we will find the appropriate value of p.

Let $p = 1$. Comparing terms at same order of μ we obtain at the order μ^1,

$$D_0^2\varphi_0 + \varphi_0 = 0,$$

whose solution is given by

$$\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$$

At the order μ^2 the multiple-scale method yields $D_1 a_0 = 0$. At order μ^3 we have

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + \frac{1}{3!}\varphi_0^3 + F_0(e^{4iT_0} + e^{-4iT_0})\varphi_0 \\ &= \left[-2iD_2a_0 + \alpha_1 a_0 + \frac{1}{2}|a_0|^2 a_0\right] e^{iT_0} + \left[\frac{1}{6}a_0^3 + F_0\bar{a}_0\right] e^{3iT_0} + \\ &+ F_0 a_0 e^{5iT_0} + c.c. \end{aligned} \tag{5.108}$$

whose nonsecularity condition is given by

$$-2iD_2a_0 + \alpha_1 a_0 + \frac{1}{2}|a_0|^2 a_0 = 0, \tag{5.109}$$

and a particular solution is given by

$$\varphi_1 = a_1 e^{5iT_0} + b_1 e^{3iT_0} + c.c.,$$

where

$$a_1 = -\frac{1}{24}F_0 a_0, \quad b_1 = -\frac{1}{48}a_0^3 - \frac{1}{8}F_0\bar{a}_0.$$

At this point we have not as yet arrived at the driven equation, thus we will have to go further to higher orders of μ. At the order μ^4 we obtain $D_3 a_0 = 0$. The order μ^5 yields

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (-D_2^2 - 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 - \frac{1}{3!}\alpha_1\varphi_0^3 + \frac{1}{2}\varphi_0^2\varphi_1 - \\ &- \frac{1}{5!}\varphi_0^5 + F_0\left(e^{4iT_0} + e^{-4iT_0}\right)\left(\varphi_1 - \frac{1}{3!}\varphi_0^3\right). \end{aligned} \tag{5.110}$$

The nonsecularity condition is given by

$$-D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 - \frac{1}{2}\alpha_1 |a_0|^2 a_0 + \frac{1}{2}\bar{a}_0^2 b_1 - \frac{1}{12}|a_0|^4 a_0 + F_0\left(a_1 + \bar{b}_1 - \frac{1}{6}\bar{a}_0^3\right) = 0.$$

If we substitute for a_1, b_1 and make use of (5.109) we obtain

$$-2iD_4 a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right) a_0 - \frac{1}{4}\alpha_1 |a_0|^2 a_0 - \frac{1}{32}|a_0|^4 a_0 - \frac{1}{4}F_0\bar{a}_0^3 = 0. \tag{5.111}$$

Equations (5.109) and (5.111) result from a multiple scale expansion of

$$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right) a_0 + \left(\frac{1}{2}\mu^2 - \frac{1}{4}\alpha_1\mu^4\right)|a_0|^2 a_0 + \\ &- \frac{1}{32}\mu^4 |a_0|^4 a_0 - \frac{1}{4}\mu^4 F_0 \bar{a}_0^3 = 0. \end{aligned} \tag{5.112}$$

Let $p = 2$. At the order μ^2 we have $\varphi_0 = a_0(T_1, T_2, \ldots)e^{iT_0} + c.c.$ and order μ^3 gives $D_1 a_0 = 0$. At the order μ^4 we have

$$\begin{aligned} D_0^2\varphi_1 + \varphi_1 &= (-2D_0D_2 + \alpha_1)\varphi_0 + F_0(e^{4iT_0} + e^{-4iT_0})\varphi_0 \\ &= (-2iD_2 a_0 + \alpha_1 a_0)e^{iT_0} + F_0(a_0 e^{5iT_0} + a_0 e^{-3iT_0}) + c.c., \end{aligned} \tag{5.113}$$

whose nonsecularity condition is given by

$$-2iD_2 a_0 + \alpha_1 a_0 = 0, \tag{5.114}$$

and a particular solution is given by

$$\varphi_1 = a_1 e^{5iT_0} + b_1 e^{3iT_0} + c.c.,$$

where

$$a_1 = -\frac{1}{24}F_0 a_0, \quad b_1 = -\frac{1}{8}F_0\bar{a}_0.$$

The order μ^5 gives $D_3 a_0 = 0$. At the order μ^6 we have

$$\begin{aligned} D_0^2\varphi_2 + \varphi_2 &= (D_2^2 + 2D_0D_4 + \alpha_2)\varphi_0 + (-2D_0D_2 + \alpha_1)\varphi_1 + \frac{1}{3!}\varphi_0^3 + \\ &+ F_0(e^{4iT_0} + e^{-4iT_0})\varphi_1 \\ &= \left[-D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0(a_1 + \bar{b}_1)\right] e^{iT_0} + \\ &+ (-10iD_2 a_1 + \alpha_1 a_1)e^{5iT_0} + \left(-6iD_2 b_1 + \alpha_1 b_1 + \frac{1}{6}a_0^3\right) e^{3iT_0} + \\ &+ F_0(a_1 e^{9iT_0} + b_1 e^{7iT_0}) + c.c. \end{aligned} \tag{5.115}$$

Suppressing secular terms we obtain

$$-D_2^2 a_0 - 2iD_4 a_0 + \alpha_2 a_0 + \frac{1}{2}|a_0|^2 a_0 + F_0(a_1 + \bar{b}_1) = 0.$$

Substituting for a_1, b_1 and using (5.114) we have

$$-2iD_4 a_0 + \left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right) a_0 + \frac{1}{2}|a_0|^2 a_0 = 0. \qquad (5.116)$$

A particular solution of (5.115) is given by

$$\varphi_2 = a_2 e^{9iT_0} + b_2 e^{7iT_0} + c_2 e^{5iT_0} + d_2 e^{3iT_0} + c.c.,$$

where

$$a_2 = \frac{1}{80 \times 24} F_0^2 a_0, \quad b_2 = \frac{1}{8 \times 48} F_0^2 \bar{a}_0, \quad c_2 = -\frac{1}{6 \times 24} F_0 \alpha_1 a_0, \quad d_2 = \frac{1}{16} F_0 \alpha_1 \bar{a}_0 - \frac{1}{48} a_0^3.$$

At the order μ^7 we arrive at $D_5 a_0 = 0$. At the order μ^8 we obtain

$$\begin{aligned} D_0^2 \varphi_3 + \varphi_3 &= (-2D_0 D_6 - 2D_2 D_4 + \alpha_3)\varphi_0 + (-D_2^2 - 2D_0 D_4 + \alpha_2)\varphi_1 + \\ &+ (-2D_0 D_2 + \alpha_1)\varphi_2 - \frac{1}{3!}\alpha_1 \varphi_0^3 + \frac{1}{2}\varphi_0^2 \varphi_1 - \frac{1}{5!}\varphi_0^5 + \\ &+ F_0(e^{4iT_0} + e^{-4iT_0})\left(\varphi_2 - \frac{1}{3!}\varphi_0^3\right) \end{aligned} \qquad (5.117)$$

Substitution of $\varphi_0, \varphi_1, \varphi_2$ yields the nonsecularity condition

$$-2iD_6 a_0 - 2D_2 D_4 a_0 + \alpha_3 a_0 + \frac{1}{2}\bar{a}_0 b_1 - \frac{1}{12}|a_0|^4 a_0 - \frac{1}{2}\alpha_1 |a_0|^2 a_0 + F_0(c_2 + \bar{d}_2 - \frac{1}{6}\bar{a}_0^3) = 0.$$

Equivalently,

$$-2iD_6 a_0 + \left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 - \frac{1}{36}F_0^2\alpha_1 + \alpha_3\right) a_0 - \frac{1}{4}\alpha_1 |a_0|^2 a_0 - \frac{1}{12}|a_0|^4 a_0 - \frac{1}{12}F_0 \bar{a}_0^3 = 0. \qquad (5.118)$$

The equations (5.114), (5.116) and (5.118) result from a multiple scale expansion of

$$\begin{aligned} &-2i\frac{\partial a_0}{\partial \tau} + \left[\mu^2\alpha_1 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right) + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 - \frac{1}{36}F_0^2\alpha_1 + \alpha_3\right)\right] a_0 + \\ &+ \left[\frac{1}{2}\mu^4 - \frac{1}{4}\mu^6\alpha_1\right]|a_0|^2 a_0 - \frac{1}{12}\mu^6 |a_0|^4 a_0 - \frac{1}{12}\mu^6 F_0 \bar{a}_0^3 = 0. \end{aligned} \qquad (5.119)$$

Discussion.

When the pendulum is driven at a frequency $4\omega_0$, the amplitude of nonlinear oscillations obeys the following equations:

- $p = 1:$ $$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right) a_0 + \left(\frac{1}{2}\mu^2 - \frac{1}{4}\alpha_1\mu^4\right)|a_0|^2 a_0 -$$ $$-\frac{1}{32}\mu^4 |a_0|^4 a_0 - \frac{1}{4}\mu^4 F_0 \bar{a}_0^3 = 0,$$

- $p=2:$ $$-2i\frac{\partial a_0}{\partial\tau}+\mu^2\alpha_1 a_0+\mu^4\left(\frac{1}{4}\alpha_1^2+\alpha_2-\frac{1}{6}F_0^2\right)a_0+\mu^6\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2-\right.$$ $$\left.-\frac{1}{36}F_0^2\alpha_1+\alpha_3\right)a_0+\left(\frac{1}{2}\mu^4-\frac{1}{4}\mu^6\alpha_1\right)|a_0|^2a_0-\frac{1}{12}\mu^6|a_0|^4a_0-\frac{1}{12}\mu^6F_0\bar{a}_0^3=0.$$

In order to determine which p to choose, we examine stationary solutions, $\dfrac{\partial a_0}{\partial\tau}=0$. (Notice that there is no linear instability window here.) For $p=1$ we write $a_0=Ae^{i\theta}$ which produces

$$\begin{aligned}&\alpha_1+\mu^2\left(\frac{1}{4}\alpha_1^2+\alpha_2-\frac{1}{6}F_0^2\right)+\left(\frac{1}{2}-\frac{1}{4}\alpha_1\mu^2\right)A^2-\frac{1}{32}\mu^2A^4-\\ &-\frac{1}{4}\mu^2F_0A^2e^{-4i\theta}=0.\end{aligned}\tag{5.120}$$

By setting to zero coefficient of the imaginary part we obtain $\sin 4\theta=0$ which gives $\theta=\frac{\pi}{2}n$, $n=0,\pm1,\pm2,\cdots$. Hence $\cos 4\theta=(-1)^n$. The real part is then given by

$$\alpha_1+\mu^2\left(\frac{1}{4}\alpha_1^2+\alpha_2-\frac{1}{6}F_0^2\right)+\left(\frac{1}{2}-\frac{1}{4}\alpha_1\mu^2\right)A^2-\frac{1}{32}\mu^2A^4-\frac{1}{4}\mu^2F_0A^2(-1)^n=0.$$

By making an expansion $A=A_0+\mu^1A_1+\mu^2A_2+\cdots$, and comparing coefficients at like powers of μ we find, at μ^0,

$$\alpha_1+\frac{1}{2}A_0^2=0,$$

whence $\alpha_1\le 0$. If $\alpha_1<0$, this gives $A_0=\sqrt{-2\alpha_1}$; hence $a_0=O(1)$ as it is supposed to be. If $\alpha_1=0$, then $A_0=0$ and

$$A_1^2=-2\left(\alpha_2-\frac{1}{6}F_0^2\right),$$

which is real if $\alpha_2\le\frac{1}{6}F_0^2$. If $\alpha_2<\frac{1}{6}F_0^2$, we have $a_0=O(\mu)$. If $\alpha_2=\frac{1}{6}F_0^2$ then $A_1=0$ and a_0 will be of order μ^2 or smaller.

When $p=2$, we have

$$\begin{aligned}&\alpha_1+\mu^2\left(\frac{1}{4}\alpha_1^2+\alpha_2-\frac{1}{6}F_0^2\right)+\mu^4\left(\frac{1}{8}\alpha_1^3+\frac{1}{2}\alpha_1\alpha_2-\frac{1}{36}F_0^2\alpha_1+\alpha_3\right)+\\ &+\frac{1}{2}\mu^2A^2-\frac{1}{4}\mu^4\alpha_1A^2-\frac{1}{12}\mu^4A^4-\frac{1}{12}\mu^4F_0A^2(-1)^n=0.\end{aligned}$$

By expanding A in a series in μ and comparing coefficients at like powers of μ, the order μ^0 gives $\alpha_1=0$. At the order μ^2 we then have

$$A_0^2=-2\left(\alpha_2-\frac{1}{6}F_0^2\right),$$

whence $\alpha_2-F_0^2/6\le 0$. If $\alpha_2<\frac{1}{6}F_0^2$, this gives $A_0\ne 0$ and hence $a_0=O(1)$ as it is supposed to be. If $\alpha_2=\frac{1}{6}F_0^2$, we have $A_0=0$ and $A_1^2=-2\alpha_3$ (whence $\alpha_3\le 0$) giving us $a_0=O(\mu)$ or smaller.

In conclusion, when $\alpha_1 < 0$, the driver will excite a stationary solution of order μ ($p = 1$). If $\alpha_1 = 0$, and $\alpha_2 < F_0^2/6$, the excited stationary solution will be $\sim \mu^2$ ($p = 2$). Notice that the linearised versions of these equations are undriven and will not exhibit parametric resonance. Small perturbations will die off if we include a linear friction term. It would be interesting to understand how the *nonlinear* (finite amplitude) solutions will be affected by the linear dissipation. If they decay similarly to small perturbations, an interesting problem to study would be the pendulum with *nonlinear* friction. The analysis of these issues is beyond the scope of this work, however.

5.6 Conclusions.

In this chapter we studied the equation of the pendulum in various frequency regimes. Since we are interested in *weakly* nonlinear effects, we assumed small solution of order μ^p ($p > 0$):

$$\varphi = \mu^p(\varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots).$$

The evolution of the amplitude of the main harmonic in various frequency regimes and for various orders of smallness of φ, is described by the following equations:

$N = 1$:

- $p = 1:\quad -2i\dfrac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \dfrac{1}{2}\mu^2|a_0|^2 a_0 + \mu^2 F_0\bar{a}_0 = 0,$
- $p = 2:\quad -2i\dfrac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\dfrac{\alpha_1^2}{2} - \dfrac{3}{8}F_0^2 + \alpha_2\right) a_0 + \dfrac{\mu^4}{2}|a_0|^2 a_0 + \mu^2 F_0\bar{a}_0 = 0.$

$N = 2$:

- $p = 2:\quad -2i\dfrac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\dfrac{1}{4}\alpha_1^2 + \alpha_2 + \dfrac{2}{3}F_0^2\right) a_0 + \dfrac{1}{2}\mu^4|a_0|^2 a_0 + \mu^4 F_0^2\bar{a}_0 = 0,$

$N = 3$:

- $p = 3:\quad -2i\dfrac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\dfrac{1}{4}\alpha_1^2 + \alpha_2 + \dfrac{9}{16}F_0^2\right) a_0 + \mu^6\left(\dfrac{1}{8}\alpha_1^3 + \dfrac{1}{2}\alpha_1\alpha_2 + \right.$
$\left. + \alpha_3 + \dfrac{117}{128}F_0^2\alpha_1\right) a_0 + \dfrac{1}{2}\mu^6|a_0|^2 a_0 + \left(\dfrac{9}{8}\right)^2 \mu^6 F_0^3\bar{a}_0 = 0.$

$N = 4$:

- $p = 4:$

$$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 + \frac{8}{15}F_0^2\right)a_0 + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 + \alpha_3 + \right.$$
$$\left. + \frac{188}{15\times 15}F_0^2\alpha_1\right)a_0 + \mu^8\left(\alpha_4 + \frac{5}{64}\alpha_1^4 + \frac{3}{8}\alpha_1^2\alpha_2 + \frac{4223}{75\times 45}F_0^2\alpha_1^2 + \frac{1}{4}\alpha_2^2 + \frac{1}{2}\alpha_1\alpha_3 + \right.$$
$$\left. + \frac{188}{15\times 15}F_0^2\alpha_2 + \frac{2192}{75\times 45}F_0^4\right)a_0 + \frac{1}{2}\mu^8|a_0|^2 a_0 + \left(\frac{4}{3}\right)^2\mu^8 F_0^4\bar{a}_0 = 0.$$

$N = 1/2$:

- $p = 1:$

$$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right)a_0 + \left(\frac{1}{2}\mu^2 - \frac{1}{4}\alpha_1\mu^4\right)|a_0|^2 a_0 -$$
$$- + \frac{1}{32}\mu^4|a_0|^4 a_0 - \frac{1}{4}\mu^4 F_0\bar{a}_0^3 = 0,$$

- $p = 2:$

$$-2i\frac{\partial a_0}{\partial \tau} + \mu^2\alpha_1 a_0 + \mu^4\left(\frac{1}{4}\alpha_1^2 + \alpha_2 - \frac{1}{6}F_0^2\right)a_0 + \mu^6\left(\frac{1}{8}\alpha_1^3 + \frac{1}{2}\alpha_1\alpha_2 - \right.$$
$$\left. - \frac{1}{36}F_0^2\alpha_1 + \alpha_3\right)a_0 + \left(\frac{1}{2}\mu^4 - \frac{1}{4}\mu^6\alpha_1\right)|a_0|^2 a_0 - \frac{1}{12}\mu^6|a_0|^4 a_0 - \frac{1}{12}\mu^6 F_0\bar{a}_0^3 = 0.$$

In the cases $N = 1, 2, 3$ and 4, the zero solution is parametrically unstable within a certain frequency window, and this instability saturates at a nonlinear stationary solution with the amplitude $\sim \mu^N$. On the edges of the instability window, the driver will excite oscillations with smaller amplitudes (μ^{N+1} and smaller). We are not going to consider these marginal oscillations in what follows; we concentrate on the generic solutions with relatively large amplitude.

The driving frequency $\sim 4\omega_0$ ($N = 1/2$) will also excite stationary oscillations. Generically, the amplitude of these oscillations is $\sim \mu$, but in certain special cases oscillations with smaller amplitudes ($\sim \mu^2$ and smaller) will be excited.

Chapter 6

The parametrically driven linear Klein-Gordon equation.

In the chapters 4 and 5 we studied a finite dimensional system, oscillator. In this chapter we extend the multiple-scales method to the case when solutions depend on x, that is, we now take into consideration the dispersion term φ_{xx} so that the equation becomes the parametrically driven linear Klein-Gordon equation:

$$\varphi_{tt} - \varphi_{xx} + \omega_0^2 \left[1 - 2F_0\varepsilon^2 \cos\left(\frac{2}{N}\Omega t\right)\right] \varphi = 0. \tag{6.1}$$

(This is tantamount to proceeding from one oscillator to an infinite chain of coupled oscillators with the spacing of the chain much smaller than the characteristic wavelength of the wave arising in this system.)

In order to apply the multiple-scales method to partial differential equations, some new ideas must be added to the ones discussed in the case of ordinary differential equations. In the case of PDEs, this method requires that both x and t be extended to sets $X_0, X_1, X_2, \ldots$ and $T_0, T_1, T_2, \ldots$ where $X_n = \varepsilon^n x$, $T_n = \varepsilon^n t$ and ε is a parameter characterising the smallness of the perturbing terms. Accordingly, the dependent variable $\varphi(x,t)$ may now be regarded as a function of the extended set of independent variables:

$$\varphi(x,t) = \varphi(X_0, X_1, \ldots, T_0, T_1, \ldots).$$

We shall here consider the linear Klein-Gordon equation and proceed to its nonlinear generalisation in the next chapter.

We proceed in a standard way. First of all, we rewrite eq.(6.1) in the form

$$\varphi_{tt} - \varphi_{xx} + \Omega^2\varphi = (\Omega^2 - \omega_0^2)\varphi + 2\omega_0^2 F_0 \varepsilon^2 \cos\left(\frac{2}{N}\Omega t\right)\varphi \tag{6.2}$$

and make the transformations $\tau = \Omega t$ and $\chi = \Omega x$ which takes it to

$$\varphi_{\tau\tau} - \varphi_{\chi\chi} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + 2\frac{\omega_0^2}{\Omega^2} F_0 \varepsilon^2 \cos\left(\frac{2}{N}\tau\right)\varphi. \tag{6.3}$$

Denoting $\mu^2 = \frac{\omega_0^2}{\Omega^2}\epsilon^2$, we have

$$\varphi_{\tau\tau} - \varphi_{\chi\chi} + \varphi = \left(1 - \frac{\omega_0^2}{\Omega^2}\right)\varphi + 2\mu^2 F_0 \cos\left(\frac{2}{N}\tau\right)\varphi. \tag{6.4}$$

In order to have a parametric instability, we need to assume that the driving frequency is close to a fraction of the natural frequency of the system:

$$1 - \frac{\omega_0^2}{\Omega^2} = \alpha_1\mu^2 + \alpha_2\mu^4 + \alpha_3\mu^6 + \cdots. \tag{6.5}$$

Expanding φ in a series in μ^2

$$\varphi(\tau,\chi) = \varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots, \tag{6.6}$$

and introducing multiple scales $T_n = \mu^n\tau$ and $X_n = \mu^n\chi$, the equation (6.4) becomes

$$\begin{aligned}
& [D_0^2 - \partial_0^2 + \mu(2D_0D_1 - 2\partial_0\partial_1) + \mu^2(D_1^2 - \partial_1^2 + 2D_0D_2 - 2\partial_0\partial_2) + \mu^3(2D_0D_3 - 2\partial_0\partial_3 + \\
+ \; & 2D_1D_2 - 2\partial_1\partial_2) + (D_2^2 - \partial_2^2 + 2D_0D_4 - 2\partial_0\partial_4 + 2D_1D_3 - 2\partial_1\partial_3) + \mu^5(2D_0D_5 - 2\partial_0\partial_5 + \\
+ \; & 2D_1D_4 - 2\partial_1\partial_4 + 2D_2D_3 - 2\partial_2\partial_3) + \mu^6(D_3^2 - \partial_3^2 + 2D_0D_6 - 2\partial_0\partial_6 + 2D_1D_5 - \\
- \; & 2\partial_1\partial_5 + 2D_2D_4 - 2\partial_2\partial_4) + \cdots + 1](\varphi_0 + \mu^2\varphi_1 + \mu^4\varphi_2 + \cdots) = \\
= \; & (\mu^2\alpha_1 + \mu^4\alpha_2 + \cdots)(\varphi_0 + \mu^2\varphi_1 + \cdots) + 2\mu^2 F_0 \cos\left(\frac{2}{N}\tau\right)(\varphi_0 + \mu^2\varphi_1 + \cdots),
\end{aligned} \tag{6.7}$$

where $D_n = \partial/\partial T_n$ and $\partial_n = \partial/\partial X_n$. The equation arising at the zeroth order of μ is

$$D_0^2\varphi_0 - \partial_0^2\varphi_0 + \varphi_0 = 0,$$

whose solution is given by

$$\varphi_0 = a_0(T_1, T_2, \ldots, X_1, X_2, \ldots)e^{i(\omega T_0 - kX_0)} + c.c. \tag{6.8}$$

where

$$\omega^2 = 1 + k^2. \tag{6.9}$$

The group velocity of the packet (6.8) is $d\omega/dk = k(1+k^2)^{-1/2}$. In our subsequent study, we will be interested in nonpropagating breathers ($d\omega/dk = 0$), so we set $k = 0$ in eqs.(6.8)-(6.9):

$$\varphi_0 = a_0(T_1, T_2, \ldots, X_1, X_2, \ldots)e^{iT_0} + c.c.$$

6.1 The driving frequency $\sim 2\omega_0$.

In the chapter (4) we saw that in the case of the linear osillator, expansions over *all* scales and over *even* scales only, produce equivalent results. (The reason is that the smallness parameter enters as μ^2 not as μ.) It is tempting to think that in the case of the Klein-Gordon equation, these two types of expansions will produce equivalent results as well. Here we check this hypothesis.

All n. We consider (6.7) and compare coefficients at same powers of μ. At the first order of μ we have $2D_0D_1\varphi_0 = 0$ which gives us $D_1a_0 = 0$. Next, at the order μ^2 we have

$$\begin{aligned} D_0^2\varphi_1 - \partial_0^2\varphi_1 + \varphi_1 &= (\partial_1^2 - 2D_0D_2)\varphi_0 + \alpha_1\varphi_0 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \partial_1^2 a_0 + \alpha_1 a_0 + F_0\bar{a}_0)e^{iT_0} + F_0a_0e^{3iT_0} + c.c. \end{aligned} \tag{6.10}$$

In order to get rid of secular terms we set the coefficient of e^{iT_0} to zero. This gives

$$-2iD_2a_0 + \partial_1^2 a_0 + \alpha_1 a_0 + F_0\bar{a}_0 = 0. \tag{6.11}$$

This equation arises in the multiple scale expansion of

$$-2i\frac{\partial a_0}{\partial \tau} + \frac{\partial^2 a_0}{\partial \chi^2} + \mu^2\alpha_1 a_0 + \mu^2 F_0\bar{a}_0 = 0. \tag{6.12}$$

Stability. To analyse stability of its trivial solution, we differentiate eq.(6.12) with respect to τ to obtain

$$2i\frac{\partial^2 a_0}{\partial \tau^2} = \left(\frac{\partial^2}{\partial \chi^2} + \mu^2\alpha_1\right)\frac{\partial a_0}{\partial \tau} + \mu^2 F_0\frac{\partial \bar{a}_0}{\partial \tau}.$$

Using (6.12) we find that

$$-4\frac{\partial^2 a_0}{\partial \tau^2} = \left(\frac{\partial^2}{\partial \chi^2} + \mu^2\alpha_1\right)^2 a_0 - \mu^4 F_0^2 a_0,$$

whose solution is given by

$$a_0 = e^{i(\omega\tau - k\chi)}$$

where ω and k satisfy the dispersion relation

$$4\omega^2 = (\mu^2\alpha_1 - k^2)^2 - \mu^4 F_0^2.$$

Instability occurs when $\omega^2 < 0$, that is, when

$$|\mu^2\alpha_1 - k^2| < \mu^2 F_0.$$

This inequality defines a range of unstable wavenumbers:

$$-\mu^2 F_0 < \mu^2\alpha_1 - k^2 < \mu^2 F_0. \tag{6.13}$$

Expanding k in terms of μ,

$$k = k_0 + \mu^1 k_1 + \mu^2 k_2 + \cdots$$

and substituting into (6.13), we have $k_0 = 0$ (as it should be since we assumed that a_0 does not depend on X_0), and

$$-F_0 < \alpha_1 - k_1^2 < F_0. \tag{6.14}$$

Therefore, unstable wavenumbers can have $k_1 \neq 0$ and so growing perturbations depend on X_1: $a_0 = a_0(T_2, \ldots, X_1, \ldots)$. Eq.(6.14) implies that the trivial solution is unstable if

$$\alpha_1 > -F_0, \tag{6.15}$$

or equivalently,

$$\Omega^2 > \omega_0^2(1 - F_0\varepsilon^2). \tag{6.16}$$

Even n. Making use of (6.7) and remembering that all the derivatives with respect to odd n will vanish, we compare terms at same order of μ. At order μ^0 we still have

$$D_0^2\varphi_0 - \partial_0^2\varphi_0 + \varphi_0 = 0,$$

whose solution is now

$$\varphi_0 = a_0(T_2, T_4, \ldots, X_2, X_4 \ldots)e^{iT_0} + c.c.$$

At order μ^2 we obtain

$$\begin{aligned} D_0^2\varphi_1 - \partial_0^2\varphi_1 + \varphi_1 &= (2\partial_0\partial_2 - 2D_0D_2)\varphi_0 + \alpha_1\varphi_0 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_0 \\ &= (-2iD_2a_0 + \alpha_1a_0 + F_0\bar{a}_0)e^{iT_0} + F_0a_0e^{3iT_0} + c.c. \end{aligned} \tag{6.17}$$

For nonsecularity we require that

$$-2iD_2a_0 + \alpha_1a_0 + F_0\bar{a}_0 = 0. \tag{6.18}$$

This equation describes only spatially homogeneous oscillations. Since we wish to to accomodate spatially inhomogeneous solutions, we have to go to higher orders. A particular solution of (6.17) is given by

$$\varphi_1 = a_1e^{3iT_0} + c.c.,$$

where $a_1 = -\frac{1}{8}F_0a_0$. At the order μ^4 we get

$$D_0^2\varphi_2 - \partial_0^2\varphi_2 + \varphi_2 = \tag{6.19}$$

$$\begin{aligned} &= (\partial_2^2 - D_2^2 - 2D_0D_4)\varphi_0 - 2D_0D_4\varphi_1 + \alpha_1\varphi_1 + \alpha_2\varphi_0 + F_0(e^{2iT_0} + e^{-2iT_0})\varphi_1 \\ &= (\partial_2^2a_0 - D_2^2a_0 - 2iD_4a_0 + \alpha_2a_0 + F_0a_1)e^{iT_0} + (-6iD_2a_1 + \alpha_1a_1)e^{3iT_0} + \\ &+ F_0a_1e^{5iT_0} + c.c. \end{aligned} \tag{6.20}$$

To suppress secular terms we set to zero coefficients of e^{iT_0}:

$$\partial_2^2a_0 - D_2^2a_0 + \alpha_2a_0 - 2iD_4a_0 + F_0a_1 = 0. \tag{6.21}$$

Substituting from eq.(6.18), we then arrive at

$$-2iD_4a_0 + \partial_2^2a_0 + \left(\alpha_2 + \frac{1}{4}\alpha_1^2 - \frac{3}{8}F_0^2\right)a_0 = 0, \tag{6.22}$$

which is a dispersive PDE.

Equations (6.18) and (6.22) result from a multiple-scale expansion of

$$-2i\frac{\partial a_0}{\partial\tau} + \frac{\partial^2a_0}{\partial\chi^2} + \mu^2\alpha_1a_0 + \mu^4\left(\alpha_2 + \frac{1}{4}\alpha_1^2 - \frac{3}{8}F_0^2\right)a_0 + \mu^2F_0\bar{a}_0 = 0. \tag{6.23}$$