The Right Exactness of the Smooth Right Puppe Sequence

M.Sc. Thesis

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Introduction

Historical Background

The notion of homotopy can be said to have its origins in the work of Jordan [15] – for the unit interval at least. The fundamental group was introduced by Poincaré in [21], who gave examples of its calculation, and Čech introduced the more general notion of homotopy groups in [4], but the modern notation is due to Hurewicz [12].

The right exact Puppe sequence is really due to Barratt [1], but we adopt the commonly used name 'Puppe', after D. Puppe, whose exposition of this sequence in [22] is often followed today in treatments of the subject. A less general form of the left exact Puppe sequence was first derived by Peterson [20].

The idea of a smooth neighbourhood deformation retract was developed by N. E. Steenrod, together with M. Rothenberg in [23], and elaborated upon by Steenrod in [27] (which we use as one of our main references), where one of his objectives was to define a category of pairs (X, A), where $A \subseteq X$ has the homotopy extension property. This idea is closely related to what has become known as a 'Strøm structure'. Strøm structures were first defined by A. Strøm in [29], the second of his two important papers on cofibrations.

The notion of 'smooth space' is based on work by A. Frohlicher and A. Kriegl [10], and L. Lawvere, S. Schanuel and W. R. Zame [16], and the first homotopy theory carried out in this category was due to Cherenack in [5], where he proves some general results regarding the category **SMTH** of smooth spaces, and shows that some of the basic objects needed for the study of homotopy theory in this category (henceforth called 'smooth homotopy') exist in a way similar to the usual continuous constructions. In particular, he shows how to construct smooth homotopy groups. A later paper [6], by the same author, proves that the left Puppe sequence (i.e. the dual of the one described in [22]) exists in the category of smooth spaces, and is left exact.

A further exposition of some other aspects of smooth homotopy is given by Cherenack, Dugmore and Rosset in [7], where a certain notion of 'smooth CWR-complex' is introduced, and the long exact sequence of smooth homotopy groups of a Hurewicz fibration is derived.

Our basic aim is to continue this exposition of homotopy theory in the category **SMTH**, concentrating on the smooth right Puppe sequence, and notions of smooth neighbourhood deformation retracts.

Overview

In [6], Cherenack shows that the left Puppe sequence is left exact, and in [5], Cherenack gives a proof that under an hypothesis regarding quotient maps, part of the smooth right Puppe sequence exists and is right exact.

It was our aim, in this thesis, to give a proof that the smooth right Puppe sequence exists and is right exact, following the methods used by Whitehead in [30], and where he shows that the usual continuous right Puppe sequence exists and is right exact. We have only partially been able meet this aim. We have attempted to follow the general approach of Steenrod [27], where he defines neighbourhood deformation retracts, but there are some difficulties involved in the theory of smooth neighbourhood deformation retracts that have made it necessary for us to assume the existence of a 'suitable' smooth structure on products such as $I \times X \times Y$, where (X, A), and (Y, B) are smooth neighbourhood deformation retracts, such that the product, defined as $(X \times Y, A \times Y \cup X \times A)$, is an SNDR pair under this 'suitable' product structure.

This enables us to develop the theory of smooth neighbourhood deformation retracts in a similar way to the theory of continuous neighbourhood deformation retracts.

Under this assumption, in Chapter 4, we derive many results which correspond closely to the continuous ones, but from Theorem 4.10 onwards, we are required to assume, in addition, that the structure on some of the product pairs we examine corresponds with the usual smooth product structure. We suspect that our assumption of Chapter 2 is not necessary for the rsults of Chapter 4 to hold, but the route we have chosen to reach the right Puppe sequence relies on knowledge about the products of smooth neighbourhood deformation retracts.

Thus, we have laid the groundwork for a proof that the smooth right Puppe sequence exists, and is right exact, but a complete proof would require a proof of the assumption of Chapter 2, or for us to know that our assumption is not unduly restrictive.

The following breakdown of chapters shows our general approach, and gives an indication of where the assumptions are required.

Chapter 1 is a summary of the basic ideas of smooth spaces. We follow an approach to smooth homotopy that, on the surface, seems slightly different from the approach in [5], [6], [7]. In this chapter, we sketch some simple proofs that show that our approach to smooth homotopy is the same, in most important respects, to the approach in [5], [6], [7]. Next we discuss the smooth structure that is given to spaces such as coproduct spaces, quotient spaces, and subspaces of smooth spaces.

We end this chapter with a short discussion of the smooth homotopy extension problem, as it arises in the smooth setting. None of the results of this chapter are dependent upon our hypothesis.

In Chapter 2 we look at smooth neighbourhood deformation retracts. We follow the approach used by Steenrod [27], where he proves corresponding results for the usual continuous case.

Smooth neighbourhood deformation retracts are slightly problematical, and thus we make an assumption about the existence of a smooth structure on products of SNDR pairs, which allows us to prove some necessary results regarding smooth neighbourhood deformation retracts, and their relation to smooth cofibrations.

In Chapter 3 we define the smooth suspension functor, and a few other basic objects that we need later. We then define smooth H'-spaces, and sketch the proofs of a few results which follow in a similar way to the corresponding continuous results. The results of this chapter are independent of the hypothesis of Chapter 2.

In Chapter 4 we derive the smooth right Puppe sequence, using results obtained in Chapter 2. We follow the method of Whitehead [30] fairly closely, but we elaborate on some of the comments that are not fully detailed in [30]. Parts of this chapter rely on the existence of a 'suitable' product structure on products of SNDR pairs. In particular, results 4.10 to 4.15 need this product structure to be the usual one, although we are able to show that (I_f, X), where I_f is the smooth mapping cylinder of a smooth map $f\colon X \to Y$, is an SNDR pair, independently of any assumptions made.

For the final results of this chapter, we rely on Corollary 4.15, and thus we assume that certain product pairs are SNDR pairs under the usual product structure.

Chapter 5 is meant to give some justification for the study of the homotopy of smooth spaces. We give examples of spaces whose usual continuous fundamental group and smooth fundamental group are not isomorphic. In this chapter we also look at homeomorphic images of the n-sphere, and show that if their Hausdorff dimension is greater than n, then they must have non-isomorphic smooth and continuous fundamental groups. Nothing in this chapter depends on the assumptions of Chapter 2 or Chapter 4.

In Chapter 6 we analyse the problem of smooth neighbourhood deformation retracts in more detail. We give a number of different approaches to products of smooth NDR pairs, and show where problems arise. In particular, we show that for a certain type of 'restricted' smooth NDR pair, some of the usual results hold. We also show that when (X, A) has a certain smooth structure, the results of Chapter 2 all hold. Unfortunately, SNDR pairs of this sort have a rather uninteresting smooth structure, and certain of the fundamental pairs that we deal with, such as $(I, 0)$, do not have the required smooth structure.

Whitehead chooses the category K, of compactly generated spaces in which to develop the theory. This category was originally proposed by Steenrod in [27] as a convenient category in which to develop algebraic topology. We use the category **SMTH**, which was shown by Frohlicher and Kriegl [10] to have most of the useful properties of Steenrod's category, such as completeness, cocompleteness, and Cartesian closedness. It is also topological over sets.

We normally follow the notation used by Whitehead, although sometimes we prefer Rotman's [24] notation, and use it. The main point where our notation differs from that of Whitehead is in the definition of 'smooth suspension'. Whitehead gives the n-th supension of a space X, the notation $S^n X$. We adapt the quite standard convention of calling the n-th smooth suspension of some smooth space X, $\Sigma^n X$. This is mainly to avoid confusion, since we do not have smooth isomorphisms which relate the n-sphere to the n-th suspension of the 0-sphere, as in the continuous case.

In all other cases we try follow the notation that has become most common in the continuous setting, and this notation usually coincides with the notation in Whitehead [30].

For categorical notions we follow Mac Lane [17], and we omit definitions and results that can be found in [17].

Chapter 1

Preliminaries

In the two papers by Cherenack [5], [6], and the paper by Cherenack, Dugmore and Rosset [7], it is shown how the techniques of algebraic topology, and in particular homotopy theory, can be applied to the category of smooth spaces. The category **SMTH** of smooth spaces is based on work by Frohlicher and Kriegl [10], and Lawvere, Schanuel and Zame [16]. The category **SMTH** is shown to be complete, co-complete, Cartesian closed and topological over sets, by Frohlicher and Kriegel [10]. Cherenack [5] shows, in particular, that smooth homotopy groups exist, and in [6] it is shown that the smooth left Puppe sequence exists, and is left exact. For completeness, we define the category **SMTH**, and then indicate where our treatment of smooth homotopy differs from that in [5], and [6].

1.1 The Category of Smooth Spaces and Smooth Homotopy Groups

Definition 1.1. *An object in* **SMTH** *is a triple* (X, C_X, F_X) *where* X *is a set, called the underlying set, a set* C_X *of maps* $c\colon \mathbb{R} \to X$*, called the structure curves, and a set* F_X *of maps* $f\colon X \to \mathbb{R}$*, called the structure functions, which satisfy the following conditions:*

(1) *Given* $c \in C_X$ *and* $f \in F_X$*,* $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$*. In other words, the composite of a structure curve and a structure function is a smooth map from* $\mathbb{R}$ *to* $\mathbb{R}$*.*
(2) $\Gamma C_X = \{f\colon X \to \mathbb{R} | f \circ c \text{ smooth for all } c \in C_X\} = F_X$
(3) $\Phi F_X = \{c\colon \mathbb{R} \to X | f \circ c \text{ smooth for all } f \in F_X\} = C_X$

A morphism $g : (X, C_X, F_X) \to (Y, C_Y, F_Y)$ *in* **SMTH** *is a map* $g\colon X \to Y$*, such that* $g \circ C_X \subset C_Y$ *or equivalently* $F_Y \circ g \subset F_X$*. We call the objects of* **SMTH** *smooth spaces.*

We will usually write X for a smooth space (X, C_X, F_X), unless it is necessary to emphasise the particular smooth structure.

If we start with an arbitrary collection C of curves $c\colon \mathbb{R} \to X$, we can define $F_X = \Gamma C$, and $C_X = \Phi F_X$. We call this smooth structure the smooth structure generated by the set of curves C.

We can define a smooth structure on X in a similar way if we start with a collection F of functions $f\colon X \to \mathbb{R}$. This is called the smooth structure on X generated by the set of functions F.

In [10] it is shown that $\Phi\Gamma C_X = C_X$, and $\Gamma\Phi F_X = F_X$, as required by the above construction.

Let A, and $\{A_i\}_{i=1}^n$ be smooth spaces. Suppose we have a set of smooth maps $\{f_i\colon A \to A_i\}_{i=1}^n$. We define the initial smooth structure on A as follows. The structure curves on A are those curves $c\colon \mathbb{R} \to A$ for which $c \circ f_i$ is smooth for each f_i.

Similarly, given a set of maps $c_i\colon A_i \to A$, we can define the final smooth structure on A.

A smooth structure (C_1, F_1) on X is called finer than another smooth structure (C_2, F_2) if $F_2 \subseteq F_1$.

In [5] Cherenack shows how to define the *underlying topology* generated by a smooth structure. If (C_1, F_1) is a finer smooth structure on X than (C_2, F_2), then the topology generated by (C_1, F_1) is finer than the topology generated by (C_2, F_2). (See [5]).

We use the notation I for the closed unit interval $[0,1]$. Let F^* denote the set of functions on I that are smooth on $(0,1)$, and right smooth at 0, and left smooth at 1. We give I the smooth structure generated by the set of functions F^*. In Section 1.2 we verify that this smooth structure coincides with the smooth subspace structure defined there.

The cartesian product is used throughout our work, and so we briefly describe its smooth structure. Let $\{(X_i, C_{X_i}, F_{X_i})\}_{i=1}^n$, for $n \in \mathbb{N}$, be a collection of smooth spaces. Their cartesian product, $\prod_{i=1}^n X_i$, is given the initial smooth structure generated by the set of projection maps

$$\{p_j\colon \prod_{i=1}^n X_i \to X_j\}, \text{ for } j \in \mathbb{N}.$$

Thus a structure curve $c\colon \mathbb{R} \to \prod_{i=1}^n X_i$ is simply a curve of the form

$$c(t) = (c_1(t), \ldots, c_n(t)),$$

where each c_i, for $i \in \mathbb{N}$, is a structure curve on X_i. The structure functions $f \in F_{(\prod_{i=1}^n X_i)}$ are those functions for which $f \circ c_i$, $c_i \in C_{X_i}$, is smooth for each $i = 1, \ldots n$. Thus if $f_j \in F_{X_j}$ then $f_j \circ p_j$ is a structure function on the product. It is easily seen that the smooth structure defined here gives us the categorical product.

Definition 1.2. *If X is a smooth space, and x_0, $x_1 \in X$ then we say that x_0 is smoothly path-connected to x_1 if there is a smooth path $c\colon I \to X$ such that $c(0) = x_0$ and $c(1) = x_1$. We write $x_0 \simeq x_1$. The relation $\simeq$ is called smooth homotopy when it is applied to hom-sets.*

Given smooth spaces X and Y, it can be shown that the hom-set of smooth mappings between X and Y has a smooth structure determined by the smooth structures on X and Y. See Frohlicher and Kriegl [10].

Definition 1.3. *If X and Y are smooth spaces, and A is a smooth subspace of X, then a smooth homotopy $H: I \times X \longrightarrow Y$ is called a smooth homotopy relative to A if $H(t,a) = a$ for $a \in A$. If $H(0,x) = f(x)$, and $H(1,x) = g(x)$, then we write $f \simeq_H g$ (rel A). We sometimes drop the subscript H if it is clear to which smooth homotopy we are referring.*

In [5], and [6], the above definition is phrased in terms of $\mathbb{R}$ instead of I, with the endpoints of the smooth path being -1 and 1. We will demonstrate, aside from [5], that with the above definition, the smooth homotopy groups exist, and are isomorphic to those defined in [5], [6].

Lemma 1.4. *The relation $\simeq$ is transitive.*

Proof. Let X be a smooth space, and let $f,g: I \longrightarrow X$, with $f(1) = g(0)$. Define a smooth braking function $\alpha: \mathbb{R} \longrightarrow \mathbb{R}$ with the following properties:

(1) $\alpha(t) = 0$ for $t \leq \frac{1}{4}$,
(2) $\alpha(t) = 1$ for $t \geq \frac{3}{4}$.

We will show in Section 1.3 that such a function exists. Next we define the composition $f * g$ of f and g as follows:

$$f * g(t) = \begin{cases} f(\alpha(2t)) \text{ if } 0 \leq t \leq \dfrac{1}{2} \\ g(\alpha(2t-1)) \text{ if } \dfrac{1}{2} < t \leq 1. \end{cases}$$

It is clear that $f * g$ is a path from $f(0)$ to $g(1)$. The braking function α ensures that the path is constant for t in a neighbourhood of $\frac{1}{2}$, and hence the path $f * g$ is smooth.

Definition 1.5.

(1) *We form the category of smooth spaces with basepoint, and basepoint preserving maps, denoted* **SMTH**$_*$. *We will usually denote the basepoint of a smooth pointed space X by 0_X. The inclusion map $i: A \longrightarrow X$ is basepoint preserving, so that $i(0_A) = 0_X$. In particular, we give the unit interval, I, the basepoint 0.*
(2) *We let* **hSMTH**$_*$ *denote the category whose objects are the same as those in* **SMTH**$_*$ *and whose morphisms are equivalence classes of smooth pointed maps under the equivalence relation $\simeq$ defined above.*
(3) *We define the smooth loop space of a pointed smooth space $(X, 0_X)$, to be $\Omega X = \{c: I \longrightarrow (X, 0_X) | c \text{ a smooth map}, c(0) = 0_X, c(1) = 0_X\}$. Since* **SMTH** *is a cartesian closed category (see Frohlicher and Kriegl [10]), ΩX has a natural smooth structure.*

Lemma 1.6. *Let $(X, 0_X)$ be an object in* **SMTH**$_*$. *We define $\pi_1(X, 0_X) = \Omega X / \simeq$, and there is a composition induced by the composition on the path space, $\pi_1(X, 0_X)$, making it* *into a group, where $\simeq$ refers to the relation defined in Definition 1.2. We call $\pi_1(X, 0_X)$ the smooth fundamental group.*

Proof. Firstly, let α be as in Lemma 1.4. We define the composition on $\pi_1(X, 0_X) = \Omega X/\simeq$ as follows.

Let $[f], [g] \in \pi_1(X, 0_X)$. Then the composition $*$ of $[f]$ and $[g]$ is defined by

$$[f] * [g] = [f * g],$$

where $[f * g]$ is the equivalence class of the composition described in Lemma 1.4.

The composition on $\pi_1(X, 0_X)$ is well-defined:

Suppose $f, f', g, g' \in \pi_1(X, 0_X)$ with:

$$h : f \simeq f' \text{ (rel } \dot{I}) \text{ and } j : g \simeq g' \text{ (rel } \dot{I}),$$

where $\dot{I} = \{0, 1\}$. We need to show $[f * g] = [f' * g']$. Define a homotopy $k \colon I \times I \to X$ as follows:

$$k(s,t) = \begin{cases} h(s, \alpha(2t)) \text{ if } 0 \leq t \leq \dfrac{1}{2} \\ j(s, \alpha(2t-1)) \text{ if } \dfrac{1}{2} < t \leq 1 \end{cases}$$

To see that k is smooth, note that, for $t \neq \frac{1}{2}$, k is locally the composition of smooth functions, and is thus smooth. For t in a neighbourhood of $\frac{1}{2}$, the braking function α ensures that k is constant, and thus is smooth across the join at $t = \frac{1}{2}$.

Next fix $s \in I$. Again k is smooth, as the braking function α ensures smoothness across the join at $t = \frac{1}{2}$. We easily observe that k is the required homotopy:

$$k(0,t) = \begin{cases} h(0, \alpha(2t)) = f(\alpha(2t)) \text{ if } 0 \leq t \leq \dfrac{1}{2} \\ j(0, \alpha(2t-1)) = g(\alpha(2t-1)) \text{ if } \dfrac{1}{2} < t \leq 1 \end{cases}$$

and

$$k(1,t) = \begin{cases} h(1, \alpha(2t)) = f'(\alpha(2t)) \text{ if } 0 \leq t \leq \dfrac{1}{2} \\ j(1, \alpha(2t-1)) = g'(\alpha(2t-1)) \text{ if } \dfrac{1}{2} < t \leq 1 \end{cases}$$

Thus $f * g \simeq f' * g'$ (rel $\dot{I}$), and so $[f * g] = [f' * g']$.

The result now follows in much the same way as that in [6]. $\square$

We now show that our smooth fundamental group is isomorphic to those defined in Cherenack [5], [6]. Note that in [5], [6], a smooth homotopy between two smooth maps is defined as follows.

Given two smooth mappings $f \colon X \to Y$, and $g \colon X \to Y$, for smooth spaces X, and Y, a smooth homotopy H between f and g is defined to be a smooth mapping $H \colon \mathbb{R} \times X \to Y$, with $H(-1, x) = f(x)$, $H(1, x) = g(x)$, and $H(t, 0_X) = 0_Y$, for $t \in \mathbb{R}$, and $x \in X$.

Lemma 1.7. *The smooth fundamental group defined in Lemma 1.6 is isomorphic to those defined in Cherenack [5], [6], for any smooth space X.*

Proof. Let X be a smooth space. Let $\pi_1(X)$ denote our usual definition of the fundamental group of X. Let $\overline{\pi_1}(X)$ denote the smooth fundamental group defined by Cherenack in [5], [6]. Recall that Cherenack uses the interval $[-1, 1]$ for his smooth homotopies. Define a smooth braking function $\beta \colon \mathbb{R} \to \mathbb{R}$ with the following properties:

(1) $\beta(t) = 0$ for $t \leq -\frac{3}{4}$,
(2) $\beta(t) = 1$ for $t \geq \frac{3}{4}$, and
(3) $\beta(t) \simeq t$ (rel 0).

Again, we show the existence of such a smooth function in Section 1.3. Now define

$$\xi \colon \pi_1(X) \to \overline{\pi_1}(X)$$

by $\xi[c] = [\bar{c}]$, where $[c] \in \pi_1(X)$, and $[\bar{c}] \in \overline{\pi_1}(X)$. We define $\bar{c} \colon \mathbb{R} \to X$ by

$$\bar{c}(t) = \begin{cases} c(\beta(0)) \text{ for } t \leq -1 \\ c(\beta(\frac{t+1}{2})) \text{ for } -1 < t < 1 \\ c(\beta(1)) \text{ for } t \geq 1 \end{cases}$$

The function $\bar{c}$ is smooth on each piece, and the braking function β makes $\bar{c}$ constant across the three pieces defining it, and is thus smooth.

Define $\nu \colon \overline{\pi_1}(X) \to \pi_1(X)$ by $\nu([l]) = [l^*]$, where $[l] \in \overline{\pi_1}(X)$, and $[l^*] \in \pi_1(X)$. Here we define $l^* \colon I \to X$ by $l^*(t) = l(2t - 1)$ for $t \in I$.

It is easy to see that ν is well defined. We verify that ξ is also well defined.

Suppose $[c_1], [c_2] \in \pi_1(X)$, with $[c_1] = [c_2]$. This implies that

$$c_1 \simeq_H c_2 \text{ (rel } \{0, 1\}),$$

where $H \colon I \times I \to X$ is the smooth homotopy. Define $L \colon \mathbb{R} \times \mathbb{R} \to X$ by

$$L(s,t) = \begin{cases} \xi(c_1(t)) \text{ for } s \leq -1 \\ \xi\Big(H\Big(\beta(s), t\Big) \text{ for } -1 < s < 1 \\ \xi(c_2(t)) \text{ for } s \geq 1 \end{cases}$$

Again, β ensures that L is smooth, and it is easy to verify that L is the smooth homotopy (in Cherenack's sense) giving $\xi[c_1] \simeq_L \xi(c_2)$. Hence ξ is well defined. Note that ν is a left inverse to ξ, since, if $[c] \in \pi_1(X)$, then $\nu \circ \xi[c] = \nu[\bar{c}] = [\bar{c}^*]$, where

$$\bar{c}^*(t) = c(\beta(\frac{(2t-1)+1}{2})) = c(\beta(t)) \simeq c(t) \text{ (rel 0)},$$

for $t \in I$.

Using techniques similar to above, it is now easy to verify that ξ defines an isomorphism of groups. □

1.2 Smooth Subspaces, Quotient and Coproduct Spaces in SMTH

We will be dealing with a smooth version of the homotopy extension problem, in the form of a certain notion of 'smooth cofibration'. Because homotopy extension problems involve looking at subspaces of a smooth space, we will now look at how subspaces and quotient spaces arise in the category **SMTH**. Coproduct spaces are important when we look at smooth mapping cylinders, and smooth mapping cones. These constructions in fact involve both quotients and coproducts.

Let us investigate smooth subspaces first. Let (A, C_A, F_A) be a subspace of the smooth space (X, C_X, F_X). We will always give A the initial smooth structure generated by the set of maps $\{f \circ i \colon A \to \mathbb{R} | f \in F_X\}$. This is defined as follows:

(1) The structure curves on A are defined to be $C_A = \{c \colon \mathbb{R} \to A | i \circ c \in C_X\}$.
(2) The structure functions on A are defined to be $F_A = \Phi C_A$.

With this definition of smooth subspace structure, we now note that our initial definition of the smooth structure on I coincides with the structure that I would inherit as a smooth subspace of $\mathbb{R}$. Let us denote, for the moment, the unit interval under the structure defined in Section 1.1 by (I^*, C_{I^*}, F_{I^*}), and the unit interval under the subspace structure defined above by (I, C_I, F_I). We show that $C_I = C_{I^*}$.

Let $c \in C_I$. Then, using the chain rule, it is easy to see that $c \circ f^*$ is a smooth real function, for $f^* \in F_{I^*}$. Recall that f^* is smooth on $(0,1)$ and right smooth at 0 and left smooth at 1. Thus $C_I \subseteq C_{I^*}$, and so $F_{I^*} = \Gamma C_{I^*} \subseteq \Gamma C_I = F_I$.

For the reverse inclusion, let $c^* \in C_{I^*}$. Then it is clear that $c^*(\mathbb{R}) \subseteq I$. We must show that c^* is smooth under the subspace structure on I. But the identity map $1 \colon I \to \mathbb{R}$ is smooth on $(0,1)$, right smooth at 0 and left smooth at 1, and so $1 \in F_{I^*} \subseteq F_I$. Thus $1 \circ c^* \colon \mathbb{R} \to \mathbb{R}$ must be smooth. Thus c^* must be a smooth mapping into the unit interval, and so $c^* \in C_I$. Thus we have $C_{I^*} \subseteq C_I$ and thus $F_I = \Gamma C_I \subseteq \Gamma C_{I^*} = F_{I^*}$. Hence the two smooth structures coincide.

We turn to quotient spaces next. Let (X, C_X, F_X) be a smooth space, and let $\simeq$ be an arbitrary equivalence relation on X. Let $q \colon X \to X/\simeq$ be the quotient map from X to the quotient space $X/\simeq$. Let $C_{X/\simeq}$ denote the stucture curves on $X/\simeq$, and $F_{X/\simeq}$ denote the structure functions. We define these curves and functions as follows.

Definition 1.8. *Under the hypotheses regarding the equivalence relation $\simeq$, and the smooth space X, the quotient space $X/\simeq$ has the following smooth structure.*

(1) $F_{X/\simeq} = \{f \colon X/\simeq \to \mathbb{R} | f \circ q \colon X \to \mathbb{R}$ *is smooth on* $X\}$.
(2) $C_{X/\simeq} = \Gamma(F_{X/\simeq})$.

Note that this is the final smooth structure on $X/\simeq$ generated by the set of maps

$$C = \{q \circ c \colon \mathbb{R} \to X/\simeq | c \in C_X\}.$$

Smooth coproduct spaces have simple characterisations for both their structure curves, and their structure functions.

Let (X, C_X, F_X) and (Y, C_Y, F_Y) be smooth spaces. Let $(X \sqcup Y, C_{X\sqcup Y}, F_{X\sqcup Y})$ denote the coproduct space of X and Y. Then the disjoint union, $X \sqcup Y$, of X and Y is given a smooth structure as follows:

(1) $C_{X\sqcup Y} = \{c \colon \mathbb{R} \to X \sqcup Y | c \in C_X, \text{ or } c \in C_Y\}$.
(2) $F_{X\sqcup Y} = \{(f \sqcup g) \colon X \sqcup Y \to \mathbb{R} | (f, g) \in F_X \times F_Y\}$.

It is clear that $\Gamma\Phi F_{X\sqcup Y} = C_{X\sqcup Y}$, and $\Phi\Gamma C_{X\sqcup Y} = F_{X\sqcup Y}$. For more detail on smooth coproduct spaces, see [10], [5], [6].

Definition 1.9. *Given a smooth map $f \colon X \to Y$, we define the adjunction space $X \sqcup_f Y$, by first forming the coproduct space $X \sqcup Y$, and then forming the quotient space by identifying x with $f(x)$. This space is given the quotient space smooth structure.*

Finally, we prove a simple result, which is useful for determining when functions of the form $f \colon X/A \to Z$ are smooth, for smooth spaces X, Z and A a smooth subspace of X.

Proposition 1.10. *Let X, Z be smooth spaces, and let A be a smooth subspace of X. Let $q \colon X \to X/A$ be the quotient map, which identifies the points in A. Suppose $f \colon X/A \to Z$ is a smooth mapping. Then f is smooth if and only if $fq \colon X \to Z$ is smooth.*

Proof. If f is smooth, then fq is smooth, since the quotient map is smooth.

Conversely, suppose fq is smooth. Let $h \colon Z \to \mathbb{R}$ be a structure function on Z. We have the following diagram:

$$X \xrightarrow{q} X/A \xrightarrow{f} Z \xrightarrow{h} \mathbb{R}$$

Now, $hf \colon X/A \to \mathbb{R}$ is a function on X/A, and hf is smooth if and only if $hfq \colon X \to \mathbb{R}$ is smooth. But this is the composite of smooth functions, h and fq, and is thus smooth. Thus hf is smooth and so f is smooth. □

1.3 Smooth Braking Functions

In this section we will show how to construct smooth braking functions, following the method of Hirsch [11]. The function α in Lemma 1.3 is one example of a smooth braking function. These functions are crucial to the theory of smooth spaces, because they allow us to convert piecewise smooth functions to smooth functions, by 'smoothing' across the joins of the smooth pieces.

Recall that the function $\phi\colon \mathbb{R} \to \mathbb{R}$, given by

$$\phi(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ e^{-\frac{1}{x^2}} \text{ if } x > 0. \end{cases}$$

is smooth (See Binmore [2]).

Now let us construct a smooth function $\alpha\colon \mathbb{R} \to \mathbb{R}$ with the following properties: Let $0 \leq a < b$.

(1) $\alpha(t) = 0$ for $t \leq a$,
(2) $0 < \alpha(t) < 1$ for $a < t < b$,
(3) $\alpha(t) = 1$ for $t \geq b$.

First we construct a smooth function, called a 'bump function' in Hirsch [11]. We start with the smooth function $\phi\colon \mathbb{R} \to \mathbb{R}$, defined as above. Now define a new function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$\gamma(x) = \phi(x-a)\phi(b-x).$$

Let $h\colon \mathbb{R} \to I$ be given by

$$h(x) = \frac{\int_x^b \gamma(x)dx}{\int_a^b \gamma(x)} dx.$$

Then our function α is given by $\alpha(t) = 1 - h(t)$. The bump function $j\colon \mathbb{R} \to I$, given by $j(x) = h(|x|)$ has the following properties:

(1) $j(x) = 1$ if $|x| \leq a$,
(2) $0 < j(x) < 1$ if $a < |x| < b$,
(3) $j(x) = 0$ if $|x| \geq b$.

The function α will be used by us whenever a function needs to be smoothed at some point. Sometimes we shift α along the x-axis so that a can be negative.

When we refer to the 'usual' smooth braking function, we mean the smooth braking function $\alpha\colon \mathbb{R} \to \mathbb{R}$ with the properties

(1) $\alpha(t) = 0$ if $t < \frac{1}{4}$, and
(2) $\alpha(t) = 1$ if $t > \frac{3}{4}$.

Note that the smooth homotopy $k\colon I \times \mathbb{R} \to \mathbb{R}$ given by $k(s,t) = (1-s)\beta(t) + st$ is the smooth homotopy required in Lemma 1.7.

1.4 The Smooth Homotopy Extension Problem

Finally, we consider the smooth homotopy extension problem. This problem is fundamental to all that follows. We may state the problem as follows:

Problem. *Suppose that X is a smooth space, with a smooth subspace A, and that Y is a smooth space. Let $h\colon 0 \times X \cup I \times A \to Y$ be a smooth mapping. Does the diagram*

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{h} & Y \\ \big\downarrow i & \nearrow_{H} & \\ I \times X & & \end{array}$$

have a completion H making it commute?

Define $f\colon X \to Y$ to be $f(x) = h(0,x)$, for $x \in X$, and $g\colon A \to Y$ to be $g(x) = h(1,x)$ for $x \in A$. Then h is a smooth homotopy, giving $f|_A \simeq_h g$. The existence of a completion H above implies that we may extend g to the whole of X, with $f \simeq_H g$, since

(1) $H(0,x) = Hi(0,x) = h(0,x) = f(x)$, for $x \in X$, and
(2) $H(1,x) = Hi(1,x) = h(1,x) = g(x)$, for $x \in A$.

This works in reverse too. Let $f\colon X \to Y$, $g\colon A \to Y$ and $K\colon I \times A \to Y$ be such that $f|_A \simeq_K g$. Then let $\alpha\colon \mathbb{R} \to \mathbb{R}$ be the usual smooth braking function. Define $h\colon 0 \times X \cup I \times A \to Y$ by

$$h(t,x) = \begin{cases} f(x) \text{ if } t = 0 \\ K(\alpha(t), x) \text{ if } t > 0 \end{cases}$$

Note that $K(\alpha(t), x)$ is a smooth homotopy that performs the same task as the smooth homotopy $K(t,x)$.

Now h is smooth, since the two pieces defining it coincide for t in a neighbourhood of 0. Thus, as we noted before, if the smooth homotopy extension problem

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{h} & Y \\ \big\downarrow i & \nearrow_{H} & \\ I \times X & & \end{array}$$

has a solution, we may extend g to X, with $f \simeq g$.

It is this ability to extend functions from a subspace to the entire space that makes the smooth (and the usual continuous) homotopy extension property important.

If the smooth homotopy extension property always has a solution for the pair (X, A) then we will call $i\colon A \to X$ a smooth cofibration, and we will prove that, under an assumption made in Chapter 2, when A is closed in the topology generated by the smooth structure on X, then, this is equivalent to the assertion that A is a smooth neighbourhood deformation retract of X, which we define in the next chapter.

Chapter 2

Smooth Neighbourhood Deformation Retracts

In this chapter we define a notion of smooth neighbourhood deformation retract (SNDR). We have been unable to prove completely that the required properties of smooth neighbourhood deformation retracts hold in the smooth case, and so we have had to make the assumption that a 'nice' smooth structure exists on products such as $I \times X \times Y$, so that we may show that $(X \times Y, A \times Y \cup X \times B)$ is an SNDR pair with respect to this smooth structure.

In Chapter 6 we investigate the problem of SNDR pairs in more depth, giving a number of different possible approaches that may yield a solution. In particular we discuss ways of restricting the class of SNDR pairs so that it is closed under products, and ways of changing the smooth structure on a pair to make the class of SNDR pairs closed under products.

2.1 SNDR pairs and SDR pairs

In this section we present the obvious modification of ordinary continuous NDR and DR pairs, to get smooth NDR and smooth DR pairs.

Definition 2.1. *Let A be a smooth subspace of a smooth space X. We call A a smooth neighbourhood deformation retract (SNDR) in X if there exists*

(1) *A smooth mapping $u\colon X \to I$*
(2) *A smooth homotopy $h\colon I \times X \to X$*

with the following properties:

(1) $A = u^{-1}(0)$,
(2) $h(0,x) = x$ *for all* $x \in X$,
(3) $h(t,x) = x$ *for* $(t,x) \in I \times A$,
(4) $h(1,x) \in A$ *for all* x *with* $ux < 1$.

The pair (X,A) is called an SNDR pair. If, in addition h may be chosen such that $h(1 \times X) \subseteq A$, then we call A a smooth deformation retract (SDR) of X, and (X,A) is an SDR pair. If h defines an SDR, then we call $h(1,x)$ a smooth deformation retraction.

We call (u,h) a representation for the SNDR (or SDR) pair.

The above definition is the smooth analogue of the definition of NDR and DR pairs described in [27]. Strøm [28], [29] gives an equivalent definition, which is used by James [14] to prove results found in [27], and other useful results regarding cofibrations. We will discuss Strøm's variant in more depth in Chapter 6.

Example 2.2. *For any smooth space X,*

(1) $(X, \emptyset)$ *is an SNDR pair.*
(2) (X, X) *is an SDR pair.*
(3) $(I, 0)$ *is an SDR pair.*

Proof.

(1) Define $ux = 1$, $h(t, x) = x$ for all $x \in X$, $t \in I$.
(2) Define $ux = 0$, $h(t, x) = x$ for all $x \in X$, $t \in I$.
(3) Define a representation (u, h) for the pair $(I, 0)$ by

$$u\colon I \to I \text{ given by } u(s) = s.$$

$$h\colon I \times I \to I \text{ given by } h(t, s) = (1-t)s.$$

It is trivial to verify that (u, h) gives a representation for $(I, 0)$ as an SDR pair.

□

2.2 SNDR Pairs and Smooth Cofibrations

We now turn our attention to smooth cofibrations. In the standard continuous theory NDR pairs are equivalent to closed cofibrations (see [27]). In our situation, a similar result holds, under an appropriate assumption regarding products.

Definition 2.3. *Suppose A is a smooth subspace of a smooth space X, and Y is a smooth space. If the commutative diagram*

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{h} & Y \\ \big\downarrow i & \nearrow & \\ I \times X & & \end{array}$$

may be completed for every h and every Y, we say that $i\colon A \hookrightarrow X$ is a smooth cofibration. We call (X, A) is a smooth cofibred pair.

Definition 2.4. *Suppose A is a smooth subspace of a smooth space X. Then A is called a smooth retract of X if there exists a smooth map $r\colon X \to A$ such that $ri = 1_A$. We call r a smooth retraction of X to A*

The next lemma shows the simple relation between smooth cofibred pairs and smooth retracts.

Lemma 2.5. *If A is a smooth subspace of a smooth space X, then the inclusion map $i\colon A \hookrightarrow X$ is a smooth cofibration if and only if $0 \times X \cup I \times A$ is a smooth retract of $I \times X$.*

Proof. Suppose $0 \times X \cup I \times A$ is a smooth retract of $I \times X$. We want to complete the diagram

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{h} & Y \\ {\scriptstyle j}\downarrow & \nearrow_{k} & \\ I \times X & & \end{array}$$

But there exists $r\colon I \times X \to 0 \times X \cup I \times A$ such that $rj = 1$.
Define $k = hr$. The diagram clearly commutes, and k is smooth, since both h and r are.

Conversely, suppose $i\colon A \hookrightarrow X$ is a smooth cofibration. We may then find a map r, such that the diagram

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{1} & 0 \times X \cup I \times A \\ {\scriptstyle j}\downarrow & \nearrow_{r} & \\ I \times X & & \end{array}$$

commutes. Thus $rj = 1$, so $0 \times X \cup I \times A$ is a smooth retract of $I \times X$. □

We can not go much further without knowing that SNDR pairs are closed under products. Thus we now state our hypothesis.

HYPOTHESIS. *There exists a 'suitable' smooth product structure on $I \times X \times Y$ such that if (X, A) and (Y, B) are SNDR pairs, then so is their product*

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y),$$

under this new smooth product structure on $I \times X \times Y$. If one is an SDR and the other is an SNDR, then their product is an SDR under the new structure on $I \times X \times Y$.

By 'suitable' in the above hypothesis, we mean that we would like the structure to be finer than the usual smooth structure on $I \times X \times Y$ (i.e. there are more structure functions), but otherwise as coarse as possible. In addition, we assume that it satisfies the following conditions.

(1) There exists a representation, (u, h), under the new smooth product structure on $I \times X \times Y$, giving $(X \times Y, A \times Y \cup X \times B)$ as an SNDR pair.
(2) The representation (u, h) of $(X \times Y, A \times Y \cup B \times X)$ is such that

$$h(1, \_, \_)\colon 1 \times X \times Y \to X \times Y$$

is smooth in the usual smooth structure on $X \times Y$.

Note that an SNDR pair as above is not an SNDR pair in the usual sense. If the smooth structure required above is, in fact, the usual smooth product structure, then the pair is obviously an SNDR pair as defined in Definition 2.1.

We will sometimes refer to product SNDR pairs, under the 'suitable' smooth structure described above, as 'generalized' SNDR pairs.

The following example gives an indication of what sort of structure is required in the case where the two SNDR pairs are $(I, 0)$.

Example 2.6.

We know from Example 2.2 that $(I, 0)$ is an SDR pair. We construct a 'representation' for $(I \times I, 0 \times I \cup I \times 0)$ as an SNDR pair, and then we investigate what properties the structure on $I \times (I \times I)$ needs in order for this 'representation' to be smooth.

Let $u\colon (I \times I) \to I$ be given by $u(r,s) = rs$. Then u is smooth, and $u^{-1}(0 \times I \cup I \times 0) = 0$.

Let $h\colon I \times (I \times I) \to I \times I$ be given by

$$h(t,r,s) = \begin{cases} \text{(a) } (r,0) \text{ if } s = 0, t = 1 \\ \text{(b) } (0,s) \text{ if } r = 0, t = 1 \\ \text{(c) } (0,0) \text{ if } r, s \neq 0, t = 1 \\ \text{(d) } (e^{-\frac{st}{1-t}} r, e^{-\frac{rt}{1-t}} s) \text{ otherwise.} \end{cases}$$

We have

(1) $h(0,r,s) = (r,s)$,
(2) $h(1,r,s) = (0,0)$,
(3) $h(t,0,s) = (0,s)$,
(4) $h(t,r,0) = (r,0)$.

Thus (u, h) do all that is required of a representation of $(I \times I, 0 \times I \cup I \times 0)$ as an SDR pair.

Let us now investigate the smoothness of h. Note that for $r,\ s \neq 0,\ t \neq 1$, $h(t,r,s)$ is given by lines (c) and (d), and h is smooth here, since each coordinate of h behaves much like the function ϕ of Section 1.3.

Let us look at smoothness between lines (d) and (a) and (b) in the definition of h. Let $c\colon \mathbb{R} \to I \times (I \times I)$, given by $c(\lambda) = (t(\lambda), r(\lambda), s(\lambda))$, be a curve which is smooth in the usual way in each of its coordinates. Then $h \circ c\colon \mathbb{R} \to I \times (I \times I)$ is given by

$$h(t(\lambda), r(\lambda), s(\lambda)) = \begin{cases} \text{(a) } (r(\lambda), 0) \text{ if } s(\lambda) = 0, t(\lambda) = 1 \\ \text{(b) } (0, s(\lambda)) \text{ if } r(\lambda) = 0, t(\lambda) = 1 \\ \text{(c) } (0,0) \text{ if } r(\lambda), s(\lambda) \neq 0, t(\lambda) = 1 \\ \text{(d) } (e^{-\frac{s(\lambda)t(\lambda)}{1-t(\lambda)}} r(\lambda), e^{-\frac{r(\lambda)t(\lambda)}{1-t(\lambda)}} s(\lambda)) \text{ otherwise.} \end{cases}$$

For lines (d) and (a) to 'meet up' smoothly, we need $s(\lambda)$ to approach 0 'much faster' than $t(\lambda)$ approaches 1. Similarly, for lines (d) and (b) to 'meet up' smoothly, we need $r(\lambda)$ to approach 0 'much faster' than $t(\lambda)$ approaches 1.

Thus, if we choose $(r(\lambda), s(\lambda))$ to be a usual structure curve on $I \times I$, we can only allow smooth curves $t(\lambda)$ into the first coordinate which approach 1 'more slowly' than both $r(\lambda)$ and $s(\lambda)$ approach 0.

This gives some indication of what the structure curves on $I \times (I \times I)$ might look like, to give us the required SDR pair. The fact that we can choose $r(\lambda)$ and $s(\lambda)$ independently indicates that $h(1, r(\lambda), s(\lambda))$ is smooth under the usual smooth structure, and $h(1, r(\lambda), s(\lambda))$ reduces to lines (a), (b) or (c), and is thus smooth under the usual smooth structure on $I \times I$.

Chapter 6 is devoted entirely to analysing attempts to find a class of SNDR pairs that is closed under the formation of products, and so we leave all other comments regarding this hypothesis to that chapter.

Theorem 2.7. *If (X, A) is a smooth pair, and A is closed in the underlying topology on X, then the following are equivalent:*

(1) *(X, A) is an SNDR,*
(2) *$(I \times X, 0 \times X \cup I \times A)$ is an SDR, for a suitable smooth structure on $I \times I \times X$, (i.e. the product pair is a generalized SDR pair),*
(3) *$0 \times X \cup I \times A$ is a smooth retract of $I \times X$.*
(4) *(X, A) is a smooth cofibred pair.*

Proof. Assume (1). $(I, 0)$ is an SDR, so the above hypothesis implies that $(I, 0) \times (X, A)$ is an SDR, under the new smooth product structure on $I \times I \times X$, so (1) implies (2).

(2) implies (3) trivially, since by hypothesis, the smooth structure on $I \times I \times X$ is such that $h(1, \_, \_)$ is smooth under the usual smooth structure on $I \times X$.

The equivalence of (3) and (4) is Lemma 2.5 .

We need now only show that (3) implies (1). Let r be a smooth retraction of $I \times X$ into $0 \times X \cup I \times A$, and let $p\colon I \times X \to X$ be the projection onto the second coordinate. Let $h\colon I \times X \to X$ be defined by $h(t, x) = pr(t, x)$. Then,

$$h(0, x) = pr(0, x) = p(0, x) = x \text{ for all } x.$$

If $x \in A$, then $h(t, x) = pr(t, x) = p(t, x) = x$. We now construct u:
Let $w\colon I \times X \to I$ be the projection onto the first coordinate of $I \times X$. We define the function $f\colon \mathbb{R} \to \mathbb{R}$, as in Section 1.3:

$$f(t) = \begin{cases} 0 \text{ if } t \leq 0 \\ e^{-\frac{1}{t^2}} \text{ if } t > 0 \end{cases}$$

Now define:

$$ux = \frac{\int_0^1 f(s - wr(1,x)wr(s,x))ds}{\int_0^1 f(s)ds}$$

It is clear that u is a smooth mapping. If $x \in A$, then $wr(1,x) = 1$, and $wr(s,x) = s$, so we have $ux = 0$.
If $x \notin A$, then there exists a neighbourhood V of $(0,x)$ in $I \times X$ such that $rV \subseteq 0 \times (X - A)$, and there is a neighbourhood U in I such that $(U \times x) \subseteq V$. Thus $wr(s,x) = 0$, for all $s \in U$. So we get

$$\begin{aligned} ux &= \frac{\int_0^1 f(s - wr(1,x)wr(s,x))ds}{\int_0^1 f(s)ds} \\ &= \frac{\int_{I-U} f(s - wr(1,x)wr(s,x))ds + \int_U f(s)ds}{\int_0^1 f(s)ds} \\ &> 0, \end{aligned}$$

since $\int_U f(s)ds \neq 0$. Thus $u^{-1}(0) = A$.
If x is such that $ux < 1$, then, since we have

$$ux = \frac{\int_0^1 f(s - wr(1,x)wr(s,x))ds}{\int_0^1 f(s)ds}$$

there must be a neighbourhood U of I such that $wr(1,x)(wr(s,x)) > 0$, for $s \in U$. Thus $wr(1,x) > 0$. But $wr(1,x) > 0$ implies $r(1,x) \in I \times A$, and $h(1,x) \in A$. □

It seems likely that we may be able to generalise the above theorem, so that we may assume that all SNDR pairs involved are 'generalised' SNDR pairs (i.e. SNDR pairs under a 'suitable' smooth structure as described in Section 2.2). This would require some additional work, in that we would probably have to generalise the notion of smooth cofibration. (See Chapter 6, where we prove a similar result to the above, but for a notion of a *restricted* SNDR pair.)

2.3 Some Miscellaneous Results

This section contains some general results that we will need to use in Chapter 3.

Lemma 2.8. *If $A \subset B \subset X$ and (B,A) and (X,B) are SNDR's, then (X,A) is an SNDR.*

Proof. This result follows just as in [27]: By Theorem 2.7 we may find smooth retractions

$$f \colon I \times X \to 0 \times X \cup I \times B, \text{ and}$$
$$g \colon I \times B \to 0 \times B \cup I \times A.$$

Extend g to $g' \colon 0 \times X \cup I \times B \to 0 \times X \cup I \times A$ by setting $g'(0,x) = (0,x)$. Now, g' coincides with g on the neighbourhood $0 \times B - 0 \times A$, where both are the identity map, and thus g' is smooth. Thus g' is a smooth retraction, and so is

$$g'f \colon I \times X \to 0 \times X \cup I \times A.$$

Thus (X,A) is an SNDR pair. □

Lemma 2.9. *If (X, A) and (Y, B) are SNDR pairs, then so also are the nine nontrivial pairs formed by these spaces,*

$$X \times Y,\ X \times B \cup A \times Y,\ X \times B,\ A \times Y,\ A \times B,$$

except for the pair $(X \times Y, X \times B \cup A \times Y)$, which is a generalized SNDR pair.

Proof. This result is proved in exactly the same way as in [27]:

The previous lemma applies to the inclusions

$$A \times B \subset X \times B \subset X \times Y.$$

Let (u, h) be a representation for $(X \times B, A \times B)$ as an SNDR pair. We extend this representation over $A \times (Y - B)$ by letting u be zero there, and h be constant. Then u, and h are still smooth, since their new values coincide with their old values on $A \times B$. Thus these new representations represent $(X \times B \cup A \times Y, A \times Y)$ as an SNDR pair.

In a similar way, $(X \times B \cup A \times Y, X \times B)$ is an SNDR pair.

By the previous lemma, the remaining smooth pairs, except the pair $(X \times Y, X \times B \cup A \times Y)$, are SNDR pairs.

Finally, the pair $(X \times Y, X \times B \cup A \times Y)$ is an SNDR pair under the smooth product structure that we assume exists on $I \times X \times Y$. $\square$

Definition 2.10. *A smooth mapping of pairs $f\colon (X, A) \to (Y, B)$ is called a smooth relative isomorphism if $f|_{(X-A)}$ is an isomorphism with $(Y - B)$ in the category* **SMTH**, *$fX = Y$ and f satisfies the following condition:*

Given a smooth map $g\colon X \to W$, (respectively $g\colon I \times X \to W$, constant, or the identity on A, (resp. $I \times A$) there exists a smooth map $g'\colon Y \to W$ such that $g = g' \circ f$, where W is some smooth space (resp. $g'\colon I \times Y \to W$, such that $g = g' \circ (1 \times f)$).

Notice that if A is a smooth subspace of a smooth space X, then the quotient map $q\colon X \to X/A$ is a smooth relative isomorphism from (X, A) to X/A, since in a Cartesian closed category like **SMTH**, quotients and products commute.

Lemma 2.11. *If $f\colon (X, A) \to (Y, B)$ is a smooth relative isomorphism and (X, A) is an SNDR pair, then (Y, B) is an SNDR pair, and a representation u, h of (X, A) induces a representation v, j of (Y, B), such that the following diagrams commute:*

$$\begin{array}{ccc} X & \xrightarrow{u} & I \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle v} & \\ Y & & \end{array} \qquad\qquad \begin{array}{ccc} I \times X & \xrightarrow{h} & X \\ {\scriptstyle 1\times f}\downarrow & & \downarrow{\scriptstyle f} \\ I \times Y & \xrightarrow{j} & Y \end{array}$$

Proof. Define a representation $v\colon Y \to I$, and $j\colon I \times X \to X$ as follows. By definition, $u\colon X \to I$ is constant on A, so define $v\colon Y \to I$ to be the smooth map u' such that $u = u' \circ f$. Then, if $y \in B$, there exists $x \in A$ such that $ux = 0$. Thus $0 = u'(f(x)) = u'(y) = v(y)$. By definition of v, the first diagram above commutes.

In a similar way we define $j\colon I \times Y \to Y$ by $j = h'\colon I \times Y \to Y$, where h' is a smooth map such that $h' \circ (1 \times f) = f \circ h$.

Suppose $y \in B$. Then there exists $x \in A$ such that $f(x) = y$. Thus

$$
\begin{aligned}
j(t,y) &= j \circ (1 \times f)(t,x) \\
&= f \circ h(t,x) \\
&= f(x) \\
&= y.
\end{aligned}
$$

One can verify the other conditions on an SNDR pair in a similar way. □

Lemma 2.12. *Let (X, A) be an SNDR pair, and let $h\colon A \to Y$ be a smooth mapping. Then $(Y \sqcup_h X, Y)$ is an SNDR pair.*

Proof. We form the disjoint union of X and Y, giving $Y \sqcup X$ and $Y \sqcup A$ the coproduct smooth structure defined in Section 1.2.

Let u and k represent (X, A) as an SNDR pair. Define u on Y by $u(Y) = 0$, and define k on $I \times Y$ by $k(t, y) = y$ for all (t, y). By the description smooth structure on $Y \sqcup X$ in Section 1.2, u is smooth, since it is smooth on both Y and X. Similarly, k is smooth since it is smooth on both $I \times Y$ and $I \times X$. Thus $(Y \sqcup X, Y \sqcup A)$ is an SNDR pair.

The quotient map $f\colon (Y \sqcup X, Y \sqcup A) \to (Y \sqcup_h X, Y)$ is smooth by definition, and quotient maps can be shown to be relative isomorphisms. Thus $(Y \sqcup_h X, Y)$ is an SNDR pair by Lemma 2.11. □

Chapter 3

The Smooth Suspension Functor, and Smooth H'-Spaces

The aim of this chapter is to define the smooth suspension and anti-suspension functors, and then to show that the suspension of an object X is, in fact an H'-space. This fact will be used in Chapter 4 to construct a sequence that is exact if and only if the right Puppe sequence is exact, under the assumption of Chapter 2.

Many of the results in this chapter follow in an obvious way from the corresponding continuous results, since the arguments are mainly categorical in nature. For the smooth versions of such results, we either give a brief sketch of the proof, or give references to where the continuous results may be found.

In this chapter we work in the categories $\mathbf{SMTH}_*$ and $\mathbf{hSMTH}_*$. The objects in these categories are the same, but where confusion is likely to arise regarding the morphisms, we specify which category we are considering.

In this chapter, and the next, we use the following (fairly standard) abbreviated notation for hom-sets arising from $\mathbf{hSMTH}_*$:

$$[X, Y] := \mathbf{hSMTH}_*(X, Y),$$

where X and Y are smooth spaces with basepoint.

We follow Spanier [25], and Rotman [24] for the theory of smooth H'-spaces, and we follow Whitehead [30] when we discuss the relation between the smooth suspension and anti-suspension functors.

3.1 The Smooth Suspension Functor

Definition 3.1. *Let X, Y be smooth spaces, with basepoints x_0, y_0, respectively. Then we make the following definitions:*

(1) *The smooth space $X \vee Y$ is the smooth space obtained from $X \sqcup Y$ by identifying x_0 and y_0, and taking the identified set as basepoint.*
(2) *The smash product (or reduced join) of X and Y is the smooth space $X \wedge Y = X \times Y/(X \vee Y)$, taking the collapsed set as basepoint.*
(3) *Let $T = (I, 0)$. Then define the cone on X by*

$$TX = T \wedge X = T \times X/(T \vee X).$$

(4) *We define the suspension of X to be*

$$\Sigma X = (I/\dot{I}) \wedge X,$$

where $\dot{I} = \{0, 1\}$.
(5) *We define the n-th smooth homotopy group to be $\pi_n(X) = [\Sigma^n S^0, X]$, for $n \in \mathbb{N}$, and X a smooth space.*

Note that in the smooth situation, we do not have the isomorphism $I/\dot{I} \cong S^1$. This is because there is a 'singularity' at the point of identification. This implies that in the smooth case we do not necessarily have the isomorphisms $\Sigma S^n \cong S^{n+1}$, as we do in the continuous case. This does not have serious implications in our work.

In Cherenack [6] it is shown that Σ defines a functor

$$\Sigma\colon \mathbf{hSMTH}_* \to \mathbf{hSMTH}_*,$$

and that (Σ, Ω) form an adjoint pair. Thus, we may write our original definition of the fundamental group of a smooth space, X, as follows

$$\pi_1(X) = [S^0, \Omega X] = [\Sigma S^0, X],$$

where $[S^0, \Omega X]$ is just another way of writing our original definition, since $0 \in S^0$ always gets mapped to the constant loop at 0_X, and 1 'picks out' a smooth loop in X.

3.2 Smooth H'-Spaces

In this section we define smooth H'-spaces, and show that for any smooth space X, ΣX is a smooth H'-space.

Definition 3.2. *Let X be a pointed smooth space. Then X is called a smooth H'-space if X is equipped with a smooth multiplication*

$$m\colon X \to X \vee X$$

that satisfies the following:

(1) *Smooth homotopy identity. If $c\colon X \to X$ is the constant map, then the following diagrams commute in* $\mathbf{hSMTH}_*$*:*

$$\begin{array}{ccc} X \vee X & \xrightarrow{(c,1)} & X \\ m \uparrow & \nearrow_{1_X} & \\ X & & \end{array}$$

and

$$\begin{array}{ccc} X \vee X & \xrightarrow{(1,c)} & X \\ m \uparrow & \nearrow_{1_X} & \\ X & & \end{array}$$

commute in $\mathbf{hSMTH}_*$.

(2) *Smooth homotopy associativity. The square*

$$\begin{array}{ccc} X & \xrightarrow{m} & X \vee X \\ \downarrow m & & \downarrow (1 \vee m) \\ X \vee X & \xrightarrow{(m \vee 1)} & X \vee X \vee X \end{array}$$

commutes in $\mathbf{hSMTH}_*$.

(3) *Smooth homotopy inverse. There exists a smooth map $e\colon X \to X$ such that the following diagrams*

$$\begin{array}{ccc} X \vee X & \xrightarrow{(1,e)} & X \\ m \uparrow & \nearrow_{c} & \\ X & & \end{array}$$

and

$$\begin{array}{ccc} X \vee X & \xrightarrow{(e,1)} & X \\ m \uparrow & \nearrow_{c} & \\ X & & \end{array}$$

commute in $\mathbf{hSMTH}_*$.

There is a dual definition of H-spaces, defined in the obvious way, with all arrows reversed in the above diagrams.

Our aim is to prove that ΣX is an H'-space, and thus show that

$$[\Sigma X, Y] = \{f\colon \Sigma X \to Y | f \text{ is smooth }\}$$

is a group, for all smooth spaces X and Y. To do this, we first prove the following important result.

Lemma 3.3. *If Y is a smooth space, then ΩY is an H-space.*

Proof. From Lemma 1.5, we know that the fundamental group of Y, namely $\Omega Y/\simeq$, is a group. This implies that ΩY satisfies the group axioms up to smooth homotopy, and thus is an H-space if Y is an object in $\mathbf{hSMTH}_*$. See Spanier [25] for a direct proof of this result for the continuous case. The proof in Spanier is easily adapted to the smooth case, if we use a braking function to smooth functions which are defined in a piecewise way. □

Note that, for smooth spaces X and Y, $[X, \Omega Y]$ is a group. This is a general categorical result. See, for example, Rotman [24].

We now use the fact that (Σ, Ω) is an adjoint pair to sketch a proof, following Rotman, that for smooth spaces X and Y, $[\Sigma X, Y]$ has a natural group structure, and hence that ΣX is an H'-space.

Lemma 3.4. *If X is a smooth space, then ΣX is an H'-space in* $\mathbf{hSMTH}_*$.

Proof. Let X be a smooth space. For every smooth space Y, we have a natural bijection

$$\tau\colon [\Sigma X, Y] \to [X, \Omega Y]$$

since (Σ, Ω) form an adjoint pair. See Cherenack [6] for an explicit description of τ.

We use the group structure, on $[X, \Omega Y]$, via the bijection τ, to define a group structure on $[\Sigma X, Y]$.

Let $f, g \in [\Sigma X, Y]$. Define

$$[f].[g] = [\tau[f] * \tau[g]],$$

where $*$ is the usual composition of loops. This definition makes τ an isomorphism of groups.

The functor $[\Sigma X, -]$ is thus a functor into the category **Grp**.

By the Yoneda lemma, (see Mac Lane [17]) we deduce that ΣX is an H'-space. □

3.3 The Smooth Anti-Suspension Operator

We now define the smooth antisuspension operator, $-\Sigma$, and prove some properties of $-\Sigma$. In this section, the mappings we consider are morphisms in $\mathbf{SMTH}_*$.

Definition 3.5. *Let the smooth antisuspension operator, $-\Sigma$ assign to each map $f\colon X \to Y$, a new map $-\Sigma f\colon \Sigma X \to \Sigma Y$, as follows:*

$$(-\Sigma f)(\bar{t} \wedge x) = \overline{1-t} \wedge f(x),$$

where $\bar{t} \wedge x$ denotes the image of (t, x) in ΣX.

To see that $-\Sigma f$ is smooth for all smooth f, note that the map $q\colon I \times X \to \Sigma X$ given by $q(t, x) = \bar{t} \wedge x$ is smooth, since it is a composite of quotient maps. Thus $\bar{t} \wedge x$ is smooth in (t, x). Then $-\Sigma f$ is the map induced on the quotient ΣX, and is thus smooth.

Lemma 3.6. *The smooth antisuspension operator has the following properties:*

(1) *If we have*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

then

$$(-\Sigma f) \circ (-\Sigma g) = (\Sigma f) \circ (\Sigma g) \colon \Sigma X \to \Sigma Z.$$

(2) *The smooth homotopy class of $-\Sigma f$ in the group $[\Sigma X, \Sigma Y]$ is minus that of Σf.*

Proof.

(1) Let $\bar{t} \wedge x \in \Sigma X$. Then

$$-\Sigma f(\bar{t} \wedge x) = \overline{1-t} \wedge f(x),$$

so

$$(-\Sigma g) \circ (-\Sigma f)(\bar{t} \wedge x) = \bar{t} \wedge g \circ f(x) = (\Sigma g) \circ (\Sigma f)(\bar{t} \wedge x).$$

(2) For the proof of this assertion, let us scrutinise the mapping τ, of Lemma 3.4 in more detail. Let $f \colon X \to Y$.

Let us use the notation of Rotman [24] and denote $\tau(\Sigma f)$ by

$$\tau(\Sigma f) = (\Sigma f)^{\sharp} \colon X \to \Omega \Sigma Y,$$

where $(\Sigma f)^{\sharp}(x) = (\Sigma f)_x$, and $(\Sigma f)_x(t) = \Sigma f(\bar{t}, x) = \bar{t} \wedge f(x)$.

Similarly, we have

$$(-\Sigma f)^{\sharp}_x(t) = \overline{1-t} \wedge f(x).$$

For more detail on the map τ, see Cherenack [6], where the map is discussed for the smooth situation, or Rotman [24], where the continuous case is discussed. The composite map that we are interested in is now given by

$$(\Sigma f_x)^{\sharp} * (-\Sigma f_x)^{\sharp}(t) = \begin{cases} \overline{\alpha(2t)} \wedge f(x) \text{ if } 0 \leq t \leq \frac{1}{2} \\ \overline{\alpha(2-2t)} \wedge f(x) \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

by definition of the composition of loops. This map is smoothly homotopic to the constant map at the point $(\bar{0} \wedge f(x)) \in \Sigma Y$, which is the basepoint of ΣY, and hence is the identity element of the group $[\Sigma X, \Sigma Y]$. This homotopy $H \colon I \times I \to \Sigma Y$, may be given by

$$H(s,t) = \begin{cases} \overline{\alpha((1-s)2t)} \wedge f(x) \text{ if } 0 \leq t \leq \frac{1}{2} \\ \overline{\alpha((1-s)(2-2t))} \wedge f(x) \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Note that $H(0,t) = (\Sigma f_x)^{\sharp} \circ (-\Sigma f_x)^{\sharp}(t)$, and $H(1,t) = (\bar{0} \wedge f(x))$ for $0 \leq t \leq 1$.

□

It is this result that we were aiming for in this chapter. In Chapter 4, Lemma 3.6 will play an important role in determining the exactness of the smooth right Puppe sequence.

Chapter 4

The Smooth Right Exact Puppe Sequence

We now follow the approach of Whitehead [30], and we use the notation in [30] for constructions in the smooth case that are analogous to constructions in the continuous case.

In this chapter we will show how to derive the smooth right Puppe sequence, and prove that it is right exact, under the hypothesis of Chapter 2.

We have generally tried explicitly to spell out most of the details in this chapter, for two main reasons. Firstly the smooth constructions often require careful checking to ensure that we do, in fact, have smoothness, and secondly because many of the comments that Whitehead gives without proof are not entirely obvious, especially in the smooth situation.

4.1 The Smooth Mapping Cylinder and Mapping Cone

In this section we define the smooth mapping cylinder, and the smooth mapping cone of a smooth map f. We then use some of the results of Chapter 2 that link SNDR pairs, and smooth homotopy equivalences. The section ends with a smooth right exact sequence involving mapping cones.

We work in the category **SMTH** for now. Later, we move over to the category **SMTH**$_*$, where all spaces have a basepoint, and note that our results carry over to that category.

Definition 4.1. *Let $f\colon X \to Y$ be a smooth map.*

(1) *The smooth mapping cylinder I_f of f is defined by:*

$$I_f = I \times X \sqcup_h Y,$$

where $h\colon 1 \times X \to Y$ is given by $h(1,x) = f(x)$.

(2) *The smooth mapping cone of f, T_f is defined to be the quotient $T_f = I_f/(0 \times X)$.*

Let $< t, x >$ be the image of (t, x) in I_f under the quotient map, and $< y >$, the image of y under this map. We identify $x \in X$ with $< 0, x >$, (and use the notation $X \subset I_f$, $0 \times X \subset I_f$ interchangeably) and $y \in Y$ with $< y > \in I_f$ obtaining inclusions: $i\colon X \hookrightarrow I_f$, and $j\colon Y \hookrightarrow I_f$. One can quite easily show that $i\colon X \hookrightarrow i(X)$ and $j\colon Y \hookrightarrow j(Y)$ are smooth isomorphisms. We may define a projection map $\tilde{f}\colon I_f \to Y$ by:

$$\tilde{f}(< t, x >) = < f(x) > \text{ , and} \tilde{f}(y) = < y > .$$

We note that $\tilde{f}$ is smooth. Let $p\colon I \times X \sqcup Y \to I_f$ be the quotient map. Then we have

(1) $p(1, x) = < f(x) > = < 1, x >$, for $x \in X$,
(2) $p(t, x) = (t, x) = < t, x >$, for $x \in X$ and $0 \leq t < 1$,
(3) $p(y) = < y >$, for $y \in Y$.

Let $c\colon \mathbb{R} \to I \times X \sqcup Y$ be a structure curve. Then, by definition of the smooth structure on $I \times X \sqcup Y$, either $c(\mathbb{R}) \subseteq I \times X$, or $c(\mathbb{R}) \subseteq Y$. Let $c(s)$ be given by either $c(s) = (t(s), x(s))$, or $c(s) = y(s)$ Let $g\colon Y \to \mathbb{R}$ be a structure function. Then $\tilde{f}$ is smooth if and only if $g\tilde{f}qc\colon \mathbb{R} \to \mathbb{R}$ is a smooth real function. (See Section 1.2). But we have

(1) $g \circ \tilde{f} \circ p(t(s), x(s)) = g\tilde{f}(f(x(s))) = gf(x(s))$, if $t(s) = 1$,
(2) $g \circ \tilde{f} \circ p(t(s), x(s)) = g\tilde{f}(< t(s), x(s) >) = gf(x(s))$, if $0 \leq t(s) < 1$,
(3) $g \circ \tilde{f} \circ p(y(s)) = g\tilde{f}(y(s)) = g(y(s))$,

and so $g \circ \tilde{f} \circ p \circ c$ is smooth on each component of $I \times X \sqcup Y$. Thus $\tilde{f}$ is smooth.

Note that

$$\tilde{f} \circ i(x) = \tilde{f}(0, x) = < f(x) > = f(x),$$

and so, $\tilde{f} \circ i = f$. Also,

$$\tilde{f} \circ j(y) = \tilde{f}(y) = < y > = y.$$

Thus $\tilde{f}$ is a retraction of I_f into Y.

Note that

$$1_{I_f} \simeq_H j \circ \tilde{f}\colon I_f \to I_f, \text{ (rel } Y,$$

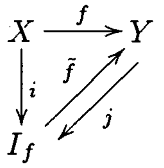
if we define the smooth homotopy $H\colon I \times I_f \to I_f$ as follows:

(1) $H(s, < t, x >) = < (1 - s)t + s, x >$, and
(2) $H(s, < y >) = < y >$.

In the proof of Lemma 4.5 we use H, and show that it is smooth. Observe that

$$h(0, < t, x >) = < t, x >,$$
$$h(s, < y >) = < y >,$$
$$h(s, < 1, x >) = < 1, x >,$$
$$h(1, < t, x >) = < 1, x > = < f(x) > = j \circ \tilde{f}(< f(x) >),$$
$$h(1, < y >) = < y > = j \circ \tilde{f}(< y >).$$

This gives us the homotopy commutative diagram



Thus $f\colon X \to Y$ is smooth homotopically equivalent to $i\colon X \hookrightarrow I_f$.

The following result is important. We show that (I_f, X) is an SNDR pair, and give two alternative proofs of the result. The first is an explicit construction of a representation for the SNDR pair, and this proof is independent of the assumption of Chapter 2.

The other is a direct proof that $i\colon X \to I_f$ is a smooth cofibration. The second proof is based on a proof of the corresponding result for the continuous situation, which can be found in Spanier [25], and relies on Theorem 2.7 which, in turn, relies on the assumption made in Chapter 2.

Lemma 4.2. *Let X, Y be smooth spaces, and $f\colon X \to Y$ be a smooth map. Then (I_f, X) is an SNDR pair.*

Proof. We prove this result directly. Define two smooth braking functions as follows. Let $\alpha_1\colon \mathbb{R} \to \mathbb{R}$ satisfy:

(1) $\alpha_1(t) = 0$ if $t = 0$,
(2) $0 < \alpha_1(t) < 1$ if $0 < t \leq \frac{3}{4}$,
(3) $\alpha_1(t) = 1$ if $t \geq \frac{3}{4}$.

Let $\alpha_2\colon \mathbb{R} \to \mathbb{R}$ satisfy:

(1) $\alpha_2(t) = 0$ if $0 \leq t \leq \frac{3}{4}$,
(2) $\alpha_2(t) = 1$ if $\frac{7}{8} \leq t \leq 1$.

Now define $u\colon I_f \to I$ by $u < t, x >= \alpha_1(t)$, and $u < y >= 1$, and define define $h\colon I \times I_f \to I_f$ by

(1) $h(s, < t, x >) =< (1-s)t + s\alpha_2(t), x >$, if $0 \leq t < 1$, and
(2) $h(s, < y >) =< y >$, otherwise.

The map u is clearly smooth. Let us consider the smoothness of h.

Let $q\colon I \times X \sqcup Y \to I_f$ be the quotient map. Note that

$$I \times (I \times X \sqcup Y) = I \times (I \times X) \sqcup (I \times Y),$$

and, since **SMTH** is Cartesian closed, q induces the obvious quotient map

$$\overline{q}\colon I \times (I \times X \sqcup Y) \to I \times I_f.$$

We have the following diagram

$$I \times (I \times X) \sqcup (I \times Y) \xrightarrow{\overline{q}} I \times I_f \xrightarrow{h} I_f$$

By our discussion of smooth quotient spaces in Chapter 1, we know that h is smooth if and only if $h \circ q$ is smooth. But we can easily see that $h \circ q|_{I\times(I\times X)}$ is smooth, and $h \circ q|_{I\times Y}$ is smooth, and so by our discussion of smooth coproduct spaces in Chapter 1, $h \circ q$ is smooth.

It remains now only to verify that (u, h) is, in fact, the required representation. Firstly, $u^{-1}(X) = u^{-1}(0 \times X) = 0$. Next, note that $h(0, < t, x >) = < t, x >$, and $h(0, < y >) = y$. Also, $h(s, < 0, x >) = < 0, x >$.

If $u(< t, x >) < 1$, then $t < \frac{3}{4}$, and so $\alpha_2(t) = 0$. Thus $h(1, < t, x >) = < 0, x >$. □

We now present an alternate proof of this result, by adapting a proof of Spanier [25]. This proof is instructive because it shows us how to 'extend' functions from the subspace X of I_f to the whole of I_f.

Lemma 4.3. *Let X, Y be smooth spaces, and $f: X \to Y$ be a smooth map. Then the inclusion $i: X \hookrightarrow I_f$ is a smooth cofibration, and hence (I_f, X) is an SNDR pair.*

Proof. Let W be a smooth space. Suppose $g: I_f \to W$ is a smooth map, and that $G: I \times (0 \times X) \to W$ is a smooth homotopy, with

$$G(0, x) = g(< 0, x >), \text{ for } x \in X.$$

Note that, as in Section 1.4, we may replace G by another smooth homotopy $G': I \times (0 \times X) \to W$, given by $G'(s, x) = G(\alpha(s), x)$, where α is the usual smooth braking function. This smooth homotopy does the same job as the smooth homotopy G.

We construct a homotopy $H: I \times I_f \to W$ as follows:

$$H(s, < y >) = g(< y >), \text{ for } y \in Y,\ s \in I$$

$$H(s, < t, x >) = \begin{cases} g(< \alpha(\frac{2t - s}{2 - s}), x >) \text{ if } 0 \leq s \leq 2t \leq 2,\ x \in X \\ G(\alpha(\frac{s - 2t}{1 - t}), x) \text{ if } 0 \leq 2t \leq s \leq 1,\ x \in X. \end{cases}$$

It is easy to verify that this is the required homotopy, and α ensures that the homotopy is smooth. □

The next few results show the relation between SNDR pairs, the mapping cylinder, and smooth deformations.

Definition 4.4. *If an inclusion map $i: A \hookrightarrow X$ has a smooth right homotopy inverse, then we say that X is smoothly deformable into A.*

Lemma 4.5. *A map $f: X \to Y$ has a smooth right homotopy inverse if and only if I_f is smoothly deformable into X.*

Proof. The proof of this result is almost identical to the proof of the continuous result in Whitehead [30], but we must verify smoothness of the maps involved. Suppose that f has a smooth right homotopy inverse. Then there exists $g: Y \to X$ such that $f \circ g \simeq 1_Y$. Define $q = g \circ \tilde{f}: I_f \to X$, where $\tilde{f}$ is the projection map of Definition 4.1. We have $j \circ \tilde{f} \simeq_H 1_{I_f}$, with the smooth homotopy $H: I \times I_f \to I_f$ given by

(1) $H(s, < t, x >) = < (1 - s)t + s, x >$ for $(t, x) \in I \times X$,
(2) $H(s, < y >) = < y >$, for $y \in Y$,

as before. We verify smoothness of H. Consider the following diagram,

$$\mathbb{R} \xrightarrow{c} I \times (I \times X) \sqcup I \times Y \xrightarrow{\tilde{q}} I \times I_f \xrightarrow{H} I_f \xrightarrow{g} \mathbb{R}$$

where $c(\lambda)$ is given by either

(1) $c(\lambda) = (s(\lambda), t(\lambda), x(\lambda))$ or,
(2) $c(\lambda) = (s(\lambda), y(s))$.

In the above diagram, $\tilde{q}$ is the quotient map giving us $I \times I_f$, and g is a structure function on I_f. Now, H is smooth if and only if $H \circ \tilde{q}$ is smooth. (See Section 1.2). To see that $H \circ \tilde{q}$ is smooth, we observe that either

(1) $gH\tilde{q}(s(\lambda), t(\lambda), x(\lambda)) = g(< (1 - s(\lambda))t(\lambda) + s(\lambda), x(\lambda) >)$, or
(2) $gH\tilde{q}(s(\lambda), y(\lambda)) =< y(\lambda) >$.

From this we see that $H \circ \tilde{q}$ is smooth on each of the components of $I \times (I \times X) \sqcup I \times Y$. Thus by our discussion in Section 1.2, we conclude that H is smooth. It is a trivial matter to verify that H is the required smooth homotopy.

We also clearly have $j \circ f \simeq_J i$, where j is the inclusion of Y in I_f, as in Definition 4.1. (The smooth homotopy $J\colon I \times X \to I_f$ may be given by $J(t, x) =< t, x >$). This gives

$$i \circ q \simeq j \circ f \circ g \circ \tilde{f} \simeq j \circ \tilde{f} \simeq 1_{I_f}.$$

Conversely, suppose there exists $q\colon I_f \to X$ such that $i \circ q \simeq 1_{I_f}$. Let $g = q \circ j\colon Y \to X$. Then we have

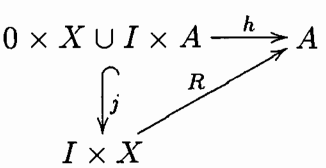
$$f \circ g = f \circ q \circ j = \tilde{f} \circ i \circ q \circ j \simeq \tilde{f} \circ j = 1_Y.$$

□

Lemma 4.6. *If (X, A) is an SNDR pair, then the inclusion map $i\colon A \hookrightarrow X$ has a smooth left homotopy inverse if and only if A is a smooth retract of X.*

Proof. Suppose A is a smooth retract of X. Then there exists a smooth map $r\colon X \to A$, such that $ri = 1_A$. Thus r is a left homotopy inverse of i.

Conversely, suppose $i\colon A \hookrightarrow X$ has a left homotopy inverse. Then there exists a smooth map $q\colon X \to A$ such that $qi \simeq_L 1_A$, and $q|_A = qi$. Since (X, A) is an SNDR pair, the following smooth homotopy extension problem has a solution R:



The map h above is the map defined by L, q, as follows.

(1) $h(0, x) = q(x)$, for $(0, x) \in 0 \times X$, and
(2) $h(t, a) = L(\alpha(t), a)$, for $(t, a) \in I \times A$,

where $\alpha\colon \mathbb{R} \to \mathbb{R}$ is the usual smooth braking function. Thus the map R is a smooth homotopy, with:

(1) $R(0,x) = R(j(0,x)) = h(0,x) = q(x)$, and
(2) $R(1,a) = R(j(1,a)) = L(\alpha(1),a) = a$, for $x \in A$.

Thus we may take as our smooth retraction, $r = R(1,x)\colon X \to A$. From point (2) above, we have $r(x) = x$, for $x \in A$. □

Lemma 4.7. *A map $f\colon X \to Y$ has a smooth left homotopy inverse if and only if the inclusion map $i\colon X \hookrightarrow I_f$ has a smooth left homotopy inverse.*

Proof. Suppose f has a smooth left homotopy inverse. Then there exists a map $g\colon Y \to X$, such that $g \circ f \simeq 1_X$. Define $q = g \circ \tilde{f}$. Then $q \circ i = g \circ \tilde{f} \circ i = g \circ f \simeq 1_X$.

Conversely, suppose i has a left homotopy inverse. Then there exists a smooth map

$$q\colon I_f \to X,$$

such that $q \circ i \simeq 1_X$. Define $g = q \circ j\colon Y \to X$. Then we have $g \circ f = q \circ j \circ f \simeq q \circ i \simeq 1_X$. □

Corollary 4.8. *A map $f\colon X \to Y$ has a smooth left homotopy inverse if and only if X is a smooth retract of I_f.*

Proof. Note that by Lemma 4.2, (I_f, X) is an SNDR pair. From Lemma 4.7, f has a smooth left homotopy inverse if and only if the inclusion map $i\colon X \to I_f$ has a smooth left homotopy inverse. By Lemma 4.6 this is equivalent to X being a smooth retract of I_f. □

Definition 4.9. *A subspace A of a smooth space X is called a strong smooth deformation retract of X if there is a smooth homotopy $F\colon I \times X \to X$ with the following properties:*

(1) *$F(0,x) = x$ for $x \in X$,*
(2) *$F(t,a) = a$ for $a \in A$, and $t \in I$,*
(3) *$F(1 \times X) \subset A$.*

Note that the only difference between a strong smooth deformation retract and an SDR pair is that a strong smooth deformation retract does not require the existence of a smooth map u with $u^{-1}(0) = A$. If such a map does exist, then the two notions are equivalent.

Before we prove the next theorem, define $\dot{I} = \{0,1\}$. Then $(I, \dot{I})$ is an SNDR pair. We may construct the representation of $(I, \dot{I})$ as follows. Let $u\colon \mathbb{R} \to \mathbb{R}$ be a smooth braking function with the following properties:

(1) $u(t) = 0$ for $t = 0$,
(2) $0 < u(t) \leq 1$ for $0 < t < \frac{1}{4}$,
(3) $u(t) = 1$ for $\frac{1}{4} < t < \frac{3}{4}$,
(4) $0 < u(t) \leq 1$ for $\frac{3}{4} < t < 1$,
(5) $u(t) = 0$ for $t = 1$.

Then u satisfies the condition that $u^{-1}(0) = \dot{I}$. Define another smooth braking function $\alpha\colon \mathbb{R} \to \mathbb{R}$, with the properties:

(1) $\alpha(t) = 0$ for $t \leq \frac{1}{4}$,
(2) $\alpha(t) = 1$ for $t \geq \frac{3}{4}$.

We may then define a smooth homotopy $h\colon \mathbb{R} \times I \to I$ by setting $h(t,s) = (1-t)s + t\alpha(s)$. Note that $h(0,s) = s$, $h(t,0) = 0$, $h(t,1) = 1$. Also, if $u(s) < 1$ then $s \in (0, \frac{1}{8})$, or $s \in (\frac{7}{8}, 1)$,and $h(1,s) = \alpha(s)$. But for $s \in (0,\frac{1}{8}) \cup (\frac{7}{8}, 1)$, $\alpha(s) = 0$, or $\alpha(s) = 1$. Thus $h(1,s) \subset \dot{I}$.

Notice that up until this point, all the results in this chapter have been independent of the assumption of Chapter 2. From now on it is necessary for us to use the assumption of Chapter 2, and in addition we need to assume that the product structure required on $I \times X \times Y$ is the usual smooth product structure, making $(X \times Y, A \times Y \cup X \times B)$ an SNDR pair in the usual sense, for the SNDR pairs (X, A) and (Y, B) that we encounter.

Theorem 4.10. *If (X, A) is an SNDR pair, then the inclusion map $i\colon A \hookrightarrow X$ is a smooth homotopy equivalence if and only if A is a strong deformation retract of X.*

Proof. Suppose, firstly that $i\colon A \hookrightarrow X$ is a smooth homotopy equivalence. Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ be a smooth braking function with the properties:

(1) $\alpha(t) = 0$ for $t \leq \frac{1}{4}$
(2) $\alpha(t) = 1$ for $t \geq \frac{3}{4}$

Let $f\colon X \to A$ be a smooth homotopy inverse of i. Thus we may find smooth homotopies $F\colon 1_A \simeq f \circ i$, and $G\colon 1_X \simeq i \circ f$. Now let $\dot{I}$ denote the subset $\{0,1\}$ of I. Define $P = I \times X$, and $Q = \dot{I} \times X \cup I \times A$. Since (X, A) is an SNDR pair, and $(I, \dot{I})$ is an SNDR pair, $(P, Q) = (I, \dot{I}) \times (X, A)$ is an SNDR pair, by the assumption that there exists a suitable smooth structure on the product $P \times X$, which coincides with the usual smooth product structure.

We may assume that f is a smooth retraction, since (X, A) is an SNDR pair, and is thus homotopic to a retraction $r\colon X \to A$, by Lemma 4.5, giving us

$$i \circ r \simeq i \circ f \simeq 1_X.$$

Define $H\colon 0 \times P \cup I \times Q \to X$ by:

(1) $H(s, 0, x) = x$
(2) $H(s, 1, x) = G(\alpha(1-s), f(x))$
(3) $H(s, t, a) = G(\alpha(1-s)\alpha(t), a)$
(4) $H(0, t, x) = G(\alpha(t), x)$.

To observe that H is smooth, we consider the places where the various functions defining H overlap:

(a) Consider parts (1) and (3) of the definition of H. Suppose $x \in A$. Then $\alpha(t) = 0$ for t in a neighbourhood of 0, so $H(s,t,a) = G(0,a) = a$, by definition of G, and so parts (1) and (3) are constant, and hence smooth in a neighbourhood of the join.

(b) Consider parts (1) and (4) of the definition. Again, $\alpha(t) = 0$ for t in a neighbourhood of 0, so (1) and (4) coincide.

(c) In parts (2) and (3) of the definition, $\alpha(t) = 1$ for t in a neighbourhood of 1, so part (3) becomes $G(1-s, a)$. But $x \in A$ implies that $f(x) = x$, since f is a smooth retraction, so parts (2) and (3) agree on this neighbourhood.

(d) For the join between parts (2) and (4), observe that for s in a neighbourhood of 0, and t in a neighbourhood of 1, part (2) becomes $G(1, f(x))$, and part (4) becomes $G(1, x)$. But $G(1,x) = i \circ f(x) = f(x)$, and $G(1, f(x)) = i \circ f(x) = f(x)$ since $f(x) \in A$.

(e) Finally, we consider the join between parts (3) and (4). For s in a neighbourhood of 0, (3) becomes $G(\alpha(t), a)$, so part (4) coincides with (3) if $x \in A$, and s is in a neighbourhood of 0.

Note that H is smooth in the last component, since each part defining H is smooth in the last component, and in the cases where two parts overlap, the two functions are equal, and so the overlap is smooth in x.

Since, by our assumption above, (P, Q) is an SNDR pair, the map $Q \hookrightarrow P$ is a smooth cofibration, and we may complete the following diagram:

$$\begin{array}{ccc} 0 \times P \cup I \times Q & \xrightarrow{H} & X \\ \big\downarrow & \nearrow_{\tilde{H}} & \\ I \times X & & \end{array}$$

It is trivial to verify that the end value of $\tilde{H}$ is the required (strong) retracting deformation. □

Note that (I_f, Y) is an SDR pair, since we may define a representation (u, h), by

(1) Let $u \colon I_f \to I$ be given by $u < t, x >= t$, and $u < y >= 1$, for $< t, x >\in I_f$ and $y \in Y$.

(2) Let $h \colon I \times I_f \to I_f$ be given by

$$h(s, < t, x >) =< (1-s)t + s, x >,$$

and $h(s, < y >) =< y >$.

We can verify that u and h are smooth by using a similar technique to that used in Lemma 4.5. Thus Theorem 4.10 gives the result.

We may deduce:

Corollary 4.11. *A map $f \colon X \to Y$ is a smooth homotopy equivalence if and only if X is a strong deformation retract of I_f.*

Proof. If we identify X with $0 \times X$ in I_f, as usual, then since (I_f, X) is an SNDR pair, by Lemma 4.2, the inclusion map $i \colon X \to I_f$ is a smooth homotopy equivalence, if and only if X is a strong deformation retract of I_f, by Lemma 4.10. But, as we noted before, i and f are smooth homotopically equivalent, and so f is a smooth homotopy equivalence if and only if X is a strong deformation retract of I_f. $\square$

We now prove three more results, and then we have enough machinery to begin the construction of the smooth right Puppe sequence.

Theorem 4.12. *Let (X, A) be an SNDR pair, and let $h\colon A \to B$, $Y = X \sqcup_h B$, and let $f\colon X \to Y$ be the quotient map that identifies x_1, $x_2 \in X$, if $h(x_1) = h(x_2)$. Also define $Z = I_f$, and $C = I_h$. Then C can be viewed as a subspace of Z, and, if we make the usual inclusions of A and X in the respective mapping cylinders, then $A = C \cap X$. Then $(Z, X \cup C)$ is an SDR pair.*

Proof. Define $P = I \times X$, and $Q = 0 \times X \cup I \times A$. For the following argument to go through, we are again required to assume that for the SNDR pairs (X, A) that we will use, the 'suitable' product structure on

$$(P, Q) = (I, 0) \times (X, A)$$

is, in fact, the usual product structure. Then (P, Q) is an SDR pair under the usual product structure, and thus $(P \sqcup B, Q \sqcup B)$ is also an SDR pair. We have the following commutative diagram:

$$\begin{array}{ccc} (I \times X) \sqcup X \sqcup B & \xrightarrow{\ l\ } & (I \times X) \sqcup B \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle q} \\ (I \times X) \sqcup Y & \xrightarrow{\ p\ } & Z \end{array}$$

where l is defined as follows:

(1) $l|(I \times X) \sqcup B$ is the identity,
(2) $l(x) = (1, x)$.

The map f_1 is defined by:

(1) $f_1|(I \times X)$ is the identity,
(2) $f_1|X \sqcup B\colon X \sqcup B \to Y$ is the quotient map.

The map $p\colon (I \times X) \sqcup Y \to Z$ is the quotient map, and q is given by:

(1) $q|(I \times X) = p|(I \times X)$, and
(2) $q|B$ is the composite of the inclusions

$$B \hookrightarrow Y \hookrightarrow Z.$$

The maps p and f_1 are quotient maps, and so q is an quotient map. Also, $q^{-1}(X \cup C) = Q \sqcup B$, and q maps

$$(P \sqcup B) - (Q \sqcup B) = P - Q$$

smooth isomorphically to $Z - (X \cup C)$, since the points in $P - Q$ are left fixed by q (i.e. no real identification takes place). The result now follows from Lemma 2.6, since, as we commented before, quotient maps are relative smooth isomorphisms. $\square$

Corollary 4.13. *Let (X, A) be an SNDR pair. If $h\colon A \to B$ is a smooth homotopy equivalence, then $f\colon X \to X \sqcup_h B$ is also a smooth homotopy equivalence.*

Proof. Again, we follow the proof in Whitehead: We know that A a strong deformation retract of C, by Lemma 4.11, and so (C, A) is an SDR pair, since we may define $u\colon C \to I$ by $u(< t, a >) = t$ and $u < b >= 1$, for $a \in A$, $b \in B$, and $t \in I$. Thus $(X \cup C, X)$ is an SDR pair. But, we have seen that $(Z, X \cup C)$ is an SDR pair. From Lemma 2.9 we deduce that (Z, X) is an SDR pair, and thus by Corollary 4.10 $f\colon X \to Y$ is a smooth homotopy equivalence. □

From now on, we will work in the category **SMTH**$_*$, where all spaces have basepoints, and all smooth mappings are basepoint preserving.

In particular, we identify the points $I \times 0_X$ in I_f, and T_f, and take the collapsed set as basepoint.

As in the continuous case, one can show that this process does not affect the smooth homotopy type of the spaces, and all the previous results of this chapter remain true in **SMTH**$_*$.

Definition 4.14. *Let $(X, 0_X)$ be a smooth space, in* **hSMTH**$_*$. *Suppose that there exists a smooth homotopy $H\colon I \times X \to X$ such that*

(1) $H(0, x) = 0_X$,
(2) $H(1, x) = x$,
(3) $H(t, 0_X) = 0_X$,

for $x \in X$. Then we say that X is contractible.

Note that if X is contractible, then $f\colon X \to 0_X$ is a smooth homotopy equivalence. This is immediate, since $f = H(0, \_)\colon X \to 0_X$, and H gives the smooth homotopy to the identity map on X.

Corollary 4.15. *If (X, A) is an SNDR pair, and A is contractible, then the quotient map $p\colon X \to X/A$ is a smooth homotopy equivalence.*

Proof. Since A is contractible, $h\colon A \to 0_A = 0_X$ is a smooth homotopy equivalence. But $X \sqcup_h 0_x \approx X/A$. Thus we apply Corollary 4.13 to deduce that $p\colon X \to X \sqcup_h 0_X = X/A$ is a smooth homotopy equivalence. □

Lemma 4.16. *If (X, A) is an SNDR pair, $i\colon A \hookrightarrow X$, and $p\colon X \to X/A$ is the quotient map, then the sequence:*

$$A \xrightarrow{i} X \xrightarrow{p} X/A$$

is right exact.

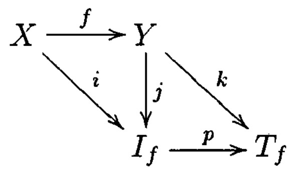
Proof. We must show that, given a smooth space W, the following diagram is right exact in the category of sets with basepoint:

$$[X/A, W] \xrightarrow{\bar{p}} [X, W] \xrightarrow{\bar{i}} [A, W].$$

Firstly, let $f\colon X/A \to W$ be an element of $[X/A, W]$. Then $\bar{f}p\colon X \to W$ has the property that $\bar{f}p(A) = 0_W$. Thus $fpi\colon A \to W$ is the constant map at 0_W. This implies that im $\bar{p} \subseteq$ ker $\bar{i}$.

To see the reverse inclusion, let $g\colon X \to W$ be an element of $[X, W]$, with $g|_A \simeq 0_W$ (rel 0_W). Since (X, A) is an SNDR pair, $i\colon A \to X$ is a smooth cofibration, and thus we may extend 0_W to a smooth map $g'\colon X \to W$, such that $g' \simeq g$. But g' is constant on A, and so by a remark in Section 2.3, we can find a smooth map $g_1 : X/A \to W$ such that $g_1 \circ p = g'$. Thus ker $\bar{i} \subseteq$ im $\bar{p}$. □

Lemma 4.17. *Let $f\colon X \to Y$ be a smooth map. There is a smooth homotopy commutative diagram*



where i, j, k are all inclusions, and p is the quotient map. Also, the sequence

$$X \xrightarrow{f} Y \xrightarrow{k} T_f$$

is right exact.

Proof. Let $x \in X$. Then $j \circ f(x) =< f(x) >= h(1, x)$, where h is defined as in Definition 4.1. We also have $i(x) = (0, x)$. We may extend h to $I \times X$ by defining $h(t, x) =< t, x >$, giving us a smooth homotopy $h\colon i \simeq j \circ f$.

Now let $y \in Y$. Then $p \circ j(y) = k(y)$, since the quotient map p leaves $j(Y)$ fixed.

For the last part of the lemma, note that by Lemma 4.16 , the sequence

$$X \longrightarrow I_f \longrightarrow T_f$$

is right exact. Since $j: Y \to I_f$ is a smooth homotopy equivalence, by Theorem 4.10, the sequence

$$X \xrightarrow{f} Y \xrightarrow{k} T_f$$

is also right exact. □

The next result follows by iterating the above procedure, exactly as in Whitehead [30]:

Lemma 4.18. *Let $f\colon X \to Y$ be a smooth map. Then there is an infinite right exact sequence*

$$X \xrightarrow{f} Y \xrightarrow{f^1} T_f \longrightarrow \cdots \longrightarrow T_{f^{n-2}} \xrightarrow{f^n} T_{f^{n-1}} \longrightarrow \cdots$$

where the f^n, $n \geq 1$ are inclusion maps.

4.2 The Smooth Right Puppe Sequence

We may now begin the construction of the smooth right Puppe sequence. We apply the results of the previous section to a sequence of inclusion and quotient maps in an iterative manner.

Lemma 4.19. *If (Y, X) is an SNDR pair, and $f\colon X \hookrightarrow Y$, $k\colon Y \hookrightarrow T_f$ are inclusions, and $p\colon Y \to Y/X$, $p_1\colon T_f \to Y/X$ are quotient maps, then there is a commutative diagram,*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{k} & T_f \\ & & & \searrow^{p} & \downarrow{\scriptstyle p_1} \\ & & & & Y/X \end{array}$$

and p_1 is a smooth homotopy equivalence. In addition, the sequence

$$X \xrightarrow{f} Y \xrightarrow{p} Y/X$$

is right exact.

Proof. Note that T_f is the subspace $Y \cup TX$ of TY, and (T_f, TX) is an SNDR pair. (If u, h represent (Y, X) as an SNDR pair, then just extend u to T_f by making u equal to 0 on TX, and extend h to T_f by making h constant on TX. These are obviously smooth maps). The quotient map from T_f to T_f/TX is a smooth homotopy equivalence, by Corollary 4.15. But T_f/TX is smoothly isomorphic to Y/X. (This is easy to show-the map which takes $< y >\in T_f/TX$ to $< y >\in Y/X$ is an isomorphism). This gives us the commutativity of the diagram, which, in turn (or using Lemma 4.16) gives the exactness of the sequence

$$X \xrightarrow{f} Y \xrightarrow{p} Y/X,$$

since, in the first case, the top line of the commutative diagram is exact by Lemma 4.17. □

Corollary 4.20. *There is a commutative diagram*

$$\begin{array}{ccccc} Y & \xrightarrow{k} & T_f & \xrightarrow{l} & T_k \\ & & & \searrow^{q} & \downarrow{\scriptstyle q_1} \\ & & & & \Sigma X \end{array}$$

with q_1 a smooth homotopy equivalence, and the sequence

$$Y \xrightarrow{k} T_f \xrightarrow{q} \Sigma X$$

is right exact.

Proof. Notice that the top line of the commutative diagram is made up of the second, third and fourth term of the sequence in Lemma 4.18. Now, (T_f, Y) is an SNDR pair, since we may define $u\colon T_f \to \mathbb{R}$ as $u(< s, x >) = 1 - s$, and $u(< y >) = 0$, and we can define $h\colon I \times T_f \to T_f$ by $h(t, < s, x >) =< (s - (s-1)t), x >$ and $h(t, < y >) =< y >$. Thus the map k satisfies the hypotheses of Lemma 4.16. Finally note that $T_f/Y = (Y \cup_f TX)/Y$ is smoothly isomorphic to ΣX. Thus Lemma 4.16 gives the result. □

We now sketch a proof that the sequence

$$X \xrightarrow{f} Y \xrightarrow{k} T_f \xrightarrow{q} \Sigma X \longrightarrow \cdots$$

$$\longrightarrow \Sigma^{n-1}T_f \xrightarrow{\Sigma^{n-1}q} \Sigma^n f \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n k} \Sigma^n T_f \xrightarrow{\Sigma^n q} \Sigma^{n+1} \longrightarrow \cdots \tag{2-1}$$

is right exact if and only if the sequence

$$X \xrightarrow{f} Y \xrightarrow{k} T_f \xrightarrow{q} \Sigma X \xrightarrow{-\Sigma f}$$

$$\Sigma Y \xrightarrow{-\Sigma k} \Sigma T_f \xrightarrow{-\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots \tag{2-2}$$

is. Here we have replaced the maps $\Sigma^n f$, $\Sigma^n k$, and $\Sigma^n q$ in (2-1) by their negatives for odd n to get (2-2).

Firstly, consider the following part of sequence (2-1).

$$\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma k} \Sigma T_f. \tag{2-3}$$

Suppose that the following part of sequence (2-2) is right exact.

$$\Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma k} \Sigma T_f. \tag{2-4}$$

To show that (2-3) is right exact, let W be a smooth space. We need to show that the induced sequence

$$[\Sigma T_f, W] \xrightarrow{\overline{-\Sigma k}} [\Sigma Y, W] \xrightarrow{\overline{-\Sigma f}} [\Sigma X, W].$$

is right exact in the category of pointed sets.

Suppose, firstly, that $f\colon \Sigma T_f \to W$ is a smooth map. Then $f\circ(\Sigma k)\circ(\Sigma f) = *$, since by Lemma 3.6, part (1), we have $(-\Sigma k)\circ(-\Sigma f) = \Sigma k \circ \Sigma f$, and (2-4) is right exact. Thus im $\overline{\Sigma k} \subseteq$ ker $\overline{\Sigma f}$.

For the reverse inclusion, suppose that $g\colon \Sigma Y \to W$ is a smooth map such that $g\circ \Sigma f = *$. Then we have $g\circ(-\Sigma f) = *$, since by definition,

$$g\circ \Sigma f(\bar{t}\wedge x) = g(\bar{t}\wedge f(x)),$$

and

$$g\circ(-\Sigma f)(\bar{t}\wedge x) = g(\overline{1-t}\wedge f(x)).$$

Thus there exists a smooth map $g^*\colon \Sigma T_f \to W$ such that $g^*\circ(-\Sigma k) = g$. Now define $g^{**}\colon \Sigma T_f \to W$ by $g^{**}(\bar{t}\wedge x) = g^*(\overline{1-t}\wedge x)$, for $x\in T_f$. Thus

$$\begin{aligned} g^{**}(\Sigma k(\bar{t}\wedge y)) &= g^{**}(\bar{t}\wedge k(y)) \\ &= g^*(\overline{1-t}\wedge k(y)) \\ &= g^*(-\Sigma k(\bar{t}\wedge y)) \\ &= g(\bar{t}\wedge y), \end{aligned}$$

for $\bar{t}\wedge y \in \Sigma Y$.

Thus we have ker $\overline{\Sigma f} \subseteq$ im $\overline{\Sigma k}$.

The converse, where we assume that (2-3) is right exact, is similar.

We now consider the parts of sequences (2-1) and (2-2) given by

$$\Sigma T_f \xrightarrow{-\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y, \tag{2-5}$$

and

$$\Sigma T_f \xrightarrow{\Sigma q} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y. \tag{2-6}$$

Suppose that (2-5) is right exact. Let $f\colon \Sigma^2 Y \to W$ be a smooth map. Then, since (2-5) is right exact, we have $f\circ(\Sigma^2 f)\circ(-\Sigma q) = *$. Again, by definition of the anti-suspension operator, this implies that $f\circ(\Sigma^2 f)\circ(\Sigma q) = *$, and so we have im $\overline{\Sigma^2 f} \subseteq$ ker $\overline{\Sigma q}$.

For the reverse inclusion, suppose $g\colon \Sigma^2 X \to W$ is a smooth map such that $g\circ(\Sigma q) = *$. Then as before, we have $g\circ(-\Sigma q) = *$, and so there exists a map $g^*\colon \Sigma^2 Y \to W$ such that $g^*\circ(\Sigma^2 f) = g$. As before we may define a smooth map $g^{**}\colon \Sigma^2 Y \to W$ such that $g^{**}\circ(\Sigma^2 f) = g$. Thus we have ker $\overline{\Sigma q} \subseteq$ im $\overline{\Sigma^2 f}$. □

To prove that (2-1) is right exact if and only if (2-2) is, we can apply the above techniques to all short exact sequences in (2-1) and (2-2). □

We will show that (2-2) is right exact, by showing that it is smooth homotopically equivalent to the sequence in Lemma 4.18. But first we return to the construction in Corollary 4.20:

$$\begin{array}{ccccc} Y & \xrightarrow{k} & T_f & \xrightarrow{l} & T_k \\ & & & \searrow^{q} & \downarrow{\scriptstyle q_1} \\ & & & & \Sigma X \end{array}$$

Observe that $q_1 = r_1^{-1} \circ q'$, where $q' \colon T_k \to T_k/TY$ is the quotient map, and $r_1 \colon \Sigma X = TX/X \to T_k/TY$ is the smooth isomorphism induced by:

$$(TX, X) \to (T_f, Y) = (T_f, T_f \cap TY) \hookrightarrow (T_f \cup TY, TY) = (T_k, TY).$$

We apply the construction of Corollary 4.20, but starting at the term T_k of the sequence in Lemma 4.18. This gives a commutative diagram

$$\begin{array}{ccccc} T_f & \xrightarrow{l} & T_k & \xrightarrow{m} & T_l \\ & & & \searrow^{\tilde{q}} & \downarrow{\scriptstyle q_2} \\ & & & & \Sigma Y \end{array}$$

Note that q_2 is the composite $r_2^{-1} \circ \hat{q}$, where

$$\hat{q} \colon T_l \to T_l/TT_f$$

is the quotient map, and

$$r_2 \colon \Sigma Y = TY/Y \to T_l/TT_f$$

is the smooth isomorphism induced by :

$$(TY, Y) \to (T_k, T_f) = (T_k, T_k \cap TT_f) \hookrightarrow (TT_f \cup T_k, TT_f) = (T_l, TT_f).$$

The map $\tilde{q}$ is the composite ${r_1'}^{-1} \circ {q_2}'$, where

$${q_2}' \colon T_k \to T_k/T_f$$

is the quotient map, and the smooth isomorphism

$${r_1'}^{-1} \colon \Sigma Y = TY/Y \to T_k/T_f$$

is induced by the inclusion

$$(TY, Y) \to (T_k, T_f).$$

For notational convenience, we now define two subsets of the smooth suspension of a smooth space.

Definition 4.21. *Let X be a smooth space. Then we define two subsets of ΣX as follows.*

(1) $T_+X = \{\bar{t} \wedge x | 0 \leq t \leq \frac{1}{2}\}$, *and*
(2) $T_-X = \{\bar{t} \wedge x | \frac{1}{2} \leq t \leq 1\}$.

Lemma 4.22. *The diagram*

$$\begin{array}{ccc} T_k & \xrightarrow{m} & T_l \\ \Big\downarrow{\scriptstyle q_1} & & \Big\downarrow{\scriptstyle q_2} \\ \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y \end{array}$$

is smooth homotopy commutative.

Proof. Firstly, we adjoin the map $\tilde{q}\colon T_k \to \Sigma Y$ to the given diagram, obtaining a new diagram:

$$\begin{array}{ccc} T_k & \xrightarrow{m} & T_l \\ \Big\downarrow{\scriptstyle q_1} & \searrow^{\tilde{q}} & \Big\downarrow{\scriptstyle q_2} \\ \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y. \end{array}$$

The right triangle is smooth homotopy commutative, by the discussion above, so we need only prove the left hand triangle smooth homotopy commutative. Note that q_1 maps TY to the basepoint, and

$$q_1|_{T_f} : Y \cup_f TX = T_f \to \Sigma X$$

is defined by:

(1) $y \to *$
(2) $t \wedge x \to \bar{t} \wedge x$

Also, $\tilde{q}$ maps T_f to the basepoint, and

$$\tilde{q}(t \wedge y) = \bar{t} \wedge y$$

.

Now define a map $\psi\colon T_k \to \Sigma Y$ as follows:

The restriction of ψ to TY is the smooth isomorphism with T_+Y, namely,

$$\psi(t \wedge y) = \overline{\frac{1}{2}t} \wedge y.$$

The restriction of ψ to $T_f = Y \cup_f TX$ is given by:

(1) $y \mapsto \overline{\frac{1}{2}t \wedge y},$
(2) $t \wedge x \mapsto \overline{1 - \frac{1}{2}t} \wedge f(x).$

Contracting T_+Y to a point, we get a smooth homotopy of ψ to a map ψ_1, such that $\psi_1(TY) = *$, and $\psi_1|_{T_f}$ is defined by:

(1) $y \mapsto *,$
(2) $t \wedge x \mapsto \overline{1 - t} \wedge f(x).$

This map is $(-\Sigma f) \circ q_1$.

On the other hand, if we smoothly contract T_-Y to a point, then we obtain a smooth homotopy of ψ to a map ψ_2 such that $\psi_2(T_f) = *$, and

$$\psi_2(t \wedge y) = \bar{t} \wedge y.$$

This map is $\tilde{q}$. □

If we iterate the above construction of q_1, we may construct the following sequence:
(2-3)

$$\begin{array}{ccccccccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{f^1} & T_f & \xrightarrow{f^2} & T_{f^1} & \xrightarrow{f^3} & T_{f^2} & \xrightarrow{f^4} & T_{f^3} & \xrightarrow{f^5} & T_{f^4} & \xrightarrow{f^6} & T_{f^5} \longrightarrow \cdots \\
 & & & & & \searrow^{q} & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_3} & & \downarrow{\scriptstyle q_4} & & \downarrow{\scriptstyle q_5} \\
 & & & & & & \Sigma X & \xrightarrow{-\Sigma f} & \Sigma Y & \xrightarrow{-\Sigma f^1} & \Sigma T_f & \xrightarrow{-\Sigma f^2} & \Sigma T_{f^1} & \xrightarrow{-\Sigma f^3} & \Sigma T_{f^2} \longrightarrow \cdots \\
 & & & & & & & & & & & \searrow^{-\Sigma q} & \downarrow{\scriptstyle \Sigma q_1} & & \downarrow{\scriptstyle \Sigma q_2} \\
 & & & & & & & & & & & & \Sigma^2 X & \xrightarrow{\Sigma^2 f} & \Sigma^2 Y \longrightarrow \cdots
\end{array}$$

Now we may prove our main result:

Theorem 4.23. *The sequence (2-3) is right exact, and hence the right Puppe sequence, (2-1) is right exact.*

Proof. The proof of this result now follows by putting together our preceding lemmas.

Firstly, from the fact that the sequence (2-1) is right exact if and only if the sequence (2-2) is right exact, we may deduce that the diagrams

$$\begin{array}{ccc}
T_{f^n} & \xrightarrow{f^{n+2}} & T_{f^{n+1}} \\
\downarrow{\scriptstyle q^n} & & \downarrow{\scriptstyle q^{n+1}} \\
\Sigma T_{f^{n-3}} & \xrightarrow{-\Sigma f^{n-3}} & \Sigma T_{f^{n-2}}
\end{array}$$

are smooth homotopy commutative.

Thus, the top row of squares in (2-3) are smooth homotopy commutative. From Corollary 4.20, the following diagram is smooth homotopy commutative.

$$\begin{array}{ccccc}
Y & \xrightarrow{k} & T_f & \xrightarrow{l} & T_k \\
 & & & \searrow^{q} & \downarrow{\scriptstyle q_1} \\
 & & & & \Sigma X
\end{array}$$

Thus, by induction, we may deduce that the diagram (2-3) is smooth homotopy commutative.

Since the bottom row of the sequence (2-3) is right exact, so is the top row, which is sequence (2-1). □

We may immediately deduce the following corollary.

Corollary 4.24. *Given a smooth map $f\colon X \to Y$ between smooth spaces, and another smooth space W, the following sequence is exact.*

$$\cdots \longrightarrow [\Sigma^{n+1}X, W] \longrightarrow [\Sigma^n T_f, W] \longrightarrow [\Sigma^n Y, W] \longrightarrow [\Sigma^n X, W] \longrightarrow$$

$$\cdots \longrightarrow [\Sigma X, W] \xrightarrow{q^*} [T_f, W] \xrightarrow{k^*} [Y, W] \xrightarrow{f^*} [X, W].$$

This is an exact sequence of groups as far as $[\Sigma X, W]$, and the morphisms to this point are homomorphisms of groups.

Let us briefly recap on the hypotheses needed to arrive at the smooth right exact Puppe sequence. From Theorem 2.7 onwards, we have made the assumption that given two SNDR pairs (X, A) and (Y, B), there exists a 'suitable' smooth structure on $I \times X \times Y$, making $(X \times Y, A \times Y \cup X \times B)$ an SNDR pair with respect to this smooth structure.

From Theorem 4.10 onwards, we have assumed, in addition, that for all the SNDR pairs (X, A) we consider, the smooth structure on $I \times I \times X$ that makes $(I \times X, 0 \times X \cup I \times A)$ an SNDR pair is, in fact, the usual smooth structure on $I \times I \times X$.

Let us just note exactly where these assumptions are needed in the construction of the righ Puppe sequence.

Theorem 4.10, Corollary 4.11, Theorem 4.12, Corollary 4.13 and Corollary 4.15 all rely on the fact that $(I \times X, 0 \times X \cup I \times A)$ is an SNDR pair in the usual sense (i.e. the product structure needed on $I \times X$, to make $(I \times X, 0 \times X \cup I \times A)$ an SNDR pair is the usual one, for the SNDR pairs (X, A) that we consider.

It is only Corollary 4.15 that we use later to deduce that the various quotient maps in the right Puppe sequence (such as q_1, q_2, Σq_1, etc.) are smooth homotopy equivalences.

Note that all results proved before Theorem 4.10 in this chapter are independent of any assumption regarding product structures.

For our derivation of the smooth right exact Puppe sequence to be complete, we would need to clarify under what circumstances our hypotheses hold. Chapter 6 indicates possible approaches to this problem.

Chapter 5

Examples and Applications

In this chapter, we investigate the smooth homotopy groups of certain spaces. In particular, we look at spaces where the smooth and continuous homotopy groups are not isomorphic. We also investigate how differences in the smooth and continuous homotopy groups, in certain cases can be linked to the notion of "Hausdorff dimension".

The aim of this chapter is to give some justification for the study of the homotopy of smooth spaces. Spaces similar to the spaces in **SMTH** are being studied by theoretical physicists. For instance R. Penrose and W. Rindler in [19] define a class of spaces that are similar to smooth spaces, except that the set C_X of structure curves into a space X is not considered. This seems to indicate that spaces with smooth structures may be useful in certain areas of theoretical physics, but does not necessarily mean that the homotopy theory of smooth spaces is interesting.

We present the following examples of spaces where the smooth and continuous homotopy theories do not coincide, in the hope that it will justify, to some extent, the study of smooth homotopy.

For this chapter only, given a smooth space X, we denote the usual continuous n-th homotopy group by $\pi_n(X)$, as is the usual notation for continuous homotopy groups. The n-th smooth homotopy group of X will be denoted by $\pi_n^s(X)$.

5.1 Examples of Spaces Whose Smooth and Continuous Fundamental Groups Differ

In [5], Cherenack gives an example of a smooth space whose smooth and continuous fundamental groups differ. In this section we present some more examples, indicating that for a space X, any of the following may be true:

(1) $\pi_1(X) < \pi_1^s(X)$, where '$<$' indicates 'subgroup',
(2) $\pi_1^s(X) < \pi_1(X)$, or
(3) $\pi_1(X) = \pi_1^s(X)$.

We note that (3) can occur, even for some spaces X that intuitively seem to have some degree of 'non-smoothness'.

Proposition 5.1. *The space S constructed below has $\pi_1(S) = \mathbb{Z}$, and $\pi_1^s(S) = 1$*

Proof. First, we construct the space S, and calculate its usual fundamental group. Let

$$f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t)$$

$1 < s < 2$, $\lambda > 1$. It is well known that f is continuous and nowhere differentiable. See Falconer [9]. Now define a new function

$$g : [0,1] \to \mathbb{R}^2$$

by

$$g(t) = (t, f(t))$$

for $t \in [0,1]$. g is also continuous, since each component is, and g is nowhere differentiable, since f is nowhere differentiable. Note that g is one-to-one. Since $[0,1]$ is compact, and $\mathbb{R}^2$ is Hausdorff, g is a homeomorphism. Let $\bar{S}$ denote the set $g([0,1])$. We give this set the smooth subspace structure that it inherits from $\mathbb{R}$. Define an equivalence relation, $\sim$ on $\bar{S}$ which identifies the points $g(0) = g(1)$. Let $S = \bar{S}/\sim$, with the induced quotient smooth structure. S is homeomorphic to S^1, since $\bar{S} \cong [0,1]$. Thus $\pi_1(S) = \pi_1(S^1) = \mathbb{Z}$.

Now we construct the smooth fundamental group of S. Suppose $c\colon \mathbb{R} \to \bar{S}$ be a structure curve. Then, by definition of the smooth subspace structure, c must be a smooth mapping into $\mathbb{R} \times \mathbb{R}$. If $c(s) = (t(s), f(t(s))$, we must have smoothness in each coordinate. But f is nowhere differentiable, and so for $f(t(s))$ to be smooth, we must have $t(s)$ constant. This implies that c is constant.

Let $C_{\bar{S}}$ denote the set of structure curves of $\bar{S}$, as usual. We construct the set of structure functions $F_{\bar{S}}$ by defining $F_{\bar{S}} = \Gamma C_{\bar{S}}$, and note that $\Gamma C_{\bar{S}} = \{f\colon \bar{S} \to \mathbb{R}\}$. So we see that all functions on $\bar{S}$ are structure functions, and so all functions $f\colon S \to \mathbb{R}$ are structure functions.

We may now reverse the above construction on S, to observe that the only structure curves on S are the constant ones.

Now suppose that $[\omega] \in \pi_1^s(S)$. Then we have the composite

$$\mathbb{R} \xrightarrow{\ c\ } \Sigma S^0 \xrightarrow{\ \omega\ } S$$

(Recall that $\pi_1^s(X) \cong [\Sigma S^0, X]$). The composite ωc defines a structure curve on S, and so it is constant, for all curves c. But ΣS^0 does not have all structure curves constant, and so ω must be constant.

We deduce that $\pi_1^s(S) = \{\omega_0\}$, the class of the constant loop. □

One might now ask if there are spaces with non-trivial smooth fundamental groups which differ from their continuous fundamental group. Consider $S \times S^1$. For this space, $\pi_1(S \times S^1) = \mathbb{Z} \times \mathbb{Z}$, and $\pi_1^s(S \times S^1) = \mathbb{Z}$. This follows because of the above example, and the fact that smooth compact manifolds without boundary have isomorphic smooth and continuous homotopy groups (see Cherenack [5]), and the fact that $\pi_1^s(X \times Y) = \pi_1^s(X) \times \pi_1^s(Y)$.

For S we had $\pi_1^s(S) < \pi_1(S)$. This is not the case in general:

Proposition 5.2. *The space C constructed below has $\pi_1(C) = 1$ and $\pi_1^s(C) = \mathbb{Z}$.*

Proof. Let $C = (S^1 \times \bar{S})/\approx$ where $\approx$ is the equivalence relation: $(s_1, g(1)) = (s_2, g(1))$ for all $s_1, s_2 \in S^1$, where g is defined as in the previous proposition. C is homeomorphic to the closed unit disk, so $\pi_1(C) = 1$.

Now we calculate $\pi_1^s(C)$. This is calculated in a similar way to the calculation in the previous proposition:

Let $c\colon \mathbb{R} \to C$ be a structure curve, given by $c(s) =< x(s), y(s) >$. As we noted before, c is constant in the second coordinate, since any curve $y\colon \mathbb{R} \to \bar{S}$ is constant. Let the loop $\gamma\colon \Sigma S^0 \to C$ be given by $\gamma(t) =< x'(t), y'(t) >$. A structure curve $c'\colon \mathbb{R} \to \Sigma S^0$ defines a structure curve into C, by $\gamma(c(t) =< x'(c(t)), y'(c(t)) >$. For this to be smooth for all curves c', we must have γ constant in its second coordinate. Thus we may associate each $\gamma \in \pi_1^s(C)$ with a loop $\gamma' \in \pi_1^s(S^1)$. It is now easy to verify that $\pi_1^s(C) = \pi_1^s(S^1) = \mathbb{Z}$. $\square$

Corollary 5.3. *Let C and S be as defined above. Then*

$$\pi_1^s(C \times S) = \pi_1(C \times S) = \mathbb{Z}.$$

This section seems to indicate that the relationship between smooth and continuous homotopy groups of a smooth space X may not be very simple.

5.2 Smooth Homotopy Groups and the Hausdorff Dimension

In this section, we briefly define the notion of 'Hausdorff dimension', following Falconer [9], and state some results proven in [9]. We then show that for the n-sphere, there is a link between the n-th homotopy groups and Hausdorff dimension.

When a space X has non-isomorphic smooth and continuous homotopy groups, then this seems to indicate that X has some degree on 'non-smoothness'. The Hausdorff dimension of a space X is a measure of the 'irregularity' of X. Spaces whose Hausdorff dimension is non-integral, and spaces with differing smooth and continuous homotopy groups both seem to exhibit some degree of 'non-smoothness'.

One might hope to find a link between these two different indications of 'non-smoothness', but the three examples given in the previous section seem to indicate that this is not a simple task. All three spaces above have non-integral Hausdorff dimensions, (see Falconer [9]), but there doesn't seem to be a simple relation between the smooth and continuous fundamental groups. Nevertheless, the final result in this section indicates, for homeomorphic images of the n-sphere at least, that there is some link between these two, seemingly different ways of measuring "non-smoothness".

We begin by summarizing the properties of Hausdorff dimension that we will need to use. We omit proofs of results concerning Hausdorff dimension that can be found in [9].

Recall that for $U \subset \mathbb{R}^n$, the diameter of U is defined to be

$$|U| = \sup\{|x - y|: x, y \in \mathbb{R}^n\}.$$

If $\{U_i\}$ is a countable collection of subsets of $\mathbb{R}^n$ that cover X, each with diameter at most δ, then we call $\{U_i\}$ a δ-cover of X.

Definition 5.3. *Let $X \subset \mathbb{R}^n$, and $s \geq 0$. For $\delta > 0$, we define*

$$H^s_\delta(X) = inf\{\sum_{i=1}^{\infty} |U|^s_i : \{U_i\} \text{ is a } \delta - \text{cover of } X\}.$$

From this, we define the $s-$dimensional Hausdorff measure to be

$$H^s(X) = \lim_{\delta \to 0} H^s_\delta(X).$$

Falconer [9] shows that this definition makes sense for any X, although $H^s(X)$ may be 0 or ∞. It is also shown in [9] that H^s defines a measure on $\mathbb{R}^n$. The Hausdorff dimension is defined as follows.

Definition 5.4. *Let $X \subset \mathbb{R}^n$. Then the Hausdorff dimension, $dim_H X$, of X is defined as*

$$dim_H X = inf\{s: H^s(X) = 0\} = sup\{s: H^s(X) = \infty\}.$$

Definition 5.5. *Let $f: A \to B$ be a continuous mapping, where $A, B \subset \mathbb{R}^n$, for some $n \in \mathbb{N}$. Suppose that f satisfies the following condition.*

$$|f(x) - f(y)| \leq c|x - y|,$$

for $x, y \in A$, and $c \in \mathbb{R}$. Then f is called a Lipschitz mapping.

The next result is proved in Falconer [9].

Proposition 5.6. *If $A \subset \mathbb{R}^n$, some $n \in \mathbb{N}$. If $f: A \to \mathbb{R}^m$, $m \in \mathbb{N}$ is a Lipschitz mapping, then $dim_H f(A) \leq dim_H A$.*

This result seems to indicate that our notion of smoothness is stronger than the notion of smoothness that is measured by the Hausdorff dimension, since any smooth real function with bounded first derivative is a Lipschitz mapping, but Lipschitz mappings are not necessarily smooth. Thus a connection between these two types of smoothness is likely to be somewhat tenuous. The next result is interesting in that it gives us an idea of how to construct smooth spaces whose higher smooth and continuous homotopy groups differ.

Henceforth, the symbol $\cong$, means 'homeomorphic as topological spaces', and $\approx$ means 'isomorphic as groups'.

Proposition 5.7. *Let $A \subset \mathbb{R}^m$, and suppose $A \cong S^n$, for $m, n \in \mathbb{N}$. (As usual, S^n denotes the n-sphere). If we have*

$$\pi_n(A) \approx \pi_n^s(A),$$

then $dim_H A \leq n$.

Proof. All homotopies in this proof are assumed to be continuous homotopies, unless otherwise stated. Let $\bar{f}: S^n \to A$ be a homeomorphism. We may replace $\bar{f}$ by a homeomorphism $f: \Sigma^n S^0 \to A$, since in the continuous situation $\Sigma^n S^0 \cong S^n$. Then $[f] \in \pi_n(A)$, with $[f] \neq [*]$, for otherwise

$$1_{\Sigma^n S^0} = f^{-1} f \simeq f^{-1} * = *,$$

which is a contradiction.

Now, $\pi_n(A) \approx \pi_n^s(A)$, so let

$$\gamma: \pi_n(A) \to \pi_n^s(A),$$

be an isomorphism of groups. Thus the map

$$\gamma f = g: \Sigma^n S^0 \to A$$

is a smooth map, with $[g] \neq *$.

We claim that $g(\Sigma^n S^0) = A$. To see this, let us suppose the converse. Then there exists a point $x_0 \in A$, such that $x_0 \in A - g(\Sigma^n S^0)$. But $A - x_0 \cong \Sigma^n S^0 - f^{-1}(x_0) \cong \mathbb{R}^n$. Let $q: A - x_0 \to \mathbb{R}^n$ be the homeomorphism $A - x_0 \cong \mathbb{R}^n$. Define

$$h = q \circ g: \Sigma^n S^0 \to \mathbb{R}^n.$$

Let us note that h is nullhomotopic, since

$$\pi_n(\mathbb{R}^n) = \pi_n^s(\mathbb{R}^n) = 1.$$

But this implies that

$$g = q^{-1} h \simeq q^{-1} * = *,$$

and so g is nullhomotopic, which is a contradiction. Thus we have shown that $g(\Sigma^n S^0) = A$.

Note that we can get the smooth n-th (reduced) suspension of S^0, $\Sigma^n S^0$ from $I^n \times \{0,1\}$, by performing a number of identifications.

Let $q_1: I^n \times \{0,1\} \to \Sigma^n S^0$ be this quotient map. Since g is smooth, so is

$$g \circ q_1: I^n \times \{0,1\} \to A.$$

But $I^n \times \{0,1\} \subset \mathbb{R}^{n+1}$, $A \subseteq \mathbb{R}^m$ and $I^n \times \{0,1\}$ is compact, so $g \circ q_1$ is a smooth map, in the standard sense, from a compact subset of $\mathbb{R}^{n+1}$ to $\mathbb{R}^m$. This implies that $g \circ q_1$ is a Lipschitz mapping.

From Proposition 5.6 we may deduce that

$$dim_H(g \circ q_1(I^n \times \{0,1\})) = dim_H(A) \leq dim_H(I^n \times \{0,1\}).$$

But $dim_H(I^n \times \{0,1\}) = n$, since by the product formula for Hausdorff dimension, we have $dim_H(I^n \times \{0,1\}) \leq n + 0 = n$, and since $I^n \times \{0,1\}$ contains an open ball of dimension n, $dim_H(I^n \times \{0,1\}) \geq n$. For these last two inequalities see the results in Falconer [9], page 92 and page 28, respectively. □

This proposition gives us a clue as to how to construct a space whose n-th smooth homotopy group is not isomorphic to its n-th continuous homotopy group. All we need to do is construct a space which is homeomorphic to S^n, but which has Hausdorff dimension greater than to n. The space in Proposition 5.1 can be shown to be such a space.

Chapter 6

The Smooth NDR Pair Problem

In this chapter we analyse the hypothesis of Chapter 2. The basic questions we are attempting to answer are:

(1) Does our hypothesis hold under the usual conditions? In other words, is the class of SNDR pairs closed under the formation of products, where all product structures are the usual ones?

(2) Is there a sub-class of SNDR pairs that is closed under usual products? Is this sub-class large enough to enable us to derive the smooth right exact Puppe sequence?

(3) If the class of SNDR pairs is not closed under the formation of products, can we weaken the notion of SNDR pair to include all products? Do these SNDR pairs have the useful properties of SNDR pairs?

(4) Can we change the product structure on SNDR pairs so that products do exist? Does this product structure give us useful SNDR pairs?

6.1 Modifying the Standard Proof

In this approach we keep the definition of SNDR pair as defined in Chapter 2, and try to modify the proof of the corresponding continuous result, with all smooth structures the usual ones.

Suppose we wish to follow a proof similar to the one given by Steenrod in [27], where he proves this result for the usual continuous NDR pairs. One is soon confronted by the function $q\colon I \times X \times Y \to X \times Y$, given by

$$q(t,x,y) = \begin{cases} \text{(a) } (h(t,x), j(t,y)) \text{ if } (x,y) \in A \times B \\ \text{(b) } (h(t,x), j(\alpha(\frac{ux}{vy})t, y)) \text{ if } vy \geq ux,\ vy > 0 \\ \text{(c) } (h(\alpha(\frac{vy}{ux})t, x), j(t,y)) \text{ if } ux \geq vy,\ ux > 0 \end{cases}$$

To show that q is smooth, we let $c\colon \mathbb{R} \to I \times X \times Y$ be a structure curve given by $c(s) = (t(s), x(s), y(s))$, and $f\colon X \times Y \to \mathbb{R}$ be a structure function. The task of showing q smooth now becomes the task of showing that $fqc\colon \mathbb{R} \to \mathbb{R}$, given by

$$fqc(s) = \begin{cases} \text{(a) } f(h(t(s),x(s)), j(t(s),y(s))) \text{ if } (x(s),y(s)) \in A \times B \\ \text{(b) } f(h(t(s),x(s)), j(\alpha(\frac{ux(s)}{vy(s)})t(s), y(s))) \text{ if } vy(s) \geq ux(s),\ vy(s) > 0 \\ \text{(c) } f(h(\alpha(\frac{vy(s)}{ux(s)})t(s), x(s)), j(t(s),y(s))) \text{ if } ux(s) \geq vy(s),\ ux(s) > 0 \end{cases}$$

is a smooth real function. It is easily seen to be smooth on the regions defined by lines (b) and (c) of the definition. The problem is that as $ux(s)$ and $vy(s)$ approach zero, we have no control over the behaviour of the terms $\frac{ux(s)}{vy(s)}$ and $\frac{vy(s)}{ux(s)}$. This is not a problem if one only requires continuity of q, because the homotopies h and j ensure continuity when ux and vy, respectively, approach zero. It is this observation that leads us to define the first coordinate independence property in Section 6.2.

We also tried to replace the occurences of $\alpha(\frac{ux}{vy})$ and $\alpha(\frac{vy}{ux})$ by functions $\beta_1, \beta_2 \colon I \times X \times Y \to \mathbb{R}$, which were more manageable as ux and vy approached $A \times B$. One example of β_1 and β_2 was

$$\beta_1(t,x,y) = \begin{cases} t \text{ if } y \in B \\ t(\alpha(\dfrac{ux}{vy}) - 1)e^{-\alpha(\frac{ux}{vy})e^{\frac{1}{vy^2}}} + t \text{ otherwise,} \end{cases}$$

and

$$\beta_2(t,x,y) = \begin{cases} t \text{ if } x \in A \\ t(\alpha(\dfrac{vy}{ux}) - 1)e^{-\alpha(\frac{vy}{ux})e^{\frac{1}{ux^2}}} + t \text{ otherwise.} \end{cases}$$

All the different functions we tried for β_1 and β_2 had the same problem as the original functions given by Steenrod. The problem in the above examples occur in β_1 when $vy \to 0$, for then $-e^{\frac{1}{vy^2}}$ approaches $-\infty$ quickly, but we are still essentially multiplying this value by ux which approaches zero.

In fact, one cannot find functions β_1 and β_2 that are smooth everywhere, and still do all that is required of them in the above 'proof', since we need

(1) $\beta_1(t,x,y) = t$ for ux near vy,
(2) $\beta_1(t,x,y) = 0$ for ux near 0.

Thus we abandoned this approach to proving the existence of products.

6.2 Restricting the Class of SNDR Pairs

In this section we define a slightly different notion of smooth neighbourhood deformation retract. We call these restricted smooth neighbourhood deformation retracts, or R-SNDR's for short. We can show that this class of SNDR's is closed under the formation of products, but it is not known whether the class is large enough to be useful. In particular, we have been unable to find a representation of $(I,0)$ as an R-SNDR. We show that if there does exist a representation for $(I,0)$ as an R-SNDR pair, then R-SNDR's are equivalent to a certain type of 'restricted' cofibration.

In the continuous theory of NDR and DR pairs, we may note that, given an NDR pair (X,A), with representation (u,h), then if $a \in A$, $h(t,x)$ is independent of the first coordinate, t. This is clearly true for SNDR and SDR pairs too. This prompts us to ask the following question. Given an SNDR pair (X,A) with representation (u,h), a structure curve $c \colon \mathbb{R} \to I \times X$, given by $c(s) = (t(s), x(s))$ with $c(s_0) \in A$, $s_0 \in \mathbb{R}$ and a structure function $f\colon X \to \mathbb{R}$, are the derivatives

$$\lim_{s \to s_0} \frac{d^n}{ds^n} fh(t(s), x(s)), \tag{2–1}$$

independent of the t coordinate? It is not known whether every SNDR pair (X, A) has at least one such representation. This prompts the following definition.

Definition 6.1. *Suppose that X,Y are smooth spaces and that A is a smooth subspace of X. Suppose further that we have a smooth map $h\colon I \times X \to Y$ that satisfies the following property.*
Let $c\colon \mathbb{R} \to I \times X$ be a map, given by $c(s) = (t(s), x(s))$, where $x(s)$ is smooth for all $s \in \mathbb{R}$, and $t(s)$ is smooth for all s such that $x(s) \in X - A$. Let $f\colon I \times X \to \mathbb{R}$ be a structure function of $I \times X$. If $\frac{d^n}{ds^n} fhc(s)$ exists for all $s \in \mathbb{R}$, and $n \in \mathbb{N}$ then we say that h has the first coordinate existence property with respect to A.

The above condition implies that the derivatives $\frac{d^n}{ds^n} fhc(s)$ are smooth for all $s \in \mathbb{R}$ and $n \in \mathbb{N}$. We may immediately note that if a smooth map $h : I \times X \to Y$ has the first coordinate independence property with respect to A, and $g : Y \to Z$ is a smooth map for some smooth space Z, then gh has the first coordinate existence property at A. To see this, let c be a map as in the above definition, and $f\colon Z \to \mathbb{R}$ be a structure function on Z. Then fg is a structure function on Y, and so $\frac{d^n}{ds^n}(fg)hc(s)$ exists, and is smooth for all $n \in \mathbb{N}, x(s) \in A$.

If h above is a homotopy $f \simeq_h f'$, with the first coordinate independence property with respect to A, then by taking $n = 0$, we observe that h is a smooth homotopy (rel A).

Definition 6.2. *Let X be a smooth space, and A a smooth subspace of X. If (X, A) is an SNDR (SDR) pair, with representation (u, h), and we may choose h to have the first coordinate existence property then we call (X, A) a restricted smooth neighbourhood deformation retract (restricted smooth deformation retract) pair. We use the abbreviation R-SNDR (R-SDR).*

Note that $(X, \emptyset)$ is an R-SDR pair, and (X, X) is an R-SNDR pair. Unfortunately we have no non-trivial examples of R-SNDR or R-SDR pairs.

The class of R-SNDR pairs is closed under the formation of products, as the next theorem demonstrates.

Theorem 6.3. *If (X, A) and (Y, B) are R-SNDR pairs, then so is their product*

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

If one is an R-SDR and the other is an R-SNDR, then their product is an R-SDR.

Proof. Our proof is based on the proof of the usual continuous result found in [27].

Let $u\colon X \to I$ and $h\colon I \times X \to X$ A as an R-SNDR in X, and let $v\colon Y \to I$ and $j\colon I \times Y \to Y$ represent B as an R-SNDR in Y.

Define $w : X \times Y \to I$ by $w(x,y) = (ux)(vy)$. It is clear that w is smooth and that $w^{-1}(0)$ is $X \times B \cup A \times Y$.

Define a smooth braking function $\alpha\colon \mathbb{R} \to \mathbb{R}$ with the properties:

(1) $\alpha(t) = 0$ for $t \leq \frac{1}{4}$,
(2) $\alpha(t) = 1$ for $t \geq \frac{3}{4}$.

Define $q\colon I \times X \times Y \to X \times Y$ as follows:

$$q(t,x,y) = \begin{cases} \text{(a) } (h(t,x), j(t,y)) \text{ if } ux = vy = 0 \\ \text{(b) } (h(t,x), j(\alpha(\dfrac{ux}{vy})t, y)) \text{ if } vy \geq ux,\ vy > 0 \\ \text{(c) } (h(\alpha(\dfrac{vy}{ux})t, x), j(t,y)) \text{ if } ux \geq vy,\ ux > 0 \end{cases}$$

Let us show that q is smooth. Note, firstly, that q is smooth on each of the pieces that define it, since it is simply the composite of smooth functions there. Next, observe that for ux in a neighbourhood of vy, with $ux \neq 0$, and $vy \neq 0$ we have $\alpha(\frac{ux}{vy}) = \alpha(\frac{vy}{ux}) = 0$ and so parts (b) and (c) of the definition coincide in some neighbourhood. Thus we need only show smoothness of q as both ux and vy approach 0.

Now, if q is smooth in each of its two coordinates, then it is smooth. Let us consider the coordinate involving the smooth homotopy j. The other case is similar. To this end, let $c\colon \mathbb{R} \to I \times X \times Y$ be a structure curve given by $c(s) = (t(s), x(s), y(s)$, and let $f\colon Y \to \mathbb{R}$ be a structure function. Define two regions as follows:

(1) $R_1 = \{s | vy(s) \geq ux(s),\ vy(s) > 0\}$,
(2) $R_2 = \mathbb{R} - R_1$.

Then $fp_2qc(s)$ is given by

$$fq(t(s), x(s), y(s)) = \begin{cases} \text{(a) } fj(t(s), y(s)) \text{ if } ux(s) = vy(s) = 0 \\ \text{(b) } fj(\alpha(\dfrac{ux(s)}{vy(s)})t(s), y(s))) \text{ if } vy(s) \geq ux(s),\ vy(s) > 0 \\ \text{(c) } fj(t(s), y(s))) \text{ if } ux(s) \geq vy(s),\ ux(s) > 0, \end{cases}$$

where p_2 is the projection onto the second coordinate. Now, for $s \in R_2$, $fqc(s)$ is given by $fj(t(s), y(s))$ which is smooth. Our only problem with smoothness occurs in R_1 when $ux(s)$ and $vy(s)$ approach 0. But in R_1, $(\frac{ux(s)}{vy(s)})t(s))$ is smooth, and so by the first coordinate independence property of j, we know that

$$\frac{d^n}{ds^n} fj(\alpha(\frac{ux(s)}{vy(s))} t(s), y(s))$$

exists and is smooth for $n \in \mathbb{N}$, and $s \in \mathbb{R}$ such that $y(s) \in B$. Thus q is smooth.

The case involving the smooth homotopy h is similar.

Let us now verify that q is the required smooth homotopy. When $t = 0$, all three lines defining q reduce to $(h(0,x), j(0,y))$. If $x \in A$ then q is given by line (b) which reduces to $(h(t,x), j(0,y))$. The case when $y \in B$ is similar. If 0¡$w(x,y) < 1$, and $t = 1$, then either $0 < ux < 1$ or $0 < vy < 1$. Suppose $0 < ux < 1$. Then either $ux \leq vy$ or $vy < ux$. If $ux \leq vy$, then q is given by line (b). Line (b) reduces to $(h(1,x), j(\alpha(\frac{ux}{vy}), y)) \in A \times Y$. If $vy < ux$, then line (c) applies, and q reduces to $(h(\alpha(\frac{vy}{ux}), x), j(1,y)) \in X \times B$.

We must now show that q has the first coordinate existence property. Let $c\colon \mathbb{R} \to I \times X \times Y$ be given by $c(s) = (t(s), x(s), y(s))$. Suppose that $t(s)$ is smooth for all s such that $(x(s), y(s)) \notin X \times B \cup Y \times A$. Suppose that $c(s_0) \in I \times X \times B \cup Y \times A$.

If $(x(s_0), y(s_0)) \in A \times B$, then h has the first coordinate existence property at $x(s_0)$, and j has the first coordinate existence property at $y(s_0)$, so both coordinates of q have this property, and hence q has the first coordinate existence property at these points.

If $(x(s_0), y(s_0)) \in A \times Y$, with $y(s_0) \notin B$, then q is given by line (b) of its definition, which is given by

$$q(t(s_0), x(s_0), y(s_0)) = (h(t(s_0), x(s_0)), j(0, y(s_0)).$$

Now the coordinate involving h of q has the first coordinate existence property at $x(s_0)$. The other coordinate is independent of $t(s)$ for s in some neighbourhood of s_0, and so has the first coordinate existence property at $y(s_0)$. Thus q has this property in this case. The other case is similar.

To see the last part of the theorem, suppose u, h represent (X, A) as an R-SDR. If we replace u by $u' = \frac{1}{2}u$, then u', h also represent (X, A) as an R-SDR. Making the above constructions now with u' in place of u, it follows that $w(x,y) < 1$ for all (x,y), so $q(1,x,y) \in X \times B \cup A \times Y$. Thus the product pair is an R-SDR. $\square$

We can now show that, if $(I, 0)$ has a representation as an R-SDR pair, then R-SNDR pairs are equivalent to closed smooth 'restricted cofibrations', where 'restricted cofibrations' are a stronger form of smooth cofibration.

Definition 6.4. *Suppose that A is a smooth subspace of a smooth space X, and that (X, A) is a smooth cofibred pair. If, in addition the following diagram,*

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{1} & 0 \times X \cup I \times A \\ \big\downarrow j & \nearrow_{r} & \\ I \times X & & \end{array}$$

where $j\colon 0 \times X \cup I \times A \hookrightarrow I \times X$ is the inclusion, may be completed with a map r, such that pr has the first coordinate existence property. Then we say that $i\colon A \hookrightarrow X$ is a smooth restricted cofibration (smooth R-cofibration, for short). We call (X, A) a smooth R-cofibred pair.

The following lemma is the analogue of Lemma 2.5 in Section 2.2.

Lemma 6.5. *If A is a smooth subspace of a smooth space X, then the inclusion map $i\colon A \hookrightarrow X$ is a smooth R-cofibration if and only if $0 \times X \cup I \times A$ is a smooth retract of $I \times X$, where we may pick the smooth retraction r such that pr has the first coordinate existence property, where p is the projection onto the second coordinate.*

Proof. Suppose that $0 \times X \cup I \times A$ is a smooth retract of $I \times X$, with a retraction $r\colon I \times X \to 0 \times X \cup I \times A$ such that pr has the first coordinate existence property, where p is the projection onto the second coordinate. We know that the inclusion map $i\colon 0 \times X \cup I \times A \to I \times X$ is a smooth cofibration. We must verify that (X, A) is, in fact a smooth R-cofibred pair. We may complete the diagram

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{1} & 0 \times X \cup I \times A \\ {\scriptstyle j}\big\downarrow & \overset{k}{\nearrow} & \\ I \times X & & \end{array}$$

by defining $k = 1r = r$. By hypothesis, pr has the first coordinate existence property.

Conversely, suppose that (X, A) is a smooth R-cofibred pair. Then, by hypothesis, the following diagram may be completed

$$\begin{array}{ccc} 0 \times X \cup I \times A & \xrightarrow{1} & 0 \times X \cup I \times A \\ {\scriptstyle j}\big\downarrow & \overset{k}{\nearrow} & \\ I \times X & & \end{array}$$

by a r such that pr has the first coordinate existence property. □

Theorem 6.6. *If X is a smooth space, and A is a smooth subspace which is closed in the underlying topology, and $p\colon I \times X \to X$ is the projection onto the second coordinate then the following are equivalent:*

(1) *(X, A) is a R-SNDR,*
(2) *$0 \times X \cup I \times A$ is a smooth retract of $I \times X$, and we may choose this smooth retraction so that pr has the first coordinate existence property.*
(3) *(X, A) is an smooth R-cofibred pair.*

Proof. Assume (1). Suppose $(I, 0)$ is an R-SDR, so Theorem 6.3 says that $(X \times I, 0 \times X \cup I \times A)$ is an R-SDR. We may construct a representation (w, q) for this R-SDR as follows,

$$q(t,s,x) = \begin{cases} \text{(a) } (h(t,s), j(t,x)) \text{ if } us = vx = 0 \\ \text{(b) } (h(t,s), j(\alpha(\frac{us}{vx})t, x)) \text{ if } vx \geq us,\ vx > 0 \\ \text{(c) } (h(\alpha(\frac{vx}{us})t, s), j(t,x)) \text{ if } us \geq vx,\ us > 0 \end{cases}$$

where (u, h) is a representation for $(I, 0)$ as an R-SNDR pair, and (v, j) is a representation for (X, A) as an R-SDR pair. We take our retraction to be $r(s, x) = q(1, s, x)$. Note that pr is given by

$$pq(1,s,x) = \begin{cases} \text{(a) } j(1,x)) \text{ if } us = vx = 0 \\ \text{(b) } j(\alpha(\frac{us}{vx}), x)) \text{ if } vx \geq us,\ vx > 0 \\ \text{(c) } j(1,x)) \text{ if } us \geq vx,\ us > 0 \end{cases}$$

which has the first coordinate existence property, since j hsa this property. Thus (1) implies (2).

The equivalence of (2) and (3) is Lemma 6.5

We need now only show that (2) implies (1). Let r be a smooth retraction of $I \times X$ into $0 \times X \cup I \times A$, with the first coordinate existence property and let $p\colon I \times X \to X$ be the projection onto the second coordinate. Let $h\colon I \times X \to X$ be defined by $h(t, x) = pr(t, x)$. By hypothesis, pr has the first coordinate existence property, so h has this property. The rest of the theorem follows as before, telling us that (X, A) is an R-SNDR pair, where the smooth homotopy h has the first coordinate existence property. This implies that (X, A) is an R-SNDR pair.

This section has shown that if we can find a representation for $(I, 0)$ as an R-SDR pair, then we have a class of SNDR's that gives us the two most important properties that we require, namely closure under products, and equivalence to some sort of cofibration. We can verify that the remaining results of Chapter 2 go through, in a suitably adjusted form.

For this approach to be useful, one would have to find the required representation for $(I, 0)$ as an R-SDR, and show that most of the other useful SNDR pairs are also R-SNDR pairs. This does not seem to be an easy task.

6.3 Adjusting the Smooth Structure on Pairs

Another approach we tried was to adjust the smooth structure on pairs (X, A), where A is a smooth subspace of a smooth space X. The construction of q in Section 6.1 runs into problems when ux and vy both approach zero. In other words, the problems arise at certain parts of the subspace $A \times Y \cup X \times B$ of $X \times Y$. This prompted us to try the following smooth structure on (X, A).

Definition 6.7. *Given a smooth subspace (A, C_A, F_A) of a smooth space (X, C_X, F_X), we give X a new smooth structure (X, C_X^*, F_X^*) as follows. Give the complement of A, $X - A$, the usual subspace structure, denoted $(X - A, C_{X-A}, F_{X-A})$.*

(1) *The elements of F_X^* are all maps $f\colon X \to \mathbb{R}$, such that $f|A \in F_A$ and $f|X - A \in F_{X-A}$.*
(2) $C_X = \Gamma F_X$.

We call this structure the smooth pair structure for a smooth pair (X, A).

We denote the space X with the pair structure generated by A by X_A. The definition of SNDR pair is then adjusted as follows.

Definition 6.8. *Let A be a smooth subspace of a smooth space X. We call A an alternative smooth neighbourhood deformation retract (A-SNDR) in X if there exists*

(1) *A smooth mapping $u\colon X \to I$*
(2) *A smooth homotopy $h\colon I \times X_A \to X$*

with the following properties:

(1) $A = u^{-1}(0)$,
(2) $h(0, x) = x$ *for all* $x \in X$,
(3) $h(t, x) = x$ *for* $(t, x) \in I \times A$,
(4) $h(1, x) \in A$ *for all* x *with* $ux < 1$.

The pair (X, A) is called an A-SNDR pair.

We define A-SDR's in a similar way.

The above structure essentially turns X into a coproduct space $X - A \sqcup A$. The construction q in Section 6.1 is now clearly smooth, since it is smooth when restricted to the subspace, and its complement.

The main problem with this structure is that it seems to trivialise the notion of SNDR pair, because if A is non-empty, we may always define a smooth retract $r\colon X_A \to A$ by $r(a) = a$ for $a \in A$, and $r(x) = a_0$ for $x \in X - A$, and some $a_0 \in A$.

Hence, if all the results of Chapter 2 that we need go through under this smooth pair structure, we will get an equivalence between A-SNDR pairs (X, A), and smooth retractions $r\colon I \times X \to 0 \times X \cup I \times A$. Thus all pairs (X, A) would be A-SNDR pairs.

It seems that we need a smooth structure that is coarser (i.e. has less structure functions) than the structure above, but finer than the usual structure on X. It is not clear how one might define such an 'in-between' smooth structure.

Besides the fact that the above approach trivialises, any approach that changes the smooth structure on (X, A) is likely to be somewhat unsatisfactory, because it seems probable that this non-standard smooth structure will carry over to the smooth Puppe sequence, giving non-standard smooth structures on the mapping cones and smooth suspensions in the sequence.

6.4 Adjusting the Smooth Structure on Products

Although the category **SMTH** has most of the useful properties cited by Steenrod in [27] as desirable properties for a category of topological spaces, as we noted in the introduction, our difficulties with the products of smooth retractions seems to indicate that **SMTH** may not have the property (also mentioned in [27]) of being 'well-behaved' under the formation of products. This, in essence, is our hypothesis of Chapter 2.

This suggests the idea of trying to find a 'convenient category of smooth spaces' which is analogous to Steenrod's 'convenient category of topological spaces', in which to do smooth homotopy.

Since most of the other properties of Steenrod's category (which is the category of compactly generated spaces) are also properties of **SMTH**, such as Cartesian closedness, this approach essentially reduces to the problem of finding a suitable product structure, that coincides with the usual Cartesian product in most important situations, such as a smooth homotopy $h\colon I \times X \to X$. We would hope that the product structure on $I \times X$ coincided with the Cartesian product, so that the notion of smooth homotopy remained unaffected.

The example given in Chapter 2 gives the unit interval I a slightly unusual smooth structure which induces a smooth structure on $I \times I$, so that $(I \times I, 0 \times I \cup I \times 0)$ has a representation as an SNDR pair. This is not ideal, as we would like to keep the usual structure on I, since this object is fundamental to almost all aspects of smooth homotopy theory.

This approach is the most similar to the situation in the topological case, and for this reason it seems as though this approach may be the most likely one to work. Unfortunately, the task of finding exactly the right subcategory in which to work does not seem to be a easy one.

6.5 Other Approaches

There are at least two other approaches that might lead to useful definitions of smooth neighbourhood deformation retracts. They are as follows.

Strøm's Variant. In [29], Strøm gives a definition of a structure on a pair (X, A), which turns out to be equivalent to the notion of NDR pair when A is closed. We may adapt his definition to the smooth case:

Definition 6.9. *Let A be a smooth subspace of a smooth space X. Suppose there exist maps $u\colon X \to I$, and $h\colon I \times X \to X$, such that*

(1) $ux = 0$ *if* $x \in A$.
(2) $h(0, x) = x$ *if* $x \in X$,
(3) $h(t, a) = a$ *if* $a \in A$,
(4) $h(t, x) \in A$ *if* $t > ux$.

The construction of a representation for the product Strøm structure on $(X \times Y, A \times Y \cup X \times B)$ is simple in the continuous case. Unfortunately, this construction relies on the non-smooth function 'min'. The results that link pairs admitting a Strøm structure with cofibred pairs also rely on non-smooth functions such as 'min' and 'max'. We do not know if it is possible to construct a representation for the product pair using only smooth functions.

Weakening the Structure on SNDR Pairs. Another approach we tried was to enlarge the class of SNDR pairs, in the hope that it would become large enough to include any product SNDR's that may not have been in this class before. We tried weakening the conditions on h in a representation (u, h), so that they only hold up to smooth homotopy. One approach was as follows.

Definition 6.10. *Let A be a smooth subspace of a smooth space X. We call A a weak smooth neighbourhood deformation retract (W-SNDR) in X if there exists*

(1) *A smooth mapping $u: X \to I$*
(2) *A smooth homotopy $h: I \times X \to X$*

with the following properties:

(1) $A = u^{-1}(0)$,
(2) $h(0, x) \simeq x$ *for all* $x \in X$,
(3) $h(t, x) \simeq x$ *for* $(t, x) \in I \times A$,
(4) $h(1, x) \in A$ *for all* x *with* $ux < 1$.

We were unable to prove any useful results with the definition of W-SNDR pair as it stands, but if one combines this with the definition of R-SNDR pair, by insisting that the h in the above definition have the first coordinate existence property, then one can prove that the class of such pairs is closed under the formation of products. In addition, we may construct a representation for $(I, 0)$ as follows.

Define $u: I \to I$, by $u(s) = s$. Define $h: I \times I \to I$ by $h(t, s) = \alpha(s)(1 - t)$. One can readily verify that (u, h) is a representation of $(I, 0)$ as required. Note, in particular, that h is independent of t for s near 0, and thus h has the first coordinate existence property.

We were unable to find a notion of smooth retraction or smooth cofibration that was equivalent to this type of SNDR pair, and we were thus forced to abandon this approach.

Although we were unable to find an approach that worked completely, the failed attempts discussed in this chapter were instructive, in that they highlighted the importance of each aspect of the definition of an SNDR pair.

The assumption we have made in Chapter 2 is effectively a 'distillation' of our failed attempts, into what we consider to be the bare minimum required for the later results in Chapter 2.

Chapter 4 requires slightly more specific knowledge about when the generalised notion of SNDR pair coincides with the usual notion of SNDR pair, and this may be a good starting point for a future study of smooth neighbourhood deformation retracts.

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