Volatility Level Dependence and the CEV Market Model

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

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Abstract

Interest-rate volatility is known to be level-dependent. However, Filipovic, Larsson and Trolle (2017) found that volatility becomes more level-dependent as the interest rate approaches the zero lower bound. This varying volatility level-dependence feature motivates the use of CEV market model to model the interest rate. In this dissertation, we compare the lognormal forward LIBOR market model, the CEV market model and the normal market model through regression analysis, hedging analysis and calibration analysis to assess their performance. The investigation is performed using EURIBOR 10-year interest-rate caps with various strike rates. This research work has a significant impact as the industry often needs to hedge interest-rate caps. We show that although the CEV market model best calibrates to market prices, the normal market model is the best in terms of hedging interest-rate caps.

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1 Introduction

Recent empirical findings by Filipovic, Larsson and Trolle (2017) suggest that the correlation between changes to interest rate and changes to interest-rate volatility is level dependent. They found that when interest rates are high, the correlation is low (i.e., when interest rates are high, changes to interest-rate volatilities have little correlation with changes to interest rates). However, when interest rates decrease, the correlation increases (i.e., when interest rates are low, changes to interest-rate volatilities have positive correlation with changes to interest rates). This empirical finding is important as it suggests that a good interest-rate model should correctly incorporate this varying volatility level-dependence. As a result, this dissertation aims to investigate a generalisation of the forward LIBOR market model (LMM) (Brace, Gatarek and Musiela, 1997), namely the constant elasticity of variance (CEV) market model (Andersen and Andreasen, 2000), which is able to capture the varying volatility level-dependence.

In the original LMM, it assumes that volatility level-dependence is strong and remains unchanged with changing forward rate levels. However, this assumption is problematic as there is little level-dependence when interest rates are high and more level-dependence when interest rates are low. This, in turn, will result in an incorrect hedging strategy. The CEV-generalisation of LMM allows the model to change from high level-dependence to no dependence with a change of exponent parameter. By calibrating the CEV market model, it is possible to incorporate correct level-dependence within an interest-rate regime. This, in turn, should allow one to find better hedging strategy.

This dissertation aims to investigate the role that the CEV market model might play in handling volatility level-dependence. This is done by calibrating the CEV market model such that there is no correlation between changes to CEV implied volatilities and changes to rates through regression analysis. Moreover, hedging analysis is performed to calibrate the CEV market model such that it produces the best hedging performance. The hedging performance is compared against the original LMM which assumes strong level-dependence and the normal model (Bachelier, Cootner *et al.*, 1964) which assumes no level-dependence.

The rest of this dissertation is structured as follows. In Section 2, we review the popular LIBOR market model and a generalisation of the LIBOR market model, namely the CEV market model. Then we briefly review the findings by Filipovic *et al.* (2017) about varying volatility level-dependence. Finally, we provide the pricing formula for interest-rate caps under these models and discuss different volatility level-dependence assumptions under different models. In Section 3, we introduce the methodologies which we will use to assess which interest-rate model best describes the relationship between interest rate and interest-rate volatility. Moreover, we also explain how the data were cleaned and bootstrapped to produce the necessary zero-coupon bond prices. In Section 4, we present the results obtained by performing the methodologies described in the previous section. Finally, Section 5 concludes.

2 Literature review

2.1 The LIBOR market model and the CEV market model

The lognormal forward LIBOR market model (LMM) is a popular financial model of simple forward rates. As opposed to an instantaneous short rate model or an instantaneous forward rate model (such as the Heath-Jarrow-Morton model), the LMM models observable simple forward rates, often the LIBOR rates in foreign markets. Moreover, it has the advantage that it is consistent with Black's model, which is the industry standard formula, when pricing interest-rate derivatives such as caps (Brigo and Mercurio, 2006).

Similar to Andersen and Andreasen (2000), we consider an increasing maturity structure $0 = T_0 < T_1 < \ldots < T_{M+1}$ where $M + 1$ is the number of caplets within a cap. With $P(t, T)$ denoting the time t price of zero-coupon bond maturing at time T, we define discrete forward rates on the maturity structure as,

$$L_k(t) = \frac{1}{\delta_k} \left(\frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right), \qquad k \in \{1, \ldots, M\} \tag{2.1}$$

where $\delta_k = T_{k+1} - T_k$. We then define forward rate dynamics by

$$dL_k(t) = \sigma_k(t) L_k(t) dW_{k+1}(t), \qquad k \in \{1, \ldots, M\}$$

where W_{k+1} is a standard Brownian motion under forward measure $\mathbb{Q}^{k+1}$ which is obtained by using the T_{k+1} maturity zero-coupon bond as the numeraire.

The above model is known as lognormal forward-LIBOR model (LFM). It is used to price interest-rate caps with Black's cap formula. A similar model known as lognormal forward-swap model (LSM) is used to price swaptions with Black's swaption formula. We do not explore LSM here as we only focus on interest-rate cap in this dissertation.

In addition, we also need to consider the money market account (bank account) which represents the riskless investment. The bank account, B, is defined by the following differential equation:

$$dB(t) = r_t B(t) dt, \qquad B_0 = 1$$

where r_t represents the instantaneous rate earned in the bank account. As a consequence, the value of bank account, $B(t)$, at time t can be calculated as

$$B(t) = \exp\left(\int_0^t r_s ds \right).$$

As per Andersen and Andreasen (2000), we assume a unique $\mathbb{Q}^{k+1}$ exists for all k. Absence of arbitrage implies that $\frac{P(t,T_k)}{P(t,T_{k+1})}$ is a martingale under forward measure $\mathbb{Q}^{k+1}$ and thus $L_k(t)$ is also a martingale under forward measure $\mathbb{Q}^{k+1}$. In addition, $\sigma_k(t)$ is assumed to be a bounded deterministic function. Then we can define the CEV market model by the following forward rate dynamics,

$$dL_k(t) = \sigma_k(t)L_k^{\gamma}(t)dW_{k+1}(t), \qquad k \in \{1, \dots, M\}$$

where $\gamma > 0$. When $\gamma = 1$, this results in the original LMM.

It is often the case that the displaced diffusion (DD) model and the CEV model are used as alternatives to the lognormal model in modelling stock prices and interest rates. This is due to the fact that these two models are able to incorporate monotonically decreasing smile which is often observed in real world (Joshi and Rebonato, 2003). The displaced diffusion model can be defined by the following forward rate dynamics,

$$dL_k(t) = \sigma_k(t)\left(L_k(t) + a_k(t)\right)dW_{k+1}(t), \qquad k \in \{1, \dots, M\}$$

where $a_k(t)$ is the constant displacement coefficient.

Svoboda-Greenwood (2006) have thoroughly investigated the similarities between the DD model and the CEV model. In addition, Svoboda-Greenwood (2009) showed how the DD model can be used to approximate the CEV model. However, despite the close relationship between the two models, we do not consider the DD model in this dissertation as it has a distinct difference to the CEV model. Unlike the CEV model, the DD model does not incorporate varying volatility level-dependence. This result will be shown in more details in Section 2.4.

2.2 Interest-rate volatility level dependence

Piazzesi (2010, Chapter 7.7, 750) found that there is a positive correlation between volatility and level of interest rates. More specifically, he estimated the squared residuals of a given yield from vector autoregression (VAR) model. Thereafter, he found that when the squared VAR-residuals are regressed on the level of the same yield, the regression coefficient is positive and significant.

Recently, Filipovic *et al.* (2017) found that when interest rates are near the zero lower bound (ZLB), volatility becomes more level-dependent. They illustrated this result by performing regression analysis on weekly changes in the 3-month normal implied volatility of the swap rate and weekly changes in swap rate level. They found that when swap rates are between 0% and 1%, the regression coefficients are large (between 0.48 and 1.2) and the coefficients of determination, R^2s, are also large (between 0.44 and 0.54). In other words, when swap rates are near the zero lower bound, there exists a significant and positive relation between swap rate and volatility changes. However, as swap rate increases in level, the relation between swap rate and volatility changes weakens. In other words, there is very little volatility level-dependence when swap rates are at moderate levels. For instance, when swap rates are between 4% and 5%, the regression coefficients are very small (between -0.07 and 0.08) and the R^2s are close to zero (between 0.00 and 0.03).

2.3 Interest-rate cap pricing

A caplet is an interest-rate derivative which limits the risk of rising interest rate. It has a payoff function of $\delta_k(L_k(T_k) - H)^+$ at time T_{k+1} where H denotes the strike rate and $(\cdot)^+ = \max(\cdot, 0)$. An interest-rate cap is a combination of caplets with different maturity times. Usually the constituent caplets have tenors of 3 or 6 months. With the CEV market model, there is a closed-form solution for the caplet price. The following theorem from Andersen and Andreasen (2000) provides the caplet pricing formula under the CEV market model and the LMM where the proof could be found in the same paper.

Theorem 2.1. *(Andersen and Andreasen, 2000)*
Let $C_k(t)$ denote the price of a LIBOR caplet with strike H and payment time T_{k+1}. Let $\Phi(\cdot)$ be the standard normal cumulative distribution function, and $\chi^2(\cdot, \theta, \lambda)$ be the cumulative distribution function of a non-central χ^2 distributed random variable with non-centrality parameter λ and degrees of freedom θ. Define

$$v_k(t, T_k) := \int_t^{T_k} \sigma(s)^2 ds; \qquad a := \frac{H^{2(1-\gamma)}}{(1-\gamma)^2 v_k(t, T_k)}; \qquad b := \frac{1}{1-\gamma};$$
$$c := \frac{L_k(t)^{2(1-\gamma)}}{(1-\gamma)^2 v_k(t, T_k)}; \qquad x_\pm := \frac{\ln\left(L_k(t)/H\right) \pm \frac{1}{2} v_k(t, T_k)}{\sqrt{v_k(t, T_k)}}.$$

Assuming the forward rate dynamics are as specified per (2.1), the arbitrage-free value of $C_k(\cdot)$ is given by the following:

1. *For $0 < \gamma < 1$ and an absorbing boundary at the level $L_k = 0$:*
$$C_k(t) = \delta_k P(t, T_{k+1}) \left\{ L_k(t) \left(1 - \chi^2(a, b+2, c)\right) - H\chi^2(c, b, a) \right\}.$$

2. *For $\gamma = 1$:*
$$C_k(t) = \delta_k P(t, T_{k+1}) \left\{ L_k(t)\Phi(x_+) - H\Phi(x_-) \right\}.$$

3. *For $\gamma > 1$:*
$$C_k(t) = \delta_k P(t, T_{k+1}) \left\{ L_k(t) \left(1 - \chi^2(c, -b, a)\right) - H\chi^2(a, 2-b, c) \right\}.$$

In addition, the caplet price with Gaussian forward rate dynamic could also be computed with a closed-form solution. The following proposition which uses a similar framework as the above theorem gives the caplet price formula under the Gaussian forward rate dynamic. This forward rate dynamic also gives rise to the normal market model.

Proposition 2.2. *Consider the framework of Theorem 2.1 and assume the forward rate dynamics are as specified by an initial value $L_k(0)$ and*

$$dL_k(t) = \sigma dW_k(t), \qquad t \geq 0.$$

Then, the arbitrage-free value of $C_k(t)$ is given by the following:

$$C_k(t) = \delta_k P(t, T_{k+1})\sigma\sqrt{T_k - t}\,\{\eta(t, T_k)\Phi(\eta(t, T_k)) + \phi(\eta(t, T_k))\}\,,$$

where $\eta(t, T_k) := (L_k(t) - H)/(\sigma\sqrt{T_k - t})$ and ϕ, Φ are the standard normal probability density function (PDF) and cumulative distribution function (CDF) respectively.

The proof for the above proposition can be found in Appendix A. Note that the normal market model is just the CEV market model with $\gamma = 0$.

Since a cap is essentially a combination of caplets with different maturity times, the price of a cap can be calculated as the sum of all the constituent caplets. However, note that the first caplet for a cap is deterministic as the LIBOR rate is known when entering the cap.

2.4 Volatility level-dependence assumption under different models

Different assumptions of volatility level-dependence can be explained by investigating the relationship between the level of interest rate and the diffusion coefficient in each model. As previously mentioned, the diffusion coefficients for the LMM, the CEV market model and the normal market model are σL, σL^γ and σ respectively. Figure 2.1 below illustrates how the diffusion coefficient changes as the level of interest rate changes. Note that a constant σ is assumed for all models.

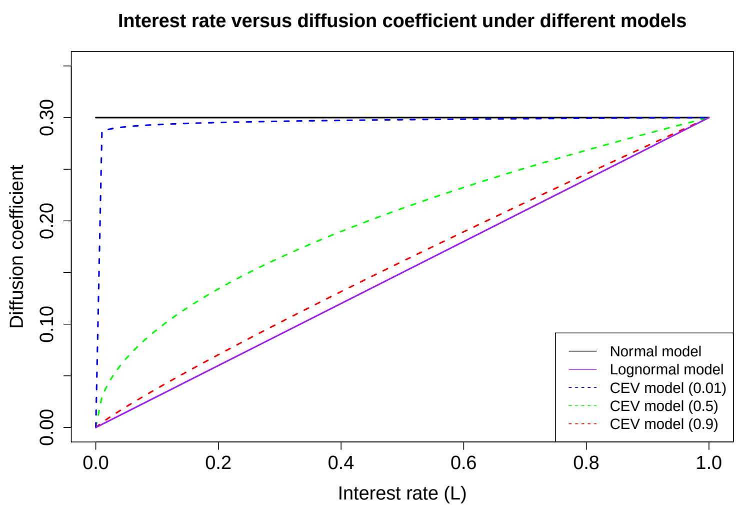


Fig. 2.1: Relationship between interest rate and diffusion coefficient under different models (The numbers in the brackets are the γ parameter values)

From the above figure, it can be seen that the diffusion coefficient for the LMM increases linearly as the interest rate increases. This shows that the LMM assumes a strong and constant volatility level-dependence. In contrary, the diffusion coefficient for the normal market model remains unchanged as interest rate changes.

The independence of the diffusion coefficient to interest rate level under the normal market model shows that the volatility is assumed to have no level-dependence.

The CEV market model, on the other hand, assumes volatility changes from strong level-dependence to weak level-dependence as interest rate increases. The degree of strong level-dependence and the speed for which strong level-dependence weakens depend on the magnitude of the γ parameter. As γ increases, the strong level-dependence, when the rate level is low, becomes weaker. Meanwhile, the speed for which strong level-dependence weakens becomes slower. As a consequence, the varying volatility level-dependence can be changed by tuning the γ parameter appropriately.

Lastly, recall the displaced diffusion model which we mentioned in Section 2.1. Its diffusion coefficient is $\sigma(L + a)$. It is clear that similar to LMM, the diffusion coefficient increases linearly as the interest rate increases. The only difference to LMM is that the relationship between diffusion coefficient and interest rate is shifted up/down in parallel depending on the sign of parameter a. As a result, it does not incorporate varying volatility level-dependence feature. Thus, we do not consider the displaced diffusion model in this dissertation.

3 Methodology

In this section, we describe how regression analysis is performed to assess which interest-rate model best describes the relationship between interest rate and interest-rate volatility in the market. This is done by finding which model gives the lowest correlation between changes to interest rate and changes to implied volatility. We then describe how the hedging strategy is performed to assess which interest-rate model gives the best hedging performance for interest-rate caps over a certain time period. Thereafter, we describe the daily cross-sectional hedging analysis which assesses models based on hedging caps with different strike rates on a daily basis. Lastly, we describe the daily calibration to market prices methodology to assess which model best fits the market prices of interest-rate caps on a daily basis.

3.1 Regression

The regression analysis involves regressing weekly changes to the implied volatilities of caps on weekly changes to swap rates. The linear model takes the form of

$$\Delta\sigma = \beta_0 + \beta_1 \Delta L + \epsilon$$

where $\Delta\sigma$ is the weekly change in implied volatility, β_0 is the regression intercept, β_1 is the regression coefficient, ΔL is the weekly change in swap rate and ϵ is the error term. In Filipovic *et al.* (2017), it was found that there is a positive correlation under the normal model suggesting a positive correlation between interest rate and interest-rate volatility. The normal implied volatility was used because the normal model assumes volatility remains constant when interest rate changes. If the normal implied volatility turns out to be changing, it means that the model has incorrectly assumed constant volatility. As a result, a positive correlation under the normal model means that the actual volatility should increase when interest rate increases. Thus the normal implied volatility gives a means of measuring correlation between changes to volatility and changes to interest rates.

Moreover, note that the correlations under different models may have different meanings for the relationship between interest rate and volatility. The implied volatility of any model is a relative measure of volatility compared to what is being assumed by the model. For instance, the normal model assumes volatility is independent of interest rate. Therefore, if the normal implied volatility turns out to be increasing (or decreasing) as interest rate increases, this means that the actual volatility should increase (or decrease) when interest rate increases. On the other hand, the lognormal model assumes that volatility increases linearly as interest rate increases. If the lognormal implied volatility turns out to be decreasing as interest rate increases, it means that the actual volatility is not increasing as much as being assumed in the model or the volatility may even be decreasing as interest rate increases depending on how negative the correlation is.

Given the above explanation for correlation between implied volatility changes and interest rate changes, the aim of regression analysis is to find which model gives the lowest correlation. The reason for this is if there is no correlation between changes to implied volatilities and changes to interest rates, it means that the model has correctly captured the relationship between interest rate and interest-rate volatility.

If the normal model gives a positive correlation while the lognormal model gives a negative correlation, this may suggest that the CEV market model may perform better than both models by achieving no correlation. This is because the CEV market model may move between the normal model and the lognormal model by changing the γ parameter. If this is the case, then regression analysis becomes an optimisation problem as the CEV parameter, γ, is calibrated to achieve no correlation. By doing so, the CEV market model (with the right γ parameter) has correctly captured the volatility level-dependence. The calibration procedure is done in four steps:

- Given the lognormal implied volatilities (Black implied volatilities) from the market data, cap prices can be computed using the lognormal cap pricing formula from Theorem 2.1.
- Then the CEV implied volatilities can be inverted from the cap prices, given a particular CEV parameter, using the CEV cap pricing formula.
- The correlation may then be computed for this particular CEV market model.
- As a result, this becomes an optimisation problem (i.e., find the CEV parameter γ such that it minimises the correlation).

3.2 Hedging

Since caplets are essentially options on the forward rate, it is possible to hedge caplets by taking appropriate positions in the forward market. Given the fact that the futures markets are relatively liquid, caplets are commonly hedged using future contracts (Gupta and Subrahmanyam, 2005).

More specifically, since a caplet has a payoff function of $(L_k - H)^+$, it can be hedged by using a FRA (forward rate agreement) which has a payoff function of $(L_k - H)$. Furthermore, a FRA can be decomposed into the short-dated bond, $P(t, T_k)$, and the long-dated bond $P(t, T_{k+1})$. In addition, it is also necessary to hedge the long-dated bond which acts as the discounting factor for the payoff. This discounting factor can be found in the pricing formula for caplet. Therefore, in order to hedge a caplet, a combination of short-dated bond and long-dated bond is used.

3.2.1 Deriving the hedging portfolio

Since the replicating portfolio for a cap is simply a combination of replicating portfolios for caplets with different maturity times, it is therefore enough to understand how to replicate a single caplet. Musiela and Rutkowski (1997) explained how to build a hedging portfolio using bonds under the LIBOR market model. As a result, we will derive the hedging portfolio for a caplet under the LIBOR market model. Thereafter, the derivation can be easily extended to the normal model or the CEV market model.

Let us denote the forward price of j^{th} caplet of a cap with settlement time T_j by $F(t, T_j)$. Then from Theorem 2.1, it is easy to see that

$$F(t, T_j) = \delta_{j-1}(L_{j-1}(t)\Phi(x_+) - H\Phi(x_-)).$$

By Itô's formula, we can then derive

$$dF(t, T_j) = \delta_{j-1}\Phi(x_+)dL_{j-1}(t).$$

Now consider a self-financing strategy in the T_j-forward market where values of all securities are expressed in units of T_j-maturity zero-coupon bonds. Since the price of j^{th} caplet at time 0 is $C_j(0) = P(0, T_j)F(0, T_j)$, we need to invest $F(0, T_j)$ units in T_j-maturity zero-coupon bond.

Subsequently, at any time $t \leq T_{j-1}$, we take $\psi_t^j = \Phi(x_+)$ positions in one-period forward swaps over the period $[T_{j-1}, T_j]$. The associated gains process ($\hat{G}$), which describes the total profit or loss generated by a portfolio, in the T_j-forward market, satisfies $\hat{G}_0 = 0$ and

$$d\hat{G}_t = \delta_{j-1}\psi_t^j dL_{j-1}(t) = \delta_{j-1}\Phi(x_+)dL_{j-1}(t) = dF(t, T_j).$$

Consequently,

$$F(T_{j-1}, T_j) = F(0, T_j) + \int_0^{T_{j-1}} \delta_{j-1}\psi_t^j dL_{j-1}(t) = F(0, T_j) + \hat{G}_{T_{j-1}}.$$

Since the payoff of the caplet is known at time T_{j-1}, therefore the caplet is completely specified by its forward price $F(T_{j-1}, T_j) = C_j(T_{j-1})/P(T_{j-1}, T_j)$. As a result, it is clear to see that the strategy ψ_t^j replicates the j^{th} caplet.

Additionally, the replicating strategy also involves a second component, η_t^j, which represents the number of forward contracts with settlement time T_j on T_j-maturity bond. Let $F_B(t, T_j, T_j)$ denote the forward price of T_j-maturity bond on settlement date T_j at time t. Then $F_B(t, T_j, T_j) = 1$ whenever $t \leq T_j$ and thus $dF_B(t, T_j, T_j) = 0$. As a result, the T_j-forward value of our strategy is

$$\hat{V}_t(\psi^j, \eta^j) = \delta_{j-1}\psi_t^j L_{j-1}(t) + \eta_t^j = F(t, T_j)$$

and

$$d\hat{V}_t(\psi^j, \eta^j) = \delta_{j-1}\psi_t^j dL_{j-1}(t) + \eta_t^j dF_B(t, T_j, T_j) = \delta_{j-1}\Phi(x_+)dL_{j-1}(t).$$

Alternatively, we could also perform the replication in the bond market. It is easy to see that

$$C_j(t) = P(t,T_j)\hat{V}_t(\psi^j, \eta^j).$$

Therefore the caplet price can be expressed in terms of two simple bonds by

$$\begin{aligned} C_j(t) &= P(t,T_j)\delta_{j-1}\psi_t^j L_{j-1}(t) + P(t,T_j)\eta_t^j \\ &= P(t,T_j)\delta_{j-1}\psi_t^j \left(\frac{1}{\delta_{j-1}} \frac{P(t,T_{j-1})}{P(t,T_j)} - 1 \right) + P(t,T_j)\eta_t^j \\ &= \psi_t^j (P(t,T_{j-1}) - P(t,T_j)) + P(t,T_j)\eta_t^j. \end{aligned}$$

and thus

$$\begin{aligned} dC_j(t) &= \psi_t^j d(P(t,T_{j-1}) - P(t,T_j)) + \eta_t^j dP(t,T_j) \\ &= \Phi(x_+) d(P(t,T_{j-1}) - P(t,T_j)) + \eta_t^j dP(t,T_j). \end{aligned}$$

where ψ_t^j and η_t^j represent the number of units held in $(P(t,T_{j-1}) - P(t,T_j))$ and $P(t,T_j)$ at time t respectively.

From the above derivation, it can be seen that ψ_t^j is simply $\frac{\partial F}{\partial L_{j-1}(t)}$ where $F = \frac{F(t,T_j)}{\delta_{j-1}}$. Therefore, we simply need to calculate $\frac{\partial F}{\partial L_{j-1}(t)}$ for the normal model and the CEV market model to derive the hedging strategy. This is the delta hedge ratio. Since the CEV market model involves non-central chi-square distribution, ψ_t^j will be computed using finite-difference approximation by

$$\frac{\partial F}{\partial L_{j-1}(t)} = \frac{F(L_{j-1}(t) + \Delta L) - F(L_{j-1}(t))}{\Delta L}.$$

Once ψ_t^j is computed, η_t^j may be computed from ψ_t^j and the caplet price $C_j(t)$ by

$$\eta_t^j = \frac{C_j(t) - \psi_t^j (P(t,T_{j-1}) - P(t,T_j))}{P(t,T_j)}.$$

Note that since the first caplet of a cap is deterministic, this caplet will be hedged trivially by cash. More specifically, an amount of $\delta_0(L_0(0) - H)^+$ will be invested in the bank account.

3.2.2 Hedging strategy

In theory, the hedging portfolio should be rebalanced continuously as the market condition constantly changes. In practice, it is impossible to perform continuous rebalancing. Furthermore, portfolio rebalancing requires transaction costs which could be expensive if the rebalancing happens frequently. Therefore, we decided to perform weekly rebalancing for this dissertation.

For each cap, a hedging portfolio is built on the first day and held for five trading days. To assess the performance of the hedging strategy, the hedging portfolio value is compared against the actual cap price after one week. If the underlying model is accurate, the hedging portfolio value should be close to the actual cap price after one week.

The hedging portfolio value will change due to change in bond prices. In reality, these new bond prices will be quoted in the market. However, we do not have such data. Instead, we only have bond prices with fixed maturities. In other words, we only have the 10-year zero-coupon bond price, but not the 10-year-less-1-week zero-coupon bond price. To overcome this problem, we compute the new bond price by performing linear interpolation on the natural logarithm (i.e., $\ln$) of the bond price since the bond price is an exponential function of interest rate. More specifically, the new bond price is computed as

$$\begin{aligned}\ln\left\{P\left(t+\frac{1}{50},T_k\right)\right\} &= \ln\left\{P\left(t+\frac{1}{50},T_{k-1}+\frac{1}{50}\right)\right\}\times\frac{1}{50}\\ &+\ln\left\{P\left(t+\frac{1}{50},T_k+\frac{1}{50}\right)\right\}\times\frac{49}{50}\end{aligned}$$

assuming, 50 weeks in the financial trading year.

Unlike the bond markets, the interest-rate cap market only quotes prices of caps with fixed maturity periods rather than fixed maturity dates. In other words, the market only quotes the price of 10-year cap everyday. However, to evaluate the hedging performance, we need the price of 10-year-less-1-week cap after one week. As a result, this new cap price, which is compared against the new hedging portfolio value, must be approximated. From the pricing formula, it is clear that the 10-year-less-1-week implied volatility is required to compute the new cap price. Since the interest-rate cap market does not quote such implied volatility, we will use the 10-year implied volatility to approximate the new cap price. The reason for using such implied volatility with the same maturity is that implied volatility remains fairly constant over a short time period. Fan, Gupta and Ritchken (2003) and Gupta and Subrahmanyam (2005) also used this method to compute the new option prices after a week. Hence, to calculate the new cap price, we put the new bond price with a shorter term and 10-year implied volatility into the appropriate pricing formula.

To assess the hedging performance of the model, we record the profit and loss (P&L) after a week. This is simply the difference between the hedging portfolio value and the cap price after a week. On each day, we construct a hedging portfolio for a new cap with 10-year maturity. Note that we are not hedging one particular cap every week, but rather hedging a different cap everyday and assess the hedging performance after a week for each cap.

For comparison purposes, we also perform the unhedged strategy to assess the relative performance of hedging. The unhedged strategy is similar to the hedging strategy. The only difference is that the initial value of cap is not invested in the hedging portfolio. Therefore, after a week, the difference between the initial cap price and the new cap price is recorded as P&L for the unhedged strategy.

To assess the overall hedging performance for a model, we use a hedging-equivalent R^2 as a performance measure. A similar measure was used by Fan, Gupta and Ritchken (2003, 2007) and Driessen, Klaassen and Melenberg (2003).

This R^2 is calculated as

$$R^2 = 1 - \frac{SS_{hedged}}{SS_{unhedged}}$$

where the variance of hedging errors SS_{hedged} is calculated as

$$SS_{hedged} = \sum_i (PL_{hedged}(i))^2$$

where PL_{hedged} is the difference between the cap price and the hedging portfolio value after a week and i is the trading day when we calculate profit and loss for a cap. Similarly, the variance of changes in unhedged investment $SS_{unhedged}$ is calculated as

$$SS_{unhedged} = \sum_i (PL_{unhedged}(i))^2$$

where $PL_{unhedged}$ is the difference between the new cap price after a week and the initial cap price. The ratio of the two variances measures how much variability in the cap is removed by the delta-hedging strategy. If the hedging strategy performs well under a certain model, then SS_{hedged} will be small. As a result, the ratio of variances will be close to 0 and the R^2 will be close to 1.

The aim of the hedging strategy is to find which model produces the lowest hedging error and thus the highest R^2. If volatility exhibits strong level-dependence when interest rate is low and weaker level-dependence as interest rate increases, we should expect the CEV market model to produce a higher R^2. In that case, we would like to investigate for which γ parameter does it give the highest R^2.

3.3 Daily cross-sectional hedging

In this section, we introduce the daily cross-sectional hedging analysis. This analysis is similar to the previously described hedging analysis. The difference is this analysis assesses interest-rate models, on a daily basis, based on performance of hedging caps across different strike rates. Moreover, it also gives some insights into the results obtained from the previous hedging analysis.

On each day, the market quotes cap prices for different strike rates. The idea for daily cross-sectional hedging analysis is to hedge all the market quoted caps altogether and assess which interest-rate model gives the best hedging performance. More specifically, hedging portfolios are constructed for different caps under a specific model each day. Then the hedging performance of the model is analysed after a week. The hedging performance is analysed by the mean percentage hedging error (*MPHE*) which is calculated as

$$MPHE = \frac{1}{k} \sum_k \frac{|HP_{new}(k) - Cap_{new}(k)|}{Cap_{new}(k)}$$

where $HP_{new}(k)$ is the value of hedging portfolio for the cap with strike rate k after a week and $Cap_{new}(k)$ is the price of the cap with strike rate k after a week. The best model for hedging the caps on this day is then recorded. We then proceed to second day and build hedging portfolios under different models again. The hedging performance is then analysed again after a week and the best model is recorded for the second day. This iterative procedure is repeated for all days. In the end, we will have a best model for each day. By doing this, we will gain some insights into the average hedging performance for caps across different strike rates on a daily basis.

3.4 Daily calibration to market prices

Since the market quotes interest-rate caps for different strike rates, it might be worth investigating which model best fits the market prices and whether this model has any relationship with the model that best hedges interest-rate caps.

The calibration to market prices is performed by minimising the fitting error by tuning the parameters in the model. More specifically, we minimise the sum of squares of fitting errors (*SSE*) which is calculated as

$$SSE = \sum_k (Cap(k) - \widehat{Cap}(k))^2$$

where $Cap(k)$ is the market quoted cap price with strike rate k and $\widehat{Cap}(k)$ is the model fitted cap price with strike rate k. For the normal model and the lognormal model, the only parameter that is available to tune is the σ parameter. On the other hand, both σ and γ parameters can be tuned for the CEV market model. Such *SSE* is computed and minimised for each model on each day . The model with the lowest *SSE* is the one that best fits the market prices.

The parameter that we are interested in is the γ parameter in the CEV market model. We want to investigate whether the γ parameter values obtained from calibration has any relationship with the γ parameter values obtained from the hedging analysis and daily cross-sectional hedging analysis.

3.5 Data cleaning

In this dissertation, EURIBOR data from 2006-07-10 to 2015-07-29 were used to analyse the performance of different models. These data comprise interest-rate cap lognormal implied volatilities for different strike rates. The dataset includes 13 different strike rates: 1%, 1.75%, 2%, 2.25%, 2.5%, 3%, 3.5%, 4%, 5%, 6%, 7%, 8% and 9%. However, the dataset only has lognormal implied volatilities from 2012 onwards for caps with strike rate 1%. The implied volatilities can be used to calculate the cap prices. Moreover, the dataset also consists of interest rates for different instruments including 6-month deposit, 6×12 FRA, 12×18 FRA and swaps with maturity times ranging from 2 up to 10 years. These rates will be used to bootstrap spot rates in order to calculate zero-coupon bond prices with different maturity times.

The interest-rate data need to be cleaned first as some data were missing on some days. This is done by making sure all interest rates for different instruments exist on a given day. If this is not the case, then the data for this day will be ignored from the dataset. After cleaning, 2308 out of 2363 days worth of interest-rate data remained in the dataset. As a result, 2.33% of data were lost. Figure 3.1 below shows the 10-year swap rates (it is also the at-the-money strike rate for 10-year cap) over the time period after the data were cleaned. The minimum, maximum and average swap rates over this period are 0.4507%, 5.094% and 2.9311% respectively. Note that the swap rates have been steadily decreasing over this period.

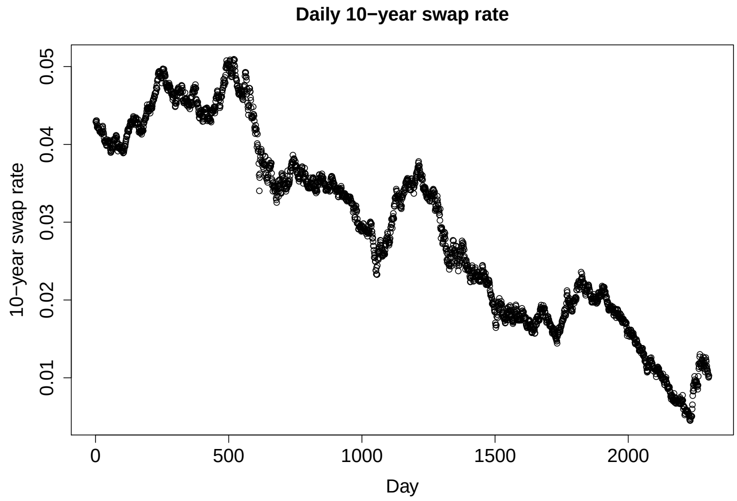


Fig. 3.1: Daily 10-year swap rates from 2006-07-10 to 2015-07-29

Once the interest-rate data are cleaned, they need to be synchronised with the lognormal implied volatility data. This is done by making sure that both interest rate and implied volatility data exist on a given day. After synchronising, 2303 out of 2308 days worth of data remained in the dataset. As a result, a further 0.22% of data were lost. Figure 3.2 below shows the lognormal implied volatilities for 10-year at-the-money cap. The minimum, maximum and average implied volatilities over this period are 11.3%, 130.12% and 30.89% respectively. From the figure, it can be seen that the lognormal implied volatilities are below 60% most of the time. However, it increases to a maximum implied volatility of 130.12% on 2015-04-29. Thereafter, it decreases back to about 60%. Furthermore, note that there is a negative correlation between the 10-year swap rates and the lognormal implied volatilities.

3.6 Bootstrapping

Once the interest-rate data are cleaned, they can be used to bootstrap spot rates in order to calculate the daily zero-coupon bond prices. In this dissertation, bootstrapping was done in a theoretical way which ignores day-count convention. The dataset consists of 6-month deposit rates, 6×12 FRA rates, 12×18 FRA rates and 2-10 year swap rates. These can be used to bootstrap zero-coupon bond prices up to maturity of 10 years.

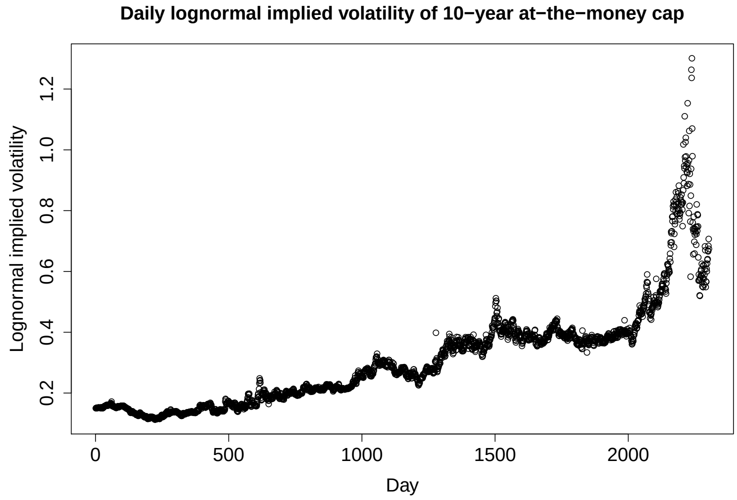


Fig. 3.2: Daily lognormal implied volatilities for 10-year at-the-money cap from 2006-07-10 to 2015-07-29

More specifically, this is done in the following procedure: Let $P(t, T_k)$ be the bond price with current time being t and maturity time being T_k. Without loss of generality, we could take $t = 0$. Moreover, let $0 = T_0 < T_1 < \ldots < T_{20} = 10$ with $\delta_k = T_{k+1} - T_k = 0.5$. Furthermore, let $L_{T_k,T_{k+1}}$ be the forward rate which applies over time period $[T_k, T_{k+1}]$ and let S_{T_k} be the swap rate for swap with maturity time T_k. Then $P(0, 0.5)$ can be calculated from 6-month deposit rate as

$$P(0, T_1) = P(0, 0.5) = \frac{1}{1 + \delta_0 \times L_{0,0.5}}.$$

$P(0, 1), P(0, 1.5)$ can be calculated from the FRA rates as

$$P(0, T_2) = P(0, 1) = \frac{P(0, 0.5)}{1 + \delta_1 \times L_{0.5,1}},$$

$$P(0, T_3) = P(0, 1.5) = \frac{P(0, 1)}{1 + \delta_2 \times L_{1,1.5}}.$$

$P(0, 2)$ can be calculated from the 2-year swap rate S_{T_4} as

$$P(0, T_4) = P(0, 2) = \frac{1 - \sum_{i=1}^{3} S_{T_4} \times \delta_{i-1} \times P(0, T_i)}{1 + \delta_3 \times S_{T_4}}.$$

Since we do not have 2.5-year swap rate, $P(0, 2.5)$ can not be calculated similarly to $P(0, 2)$. Instead, we make a simplifying assumption about the forward rate which applies over time period 2 and 3 year. We assume that $L = L_{2,2.5} = L_{2.5,3}$. By doing so, we can calculate $P(0, 2.5)$ and $P(0, 3)$ simultaneously by

$$P(0, T_5) = P(0, 2.5) = \frac{P(0, 2)}{1 + \delta_4 \times L},$$

$$P(0, T_6) = P(0, 3) = \frac{P(0, 2)}{(1 + \delta_4 \times L)^2} = \frac{P(0, 2.5)}{1 + \delta_5 \times L},$$

where the forward rate L can be calculated from the 3 year swap rate S_{T_6} as

$$S_{T_6} = \frac{1 - P(0, 3)}{\sum_{i=1}^{6} \delta_{i-1} \times P(0, T_i)}.$$

In a similar way, $P(0, 3.5)$ and $P(0, 4)$ can be calculated from the 4-year swap rate and $P(0, 4.5)$ and $P(0, 5)$ can be calculated from the 5-year swap rate. This iterative process is repeated until we have $P(0, 10)$. In the end, we will obtain 20 semi-annual bond prices on each day.

4 Results

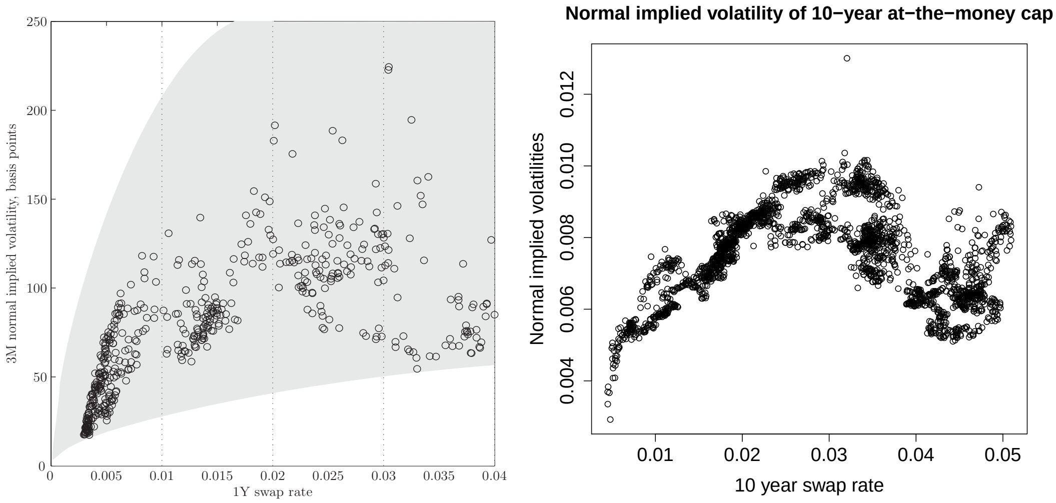
In this section, we present the findings about the relationship between normal implied volatility of at-the-money cap and swap rate. A similar finding was presented in Filipovic *et al.* (2017). We then present the regression analysis results for three different models to gain insight into which model best describes the relationship between interest rate and interest-rate volatility. Thereafter, we present the hedging and daily cross-sectional hedging results to find the best model in terms of hedging interest-rate caps. Finally, we find the best model that fits the market prices of interest-rate caps across different strike rates and investigate the relationship between this model and the model that best hedges interest-rate caps.

4.1 Normal implied volatility versus swap rate

As previously mentioned, Filipovic *et al.* (2017, 657) found that "volatility becomes compressed and gradually more level-dependent as interest rates approach the ZLB". They illustrated this finding by plotting the normal implied volatility (NIV) of 3-month at-the-money (ATM) swaption on 1-year swap against the level of 1-year swap rate. Their plot is shown below in Figure 4.1a. From their plot, it can be seen that when the 1-year swap rate is below 1%, volatility is more level dependent. However, as swap rate increases above 1%, volatility exhibits little level-dependence.

Recall the reason for using normal implied volatility as the means to investigate the relationship between volatility and interest rate was justified in Section 3.1. Similar to Filipovic *et al.* (2017), we can plot the same graph but adapted to our data using 10-year interest-rate cap. Figure 4.1b below plots the normal implied volatility of 10-year at-the-money cap against the level of 10-year swap rate. The plot shows that volatility exhibits strong positive level-dependence when the swap rate is below 2.5%. However, when the swap rate increases above 2.5%, the volatility exhibits weaker level-dependence. Moreover, note that the volatility is negatively correlated with the swap rate. This is illustrated more clearly in Figure 4.2. Figure 4.2a shows that when the swap rate is below 2.5%, the normal implied volatility is positively correlated with the swap rate and it has a correlation coefficient of 0.2136. However, Figure 4.2b shows that when the swap rate is above 2.5%, the normal implied volatility is negatively correlated with the swap rate and it has a correlation coefficient of -0.1299.

Although this result may suggest that the level-dependence of volatility decreases as the swap rate increases, it is important to note that the result may not



(a) 3-month NIV of 1-year swap rate

(b) NIV of 10-year ATM Cap

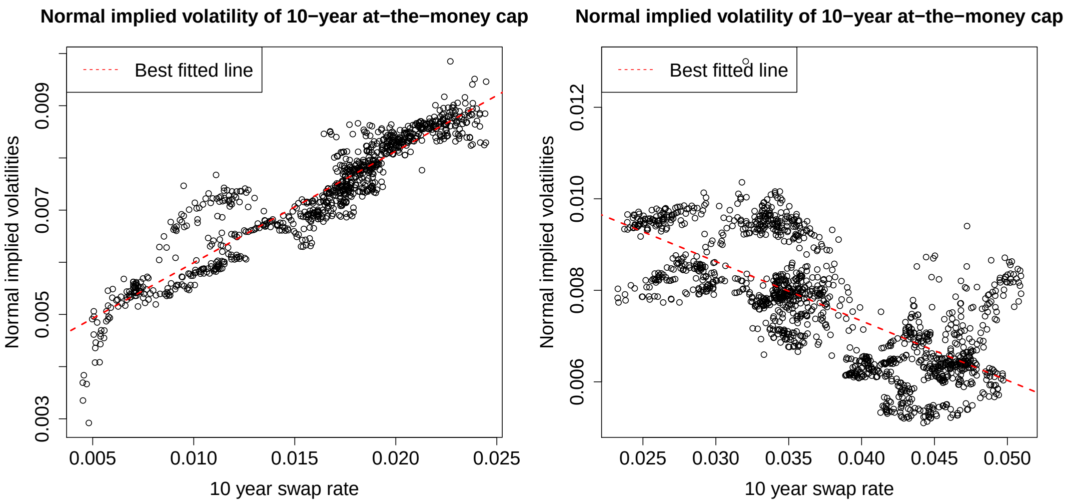
Fig. 4.1: Normal implied volatility plots comparison between 3-month at-the-money swaption on 1-year swap and 10-year at-the-money cap

be describing the relationship between volatility and swap rate correctly. Note that each implied volatility on the plot is the implied volatility of a different cap with a different strike rate since at-the-money cap is being considered. Therefore, the x-axis is not only the level of 10-year swap rate but also the strike rate for daily 10-year at-the-money cap. As a result, the plot is not illustrating purely the relationship between volatility and swap rate. When considering hedging interest-rate cap, we need to hedge an interest-rate cap with a fixed strike rate. Consequently, we should rather consider a fixed strike rate cap rather than at-the-money cap to investigate the relationship between volatility and swap rate.

Figure 4.3 below plots the normal implied volatility of 10-year cap with a fixed strike rate of 1.75%. It can be seen that the strong volatility level-dependence feature no longer exists when the swap rate is close to the zero lower bound. Furthermore, note that the volatility is not necessarily increasing when the swap rate is increasing which was observed before. However, there is a strong positive relationship between normal implied volatility and swap rate when the swap rate is between 1.5% and 2.5%. Nonetheless, this relationship may need further investigation as it may only be a feature of this particular interest-rate market over this particular time period. Additionally, the same plots are being conducted for caps with fixed strike rates 2% and 2.25%, similar trends were observed. The plots can be found in Appendix B.

4.2 Regression

Firstly, we implement the regression analysis described in Section 3.1 to find the correlation between weekly changes in swap rates and weekly changes in implied



(a) NIV of 10-year ATM cap (strikes below 2.5%)

(b) NIV of 10-year ATM cap (strikes above 2.5%)

Fig. 4.2: Normal implied volatility of 10-year at-the-money cap plots comparison between swap rate below 2.5% and above 2.5%

volatilities.

Initially, we perform the regression analysis on part of the dataset which runs from 2013-01-01 to 2015-07-29 where the swap rates are below 2.5%. We focus particularly on 10-year interest-rate caps with strike rates 1%, 1.75% and 2%. From Table 4.1, it can be seen that there is little correlation between weekly changes in swap rates and weekly changes in normal implied volatilities when the strike rate is 1%. However, there is a negative correlation when strike rates are 1.75% and 2%. This suggests that there is little correlation between swap rate and volatility when the caps strike at 1%. As the caps become more out-the-money, their correlations become more negative. A scatter plot, Figure C.1, which shows the weekly changes in swap rates and weekly changes in normal implied volatilities can be found in Appendix C.

Strikes	Lognormal	CEV	Normal
1%	-0.4735141	-0.2396306 (0.001)	0.06766311
1.75%	-0.5005883	-0.2552083 (0.001)	-0.1135972
2%	-0.4895537	-0.2471741 (0.001)	-0.1498042

Tab. 4.1: Correlation between weekly changes in swap rates and changes in implied volatilities under different models with different strike rates (2013-01-01 to 2015-07-29)

For all strike rates, the normal model has the least correlations which suggest that the normal model is the better model to describe the relationship between volatility and swap rate for this particular part of dataset. The reason for this is

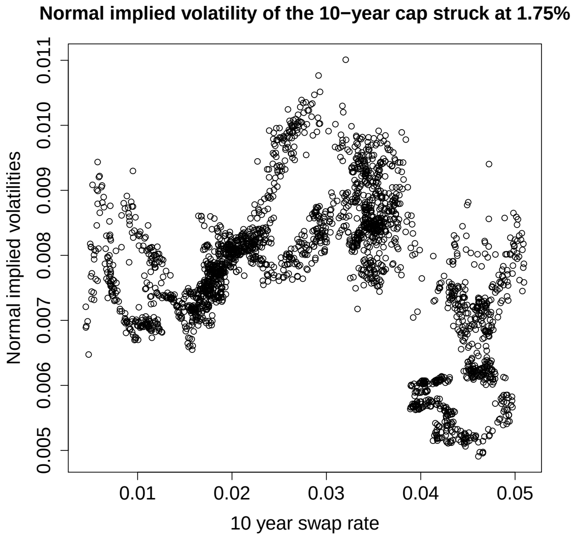


Fig. 4.3: Normal implied volatility of the 10-year cap with strike rate 1.75% (2006-07-10 to 2015-07-29)

if the model correctly describes the relationship between interest rate and interest-rate volatility, then implied volatilities should not change as swap rate changes because the model has already factored in their relationship. Therefore, the best model should have the lowest correlation between swap rate changes and implied volatility changes.

In contrast, the lognormal model always has the highest correlations which suggest that it is the worst model to represent the dataset. This should not be surprising as there is no evidence for strong positive correlation between volatility and swap rate for this dataset. This may also be explained by the negative correlations which suggest that the lognormal model is overestimating its assumed positive correlation.

The correlations for the CEV market model (with a γ parameter of 0.001) lie between the lognormal model and the normal model. This suggests that the CEV market model is better than the lognormal model but worse than the normal model. This result is intuitive as the CEV market model can move towards the normal model or the lognormal model by changing the γ parameter from 0 to 1.

Furthermore, note that the correlations for the CEV market model do not converge to the normal model as γ tends to 0. This is shown in Figure 4.4 below. Although the correlation for the CEV market model increases as γ tends to 0, it does not converge to the correlation for the normal model. The reason for this may be explained by Figure 2.1 in Section 2.4 where we showed the relationship between diffusion coefficient and interest rate. From the figure, it can be seen that as γ increases towards 1, the CEV market model will converge to the lognormal model. However, as γ decreases towards 0, even though the diffusion coefficient will converge to the diffusion coefficient of the normal model when interest rate is relatively high, there will always be an increasing state from 0 to σ. As γ decreases, the diffusion coefficient will increase more rapidly. This discontinuity may justify why the CEV market model can not converge to the normal model.

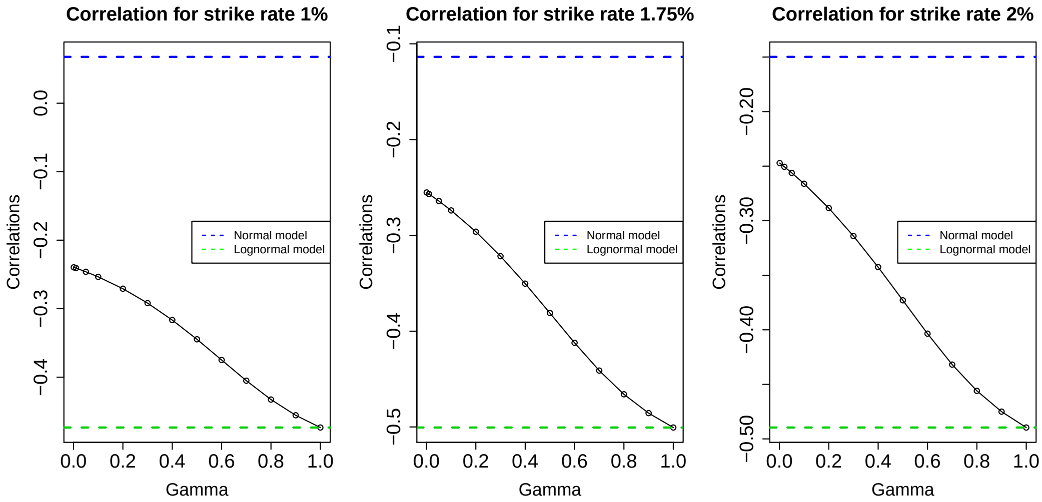


Fig. 4.4: Relation between correlation and γ parameter (2013-01-01 to 2015-07-29)

From Figure 4.3, observe that there is a positive correlation between swap rates and normal implied volatilities when swap rates are between 1.5% and 2.5%. Given this positive correlation, we decided to perform the same regression analysis on this part of dataset. This dataset runs from 2013-01-01 to 2014-06-30.

From Table 4.2, it can be seen that there is a positive correlation between weekly changes in swap rates and weekly changes in normal implied volatilities for all strike rates. This suggests that there is a positive correlation between swap rate and volatility. Similar to previous result, the lognormal model still has high negative correlations which suggest that it is the worst model. In contrast to the previous result, the CEV market model has very low correlations which suggest that the CEV market model best describes the relationship between swap rate and volatility. A scatter plot, Figure C.2, which shows the weekly changes in swap rates and weekly changes in normal implied volatilities can be found in Appendix C.

Strikes	Lognormal	CEV	Normal
1%	-0.3458637	-0.001706884 (0.31)	0.2358751
1.75%	-0.42066	0.0005464103 (0.15)	0.1528152
2%	-0.4329792	0.002012879 (0.09)	0.1110982

Tab. 4.2: Correlation between weekly changes in swap rates and changes in implied volatilities under different models with different strike rates (2013-01-01 to 2014-06-30)

Equivalently, Figure 4.5 shows the relationship between the correlation and the γ parameter for the CEV market model on the new dataset. It is clear that for certain γ parameter, the CEV market model can achieve 0 correlation which was not the case for the previous dataset. Furthermore, the CEV market model still cannot converge to the normal model even though it is closer to the normal model in this case.

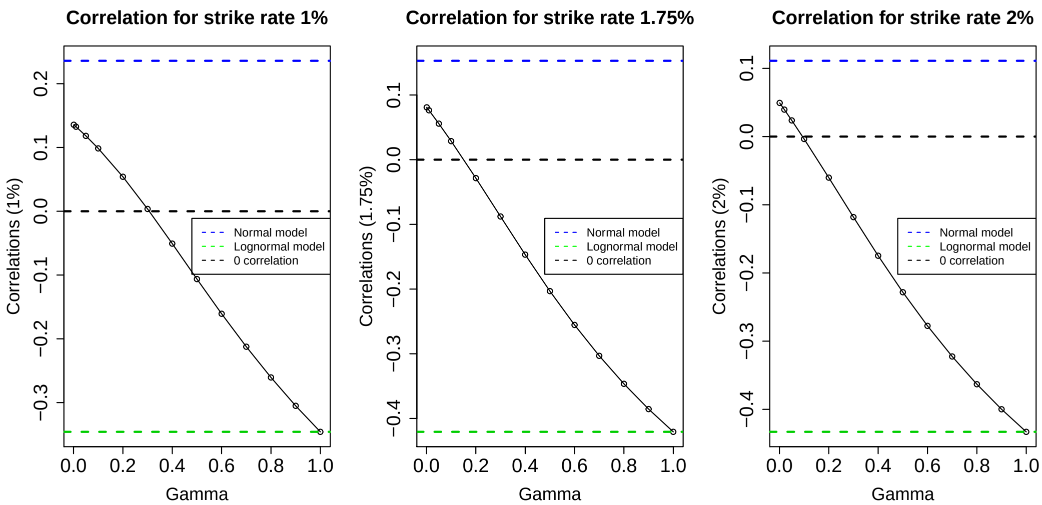


Fig. 4.5: Relation between correlation and γ parameter (2013-01-01 to 2014-06-30)

In addition, note that the correlations between weekly changes in swap rates and weekly changes in normal implied volatilities decrease as the strike rate increases. This suggests that as strike rate increases, the correlation between volatility and swap rate decreases. Furthermore, the γ parameter for CEV market model decreases as the strike rate increases in order to achieve low correlations. This finding is intuitive as the normal implied volatilities result shows that there is less correlation when strike rate increases. Therefore, the CEV market model must move towards the normal model in order to achieve low correlation.

Lastly, we perform the regression analysis on the entire dataset time period which runs from 2006-07-10 to 2015-07-29. Since the dataset only contains lognormal implied volatility from 2012 onwards for caps that strike at 1%, we will perform the regression analysis for caps that strike at 1.75% and 2% to assess the models on the entire time period.

Table 4.3 below shows that the lognormal model is still the worst model. For strike rate 1.75%, the CEV market model with a γ parameter of 0.001 is better than the normal model. However, it is only slightly better. For strike rate 2%, the normal model achieves very low correlation and thus it is better than the CEV market model with a γ parameter of 0.001. A scatter plot, Figure C.3, which shows the weekly changes in swap rates and weekly changes in normal implied volatilities can be found in Appendix C.

Similarly, Figure 4.6 below illustrates the relationship between the correlation and the γ parameter. Same finding was observed. The CEV market model cannot converge to the normal model.

Strikes	Lognormal	CEV	Normal
1.75%	-0.3336573	-0.05303952 (0.001)	0.06419583
2%	-0.3280402	-0.06515799 (0.001)	0.0292108

Tab. 4.3: Correlation between weekly changes in swap rates and changes in implied volatilities under different models with different strike rates (2006-07-10 to 2015-07-29)

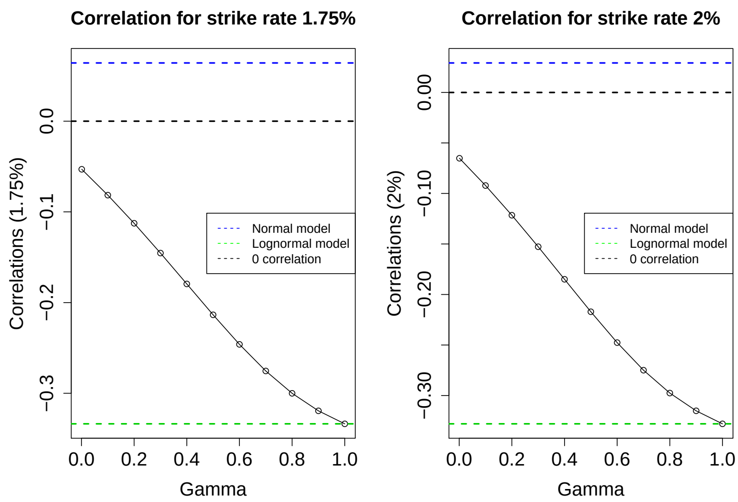


Fig. 4.6: Relation between correlation and γ parameter (2006-07-10 to 2015-07-29)

4.3 Hedging

We now implement the hedging procedure described in Section 3.2 to find out which model produces the best hedging strategy. Firstly, the hedging test was performed to ensure that the hedging algorithm is correctly implemented. The details of the hedging test are discussed in Appendix D. The results show that the hedging algorithm is correctly implemented.

Now we proceed to do the actual hedging strategy. Similar to the regression analysis, we first perform the hedging strategy on part of the dataset which runs from 2013-01-01 to 2015-07-29. From Table 4.4, it can be seen that the normal model obtains the highest R^2s for all strike rates which means that it produces the least hedging errors. Lognormal model always produces the lowest R^2s which suggest that it is the worst model. The R^2s for the CEV market model lie between the lognormal model and the normal model which suggest that it is better than the lognormal model and worse than the normal model. Consequently, the hedging result coincides with the result obtained from the regression analysis.

In addition, note that the R^2s for the CEV market model do not converge to the ones for normal model. This is shown in Figure 4.7 below which illustrates the relationship between R^2 and γ parameter. It is clear that the R^2 increases as γ decreases. However, observe that the R^2 is a concave function of γ. As a result, the

Strikes	Lognormal	CEV	Normal
1%	0.7922669	0.852124 (0.001)	0.8812281
1.75%	0.4548518	0.6517967 (0.001)	0.7157613
2%	0.2958256	0.5449949 (0.001)	0.6184676

Tab. 4.4: Hedging result: R^2 under different models for 10-year cap with different strike rates from 2013-01-01 to 2015-07-29

CEV market model cannot perform as well as the normal model.

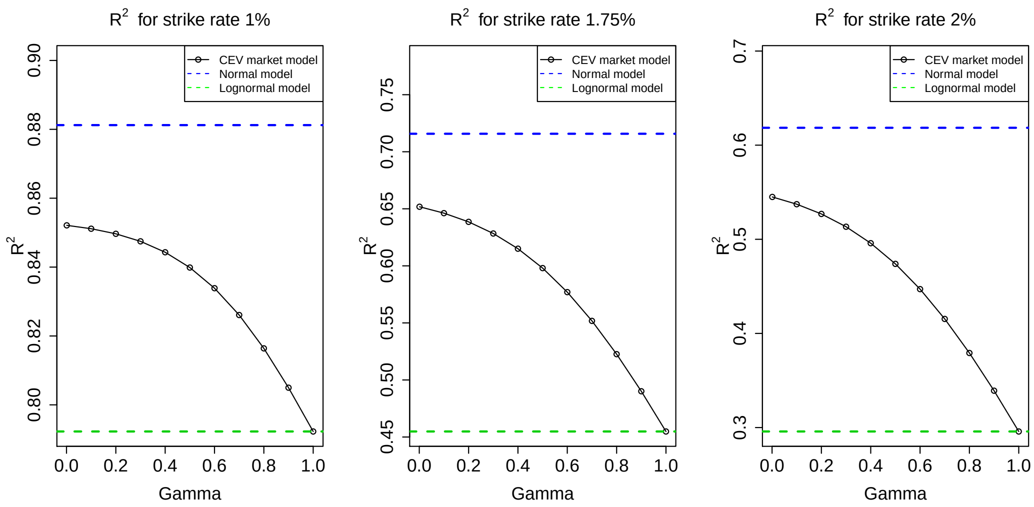


Fig. 4.7: Relation between R^2 and γ parameter (2013-01-01 to 2015-07-29)

For a similar reason mentioned in Section 4.2, we perform the same hedging procedure on the dataset from 2013-01-01 to 2014-06-30. From Table 4.5, it can be seen that the CEV market model (with a different γ parameter for different strike rate) always obtains higher R^2s than the normal model. However, the CEV market model is only slightly better than the normal model. On average, the R^2s for CEV market model are only 0.44% higher than the ones for the normal model. Furthermore, note that the R^2 for the lognormal model is slightly higher than the R^2 for normal model when the strike rate is 1% and much lower when the strike rates are 1.75% and 2%.

Strikes	Lognormal	CEV	Normal
1%	0.9436435	0.9493547 (0.44)	0.9428467
1.75%	0.8680087	0.8963637 (0.15)	0.8929602
2%	0.8231163	0.8655647 (0.05)	0.8633758

Tab. 4.5: Hedging result: R^2 under different models for 10-year cap with different strike rates from 2013-01-01 to 2014-06-30

Equivalently, Figure 4.8 below illustrates the relationship between R^2 and γ

parameter for CEV market model. The R^2 is still a concave function of the γ parameter. However, the maximum R^2 achieved by the CEV market model is higher than the one for normal model.

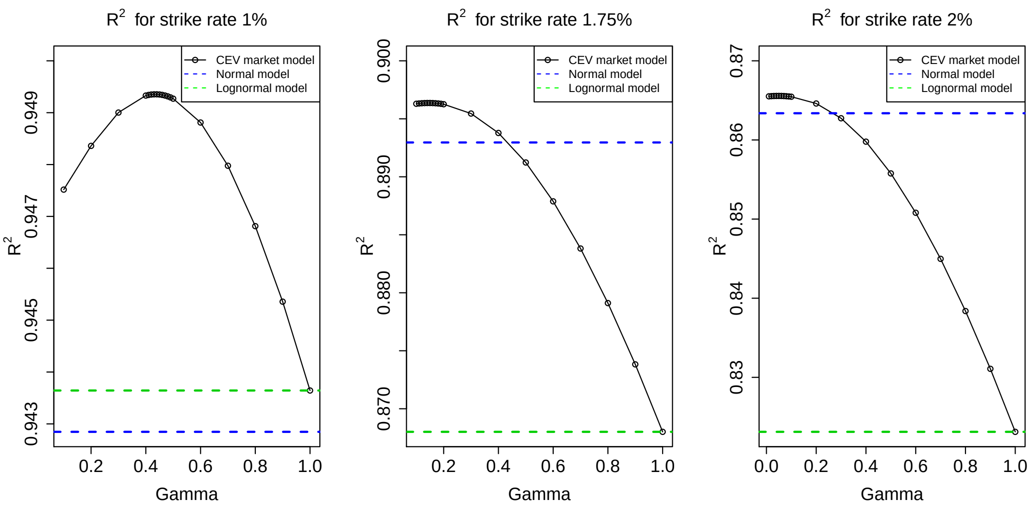


Fig. 4.8: Relation between R^2 and γ parameter (2013-01-01 to 2014-06-30)

We now implement the hedging algorithm on the entire dataset time period which runs from 2006-07-10 to 2015-07-29. Table 4.6 below shows that the lognormal model has the lowest R^2 as usual. The normal model gives the highest R^2 for both strike rates. This should be expected as over the entire time period, there is little correlation between volatility and interest rate. The CEV market model with a γ parameter of 0.001 has a slightly lower R^2. This also shows that over the entire period, volatility is less correlated with interest rate.

Strikes	Lognormal	CEV	Normal
1.75%	0.9063882	0.934812 (0.001)	0.9413262
2%	0.8875712	0.9221124 (0.001)	0.9292763

Tab. 4.6: Hedging result: R^2 under different models for 10-year cap with different strike rates from 2006-07-10 to 2015-07-29

Similarly, Figure 4.9 illustrates the relationship between R^2 and the γ parameter. Similary trend was observed. The R^2 for the CEV market model will increase towards the R^2 for the normal model as γ decreases. However, it will never converge to the normal model.

Finally, note that the hedging performance differs for caps with different strike rates. When the caps are in-the-money, it produces higher R^2. As the caps become more out-the-money, the R^2 becomes smaller which means the hedging strategy is performing worse. The underlying reason for this is the interest-rate cap is more sensitive to implied volatility when it is out-the-money and less sensitive when it is in-the-money. This is illustrated in Figure 4.10. Therefore, when implied volatility

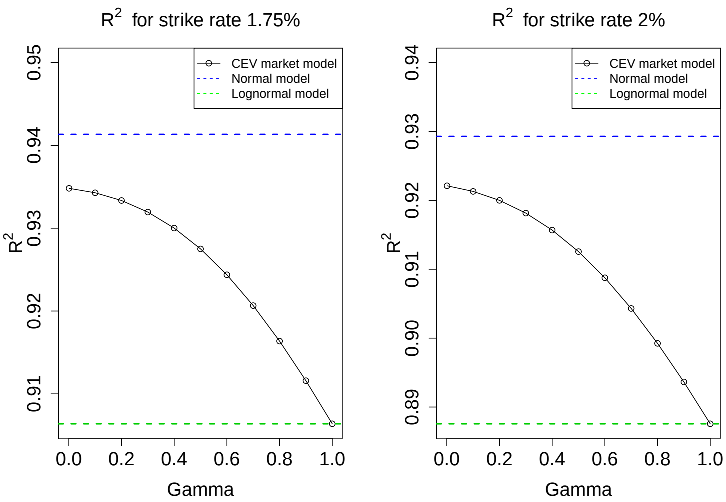


Fig. 4.9: Relation between R^2 and γ parameter (2006-07-10 to 2015-07-29)

changes, the price of in-the-money cap will have a smaller change compare to the price of out-the-money cap. As a result, in-the-money cap is easier to hedge.

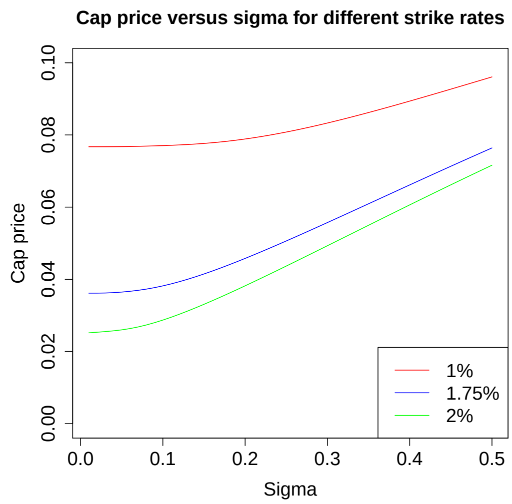


Fig. 4.10: Relation between cap price and σ parameter (implied volatility) for caps with different strike rates

From the above discussions, it is clear that the lognormal model is the worst model in hedging interest-rate caps. Table 4.7 shows the profit or loss the hedging strategy would incur if \$1 nominal of 10-year interest-rate cap is short and hedged using different models. If we were to short \$1 million nominal in 10-year interest-rate caps which strike at 1.75% over 2006-07-10 to 2015-07-29, we would lose \$63626.01 using the lognormal model. However, we would make a profit of \$36219.1 under the normal model and \$53976.97 under the CEV market model with a γ parameter of 0.001. Furthermore, note that as strike rates increases, all models would perform relatively worse. This result is intuitive as the caps are more out-the-money as strike rate increases and we know that out-the-money caps are more difficult to hedge as we explained previously.

Strikes	Lognormal	CEV	Normal
1.75%	-0.06362601	0.05397697 (0.001)	0.0362191
2%	-0.1057752	0.01328369 (0.001)	-0.004294164

Tab. 4.7: Hedging profit or loss under different models for 10-year cap with different strike rates from 2006-07-10 to 2015-07-29

4.4 Daily cross-sectional hedging

We now implement the daily cross-sectional hedging analysis described in Section 3.3 to investigate which model is the best in terms of average hedging performance across caps with different strike rates. We decided to implement this method on the dataset from 2013-01-01 to 2014-06-30 as the previous regression analysis and hedging analysis shows that the CEV market model performs the best on this part of dataset. On each day, the dataset comprises 13 different strike rates which were mentioned in Section 3.5. Therefore, the cross-sectional hedging is performed on these 13 caps each day.

Besides the procedure described in Section 3.3, we decided to use a certain set of γ parameters for the CEV market model in finding the best model. The reason behind this is due to computational inefficiency in computing the mean percentage hedging error. Moreover, the goal of this method is to investigate the range of γ parameter. By selecting an appropriate set of γ parameters, it suffices to meet this goal. As a consequence, we used a γ parameter set of $\{0.001, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$ for the CEV market model. On a 372-day period, it takes about 1.5 hours to compute the best model for each day.

By implementing the daily cross-sectional hedging analysis over this 372-day period, we found that the normal model, the lognormal model and the CEV market model perform the best on 203 days, 94 days and 75 days respectively. Figure 4.11 below shows the relationship between the mean percentage hedging error and the γ parameter on three typical days when the three models perform the best. Note that the mean percentage hedging error range for each day may vary. The fact that the normal model performs the best on most days may explain why the normal model performs well for the previous hedging analysis. Moreover, Figure 4.12 below shows the number of days for each γ parameter to be the best CEV market model. It is clear that over the 75 days where the CEV market model performs the best, the γ parameter can range anywhere from 0 to 1.

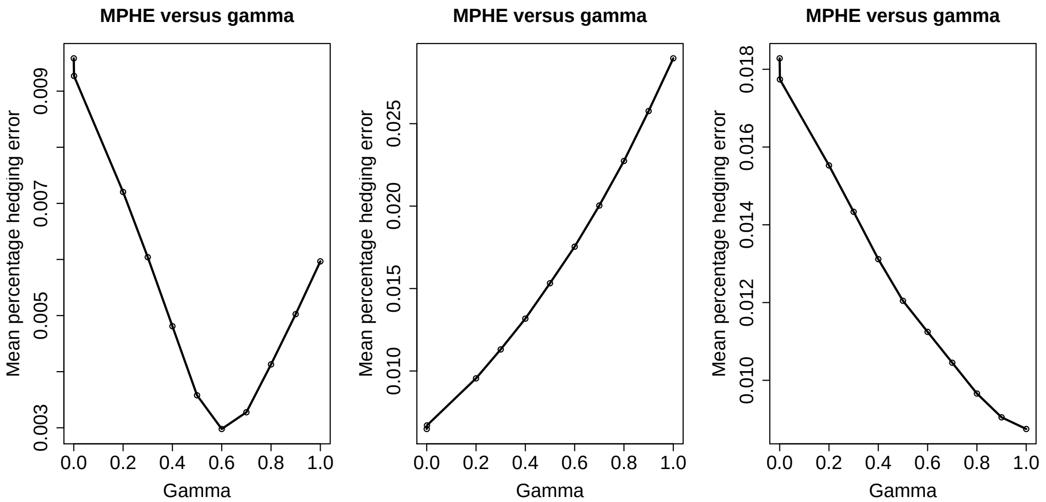


Fig. 4.11: Relationship between mean percentage error and gamma on 3 different days where different model performs the best on each day (Left: CEV market model; Middle: normal model; Right: lognormal model)

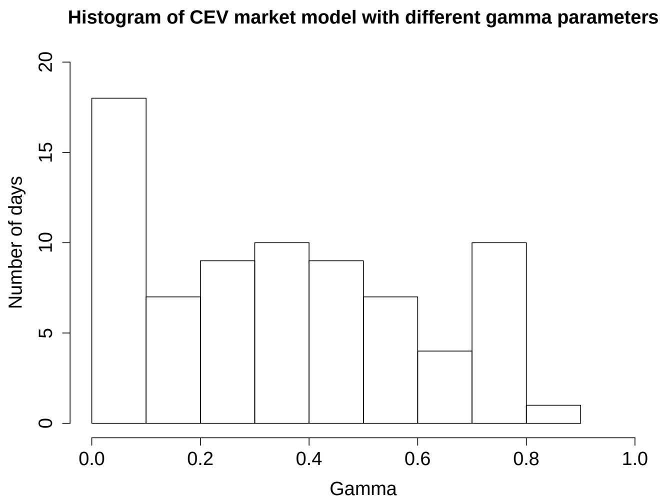


Fig. 4.12: Number of days for which CEV market model performs the best with different gamma parameters

4.5 Daily calibration to market prices

We now implement the calibration methodology described in Section 3.4 to assess which model best fits the market prices. Moreover, we would like to investigate the relationship between the γ parameter values obtained from calibration and the γ parameter values obtained from both the hedging analysis and the cross-sectional hedging analysis. Similar to reasons mentioned in Section 4.4, we will implement the calibration on dataset from 2013-01-01 to 2014-06-30 and the calibration is performed by using the 13 caps with different strike rates in the dataset.

Table 4.8 shows the summary result of the mean percentage hedging error over the 372-day period. From the table, it is clear that the normal model performs poorly in terms of calibrating to market prices. The minimum *MPHE* is as high as 0.2760 and the maximum *MPHE* reaches 0.4103. The average *MPHE* over 372 days is 0.3510. On the other hand, the lognormal model performs better than the normal model. Both the minimum *MPHE* (0.0692) and the maximum *MPHE* (0.3567) are lower than the ones for normal model. The average *MPHE* amounts to 0.2186. However, its performance is still poor for fitting market prices. Lastly, the CEV market model performs significantly better in terms of calibration. The minimum *MPHE* can reach to a low level of 0.0313 and the maximum *MPHE* (0.1613) is significantly lower than the normal model and the lognormal model. The average *MPHE* amounts to 0.0721 which is also significantly lower than the normal model and the lognormal model. As a result, the CEV market model is able to fit the market prices more closely. However, this should not be too surprising as the CEV market model has 2 parameters that are free to change, namely the σ and the γ parameters.

	Lognormal	CEV	Normal
Minimum	0.06921496	0.03131126	0.2759917
Maximum	0.3566532	0.161332	0.4103103
Average	0.2185583	0.07205891	0.3509968

Tab. 4.8: Summary result of the mean percentage hedging error (*MPHE*) of calibration across different caps for different models

Table 4.9 below summaries the calibration parameter values for different models. For the purpose of comparing calibration parameter values to the hedging parameter values, we focus on the range of γ parameter values here. From the table, it can be seen that the γ parameter values range from 0.41 to 0.75 and the average value is 0.58. However, if we compare this range to the one that was obtained from the daily cross-sectional hedging analysis, we found that they are very different. Firstly, the daily cross-sectional hedging analysis shows that the normal model performs the best on 203 days which means that the γ parameter should be 0. Secondly, the lognormal model performs the best on 94 days which means that the γ parameter should be 1. Lastly, although there are 75 days where the CEV market model performs the best, the γ parameter values can range anywhere from 0 to 1. More specifically, only 20 days out of 75 days will have a γ parameter value that is in the range. As a result, only 5% of the 372 days will have a γ parameter value that is in the range found by calibration.

	Lognormal	CEV		Normal
	Sigma	Sigma	Gamma	Sigma
Minimum	0.2947412	0.03582628	0.4054969	0.007122559
Maximum	0.3837598	0.1552456	0.7473782	0.01049268
Average	0.3310388	0.07944728	0.5840092	0.008544984

Tab. 4.9: Summary result of calibration parameter values for different models

Moreover, recall the hedging performance illustrated in Table 4.5 from Section 4.3. We have shown that the CEV market model with a γ parameter of 0.44 performs the best when hedging interest-rate cap with strike rate 1%. However, note that this is the only case where the γ parameter is in the range of calibration γ parameter values. As strike rate increases, the γ parameter value no longer belong to the range.

By comparing the calibration γ parameter value range to the values obtained from hedging analysis and the daily cross-sectional hedging analysis, we found that the best model in terms of calibration may not necessarily be the best model for hedging. In other words, if we were to calibrate the CEV market model to the market instruments today and build hedging portfolio for interest-rate cap from this model, it is highly likely that the hedging performance will be poor.

5 Conclusion

The motivation for this dissertation comes from Filipovic *et al.* (2017) where they found that volatility becomes more level-dependent when the interest rate is close to the zero lower bound. This motivates the use of the CEV market model which can capture varying level-dependence across interest-rate level. However, we have shown that the varying level-dependence feature does not exist when considering fixed strike rate interest-rate caps.

Despite this fact, we perform regression analysis and implement a hedging strategy on 10-year caps. We found that the normal market model performs the best when considering interest-rate caps from 2013-01-01 to 2015-07-29 where swap rates are below 2.5%. On the other hand, the CEV market model performs the best when considering interest-rate caps from 2013-01-01 to 2014-06-30 where swap rates are between 1.5% and 2.5%. This part of dataset exhibits positive correlation between volatility and swap rate. However, the CEV market model is only slightly better than the normal market model.

Moreover, we have shown that the normal market model performs the best on most days when performing daily cross-sectional hedging analysis. This explains why the normal market model performs the best for the hedging analysis. The calibration methodology showed that the model that best fits the market prices may not necessarily provide a good hedge for interest-rate caps.

In addition, we found that the CEV market model does not converge to the normal market model as γ tends to 0. This is shown by both regression and hedging results. We also found that the hedging performance is worse when strike rate increases (i.e., when caps are more out-the-money).

In conclusion, the normal market model is considered to be the most effective model for hedging interest-rate caps.

Bibliography

Andersen, L. and Andreasen, J. (2000). Volatility skews and extensions of the Libor market model, *Applied Mathematical Finance* **7**(1): 1–32.

Bachelier, L., Cootner, P. H. *et al. (1964). The random character of stock market prices,* Theorie de la speculation, Gauthiers, MA *pp. 17–18.*

Brace, A., Gatarek, D. and Musiela, M. (1997). The market model of interest rate dynamics, Mathematical Finance **7***(2): 127–155.*

Brigo, D. and Mercurio, F. (2006). Interest rate models – theory and practice: with smile, inflation and credit, *Springer.*

Driessen, J., Klaassen, P. and Melenberg, B. (2003). The performance of multi-factor term structure models for pricing and hedging caps and swaptions, Journal of Financial and Quantitative Analysis **38***(3): 635–672.*

Fan, R., Gupta, A. and Ritchken, P. (2003). Hedging in the possible presence of unspanned stochastic volatility: Evidence from swaption markets, Journal of Finance **58***(5): 2219–2248.*

Fan, R., Gupta, A. and Ritchken, P. (2007). On pricing and hedging in the swaption market: How many factors, really?, Journal of Derivatives **15***(1): 9–33.*

Filipovic, D., Larsson, M. and Trolle, A. B. (2017). Linear-rational term structure models, Journal of Finance **72***(2): 655–704.*

Gupta, A. and Subrahmanyam, M. G. (2005). Pricing and hedging interest rate options: Evidence from cap–floor markets, Journal of Banking & Finance **29***(3): 701–733.*

Joshi, M. and Rebonato, R. (2003). A stochastic-volatility, displaced-diffusion extension of the Libor market model, Quantitative Finance **3***(6): 458–469.*

Musiela, M. and Rutkowski, M. (1997). Martingale methods in financial modelling, *Springer.*

Piazzesi, M. (2010). Affine term structure models, Handbook of Financial Econometrics, *Elsevier, pp. 691–766.*

Svoboda-Greenwood, S. (2006). On the similarity between displaced diffusion and CEV processes.

Svoboda-Greenwood, S. (2009). Displaced diffusion as an approximation of the constant elasticity of variance, Applied Mathematical Finance **16***(3): 269–286.*

A Proof for Proposition 2.2

We now provide the proof of Proposition 2.2 from Section 2.3 where it was used to obtain the caplet pricing formula for the normal market model.

Proof. We have directly $L_k(t) = L_k(0) + \sigma W_k(t)$ for any $t \geq 0$. Now, we compute $\mathbb{E}_{\mathbb{Q}^k}\left\{(L_{T_k} - H)^+ | \mathcal{F}_t\right\}$ as follows:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^k}\left\{(L_{T_k} - H)^+ | \mathcal{F}_t\right\} &= \mathbb{E}_{\mathbb{Q}^k}\left\{(L_k(0) + \sigma W_t + \sigma(W_{T_k} - W_t) - H)^+ | \mathcal{F}_t\right\} \\
&= \int_{-\eta(t,T_k)}^{\infty} \left\{(L_k(t) - H + \sigma\sqrt{T_k - t}x)(2\pi)^{-1/2} \exp\{-x^2/2\}\right\} dx \\
&= (L_k(t) - H)\Phi\left(\eta(t, T_k)\right) + \sigma(T_k - t)^{1/2}\phi\left(\eta(t, T_k)\right) \\
&= \sigma(T_k - t)^{1/2}\left\{\eta(t, T_k)\Phi(\eta) + \phi\left(\eta(t, T_k)\right)\right\}
\end{aligned}$$

We then evaluate this expectation at time T_k and discount back to time t from payment time T_{k+1} and conclude. $\square$

B Normal implied volatility versus swap rate for 10-year cap with different strike rates

The plots below show the relationship between the normal implied volatility and the swap rate for 10-year caps which strike at 2% and 2.25%. We observe similar trends comparing to Figure 4.3 where the 10-year cap strikes at 1.75%.

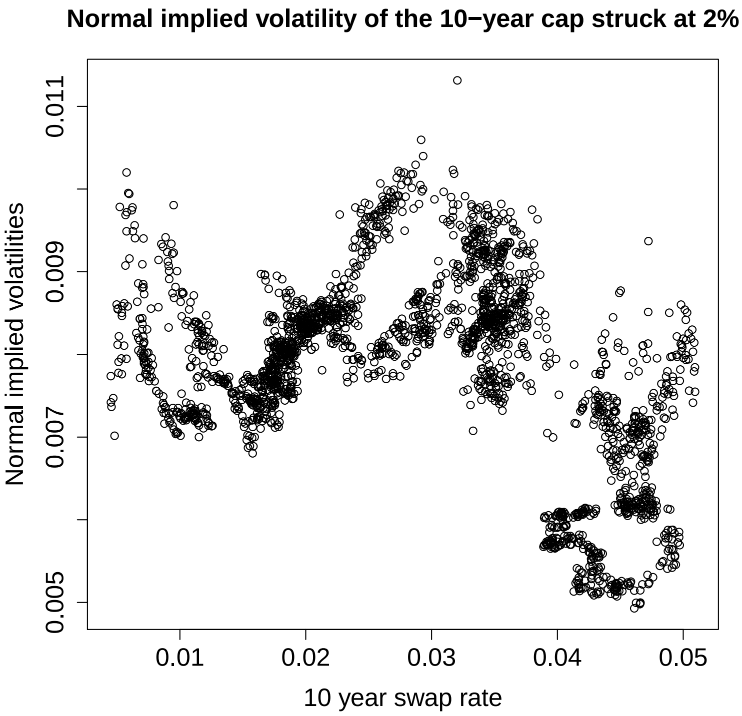


Fig. B.1: Normal implied volatility of the 10-year cap with strike rate 2% (2006-07-10 to 2015-07-29)

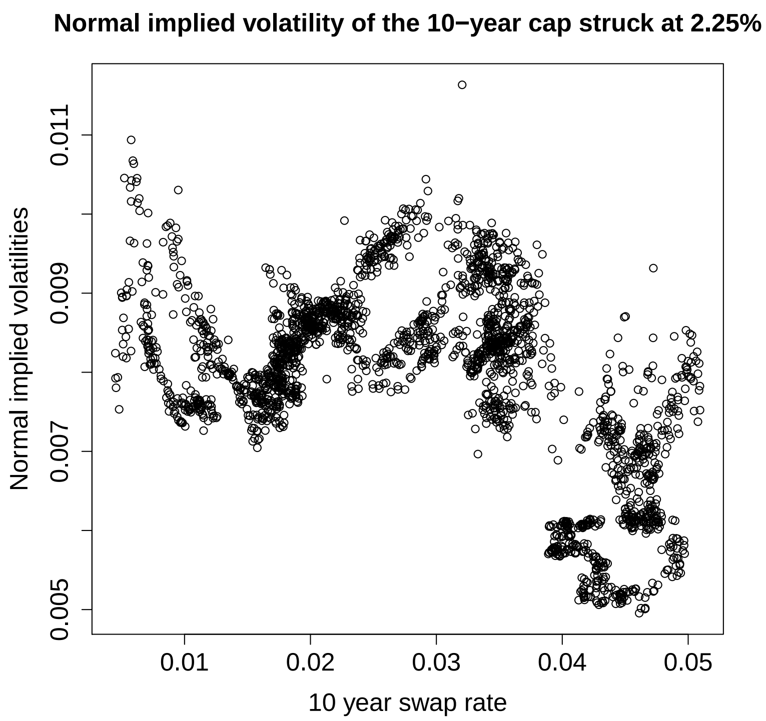


Fig. B.2: Normal implied volatility of the 10-year cap with strike rate 2.25% (2006-07-10 to 2015-07-29)

C Regression analysis graphs

The plots below illustrate the relationship between weekly changes in implied volatilities and weekly changes in swap rates under different models. Figure C.1, Figure C.2 and Figure C.3 are plots for time period 2013-01-01 to 2015-07-29, 2013-01-01 to 2014-06-30 and 2006-07-10 to 2015-07-29 respectively.

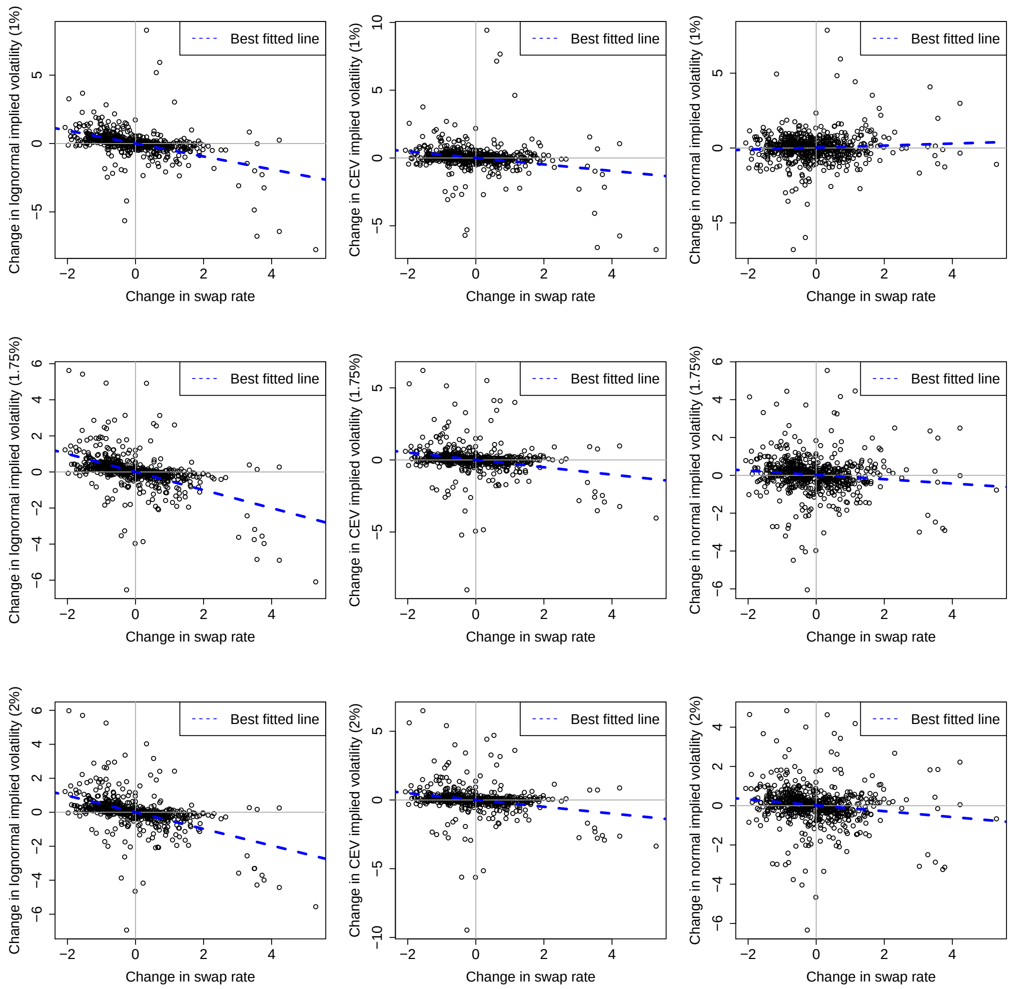


Fig. C.1: Weekly changes in implied volatilities against changes in swap rates (2013-01-01 to 2015-07-29)

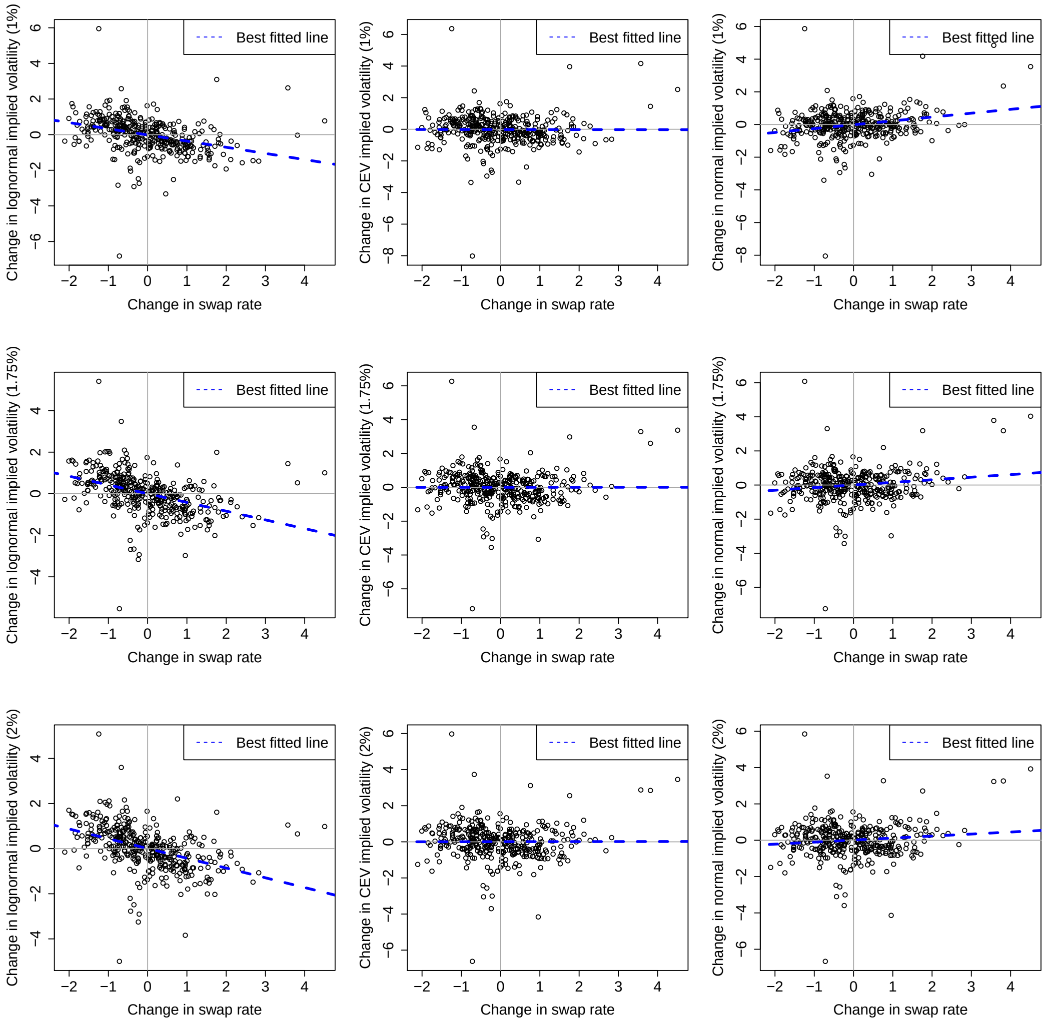


Fig. C.2: Weekly changes in implied volatilities against changes in swap rates (2013-01-01 to 2014-06-30)

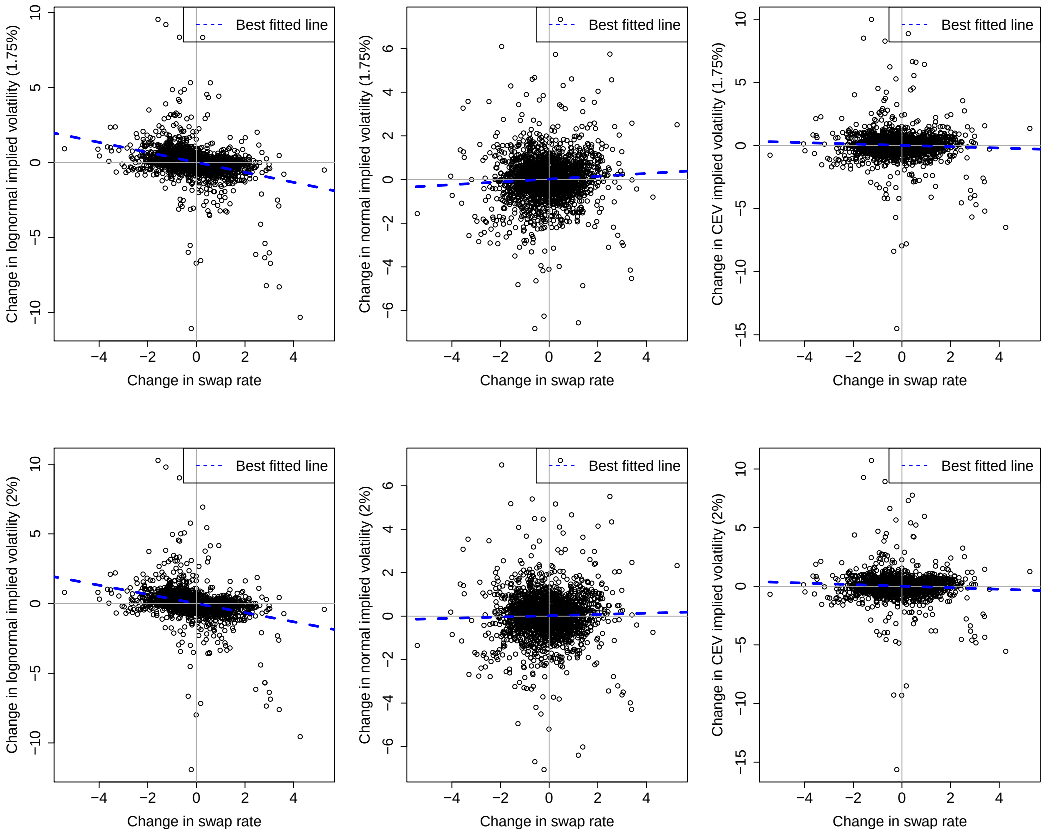


Fig. C.3: Weekly changes in implied volatilities against changes in swap rates (2006-07-10 to 2015-07-29)

D Hedging test results

A hedging test was implemented to ensure that the hedging algorithm is correctly implemented. The difference between the hedging test and the actual hedging strategy is that instead of using the new implied volatility after a week, the old implied volatility is used to calculate the new cap price. If the model correctly captures the relationship between swap rate and volatility, then the implied volatility would remain the same after a week. This means that the hedging portfolio based on the model should have the same value as the new cap price. As a result, if the R^2 from the hedging test is close to 1, this means that the hedging strategy is correctly implemented. The only reason for the R^2 not being exactly 1 is because the hedging is not done continuously but rather discretely. If the delta hedging is done daily rather than weekly, then R^2 will be closer to 1.

Strikes	Lognormal	CEV	Normal
1%	0.999591	0.9995807 (0.001)	0.9979394
1.75%	0.9988402	0.9991637 (0.001)	0.9974316
2%	0.9985113	0.9990311 (0.001)	0.9972629

Tab. D.1: Hedging test result: R^2 under different models for 10-year cap with different strike rates from 2013-01-01 to 2015-07-29

Strikes	Lognormal	CEV	Normal
1%	0.9995852	0.9994647 (0.44)	0.9989907
1.75%	0.9990553	0.998731 (0.15)	0.9983095
2%	0.9988537	0.9984635 (0.05)	0.9980966

Tab. D.2: Hedging test result: R^2 under different models for 10-year cap with different strike rates from 2013-01-01 to 2014-06-30

Strikes	Lognormal	CEV	Normal
1.75%	0.9983339	0.9981114 (0.001)	0.9977989
2%	0.9982893	0.9980112 (0.001)	0.9977242

Tab. D.3: Hedging test result: R^2 under different models for 10-year cap with different strike rates from 2006-07-10 to 2015-07-29

Table D.1, Table D.2 and Table D.3 show the hedging test results for the three different time periods. From the tables, it can be seen that the R^2s for the hedging test are all very close to 1. This means that the hedging strategy is implemented correctly.