FUNCTIONS OF OPERATORS AND THE CLASSES ASSOCIATED WITH THEM

by

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A THESIS PREPARED UNDER THE SUPERVISION OF
DR. R.W. CROSS IN FULFILMENT OF THE REQUIREMENTS
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To The Father...

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DECLARATION

This thesis represents the author's own work both in concept and execution with such excerpts from the work of others as may be pertinent. We note however that due to the analogy of this work with work done earlier by Dr. R. W. Cross, there are isolated results where there is a measure of overlap. Where this is the case it is expressly stated.

ABSTRACT

The important classes of *normally solvable*, Φ_+ (Φ_-) and *strictly singular* (*strictly cosingular*) operators have long been studied in the setting of bounded or closed operators between Banach spaces. Results by Kato, Lacey, et al (see Goldberg [16; III.1.9, III.2.1 and III.2.3]) led to the definition of certain norm related functions of operators (Γ, Δ and Γ_0) which provided a powerful new way to study the classes of Φ_+ and strictly singular operators (see for example Gramsch[19], Lebow and Schechter[28] and Schechter[36]). Results by Brace and R.-Kneece[4] among others led to the definition of analogous functions (Γ' and Δ') which were used to study Φ_- and strictly cosingular operators (see for example Weis, [37] and [38]). Again this problem was considered mainly for the case of bounded operators between Banach spaces. This thesis represents a contribution to knowledge in the sense that by considering the functions Γ', Δ', Γ_0' as well as the minimum modulus function γ in the more general setting of unbounded linear operators between normed linear spaces, we obtain the classes of F_- and *Range Open* operators which turn out to be closely related to the classes of Φ_- and normally solvable operators respectively. We also define unbounded strictly cosingular operators and find that many of the classical results on Φ_-, normally solvable and bounded strictly cosingular operators go through for F_-, range open and unbounded strictly cosingular operators respectively. This ties up with work done by R. W. Cross and provides a workable framework within which to study Φ_- and Φ_+ type operators in the much more general setting of unbounded linear operators between normed linear spaces.

Chapter I serves as an introduction, providing the foundation for work done in subsequent chapters.

In Chapter II we consider the basic properties of the functions Γ', Δ' and Γ_0' as well as characterising their respective kernels.

Chapter III is devoted mainly to the minimum modulus function γ and the related class of range open operators.

In Chapter IV we make use of Γ_0' in introducing and studying the concepts of semi-continuous and semi-precompact operators

In Chapter V we study the class of unbounded strictly cosingular operators by means of Δ' as well as proving some results on semi-bounded strictly cosingular operators

Chapter VI is devoted to the study of F_- operators. We also establish some perturbation results for F_+ and F_- operators.

In the final chapter, Chapter VII, we consider the instability of a class of non-semi-Fredholm type operators under compact perturbations. Chapter VII provides a generalisation of earlier results by the author[26] (see appendix).

LIST OF SYMBOLS

	Page no.
$\|.\|$	12
$\mathbb{R}$, $\mathbb{C}$	12
$\lambda, \mu, ..$	12
$x, y, z, ..$	12
$X, Y, Z, ..$	12
B_X, U_X	12
$d(x,y)$	12
$d(x,M)$	13
span K, $\overline{\text{span}}$ K	13
$A + B$, $A \oplus B$	13
$x + M$	13
X/M	13
$dim\ M$	14
$cod_X M$	14
$\{s_n\}$	14
$\mathbb{N}$	14
T	16
$D(T)$, $N(T)$, $R(T)$	16
$L(X,Y)$, $L[X,Y]$	16
$B(X,Y)$, $B[X,Y]$	16
$\|T\|$	17
T^{-1}	17
$X \approx Y$ $(X \equiv Y)$	18
$G(T)$	18
$C(X,Y)$, $C[X,Y]$	18
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$\Phi_+(X,Y)$, $\Phi_+(X,Y)$	18
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$\Phi(X,Y)$, $\Phi(X,Y)$	18
$x_n \to x$	19
$\tilde{T}$	19
$\hat{T}$	19
$a(T)$, $b(T)$, $\bar{b}(T)$	20
$k(T)$, $\bar{k}(T)$	20
X'	21
J_M^X, J_M	21
$\tilde{X}$	21
$J_X^{X''}$, J_X	21
$\bar{T}$	22
$\sigma(X,X')$, $\sigma(X',X)$	22
$M^{\perp}$, ${}^{\perp}K$	23
x'	23
$C(\mathcal{P})$, $\mathcal{P}$	24
δ_{nk}	24
Q_M^X, Q_M	26
P	26
α_X, β_X	28
X_1, $\|.\|_1$	28
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V_j, $V_j(X,Y)(j = 0,1,\ldots,5)$	127
SF, $SF(X,Y)$, $SF[X,Y]$	135
$C_\alpha(X,Y)$, $C_\alpha[X,Y]$	140
$K_\alpha(X,Y)$, $K_\alpha[X,Y]$	140
$[a,b]$	140
τ, $T_{\tau,p,q}$, $T_{0,\tau,p,q}$	140
τ^*	147
U_λ, V_μ	148
U_j $(j = 0,1,\ldots,5)$	149

CHAPTER 1

INTRODUCTION

§1 NORMED LINEAR SPACES

I.1.1 **DEFINITION.** Let X be a vector space over the field of real (complex) numbers. A *norm* on X, denoted by $\|.\|$, is a real valued function with the following properties:

I) $\|x\| \geq 0$ for all $x \in X$;

II) $\|x\| = 0$ implies $x = 0$;

III) $\|\lambda x\| = |\lambda|.\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}(\lambda \in \mathbb{C})$ where $\mathbb{R}(\mathbb{C})$ denotes the real (complex) numbers;

IV) $\|x + y\| \leq \|x\| + \|y\|$ for all $x,y \in X$.

The vector space X, together with a norm on X, will be referred to as a *normed linear space*. If the scalars over X are real (complex), X is called a real (complex) normed linear space.

We note that in general $X,Y,Z,\ldots$ will denote normed linear spaces different from the zero space.

I.1.2 **DEFINITION.** For a normed linear space X we define the *closed* and *open unit balls* of X, denoted by B_X and U_X respectively, to be the sets $B_X = \{x | x \in X, \|x\| \leq 1\}$ and $U_X = \{x | x \in X, \|x\| < 1\}$.

I.1.3 **DEFINITION.** Let X be a normed linear space. The *metric induced by the norm*, say d, is defined by $d(x,y) = \|x - y\|$ for $x,y \in X$. Unless otherwise stated, the topology on X will be the metric topology determined by d. If X is a complete metric space with respect to d, X is referred to as a *complete normed linear space* or a *Banach space*.

Note that for $x \in X$ and M a subset of X, $d(x,M)$ will denote the distance from x to M, that is $\inf_{m \in M} \|x - m\| = d(x,M)$.

I.1.4 **DEFINITION.** Let X be a normed linear space. If M is a vector subspace of X and the norm on M is taken as the restriction of the norm on X to M, then M is said to be a *subspace* of X. For a subset K of X, *span* K will denote the subspace of X generated by taking linear combinations of elements of K. The closure of *span* K will be denoted by $\overline{\mathit{span}}\ K$. A linearly independent subset N of X such that *span* $N = X$ is called a *Hamel base*.

I.1.5 **PROPOSITION.** *Every vector space has a Hamel base.*

Proof. See for example [35]. □

I.1.6 **DEFINITION.** Let X be a normed linear space and let M be a closed subspace of X. We say that M is *complemented* in X if there exists another closed subspace N of X such that $N \cap M = \{0\}$ and $N + M = X$. N is referred to as a *complement* of M. In general if A and B are subspaces of X with $A \cap B = \{0\}$, $A + B$ will be denoted by $A \oplus B$.

I.1.7 **DEFINITION.** Let X be a normed linear space and M a subspace of X. Define an equivalence relation R on X as follows: xRy if and only if $x - y \in M$ where $x,y \in X$. Denoting the set of equivalence classes in X by X/M we see that X/M is a vector space where vector addition and scalar multiplication are defined as follows:

Let $x + M$ denote the class of elements equivalent to $x \in X$. Then for $x,y \in X$, $(x + M) + (y + M) = (x + y) + M$ and $\lambda(x + M) = \lambda x + M$ where λ is a scalar. If M is closed $d(x,M)$ defines a norm on X/M where $x \in X$. We refer to X/M as a *quotient space* and to $d(x,M)$ as the *quotient norm*.

I.1.8 **DEFINITION.** Let M be a subspace of the normed linear space X. We will denote the *dimension* of M by $\dim M$. The *codimension* of M in X, denoted by $\mathrm{cod}_X M$, is defined to be the dimension of the vector space X/M. If there is no danger of confusion we will write $\mathrm{cod}\, M$ instead of $\mathrm{cod}_X M$.

I.1.9 **THEOREM** [16; IV.2.8]. *Let* M *be a closed finite codimensional subspace of* X.

I) *For any subspace* V *of* X, *there exists a finite dimensional subspace* $N \subset V$ *such that* $\bar{V} = (\bar{V} \cap M) \oplus N$.

II) *If* V *is dense in* X, $V \cap M$ *is dense in* M.

Proof. Using [35; Corollary V.7.29] instead of [16; II.1.14] we note that the proof in [16] goes through for normed linear spaces as well. □

I.1.10 **THEOREM** ([23]; cf.[16; V.1.1]) *Let* M *and* N *be subspaces of* X *with* $\dim M > \dim N$. *Then there exists an* $m \in M$, $m \neq 0$, *such that* $\|m\| = d(m,N)$.

§2 COMPLETE NORMED LINEAR SPACES

I.2.1 **THEOREM.** *If* X *is a Banach space and* M *a closed subspace of* X, *then* X/M *and* M *are Banach spaces.*

Proof. Follows directly from [16; I.2.8] and [24; 2.3-1]. □

I.2.2 **DEFINITION.** An *infinite series* $\sum_{i=1}^{\infty} x_i$ of elements x_i of the normed linear space X is said to *converge* in X if there exists $x \in X$ such that the sequence $\{s_n\} \subset X$ defined by $s_n = \sum_{i=1}^{n} x_i$ for each $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the natural numbers), converges to x. We write $x = \sum_{i=1}^{\infty} x_i$. We say that the series *converges absolutely* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

I.2.3 **THEOREM** [16; I.2.5]. *A normed linear space is complete if and only if each absolutely convergent series in* X *converges in* X.

§3 FINITE DIMENSIONAL NORMED LINEAR SPACES

Proofs of the first three results may be found in Chapter I of [12].

I.3.1 **PROPOSITION.** *Finite dimensional normed linear spaces are complete.*

I.3.2 **PROPOSITION.** *Finite dimensional subspaces of a normed linear space are closed.*

I.3.3 **PROPOSITION.** *A normed linear space* X *is finite dimensional if and only if* B_X *is compact (totally bounded).*

The statement in the bracket follows when the result in [12] is considered alongside [16; I.4.6].

I.3.4 **PROPOSITION** [16; I.4.12]. *The sum of two closed subspaces of a normed linear space is closed whenever one of the subspaces is finite dimensional.*

§4. LINEAR OPERATORS

I.4.1 **DEFINITION.** Let X and Y be normed linear spaces over the same field of scalars. A function T with domain in X and range in Y is called a *linear operator* if for all x,y in the domain of T and all scalars λ,μ we have that

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty.$$

We will denote the *domain*, *null space* and *range* of a linear operator T by $D(T)$, $N(T)$ and $R(T)$ respectively. T is called *injective* if $N(T) = \{0\}$, *surjective* if $R(T) = Y$ and *bijective* if T is both injective and surjective. We will denote the class of all linear operators with domain in X and range in Y by $L(X,Y)$. The class of elements of $L(X,Y)$ which are defined everywhere on X is denoted by $L[X,Y]$.

We note that we will sometimes refer to a linear operator as an operator or a map.

I.4.2 **THEOREM** [16; I.3.2]. *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) T *is continuous at a point.*

II) T *is uniformly continuous on* $D(T)$.

III) *There exists a number* $k \geq 0$ *such that* $\|Tx\| \leq k\|x\|$ *for all* $x \in D(T)$.

The class of all continuous operators in $L(X,Y)$ will be denoted by $B(X,Y)$ with $L[X,Y] \cap B(X,Y)$ being denoted by $B[X,Y]$. We adopt the following convention: We refer to an operator T in $L(X,Y)$ as being *bounded* if and only if $T \in B[X,Y]$.

I.4.3 **DEFINITION.** Let $T \in B(X,Y)$. We define the *norm of* T, denoted by $\|T\|$, as follows:

$$\|T\| = \sup_{\substack{\|x\|=1 \\ x\in D(T)}} \|Tx\|.$$

It is easy to see that this defines a norm on the vector space $B[X,Y]$.

I.4.4 *Remark.* Let $T \in B(X,Y)$. Then

I) T maps a dense subspace of $D(T)$ onto a dense subspace of $R(T)$;

II) $N(T)$ is relatively closed in $D(T)$.

Both of these are easy consequences of III) of Theorem I.4.2.

I.4.5 **THEOREM** [16; I.3.5 and I.5.8]. *If X and Y are normed linear spaces over the same field of scalars then Y is complete if and only if $B[X,Y]$ is complete.*

I.4.6 **DEFINITION.** An operator T in $L(X,Y)$ is defined to be *open* if there exists $\lambda > 0$ such that

$$\lambda B_Y \subset TB_{D(T)} \quad \text{or equivalently} \quad \lambda U_Y \subset TU_{D(T)}.$$

Note that T is open if and only if T maps sets which are open in $D(T)$ onto sets open in Y.

I.4.7 **DEFINITION.** For an injective operator $T \in L(X,Y)$, we define the *inverse of* T, denoted by T^{-1}, to be the operator with domain $R(T)$ and

$$T^{-1}(Tx) = x \quad \text{for any} \quad Tx \in R(T).$$

Clearly $T^{-1} \in L(Y,X)$ with $R(T^{-1}) = D(T)$.

I.4.8 **THEOREM** [16; I.3.7]. *Let $T \in L(X,Y)$. Then T^{-1} exists and is continuous if and only if there is some $k > 0$ such that*

$$\|Tx\| \geq k\|x\| \quad \text{for each} \quad x \in D(T).$$

I.4.9 **DEFINITION.** Let $T \in L(X,Y)$. If T^{-1} exists and both T and T^{-1} are continuous, then T is called an *isomorphism*. If $\|Tx\| = \|x\|$ for each $x \in D(T)$, T is called an *isometry*. The two spaces X and Y are said to be *isomorphic* (*isometric*), denoted by $X \approx Y$ $(Y \equiv Y)$, if there exists an isomorphism (isometry) from X onto Y.

I.4.10 **PROPOSITION** [16]. *If a normed linear space X is isomorphic to a Banach space then X is also a Banach space.*

I.4.11 **DEFINITION.** Let $T \in L(X,Y)$. T is said to be a *closed operator* if the space

$$G(T) = \{(x,Tx)\,|\,x \in D(T)\}$$

is a closed subspace of $X \times Y$ where the norm on $X \times Y$ is defined by $\|(x,y)\| = \|x\| + \|y\|$ for $x \in X$ and $y \in Y$. $G(T)$ is referred to as the graph of T. T is called *closable* if there exists a linear extension of T which is a closed operator. If T is a closed operator with $R(T)$ a closed subspace of Y, T is said to be *normally solvable*. If T is normally solvable with finite dimensional null space (finite codimensional range) then T is defined to be a $\phi_+(\phi_-)$*-operator*. The classes of closed, normally solvable, ϕ_+ and ϕ_--operators in $L(X,Y)$ will be denoted by $C(X,Y)$, $NS(X,Y)$, $\phi_+(X,Y)$ and $\phi_-(X,Y)$ respectively with the intersections of these classes with $L[X,Y]$ denoted by $C[X,Y]$, $NS[X,Y]$, $\phi_+[X,Y]$ and $\phi_-[X,Y]$ respectively. We will denote $\phi_+(X,Y) \cap \phi_-(X,Y)$ by $\phi(X,Y)$ and $L[X,Y] \cap \phi(X,Y)$ by $\phi[X,Y]$.

Note that we will sometimes refer to ϕ, ϕ_+ and ϕ_- operators as Fredholm, upper-semi-Fredholm and lower-semi-Fredholm operators respectively.

I.4.12 *Remark.*

I) T is closed if and only if for each sequence $\{x_n\} \subset D(T)$ such that $x_n \to x \in X$ and $Tx_n \to y \in Y$ we have that $x \in D(T)$ and $Tx = y$.

II) $N(T)$ is closed if T is closed.

III) If T is continuous then T is closed if $D(T)$ is closed.

IV) If T is injective, T is closed if and only if T^{-1} is closed.

I.4.13 **THEOREM** [16; II.2.11]. *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) T *is closable.*

II) T *has a minimal closed extension (denoted by* $\tilde{T}$*); ie there exists a closed linear extension of* T*, say* $\tilde{T}$*, such that any closed linear extension of* T *is also a closed linear extension of* $\tilde{T}$.

I.4.14 **PROPOSITION.** *Let* $T \in L(X,Y)$ *be closable. Then* $G(\tilde{T}) = \overline{G(T)}$.

Proof. This follows from the way $\tilde{T}$ is defined in the proof of [16; II.2.11].

I.4.15 **DEFINITION.** Let $T \in L(X,Y)$ with $N(T)$ closed (hence $X/N(T)$ is a normed linear space). The *induced injective operator* or *injective component* of T, denoted by $\hat{T}$, is defined as follows:

$$\hat{T}(x + N(T)) = Tx \quad \text{for each} \quad x \in D(T).$$

Observe that $\hat{T} \in L(X/N(T),Y)$ with $R(\hat{T}) = R(T)$ and $D(\hat{T}) = D(T)/N(T)$.

I.4.16 **PROPOSITION** [16; II.4.7]. *Let* $T \in L(X,Y)$ *with* $N(T)$ *closed. Then*

I) T *is closed if and only if* $\hat{T}$ *is closed;*

II) T *is continuous if and only if* $\hat{T}$ *is continuous, in which case* $\|T\| = \|\hat{T}\|$.

I.4.17 **PROPOSITION** [16; IV.1.13]. *Let* $T \in C(X,Y)$ *with* X *and* Y *complete. If* $R(T)$ *has finite codimension in* Y, *then* $R(T)$ *is closed.*

I.4.18 **DEFINITION.** Let $T \in L(X,Y)$. We let $a(T) = \dim N(T)$, $b(T) = \mathrm{cod}_Y R(T)$ and $\bar{b}(T) = \mathrm{cod}_Y \overline{R(T)}$. If $a(T)$ and $b(T)$ are not both infinite we say that T has an *index*, denoted by $k(T)$, where $k(T) = a(T) - b(T)$. Similairly if $a(T)$ and $\bar{b}(T)$ are not both infinite, $\bar{k}(T) = a(T) - \bar{b}(T)$ is defined to be the *reduced index* of T. (Note that for any real number r we let $\infty - r = \infty$ and $r - \infty = -\infty$.)

I.4.19 **THEOREM** ([21]; cf.[16; V.1.6]). *Let* X *and* Y *be complete and let* $T \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$. *For any* $B \in B(X,Y)$ *such that* $D(B) \supset D(T)$ *there exists* $\rho > 0$ *such that for* $|\lambda| < \rho$,

I) $T + \lambda B$ *is normally solvable;*

II) $a(T + \lambda B) \leq a(T)$ *and* $b(T + \lambda B) \leq b(T)$;

III) $k(T + \lambda B) = k(T)$.

1.4.20 **PROPOSITION** [16; V.1.7]. *Let* X *and* Y *be complete,* $T \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$ *and* $B \in B(X,Y)$ *with* $D(B) \supset D(T)$. *Then there exists* $\rho > 0$ *such that* $a(T + \lambda B)$ *and* $b(T + \lambda B)$ *are constant in the annulus* $0 < |\lambda| < \rho$.

I.4.21 **THEOREM** [16; V.1.8]. *Let* X *and* Y *be complete,* $T \in L(X,Y)$ *and* $B \in B(X,Y)$ *with* $D(B) \supset D(T)$. *Define* U *to be the set of scalars* λ *for which* $T + \lambda B \in \Phi_+(X,Y) \cup \Phi_-(X,Y)$. *Then*

I) U *is open;*

II) *If* C *is a component of* U *(a largest connected subset of* U*), then on* C, *with the possible exception of isolated points,* $a(T + \lambda B)$ *and* $b(T + \lambda B)$ *have constant values* n_1 *and* n_2 *respectively. At the isolated points* $a(T + \lambda B) > n_1$ *and* $b(T + \lambda B) > n_2$.

§5 DUAL SPACES AND THE COMPLETION OF A NORMED LINEAR SPACE

I.5.1 **DEFINITION.** For a normed linear space X, we define the *dual space* of X, denoted by X', to be the space $B[X,\mathbb{R}]$ if X is a real normed linear space and $B[X,\mathbb{C}]$ if X is a complex normed linear space. Elements of X' will be referred to as *bounded linear functionals*. Note that X' is complete by Theorem I.4.5.

I.5.2 **DEFINITION.** Let M be a subspace of X. We define the *injection of* M *into* X, denoted by J_M^X, to be the operator defined by $J_M^X m = m$ where $m \in M$. (We will use J_M instead of J_M^X if there is no danger of confusion.)

I.5.3 **PROPOSITION.** *Every normed linear space* X *is a dense subspace of a Banach space* $\tilde{X}$. *(We shall refer to* $\tilde{X}$ *as the completion of* X.*)*

The above proposition is a trivial consequence of the fact that X can be regarded as a subspace of X''. Hence we can let $\tilde{X} = \overline{(J_X^{X''} X)}$. We will denote the injection of X into $\tilde{X}$ by J_X.

I.5.4 **DEFINITION.** We define a normed linear space X to be *reflexive* if $X'' = J_X^{X''} X$.

I.5.5 **PROPOSITION.** *Let* X *be a normed linear space and let* M *be a closed subspace of* $\tilde{X}$. *Then* $cod_{\tilde{X}} M = \infty$ *if and only if* $cod_X(M \cap X) = \infty$.

Proof. Suppose $cod_{\tilde{X}} M = \infty$ and $cod_X M \cap X < \infty$. Hence there exists a finite dimensional subspace F of X so that $X = F + (M \cap X)$. Noting that both F and $(M \cap X)^{\sim}$ can be regarded as subspaces of $\tilde{X}$ we have $F + (M \cap X)^{\sim} \subset \tilde{X}$. since $(M \cap X)^{\sim}$ is a complete and hence closed subspace of $\tilde{X}$, $F + (M \cap X)^{\sim}$

is a closed and hence complete subspace of $\tilde{X}$ by Proposition I.3.4. But $X \subset F + (M \cap X)^{\sim}$ and since $F + (M \cap X)^{\sim}$ is complete, $\tilde{X} \subset F + (M \cap X)^{\sim}$. Hence $\tilde{X} = F + (M \cap X)^{\sim}$. Moreover M is a closed and hence complete subspace of $\tilde{X}$ with $M \cap X \subset M$. Consequently $(M \cap X)^{\sim} \subset M$ and so $\tilde{X} = M + F$. Therefore $cod_{\tilde{X}} M \leq dim\ F < \infty$; a contradiction. Thus $cod_X(M \cap X) = \infty$ if $cod_{\tilde{X}} M = \infty$. Conversely if $cod_{\tilde{X}} M < \infty$, then $cod_X(M \cap X) < \infty$. □

The above proposition is by the author.

I.5.6 **THEOREM** [16; II.2.1]. *Let* Y *be complete and* $T \in B(X,Y)$. *If* T *is defined on a dense subspace, say* M, *of* X, *then there exists a unique continuous linear extension of* T *to all of* X, *say* T_1, *and* $\|T_1\| = \|T\|$. *Also* $M' \equiv X'$.

Considering the previous theorem we note that for any normed linear space X, $X' \equiv (\tilde{X})'$ and hence we will use these two concepts interchangably. We also adopt the following convention:

If X and Y are normed linear spaces and $T \in B(X,Y)$, then $\bar{T}$ will denote the *unique continuous linear extension of* $J_Y T J_X^{-1}$ (that is T regarded as an element of $B(\tilde{X},\tilde{Y})$), to $D(T)^{\sim}$ where $D(T)^{\sim}$ is regarded as a subspace of $\tilde{X}$.

I.5.7 **DEFINITION**. Let X be a normed linear space. We define the *weak topology* on X to be the coarsest topology on X such that all elements of X' are continuous on X. Analogously we define the *weak* topology* on X' to be the coarsest topology on X' such that all the elements of X are continuous on X'. The weak and weak* topologies are denoted by $\sigma(X,X')$ *and* $\sigma(X',X)$ respectively.

I.5.8 **DEFINITION.** Let M be a subset of X and K a subset of X'. We define the *orthogonal complement of* M *in* X', denoted by $M^{\perp}$, to be the set

$$M^{\perp} = \{x' \mid x' \in X',\ x'm = 0 \text{ for all } m \in M\}$$

Analogously the *orthogonal complement of* K *in* X is the set

$$^{\perp}K = \{x \mid x \in X,\ x'x = 0 \text{ for all } x' \in K\}$$

I.5.9 *Remarks*

I) $^{\perp}K$ and $M^{\perp}$ are closed subspaces of X and X' respectively [16; II.3.3].

II) If M is a subspace of X, then $^{\perp}(M^{\perp}) = \bar{M}$ [16; II.3.4].

III) If F is a finite dimensional subspace of X', then $(^{\perp}F)^{\perp} = F$ [16; II.3.6].

IV) If K is a subspace of X', then $K = (^{\perp}K)^{\perp}$ if and only if K is $\sigma(X',X)$-closed. (Follows from the discussion on polar sets in Chapter II of [35].)

I.5.10 **THEOREM** [16; I.6.4]. *Let* M *be a subspace of the normed linear space* X. *Then*

I) $X'/M^{\perp}$ *is isometric to* M' *under the map* U *defined by* $U(x' + M^{\perp}) = x'_M$ *where* $x' + M^{\perp} \in X'/M^{\perp}$ *and* x'_M *is the restriction of* x' *to* M;

II) $(X/\bar{M})'$ *is isometric to* $M^{\perp} = \bar{M}^{\perp}$ *under the map* V *defined by*

$$(Vz')x = z'(x + \bar{M}) \text{ where } z' \in (X/\bar{M})' \text{ and } x + \bar{M} \in X/\bar{M}.$$

I.5.11 **PROPOSITION** [35; Lemma II.3.5]. *Let* X *be a normed linear space and* $x'_1, x'_2, \ldots, x'_n$ *elements of* X'. *If* $x' \in X'$ *such that* $N(x') \subset \bigcap_{i=1}^{n} N(x'_n)$ *then* x' *is a linear combination of* $x'_1, x'_2, \ldots, x'_n$.

§6 SEPARABLE NORMED LINEAR SPACES

I.6.1 **DEFINITION.** A normed linear space X is defined to be *separable* if there exists a countable dense subset of X.

I.6.2 **THEOREM** [13; V.7.14]. *A Banach space X is separable if and only if it is isometric to a closed subspace of $C(\mathcal{P})$ where $\mathcal{P}$ is the Cantor perfect set.*

I.6.3 **COROLLARY.** *Every subspace of a separable normed linear space is separable.*

Proof. Suppose X is separable and that M is a subspace of X. It can easily be verified that $\tilde{X}$ is separable and that $\tilde{M}$ can be regarded as a closed subspace of $\tilde{X}$. Hence $\tilde{M}$ is separable by Theorem I.6.2. Let $\{\tilde{z}_n\}$ be a countable dense subset of $\tilde{M}$ and for each $n \in \mathbb{N}$ select $\{x_{n_k}\} \subset M$ such that $x_{n_k} \to \tilde{z}_n$. Then $\bigcup_{n=1}^{\infty} \{x_{n_k}\} \subset M$ is a countable dense subset of $\tilde{M}$ and hence also of M. □

I.6.4 **DEFINITION.** A subset M of X is said to be *fundamental* if $\overline{\text{span}\, M} = X$. A subset K of X' is said to be *total* if $x = 0$ whenever $x'x = 0$ for every $x' \in K$.

I.6.5 **DEFINITION.** We say that sequences $\{x_n\} \subset X$ and $\{x'_n\} \subset X'$ are *biorthogonal* if $x'_n(x_k) = \delta_{nk}$.

I.6.6 **THEOREM** [20; 14.1.5]. *Every separable normed linear space X admits of a fundamental and total biorthogonal system $(\{x_n\};\{x'_n\})$.*

I.6.7 **DEFINITION.** Let M be a closed subspace of X. We say that M is *quasicomplemented* in X if there exists a closed subspace N of X with $M \cap N = \{0\}$ and $\overline{M \oplus N} = X$.

I.6.8 **THEOREM** [30]. *Every closed subspace of a separable normed linear space is quasicomplemented.*

§7 THE HAHN-BANACH AND BANACH-STEINHAUS THEOREMS.

I.7.1 **PROPOSITION** [16]. *Let M be a subspace of X and let $m' \in M'$. Then there exists an extension $x' \in X'$ of m' such that $\|x'\| = \|m'\|$.*

I.7.2 **COROLLARY** [16]. *Let M be a subspace of X. Given $x \in X$ with $d(x,M) > 0$, there exists $x' \in X'$ such that*

$$\|x'\| = 1, \ x' \in M^{\perp} \ \text{and} \ x'x = d(x,M).$$

I.7.3 **COROLLARY** [16]. *Given $x \in X$, there exists $x' \in X'$ such that $\|x'\| = 1$ and $x'x = \|x\|$.*

Proof. Let $M = \{0\}$ in the previous corollary. □

I.7.4 **COROLLARY** [16]. *For any $x \in X$,*

$$\|x\| = \sup_{\substack{\|x'\|=1 \\ x' \in X'}} |x'x|.$$

I.7.5 **THEOREM** [16; II.1.12]. *Suppose K is a subset of X such that*

$$\sup_{k \in K} |x'k| < \infty \quad \text{for each } x' \in X'.$$

Then K is bounded.

§8 TOPOLOGICAL COMPLEMENTATION AND THE QUOTIENT MAP

I.8.1 **DEFINITION.** Let M be a subspace of X. An operator P from X onto M is called a *projection from* X *onto* M if $P = P^2$. The operator Q_M^X from X onto X/M defined by $Q_M^X(x) = x + M$ for each $x \in X$ is said to be the *quotient map* from X onto X/M. If there is no danger of confusion we will use Q_M instead of Q_M^X. Where convenient we will also denote the equivalence class of x in X/M by $Q_M x$ instead of $x + M$.

I.8.2 **PROPOSITION.** *Let* M *be a closed subspace of* X *(so that* X/M *is a normed linear space). Then* Q_M^X *is both bounded and open.*

Proof. The fact that Q_M^X is bounded follows trivially from the fact that $\|Q_M^X x\| = d(x,M) \leq \|x\|$ for each $x \in X$ (in fact $\|Q_M^X\| \leq 1$). Now select $x + M \in U_{X/M}$ arbitrarily. Then $\|x + M\| = d(x,M) = \inf_{m\in M}\|x - M\| < 1$ by definition. Hence there exists $m_0 \in M$ such that $\|x - m_0\| < 1$; that is $x - m_0 \in U_X$. Observing that since $m_0 \in M$, $Q_M^X(x - m_0) = Q_M^X x = x + M$ we conclude that $Q_M^X\, U_X = U_{X/M}$ and hence that Q_M^X is open. □

I.8.3 **DEFINITION.** Let M be a closed subspace of X. We say that M is *topologically complemented* in X if there exists a bounded projection P from X onto M. Note that if this is the case M is complemented by $N(P)$. $N(P)$ is called a *topological complement* of M.

I.8.4 **PROPOSITION.**

I) *If* M *and* N *are complementary (and hence closed) subspaces of* X *they are topological complements whenever one is finite dimensional.*

II) *If* M *is finite dimensional (closed and finite codimensional) in* X, *it is topologically complemented.*

Proof.

I) Let M and N be complementary subspaces of X with $\dim N = n < \infty$. Let $\{x_1, x_2, \dots, x_n\}$ be a base for N and let $M_k = \operatorname{span}\{x_1, x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_n\} \oplus M$ for $1 \le k \le n$. Note that each M_k is closed by Proposition I.3.4. By Corollary I.7.2 it follows that we can select $\{z_1', z_2', \dots, z_n'\}$ so that $z_k' x_k \neq 0$ and $z_k' M_k = 0$ for each $1 \le k \le n$. Now let $x_k' = z_k'/(z_k' x_k)$ and define P as follows:

$$Px = \sum_{k=1}^{n} x_k'(x) x_k \quad \text{for each } x \in X.$$

It is now easy to verify that P is a bounded projection from X onto N with $N(P) = M$.

II) If M is closed with $\operatorname{cod}_X M < \infty$ select any finite dimensional subspace N such that $M + N = X$ and $M \cap N = \{0\}$. The assertion then follows from Proposition I.3.2 and I). If $\dim M < \infty$, this is just [16; II.1.16]. □

I.8.5 **PROPOSITION** *Let* M *and* N *be subspaces of* X *which are topological complements of each other. Then* $Q_M^X J_N$ *is an isomorphism from* N *onto* X/M.

Proof. Since M and N are topological complements, there exists a bounded projection P from X onto N with $N(P) = M$. By Proposition I.4.16 $\hat{P}$ is a bounded map from X/M onto N. In fact $(\hat{P})^{-1}$ is also bounded since $\|Q_M^X n\| = d(n, M) \le \|n\|$ for any $n \in N$. Hence $\hat{P}$ is an isomorphism from X/M onto N. The proposition follows on noticing that $(\hat{P})^{-1} = Q_M^X J_N$. □

I.8.6 **PROPOSITION.** *Let* M *be a closed subspace of* X. *Then*

I) *for any closed subspace* K *of* X *such that* $K \supset M$, $X/K \equiv (X/M)/(K/M)$;

II) *for any closed subspace* N *of* (X/M) *there exists a closed subspace* K *of* X *such that* $K \supset M$ *with* $Q_K^X = Q_N^{X/M} Q_M^X$.

Proof. I) Note that since $K \supset M$ we obtain

$$\begin{aligned}\|Q^{X/M}_{K/M}\, Q^X_M x\| = \inf_{k\in K}\|Q^X_M x - Q^X_M k\| = \inf_{k\in K}\|Q^X_M(x - k)\| &= \inf_{k\in K}(\inf_{m\in M}\|(x - k) - m\|)\\ &= \inf_{k\in K}\|x - k\|\\ &= \|Q^X_K x\| \quad \text{for each } x \in X.\end{aligned}$$

II) Let N be a closed subspace of (X/M). Since Q^X_M is bounded, $(Q^X_M)^{-1}N = K$ is closed ($(Q^X_M)^{-1}$ taken in the set theoretic sense). Since $0 \in N$ and $N(Q^X_M) = M$ we trivially have $M \subset K$. Considering I) it can now easily be verified that K is the required subspace. □

§9 THE GENERALISED OPEN MAPPING AND CLOSED GRAPH THEOREMS

I.9.1 **DEFINITION** [5]. A normed linear space X is defined to be an *operator range* if it is the image of a bounded linear operator defined on a Banach space.

I.9.2 *Remark*. Note that if X is an operator range we can assume without loss of generality that X is the injective bounded image of a Banach space. This follows from Theorem I.2.1 and Proposition I.4.16. We will denote the bounded injective map from a Banach space onto X by α_X and the Banach space on which α_X is defined by X_1. X_1 will be referred to as the *pre-image space* of X. We also denote α_X^{-1} by β_X. Observe that since α_X is bounded, both α_X and β_X are closed operators with β_X open. Finally we remark that X is an operator range if and only if there exists a stronger norm on X, say $\|.\|_1$, under which X is complete [5; 2.1]. We can assume the norm on X_1 to be $\|.\|_1$ since by an application of the closed graph theorem, the pre-image space of X turns out to be unique up to an isomorphism.

I.9.3 **PROPOSITION.** *Let* M *be a closed subspace of an operator range* X. *Then* M *and* X/M *are also operator ranges*

Proof. Note that since M is closed and α_X bounded $\alpha_X^{-1}M = \beta_X M$ is a closed subspace of X_1. Hence $\beta_X M$ and $X_1/\beta_X M$ are complete by Theorem I.2.1. Now let $M_1 = \beta_X M$, $(X/M)_1 = X_1/\beta_X M$, $\alpha_M = \alpha_X J_{\beta_X M}$ and $\alpha_{(X/M)} = (Q_M \alpha_X)^\wedge$. □

The following Lemma can be easily verified:

I.9.4 **LEMMA.** *Let* X *be an operator range and* Z *a normed linear space. Then*

I) *if* $T \in C(Z,X)$, $\beta_X T \in C(Z,X_1)$;

II) *if* $T \in C(X,Z)$, $T\alpha_X \in C(X_1,Z)$.

I.9.5 **THEOREM** [27] (*Generalised Open Mapping Theorem*). *Let* X *be an operator range and* Y *complete (of the second category). If* $T \in C(X,Y)$ *with* $R(T) = Y$, *then* T *is an open map.*

Proof. From the Lemma $T\alpha_X \in C(X_1,Y)$ with $R(T\alpha_X) = Y$. Hence $T\alpha_X$ is open by the open mapping theorem (cf[16; II.1.8]). But then $T = (T\alpha_X)\beta_X$ is open since it is just the composition of two open maps. □

I.9.6 **THEOREM** [27] (*Generalised Closed Graph Theorem*). *Let* X *be complete,* Y *an operator range and* $T \in C[X,Y]$. *Then* T *is a bounded operator.*

Proof. By Lemma I.9.4 $\beta_Y T \in C[X,Y_1]$ and so $\beta_Y T$ is bounded by the closed graph theorem (cf[16; II.1.9]). But then $\alpha_Y(\beta_Y T) = T$ is bounded. □

§10 THE ADJOINT OF A LINEAR OPERATOR

I.10.1 **DEFINITION.** Let $T \in L(X,Y)$. The *adjoint* T' of T is defined as follows: $D(T') = \{y' \mid y' \in Y',\ y'TJ_{D(T)}$ is continuous on $D(T)\}$. We define T' to be the operator which maps each $y' \in D(T')$ onto $y'TJ_{D(T)} \in (D(T))'$. Note that $D(T')$ is a subspace of Y' and that T' is linear.

The above is a natural generalisation by the author of the definition of the adjoint (conjugate) operator given in [16]. Considering Theorem I.5.6 we note that the two definitions are equivalent if $D(T)$ is dense in X.

I.10.2 **PROPOSITION** [16]. *Let* $T \in L(X,Y)$. *Then* T' *is a closed linear operator.*

I.10.3 **THEOREM.** *Let* $T \in L(X,Y)$. *Then*

I) $D(T') = Y'$ *if and only if* T *is continuous. If that is the case then* T' *is also continuous and* $\|T'\| = \|T\|$ [16; II.2.8].

II) *If* $T' \in L[Y',(D(T))']$ *then* T' *is also* $\sigma(Y',Y)$ *to* $\sigma((D(T))',\ D(T))$ *continuous* [20; 8.6.1].

I.10.4 **PROPOSITION** [16; II.3.7]. *Let* $T \in L(X,Y)$. *Then*

I) $\overline{R(T)}^{\perp} = R(T)^{\perp} = N(T')$;

II) $\overline{R(T)} = {}^{\perp}N(T')$.

Hence T *has dense range if and only if* T' *is injective.*

I.10.5 **PROPOSITION** [16; II.3.8]. *Let* $T \in L(X,Y)$. *Then*

I) ${}^{\perp}R(T') \supset N(T)$ *with* $N(T) = N(TJ_{D(T)}) = {}^{\perp}R(T')$ *if* $D(T')$ *is total;*

II) $\overline{R(T')} \subset N(TJ_{D(T)})^{\perp}$.

Hence if $R(T')$ *is total, then* T *is injective.*

I.10.6 **THEOREM.** *Let* $T \in L(X,Y)$ *with* $D(T)$ *dense in* X. *If* T *and* T' *each have an inverse, then* $(T^{-1})' = (T')^{-1}$.

Proof. Note that by the hypothesis $D(T)$ is dense in X and since T' is injective, $R(T) = D(T^{-1})$ is dense in Y. However, as was noted earlier, if this is the case our definition of the adjoint agrees with that given in Goldberg [16]. The result now follows trivially from [16; II.3.9]. □

I.10.7 **PROPOSITION** [16; II.3.11 *and* II.3.13]. *Let* $T \in L(X,Y)$. *Then*

I) T' *is surjective if and only if* T *has a continuous inverse;*

II) T' *has a continuous inverse if* Y *is complete and* T *surjective.*

I.10.8 **THEOREM.** *Let* X *be complete and* $T \in C(X,Y)$. *If* T' *has a continuous inverse, then* $TB_{D(T)} \supset rU_Y$ *where* $r = 1/\|(T')^{-1}\|$. *Thus* T *is an open map.*

Proof. Note that $\overline{D(T)}$ is still complete and hence regarding T as an element of $L(\overline{D(T)},Y)$ we observe that T is still closed. Since in this case our definition of the adjoint is equivalent to that in [16], the result now follows trivially from [16; II.4.3]. □

I.10.9 **PROPOSITION.** *Let* $T \in L(X,Y)$ *and* $B \in L(Y,Z)$ *such that* $D(B)$ *contains* $R(T)$ *and is dense in* Y. *Then* $(BT)'$ *is an extension of* $T'B'$ *with* $(BT)' = T'B'$ *if* B *is continuous.*

Proof. Note that since $D(B)$ is dense in Y, $(D(B))' = Y'$ and so $B' \in C(Z',Y')$. The first statement follows easily from the observation that $D((BT)') \supset D(T'B')$. Now suppose that B is continuous. Then $D(B') = Z'$ by Theorem I.10.3. Select an arbitrary $z' \in D((BT)')$. Then $z'BTJ_{D(TB)} = z'BTJ_{D(T)}$ is continuous by definition.

But $z' \in D(B')$ and so $z'BTJ_{D(T)} = B'z'TJ_{D(T)}$. We conclude that $B'z' \in D(T')$ and hence that $D((BT)') \subset D(T'B')$. But then $D((BT)') = D(T'B')$ and so $T'B' = (BT)'$. □

I.10.10 *Remark.* Let M be a subspace of Y.

I) Then we have up to an isometry that $(J_M^Y)' = Q_{M^\perp}^{Y'}$ and if M is closed $(Q_M^Y)' = J_{M^\perp}^{Y'}$. (This follows from a careful consideration of Theorem I.5.10).

II) For $T \in L(X,Y)$ and K a subspace of $D(T)$, $(TJ_K^X)' = (TJ_{D(T)}J_K^{D(T)})'$ is an extension of $Q_{K^\perp}^{(D(T))'}T'$ and if M is closed, $(Q_M^Y\ T)' = T'J_{M^\perp}^{Y'}$. (This follows from I) and Proposition I.10.9.)

I.10.11 **PROPOSITION** [8; 1.7]. *Let* $T \in L(X,Y)$ *and let* M *be a relatively closed subspace of* $D(T)$. *If* $\mathrm{cod}_{D(T)}M < \infty$, *then* $(TJ_M^X)' = Q_{M^\perp}^{(D(T))'}T'$.

Proof. Without loss of generality let $D(T) = X$. Considering Remark I.10.10 we only need to prove that $D(Q_{M^\perp}^{X'}T') \supset D((TJ_M^X)')$. Select $y' \in D((TJ_M^X)')$ arbitrarily and let F be a finite dimensional subspace of X so that $X = M \oplus F$. Since F is closed by Proposition I.3.2, there exists a bounded projection P from X onto M by Proposition I.8.4. Now let

$$y'T = y'TJ_M^X\,P + y'TJ_F^X(I-P). \tag{1}$$

We already know that $y'TJ_M^X$ is continuous and hence considering $y'TJ_F^X$ we observe that $G(y'TJ_F^X)$ and $D(y'TJ_F^X) = F$ are finite dimensional. Consequently, from Propositions I.3.1, I.3.2 and the closed graph theorem we conclude that $y'TJ_F^X$ is continuous. But then $y'T$ is continuous by (1) and so $y' \in D(T') = D(Q_{M^\perp}^{X'}T')$. Hence the result follows. □

I.10.12 **PROPOSITION** [16; II.4.7]. *Suppose* $T \in L(X,Y)$ *with* $N(T)$ *a closed subspace of* X. *Then* $D(T') = D((\hat{T})')$.

§11 SOME OPERATOR IDEALS

I.11.1 **DEFINITION.** Let $T \in L(X,Y)$. We define T to be a *compact operator* if $\overline{TB_{D(T)}}$ is compact and *precompact* if $\overline{TB_{D(T)}}$ is totally bounded. The classes of compact and precompact operators in $L(X,Y)$ will be denoted by $K(X,Y)$ and $PK(X,Y)$ respectively. We will denote $K(X,Y) \cap L[X,Y]$ and $PK(X,Y) \cap L[X,Y]$ by $K[X,Y]$ and $PK[X,Y]$ respectively.

I.11.2 **DEFINITION.** An absolutely convex subset V of X is called a *disk* if for each $\sigma(X,X')$ neighbourhood W of zero there exists λ such that $\lambda W \supset V$. Note that the function $p(x) = \inf\{q > 0 | x \in qV\}$ for each $x \in \mathit{span}V$ defines a norm on $\mathit{span}V$ [20; 8.3]. We shall denote the subspace $\mathit{span}V$ normed by this norm by X_V. V is called a *Banach disk* if X_V is a Banach space. Now let $T \in L(X,Y)$. Then T is called a *nuclear operator* if we can find a bounded sequence $\{x'_n\} \subset X'$, a Banach disk V in Y, a bounded sequence $\{y_n\}$ in Y_V and a sequence $\{\lambda_n\} \in \ell_1$ (that is $\sum_{n=1}^{\infty} |\lambda_n| < \infty$) such that for every $x \in D(T)$

$$Tx = \sum_{n=1}^{\infty} \lambda_n x'_n(x) y_n .$$

Here the series is understood to converge in Y. (Nuclear operators are treated in more generality in [20], however for our purposes the above definition wil suffice.) We denote the class of all nuclear operators in $L(X,Y)$ by $N(X,Y)$ with $L[X,Y] \cap N(X,Y)$ being denoted by $N[X,Y]$.

I.11.3 **DEFINITION.** Let A and E denote two classes of operators. We say that $A(X,Y)$ *is a left (right) ideal with respect to the class* E if for every $T \in A(X,Y)$, $ST \in A(X,Z)$ $(TS \in A(Z,Y))$ whenever $S \in E(Y,Z)$ $(S \in E(Z,X))$ where S and T denote linear operators.

We say that $A(X,Y)$ *is an ideal with respect to the class* E if it is both a left and a right ideal with respect to E.

We note the following results:

I.11.4 *Remarks.* [16; III.1.4]

I) Every bounded finite rank operator is compact. (Follows from Proposition I.3.3.)

II) Every precompact operator is bounded. (Follows from the fact that a totally bounded set is bounded.)

III) An operator is compact (precompact) if it maps bounded sequences onto sequences which have a convergent (Cauchy) subsequence. Hence if $T \in L(X,Y)$ then T is precompact if and only if $J_Y T$ is compact.

I.11.5 **PROPOSITION** [16; III.1.11]. *A continuous linear operator is precompact if and only if its adjoint is compact.*

I.11.6 **PROPOSITION** [16; III.1.12]. *If the range of a precompact operator is complete, then it is finite-dimensional.*

I.11.7 **THEOREM.** *$N[X,Y]$, $K[X,Y]$ and $PK[X,Y]$ are subspaces of $B[X,Y]$ with $N[X,Y] \subset K[X,Y] \subset PK[X,Y]$.*

Proof. Immediate from [16; III.2.4] and [20; 17.3.1 and 17.3.4]. □

I.11.8 **THEOREM.** *The classes $N[X,Y]$, $K[X,Y]$ and $PK[X,Y]$ are operator ideals with respect to the bounded operators.*

Proof. Consider [16; III.2.5] and [20; 17.3.1]. □

I.11.9 **THEOREM** ([25]; cf.[16; III.2.3]). *Let* $T \in B(X,Y)$. *Then* T *is precompact if and only if for every* $\epsilon > 0$ *there exists a finite codimensional subspace* N *of* $D(T)$ *such that* $\|TJ_N\| \leq \epsilon$.

I.11.10 **THEOREM** [16; III.1.9]. *Let* $T \in L(X,Y)$. *Suppose* T *does not have a continuous inverse when restricted to any closed finite codimensional subspace of* X. *Then for any* $\epsilon > 0$ *there exists an infinite dimensional subspace* M *of* $D(T)$ *so that* TJ_M *is precompact with* $\|TJ_M\| < \epsilon$.

CHAPTER II

BASIC PROPERTIES OF THE FUNCTIONS Γ', Δ' AND Γ_0'

This chapter is devoted mainly to the functions Γ', Δ' and Γ_0'. We will however prove some results concerning Γ, Δ and Γ_0 which will be needed in subsequent chapters.

§1 INVARIANCE PROPERTIES

II.1.1 **DEFINITION.** For an arbitrary normed linear space X we denote the class of all *closed infinite codimensional subspaces of* X by $\mathcal{I}_c(X)$, the class of *finite dimensional* subspaces by $\mathcal{F}(X)$, the class of *infinite dimensional* subspaces by $\mathcal{I}(X)$ and the class of *finite codimensional subspaces of* X by $\mathcal{F}_c(X)$. Now for any $T \in L(X,Y)$ let $\Gamma(T) = \Delta(T) = \Gamma_0(T) = 0$ if $D(T) \in \mathcal{F}(X)$. If $D(T) \in \mathcal{I}(X)$ let

$$\Gamma(T) = \inf_{M \in \mathcal{I}(D(T))} \|TJ_M\|$$

$$\Delta(T) = \sup_{M \in \mathcal{I}(D(T))} \inf_{N \in \mathcal{I}(M)} \|TJ_N\|$$

$$\text{and} \quad \Gamma_0(T) = \inf_{E \in \mathcal{F}_c(D(T))} \|TJ_E\|.$$

By analogy if Y is finite dimensional let $\Gamma'(T) = \Delta'(T) = \Gamma_0'(T) = 0$. If not then let

$$\Gamma'(T) = \inf_{M \in \mathcal{I}_c(\tilde{Y})} \|Q_M J_Y T\|$$

$$\Delta'(T) = \sup_{M \in \mathcal{I}_c(Y)} \inf_{\substack{N \in \mathcal{I}_c(\tilde{Y}) \\ N \supset M}} \|Q_N J_Y T\|$$

$$\text{and} \quad \Gamma_0'(T) = \inf_{F \in \mathcal{F}(Y)} \|Q_F T\|.$$

II.1.2 **LEMMA.** *Let* $T \in L(X,Y)$ *with* Y *infinite dimensional. Then*

$$\inf_{\substack{N \in \mathcal{S}_c(\tilde{Y}) \\ N \supset M}} \|Q_N J_Y T\| = \Gamma'(Q_M T) \quad \text{for any} \quad M \in \mathcal{S}_c(Y)$$

and hence $\Delta'(T) = \sup_{M \in \mathcal{S}_c(Y)} \Gamma'(Q_M T)$.

Proof. Select $M \in \mathcal{S}_c(Y)$ and let $N \in \mathcal{S}_c(\tilde{Y})$ be arbitrary. Since N is closed by definition, we conclude that $M \subset N$ if and only if $\overline{M}^{\tilde{Y}} \subset N$. Considering Proposition I.8.6 we see that the elements of $\mathcal{S}_c(\tilde{Y})$ which contain M can be identified with elements of $\mathcal{S}_c(\tilde{Y}/\overline{M}^{\tilde{Y}})$. Observe that since M is dense in $\overline{M}^{\tilde{Y}}$, $\|Q_M^Y y\| = \|Q_{\overline{M}}^{\tilde{Y}} J_Y y\|$ for each $y \in Y$. Hence $Y/M \equiv J_Y Y + \overline{M}^{\tilde{Y}} \in \tilde{Y}/\overline{M}^{\tilde{Y}}$. But by the continuity of $Q_{\overline{M}}^{\tilde{Y}}$, $J_Y Y + \overline{M}^{\tilde{Y}}$ is dense in $\tilde{Y}/\overline{M}^{\tilde{Y}}$ since $J_Y Y$ is dense in $\tilde{Y}$ (Remark I.4.4). As $\tilde{Y}/\overline{M}^{\tilde{Y}}$ is complete by Theorem I.2.1, we deduce that $(Y/M)^{\sim} \equiv \tilde{Y}/\overline{M}^{\tilde{Y}}$. Therefore again considering Proposition I.8.6 and the observation made earlier in the proof we see that

$$\begin{aligned}\inf_{\substack{N \in \mathcal{S}_c(\tilde{Y}) \\ N \supset M}} \|Q_N J_Y T\| &= \inf_{K \in \mathcal{S}_c(\tilde{Y}/\overline{M})} \|Q_K^{\tilde{Y}/\overline{M}}\, Q_{\overline{M}}^{\tilde{Y}}\, J_Y T\| = \inf_{K \in \mathcal{S}_c(Y/M)^{\sim}} \|Q_K^{(Y/M)^{\sim}} J_{Y/M}\, Q_M^Y\, T\| \\ &= \Gamma'(Q_M T).\end{aligned}$$

Taking the supremum over all $M \in \mathcal{S}_c(Y)$ we obtain the result. □

II.1.3 **PROPOSITION.** *Let* $T \in L(X,Y)$ *and* $F \in \mathcal{F}(Y)$. *Then*

I) $\Gamma'(T) = \Gamma'(Q_F T)$;

II) $\Delta'(T) = \Delta'(Q_F T)$;

III) $\Gamma_0'(T) = \Gamma_0'(Q_F T)$.

Proof. Without loss of generality let Y be infinite dimensional.

I) and II). Let $N \in \mathcal{I}_c(Y)$ be arbitrary. Since $\dim F < \infty$, $N + F \in \mathcal{I}_c(Y)$ (Proposition I.3.4) and so by the lemma

$$\Gamma'(Q_{N+F}T) = \inf_{\substack{M\in\mathcal{I}_c(\tilde{Y})\\ M \supset N+F}} \|Q_M J_Y T\| \geq \inf_{\substack{M\in\mathcal{I}_c(\tilde{Y})\\ M \supset N}} \|Q_M J_Y T\| = \Gamma'(Q_N T) \tag{1}$$

Therefore as equality holds if $\Gamma'(Q_N T) = \infty$ suppose $\Gamma'(Q_N T) < \infty$. Select $K \in \mathcal{I}_c(\tilde{Y})$, $K \supset N$ such that

$$\|Q_K J_Y T\| \leq \Gamma'(Q_N T) + \epsilon \tag{2}$$

for some arbitrarily chosen $\epsilon > 0$. Now since $K + F \in \mathcal{I}_c(\tilde{Y})$ with $K \subset K + F$ and $N + F \subset K + F$ we have by (1) and (2) that

$$\Gamma'(Q_{N+F}T) \leq \|Q_{K+F}J_Y T\| \leq \|Q_K J_Y T\| \leq \Gamma'(Q_N T) + \epsilon \leq \Gamma'(Q_{N+F}T) + \epsilon.$$

As $\epsilon > 0$ was arbitrarily chosen we conclude that

$$\Gamma'(Q_{N+F}T) = \Gamma'(Q_N T). \tag{3}$$

Letting $N = \{0\}$ we obtain I). Alternatively noting from Proposition I.8.6 that $\mathcal{I}_c(Y/F)$ can be identified with elements of $\mathcal{I}_c(Y)$ which contain F we obtain II) by taking the supremum over all $N \in \mathcal{I}_c(Y)$. (Observe that $\{N\colon N \in \mathcal{I}_c(Y),\ N \supset F\} = \{N + F\colon N \in \mathcal{I}_c(Y)\}$ by Proposition I.3.4.)

III) From Propositions I.3.2 and I.8.7 we observe that $\mathcal{F}(Y/F)$ can be identified with elements of $\mathcal{F}(Y)$ which contain F and so, as with Γ', we have

$$\Gamma'_0(Q_F T) = \inf_{\substack{K \in \mathcal{F}(Y)\\ K \supset F}} \|Q_K T\| \geq \inf_{K \in \mathcal{F}(Y)} \|Q_K T\| = \Gamma'_0(T). \tag{4}$$

As before equality holds if $\Gamma'_0(T) = \infty$ and so let $\Gamma'_0(T) < \infty$. For an arbitrary $\epsilon > 0$ we may select $M \in \mathcal{F}(Y)$ so that

$$\|Q_M T\| \leq \Gamma'_0(T) + \epsilon \tag{5}$$

But now $M + F \in \mathcal{F}(Y)$ with $F \subset M + F$ and so from (4) and (5) we have

$$\Gamma'_0(Q_F Y) \leq \|Q_{M+F}T\| \leq \|Q_M T\| \leq \Gamma'_0(T) + \epsilon \leq \Gamma'_0(Q_F T) + \epsilon.$$

Since $\epsilon < 0$ was arbitrary we conclude that $\Gamma'_0(Q_F T) = \Gamma'_0(T)$. □

II.1.4 **PROPOSITION** [6]. *Let* $T \in L(X,Y)$ *and* $E \in \mathcal{F}_c(D(T))$. *Then*

I) $\Gamma(T) = \Gamma(TJ_E)$;

II) $\Delta(T) = \Delta(TJ_E)$;

III) $\Gamma_0(T) = \Gamma_0(TJ_E)$.

Proof. Without loss of generality let $\dim D(T) = \infty$.

I) By the definition

$$(1) \qquad \Gamma(TJ_E) = \inf_{M\in\mathcal{I}(E)} \|TJ_M\| \geq \inf_{M\in\mathcal{I}(D(T))} \|TJ_M\| = \Gamma(T).$$

Hence if $\Gamma(T) = \infty$ we have equality. Let $\Gamma(T) < \infty$ and select $K \in \mathcal{I}(D(T))$ so that for some arbitrarily chosen $\epsilon > 0$,

$$\|TJ_K\| \leq \Gamma(T) + \epsilon.$$

But since $E \in \mathcal{F}_c(D(T))$, $E \cap K \in \mathcal{I}(E)$ whence

$$\Gamma(TJ_E) \leq \|TJ_{E\cap K}\| \leq \|TJ_K\| \leq \Gamma(T) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary we conclude from the above and from (1) that $\Gamma(TJ_E) = \Gamma(T)$.

II) Notice that $\Delta(T) = \sup\limits_{M\in\mathcal{I}(D(T))} \Gamma(TJ_M)$. Now since $M \cap E \in \mathcal{F}_c(M)$ for any $M \in \mathcal{I}(D(T))$, we conclude from I) that

$$\Delta(T) = \sup_{M\in\mathcal{I}(D(T))} \Gamma(TJ_M) = \sup_{M\in\mathcal{I}(D(T))} \Gamma(TJ_{M\cap E}) = \sup_{K\in\mathcal{I}(E)} \Gamma(TJ_K) = \Delta(TJ_E).$$

III) From the definition we conclude that since $\mathcal{F}_c(E) \subset \mathcal{F}_c(D(T))$, we have

$$(2) \qquad \Gamma_0(TJ_E) = \inf_{K \in \mathcal{F}_c(E)} \|TJ_K\| \geq \inf_{K \in \mathcal{F}_c(D(T))} \|TJ_K\| = \Gamma_0(T).$$

If $\Gamma_0(T) = \infty$ equality holds and so let $\Gamma_0(T) < \infty$. Select $M \in \mathcal{F}_c(D(T))$ such that $\|TJ_M\| \leq \Gamma_0(T) + \epsilon$ for some arbitrarily chosen $\epsilon > 0$. Note that $M \cap E \in \mathcal{F}_c(E)$ and so by (2) we have

$$\Gamma_0(T) \leq \Gamma_0(TJ_E) \leq \|TJ_{M\cap E}\| \leq \|TJ_M\| \leq \Gamma_0(T) + \epsilon.$$

Since ϵ was arbitrary $\Gamma_0(T) = \Gamma_0(TJ_E)$. □

II.1.5 **PROPOSITION.** *Let* $T \in L(X,Y)$ *such that* $N(T)$ *is closed and let* M *be a closed subspace of* Y. *Then*

I) $\Gamma'(Q_M T) = \Gamma'(Q_M \hat{T})$;

II) $\Delta'(Q_M T) = \Delta'(Q_M \hat{T})$;

III) $\Gamma_0'(Q_M T) = \Gamma_0'(Q_M \hat{T})$.

Proof. We show that $\|Q_K J_Y T\| = \|Q_K J_Y \hat{T}\|$ if either $Q_K J_Y T$ or $Q_K J_Y \hat{T}$ is continuous where K is an arbitrary closed subspace of $\tilde{Y}$ from which the result follows trivially. If $Q_K J_Y T$ is continuous

$$\|Q_K J_Y \hat{T}\, Q_{N(T)} x\| = \|Q_K J_Y\, Tz\| \leq \|Q_K J_Y T\|\, \|z\| \quad \text{for any} \quad z \in x + N(T).$$

Hence $\|Q_K J_Y \hat{T}\, Q_{N(T)} x\| \leq \|Q_K J_Y T\| \inf\limits_{z \in x+N(T)} \|z\| = \|Q_K J_Y T\|\, \|Q_{N(T)} x\|$ and so $\|Q_K J_Y \hat{T}\| \leq \|Q_K J_Y T\|$. Conversely if $Q_K J_Y \hat{T}$ is continuous, then

$$\|Q_K J_Y\, Tx\| = \|Q_K J_Y\, \hat{T}\, Q_{N(T)} x\| \leq \|Q_K J_Y\, \hat{T}\|\, \|Q_{N(T)} x\| \leq \|Q_K J_Y\, \hat{T}\|\, \|x\|.$$

Thus $\|Q_K J_Y T\| \leq \|Q_K J_Y \hat{T}\|$ and therefore if either $Q_K J_Y T$ or $Q_K J_Y \hat{T}$ is continuous, then $\|Q_K J_Y T\| = \|Q_K J_Y \hat{T}\|$. □

Lemma II.1.2 is a generalisation by the author with Propositions II.1.3 and II.1.5 also due to the author. Proposition II.1.4 is due to R.W.Cross [6].

§2 INEQUALITIES

II.2.1 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then*

I) $\Gamma'(T) \leq \Gamma'(Q_M T)$, $\Gamma_0'(Q_M T) \leq \Gamma_0'(T)$ *and* $\Delta'(Q_M T) \leq \Delta'(T)$ *for* $M \in \mathcal{I}_c(Y)$;

II) $\Gamma'(T) \leq \Delta'(T) \leq \Gamma_0'(T)$ $(\leq \|T\|$ *if* T *is continuous*$)$.

Proof. I) Let $M \in \mathcal{I}_c(Y)$. From Lemma II.1.2 we conclude that

$$\Gamma'(T) = \inf_{K \in \mathcal{I}_c(\tilde{Y})} \|Q_K J_Y\, T\| \leq \inf_{\substack{K \in \mathcal{I}_c(\tilde{Y}) \\ K \supset M}} \|Q_K J_Y\, T\| = \Gamma'(Q_M T).$$

For $\Delta'(T)$ we have that

$$\Delta'(T) = \sup_{K\in\mathcal{J}_c(Y)} \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset K}} \|Q_N J_Y\, T\| \geq \sup_{\substack{K\in\mathcal{J}_c(Y)\\ K\supset M}} \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset K}} \|Q_N J_Y\, T\| = \Delta'(Q_M T)$$

Now for any $F \in \mathcal{F}(Y)$ we have that $M + F \in \mathcal{J}_c(Y)$ by Proposition I.3.4 and so since $F \subset M + F$, $\|Q_{M+F}T\| \leq \|Q_F T\|$ if the norms exist. Hence

$$\Gamma_0'(Q_M T) = \inf_{F\in\mathcal{F}(Y)} \|Q_{M+F}T\| \leq \inf_{F\in\mathcal{F}(Y)} \|Q_F T\| = \Gamma_0'(T).$$

II) Since $\Gamma'(T) = \Delta'(T) = \Gamma_0'(T) = 0$ if Y is finite dimensional, we can assume Y to be infinite dimensional. From I) $\Gamma'(T) \leq \Gamma'(Q_M T)$ for any $M \in \mathcal{J}_c(Y)$ and so $\Gamma'(T) \leq \sup_{K\in\mathcal{J}_c(Y)} \Gamma'(Q_K T) = \Delta'(T)$. Observe that $\mathcal{F}(Y) \in \mathcal{J}_c(\tilde{Y})$ (Proposition I.3.2) whence

$$\Gamma'(T) = \inf_{K\in\mathcal{J}_c(\tilde{Y})} \|Q_K J_Y\, T\| \leq \inf_{F\in\mathcal{F}(Y)} \|Q_F J_Y\, T\| = \inf_{F\in\mathcal{F}(Y)} \|Q_F T\| = \Gamma_0'(T).$$

Hence for any $N \in \mathcal{J}_c(Y)$ we have that $\Gamma'(Q_N T) \leq \Gamma_0'(Q_N T) \leq \Gamma_0'(T)$ and so $\Delta'(T) = \sup_{N\in\mathcal{J}_c(Y)} \Gamma'(Q_N T) \leq \Gamma_0'(T)$. □

II.2.2 **COROLLARY.** *Let* $T \in L(X,Y)$ *with* $\dim Y = \infty$. *Then* $\Gamma'(T) = \inf_{N\in\mathcal{J}_c(\tilde{Y})} \Gamma_0'(Q_N J_Y T)$ *and so* $\Delta'(T) = \sup_{K\in\mathcal{J}_c(Y)} \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset K}} \Gamma_0'(Q_N J_Y T)$.

Proof. We first consider $\Gamma'(T)$. Observe that for any $N \in \mathcal{J}_c(\tilde{Y})$, $\|Q_N J_Y T\| \geq \Gamma_0'(Q_N J_Y T)$ if the norm exists and so

(1) $$\Gamma'(T) = \inf_{N\in\mathcal{J}_c(\tilde{Y})} \|Q_N J_Y T\| \geq \inf_{N\in\mathcal{J}_c(\tilde{Y})} \Gamma_0'(Q_N J_Y T).$$

Now suppose $\inf_{N\in\mathcal{J}_c(\tilde{Y})} \Gamma_0'(Q_N J_Y T) < \infty$ since if this was not the case equality follows trivially. Hence let $\epsilon > 0$ be arbitrary and choose $K \in \mathcal{J}_c(\tilde{Y})$ such that $\Gamma_0'(Q_K J_Y T) \leq \inf_{N\in\mathcal{J}_c(\tilde{Y})} \Gamma_0'(Q_N J_Y T) + \epsilon$.

From (1) and Proposition II.2.1 we conclude that

$$\Gamma'(T) + \epsilon \geq \inf_{N\in\mathcal{J}_c(\tilde{Y})} \Gamma_0'(Q_N J_Y T) + \epsilon \geq \Gamma_0'(Q_K J_Y T) \geq \Gamma'(Q_K J_Y T) \geq \Gamma'(J_Y T) = \Gamma'(T).$$

Since $\epsilon > 0$ was chosen arbitrarily, we have $\Gamma'(T) = \inf_{N\in\mathcal{J}_c(\tilde{Y})} \Gamma_0'(Q_N J_Y T)$. The second assertion now follows trivially from Lemma II.1.2. $\square$

II.2.3 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then* $\Gamma'(J_Y T) = \Gamma'(T)$, $\Delta'(J_Y T) \geq \Delta'(T)$ *and* $\Gamma_0'(J_Y T) \leq \Gamma_0'(T)$.

Proof. Since $\mathcal{F}(\tilde{Y}) \supset \mathcal{F}(Y)$ we have $\Gamma_0'(J_Y T) \leq \Gamma_0'(T)$ with $\Gamma'(T) = \Gamma'(J_Y T)$ by definition. Considering Δ' we note from Proposition I.5.5 that for any subspace M of $\tilde{Y}$, $\bar{M} \in \mathcal{J}_c(\tilde{Y})$ if and only if $\bar{M} \cap Y \in \mathcal{J}_c(Y)$ and so since $\inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset M}} \|Q_N J_Y T\| \geq \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset M\cap Y}} \|Q_N J_Y T\|$ for any $M \in \mathcal{J}_c(\tilde{Y})$, we have that

$$\begin{aligned}\Delta'(T) = \sup_{K\in\mathcal{J}_c(Y)} \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset K}} \|Q_N J_Y T\| &= \sup_{M\in\mathcal{J}_c(\tilde{Y})} \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset M\cap Y}} \|Q_N J_Y T\| \\ &\leq \sup_{M\in\mathcal{J}_c(\tilde{Y})} \inf_{\substack{N\in\mathcal{J}_c(\tilde{Y})\\ N\supset M}} \|Q_N J_Y T\| \\ &= \Delta'(J_Y T).\end{aligned}$$

$\square$

II.2.4 **THEOREM.** *Let* $T,B \in L(X,Y)$. *Then*

I) $\Gamma'(T + B) \leq \Delta'(J_Y T) + \Gamma'(B)$;

II) $\Delta'(T + B) \leq \Delta'(J_Y T) + \Delta'(B)$;

III) $\Gamma_0'(T + B) \leq \Gamma_0'(T) + \Gamma_0'(B)$.

Proof. I) and II). Suppose $T, B \in L(X,Y)$ and suppose $\Delta'(J_Y T) < \infty$ and $\Gamma'(B) < \infty$ since on considering Proposition II.2.1 we note that the result is trivial otherwise.

Without loss of generality let $\dim Y = \infty$. Fix $M \in \mathcal{I}_c(Y)$ and let N be an arbitrary element of $\mathcal{I}_c(\tilde{Y})$ such that $N \supset M$. Now let $\epsilon > 0$ be arbitrary and choose $V \in \mathcal{I}_c(\tilde{Y})$ such that $V \supset N$ and

$$\|Q_V J_Y T\| \leq \inf_{\substack{U \in \mathcal{I}_c(\tilde{Y}) \\ U \supset N}} \|Q_U J_Y T\| + \epsilon \leq \Delta'(J_Y T) + \epsilon.$$

Therefore as $\|Q_V J_Y B\| \leq \|Q_N J_Y B\|$ (if the norms exist), we obtain

$$\begin{aligned} \inf_{\substack{U \in \mathcal{I}_c(\tilde{Y}) \\ U \supset N}} \|Q_U J_Y (T+B)\| \leq \|Q_V J_Y (T+B)\| &\leq \|Q_V J_Y T\| + \|Q_V J_Y B\| \\ &\leq \inf_{\substack{U \in \mathcal{I}_c(\tilde{Y}) \\ U \supset N}} \|Q_U J_Y T\| + \epsilon + \|Q_N J_Y B\| \\ &\leq \Delta'(J_Y T) + \epsilon + \|Q_N J_Y B\| \end{aligned}$$

and so since $\epsilon > 0$ was arbitrarily chosen we have

(1)
$$\begin{aligned} \inf_{\substack{N \in \mathcal{I}_c(\tilde{Y}) \\ N \supset M}} \|Q_N J_Y (T+B)\| &= \inf_{\substack{N \in \mathcal{I}_c(\tilde{Y}) \\ N \supset M}} \inf_{\substack{U \in \mathcal{I}_c(\tilde{Y}) \\ U \supset N}} \|Q_U J_Y (T+B)\| \\ &\leq \Delta'(J_Y T) + \inf_{\substack{N \in \mathcal{I}_c(\tilde{Y}) \\ N \supset M}} \|Q_N J_Y B\|. \end{aligned}$$

If we let $M = \{0\}$ we obtain

$$\Gamma'(T + B) \leq \Delta'(J_Y T) + \Gamma'(B).$$

Considering (1) and taking the supremum over all $M \in \mathcal{I}_c(Y)$ we get $\Delta'(T + B) \leq \Delta'(J_Y T) + \Delta'(B)$.

III) Finally for Γ_0' suppose $\Gamma_0'(T) < \infty$ and $\Gamma_0'(B) < \infty$ since the result is trivial otherwise. Now let $\epsilon > 0$ be arbitrary and choose $F, K \in \mathcal{F}(Y)$ so that $\|Q_F T\| \leq \Gamma_0'(T) + \epsilon/2$ and $\|Q_K B\| \leq \Gamma_0'(B) + \epsilon/2$.

However $F + K \in \mathcal{F}(Y)$ with $\|Q_{F+K}T\| \leq \|Q_F T\|$ and $\|Q_{F+K}B\| \leq \|Q_K B\|$. Consequently

$$\Gamma_0'(T + B) = \inf_{N\in\mathcal{F}(Y)} \|Q_N(T + B)\| \leq \|Q_{F+K}(T + B)\| \leq \|Q_{F+K}T\| + \|Q_{F+K}B\|$$
$$\leq \Gamma_0'(T) + \Gamma_0'(B) + \epsilon.$$

Thus $\Gamma_0'(T + B) \leq \Gamma_0'(T) + \Gamma_0'(B)$ since $\epsilon > 0$ was chosen arbitrarily. □

II.2.5 **PROPOSITION.** *Let* $T,B \in L(X,Y)$ *with* $\operatorname{cod}_{\tilde{Y}} Y < \infty$. *Then* $\Gamma'(T + B) \leq \Delta'(T) + \Gamma'(B)$ *and* $\Delta'(T + B) \leq \Delta'(T) + \Delta'(B)$.

Proof. The proof is similar to that for Theorem II.2.4 except that we now select $V \in \mathcal{J}_c(\tilde{Y})$ such that $V \supset N \cap Y$ and $\|Q_V J_Y T\| \leq \inf_{\substack{U\in\mathcal{J}_c(\tilde{Y})\\ U\supset N\cap Y}} \|Q_U J_Y T\| + \epsilon \leq \Delta'(T) + \epsilon$. However since $\operatorname{cod}_{\tilde{Y}} Y < \infty$ there exists $F \in \mathcal{F}(\tilde{Y})$ such that $V + F \supset N$. We still have $V + F \in \mathcal{J}_c(\tilde{Y})$ by the finite dimensionality of F. The fact that $N \subset V + F$ and $V \subset V + F$ implies that $\|Q_{V+F}J_Y B\| \leq \|Q_N J_Y B\|$ (if the norms exist) and that $\|Q_{V+F}J_Y T\| \leq \|Q_V J_Y T\| \leq \Delta'(T) + \epsilon$. The rest of the proof is the same except that we now use $V + F$ instead of V. □

II.2.6 **THEOREM** [6]. *Let* $T,B \in L(X,Y)$. *Then*

I) $\Gamma(T + B) \leq \Gamma(T) + \Delta(B)$ *if* $D(T + B) \in \mathcal{F}_c(D(T))$;

II) $\Delta(T + B) \leq \Delta(T) + \Delta(B)$;

III) $\Gamma_0(T + B) \leq \Gamma_0(T) + \Gamma_0(B)$.

Proof. I) Suppose $D(T + B) \in \mathcal{F}_c(D(T))$. Without loss of generality let $\Gamma(T) < \infty$, $\Delta(B) < \infty$ and $\dim D(T + B) = \infty$ since the assertion is trivial otherwise. Set $T_1 = T|_{D(T + B)}$, the restriction in $L(X,Y)$ of T to $D(T + B)$, and select $N \in \mathcal{J}(D(T + B))$ so that for some arbitrary $\epsilon > 0$,

(1) $$\|T_1 J_N\| \leq \Gamma(T_1) + \epsilon/2.$$

Now choose $V \in \mathcal{J}(N)$ such that

$$(2) \qquad \|BJ_V\| \leq \Gamma(BJ_N) + \epsilon/2 \leq \sup_{K \in \mathcal{J}(D(B))} \Gamma(BJ_K) + \epsilon/2 = \Delta(B) + \epsilon/2.$$

(Note that $\mathcal{J}(D(T + B)) \subset \mathcal{J}(D(B))$ as $D(T + B) \subset D(B)$). From (1) and (2) we now conclude that

$$\begin{aligned}\Gamma(T + B) \leq \|(T + B)J_V\| \leq \|T_1 J_V\| + \|BJ_V\| &\leq \|T_1 J_N\| + \Delta(B) + \epsilon/2\\ &\leq \Gamma(T_1) + \Delta(B) + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary and since $\Gamma(T) = \Gamma(T_1)$ by Proposition II.1.4, we have $\Gamma(T + B) \leq \Gamma(T) + \Delta(B)$.

II) Without loss of generality let $dim\ D(T + B) = \infty$ and let $N \in \mathcal{J}(D(T + B))$ be arbitrary. By I) we then have

$$\begin{aligned}\Gamma((T + B)J_N) \leq \Gamma(TJ_N) + \Delta(BJ_N) &\leq \Gamma(TJ_N) + \sup_{M \in \mathcal{J}(N)} \Gamma(BJ_M) \cdot\\ &\leq \sup_{K \in \mathcal{J}(D(T))} \Gamma(TJ_K) + \sup_{M \in \mathcal{J}(D(B))} \Gamma(BJ_M)\\ &\leq \Delta(T) + \Delta(B).\end{aligned}$$

Taking the supremum over all $N \in \mathcal{J}(D(T + B))$ we obtain $\Delta(T + B) \leq \Delta(T) + \Delta(B)$.

III) Let $\Gamma_0(T) < \infty$ and $\Gamma_0(B) < \infty$ since the assertion is trivial otherwise. Hence select $E_1 \in \mathcal{F}_c(D(T))$ and $E_2 \in \mathcal{F}_c(D(B))$ such that

$$\|TJ_{E_1}\| \leq \Gamma_0(T) + \epsilon/2 \quad \text{and} \quad \|BJ_{E_2}\| \leq \Gamma_0(B) + \epsilon/2$$

for some arbitrary $\epsilon > 0$. Notice that $E = E_1 \cap E_2 \in \mathcal{F}_c(D(T) \cap D(B)) = \mathcal{F}_c(D(T + B))$ and so

$$\Gamma_0(T + B) \leq \|(T + B)J_E\| \leq \|TJ_E\| + \|BJ_E\| \leq \|TJ_{E_1}\| + \|BJ_{E_2}\| \leq \Gamma_0(T) + \Gamma_0(B) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary we conclude that III) holds. □

The following Lemma as well as Theorem II.2.9, proved in the classical setting of Banach spaces and bounded operators in [38], was generalised with the assistance of Dr. R.W. Cross.

II.2.7 **LEMMA.** *Let* N *be a subspace of* Y *and* $M \in \mathcal{J}_c(\tilde{Y})$ *such that* $\mathrm{cod}_Y \bar{N}^Y < \mathrm{cod}_{\tilde{Y}} M = \infty$. *Then for any* $\epsilon > 0$ *there exists* $0 \neq n \in N$ *such that* $\|Q_M J_Y n\| \geq (1 - \epsilon)\, \|n\|$.

Proof. Suppose N is a subspace of Y and $M \in \mathcal{J}_c(\tilde{Y})$ such that $\mathrm{cod}_Y \bar{N}^Y < \mathrm{cod}_{\tilde{Y}} M = \infty$. By Theorem I.5.10

$$\dim N^{\perp} = \dim(Y/\bar{N})' = \dim Y/\bar{N} = \mathrm{cod}_Y \bar{N} < \mathrm{cod}_{\tilde{Y}} M = \dim M^{\perp \tilde{Y}}$$

and so by Theorem I.1.10 there exists $f \in M^{\perp \tilde{Y}}$ such that $f \neq 0$ and $\|f\| = d(f, N^{\perp}) = \|Q_{N^{\perp}} f\|$. But $Y'/N^{\perp} \equiv N'$ by Theorem I.5.10 and so $\|f\| = \|Q_{N^{\perp}} f\| = \sup\limits_{0 \neq n \in N} \dfrac{|f(n)|}{\|n\|}$. Hence there exists $n \in N$ such that $n \neq 0$ and $|f(n)| \geq \|f\|(1 - \epsilon)\, \|n\|$. However as $f \in M^{\perp \tilde{Y}}$ we have that

$$\|f\|(1 - \epsilon)\|n\| \leq |f(n)| = \inf_{m \in M} |f(n - m)| \leq \|f\| \inf_{m \in M} \|n - m\| \leq \|f\|\, \|Q_M J_Y n\|.$$

Hence $(1 - \epsilon)\, \|n\| \leq \|Q_M J_Y n\|$. □

II.2.8 **DEFINITION.** Let $T \in L(X, Y)$. We define $\gamma(T)$ as follows:

$$\gamma(T) = \sup\{\lambda \in \mathbb{R} : \|Tx\| \geq \lambda . d(x, N(T)) \text{ for all } x \in D(T)\}.$$

$\gamma(T)$ is referred to as the *minimum modulus of* T.

II.2.9 **THEOREM.** *Let* $T \in L(X, Y)$.

I) Suppose $\dim D(T) = \infty$. *Then* $\gamma(T) \leq \Gamma(T)$ *if* $a(T) < \infty$.

II) Suppose $\dim Y = \infty$. *Then* $\gamma(T) \leq \Gamma'(T)$ *if* $\bar{b}(T) < \infty$.

Proof. Let $\gamma(T) > 0$ since the theorem is trivial otherwise.

I) Suppose $\dim D(T) = \infty$ and $a(T) < \infty$. Let $M \in \mathcal{J}(D(T))$. By Theorem I.1.10 there exists $0 \neq m \in M$ such that $\|m\| = d(m, N(T))$. Hence $\|Tm\| \geq \lambda . d(m, N(T)) = \lambda \|m\|$ for every $\lambda \in (0, \gamma(T))$. Consequently $\dfrac{\|Tm\|}{\|m\|} \geq \gamma(T)$ and so $\|TJ_M\| \geq \gamma(T)$ (if the norm exists). Taking the infinum over all $M \in \mathcal{J}(D(T))$ we get $\Gamma(T) \geq \gamma(T)$.

II) Suppose $dim\ Y = \infty$ and $\bar{b}(T) < \infty$. Let $M \in \mathcal{I}_c(\tilde{Y})$. Then $cod_Y \overline{R(T)} < cod_{\tilde{Y}} M$. Hence for an arbitrary $\epsilon > 0$ there exists $Tx \in R(T)$ such that $Tx \neq 0$ and

$$(1) \qquad \|Q_M J_Y Tx\| \geq (1 - \epsilon)\|Tx\| > 0.$$

If $d(x,N(T)) > 0$ we can select $n \in N(T)$ such that $(1 + \epsilon)^{-1}\|x - n\| \leq d(x,N(T))$. Now select an arbitrary $\lambda \in (0,\gamma(T))$. Then $(1 + \epsilon)^{-1}.\lambda.\|x - n\| \leq \lambda.d(x,N(T)) \leq \|Tx\|$. If $d(x,N(T)) = 0$ select $n \in N(T)$ such that $(1 + \epsilon)^{-1}\lambda\ \|x - n\| \leq \|Tx\| = \|T(x - n)\|$. Hence we have

$$(2) \qquad \|Q_M J_Y\ T(x - n)\| = \|Q_M J_Y Tx\| \geq (1 - \epsilon)\|Tx\| = (1 - \epsilon)\|T(x - n)\| \\ \geq \frac{1 - \epsilon}{1 + \epsilon}.\lambda.\|x - n\|.$$

But from (1) we see that since $Tx \neq 0$, $x \notin N(Q_M J_Y T)$ and thus $(x - n) \notin N(Q_M J_Y T)$. We now conclude from (2) that $\|Q_M J_Y T\| \geq \frac{1 - \epsilon}{1 + \epsilon}.\lambda$ and therefore $\|Q_M J_Y T\| \geq \frac{1 - \epsilon}{1 + \epsilon}.\gamma(T)$ if the norm exists. Hence taking the infinum over all $M \in \mathcal{I}_c(\tilde{Y})$ we get $\Gamma'(T) \geq \frac{1 - \epsilon}{1 + \epsilon}\gamma(T)$. However $\epsilon > 0$ was chosen arbitrarily and so $\Gamma'(T) \geq \gamma(T)$ □

II.2.10 **DEFINITION.** Let $T \in L(X,Y)$. We define T to be *partially continuous* if there exists $E \in \mathcal{F}_c(D(T))$ such that TJ_E is continuous, *partially bounded* if $D(T) = X$. T is called *semi-continuous* if there exists $F \in \mathcal{F}(Y)$ such that $Q_F T$ is continuous, *semi-bounded* if $D(T) = X$. We denote the class of partially continuous (semi-continuous) operators by $PB(X,Y)$ $(SB(X,Y))$, with $PB(X,Y) \cap L[X,Y]$ $(SB(X,Y) \cap L[X,Y])$ being denoted by $PB[X,Y]$ $(SB[X,Y])$.

II.2.11 **LEMMA** [38; 3.5]. *Let* $T \in B[X,Y]$ *and* $S \in B[Y,Z]$ *with* X, Y *and* Z *Banach spaces. Then* $\Gamma'(ST) \leq \Gamma'(S).\Delta'(T)$.

II.2.12 **THEOREM.** *Let* $T \in PB(X,Y)$ *and* $S \in PB(Y,Z)$ *with* $D(S)$ *dense in* Y. *Then*

I) $\Gamma'(ST) \leq \Gamma'(S).\Delta'(J_Y T)$

II) $\Delta'(ST) \leq \Delta'(S).\Delta'(J_Y T)$.

Proof. I) Suppose $T \in PB(X,Y)$ and $S \in PB(Y,Z)$ with $D(S)$ dense in Y. Since $\Gamma'(J_Z ST) = \Gamma'(ST)$ and $\Gamma'(T) = \Gamma'(J_Y T)$ we can assume Y and Z to be complete without loss of generality. By hypothesis there exists $E \in \mathcal{F}_c(D(T))$ such that TJ_E or equivalently $T|_E$ is continuous where $T|_E$ denotes the restriction of T to E. Now let $F \in \mathcal{F}(D(T))$ such that $\overline{E}^{D(T)}$ and F are topologically complemented in $D(T)$ (Proposition I.8.4). Next let $A \in B(X,Y)$ be defined by $A|_{\overline{E} \cap D(T)} = \overline{(T|_E)}J_{D(T)}$ and $A|_F = 0$. (Note that A is continuous since $A|_{\overline{E} \cap D(T)}$ is and since there is a continuous projection from $D(T)$ onto $\overline{E}^{D(T)}$.) Then $K = T - A$ is a finite rank operator and $T = A + K$. Hence $Q_{R(K)}T = Q_{R(K)}(A + K) = Q_{R(K)}A$ and so T, and similarly S and ST are semicontinuous. Now let $M \in \mathcal{F}(Y)$ and $V \in \mathcal{F}(Z)$ be such that $Q_V S$ and $Q_M T$ are continuous. We now show that we can select $W \in \mathcal{F}(Z)$ such that $Q_W S$ is continuous with $N(\overline{Q_W S}) \supset M$. Let $W = Q_V^{-1}((\overline{Q_V S})M)$. Note that by choice of W, $W \supset V$ and since $M \in \mathcal{F}(Y)$ and $V \in \mathcal{F}(Z)$, $(\overline{Q_V S})M \in \mathcal{F}(Z/V)$ and $W \in \mathcal{F}(Z)$. Observe that $D(S) = D(Q_V S)$ is dense in Y and therefore that $D(\overline{Q_V S}) = Y \supset M$. Hence for W, $Q_W S$ is continuous since $W \supset V$, and $N(\overline{Q_W S}) = N \supset M$ since $D(\overline{Q_V S}) \supset M$ and $\overline{Q_W S} = Q^{Z/V}_{\overline{(Q_V S)M}}.\overline{Q_V S}$ (Theorem I.5.6 and Proposition I.8.6).

Now let T_1 be the restriction of T to $D(ST)$. Then $Q_M T_1$ is still continuous. Consider the following: $(\overline{Q_W S})^\wedge.Q^{Y/M}_{N/M}.Q^Y_M.T_1$. For an arbitrary $x \in D(ST)$ $Q_M T_1$ continuously maps x onto $Tx + M$ where $Tx \in D(S)$.

Notice that $Q_{N/M}^{Y/M}$ is a map from Y/M onto $(Y/M)/(N/M)$ and since $M \subset N$, $(Y/M)/(N/M) \equiv Y/N$ by Proposition I.8.6. Consequently $Q_{N/M}^{Y/M}$ continuously maps $Tx + M$ onto $Tx + N$. But $Tx \in D(S)$ and $N = N(\overline{Q_W S})$ and so $(\overline{Q_W S})\hat{}$ continuously maps $Tx + N$ onto STx. Hence $Q_W ST = (\overline{Q_W S})\hat{}.Q_{N/M}^{Y/M}.Q_M T_1$ is continuous. Let $U = (\overline{Q_W S})\hat{}.Q_{N/M}^{Y/M}$. Since $D(S)$ is dense in Y, $\overline{Q_W S}$ and thus $U = (\overline{Q_W S})\hat{}.Q_{N/M}^{Y/M}$ is a bounded operator. Therefore as $\overline{Q_W ST}$ is unique, $\overline{Q_W ST} = U.\overline{Q_M T_1}$.

Now observe that for say $A \in L(Y,Z)$ and $B \in B(X,Y)$, $\Gamma'(B) = \Gamma'(\bar{B})$, $\Delta'(B) = \Delta'(\bar{B})$ and $\Gamma'(AB) \leq \Gamma'(A).\|B\|$ since $\|Q_K B\| = \|\overline{Q_K B}\| = \|Q_K \bar{B}\|$ for each $K \in \mathcal{P}_c(Y)$ and $\|Q_P AB\| \leq \|Q_P A\|.\|B\|$ for each $P \in \mathcal{P}_c(Z)$. Finally since $\|Q_K T_1\| \leq \|Q_K T\|$ for each $K \in \mathcal{P}_c(Y)$ such that $K \supset M$, $\Delta'(Q_M T_1) \leq \Delta'(Q_M T)$. Consequently considering $\overline{Q_M T_1}$ as an element of $B[D(ST)^{\sim},Y]$ we can now apply the Lemma and Propositions II.1.3 and II.1.5 to obtain the following:

$$\begin{aligned}
\Gamma'(ST) = \Gamma'(Q_W ST) = \Gamma'(\overline{Q_W ST}) &= \Gamma'(U.(\overline{Q_M T_1})) \leq \Gamma'(U).\Delta'(\overline{Q_M T_1}) \\
&= \Gamma'((\overline{Q_W S})\hat{}.Q_{N/M}^{Y/M}).\Delta'(Q_M T_1) \\
&\leq \Gamma'((\overline{Q_W S})\hat{}).\|Q_{N/M}^{Y/M}\|.\Delta'(Q_M T) \\
&\leq \Gamma'(\overline{Q_W S}).\Delta'(T) \\
&= \Gamma'(Q_W S).\Delta'(T) \\
&= \Gamma'(S).\Delta'(T).
\end{aligned}$$

Since initially we assumed Y and Z to be complete, we have

$$\Gamma'(ST) = \Gamma'(J_Z ST) \leq \Gamma'(J_Z S).\Delta'(J_Y T) = \Gamma'(S).\Delta'(J_Y T).$$

II) Suppose $\dim Z = \infty$. We note from I) that $\Gamma'(Q_K ST) \geq \Gamma'(Q_K S).\Delta'(J_Y T)$ for each $K \in \mathcal{P}_c(Z)$. Hence

$$\Delta'(ST) = \sup_{K \in \mathcal{P}_c(Z)} \Gamma'(Q_K ST) \leq \sup_{K \in \mathcal{P}_c(Z)} \Gamma'(Q_K S).\Delta'(J_Y T) = \Delta'(S).\Delta'(J_Y T).$$

If $\dim Z < \infty$, $\Delta'(ST) = \Gamma'(ST) = 0$ and $\Delta'(S) = \Gamma'(S) = 0$ and so the result follows from I). □

II.2.13 THEOREM. *Let $T \in L(X,Y)$ and $S \in L(Y,Z)$ with $D(S)$ dense in Y. Then*

I) $\Gamma_0'(J_Z ST) \leq \Gamma_0'(J_Z S).\Gamma_0'(T)$ *with* $\Gamma_0'(ST) \leq \Gamma_0'(S).\Gamma_0'(T)$ *if* $D(S) = Y$;

II) $\Gamma'(ST) \leq \Gamma'(S).\Gamma_0'(T)$;

III) $\Delta'(ST) \leq \Delta'(S).\Gamma_0'(T)$;

except in the case $0.\infty$.

Proof. Without loss of generality let $\dim Z = \infty$.

I) Suppose $T \in L(X,Y)$ and $S \in L(Y,Z)$ with $D(S)$ dense in Y. We first show that $\Gamma_0'(ST) \leq \Gamma_0'(S).\Gamma_0'(T)$ if $D(S) = Y$. If either $\Gamma_0'(S) = \infty$ or $\Gamma_0'(T) = \infty$, the assertion is trivial and so assume $\Gamma_0'(S) < \infty$ and $\Gamma_0'(T) < \infty$. Notice that for say $B \in B(X,Y)$

$$(1) \qquad \Gamma_0'(SB) = \inf_{F \in \mathcal{F}(Z)} \|Q_F SB\| \leq \inf_{F \in \mathcal{F}(Z)} \|Q_F S\|.\|B\| = \Gamma_0'(S).\|B\|.$$

Let $\epsilon > 0$ be arbitrary and choose $K \in \mathcal{F}(Y)$ such that

$$(2) \qquad \|Q_K T\| \leq \Gamma_0'(T) + \epsilon.$$

Since $\Gamma_0'(S) < \infty$ we conclude that there exists $M \in \mathcal{F}(Z)$ such that $Q_M S$ is continuous. Moreover $SK \in \mathcal{F}(Z)$ as $K \in \mathcal{F}(Y)$. Now let $W = SK + M$. Then since $W \supset M$, $\|Q_W S\| \leq \|Q_M S\|$ and so $Q_W S$ is continuous. Also $K \subset N(Q_W S) = N$ since $SK \subset W$ and $\|Q_N T\| \leq \|Q_K T\|$ since $K \subset N$. Consequently as $W \in \mathcal{F}(Z)$ we have by (1), (2) and Propositions II.1.3 and II.1.5 that

$$\Gamma_0'(ST) = \Gamma_0'(Q_W ST) = \Gamma_0'((Q_W S)^\wedge.Q_N T) \leq \Gamma_0'((Q_W S)^\wedge)\ \|Q_N T\| \leq \Gamma_0'(Q_W S).\|Q_K T\|$$
$$\leq \Gamma_0'(S).(\Gamma_0'(T) + \epsilon).$$

Since $\epsilon > 0$ was chosen arbitrarily we obtain $\Gamma_0'(ST) \leq \Gamma_0'(S).\Gamma_0'(T)$.

Now suppose $D(S) \neq Y$. As before we may assume that $\Gamma_0'(J_Z S) < \infty$ since the assertion is trivial otherwise. Hence there exists $M \in \mathcal{F}(\tilde{Z})$ such that $Q_M J_Z S$ is continuous.

Letting T_1 denote the restriction of T to $D(ST)$ we note that $\overline{(Q_MJ_ZS)}J_YT_1 = Q_MJ_ZST$. Arguing as in the proof of Theorem II.2.12 we see that $\Gamma_0'(Q_MJ_ZS) = \Gamma_0'(\overline{Q_MJ_ZS})$ and $\Gamma_0'(T_1) \leq \Gamma_0'(T)$ and hence since $(\overline{Q_MJ_ZS})$ is bounded we conclude from the first part of the proof and Propositions II.1.3 and II.2.3 that

$$\begin{aligned}\Gamma_0'(J_ZST) = \Gamma_0'(Q_MJ_ZST) = \Gamma_0'((\overline{Q_MJ_ZS})J_YT_1) &\leq \Gamma_0'(\overline{Q_MJ_ZS}).\Gamma_0'(J_YT_1)\\ &\leq \Gamma_0'(Q_MJ_ZS).\Gamma_0'(T_1) \leq \Gamma_0'(J_ZS).\Gamma_0'(T).\end{aligned}$$

II) Let $M \in \mathcal{J}_c(\tilde{Z})$ be arbitrary. We then have $\Gamma_0'(Q_MJ_ZST) \leq \Gamma_0'(Q_MJ_ZS).\Gamma_0'(T)$ by I) and so from Proposition II.2.2

$$\Gamma'(ST) = \inf_{M\in\mathcal{J}_c(\tilde{Z})} \Gamma_0'(Q_MJ_ZST) \leq \inf_{M\in\mathcal{J}_c(\tilde{Z})} \Gamma_0'(Q_MJ_ZS).\Gamma_0'(T) = \Gamma'(S).\Gamma_0'(T).$$

III) Let $M \in \mathcal{J}_c(Z)$ be arbitrary. By II) we have that $\Gamma'(Q_MST) \leq \Gamma'(Q_MS).\Gamma_0'(T)$. Hence taking the supremum over all $M \in \mathcal{J}_c(Z)$, we have $\Delta'(ST) \leq \Delta'(S).\Gamma_0'(T)$. □

As was noted Theorem II.2.6 is a generalisation by R.W. Cross with Lemma II.2.7 and Theorem II.2.9 generalisations due mainly to the same. Theorem II.2.9 I) generalises a result by B. Gramsch [19] whereas II) of the same theorem generalises a result by Weis [38]. All other results are generalisations by the author. The proofs of the first part of Theorem II.2.4 as well as that of Lemma II.2.7 and Theorem II.2.9 are essentially the same as those used by Weis [38], suitably changed to hold for the more general case.

§3 ZERO SETS

II.3.1 **DEFINITION.** Let $T \in L(X,Y)$. T is defined to be *partially precompact* (*semi-precompact*) if there exists $E \in \mathcal{F}_c(D(T))$ $(F \in \mathcal{F}(Y))$ such that TJ_E $(Q_F T)$ is precompact. We will denote the class of all partially precompact (semi-precompact) operators in $L(X,Y)$ by $PPK(X,Y)$ $(SPK(X,Y))$ and in $L[X,Y]$ by $PPK[X,Y]$ $(SPK[X,Y])$.

II.3.2 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent.*

I) $T \in SPK(X,Y)$.

II) $\Gamma_0'(T) = 0$.

If T *is continuous then* $T \in PK(X,Y)$ *if and only if* $\Gamma_0'(T) = 0$.

Proof. I) ⇒ II) Since T is semi-precompact if and only if there exists an $N \in \mathcal{F}(Y)$ such that $Q_N T$ is precompact and since $\Gamma_0'(T) = \Gamma_0'(Q_N T)$ by Proposition II.1.3, we can assume T to be precompact. Now let $\epsilon > 0$ be arbitrary. From the definition of precompactness it now follows that there exist $\{x_1,\dots,x_n\} \subset B_{D(T)}$ such that for any $x \in B_{D(T)}$ there is an x_k, $1 \leq k \leq n$, with $\|Tx - Tx_k\| < \epsilon$. Consider $F = span\{Tx_1,\dots,Tx_n\}$. For an arbitrary $x \in B_{D(T)}$ there exists k, $1 \leq k \leq n$, such that $\|Q_F Tx\| \leq \|Tx - Tx_k\| < \epsilon$ and so we conclude that $\|Q_F T\| \leq \epsilon$. Hence $\Gamma_0'(T) = 0$.

II ⇒ I) Suppose T is continuous. Observing that for any $M \in \mathcal{F}(Y)$ $dim\ M = dim\ M' = dim\ Y'/M^{\perp} = cod\ M^{\perp} < \infty$ by Theorem I.5.10, we conclude that II) implies that for each $\epsilon < 0$ there exists $M \in \mathcal{F}(Y)$ $(cod\ M^{\perp} < \infty)$ such that $\|Q_M T\| = \|(Q_M T)'\| = \|T' J_{M^{\perp}}\| \leq \epsilon$ and hence T' is compact by Theorem I.11.9. But then T is precompact by Proposition I.11.5. Now if T is not continuous, then by II) there exists $N \in \mathcal{F}(Y)$ such that $Q_N T$ is continuous. But since $\Gamma_0'(T) = \Gamma_0'(Q_N T)$ by Proposition II.1.3, II) still holds for $Q_N T$ and so by what we've just shown $Q_N T$ is precompact. □

The following proposition appears in [6] and is a generalisation of the result for continuous operators (Theorem I.11.9).

II.3.3 **PROPOSITION** [6]. *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $T \in PPK(X,Y)$.

II) $\Gamma_0(T) = 0$.

If T *is continuous,* $T \in PK(X,Y)$ *if and only if* $\Gamma_0(T) = 0$.

Proof. Noting that if either I) or II) holds then $T \in PB(X,Y)$, the proposition now follows from a consideration of Theorem I.11.9 and Proposition II.1.4. □

II.3.4 **DEFINITION.** Let $T \in L(X,Y)$. T is defined to be F_- (F_+) if there exists $F \in \mathcal{F}(Y)$ $(E \in \mathcal{F}_c(D(T)))$ such that $(Q_F T)' = T' J_{F^\perp}$ has a continuous inverse (TJ_E has a continuous inverse). The class of all F_- (F_+) operators in $L(X,Y)$ will be denoted by $F_-(X,Y)$ $(F_+(X,Y))$ with $F_-(X,Y) \cap L[X,Y]$ $(F_+(X,Y) \cap L[X,Y])$ being denoted by $F_-[X,Y]$ $(F_+[X,Y])$.

For the sake of completeness we prove the following result due to R.W. Cross ([6; 2.2] and [8; 1.18]).

II.3.5 **THEOREM.** *Let* $T \in L(X,Y)$ *with* $\dim D(T) = \infty$. *Then the following are equivalent:*

I) $T \notin F_+$.

II) $T' \notin \Phi_-$.

III) *There is no* $E \in \mathcal{F}_c(D(T))$ *such that* $TJ_{\bar{E}}$ *has a continuous inverse.*

IV) *There exists* $M \in \mathcal{I}(D(T))$ *such that* TJ_M *is precompact.*

V) *For every* $\epsilon > 0$ *there exists* $M \in \mathcal{I}(D(T))$ *such that* $\|TJ_M\| \leq \epsilon$; *that is* $\Gamma(T) = 0$.

Proof. I) $\Rightarrow$ III). This is an easy consequence of the definition of F_+ operators.

III) $\Rightarrow$ IV). This follows on considering Theorem I.11.10.

IV) $\Rightarrow$ V). Assuming there exists $M \in \mathscr{I}(D(T))$ such that TJ_M is precompact, it follows from Proposition II.3.3 that for any $\epsilon > 0$ we can select $N \in \mathscr{F}_c(M) \subset \mathscr{I}(D(T))$ such that $\|TJ_N\| \leq \epsilon$.

V) $\Rightarrow$ I) Assume that V) holds with $T \in F_+$. Hence there exists $E \in \mathscr{F}_c(D(T))$ such that TJ_E has a continuous inverse, that is there exists $k > 0$ such that

$$\|TJ_E x\| \geq k\|x\| \quad \text{for every} \quad x \in E.$$

Select $M \in \mathscr{I}(D(T))$ such that $\|TJ_M\| \leq k/2$. Hence $M \cap E \in \mathscr{I}(D(T))$ with

$$k\|x\| \leq \|TJ_{M\cap E}x\| = \|TJ_M x\| \leq k/2\ \|x\|$$

for each $x \in M \cap E$. An obvious contradiction. Consequently $T \notin F_+$.

We have already shown I), III), IV) and V) to be equivalent. To conclude we show that III) and II) are equivalent.

II) $\Leftrightarrow$ III) Suppose there exists a relatively closed finite codimensional subspace of $D(T)$, say E, so that TJ_E has a continuous inverse. But then by Propositions I.10.7 and I.10.11 $Q^{(D(T))'}_{E^\perp} T'$ is surjective. Since $\dim E^{\perp D(T)} = \mathrm{cod}_{D(T)} E < \infty$ by Theorem I.5.10, we conclude that $b(T') < \infty$ and hence $T' \in \Phi_-$ by Propositions I.4.17 and I.10.2.
Conversely suppose $T' \in \Phi_-$. Then $b(T') < \infty$ by definition and so there exists $F \in \mathscr{F}((D(T))')$ such that $Q_F T'$ is surjective. However $F = ({}^\perp F)^\perp$ by Remark I.5.9 with $\mathrm{cod}_{D(T)} {}^\perp F = \dim({}^\perp F)^\perp = \dim F < \infty$ by Theorem I.5.10. Hence $Q_F T' = Q_{({}^\perp F)^\perp} T' = (TJ_{{}^\perp F})'$ by Proposition I.10.11 and so $TJ_{{}^\perp F}$ has a continuous inverse by Proposition I.10.7. $\square$

The construction in the following theorem is adapted from Brace and R.-Kneece [4]. The result generalises the corresponding result for bounded operators between Banach spaces by Weis [38; 3.2] and complements a result for *DF*-spaces by Wrobel [40; Corollaries 2 and 3].

II.3.6 **THEOREM.** *Let* $T \in L(X,Y)$ *with* Y *infinite dimensional. Then the following are equivalent:*

I) $T \notin F_-$.

II) $T' \notin F_+$;

III) $T' \notin \Phi_+$.

IV) $\Gamma(T') = 0$ *and* $\dim D(T') = \infty$.

V) $\Gamma'(T) = 0$.

VI) *There exists* $M \in \mathscr{J}_c(\tilde{Y})$ *such that* $Q_M J_Y T$ *is a compact (nuclear) operator.*

Proof. Without loss of generality suppose $D(T) = X$. The equivalence of II) and IV) is immediate from Theorem II.3.5.

VI) $\Rightarrow$ V). Suppose there exists $M \in \mathscr{J}_c(\tilde{Y})$ such that $Q_M J_Y T$ is a nuclear operator. But then $Q_M J_Y T$ is compact and so for an arbitrary $\epsilon > 0$, there exists $F \in \mathscr{F}(\tilde{Y}/M)$ such that $\|Q_F^{\tilde{Y}/M}(Q_M^{\tilde{Y}}\, J_Y T)\| \leq \epsilon$ (Theorem II.3.2). By Proposition I.8.6 there exists K, a closed subspace of $\tilde{Y}$ such that $Q_K^{\tilde{Y}}\, J_Y T = Q_F^{\tilde{Y}/M}(Q_M^{\tilde{Y}}\, J_Y T)$. Now since $\dim(Q_F(\tilde{Y}/M)) = \infty$, $K \in \mathscr{J}_c(\tilde{Y})$.

V) $\Rightarrow$ II) Suppose that V) holds and that $T' \in F_+(Y',X')$. Firstly, since there exists $N \in \mathscr{J}_c(\tilde{Y})$ such that $Q_N J_Y T$ is continuous, $(Q_N J_Y T)' = (J_Y T)' J_{N^\perp} = T' J_{N^\perp}$ is bounded (Theorem I.10.3 and Remark I.10.10). Hence $N^\perp \subset D(T')$ and since $\dim N^\perp = \operatorname{cod}_{\tilde{Y}} N = \infty$, $\dim D(T') = \infty$.

Now let $E \in \mathcal{F}_c(D(T'))$ such that $T'J_E$ has a continuous inverse. Then there exists $k > 0$ such that $\|T'J_E y'\| \geq k\|y'\|$ for each $y' \in E$. However by V) we can choose $M \in \mathcal{I}_c(\tilde{Y})$ such that $\|Q_M J_Y T\| \leq k/2$. Hence $dim\ M^{\perp\tilde{Y}} = cod_{\tilde{Y}} M = \infty$ with $M^{\perp\tilde{Y}} \subset D(T')$ since $\|T'J_{M^\perp}\| = \|(Q_M J_Y T)'\| = \|Q_M J_Y T\| \leq k/2$ by Theorem I.10.3 and Remark I.10.10. But now $M^{\perp} \cap E \in \mathcal{I}(D(T'))$ with

$$k\|y'\| \leq \|T'J_E y'\| = \|T'J_{M^\perp} y'\| \leq k/2\ \|y'\| \quad \text{for each} \quad y' \in M^{\perp\tilde{Y}} \cap E.$$

An obvious contradiction. Hence $T' \notin F_+(Y',X')$.

II) ⇒ I) Suppose $T \in F_-(X,Y)$. Then there exists $F \in \mathcal{F}(Y)$ such that $(Q_F T)' = T'J_{F^\perp}$ has a continuous inverse. But $dim\ F = dim\ F' = cod\ F^\perp < \infty$ and so $F^\perp \cap D(T') \in \mathcal{F}_c(D(T'))$ implying that $T' \in F_+(Y',X')$.

I) ⇒ VI). Assume that I) holds. By the hypothesis T' does not have a continuous inverse and so there exists $y_1' \in Y'$ such that $\|y_1'\| = 1$ and $\|T'y_1'\| \leq 2^{-1}.6^{-1}$. Since $\|y_1'\| = 1$ there exists $y_1 \in Y$ such that $\|y_1\| = 1$ and $1/2 \leq y_1'(y_1) \leq 1$. Letting $F = span\{y_1\}$ we note that $T'J_{F^\perp} = (Q_F T)'$ does not have a continuous inverse and so there exists $y_2' \in (span\{y_1\})^\perp$ such that $\|y_2'\| = 1$ and $\|T'y_2'\| \leq 2^{-1}.6^{-2}$. Since $\|y_2'\| = 1$ there exists $y_2 \in Y$ such that $\|y_2\| = 1$ and $1/2 \leq y_2'(y_2) \leq 1$. Continuing inductively we obtain $\{y_n\} \subset Y$ and $\{y_n'\} \subset D(T')$ such that

(1) $\|y_n\| = \|y_n'\| = 1$ for every $n \in \mathbb{N}$,

(2) $\|T'y_n'\| \leq 2^{-1}.6^{-n}$ for every $n \in \mathbb{N}$,

(3) $y_k'(y_i) = 0$ for $i < k$ with $i,k \in \mathbb{N}$,

(4) $1/2 \leq y_n'(y_n) \leq 1$ for every $n \in \mathbb{N}$.

Now let $\lambda_k = 1/(y_k'(y_k))$. Then $1 \leq \lambda_k \leq 2$ for every $k \in \mathbb{N}$. Define $\{v_n\} \subset Y$ by $v_1 = \lambda_1 y_1$ and $v_n = \lambda_n[y_n - \sum_{i=1}^{n-1} (y_i'(y_n))v_i]$ for $n \geq 2$.

We show that

$$(5) \qquad y_k'(v_n) = \delta_{kn} \quad \text{for} \quad k,n \in \mathbb{N}.$$

Suppose $n < k$:

Observe that $v_n \in span\,\{y_1,\dots,y_n\}$. However since $k > n$ we have by (3) that $span\{y_1,\dots,y_n\} \subset N(y_k')$ and so $y_k'(v_n) = 0$.

Suppose $n = k$:

If $n = 1$, then it is obvious that $y_1'(v_1) = 1$. Suppose $n > 1$. Then by what we've just shown $\{v_1,\dots,v_{n-1}\} \subset N(y_n')$ and so

$$y_n'(v_n) = \lambda_n y_n'[y_n - \sum_{i=1}^{n-1} (y_i'(y_n))v_i] = \lambda_n y_n'(y_n) = 1.$$

Suppose $n > k$:

First let $n = k + 1$. By what we've just shown we have

$$\begin{aligned} y_k'(v_{k+1}) &= \lambda_{k+1}[y_k'(y_{k+1}) - \sum_{i=1}^{k} (y_i'(y_{k+1}))\cdot y_k'(v_i)] \\ &= \lambda_{k+1}[y_k'(y_{k+1}) - y_k'(y_{k+1})] = 0. \end{aligned}$$

Continuing inductively we prove (5).

We now establish that

$$(6) \qquad \|v_n\| \leq 2.3^n \quad \text{for every} \quad n \in \mathbb{N}.$$

The proof is by induction. Note that

$$\|v_1\| = \|\lambda_1 y_1\| \leq |\lambda_1|.\|y_1\| \leq 2.1 \leq 2.3.$$

Suppose that (6) has been verified up to $k \in \mathbb{N}$. Then

$$\begin{aligned} \|v_{k+1}\| &= |\lambda_{k+1}|.\|y_{k+1} - \sum_{i=1}^{k} (y_i'(y_{k+1}))v_i\| \\ &\leq 2(\|y_{k+1}\| + \sum_{i=1}^{k} \|y_i'\|\|y_{k+1}\|\|v_i\|) \\ &\leq 2(1 + \sum_{i=1}^{k} 2.3^i) \\ &\leq 2(1 + \sum_{i=0}^{k} 2.3^i) = 2.3^{k+1}. \end{aligned}$$

Now let $u_i = v_i.2^{-1}.3^{-i}$. Then $\|u_i\| \leq 1$ for every $i \in \mathbb{N}$. Define a nuclear operator $K \in B[X,\tilde{Y}]$ as follows:

$$Kx = \sum_{i=1}^{\infty} (1/2^i)(x_i'(x))u_i \quad \text{where} \quad x_i' = 2.6^i.T'y_i' \quad \text{for each} \quad i \in \mathbb{N}.$$

For any $x \in X$ we have

$$\|Kx\| \leq \sum_{i=1}^{\infty} (1/2^i)\|x_i'\| \, \|x\| \, \|u_i\| \leq \sum_{i=1}^{\infty} (1/2^i)2.6^i\|T'y_i'\| \, \|x\| \leq \|x\| \sum_{i=1}^{\infty} (1/2^i) = \|x\|.$$

Hence Kx is absolutely convergent for any $x \in X$ and so Kx exists by the completeness of $\tilde{Y}$. Also $\|K\| \leq 1$. Observe that for $y' \in Y'$,

$$K'y' = \sum_{i=1}^{\infty} (1/2^i)(y'(u_i)).x_i'$$

and so K' is still a nuclear operator with $\|K'\| \leq 1$.

We show that K' agrees with T' on $W = span\{y_1', y_2', \ldots\}$. Let $y' \in W$. Then $y' = \sum_{i=1}^{m} \mu_i y_i'$ for some $m \in \mathbb{N}$. From (5) it follows that $\mu_i = y'(v_i)$ for each $1 \leq i \leq m$. Thus

$$T'y' = \sum_{i=1}^{m} \mu_i T'y_i' = \sum_{i=1}^{m} y'(v_i)(T'y_i') = \sum_{i=1}^{m} \frac{2.3^i}{2.6^i} .y'(v_i/(2.3^i)).(2.6^i.T'y_i')$$
$$= \sum_{i=1}^{m} (1/2^i)(y'(u_i))x_i'.$$

In fact since $y'(v_k) = 0$ for every $k > m$ by (5), we have that

$$T'y' = \sum_{i=1}^{\infty} (1/2^i)(y'(u_i)).x_i' = K'y'.$$

Hence $N(T' - K') \supset W$ and so $N(T' - K')$ is infinite dimensional. Observe that $(J_Y T)' = T'$ and that $(J_Y T - K)' = T' - K'$. Therefore $T' - K'$ is the adjoint of an operator in $L(X,\tilde{Y})$ and so $N(T' - K') = \overline{R(J_Y T - K)}^{\perp}$ by Proposition I.10.4. Let $M = {}^{\perp\tilde{Y}}(N(T' - K')) = \overline{R(J_Y T - K)}$ (Proposition I.10.4). Then $Q_M(J_Y T - K) = 0$ since $R(J_Y T - K) \subset M$. Therefore $Q_M J_Y T J_{D(T)}$ agrees with the nuclear operator $Q_M K J_{D(T)}$.

Finally observe that $\dim N(T' - K') = \dim M^{\perp} = \dim(\tilde{Y}/M)' = \dim(\tilde{Y}/M) = \infty$ and so $M \in \mathcal{I}_c(\tilde{Y})$. (Note that $Q_M J_Y T$ is compact by Theorem I.11.7).

II) $\Leftrightarrow$ III) Suppose $T' \in \Phi_+$. By definition $R(T')$ is closed and hence complete with $\alpha(T') < \infty$. Since $\alpha(T') < \infty$, there exists a topological complement of $N(T')$ in Y', say E (Proposition I.8.4). As E is closed, and thus complete, $T'J_E$ is still a closed operator and so since $T'J_E$ is injective with $R(T'J_E) = T'(E) = T'(E \oplus N(T')) = R(T')$, we conclude from the open mapping theorem that $T'J_E$ has a continuous inverse. Hence $T' \in F_+$ as $cod_{Y'} E = \alpha(T') < \infty$.

Conversely assume $T' \in F_+$. By III) of Theorem II.3.5 there exists $E \in \mathcal{F}_c(D(T'))$ such that $T'J_{\bar{E}}$ has a continuous inverse. We show that $R(T'J_{\bar{E}})$ is closed. Let $x' \in \overline{R(T'J_{\bar{E}})}$ and select $\{y'_n\} \subset \bar{E} \cap D(T')$ such that $T'y'_n \to x'$. But $T'J_{\bar{E}}$ has a continuous inverse and so $\{y'_n\}$ is Cauchy in $\bar{E}$. However $\bar{E}$ is a closed and therefore complete subspace of Y' and so there exists $y' \in \bar{E}$ with $y'_n \to y'$. Since T' is closed we conclude that $T'y' = x'$ and hence that $x' \in R(T'J_{\bar{E}})$ since $y' \in \bar{E}$. Consequently $R(T'J_{\bar{E}})$ is closed. As $E \in \mathcal{F}_c(D(T'))$, there exists $F \in \mathcal{F}(D(T'))$ with $R(T') = R(T'J_{\bar{E}}) + T'F$ and so $R(T')$ is closed by Proposition I.3.4. Now since $\alpha(T') < \infty$ if $T' \in F_+$ we conclude that $T' \in \Phi_+$. □

We will see later that $T' \in \Phi_-$ if and only if $T' \in F_-$ and hence that $\Gamma'(T') = 0$ if and only if $\Gamma(T) = 0$ by Theorem II.3.5.

II.3.7 **DEFINITION.** Let $T \in L(X,Y)$. Then T is defined to be *strictly singular* (*strictly cosingular*) if there is no $M \in \mathcal{I}(D(T))$ ($M \in \mathcal{I}_c(Y)$) such that TJ_M ($(Q_M T)'$) has a continuous inverse.

The class of strictly singular (strictly cosingular) operators in $L(X,Y)$ will be denoted by $SS(X,Y)$ $(SC(X,Y))$ whereas $SS(X,Y) \cap L[X,Y]$ $(SC(X,Y) \cap L[X,Y])$ will be denoted by $SS[X,Y]$ $(SC[X,Y])$.

We remark that strictly singular operators were first defined by T. Kato [21], whereas strictly cosingular operators were defined by Pelczynski [33]. The definition we give for strictly singular operators is a generalisation of the original definition in the sense that we state the definition for arbitrary operators (not necessarily continuous). It will be shown in Chapter V that the definition given for strictly cosingular operators corresponds to that given by Pelczynski in the classical setting of bounded operators between Banach spaces. The following theorem was first established by Brace and Royce-Kneece [4] in the classical case. The result below generalises and complements the results by Brace and R.-Kneece and Wrobel [40].

II.3.8 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $T \in SC(X,Y)$.

II) *For every* $M \in \mathcal{J}_c(Y)$ *there exists* $N \in \mathcal{J}_c(\tilde{Y})$ *such that* $N \supset M$ *with* $Q_N J_Y T$ *a compact (nuclear) operator.*

III) *For every* $\epsilon > 0$ *and for every* $M \in \mathcal{J}_c(Y)$ *there exists* $N \in \mathcal{J}_c(\tilde{Y})$ *such that* $N \supset M$ *with the norm of* $Q_N J_Y T$ *not exceeding* ϵ *(ie.* $\Lambda'(T) = 0$*).*

Proof. Without loss of generality suppose that $\dim Y = \infty$.

I) ⇒ II). Let $T \in SC(X,Y)$ and let $M \in \mathcal{J}_c(Y)$ be arbitrary. For any $F \in \mathcal{F}(Y/M)$ we have by Proposition I.8.6 that there exists a closed subspace K of Y such that $Q_F^{Y/M}.Q_M^Y = Q_K^Y$. Now since $\dim Y/M = \infty$ and $F \in \mathcal{F}(Y/M)$, $K \in \mathcal{J}_c(Y)$ and so $(Q_F^{Y/M}(Q_M^Y.T))' = (Q_K^Y.T)'$ does not have a continuous inverse. Since $F \in \mathcal{F}(Y/M)$ was chosen arbitrarily we have by Proposition I.8.6 and Theorem II.3.6 that II) holds.

II) ⇒ III) Let $M \in \mathscr{P}_c(Y)$ and $\epsilon > 0$ be chosen arbitrarily. By II) there exists $N \in \mathscr{P}_c(\tilde{Y})$ such that $N \supset M$ with $Q_N J_Y T$ a nuclear operator. But since $Q_N J_Y T$ is nuclear, it is also precompact and so by Theorem II.3.2 there exists $F \in \mathscr{F}(\tilde{Y}/N)$ such that $\|Q_F^{\tilde{Y}/N}(Q_N^{\tilde{Y}}.J_Y T)\| \leq \epsilon$. By Proposition I.8.6 there exists K, a closed subspace of $\tilde{Y}$, such that $Q_K^{\tilde{Y}} = Q_F^{\tilde{Y}/N}.Q_N^{\tilde{Y}}$ with $K \supset N \supset M$. Now since $N \in \mathscr{P}_c(\tilde{Y})$ and $F \in \mathscr{F}(\tilde{Y}/N)$, $K \in \mathscr{P}_c(\tilde{Y})$.

III) ⇒ I). Assume that III) holds and that $T \notin SC(X,Y)$. That is there exists $M \in \mathscr{P}_c(Y)$ such that $(Q_M T)'$ has a continuous inverse. Hence there exists $k > 0$ such that $\|T'y'\| \geq k\|y'\|$ for each $y' \in M^{\perp} \cap D(T')$. Now by III) there exists $N \in \mathscr{P}_c(\tilde{Y})$ such that $N \supset M$ with $\|Q_N J_Y T\| \leq k/2$. Since $Q_N J_Y T$ is continuous, $(Q_N J_Y T)' = T' J_{N^{\perp}}$ is bounded (Theorem I.10.3) and so $N^{\perp} \subset D(T')$. Also as $N \in \mathscr{P}_c(\tilde{Y})$ and $N \supset M$, $N^{\perp} \in \mathscr{P}(Y')$ with $N^{\perp} \subset M^{\perp}$. Thus $N^{\perp} \in \mathscr{P}(M^{\perp} \cap D(T'))$ with $\|T' J_{N^{\perp}}\| = \|Q_N J_Y T\| \leq k/2$ by Theorem I.10.3. But then we have that

$$k\|n'\| \leq \|T'n'\| = \|T' J_{N^{\perp}} n'\| \leq k/2.\|n'\| \quad \text{for every } n' \in N^{\perp}.$$

This is obviously a contradiction and so $T \in SC(X,Y)$. □

Considering Theorems II.3.6 and II.3.8 we observe that the definition of Γ' and Δ' in terms of the completion $\tilde{Y}$ of Y $(T \in L(X,Y))$ seems to be the most suitable way to make them usefully applicable to all operators, both bounded and unbounded, since in this case the important classes of F_- and strictly cosingular operators can be characterised in terms of the functions Γ' and Δ' respectively. Therefore a study of these functions will then yield results concerning F_- and strictly cosingular operators. Again for the sake of completeness we state the following theorem, a generalisation by R.W. Cross [7; 14] of [16; III.2.1].

II.3.9 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent.*

I) $T \in SS(X,Y)$.

II) *For every* $M \in \mathcal{I}(D(T))$ *there exists* $N \in \mathcal{I}(M)$ *such that* TJ_N *is precompact.*

III) *For every* $\epsilon > 0$ *and for every* $M \in \mathcal{I}(D(T))$ *there exists* $N \in \mathcal{I}(M)$ *such that the norm of* TJ_N *does not exceed* ϵ *(ie.* $\Delta(T) = 0$*).*

Proof. I) ⇒ II). This is a consequence of Theorem II.3.5.

II) ⇒ III). Let $M \in \mathcal{I}(D(T))$ be arbitrary and select $N \in \mathcal{I}(M)$ so that TJ_N is precompact. It follows from Proposition II.3.3 that for an arbitrary $\epsilon > 0$ we can select $K \in \mathcal{F}_c(N) \subset \mathcal{I}(M)$ so that $\|TJ_K\| \leq \epsilon$.

III) ⇒ I) Suppose that III) holds and that $T \notin SS(X,Y)$. Hence select $M \in \mathcal{I}(D(T))$ such that TJ_M has a continuous inverse. Then there is some $k > 0$ with $\|Tx\| \geq k\|x\|$ for each $x \in M$. Now select $N \in \mathcal{I}(M)$ such that $\|TJ_N\| \leq k/2$. But then

$$k\|x\| \leq \|Tx\| = \|TJ_N x\| \leq k/2.\|x\| \quad \text{for each } x \in N,$$

an obvious contradiction. Consequently $T \in SS(X,Y)$ if III) holds. □

II.3.10 **PROPOSITION.** *Consider* $B[X,Y]$. *Then* $Z(\Gamma') \cap B[X,Y]$, $Z(\Delta') \cap B[X,Y]$ *and* $Z(\Gamma'_0) \cap B[X,Y]$ *are closed subsets of* $B[X,Y]$ *where* $Z(f)$ $(f \in \{\Gamma', \Delta', \Gamma'_0\})$ *denotes the kernel of* f *in* $L(X,Y)$.

Proof. We give a proof for $Z(\Gamma')$, the proofs for the other two cases being entirely analogous. Suppose $T \in \overline{Z(\Gamma') \cap B[X,Y]}$. Then there exists $\{T_n\} \subset Z(\Gamma') \cap B[X,Y]$ such that $T_n \to T$ and so by Theorem II.2.4
$\Gamma'(T) \leq \Delta'(J_Y(T - T_n)) + \Gamma'(T_n) = \Delta'(J_Y(T - T_n)) \leq \|T - T_n\|$ for each $n \in \mathbb{N}$. Since $\|T - T_n\| \to 0$ as $n \to \infty$, we conclude that $\Gamma'(T) = 0$ and hence that $Z(\Gamma') \cap B[X,Y]$ is closed. □

II.3.11 *Remark.* Consider $B[X,Y]$.

I) Recalling that $PK[X,Y]$ denotes the bounded precompact operators, we conclude from Theorems II.2.4 and II.3.2 that Γ_0' defines a norm on $B[X,Y]/PK[X,Y]$. In fact, by Theorem II.2.13, $B[X,X]/PK[X,X]$ is a normed algebra with respect to Γ_0'. We deduce from Theorems II.2.4 and II.3.2 that $\Gamma_0'(T) = \Gamma_0'(T + B) \leq \|T + B\|$ for $T \in B[X,Y]$ and B any element of $PK[X,Y]$. Hence the induced quotient norm on $B[X,Y]/PK[X,Y]$ is stronger than Γ_0'. Consequently, since $B[X,Y]/PK[X,Y]$ is a Banach space with respect to the quotient norm when Y is complete (Theorems I.2.1 and I.4.5), $(B[X,Y]/PK[X,Y],\Gamma_0')$ is an operator range when Y is complete.

II) Let Y be complete. As in I) we can deduce from Theorems II.2.4 and II.3.8 that Δ' defines a norm on $B[X,Y]/SC[X,Y]$ and that $(B[X,Y]/SC[X,Y],\Delta')$ is an operator range. From Theorem II.2.12 we conclude that $B[Y,Y]/SC[Y,Y]$ is a normed algebra with respect to the norm Δ'.

Theorems II.3.2, II.3.6 and II.3.8 are generalisations by the author with the fact that I) implies VI) in Theorem II.3.6 being established with the assistance of Dr. R.W. Cross. Remark II.3.11 is also by the author.

CHAPTER III

PROPERTIES OF THE γ FUNCTION

§1 SOME INEQUALITIES

III.1.1 **DEFINITION.** Let $T \in L(X,Y)$. We define T to be a *range open operator* if T is an open map when considered as an element of $L(X,R(T))$. In other words if there is some $\lambda \in \mathbb{R}$ such that $\lambda TB_X \supset B_{R(T)}$ (or equivalently $\lambda TU_X \supset U_{R(T)}$). If T is a range open operator, then this property will be denoted by $T \in RO$.

III.1.2 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then* $T \in RO$ *if and only if* $\gamma(T) > 0$.

Proof. Observe that $T \in RO$

$\Leftrightarrow$ there exists $\lambda \in (0,\infty)$ such that $\lambda TU_X \supset U_{R(T)}$

$\Leftrightarrow$ there exists $\lambda \in (0,\infty)$ such that for each $x \in D(T)$ with $\|Tx\| < 1$, $\|x + n\| < \lambda$ for some $n \in N(T)$

$\Leftrightarrow$ there exists $\lambda \in (0,\infty)$ such that for each $x \in D(T)$ with $\|Tx\| < 1$, $d(x,N(T)) < \lambda$

$\Leftrightarrow$ there exists $\lambda \in (0,\infty)$ such that for each $x \in D(T)$ with $\|Tx\| = 1$, $d(x,N(T)) \leq \lambda$

$\Leftrightarrow$ there exists $\lambda \in (0,\infty)$ such that for each $x \in D(T)$, $\|Tx\| \geq (1/\lambda).d(x,N(T))$

$\Leftrightarrow$ $\gamma(T) > 0$. □

III.1.3 **PROPOSITION.** *Let* $T \in L(X,Y)$ *with* N *a closed subspace of* Y *such that* $N \cap R(T) = \{0\}$. *Then* $\gamma(T) \geq \gamma(Q_N T)$.

Proof. The proposition is trivial if $\gamma(Q_N T) = 0$. Hence suppose $\gamma(Q_N T) > 0$. Note that $N(T) = N(Q_N T)$ since $N \cap R(T) = \{0\}$. Observe that for any $x \in D(T)$, $\|Tx\| \geq \|Q_N Tx\|$ and so for any $\lambda \in (0,\gamma(Q_N T))$ the fact that $\|Q_N Tx\| \geq \lambda.d(x,N(T))$ for all $x \in D(T)$ implies that $\|Tx\| \geq \lambda.d(x,N(T))$ for all $x \in D(T)$. Hence $\gamma(T) \geq \gamma(Q_N T)$. □

III.1.4 **PROPOSITION.** *Let* $T \in L(X,Y)$ *and let* N *be a subspace of* $R(T)$. *Then* $\gamma(Q_{\bar{N}} T) > 0$ *if* $\gamma(T) > 0$.

Proof. Suppose $T \in L(X,Y)$ with $\gamma(T) > 0$. Then $T \in RO$ by Proposition III.1.2. Let N be an arbitrary subspace of $R(T)$ and let N_R denote $\bar{N} \cap R(T)$. Then $Q_{N_R}^{R(T)}$ is an open map and so, considering T as an element of $L(X,R(T))$, $Q_{N_R}^{R(T)}T$ is open since it is the composition of two open maps. But since N_R is dense in $\bar{N}$, $R(Q_{N_R}^{R(T)}T) \doteq R(T)/N_R \equiv R(T) + \bar{N} = R(Q_{\bar{N}}^{Y}.T)$, and so by definition, $Q_{\bar{N}} T \in RO$. Hence $\gamma(Q_{\bar{N}} T) > 0$ by Proposition III.1.2. □

III.1.5 **PROPOSITION.** *Let* $T \in L(X,Y)$ *and* M *an arbitrary subspace of* X. *Then* $\gamma(TJ_{M+N(T)}) \geq \gamma(T)$.

Proof. The result is trivial if $\gamma(T) = 0$, and so suppose $\gamma(T) > 0$. Now let $\lambda \in (0,\gamma(T))$. Hence $\|Tx\| \geq \lambda.d(x,N(T))$ for each $x \in D(T)$.

But then

$$\|Tx\| \geq \lambda.d(x,N(T)) \quad \text{for each} \quad x \in D(T)\cap(M + N(T)).$$

Since $N(TJ_{M+N(T)}) = N(T)$ we conclude that $\gamma(TJ_{M+N(T)}) \geq \lambda$. Now as λ was an arbitrary element of $(0,\gamma(T))$, we deduce that $\gamma(TJ_{M+N(T)}) \geq \gamma(T)$. □

III.1.6 **THEOREM.** *Let $T \in L(X,Y)$ be closable. Then $\gamma(T) \leq \gamma(\tilde{T})$ with equality holding if $N(T)$ is dense in $N(\tilde{T})$.*

Proof. Assume that $T \in L(X,Y)$ is closable. We will show that

$$\gamma(T) = \sup\{\lambda \in \mathbb{R}\colon \|\tilde{T}x\| \geq \lambda.d(x,N(T)) \text{ for all } x \in D(\tilde{T})\}$$

from which the result will follow. Observe that since $D(\tilde{T}) \supset D(T)$,

$$(1)\qquad \begin{aligned} \gamma(T) &= \sup\{\lambda \in \mathbb{R}\colon \|Tx\| \geq \lambda.d(x,N(T)) \text{ for all } x \in D(T)\} \\ &\geq \sup\{\lambda \in \mathbb{R}\colon \|\tilde{T}x\| \geq \lambda.d(x,N(T)) \text{ for all } x \in D(\tilde{T})\}. \end{aligned}$$

Let $\delta = \sup\{\lambda \in \mathbb{R}\colon \|\tilde{T}x\| \geq \lambda.d(x,N(T)) \text{ for all } x \in D(\tilde{T})\}$. Now if $\delta = \infty$, then trivially $\delta = \gamma(T)$ by (1). Hence suppose $\delta < \infty$ and let $\epsilon > 0$ be arbitrary. It follows that we can select $\tilde{z} \in D(\tilde{T})\backslash N(T)$ so that

$$(2)\qquad \|\tilde{T}\tilde{z}\| < (\delta + \epsilon).d(\tilde{z},N(T)).$$

From Proposition I.4.14 we note that the graph $G(T)$ of T is dense in $G(\tilde{T})$. Hence there exists $\{(x_n,Tx_n)\} \in G(T)$ such that

$$(x_n,Tx_n) \to (\tilde{z},\tilde{T}\tilde{z}).$$

Consequently

$$(3)\qquad \|x_n - \tilde{z}\| + \|Tx_n - \tilde{T}\tilde{z}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Considering (3), we can write

$$(4)\qquad d(x_n,N(T)) + \|x_n - \tilde{z}\| \geq d(\tilde{z},N(T)) \quad \text{and} \quad \|\tilde{T}\tilde{z}\| \geq (\|Tx_n\| - \|\tilde{T}\tilde{z} - Tx_n\|)$$

for all $n \in \mathbb{N}$. From (2), (3) and (4) it now follows that we can choose $k \in \mathbb{N}$ such that $d(x_k,N(T))(\delta + \epsilon) > \|Tx_k\|$ and hence we conclude that $\gamma(T) \leq \delta + \epsilon$. Since $\epsilon > 0$ was chosen arbitrarily we have $\gamma(T) \leq \delta$. Considering (1) we deduce that

$$(5)\qquad \gamma(T) = \delta = \sup\{\lambda \in \mathbb{R}\colon \|\tilde{T}x\| \geq \lambda.d(x,N(T)) \text{ for all } x \in D(\tilde{T})\}.$$

Now if $N(T)$ was dense in $N(\tilde{T})$ we would have

$d(x,N(T)) = d(x,\overline{N(T)}) = d(x,N(\tilde{T}))$ for all $x \in D(\tilde{T})$ and so

$$\gamma(T) = \delta = \gamma(\tilde{T}).$$

If however $\overline{N(T)} \subsetneq N(\tilde{T})$, then $d(x(N(\tilde{T})) \leq d(x,\overline{N(T)}) = d(x,N(T))$ and so from (5) we conclude that $\gamma(T) = \delta \leq \gamma(\tilde{T})$. □

Theorem III.1.6 is due to the author. We note that the proof of Proposition III.1.5 is essentially contained in [16; IV.2.9]. The author is unaware of Propositions III.1.3 and III.1.4 having been proved before in the general case.

§2 THE RELATIONSHIP BETWEEN $\gamma(T)$, $\gamma(TG)$ AND $\gamma(T')$.

III.2.1 **DEFINITION**. Let $T \in L(X,Y)$. We define X_T to be $D(T)$ renormed with the stronger norm $\|.\|_T = \|.\| + \|T.\|$. The identity map from X_T onto $D(T)$ will be denoted by G_T (G if there is no danger of confusion) and $TJ_{D(T)}G_T$ by TG_T. Note that G_T is injective with $G_T \in B[X_T, D(T)]$ and $TG_T \in B[X_T, Y]$.

III.2.2 **THEOREM**. *Let* $T \in L(X,Y)$.

I) *If* $T = 0$, *then* $\gamma(T) = \gamma(TG) = \infty$.

II) *If* $T \neq 0$, *then* $\gamma(T) = \dfrac{\gamma(TG)}{1 - \gamma(TG)}$ *where we agree to have* $1/0 = \infty$.

Proof. I) Assuming that $T = 0$ we trivially have that $TG = 0$ and hence from the definition of the minimum modulus we conclude that $\gamma(T) = \gamma(TG) = \infty$.

II) Suppose $T \neq 0$. Notice that for any $x \in D(T)$,

$$\begin{aligned} d(G^{-1}x, N(TG)) &= \inf_{z \in N(T)} \|G^{-1}x - G^{-1}z\|_T \\ &= \inf_{z \in N(T)} (\|x - z\| + \|Tx - Tz\|) \\ &= \inf_{z \in N(T)} \|x - z\| + \|Tx\| \\ &= d(x, N(T)) + \|Tx\| \end{aligned}$$

and so, since $T \neq 0$ and thus $TG \neq 0$, it follows from the above that $\gamma(TG) \leq 1$. We will first show that $\gamma(T) \geq \dfrac{\gamma(TG)}{1 - \gamma(TG)}$. If $\gamma(TG) = 0$ this is trivial. Hence suppose that $\gamma(TG) > 0$.

Choose $\lambda \in (0,\gamma(TG))$ arbitrarily. Then

$$\|Tx\| \geq \lambda.d(G^{-1}x,N(TG)) = \lambda.(d(x,N(T)) + \|Tx\|) \quad \text{for each} \quad x \in D(T),$$

by definition. Consequently

$$\|Tx\| \geq \frac{\lambda}{1-\lambda}.d(x,N(T)) \quad \text{for each} \quad x \in D(T)$$

and therefore $\gamma(T) \geq \frac{\lambda}{1-\lambda}$. Since $\lambda \in (0,\gamma(TG))$ was chosen arbitrarily, we have that $\gamma(T) \geq \frac{\gamma(TG)}{1-\gamma(TG)}$.

Conversely if $\gamma(T) = 0$, then trivially $\frac{\gamma(TG)}{1-\gamma(TG)} \geq \gamma(T)$. Therefore suppose that $\gamma(T) > 0$. Now choose a monotone increasing sequence $\{\delta_n\} \subset (0,1)$ such that $\frac{\delta_n}{1-\delta_n} \to \gamma(T)$ as $n \to \infty$. (Note that $\gamma(T) > \frac{\delta_n}{1-\delta_n} > 0$ for each $n \in \mathbb{N}$.) Hence

$$\|Tx\| \geq \frac{\delta_n}{1-\delta_n}.d(x,N(T)) \quad \text{for each} \quad x \in D(T) \quad \text{and} \quad n \in \mathbb{N}$$

and so

$$\|Tx\| \geq \delta_n.(d(x,N(T)) + \|Tx\|) \quad \text{for each} \quad x \in D(T) \quad \text{and} \quad n \in \mathbb{N}.$$

It now follows that $\gamma(TG) \geq \delta_n$ for each $n \in \mathbb{N}$, whence $\frac{\gamma(TG)}{1-\gamma(TG)} \geq \frac{\delta_n}{1-\delta_n}$ for each $n \in \mathbb{N}$. We conclude that $\frac{\gamma(TG)}{1-\gamma(TG)} \geq \gamma(T) = \lim_{n\to\infty} \frac{\delta_n}{1-\delta_n}$ and so from the first part of the proof it follows that $\frac{\gamma(TG)}{1-\gamma(TG)} = \gamma(T)$. □

III.2.3 **COROLLARY.** *Let* $T \in L(X,Y)$. *Then* $T \in RO$ *if and only if* $TG \in RO$.

Proof. From Theorem III.2.2 it follows that $\gamma(T) > 0$ if and only if $\gamma(TG) > 0$ and hence the Corollary follows on considering Proposition III.1.2.□

III.2.4 **COROLLARY.** *Let* $T \in L(X,Y)$ *such that* $T \neq 0$. *Then the following are equivalent:*

I) $N(T)$ *is dense in* $D(T)$.

II) $\gamma(TG) = 1$.

Proof. Suppose $T \in L(X,Y)$ such that $T \neq 0$. On considering the definition of $\gamma(T)$ we notice that $\gamma(T) = \infty$ if and only if for each $x \in D(T)$, $d(x,N(T)) = 0$. But this is the case if and only $N(T)$ is dense in $D(T)$. Since in this case $\gamma(T) = \infty$ if and only if $\gamma(TG) = 1$ by Theorem III.2.2, we are done. □

Theorem III.2.2 and Corollaries III.2.3 and III.2.4 are the authors own.

III.2.5 **LEMMA.** *Let* $T \in L(X,Y)$. *Then* $T' = (G^{-1})'.(TG)'$.

Proof. Note that $G_T \in L[X_T,D(T)]$ and hence the lemma is a special case of Proposition I.10.9. □

III.2.6 **THEOREM.** *Let* $T \in L(X,Y)$. *Consider the following statements*:

I) $\gamma(T) > 0$.

II) $\gamma(T') > 0$.

III) $R(T') = N(TJ_{D(T)})^{\perp}$.

IV) $R(T)$ *is closed (ie.* $R(T) = {}^{\perp}N(T')$ *by Proposition I.10.4).*

Then:

a) if X *and* Y *are normed linear spaces, then I) and III) are equivalent, either of which imply II);*

b) if X *is complete and* T *closed, then I), II) and III) are equivalent, any of which imply IV);*

c) if Y *is complete, IV) implies II);*

d) if X *is an operator range,* Y *complete and* T *closed, IV) implies I) and hence III);*

e) if both X *and* Y *are complete and* T *is closed, all four conditions are equivalent.*

Proof.

a) Let $T \in L(X,Y)$ with X and Y normed linear spaces.

III) ⇒ II). If $R(T') = N(TJ_{D(T)})^{\perp}$ then $R(T')$ is closed and so the fact that $\gamma(T') > 0$ is an immediate consequence of *d*) (proved later).

I) ⇔ III). Assume that $\gamma(T) > 0$. First suppose that $N(T)$ is closed. Then since $d(x,N(T)) = \|Q_{N(T)}x\|$ and $Tx = \hat{T}.Q_{N(T)}x$ for each $x \in D(T)$, we conclude from the definition of $\gamma(T)$ that $\gamma(\hat{T}) = \gamma(T) > 0$ and hence that $\hat{T}$ has a continuous inverse. But then $R((\hat{T})') = (D(T)/N(T))'$ by Proposition I.10.7. By Theorem I.5.10 $(D(T)/N(T))'$ is isometric to $N(TJ_{D(T)})^{\perp}$ under the map V given by $(Vz')x = z'(Q_{N(T)}x)$ where $z' \in (D(T)/N(T))'$ and $x \in D(T)$. Hence

$$V(R((\hat{T})')) = N(TJ_{D(T)})^{\perp}.$$

However since $D(T') = D((\hat{T})')$ by Proposition I.10.12 and since $(\hat{T})'y'(Q_{N(T)}x) = T'y'x$ for $y' \in D(T')$, we conclude from the definition of V that

$$R(T') = V(R((\hat{T})')) = N(TJ_{D(T)})^{\perp}.$$

Now suppose $N(T)$ is not closed. As $\gamma(T) > 0$ we note from Theorem III.2.2 that $\gamma(TG) > 0$. Then $R((TG)') = N(TG)^{\perp}$ by what we have just shown. From the Lemma we deduce that

$$(1) \qquad R(T') = R((G^{-1})'J_{R((TG)')}) = R((G^{-1})'J_{N(TG)^{\perp}})$$

Since G is bounded, $N = G^{-1}(\overline{N(TJ_{D(T)})})$ is closed. Notice that $N(Q_N G^{-1}) = \overline{N(TJ_{D(T)})}$ and that $Q_N G^{-1}$ is the composition of two open maps. Hence $Q_N G^{-1}$ is an open map and so $\gamma(Q_N G^{-1}) > 0$ by Proposition III.1.2. But then we can deduce from the first part that

$$(2) \qquad R((G^{-1})'J_{N^{\perp}}) = R((Q_N G^{-1})') = \overline{N(TJ_{D(T)})}^{\perp} = N(TJ_{D(T)})^{\perp}.$$

However $N \supset N(TG)$ and so $N^{\perp} \subset N(TG)^{\perp}$. Therefore $(G^{-1})'J_{N^{\perp}}$ is a restriction of $(G^{-1})'J_{N(TG)^{\perp}}$ and so, comparing (1) and (2), we conclude that $R(T') \supset N(TJ_{D(T)})^{\perp}$. But $R(T') \subset N(TJ_{D(T)})^{\perp}$ by Proposition I.10.5 and so $R(T') = N(TJ_{D(T)})^{\perp}$.

Conversely suppose $R(T') = N(TJ_{D(T)})^{\perp}$ with $\gamma(T) = 0$. From the definition of $\gamma(T)$ we note that if this is the case, there exists $\{x_n\} \subset D(T)$ such that $d(x_n, N(T)) = 1$ for each $n \in \mathbb{N}$ with $\|Tx_n\| \to 0$. Let $z_n = x_n / \sqrt{\|Tx_n\|}$ (note that $\{x_n\} \not\subset N(T)$ since $d(x_n, N(T)) = 1$ for each $n \in \mathbb{N}$). Then $\|z_n + \overline{N(TJ_{D(T)})}\| = d(z_n, N(T)) = d(x_n, N(T))/\sqrt{\|Tx_n\|} \to \infty$ whereas $\|Tz_n\| \to 0$. Thus $y'Tz_n \to 0$ for each $y' \in D(T')$, whence $x'z_n \to 0$ for each $x' \in N(TJ_{D(T)})^{\perp}$. Considering Theorem I.5.10 we conclude that $q'(z_n + \overline{N(TJ_{D(T)})}) \to 0$ for each $q' \in (D(T)/\overline{N(TJ_{D(T)})})'$. But then $z_n + \overline{N(TJ_{D(T)})}$ is bounded by Theorem I.7.5, which is a contradiction. Consequently $\gamma(T) > 0$ whenever $R(T') = N(TJ_{D(T)})^{\perp}$.

b) Let X be complete and T closed.

I) $\Rightarrow$ IV) Let $\gamma(T) > 0$. Since T is closed, $N(T)$ is closed, and so, as in the proof of a), we conclude that $\gamma(T) = \gamma(\hat{T}) > 0$ and that $\hat{T}$ has a continuous inverse. Observe that $X/N(T)$ is still a Banach space and that $\hat{T}$ is closed (Theorem I.2.1 and Proposition I.4.16). Now let $y \in \overline{R(T)} = \overline{R(\hat{T})}$ and select $\{Q_{N(T)}x_n\} \subset D(\hat{T})$ such that $\hat{T}Q_{N(T)}x_n \to y$. Since $\hat{T}$ has a continuous inverse, $\{Q_{N(T)}x_n\}$ is a Cauchy and hence convergent sequence in the Banach space $X/N(T)$. Say $Q_{N(T)}x_n \to Q_{N(T)}x$. From the fact that $\hat{T}$ is closed it now follows that $y = \hat{T}Q_{N(T)}x \in R(\hat{T}) = R(T)$. Consequently $R(T)$ is closed.

III) $\Rightarrow$ I) Suppose $\gamma(T') > 0$. Then $R(T')$ is closed by what we have just proved. Now let T_1 be T considered as an element of $L(\overline{D(T)}, \overline{R(T)})$. Observe that T_1 is still closed with T_1' injective (Proposition I.10.4). For any $y' \in Y'$, $y' \in D(T')$ if and only if $y'J_{\overline{R(T)}} \in D(T_1')$ with $y'TJ_{D(T)} = y'J_{\overline{R(T)}}T_1J_{D(T_1)}$. But by the Hahn-Banach theorem each $r' \in (\overline{R(T)})'$ can be represented as a restriction of some $y' \in Y'$.

Consequently $R(T_1') = R(T')$ and so, since $R(T_1')$ is therefore closed, we conclude from the injectivity of T_1' and the open mapping theorem that T_1' has a continuous inverse. But then T_1 is an open map by Theorem I.10.8 and so $T \in RO$. Hence $\gamma(T) > 0$ by Proposition III.1.2.

c) Let Y be complete.

IV) ⇒ II) Suppose $R(T)$ is closed and hence complete. Defining T_1 as in the proof of b) we note that $R(T') = R(T_1')$. Since $R(T_1) = R(T)$ is complete, we conclude from Proposition I.10.7 that T_1' has a continuous inverse, that is $\gamma(T_1') > 0$. Considering *e*) we conclude that $\gamma(T') > 0$.

d) Suppose X is an operator range, Y complete and T closed.

IV) ⇒ I) Observe that if $R(T)$ is closed, $R(T)$ is a Banach space by virtue od the fact that Y is a Banach space. Hence, regarding T as an element of $C(X,R(T))$, we apply the generalised open mapping theorem to obtain $T \in RO$. But then $\gamma(T) > 0$ by Proposition III.1.2.

e) This is an immediate consequence of *a*), b) and *d*). □

III.2.7 **COROLLARY.** *Let* $T \in L(X,Y)$. *Then*

I) $\alpha(T') = \bar{b}(T)$ [16];

II) $\bar{b}(T') \geq \alpha(T)$ *with* $b(T') = \bar{b}(T') = \alpha(T)$ *if* $\gamma(T) > 0$.

Proof. Without loss of generality let $D(T) = X$.

I) By Theorem I.5.10 and Proposition I.10.4

$$\bar{b}(T) = \dim(Y/\overline{R(T)}) = \dim(Y/\overline{R(T)})' = \dim R(T)^{\perp} = \dim N(T') = \alpha(T')$$

II) By Proposition I.10.5 $\overline{R(T')} \subset N(T)^{\perp}$ and so by Theorem I.5.10

$$\bar{b}(T') = \dim(X'/\overline{R(T')}) \geq \dim(X'/N(T)^{\perp}) = \dim(N(T))' = \dim N(T) = \alpha(T).$$

Now if $\gamma(T) > 0$, $R(T') = N(T)^{\perp}$ by Theorem III.2.6 and so

$$\bar{b}(T') = b(T') = \dim(X'/N(T)^{\perp}) = \dim N(T) = \alpha(T).$$ □

III.2.8 **THEOREM.** *Let* $T \in L(X,Y)$. *Then* $\gamma(T') = \gamma(T)$ *whenever* $\gamma(T) > 0$.

Proof. Without loss of generality assume $D(T) = X$. Suppose $\gamma(T) = \infty$. Considering Theorem III.2.2 and Corollary III.2.4 we conclude that this can only be the case if $N(T)$ is dense in X. Since for any $y' \in D(T')$ we have $N(y'T) \supset N(T)$ and since $y'T$ is continuous, $N(T'y')$ is closed. Consequently $N(T'y') \supset \overline{N(T)} = X$. Hence $T' = 0$ and therefore $\gamma(T') = \infty$. Now let $0 < \gamma(T) < \infty$. Because $\gamma(T) < \infty$, we conclude from Theorem III.2.2 and Corollary III.2.4 that $\overline{N(T)} \neq X$. Noting that

$d(x,N(T)) = d(x,\overline{N(T)}) = \|Q_{\overline{N(T)}}x\|$ for each $x \in X$ we conclude that since $\overline{N(T)} \subsetneq X$, we have

$$\gamma(T) = \sup\{\lambda \in \mathbb{R}\colon \|Tx\| \geq \lambda.\|Q_{\overline{N(T)}}x\| \text{ for each } x \in X\backslash\overline{N(T)}\} \tag{1}$$

where $X\backslash\overline{N(T)}$ denotes the complement of $\overline{N(T)}$ in X. Hence we can write

$$\gamma(T) = \inf_{x\in X\backslash\overline{N(T)}} \frac{\|Tx\|}{\|Q_{\overline{N(T)}}x\|}. \tag{2}$$

Now let $\epsilon > 0$ be arbitrary and select $x \in X\backslash\overline{N(T)}$ such that

$$\gamma(T) + \epsilon \geq \frac{\|Tx\|}{\|Q_{\overline{N(T)}}x\|}. \tag{3}$$

By Theorems III.2.6 and I.5.10 we have that $R(T') = N(T)^{\perp} \equiv (X/\overline{N(T)})'$ and so by Corollary I.7.3 there exists $T'y' \in R(T')$ such that

$$\|T'y'\| = 1 \quad \text{and} \quad T'y'x = \|Q_{\overline{N(T)}}x\|. \tag{4}$$

From Theorem I.5.10 and Proposition I.10.4 we see that for any $z' \in Y'$

$$d(z',N(T')) = d(z',R(T)^{\perp}) = \|Q_{R(T)^{\perp}}z'\| = \|z'J_{R(T)}\|. \tag{5}$$

We now conclude from (3), (4) and (5) that

$$d(y',N(T')) = \|y'J_{R(T)}\| \geq \frac{|y'Tx|}{\|Tx\|} = \|Q_{\overline{N(T)}}x\|/\|Tx\| \geq \frac{1}{\gamma(T) + \epsilon}.$$

Hence $\gamma(T) + \epsilon \geq \dfrac{\|T'y'\|}{d(y',N(T'))}$ since $\|T'y'\| = 1$. (Note that $y' \in D(T')\setminus N(T')$). As with $\gamma(T)$ we now have that

$$\gamma(T') = \inf_{z'\in D(T')\setminus N(T')} \frac{\|T'z'\|}{d(z',N(T'))}$$ since $N(T')$ is closed and $T' \neq 0$.

Consequently $\gamma(T) + \epsilon \geq \gamma(T')$ and since $\epsilon > 0$ was chosen arbitrarily,

$$\gamma(T) \geq \gamma(T'). \tag{6}$$

Conversely, let $\delta > 0$ be arbitrary and choose $z' \in D(T')\setminus N(T')$ such that $\gamma(T') + \delta > \dfrac{\|T'z'\|}{d(z',N(T'))} = \dfrac{\|T'z'\|}{\|z'J_{R(T)}\|}$. Now select $\mu > 0$ such that

$$\gamma(T') + \delta \geq \frac{\|T'z'\|}{\|z'J_{R(T)}\| - \mu} > 0. \tag{7}$$

Let $Tx \in R(T)$ such that

$$\|Tx\| = 1 \quad \text{and} \quad |z'Tx| \geq \|z'J_{R(T)}\| - \mu > 0. \tag{8}$$

Note that $x \notin N(T'z') \supset \overline{N(T)}$ and so

$$d(x,\overline{N(T)}) = \|Q_{\overline{N(T)}}x\| > 0. \tag{9}$$

But since $R(T') = N(T)^{\perp} \equiv (X/\overline{N(T)})'$, we have by (2), (7), (8), (9) and Corollary I.7.4 that

$$\frac{1}{\gamma(T)} \geq \|Q_{\overline{N(T)}}x\|/\|Tx\| = \|Q_{\overline{N(T)}}x\| = \sup_{\substack{r'\in R(T')\\ \|r'\|=1}} |r'x| \geq \frac{|z'Tx|}{\|T'z'\|}$$
$$\geq \frac{\|z'J_{R(T)}\| - \mu}{\|T'z'\|} \geq \frac{1}{\gamma(T') + \delta}.$$

Hence $\gamma(T') + \delta \geq \gamma(T)$. Considering (6) and the fact that $\delta > 0$ was chosen arbitrarily, we conclude that $\gamma(T') = \gamma(T)$. □

III.2.9 **COROLLARY.** *Let* $T \in L(X,Y)$ *with* $\gamma(T) > 0$ *and let* $S \in L(X,Y)$ *with* $D(S) \supset D(T)$. *If* $\|S\| < \gamma(T)$ *or* $\|SJ_{D(T)}G_T\| < \gamma(TG_T)$, *then*

I) $a(T + S) \leq a(T)$;

II) $\bar{b}(T + S) \leq \bar{b}(T)$.

Proof. We prove only the case where $\|S\| < \gamma(T)$. The proof of the case where $\|SJ_{D(T)}G_T\| < \gamma(TG_T)$ is similar and follows from the fact that $\alpha(T + S) = \alpha((T + S)G_T)$ and $\bar{b}(T + S) = \bar{b}((T + S)G_T)$.

I) Choose $\lambda \in (0,\gamma(T))$ such that $\|S\| < \lambda$. For $0 \neq x \in N(T + S)$ we then have $\lambda.d(x,N(T)) \leq \|Tx\| = \|Sx\| \leq \|S\|\,\|x\| < \lambda.\|x\|$. Hence $d(x,N(T)) < \|x\|$ and therefore, as x was an arbitrary non-zero element of $N(T + S)$, we have that $\alpha(T + S) \leq \alpha(T)$ by Theorem I.1.10.

II) Let $S_1 = SJ_{D(T)}$. Then

$$\gamma(T') = \gamma(T) > \|S\| \geq \|S_1\| = \|S_1'\|$$

by Theorem III.2.8. It now follows from I) and Corollary III.2.7 that

$$\bar{b}(T + S) = \alpha((TJ_{D(T)} + S_1)') = \alpha(T' + S_1') \leq \alpha(T') = \bar{b}(T). \qquad \square$$

III.2.10 **COROLLARY.** *Let* $T \in L(X,Y)$ *with* $0 < \gamma(T) < \infty$. *Then* $\gamma(TG_T) = \gamma((TG_T)') = \gamma(T'G_{T'})$.

Proof. Suppose $0 < \gamma(T) < \infty$. Then $\gamma(T) = \gamma(T')$ by Theorem III.2.8 and hence

$$\frac{\gamma(TG_T)}{1 - \gamma(TG_T)} = \gamma(T) = \gamma(T') = \frac{\gamma(T'G_{T'})}{1 - \gamma(T'G_{T'})}$$

by Theorem III.2.2. Consequently $\gamma(TG_T) = \gamma(T'G_{T'})$. Now since $\gamma(T) > 0$ if and only if $\gamma(TG_T) > 0$ we have $\gamma((TG_T)') = \gamma(TG_T) = \gamma(T'G_{T'})$ by Theorem III.2.8. $\square$

We remark that in Goldberg's book [16] the treatment of $\gamma(T)$ is restricted to the case where $N(T)$ is closed (cf. [16; IV.1.3]). Hence Theorem III.2.8 and Corollary III.2.9 are generalisations by the author of [16; IV.1.8 and V.1.2]. We note that the proof of Theorem III.2.8, due to the author, is an alternative to that for [16; IV.1.8].

The fact that when X is complete and T closed II) implies I) in Theorem III.2.6, was proved by R.W. Cross and privately communicated to the author. This fact, when considered alongside Theorem III.2.8, generalises a result by Kato (cf.[16; IV.1.9]). Note that *a*) and c) of Theorem III.2.6 are due to the author whereas the facts that IV) implies I) in *d*) and that I) implies IV) in b) were jointly obtained by the author and Dr. R. W. Cross. Observe that Theorem III.2.6 contains [16; II.3.11, II.3.13, IV.1.2 and IV.1.6] as special cases. Also note that II) of Corollary III.2.7 is a generalisation by the author of [16; IV.2.3]. The proofs of Corollaries III.2.7 and III.2.9 are essentially the same as in [16]. Finally observe that Corollary III.2.10 and, as was noted earlier, Theorem III.2.2 and Corollaries III.2.3 and III.2.4 are the authors own.

§3 PRODUCTS OF RANGE OPEN OPERATORS

III.3.1 *Remark* [16; II.5 (I_2,III_1)]. The product of range open operators need not be range open. Let $Y = \ell_2$ and B be a Hamel base for ℓ_2 with all elements in B having norm 1 (Proposition I.1.5). Let X be ℓ_2 renormed by $|||.|||$ where $|||.|||$ is defined as follows:

$$|||\sum_{i=1}^{n} \lambda_i b_i||| = \sum_{i=1}^{n} |\lambda_i| \quad \text{where } b_i \in B \text{ for each } i.$$

Clearly $|||x||| \geq \|x\|_2$ for each $x \in \ell_2$. Hence the identity map, say T, from X onto ℓ_2, is bounded. We show that X is not complete and hence that T does not have a bounded inverse. Let $\{x_n\}$ be any infinite countable subset of B and let $z_n = \sum_{i=1}^{n} i^{-2} x_i$. Then $\{z_n\}$ is a Cauchy sequence in X which does not converge. Consequently X is not complete. Since we now have that T does not have a bounded inverse, $T \notin RO$. However the continuous extension of T to $\tilde{X}$, $\bar{T}$, is a bounded map from a Banach space onto a Banach space and so $\bar{T} \in RO$ by the open mapping theorem. Also $J_X \in RO$. But as was noted earlier, $T = \bar{T}.J_X \notin RO$.

III.3.2 **LEMMA.** *Let* $T \in NS(X,Y)$ *with* $\gamma(T) > 0$. *If* M *is a subspace of* X *such that* $M + N(T)$ *is closed then* TM *is closed.*

Proof. Assume $\gamma(T) > 0$ for $T \in NS(X,Y)$. Let M be a subspace of X such that $M + N(T)$ is closed and let $y \in \overline{TM}$. Select $\{m_n\} \subset M \cap D(T)$ such that $Tm_n \to y$. Since $R(T)$ is closed, $y \in R(T)$ and hence $y = Tx$ for some $x \in D(T)$. We obtain $\hat{T}.Q_{N(T)}m_n = Tm_n \to \hat{T}.Q_{N(T)}x$. As in the proof of Theorem III.2.6 we conclude that $(\hat{T})^{-1}$ is continuous. Hence

$$Q_{N(T)}m_n \to Q_{N(T)}x \in D(\hat{T}) = D(T)/N(T).$$

We can therefore choose a sequence $\{k_n\}$ in $N(T)$ such that $m_n + k_n \to x$ and hence since $M + N(T)$ is closed and $\{m_n + k_n\} \subset M + N(T)$, it follows that $x \in M + N(T)$. Consequently $y = Tx \in TM$, proving that TM is closed. □

III.3.3 **LEMMA.** *Suppose* $T \in L(X,Y)$, $B \in L(Z,X)$, $\overline{D(T)} = X$ *and* $R(B)$ *closed with* $k(T)$ *and* $k(B)$ *both finite. Then* $k(TB) = k(T) + k(B)$.

Proof. Assume $T \in L(X,Y)$ and $B \in L(Z,X)$ with $R(B)$ closed, $D(T)$ dense in X and $k(T)$ and $k(B)$ both finite. It follows that $\alpha(B) < \infty$ implying that $N(B)$ is finite dimensional and hence closed. Consequently $N(TB)/N(B)$ is a normed space. Now define $\eta \in L(N(TB)/N(B), R(B) \cap N(T))$ by $\eta(x + N(B)) = Bx$. It is clear that η is a linear bijection from $N(TB)/N(B)$ onto the finite dimensional space $N_1 = R(B) \cap N(T)$ and hence it becomes clear that

(1) $$\alpha(TB) = \alpha(B) + n_1 \text{ where } n_1 = \dim N_1.$$

Now let N_2 be a subspace of $N(T)$ such that $N(T) = N_1 \oplus N_2$. Then

(2) $$\alpha(T) = n_1 + n_2 \text{ with } n_2 = \dim N_2.$$

Notice that $R(B)$ and N_2 are linearly independent since if $Bx \in R(B) \cap N_2 \subset N(T)$, then obviously $Bx \in R(B) \cap N(T) \cap N_2 = N_1 \cap N_2 = \{0\}$.

Since $R(B)$ is closed by hypothesis it follows from the finite dimensionality of N_2 that $R(B) \oplus N_2$ is closed (Proposition I.3.4). By hypothesis $D(T)$ is dense in X and hence it follows from Theorem I.1.9 that

$$R(B) \oplus N_2 \oplus N_3 = X \tag{3}$$

for some finite dimensional subspace N_3 of $D(T)$. Consequently

$$b(B) = n_2 + n_3 \quad \text{with} \quad n_3 = \dim N_3. \tag{4}$$

Now $N(T) = N_1 \oplus N_2 \subset R(B) \oplus N_2$. This together with (3) implies that T is injective on all of N_3 and hence since $TX = TR(B) \oplus TN_3$, the injectivity of T on N_3 implies that

$$b(TB) = b(T) + n_3. \tag{5}$$

From (1), (2), (4) and (5) it follows that

$$\begin{aligned} k(TB) &= a(B) + n_1 - b(T) - n_3 \\ &= a(B) + a(T) - n_2 - b(T) - n_3 \\ &= a(B) + a(T) - b(B) - b(T) \\ &= k(T) + k(B) \end{aligned}$$

hence proving the lemma. □

The following theorem was first proved by Gohberg and Krein [14] for T and B Fredholm operators with X, Y and Z complete. We give a generalisation of the version in Goldberg [16] where T is a Φ_+ operator, B closed and X and Y complete, with Z required to be complete for the last part of the result.

III.3.4 **THEOREM.** *Let* $T \in \Phi_+(X,Y)$ *with* $\gamma(T) > 0$ *and* $B \in L(Z,X)$. *Then*

I) TB *is closed whenever* B *is closed,*

II) TB *is normally solvable whenever* B *is normally solvable,*

III) TB *is a Fredholm operator with* $k(TB) = k(T) + k(B)$ *whenever* T *and* B *are Fredholm operators with* $D(T)$ *dense in* X.

Proof. I) Suppose $TBz_n \to y$ and $z_n \to z$. Then $TBz_n \to Tx$ for some $x \in D(T)$ since $R(T)$ is closed. As in the proof of Theorem III.2.6, we conclude that $(\hat{T})^{-1}$ is continuous. Consequently $Q_{N(T)}Bz_n \to Q_{N(T)}x$. Choose a sequence $\{w_n\} \subset N(T)$ such that $Bz_n + w_n \to x$. Suppose that $\{w_n\}$ is an unbounded sequence. Select a subsequence $\{w_{k'}\}$ of $\{w_k\}$ such that $\|w_{k'}\| \to \infty$. Then

$$B(z_{k'}/\|w_{k'}\|) + w_{k'}/\|w_{k'}\| \to 0.$$

Since $\{w_{k'}/\|w_{k'}\|\} \subset N(T)$, which is finite dimensional, we can choose a convergent subsequence $\{w_{k''}/\|w_{k''}\|\}$ of $\{w_{k'}/\|w_{k'}\|\}$ (Proposition I.3.3). Say $w_{k''}/\|w_{k''}\| \to v$. Then $B(z_{k''}\|w_{k''}\|) \to -v$ where $z_{k''}/\|w_{k''}\| \to 0$. From the fact that B is closed, we conclude that $-v = 0$. Hence $w_{k''}/\|w_{k''}\| \to 0$. A contradiction since $\{w_{k''}/\|w_{k''}\|\}$ is a sequence of norm one elements. Therefore $\{w_k\}$ is bounded. Since $N(T)$ is finite dimensional, we can select a subsequence $\{w_{k_n}\}$ such that $w_{k_n} \to w \in N(T)$. Thus $Bz_{k_n} \to x - w$. As B is closed, we have $z \in D(B)$ and $Bz = x - w$. Then, since T is closed, $Bz \in D(T)$ and $TBz = Tx = y$, showing that TB is closed.

II) This follows from I) and Lemma III.3.2 since if $R(B)$ is closed, $R(B) + N(T)$ is closed by the finite dimensionality of $N(T)$ (Proposition I.3.4).

III) If B is normally solvable, TB is normally solvable by II). Hence if in addition $D(T)$ is dense in X with $k(T)$ and $k(B)$ both finite, $k(TB) = k(T) + k(B)$ by Lemma III.3.3 and hence the result follows. □

III.3.5 **DEFINITION**. Let $T \in L(X,Y)$ and suppose $X = Y$. Given a polynomial $p(\lambda) = \sum_{k=0}^{n} a_k\lambda^k$, define $p(T) = \sum_{k=0}^{n} a_kT^k$, where T^0 is the identity operator I defined on all of X. The domain of $p(T)$ is the domain of T^n.

III.3.6 **COROLLARY**. *Let* $T \in L(X,X)$. *If there exists a scalar* λ_0 *such that* $\lambda_0 I - T \in \Phi_+$ *with* $\gamma(\lambda_0 I - T) > 0$, *then* $p(T)$ *is closed for any polynomial* p.

Proof. Let the degree of p be n. We prove the corollary by induction. The corollary is trivial for the case where $n = 0$. Suppose it holds for $n = k$ and let q be a polynomial of degree $k + 1$. Now let

$$q(\lambda) = (\lambda_0 - \lambda)\, r(\lambda) + c$$

where r is a polynomial of degree k and c is a constant. Hence

$$q(T) = (\lambda_0 I - T).r(T) + cI.$$

By the induction hypothesis $r(T)$ is closed, and so by I) of Theorem III.3.4 $(\lambda_0 I - T).r(T)$ is closed. We conclude that $q(T)$ is closed. □

Theorem III.3.4 was proved with the assistance of R.W. Cross. Corollary III.3.6 generalises [16; IV.2.12] where X is required to be complete. The proof is the same with Theorem III.3.4 being used instead of [16; IV.2.7]. Lemma III.3.2 generalises [16; IV.2.9] and was noted by J.Jaftha and privately communicated to the author. Lemma III.3.3 is a generalisation by the author and is essentially contained in the proof of [16; IV.2.7].

CHAPTER IV

SEMI-PRECOMPACT AND SEMI-CONTINUOUS OPERATORS

§1 SEMI- AND PARTIALLY CONTINUOUS OPERATORS

IV.1.1 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent*

I) $T \in SB(X,Y)$.

II) $T \in SB(X,\overline{R(T)})$.

III) $\Gamma_0'(T) < \infty$

IV) $T = A + S$ *where* A *is continuous and* S *a finite rank operator.*

V) $D(T')$ *is* $\sigma(Y',Y)$*-closed and finite codimensional in* Y'.

Proof. I) ⟺ III) The equivalence of I) and III) follows from the definitions of Γ_0' and of semi-continuity.

I) ⟺ IV). Suppose $T \in SB(X,Y)$. Without loss of generality assume $D(T) = X$. By definition there exists $F \in \mathcal{F}(Y)$ such that $Q_F T$ is bounded. Now since F is finite dimensional, there exists $E \in \mathcal{F}_c(Y)$ such that E is a topological complement of F in Y (Proposition I.8.4). Therefore $E \approx Y/F$ under the isomorphism $Q_F J_E$ (Proposition I.8.5). Also $cod_{R(T)} E \cap R(T) < \infty$ since $cod_Y E < \infty$, whence $cod_X T^{-1}(E \cap R(T)) < \infty$ where T^{-1} is taken in the set theoretic sense. Write $N = T^{-1}(E \cap R(T))$. Recall that $Q_F J_E$ is an isomorphism, and so from the fact that $Q_F J_E T|_N$ is continuous, we conclude that $T|_N$ is continuous. Now since $N \in \mathcal{F}_c(X)$, $\bar{N} \in \mathcal{F}_c(X)$ and since $\bar{N}$ is closed, there exists $K \in \mathcal{F}(X)$ such that $\bar{N}$ and K are topological complements in X. Now define $A \in B[X,\tilde{Y}]$ by $A|_{\bar{N}} = (\overline{T|_N})|_{\bar{N}}$ and $A|_K = 0$. Note that A is continuous since $A|_{\bar{N}}$ is continuous and since there exists a bounded projection from X onto $\bar{N}$. Select an arbitrary $x \in \bar{N}$.

Then there exists $\{x_n\} \subset N$ such that $x_n \to x$ and hence $Ax_n = Tx_n \to Ax \in \tilde{Y}$. But since Q_FT is bounded, $Q_FTx_n \to Q_FTx$. Choose a sequence $\{f_n\} \subset F$ such that $Tx_n + f_n \to Tx \in Y$. But then $f_n = f_n + Tx_n - Ax_n \to Tx - Ax \in \tilde{Y}$ and since $\{f_n\} \subset F$, $Tx - Ax \in F \subset Y$. Hence $Ax \in Y$. Consequently $A \in B[X,Y]$. Since A agrees with T on $N \in \mathcal{F}_c(X)$, $T - A$ is a finite rank operator with $T = A + T - A$.

Conversely note that if $T = A + S$ with $A \in B(X,Y)$ and $dim\ R(S) < \infty$, then $Q_{R(S)}T = Q_{R(S)}A$ and so $T \in SB(X,Y)$.

I) ⇔ II) Trivially $T \in SB(X,Y)$ if $T \in SB(X,\overline{R(T)})$. Conversely suppose $T \in SB(X,Y)$. Repeating the construction in the first part of the proof we note that $TN = AN \subset A\bar{N} = R(A) \subset Y$. However as N is dense in $\bar{N}$, $AN = TN$ is dense in $A\bar{N}$ by continuity. Hence $\overline{R(T)} \supset \overline{TN} \supset R(A)$. Consequently $A \in B(X,\overline{R(T)})$ and trivially $R(T - A) \subset \overline{R(T)}$. Thus $R = R(T - A) \in \mathcal{F}(\overline{R(T)})$ such that $Q_RT = Q_RA$ is continuous.

I) ⇒ V). Suppose $T \in SB(X,Y)$. Now select $F \in \mathcal{F}(Y)$ such that Q_FT is continuous. Then $(Q_FT)' = T'J_{F^\perp}$ is bounded by Theorem I.10.3. But $F^\perp$ is $\sigma(Y',Y)$-closed and finite codimensional in Y' (Remark I.5.9 and Theorem I.5.10). Hence since $F^\perp \subset D(T')$, $D(T')$ is closed and finite codimensional in Y' (Proposition I.3.4). Now consider $Q^{Y'}_{F^\perp} = (J^Y_F)'$ where $F' \equiv Y'/F^\perp$ by Theorem I.5.10. Note that $D(T')/F^\perp$ is a finite dimensional and thus a $\sigma(Y'/F^\perp, F)$-closed subspace of $Y'/F^\perp$ (Remark I.5.9). Consequently as $Q^{Y'}_{F^\perp}$ is $\sigma(Y',Y)$ to $\sigma(Y'/F^\perp, F)$ continuous by Theorem I.10.3, $(Q^{Y'}_{F^\perp})^{-1}(D(T')/F^\perp) = D(T')$ is $\sigma(Y',Y)$-closed where $(Q^{Y'}_{F^\perp})^{-1}$ is taken in the set theoretic sense.

V) $\Rightarrow$ I) Suppose $D(T')$ is $\sigma(Y',Y)$-closed with $\operatorname{cod}_{Y'} D(T') < \infty$. Write $N = D(T')$. Then $N = ({}^{\perp}N)^{\perp}$ with ${}^{\perp}N \in \mathcal{F}(Y)$ since $N \in \mathcal{F}_c(Y')$ (Remark I.5.9 and Theorem I.5.10). Now $T'J_N = T'J_{({}^{\perp}N)^{\perp}} = (Q_{{}^{\perp}N}T)'$ is a bounded operator and hence $Q_{{}^{\perp}N}T$ is continuous (Theorem I.10.3 and Remark I.10.10). □

The following characterisations for partial continuity were obtained by R.W. Cross.

IV.1.2 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $T \in PB(X,Y)$.

II) *For every* $M \in \mathcal{I}(D(T))$ *there exists* $N \in \mathcal{I}(M)$ *such that* TJ_N *is continuous* [7; *Theorem* 4].

III) $\Delta(T) < \infty$ [6; 4.3].

IV) $\Gamma_0(T) < \infty$ [6; 4.4].

Proof. Without loss of generality let $\dim D(T) = \infty$.

I) $\Rightarrow$ IV). Suppose $T \in PB(X,Y)$ and select $E \in \mathcal{F}_c(D(T))$ such that TJ_E is continuous. Then $\Gamma_0(T) \leq \|TJ_E\| < \infty$.

IV) $\Rightarrow$ III). We show that $\Delta(T) \leq \Gamma_0(T)$. This is trivial if $\Gamma_0(T) = \infty$ and so let $\Gamma_0(T) < \infty$. Now select $E \in \mathcal{F}_c(D(T))$ such that $\|TJ_E\| \leq \Gamma_0(T) + \epsilon$ for some arbitrary $\epsilon > 0$. Considering Proposition II.1.4 we obtain

$$\Delta(T) = \Delta(TJ_E) \leq \|TJ_E\| \leq \Gamma_0(T) + \epsilon$$

and hence $\Delta(T) \leq \Gamma_0(T)$ since $\epsilon > 0$ was arbitrary.

III) $\Rightarrow$ II). Let $M \in \mathcal{I}(D(T))$ be arbitrary and suppose $\Delta(T) < \infty$. Then $\Gamma(TJ_M) \leq \sup_{K \in \mathcal{I}(D(T))} \Gamma(TJ_K) = \Delta(T) < \infty$ and so by the definition of $\Gamma(TJ_M)$ there must be some $N \in \mathcal{I}(M)$ such that TJ_N is continuous.

II) $\Rightarrow$ I). Suppose that II) holds and that $T \notin PB(X,Y)$. Now if $G_T \in F_+$ we could select $E \in \mathcal{F}_c(X_T)$ such that $(G_TJ_E)^{-1}$ is continuous. But then $G(E) \in \mathcal{F}_c(D(T))$ with $TJ_{G(E)} = (TG)(GJ_E)^{-1}$ continuous. We conclude that $G_T \notin F_+$ if $T \notin PB(X,Y)$. Considering Theorem II.3.5 we select $K \in \mathcal{I}(X_T)$ such that G_TJ_K is precompact. By II) we can now choose $N \in \mathcal{I}(GK)$ so that TJ_N is continuous and hence that $GJ_{G^{-1}(N)}$ has a continuous inverse. But $G^{-1}(N) \subset K$ and so $GJ_{G^{-1}(N)}$ is precompact as well. Therefore $GJ_{G^{-1}(N)}$ is a precompact isomorphism from $G^{-1}(N)$ onto N and so $\dim N < \infty$ (Proposition I.3.3). But this is obviously a contradiction and hence we conclude that $T \in PB(X,Y)$ if II) holds. □

Comparing partial and semi-continuity we get the following:

IV.1.3 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then* $T \in PB(X,Y)$ *if and only if* $T \in SB(X,\tilde{Y})$.

Proof. If $T \in SB(X,\tilde{Y})$, then $T = A + S$ where $A \in B(X,\tilde{Y})$ and $\dim R(S) < \infty$ by Theorem IV.1.1. Taking the restriction of T to $N(S)$ we conclude that $T|_{N(S)} = A|_{N(S)}$ and hence that $T \in PB(X,Y)$. Conversely if $T \in PB(X,Y)$ then there exists $N \in \mathcal{F}_c(X)$ such that $T|_N$ is continuous. As in Theorem IV.1.1 we can construct $A \in B(X,\tilde{Y})$ such that $A|_{\bar{N}} = (\overline{T|_N})|_{\bar{N}}$ and hence $T = A + T - A$ where $\dim R(T - A) < \infty$. The result now follows from Theorem IV.1.1. □

IV.1.4 **COROLLARY.** *Let* $T \in L(X,Y)$. *Then* $T \in PB(X,Y)$ *if and only if* $D(T')$ *is* $\sigma(Y',\tilde{Y})$*-closed and finite codimensional in* Y'.

IV.1.5 **COROLLARY.** *The adjoints of partially continuous operators are continuous.*

Proof. Follows from Corollary IV.1.4, the closed graph theorem and the fact that T' is closed. □

IV.1.6 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $T \notin PB(X,Y)$.

II) $G_T \notin F_+$.

III) *For each* $\epsilon > 0$ *there exists* $N \in \mathcal{S}(D(T))$ *such that* $(TJ_N)^{-1}$ *exists and is precompact with norm not exceeding* $\epsilon > 0$.

IV) *There exists* $N \in \mathcal{S}(D(T))$ *such that* $(TJ_N)^{-1}$ *exists and is precompact.*

V) *For each* $\epsilon > 0$ *there exists* $N \in \mathcal{S}(D(T))$ *such that* $(TJ_N)^{-1}$ *exists and has norm not exceeding* ϵ [6; 4.3 *and* 4.6].

Proof. I) $\Leftrightarrow$ II) The equivalence of I) and II) follows from the fact that for $N \in \mathcal{F}_c(D(T))$, TJ_N is continuous if and only if $G_TJ_{G^{-1}N}$ is an isomorphism.

II) $\Rightarrow$ III) Note that if $\dim X_T < \infty$ then $\{0\} \in \mathcal{F}_c(X_T)$ with $G_TJ_{\{0\}}$ having a continuous inverse. But this is a contradiction since $G_T \notin F_+$. Hence $\dim X_T = \infty$. By Theorem II.3.5 it follows that we can select $M \in \mathcal{S}(X_T)$ with G_TJ_M precompact. Let $\epsilon > 0$ be arbitrary. Considering Proposition II.3.3 we note that we could have chosen M so that $\|G_TJ_M\| \leq \dfrac{\epsilon}{1+\epsilon}$ as well. Hence $\|x\| \leq \dfrac{\epsilon}{1+\epsilon}(\|x\| + \|Tx\|)$ for each $x \in GM$ and so $\|x\| \leq \epsilon\|Tx\|$ for each $x \in GM$. Hence $(TJ_{GM})^{-1}$ exists and has norm not exceeding ϵ. Now let $\{Tx_n\}$ be an arbitrary bounded sequence in TGM. Say $\{Tx_n\}$ is bounded by λ. Then $\{(TJ_{GM})^{-1}Tx_n\} = \{z_n\} \subset GM$ is bounded by $\epsilon.\lambda$ since $\|(TJ_{GM})^{-1}\| \leq \epsilon$. Hence $\{G_T^{-1}z_n\} \subset M$ is bounded by $\lambda(1+\epsilon)$. However since G_TJ_M is precompact, we conclude that $\{z_n\}$ has a Cauchy subsequence and so $(T|_{GM})^{-1}$ is precompact .

III) ⇒ IV) This follows trivially from III).

IV) ⇒ V) Suppose IV) holds. Let $N \in \mathcal{I}(D(T))$ be such that $(TJ_N)^{-1}$ exists and is precompact and let $\epsilon > 0$ be arbitrary. Considering Proposition II.3.3 we conclude that V) holds.

V) ⇒ I) Suppose that V) holds and that $T \in PB(X,Y)$. By definition there exists $E \in \mathcal{F}_c(D(T))$ such that $\|TJ_E\| \leq k$ for some $k > 0$. By V) there exists $N \in \mathcal{I}(D(T))$ such that $(TJ_N)^{-1}$ exists with $\|(TJ_N)^{-1}\| \leq (1/2k)$. Hence $N \cap E \in \mathcal{I}(D(T))$ such that

$$\|x\| \leq (1/2k)\|TJ_{N\cap E}x\| \leq k(1/2k)\ \|x\| = 1/2\ \|x\| \quad \text{for each } x \in N \cap E;$$

an obvious contradiction. Hence V) implies I). □

IV.1.7 THEOREM. *Let* $T \in L(X,Y)$ *such that either* $G_T \in C(\tilde{X}_T, D(T))$, *or* $\bar{G}_T \in B[\tilde{X}_T, (D(T))^{\sim}]$ *is injective. Then the following are equivalent:*

I) $T \in PB(X,Y)$;

II) $T \in SB(X,Y)$;

III) $T \in SB(X,R(T))$;

IV) $G_T \in F_+$;

V) $G_T \in F_-$.

Proof. I) ⇒ IV). This follows from Proposition IV.1.6.

IV) ⇒ V) Note that by definition $G_T \in B[X_T, D(T)]$ and hence $\bar{b}(G_T) = 0$. Now since $G_T \in F_+$, $(G_T)' \in \Phi_-$ by Theorem II.3.5 and since $\bar{b}(G_T) = \alpha((G_T)') = 0$ by Corollary III.2.7, we conclude that $(G_T)' \in \Phi_+$. Hence $G_T \in F_-$ by Theorem II.3.6.

V) ⇒ III) Assume $G_T \in F_-$. First suppose $G_T \in C(\tilde{X}_T, D(T))$. Since $G_T \in F_-$ there exists $K \in \mathcal{F}(D(T))$ such that $(Q_K^{D(T)} G_T)'$ has a continuous inverse. However as $Q_K^{D(T)} \in \Phi_+$ with $\gamma(Q_K^{D(T)}) > 0$, it follows that $Q_K^{D(T)} G_T \in C(\tilde{X}_T, D(T)/K)$ by Theorem III.3.4. But then $Q_K^{D(T)} G_T$ is an open map by Theorem I.10.8 and so since $\gamma(Q_K^{D(T)} G_T) = \gamma((Q_K^{D(T)} G_T)^\wedge)$, $(Q_K^{D(T)} G_T)^\wedge$ is still open by Proposition III.1.2. Thus $(Q_K^{D(T)} G_T)^\wedge$ is an isomorphism from $X_T/G^{-1}(K)$ onto $D(T)/K$ (Proposition I.4.16). Now suppose $\bar{G}_T \in B[\tilde{X}_T, (D(T))^\sim]$ is injective. Noting that for $K \in \mathcal{F}(D(T))$ $(Q_K^{D(T)} G_T)' = (Q_K^{D(T)^\sim} . \bar{G}_T)'$, we can argue as before to show that $(Q_K^{D(T)^\sim} . \bar{G}_T)^\wedge$ is an isomporphism from $\tilde{X}_T/G^{-1}(K)$ onto $D(T)^\sim/K$. Hence as $(Q_K^{D(T)^\sim} . \bar{G}_T)^\wedge . J_{X_T/G^{-1}(K)} = J_{D(T)/K}(Q_K^{D(T)} G_T)^\wedge$, $(Q_K^{D(T)} G_T)^\wedge$ is an isomorphism from $X_T/G^{-1}(K)$ onto $D(T)/K$. Now consider the following:

$$(1) \qquad (Q_{TK} T G_T)^\wedge . Q^{X_T/G^{-1}(K)}_{N/G^{-1}(K)} . [(Q_K G_T)^\wedge]^{-1} . Q_K$$

$$\text{where } N = N(Q_{TK} T G_T) \text{ and } Q_K^{D(T)} = Q_K.$$

Note that Q_K is bounded and maps each $x \in D(T)$ onto $x + K \in D(T)/K$. $[(Q_K G_T)^\wedge]^{-1}$ is bounded on $D(T)/K$ and maps each $x + K \in D(T)/K$ onto $G_T^{-1}x + G^{-1}K \in X_T/G^{-1}(K)$. Again $Q^{X_T/G^{-1}(K)}_{N/G^{-1}(K)}$ is bounded and maps each $G_T^{-1}x + G^{-1}K$ onto $G_T^{-1}x + N = G_T^{-1}x + N(Q_{TK} T G_T) \in X_T/N$ (observe that $(X_T/N) \equiv (X_T/G^{-1}(K))/(N/G^{-1}(K))$ by Proposition I.8.6 since $G_T^{-1}(K) \subset N(Q_{TK} T G_T)$). Finally $(Q_{TK} T G_T)^\wedge$ is bounded and maps each $G_T^{-1}x + N(Q_{TK} T G_T)$ onto $Q_{TK} T x$. We deduce that the continuous map in (1) is nothing else than $Q_{TK} T x$ where $TK \in \mathcal{F}(R(T))$.

III) ⇒ II). This follows from the definition of semi-continuity.

II) ⇒ I) Follows from Proposition IV.1.3. □

IV.1.8 *Examples*. I) There exists a partially continuous operator which is not semi-continuous. Let $X = c_0$. If $e_n = (0,0,\dots,0,1,0,\dots)$ (1 in the nth position) and f_n is the bounded linear functional on c_0 defined by $f_n(e_k) = \delta_{nk}$, we define a bounded linear operator from c_0 into ℓ_2 as follows:

$$Ax = \sum_{n=1}^{\infty} \frac{1}{n} f_n(x)e_n \quad \text{for each} \quad x \in c_0.$$

Defining $A_k = \sum_{n=1}^{k} \frac{1}{n} f_n(\cdot)e_n$ we note that

$$\|(A - A_k)x\|_2 = \|\sum_{n=k+1}^{\infty} \frac{1}{n} f_n(x)e_n\|_2$$

$$= (\sum_{n=k+1}^{\infty} (1/n^2)|f_n(x)|^2)^{1/2}$$

$$\leq (\sum_{n=k+1}^{\infty} 1/n^2)^{1/2} . \sup_{n\in\mathbb{N}}|f_n(x)|$$

$$= (\sum_{n=1}^{\infty} 1/n^2)^{1/2} . \|x\|_0 \quad \text{for each} \quad x \in c_0.$$

Note that Ax exists for each $x \in c_0$ by Theorem I.2.3. In fact A is bounded and is the limit of $\{A_k\}$ in $B[c_0,\ell_2]$ and so A is compact (Proposition II.3.10 and Remark I.11.4). Now let A_0 be the restriction of A to a dense subspace of codimension 1 in c_0 and let $Y = R(A_0)$. Choose $x_0 \in c_0\backslash D(A_0)$ and define $T \in L[c_0,Y]$ by $T|_{D(A_0)} = A_0$ and $Tx_0 = 0$. Trivially $T \in PB(X,Y)$. Now suppose $T \in SB[X,Y]$. Select $\{x_n\} \subset D(A_0)$ such that $x_n \to x_0$. Hence $Tx_n = Ax_n \to Ax_0$. However since $T \in SB[X,Y]$ there exists $F \in \mathcal{F}(Y)$ such that Q_FT is bounded and so $Q_FTx_n \to Q_FTx_0 = 0$.

Consequently there exists $\{f_n\} \subset F$ such that $f_n + Tx_n = f_n + Ax_n \to 0$. Hence $f_n \to -Ax_0$. We conclude that $Ax_0 \in F \subset Y$. However since A is injective with $x_0 \notin D(A_0)$, we see that $Ax_0 \notin R(A_0) = Y$. An obvious contradiction. Now since A is compact and $AJ_{D(A_0)} = TJ_{D(A_0)}$, we conclude from Proposition II.3.3 that $\Gamma_0(T) = 0$. But since $T \notin SB[X,Y]$, $\Gamma_0'(T) = \infty$ by Theorem IV.1.1.

II) There exists $S \in SB[X,Y]$ such that $S \notin SB[X,R(S)]$. Let $X = c_0$, $Y = \ell_2$ and $S = J_{R(T)}T$ in the previous example. Then $S \in SB[X,R(T)^{\sim}] \subset SB[X,Y]$ by Proposition IV.1.3. However $S \notin SB[X,R(S)]$ as was shown.

We shall see in Chapter V that for any $T \in L(X,Y)$, $\Delta'(T) < \infty$ need not imply that $T \in SB(X,Y)$. Finally we ask ourselves the question. When does partial continuity imply continuity? The following result is due to R.W. Cross.

IV.1.9 **PROPOSITION.** *Let* $T \in PB(X,Y)$.

I) *If there exists a closed subspace* M *of* $D(T)$ *such that* $\mathrm{cod}_{D(T)}M < \infty$ *and* TJ_M *is continuous, then* T *is continuous* [6; 2.16].

II) *If* X_T *is complete, then* T *is continuous* [7; Corollary 11].

III) *If* J_YT *is closable in* $L(X,\tilde{Y})$ *then* T *is continuous.*

Proof. I) Suppose $T \in PB(X,Y)$ and that there exists $M \in \mathcal{F}_c(D(T))$ such that M is closed with TJ_M continuous. Let P be a projection from $D(T)$ onto M with say $F = N(P)$ (Proposition I.8.4). Note that $\dim F = \mathrm{cod}_{D(T)}M < \infty$ and so $G(J_{TF}^{-1}\, TJ_F)$ is a finite dimensional and hence complete subspace of $TF \times F$ (Proposition I.3.1). Consequently $J_{TF}^{-1}\, TJ_F \in C[F,TF]$. Since both F and TF are complete (Proposition I.3.1) we conclude from the closed graph theorem that TJ_F is continuous. Now for any $x \in D(T)$ we have $(I-P)x \in F$ and $Px \in M$ and hence

$$\|Tx\| \leq \|TJ_MPx\| + \|TJ_F(I-P)x\| \leq (\|TJ_M\|\,\|P\| + \|TJ_F\|\,\|I-P\|)\|x\|.$$

II) Suppose $T \in L(X,Y)$ with X_T complete. If $\{x_n\} \subset D(T)$ such that $x_n \to x \in X$ and $Tx_n \to \tilde{y} \in Y$ then since both $\{x_n\}$ and $\{Tx_n\}$ must be Cauchy we conclude that $\{G_T^{-1}x_n\} \subset X_T$ is Cauchy. By the completeness of X_T, $x \in D(T)$ and $Tx = \tilde{y}$. Hence $J_Y T$ is closed and so by part III) T is continuous.

III) Let $T \in PB(X,Y)$ with $J_Y T$ closable in $L(X,\tilde{Y})$. Select $M \in \mathcal{F}_c(D(T))$ such that TJ_M is continuous. Noting that $(\overline{TJ_M})J_{\bar{M}}^{\tilde{M}} = (J_Y T)^\sim J_{\bar{M}}^X$ we conclude from I) that $(J_Y T)^\sim$ and hence T is continuous. □

Theorem IV.1.2, Propositions IV.1.3 and IV.1.9, Corollary IV.1.5 as well as I) of Example IV.1.8 are by R.W. Cross. Proposition IV.1.6 was established independently by the author and is equivalent to a combination of [6; 4.3] and [6; 4.6]. All other results in this section are by the author.

§2 SEMI-PRECOMPACT OPERATORS

IV.2.1 **PROPOSITION.** *$SPK[X,Y]$ is an operator ideal with respect to the semi-bounded operators and is closed under addition.*

Proof. The result follows on considering Theorems II.2.4, II.2.13, II.3.2 and IV.1.1. □

IV.2.2 **PROPOSITION.** *Let $T \in L(X,Y)$. Then $T \in SPK(X,Y)$ if and only if $T = A + S$ where A is a precompact and S a finite rank operator in $L(X,Y)$.*

Proof. Suppose $T = A + S$ where $A \in PK(X,Y)$ and $\dim R(S) < \infty$. Let $R = R(S)$. Then $Q_R T = Q_R A \in PK(X,Y/R)$ (Theorem I.11.8). Hence $T \in SPK(X,Y)$.

Conversely suppose $T \in SPK(X,Y)$. Then $T = A + S$ where A is continuous and $\dim R(S) < \infty$ by Theorem IV.1.1. Now considering Proposition II.1.3 we have $\Gamma_0'(T) = \Gamma_0'(Q_R T) = \Gamma_0'(Q_R A|_{D(T)}) = \Gamma_0'(A|_{D(T)})$ where $R = R(S)$. Hence $A|_{D(T)}$ is precompact by Theorem II.3.2 with $T = A|_{D(T)} + T - A$. □

IV.2.3 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $T \in PPK(X,Y)$.

II) $TG \in PK[X_T, Y]$.

III) $T \in SPK(X,\tilde{Y})$.

Proof. I) ⇔ II) Suppose $T \in PPK(X,Y)$. Then since $\|TGJ_M\| \le \|TJ_{GM}\|\,\|G_T\| \le \|TJ_{GM}\|$ (if the norm exists) for each $M \in \mathcal{F}_c(X_T)$, we conclude that $\Gamma_0(TG) \le \Gamma_0(T)$ and hence that TG is precompact by Proposition II.3.3.

Conversely suppose TG is precompact. From Proposition II.3.3 it follows that for some arbitrary $\epsilon > 0$, there exists $M \in \mathcal{F}_c(X_T)$ such that

$$\|Tx\| \le \frac{\epsilon}{1+\epsilon}\,(\|x\| + \|Tx\|) \quad \text{for each} \quad G_T^{-1}x \in M.$$

Hence

$$\|Tx\| \le \epsilon\,\|x\| \quad \text{for each} \quad x \in GM$$

and therefore $\|TJ_{GM}\| \le \epsilon$ where $GM \in \mathcal{F}_c(D(T))$. Since $\epsilon > 0$ was arbitrary we conclude from Proposition II.3.3 that $T \in PPK(X,Y)$.

I) ⇔ III) Let $T \in PPK(X,Y)$. Then $T \in PB(X,Y)$ and so $T \in SB(X,\tilde{Y})$ by Proposition IV.1.3. Hence there exists $F \in \mathcal{F}(\tilde{Y})$ such that $Q_F J_Y T$ is continuous. However for any $M \in \mathcal{F}_c(D(T))$ for which TJ_M is continuous, we have $\|Q_F J_Y T J_M\| \le \|TJ_M\|$ and therefore $\Gamma_0(Q_F J_Y T) \le \Gamma_0(T) = 0$. We conclude from Proposition II.3.3 that $Q_F J_Y T$ is precompact and hence that $T \in SPK(X,\tilde{Y})$.

Conversely suppose $T \in SPK(X,\tilde{Y})$. Hence $\Gamma_0'(J_YT) = 0$ by Theorem II.3.2. But $T \in SB(X,\tilde{Y})$ and so $T \in PB(X,Y)$ by Proposition IV.1.3. Let $M \in \mathcal{F}_c(D(T))$ such that TJ_M is continuous. For any $F \in \mathcal{F}(\tilde{Y})$ for which Q_FJ_YT is continuous we have that $\|Q_FJ_YTJ_M\| \leq \|Q_FJ_YT\|$ and hence $\Gamma_0'(J_YTJ_M) \leq \Gamma_0'(J_YT) = 0$. Consequently J_YTJ_M is compact by Theorem II.3.2 and the completeness of $\tilde{Y}$. Thus TJ_M is precompact, implying that $T \in PPK(X,Y)$. □

IV.2.4 *Remarks*. I) There exists a partially precompact operator which is not semi-precompact. (Consider Example IV.1.8 I).)

II) $PPK[X,Y]$ is an operator ideal with respect to the partially bounded operators and is closed under addition. (Consider Propositions IV.1.3, IV.2.1 and IV.2.3).

III) All finite rank operators are semi-precompact. (Follows from the definition of Γ_0').

IV.2.5 **PROPOSITION**. *Let* $T \in L(X,Y)$. *Then*

I) $\Gamma_0(T) \geq \Gamma_0'(T')$ *and* $\Gamma_0'(T) \geq \Gamma_0(T')$;

II) $\Gamma_0(T) = \Gamma_0'(T')$ *if* T *is continuous.*

Proof. Without loss of generality let $D(T) = X$.

I) The inequalities are trivial if $\Gamma_0(T) = \infty = \Gamma_0'(T)$. Hence suppose $\Gamma_0(T) < \infty$ $(\Gamma_0'(T) < \infty)$. Let $\epsilon > 0$ be arbitrary and choose $M \in \mathcal{F}_c(X)$ $(F \in \mathcal{F}(Y))$ so that

$$(1) \qquad \Gamma_0(T) + \epsilon \geq \|TJ_M\| \quad (\Gamma_0'(T) + \epsilon \geq \|Q_FT\|).$$

From Theorem I.5.10 we conclude that $M^{\perp} \in \mathcal{F}(X')$ $(F^{\perp} \in \mathcal{F}_c(Y'))$.

Now since $(TJ_M)'$ is an extension of $(J_M)'\ T' = Q^{X'}_{M^\perp} T'$ we deduce from (1) that

$$\Gamma_0(T) + \epsilon \ge \|TJ_M\| = \|(TJ_M)'\| \ge \|Q^{X'}_{M^\perp} T'\| \ge \Gamma_0'(T')$$

and hence that $\Gamma_0(T) \ge \Gamma_0'(T')$ since $\epsilon > 0$ was chosen arbitrarily. (By analogy since $F^\perp \in \mathcal{F}_c(Y')$, we have $F^\perp \cap D(T') \in \mathcal{F}_c(D(T'))$ and so from (1) it follows that

$$\Gamma_0'(T) + \epsilon \ge \|Q_F T\| = \|(Q_F T)'\| = \|T' J_{F^\perp}\| \ge \Gamma_0(T').$$

Again, since $\epsilon > 0$ was chosen arbitrarily, we conclude that $\Gamma_0'(T) \ge \Gamma_0(T')$.)

II) Let T be continuous. Considering I) we see that we need only show that $\Gamma_0'(T') \ge \Gamma_0(T)$. Let $\epsilon > 0$ be arbitrary and select $F \in \mathcal{F}(X')$ so that

(2) $$\Gamma_0'(T') + \epsilon \ge \|Q_F T'\|.$$

Note that $F = ({}^\perp F)^\perp$ by the finite dimensionality of F (Remark I.5.9) and that ${}^\perp F \in \mathcal{F}_c(X)$ by Theorem I.5.10. Also since T is continuous, $(TJ_{{}^\perp F})' = (J_{{}^\perp F})'T' = Q^{X'}_{({}^\perp F)^\perp} T' = Q^{X'}_F\, T'$ and so by (2) we have

$$\Gamma_0'(T') + \epsilon \ge \|Q^{X'}_F\, T'\| = \|(TJ_{{}^\perp F})'\| = \|TJ_{{}^\perp F}\| \ge \Gamma_0(T).$$

Since $\epsilon > 0$ was chosen arbitrarily, $\Gamma_0'(T') \ge \Gamma_0(T)$ as was required. □

IV.2.6 **COROLLARY**. *Let* $T \in L(X,Y)$. *Then* $T \in SPK(X,Y)$ *if and only if* T' *is compact with* $D(T')$ $\sigma(Y',Y)$*-closed and finite codimensional in* Y'.

Proof. Suppose $T \in SPK(X,Y)$. Then $T \in SB(X,Y)$ and so T' is continuous with $D(T')$ $\sigma(Y',Y)$-closed and finite codimensional in Y' by Theorem IV.1.1. But $\Gamma_0'(T) \ge \Gamma_0(T') \ge 0$ by Proposition IV.2.5 and so by Theorem II.3.2 $0 = \Gamma_0'(T) = \Gamma_0(T')$. Hence T' is compact (Proposition II.3.3 and Remark I.11.4). Conversely let T' be compact with $D(T')$ $\sigma(Y',Y)$-closed and finite codimensional in Y'. Hence $T'J_N$ is bounded where $N = D(T')$. Note that as in Theorem IV.1.1, $N = ({}^\perp N)^\perp$.

From Theorem I.10.3 we conclude that since $T'J_N = T'J_{({}^{\perp}N)^{\perp}} = (Q_{{}^{\perp}N}T)'$ is bounded, $Q_{{}^{\perp}N}T$ is continuous. Since $T'J_N$ is compact, we have by Theorem II.3.2 and Proposition IV.2.5 that

$$0 = \Gamma_0'(T'J_N) = \Gamma_0(Q_{{}^{\perp}N}T).$$

Hence $Q_{{}^{\perp}N}T$ is precompact by Proposition II.3.3. As ${}^{\perp}N \in \mathcal{F}(Y)$ (Theorem I.5.10), we are done. $\square$

IV.2.7 COROLLARY. *Let* $T \in L(X,Y)$. *Then* $T \in PPK(X,Y)$ *if and only if* T' *is compact with* $D(T')$ $\sigma(Y',\tilde{Y})$*-closed and finite codimensional in* Y'.

Proof. Follows from Proposition IV.2.3 and Corollary IV.2.6. $\square$

All results in this section are by the author.

CHAPTER V

STRICTLY COSINGULAR OPERATORS

§1 CONTINUOUS AND SEMI-CONTINUOUS STRICTLY COSINGULAR OPERATORS

V.1.1 **THEOREM.** *Let* $T \in L(X,Y)$. *Consider the following statements:*

I) $T \in SC(X,Y)$.

II) *There is no* $M \in \mathscr{J}_c(Y)$ *such that* $Q_M T$ *is an open map.*

III) *There is no* $M \in \mathscr{J}_c(Y)$ *such that* $Q_M T$ *is surjective.*

IV) *There is no* $M \in \mathscr{J}_c(Y)$ *such that* $\gamma(Q_M T) > 0$ *and* $\bar{b}(Q_M T) = 0$.

V) *There is no* $M \in \mathscr{J}_c(Y)$ *such that* $R(T'J_{M^\perp}) = N(Q_M T J_{D(T)})^\perp$ *and* $a(T'J_{M^\perp}) = 0$.

a) *In general IV) and V) are equivalent, either of which imply II), I) implies IV) and III) implies II).*

b) *If* X *is complete and* $T \in SB[X,Y]$ *then I), II), IV) and V) are all equivalent.*

c) *If* Y *is complete I) implies III) and if in addition* X *is an operator range and* $T \in SB[X,Y]$, *then II) and III) are equivalent.*

d) *If* X *and* Y *are complete and* $T \in SB[X,Y]$, *all five statements are equivalent.*

Proof. *a)* IV) ⇔ V) This follows easily from Theorem III.2.6 and Corollary III.2.7.

IV) ⇒ II) This is a trivial consequence of Proposition III.1.2.

I) ⇒ IV) Suppose there exists $M \in \mathscr{J}_c(Y)$ such that $\gamma(Q_M T) > 0$ and $\bar{b}(Q_M T) = 0$. Then $(Q_M T)' = T'J_{M^\perp}$ is injective by Proposition I.10.4 and also range open by Theorem III.2.6. Hence $T'J_{M^\perp}$ has a continuous inverse and so $T \notin SC(X,Y)$.

III) ⇒ II) This follows trivially from the fact that an open map is surjective.

b) Assume $T \in SB[X,Y]$ with X complete. To establish the equivalence of I), II), IV) and V) we see from *a*) that we need only show that II) implies I).
II) $\Rightarrow$ I) Suppose $T \notin SC[X,Y]$. Then there exists $M \in \mathcal{I}_c(Y)$ such that $(Q_M T)' = T' J_{M^\perp}$ has a continuous inverse. But since T is semi-bounded, there exists $F \in \mathcal{F}(Y)$ such that $Q_F T$ is bounded. Consequently since $F \subset M + F$ and $M \subset M + F$, $Q_{M+F}T$ is bounded with $(Q_{M+F}T)' = T' J_{(M+F)^\perp}$ having a continuous inverse since $(M + F)^\perp \subset M^\perp$ (ie. $T' J_{(M+F)^\perp}$ is a restriction of $T' J_{M^\perp}$). Considering Theorem I.10.8 we conclude that $Q_{M+F}T$ is an open map.

c) I) $\Rightarrow$ III) Let Y be complete. Suppose there exists $M \in \mathcal{I}_c(Y)$ such that $Q_M T$ is surjective. Then $(Q_M T)' = T' J_{M^\perp}$ has a continuous inverse by Proposition I.10.7. Hence $T \notin SC(X,Y)$.
II) $\Leftrightarrow$ III) Let Y be complete, X an operator range and $T \in SB[X,Y]$. Suppose there is some $M \in \mathcal{I}_c(Y)$ such that $Q_M T$ is surjective. Selecting $F \in \mathcal{F}(Y)$ such that $Q_F T$ is bounded, we note that $Q_{M+F}T$ is both bounded and surjective. But then $Q_{M+F}T$ is an open map by the generalised open mapping theorem. Hence II) $\Rightarrow$ III). The converse is a consequence of *a*).

d) This follows from a consideration of *a*), *b*) and *c*). □

V.1.2 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Then* $T \in SC(X,Y) \cap SB(X,Y)$ *if and only if* $T = A + S$ *where* A *is a continuous strictly cosingular operator and* S *is a finite rank operator.*

Proof. If $T = A + S$ where A is continuous and strictly cosingular and $\dim R(S) < \infty$, then $J_Y S \in SC(X,\tilde{Y})$ since $\Delta'(J_Y S) = \Delta'(Q_R J_Y S) = 0$ by Proposition II.1.3 where $R = R(J_Y S)$. Hence by Theorems II.2.4 and II.3.8 $\Delta'(T) \leq \Delta'(A) + \Delta'(J_Y S) = 0$ and so T is strictly cosingular. Conversely let $T \in SC(X,Y)$. Since $T \in SB(X,Y)$ by hypothesis, we have by Theorem IV.1.1 that $T = A + S$ where $A \in B(X,Y)$ and $\dim R(S) < \infty$.

As before $\Delta'(A|_{D(T)}) \leq \Delta'(T) + \Delta'(J_YS) = 0$ and hence $A|_{D(T)}$ is strictly cosingular with $T = A|_{D(T)} + S$. $\square$

V.1.3 *Example. There exists a strictly cosingular operator which is not the sum of a continuous and a finite rank operator.* Let T, X and Y be as in Example IV.1.8 I). Note that we showed that $\Gamma_0(T) = 0$. From Theorem II.3.2 and Propositions II.3.3 and IV.2.3 we conclude that $\Gamma_0'(J_YT) = 0$. It now follows from Propositions II.2.1 and II.2.3 that $0 = \Gamma_0'(J_YT) \geq \Delta'(J_YT) \geq \Delta'(T)$, that is $\Delta'(T) = 0$. However as was noted in Example IV.1.8 I), $\Gamma_0'(T) = \infty$, that is T is not the sum of a continuous and a finite rank operator (Theorem IV.1.1).

V.1.4 *Remarks.* Let $T \in L(X,Y)$.

I) If $J_YT \in SC(X,\tilde{Y})$, then $T \in SC(X,Y)$. This follows from Proposition II.2.3 and Theorem II.3.8.

II) Let $T \in B(X,Y)$ with Y complete. Then $T \in SC(X,Y)$ if and only if $\bar{T} \in SC(\tilde{X},Y)$. This follows from the fact that for each $M \in \mathcal{I}_c(Y)$, $\overline{Q_MT} = Q_M\bar{T}$ since the extension is unique. Hence $\|Q_MT\| = \|Q_M\bar{T}\|$ for each $M \in \mathcal{I}_c(Y)$ (Theorem I.5.6) and so $\Delta'(T) = \Delta'(\bar{T})$.

III) $SPK(X,Y) \subset PPK(X,Y) \subset SC(X,Y)$. This follows from Proposition IV.2.3 and the fact that $\Gamma_0'(J_YT) \geq \Delta'(J_YT) \geq \Delta'(T)$ by Propositions II.2.1 and II.2.3.

IV) $SPK(X,Y) \subset PPK(X,Y) \subset SS(X,Y)$. This is a consequence of the above and the fact that in the proof of Theorem IV.1.2 we showed that $\Gamma_0(T) \geq \Delta(T)$.

All results in this section are by the author.

§2 IDEAL PROPERTIES

V.2.1 PROPOSITION. $SC(X,Y)$ *is a right ideal with respect to* $PB(Z,X)$.

Proof. Let $T \in SC(X,Y)$ and $S \in PB(Z,X)$. First suppose that $D(T)$ is dense in X. From the definition of Δ' we conclude that $\Delta'(T.J_X^{-1}) = \Delta'(T)$. Hence since $D(T)$ is still dense in $\tilde{X}$, we conclude from Theorem II.2.13 that

$$\Delta'(TS) = \Delta'(TJ_X^{-1}.J_XS) \leq \Delta'(T).\Gamma_0'(J_XS).$$

It now follows from Theorems II.3.8 and IV.1.1 and Proposition IV.1.3 that $TS \in SC$. Now suppose $D(T)$ is not dense in X. Let $T_1 = TJ_{D(T)}$ and let S_1 be $SJ_{D(TS)}$ considered as an element of $L[D(TS),D(T)]$. From the definition of partial continuity we note that a restriction of a partially continuous operator, and in particular S_1, is still partially continuous. From the first part of the proof we now conclude that T_1S_1, and hence TS is strictly cosingular. □

V.2.2 PROPOSITION. *Let* Y *be complete. Then* $SC(X,Y)$ *is closed under addition and* $SC(X,Y) \cap PB(X,Y)$ *is an ideal with respect to the densely defined partially continuous operators.*

Proof. Let Y be complete. The fact that $SC(X,Y)$ is closed under addition now follows from Theorem II.2.4. In order to prove the second assertion we see from Proposition V.2.1 that we need only show that $SC(X,Y) \cap PB(X,Y)$ is a left ideal with respect to the densely defined partially continuous operators. Note that for any $S \in L(Y,Z)$ we have by Theorem IV.1.1 and Proposition IV.1.3 that $S \in PB(Y,Z)$ if and only if $\Gamma_0'(J_ZS) < \infty$. Since $\Delta'(S) \leq \Delta'(J_ZS) \leq \Gamma_0'(J_ZS)$ by Propositions II.2.1 and II.2.3, the result now follows from Theorems II.2.12 and II.3.8. □

V.2.3 **COROLLARY.** *If* X *is complete, then* $SC[X,Y] \cap SB[X,Y]$ *is an operator ideal with respect to the semi-bounded operators.*

Proof. Since $SB[Z,X] \supset PB[Z,X]$ (Proposition IV.1.3) it follows from Proposition V.2.1 that we need only show that $SC[X,Y] \cap SB[X,Y]$ is left ideal if X is complete. Hence suppose X is complete, $T \in SC[X,Y] \cap SB[X,Y]$ and $S \in SB[Y,Z]$. Considering Theorems II.2.13 and IV.1.1 we conclude that $ST \in SB[X,Z]$. Assume S to be bounded and suppose $ST \notin SC[X,Z]$. Then by Theorem V.1.1 there exists $M \in \mathcal{I}_c(Z)$ so that $Q_M ST$ is an open map. However since $Q_M ST$ is surjective, so is $Q_M S$. Hence $(Q_M S)\hat{}$ is a bounded bijection from Y/N onto Z/M where $N = N(Q_M S) \in \mathcal{I}_c(Y)$ (Proposition I.4.16). We conclude that $Q_N T = [(Q_M S)\hat{}]^{-1}.Q_M ST$ is open since it is the composition of two open maps. Thus $T \notin SC[X,Y]$ by Theorem V.1.1. Now let $S \in SB[Y,Z]$. By Theorem IV.1.1 $S = A + B$ where $A \in B[Y,Z]$ and $\dim R(B) < \infty$. Therefore by what we have just shown $AT \in SC[X,Z]$ whereas $\dim R(BT) < \infty$. Thus $ST \in SC[X,Z]$ by Proposition V.1.2. □

All results in this section are by the author.

§3 ADJOINTS OF STRICTLY COSINGULAR OPERATORS

V.3.1 **PROPOSITION.** *Let* $T \in L(X,Y)$.

I) *If* T *is continuous, then* T *is strictly singular if* T' *is strictly cosingular.*

II) *If* $D(T') \in \mathcal{F}_c(Y')$, *then* T *is strictly cosingular if* T' *is strictly singular.*

Proof. Without loss of generality let $D(T) = X$.

I) Let $T \in B[X,Y]$ and suppose that there exists $M \in \mathcal{I}(X)$ such that TJ_M has a continuous inverse. Then $(TJ_M)' = Q^{X'}_{M^\perp}T'$ is surjective by Proposition I.10.7. Since $dim\ M = dim\ M' = dim\ X'/M^\perp = \infty$ by Theorem I.5.10, $T' \notin SC[Y',X']$ by Theorem V.1.1.

II) Let $D(T') \in \mathcal{F}_c(Y')$ and suppose $T \notin SC[X,Y]$. By definition there exists $M \in \mathcal{I}_c(Y)$ such that $(Q_MT)' = T'J_{M^\perp}$ has a continuous inverse. Since $dim\ M^\perp = dim\ Y/M = \infty$ (Theorem I.5.10), $M^\perp \cap D(T') \in \mathcal{I}(D(T'))$ and so $T' \notin SS(Y',X')$. □

The above Proposition is by the author and generalises the well known classical result.

V.3.2 **DEFINITION** [39]. A normed linear space X is said to be *superprojective* if for every $M \in \mathcal{I}_c(X)$, there exists $N \in \mathcal{I}_c(X)$ with $N \supset M$ such that N is topologically complemented in X.

V.3.3 **THEOREM**. *Let* $T \in SB[X,Y]$ *with* X *a superprojective Banach space. Then* $T \in SC[X,Y]$ *if* $T' \in SC(Y',X')$.

Proof. First assume $T \in B[X,Y]$. Suppose $T \notin SC[X,Y]$. By Theorem V.I.I there is some $M \in \mathcal{I}_c(Y)$ such that Q_MT is a bounded open map. Noting that $\gamma(Q_MT) = \gamma((Q_MT)^\wedge)$ we conclude from Proposition III.1.2 that $(Q_MT)^\wedge$ is open and hence that $(Q_MT)^\wedge$ is an isomorphism from X/N onto Y/M where $N = N(Q_MT)$ (Proposition I.4.16). Consequently $N \in \mathcal{I}_c(X)$ and so there exists $W \in \mathcal{I}_c(X)$ with $W \supset N$ such that W is topologically complemented in X. Let $X = W \oplus V$ where V is a topological complement of W.

We note that $W/N \in \Phi_c(X/N)$ since $dim\ (X/N)/(W/N) = dim\ X/W$ by Proposition I.8.6 and hence as $(Q_M T)^{\wedge}$ is an isomporphism, $(Q_M T)^{\wedge}(W/N) = TW/M \in \Phi_c(Y/M)$. Therefore $K = Q_M^{-1}(TW/M) \in \Phi_c(Y)$ where Q_M^{-1} is taken in the set theoretic sense. Now since $M \subset K$, $Q_K^Y T = Q_{K/M}^{Y/M} \cdot (Q_M^Y T)$ (Proposition I.8.6) and so $Q_K T$ is open since it is just the composition of two open maps. We show that $N(Q_K T) = W$. Suppose $Q_K T x = 0$. Hence $Tx \in K(= Q_M^{-1}(TW/M))$ and so $Q_M Tx \in Q_M TW$. Now since $(Q_M T)^{\wedge}$ is an isomorphism, we conclude that $x + N \in (W/N)$. But $N \subset W$ and so $x \in W$, that is $N(Q_K T) \subset W$. But $W \subset N(Q_K T)$ by choice of K and so equality holds. Consequently, since W is topologically complemented by V, $Q_K T J_V$ is a bounded bijection. Recalling that X and therefore X/N is complete and that $(Q_M T)^{\wedge}$ is an isomorphism from X/N onto Y/M, we conclude that Y/M is complete. Therefore by Theorem I.2.1 and Proposition I.8.6 $Y/K \equiv (Y/M)/(K/M)$ is complete. Since V is a closed and therefore complete subspace of X, we conclude from the open mapping theorem that $Q_K T J_V$ is an isomorphism. Hence $(Q_K T J_V)' = (J_V)' T' (Q_K)' = Q_{V^\perp} . T' J_{K^\perp}$ and therefore $Q_{V^\perp} . T'$ is surjective. Moreover $cod\ V^{\perp} = dim\ V = cod\ W = \infty$ and so $T' \notin SC[Y', X']$ by Theorem V.1.1. Now let $T \in SB[X,Y]$ with $T' \in SC(Y', X')$. Hence $T' J_{D(T')} \in SC[D(T'), X'] \cap B[D(T'), X']$ (Corollary IV.1.5). Note that $D(T')$ is $\sigma(Y', Y)$-closed with ${}^{\perp}D(T') \in \mathcal{F}(Y)$ by Theorems I.5.10 and IV.1.1 and Remark I.5.9. Since $(Q_{{}^{\perp}D(T')} T)' = T' J_{({}^{\perp}D(T'))^{\perp}} = T' J_{D(T')}$ (Remark I.5.9), $Q_{{}^{\perp}D(T')} T$ is bounded by Theorem I.10.3. We conclude from the first part of the proof that $Q_{{}^{\perp}D(T')} T$ is strictly cosingular. A consideration of Proposition II.1.3, Theorem II.3.8 and the fact that ${}^{\perp}D(T') \in \mathcal{F}(Y)$ now yields the result.

V.3.4 **COROLLARY.** *Let* $T \in SB[X,Y]$ *with* X *and* Y *reflexive and* Y' *superprojective. Then* $T' \in SC(Y', X')$ *if* $T \in SC[X,Y]$.

Proof. First suppose that T is bounded. Then $T'' = J_Y^{Y''}.T.J_X^{X''-1}$ and so $T'' \in SC[X'',Y'']$ if and only if $T \in SC[X,Y]$. Consequently $T \in SC[X,Y]$ implies $T' \in SC[Y',X']$ by Theorem V.3.3. Now let $T \in SB[X,Y] \cap SC[X,Y]$. Then $T = A + S$ where $A \in B[X,Y] \cap SC[X,Y]$ and $dim\ R(S) < \infty$ by Proposition V.1.2. Hence $A' \in SC[Y',X']$ by the first part of the proof with $dim\ R(S') < \infty$ since $cod\ N(S') = cod\ R(S)^{\perp} = dim\ R(S) < \infty$ by Theorem I.5.10 and Proposition I.10.4. The corollary now follows from Proposition V.1.2 and the observation that $T' = A' + S'$. □

The author is unaware of Theorem V.3.3 and Corollary V.3.4 having been proved before and believes these results to be his own.

CHAPTER VI

F_- OPERATORS

Observe that for any $T \in L(X,Y)$, $T' = (J_Y T)'$ and hence $\bar{b}(T) = \alpha(T') = \bar{b}(J_Y T)$ by Corollary III.2.7. Since $\bar{b}(T) = \bar{b}(J_Y T)$ we will use these two concepts interchangably for the rest of this manuscript.

§1 CHARACTERISATIONS OF F_- OPERATORS

VI.1.1 *Remark.* Let $T \in L(X,Y)$.

I) If $\dim Y = \infty$, then $T \in F_-(X,Y)$ if and only if $\Gamma'(T) > 0$ (Theorem II.3.6).

II) $T \in F_-(X,Y)$ if $\dim Y < \infty$. This follows from the fact that $Y \in \mathcal{F}(Y)$ with $(Q_Y T)' = T' J_{Y^\perp} = T' J_{\{0\}}$ an isometry from the zero-dimensional subspace $\{0\}$ of Y onto the zero-dimensional subspace $\{0\}$ of $R(T')$.

III) $T \in F_-(X,Y)$ if and only if $J_Y T \in F_-(X,\tilde{Y})$. This follows from I), II) and the fact that $\Gamma'(T) = \Gamma'(J_Y T)$ (Proposition II.2.3).

IV) $T \in F_+$ if $\dim D(T) < \infty$. Note that if $\dim D(T) < \infty$, $\{0\} \in \mathcal{F}_c(D(T))$ with $TJ_{\{0\}}$ an isometry.

V) $T \in F_-$ if and only if $T' \in \Phi_+(F_+)$. If $\dim Y (= \dim Y') < \infty$, then $T \in F_-$ and $T' \in F_+$ by II) and IV). Since $\dim R(T') < \infty$, $R(T')$ is closed and hence $T' \in \Phi_+$ as $\alpha(T') \leq \dim Y' < \infty$. The rest follows from Theorem II.3.6.

VI) $T \in F_+$ and only if $T' \in \Phi_-$. The proof is analogous to that for V) with use being made of Theorem II.3.5 instead of II.3.6.

VII) $\bar{b}(T) < \infty$ if $T \in F_-$. Note that $T \in F_-$ implies $T' \in \Phi_+$ by V) and hence $\alpha(T') = \bar{b}(T) < \infty$.

VI.1.2 **PROPOSITION.** *Let* $T \in L(X,Y)$. *If* M *is an arbitrary closed subspace of* Y, *then* $Q_M T \in F_{-}(X,Y/M)$ $(Q_M TG_{Q_M T} \in F_{-}[X_{Q_M T}, Y/M])$ *if* $T \in F_{-}(X,Y)$ $(TG_T \in F_{-}[X_T, Y])$.

Proof. Without loss of generality let $\dim Y = \infty$. If $\dim Y/M < \infty$ the proposition is trivial and so let $M \in \mathcal{I}_c(Y)$. Now if $T \in F_{-}(X,Y)$ then $0 < \Gamma'(T) \le \Gamma'(Q_M T)$ by Proposition II.2.1 and so $Q_M T \in F_{-}(X,Y/M)$. Analogously if $TG_T \in F_{-}[X_T, Y]$ then $Q_M TG_T \in F_{-}[X_T, Y/M]$. Note that $\|x\| + \|Q_M Tx\| \le \|x\| + \|Tx\|$ for each $x \in D(T)$ and so the identity map, say V, from X_T onto $X_{Q_M T}$ is bounded. Hence by Theorem II.2.13

$$0 < \Gamma'(Q_M TG_T) = \Gamma'(Q_M TG_{Q_M T}.V) \le \Gamma'(Q_M TG_{Q_M T}).\Gamma'_0(V)$$

(note that $\Gamma'_0(V) < \infty$ and $\Gamma'(Q_M TG_{Q_M T}) < \infty$ since both V and $Q_M TG_{Q_M T}$ are bounded). Hence $Q_M TG_{Q_M T}$ is an F_{-} operator. □

VI.1.3 **LEMMA.** *Let* $T \in L(X,Y)$. *If* Y *is complete and* $b(T) < \infty$, *then* $T \in F_{-}(X,Y)$.

Proof. Since $b(T) < \infty$, there exists $F \in \mathcal{F}(Y)$ such that $Q_F T$ is surjective. However as Y/F is still complete by Theorem I.2.1, $(Q_F T)'$ has a continuous inverse by Proposition I.10.7. □

VI.1.4 **THEOREM.** *Let* $T \in L(X,Y)$. *Consider the following statements:*

I) $T \in F_-(X,Y)$

II) $TG \in F_-[X_T, Y]$

III) $\overline{TG} \in \Phi_-[\widetilde{X}_T, \widetilde{Y}]$

IV) $T \in \Phi_-(X,Y)$

a) *If* X *and* Y *are normed linear spaces, then II) and III) are equivalent, either of which imply I).*

b) *If* X *is complete and* T *closed then I), II) and III) are equivalent, any of which imply IV).*

c) *If* Y *is complete then IV) implies II) and hence III) and I) as well.*

d) *If both* X *and* Y *are complete and* T *is closed, then all four statements are equivalent.*

Proof. *a)* Let X and Y be arbitrary normed linear spaces.

II) $\Leftrightarrow$ III) On considering V) of Remark VI.1.1 and the fact that $(TG)' = (\overline{TG})'$ we conclude that $TG \in F_-$ if and only if $\overline{TG} \in F_-$. The result now follows on considering *d)* (proved later).

II) $\Rightarrow$ I) Suppose $TG \in F_-(X,Y)$. If $\dim Y < \infty$, $T \in F_-$ by Remark VI.1.1. Hence let $\dim Y = \infty$. We show that $\Gamma'(TG_T) \le \Gamma'(T)$ and hence that $T \in F_-(X,Y)$ (Remark VI.1.1 I)). Without loss of generality let $\Gamma'(T) < \infty$. Let $\epsilon > 0$ be arbitrary and select $M \in \mathscr{I}_c(\widetilde{Y})$ so that $\|Q_M J_Y T\| \le \Gamma'(T) + \epsilon$. But then

$$\Gamma'(TG_T) \le \|Q_M J_Y TG_T\| \le \|Q_M J_Y T\| \, \|G_T\| \le \|Q_M J_Y T\| \le \Gamma'(T) + \epsilon$$

and so $\Gamma'(TG_T) \le \Gamma'(T)$ since ϵ was chosen arbitrarily.

b) Let X be complete and T closed.

I) $\Rightarrow$ IV) Suppose $T \in F_-(X,Y)$. Then $T' \in \Phi_+$ by Remark VI.1.1 and so $\gamma(T') > 0$ by Theorem III.2.6 as $R(T')$ is closed by definition.

Consequently, again by Theorem III.2.6, $R(T)$ is closed and so since $\bar{b}(T) = b(T) < \infty$ by Remark VI.1.1 we conclude that $T \in \Phi_-(X,Y)$.

I) ⇒ II) Suppose $T \in F_-(X,Y)$. From the proof of the first part we conclude that $\bar{b}(TG) = \bar{b}(T) = b(T) < \infty$ and $\gamma(T') > 0$. But then $\gamma(T) > 0$ by Theorem III.2.6 and so $\gamma(TG) > 0$ by Theorem III.2.2. If $\dim Y < \infty$, $TG \in F_-$ by Remark VI.1.1 and if $\dim Y = \infty$ then $0 < \gamma(TG) \leq \Gamma'(TG)$ by Theorem II.2.9. Hence $TG \in F_-[X_T,Y]$ by Remark VI.1.1.

c) Let Y be complete.

IV) ⇒ II) If $T \in \Phi_-(X,Y)$, then $b(T) = b(TG) < \infty$ by definition and so from the Lemma we conclude that $TG \in F_-[X_T,Y]$.

d) This follows from a consideration of *a*), *b*) and *c*). □

The following result was proved by R.W. Cross and privately communicated to the author.

VI.1.5 **THEOREM** [9]. *Let* $T \in L(X,Y)$. *Consider the following statements:*

I) $T \in F_+(X,Y)$.

II) $TG \in F_+[X_T,Y]$.

III) $\overline{TG} \in \Phi_+[\tilde{X}_T,\tilde{Y}]$.

IV) $T \in \Phi_+(X,Y)$.

a) If X *and* Y *are normed linear spaces I), II) and III) are equivalent.*

b) If X *is complete and* T *closed, then I) implies IV).*

c) If X *is an operator range and* Y *complete then IV) implies I).*

d) If X *and* Y *are complete and* T *is closed all four statements are equivalent.*

Proof. *a*) Let X and Y be normed linear spaces.

II) ⇔ III) By analogy to the proof of Theorem VI.1.4 this follows from *d*) (proved later), Remark VI.1.1 and the fact that $(\overline{TG})' = (TG)'$.

II) ⇒ I) Again analogous to Theorem VI.1.4 this follows from Remark VI.1.1 and the fact that $\Gamma(TG_T) \le \Gamma(T)$.

I) ⇒ II) Suppose $TG_T \notin F_+[X_T, Y]$. Hence $\dim X_T = \dim D(T) = \infty$ and $\Gamma(TG) = 0$ by Remark VI.1.1 and Theorem II.3.5. Let $\epsilon > 0$ be arbitrary and select $M \in \mathcal{F}(X_T)$ so that $\|TG_TJ_M\| \le \frac{\epsilon}{1+\epsilon}$. Hence $GM \in \mathcal{F}(D(T))$ with

$$\|Tx\| \le \frac{\epsilon}{1+\epsilon}(\|x\| + \|Tx\|) \quad \text{for every} \quad x \in GM$$

and so

$$\|Tx\| \le \epsilon\|x\| \quad \text{for every} \quad x \in GM.$$

We conclude that $\|TJ_{GM}\| \le \epsilon$ and hence that $\Gamma(T) = 0$ since ϵ was chosen arbitrarily. But then $T \notin F_+(X,Y)$ by Theorem II.3.5.

b) I) ⇒ IV). Let X be complete and T closed and suppose that $T \in F_+(X,Y)$. Hence $\alpha(T) < \infty$ by definition and so we need only show that $R(T)$ is closed. Note that $T' \in \Phi_-$ by Remark VI.1.1. Therefore since $R(T')$ is closed by definition, $\gamma(T') > 0$ by Theorem III.2.6 and so $R(T)$ is closed by the same Theorem.

c) IV) ⇒ I) Let X be an operator range, Y complete and suppose that $T \in \Phi_+(X,Y)$. If $\dim D(T) < \infty$, $T \in F_+$ by Remark VI.1.1. Hence let $\dim D(T) = \infty$. Then since $R(T)$ is closed by definition, $\gamma(T) > 0$ by Theorem III.2.6. As $\alpha(T) < \infty$ we conclude from Theorems II.2.9 and II.3.5 that $T \in F_+(X,Y)$.

d) This is a trivial consequence of *a*), *b*) and *c*). □

VI.1.6 *Remark.* Considering Theorems III.2.6, VI.1.4 and VI.1.5 we see that corresponding to the classical classes of normally solvable, Φ_- and Φ_+ operators we have the classes of range open, F_- and F_+ operators respectively with equivalence holding in the classical setting of closed operators between Banach spaces.

VI.1.7 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $T \in F_-$.

II) $\hat{T} \in F_-$ *if* $N(T)$ *is closed.*

III) $Q_F T \in F_-$ *for every (for some)* $F \in \mathcal{F}(Y)$.

IV) $T + B \in F_-$ *for every (for some)* $B \in L(X,Y)$ *such that* $D(B) \supset D(T)$ *and* $J_Y B \in SC(X,\tilde{Y})$.

V) $T + B \in F_-$ *for every (for some)* $B \in PPK(X,Y)$ *such that* $D(B) \supset D(T)$.

VI) $T + B \in F_-$ *for every (for some)* $B \in PK(X,Y)$ *such that* $D(B) \supset D(T)$.

VII) $T + B \in F_-$ *for every (for some)* $B \in L(X,Y)$ *such that* $\dim R(B) < \infty$ *and* $D(B) \supset D(T)$.

VIII) $\bar{b}(J_Y T - B) < \infty$ *for every* $B \in L(X,\tilde{Y})$ *such that* B *is compact (nuclear) with* $D(B) \supset D(T)$.

Proof. Without loss of generality let $\dim Y = \infty$.

I) $\Leftrightarrow$ II) Let $N(T)$ be closed. The equivalence now follows from Theorem II.3.6 and Proposition II.1.5.

I) $\Leftrightarrow$ III) This follows from Theorem II.3.6 and Proposition II.1.3.

I) $\Leftrightarrow$ IV) Let $T \in L(X,Y)$ and let $B \in L(X,Y)$ be such that $D(B) \supset D(T)$ and $J_Y B \in SC(X,\tilde{Y})$. Hence $\Delta'(J_Y B) = 0$ by Theorem II.3.8. But by Theorem II.2.4

$$\begin{aligned}\Gamma'(T) = \Gamma'(T + B - B) &\leq \Gamma'(T + B) + \Delta'(J_Y B) = \Gamma'(T + B) \\ &\leq \Gamma'(T) + \Delta'(J_Y B) = \Gamma'(T).\end{aligned}$$

Hence $\Gamma'(T) = \Gamma'(T + B)$ and so $T \in F_-$ if and only if $T + B \in F_-$.

I) $\Leftrightarrow$ V) Let $B \in L(X,Y)$. Then $B \in PPK(X,Y)$ if and only if $\Gamma_0'(J_Y B) = 0$ by Proposition IV.2.3 and so since

$$\Gamma'(T + B) \le \Gamma'(T) + \Delta'(J_Y B) \le \Gamma'(T) + \Gamma_0'(J_Y T)$$

by Theorem II.2.4 and Proposition II.2.1 we can construct a proof analogous to the above.

I) $\Leftrightarrow$ VI) As before this follows from the fact that

$$\Gamma'(T + B) \le \Gamma'(T) + \Gamma_0'(J_Y B) \le \Gamma'(T) + \Gamma_0'(B)$$

(Proposition II.2.3) for $T,B \in L(X,Y)$.

I) $\Leftrightarrow$ VII) Observing that the class of finite rank operators in $L(X,Y)$ is contained in $SC(X,\tilde{Y})$ (Theorem II.3.8), we can use the inequality

$$\Gamma'(T + B) \le \Gamma'(T) + \Delta'(J_Y B)$$

to construct a proof analogous to the above.

VI) $\Rightarrow$ VIII) We see from Remark VI.1.1 that $T \in F_-(X,Y)$ if and only if $J_Y T \in F_-(X,\tilde{Y})$ and so VIII) follows from VI) and Remark VI.1.1 VII).

VIII) $\Rightarrow$ I) Suppose $T \notin F_-(X,Y)$. If $\bar{b}(T + 0) = \bar{b}(T) = \infty$ we are done. Hence suppose $\bar{b}(T) < \infty$. We construct $B \in B[X,\tilde{Y}]$ such that $\bar{b}(J_Y T - B) = \infty$. Let $\{a_n\}$ be a sequence of integers defined inductively by

(1) $$a_1 = 2, \quad a_n = 2(1 + \sum_{k=1}^{n-1} a_k) \quad \text{for } n = 2,3,\ldots$$

We claim that there exists sequences $\{y_k\} \subset Y$ and $\{y_k'\} \subset Y'$ such that

(2) $$\|y_k\| \le a_k, \quad \|y_k'\| = 1, \quad \|T'y_k'\| < 1/(2^k a_k)$$
$$y_j'(y_k) = \delta_{jk}, \quad j,k = 1,2,\ldots .$$

Assuming this for the moment we define $B \in L[D(T),\tilde{Y}]$ as follows:

$$Bx = \sum_{k=1}^{\infty} T'y_k'(x)y_k \quad \text{for each} \quad x \in D(T).$$

Observe that for any $x \in D(T)$

$$\|Bx\| \leq \sum_{k=1}^{\infty} \|T'y_k'\|.\|x\|.\|y_k\| \leq \|x\|(\sum_{k=1}^{\infty} 1/2^k)$$

and so Bx exists by the completeness of $\tilde{Y}$ (Theorem I.2.3). Letting $z_k = y_k/a_k$ and $x_k' = a_k 2^k.T'y_k'$ for each $k \in \mathbb{N}$, we note that $B = \sum_{k=1}^{\infty} (1/2^k)x_k'(.)z_k$ is nuclear and hence compact by Theorem I.11.7. If $D(T) \subsetneq X$ we can consider x_k' to be an extension of $a_k 2^k.T'y_k'$ to all of X (Proposition I.7.1) and hence without loss of generality we may assume $B \in N[X,\tilde{Y}]$. Now for each $x \in D(T)$ we have by (2) that

$$y_k'(Bx) = T'y_k'(x) = y_k'(Tx) \quad \text{for} \quad k = 1,2,\ldots$$

and so each y_k' annihilates $\overline{R(J_Y T - B)}$. Since the y_k''s are linearly independent by (2), we conclude that $\bar{b}(J_Y T - B) = \infty$. It remains to find sequences satisfying (2). By assumption $T \notin F_-$ and so since $T \in F_-$ if and only if $T' \in F_+$ by Remark VI.1.1, we deduce that $T' \notin F_+$. Hence there exists $y_1' \in Y'$ so that $\|y_1'\| = 1$ with $\|T'y_1'\| < 1/4$. If this was not the case then T' would have a continuous inverse; a contradiction. Select $y_1 \in Y$ such that $y_1'(y_1) = 1$ and $\|y_1\| < 2$. Suppose $y_1,\ldots,y_{n-1}$, $y_1',\ldots,y_{n-1}'$ have been found satisfying (2). Considering $T'|_{(span\{y_1;\ldots;y_{n-1}\})^\perp} = T_{n-1}'$ we note that as before T_{n-1}' does not have a continuous inverse since $(span\{y_1;\ldots;y_{n-1}\})^\perp \in \mathcal{F}_c(Y')$ (Theorem I.5.10). Consequently there exists $y_n' \in (span\{y_1,\ldots,y_{n-1}\})^\perp$ such that $\|y_n'\| = 1$ and $\|T'y_n'\| < 1/(2^n a_n)$. Select $y \in Y$ such that $y_n'(y) = 1$ and $\|y\| < 2$. Let

$$y_n = y - \sum_{k=1}^{n-1} y_k'(y)y_k.$$

Then

$$\|y_n\| \le \|y\|(1 + \sum_{k=1}^{n-1} \|y_k\|) \le 2(1 + \sum_{k=1}^{n-1} a_k) = a_n$$

by (1) and (2). Also $y_n'(y_n) = y_n'(y) = 1$ with $y_n'(y_k) = 0$ for $1 \le k < n$ by the way y_n' and y_n were chosen. Finally $y_k'(y_n) = y_k'(y) - y_k'(y) = 0$ for $1 \le k < n$ and so by induction the result follows. □

VI.1.8 **THEOREM.** *Let* $T \in L(X,Y)$. *Then the following are equivalent:*

I) $TG_T \in F_-$.

II) $\hat{T}G_{\hat{T}} \in F_-$ *if* $N(T)$ *is closed.*

III) $(T + B)G_{T+B} \in F_-$ *for each (for some)* $B \in B(X,Y)$ *such that* $D(B) \supset D(T)$ *and* $J_Y B \in SC(X,\tilde{Y})$.

IV) $(T + B)G_{T+B} \in F_-$ *for each (for some)* $B \in PK(X,Y)$ *such that* $D(B) \supset D(T)$.

Proof. Without loss of generality let $\dim Y = \infty$.

I) ⇔ II) Let $TG_T \in F_-$ with $N(T)$ closed. Then $(TG_T)^\wedge \in F_-$ if and only if $TG_T \in F_-$ by Theorem VI.1.7. The equivalence now follows from the fact that $(TG_T)^\wedge = \hat{T}G_{\hat{T}}$.

I) ⇔ III) Suppose $B \in B(X,Y) \cap SC(X,\tilde{Y})$ such that $D(B) \supset D(T)$. Since $SC(X,\tilde{Y})$ is a right ideal by Proposition V.2.1, $J_Y BG_T \in SC[X_T,\tilde{Y}]$. By Theorem VI.1.7 we now have that $TG_T \in F_-$ if and only if $(T + B)G_T \in F_-$. But since B is continuous we have that

$$\|x\| + \|(T + B)x\| \le \|x\| + \|Tx\| + \|B\|\|x\| \le (1 + \|B\|)(\|x\| + \|Tx\|)$$

for each $x \in D(T)$ and similarly

$$\|x\| + \|Tx\| \le (1 + \|B\|)(\|x\| + \|(T + B)x\|).$$

Hence $X_T \approx X_{T+B}$ and so $(T + B)G_T \in F_-$ if and only if $(T + B)G_{T+B} \in F_-$.

I) $\Leftrightarrow$ IV) Noting that $PK(X,Y) \subset SC(X,\check{Y})$ (Remarks I.11.4 and V.1.4) we can construct a proof analogous to the above. □

As was noted Theorem VI.1.5 is due to R.W. Cross as is Lemma VI.1.3. The equivalence of I) and VIII) in Theorem VI.1.7 was jointly established by the author and Dr. R.W. Cross. All other results in this section are by the author. We remark that the construction used in Theorem VI.1.7 is essentially that used by Lebow and Schechter [28; Theorem 5.4].

§2 THE RELATIONSHIP BETWEEN RANGE OPEN, F_- AND F_+ OPERATORS

VI.2.1 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Consider the following statements:*

I) $T \in F_-$.

II) *There exists* $K \in \mathcal{F}(Y)$ *such that* $\gamma(Q_K T) > 0$ *with* $\bar{b}(T) < \infty$.

III) $\gamma(T) > 0$ *with* $\bar{b}(T) < \infty$.

IV) $R(T') = N(TJ_{D(T)})^{\perp}$ *with* $\alpha(T') < \infty$.

a) *If* X *and* Y *are normed linear spaces, III) and IV) are equivalent, III) implies II) and II) implies I).*

b) *If* X *is complete and* T *closed then I), II), III) and IV) are all equivalent.*

Proof. a) III) $\Leftrightarrow$ IV) This follows easily from Theorem III.2.6 and Corollary III.2.7.

III) $\Rightarrow$ II) $\Rightarrow$ I). We trivially have that III) implies II) and hence suppose that $\bar{b}(T) < \infty$ and that there is some $K \in \mathcal{F}(Y)$ such that $\gamma(Q_K T) > 0$. But then $R((Q_K T)') = R(T' J_{K^\perp})$ is closed by Theorem III.2.6. Now since $\operatorname{cod} K^{\perp} = \dim K < \infty$ (Theorem I.5.10), there is some $N \in \mathcal{F}(Y')$ such that $R(T') = R(T' J_{K^\perp}) + T'N$. Hence $R(T')$ is closed (Proposition I.3.4). But since $\alpha(T') = \bar{b}(T) < \infty$ (Corollary III.2.7), $T' \in \Phi_+$ and so $T \in F_-$ by Remark VI.1.1.

b) I) $\Rightarrow$ III) Suppose that X is complete with T a closed F_- operator. As $T' \in \Phi_+$ by Remark VI.1.1, $\gamma(T') > 0$ since $R(T')$ is closed (Theorem III.2.6). Hence $\gamma(T) > 0$ by Theorem III.2.6. Trivially $T \in F_-$ implies that $\bar{b}(T) < \infty$ (Remark VI.1.1) and so we are done. □

VI.2.2 **PROPOSITION.** *Let $T \in L(X,Y)$. Consider the following statements:*

I) $T \in F_+$.

II) There exists $K \in \mathcal{F}(Y)$ such that $\gamma(Q_K T) > 0$ with $\alpha(T) < \infty$.

III) $\gamma(T) > 0$ with $\alpha(T) < \infty$.

IV) $R(T') = N(TJ_{D(T)})^{\perp}$ with $b(T') < \infty$.

a) If X and Y are normed linear spaces, III) and IV) are equivalent, III) implies II) and II) implies I).

b) If X is complete and T closed then all the above are equivalent.

Proof. *a)* III) $\Leftrightarrow$ IV) This is an easy consequence of Theorem III.2.6 and Corollary III.2.7.

III) $\Rightarrow$ II) $\Rightarrow$ I). We trivially have that III) implies II) and so suppose that there exists $K \in \mathcal{F}(Y)$ such that $\gamma(Q_K T) > 0$ with $\alpha(T) < \infty$. Then $\alpha(Q_K T) < \infty$ since $K \in \mathcal{F}(Y)$ with $R((Q_K T)') = R(T' J_{K^\perp})$ closed by Theorem III.2.6. Moreover as $cod\ K^{\perp} = dim\ K < \infty$ we deduce that $cod_{R(T')} R(T' J_{K^\perp}) < \infty$ and hence that $R(T')$ is closed (Proposition I.3.4). Considering Corollary III.2.7 we see that $b(T' J_{K^\perp}) = \alpha(Q_K T) < \infty$ and so since $R(T') \supset R(T' J_{K^\perp})$, $b(T') < \infty$. Hence $T' \in \Phi_-$ and so $T \in F_+$ by Remark VI.1.1.

b) I) $\Rightarrow$ III) Let X be complete and T closed. Suppose $T \in F_+$. Then $T' \in \Phi_-$ by Remark VI.1.1. Hence $\gamma(T') > 0$ by Theorem III.2.6 as $R(T')$ is closed by definition. Moreover $\gamma(T) > 0$ by the same theorem and so since $T \in F_+$ implies $\alpha(T) < \infty$, we are done. □

VI.2.3 **PROPOSITION.** *Let* $T \in L(X,Y)$. *Consider the following statements:*

I) $T \in F_+$ *and* $\bar{b}(T) < \infty$.

II) $T \in F_-$ *and* $a(T) < \infty$.

In general I) implies II) with the statements being equivalent if X *is complete and* T *closed.*

Proof. I) ⇒ II). Suppose $T \in F_+$ with $\bar{b}(T) < \infty$. Then $T' \in \phi_-$ by Remark VI.1.1 and since $a(T') = \bar{b}(T) < \infty$, $T' \in \phi_+$. Hence $T \in F_-$ by Remark VI.1.1 with $a(T) < \infty$ since $T \in F_+$.

II) ⇒ I) Suppose X is complete and T closed. Let $T \in F_-$ with $a(T) < \infty$. By Proposition VI.2.1 we then have $\gamma(T) > 0$ with $\bar{b}(T) < \infty$ and $a(T) < \infty$. The result follows on considering Proposition VI.2.2. □

All results in this section are by the author.

§3 PERTURBATION RESULTS FOR F_+ AND F_- OPERATORS

VI.3.1 **PROPOSITION.** *Let* $T \in L(X,Y)$. *If* $T \in F_+(F_-)$ *and* $B \in B(X,Y)$ *such that* $D(B) \supset D(T)$, *then there exists* $\rho > 0$ *such that* $R((T + \lambda B)')$ *is closed with* $b((T + \lambda B)')$ *and* $\bar{b}(T + \lambda B)$ *constant in the annulus* $0 < |\lambda| < \rho$.

Proof. Let $T \in F_+(F_-)$ and let $B_1 = BJ_{D(T)}$. Then $T' \in \phi_-(\phi_+)$ by Remark VI.1.1 and so, since $T' + \lambda B_1' = (T + \lambda B)'$ for each λ, it follows from Proposition I.4.20 that there exists $\rho_1 > 0$ with $a(T' + \lambda B_1') = \bar{b}(T + \lambda B)$ and $b(T' + \lambda B_1')$ constant in the annulus $0 < |\lambda| < \rho_1$. Considering Theorem I.4.19 we see that since $T' \in \phi_-(\phi_+)$, there exists $\rho_2 > 0$ such that $R(T' + \lambda B_1')$ is closed for each $|\lambda| < \rho_2$. Hence let $\rho = min\{\rho_1, \rho_2\}$. □

VI.3.2 **COROLLARY.** *Let* $T \in C(X,Y)$ *and* X *complete. If* $T \in F_+(F_-)$ *and* $B \in B(X,Y)$ *such that* $D(B) \supset \overline{D(T)}$, *then there exists* $\rho > 0$ *such that* $\gamma(T + \lambda B) > 0$ *(and thus* $R(T + \lambda B)$ *closed) and such that* $\alpha(T + \lambda B)$ *and* $b(T + \lambda B)$ *are constant in the annulus* $0 < |\lambda| < \rho$.

Proof. Let $T \in F_+(F_-)$ and $B \in B(X,Y)$ such that $D(B) \supset \overline{D(T)}$. Since T is closed, we note that $T + \lambda B$ is closed for any λ. Let $B_1 = BJ_{D(T)}$ and let $\rho > 0$ be as in Proposition VI.3.1. Then since $R(T' + \lambda B_1')$ is closed for each $0 < |\lambda| < \rho$, $\gamma(T' + \lambda B_1') > 0$ and hence $\gamma(T + \lambda B) > 0$ by Theorem III.2.6 $(T' + \lambda B_1' = (T + \lambda B)')$. By Corollary III.2.7 we then have that $b(T' + \lambda B_1') = \alpha(T + \lambda B)$ for each $0 < |\lambda| < \rho$. Moreover as $\gamma(T + \lambda B) > 0$ and $T + \lambda B$ is closed for each $0 < |\lambda| < \rho$, we see from Theorem III.2.6 that $R(T + \lambda B)$ is closed for each $0 < |\lambda| < \rho$ and hence the result follows. □

VI.3.3 **PROPOSITION.** *Let* $T \in L(X,Y)$ *and* $B \in B(X,Y)$ *such that* $D(B) \supset D(T)$. *Define* U *to be the set of scalars* λ *for which* $T + \lambda B \in F_+ \cup F_-$. *Then*

I) U *is open;*

II) *if* C *is a component of* U, *we have that* $\bar{b}((T + \lambda B)') = b((T + \lambda B)')$ *and* $\bar{b}(T + \lambda B)$ *have constant values,* n_1 *and* n_2 *respectively, on* C *with the possible exception of isolated points. At the isolated points* $\bar{b}((T + \lambda B)') = b((T + \lambda B)') > n_1$ *and* $\bar{b}(T + \lambda B) > n_2$.

Proof. Considering Remark VI.1.1 we see that U corresponds to the set of scalars λ for which $(T + \lambda B)' \in \Phi_- \cup \Phi_+$. Let $B_1 = BJ_{D(T)}$. Since $(T + \lambda B)' = T' + \lambda B_1'$ and hence $\alpha(T' + \lambda B_1') = \bar{b}(T + \lambda B)$ for each λ, the result now follows by applying Theorem I.4.21 to T' and B_1'. □

VI.3.4 **COROLLARY.** *Let* $T \in C(X,Y)$ *with* X *complete and* $B \in B(X,Y)$ *such that* $D(B) \supset \overline{D(T)}$. *Defining* U *as in Proposition VI.3.3 we have that if* C *is a component of* U, *then on* C, *with the possible exception of isolated points,* $\alpha(T + \lambda B)$ *and* $\bar{b}(T + \lambda B) = b(T + \lambda B)$ *have constant values* n_1 *and* n_2 *respectively. At the isolated points* $\alpha(T + \lambda B) > n_1$ *and* $\bar{b}(T + \lambda B) = b(T + \lambda B) > n_2$.

Proof. Note that U corresponds to the set of scalars λ for which $(T + \lambda B)' \in \Phi_- \cup \Phi_+$ (Remark VI.1.1). Hence as in Corollary VI.3.2 we can show that $\gamma(T + \lambda B) > 0$ whence $R(T + \lambda B)$ is closed for each $\lambda \in U$ by Theorem III.2.6. The result now follows from Proposition VI.3.3 and Corollary III.2.7.

□

VI.3.5 **THEOREM.** *Let* $T \in L(X,Y)$ *and* $B \in B(X,Y)$ *such that* $D(B) \supset D(T)$. *If* $T \in F_-(F_+)$ *and* $\Delta'(J_Y B) < \Gamma'(T)$ $(\Delta(B) < \Gamma(T))$, *then*

I) $T + B \in F_-(F_+)$;

II) $k(T') = k((T + B)')$;

III) $b((T + \lambda B)')$ *and* $\bar{b}(T + \lambda B)$ *have constant values* n_1 *and* n_2 *respectively for each* λ *such that* $|\lambda|\Delta'(J_Y B) < \Gamma'(T)$ $(|\lambda|\Delta(B) < \Gamma(T))$ *except perhaps for isolated points. At the isolated points* $\infty > b((T + \lambda B)') > n_1$ *and* $\bar{b}(T + \lambda B) > n_2$ $(b((T + \lambda B)') > n_1$ *and* $\infty > \bar{b}(T + \lambda B) > n_2)$.

Proof.

I) Let $T \in F_-$ and $\Delta'(J_Y B) < \Gamma'(T)$. By Theorem II.2.4

$$0 < \Gamma'(T) = \Gamma'(T + B - B) \leq \Gamma'(T + B) + \Delta'(J_Y B).$$

Hence $0 < \Gamma'(T) - \Delta'(J_Y B) \leq \Gamma'(T + B)$ and so $T + B \in F_-$ by Remark VI.1.1 (note that $dim\ Y = \infty$ since $\Gamma'(T) > 0$). If $T \in F_+$ we can construct an analogous proof by using the inequality $\Gamma(T + B) \leq \Gamma(T) + \Delta(B)$ in Theorem II.2.6.

II) Note that since $\Delta'(J_Y\lambda B) \leq \Delta'(J_YB)$ ($\Delta(\lambda B) \leq \Delta(B)$) for each λ such that $|\lambda| \leq 1$, we conclude from I) that $T + \lambda B \in F_-(F_+)$ for each scalar λ such that $|\lambda| \leq 1$. Hence $(T + \lambda B)' = T' + \lambda B_1' \in \phi_+(\phi_-)$ for each $|\lambda| \leq 1$ by Remark VI.1.1 where $B_1 = BJ_{D(T)}$. Define a function $\Phi(\lambda) = k(T' + \lambda B_1')$ from the closed unit ball of the scalars into $\mathbb{Z}$ where the scalars have the usual topology and $\mathbb{Z}$ is the integers together with $-\infty$ $(+\infty)$ under the discrete topology. Applying Theorem I.4.19 we conclude that Φ is continuous and hence that Φ is a constant function [22; 3A]. Consequently $k(T') = k(T' + B_1')$.

III) Since the set of scalars being considered is connected and open, this is a consequence of I), Proposition VI.3.3 and the fact that $\bar{b}(T + \lambda B) < \infty$ if $T + \lambda B \in F_-$ ($b((T + \lambda B)') < \infty$ if $T + \lambda B \in F_+$ since then $(T + \lambda B)' \in \phi_-$). □

VI.3.6 **COROLLARY**. *Let* $T \in C(X,Y)$, $B \in B(X,Y)$ *such that* $D(B) \supset \overline{D(T)}$ *and* X *complete. If* $T \in F_-(F_+)$ *and* $\Delta'(J_YB) < \Gamma'(T)$ ($\Delta(B) < \Gamma(T)$), *then*

I) $T + B \in F_-(F_+)$.

II) $k(T) = k(T + B)$ *with* T *and* $T + B$ *normally solvable.*

III) $a(T + \lambda B)$ *and* $b(T + \lambda B)$ *have constant values* n_1 *and* n_2 *respectively for each* λ *such that* $|\lambda|\Delta'(J_YB) < \Gamma'(T)$ $(|\lambda|\Delta(B) < \Gamma(T))$, *except perhaps for isolated points. At the isolated points* $\infty > a(T + \lambda B) > n_1$ *and* $b(T + \lambda B) > n_2$ ($a(T + \lambda B) > n_1$ *and* $\infty > b(T + \lambda B) > n_2$).

Proof. As in Corollaries VI.3.2 and VI.3.4 we can show that $T + \lambda B$ is closed for each λ. Hence as before if $T + \lambda B \in F_-(F_+)$, $(T + \lambda B)' \in \phi_+(\phi_-)$ and so $\gamma(T + \lambda B) > 0$ with $R(T + \lambda B)$ closed. Considering Corollary III.2.7 and Theorem VI.3.5, the result follows. □

VI.3.7 *Remark.* If $T \in F_-(X,Y)$ $(F_+(X,Y))$ and $B \in L(X,Y)$ such that $D(B) \supset D(T)$ and $\Delta'(J_Y B) < \Gamma'(T)$ $(\Delta(B) < \Gamma(T))$ then $T + B \in F_-(F_+)$. Consequently $SC[X,\tilde{Y}] \subset P(F_-)$ where $P(F_-)$ denotes the class of operators such that for $A \in P(F_-)$, $A + S \in F_-$ for any $S \in F_-(X,Y)$. (If $\dim Y < \infty$ this follows from Remark VI.1.1 and if $\dim Y = \infty$ this can be proved along similar lines to the proof for I) of Theorem VI.3.5). We note that Cross [6] showed that $P(F_+) = SS(X,Y)$. Selecting $T \in F_+$ and $B \in SS(X,Y)$ arbitrarily we note from Remark VI.1.1 that we can assume $D(T + B) \in \mathscr{I}(D(T))$ since $T + B \in F_+$ otherwise. By the definition of Γ we note that $0 < \Gamma(T) \leq \Gamma(T|_{D(T+B)})$. Hence $T|_{D(T+B)} \in F_+$ by Theorem II.3.5 and so as in I) of Theorem VI.3.5 we can show that $T + B \in F_+$. Conversely suppose $SS(X,Y) \subsetneq P(F_+)$ and select $B \in P(F_+)\backslash SS(X,Y)$. But then $\Delta(B) = \sup_{M \in \mathscr{I}(D(B))} \Gamma(BJ_M) > 0$. Choose $K \in \mathscr{I}(D(B))$ such that $\Gamma(BJ_K) > 0$ and let $T = B|_K$. Then $T \in F_+$ with $\Gamma(T - B) = 0$ and hence $T - B \notin F_+$ as $D(T + B) = K$ is infinite dimensional (Theorem II.3.5). This is a contradiction and so we are done.

VI.3.8 **COROLLARY.** *Let $T \in L(X,Y)$ with $D(T)$ dense in X and $X = Y$. Suppose there exists λ_0 such that $(\lambda_0 I - T)^{-1} \in SS(Y,X) \cap B(Y,X)$ $(SC(Y,\tilde{X}) \cap B(Y,X))$ with $R(\lambda_0 I - T)$ dense in Y. Then for every λ, $(\lambda I - T) \in F_+ \cap F_-$ with $b((\lambda I - T)') = \bar{b}(\lambda I - T) < \infty$. If $\lambda_0 I - T$ is surjective then $\lambda I - T$ is also closed for each λ. (Note that $I = I_X$.)*

Proof. By the hypothesis $R(\lambda_0 I - T)$ is dense in Y and so $(\lambda_0 I - T)'$ is injective by Proposition I.10.4. Hence $[(\lambda_0 I - T)^{-1}]' = [(\lambda_0 I - T)']^{-1}$ by Theorem I.10.6. Also since $(\lambda_0 I - T)$ has a continuous inverse, $(\lambda_0 I - T) \in RO$ and so $\gamma(\lambda_0 I - T) > 0$. Therefore

(1) $\alpha((\lambda_0 I - T)') = \bar{b}(\lambda_0 I - T) = 0$ and $b((\lambda_0 I - T)') = \alpha(\lambda_0 I - T) = 0$

by Corollary III.2.7.

Observe that for each λ, $(\lambda I - T)' = \lambda I' - T' = \lambda I_{X'} - T'$ and since $\lambda_0 I' - T'$ is bijective by (1), we can write

(2) $$(\lambda I - T)' = (I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1})(\lambda_0 I' - T').$$

Let I_R denote the restriction of I to the dense subspace $R(\lambda_0 I - T)$ of X. Then $(I_R)' = I'$ by Theorem I.5.6 and so since $(\lambda_0 I - T)^{-1} \in SS(Y,X) \cap B(Y,X)$ $(SC(Y,\tilde{X}) \cap B(Y,X))$ it follows from Theorem VI.3.5 that $I_R + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1} \in F_+(F_-)$ with $k(I') = k((I_R + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1})')$. However

$$\begin{aligned}(I_R + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1})' &= I' + (\lambda - \lambda_0)[(\lambda_0 I - T)^{-1}]' \\ &= I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1}\end{aligned}$$

and therefore

(3) $$0 = k(I') = k(I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1}),$$

whence $I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1} \in \Phi_- \cap \Phi_+$

since $I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1} \in \Phi_-(\Phi_+)$ by Remark VI.1.1.Note that $I + (\lambda - \lambda_0)(\lambda_0 I - T)^{-1}$ is continuous and thus $I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1}$ is bounded. Considering (3) $\gamma(I' + (\lambda - \lambda_0)[\lambda_0 I' - T']^{-1}) > 0$ by Theorem III.2.6 and so by (1), (2), (3) and Theorem III.3.4 $(\lambda I - T)' \in \Phi_+ \cap \Phi_-$ with $k((\lambda I - T)') = 0$ for each λ. That is $\lambda I - T \in F_+ \cap F_-$ (Remark VI.1.1) with $b(\lambda I' - T') = \alpha(\lambda I' - T') = \bar{b}(\lambda I - T) < \infty$ for each λ. The fact that $\lambda I - T$ is closed if $\lambda_0 I - T$ is surjective follows from Corollary III.3.6. (Note that if $\lambda_0 I - T$ is surjective, the fact that $(\lambda_0 I - T)^{-1}$ is bounded implies that $\lambda_0 I - T$ is closed by Remark I.4.12.) $\square$

All results in this section except the fact that $P(F_+(X,Y)) = SS(X,Y)$ are generalisations by the author. Proposition VI.3.1 and Corollary VI.3.2 generalise [16; V.1.7] whereas Proposition VI.3.3 and Corollary VI.3.4 generalise [16; V.1.8]. Theorem VI.3.5 and Corollary VI.3.6 contain and generalise results by Kato (cf. [16; V.1.6 and V.2.1]), Weis [38; 3.8] and Schechter [36; 2.12]. Corollary VI.3.8 generalises [16; V.2.3].

CHAPTER VII

THE INSTABILITY OF A CLASS OF NON-SEMI-FREDHOLM TYPE OPERATORS UNDER COMPACT PERTURBATIONS

All results in this chapter are by the author and generalise and complement results by the author which are due for publication ([26]; see appendix).

§1 BASIC THEOREMS

The first result we prove was initially proved by Goldman ([17]; cf[16; V.2.6]) for normally solvable operators between Banach spaces and subsequently generalised by the author [26; 2.2] to normally solvable operators between operator ranges. The proof is similar to that appearing in Goldberg [16], suitably changed to hold for the more general case.

VII.1.1 **THEOREM.** *Let* $T \in L(X,Y)$ *with* Y *an operator range. If* $\gamma(T) > 0$ *and the reduced index of* T, $\bar{k}(T)$, *does not exist, then there is a compact (nuclear) operator* $B \in B[X,Y]$ *such that* $\gamma(T + \lambda B) = 0$ *for each* $\lambda \neq 0$. *If* $N(T)$ *is separable, then* B *may be chosen so that* $T + \lambda B$ *is injective as well.*

Proof. Since $\dim \overline{R(T)}^{\perp} = \dim(Y/\overline{R(T)})' = \bar{b}(T) = \infty$ there exists an infinite linear independent set $\{y_1', y_2', \ldots\} \subset \overline{R(T)}^{\perp}$. Select $y_1 \in Y$ so that $y_1'(y_1) \neq 0$. For $n = 1,2,\ldots$ let y_{n+1} be an element in ${}^{\perp}\mathrm{span}\{y_1', y_2', \ldots, y_n'\} = \bigcap_{i=1}^{n} N(y_i')$ but not in $N(y_{n+1}')$. The existence of y_{n+1} is assured by the fact that if $N(y_{n+1}') \subset \bigcap_{i=1}^{n} N(y_i')$, then y_{n+1}' would be a linear combination of $y_1', y_2', \ldots, y_n'$ (Proposition I.5.11).

Hence we can choose $\{y_k\} \subset Y$ so that

$$(1) \qquad \begin{aligned} y_k'R(T) &= 0 \quad \text{for} \quad k \geq 1; \\ y_k'(y_n) &= 0 \quad \text{for} \quad n > k \geq 1; \\ y_k'(y_k) &\neq 0 \quad \text{for} \quad k \geq 1. \end{aligned}$$

Let $\{x_1, x_2, \ldots\}$ be an infinite linearly independent set in $N(T)$. Since $X_0 = \overline{span}\{x_1, x_2, \ldots\}$ is separable, there exists a countable total set $\{v_k'\} \subset X_0'$ (Theorem I.6.6). Without loss of generality we may assume $v_k' \neq 0$ where $k \in \mathbb{N}$. For each $k \in \mathbb{N}$ let x_k' be an extension of v_k' to all of X. Consider the following:

$$(2) \qquad Bx = \sum_{k=1}^{\infty} \frac{x_k'(x) y_k}{2^k \|x_k'\| \, \|\beta_Y y_k\|_1} \quad \text{for each} \quad x \in X.$$

Hence

$$\|\beta_Y Bx\|_1 \leq \sum_{k=1}^{\infty} \frac{\|x_k'\| \, \|x\| \, \|\beta_Y y_k\|_1}{2^k \|x_k'\| \, \|\beta_Y y_k\|_1} = \|x\| \sum_{k=1}^{\infty} 1/2^k \quad \text{for each} \quad x \in X$$

and so, by the completeness of the preimage space Y_1 of Y and the absolute convergence of $\beta_Y Bx$ for each $x \in X$, we conclude that $\beta_Y Bx$ and therefore $Bx = \alpha_Y \beta_Y Bx$ exists for each $x \in X$. Note that $\beta_Y B$ is a nuclear operator and hence compact (Theorem I.11.7). Since α_Y is bounded we conclude from Theorem I.11.8 that $\alpha_Y(\beta_Y B) = B$ is still compact (nuclear).

Now suppose $Bx \in R(B) \cap R(T)$. By (1) and (2) we then have

$$0 = y_1'Bx = \frac{x_1'(x) \cdot y_1'(y_1)}{2\|x_1'\| \, \|\beta_Y y_1\|_1} \quad \text{whence} \quad x_1'(x) = 0.$$

Consequently

$$0 = y_2'Bx = \frac{x_2'(x) \cdot y_2'(y_2)}{4\|x_2'\| \, \|\beta_Y y_2\|_1} \quad \text{whence} \quad x_2'(x) = 0.$$

Continuing inductively we see that $x_k'(x) = 0$ for each $k \in \mathbb{N}$ and so $Bx = 0$. Hence $R(B) \cap R(T) = \{0\}$ and so

$$(3) \qquad N(T + \lambda B) = N(T) \cap N(B) \quad \text{for each} \quad \lambda \neq 0.$$

Let $x \in X_0 \subset \overline{N(T)}$ with $Bx = 0$. As before we can show that $x_k'(x) = 0$ for each $k \in \mathbb{N}$ and since $\{x_k'\}$ is total on X_0, we conclude that $x = 0$. Hence B is injective on X_0. Assume that $\gamma(T + \lambda B) > 0$ for some $\lambda \neq 0$. Therefore for any $\mu \in (0,\gamma(T + \lambda B))$ we have that

$$(4) \qquad \|T + \lambda B\| \geq \mu.d(x,N(T + \lambda B)) \quad \text{for each } x \in D(T).$$

Now let B_1 be the restriction of B to $N(T)$. From (3) and (4) we see that $N(B_1) = N(T) \cap N(B) = N(T + \lambda B)$ and so

$$(5) \qquad \|B_1 x\| = \frac{1}{|\lambda|}.\|(T + \lambda B)x\| \geq \frac{\mu}{|\lambda|}.d(x,N(B_1)) \quad \text{for each } x \in N(T).$$

Thus $\gamma(B_1) \geq \frac{\mu}{|\lambda|} > 0$. Considering $\bar{B}_1$ as an element of $B[N(T)^\sim,\tilde{Y}]$ we have that $\bar{B}_1$ is still compact since $\Gamma_0(B_1) = \Gamma_0'(B_1') = \Gamma_0'((\bar{B}_1)') = \Gamma_0(\bar{B}_1)$ by Proposition IV.2.5. Also $\gamma(\bar{B}_1) \geq \gamma(B_1) > 0$ by Theorem III.1.6. Hence $R(\bar{B}_1)$ is closed by Theorem III.2.6 and thus finite dimensional by Proposition I.11.6. But this is a contradiction since $\bar{B}_1$ is injective on $X_0 \subset N(T)^\sim$. Consequently $\gamma(T + \lambda B) = 0$ for each $\lambda \neq 0$. Now if $N(T)$ was separable, we could have chosen X_0 so that $X_0 = \overline{N(T)}$ and so as B is injective on X_0, we have that $N(T + \lambda B) = N(T) \cap N(B) \subset X_0 \cap N(B) = \{0\}$. □

VII.1.2 **COROLLARY.** *Let* $T \in L(X,Y)$. *If* $\gamma(T) > 0$ *and* $\bar{k}(T)$ *does not exist, then there exists a compact (nuclear) operator* $B \in B[X,\tilde{Y}]$ *such that* $\gamma(J_Y T + \lambda B) = 0$ *for each* $\lambda \neq 0$. *If* $N(T)$ *is separable, then* B *may be chosen such that* $J_Y T + \lambda B$ *is injective as well.*

Proof. Observe that $\gamma(J_Y T) = \gamma(T)$, $a(J_Y T) = a(T)$ and $\bar{b}(J_Y T) = \bar{b}(T)$. Since $\tilde{Y}$ is an operator range, the corollary now follows from Theorem VII.1.1. □

Considering Theorem III.2.6 we see that if $T \in L(X,Y)$ is closed with X and Y complete then T is normally solvable if and only if $\gamma(T) > 0$ and so Theorem VII.1.1 and Corollary VII.1.2 contain the result by Goldman.

VII.1.3 **DEFINITION.** Let $T \in L(X,Y)$. If $\{T_n\} \subset L(X,Y)$ such that $D(T_n) \supset D(T)$ with $T_n - T$ continuous for each $n \in \mathbb{N}$ then we say that T_n *converges to* T, denoted $T_n \to T$, if $\|T_n - T\| \to 0$ as $n \to \infty$.

The following theorem was first proved for bounded operators between Hilbert spaces by R. Bouldin [3]. This result was then generalised to the Banach space case by M. Gonzales and V.M. Onieva [18] and later further generalised by the author [26] to closed and α-compact operators from an operator range into a Banach space. The result in its present form is also due to the author and contains the results by Bouldin and Gonzales and Onieva since for closed operators between Banach spaces the classes $F_+ \cup F_-$ and $RO \cup F_+ \cup F_-$ agree with the semi-Fredholm and normally solvable operators respectively (Theorems III.2.6, VI.1.4 and VI.1.5). Note that $L(X,Y) \setminus F_+ \cup F_-$ and $L(X,Y) \setminus RO \cup F_+ \cup F_-$ denote the complements of $F_+ \cup F_-$ and of $RO \cup F_+ \cup F_-$ in $L(X,Y)$ respectively.

VII.1.4 **THEOREM.** *Let* $T \in L(X,Y)$. *Then*

I) $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if* $J_Y T$ *is the limit in* $L(X,\tilde{Y})$ *of a sequence of operators* $\{T_n\} \subset L(X,\tilde{Y}) \setminus RO \cup F_+ \cup F_-$ *with* $D(T) = D(T_n)$ *for each* $n \in \mathbb{N}$.

II) $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if there exists a compact (nuclear) operator* $B \in B[X,\tilde{Y}]$ *and* $A \in L(X,\tilde{Y}) \setminus RO \cup F_+ \cup F_-$ *such that* $J_Y T = A + B$.

III) $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if there exists a compact (nuclear) operator* $B \in B[X,\tilde{Y}]$ *and* $A \in L(X,\tilde{Y})$ *such that* $J_Y T = A + B$ *with* $\alpha(A) = \bar{b}(A) = \infty$.

Proof. From II) and IV) of Remark VI.1.1 we conclude that $\dim D(T) = \dim Y = \infty$.

I) Let $T \in L(X,Y) \setminus F_+ \cup F_-$. Suppose $T \notin RO$. Then let $T_n = J_Y T$ for each $n \in \mathbb{N}$. Now suppose $T \in RO$ $(\gamma(T) > 0)$. Then $\bar{k}(T)$ does not exist since otherwise $T \in F_+ \cup F_-$ by Propositions VI.2.1 and VI.2.2. We conclude from Corollary VII.1.2 that there exists $B \in K[X,\tilde{Y}]$ such that $J_Y T + \lambda B \notin RO$ $(\gamma(J_Y T + \lambda B) = 0)$ for each $\lambda \neq 0$. Select a sequence of non-zero scalars $\{\lambda_n\}$ such that $\lambda_n \to 0$. Now let $T_n = J_Y T + \lambda_n B$. Obviously $T_n \to J_Y T$ with $T_n \notin RO$ and $D(T) = D(T_n)$ for each $n \in \mathbb{N}$. It remains to prove that $J_Y T + \lambda_n B \notin F_+ \cup F_-$. Since B is compact, $B \in SS(X,\tilde{Y}) \cap SC(X,\tilde{Y})$ by Remark V.1.4 and so if $J_Y T + \lambda_n B \in F_+ \cup F_-$ then $J_Y T \in F_+ \cup F_-$ by Theorem VI.3.5. Hence $T \in F_+ \cup F_-$ by Remark VI.1.1, a contradiction.

Conversely suppose $T \in F_+ \cup F_-$ and that there exists $\{T_n\} \subset L(X,\tilde{Y}) \setminus RO \cup F_+ \cup F_-$ such that $T_n \to J_Y T$ and $D(T) = D(T_n)$ for each $n \in \mathbb{N}$. If $T \in F_+$ then there exists $k \in \mathbb{N}$ so that $\|J_Y T - T_k\| < \Gamma(T)/2 = \Gamma(J_Y T)/2$ $(\Gamma(T) > 0$ by Theorem II.3.5) and hence

$$\Delta(J_Y T - T_k) \leq \|J_Y T - T_k\| < \Gamma(J_Y T)/2.$$

But then $\Gamma(T_k) > 0$ by Remark VI.3.7 and so $T_k \in F_+$; a contradiction. Analogously if $T \in F_-$, we obtain some $k \in \mathbb{N}$ such that $T_k \in F_-$. Hence the result follows.

II) Let $T \in L(X,Y) \setminus F_+ \cup F_-$. If $T \notin RO$ then let $J_Y T = A$ and $B = 0$. If $T \in RO$ then as in I) we can show that $\bar{k}(T)$ does not exist and so by Corollary VII.1.2 there is some nuclear operator $B \in B[X,\tilde{Y}]$ with $J_Y T - B = A \notin RO$. Again as in I) we can conclude from Theorem VI.3.5 that since B is compact, $T \notin F_+ \cup F_-$ implies $J_Y T - B \notin F_+ \cup F_-$.

Conversely suppose $J_Y T = A + B$ where $A \in L(X,\tilde{Y}) \setminus RO \cup F_+ \cup F_-$ and $B \in B[X,\tilde{Y}]$ compact (nuclear). Since B is compact, we conclude from Remark V.1.4 that $\Delta'(B) = \Delta(B) = 0$ and hence by Theorem VI.3.5, $J_Y T \in F_+ \cup F_-$ implies $J_Y T - B = A \in F_+ \cup F_-$. We conclude that $J_Y T \notin F_+ \cup F_-$ whence $T \notin F_+ \cup F_-$ (Remark VI.1.1).

III) Suppose $T \in F_+ \cup F_-$. Then $J_Y T \in F_+ \cup F_-$ (Remark VI.1.1). Considering Theorem VI.3.5 and the fact that $K[X,\tilde{Y}] \subset SS[X,\tilde{Y}] \cap SC[X,\tilde{Y}]$, we conclude that $J_Y T - B \in F_+ \cup F_-$ for every compact (nuclear) operator $B \in B[X,\tilde{Y}]$. Hence either $\alpha(J_Y T - B) < \infty$ or $\bar{b}(J_Y T - B) < \infty$ by Remark VI.1.1.

Conversely suppose $T \notin F_+ \cup F_-$. As in Theorem VI.1.7 we can construct a compact (nuclear) operator $B_1 \in B[X,\tilde{Y}]$ such that $\bar{b}(J_Y T - B_1) = \infty$. Now let $\alpha(J_Y T - B_1) < \infty$ since we are done otherwise. Suppose $J_Y T - B_1 \in F_+$. Then since B_1 is compact, $\Delta(B_1) = 0$ (Remark V.1.4) and so $J_Y T \in F_+$ by Theorem VI.3.5. But then $T \in F_+$; a contradiction. Hence $J_Y T - B_1 \notin F_+$ and so $J_Y T - B_1$ does not have a continuous inverse. Consequently select $x_1 \in D(T) = D(J_Y T - B_1)$ such that $\|x_1\| = 1$ and $\|(J_Y T - B_1)x_1\| \le \frac{1}{2}$. Now choose $x_1' \in X'$ so that $x_1'(x_1) = \|x_1'\| = 1$. Suppose $\{x_1, x_2, \dots, x_{n-1}\} \subset D(J_Y T - B_1)$ and $\{x_1', x_2', \dots, x_{n-1}'\} \subset X'$ have been constructed satisfying

$$(1) \qquad \|x_k\| = 1, \quad \|(J_Y T - B_1)x_k\| \le 2^{1-2k},$$
$$\|x_k'\| \le 2^{k-1} \quad \text{and} \quad x_k'(x_j) = \delta_{kj} \quad \text{for} \quad 1 \le k, j \le n-1.$$

Since $J_Y T - B_1 \notin F_+$, the restriction of $J_Y T - B_1$ to the finite codimensional subspace $\bigcap_{i=1}^{n-1} N(x_i') = {}^{\perp}\{x_1', \dots, x_{n-1}'\}$ does not have a continuous inverse. Hence there exists $x_n \in D(J_Y T - B_1) \cap {}^{\perp}\{x_1', \dots, x_{n-1}'\}$ such that $\|x_n\| = 1$ and $\|(J_Y T - B_1)x_n\| \le 2^{1-2n}$. Select $z' \in X'$ such that $z'(x_n) = \|z'\| = 1$. Now let $x_n' = z' - \sum_{k=1}^{n-1} z'(x_k)x_k'$. It is then easy to show that $x_n'(x_k) = \delta_{nk}$ for $1 \le k \le n$ where

$$\|x_n'\| \le \|z'\| + \sum_{k=1}^{n-1} \|z'\| \, \|x_k\| \, \|x_k'\| \le 1 + \sum_{k=1}^{n-1} 2^{k-1} = 2^{n-1}.$$

And so by induction we can construct sequences $\{x_1, x_2, \dots\} \subset D(J_Y T - B_1)$ and $\{x_1', x_2', \dots\} \subset X'$ such that (1) is satisfied for all $n \in \mathbb{N}$.

Now define

$$(2) \qquad B_2x = \sum_{k=1}^{\infty} x_k'(x)(J_YT - B_1)x_k \quad \text{for all} \quad x \in X.$$

Since $\tilde{Y}$ is complete and B_2x converges absolutely for each $x \in X$, $B_2x \in \tilde{Y}$ exists for each x. Letting $z_k' = x_k'/2^{k-1}$ and $y_k = 2^{2k-1}.(J_YT - B_1)x_k$ we note that $B_2 = \sum_{k=1}^{\infty} (1/2^k).z_k'(.)y_k$. Consequently $B_2 \in B[X,\tilde{Y}]$ is a nuclear and hence compact operator by Theorem I.11.7. Note that for each $x_k \in D(J_YT - B_1)$, $B_2x_k = (J_YT - B_1)x_k$ and so since $\{x_k\} \subset D(J_YT - B_1)$ is linearly independent by (1), $\alpha(J_YT - B_1 - B_2) = \infty$. Moreover

$$R(B_2) \subset \overline{span\{(J_YT - B_1)x_1,(J_YT - B_1)x_2,\dots\}} \subset \overline{R(J_YT - B_1)}.$$

Consequently since $\bar{b}(J_YT - B_1) = \infty$, $\bar{b}(J_YT - B_1 - B_2) = \infty$. Now let $B = B_1 + B_2$ and $A = J_YT - B_1 - B_2$. Note that $B_1 + B_2$ is still nuclear and hence compact (Theorem I.11.7). □

The construction in Theorem VII.1.4 is essentially the same as that used by Gohberg, Markus and Feldman [15; 4.1], suitably changed to hold for the more general case.

VII.1.5 **PROPOSITION.** *Let* $T \in L(X,Y)$ *with* Y *an operator range. Then*

I) $T \in L(X,Y)\setminus F_+ \cup F_-$ *if and only if* T *is the limit in* $L(X,Y)$ *of a sequence of operators* $\{T_n\} \subset L(X,Y) \setminus RO \cup F_+ \cup F_-$ *with* $D(T_n) = D(T)$ *for each* $n \in \mathbb{N}$.

II) $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if there exists a compact (nuclear) operator,* $B \in B[X,Y]$ *and* $A \in L(X,Y) \setminus RO \cup F_+ \cup F_-$ *such that* $T = A + B$.

Proof. The proof is analogous to that for I) and II) of Theorem VII.1.4 with Theorem VII.1.1 being used instead of Corollary VII.1.2. □

VII.1.6 *Remark.* Let Y be an operator range. By Theorem VI.3.5 we conclude that if $T \in F_+(X,Y)$ $(F_-(X,Y))$ and $B \in B[X,Y]$ such that $\|B\| < \Gamma(T)$ $(\|B\| < \Gamma'(T))$ then $\Delta(B) \leq \|B\| < \Gamma(T)$ $(\Delta'(J_Y B) < \Gamma'(T))$ and so $T + B \in F_+(F_-)$. Hence $(F_+ \cup F_-) \cap B[X,Y]$ is open and consequently $B[X,Y] \setminus F_+ \cup F_-$ is closed. From I) of Proposition VII.1.5 we conclude that

$\overline{B[X,Y] \setminus RO \cup F_+ \cup F_-} = B[X,Y] \setminus F_+ \cup F_-$.

§2 RESULTS ON INSTABILITY

VII.2.1 **DEFINITION.** Let $V_0, V_1, \dots, V_5$ be defined by

$$V_0 = \{T \in L(X,Y) \setminus RO \cup F_+ \cup F_- : \ a(T) = \bar{b}(T) < \infty \}$$
$$V_1 = \{T \in L(X,Y) \setminus RO \cup F_+ \cup F_- : \ 0 \neq |a(T) - \bar{b}(T)| < \infty \}$$
$$V_2 = \{T \in L(X,Y) \setminus RO \cup F_+ \cup F_- : \ a(T) - \bar{b}(T) = -\infty \}$$
$$V_3 = \{T \in L(X,Y) \setminus RO \cup F_+ \cup F_- : \ a(T) - \bar{b}(T) = +\infty \}$$
$$V_4 = \{T \in L(X,Y) \setminus RO \cup F_+ \cup F_- : \ a(T) = \infty = \bar{b}(T) \}$$
$$V_5 = \{T \in RO(X,Y) : \ a(T) = \infty = \bar{b}(T) \}$$

Where there is danger of confusion we will write $V_j(X,Y)$, $j = 0,1,\dots,5$, to denote that $V_j \subset L(X,Y)$.

VII.2.2 **LEMMA.** *Let Z be an infinite dimensional normed linear space.*

I) *If $M \in \mathcal{S}_c(Z)$ then there exists an infinite dimensional and separable closed subspace F of Z such that $F \cap M = \{0\}$ and $\overline{F \oplus M}$ is infinite codimensional.*

II) *If $M \in \mathcal{S}(Z)$ then there exist two separable subspaces, P and Q, of M and a fundamental and total biorthogonal system $(\{x_n\}; \{p_n'\})$ in P such that P is infinite dimensional, Q may be chosen either finite or infinite dimensional, $P \cap Q = \{0\}$ and each p_n' has an extension to all of Z which annihilates Q.*

Proof. I) We first construct $E \in \mathcal{I}_c(Z)$ such that $M \subset E$ and $cod_E M = \infty$. Select $\{x_n\} \subset Z$ and $\{x'_n\} \subset M^\perp$ inductively as follows:

$$x'_1 \in M^\perp \text{ and } x_1 \in N(x'_1) \setminus M$$

with

$$x'_n \in E^\perp_{n-1} \setminus span\{x'_1,\dots,x'_{n-1}\} \text{ and } x_n \in G_n \setminus E_{n-1}$$

where $E_{n-1} = M \oplus span\{x_1,\dots,x_{n-1}\}$ and $G_n = \bigcap_{i=1}^{n} N(x'_i)$ for $n \geq 2$ and $G_1 = N(x'_1)$. Observe that

$$M \subset E_1 \subset \dots \subset E_n \subset \dots \text{ and } G_1 \supset G_2 \supset \dots \supset G_n \supset \dots .$$

Notice that $E_1 \subset G_1$ whence $E_2 \subset G_2$ and so by induction $E_n \subset G_n$ for each $n \in \mathbb{N}$. Consequently $E_n \subset G_m$ for any $n,m \in \mathbb{N}$. Now let E and G be as follows:

$$E = \overline{\bigcup_{i=1}^{\infty} E_i} \text{ and } G = \bigcap_{i=1}^{\infty} G_i .$$

Obviously $M \subset E \subset G$ with $cod_Z E \geq cod_Z G = \infty$ and $cod_E M = \infty$. Select a linearly independent sequence $\{y_n\}$ in $E \setminus M$. Since $\overline{span\{y_1,y_2,\dots\}}$ is a separable space, $M \cap \overline{span\{y_1,y_2,\dots\}}$ has a quasicomplement F in $\overline{span\{y_1,y_2,\dots\}}$ (Theorem I.6.8). Then $F \cap M = \{0\}$ and $cod_Z \overline{F \oplus M} \geq cod_Z E = \infty$.

II) Let $M \in \mathcal{I}(Z)$. Select a separable subspace R of M with $R \in \mathcal{I}(M)$. By Theorem I.6.6 there exists a fundamental and total biorthogonal system $(\{r_n\};\ \{r'_n\})$ in R. Let P, Q denote the subspaces $P = \overline{span}\{r_{2n}: n \in \mathbb{N}\}$ and $Q = \overline{span}\{r_{2n-1}: n \in \mathbb{N}\}$. Clearly $P \cap Q = \{0\}$. For any $n \in \mathbb{N}$ we define $x_n = r_{2n}$ and $p'_n = r'_{2n}|_P$ and let x'_n be an extension of r'_{2n} to all of Z. Hence $(\{x_n\};\ \{p'_n\})$ is a fundamental and total biorthogonal system in P with x'_n an extension of p'_n to all of Z which annihilates Q. Alternatively let $P = \overline{span}\{r_{k+1},\ r_{k+2},\dots\}$ and $Q = span\{r_1,r_2,\dots,r_k\}$. For every $n \in \mathbb{N}$ we then let $x_n = r_{k+n}$, $p'_n = r'_{k+n}|_P$ and with x'_n an extension of r'_{k+n} to all of Z which annihilates Q. □

VII.2.3 **THEOREM.** *Consider* $L(X,Y)$ *and let* Y *be an operator range*

I) If X *and* Y *are separable then* $V_4 \cup V_5 \subset V_j + K[X,Y]$ *for* $j = 0;1$.

II) If X *is separable then* $V_4 \cup V_5 \subset V_2 + K[X,Y]$.

III) If Y *is separable then* $V_4 \cup V_5 \subset V_3 + K[X,Y]$.

IV) $V_4 \cup V_5 \subset V_4 + K[X,Y]$.

Proof. Assume $T \in V_4 \cup V_5$. Let P and Q be subspaces of $N(T)$ such that $P \cap Q = \{0\}$ with $P \in \mathcal{I}(N(T))$ separable and $(\{x_n\}, \{p_n'\})$ a fundamental and total biorthogonal system in P such that each p_n' has an extension, say x_n', which annihilates Q. The existence of P and Q follows from the Lemma. Note that the restriction of x_n' to $D(T)$ still satisfies the requirements that it is an extension of p_n' to all of $D(T)$ which annihilates Q and so without loss of generality we can assume that $D(T) = X$. Again by the Lemma we now have that there exists $F \in \mathcal{I}(Y)$ such that F is separable and closed with $F \cap \overline{R(T)} = \{0\}$. Since F is separable, there exists a fundamental and total biorthogonal system $(\{y_n\}, \{f_n'\})$ in F (Theorem I.6.6). For each $n \in \mathbb{N}$ let y_n' be an extension of f_n' to all of Y. Now define $B \in K[X,Y]$ as follows:

$$Bx = \sum_{i=1}^{\infty} \lambda_i x_i'(x) y_i \quad \text{for each} \quad x \in X$$

where $\lambda_i = (\|x_i'\| . \|\beta_Y y_i\|_1 . 2^i)^{-1}$ for each $i \in \mathbb{N}$. Note that by choice of λ_i and by the completeness of Y_1, the pre-image space of Y, $\beta_Y Bx$ converges absolutely for each $x \in X$ and hence $\beta_Y Bx \in Y_1$, and so $Bx \in Y$, exists for each $x \in X$ (Theorem I.2.3). In fact $\beta_Y B$ is nuclear and hence $\alpha_Y(\beta_Y B)$ is nuclear since α_Y is bounded (Theorem I.11.8). Consequently B is compact (Theorem I.11.7). The adjoint of B is

$$B'y' = \sum_{i=1}^{\infty} \lambda_i y'(y_i) x_i' \quad \text{for each} \quad y' \in Y'.$$

Since $R(B) \subset \overline{\mathrm{span}\{y_1, y_2, \ldots\}} \subset F$, we have $R(B) \cap R(T) = \{0\}$ and hence $N(T + B) = N(T) \cap N(B)$. Notice that for any $x \in N(T + B)$,

$$y_n'(Bx) = \lambda_n x_n'(x) = 0 \quad \text{for each} \quad n \in \mathbb{N}$$

and so each x_n' annihilates $N(T + B)$. However as $\{x_n'\}$ is total over P it follows that $P \cap N(T + B) = \{0\}$. From the construction of B it follows that $Q \subset N(B)$ and so $Q \subset N(T) \cap N(B) = N(T + B)$. Hence

$$\dim Q \leq \alpha(T + B) \leq \mathrm{cod}_{N(T)} P. \tag{1}$$

Now if $y' \notin N(B')$ then there exists $k \in \mathbb{N}$ such that $y'(y_k) \neq 0$, whence

$$J_X^{X''} x_k(B'y') = y'(Bx_k) = \lambda_k y'(y_k) \neq 0.$$

Moreover since $x_k \in N(T)$, $J_X^{X''} x_k \in R(T')^{\perp}$ and so we deduce that $R(T') \cap R(B') = \{0\}$. Therefore $N(T' + B') = N(T') \cap N(B')$. Thus for each $y' \in N(T' + B')$ it now follows that

$$J_X^{X''} x_n(B'y') = y'(Bx_n) = \lambda_n y'(y_n) = 0 \quad \text{for each} \quad n \in \mathbb{N}.$$

Consequently each $y' \in N(T' + B')$ annihilates the fundamental sequence $\{y_n\}$ in F as well as $R(T)$ since by Proposition I.10.4

$N(T' + B') \subset N(T') = R(T)^{\perp}$. Hence $N(T' + B') \subset (\overline{R(T) \oplus F})^{\perp}$ and since $R(B) \subset F$, we also have

$$(\overline{R(T) \oplus F})^{\perp} = \overline{R(T)}^{\perp} \cap F^{\perp} \subset R(T)^{\perp} \cap R(B)^{\perp} = N(T') \cap N(B') = N(T' + B')$$

(Proposition I.10.4). It now follows that

$$N(T' + B') = (\overline{R(T) \oplus F})^{\perp} \equiv (Y/\overline{R(T) \oplus F})'$$

by Theorem I.5.10 and so since $(T + B)' = T' + B'$ by the continuity of B, we have by Corollary III.2.7 that

$$\overline{b}(T + B) = \alpha(T' + B') = \mathrm{cod}_Y(\overline{R(T) \oplus F}). \tag{2}$$

We show that $\gamma(T + B) = 0$. Suppose the contrary. Then letting B_1 be the restriction of B to $N(T) \in \mathcal{S}(X)$ we see that for any $\mu \in (0, \gamma(T + B))$

$$\|B_1 x\| = \|(T + B)x\| \geq \mu . d(x, N(T + B)) = \mu . d(x, N(T) \cap N(B))$$

$$\text{for each} \quad x \in N(T).$$

Therefore $\gamma(B_1) \geq \mu > 0$ since $N(B_1) = N(T) \cap N(B)$. By Theorem III.1.6 $\gamma(\bar{B}_1) \geq \gamma(B_1) > 0$ where $\bar{B}_1$ is the extension in $L(N(T)^{\sim}, \tilde{Y})$ of B_1 to all of $N(T)^{\sim}$. Thus $R(\bar{B}_1)$ is closed by Theorem III.2.6 and so $R(\bar{B}_1)$ is finite dimensional since $\bar{B}_1$ is still compact (Proposition I.11.6). (Note that since B_1 is precompact and $(\bar{B}_1)' = B_1'$, $\Gamma_0(\bar{B}_1) = \Gamma_0'(B_1') = \Gamma_0(B_1) = 0$ by Proposition IV.2.5). But this is impossible since for each $x_n \in N(T)$ we have that $B_1 x_n = \lambda_n y_n$ and hence $span\{y_1, y_2, \ldots\} \subset R(B_1)$. We conclude that $\gamma(T + B) = 0$. Now since $\alpha(T) = \bar{b}(T) = \infty$, $T \notin F_+ \cup F_-$. Suppose $T + B \in F_+ \cup F_-$. Then $T = T + B - B \in F_+ \cup F_-$ by Theorem VI.3.5 and the compactness of B. Consequently $T + B \notin F_+ \cup F_-$ and so $T + B \in L(X,Y) \setminus RO \cup F_+ \cup F_-$ since $\gamma(T + B) = 0$.

Finally for $j = 0,1,2,3,4$ we select P, Q and F as follows:

$\underline{j = 0:}$ Let X and Y be separable and $P = N(T)$, $Q = \{0\}$ and F a quasicomplement of $\overline{R(T)}$ in Y (Corollary I.6.3 and Theorem I.6.8). From (1) and (2) it follows that $\alpha(T + B) = 0 = \bar{b}(T + B)$ and hence $T + B \in V_0$.

$\underline{j = 1:}$ Let X and Y be separable with N a finite codimensional closed subspace of Y such that $\overline{R(T)} \subset N \neq Y$. Let $P = N(T)$, $Q = \{0\}$ and F a quasicomplement of $\overline{R(T)}$ in N (Corollary I.6.3 and Theorem I.6.8). It follows from (1) and (2) that $\alpha(T + B) = 0$ and $0 < \bar{b}(T + B) = cod\ N < \infty$. Thus $T + B \in V_1$.

$\underline{j = 2:}$ Let X be separable, $P = N(T)$ and $Q = \{0\}$ (Corollary I.6.3). From the lemma it follows that we can choose F so that $cod_Y(\overline{R(T) \oplus F}) = \infty$ whence $\alpha(T + B) = 0$ and $\bar{b}(T + B) = \infty$ by (1) and (2). Therefore $T + B \in V_2$.

$\underline{j = 3:}$ Let Y be separable and F a quasicomplement of $\overline{R(T)}$ in Y (Corollary I.6.3 and Theorem I.6.8). From the lemma we conclude that we can select P and Q so that both are infinite dimensional. But then $\alpha(T + B) = \infty$ and $\bar{b}(T + B) = 0$ by (1) and (2) and therefore $T + B \in V_3$.

<u>$j = 4$:</u> By the lemma we may select P, Q and F so that $\dim P = \dim Q = cod_Y(\overline{R(T)\oplus F}) = \infty$. Then $T + B \in V_4$ by (1) and (2). □

The proofs of the basic lemma VII.2.2 and the fundamental theorem VII.2.3 are essentially the same as those used by Gonzalez and Onieva in [18], suitably changed to hold for the more general case.

VII.2.4 **THEOREM**. *Let* $T \in L(X,Y)$.

I) *If* X *and* Y *are separable then* $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if* $J_Y T \in V_j(X,\tilde{Y}) + K[X,\tilde{Y}]$ *for* $j = 0,1$.

II) *If* X *is separable then* $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if* $J_Y T \in V_2(X,\tilde{Y}) + K[X,\tilde{Y}]$.

III) *If* Y *is separable then* $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if* $J_Y T \in V_3(X,\tilde{Y}) + K[X,\tilde{Y}]$.

IV) $T \in L(X,Y) \setminus F_+ \cup F_-$ *if and only if* $J_Y T \in V_4(X,\tilde{Y}) + K[X,\tilde{Y}]$.

Proof. Observe that if Y is separable, so is $\tilde{Y}$. Now suppose that $J_Y T \in V_j(X,\tilde{Y}) + K[X,\tilde{Y}]$, $j = 0,1,2,3,4$. Then $T \in L(X,Y) \setminus F_+ \cup F_-$ by II) of Theorem VII.1.4. Conversely suppose $T \in L(X,Y) \setminus F_+ \cup F_-$. We conclude from III) of Theorem VII.1.4 that $J_Y T \in V_4(X,\tilde{Y}) \cup V_5(X,\tilde{Y}) + K[X,\tilde{Y}]$. Noting that $\tilde{Y}$ is an operator range, the result now follows from Theorem VII.2.3 and the fact that $K[X,\tilde{Y}]$ is closed under addition by Theorem II.2.4. □

VII.2.5 *Remark*. Theorems VII.2.3 and VII.2.4 can be proved for the classes of bounded nuclear operators $N[X,Y]$ and $N[X,\tilde{Y}]$ instead of $K[X,Y]$ and $K[X,\tilde{Y}]$ respectively. This follows from the fact that the operator constructed in Theorem VII.2.3 is nuclear.

Theorem VII.2.3 generalises [26; Theorem 3.4] whereas Theorem VII.2.4 generalises [26; Corollary 3.5], [18; Theorem 3.4] and [3; Theorem 3.2].

APPENDIX

ON THE INSTABILITY OF NON-SEMI-FREDHOLM CLOSED OPERATORS UNDER COMPACT PERTURBATIONS WITH APPLICATIONS TO ORDINARY DIFFERENTIAL OPERATORS

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ABSTRACT

The stability of several natural subsets of the bounded non-semi-Fredholm operators under compact perturbations were studied by R Bouldin [3] in separable Hilbert spaces and by M Gonzales and V M Onieva [18] in Banach spaces. The aim of this paper is to study this problem for closed operators in operator ranges. The main results are a characterization of the non-semi-Fredholm operators with respect to α-closed and α-compact operators as well as a generalisation of a result by M Goldman [17]. We also give some applications of the theory developed to ordinary differential operators.

For T an arbitrary element of $L(X,Y)$, $\alpha(T)$ will denote the dimension of the null space $N(T)$ of T, $b(T)$ the codimension of $R(T)$ in Y, $\bar{b}(T)$ the codimension of $\overline{R(T)}$ in Y and $k(T)$ the index of T.

2. BASIC THEOREMS WITH APPLICATIONS TO ORDINARY LINEAR DIFFERENTIAL OPERATORS

2.1 **LEMMA**. *Let Y be an operator range. If $T \in NS(X,Y)\backslash SF$ then $\beta_Y T \in NS(X,Y_1)\backslash SF$.*

Proof. Assume $T \in NS(X,Y)\backslash SF$. From the definition of β_Y it now follows that β_Y is an open everywhere defined bijection and hence β_Y maps the open set $Y\backslash R(T)$ onto the open set $Y_1\backslash\beta_Y R(T)$. Therefore $\beta_Y R(T) = R(\beta_Y T)$ is closed. It also follows that $\alpha(T) = \alpha(\beta_Y T)$ and $b(T) = b(\beta_Y T)$. It remains to be verified that $\beta_Y T$ is closed. Suppose $\{x_n\} \subset D(\beta_Y T) = D(T)$ with $x_n \to x$ and $\beta_Y T x_n \to y_1 \in Y_1$. Now since the inverse of β_Y, α_Y, is bounded it follows that $\{x_n\} \subset D(T)$ with $x_n \to x$ and $Tx_n = \alpha_Y(\beta_Y T\, x_n) \to \alpha_Y y_1 \in Y$. However T is closed and hence $x \in D(T) = D(\beta_Y T)$ and $Tx = \alpha_Y y_1$, that is $\beta_Y Tx = \beta_Y(\alpha_Y y_1) = y_1$, implying that $\beta_Y T$ is closed. □

We require the following generalisation of Goldman's result, i.e. [17] cf. [16; Theorem V.2.6].

2.2 **LEMMA**. *Let $T \in NS(X,Y)\backslash SF$ with X and Y operator ranges. Then there exists an operator $B \in K[X,Y]$ such that for any $\lambda \neq 0$, $T + \lambda B \in C(X,Y)\backslash NS$. If $N(T)$ is separable then B can be chosen so that $T + \lambda B$ is injective.*

Proof. Assume $T \in NS(X,Y)\backslash SF$ with X and Y operator ranges. Suppose first that Y is complete. Since $\dim R(T)^{\perp} = \dim(Y/R(T))' = \dim(Y/R(T)) = \infty$ [16; Theorem I.6.4] we obtain an infinite linearly independent subset $\{y_1', y_2', \dots\} \subset R(T)^{\perp}$. Choose $y_1 \in Y$ such that $y_1'y_1 \neq 0$. For $k = 1, 2, \dots$ let y_{k+1} be an element of $\bigcap_{i=1}^{k} N(y_i') = {}^{\perp}\mathrm{span}\,\{y_1', y_2', \dots, y_k'\}$ but not $N(y_{k+1}')$. The existence of y_{k+1} follows from [16; Remark II.3.6] and the linear independence of $\{y_1', y_2', \dots\}$. Hence

$$\begin{aligned} & y_j'R(T) = 0 && \text{for } j \geq 1 \\ (1) \qquad & y_j'y_i = 0 && \text{for } i > j \geq 1 \\ & y_j'y_j \neq 0 . \end{aligned}$$

Let $\{x_1, x_2, \dots\}$ be an infinite linear independent set in $N(T)$. For $X_0 = \overline{\mathrm{span}}\,\{x_1, x_2, \dots\}$, the closure of the span, we obtain a countable total set $v_1', v_2', \dots \subset X_0'$ where $v_i \neq 0$ for $i \in \mathbb{N}$ [16; Lemma V.2.5]. Letting x_i' be an extension of v_i' to all of X, we obtain the following everywhere defined compact operator:

$$(2) \qquad Bx = \sum_{i=1}^{\infty} \frac{x_i'(x)y_i}{2^i \|x_i'\| \|y_i\|} .$$

Clearly Bx exists for each $x \in X$ by the completeness of Y and the absolute convergence of the series in (2).

Suppose $Bx \in R(T) \cap R(B)$. Then by (1) and (2) it follows that

$$0 = y_1'Bx = \frac{x_1'(x)y_1'y_1}{2\|x_1'\| \|y_1\|} \quad \text{whence} \quad x_1'x = 0$$

and that

$$0 = y_2'Bx = \frac{x_2'(x)y_2'y_2}{2^2\|x_2'\| \|y_2\|} \quad \text{whence} \quad x_2'x = 0$$

Continuing inductively we conclude that $x_i'x = 0$ for $i \in \mathbb{N}$ and hence $Bx = 0$, that is $R(T) \cap R(B) = \{0\}$.

Now B is injective on X_0 since $Bx = 0 \in R(T)$ implies $x_i'x = 0$ for $i \in \mathbb{N}$ and hence $x = 0$ since $\{x_1', x_2', \ldots\}$ is total on X_0.
For the sake of a contradiction suppose that $R(T + \lambda B) = R((T + \lambda B)\alpha_X)$ is closed for some $\lambda \neq 0$. Since $T + \lambda B$ and thus $(T + \lambda B)\alpha_X$ is closed, it follows that

$$\gamma((T + \lambda B)\alpha_X) = \inf_{\beta_X x \in D((T + \lambda B)\alpha_X)} \frac{\|(T + \lambda B)\alpha_X \beta_X x\|}{d(\beta_X x, N((T + \lambda B)\alpha_X))} > 0$$

[16; Theorem IV.1.6]. Hence

$$\|(T + \lambda B)\alpha_X \beta_X x\| \geq \gamma . d(\beta_X x, N((T + \lambda B)\alpha_X)) \tag{3}$$

where $\gamma = \gamma((T + \lambda B)\alpha_X) > 0$ and $x \in D(T + \lambda B)$. Now $N(T + \lambda B) = N(B) \cap N(T)$ since $(T + \lambda B)x = 0$ implies $Bx \in R(T) \cap R(B) = \{0\}$ and hence $Tx = 0$. Defining B_1 to be the restriction of B to $N(T)$ it follows that $N(B_1) = N(T) \cap N(B)$ and hence $N(B_1\alpha_X) = N((T + \lambda B)\alpha_X)$. Therefore for $\beta_X x \in N(T\alpha_X)\backslash N(B_1\alpha_X)$ we see from (3) that

$$\|B_1\alpha_X\beta_X x\| = \frac{1}{|\lambda|} \|((T + \lambda B_1)\alpha_X\beta_X x\| \geq \frac{\gamma}{|\lambda|} d(\beta_X x, N(B_1\alpha_X)) > 0$$

which implies that $\gamma(B_1\alpha_X) > 0$. Consequently since $B_1\alpha_X$ is a continuous operator with closed domain $N(T\alpha_X)$, it is closed and hence $R(B_1) = R(B_1\alpha_X)$ is closed [16; Theorem IV.1.6]. Since B_1 is compact and $R(B_1)$ is a closed subspace of the Banach space Y, it follows that $R(B_1)$ is finite dimensional. This is a contradiction however, since B and hence B_1 is injective on $X_0 \subset N(T)$, an infinite dimensional subspace. We conclude that $R(T + \lambda B)$ is not closed.

Suppose now that Y is a non-complete operator range. It now follows from Lemma 2.1 that $T \in NS(X,Y)\backslash SF$ implies $\beta_Y T \in NS(X,Y_1)\backslash SF$ and hence we conclude that there exists $C \in K[X,Y_1]$ such that for any $\lambda \neq 0$ $R(\beta_Y T + \lambda C)$ is not closed. Now letting $B = \alpha_Y C \in K[X,Y]$ it follows that $R(T + \lambda B)$ is not closed for any $\lambda \neq 0$.

If $N(T)$ is separable, then $\{x_1, x_2, \ldots\}$ may be chosen so that $X_0 = N(T)$, in which case $T + \lambda B$ is injective since $N(T + \lambda B) = N(T) \cap N(B) = N(B) \cap X_0 = \{0\}$ as B is injective on X_0 by construction. □

We shall define convergence in $C(X,Y)$ as follows. For $\{T_n\} \subset C(X,Y)$ $T_n \to T \in C(X,Y)$ if $D(T_n) \supset D(T)$ for each $n \in \mathbb{N}$ and $\|T_n - T\| \to 0$ in $B(X,Y)$. We then obtain the following result :

2.3 **THEOREM.** *Let* $T \in C(X,Y)\backslash SF$ *where* X *and* Y *are operator ranges. Then*

I) T *is the limit in* $C(X,Y)$ *of a sequence of operators* $\{T_n\}$ *in* $C(X,Y)\backslash NS$,

II) *there exists* $B \in K[X,Y]$ *and* $A \in C(X,Y)\backslash NS$ *such that* $T = A + B$.

Proof.

I) Suppose $T \in C(X,Y)\backslash SF$. If $T \in C(X,Y)\backslash NS$ let $T = T_n$ for every $n \in \mathbb{N}$. If $T \in NS(X,Y)\backslash SF$ there exists by Lemma 2.2 $B \in K[X,Y]$ such that $T + \lambda B \in C(X,Y)\backslash NS(X,Y)$ for every $\lambda \neq 0$. Hence choosing λ_n such that $0 \neq \lambda_n \to 0$, we obtain $\{T + \lambda_n B\} \subset C(X,Y)\backslash NS(X,Y)$ with $T + \lambda_n B \to T$.

II) Assume $T \in C(X,Y)\backslash SF$. Obviously if $T \notin NS(X,Y)$, $T = T + 0$. If $T \in NS(X,Y)\backslash SF$ then by Lemma 2.2 there exists $B \in K[X,Y]$ such that $T - B = A \in C(X,Y)\backslash NS$ and $T = T - B + B = A + B$. □

2.4 **COROLLARY.** *Let* X *be an operator range and* Y *complete. Then* $T \in C(X,Y)\backslash SF$ *if and only if there exists* $B \in K[X,Y]$ *and* $A \in C(X,Y)\backslash NS$ *such that* $T = A + B$.

Proof. Suppose there exists $B \in K[X,Y]$ and $A \in C(X,Y)\backslash NS$ such that $T = A + B$. T is easily seen to be closed. Suppose $T \in SF(X,Y)$. It now follows that $T\alpha_X \in SF(X_1,Y)$, $A\alpha_X \in C(X_1,Y)\backslash NS$ and that $-B\alpha_X \in K[X_1,Y]$ and hence $(-B\alpha_X)' \in K[Y',X_1']$ with X_1 and Y both complete. Now by [16; Corollary V.2.2] it follows that $T\alpha_X - B\alpha_X = A\alpha_X \in SF(X_1,Y) \subset NS(X_1,Y)$, a contradiction. We note that if $D(T\alpha_X)$ was not dense in X_1, we could replace X_1 and X by $\overline{D(T\alpha_X)}$ and $\alpha_X\overline{D(T\alpha_X)}$ respectively and consider the restrictions of T, A and B to $\alpha_X\overline{D(T\alpha_X)}$. The converse follows immediately from Theorem 2.3. $\square$

2.5 **DEFINITION.** Let X be an operator range. We define a linear operator T to be *α-compact* (*α-closed*) if $T\alpha_X \in K(X_1,Y)$ $(T\alpha_X \in C(X_1,Y))$. We will denote the class of all α-compact (α-closed) operators by $K_\alpha(X,Y)$ $(C_\alpha(X,Y))$; $K_\alpha[X,Y]$ $(C_\alpha[X,Y])$ if $D(T) = X$.

We give an example of an α-compact operator before characterizing non-semi-Fredholm operators with respect to α-compact and α-closed operators.

2.6 *Example*. We construct a non-trivial α-compact operator. Let $1 < p < \infty$, $I = [a,b]$ be compact and $\tau = \sum\limits_{k=0}^{n} a_k D^k$ with $D = \frac{d}{dt}$, $a_k \in C^k(I)$ for $0 \leq k \leq n$ and $\frac{1}{a_n} \in \mathcal{L}_\infty(I)$ (with $a_n(t) \neq 0$ for $t \in I$). We note that by [2; Exercise 6K] $f \in \mathcal{L}_1(I)$ if $f \in \mathcal{L}_q(I)$ and hence considering for example the maximal operators $T_{\tau,p,q}$ and $T_{\tau,p,1}$ as defined in [16], it follows that $D(T_{\tau,p,q}) \subset D(T_{\tau,p,1})$. Now let q be a real number such that $q > 1$. By [16, Theorem VI.2.10] the minimal operator $T_{0,\tau,p,q}$ as defined in [16] is injective and hence since $T_{0,\tau,p,q}$ is closed by definition, it follows from [16, Corollary VI.3.3] that $T_{0,\tau,p,q}$ has a compact inverse. Now let T be $T_{0,\tau,p,q}$ considered as a map into $\mathcal{L}_1(I)$.

Note that T exists since, as was noted before, $f \in \mathcal{L}_1(I)$ if $f \in \mathcal{L}_q(I)$. It is easily seen that if $R(T_{0,\tau,p,q})$ is closed and hence complete in $\mathcal{L}_q(I)$ then $R(T)$ will be an operator range with $R(T_{0,\tau,p,q})$ as a pre-image space and since $(T_{0,\tau,p,q})^{-1}$ is compact, it follows that T^{-1} will be α-compact. It remains to prove that $R(T_{0,\tau,p,q})$ is closed. By [16; Theorem VI.2.7] it will suffice to show that $T_{\tau,p,q}$ is surjective. This is the case as can be seen from [16; Theorem VI.3.1].

2.7 **PROPOSITION.** *Let* $I = [a,b]$ *be compact and let* τ *be the differential expression* $\tau = \sum_{k=0}^{n} a_k D^k$, *where* $a_k \in \mathcal{L}_1(I)$ *for* $0 \leq k \leq n-1$ *and* $\frac{1}{a_n} \in \mathcal{L}_\infty(I)$. *Suppose* $T \in L(\mathcal{L}_p(I), \mathcal{L}_1(I))$ *is an injective closed operator which is a restriction of the maximal operator corresponding to* $(\tau,p,1)$, $1 \leq p \leq \infty$. *If* $R(T)$ *can be renormed with* $\|.\|_q$ $(q > 1)$ *such that* $R(T)$ *is closed in* $\mathcal{L}_q(I)$, *then* T^{-1} *is* α-*compact.*

Proof. Suppose $T \in L(\mathcal{L}_p(I), \mathcal{L}_1(I))$ is a closed injective operator satisfying the given hypothesis. It is easily seen that $R = R(T)$ is an operator range with $R_1 = (R(T), \|.\|_q)$ (where $q > 1$) as a pre-image space. It follows that $\beta_R T$ is closed and injective and since $f \in R(T)$ implies $f \in \mathcal{L}_q(I)$ by the hypothesis, we conclude that $\beta_R T$ is a restriction of the maximal operator $T_{\tau,p,q}$. Hence by [16; Corollary VI.3.3] $(\beta_R T)^{-1} = T^{-1}\beta_R^{-1} = T^{-1}\alpha_R$ is compact. That is, T^{-1} is α-compact. □

2.8 **THEOREM.** *Let* X *be an operator range and* Y *complete. Then* $T \in C(X,Y)\backslash SF$ *if and only if there exist operators* $B \in K_\alpha[X,Y]$ *and* $A \in C_\alpha(X,Y)$ *such that* $\alpha(A) = \bar{b}(A) = \infty$ *and* $T = A + B \in C(X,Y)$.

Proof. Throughout this proof we shall write α for α_X and β for β_X. Assume there exist $B \in K_\alpha[X,Y]$ and $A \in C_\alpha(X,Y)$ such that $T = A + B \in C(X,Y)$ and $\alpha(T-B) = \overline{b}(T-B) = \infty$. Suppose $T \in SF(X,Y)$. Then $T\alpha \in SF(X_1,Y)$ and hence considering $T\alpha$ and $-B\alpha$ as operators from $\overline{D(T\alpha)} \subset X_1$ into Y, it follows from [16; Corollary V.2.2] that $T\alpha - B\alpha \in SF(X_1,Y)$. Consequently either $\alpha(T-B) = \alpha(T\alpha-B\alpha) < \infty$ or $\overline{b}(T-B) = b(T-B) = b(T\alpha-B\alpha) < \infty$, which is a contradiction.

Conversely assume $T \in C(X,Y)\backslash SF$. *If* $T \in NS(X,Y)\backslash SF$ let $B = 0$. If $T \notin NS(X,Y)$ we will construct two α-compact operators B_1 and B_2 such that $\alpha(T - B_1 - B_2) = \overline{b}(T - B_1 - B_2) = \infty$. Hence $T = (T - B) + B$ where $B = B_1 + B_2$.

As $T \notin NS(X,Y)$, $R(T)$ is not closed. If $\overline{b}(T) = \infty$ then take $B_1 = 0$. Now suppose $\overline{b}(T) < \infty$; we shall define B_1 such that $\overline{b}(T - B_1) = \infty$. Inductively define a sequence of integers a_n as follows:

(1) $$a_1 = 2, \quad a_n = 2(1 + \sum_{k=1}^{n-1} a_k) \quad \text{for} \quad n = 2,3,\ldots$$

Assuming for the moment that there exist sequences $\{y_k\} \subset Y$ *and* $\{y_k'\} \subset D((T\alpha)') \subset Y'$ such that

(2) $$\|y_k\| \le a_k \quad \|y_k'\| = 1 \quad \|(T\alpha)'y_k'\| = \frac{1}{2^k a_k}$$
$$y_j(y_k) = \delta_{jk} \quad \text{for} \quad j,\ k = 1,\ 2,\ 3,\ldots$$

we define

$$V_n x = \sum_{k=1}^{n} (T\alpha)'y_k'\ (\beta x)\ y_k \quad \text{for} \quad n = 1,2,\ldots, \text{ and for } x \in X.$$

Now for $n > m$ and $x \in X$ we have

$$\begin{aligned} \|V_n\alpha\,\beta x - V_m\alpha\,\beta x\| = \|V_n x - V_m x\| &\le \sum_{k=m+1}^{n} \|(T\alpha)'y_k'\|\,\|\beta x\|\,\|y_k\| \\ &\le \Big(\sum_{k=m+1}^{n} 2^{-k}\Big)\,\|\beta x\| \\ &\le \|\beta x\|/2^m \to 0 \quad \text{as} \quad m \to \infty\,. \end{aligned}$$

Considering the above we observe that $\{V_n\alpha\}$ is a Cauchy sequence in the Banach space $B[X_1, Y]$ converging to the compact operator $B_1\alpha \in B[X_1, Y]$ where $B_1 x = \sum_{k=1}^{\infty} (T\alpha)'y_k'\,(\beta x)\,y_k$ for $x \in X$. Now for each $x \in D(T)$ and each k we have by (2) that

$$y_k'(B_1 x) = (T\alpha)'y_k'(\beta x) = y_k'(Tx)$$

and hence $y_k'(\overline{R(T-B_1)}) = 0$ for each k. Since $y_1, y_2, \ldots,$ are linearly independent and by (2) obviously not in $\overline{R(T-B_1)}$, it follows that $\overline{b}(T-B_1) = \infty$. Notice that if $D(T\alpha)$ was not dense in X_1, we could redefine $T\alpha$ and T to be from $\overline{D(T\alpha)}$ and $\alpha(\overline{D(T\alpha)})$ respectively, thereby ensuring the existence of $(T\alpha)'$. Considering the Hahn-Banach theorem we see that in this case we could still regard B_1 to be an element of $K_\alpha[X, Y]$. It remains to find sequences satisfying (2).

Since $R(T) = R(T\alpha)$ is not closed, $R((T\alpha)')$ is not closed [16; Theorem IV.1.2]. We verify that there exists $y_1' \in D((T\alpha)')$ such that $\|y_1'\| = 1$ and $\|(T\alpha)'y_1'\| < 1/4$. Suppose on the contrary that $\|(T\alpha)'y'\| \ge 1/4$ for all $y' \in D((T\alpha)')$ such that $\|y'\| = 1$. Then $(T\alpha)'$ has a continuous inverse and hence $R((T\alpha)')$ is closed by [16; Lemma IV.1.1], a contradiction. Also, it can be easily verified that there exists $y_1 \in Y$ with $\|y_1\| < 2$ and $y_1'(y_1) = 1$.

Suppose $y_1, y_2, \ldots, y_{n-1}, y'_1, y'_2, \ldots, y'_{n-1}$ have been found satisfying (2). Let $M = \text{span}\,\{y_1, y_2, \ldots, y_{n-1}\}$. Thus M is finite dimensional with $Y = M \oplus N$ where N is a closed subspace of Y. Hence $Y' = M^\perp \oplus N^\perp$ by [16; Lemma IV.1.11]. Now $(T\alpha)'M^\perp$ is not closed since if it was closed, $R((T\alpha)') = (T\alpha)'M^\perp + (T\alpha)'N^\perp$ would be closed by the finite dimensionality of $N^\perp$. Hence $\dim D((T\alpha)') \cap M^\perp = \infty$. We now claim that there exists $y'_n \in D((T\alpha)) \cap M^\perp$ such that $\|y'_n\| = 1$ and $\|(T\alpha)'y'_n\| < \dfrac{1}{2^n a_n}$. Notice that $y'_n(y_k) = 0$ for $k = 1, 2, \ldots, n-1$ since $y'_n \in M^\perp$. Suppose y'_n doesn't exist. Then $\|(T\alpha)'y'\| \geq \dfrac{1}{2^n a_n}$ for each $y' \in D((T\alpha)') \cap M^\perp$ such that $\|y'\| = 1$. Hence $(T\alpha)'$ restricted to $M^\perp$ has a continuous inverse and therefore $(T\alpha)'M^\perp$ is closed by [16; Lemma IV.1.1], a contradiction. Thus y'_n exists. As before we can find $y \in Y$ such that $y'_n(y) = 1$ and $\|y\| < 2$. Let $y_n = y - \sum\limits_{k=1}^{n-1} y'_k(y)\, y_k$ .Then

$$\|y_n\| \leq \|y\| \,(1 + \sum_{k=1}^{n-1} \|y_k\|) \leq 2\,(1 + \sum_{k=1}^{n-1} a_k) = a_n \quad \text{by} \quad (1).$$

It can now easily be verified that $y'_j(y_k) = \delta_{jk}$ for $j,k = 1, 2, \ldots, n$ and hence the required sequences exist by induction.

Suppose now that $\alpha(T-B_1) < \infty$ since otherwise we take $B_2 = 0$ and we are done. Assuming that there exist sequences, $\{x_k\} \subset D(T) = D(T-B_1)$ and $\{x'_{1k}\} \subset X'_1$, such that

$$(3) \qquad \|\beta x_k\| = 1 \quad \|x'_{1k}\| \leq 2^{k-1} \quad \|(T-B_1)x_k\| \leq 2^{1-2k}$$

$$x'_{1k}(\beta x_j) = \delta_{kj} \quad \text{for} \quad k,j = 1, 2, \ldots,$$

we define

$$U_n x = \sum_{k=1}^{n} x'_{1k}(\beta x)\,(T-B_1)x_k \quad \text{for} \quad n = 1, 2, \ldots, \quad \text{and} \quad x \in X.$$

It is easily seen that $\{U_n\alpha\}$ is a Cauchy sequence of finite rank operators converging in the Banach space $B[X_1,Y]$ to the compact operator $B_2\alpha$ defined as follows:

$$B_2x = \sum_{k=1}^{\infty} x'_{1k}(\beta x)\ (T-B_1)x_k \quad \text{for} \quad x \in X.$$

Notice that the completeness of Y and the absolute convergence of $B_2\alpha(\beta x) = B_2x$ for each $x \in X$ ensures the existence of B_2. Hence B_2 is an α-compact operator which coincides with $T - B_1$ on the space $\operatorname{span}\{x_1, x_2, \ldots\}$ and therefore $\alpha(T - B_1 - B_2) = \infty$ since $\{x_1, x_2, \ldots\}$ is a linearly independent set. Moreover $R(B_2) \subset \overline{R(T - B_1)}$ and since $\overline{b}(T - B_1) = \infty$ we have $\overline{b}(T - B_1 - B_2) = \infty$. It remains to find sequences satisfying (3). We claim that $R(T\alpha - B_1\alpha)$ is not closed. Suppose $T\alpha - B_1\alpha \in NS(X_1,Y)$. Considering $T\alpha - B_1\alpha$ to be from $\overline{D(T\alpha)} = \overline{D(T\alpha - B_1\alpha)}$ and taking the restriction of $B_1\alpha$ to $\overline{D(T\alpha)}$, it follows from [16; Corollary V.2.2] and the fact that $\alpha(T-B_1) = \alpha(T\alpha - B_1\alpha) < \infty$ that $T\alpha \in SF(X_1,Y)$; that is $R(T) = R(T\alpha)$ is closed. But $T \in C(X,Y)\backslash NS$ by assumption and hence $R(T\alpha - B_1\alpha)$ is not closed. Consequently there exists $x_1 \in D(T) = D(T - B_1)$ such that $\|\beta x_1\| = 1$ and $\|(T - B_1)x_1\| < \frac{1}{2}$. Assuming the contrary we have that $\|(T - B_1)x\| = \|(T\alpha - B_1\alpha)\beta x\| \geq \frac{1}{2}$ for every $\beta x \in D(T\alpha - B_1\alpha)$ such that $\|\beta x\| = 1$ and hence $T\alpha - B_1\alpha$ has a continuous inverse whence $R(T\alpha - B_1\alpha)$ is closed by [16; Lemma IV.1.1], a contradiction. Clearly there exists $x'_{11} \in X'_1$ such that $\|x'_{11}\| = 1$ and $x'_{11}(\beta x_1) = 1 = \|\beta x_1\|$. Suppose we have constructed $x_1, x_2, \ldots, x_{n-1}$ and $x'_{11}, x'_{12}, \ldots, x'_{1n-1}$ in X and X'_1, respectively, such that (3) is satisfied. Now let $M = \operatorname{span}\{x'_{11}, x'_{12}, \ldots, x'_{1n-1}\}$. It is obvious that M is finite dimensional and from [16; Theorem I.6.4 and Remark II.3.6] it follows that

$$\dim (X_1/{}^{\perp}M) = \dim(X_1/{}^{\perp}M)' = \dim({}^{\perp}M)^{\perp} = \dim M < \infty .$$

Hence ${}^{\perp}M$ is a closed subspace of X_1 with finite codimension. Consequently since $R(T\alpha - B_1\alpha)$ is not closed, the range of $T\alpha - B_1\alpha$ restricted to ${}^{\perp}M$ will also not be closed. Hence as before there exists $\beta x_n \in D(T\alpha - B_1\alpha) \cap {}^{\perp}M$ such that $\|\beta x_n\| = 1$ and $\|(T - B_1)x_n\| = \|(T\alpha - B_1\alpha)\beta x_n\| < 2^{1-2n}$ since if this was not the case, $T\alpha - B_1\alpha$ restricted to ${}^{\perp}M$ would have a continuous inverse and $(T\alpha - B_1\alpha)({}^{\perp}M)$ would therefore be closed by [16; Lemma IV.1.1]. Now let x'_1 be an arbitrary element of X'_1 such that $x'_1(\beta x_n) = 1$ and $\|x'_1\| = 1$. Then the functional

$$x'_{1n} = x'_1 - \sum_{k=1}^{n-1} x'_1(\beta x_k)x'_{1k}$$

has the properties that

$$x'_{1n}(\beta x_k) = 0 \qquad \text{for } k = 1, 2, \ldots, n-1$$

and $\|x'_{1n}\| \leq 2^{n-1}$. Now since $\beta x_n \in {}^{\perp}M$ it follows that

$$x'_{1j}(\beta x_k) = \delta_{jk} \quad \text{for } j,k = 1, 2, \ldots, n$$

and hence we can construct the necessary sequences by induction. □

2.9 *Remark.* We observe that Theorems 2.3 and 2.8 and Corollary 2.4 contain [18; Theorem 2.1]. Suppose that X is an operator range and Y is complete. Then if $T \in B[X,Y]\backslash SF$ it follows from Theorem 2.3 that there exists $\{T_n\} \subset B[X,Y]\backslash NS$ such that $T_n \to T$ in $B[X,Y]$ and since $SF[X,Y] \cap B[X,Y]$ is an open set by [27; Theorem V.1.7] it follows that $\overline{B[X,Y]\backslash NS} = B[X,Y]\backslash SF$. It now easily follows from Corollary 2.4 that $T \in B[X,Y]\backslash SF$ if and only if there exists $B \in K[X,Y]$ and $A \in B[X,Y]\backslash NS$ such that $T = A + B$. Now if X is also complete, $K_\alpha[X,Y] = K[X,Y]$ and $C_\alpha(X,Y) = C(X,Y)$ and hence it follows from Theorem 2.8 that $T \in B[X,Y]\backslash SF$ if and only if there exists $B \in K[X,Y]$ and $A \in B[X,Y]$ such that $a(A) = \bar{b}(A) = \infty$ and $T = A + B$.

2.10 **COROLLARY.** *Let X and Y be complete. Then $T \in SF(X,Y)$ if and only if for any $B \in K[X,Y]$ either $a(T + B) < \infty$ or $\bar{b}(T + B) < \infty$.*

Proof. The corollary follows immediately from Theorem 2.8 and the fact that $K[X,Y] = K_\alpha[X,Y]$ when X is complete. □

In seeking applications of Theorem 2.8 to ordinary differential operators we will make use of the form that appears in Corollary 2.10 and use it as a test for the closed range property. On defining differential operators as in [16] and letting $\tau = \sum_{k=0}^{n} a_k D^k$ where $a_k \in C^k(I)$, $0 \le k \le n$, $a_n(t) \neq 0$ for $t \in I$, and either $1 \le p,q < \infty$ or $1 < p,q \le \infty$ we obtain the following results:

2.11 **THEOREM.** *Suppose I contains one of its endpoints with T any one of the minimal or maximal operators corresponding to (τ,p,q) (or (τ^*,q',p')). Then $a(T + B) < \infty$ or $\bar{b}(T + B) < \infty$ for any $B \in K[\mathcal{L}_p(I),\mathcal{L}_q(I)]$ (or for any $B \in K[\mathcal{L}_{q'}(I),\mathcal{L}_{p'}(I)]$) if and only if all four operators are Fredholm operators and*

$$\dim \frac{D(T_{\tau,p,q})}{D(T_{0,\tau,p,q})} = k(T_{\tau,p,q}) - k(T_{0,\tau,p,q}) .$$

Proof. This follows immediately from Corollary 2.10 and [16; Theorem VI.2.7].

2.12 **THEOREM.** *Suppose I contains one of its endpoints with T any one of the minimal or maximal operators corresponding to (τ,p,q) (or (τ^*,q',p')). If either $a(T + B) < \infty$ or $\bar{b}(T + B) < \infty$ for any $B \in K[\mathcal{L}_p(I), \mathcal{L}_q(I)]$ (or for any $B \in K[\mathcal{L}_{q'}(I), \mathcal{L}_{p'}(I)]$) then $T_{\tau,p,q}$ is surjective and $T_{0,\tau,p,q}$ has a continuous inverse.*

Proof. This follows immediately from Corollary 2.10 and [16; Theorem VI.2.11].

2.13 THEOREM. *Let* T *be a differential operator corresponding to* (τ,p,q). *If* $\{U_\lambda\}$ *is a set of boundary value functionals which determine* T *and if either* $a(T_{0,\tau,p,q} + B) < \infty$ *or* $\bar{b}(T_{0,\tau,p,q} + B) < \infty$ *for any* $B \in K[\mathcal{L}_p(I),\mathcal{L}_q(I)]$, *then*

$$\dim \operatorname{span}\{U_\lambda\} = k(T_{\tau,p,q}) - k(T) = \dim \frac{D(T_{\tau,p,q})}{D(T)} < \infty .$$

Thus T *is the restriction of the maximal operator to those* $y \in D(T_{\tau,p,q})$ *which satisfy the conditions*

$$0 = U_1 y = U_2 y = \dots = U_m y$$

where $m = k(T_{\tau,p,q}) - k(T)$ *and* $\{U_1, U_2, \dots, U_m\}$ *is any linear independent subset of* $\{U_\lambda\}$.

Proof. This follows immediately from Corollary 2.10 and [16; Theorem VI.5.6].

2.14 THEOREM. *If* $\{U_\lambda\}$ *is a set of boundary value functionals which determines* $T_{0,\tau,p,q}$ *and* $\{V_\mu\}$ *is a set of boundary value functionals which determines* $T_{0,\tau^*,p',q'}$ *and if either* $a(T_{0,\tau,p,q}+B) < \infty$ *or* $\bar{b}(T_{0,\tau,p,q}+B) < \infty$ *for any* $B \in K[\mathcal{L}_p(I), \mathcal{L}_q(I)]$, *then*

$$\dim \operatorname{span}\{U_\lambda\} = \dim \operatorname{span}\{V_\mu\} = k(T_{\tau,p,q}) - k(T_{0,\tau,p,q}) < \infty.$$

Proof. This follows from Corollary 2.10 and [16; Corollary VI.5.7]. □

3. RESULTS ON INSTABILITY

We note that for any linear operator T, $\alpha(T') = \bar{b}(T)$ if T' exists [16; Theorem IV.2.3]. For any $T \in L(X,Y)$ ($D(T)$ not necessarily dense in X) we will consider T' to be the map $(T_{\overline{D(T)}})'$ where $T_{\overline{D(T)}}$ is T considered as an element of $L(\overline{D(T)},Y)$.

3.1 **DEFINITION.** *Let* $U_0, U_1, \ldots, U_5$ *be defined by*

$U_0 = \{T \in C(X,Y)\backslash NS \quad : \alpha(T) = \alpha(T') < \infty\}$

$U_1 = \{T \in C(X,Y)\backslash NS \quad : 0 \neq |\alpha(T) - \alpha(T')| < \infty\}$

$U_2 = \{T \in C(X,Y)\backslash NS \quad : \alpha(T) - \alpha(T') = -\infty\}$

$U_3 = \{T \in C(X,Y)\backslash NS \quad : \alpha(T) - \alpha(T') = \infty\}$

$U_4 = \{T \in C(X,Y)\backslash NS \quad : \alpha(T) = \infty = \alpha(T')\}$

$U_5 = \{T \in NS(X,Y) \quad : \alpha(T) = \infty = \alpha(T')\}.$

3.2 **LEMMA.** *Let* X *be a separable normed space. Then* X *admits of a fundamental and total biorthogonal system.*

The above lemma is proved in more generality in [20; Corollary 14.1.5]. Using Lemma 3.2 we obtain the following generalisation of [18; Lemma 3.3].

3.3 **LEMMA.** *Let* Z *be an infinite dimensional normed linear space with* M *a closed subspace of* Z .

I) *If* M *is infinite codimensional then there exists an infinite dimensional and separable closed subspace* F *of* Z *such that* $F \cap M = \{0\}$ *and* $\overline{F \oplus M}$ *is infinite codimensional.*

II) *If M is infinite dimensional then there exists two separable closed subspaces, P and Q, of M and a fundamental and total biorthogonal system $(\{x_n\}, \{p_n'\})$ in P such that P is infinite dimensional, Q may be chosen either finite or infinite dimensional, $P \cap Q = \{0\}$ and each of the p_n''s has an extension to all of Z which annihilates Q.*

Proof. The proof is identical to that used by Gonzalez and Onieva [18] with use being made of Lemma 3.2 instead of [29; Theorem 1.f.4] to ensure that the lemma holds for normed linear spaces and not just Banach spaces.

3.4 **THEOREM.** *Let X and Y be operator ranges and let*

$$D = \{T \in C(X,Y) : a(T) = \infty = a(T')\}.$$

I) *If X and Y are separable then $D \subset U_j + K[X,Y]$ for $j = 0, 1$.*

II) *If X is separable then $D \subset U_2 + K[X,Y]$.*

III) *If Y is separable then $D \subset U_3 + K[X,Y]$.*

IV) $D \subset U_4 + K[X,Y]$.

Proof. Assume $T \in D$. Without loss of generality let $D(T)$ be dense in X. Let P and Q be closed subspaces of $N(T)$ such that $P \cap Q = \{0\}$, with P infinite dimensional and separable. Let $(\{x_n\}, \{p_n'\})$ be a fundamental and total biorthogonal system in P and let x_n' be an extension of p_n' to all of X which annihilates Q. The existence of P and Q is established by Lemma 3.3.

Now let F be an infinite dimensional and separable closed subspace of Y such that $\overline{R(T)} \cap F = \{0\}$. As F is separable we see by Lemma 3.2 that there exists a fundamental and total biorthogonal system $(\{y_n\}, \{f_n'\})$ in F. Let y_n' be an extension of f_n' to all of Y.

Now define the operator $Bx = \sum_{i=1}^{\infty} \lambda_i x_i'(x)\, y_i$ for $x \in X$, where $\lambda_i = (\|x_i'\| \|\beta_Y y_i\|_1 \, 2^i)^{-1} > 0$ for all i. Notice that by choice of λ_i and by the completeness of Y_1, we have that $\beta_Y Bx = \sum_{i=1}^{\infty} \lambda_i x_i'(x)\, \beta_Y y_i$ exists for each $x \in X$ and that $\beta_Y B \in K[X, Y_1]$. Hence $\alpha_Y(\beta_Y B) = B \in K[X,Y]$ and $\beta_Y B \alpha_X \in K[X_1, Y_1]$. The conjugate of B is

$$B'y' = \sum_{i=1}^{\infty} \lambda_i y'(y_i) x_i' \quad \text{for } \; y' \in Y' .$$

Since $R(B) \subset \overline{\mathrm{span}}\, \{y_n\} \subset F$, we have that $R(B) \cap R(T) = \{0\}$ and hence $N(T+B) = N(T) \cap N(B)$. For any $x \in N(T+B)$ it now follows that $y_n'(Bx) = \lambda_n x_n'(x) = 0$ for any $n \in \mathbb{N}$ whence it follows that each x_n' annihilates $N(T+B)$. However as $\{p_n'\}$ and thus $\{x_n'\}$ is total over P it follows that $P \cap N(T+B) = \{0\}$. From the construction of B it now follows that $Q \subset N(B)$ and hence $Q \subset N(T) \cap N(B) = N(T+B)$. Therefore

$$\dim Q \;\le\; \alpha(T+B) \;\le\; \mathrm{cod}_{N(T)} P . \tag{1}$$

Alternatively if $y' \notin N(B')$ then there is an integer k so that $y'(y_k) \neq 0$ whence $J_X^{X''} x_k \,(B'y') = y'(Bx_k) = \lambda_k y'(y_k) \neq 0$ where $J_X^{X''}$ is the natural map of X into X''. However since $x_k \in N(T)$ it follows that $J_X^{X''} x_k \in R(T')^{\perp}$ from which we deduce that $R(T') \cap R(B') = \{0\}$ and hence that $N(T' + B') = N(T') \cap N(B')$. Moreover for each $y' \in N(T' + B')$ it now follows that $J_X^{X''} x_n (B'y') = y'(Bx_n) = \lambda_n y'(y_n) = 0$ for each $n \in \mathbb{N}$. Thus each $y' \in N(T' + B')$ annihilates the fundamental sequence $\{y_n\}$ in F as well as $R(T)$ since $y' \in N(T' + B') \subset N(T') = R(T)^{\perp}$ by [16; Theorem II.3.7]. It therefore follows that $N(T' + B') \subset (R(T) \oplus F)^{\perp}$ and since $R(B) \subset F$, $(R(T) \oplus F)^{\perp} \subset R(T)^{\perp} \cap R(B)^{\perp} = N(T') \cap N(B') = N(T' + B')$, and hence $N(T' + B') = (R(T) \oplus F)^{\perp} \equiv (Y/\overline{R(T) \oplus F})'$. Consequently

$$\alpha(T' + B') = \mathrm{cod}_Y \, \overline{R(T) \oplus F} . \tag{2}$$

We now show that $R(T+B)$ is not closed. Suppose the contrary. Since $T + B \in NS(X,Y)$ it can easily be shown that $\beta_Y T\alpha_X + \beta_Y B\alpha_X \in NS(X_1,Y_1)$. Then letting $T_1 = \beta_Y T\alpha_X$ and $B_1 = \beta_Y B\alpha_X$ it follows that $N(T_1 + B_1) = N(T_1) \cap N(B_1)$ since $N(T+B) = N(T) \cap N(B)$. Moreover

$$\|(T_1 + B_1)x\| \geq \gamma . d(x, N(T_1 + B_1)) = \gamma . d(x, N(T_1) \cap N(B_1))$$

for $x \in D(T_1) = D(T_1 + B_1) \subset X_1$ where $\gamma = \gamma(T_1 + B_1) > 0$ [16; Theorem IV.1.6]. Now let $B_{N(T_1)}$ be the restriction of B_1 to $N(T_1)$. For every $x \in N(T_1)$ we obtain $\|B_{N(T_1)}x\| = \|(T_1 + B_1)x\| \geq \gamma . d(x, N(B_{N(T_1)}))$ and hence $\gamma(B_{N(T_1)}) \geq \gamma > 0$. Thus $R(B_{N(T_1)})$ is closed by [16; Theorem IV.1.6] and consequently since B_1 and therefore $B_{N(T_1)}$ is compact, $R(B_{N(T_1)})$ must be finite dimensional. However, for each $x_n \in N(T)$ we have $B_{N(T_1)}(\beta_X x_n) = (\beta_Y B\alpha_X)(\beta_X x_n) = \beta_Y(Bx_n) = \lambda_n(\beta_Y y_n)$. Hence $R(B_{N(T_1)})$ contains the infinite linearly independent set $\{\beta_Y y_n\}$. We conclude that $R(T+B)$ is not closed.

Finally for U_j, $j = 0, 1, 2, 3, 4$ we choose P, Q and F as follows:

$\underline{j = 0}$. Let X and Y be separable, and let $P = N(T)$, $Q = \{0\}$ and F a quasicomplement of $\overline{R(T)}$ [30]. It follows from (1) and (2) that $\alpha(T+B) = 0 = \alpha(T' + B')$ and hence $T + B \in U_0$.

$\underline{j = 1}$. Let X and Y be separable and let N be a finite codimensional closed subspace of Y so that $\overline{R(T)} \subset N \neq Y$. Letting $P = N(T)$, $Q = \{0\}$ and F a quasicomplement of $\overline{R(T)}$ in N, it follows from (1) and (2) that $\alpha(T+B) = 0$ and $0 < \alpha(T' + B') = cod\ N < \infty$, and hence $T + B \in U_1$.

$\underline{j = 2}$. Let X be separable, $P = N(T)$ and $Q = \{0\}$. From Lemma 3.3 it now follows that we can choose F so that $\overline{R(T) \oplus F}$ is infinite codimensional and hence from (1) and (2) we have that $\alpha(T+B) = 0$ and $\alpha(T' + B') = \infty$. Thus $T + B \in U_2$.

$\underline{j = 3}$. Let Y be separable and let F be a quasicomplement of $\overline{R(T)}$ [30]. From Lemma 3.3 we see that P and Q may be chosen so that they are of infinite dimension and hence $\alpha(T + B) = \infty$ and $\alpha(T' + B') = 0$. Therefore $T + B \in U_3$.

$\underline{j = 4}$. Applying Lemma 3.3 we choose P and Q of infinite dimension and F so that $\overline{R(T) \oplus F}$ has infinite codimension. Then $\alpha(T + B) = \alpha(T' + B') = \infty$ and hence $T + B \in U_4$. □

3.5 **COROLLARY.** *Let* X *and* Y *be complete.*

I) *If* X *and* Y *are separable then*

$$U_j + K[X,Y] = C(X,Y) \backslash SF \quad \text{for } j = 0,1 .$$

II) *If* X *is separable then* $U_2 + K[X,Y] = C(X,Y) \backslash SF$.

III) *If* Y *is separable then* $U_3 + K[X,Y] = C(X,Y) \backslash SF$.

IV) $U_4 + K[X,Y] = C(X,Y) \backslash SF$.

Proof. From Corollary 2.4 it follows that

$$U_j + K[X,Y] \subset C(X,Y) \backslash SF \quad \text{for } j = 0,\ 1,\ 2,\ 3,\ 4 .$$

Now since X and Y are complete, it follows from Theorem 2.8 that $C(X,Y) \backslash SF \subset D + K[X,Y]$ where $D = \{T \in C(X,Y) : \alpha(T) = \infty = \alpha(T')\}$. The corollary now follows on applying Theorem 3.4. □

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BIBLIOGRAPHY

1. S. BANACH, "Théorie des Opérations Linéaires", Monografje Matematyczne, Warsaw, 1932.

2. R.G. BARTLE, "The Elements of Integration", John Wiley and Sons, New York, 1966.

3. R. BOULDIN, The Instability of non-semi-Fredholm Operators under Compact Perturbations, *J. Math. Anal. Appl.* **87**(1982), 632 - 638.

4. J.W. BRACE and R. ROYCE KNEECE, Approximation of Strictly Singular and Strictly Cosingular Operators using Nonstandard Analysis, *Trans. Amer. Math. Soc.* **168**(1972), 483 - 496.

5. R.W. CROSS, On the Continuous Linear Image of a Banach Space, *J. Austral. Math. Soc.* (*Series A*) **29**(1980), 219 - 234.

6. R.W. CROSS, Properties of some Norm Related Functions of Unbounded Linear Operators, *Math.* Z. (to appear).

7. R.W. CROSS, Some Continuity Properties of Linear Transformations in Normed Spaces, *Glasgow Math. J.* **30**(1988), 243 - 247.

8. R.W. CROSS, The Essential State Diagram of a Linear Operator (unpublished).

9. R.W. CROSS, Unbounded Linear Transformations of the Upper Semi-Fredholm Type in Normed Spaces (unpublished).

10. R.W. CROSS, Unbounded Strictly Singular Operators, *Indag. Math.* (to appear).

11. R.W. CROSS and L.E. LABUSCHAGNE, Partially Continuous and Semicontinuous Linear Operators in Normed Spaces (unpublished).

12. J. DIESTEL, "Sequences and Series in Banach Spaces", Springer-Verlag, New York, 1984.

13. N. DUNFORD and J.T. SCHWARTZ, "Linear Operators. I: General Theory", Pure and Appl. Math., vol. 7, Interscience, New York, 1958.

14. I.C. GOHBERG and M.G. KREIN, Fundamental Theorems on Deficiency Numbers, Root Numbers, and Indices of Linear Operators (Russian), *Usp. Mat. Nauk* **12**(1957), 43 - 118. Translated in *Amer. Math. Soc. Transls.* (*Series* 2) **13**.

15. I.C. GOHBERG, A.S. MARKUS and I.A. FELDMAN, Normally Solvable Operators and Ideals associated with them, *Amer. Math. Soc. Transls.* (*Series* 2) **61**(1967), 63 - 84.

16. S. GOLDBERG, "Unbounded Linear Operators", McGraw-Hill, New York, 1966.

17. M.A. GOLDMAN, On the Stability of the Property of Normal Solvability of Linear Equations (Russian), *Dokl. Akad. Nauk SSSR(N.S.)* **100**(1955), 201 - 204.

18. M. GONZALES and V.M. ONIEVA, On the Instability of Non-semi-Fredholm Operators under Compact Perturbations, *J. Math. Anal. Appl.* **114**(1986), 450 - 457.

19. B. GRAMSCH, "Über analytische Störungen und den Index von Fredholm-operatoren auf Banachräumen", Department of Mathematics, University of Maryland, 1969.

20. H. JARCHOW, "Locally Convex Spaces", B.G. Teubner, Stuttgart, 1981.

21. T. KATO, Perturbation theory for Nullity, Deficiency and other quantities of Linear Operators, *J. d'Analyse Math.* **6**(1958), 273 - 322.

22. J.L. KELLEY, "General Topology", Springer-Verlag, New York-Heidelberg-Berlin, Reprint of 1955 ed. published by Van Nostrand.

23. M.G. KREIN, M.A. KRASNOSELSKII and D.P. MILMAN, On the Defect Numbers of Linear Operators in a Banach Space and on Some Geometrical Questions (Russian), *Sb. Tr. Inst. Mat. Akad. Nauk Ukr. SSR* **11**(1948), 97 - 112.

24. E. KREYSZIG, "Introductory Functional Analysis with Applications", John Wiley and Sons, New York, 1978.

25. E. LACEY, "Generalisations of Compact Operators in Locally Convex Topological Linear Spaces", Thesis, New Mexico State University, 1963.

26. L.E. LABUSCHAGNE, On the Instability of Non-semi-Fredholm Closed Operators under Compact Perturbations with Applications to Ordinary Differential Operators, *Royal Soc. Edinburgh Proc. (Series A)*, (to appear).

27. L.E. LABUSCHAGNE, "Unbounded Linear Operators in Operator Ranges", Thesis Reprints 2, Department of Mathematics, University of Cape Town, 1987.

28. A. LEBOW and M. SCHECHTER, Semigroups of Operators and Measures of Noncompactness, *J. of Funct. Anal.* **7**(1971), 1 - 26.

29. J. LINDENSTRAUSS and L. TZAFRIRI, "Classical Banach Spaces", Springer, Berlin-New York, 1977.

30. G. MACKEY, Note on a theorem of Murray, *Bull. Amer. Math. Soc.* **52**(1946), 322 - 325.

31. G. NIKAIDO, Product of Linear Operators with Closed Range, *Proc. Japan Acad. (Series A)* **62**(1986), 338 - 340.

32. G. NIKAIDO, Remarks on the Lower Bound of a Linear Operator, *Proc. Japan Acad. (Series A)* **56**(1980), 321 - 323.

33. A. PELCZYNSKI, On Strictly Singular and Strictly Cosingular Operators I, II, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **13**(1965), 31 - 41.

34. A. PIETSCH, "Operator Ideals", North-Holland, Amsterdam-New York-Oxford, 1980.

35. A.P. ROBERTSON and W.J. ROBERTSON, "Topological Vector Spaces (2nd ed.)", Cambridge University Press, Cambridge, 1973.

36. M. SCHECHTER, Quantities Related to Strictly Singular Operators, *Indiana Univ. Math. J.* **21**(1972) 1061 - 1071.

37. L. WEIS, On the Computation of some Quantities in the theory of Fredholm Operators, *Supplemento ai Rendiconti del Circolo Mat. di Palermo* (*Serie II*) **5**(1984), 109 - 133.

38. L. WEIS, "Über Strikt Singuläre und Strikt Cosinguläre Operatoren in Banachräumen", Dissertation, Bonn, 1974.

39. R.J. WHITLEY, Strictly Singular Operators and their Conjugates, *Trans. Amer. Math. Soc.* **113**(1964), 252 - 261.

40. V. WROBEL, Strictly Cosingular Operators on (DF)-Spaces, *Math. Z.* **174**(1980), 281 - 287.

41. J. ZEMANEK, Geometric Characteristics of Semi-Fredholm Operators and their Asymptotic Behaviour, *Studia Math. T. LXXX* (1984), 219 - 234.