Some Problems Involving

FREE CONVECTIVE FLOW OVER VERTICAL PLATES

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SUMMARY

A collection of thirteen papers dealing with various problems involving free convective flow over a vertical plate is presented. The main topics covered are the effects of surface slip on laminar free convective heat transfer, the laminar free convective flow of non-Newtonian power law fluids, the experimental determination of the effect of plate width on laminar free convective heat transfer, combined forced and free laminar convective flow and turbulent free convective flow. Each paper is preceded by an individual summary.

DECLARATION

It is, hereby, declared that all the work included in the present thesis is the original work of the author and that no part of it has been submitted to any other university or similar institution for consideration for the award of any degree. It is also declared that permission to include published work as part of the thesis has been obtained from the thesis Supervisor, Mr. R.M. Stegen of the Department of Mechanical Engineering, University of Cape Town.

P. H. Oosthuizen
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PREFACE

This thesis consists of a collection of thirteen separate papers, ten analytical, three experimental, all concerned with some problem involving free convection from a vertical plate. The danger that exists with a thesis of this type is that in trying to cover a fairly wide range of problems, each problem will receive only shallow treatment, the study being terminated when the first major difficulty is encountered. While it cannot be claimed that this thesis is faultless in this respect, it is, of course, hoped that, taken overall, it satisfies the degree requirements.

The form of presentation used for this thesis, i.e. a collection of self-contained papers, means that considerable repetition, particularly in the introductory passages, is inevitable. Also, although an effort was made to maintain a uniform notation, some differences do exist between the papers in this respect due, mainly, to the fairly long time over which the different papers were prepared. The symbols used are, therefore, separately defined in each paper.

Of the thirteen papers, five have been published and these are presented in their published form with diagrams interspersed with the text. The remaining papers are presented with the text first then the diagrams, each on a separate page.

Most of the papers presented that are concerned with the theoretical analysis of laminar flows utilize the

Pohlhausen integral equation approach. As applied in the present work, the velocity and temperature profiles are fully determined by the assumed boundary conditions and the momentum and energy integral equations are then used to solve for the thicknesses of the velocity and thermal boundary layers. The other possible approach to the use of the integral equation method is to assume that the two boundary layers have equal thickness and to then keep an unknown parameter in the velocity or temperature profile, using the integral equations to solve for the boundary layer thickness and this profile parameter. Nothing definite can, at this stage, be said as to which is the best approach. The second probably describes the profiles more accurately when the Prandtl number is near unity. At Prandtl numbers very different from unity, however, when the two boundary layer thicknesses are known, from available exact analytical solutions, to be very different, the first method is probably better. The ideal is, of course, a combination of the two approaches using more than two integral equations. No studies of free convection problems based on this last approach are presented here nor have any yet, to the author's knowledge, been published.

Certain general points concerning some of the papers included in the thesis may be made here. The first paper is basically an introductory one and third degree polynomials for the profiles as opposed to the fourth degree polynomials adopted in the later integral equation analyses, were used for simplicity. The second paper presents a study, using the classical boundary layer integral equations, of the effects of surface slip on the free convective heat transfer under low pressure conditions. When carrying out this study it was realized that under the conditions in which slip is significant, the Grashof number will normally be so low that the boundary layer equations are not applicable. The results should, however, qualitatively indicate the effects of slip on the actual heat transfer and should indicate under what conditions they will be significant. The theory is also capable of modification to take higher order boundary layer effects into account. The fourth and fifth papers present experimental studies of the free convective heat transfer from narrow plates. It is possible, although extremely unlikely, that the observed phenomenon is not due to plate width. Also, even if it is due to this effect, it is probable that the results are influenced by the thickness of the plate. However, the fact that the plate width must influence free convective heat transfer is obvious and often mentioned, yet no other systematic study of its influence appears, to the author, to be available. The present results seem, therefore, to be of some significance. In the seventh paper the integral equation approach for combined convection is outlined and some solutions based on simplifying assumptions are given. The justification or otherwise for the use of these assumptions really, however, only follows from the results of the eighth paper which presents a continuation of the investigation. Paper ten presents an experimental study of laminar combined convection. Certain deficiencies in the method of velocity measurement used are obvious. It does not seem possible that these deficiencies can, however, lead to errors of a sufficient magnitude to completely invalidate the results. Paper eleven presents an empirical study of transition in free convective flow. The study has purposely been kept at an empirical level throughout no deep theoretical explanation of the method used, particularly in its relation to mixing length type theories, for turbulence, being attempted.

In conclusion, it is felt that all the papers presented leave many questions unanswered but that then is the nature of most research.

C O N T E N T S

Paper No. Page

Paper 1

AN APPROXIMATE ANALYSIS OF THE LAMINAR FREE CONVECTIVE FLOW OVER A VERTICAL PLATE.

AN APPROXIMATE ANALYSIS OF THE LAMINAR FREE CONVECTIVE FLOW OVER A VERTICAL PLATE

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Summary:

An approximate analysis of the laminar free convective flow over a vertical plate with either constant wall temperature or constant wall heat flux is developed. The analysis is based on the boundary layer integral equations but differs from previous such analyses in that it is not assumed that the velocity and thermal boundary layers have the same thickness and in that the wall boundary conditions are explicitly satisfied by the assumed velocity and temperature profiles. The results given by the analysis for the constant wall temperature case show fair agreement with existing exact solutions of this problem.

Introduction:

The transfer of heat by free convection plays an important role in many engineering systems. However, the theoretical analysis of free convective flow has not received the same attention as the analysis of forced convection. This is, to some extent, surprising as laminar flow, which is, generally, more amenable to theoretical analysis than turbulent flow, occurs more extensively in free convective than in forced convective flows.

The problem of free convective flow over a vertical plate was one of the first to receive attention. The case of two-dimensional laminar flow over an isothermal plate was solved completely by Pohlhausen.[1] His method does not, however, give the results in a convenient form, nor is it easily adapted to the solution of other similar problems. Attempts continue, therefore, to be made to develop approximate methods of solution to the Pohlhausen problem, which are of a more general nature in that they are capable of application with comparatively little modification to other problems involving free convection.

The method of solution adopted in the present note is basically similar to that of Von Karman and Pohlhausen[2] for incompressible laminar pressure-gradient flow. In this method it is assumed that the boundary layer has a definite thickness and no attempt is made to derive exact expressions for the velocity and temperature distributions. Instead, approximate polynomial expressions for these distributions are used which satisfy the correct boundary conditions and which, by the application of the integrated momentum and energy equations, satisfy the mean conditions in the boundary layer. In the present note this method is applied to both the case of the isothermal plate and the case of a plate having uniform wall heat flux.

Squire[1] and Eckert[3] have applied similar methods to the case of the isothermal plate, but the method employed in the present note differs from theirs in that all the boundary conditions are explicitly satisfied by the assumed velocity and temperature profiles and it is not assumed that the velocity and temperature boundary layers have the same thickness.

Basic Equations:

The basic equations for two-dimensional free-convective laminar flow over a vertical plate are, with the boundary layer approximations applied:[1]

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0 \qquad (1)$$

$$u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y} = (\mu/\rho)\frac{\delta^2 u}{\delta y^2} + \beta g(T - T_0) \qquad (2)$$

$$u\frac{\delta T}{\delta x} + v\frac{\delta T}{\delta y} = (k/\rho C_p)\frac{\delta^2 T}{\delta y^2} \qquad (3)$$

These equations apply to the case where the fluid properties ρ, μ, β and C_p can be regarded as constant except for the density variation with temperature which produces the buoyancy forces which cause the flow.

In the above equations the notation adopted is as follows:

x = co-ordinate parallel to plate, i.e. in the direction of the flow,
y = co-ordinate at right angles to plate,
u = velocity component in the x-direction,
v = velocity component in the y-direction,
T = temperature of the fluid at any point considered,
T_0 = constant fluid temperature outside the boundary layer,
μ = coefficient of viscosity of the fluid,
ρ = density of fluid,
β = coefficient of cubical expansion of fluid,
k = coefficient of conductivity of fluid,
C_p = specific heat of fluid.

The momentum equation (2) and energy equation (3) can be integrated across the boundary layer to give, with the aid of the continuity equation (1):

$$\frac{d}{dx}\int_0^h u^2 dy = \beta g \int_0^h (T - T_0)\, dy - (\mu/\rho)\left(\frac{\delta u}{\delta y}\right)_{y=0} \qquad (4)$$

and—

$$\frac{d}{dx}\int_o^h u(T-T_o)\,dy = -(k/\rho C_p)\left(\frac{\delta T}{\delta y}\right)_{y=o} \quad (5)$$

The upper limit of the integration, h, being greater than or equal to the boundary layer thickness.

It will be assumed that the velocity and temperature distributions can be approximately fitted by third-degree polynomials, i.e. it will be assumed that:

$$u = A+B\eta+C\eta^2+D\eta^3 \quad (6)$$

and—

$$T = E+F\epsilon+G\epsilon^2+H\epsilon^3 \quad (7)$$

where A, B, . . . , H are coefficients which have to be determined and:

$$\eta = y/\delta_u \quad (8)$$

$$\epsilon = y/\delta_T \quad (9)$$

where δ_u and δ_T are the thicknesses of the velocity and temperature boundary layers respectively.

Boundary Conditions:

In order to determine the coefficients in equations (6) and (7), the boundary conditions, i.e. conditions at the edges of the boundary layer, are applied.

The boundary conditions on velocity are:

At $y = o$ $\quad u = o \quad (10)$

At $y = \delta_u$ $\quad u = o, \ \frac{\delta u}{\delta y} = o$

and also, since equation (2) applies throughout the boundary layer:

At $y = o$ $\quad \frac{\delta^2 u}{\delta y^2} = -(\beta g \rho/\mu)(T-T_o) \quad (11)$

Applying these boundary conditions to equation (6) gives:

$$u = (\beta g\rho/4\mu)(T_w-T_o)\,\delta^2_u\eta\,(1-\eta)^2 \quad (12)$$

where T_w is the temperature of the wall, i.e. the temperature at $y = o$.

Similarly the boundary conditions on temperature are:

At $y = \delta_T$: $\quad T = T_o, \ \frac{\delta T}{\delta y} = o \quad (13)$

and by applying equation (3) to conditions at the wall:

At $y = o$: $\quad \frac{\delta^2 T}{\delta y^2} = o \quad (14)$

Substituting these boundary conditions into equation (7) gives:

$$(T-T_o) = (2-3\epsilon+\epsilon^3)(E-T_o)/2 \quad (15)$$

A fourth boundary condition on temperature, which allows the coefficient E to be determined, is given by the particular problem considered, i.e. in the present note either constant wall temperature or constant wall heat flux.

Analysis with Constant Wall Temperature:

When the wall temperature is constant the fourth boundary condition on temperature becomes:

$$\text{At} \qquad y = o \qquad T = \text{constant} = T_w \qquad \text{(16)}$$

Substituting this condition into equation (15) gives the temperature distribution for this case as:

$$(T - T_o) = (2 - 3\epsilon + \epsilon^3)(T_w - T_o)/2 \qquad \text{(17)}$$

i.e. defining, for convenience:

$$\theta = (T - T_o)$$

$$\theta_w = (T_w - T_o) \qquad \text{(19)}$$

equation (17) becomes:

$$\theta = (\theta_w/2)(2 - 3\epsilon + \epsilon^3) \qquad \text{(20)}$$

Substituting equations (12) and (20) into the momentum integral equation (4) gives, when the upper limit is correctly chosen:

$$\frac{d}{dx}\left[\left(\frac{\beta g \rho \theta_w}{4\mu}\right)^2 \delta_u^5 \int_o^1 \eta^2(1-\eta)^4 d\eta\right]$$

$$= \left(\frac{\beta g \delta_T \theta_w}{2}\right) \int_o^1 (2 - 3\epsilon + \epsilon^3)\, d\epsilon - \left(\frac{\beta g \theta_w}{4}\right) \delta_u \qquad \text{(21)}$$

which can be re-arranged to give:

$$\left(\frac{\beta g \rho^2 \theta_w}{336\mu^2}\right) \delta_u^4 = \int_o^x \left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right) - 1\right] dx \qquad \text{(22)}$$

where the origin of the x co-ordinate is chosen as the point, real or hypothetical, where both boundary layers have zero thickness. This implies that, in general, the present analysis will not apply close to the leading edge of the plate.

If it is assumed that, since constant fluid properties are being considered and since the effects of conditions near the leading edge are negligible in the region in which the analysis is applicable, (δ_T/δ_u) is independent of x, then equation (22) gives:

$$\delta_u = \left\{\left(\frac{336\mu^2}{\beta g \rho^2 \theta_w}\right)\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right) - 1\right] x\right\}^{\frac{1}{4}} \qquad \text{(23)}$$

A similar procedure will be adopted with the energy integral equation (5). Consider first the integral on the left-hand side of this equation. If the velocity boundary layer δ_u is thicker than temperature boundary layer δ_T, then this becomes:

$$\int_o^h u\,\theta\,dy = \int_o^{\delta_T} u\,\theta\,dy \qquad \text{(24)}$$

since $T = T_o$, i.e. $\theta = o$, for $y > \delta_T$.

Similarly, if the temperature boundary layer is thicker than the velocity boundary layer, then the integral becomes:

$$\int_o^h u\,\theta\,dy = \int_o^{\delta_u} u\,\theta\,dy \qquad \text{(25)}$$

since $u = o$ for $y > \delta_u$.

Therefore substituting equations (12), (20), (24) and (25) into equation (5) and noting that:

$$\eta = \epsilon(\delta_T/\delta_u); \quad \epsilon = \eta(\delta_u/\delta_T) \qquad \text{(26)}$$

this becomes:

(i) For $\delta_u > \delta_T$:

$$\frac{d}{dx}\left\{\left(\frac{\beta g \rho \theta_w}{8\mu}\right)\delta_u^2\,\theta_w\,\delta_T \int_o^1 \epsilon\left(\frac{\delta_T}{\delta_u}\right)\left[1-\epsilon\left(\frac{\delta_T}{\delta_u}\right)\right]^2(2-3\epsilon+\epsilon^2)\,d\epsilon\right\} = (k/\rho C_p)\,(3\theta_w/2\delta_T) \qquad \text{(27)}$$

which can be rearranged to give, since (δ_T/δ_u) is, by assumption, independent of x:

$$\delta_u = \left\{\left(\frac{16\mu^2}{\beta g \rho^2 \theta_w}\right) x \Big/ P_r\left(\frac{\delta_T}{\delta_u}\right)^3\left[\frac{1}{5}-\frac{1}{6}\left(\frac{\delta_T}{\delta_u}\right)+\frac{3}{70}\left(\frac{\delta_T}{\delta_u}\right)^2\right]\right\}^{\frac{1}{4}} \qquad \text{(28)}$$

where P_r is the Prandtl number of the fluid, i.e. $(\mu C_p/k)$.

(ii) For $\delta_u < \delta_T$:

$$\frac{d}{dx}\left\{\left(\frac{\beta g \rho \theta_w}{8\mu}\right)\delta_u^3\,\theta_w \int_o^1 \eta(1-\eta)^2\left[2-3\eta\left(\frac{\delta_u}{\delta_T}\right)+\eta^3\left(\frac{\delta_u}{\delta_T}\right)^3\right]d\eta\right\} = (k/\rho C_p)\,(3\theta_w/2\delta_T) \qquad \text{(29)}$$

which on rearrangement gives:

$$\delta_u = \left\{\left(\frac{16\mu^2}{\beta g \rho^2 \theta_w}\right)\left(\frac{\delta_u}{\delta_T}\right) x \Big/ P_r\left[\frac{1}{6}-\frac{1}{10}\left(\frac{\delta_u}{\delta_T}\right)+\frac{1}{105}\left(\frac{\delta_u}{\delta_T}\right)^3\right]\right\}^{\frac{1}{4}} \qquad \text{(30)}$$

Combining equations (23) and (28) and equations (23) and (30) gives:

(i) For $\delta_u > \delta_T$:

$$21P_r\left(\frac{\delta_T}{\delta_u}\right)^3\left[\frac{1}{5}-\frac{1}{6}\left(\frac{\delta_T}{\delta_u}\right)+\frac{3}{70}\left(\frac{\delta_T}{\delta_u}\right)^2\right]\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right]=1 \qquad (31)$$

(ii) For $\delta_u < \delta_T$:

$$21P_r\left[\frac{1}{6}-\frac{1}{10}\left(\frac{\delta_u}{\delta_T}\right)+\frac{1}{105}\left(\frac{\delta_u}{\delta_T}\right)^3\right]\left[\frac{3}{2}-\left(\frac{\delta_u}{\delta_T}\right)\right]=\left(\frac{\delta_u}{\delta_T}\right)^2 \qquad (32)$$

These equations allow the variation of (δ_T/δ_u) with P_r to be determined. With these values δ_u and δ_T can be determined from equation (23) and all the other properties of the flow can then be determined.

Results with Constant Wall Temperature:

The variation of (δ_T/δ_u) with Prandtl Number P_r as given by equations (31) and (32) is shown in Fig. 1.

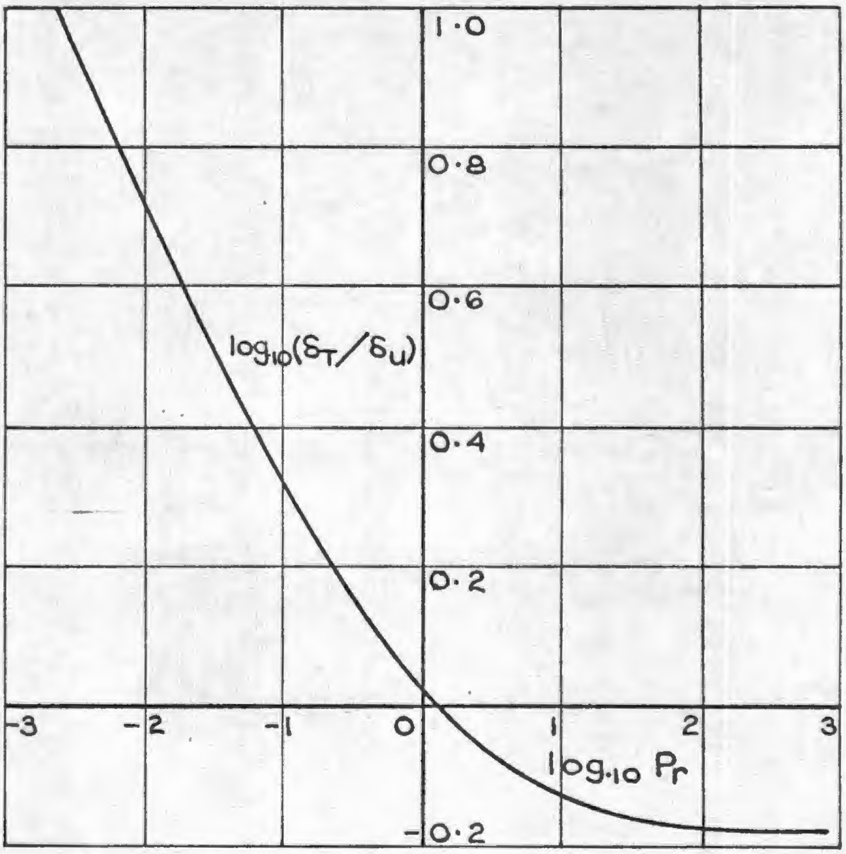

Fig. 1
Variation of boundary layer thickness ratio with Prandtl number for constant surface temperature plate.

Now equation (23) can be written as:

$$\delta_u \left(\frac{\beta g \rho^2 \theta_w}{\mu^2 x}\right)^{\frac{1}{4}} = \left\{ 336 \left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right) - 1\right] \right\}^{\frac{1}{4}} \quad \text{.............} \quad (33)$$

Therefore, by using this equation and the results given in Fig. 1, the variation of $\delta_u \left(\frac{\beta g \rho^2 \theta_w}{\mu^2 x}\right)^{\frac{1}{4}}$ and $\delta_T \left(\frac{\beta g \rho^2 \theta_w}{\mu^2 x}\right)^{\frac{1}{4}}$ with P_r can be determined and is shown in Fig. 2.

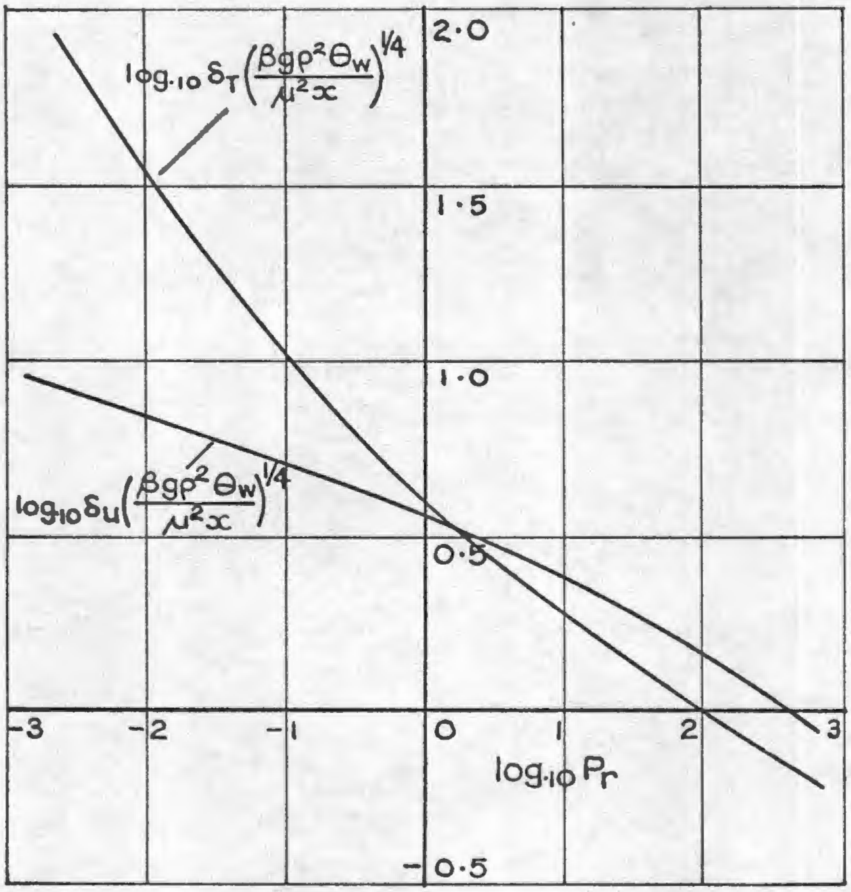

Fig. 2
Variation of non-dimensional boundary layer thickness factors with Prandtl number for constant surface temperature plate.

The local rate of heat transfer, q_x, per unit area is given by:

$$q_x = -k\left(\frac{\delta T}{\delta y}\right)_{y=o} \quad \text{.............} \quad (34)$$

Using equations (20) and (33) this becomes:

$$\frac{q_x}{k\theta_w} = \frac{3}{2}\left(\frac{\beta g\rho^2\theta_w}{\mu^2 x}\right)^{\frac{1}{4}} \Big/ \left(\frac{\delta_T}{\delta_u}\right)\left\{336\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right]\right\}^{\frac{1}{4}} \quad (35)$$

which can be written as:

$$\frac{N_x}{G_x^{\frac{1}{4}}} = 0\cdot350 \Big/ \left(\frac{\delta_T}{\delta_u}\right)\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right]^{\frac{1}{4}} \quad \text{......} \quad (36)$$

where:

N_x = local Nusselt number,
= $(q_x x/k\theta_w)$ .... (37)
G_x = local Grashof number,
= $(\beta g\rho^2\theta_w x^3/\mu^2)$ .... (38)

Using equation (36) and the results given in Fig. 1, the variation of $N_x/G_x^{\frac{1}{4}}$ with P_r can be determined and is shown in Fig. 3.

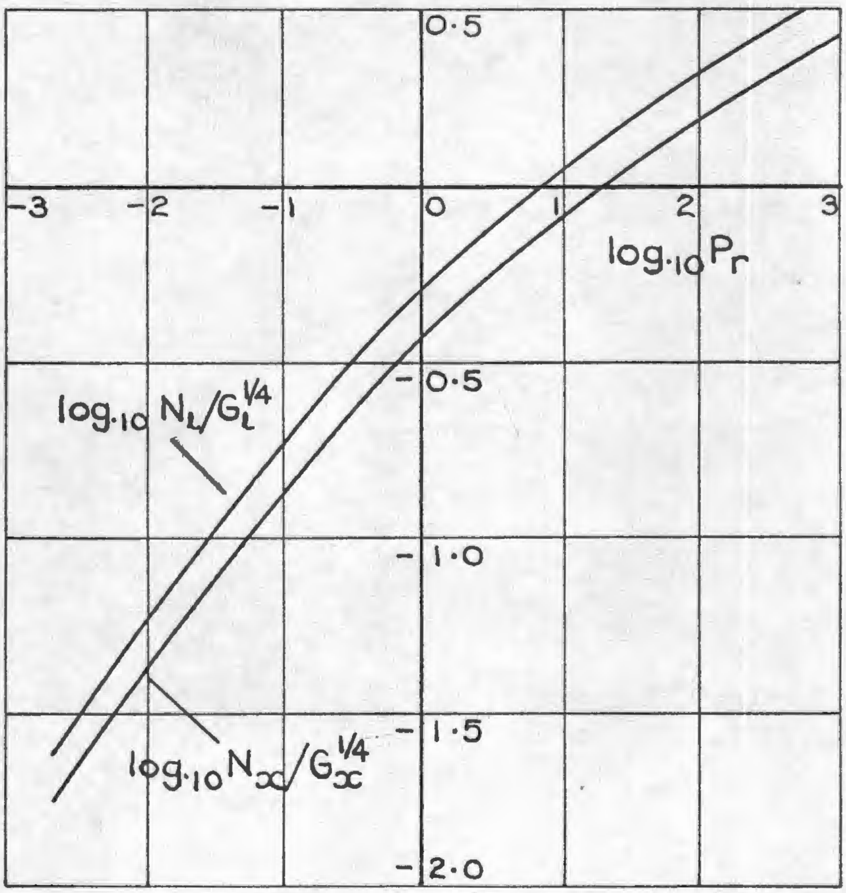

Fig. 3
Variation of local and average non-dimensional heat transfer factors with Prandtl number for constant surface temperature plate.

The mean rate of heat transfer, q_l, per unit area from a plate of length l is given by:

$$q_l = \frac{1}{l}\int_o^l q_x dx \qquad \text{(39)}$$

Using equation (35) this gives:

$$\frac{q_l}{k\theta_w} = 2\left(\frac{\beta g\rho^2\theta_w}{\mu^2 l}\right)^{\frac{1}{4}} \Big/ \left(\frac{\delta_T}{\delta_u}\right)\left\{336\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right]\right\}^{\frac{1}{4}} \qquad \text{(40)}$$

which can be written as:

$$\frac{N_l}{G_l^{\frac{1}{4}}} = 0{\cdot}467 \Big/ \left(\frac{\delta_T}{\delta_u}\right)\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right]^{\frac{1}{4}} \qquad \text{(41)}$$

where:

N_l = mean Nusselt number,

$= (q_l l/k\theta_w)$ .... (42)

G_l = mean Grashof number,

$= (\beta g\rho^2\theta_w l^3/\mu^2)$ .... (43)

It is sometimes convenient to note that:

$$(q_x/q_l) = (3/4)\,(l/x)^{\frac{1}{4}} \qquad \text{(44)}$$

and—

$$\left(\frac{N_x}{G_x^{\frac{1}{4}}}\right) \Big/ \left(\frac{N_l}{G_l^{\frac{1}{4}}}\right) = \frac{3}{4} \qquad \text{(45)}$$

$\frac{1}{4}$

The variation of N_l/G_l with P_r as given by equation (41) is also shown in Fig. 3.

Analysis with Constant Wall Heat Flux:

With constant heat flux from the wall, the fourth boundary condition on temperature becomes:

$$\text{At } y = o: \qquad -k\left(\frac{\delta T}{\delta y}\right)_{y=o} = \text{constant} = q \qquad \text{(46)}$$

Substituting this condition into equation (15) gives the temperature distribution for this case as:

$$\theta = (q\delta_T/3k)\,(2-3\epsilon+\epsilon^3) \qquad \text{(47)}$$

Therefore, under these conditions, the wall temperature is given by:

$$\theta_w = 2q\delta_T/3k \qquad (48)$$

Using equation (47) for the temperature distribution, the momentum integral equation gives, when it is again assumed that (δ_T/δ_u) is independent of x:

$$\delta_u = \left\{ \left(\frac{450\mu^2 k}{\beta g \rho^2 q}\right) \left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right) - 1\right] x \Big/ \left(\frac{\delta_T}{\delta_u}\right) \right\}^{\frac{1}{5}} \qquad (49)$$

While the energy integral equation gives:

(i) For $\delta_u > \delta_T$:

$$\delta_u = \left\{ \left(\frac{18\mu^2 k}{\beta g \rho^2 q}\right) x \Big/ P_r \left(\frac{\delta_T}{\delta_u}\right)^4 \left[\frac{1}{5} - \frac{1}{6}\left(\frac{\delta_T}{\delta_u}\right) + \frac{3}{70}\left(\frac{\delta_T}{\delta_u}\right)\right] \right\}^{\frac{1}{5}} \qquad (50)$$

(ii) For $\delta_u < \delta_T$:

$$\delta_u = \left\{ \left(\frac{18\mu^2 k}{\beta g \rho^2 q}\right) \left(\frac{\delta_u}{\delta_T}\right)^2 x \Big/ P_r \left[\frac{1}{6} - \frac{1}{10}\left(\frac{\delta_u}{\delta_T}\right) + \frac{1}{105}\left(\frac{\delta_u}{\delta_T}\right)^3\right] \right\}^{\frac{1}{5}} \qquad (51)$$

Combining equations (49) and (50) and equations (49) and (51) then gives:

(i) For $\delta_u > \delta_T$:

$$25P_r\left(\frac{\delta_T}{\delta_u}\right)^3 \left[\frac{1}{5} - \frac{1}{6}\left(\frac{\delta_T}{\delta_u}\right) + \frac{3}{70}\left(\frac{\delta_T}{\delta_u}\right)^2\right] \left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right) - 1\right] = 1 \qquad (52)$$

(ii) For $\delta_u < \delta_T$:

$$25P_r\left[\frac{1}{6} - \frac{1}{10}\left(\frac{\delta_u}{\delta_T}\right) + \frac{1}{105}\left(\frac{\delta_u}{\delta_T}\right)^3\right] \left[\frac{3}{2} - \left(\frac{\delta_u}{\delta_T}\right)\right] = \left(\frac{\delta_u}{\delta_T}\right)^2 \qquad (53)$$

The two equations, (52) and (53), which may be compared with equations (31) and (32), allow the variation of (δ_T/δ_u) with P_r to be determined. With these results the other properties of the flow can then be determined.

Results with Constant Wall Heat Flux:

The variation of (δ_T/δ_u) with P_r as given by equations (52) and (53) is shown n Fig. 4.

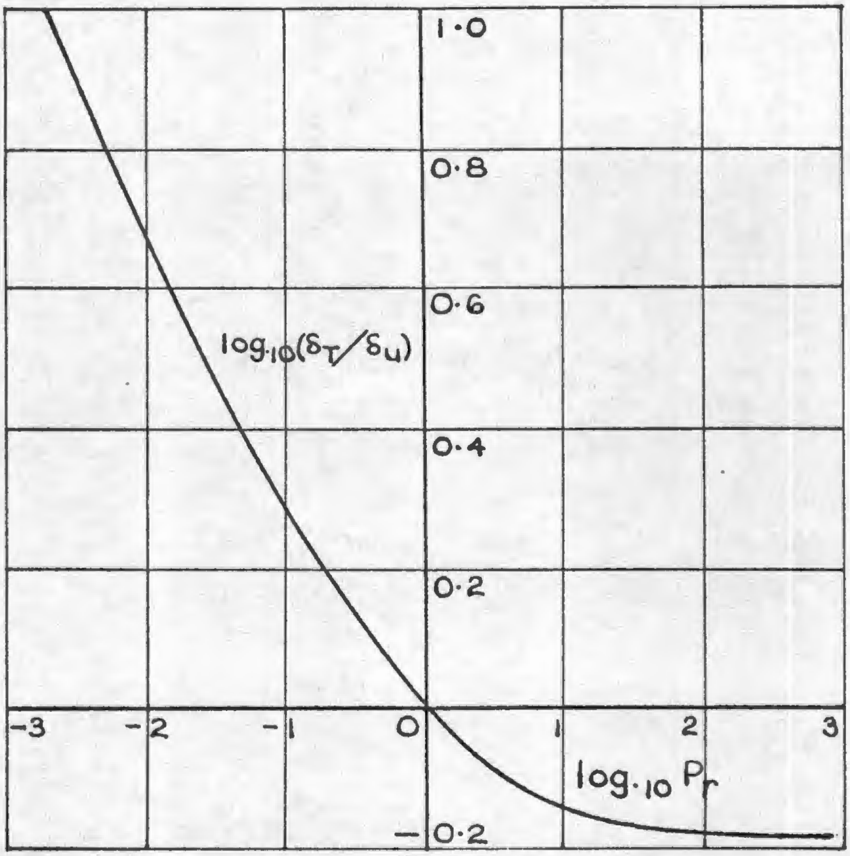

Fig. 4
Variation of boundary layer thickness ratio with Prandtl number for constant surface heat flux plate.

Equation (49) gives:

$$\delta_u \left(\frac{\beta g \rho^2 q}{\mu^2 k x} \right)^{\frac{1}{5}} = \left\{ 450 \left[\frac{3}{2} \left(\frac{\delta_T}{\delta_u} \right) - 1 \right] \Big/ \left(\frac{\delta_T}{\delta_u} \right) \right\}^{\frac{1}{5}} \quad \text{......} \quad (54)$$

which, with the aid of the results given in Fig. 4, allows the variation of $\delta_u \left(\frac{\beta g \rho^2 q}{\mu^2 k x} \right)$ and $\delta_T \left(\frac{\beta g \rho^2 q}{\mu^2 k x} \right)$ with P_r to be determined, results being shown in Fig. 5.

The local Nusselt number $(qx/k\theta_w)$ is given by equations (48) and (54) as:

$$\frac{N_x}{G_x^{\frac{1}{4}}} = 0{\cdot}360 \Big/ \left(\frac{\delta_T}{\delta_u} \right) \left[\frac{3}{2} \left(\frac{\delta_T}{\delta_u} \right) - 1 \right]^{\frac{1}{4}} \quad \text{..................} \quad (55)$$

where, as before, $G_x = (\beta g \rho^2 \theta_w x^3/\mu^2)$ is the local Grashof number. It will be noted that although θ_w is not constant N_x and G_x are still defined in terms of this parameter and not in terms of (qx/k). However,

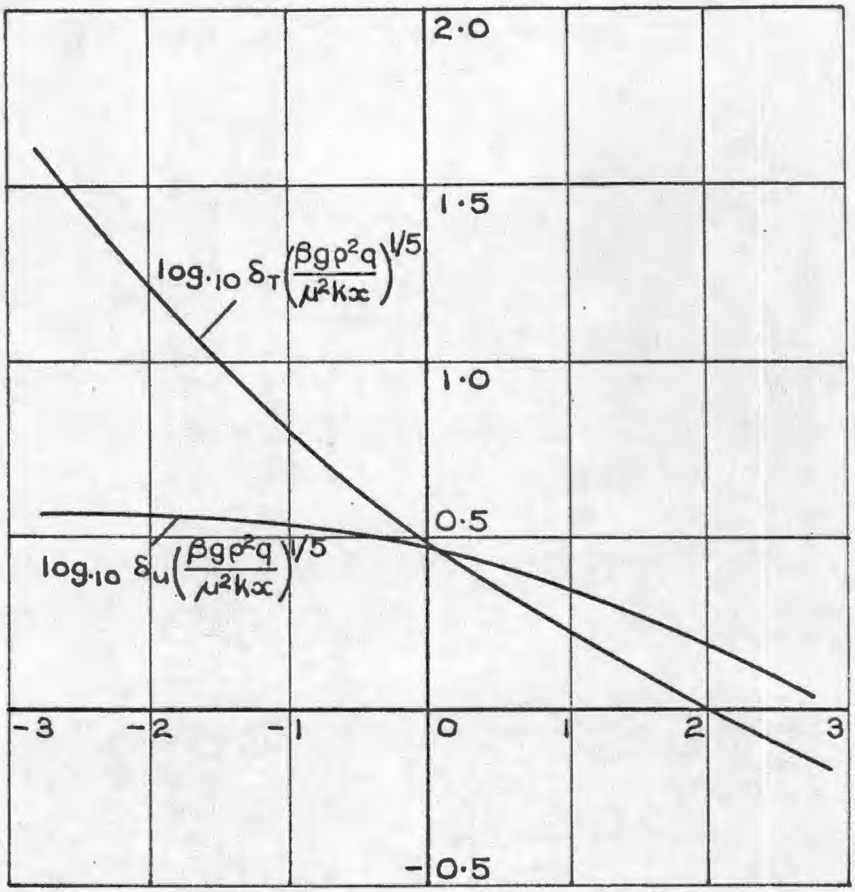

Fig. 5
Variation of non-dimensional boundary layer thickness factors with Prandtl number for constant surface heat flux plate.

although N_x and G_x vary with x, the ratio $N_x/G_x^{\frac{1}{4}}$ is independent of x.

Thus, both with constant wall temperature and constant wall heat flux N_x is proportional to $G_x^{\frac{1}{4}}$ for a given value of P_r.

The variation of $N_x/G_x^{\frac{1}{4}}$ as given by equation (55) is shown in Fig. 6.

The average Nusselt number for a plate of length l must, in the case under consideration, be defined in terms of a mean wall temperature θ_{wl}. This mean wall temperature will here be defined as:

$$\theta_{wl} = \frac{1}{l}\int_0^l \theta_w dx \qquad \text{.......................} \quad (56)$$

Substituting equations (47) and (48) then gives:

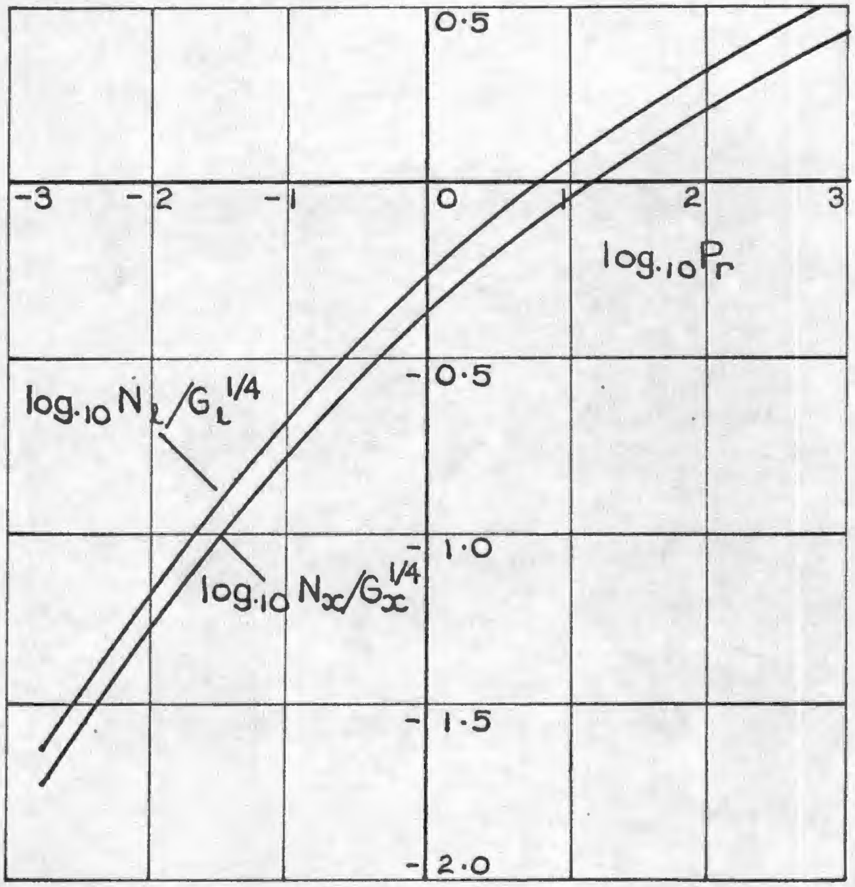

Fig. 6
Variation of local and average non-dimensional heat transfer factors with Prandtl number for constant surface heat flux plate.

$$\theta_{wl} = \frac{5}{9}\left(\frac{q}{k}\right)^{\frac{4}{5}}\left(\frac{\delta_T}{\delta_u}\right)\left\{\left(\frac{450\mu^2}{\beta g\rho^2}\right)\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right] l \Big/ \left(\frac{\delta_T}{\delta_u}\right)\right\}^{\frac{1}{5}} \quad \text{...... (57)}$$

Re-arranging this equation gives:

$$\frac{N_l}{G_l^{\frac{1}{4}}} = 0{\cdot}454 \Big/ \left(\frac{\delta_T}{\delta_u}\right)\left[\frac{3}{2}\left(\frac{\delta_T}{\delta_u}\right)-1\right]^{\frac{1}{4}} \quad \text{.................. (58)}$$

where, as before,

N_l = Mean Nusselt number,
= $ql/k\theta_{wl}$.
G_l = Mean Grashof number,
= $(\beta g\rho^2\theta_{wl}l^3/\mu^2)$.

It will be noted that with constant wall heat flux:

$$\left(\frac{N_x}{G_x^{\frac{1}{4}}}\right) \Big/ \left(\frac{N_l}{G_l^{\frac{1}{4}}}\right) = 0{\cdot}793 \quad \text{.................................. (59)}$$

The variation of $N_t/G_t^{\frac{1}{4}}$ as given by equation (58) is also shown in Fig. 6.

TABLE I

Prandtl Number	$N_t/G_t^{\frac{1}{4}}$ *from Exact Solution*	$N_t/G_t^{\frac{1}{4}}$ *from Approx. Solution*
0·73	0·478	0·456
10	1·09	1·09
100	2·06	2·00
1000	3·67	3·56

Discussion of Results:

Since a complete solution of the isothermal vertical plate problem is available, the results of this solution can be used to check the results given by the present approximate analysis. In Table I, therefore, values of $N_t/G_t^{\frac{1}{4}}$ as given by equation (41) are compared with values given by Schuh's[4] solution of the Pohlhausen equation, while in Figs. 7 and 8

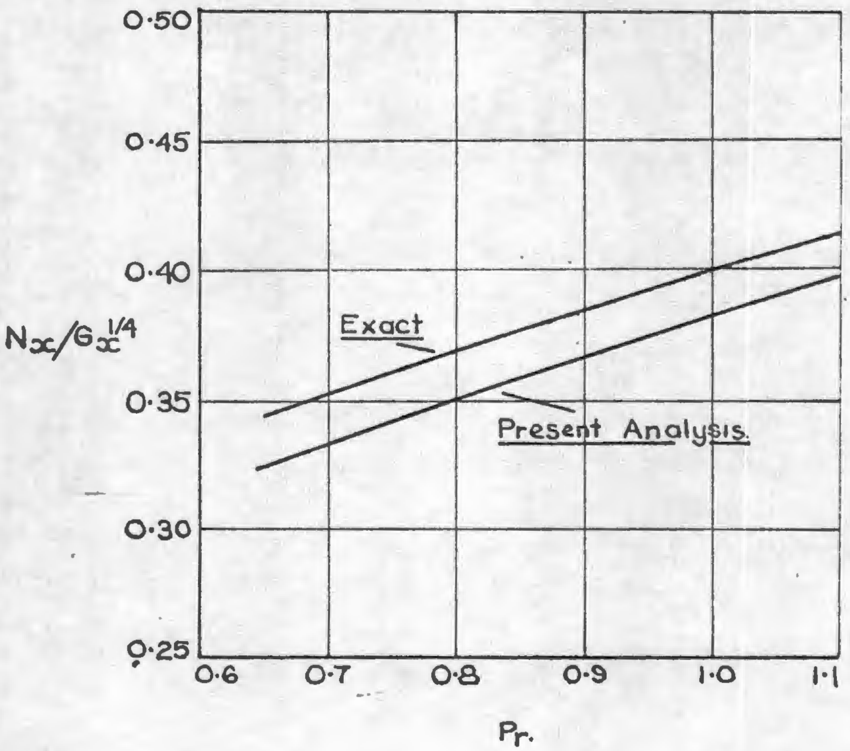

Fig. 7
Comparison of local non-dimensional heat transfer factors as given by present approximate analysis with results given by exact analysis of the constant surface temperature plate problem for Prandtl numbers of common gases.

values of $N_x/G_x^{\frac{1}{4}}$ as given by equation (36) are compared with values given by Sparrow and Gregg's[5] solution. The agreement is seen to

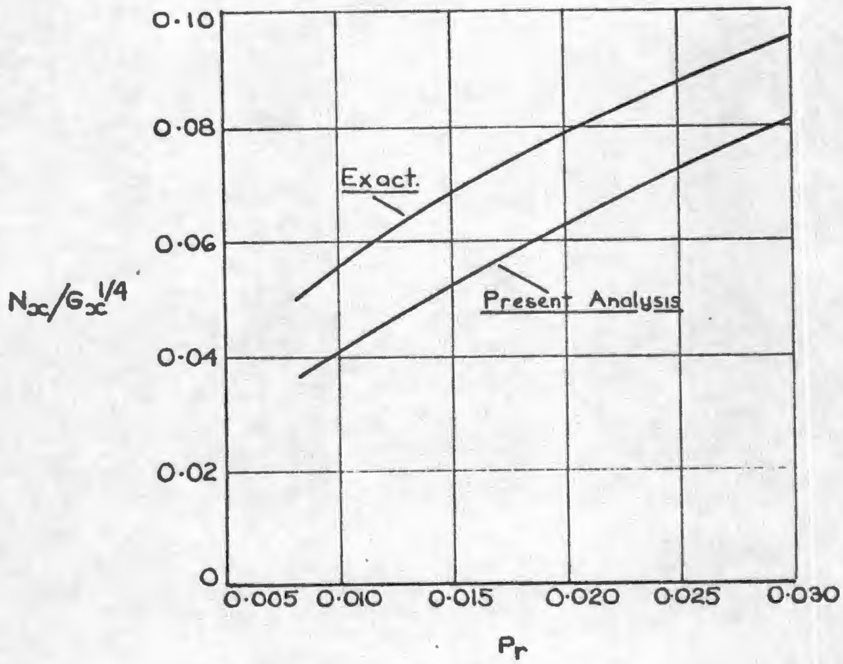

Fig. 8
Comparison of local non-dimensional heat transfer factors as given by present approximate analysis with results given by exact analysis of the constant surface temperature plate problem for Prandtl numbers of liquid metals.

be quite good except at low Prandtl numbers, but as all methods based on the boundary layer hypothesis are suspect in this region this discrepancy cannot be regarded as proving the method to be generally inaccurate. The good agreement at very large Prandtl numbers must be due partly to the fact that (δ_T/δ_u) is asymptotic to $(2/3)$ as the Prandtl number increases as opposed to the fact that (δ_T/δ_u) tends to infinity as the Prandtl number decreases.

Conclusion:

An approximate method for analysing the natural convective flow over vertical plates has been developed. The results given by this method show fair agreement with available exact solutions. The method, as with other methods based on the integral equations, can be quite easily extended to other problems. Since it is not assumed that the velocity and thermal boundary layer thicknesses are the same, it is to be expected that the results obtained by means of the present method for fluids having Prandtl numbers greatly different from unity should be better than those given by existing integral methods.

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Paper 2.

THE EFFECT OF SURFACE SLIP ON THE LAMINAR FREE CONVECTIVE HEAT TRANSFER FROM AN ISOTHERMAL VERTICAL FLAT PLATE.

Appl. Sci. Res. Vol. 16

THE EFFECT OF SURFACE SLIP ON THE LAMINAR FREE CONVECTIVE HEAT TRANSFER FROM AN ISOTHERMAL VERTICAL FLAT PLATE

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Summary

Under certain circumstances, for instance when the pressure is very low, it cannot be assumed that when a fluid flows over a body it attains, at the body surface, the local velocity and temperature of the body and slip flow then exists. The present note attempts to predict, approximately, the effects of very small amounts of slip on laminar free convective flow over a vertical plate, the main aim being to try to indicate the circumstances under which slip effects become appreciable rather than predicting very accurately the magnitude of these effects.

§ 1. *Introduction.* Under most conditions, considered to be normal in engineering practice, the equations governing fluid flow over bodies can be solved with the assumption that at the surface of the body the fluid attains the velocity and local surface temperature of the body. With axes so chosen that the body is stationary, this implies that the fluid velocity at the surface is zero. When the temperature is high or the fluid density low, however, and the mean free path of the molecules not, therefore, insignificant in comparison with the governing dimensions of the body, this assumption is not valid and slip flow exists. The velocity and temperature of the fluid at the surface of the body are given, approximately, in such circumstances, in the usual form, by[1])

$$u = l \frac{\partial u}{\partial y} + 0.75(\mu/\rho T) \frac{\partial T}{\partial x} \tag{1}$$

and

$$T = T_w + (\alpha/Pr)\, l \frac{\partial T}{\partial y} \tag{2}$$

respectively, where u is the velocity in the x-direction, x the co-ordinate parallel to body surface, y the co-ordinate normal to body surface, l the mean free path of constituent fluid molecules, T the local fluid temperature, μ the viscosity, ρ the density, T_w the local surface temperature of body, Pr the Prandtl number of fluid, α equals $2.44\gamma/(\gamma - 1)$, γ the ratio of specific heats.

In setting up the above equations it has been assumed that the fraction of the tangential molecular momentum transmitted to the wall (given the symbol f in ref. 1) is equal to 1 and that the accommodation coefficient (given the symbol α in ref. 1) is equal to 0.9.

There are certain practical problems in which heat is transferred by free convection under such conditions that the effects of surface slip cannot be neglected. In order to indicate the magnitude of the effects of this slip in free convective flow, an attempt will be made in the present note to predict the effect of small amounts of slip on the two-dimensional laminar free convective flow over a vertical plate at a constant temperature.

Since the slip is assumed to be small, the normal boundary layer equations will be used as the basis of the analysis. It will be assumed in applying these equations that l/δ is so small that terms of the order $(l/\delta)^2$ may be neglected.

The application of an order of magnitude analysis, similar to that which can be used to derive the boundary layer equations from the Navier-Stokes equations[2]), to equation (1) shows that, if terms of a lower order of magnitude than those retained in the boundary layer equations are neglected, then the second term in (1), i.e. the creeping velocity, is negligible. In applying this analysis it was noted that this second term could be written as

$$[\text{constant} \times l\sqrt{T}(\partial T/\partial x)].$$

It will also be assumed in the analysis that the temperature difference across the boundary layer is small enough for the fluid property variation across the boundary layer to be neglected except for the density change which causes the buoyancy forces and, hence, the flow.

§ 2. *Boundary layer profiles.* The approximate Von Karman integral method of solution will be adopted in the present note. In this method, polynomial expressions for the velocity and temperature profiles within the boundary layer are assumed and used to solve the integrated momentum and energy equations. In the present case, fourth order polynomials will be assumed for both the velocity and temperature profiles i.e. it will be assumed that

$$u = A + B\eta + C\eta^2 + D\eta^3 + E\eta^4 \tag{3}$$

and

$$T = F + G\varepsilon + H\varepsilon^2 + I\varepsilon^3 + J\varepsilon^4, \tag{4}$$

where

$$\eta = y/\delta_u \tag{5}$$

and

$$\varepsilon = y/\delta_T. \tag{6}$$

δ_u and δ_T being the local thicknesses of the velocity and temperature boundary layers respectively.

The coefficients $A, B, \ldots, I, J$ are determined by applying the known conditions at the inner and outer edges of the boundary layer.

Now the boundary layer equations for two-dimensional flow are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(\frac{\mu}{\rho}\right)\frac{\partial^2 u}{\partial y^2} + \beta g(T - T_1) \tag{7}$$

and

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \left(\frac{\mu}{\rho Pr}\right)\frac{\partial^2 T}{\partial y^2}, \tag{8}$$

where v is the velocity in the y-direction, β the coefficient of cubical expansion of the fluid, T_1 the constant temperature outside the boundary layer.

Applying the above equations to conditions at the wall where $v = 0$ and using (1) and (2) for the velocity and temperature at the wall, gives the following conditions on the velocity and temperature profiles at the wall:

At $y = 0$:

$$u = l\frac{\partial u}{\partial y}, \tag{9}$$

$$u\frac{\partial u}{\partial x} = \left(\frac{\mu}{\rho}\right)\frac{\partial^2 u}{\partial y^2} + \beta g(T - T_1), \tag{10}$$

$$T = T_w + \left(\frac{\alpha}{Pr}\right) l \frac{\partial T}{\partial y}, \tag{11}$$

$$u \frac{\partial T}{\partial x} = \left(\frac{\mu}{\rho Pr}\right) \frac{\partial^2 T}{\partial y^2}. \tag{12}$$

At the outer edges of the boundary layers the gradients disappear and the velocity and temperature become equal to their undisturbed values i.e. 0 and T_1 respectively. Therefore

At $y = \delta_u$:

$$u = 0; \quad \frac{\partial u}{\partial y} = 0; \quad \frac{\partial^2 u}{\partial y^2} = 0. \tag{13}$$

At $y = \delta_T$:

$$T = T_1; \quad \frac{\partial T}{\partial y} = 0; \quad \frac{\partial^2 T}{\partial y^2} = 0. \tag{14}$$

Applying the boundary conditions (9) to (14) to equations (3) and (4) and ignoring terms containing the quantity (l/δ) raised to a higher power than unity, gives the velocity and temperature profiles as:

$$\begin{aligned} u = (\beta g \rho \theta_w \delta_u^2/6\mu) \cdot \\ \cdot\{(1 - 2\alpha l/Pr\delta_T)(\eta - 3\eta^2 + 3\eta^3 - \eta^4) + \\ + (l/\delta_u)(1 - 2\eta + 2\eta^3 - \eta^4)\} \end{aligned} \tag{15}$$

and

$$\theta/\theta_w = (1 - 2\alpha l/Pr\delta_T)(1 - 2\varepsilon + 2\varepsilon^3 - \varepsilon^4), \tag{16}$$

where the following have been introduced:

$$\theta = T - T_1 \tag{17}$$

and

$$\theta_w = T_w - T_1. \tag{18}$$

§ 3. *Analysis.* Integrating (7) and (8) across the boundary layer, which is assumed, as above, to have definite thickness, gives

$$\frac{d}{dx} \int_0^{\delta_u} u^2 \, dy = -\left(\frac{\mu}{\rho}\right)\left(\frac{\partial u}{\partial y}\right)_w + \beta g \int_0^{\delta_T} (T - T_1) \, dy \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta} u\theta \,\mathrm{d}y = -\left(\frac{\mu}{\rho Pr}\right)\left(\frac{\partial T}{\partial y}\right)_{\mathrm{w}}, \qquad (20)$$

where the upper limit of the integral in (20), either δ_u or δ_T, depends on whether δ_u or δ_T is the larger, and where subscript w refers to conditions at the wall.

Substituting (15) and (16) into the above integral equations and introducing the definition

$$\Delta = \delta_T/\delta_u, \qquad (21)$$

the following equations are obtained:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\delta_u^5[A + (B - 4\alpha A/Pr\Delta)(l/\delta_u)]\} =$$
$$= (6\mu^2/\beta g\rho^2\theta_{\mathrm{w}})\,\delta_u\{(6C\Delta - 1) +$$
$$+ [2 - (6C\Delta - 1)(2\alpha/Pr\Delta)](l/\delta_u)\} \qquad (22)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\delta_T^3[Y + (Z\Delta - 4\alpha Y/Pr)(l/\delta_T)]\} =$$
$$= (12\mu^2/\beta g\rho^2\theta_{\mathrm{w}}Pr\delta_T)(1 - 2\alpha l/Pr\delta_T), \qquad (23)$$

where the following constants have been introduced:

$$A = \int_0^1 (\eta - 3\eta^2 + 3\eta^3 - \eta^4)^2\,\mathrm{d}\eta = 0.003968, \qquad (24)$$

$$B = \int_0^1 (\eta - 3\eta^2 + 3\eta^3 - \eta^4)(1 - 2\eta + 2\eta^3 - \eta^4)\,\mathrm{d}\eta =$$
$$= 0.02182, \qquad (25)$$

$$C = \int_0^1 (1 - 2\varepsilon + 2\varepsilon^3 - \varepsilon^4)\,\mathrm{d}\varepsilon = 0.3. \qquad (26)$$

Y and Z depend on Δ, their actual form depending on whether Δ is greater or less than 1. They are given by the following expressions.

For $\Delta < 1$:

$$Y = \frac{1}{\Delta^2}\int_0^1 (\varepsilon\Delta - 3\varepsilon^2\Delta^2 + 3\varepsilon^3\Delta^3 - \varepsilon^4\Delta^4)(1 - 2\varepsilon + 2\varepsilon^3 - \varepsilon^4)\,\mathrm{d}\varepsilon, \qquad (27)$$

$$Z = \frac{1}{\Delta^2}\int_0^1 (1 - 2\varepsilon\Delta + 2\varepsilon^3\Delta^3 - \varepsilon^4\Delta^4)(1 - 2\varepsilon + 2\varepsilon^3 - \varepsilon^4)\,\mathrm{d}\varepsilon. \qquad (28)$$

For $\Delta > 1$:

$$Y = \frac{1}{\Delta^3}\int_0^1 (\eta - 3\eta^2 + 3\eta^3 - \eta^4)(1 - 2\eta/\Delta + 2\eta^3/\Delta^3 - \eta^4/\Delta^4)\,\mathrm{d}\eta, \quad (29)$$

$$Z = \frac{1}{\Delta^3}\int_0^1 (1 - 2\eta + 2\eta^3 + \eta^4)(1 - 2\eta/\Delta + 2\eta^3/\Delta^3 - \eta^4/\Delta^4)\,\mathrm{d}\eta. \quad (30)$$

Now when l is negligible in comparison with δ; the above set of equations can be shown to give

$$\Delta^4 = 10A/3YPr(6C\Delta - 1), \quad (31)$$

which gives Δ as a function of Pr alone, i.e. a function of the type of fluid alone.

Now, since only the effects of small amounts of slip are being considered, δ_u and δ_T will only differ by small amounts from the values they would have under the same conditions if the slip were entirely negligible. It seems possible to assume, therefore, that, to a fair degree of accuracy, the value of Δ in the slip flow is equal to the value that would exist if the slip was negligible i.e. as given by (31). With Δ assumed to be constant and known for any fluid, Y and Z are constants for a given fluid and (22) and (23) are conveniently rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}(A\delta_u^5 + Dl\delta_u^4) = (6\mu^2/\beta g\rho^2\theta_w)(E\delta_u + Fl) \quad (32)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(G\delta_T^3 + Hl\delta_T^2) = (12\mu^2/\beta g\rho^2\theta_w Pr)[(1/\delta_T) + (Il/\delta_T^2)], \quad (33)$$

where

$$D = (B - 4\alpha A/Pr\Delta), \quad (34)$$

$$E = (6C\Delta - 1), \quad (35)$$

$$F = (2 - 2\alpha E/Pr\Delta), \quad (36)$$

$$G = Y, \quad (37)$$

$$H = (Z\Delta - 4\alpha Y/Pr), \quad (38)$$

$$I = -2\alpha/Pr, \quad (39)$$

and all now depend, by assumption, on the type of fluid alone.

Subject to the boundary condition $\delta_u = \delta_T = 0$ when $x = 0$, (32) and (33) constitute differential equations giving the variation of δ_u and δ_T respectively, with x, for any given value of l.

§ 4. *Heat transfer rate.* The local rate of heat transfer per unit area from the wall to the fluid is given by

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_w = (2k\theta_w/\delta_T)(1 - 2\alpha l/Pr\delta_T). \tag{40}$$

The local Nusselt number, defined as usual as

$$N_x = q_w x/k\theta_w \tag{41}$$

is therefore, given by

$$N_x = 2(x/\delta_T)[1 - 2(\alpha/Pr)(l/x)(x/\delta_T)]. \tag{42}$$

Hence, using (33) to determine (δ_T/x), (42) can be used to determine N_x for any value of (l/x) and of local Grashof number G_x, defined as

$$G_x = \beta g\rho^2 x^3\theta_w/\mu^2. \tag{43}$$

A practically more important parameter is the average Nusselt number for a plate of length L, which is defined as

$$N_L = \bar{q}_w L/\theta_w k, \tag{44}$$

$\bar{q}_w$ being the average rate of heat transfer per unit area from the wall to the fluid which is given by

$$\bar{q}_w = \frac{1}{L}\int_0^L q_w \, dx. \tag{45}$$

Therefore, substituting for q_w from (40) gives

$$N_L = 2\int_0^1 (L/\delta_T)[1 - 2(\alpha/Pr)(l/L)(L/\delta_T)] \, d(x/L). \tag{46}$$

Since (δ_T/L) can be determined as a function of (x/L) for any given value of G_L, the Grashof number based on plate length, which is given by

$$G_L = \beta g\rho^2 L^3\theta_w/\mu^2 \tag{47}$$

and for any given value of (l/L), the variation of N_L with G_L and (l/L) can be determined for any given fluid. Results determined in this manner, for the case of fluids with $Pr = 0.72$, are shown in fig. 1.

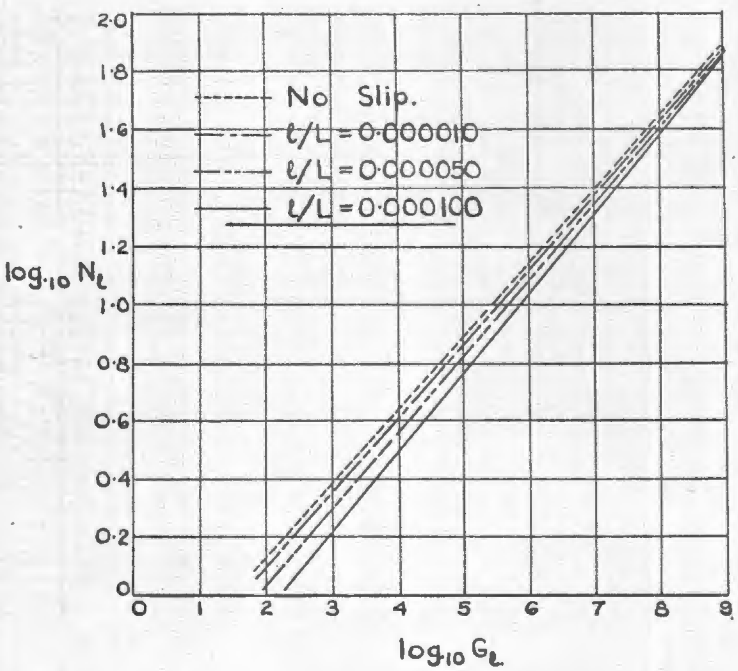

Fig. 1. Effect of slip on the variation of mean Nusselt number with Grashof number.

In carrying out the above calculation a comparatively crude numerical procedure had to be used to start the process since near the leading edge the boundary layer equations, which do not of course apply in this region, break down when slip effects are considered. Comparison of numerical results obtained, using this procedure, for the case where $(l/L) = 0$ with the simple explicit solution obtained in this case, suggest that the starting procedure has a negligible effect on the solution outside the region of the leading edge.

It will be noted that, since Δ is assumed to be constant, only the energy integral equation (33) was used in determining the above heat transfer results.

§ 5. *Numerical example.* In order to illustrate, more directly, the effects of slip consider a plate 2 ft. long suspended in air in such a manner that the air pressure can be varied. Assume the plate to be held at such a constant temperature that the mean film temperature is always 60°F. Then noting that

$$l = 1.255\mu/\rho(RT)^{0.5},$$

the variation of $\bar{q}_w$ with pressure can be determined, using the previously given results, for any chosen value of θ_w, results being shown in fig. 2 for $\theta_w = 10$°F. In determining these results it was assumed that μ, Pr and C_p were independent of pressure.

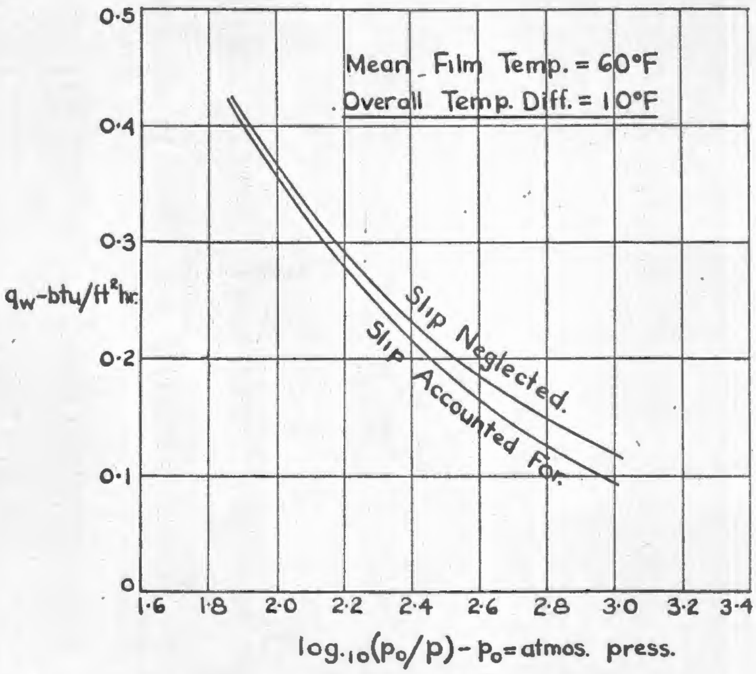

Fig. 2. Effect of reduced pressure on rate of heat transfer from 2 ft. high plate.

§ 6. *Conclusion.* A simple approximate method for predicting the effects of small amounts of surface slip on the rate of heat transfer by laminar free convection from a vertical flat plate has been derived. An illustrative example, which shows the effects of increased mean free path with decreased pressure, has been given.

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Paper 3.

HEAT TRANSFER BY LAMINAR FREE CONVECTION FROM A VERTICAL FLAT PLATE TO A POWER LAW NON-NEWTONIAN FLUID.

SUMMARY:

Assuming that the boundary layer equations are applicable to the problem under consideration, the similarity variables and equations are derived. The solution to the problem by integral equation methods is also considered and to illustrate the type of results to be expected a simplified solution, applicable under certain special conditions, is presented.

HEAT TRANSFER BY LAMINAR FREE CONVECTION FROM A VERTICAL FLAT PLATE TO A POWER LAW NON-NEWTONIAN FLUID

Introduction:

A Newtonian fluid is one in which the local shearing stress, τ, on any plane at any point in the fluid is related by a linear law to the velocity gradient, $(\partial V/\partial m)$, normal to this plane and disappears when this gradient is zero, i.e. a Newtonian fluid is one in which:

$$\tau = \mu \frac{\partial V}{\partial m} \qquad (1)$$

The coefficient of viscosity, μ, being dependent only on the local fluid properties and not, directly, therefore, on the velocity field.

A non-Newtonian fluid is, in general, any fluid in which equation (1) does not apply. It is convenient to distinguish between non-Newtonian fluids in which the shearing stress disappears with the velocity gradient and those in which there exist a shearing stress, τ_o, even when the velocity gradient is zero.

A number of papers dealing with the flow of non-Newtonian fluids through pipes have been published.[1.] Little attention has, however, been given to external flows involving such fluids. One reason for this is that, when an external flow problem is encountered in practice, the conditions are very often such that neither the boundary layer nor the viscous flow approximations can be applied. Nevertheless, at least for fluids in which the deviation from the Newtonian equation (1) is small, solutions based on the boundary layer approximations should be of some use in certain circumstances even if it is only to give a qualatative indication of the effects of the deviation from equation (1) on the problem.

In the present note, therefore, the boundary layer equations will be used as the basis of a study of the heat transfer by laminar free convection from a vertical plate to a non-Newtonian fluid. Attention will be restricted to fluids of the first type, i.e. with $\tau_0 = 0$, and, as is quite common, a simple power law relation between the shearing stress and the velocity gradient will be assumed, i.e. it will be assumed that:

$$\tau = K\left(\frac{\partial v}{\partial m}\right)^n \qquad (2)$$

In carrying out the analysis, it will also be assumed that the flow is two-dimensional and that the overall temperature changes are small enough for the changes in fluid properties through the boundary layer to be negligible and K and n in equation (2) will, therefore, be taken as constants. Attention will also be restricted to the case where the wall temperature is constant.

Similarity Considerations:

The boundary layer approximations, which are being assumed here, give the following equations for two-dimensional, constant fluid property flow over a vertical flat plate:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = (1/\rho) \frac{\partial \tau}{\partial y} + \beta g (T - T_1) \qquad (3)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = (k/\rho\, C_p) \frac{\partial^2 T}{\partial y^2} \qquad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (5)$$

where x = co-ordinate parallel to plate surface, the origin being at the leading edge of the plate,

y = co-ordinate normal to plate surface, the origin being at the surface,

u = velocity componant in x-direction,

v = velocity componant in y-direction,

ρ = density,

τ = shearing stress on xy planes,

β = coefficient of cubical expansion of fluid,

T = local fluid temperature,

T_1 = undisturbed fluid temperature,

k = conductivity,

c_p = specific heat.

The above boundary layer equations can be solved exactly in terms of two universal functions, defined below, if the velocity and temperature profiles are both similar,[2] in terms of the same similarity variable, for all values of x i.e. if, for all values of x, these profiles can be expressed in the form:

$$u = \overline{u}\,(x)\;F'\,(\eta_s) \qquad (6)$$

where the dash denotes differentiation with respect to η_s and the reason for choosing this form will become obvious at a later stage, and:

$$\theta = \theta_w\;G(\eta_s) \qquad (7)$$

where $\theta = T - T_1$

$\theta_w = T_w - T_1$

T_w = temperature of wall, here assumed constant.

The velocity coefficient $\overline{u}$ and the similarity variable, η_s, used for both velocity and temperature, are determined directly from the boundary layer equations. The functions F and G are determined by solving the equations which result on substituting equations (6) and (7) into the boundary layer equations. This is a comparatively simple task since if, as is being assumed, similarity exists, these equations reduce, on substitution, to a pair of simultaneous ordinary differential equations for F and G.

In order to determine whether similar solutions, of the form indicated in equations (6) and (7), do exist to the problem at present under consideration, it will be assumed, initially, that such solutions do exist.

Now the continuity equation (5) can be satisfied by introducing, as usual, a stream function, ψ, such that:

$$u = \frac{\partial \psi}{\partial y} \tag{8}$$

and

$$v = -\frac{\partial \psi}{\partial x} \tag{9}$$

This stream function must, since similarity is being assumed, take the form:

$$\psi = A(x)\ F(\eta_s) \tag{10}$$

Equations (6), (8) and (10), together, then give:

$$\eta_s = \bar{u}\ y/A \tag{11}$$

Substituting the above relations, together with equation (2), into the boundary layer momentum and energy equations (3) and (4) and rearranging then gives:

$$\left[(Kn/\rho)(\bar{u}^{2n+1}/A^{n+1}) \ / \ (\beta g \theta_w) \right] (F'')^{n-1}\ F'''$$

$$+ \left[(\bar{u}^2/A)\ \frac{dA}{dx} \ / \ (\beta g \theta_w) \right] FF'' - \left[\left(\bar{u}\ \frac{d\bar{u}}{dx} \right) / \ (\beta g \theta_w) \right] F'^2 + G = 0 \tag{12}$$

and:

$$(\bar{u}/A)\left(\frac{dA}{dx}\right) G'F + (k/\rho C_p)\ (\bar{u}^2/A^2)\ G'' = 0 \qquad (13)$$

If similarity does exist then all the coefficients in the above differential equations must be independent of x, i.e. considering, first, the momentum equation (12):

$$(Kn/\rho)(\bar{u}^{2n+1}/A^{n+1}) \Big/ (\beta g \theta_w) = C_1 \qquad (14)$$

$$(\bar{u}^2/A)\left(\frac{dA}{dx}\right) \Big/ (\beta g \theta_w) = C_2 \qquad (15)$$

$$\left(\bar{u}\frac{d\bar{u}}{dx}\right) \Big/ (\beta g \theta_w) = C_3 \qquad (16)$$

where C_1, C_2 and C_3 are constants.

Now the values of two of the constants can be arbitrarily chosen. Therefore, choosing $C_1 = 1$, equation (14) gives:

$$\bar{u} = (\beta g \rho \theta_w / Kn)^{1/(2n+1)}\ A^{(n+1)/(2n+1)} \qquad (17)$$

Substituting this into equation (15) and the resultant solution for A back into equation (17), and choosing, for convenience:

$$C_2 = (2n+1) \qquad (18)$$

then gives:

$$A = (\beta g \theta_w)^{(2n-1)/(2n+2)}\ (2n+2)^{(2n+1)/(2n+2)}\ (Kn/\rho)^{1/(n+1)} \quad \ldots$$

$$\ldots x^{(2n+1)/(2n+2)} \tag{19}$$

and:

$$\bar{u} = \left[(\beta g \theta_w)(2n+2)\ x\right]^{1/2} \tag{20}$$

Substituting these into equation (16) then gives:

$$C_3 = (n+1) \tag{21}$$

Now substituting equations (18) and (21) into equation (12) gives the final form of the momentum equation as:

$$(F'')^{n-1}F''' + (2n+1)FF'' - (n+1)F'^2 + G = 0 \tag{22}$$

Substituting the forms derived above for A and $\bar{u}$, i.e. equations (19) and (20), into the energy equation (13) and rearranging gives:

$$G'' + \left[(\beta g \theta_w)^{(3n-3)/(2n+2)} (\rho\ Cp/k)(Kn/\rho)^{2/(n+1)} (2n+2)^{(n-1)/(n+1)} (2n+1)\ x^{(n-1)/(n+1)}\right] G'F = 0 \tag{23}$$

It is seen, therefore, that, since constant fluid property flow is being considered, it is only when n = 1 that the coefficient in this equation becomes independent of x. Therefore, similar solutions, in the form assumed above, exist for free convective flat plate flow only for Newtonian fluids.

If, however, instead of assuming that the velocity and temperature profiles are functions of the same variable, η_s, it is assumed that, instead of equation (7), the temperature distribution is given by:

$$\theta = \theta_w \, G(\eta_{ST}) \qquad (24)$$

where, η_{ST}, is a new temperature profile variable which, it will be assumed, is related to η_s by:

$$\eta_{ST} = B(x)\,\eta_s \qquad (25)$$

Now the assumption of equation (24) in place of equation (7) does not effect any of the argument presented above as to the form of the momentum equation, nor does it alter the derived forms for η_s, $\bar{u}$ and A. Instead of equation (23), however, the energy equation now reduces to:

$$G'' + B\left[(\beta g\,\theta_w)^{(3n-3)/(2n+2)}(\rho C_p/k)(Kn/\rho)^{2/(n+1)}(2n+2)^{(n-1)/(n+1)}(2n+1)x^{(n-1)/(n+1)}\right] G'F = 0 \qquad (26)$$

the dashes in this equation referring to differentiation with respect to η_{ST}.

If, now, B is chosen such that:

$$B = \left[(\beta g\,\theta_w)^{(3n-3)/(2n+2)}(\rho C_p/k)(Kn/\rho)^{2/(n+1)}(2n+2)^{(n-1)/(n+1)}(2n+1)x^{(n-1)/(n+1)}\right]^{-1} \qquad (27)$$

equation (26) reduces to:

$$G'' + G'F = 0 \tag{28}$$

For any given value of B, equations (22) and (28) can be solved simultaneously in conjunction with equation (25) to determine the forms of F and G which are subject to the boundary conditions:

$\eta_s = \eta_{ST} = 0$:

$F = 0 \quad F' = 0 \; ; \quad G = 1$

$\eta_s = \infty$

$F' = 0$

$\eta_{ST} = \infty$

$G = 0$

Of course these forms for F and G will vary with the value of B, but the above procedure has reduced the initial partial differential equations to a pair of, more easily solved, total differential equations.

Once the form of G is known, the heat transfer rate from the wall can be found by differentiating and using equations (11), (19), (20) and (25).

Numerical results derived by using the above procedure will not be given in the present note. Instead consideration will now be given to an approximate, but somewhat simpler solution obtained by using the van Karman integral equation method for solving boundary layer flow problems. This method, as will be seen, relies in some respects on similar basic assumptions to those introduced above.

Integral Equation Solution:

In the integral method, velocity and thermal boundary layers of definite thickness are assumed and the boundary layer equations are then integrated across these boundary layers to give:

$$\frac{d}{dx}\int_{o}^{\delta_u} u^2\, dy = -(\tau_w/\rho) + \beta g \int_{o}^{\delta_T} dy \qquad (29)$$

and

$$\frac{d}{dx}\int_{o}^{\delta} u\,\theta\, dy = -(k/\rho C_p)\left(\frac{\partial\theta}{\partial y}\right)_w \qquad (30)$$

where δ_u = velocity boundary layer thickness,

δ_T = temperature boundary layer thickness.

and where subscript w indicates conditions at the wall. The upper limit in the second equation is, due to the forms assumed for the boundary layer profiles, dependent on which of δ_u or δ_T is the larger.

In order to solve these equations for δ_u and δ_T, relations for the form of variation of u and θ through the boundary layer, which satisfy the correct boundary conditions, are assumed. In the present note, fourth degree polynomials will be assumed for the variation of both u and θ i.e. it will be assumed that:

$$u = A + B\eta + C\eta^2 + D\eta^3 + E\eta^4 \tag{31}$$

and

$$\theta = F + G\epsilon + H\epsilon^2 + I\epsilon^3 + J\epsilon^4 \tag{32}$$

where

$$\eta = y/\delta_u \tag{33}$$

and:

$$\epsilon = y/\delta_T \tag{34}$$

and A, B, ..., I, J are constants to be determined by applying the boundary conditions.

It will be noted that the use of equations (31) and (32) for the velocity and temperature profiles involves the same assumption as introduced in the previous section, i.e. that the two profiles are both similar but that, since the variation of δ_u and δ_T with x is not, necessarily, the same, the similarity variable is different for velocity and temperature.

Now, since boundary layers of definite thickness and zero wall slip are being assumed, the above profiles must be such that:

At $y = 0$ i.e. $\eta = \epsilon = 0$:

$$u = 0 \tag{35}$$

$$\theta = \theta_w \tag{36}$$

At $y = \delta_u$ i.e. $\eta = 1$:

$$u = 0 \tag{37}$$

$$\frac{du}{d\eta} = 0 \tag{38}$$

$$\frac{d^2u}{d\eta^2} = 0 \tag{39}$$

At $y = \delta_T$ i.e. $\epsilon = 1$:

$$\theta = 0 \tag{40}$$

$$\frac{d\theta}{d\epsilon} = 0 \tag{41}$$

$$\frac{d^2\theta}{d\epsilon^2} = 0 \tag{42}$$

Also, applying the basic boundary layer equations (3) and (4) to conditions at the wall give:

At $y = 0$:

$$\beta g \theta_w = -(1/\rho)\left(\frac{\partial \tau}{\partial y}\right)_w \tag{43}$$

$$\frac{d^2\theta}{d\epsilon^2} = 0 \tag{44}$$

Applying the above set of boundary conditions to equations (31) and (32) gives the velocity and temperature distributions as:

$$u = (\beta g \rho \theta_w / 6nK)^{1/n} \delta_u^{(n+1)/n} (\eta - 3\eta^2 + 3\eta^3 - \eta^4) \tag{45}$$

and:

$$\theta = \theta_w (1 - 2\epsilon + 2\epsilon^3 - \epsilon^4) \tag{46}$$

Substituting these equations into equations (29) and (30), rearranging and introducing, for convenience:

$$\Delta = \delta_T / \delta_u \tag{47}$$

gives:

$$\delta_u^{(n+2)/n} \frac{d\delta_u}{dx} = (\beta g \rho \theta_w / 6nK)^{(n-2)/n} \left[n/(3n+2) \right] \left[6nK\Delta L/\rho M - K/\rho M \right] \tag{48}$$

and:

$$X\left[(2n+1)/n\right]\delta_u^{(2n+1)/n}\frac{d\delta_u}{dx} + \delta_u^{(3n+1)/n} X' \frac{d\Delta}{dx}$$

$$= (6nK/\beta g\rho\theta_w)^{1/n}\,(2k/\Delta\rho C_p) \qquad (49)$$

where:

$$L = \int_o^1 (1-2\epsilon + 2\epsilon^3 - \epsilon^4)\, d\epsilon = 0.3 \qquad (50)$$

and:

$$M = \int_o^1 (\eta-3\eta^2+3\eta^3-\eta^4)^2\, d\eta = 0.003968 \qquad (51)$$

and:

For $\Delta < 1$:

$$X = \Delta^2/15 - \Delta^3/14 + 9\Delta^4/280 - \Delta^5/180 \qquad (52)$$

$$X' = \frac{dx}{d\Delta} = 2\Delta/15 - 3\Delta^2/14 + 9\Delta^3/70 - \Delta^4/36 \qquad (53)$$

For $\Delta > 1$:

$$X = 1/20 - 1/30\Delta + 1/140\Delta^3 - 1/504\Delta^4 \qquad (54)$$

$$X' = \frac{dx}{d\Delta} = 1/30\Delta^2 - 3/140\Delta^4 + 4/504\Delta^5 \qquad (55)$$

Equations (48) and (49) constitute a solution for the variation of δ_{μ} and δ_T (or Δ) with x. When n is equal to 1, this solution is particularly simple since in this case Δ is independent of x. With these determined, the local rate of heat transfer per unit area from the wall to the fluid, q_w, can be determined by noting that:

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_w = (2k\,\theta_w/\delta_T) \tag{56}$$

and, hence, the local Nusselt number N_{ux}, defined by:

$$Nux = (q_w\, x/\theta_w\, k) \tag{57}$$

can be also determined.

Nux, obtained in this way, will, as can be shown by dimensional reasoning, be dependent on the following two dimensionless parameters:

$$G_x = \left[\beta g\,\theta_w \rho^{2/(2-n)} x^{(2+n)/(2-n)} \big/ K^{2/(2-n)}\right] \tag{58}$$

and:

$$P_x = \left[K^{1/(2-n)} \rho^{(1-n)/(2-n)} c_p x^{(2-2n)/(2-n)}/k\right] \tag{59}$$

which, when n is equal to 1, reduce to the familiar Grashof number and Prandtl number, which, in this case, is a constant for constant fluid property flow.

No general solutions to equations (48) and (49) will be presented here. Instead, a simple analytical solution, which is only applicable under certain special circumstances but which serves to illustrate the general form to be expected of the results, is presented in the next section.

Solution for Large Values of Δ :

A simple approximate solution to the derived integral equations, presented above, can be obtained if Δ is large. Such a solution will, of course, not be generally applicable but should, as mentioned above, serve to illustrate the main features of the general solution and it will, therefore, be presented here.

If Δ is large, it follows from equations (54) and (55) that X is approximately equal to 1/20 and X' approximately equal to zero. Using these values and then eliminating $(d\delta_u/dx)$ between equations (48) and (49) gives, when the contribution of the friction to the rate of momentum change is neglected in comparison with the contribution of the nett buoyancy force:

$$\Delta = C_1 \left(G_x \delta_u / x\right)^{(1-n)/2n} \Big/ P_x^{\;1/2} \tag{60}$$

where:

$$C_1 = \left[40M(3n+2)/(6n)^{1/n}(2n+1)\; L \right]^{1/2} \tag{61}$$

Substituting this back into either of (48) or (49) and integrating, on the assumption that $\delta_u = 0$ when $x = 0$, then gives:

$$(\delta_u/x) = C_2\, G_x^{(n-3)/(5n+3)}\, P_x^{n/(5n+3)} \tag{62}$$

where:

$$C_2 = \left[20(5n+3)(6n)^{1/n}/C_1(2n+1)\right]^{2n/(5n+3)} \tag{63}$$

Substituting this back into equation (60) gives:

$$\Delta = C_3\, G_x^{(3-3n)/(5n+3)} / P_x^{(2n+2)/(5n+3)} \tag{64}$$

where

$$C_3 = C_1\, C_2^{(1-n)/2n} \tag{65}$$

which, for a given fluid, since constant fluid properties are being assumed, can be written as:

$$\Delta = \text{constant}.x^{(1-n)/(5n+3)} \tag{66}$$

which shows that for a given fluid, if n is less than 1, the present solution will be associated with large values of x while if n is greater than 1 it will be associated with small values of x.

Substituting equations (62) and (64) into equation

(57) and using equation (56) gives:

$$N_{ux} = C_4 \, G_x^{\,2n/(5n+3)} \, P_x^{\,(3n+2)/(5n+3)} \tag{67}$$

where:

$$C_4 = 2/C_2C_3 = 2/C_1C_2^{\,(1+n)/2n} \tag{68}$$

Some typical results for various values of the governing parameters, obtained by using equation (67), are shown in fig.1.

In order to obtain some idea of the validity of the above approximate solution, the results obtained for the case of n = 1 can be compared with the known exact results for this case. When n = 1, equation (64) gives:

$$\Delta = C_3/P_r^{\,1/2} \tag{69}$$

where P_r is the Prandtl number of the fluid i.e. $(\mu C_p/k)$. Equation (69) shows, as is, in fact, obvious, that large values of Δ are associated with small values of P_r when n = 1.

Similarly, when n = 1, equation (67) gives:

$$(N_{ux}/G_{rx}^{\,1/4}) = C_4 \, P_r^{\,5/8} \tag{70}$$

where G_{rx} is the Grashof number i.e. $(\beta g \rho^2 \theta_w x^3/\mu^2)$.

The variation of $(N_{ux}/G_{rx}^{1/4})$ with P_r as given by equation (70) is compared in fig. 2 with the exact solution for small values of P_r due to Sparrow and Gregg.[3.] The agreement is seen to be reasonably good.

<u>Conclusion</u>:

When n is different from unity the governing equations increase considerably in complexity. Two methods of solving these equations, the first based directly on the governing partial differential equations and the other using the integral equation approach, have been indicated.

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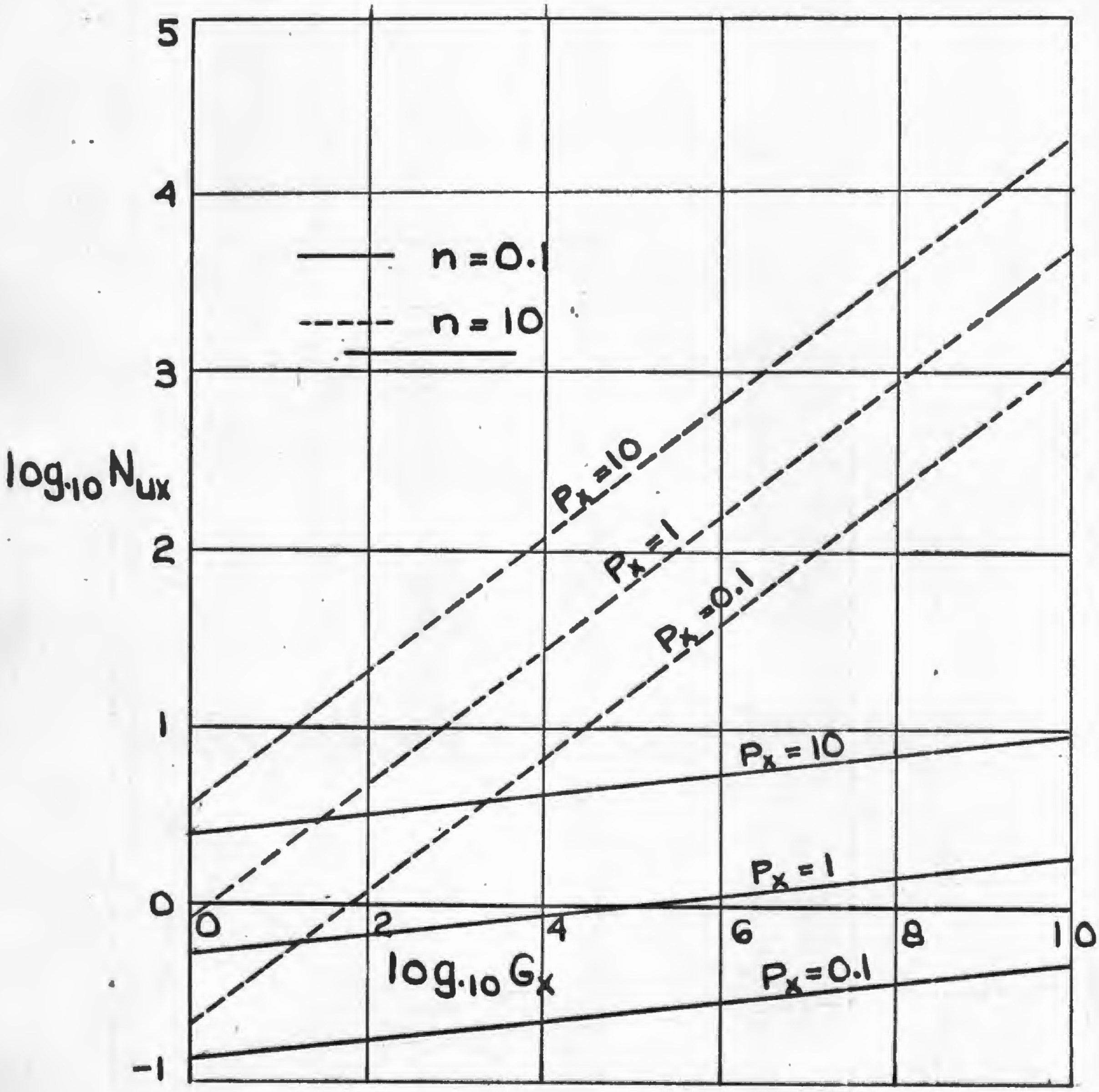

Fig. 1. Typical results given by approximate solution for large Δ .

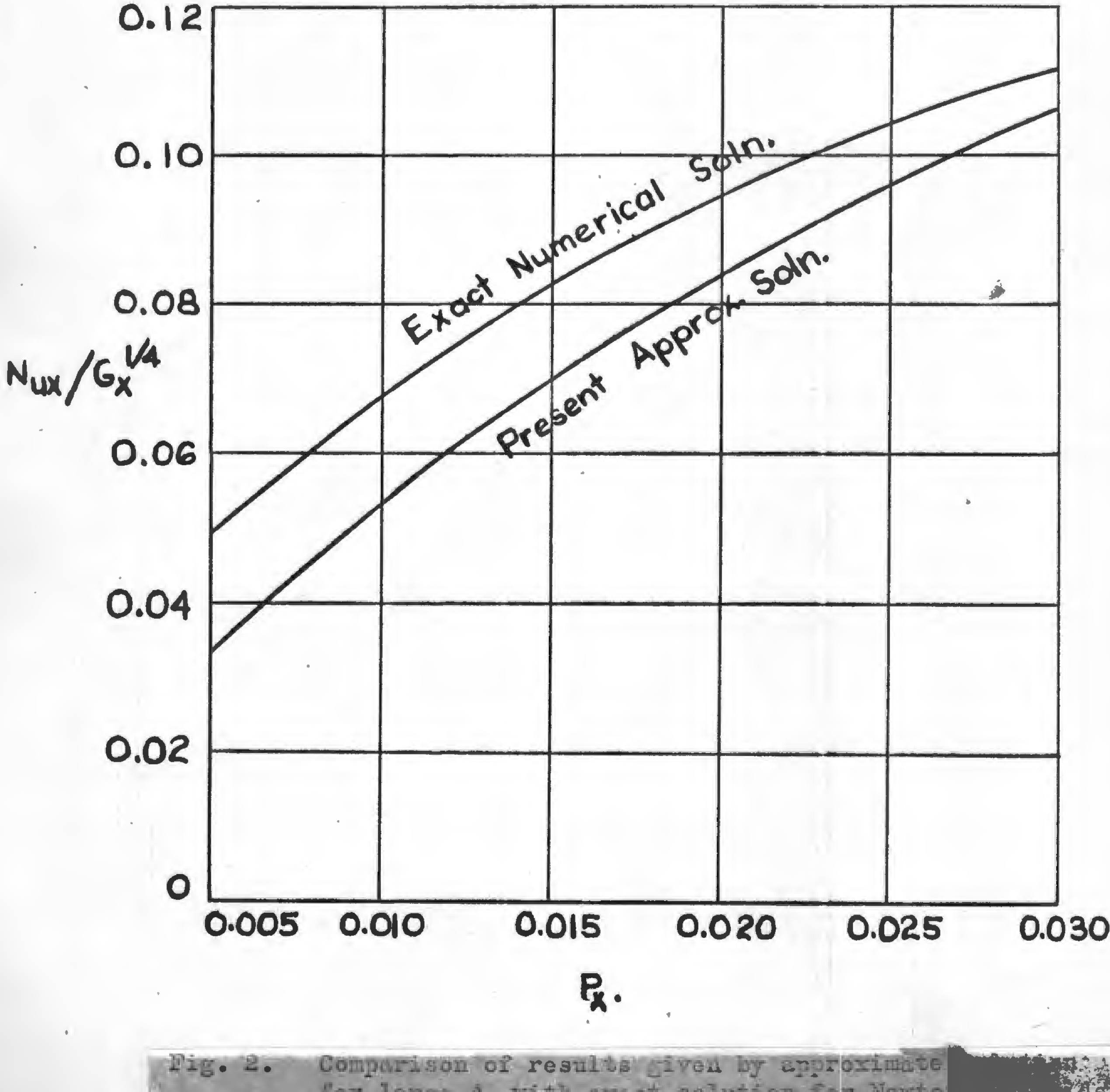

Fig. 2. Comparison of results given by approximate [redacted] for large Δ with exact solution for Newto[redacted]ds.

Paper 4.

AN EXPERIMENTAL ANALYSIS OF THE HEAT TRANSFER BY LAMINAR FREE CONVECTION FROM A NARROW VERTICAL PLATE.

An experimental analysis of the heat transfer by laminar free convection from a narrow vertical plate

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Most experimental investigations of the rate of heat loss by free convection from vertical plates have been made with very wide plates, the flow over which corresponds very nearly to the idealized two-dimensional flow which has been the subject of many theoretical analyses. In the present case, measurements of the local rates of heat loss by laminar free convection from a narrow isothermal vertical plate have been made in order to determine the effect of the plate edges. The results indicate that if N_{ux_2} is the local Nusselt number predicted for two-dimensional flow under the same physical conditions as in the actual flow in which the local Nusselt number is N_{ux}, then $N_{ux}/N_{ux_2} = f(G_{rw})$ where G_{rw} is the Grashof number based on the width of the plate. The measurements also indicate that the Grashof number at transition is also a function of G_{rw}.

INTRODUCTION

If it is assumed that the variation of fluid properties in the boundary layer is negligible and that the flow is two-dimensional (i.e., that the width has no effect on the flow) then dimensional reasoning shows that the heat transfer by free convection at a distance x from the leading edge of a vertical plate is given by:—

$$N_{ux} = f\,(G_{rx}, P_r) \qquad (1)$$

where:—

N_{ux} = Local Nusselt Number = hx/k,
G_{rx} = Local Grashof Number, = $\beta\, g\, \rho^2\, \Delta T\, x^3/\mu^2$,
P_r = Prandtl Number, = $\mu\, Cp/k$,
h = film coefficient, = $q/\Delta T$,
q = local rate of heat transfer per unit area,
ΔT = difference between surface temperature of plate and the temperature outside the boundary layer.
k = coefficient of conductivity of the fluid,
β = coefficient of cubical expansion of the fluid,
ρ = density of the fluid,
μ = coefficient of viscosity of fluid,
Cp = specific heat of fluid.

For the case of laminar boundary layer flow a number of theoretical analyses have predicted the following form for the function f of equation (1):—

$$N_{ux}/G_{rx}^{0.25} = g(P_r) \qquad (2)$$

Since the function g is dependent on the Prandtl number of the fluid alone, it is effectively constant for air and for air, therefore equation (2) reduces to:—

$$N_{ux}/G_{rx}^{0.25} = K \qquad (3)$$

where K is a constant. Values of K derived by various investigators are shown in Table 1.

TABLE 1

Investigator	Reference	K
Pohlhausen	3	0·366
Squire	7	0·397
Saunders	7	0·374
Sparrow and Gregg	1	0·353

Experimental studies of the heat transfer from vertical plates tend to support the form of equation (3) at high Grashof numbers but diverge considerably from it at low values of G_{rx}. A fair amount of variation between the results of various investigators is also to be noted.

In comparing the theoretical results with experiment it should be borne in mind that the following assumptions made in the theoretical analysis are not exactly satisfied in the experimental apparatus:—

(i) the fluid properties are constant in the boundary layer
(ii) the boundary layer has zero thickness at the leading edge of the plate
(iii) the flow in the boundary layer is two-dimensional

However, the analysis of Sparrow and Gregg[1] taking the variation of fluid properties with temperature into account, suggests that, provided the fluid properties are evaluated at the mean film temperature, assumption (i), mentioned above, is sufficiently accurate.

Similarly, Scherberg's[2] analysis and Schmidt's[3] shadowgraph photographs suggest that, except in the immediate vicinity of the leading edge, assumption (ii) will have little effect on the results.

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The purpose of the present analysis was to try to determine the significance of assumption (iii). In order to do this the heat transfer from a plate which was narrow in comparison with its overall length was measured. In all previous experimental analyses with which the author is familiar, very wide plates were used in order to try to satisfy the assumption of two-di-mensional flow. The results of the present analysis should then indicate whether the discrepancies between the previous experimental analyses and the theoretical predictions are the result of edge effects.

Now if the width, w, of the plate has an effect on the heat transfer, dimensional reasoning suggests the relation:—

$$N_{ux} = F(G_{rx}, P_r, w/x) \quad \dots\dots\dots\dots (4)$$

It is convenient to write this equation in the following form for comparison with equation (2):—

$$N_{ux}/G_{rx}^{0\cdot25} = G(G_{rx}, P_r, G_{rw}/G_{rx}) \quad \dots\dots\dots (5)$$

where:—

G_{rw} = Grashof number based on width of plate $= \beta\, g\, \rho^2\, \Delta\, T w^3/\mu^2$

For air, on the assumption that P_r is constant, equation (5) reduces to:—

$$N_{ux}/G_{rx}^{0\cdot25} = H(G_{rx}, G_{rw}/G_{rx}) \quad \dots\dots\dots\dots (6)$$

The purpose of the present analysis was to try to find the form of the function H.

EXPERIMENTAL APPARATUS

The construction of the plate used is shown in Fig 1. It consisted of an insulated board around which was wrapped a ribbon-type nicrome electrical heating element. This board was sandwiched between two 8-gauge aluminium plates, the aluminium being insulated from the electrical element by thin mica sheets. The aluminium plates were 32-in long and 8-in wide. A nosepiece ⅜-in long was fitted. The exposed faces of the aluminium plates were highly polished. High-temperature plaster was used to seal the sides and top of the plate to prevent any heat loss due to air-currents between the windings and the plates.

In order to allow some control over the plate temperature, the heating element was divided into 10 approximately equal sections and a suitable rheostat connected in parallel with each of these sections. A 10-way switch was used to allow the voltage drop across each section in turn to be measured. Using the measured value of the resistance of each section, the heat produced could then be calculated.

The electrical supply to the plate was from a constant voltage transformer, the average plate temperature being adjusted by means of a variable transformer. The circuit is shown in Fig 2.

Ten thermocouples were embedded along the centre-line of each plate at the centre of each of the 10 heater-element sections. These thermocouples could be connected, in turn, by means of a 20-way switch, to a workshop grade potentiometer which was used for all temperature measurements.

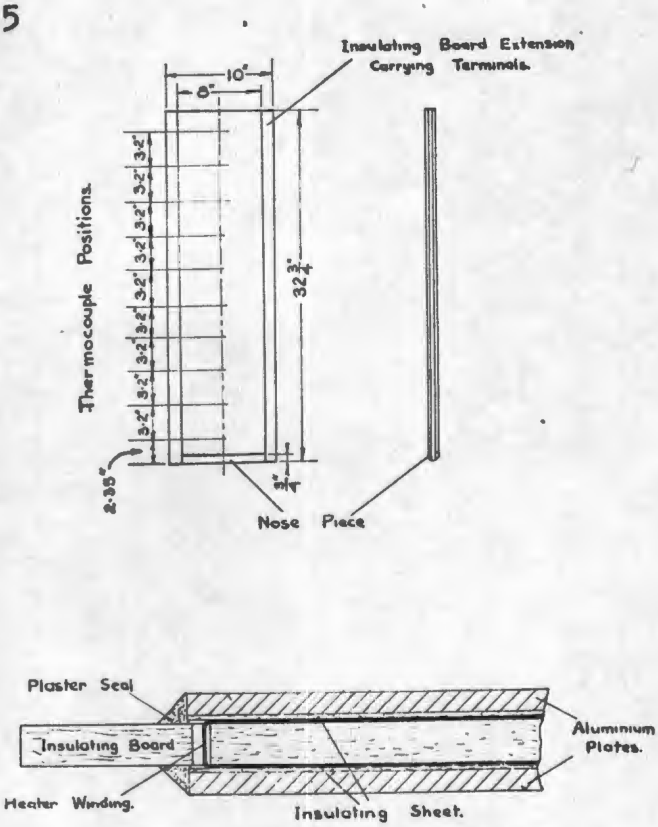

Fig 1 Constructional details of heated plate

EXPERIMENTAL PROCEDURE

The variable transformer was set to give an average temperature of the order required. Using a trial and error procedure, the 10 rheostats were then adjusted until the variation in temperature along the plate was less than about five per cent of the average temperature.

When steady conditions had been reached, which usually took about three hours, the actual temperatures at the 20 points and the voltage drops across the 10 sections were measured. Tests were conducted at mean temperature differences ranging from approximately 1°F to 170°F.

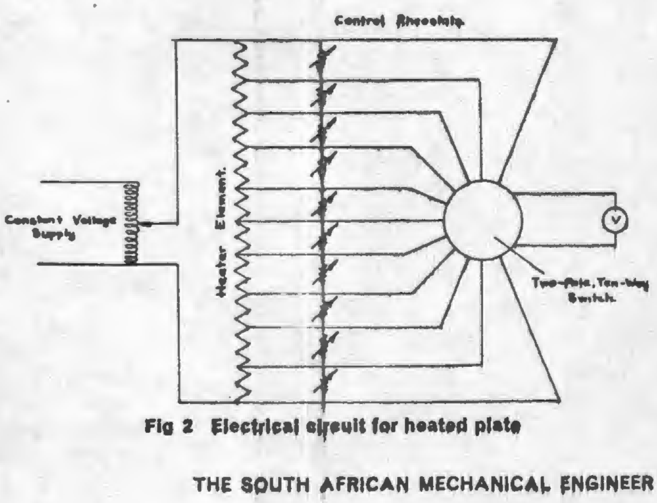

Fig 2 Electrical circuit for heated plate

DERIVATION OF RESULTS

The heat transfer from each section was calculated from the measured voltage drop and known resistance, assuming equal heat transfer from both sides of the plate. The average film coefficient for each section was then determined, using the average of the temperature on the two sides of the plates. A correction for radiation was applied to all the results, it being assumed that the emissivity of the polished aluminium surfaces was 0·05. A small correction for the effect of the unheated nose-piece was also applied to the result for the lowest section.

All required fluid properties were then evaluated at a mean film temperature based on the average of all 20 plate temperatures and the room temperatures.

Values of N_{ux} and G_{rx} could then be determined, x being taken as the distance of the thermocouple at the centre of the section considered from the leading edge.

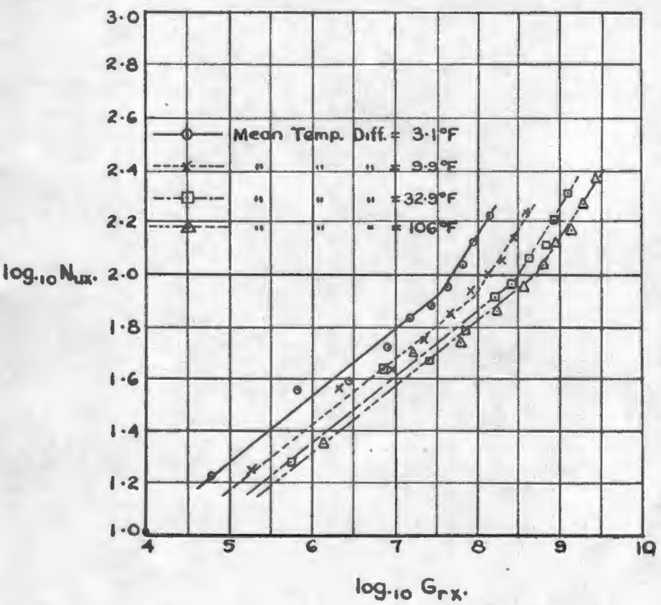

Fig 3 Variation of local Nusselt number with local Grashof number for various temperature differences

RESULTS

Typical variations of N_{ux} with G_{rx} for various average temperature differences are shown in Fig 3. Two points will be noted:—

(i) Transition from laminar to turbulent flow occurs in all cases at the value of G_x at which the transition effectively occurs, G_{xT}, being dependent on the temperature difference. This point will be discussed later.

(ii) Due, presumably, to the effect of the finite plate width, the variation of N_{ux} with G_{rx} is different for each temperature difference. It will be noted, however, that it is possible in all cases for the laminar region to express the variation, to a fair degree of accuracy, in the form:—

$$N_{ux}/G_{rx}^{0 \cdot 25} = K' \quad \ldots\ldots (7)$$

K' being dependent on the temperature difference.

Comparison of equations (6) and (7) suggests that the function H of equation (6) is independent of G_{rx} and that the heat transfer by free convection from plates of finite width can, therefore be expressed in the form:—

$$N_{ux}/G_{rx}^{0 \cdot 25} = I\ (G_{rw}) \quad \ldots\ldots (8)$$

this being for the case of air, I, in general, being dependent also on P_r.

Values of I derived from the experimental results by drawing the best straight line of slope 0·25 through the laminar flow values are shown in Fig 4.

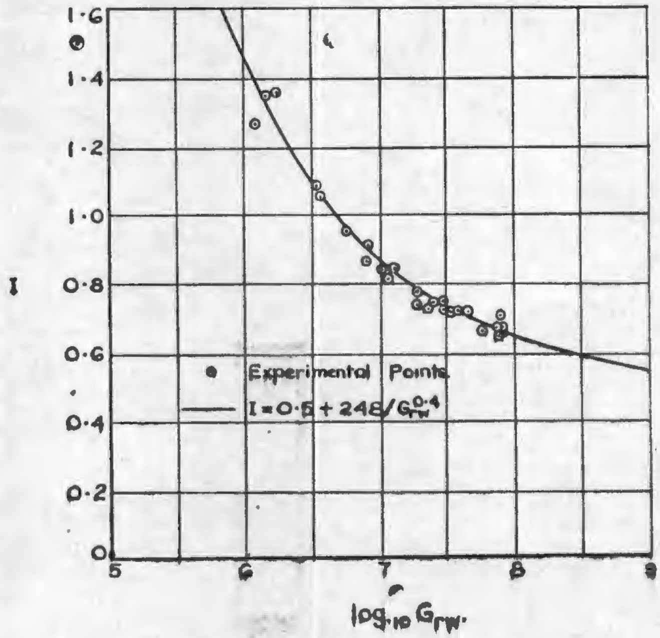

Fig 4 Variation of function I with Grashof number, based on plate width

It will be noted that the function I is approximately given by:—

$$I = C + A/G_{rw}^{B} \quad \ldots\ldots (9)$$

where A, B and C are constants.

The best values of the constants as derived from the experimental points are,

$A = 248$; $B = 0{\cdot}40$; $C = 0{\cdot}50$,

giving, therefore, the heat transfer by free convection from plates of finite width as:—

$$N_x/G_{rx}^{0 \cdot 25} = 0{\cdot}50 + 248/G_{rw}^{0 \cdot 40} \quad \ldots\ldots (10)$$

For two-dimensional flow, for which G_{rw} is effectively infinite, this reduces to:

$$N_x/G_{rx}^{0 \cdot 25} = 0{\cdot}50 \quad \ldots\ldots (11)$$

which may be compared with the theoretical values given earlier. Due to the extent of the exterpolation no great importance can, however, be attached to this result.

In most cases the average heat from a plate is of more importance than the local values. Defining, as usual, the mean Nusselt number as:—

$$N_{ul} = h_{ml}/k \quad \ldots \ldots \ldots \ldots (12)$$

where l is the length of the plate and the mean film coefficient h_m is given by:—

$$h_m = \frac{1}{l}\int_0^l h\,d\,x \quad \ldots \ldots \ldots \ldots (13)$$

equation (9) gives for plates of finite width:—

$$N_{ul}/G_{rl}^{0.25} = C' + A'/G_r^B \quad \ldots \ldots \ldots \ldots (14)$$

where $C' = 4C/3$; $A' = 4A/3$.

Using the values of A, B and C derived from the experimental results, this reduces to:—

$$N_{ul}/G_{rl}^{0.25} = 0{\cdot}67 + 330/G_{rw}^{0.40} \quad \ldots \ldots \ldots \ldots (15)$$

where G_{rl} is the Grashof number based on l, i.e. $\beta\, g\, \rho^2\, \Delta\, Tl^3/\mu^2$.

The variation of N_{ul} with G_{rl} for plates of various length to width (l/w) ratios is shown in Fig 5.

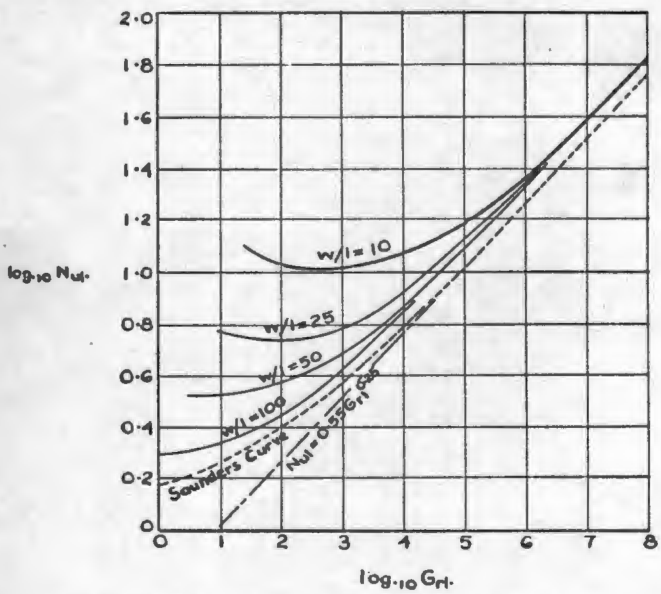

Fig 5 Predicted variation of average Nusselt number with average Grashof number for various length-to-width ratios

COMPARISON WITH PREVIOUS EXPERIMENT

As mentioned before, previous experimental studies of the heat loss by free convection from plates have all used very wide plates in order to minimize the edge effect, although the present results suggest that this effect can never entirely be eliminated at low values of G_{rx}. Typical experimental results are those of Saunders[4] using plates having (l/w) ratios ranging approximately from 0·14 to 0·33. The mean curve through his results is shown in Fig 5, together with the predicted curves. It will be noted that his curve does display the basic predicted characteristic diverging considerably from the '$Nul/G_{rl}^{0.25}$ =constant law at low values of G_rl. However, his results, like those of King[5], do not exhibit the large variation with (l/w) as predicted by the present results. As both investigators used very thin plates, a direct comparison with the present results is not possible, however, as the form of the edge of the plate must be expected to exert a considerable influence on the edge effect. It should be noted, however, that the results of Griffiths and Davies[4] and those of Weise[6] are considerably higher than those of Saunders, suggesting that they have been influenced by edge effects. Direct comparison of the present results with measurements of the heat loss from vertical cylinders, the flow about which should approximate to that over a vertical plate of infinite width, is not possible, owing to the considerable influence of the differing leading edge conditions.

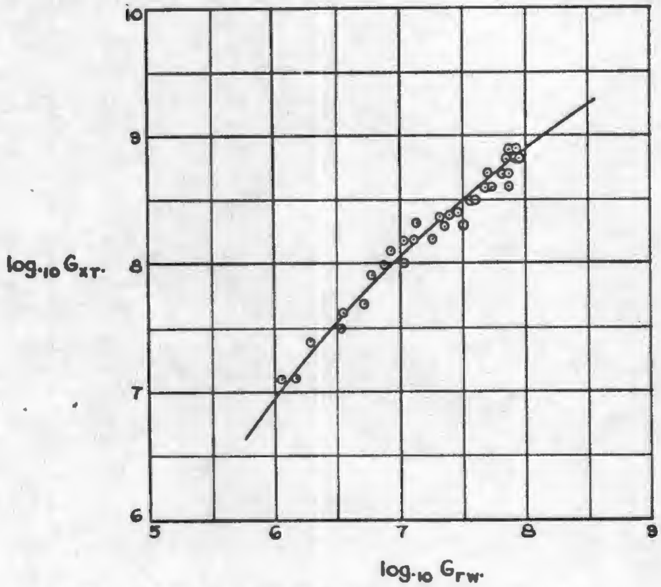

Fig 6 Variation of local Grashof number at transition with Grashof number based on plate width

VARIATION IN TRANSITION POINT

The variation of the effective transition point G_{xT}, as obtained from the experimental results with G_{rw}, is shown in Fig 6. The effective transition point was taken as the point of intersection of a straight line of slope 0·25 through the laminar flow points with a straight line drawn through the turbulent flow point on a plot of the experimental results in the form log N_{ux} against log G_{rx}.

It will be noted that G_{xT} appears to decrease with decreasing G_{rw}. However, as changes in G_{rw} were obtained by varying the mean plate temperature, and since it is to be expected that stray atmospheric turbulence will have more effect on the boundary layer characteristics, and hence transition point, at low temperatures than at high temperatures, no great accuracy can be expected of these results.

PLATE DIMENSIONS FOR MINIMUM HEAT

It will be noted that for a plate of given area, held at a fixed surface temperature, the two-dimensional theory predicts that the heat loss from the plate will be a minimum when the plate is long and narrow. However, the present analysis suggests that the edge effects will increase the heat transfer if the plate is too narrow. It is to be expected, therefore, that there is a definite length-to-width ratio for which the heat transfer will be a minimum.

For a plate of area a and length l, the width is given by:—

$$w = a/l \quad \ldots\ldots\ldots\ldots\ldots (16)$$

Substituting this into equation (14) gives the mean film coefficient h_m in terms of the constants A', B and C' of equation (9) as:—

$$h_m = G^{0.25}k\,[C'/l^{0.25} + A'l^{3B-0.25}/G^{B}a^{3B}] \quad .(17)$$

where $G = \beta\, g\, \rho^2\, \Delta T/\mu^2$

This is a minimum, i.e. the overall heat transfer is a minimum, when l is such that:—

$$l/\sqrt{a} = [0{\cdot}25\, C'/A'(3B - 0{\cdot}25)]^{\frac{1}{3B}}(Ga^{\frac{3}{2}})^{\frac{1}{3}} \quad \ldots .(18)$$

which can be re-arranged to give:—

$$(l/w)_{min} = [0{\cdot}25\, C'/A'(3B - 0{\cdot}25)]^{\frac{2}{3B}}$$

$$[\beta\, g\, \rho^2\, \Delta\, Ta^{\frac{3}{2}}/\mu^2]^{\frac{2}{3}} \quad \ldots\ldots\ldots .(19)$$

(l/w) min being the length-to-width ratio for minimum heat transfer.

Substituting the experimentally determined values of A', B and C' gives, finally:—

$$(l/w)_{min} = 5{\cdot}7 \times 10^{-6}[\beta\, g\, \rho^2\, \Delta Ta^{\frac{3}{2}}/\mu^2]^{\frac{2}{3}} \quad \ldots .(20)$$

The variation of $(l/w)_{min}$ with $(\beta\ g\ \rho^2\ \Delta a^{\frac{3}{2}}/\mu^2)$, as given by the equation, is shown in Fig 7.

FURTHER EXPERIMENTAL WORK

To try to provide further confirmation of the results obtained in the experiments described above, some measurements of the heat loss from the plate, described above, were made with the plate mounted with its short (8-in) side vertical. In this position the length-to-width ratio of the plate was approximately four.

With the plate mounted in this position, it was realized that the side and leading edge conditions differed considerably from those which existed when the plate was mounted with its long side vertical. It was hoped, however, that some correlation between the two sets of results would exist.

Only mean heat transfer values could, of course, be measured with the plate mounted in this position.

The variation of N_ul with G_rl, as obtained experimentally with the plate mounted in this position, is shown in Fig 8. The variation of N_ul with G_rl for a plate having an (l/w) ratio of four, as predicted by the previous experimental results, is also shown in this figure, equation (15) having been used to derive this curve. It will be seen that the agreement is reasonably good, particularly when the differences in edge conditions mentioned above are borne in mind. The mean curve of Saunders[4], through his results, is also given in this figure for comparison.

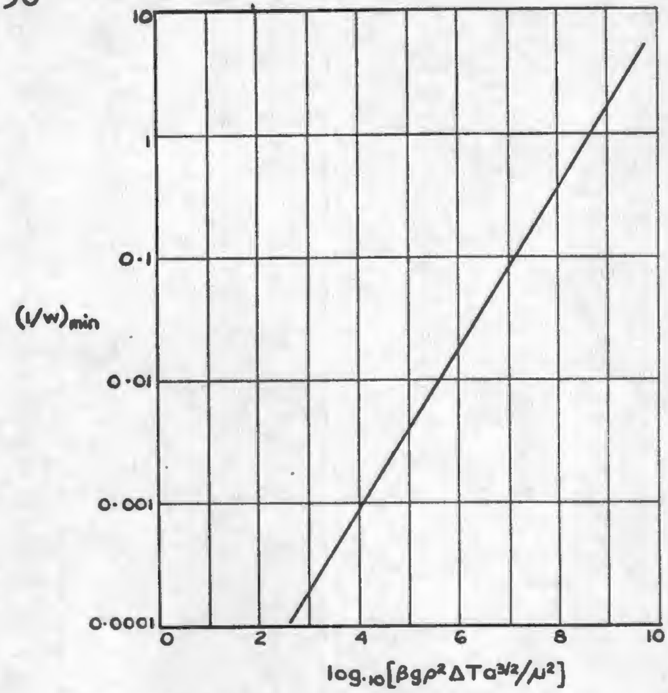

Fig 7 Predicted length-to-width ratios for minimum heat transfer

CONCLUSIONS

A series of experimental measurements of the heat loss by laminar free convection from an isothermal vertical plate, which was narrow in comparison to its overall

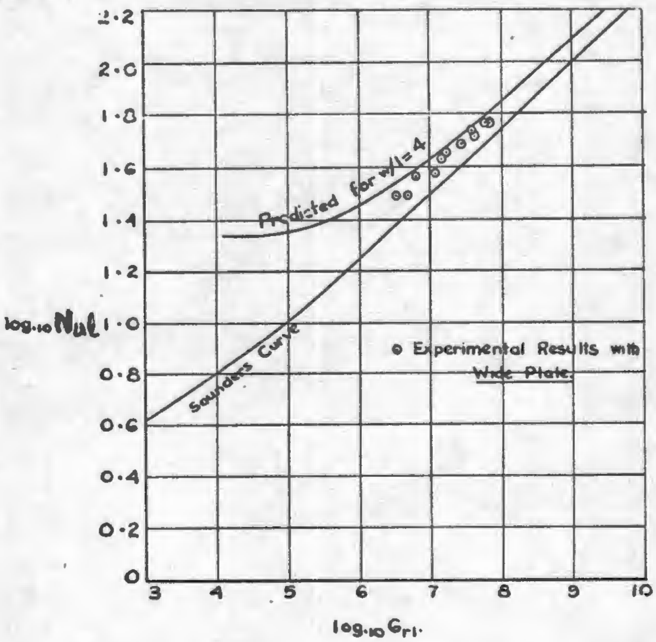

Fig 8 Comparison of results of measurements with wide plate with results derived from measurements with narrow plate

length, have been performed. It has been shown that the effect of the finite plate width is to increase, quite considerably, the rate of heat transfer above that which would exist under the same conditions if the flow were two-dimensional. It has been shown that the ratio of the heat transfer from a plate of finite width to a plate of infinite width can be expressed, approximately, in terms of the Grashof number based on the plate width alone.

The measurements also indicate that the Grashof number at transition is a function of the Grashof number based on the width of the plate.

ACKNOWLEDGEMENTS

The author would like to express his appreciation to Mr R. M. Stegen for his guidance in the design of the apparatus, to Mr J. Gordon for constructing the apparatus and to Mr P. B. Snare, who carried out some of the preliminary work as part of his undergraduate thesis.

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Paper 5.

A FURTHER EXPERIMENTAL STUDY OF LAMINAR FREE CONVECTIVE HEAT TRANSFER FROM NARROW VERTICAL PLATES IN AIR

Summary:

Measurements of the average rates of heat transfer by laminar free convection from two vertical plates, identical in all respects except their width, are described. These measurements support an earlier conclusion as to the fairly considerable effect of plate width on the rate of heat transfer per unit area under such circumstances.

A FURTHER EXPERIMENTAL STUDY OF THE LAMINAR FREE CONVECTIVE HEAT TRANSFER FROM NARROW VERTICAL PLATES IN AIR.

Introduction:

An earlier note[1.] described an experimental study in which the local rates of heat transfer by laminar free convection from a vertical plate, which was comparatively narrow, were measured. This study indicated that the effect of the finite width of the plate, i.e. the edge effect, was comparatively large and larger than that which arises in forced flow. While these measurements were quite consistent within themselves, the observed effect of finite plate width was so large that some doubt as to their accuracy must exist. The present note, therefore, describes a related experimental study which was carried out in an effort to confirm the previously mentioned results.

The experimental work described in the present note is of a simpler nature than that of ref.1 in that only average heat transfer rates were measured, whereas in ref.1 the apparatus allowed the approximate determination of local heat transfer rates. However, in the present case, two plates, having different widths but identical in all other respects, were tested and the results should, therefore, provide more convincing direct evidence of the effects of finite width.

Apparatus:

Two heated plates were constructed, both having a length of 10", but one being 30" wide and the other being 8" wide. Both plates consisted of

/a nicrome ....

a nicrome ribbon heating element wound uniformly lengthwise about a 1/16" thick core of insulating material, this assembly being clamped between two 8-gauge aluminium plates and insulated from them with mica sheets. The construction is shown in fig.1.

Both test plates carried five copper-constantan thermocouples, two in the aluminium plate on one side and three in that on the other. The thermocouples were soldered into small brass discs which were pressed into shallow holes in the aluminium plates (see fig.1) and the wires brought out at the upper end of the plate through small vertical slots milled in the aluminium plates.

The power supplied to the plate was measured with a wattmeter and was controlled by means of a variable autotransformer. The plate temperatures were measured by means of a workshop grade potentiometer.

Except for their widths, the two plates were identical in size and in all constructional details. The thermocouple wires used were drawn from the same reels and, therefore, presumably, had the same calibration. Therefore, while the assumed calibration, used in deriving the results, may not have been completely accurate at all temperatures, at a given plate temperature, and, therefore, given Grashof number (see later), the error should have been the same for both plates.

Derivation of Results:

In all cases, since only the average rate of heat transfer was being measured, the plate temperature was taken as the arithmetic mean of the five measured temperatures. Using the measured, undisturbed room temperature, the mean film temperature could then be determined

/and the fluid ....

and the fluid properties of the air at this temperature determined.

The Grashof number, defined, as usual, as:-

$$G_l = \beta g \rho^2 l^3 (T_w - T_1)/\mu^2 \qquad (1)$$

where:-

β = coefficient of cubical expansion (assumed to be equal to the reciprocal of the mean film temperature),

ρ = density,

l = length of plate (= 10" in both cases),

T_w = average plate temperature,

T_1 = undisturbed air temperature,

μ = coefficient of viscosity.

could then be determined.

In a similar way, the average Nusselt number, defined, as usual, as:-

$$N_l = Ql / k A (T_w - T_1) \qquad (2)$$

where:-

Q = total rate of heat loss by convection,

A = effective plate area,

k = conductivity.

could be determined. In deriving Q from the measured power supply, a correction for radiation was applied, it being assumed that the emissivity

/was 0.05. ....

was 0.05. Because of the comparatively low temperatures used, any error in the assumed value for emissivity could have only a small effect on Q.

Results:

The variation of N_l with G_l for the two plates is shown in fig.2, considerable differences between the two sets of results being immediately apparent. The limits on the values of G_l used were imposed by, at the lower end, the accuracy of the instrumentation, and, at the upper end, by the desire to avoid, as far as possible, any effects arising from the onset of transition to turbulent flow.

The results given in fig.2 appear to support the previous conclusion as to the comparatively large effects of finite plate width.

Discussion of Results:

As mentioned in ref.1, dimensional reasoning indicates that, in general, for free convective flow over a vertical plate:-

$$N_l = f\,(G_l,\, G_w,\, P_r) \qquad (3)$$

where P_r is the Prandtl number of the fluid and G_w is the Grashof number based on the width w, of plate and is defined, therefore, as:-

$$G_w = \beta\, g\, \rho^2\, w^3\, (T_w - T_1)\,/\,\mu^2 \qquad (4)$$

The local heat transfer rate measurements of ref.1 indicated,

/however, ...

however, that in the case of air, for which P_r is, effectively, constant, equation (3) could be written as:-

$$N_l / G_l^{0.25} = g(G_w) \qquad (5)$$

i.e., since $N_l / G_l^{0.25}$ is constant in two-dimensional flow, this equation effectively states that:-

$$N_l / N_{l2} = h(G_w) \qquad (6)$$

where N_l is the aotual Nusselt number and N_{l2} is the Nusselt number that would exist in two dimensional flow at the same value of G_l.

In order to test the above conclusion, the variation of $N_l / G_l^{0.25}$ with G_w, as obtained from the measurements described above, for both plates, is shown in fig.3. It will be noted that these measurements do, to a fair degree of accuracy, confirm equation (5).

Also shown in fig.3 is a curve which is seen to fit the results fairly well of the form:-

$$N_l / G_l^{0.25} = A + B / G_w^{n} \qquad (7)$$

where A, B and n are constants. This is of the same form as that shown to fit the results of ref.1, although the best values of A, B and n in the present case are 0.53, 150 and 0.4 respectively as compared with values of 0.67, 330 and 0.4 derived in ref.1.

/Conclusions:.........

Conclusions:

The present experimental study supports the previously drawn conclusion as to the considerable effect of plate width on the rate of heat transfer by laminar free convection from a vertical plate. These results indicate that, for a plate of finite width, the average rate of heat transfer is approximately given by:-

$$N_l / G_l^{0.25} = A + B / G_W^n$$

A, B and n being constants, tentatively given as 0.53, 150 and 0.4 respectively.

Acknowledgements:

The author would like to express his appreciation to Mr. J.R. Gordon, who was responsible for the construction of the apparatus, and to Mr. M. Bettesworth, who assisted in this task. Both are members of the technical staff of the Department of Mechanical Engineering of the University of Cape Town.

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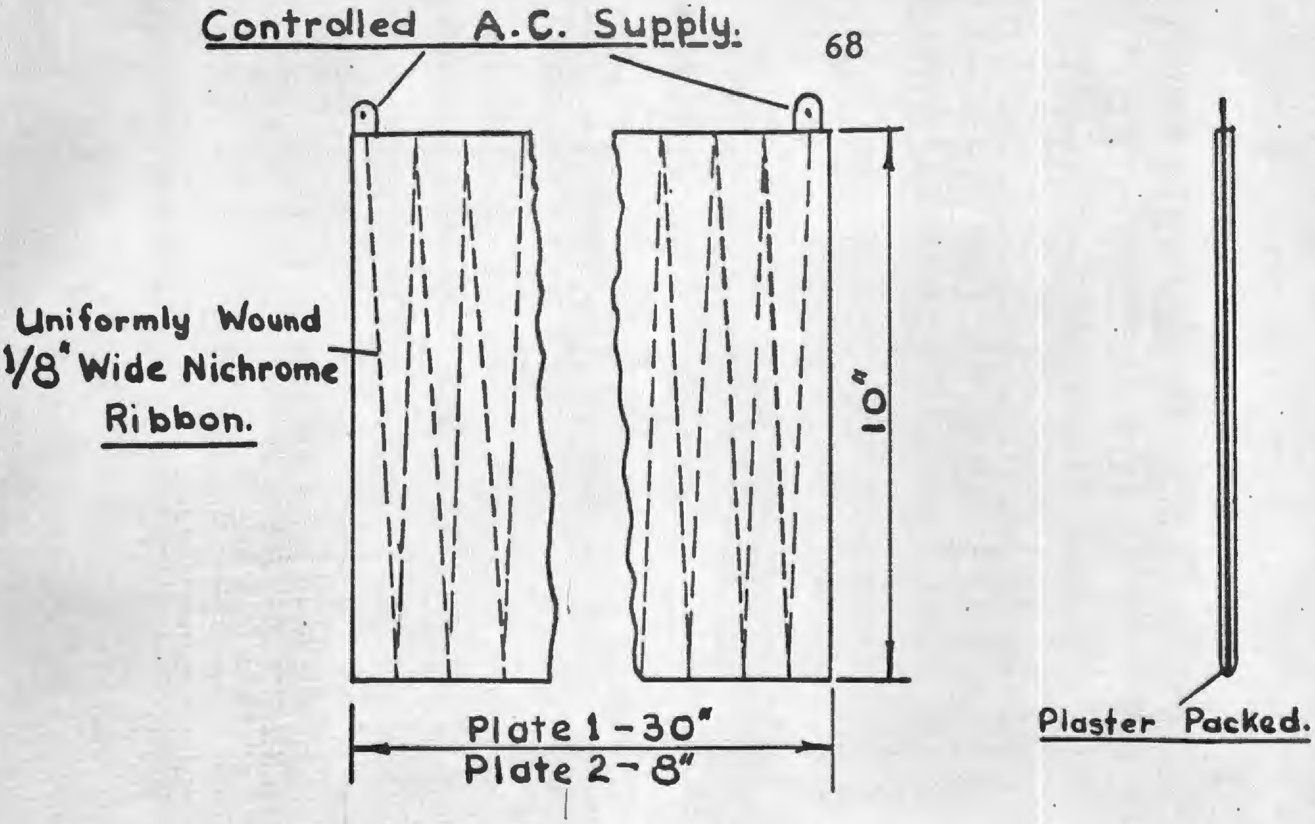

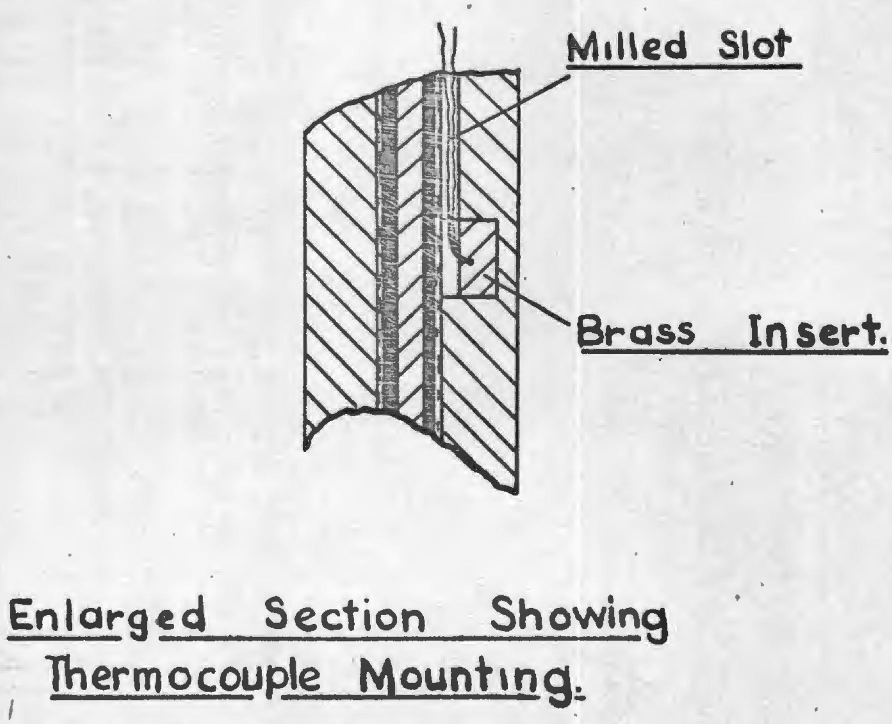

Fig. 1 Constructional details of plates.

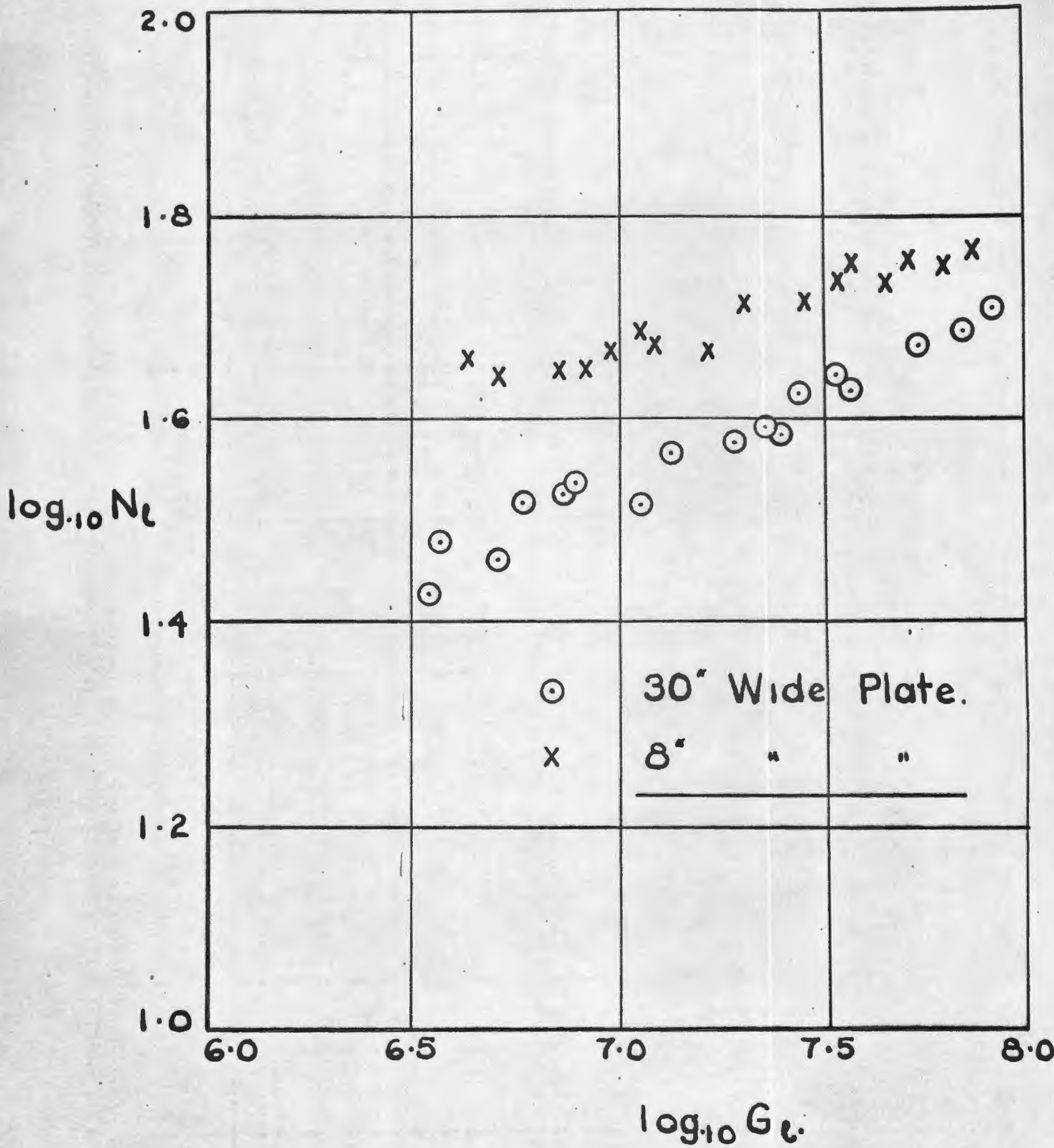

Fig. 2. Measured heat transfer results for the two plates.

Fig 2

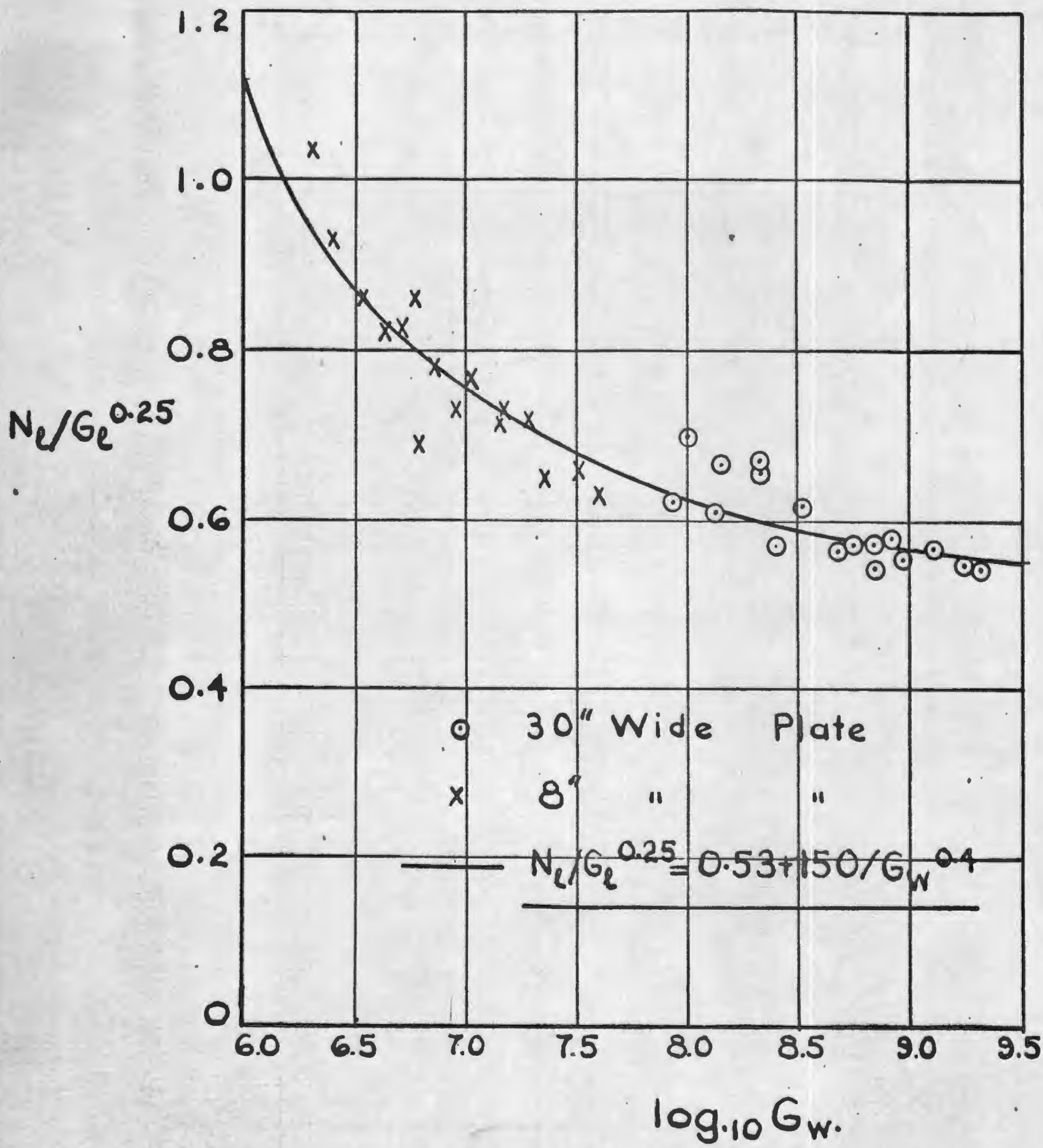

Fig. 3 Correlation of heat transfer results.

Fig 3

Paper 6.

THE GOVERNING EQUATIONS FOR COMBINED CONVECTIVE LAMINAR FLOW OVER A VERTICAL PLATE.

Summary:

It is shown that the solution for combined convection in which the flow variables are expressed relative to those that would exist in the case of either purely forced or purely free flow as a function of a parameter which combines the Reynolds and Grashof numbers, is completely general and not just applicable when either the forced or the free flow is dominant. The governing equations in terms of the combined convection parameters are derived.

<u>Notation:</u>

c = $\left[\beta g \Delta T_w / 4\nu^2\right]^{0.25}$

cp = Specific heat.

f_o = Function of η determining the stream function in purely forced flow.

$\bar{f}_o$ = Function of $\bar{\eta}$ determining the stream function in purely free flow.

G_x = Grashof number, $\beta g \Delta T_w x^3/\nu^2$.

h = Function of η and s_x determining the difference between the actual stream function and that which would exist in purely forced flow at the same values of η and R_x.

$\bar{h}$ = Function of $\bar{\eta}$ and $\bar{s}_x$ determining the difference between the actual stream function and that which would exist in purely free flow at the same values of $\bar{\eta}$ and G_x.

k = Coefficient of conductivity.

N_x = Local Nusselt number, $q_w x / \Delta T_w k$.

$N_{x\ for}$ = Value of N_x that would apply to purely forced flow at the same value of R_x as in the actual flow.

$N_{x\ free}$ = Value of N_x that would apply to purely free flow at the same value of G_x as in the actual flow.

q_w = Rate of heat transfer from the wall to the fluid per unit wall area.

P_r = Prandtl number.

R_x = Reynolds number, $u_1 x/\nu$.

/s_x = ...

a_x = G_x/R_x^2.

$\bar{a}_x$ = $R_x/G_x^{0.5}$.

T = Temperature.

T_1 = Freestream temperature.

T_w = Wall temperature.

u = Velocity component in x-direction.

u_1 = Freestream velocity.

v = Velocity component in y-direction.

x = Co-ordinate measured along plate surface in flow direction.

y = Co-ordinate measured normal to plate surface.

β = Coefficient of cubical expansion.

ΔT_w = $T_w - T_1$.

η = $y\sqrt{u_1/\nu x}$

$\bar{\eta}$ = $cy/x^{0.25}$

θ = $(T - T_1)/(T_w - T_1)$

θ_o = The value of θ, at the η considered, that would exist in purely forced flow.

$\bar{\theta}_o$ = The value of θ, at the $\bar{\eta}$ considered, that would exist in purely free flow.

ν = Coefficient of kinematic viscosity.

ρ = Density.

ϕ = Difference between θ and θ_o at η considered.

$\bar{\phi}$ = Difference between θ and $\bar{\theta}_o$ at $\bar{\eta}$ considered.

Ψ = Stream function.

THE GOVERNING EQUATIONS FOR COMBINED CONVECTIVE LAMINAR FLOW OVER A VERTICAL PLATE.

Introduction:

The governing equations of continuity, momentum and energy for laminar flow over a vertical plate when buoyancy forces cannot be neglected are:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \pm \beta g \Delta T_w \theta \tag{2}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = (k/\rho c_p)\frac{\partial^2 \theta}{\partial y^2} \tag{3}$$

the + or the - sign in equation (2) is used depending on whether the buoyancy force and the forced flow are in the same or opposite directions.

In the present note attention will be restricted to flows in which the freestream velocity and temperature and the wall temperature remain constant as do the fluid properties, the latter being implied by the use of the above equations.

Now it would appear from the form of these equations that their solution will, in general, give a local Nusselt that is dependent on both the Greshof and Reynolds numbers. However, by noting that the ratio of buoyancy to viscous forces introduced by the forced flow will be determined by the parameter:-

$$a_x = G_x / R_x^2 \tag{4}$$

Sparrow and Gregg[1.], Szewczyk[2.] and others have obtained limiting

/forms ....

forms of the solution to the above set of equations by considering a power series solution for ψ and for θ having the form:-

$$\psi = \sqrt{\nu x u_1} \left[f_o(\eta) + a_1(\eta) s_x + a_2(\eta) s_x^2 + \cdots\cdots \right] \quad (5)$$

$$\theta = \theta_o + b_1(\eta) s_x + b_2(\eta) s_x^2 + \cdots\cdots\cdots\cdots \quad (6)$$

for the case where the forced flow is dominant, i.e. s_x small Similarly, by noting that the ratio of forced flow generated viscous forces to buoyancy forces will be dependent on the parameter:-

$$\bar{s}_x = R_x / G_x^{0.5} = s_x^{-0.5} \quad (7)$$

they obtain a limiting form of solution for the case where the buoyancy forces are dominant, i.e. where $\bar{s}_x$ is small, by introducing the following power series for ψ and for θ:-

$$\psi = (4\nu c x^{0.75}) \left[\bar{f}_o(\bar{\eta}) + \bar{a}_1(\bar{\eta}) \bar{s}_x + \bar{a}_2(\bar{\eta}) \bar{s}_x^2 + \cdots\cdots \right] \quad (8)$$

$$\theta = \bar{\theta}_o + \bar{b}_1(\bar{\eta}) \bar{s}_x + \bar{b}_2(\bar{\eta}) \bar{s}_x^2 + \cdots\cdots\cdots\cdots \quad (9)$$

It should be noted that in both pairs of solutions the first terms represent the solution in purely forced flow and purely free flow respectively.

/the purpose ....

The purpose of the present note is to indicate that the above series solutions have the same form as possible general solutions (i.e. applicable for all s_x and $\bar{s}_x$) to the set of governing equations, (1) to (3).

Analysis:

As mentioned above, the purpose of the present note is to indicate the general nature of the forms introduced in the above series solutions. Consider first the case where the solution is expressed relative to the solution for purely forced flow at the same Reynolds number as in the actual flow. Using equation (5) as a guide it is assumed that, in general, ψ is given by:-

$$\psi = \sqrt{\nu x\, u_1}\left[f_o(\eta) + h\,(\eta, s_x)\right] \tag{10}$$

where, as usual, :-

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \tag{11}$$

and the continuity equation is, thereby, satisfied.
Using these equations then leads to:-

$$u/u_1 = f_o' + \frac{\partial h}{\partial\eta} \tag{12}$$

/and ....

and:-

$$v/u_1 = -\sqrt{\nu/u_1 x}\ \left[(f_o + h)/2 + s_x \frac{\partial h}{\partial s_x}\right]$$

$$+ \ (y/2x)\ (f_o' + \frac{\partial h}{\partial \eta}) \qquad (13)$$

the dashes on the f_o indicating, of course, differentiation with respect to η.

Similarly, using equation (6) as a guide, it is assumed that the temperature distribution is given by:-

$$\theta = \theta_o + \phi\ (\eta, s_x) \qquad (14)$$

Using the above equations and noting that, since f_o and θ_o are the solutions for purely forced flow:-

$$f_o'' f_o + 2f_o''' = 0 \qquad (15)$$

$$\theta_o'' + P_r \theta_o' f_o/2 = 0 \qquad (16)$$

equations (2) and (3) reduce to:-

$$\frac{\partial^3 h}{\partial \eta^3} - s_x \frac{\partial h}{\partial \eta}\frac{\partial^2 h}{\partial s_x \partial \eta} + s_x \frac{\partial h}{\partial s_x}\frac{\partial^2 h}{\partial \eta^2} - f_o' s_x \frac{\partial^2 h}{\partial s_x \partial \eta}$$

$$+ \left[(h + f_o)/2\right] \frac{\partial^2 h}{\partial \eta^2} + f_o'' s_x \frac{\partial h}{\partial s_x} + h\, f_o''/2 \pm s_x(\theta_o + \phi) = 0$$

(17) /and ...

and:-

$$s_x f_o' \frac{\partial\phi}{\partial s_x} + s_x \frac{\partial h}{\partial\eta}\frac{\partial\phi}{\partial s_x} - \left[(h + f_o)/2\right]\frac{\partial\phi}{\partial\eta} - s_x \frac{\partial h}{\partial s_x}\ \frac{\partial\phi}{\partial\eta}$$

$$- (1/P_r)\frac{\partial^2\phi}{\partial\eta^2} - \theta_o' h/2 - \theta_o' s_x \frac{\partial h}{\partial s_x} = 0 \qquad (18)$$

The boundary conditions on these equations follow from the boundary conditions on u, v and θ in equations (2) and (3) which are:-

$$y = 0 \ : \quad u = 0, \quad v = 0, \quad \theta = 1$$

$$y \rightarrow \infty : \quad u \rightarrow u_1, \ \theta \rightarrow 0 \qquad (19)$$

From these conditions and taking account of the boundary conditions on f_o and θ_o, it follows from equations (12), (13) and (14) that:-

$$\eta = 0 \ : \quad \frac{\partial h}{\partial\eta} = 0, \quad h/2 + s_x \frac{\partial h}{\partial s_x} = 0, \quad \phi = 0$$

$$\eta \rightarrow \infty : \ \frac{\partial h}{\partial\eta} \rightarrow 0, \ \phi \rightarrow 0 \qquad (20)$$

/Equations (17) ......

Equations (17) and (18) and the boundary conditions (20) indicate that equations (10) and (14) do represent a possible form of solution to the equations governing combined convective flow. They show that by using the conditions that would exist in purely forced flow for reference then the effect of buoyancy forces at any value of η will be a function of s_x alone. It also follows that since, using the definitions of the local Nusselt number and skin friction coefficient, i.e.

$$N_x = \sqrt{xu_1/\nu}\ \frac{\partial\theta}{\partial\eta}\Big|_{\eta = 0} \tag{21}$$

$$c_f = (2/\sqrt{xu_1/\nu}\ )\ \frac{\partial(u/u_1)}{\partial\eta}\Big|_{\eta = 0} \tag{22}$$

and noting that from them it follows that at the same Reynolds' number as in the actual flow:-

$$N_{x\ for} = \sqrt{xu_1/\nu}\ \theta_o'\,|_{\eta = 0}$$

$$(23)$$

$$c_{f\ for} = (2/\sqrt{xu_1/\nu})\ f_o''\,|_{\eta = 0}$$

then:-

$$(N_x - N_{x\ for})/N_{x\ for} = N(s_x) \tag{24}$$

/and (c_f - ....

and

$$(c_f - c_{f\ for})/c_{f\ for} = c\ (s_x) \qquad (25)$$

Similar expressions will also, of course, apply to the average Nusselt number and skin friction coefficient.

An alternative form of solution can be obtained by using the solution for free flow alone for reference i.e. by assuming by analogy with equations (8) and (9) that in general, for any value of $\bar{s}_x$,:-

$$\psi = (4\ \nu\ c\ x^{0.75}) \left[\bar{f}_o + \bar{h}\ (\bar{\eta}, \bar{s}_x)\right] \qquad (26)$$

and

$$\theta = \bar{\theta}_o + \bar{\phi}\ (\bar{\eta}, \bar{s}_x) \qquad (27)$$

From equation (26) it follows that, using equation (11):-

$$u = (4\ \nu\ c^2\ x^{0.5}) \left[\bar{f}_o' + \frac{\partial \bar{h}}{\partial \eta}\right] \qquad (28)$$

$$v = \left[-3\ \nu\ c\ (\bar{f}_o + \bar{h})/x^{0.25}\right] + \left[(\nu\ c^2 y/x^{0.5})(\bar{f}_o + \frac{\partial \bar{h}}{\partial \bar{\eta}})\right] + \left[(u_1/cx^{0.75})\ \frac{\partial \bar{h}}{\partial \bar{s}_x}\right] \qquad (29)$$

It having been noted that:-

$$\bar{s}_x = u_1/2\ \nu\ c^2\ x^{0.5} \qquad (30)$$

/Then, noting ....

Then, noting that since $\bar{f}_o$ and $\bar{\theta}_o$ apply in purely free convection they must satisfy the following equations:-

$$\bar{f}_o''' + 3\bar{f}_o \bar{f}_o'' - 2\bar{f}_o'^2 + \bar{\theta}_o = 0 \tag{31}$$

$$\bar{\theta}_o'' + 3P_r \bar{f}_o \bar{\theta}_o = 0 \tag{32}$$

equations (2) and (3) reduce, in this case, to:-

$$4\bar{f}_o \frac{\partial \bar{h}}{\partial \bar{\eta}} + \frac{2\partial \bar{h}}{\partial \bar{\eta}} - 3\bar{f}_o'' \bar{h} - 3\frac{\partial^2 \bar{h}}{\partial \bar{\eta}^2}(\bar{f}_o + \bar{h}) - 2\bar{f}_o \bar{s}_x \frac{\partial^2 \bar{h}}{\partial \bar{s}_x \partial \bar{\eta}}$$

$$- 2\bar{s}_x \frac{\partial \bar{h}}{\partial \bar{\eta}} \frac{\partial^2 \bar{h}}{\partial \bar{s}_x \partial \bar{\eta}} + 2\bar{s}_x \bar{f}_o'' \frac{\partial \bar{h}}{\partial \bar{s}_x} + 2\bar{s}_x \frac{\partial^2 \bar{h}}{\partial \bar{\eta}^2} \frac{\partial \bar{h}}{\partial \bar{s}_x} - \frac{\partial^3 \bar{h}}{\partial \bar{\eta}^3} \mp \bar{\phi} = 0 \tag{33}$$

and:-

$$3(\bar{f}_o + \bar{h}) \frac{\partial \bar{\phi}}{\partial \bar{\eta}} + 2\bar{s}_x \bar{f} \frac{\partial \bar{\phi}}{\partial \bar{s}_x} + 2\bar{s}_x \frac{\partial \bar{h}}{\partial \eta} \frac{\partial \bar{\phi}}{\partial \bar{s}_x} + 3\bar{h}\bar{\theta}_o$$

$$- 2\bar{s}_x \bar{\theta}_o \frac{\partial \bar{h}}{\partial \bar{s}_x} - 2\bar{s}_x \frac{\partial \bar{h}}{\partial \bar{s}_x} \frac{\partial \bar{\phi}}{\partial \eta} + (1/P_r) \frac{\partial^2 \bar{\phi}}{\partial \bar{\eta}^2} = 0 \tag{34}$$

It should be noted that it is implicit in these equations that the positive x-direction is the direction in which the buoyancy forces act while in equations (17) and (18) it is in the direction of the forced flow.

/This point is, ...

This point is, of course, only important when the flows are in opposition. In equation (33) the upper, negative, sign is the one applicable to assisting flows.

The boundary conditions as given by equation (19) reduce, in this case, to:-

$$\bar{\eta} = 0 : \quad \frac{\partial \bar{h}}{\partial \bar{\eta}} = 0, \quad \bar{s}_x \frac{\partial \bar{h}}{\partial \bar{s}_x} - 3\bar{h}/2 = 0, \quad \phi = 0$$

$$\eta \to \infty : \quad \frac{\partial \bar{h}}{\partial \bar{\eta}} \to \bar{s}_x/2, \quad \bar{\phi} \to 0 \tag{35}$$

Equations (33) and (34) together with the boundary conditions (35) indicate that equations (26) and (27) are also possible solutions to the governing equations i.e. that by using purely free flow conditions for reference, the effect of the forced flow at any value of $\bar{\eta}$ can be expressed as a function of $\bar{s}_x$ alone.

It is also worth noting that from the above using the same procedure as was used to derive equation (24):-

$$(N_x - N_{x\ free}) / N_{x\ free} = \bar{\bar{N}}(\bar{s}_x) \tag{36}$$

Thus it has been shown that general solutions to the governing set of equations can be obtained by using either the purely forced or the purely free flow conditions for reference. The two solutions are, of course,

/basically ......

basically the same and one can be derived from the other. The actual relation between them follows by equating (10) and (26) and (14) and (27) since, at fixed y and x, Ψ and θ must be the same. This procedure gives:-

$$s_x = 1/\sqrt{\bar{s}_x} \;;\; \bar{\eta}/\eta = \sqrt{2}\, s_x = \sqrt{2/\bar{s}_x}$$

$$\bar{h} = (\bar{s}_x/2\sqrt{2})(f_o + h) - \bar{f}_o$$

$$\bar{\phi} = \theta_o + \phi - \bar{\theta}_o$$

The solutions, and hence f_o, θ_o, $\bar{f}_o$ and $\bar{\theta}_o$, in purely forced and purely free flow being assumed to be known.

The advantage in having both solutions follows from the fact that since $s_x \rightarrow \infty$ as $\bar{s}_x \rightarrow 0$, the whole range is more easily covered and from the fact that flows in which separation occurs, i.e. opposing flows in which there is, over a portion of the plate, a flow adjacent to the wall in the opposite direction to the forced flow, are more easily dealt with.

Conclusions:

It has been shown that by using either the purely forced or the purely free flow conditions for reference, the solution to the combined convective flow problem can be expressed as a function of either s_x or $\bar{s}_x$. The reduced governing partial differential equations, in terms of these

/variables ....

variables, have been derived but no results of any attempt to solve them are given.

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Paper 7.

A NOTE ON THE COMBINED FREE AND FORCED CONVECTIVE LAMINAR FLOW OVER A VERTICAL ISOTHERMAL PLATE.

A note on the combined free and forced convective laminar flow over a vertical isothermal plate

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The integrated boundary layer equations for combined convective flow over a flat plate are applied to the case of fluids with Prandtl numbers near unity, it being shown that for such fluids it can be assumed that the ratio of thermal boundary layer thickness to velocity boundary layer thickness is constant. This assumption is also used for fluids having arbitrary Prandtl numbers to derive the values of (Gx/Rx^2) for which the buoyancy effects and the forced velocity effects are negligible.

INTRODUCTION

While the heat transfer from bodies by either free convection alone or by forced convection alone has been the subject of many experimental and theoretical analyses, the problem of determining the heat transfer when both types of convection are of importance has received relatively little attention. There are, however, numerous practical cases in which the heat transfer is by the combined action of free and forced convection. Many examples exist, for instance, in the air-conditioning and refrigerating industries.

In the present note an attempt has been made to consider some aspects of one of the simplest problems involving combined convection, that of the heat transfer from a vertical plate held at a constant temperature with the forced flow parallel to the plate surface and in the same direction as the buoyancy forces, i.e. assisting the free convective flow.

The method of solution adopted is the integral equation method of Von Karman and Pohlhausen. This method has been applied successfully before, both to the problem of free convection alone and to that of forced convection alone. In this method it is assumed that the changes in velocity and temperature from the values at the surface of the plate to the values which exist exactly only at an infinite distance away from the plate, occur effectively in a thin boundary layer. The velocity and temperature profiles in this thin boundary layer are fitted, approximately, by polynomial expressions which satisfy the correct boundary conditions and the integrated boundary layer equations, i.e. they satisfy the conditions at the inner and outer edges of the boundary layer and also the mean conditions in the boundary layer.

No attempt has been made in the present note to obtain the general solutions to the equations derived by the procedure outlined above. Instead, attention has been given to those aspects of the problem in which it is possible to assume that the ratio of the thermal boundary layer thickness to the velocity boundary layer thickness is constant.

BASIC ANALYSIS

It will be assumed that the velocity and temperature distributions can be approximately represented by fourth-degree polynomials, i.e. that:

$$u = A + B\eta + C\eta^2 + D\eta^3 + E\eta^4 \quad \ldots\ldots (1)$$

and

$$T = F + G\epsilon + H\epsilon^2 + I\epsilon^3 + J\epsilon^4 \quad \ldots\ldots (2)$$

where

u = velocity component in boundary layer parallel to the surface at distance y from it
T = temperature in boundary layer at distance y from the surface
η = y/δ_u
δ_u = thickness of velocity boundary layer
ϵ = y/δ_T
δ_T = thickness of thermal boundary layer
$A, B \ldots, J$ = coefficients to be determined

Now the momentum and energy equations for the type of two-dimensional flow being considered become, when the boundary layer approximations are applied:

$$u(\partial u/\partial x) + v(\partial u/\partial y) = \beta g(T - T_o) + (\mu/\rho)(\partial^2 u/\partial y^2) \quad \ldots\ldots (3)$$

$$u(\partial T/\partial x) + v(\partial T/\partial y) = (k/\rho C_p)(\partial^2 T/\partial y^2) \quad \ldots (4)$$

where

x = co-ordinate parallel to surface
y = co-ordinate normal to surface
v = velocity component in y- direction
T_o = temperature outside boundary layer
β = coefficient of cubical expansion
μ = coefficient of viscosity
ρ = density
k = coefficient of conductivity
C_p = specific heat

In deriving the above equations it has been assumed that the fluid properties are constant in the boundary

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layer except for the density change with temperature which causes the buoyancy force. For the case of gases this assumption will, strictly, limit the application of the present analysis to the case where the overall temperature difference is small. However, experience with free flow and forced flow alone suggests that it will normally be possible to remove this restriction by evaluating the fluid properties at some suitable 'mean film temperature'. In deriving the energy equation, it has also been assumed that the dissipation due to viscosity is negligible. This assumption is justified by the fact that the velocities must remain relatively small in flows in which it is necessary to take account of the buoyancy forces.

Now at the surface of the plate $u = v = o$, so that equations (3) and (4) give respectively:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = -\beta g \rho (T_w - T_o)/\mu \quad \ldots \ldots (5)$$

and

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{y=0} = 0 \quad \ldots \ldots (6)$$

where T_w = the constant surface temperature of the plate.

Using the relations (5) and (6), the boundary conditions on velocity and temperature which must be satisfied by equations (1) and (2) therefore become:

(i) on velocity:—

At $y = o$; $u = o$, $\partial^2 u/\partial y^2 = -\beta g \rho (T_w - T_o)/\mu$ . . . . . . (7)

At $y = \delta_u$; $u = u_1$, $(\partial u/\partial y) = o$, $(\partial^2 u/\partial y^2) = o$

where u_1 = constant velocity outside boundary layer.

(ii) on temperature:—

At $y = o$; $T = T_w$, $(\partial^2 T/\partial y^2) = o$ . . . . . . (8)

At $y = \delta_T$; $T = T_o$, $(\partial T/\partial y) = o$, $(\partial^2 T/\partial y^2) = o$

Substituting boundary conditions (7) into equation (1) gives:

$$u = [\beta g \rho \delta_u^2 (T_w - T_o)/6\mu] \; (\eta - 3\eta^2 + 3\eta^3 - \eta^4) + u \; (2\eta - 2\eta^3 + \eta^4) \quad \ldots \ldots (9)$$

The first term on the right-hand side of this equation gives the velocity distribution that would exist in the absence of a forced velocity, while the second term gives the velocity distribution that would exist if the buoyancy effects were negligible. It is seen, therefore, that the velocity distribution for the case of combined free and forced convection over a vertical plate is obtained by adding, algebraically, the velocity distributions that would exist with forced convection alone and with free convection alone.

Similarly, substituting the boundary conditions on temperature (8) into equation (2) gives:

$$(T - T_o) = (T_w - T_o)\,(1 - 2\epsilon + 2\epsilon^3 - \epsilon^4) \quad .(10)$$

The temperature distribution will be given by the same equation if forced convection alone or free convection alone is considered.

Writing for convenience:

$$\theta = (T - T_o) \quad \ldots \ldots (11)$$

$$\theta_w = (T_w - T_o) \quad \ldots \ldots (12)$$

$$a = \beta g \rho \theta_w / \mu \quad \ldots \ldots (13)$$

The velocity and temperature distributions become:

$$u = (a\,\delta_u^2/6)(\eta - 3\eta^2 + 3\eta^3 - \eta^4) + u_1(2\eta - 2\eta^3 + \eta^4) \quad (14)$$

and

$$\theta = \theta_w(1 - 2\epsilon + 2\epsilon^3 - \epsilon^4) \quad \ldots \ldots (15)$$

respectively.

Now integration of the momentum equation (3) across the boundary layer gives, with the aid of the continuity equation:—

$$\frac{d}{dx}\int_o^{\delta_u} (u\,u_1 - u^2)dy = -\beta g \int_o^{\delta_T} \theta dy + (\mu/\rho)\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad . (16)$$

Substituting equations (14) and (15) into equation (16) gives, after some rearrangement:

$$(d/dx)[(a\delta_u^3 u_1/945) - (37/315)u_1^2\delta u + (a^2\delta_u^5/9072)]$$
$$= (3a\mu\delta_T/10\rho) - (\mu/\rho\delta_u)[(a\delta u^2/6) + 2u_1] \quad \ldots (17)$$

Similarly, the energy equation (4) can be integrated across the boundary layer to give:

$$\frac{d}{dx}\int_o^h u\,\theta\,dy = -(k/\rho C_p)\left(\frac{\partial T}{\partial y}\right)_{y=0} \quad \ldots \ldots (18)$$

where the upper limit h of the integration depends on whether δ_T is less than or greater than δ_u. Defining, for convenience:

$$\Delta = (\delta_T/\delta_u) \quad \ldots \ldots (19)$$

the integral in equation (18) then becomes:

(i) For $\Delta < 1$:—

$$\int_o^h u\,\theta\,dy = \int_o^{\delta_T} u\,\theta\,dy \quad \ldots \ldots (20)$$

as $\theta = 0$ for $y > \delta_T$ in this case.

This integral can be evaluated by putting $\eta = \epsilon\Delta$ in the expression for the velocity distribution.

(ii) For $\Delta > 1$:—

$$\int_o^h u\,\theta\,dy = \int_o^{\delta_u} u\,\theta dy + u_1 \int_o^{\delta_T} \theta\,dy \quad \ldots \ldots (21)$$

as $u = u_1$ for $y > \delta_u$ in this case.

The first integral on the right hand side can be evaluated by putting $\epsilon = \eta/\Delta$ in the expression for the velocity distribution.

Substituting equations (14) and (15) into equation (18) and using equations (20) and (21) thus gives, by using a similar procedure to that adopted with the momentum integral:

$$(d/dx)(a\delta_u^3\theta_w L + u_1\theta_w\delta_u M) = 2(k/\rho C_p)\,(\theta_w/\Delta\delta_u) \quad (22)$$

where

(i) For $\Delta < 1$ $(\delta_T < \delta_u)$:—

$$L = (\Delta/6)(\Delta/15 - \Delta^2/14 + 9\Delta^3/280 - \Delta^4/180) \quad \ldots \ldots (23)$$

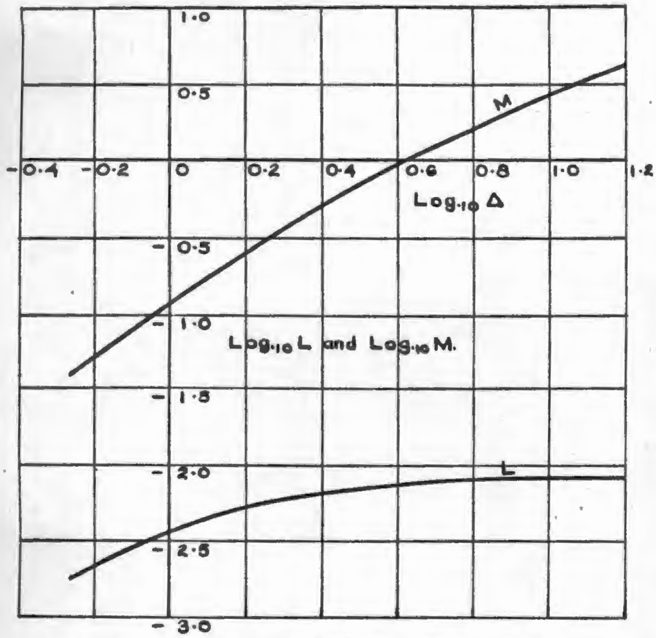

Fig 1 Variation of *L* and *M* with boundary layer thickness ratio, Δ

$$M = \Delta(2\Delta/15 - 3\Delta^3/140 + \Delta^4/180)$$

(ii) For $\Delta > 1$ $(\delta_T > \delta_u)$:—

$$L = (1/6)(1/20 - 1/30\Delta + 1/140\Delta^3 - 1/504\Delta^4) \quad \ldots \quad (24)$$

$$M = \Delta(3/10 - 3/10\Delta + 2/15\Delta^2 - 3/140\Delta^4 + 1/180\Delta^5)$$

The variation of L and M with Δ is shown in Fig 1.

Equations (17) and (22) constitute a pair of simultaneous differential equations for the unknowns δ_u and δ_T in terms of x for any given values of the flow parameters, i.e. u_1, θ_w and the fluid properties. Once δ_u and δ_T are known any other required property of the flow can be determined. In the present note the main such properties of the flow considered will be the local rate of heat transfer per unit surface area, q_x, which is given by:

$$q_x = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \quad \ldots \quad (25)$$

and the local surface shearing stress, τ_w, which is given by:

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \quad \ldots \quad (26)$$

Combining equation (15) and (25) gives:

$$N_x = 2\,x/\Delta\delta_u \quad \ldots \quad (27)$$

where $N_x = (q_x x/\theta_w k)$ is the local Nusselt number.

Similarly, combining equations (14) and (26) gives:

$$C_f = \frac{1}{3}\left(\frac{G_x}{R_x^2}\right)\left(\frac{\delta_u}{x}\right) + \left[\frac{4}{R_x\ (\delta_u/x)}\right] \quad \ldots \quad (28)$$

where $C_f = (2\tau_w/\rho u_1^2)$ is the local coefficient of shearing stress

$R_x = (\rho u_1 x/\mu)$ is the Reynolds' number based on the forced velocity u_1 and on x

$G_x = (a\rho x^3/\mu)$ is the local Grashof number.

APPROXIMATE METHOD OF SOLUTION

A simple solution to equations (17) and (22) can be obtained if it is possible to assume that Δ is independent of x, i.e. dependent on the overall flow parameters alone. Such an assumption is not, in general, applicable, as is obvious from the dissimilarity between the momentum and energy equations (3) and (4). However, as will be shown below, such an assumption leads to fairly accurate results for free convection alone and forced convection alone, which suggests that it can be applied to some particular problems involving combined convection.

With Δ assumed to be independent of x, the energy integral equation (22) can be integrated to give:

$$(3a\ L/4)\ \delta_u^4 + (u_1 M/2)\ \delta_u^2 = (2\ \mu/\rho\ P_r)\ (x/\Delta) \quad \ldots \quad (29)$$

where P_r is the Prandtl number of the fluid, i.e. $(\mu C_p/k)$.

In carrying out this integration it has been assumed that the origin can be taken as the point, real or hypothetical, at which δ_u and δ_T both have zero thickness. Such an assumption is known to be applicable when either forced convection alone or free convection alone is considered and it is to be assumed, therefore, that it will apply to the type of combined convection here being considered, in which the two flows are assisting each other. For many leading-edge conditions no such point actually exists, but in such cases it is to be expected that solutions based on the above assumption will apply everywhere except in the immediate vicinity of the leading edge. When the buoyancy forces oppose the forced flow it will be possible to make the above assumption only when the forced flow is predominant.

Equation (29) gives the variation of δ_u with x and Δ. Elimination of δ_u between this equation and the momentum integral equation (17) should then give the variation of Δ with the flow parameters. To facilitate this, equation (17) can be integrated, remembering that Δ is, by assumption, independent of x, to give:

$$(\delta_u^3/630)(u_1/a) - (37/315)(u_1/a)^2 \log_e \delta_u + (5/36\,288)\delta_u^4$$
$$= \left[\left(\frac{3}{10}\right)\Delta - \frac{1}{6}\right]\left(\frac{\mu}{\rho a}\right)x - 2\left(\frac{\mu}{\rho a}\right)\left(\frac{u_1}{a}\right)\int_0^x \frac{dx}{\delta_u^2} \quad \ldots \quad (30)$$

In carrying out this integration the same assumptions as to conditions at the leading edge have been made as were made in deriving equation (29).

SOLUTION FOR FREE CONVECTION ALONE

When the forced velocity u_1 is zero equation (29) gives:

$$\delta_u^4 = (8\mu/3a\ \rho\ P_r\ L\Delta)\ x \quad \ldots \quad (31)$$

which is more conveniently written as:

$$(\delta_u/x) = 1{\cdot}28/G_x^{0{\cdot}25}\ (P_r\ L\Delta)^{0{\cdot}25} \quad \ldots \quad (32)$$

But, with u_1 equal to zero, equation (30) becomes:

$$(5/36\,288)\delta_u^4 = [(3/10)\Delta - 1/6]\ (\mu/\rho a)\ x \quad \ldots \quad (33)$$

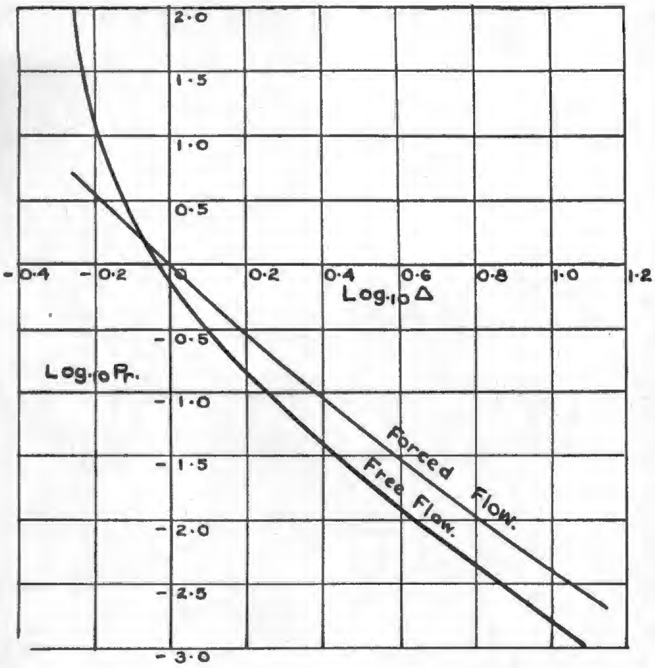

Fig 2 Variation of boundary layer thickness ratio, Δ, with Prandtl number, P_r, of fluid

Eliminating (δ_u^4/x) between equations (31) and (33) then gives:

$$P_r L\Delta\,(3\Delta/10 - 1/6) = 5/13\,608 \quad \ldots\ldots (34)$$

The variation of Δ with P_r, as given by this equation, is shown in Fig 2.

Substituting equation (32) into equation (26) gives, for free convection alone:

$$N_x/G_x^{0\cdot25} = 1\cdot56\,(P_r L)^{0\cdot25}/\Delta^{0\cdot75} \quad \ldots\ldots (35)$$

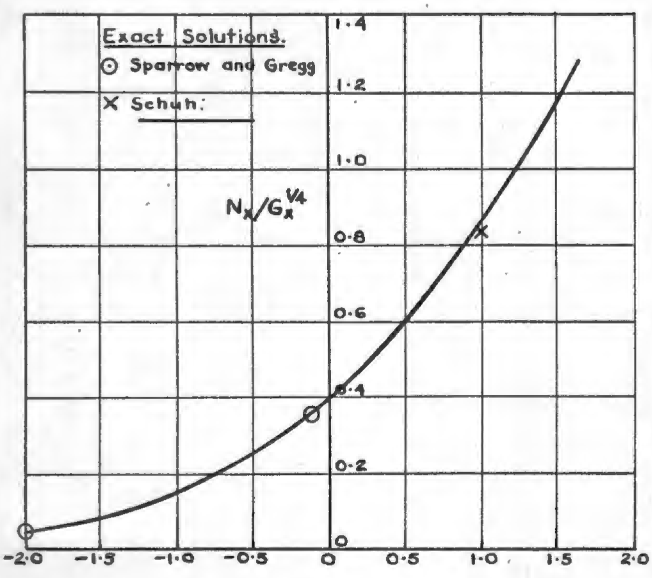

Fig 3 Results for free convection alone

This equation, in conjunction with equation (34), allows the variation of $N_x/G_x^{0\cdot25}$ with P_r to be determined. In Fig 3 results so obtained are compared with exact solutions of the present problem due to Schuh[1] and to Sparrow and Gregg[2]. Reasonably good agreement is seen to be obtained.

SOLUTION FOR FORCED CONVECTION ALONE

When the terms which arise because of the buoyancy effects (i.e. terms containing a) are negligible, equation (29) gives:

$$\delta_u^2 = (4\mu/\rho P_r\, u_1\, M\Delta)\, x \quad \ldots\ldots (36)$$

which is, for most purposes, more conveniently written as:

$$(\delta_u/x) = 2/R_x^{0\cdot50}\,(M\, P_r\, \Delta)^{0\cdot50} \quad \ldots\ldots (37)$$

In order to apply equation (30) to the case at present under consideration, it is multiplied through by $(a/u_1)^2$ and terms containing a are then neglected. This procedure gives:

$$(37/315)\,\log_e \delta_u = (2\mu/\rho u_1)\int^x (dx/\delta_u^{\,2})$$

which, in turn, gives on rearrangement:

$$\delta_u^2 = (1260\,\mu\, x/37\,\rho\, u_1) \quad \ldots\ldots (38)$$

Eliminating (δ_u^2/x) between equations (36) and (38) then gives:

$$P_r M\Delta = 37/315 \quad \ldots\ldots (39)$$

The variation of Δ with P_r, as given by this equation, is also shown in Fig 2.

Substituting equation (37) into equations (26) and (28) gives, for forced convection alone, with the aid of equation (39)—

$$N_x/R_x^{0\cdot50} = 0\cdot343/\Delta \quad \ldots\ldots (40)$$

and . . . .

$$C_f = 0\cdot686/R_x^{0\cdot50} \quad \ldots\ldots (41)$$

These results are, of course, equivalent to those obtained in Ref 3, wherein it is shown that they agree reasonably well with exact solutions.

APPLICATION TO COMBINED CONVECTION

If it is assumed that Δ is independent of x in combined convective flows, then a similar procedure to that adopted with free convection alone and with forced convection alone can be used to try to derive the relationship between Δ and P_r. However, instead of substituting equation (29) directly into equation (30), it is simpler to rearrange the former in the following form:

$$\frac{3L}{2}\,\delta_u^2 + M\left(\frac{u_1}{a}\right)\log_e \delta_u = \left(\frac{2}{P_r}\right)\left(\frac{\mu}{\rho a}\right)\frac{1}{\Delta}\int_0^x \frac{dx}{\delta_u^{\,2}} \quad .(42)$$

The integral term on the right-hand side can now be eliminated between equations (30) and (42), giving:

$$[(1/630) + (3LP_r\Delta/2)]\,(u_1/a)\delta_u^3 - [(37/315) - (MP_r\Delta)](u_1/a)^2 \log_e \delta_u + (5/36\,288)\,\delta_u^4 = [(3/10)\Delta - (1/6)]\,(\mu/\rho a)x \quad \ldots\ldots\ldots (43)$$

Equation (43) can, in turn, be combined with equation (29) to give:

$$\{(1/630) + (3LP_r\Delta/2) - (MP_r\Delta/4)[(3/10)\Delta - (1/6)]\}\,(u_1/a)\,\delta_u^3 - (37/315 - MP_r\Delta)\,(u_1/a)^2 \log_e \delta_u + \{(5/36\,288) - (3LP_r\Delta/8)\,[(3/10)\Delta - (1/6)]\}\,\delta_u^4 = 0 \quad \ldots\ldots\ldots (44)$$

But Δ is, by assumption, independent of x while δ_u will, in general, depend on x. Equation (44) implies therefore that, if Δ is independent of x, the following set of equations must, simultaneously, apply:

$$(1/630) + (3LP_r\Delta/2) - (MP_r\Delta/4)[(3/10)\Delta - (1/6)] = 0 \quad \ldots\ldots\ldots (45)$$

$$[(37/315) - MP_r\Delta] = 0 \quad \ldots\ldots\ldots (46)$$

$$(5/36\,288) - (3LP_r\Delta/8)\,[(3/10)\Delta - (1/6)] = 0 \quad \ldots (47)$$

Equation (46), which is identical to equation (39), is the equation relating Δ to P_r for forced convection alone. It could have been obtained from equation (44) by multiplying through by $(a/u_1)^2$ and setting a equal to 0. Similarly, equation (47), which is identical to equation (34), is the equation relating Δ to P_r for free convection alone. It could have been obtained from equation (44) by setting u_1 equal to zero.

In general, however, equations (45), (46) and (47) are incompatible, showing that a solution based on the assumption that Δ is independent of x is not admissible for combined convection. This result is, in fact, obvious from the differing boundary conditions on the velocity and temperature profiles and the form of the basic equations.

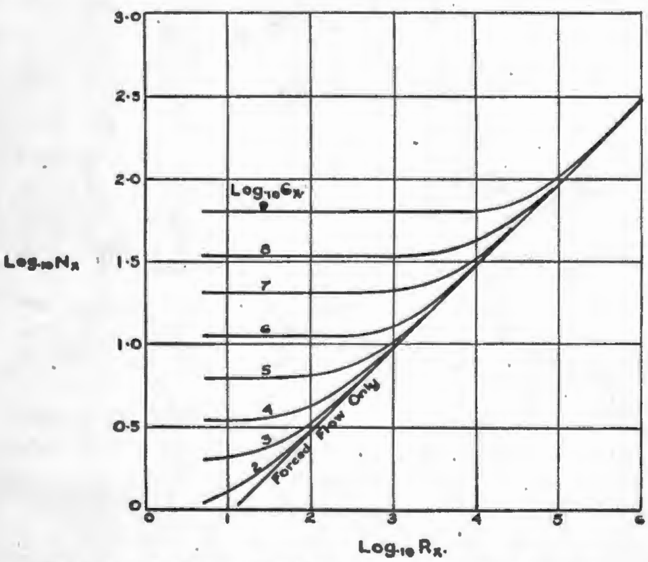

Fig 4 Predicted values of local Nusselt number for combined convection in air

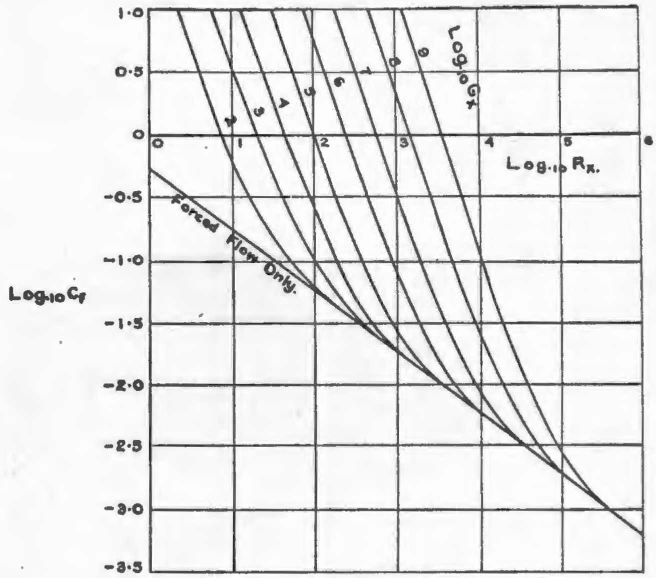

Fig 5 Predicted values of local friction coefficient for combined convection in air

RESULTS FOR FLUIDS WITH PRANDTL NUMBERS NEAR UNITY

It will be noted from Fig 2 that, for both free convection alone and forced convection alone, Δ is very nearly equal to unity for fluids having Prandtl numbers near unity. It is to be expected, therefore, that in combined convective flows with such fluids, Δ will also remain very near unity. Thus, although it is not in general possible to assume a constant value of Δ for combined convective flows, it is to be expected that a solution for such flows based on the assumption that $\Delta = 1$ will be approximately correct for the flow of fluids having Prandtl numbers near unity, and this includes all diatomic gases.

Now, equation (29) gives for combined convective flows, with Δ constant:

$$(\delta_u/x) = \{-(M/3L)\,(R_x/G_x) + [(1/9)\,(M/L)^2\,(R_x/G_x)^2 + (8/3\,P_rL\Delta)\,(1/G_x)]^{0{\cdot}50}\}^{0{\cdot}50} \quad \ldots\ldots\ldots (48)$$

The correct algebraic sign in this solution has been determined by noting that, as only the case where the free and forced flows assist each other is being considered, both R_x and G_x must be positive.

For the present case, where Δ is being assumed equal to unity, this equation reduces to:

$$(\delta_u/x) = \{-(10{\cdot}8R_x/G_x) + [(10{\cdot}8R_x/G_x)^2 + (733/P_r\,G_x)]^{0{\cdot}50}\}^{0{\cdot}50} \quad \ldots\ldots\ldots (49)$$

The values of M and L corresponding to Δ equal to unity, as given in Fig 1, have been used.

Using equation (49), (27) and (28) allows the variation of N_x and C_f with R_x and G_x to be determined for any fluid having a Prandtl number near unity. Results for air ($P_r = 0{\cdot}73$) are shown in Figs 4 and 5.

LIMITS FOR FREE AND FORCED CONVECTION

It is to be expected that, in combined convective flows in which either the forced or the free convective effect is very much more predominant than the other, Δ will be very nearly equal to the value that would exist if the predominant flow alone existed. Therefore, in order to find the conditions under which either the forced velocity or the buoyancy effects are negligible, it will be assumed that Δ is constant and equal to the value corresponding to that existing in the same fluid when the predominant flow alone exists.

Now, for combined convective flows in which Δ can be taken as constant, equations (48) and (27) give:

$$N_x = 2/\Delta\{-(M/3L)(R_x/G_x) + [1/9(M/L)^2(R_x/G_x)^2 + (8/3\,P_r\,L\Delta)(1/G_x)]^{0\cdot50}\}^{0\cdot50} \quad . . . . (50)$$

If, therefore, N_{xc} denotes the Nusselt number existing in a given combined convective flow, and if N_{xr} denotes the Nusselt number that would exist with free flow alone at the same value of G_x, then equations (35) and (50) give:

$$(N_{xr}/N_{xc}) = \{-(M/3L)(3P_rL\Delta/8)^{0\cdot50}(R_x/G_x)\,G_x^{0\cdot50} + [(3P_rL\Delta/8)(M/3L)^2(R_x/G_x)^2\,G_x + 1]^{0\cdot50}\}^{0\cdot50} \quad . (51)$$

This equation allows the values of R_x to be determined for which N_{xr} differs from N_{xc} by any specified percentage for any values of G_x and P_r, Δ being related to P_r (by assumption) by equation (34). It is seen that the solution of this equation, for any specified percentage difference, is of the form:

$$G_x/R_x^2 = f(P_r) \quad . . . . (52)$$

Values, for a five per cent difference between N_{xr} and N_{xc}, of the function f are shown in Fig 6.

For flows in which the forced velocity is predominant a similar procedure can be adopted. If, here again, subscript c denotes the conditions in the combined convective flow and if subsript f denotes the conditions that would exist with a forced flow alone, in which the buoyancy effects are completely negligible, at the same value of R_x, then equations (40) and (50) give:

$$(N_{xf}/N_{xc}) = 0\cdot172\,(R_x^3/G_x)^{0\cdot50}\{-(M/3L) + [(M/3L)^2 + (8G_x/3P_rL\Delta R_x^2)]^{0\cdot50}\}^{0\cdot50} \quad . . . . (53)$$

This equation allows the values of G_x to be dctermined for which N_{xf} differs from N_{xc} by any specified percentage for any values of R_x and P_r, Δ being related to P_r, by assumption, by equation (39). It is seen that the solution of this equation is of the same form as obtained with predominantly free convective flow, i.e.:

$$G_x/R_x^2 = g(P_r) \quad . . . . (54)$$

Values, for a five per cent difference between N_{xf} and N_{xc}, of the function g are also shown in Fig 6.

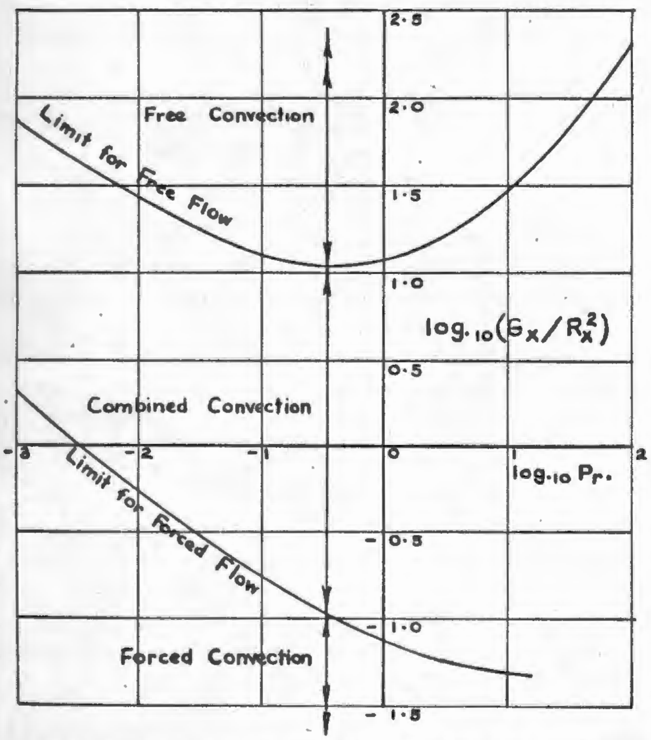

Fig 6 Predicted five per cent limits for free and forced convection alone

COMPARISON WITH A PREVIOUS ANALYSIS

Sparrow and Gregg[4], by considering a limiting form of the solution to the basic partial differential equations for combined convection, have predicted that the local Nusselt number as given by the equations appropriate to forced convection alone will be within approximately five per cent of the actual value in the combined flow when:

$$G_x/R_x^2 < 0\cdot075$$

for all Prandtl numbers. This result may be compared with the results derived in the preceding section and given in Fig 6.

CONCLUSION

The integrated boundary layer equations for combined convective flows have been derived. It has been shown that while simple solutions, in which the ratio of the thermal boundary layer thickness to the velocity boundary layer thickness is constant, can be derived for free flow alone and for forced flow alone, such a solution will not, in general, apply in combined flows. It is indicated, however, that such a solution will be approximately correct for combined flows involving fluids having Prandtl numbers near unity, and results based on this assumption have been derived for air flow. This assumption is also used, for fluids having arbitrary Prandtl numbers, to derive approximately the conditions under which either the forced flow effects or the free flow effects are negligible in comparison with the other to any required degree of accuracy.

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Paper 8.

INTEGRAL EQUATION SOLUTIONS FOR COMBINED FREE AND FORCED CONVECTIVE LAMINAR FLOW OVER A VERTICAL ISOTHERMAL PLATE.

Summary:

It has been shown, elsewhere, that a solution to the combined convection problem can be obtained by expressing this solution relative to that applicable in either purely forced or purely free flow in terms of a parameter which combines the Reynolds and Grashof numbers. The application of this result to an integral equation solution is dealt with in the present note. Numerical results for the flow of a fluid with a Prandtl number of 0.72 are given.

Notation:

A_1, A_2, A_3 = Coefficients in momentum integral equation.

C_f = Local skin friction coefficient.

= $2 \tau_w / \rho u_1^2$.

f = Function describing the difference between δu and δuo.

$\bar{f}$ = Function describing the difference between δu and $\bar{\delta} uo$

G_x = Grashof number based on x_1.

= $\beta g \rho^2 x^3 (T_w - T_1) / \mu^2$.

G_ℓ = Grashof number based on ℓ.

= $\beta g \rho^2 \ell^3 (T_w - T_1) / \mu^2$.

h = Function describing the difference between Δ and Δ_o.

$\bar{h}$ = Function describing the difference between Δ and $\bar{\Delta}_o$.

K = Function of $\bar{\Delta}_o$ given by equation (13).

ℓ = Length of plate considered.

L = Function of Δ, given by equations (5) and (6).

M = Function of Δ, given by equations (5) and (6).

N_x = Local Nusselt number.

= $q_w / (T_w - T_1) k$.

N_ℓ = Mean Nusselt number.

= $\bar{q}_w \ell / (T_w - T_1) k$.

P_r = Prandtl number.

q_w = Local rate of heat transfer per unit area from wall to fluid.

$\bar{q}_w$ = Average rate of heat transfer per unit area from wall to fluid over plate length, ℓ.

R_x = Reynolds number based on x,

= $\rho u_1 x / \mu$.

R_ℓ = Reynolds number based on ℓ.

= $\rho \, u_1 \ell / \mu$

S_x = $G_x / R_x^{\,2}$

$\bar{S}_x$ = $R_x / G_x^{\,0.5}$

S_ℓ = G_ℓ / R_ℓ^2

T = Temperature.

T_1 = Temperature outside boundary layer.

T_w = Wall temperature.

u = Velocity componant in x-direction.

u_1 = Value of u outside boundary layer.

x = Co-ordinate parallel to plate surface.

y = Co-ordinate normal to plate surface.

α = $\beta \, g \, \rho \, \theta_w / \mu$.

β = Coefficient of Cubical expansion.

δ_u = Thickness of Velocity boundary layer.

δ_{uo} = Thickness of velocity boundary layer in purely forced flow at some value of R_x as in actual flow.

$\bar{\delta}_{uo}$ = Thickness of velocity boundary layer in purely free flow at same value of G_x as in actual flow.

δ_T = Thickness of temperature boundary layer.

Δ = δ_T / δ_u

Δ_0 = Value of Δ in purely forced flow.

$\bar{\Delta}_0$ = Value of Δ in purely free flow.

ϵ = y/δ_T

η = y/δ_u .

θ = $T - T_1$.

θ_w = $T_w - T_1$.

μ = Coefficient of viscosity.

ρ = Density.

τ_w = Local surface shearing stress.

Subscripts:

for = Conditions that would exist in purely forced flow at the same value of R_x as in the actual flow.

free = Conditions that would exist in purely free flow at the same value of G_x as in the actual flow.

INTEGRAL EQUATION SOLUTIONS FOR COMBINED FREE AND FORCED CONVECTIVE LAMINAR FLOW OVER A VERTICAL ISOTHERMAL PLATE

Introduction:

In ref.1, the outline of an integral equation approach to the study of laminar combined free and forced convective flow over a vertical isothermal plate was given. In that note, approximate solutions to the derived equations were given for the case where the forced and the free flow were in the same direction, i.e. assisting each other. The purpose of the present note is to indicate how a general solution to this set of equations can be obtained, this general solution being based on the observation of ref.2 that the general combined convection solution can be expressed in terms of either the purely forced or purely free flow solutions as a function of a parameter which combined the Reynolds and Grashof numbers.

Now it was shown in ref.1 that if fourth degree polynomials were assumed for the velocity and temperature profiles, then the application of the assumed boundary conditions on these profiles led to the following expressions for these profiles:-

$$u = (\alpha \delta_u^2/6)(\eta - 3\eta^2 + 3\eta^3 - \eta^4) + U_1(2\eta - 2\eta^3 + \eta^4) \quad (1)$$

and

$$\theta = \theta_W(1 - 2\epsilon + 2\epsilon^3 - \epsilon^4) \quad (2)$$

Using these in the momentum and energy integral equations then leads to the following governing equations:-

$$\frac{d}{dx}\left[A_1 \alpha\ u_1 \delta_u^3 + A_2\ u_1^2\ \delta_u + A_3 \alpha^2\ \delta_u^5 \right]$$

$$= (0.3\alpha\mu\Delta\delta_u/\rho) - (\mu/\rho)\left[\alpha\,\delta_u/6 + 2u_1/\delta_u\right] \quad (3)$$

and:-

$$\frac{d}{dx}\left[\alpha\,\delta_u^3\,L + u_1\delta_u\ M\right] = (2\mu/\rho\ P_r\,\Delta\delta_u) \quad (4)$$

Where L and M are functions of Δ which are given by:-

(i) For $\Delta < 1$:-

$$L = \Delta^2/90 - \Delta^3/84 + 9\Delta^4/1680 - \Delta^5/1080$$

$$M = 2\Delta^2/15 - 3\Delta^4/140 + \Delta^5/180 \quad (5)$$

(ii) For $\Delta > 1$:-

$$L = 1/120 - 1/180\Delta + 1/840\Delta^3 - 1/3024\Delta^4$$

$$M = 3\Delta/10 - 3/10 + 2/15\Delta - 3/140\,\Delta^3 + 1/180\Delta^4 \quad (6)$$

The coefficients in equation (3) are given by:-

$$A_1 = 1/945 \;, \quad A_2 = -37/315 \;, \quad A_3 = 1/9072$$

It is also shown in ref.1 that the local Nusselt number and skin friction coefficient are given by:-

$$N_x = 2\,x / \Delta\,\delta_u \tag{7}$$

and:-

$$C_f = (G_x \delta_u / 3R_x^{\;2}\,x) + (4\,x / R_x \delta_u) \tag{8}$$

In the case of purely forced flow, when the buoyancy effects are assumed to be negligible, i.e. α is taken as zero, the above set of equations can be solved giving the following:-

$$\delta_{uo}/x = (2 / \sqrt{-A_2}\,) / \sqrt{R_x} \tag{9}$$

and:-

$$\Delta_o M(\Delta_o) = -A_{2/P_r} \tag{10}$$

Similarly, in purely free flow when $u_1 = 0$, the above set of equations can also be solved giving the following:-

$$\bar{\delta}_{uo}/x = K / G_x^{\;0.25} \tag{11}$$

and:-

$$3K^4 L(\bar{\Delta}_o)\bar{\Delta}_o = 8/P_r \qquad (12)$$

where:-

$$K^4 = (4/5A_3)(0.3\bar{\Delta}_o - 1/6) \qquad (13)$$

These are the reference states relative to which the present combined convective solution will be expressed. It will be noted that in both cases Δ is a constant, i.e. independent of x, equal, in the first case, to Δ_o and, in the second, to $\bar{\Delta}_o$.

Analysis with Forced Flow Reference Conditions:

The analysis of the general combined convection flow equations given in ref.2 suggests that the solution to the present set of equations, when expressed relative to the solution for purely forced flow, will be of the form:-

$$\delta_u = \delta_{uo}\left[1 + f(G_x/R_x^2)\right] = \delta_{uo}\left[1 + f(s_x)\right] \qquad (14)$$

$$\Delta = \Delta_o\left[1 + h(G_x/R_x^2)\right] = \Delta_o\left[1 + h(s_x)\right] \qquad (15)$$

Substituting these into the governing integral equations (3) and (4) and using equation (9), gives:-

$$\left[\mp 24 A_1 s_x^2 (1+f)^2/A_2 + 2A_2 s_x + 160 A_3 s_x^3(1+f)^4/A_2^2\right]\frac{df}{ds_x}$$

$$= (1 + f) \left[\pm 12 A_1 s_x (1 + f)^2/A_2 - A_2 - 80 A_3 s_x^2 (1 + f)^4/A_2^2 \right]$$

$$\pm 0.6 \Delta_o s_x (1 + f)(1 + h) \mp s_x (1 + f)/3 + A_2/(1 + f) \quad (16)$$

and:-

$$\left[2M \mp 24 L (1 + f)^2 s_x/A_2 \right] \frac{d f}{d s_x} + \left[2 M' \Delta_o (1 + f) \right.$$

$$\left. \mp 8 L' \Delta_o (1 + f)^3 s_x/A_2 \right] \frac{d h}{d s_x} = \pm 12 L (1 + f)^3/A_2$$

$$- A_2 / P_r \Delta_o s_x (1 + h)(1 + f) - M (1 + f)/s_x \quad (17)$$

Where:-

$$L' = \frac{d L}{d \Delta} \quad (18)$$

$$M' = \frac{d M}{d \Delta}$$

In these equations the positive x-direction has been taken as the direction of the forced flow. Thus, where the alternative of a + or - sign is given, the upper sign applies in those cases where the buoyancy forces and forced flow are in the same direction, i.e. in assisting flow, and the lower sign applies where they are in opposite directions, i.e. in opposing flow.

The boundary conditions on f and h in the above pair of equations follow from the fact that s_x determines the ratio of buoyancy force effects to forced flow effects and f and h represent the effects of these buoyancy forces in combined convection on the purely forced flow solution. Therefore these boundary conditions are:-

$$S_x \rightarrow 0 : \; f \rightarrow 0 \quad , \; h \rightarrow 0 \tag{19}$$

Equations (16) and (17) can be simultaneously solved by numerical methods to give the variation of f and h with S_x. To start the numerical process, a series expansion for f and h in terms of S_x, similar to that used in ref.3, was adopted in the present study. Calculations have been carried out for the case of $P_r = 0.72$, the variation of f and h with S_x for assisting and opposing flows being shown in fig. 1. The calculations for the opposing flow case were terminated near the separation point (see later).

Now, substituting equations (14) and (15) into equations (7) and (8) gives:-

$$N_x = 2_x / \Delta_o \, \delta_{uo} \; (1 + h) \; (1 + f) \tag{20}$$

and:-

$$C_f = S_x \, \delta_{uo} \; (1 + f)/3\,x \; + \; 4x/R_x \, \delta_{uo}(1 + f) \tag{21}$$

But, from these it also follows that:-

$$N_{x\ for} = 2\,x/\Delta_o\,\delta_{uo} \tag{22}$$

and

$$C_{f\ for} = 4\,x\,/R_x\,\delta_{uo} \tag{23}$$

Therefore, dividing (20) and (21) by (22) and (23) respectively and rearranging gives:-

$$N_x\,/N_{x\ for} = 1/\,(1 + f)\,(1 + h) \tag{24}$$

and

$$C_f/\,C_{f\ for} = -\,S_x\,(1 + f)/3A_2 + 1/(1 + f) \tag{25}$$

Using these equations, the calculated values of f and h can be used to obtain the variation of $N_x/N_{x\ for}$ and $C_f/C_{f\ for}$ with S_x. Results for $P_r = 0.72$ are shown in figs. 2 and 3.

The mean Nusselt number, N_ℓ, is also of considerable interest, it being defined by:-

$$N_\ell = \overline{q}_w\,\ell\,/\,(\,T_w - T_1\,)\,k \tag{26}$$

$\overline{q}_w$ being the mean rate of heat transfer, given by:-

$$\overline{q}_w = \left(\int_0^\ell q_w\,d\,x\right)\Big/\ell \tag{27}$$

From these it follows that:-

$$N_{\ell} = \int_{o}^{\ell} (N_x / x)\, d\, x \tag{28}$$

Hence, using equations (20), (22) and (9) it follows that:-

$$N_{\ell} / N_{\ell\,for} = (1/2\, S_{\ell}^{0.5}) \int_{o}^{S_{\ell}} dS_X\, S_X^{0.5} / (1 + f)\,(1 + h) \tag{29}$$

Values of $N_{\ell} / N_{\ell\, for}$ for the case $P_r = 0.72$, derived from the previously given results, are also shown in fig.2.

It is also worth nothing that in the opposing flow case it is possible for "separation" to occur, i.e. for a point to be reached beyond which there is a buoyancy controlled flow in one direction adjacent to the wall in the opposite direction to the outer forced flow. At this separation point $d\,u/ d\,y = 0$ so it follows from equation (1) that the value of S_x at separation, i.e. $S_{x\ sep}$, is given by:-

$$S_{x\ sep} \left[1 + f\ (S_{x\ sep}) \right]^2 = -3A_2 \tag{30}$$

Extrapolating the previously given variation of f with S_X gives:-

$$S_{x\ sep} = 0.228$$

The results given in fig. 1 show that, at least for a

Prandtl number of 0.72, f remains approximately four times greater than h justifying, to some extent, the approximate solution for air flow given in ref.1. This approximate solution was based on the assumption that Δ remained approximately equal to one even in combined flow and that the combined convection effects arose only, therefore, through changes in δ_u.

Analysis with Free Flow Reference Conditions:

The analysis of ref. 2 shows that the combined convection solution can also be obtained by expressing it relative to the purely free flow conditions. For the purpose of the present analysis, this implies that an alternative way of expressing the solution is:-

$$\delta_u = \bar{\delta}_{uo} \left[1 + \bar{f} \; (R_x / G_x^{0.5}) \right] = \bar{\delta}_{uo} \left[1 + \bar{f} \; (\bar{\xi}_x) \right] \quad (31)$$

and:-

$$\Delta = \bar{\Delta}_o \left[1 + \bar{h} \; (R_x / G_x^{0.5}) \right] = \bar{\Delta}_o \left[1 + \bar{h} \; (\bar{\xi}_x) \right] \quad (32)$$

Of course, this solution must be related to the previous one, which used purely forced flow reference conditions, since, at a given point in the flow, they must both give the same values of δ_u and Δ. Equating equations (14) and (31) and equations (15) and (32) gives the following relations connecting the two solutions:-

$$\overline{f} = \left(2/K \sqrt{-A_2 s_x}\right) (1 + f) - 1 \qquad (33)$$

$$\overline{h} = (\Delta_o / \overline{\Delta}_o) (1 + h) - 1 \qquad (34)$$

While, from their definitions, it follows that:-

$$\overline{s}_x = 1/ s_x \qquad (35)$$

Using these equations, the purely free flow reference condition solution can be obtained from the purely forced flow reference condition solution. The advantage in having both solutions available is that the entire range of values of s_x and $\overline{s}_x$ is more easily covered since $s_x \to \infty$ as $\overline{s}_x \to 0$ and vice versa.

Substituting equations (31) and (32) into equations (3) and (4) and rearranging gives:-

$$\left[\mp 3 A_1 \overline{s}_x K^2 (1 + \overline{f})^2 - A_2 \overline{s}_x^{\,2} - 5A_3 K^4 (1 + \overline{f})^4\right] (K \overline{s}_x/2) \frac{d\overline{f}}{d\overline{s}_x}$$
$$= \left[\mp 3A_1 \overline{s}_x K^2 (1 + \overline{f})^2 - A_2 \overline{s}_x^{\,2} - 5A_3 K^4(1 + \overline{f})^4\right] K (1 + \overline{f})/4$$
$$+ 0.3 \overline{\Delta}_o K (1 + \overline{f}) (1 + \overline{h}) - K (1 + \overline{f})/6 \mp 2 \overline{s}_x/K (1 + \overline{f}) \qquad (36)$$

and:-

$$\left[3 L K^2 (1 + \overline{f})^2 \pm M \overline{s}_x\right] (K \overline{s}_x /2) \frac{d f}{d s_x} + \left[K^3 L' (1 + \overline{f})^3\right.$$

$$\pm \; \bar{S}_x (1 + \bar{f}) K M' \Big] (\bar{\Delta}_0 \; \bar{S}_x \; /2) \frac{d \bar{h}}{d \bar{S}_x} = \Big[3 L K^2 (1 + \bar{f})^2$$

$$\pm M \bar{S}_x \Big] K (1 + \bar{f}) /4 - 2/P_r \bar{\Delta}_0 K (1 + \bar{f}) (1 + \bar{h}) \qquad (37)$$

In these equations, the positive x-direction has been taken in the direction of the buoyancy forces, the upper sign, in those terms where an alternative is given, referring to assisting flow and the lower to opposing flow,

By virtue of the same reasoning as was used to derive equation (19), the boundary conditions on $\bar{f}$ and $\bar{h}$ in equation (36) and (37) are:-

$$\bar{S}_x \rightarrow 0 \; ; \quad \bar{f} \rightarrow 0 \quad , \quad \bar{h} \rightarrow 0 \qquad (38)$$

The variation of $\bar{f}$ and $\bar{h}$ with $\bar{S}_x$ for the assisting flow of a fluid with a Prandl number of 0.72, obtained by simultaneous numerical solution of equations (36) and (37), is shown in fig.4. Also, noting that, by using the same procedure as used to obtain equation (24),:-

$$N_x / N_{x\,free} = 1/(1 + \bar{f}) (1 + \bar{h}) \qquad (39)$$

The variation of $N_x/N_{x\,free}$ obtained from the results given in fig.4 is shown in fig.5.

No results for opposing flow using the purely free flow reference conditions were obtained. This was due, mainly, to the difficulties experienced in starting the numerical solution. However, it was also noted that, since the solution would have to start at a small value of $\bar{S}_x$, it would largely be covering the separated region of the flow, $\bar{S}_x$ being equal to zero when x is infinite. Therefore, since it is doubtful whether the derived equations are applicable in this separated region, the solution, had it been obtained, would have had considerable doubt attached to it.

Limits for PurelyFree and Forced Flow:

One of the most important questions that combined convection solutions must answer is under what conditions the heat transfer can, to any prescribed degree of accuracy, be assumed to be by either purely forced or purely free convection. The answer is obtained in the present case, by noting that $(N_x - N_{x\ for})/N_{x\ for}$ and $(N_x - N_{x\ free})/N_{x\ free}$ are the relative differences between the actual N_x and those existing, under the same conditions, in the two limiting cases. Thus, for any prescribed difference, the corresponding values of S_x and $\bar{S}_x$ can be obtained. The previously given results for a fluid with a Prandtl number of 0.72 give the following for a difference of 5%:-

(i) For assisting flow, the actual N_x is within 5% of $N_{x\ for}$ when S_x is less than 0.087,

(ii) For opposing flow, the actual N_x is within 5% of $N_{x\ for}$ when S_x is less than 0.072,

(iii) For assisting flow, the actual N_x is within 5% of $N_{x\ free}$ when $\bar{S}_x$ is less than 0.34.

Conclusions:

The formulation of a method of solving the integral equations for combined convection has been given. Numerical results for the flow of a fluid with a Prandtl humber of 0.72 have been obtained and have been used to deduce under what conditions the flow can be assumed to be either purely forced or purely free convection.

References:

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2. Paper 6 of this thesis.

3. HAUPTMANN, E.G., "Laminar Boundary-Layer Flows with Small Buoyancy Effects", Int. J. Heat and Mass Transfer, vol. 8, 1965, pp. 289 - 295.

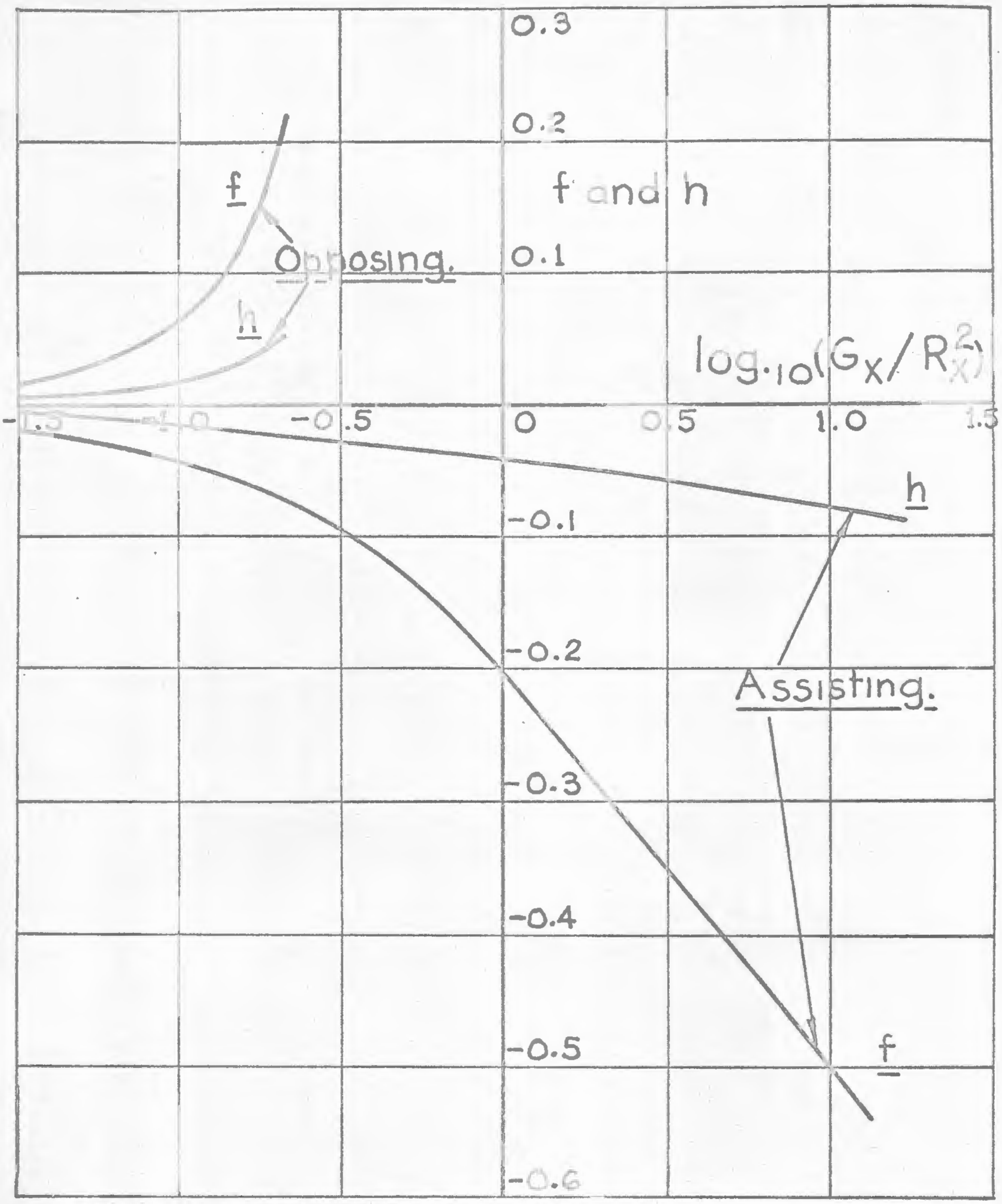

Fig. 1. Variation of the velocity boundary layer thickne[redacted] parameter, f, and thickness ratio parameter, h, in the solution expressed relative to that in purely forced flow, with δ_x = [redacted].

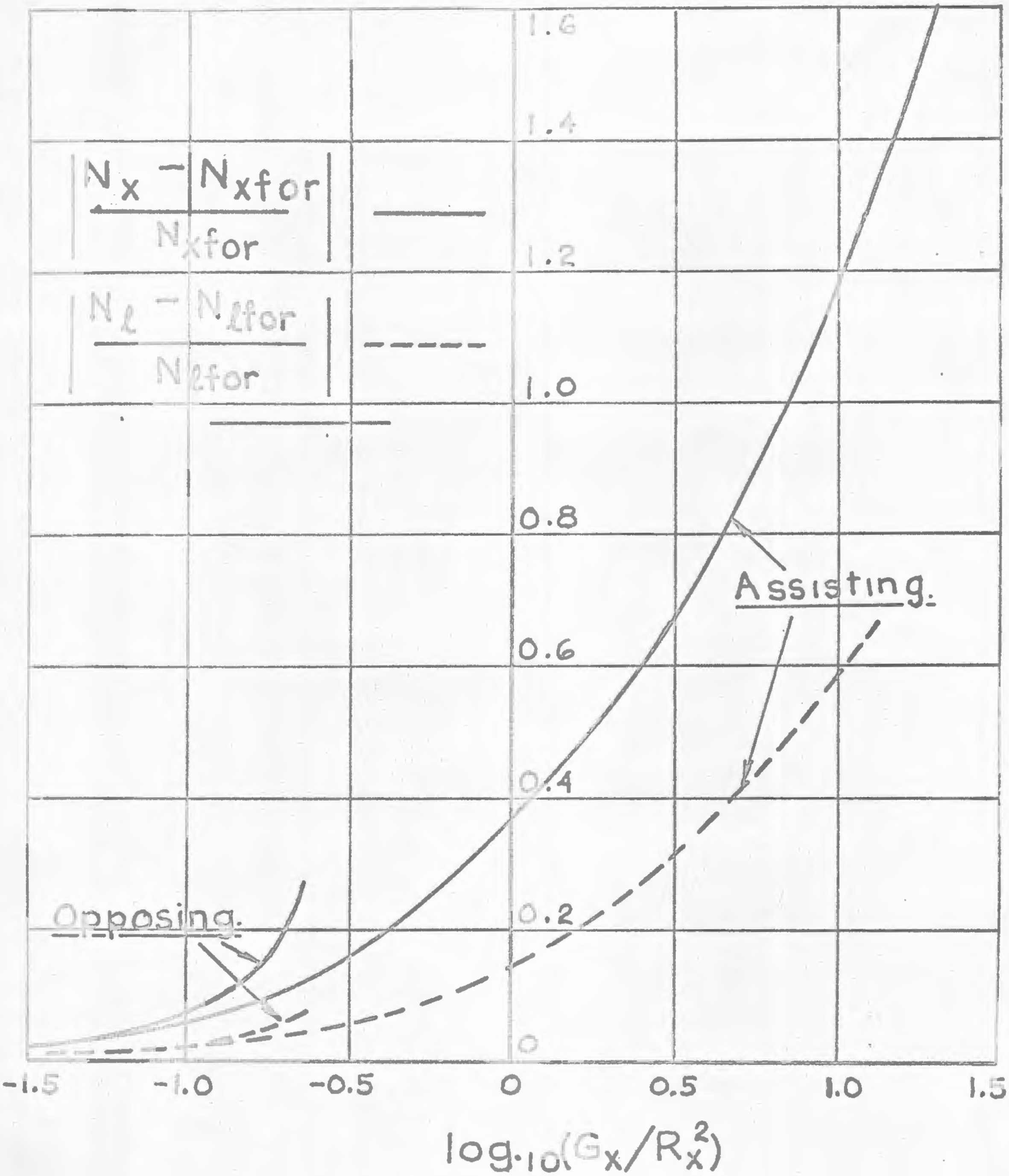

Fig. 2. Variation of local and average Nusselt number difference with S_x.

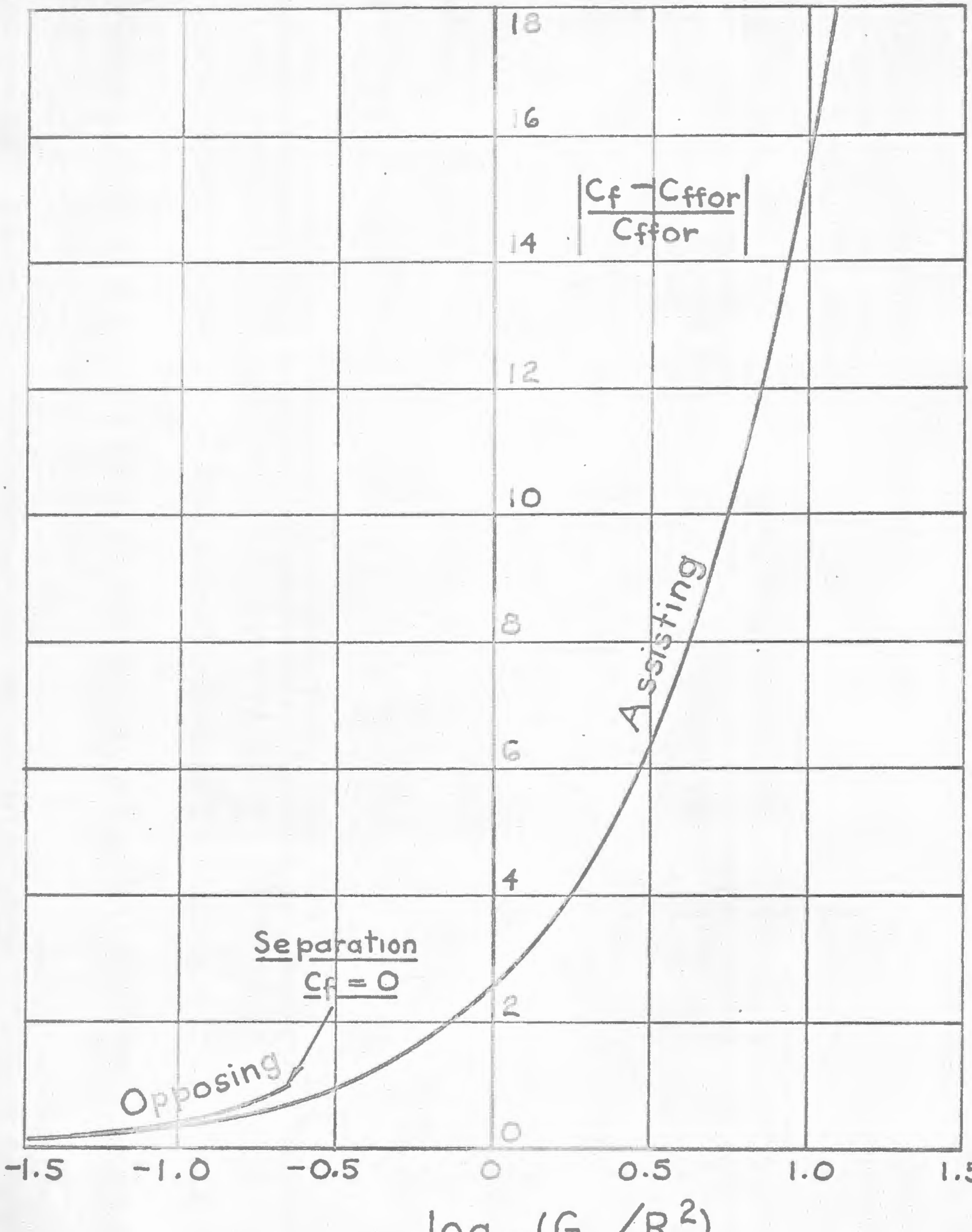

Fig. 3. Variation of local skin friction coefficient difference with S_x.

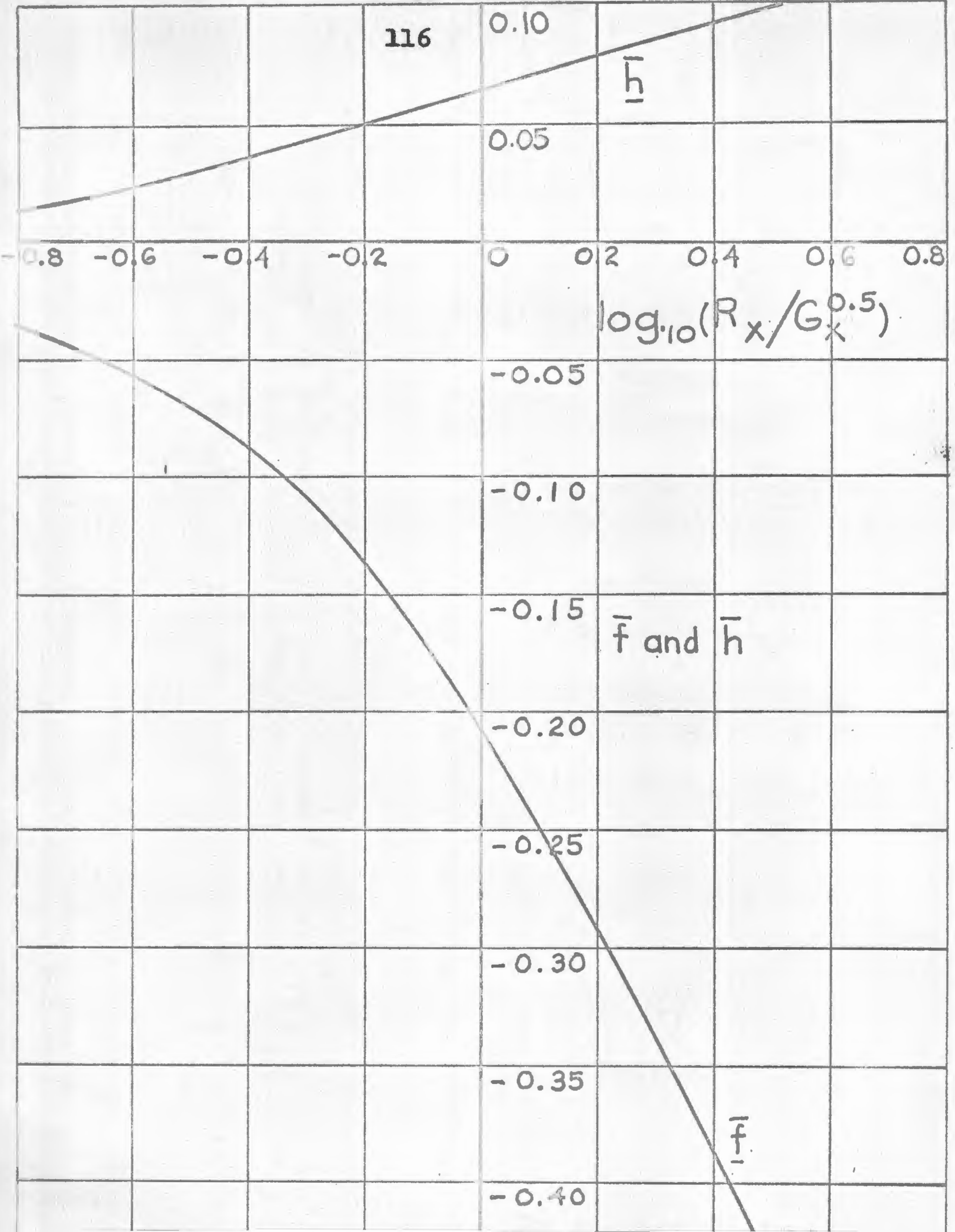

Fig. 4. Variation of the velocity boundary layer thickness parameter, $\bar{f}$, and thickness ratio parameter, $\bar{h}$, in the solution expressed relative to that in purely free flow, with $S_x = R_x / G_x^{0.5}$.

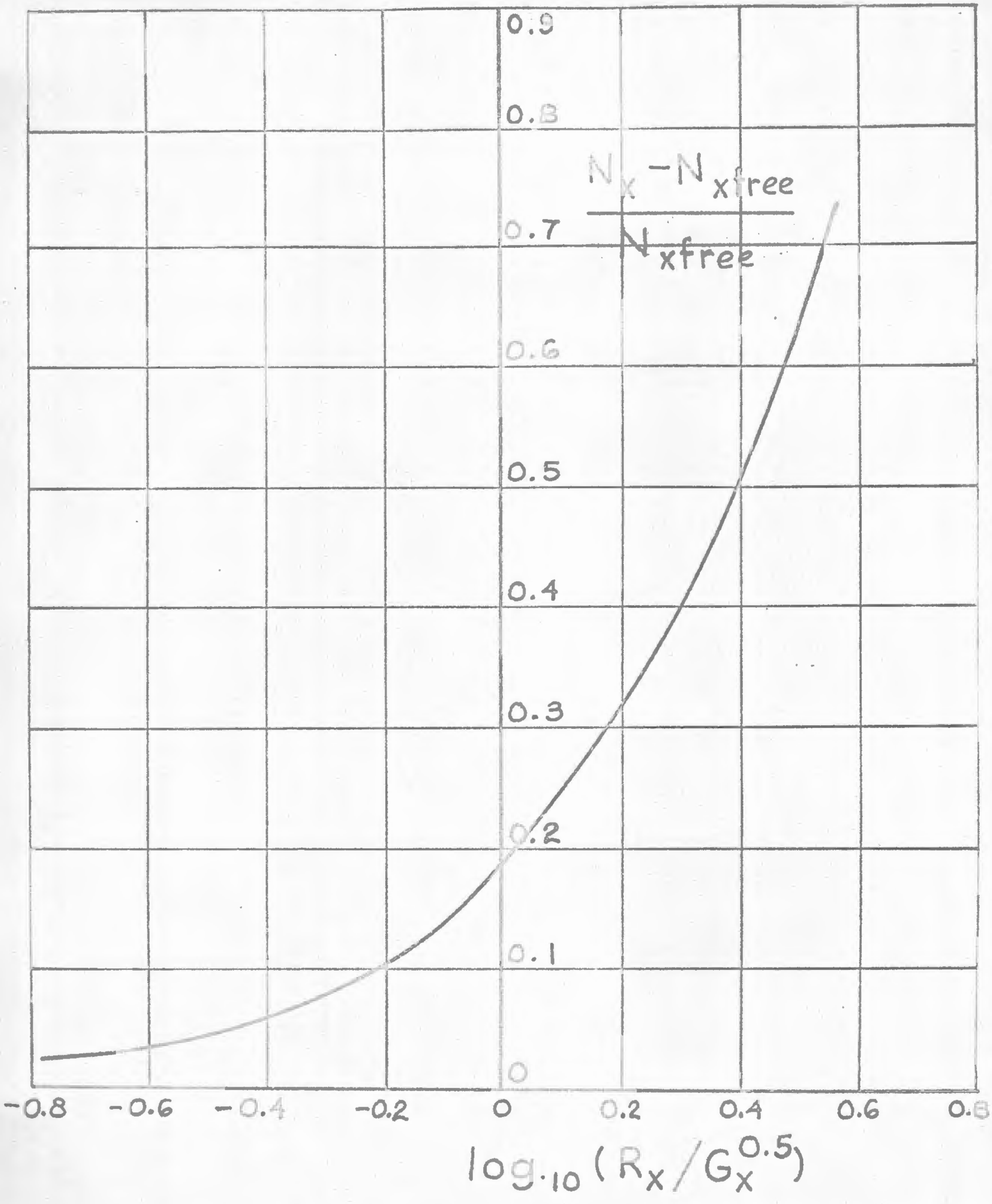

Fig. 5. Variation of local Nusselt number difference with $\bar{S}_x$ for assisting flow.

Paper 9.

A NUMERICAL STUDY OF LAMINAR COMBINED CONVECTIVE AIR FLOW OVER A VERTICAL PLATE.

SUMMARY:

The basic laminar boundary layer equations are solved, for various values of Grashof and Reynolds numbers, using a numerical step-by-step procedure. Typical results for the velocity and temperature profiles and the local Nusselt number are presented. By using the purely forced or purely free flow solution as a reference state , it is shown that the combined convective effect can be expressed in terms of a single parameter which combines the Reynolds and Grashof numbers. The present numerical results are also compared with some available analytical solutions.

A NUMERICAL STUDY OF LAMINAR COMBINED CONVECTIVE AIR FLOW OVER A VERTICAL PLATE.

Introduction:

The problems of laminar forced convective flow and laminar free convective flow over a flat plate have been the subject of a number of analyses and very good agreement has been obtained, in such cases, between experiment and theory. Comparatively little attention has, however, been given to the problem of combined convective flows in which both the forced flow and the buoyancy effects are of importance. Available solutions to this problem are based either on the approximate van Karman-Pohlhausen integral equation method[1.-4.] or on a series solution[5.6.] to the governing partial differential equations. These latter solutions start with either the forced flow or the free flow solution and express the difference between the combined flow solution and these known solutions in terms of a parameter known to characterise the ratio of buoyancy to forced flow effects, or its inverse. While a solution of this form is known to be perfectly general,[7.] the assumption of a power series in which the form of the general coefficient is unknown limits the solution to either predominantly forced or predominantly free flows. So far, only two terms[6.] in the power series are available and no indication of the range of values over which these solutions can be applied is available.

The main difficulty which, of course, characterises combined convective flow is that the boundary layer profiles are not, in general, similar[8.] and the governing partial differential equations cannot, therefore, be reduced to a set of ordinary differential equations. In view of this fact and of the deficiencies in the available solutions, mentioned above, an attempt to solve the governing equations using a simple, purely numerical technique of the type given by Wu[9]. seems to be justified.

Now it was shown in reference 7. that by considering the difference between the local velocity and temperature and the values that would exist, under the same conditions, in the limiting case of either purely forced or purely free flow, the governing boundary layer equations could be reduced to a general set of equations in two new independent variables. This implies that there is not an independent solution for every combination of Grashof and Reynolds numbers. However, the general set of equations derived in this manner is far more complex than the basic boundary layer equations and, in the present note, these latter basic equations are for simplicity solved for particular values of Grashof and Reynolds numbers. The results so obtained are then used to derive the general solution as explained later.

In the present note attention is restricted to the flow of a gas with a Prandtl number of 0.72 and whose properties are constant throughout the flow field, over a flat plate with a constant surface temperature. The numerical method of solution used is, however, quite general, being capable of taking into account the variation of any or all of these quantities.

Method of Analysis:

The equations governing the flow of a constant property fluid over a vertical flat plate are, if the boundary layer assumptions can be adopted,:-

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \beta \, g \, (T - T_1) + (\mu/\rho) \frac{\partial^2 u}{\partial y^2} \qquad (1)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = (k/\rho \, C_p) \frac{\partial^2 T}{\partial y^2} \qquad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

where x = co-ordinate along plate surface in the flow direction and measured from an origin at the leading edge of the plate,

y = co-ordinate normal to plate surface,

u = velocity componant in the x-direction at any point in the flow field,

v = velocity componant in the y-direction at any point in the flow field,

T_1 = freestream fluid temperature,

μ = coefficient of viscosity,

ρ = density,

k = coefficient of conductivity,

C_p = specific heat,

β = coefficient of cubical expansion.

For the present purposes, the above set of equations is more conveniently re-written in the following form:-

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = (G_{\ell}/R_{\ell}^{2})\bar{\theta} + (1/R_{\ell})\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \qquad (4)$$

$$\bar{u}\frac{\partial \bar{\theta}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{\theta}}{\partial \bar{y}} = (1/P_r R_{\ell})\frac{\partial^2 \bar{\theta}}{\partial \bar{y}^2} \qquad (5)$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \qquad (6)$$

where $\bar{u} = u/u_1$

$\bar{v} = v/u_1$

u_1 = freestream forced velocity,

$\bar{\theta} = (T - T_1)/(T_w - T_1)$

T_w = constant wall temperature,

$\bar{x} = x/\ell$,

$\bar{y} = y/\ell$,

ℓ = some convenient reference length,

G_{ℓ} = Grashof number based on ℓ

$= \beta g \rho^2 \ell^3 (T_w - T_1)/\mu^2$,

R_{ℓ} = Reynolds number based on u_1 and ℓ

$= \rho u_1 \ell/\mu$,

P_r = Prandtl number of fluid

$= \mu C_p/k$.

With the system of co-ordinates used above, G_ℓ will be taken as positive when the buoyancy forces are in the positive x-direction, i.e. in the same direction as the forced flow, and as negative when the buoyancy forces are in the negative x-direction, i.e. in the opposite direction to the forced flow.

In order to numerically solve the above set of equations using a step-by-step procedure, the grid system shown in fig.1 will be used. The subscript m is used to denote the grid lines in the x-direction and subscript n the grid lines in the y-direction.

For small grid spacings $\Delta\bar{x}$ and $\Delta\bar{y}$, the following numerical approximations can be deduced using Taylor's expansion[9]:-

$$\left\{ \frac{\partial \bar{u}}{\partial \bar{x}} \right\}_{m,n} = (\bar{u}_{m+1,n} - \bar{u}_{m,n}) / \Delta \bar{x} \qquad (7)$$

$$\left\{ \frac{\partial \bar{u}}{\partial \bar{y}} \right\}_{m,n} = (\bar{u}_{m,n+1} - \bar{u}_{m,n-1}) / 2\Delta \bar{y} \qquad (8)$$

$$\left\{ \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right\}_{m,n} = (\bar{u}_{m,n+1} - 2\,\bar{u}_{m,n} + \bar{u}_{m,n-1}) / \Delta \bar{y}^2 \qquad (9)$$

Similar expressions apply, of course, for the derivatives of $\bar{\theta}$. Other forms of these approximations are possible, those given above being chosen to allow the solution to develop from the known boundary conditions which are discussed later.

Using the above numerical approximations, equations (4) and (5) reduce to:-

$$\bar{u}_{m+1,n} = \bar{u}_{m,n} - (\Delta\bar{x}/2\Delta\bar{y})(\bar{v}_{m,n}/\bar{u}_{m,n})(\bar{u}_{m,n+1} - \bar{u}_{m,n-1})$$
$$+ (G_\ell \Delta\bar{x}/R_\ell^2)(\bar{\theta}_{m,n}/\bar{u}_{m,n})$$
$$+ (\Delta\bar{x}/R_\ell \Delta\bar{y}^2)(\bar{u}_{m,n+1} - 2\bar{u}_{m,n} + \bar{u}_{m,n-1})/\bar{u}_{m,n} \quad (10)$$

and:-

$$\bar{\theta}_{m+1,n} = \bar{\theta}_{m,n} - (\Delta\bar{x}/2\Delta\bar{y})(\bar{v}_{m,n}/\bar{u}_{m,n})(\bar{\theta}_{m,n+1} - \bar{\theta}_{m,n-1})$$
$$+ (\Delta\bar{x}/P_r R_\ell \Delta\bar{y}^2)(\bar{\theta}_{m,n-1} - 2\bar{\theta}_{m,n} + \bar{\theta}_{m,n-1})/\bar{u}_{m,n} \quad (11)$$

The continuity equation is written in a slightly different form having regard to the boundary conditions and to the limitation of the numerical approximation error[9]:-

$$\bar{v}_{m+1,n} = \bar{v}_{m+1,n-1} - (\Delta\bar{y}/2\Delta\bar{x})(\bar{u}_{m+1,n-1} - \bar{u}_{m,n-1} + \bar{u}_{m+1,n} - \bar{u}_{m,n}) \quad (12)$$

Using the known conditions at the surface of the plate and at the leading edge, these being listed below, the above set of equations allow the two velocity components and the temperature function to be found at all the grid points.

The boundary conditions at the plate surface, where n = o, follow from the "no-slip" assumption i.e. from the assumption that at the surface both velocity componants are everywhere zero and the fluid temperature is everywhere equal to the constant plate temperature i.e. one set of boundary conditions is:-

For all values of m:-

$$\bar{u}_{m,o} = \bar{v}_{m,o} = 0 \qquad (13)$$

$$\bar{\Theta}_{m,o} = 1$$

It will also be assumed that along the grid line normal to the surface at the leading edge, i.e. along the m = o grid line, the velocity and temperature are undisturbed except, of course, at the actual plate surface where equation(13) applies. Thus the boundary conditions at the leading edge are:-

For all values of n greater than o:-

$$\bar{u}_{o,n} = o \qquad (14)$$

$$\bar{v}_{o,n} = \bar{\Theta}_{o,n} = o$$

The validity of this leading edge condition is discussed in ref.9. It will be noted that the use of this condition in dealing with the combined convective flow problem limits the extent of the results which can be derived for opposing flows since it assumes that there is a "separation" point somewhere on the plate i.e. a point on the plate beyond which the flow adjacent to the plate is in the opposite direction to the outer forced flow. It does not allow for the possibility that this reversed flow adjacent to the surface extends all the way to the leading edge.

With the above boundary conditions, the governing equations (10) to (12) can be solved for any given values of G_ℓ, R_ℓ and P_r using the following procedure. Since equations (13) and (14) together define conditions for all values of n along the m = o grid line, equations (10) and (11) can be used to determine $\bar{u}$ and $\bar{\theta}$ along the m = 1 line. With these determined, equation (12) can, since, from equation (13), $\bar{v}_{1,o} = o$, be used to find $\bar{v}$ along this m = 1 line. With all conditions along this m = 1 line then known, the same procedure can then be used to determine conditions along the m = 2 line and so on for all m.

The above procedure allows the velocity and temperature profiles along the grid lines to be determined. Beside these profiles, the main interest of the present note lies in the local Nusselt number. Since laminar flow is being considered, the local rate of heat transfer per unit area from the plate to the fluid, q_w, is given by:-

$$q_w = -k. \left.\frac{\partial T}{\partial y}\right|_{y = o} \qquad (15)$$

Using this, the local Nusselt number N_x, defined by:-

$$N_x = q_w x / k \ (T_w - T_1) \tag{16}$$

becomes in terms of the previously introduced dimensionless variables:-

$$N_x = -\bar{x} \left. \frac{\partial \bar{\theta}}{\partial \bar{y}} \right|_{\bar{y} = o} \tag{17}$$

Hence, using the numerical approximation to obtain the derivative, the Nusselt number can be obtained from the temperature profile.

Numerical Results:

In solving the above set of equations, a fixed value of R_ℓ was chosen and the solutions for various values of G_ℓ dervied. The main calculations were carried out for $R_\ell = 10^4$ and G_ℓ = 10^9, 3×10^8, 10^8, 3×10^7, 0, $- 3 \times 10^7$, $- 5 \times 10^7$, $- 8 \times 10^7$, $- 10^8$. Here the negative Grashof numbers imply that the buoyancy forces are in the opposite direction to the forced flow, i.e. opposing flow. Some further calculations were carried out for $R_\ell = 10^3$ and G_ℓ so chosen that the combined convective parameter $G_\ell / R_\ell^{\ 2}$ (and, hence, $R_\ell / G_\ell^{0.5} = 1/ [G_\ell / R_\ell^{\ 2}]^{0.5}$) had the same values as in the first set of calculations. In both cases the results were carried out for values of $\bar{x} = R_x/R_\ell$ of from o to 1 in the assisting flow case and, in the opposing flow case, from o up to as near the point at which separation occurred as was possible before numerical instability (see later) developed.

Typical velocity and temperature profiles obtained are shown in figs. 2 and 3. With the co-ordinate system used the profiles obtained for $R_\ell = 10^4$ and 10^3 should be identical since the values of G_x/R_x^2 were the same. This was, in fact, found to be the case and the curves are, therefore, labeled with the corresponding values of (G_x/R_x^2) in order to indicate their universal nature. Comparison of the profiles for $G_x/R_x^2 = o$ with exact analytical solutions for purely forced flow[10.] show excellent agreement. In fig.4 the variation of (v_1/u_1) with $\bar{x}$ for $R_\ell = 10^4$ and the various values of G_ℓ is shown. Since (v_1/u_1) is a measure of the displacement effect of the boundary layer, these curves show clearly the effects of the buoyancy forces which in the assisting flow case tend to decrease the displacement thickness and can lead to it becoming negative, as it is, of course, in purely free flow.

The basic heat transfer results for $R_\ell = 10^4$, obtained from the temperature profiles, are shown in fig.5. The Nusselt number curves for $R_\ell = 10^3$ were identical but displaced, of course, by the factor $10^{0.5}$ which again indicates the universal nature of the results when expressed in terms of the parameter (G_x/R_x^2). Comparison of these results for $G_\ell = o$ with the exact analytical solution for purely forced flow[10.] shows the effects of the numerical approximations on the results for small R_x i.e. near the leading edge. In this region $\Delta \bar{y}$ is, of necessity, large compared with the overall boundary layer thickness with the result that the numerical approximations to the derivatives are poor. As $\bar{x}$ increases the boundary layer thickness increases and $\Delta\bar{y}$, which was kept constant in the present calculations, becomes a smaller and smaller percentage of this thickness with a consequent decreasing in the numerical error. This is shown in the $G_{\ell} = o$ results which are in very good agreement with the analytical solution for the larger values of R_{ℓ}. These basic heat transfer results also show clearly the effects of the buoyancy forces on the Nusselt number, increasing it above the value that would exist in purely forced flow in assisting flow and decreasing it in opposing flow.

In fig.6 the above heat transfer results are plotted in the form $(N_x - N_{x\,for})/N_{x\,for}$ against (G_x/R_x^2), where $N_{x\,for}$ is the value of N_x that would exist in purely forced flow at the same value of R_x as in the actual flow. In deriving these results, the numerical results for $G_{\ell} = o$ were used to obtain $N_{x\,for}$. The correlation between the assisting flow results for the various values of G_{ℓ} and between the opposing flow results is satisfactory. The small differences between the results for various values of G_{ℓ} is probably the result of the numerical approximations used. The differences between the correlated curves for assisting and opposing flows may be noted as may be the fact that separation occurs at a value of G_x/R_x^2 near 0.2.

In fig.7 the same results for assisting flow are plotted in the form $(N_x - N_{x\,free}) / N_{x\,free}$ against $(R_x/G_x^{0.5})$, acceptable correlation again being achieved. Here, $N_{x\ free}$ is the value of N_x that would exist in purely free flow at the same value of G_x as exists in the actual flow. In order to express the Nusselt number in the above form it was noted that analytical solutions give:-

$$N_{x\ for} = 0.298\ R_x^{0.5} \quad ; \quad N_{x\ free} = 0.360\ G_x^{0.25}$$

From these it follows that:-

$$N_x / N_{x\ free} = 0.827 \ (R_x/G_x^{0.5})^{0.5} \ (N_x/ N_{x\ for}) \qquad (18)$$

This equation was used to derive $(N_x/N_{x\ free})$ because it was hoped that by using $N_{x\ for}$ as given by the numerical results and not by the analytical solution, the effects of the numerical inaccuracies on the correlation would be reduced.

No attempt has been made to correlate the opposing flow results in this manner since the origin for the free flow, i.e. for the value of x used to define N_x and $R_x/G_x^{0.5}$ in the correlation, would, in the opposing flow case, have had to be taken at the "trailing" edge of the plate which does not exist in the present solution which is for a semi-infinite plate.

Comparison with Other Solutions:

The variations of $(N_x - N_{x\ for})/N_{x\ for}$ and $(N_x - N_{x\ free})/N_{x\ free}$ with (G_x/R_x^2) and $(R_x/G_x^{0.5})$ respectively as given by the series solutions of ref.6 and by the integral equation solution of ref.4 are also shown in figs. 6 and 7. The series solution is given as being applicable in both assisting and opposing flows. The integral equation solution is seen to be in good agreement with the numerical results, especially in the assisting flow case. The series solution results are seen, however, to be applicable over only a very small range of values of the correlation parameters.

Stability Considerations:

A finite difference solution is said to be stable if the errors introduced into the solution due to the numerical approximations used are not magnified as the computation proceeds thus leading to a worthless solution. The exact conditions governing the stability of the set of equations solved in the present note were not derived due to the difficulties associated with the inter-relation of the momentum and energy equations through the buoyancy term. However, experience gained in carrying out the solution suggested that, at least for the Prandtl number being considered in the present note, the stability criterion derived by Wu[9.] provides a guide to the choice of the x and y grid spacing. This criterion, expressed in the present notation, is that stability will exist provided that:-

$$\Delta \bar{x} / \Delta \bar{y}^2 < \bar{u}_{m,n} \; R_\ell / 2 \qquad (19)$$

Since the wall points are given as boundary conditions, the lowest value of $\bar{u}_{m,n}$ will normally be $\bar{u}_{m,1}$ and this criterion then reduces to:-

$$\Delta \bar{x} / \Delta \bar{y}^2 < \bar{u}_{m,1} R_\ell / 2 \qquad (20)$$

In applying this criterion to the present case it was modified by the introduction of a coefficient, therefore, becoming:-

$$\Delta \bar{x} / \Delta \bar{y}^2 < K \bar{u}_{m,1} R_\ell / 2 \qquad (21)$$

In the calculations, $\Delta \bar{y}$ was kept constant and $\Delta \bar{x}$ varied in steps to suit this criterion, $\bar{u}_{m,1}$ in most cases, except in assisting flows with large values of G_ℓ / R_ℓ^2, decreasing with increasing $\bar{x}$ necessitating a decrease in $\Delta \bar{x}$. In opposing flows the solution could not proceed through the separation point since here $\bar{u}_{m,1} = o$, the solution here being terminated before this point was reached.

Conclusions:

Using a numerical procedure similar to that introduced by Wu[9.], the boundary layer equations for laminar combined convection have been solved for a series of values of the governing parameters. The results of these calculations show clearly the combined convection effects and it is shown that they can be correlated by plotting them relative to the values that would exist in purely forced or purely free flow.

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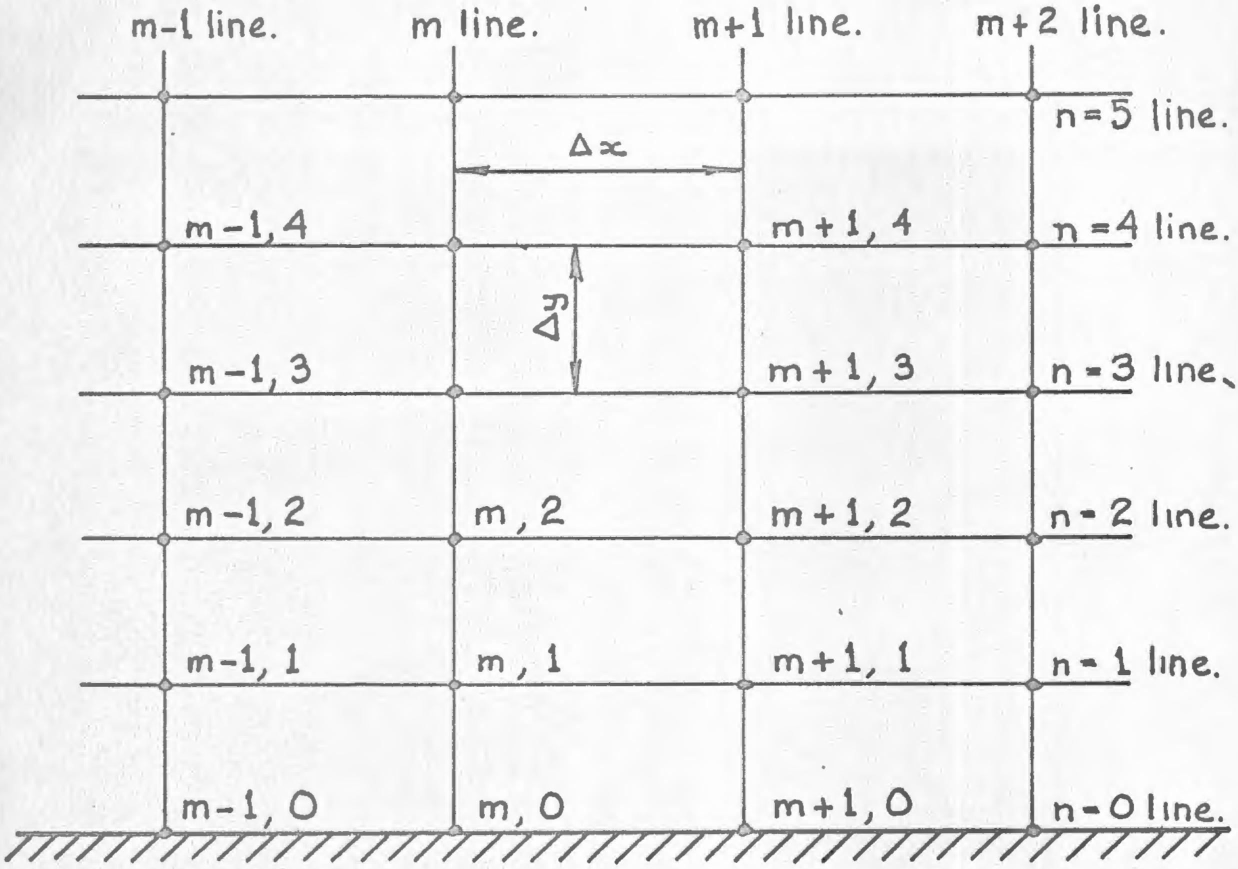
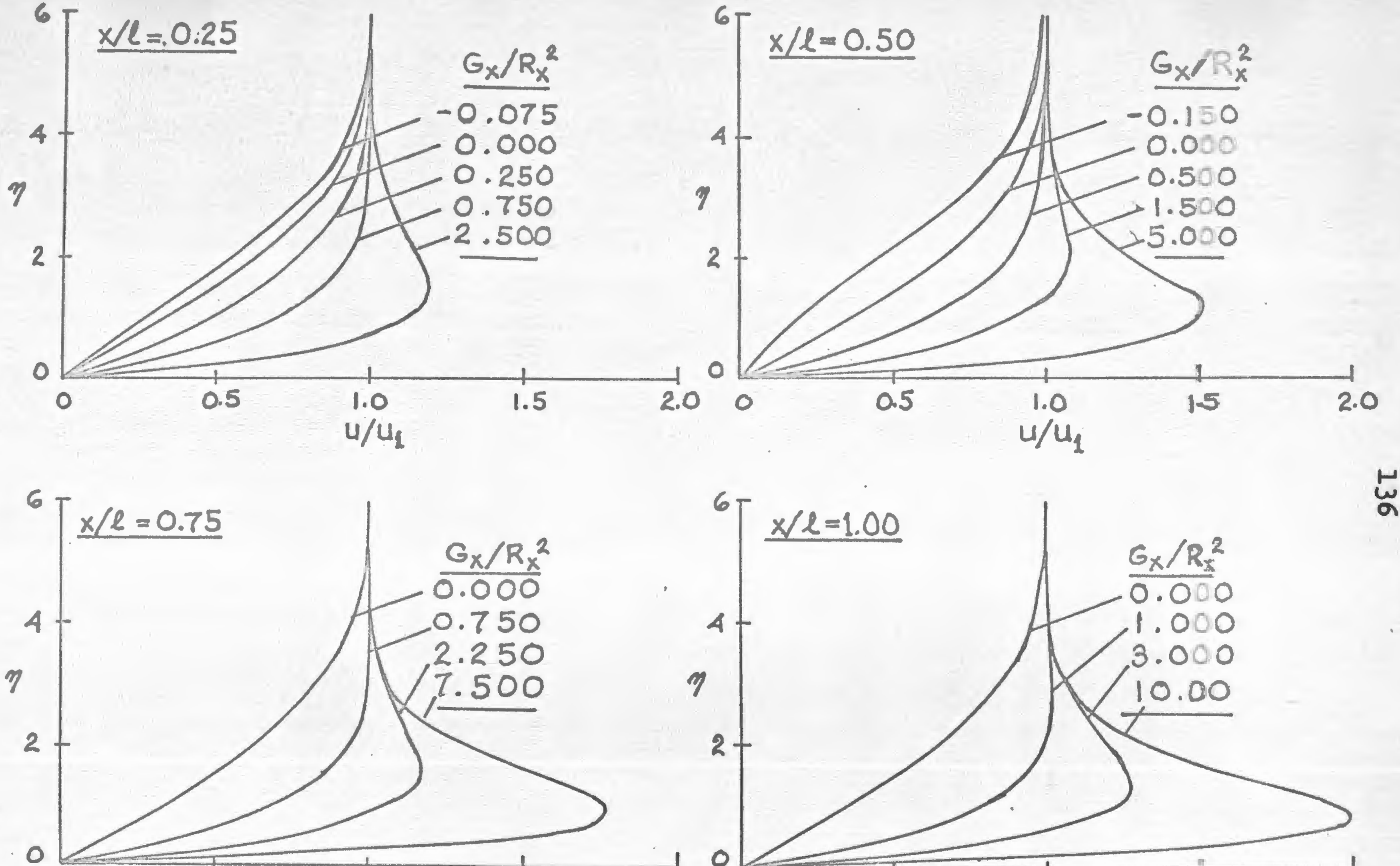


Fig. 1. Grid System Used.


x/ℓ = 0.25
G_x/R_x^2
-0.075
0.000
0.250
0.750
2.500
x/ℓ = 0.50
G_x/R_x^2
-0.150
0.000
0.500
1.500
5.000
x/ℓ = 0.75
G_x/R_x^2
0.000
0.750
2.250
7.500
x/ℓ = 1.00
G_x/R_x^2
0.000
1.000
3.000
10.00
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0.5
1.0
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2
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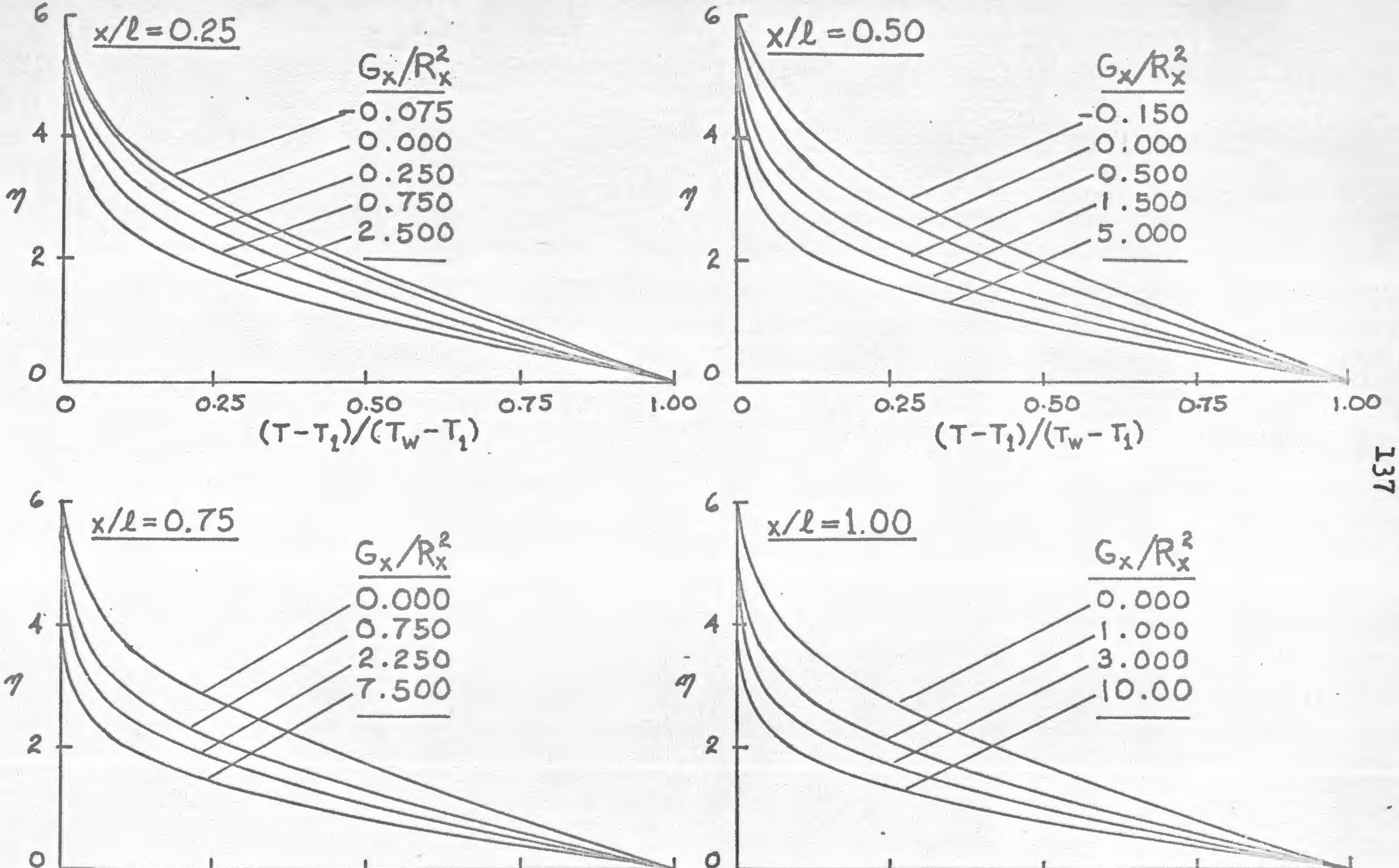
6


x/ℓ = 0.25
G_x/R_x^2
-0.075
0.000
0.250
0.750
2.500
η
0
2
4
6
0
0.25
0.50
0.75
1.00
$(T-T_1)/(T_w-T_1)$
x/ℓ = 0.50
G_x/R_x^2
-0.150
0.000
0.500
1.500
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0
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0
0.25
0.50
0.75
1.00
$(T-T_1)/(T_w-T_1)$
x/ℓ = 0.75
G_x/R_x^2
0.000
0.750
2.250
7.500
η
0
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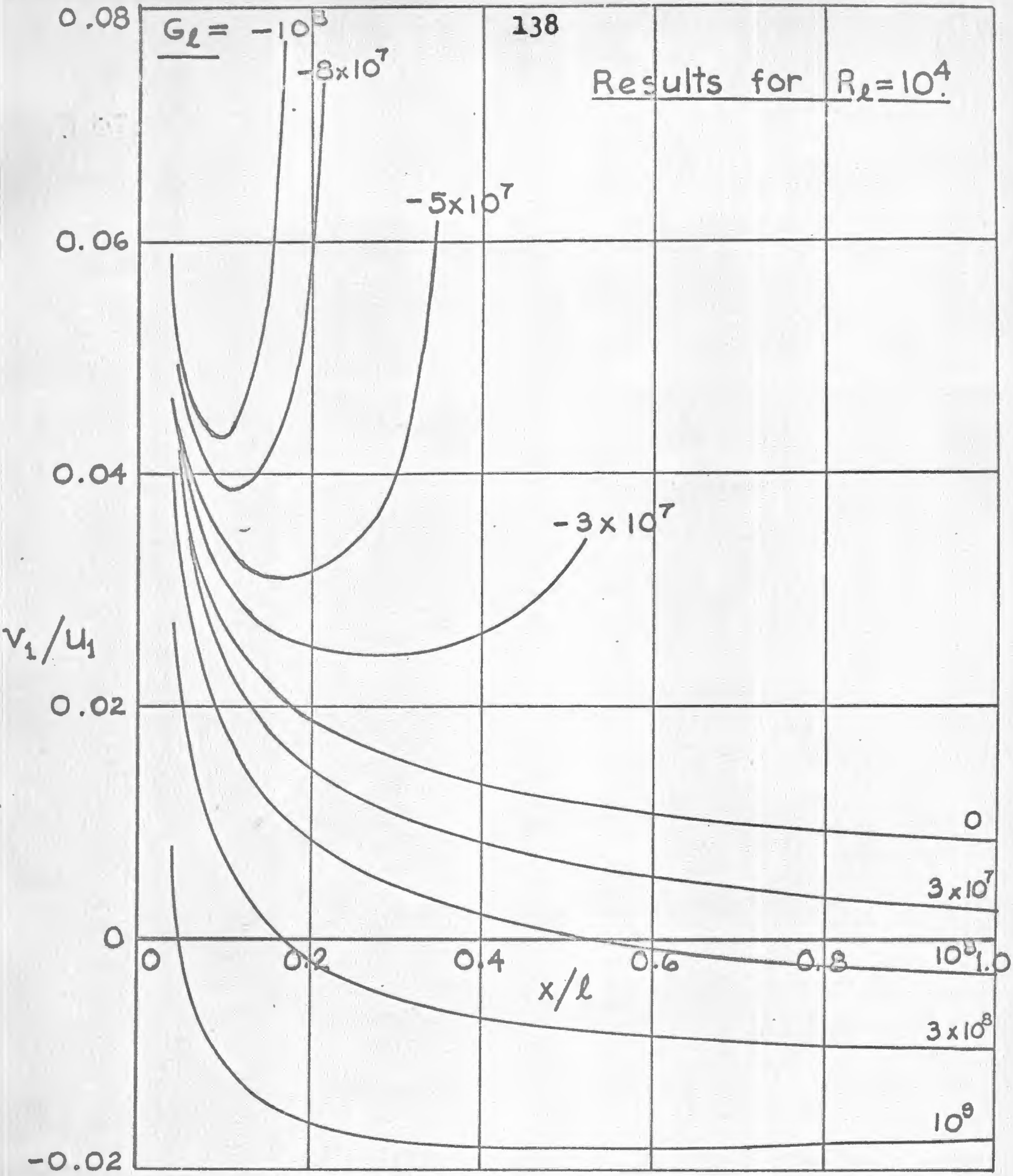

Fig. 4. Variation of the normal component of the freestream velocity with distance along plate.

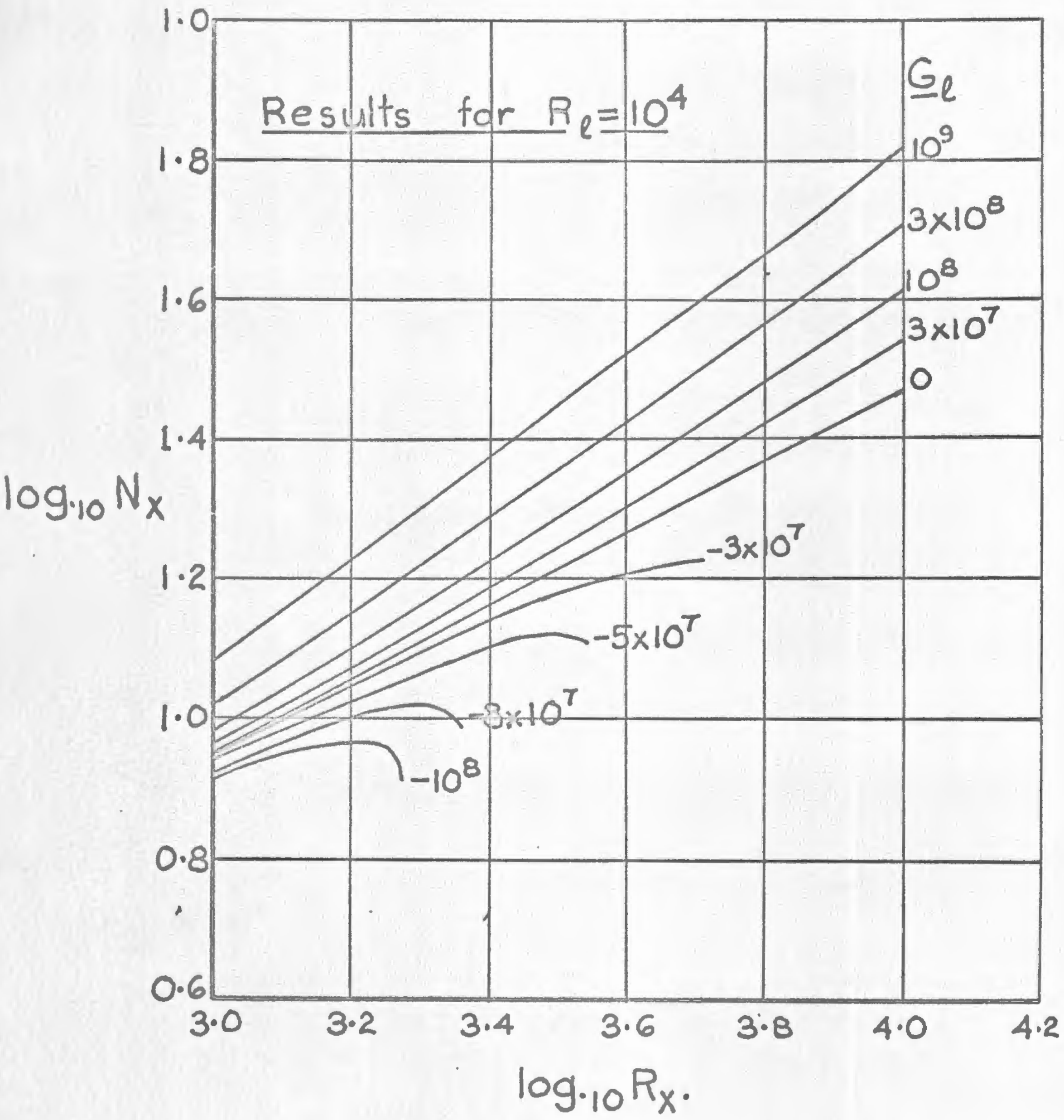

Fig. 5. Basic heat transfer results.

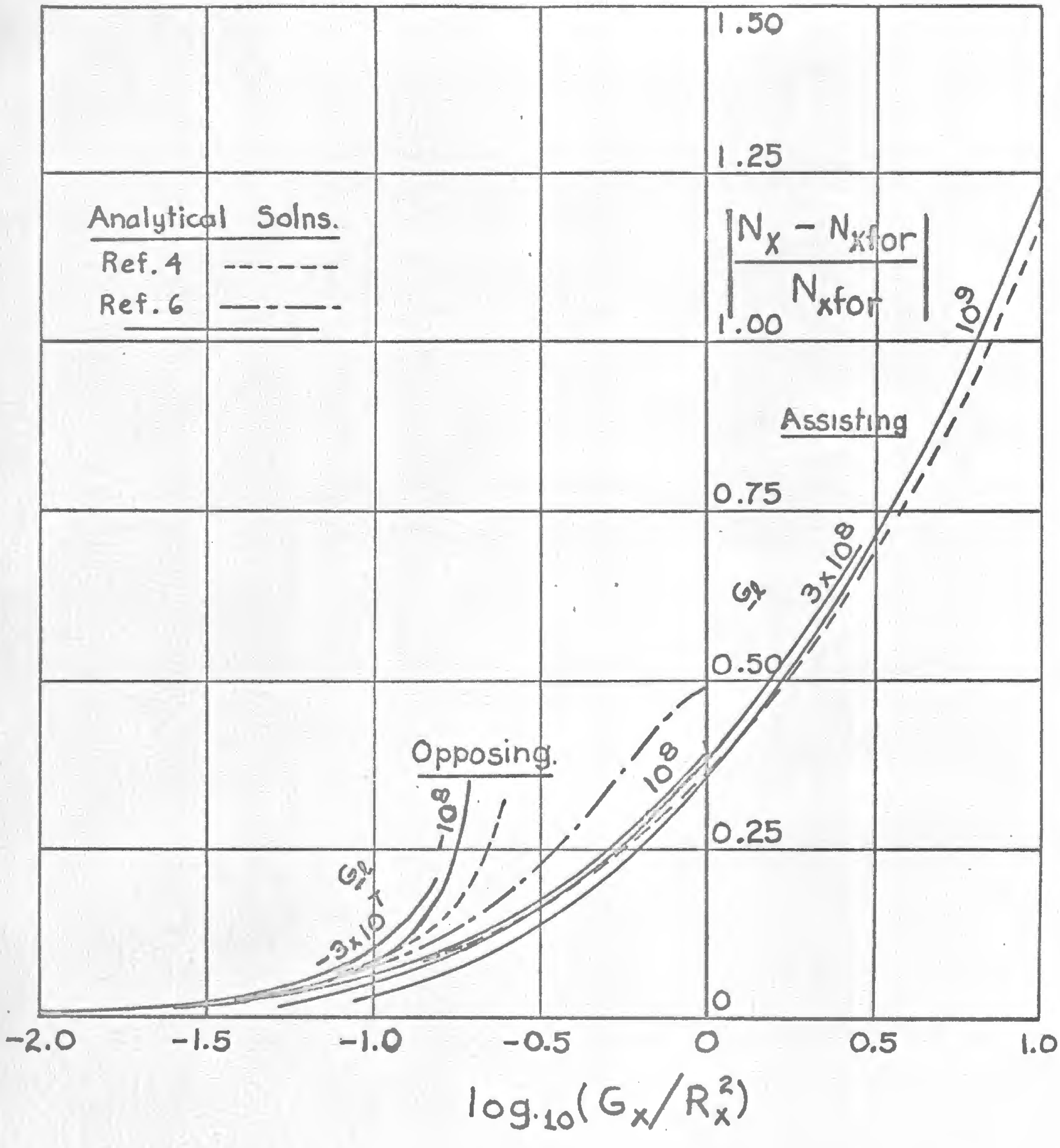

Fig. 8. Correlation of assisting and opposing flow heat transfer results in terms of the purely forced flow values and the parameter (G_x/R_x^2).

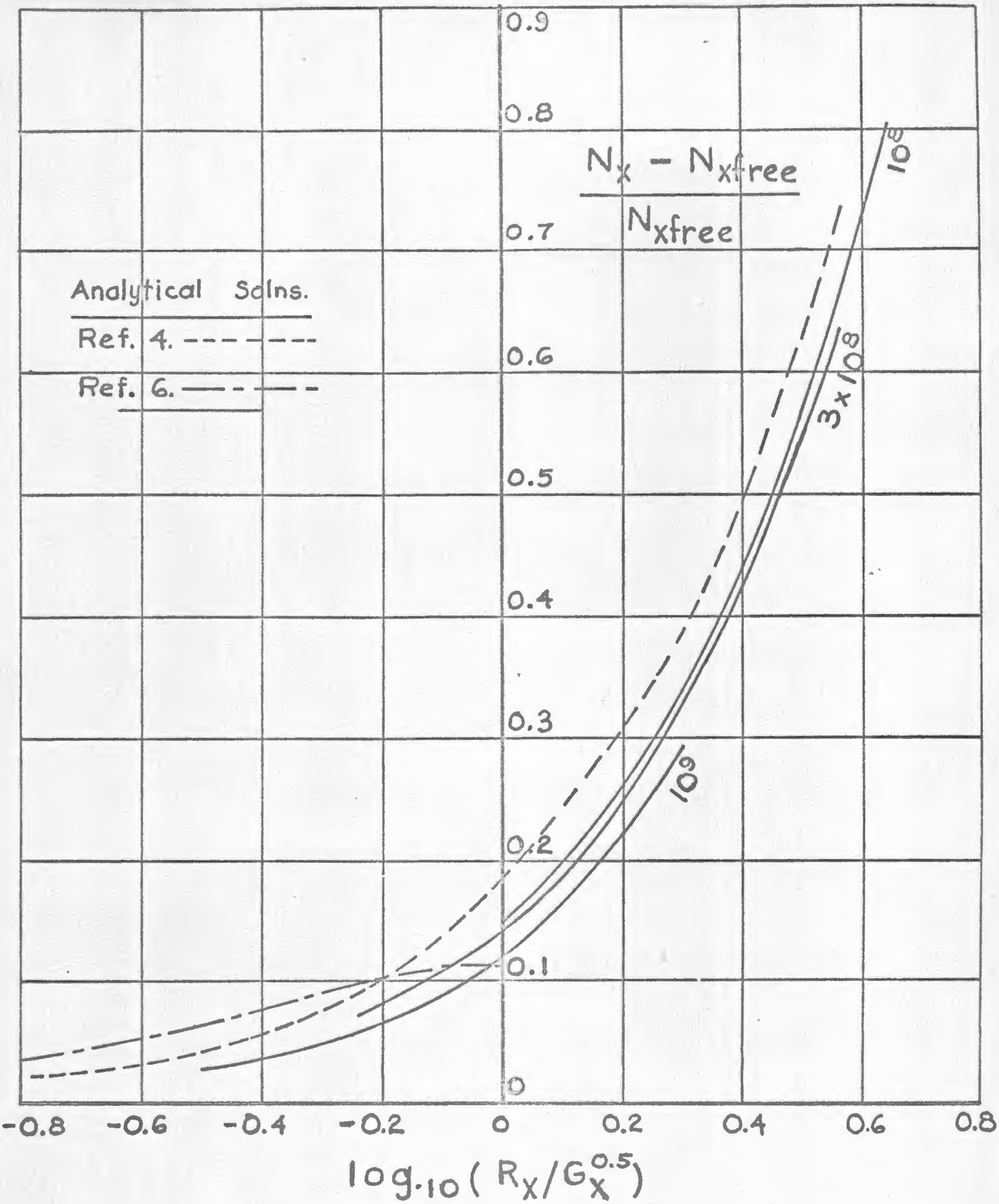

Fig. 7. Correlation of assisting flow heat transfer results in terms of the purely free flow values and the parameter (R /G $^{0.5}$).

Paper 10.

AN EXPERIMENTAL STUDY OF LAMINAR COMBINED CONVECTIVE HEAT TRANSFER FROM A VERTICAL PLATE WITH UNIFORM HEAT FLUX.

SUMMARY:

Heat transfer rates by combined convection from a vertical plate have been measured over a Reynolds number range of approximately 1900 to 3700 and a Grashof number range of approximately $10^{7.3}$ to $10^{7.9}$ with the plate so arranged that the buoyancy forces were, in one set of tests, assisting the forced flow and in another set, opposing it. The results show that the actual heat transfer rate in both cases is greater than that given by either the free or the forced flow relations. The results for all values of the flow parameters are shown to correlate using a suitable form of plotting.

NOTATION:

G_{ℓ} = Grashof number based on overall plate length.

ℓ = Overall length of plate.

N_{ℓ} = Nusselt number based on mean temperature difference and overall plate length.

$N_{\ell for}$ = Nusselt number that would exist in pure forced flow at the same Reynolds number as in the actual flow,

$N_{\ell free}$ = Nusselt number that would exist in pure free flow at the same Grashof number as in the actual flow.

R_{ℓ} = Reynolds number based on freestream velocity and overall plate length.

AN EXPERIMENTAL STUDY OF LAMINAR COMBINED CONVECTIVE HEAT TRANSFER FROM A VERTICAL PLATE WITH UNIFORM HEAT FLUX

Introduction:

Numerous practical heat transfer problems involve forced flows of such low magnitudes that the actual heat transfer is by the combined action of the forced flow and the buoyancy forces arising due to the fluid density changes with temperature. Several papers dealing with the theoretical analysis of heat transfer by combined convection from an isothermal vertical plate under such circumstances that the flow in the boundary layer is laminar have been published over the past few years. Very little supporting experimental work has, however, been done in this field although the gross assumptions or obviously limited range of application of these available analyses is such that their experimental verification must be regarded as necessary.

The work described in the present note is of an essentially simple nature in that only the total heat transfer from the plate, and not the details of the flow field, was measured and that a comparatively small range of values of the governing parameters was covered. This range was, however, so chosen that neither the forced or the free flow effects were predominant, these being the circumstances under which the least theoretical work has been done.

Apparatus:

The heated plate used in the present study was $13\frac{3}{4}$" high and 9" wide. The plate construction is indicated in fig. 1 and consisted of a thin insulating board core on which was uniformly wrapped, lengthwise as indicated in the figufe, a nichrome ribbon heating element. This core was clamped between two one-eighth of an inch thick aluminium plates and insulated from them with thin mica sheets.

A total of ten copper-constantan thermocouples were fitted to the aluminium plates, their positions being given in fig.1. The thermocouple junctions were soldered into brass plugs which were pressed into shallow holes end-milled into the aluminium plates. The thermo-couple leads were then brought out along slots in the plates and through holes in the sidewalls of the tunnel. These holes were then sealed with modelling clay.

The assembled plate was mounted by means of a symmetrical sting arrangement screwed to the plate in a small variable speed, open-return wind-tunnel with a 9" square working section, the plate completely spanning this section. The general arrangement of the tunnel is indicated in fig.2. After passing through the working section the air-stream was ducted through a $1\frac{1}{2}$" diameter pipe, as shown in the figure, before passing through the driving fan. A traversing, micrometer mounted Pitot tube and two static tappings were fitted to this pipe following the arrangement indicated in ref.1. Prior to the heat transfer tests, the relationship between the mean and the centre-line velocities in this pipe at various flow rates was determined from Pitot tube traverses using the procedure given in ref.1. It was found that, for the flow rates covered in the tests, this ratio remained, effectively constant. With the relationship between mean and centre-line velocities determined, it was only necessary to measure the centre-line velocity during the heat transfer tests.

The tunnel was mounted with its axis vertical in all tests. It could, however, be held in this position with the fan end either at the top or the bottom so that the forced flow was either assisting opposing the buoyancy forces.

The electrical input to the plate heater element, which was supplied via a constant voltage transformer, was controlled by means of a variable auto-transformer. Measurements taken prior to the main heat transfer tests had confirmed that the resistance of the heating element remained, effectively, constant over the temperature range to be covered during the heat transfer tests. In these tests, therefore, the heat input to the plate was determined by measuring the voltage drop across the element and using the value of the resistance given by the preliminary tests.

The plate temperature was measured using a workshop grade potentiometer with a thermocouple in the air at the entrance to the tunnel as the cold junction. The actual air temperature at this latter point was measured using a high sensitivity thermometer.

Derivation of Results:

In the heat transfer tests, the tunnel was run at five nominally constant speeds in the assisting flow configuration and four such speeds in the opposing flow configuration, the variation in speed between the tests in each nominally constant speed group being limited to about 10%. Tests were carried out at each constant speed for various rates of heat input to the plate. The temperature range covered was limited at the upper end by the desire to avoid, if possible the transition to turbulence and at the lower end by the sensitivity of the potentiometer used for the temperature measurements.

Since the thermocouples were not evenly spaced along the plate, the mean plate temperature was taken as the areawise mean temperature. The free stream temperature was taken as the temperature of the air at the inlet to the tunnel. All fluid properties required for the determination of the flow parameters, i.e. Reynolds, Grashof and Nusselt numbers, were evaluated at the mean film temperature i.e. the mean of the average plate temperature and the freestream temperature.

The freestream velocity in the working section was obtained from the measured centre-line velocity in the pipe section by first calculating the mean velocity in this section and then applying the continuity equation and neglecting the displacement effect of the boundary layer on the walls of the tunnel, which were parallel. Since the boundary layer growth in the contraction cone of the tunnel must remain small, the actual freestream velocity calculated in this manner must be that at the leading edge of the plate, the effect of the wall boundary layers and that on the plate being to introduce longitudinal pressure gradients.

The actual convective heat transfer from the plate to the fluid was taken as the total heat input to the element less the heat lost by radiation from the plate surfaces. Thie latter quantity was calculated assuming the aluminium surface to have an emissivity of 0.08[2].

With the fluid and flow variables derived as outlined above, the Nusselt, Grashof and Reynolds numbers based on the overall plate length could be derived.

In addition to the above tests with forced flow present, the heat transfer by pure free convection was also measured for various heat inputs with the tunnel mounted in both the assisting and opposing flow configurations. The Nusselt and Grashof numbers for these tests were determined using the procedure outlined above.

Results for Assisting Flow:

The variation of the measured Nusselt number, N_ℓ, with Grashof number, G_ℓ, for the various nominally constant Reynolds numbers, R_ℓ, for the tunnel mounted such that the free and forced flows are in the same direction, i.e. assisting each other, is shown in fig.3.

The results for pure free convection, i.e. with $R_\ell = 0$, are fitted by the equation:-

$$N_\ell = 0.6 \; G_\ell^{0.25} \qquad (1)$$

The coefficient 0.6 in this correlation is somewhat higher than that normally accepted but not to an unreasonable degree.

For the Grashof and Reynolds number range covered by the tests, the Nusselt numbers for pure free convection are considerably higher, roughly by a factor of two, than those that would exist in purely forced flow. Thus the experimental results show that the actual heat transfer, which is by combined convection, is considerably higher than that predicted by either the forced flow or the free flow relations.

Now, it is shown in ref.3 that in combined convection it should be possible to correlate the results in either of the following forms:-

$$(N_{\ell} - N_{\ell for})/N_{\ell for} = f\,(G_{\ell}/R_{\ell}^{2}) \tag{2}$$

or

$$(N_{\ell} - N_{\ell free})/N_{\ell free} = g\,(R_{\ell}/G_{\ell}^{0.5}) \tag{3}$$

Therefore using equation (1) to obtain $N_{\ell free}$ and assuming $N_{\ell for}$ to be given by[4.]:-

$$N_{\ell for} = 0.595\,R_{\ell}^{0.5} \tag{4}$$

The variation of $(N_{\ell} - N_{\ell for})/N_{\ell for}$ and $(N_{\ell} - N_{\ell free})/N_{\ell free}$ with (G_{ℓ}/R_{ℓ}^{2}) and $(R_{\ell}/G_{\ell}^{0.5})$ respectively as given by the experimental results is shown in figs. 4 and 5. These results indicate that, as predicted, although neither the forced nor the free flows is predominant, the results can be correlated in this manner.

Results for Opposing Flow:

The variation of measured N_ℓ with G_ℓ for the various values of R_ℓ , with the tunnel so mounted that the buoyancy forces are in the opposite direction to the forced flow is shown in fig.6.

The results for free flow alone are fitted, assuming the $G_\ell^{0.25}$ relation to apply, by:-

$$N_\ell = 0.64\ G_\ell^{0.25} \qquad (5)$$

The large difference between this result and that obtained in the assisting flow position must be due either to the different shape of the leading edge in the two positions or due to the interference of the sting which is at the leading edge when the plate is in the opposing flow position. Previous work has suggested that the difference in the two leading shapes should have only a small effect on the results and the sting interference effect is probably, therefore, the major one. The difference between the coefficient in equqtion (5) and accepted values is so great that it must throw some doubt on the present results for opposing flow. However, since it is mainly the end correlation of the results which is being sought and since this correlation involves only differences of the actual heat transfer rates and those which exist in either pure free or pure forced flow, it is hoped that the results will, at least, be of some qualitative importance.

Consideration of the opposing flow results given in fig. 6 shows that in this configuration, as with assisting flow, the actual heat transfer by combined convection is greater than that which would exist in pure forced or free flow at the same Reynolds and Grashof numbers respectively. In view of the theoretical predictions that in opposing flow the heat transfer will be less than the greater of the heat transfer results given by the pure forced flow and the pure free flow relations, this result obviously calls for some explanation. This explanation probably lies in the fact that for the values of the flow parameters used in the present tests there will be a fairly large length of the plate where the flow is "separated" from the surface i.e. where there is a buoyancy force dominated flow in one direction adjacent to the surface and a forced flow in the opposite direction in the outer part of the boundary layer. The theoretical solutions available are not applicable to such flows. Another source of explanation lies in the fact that the theoretical solutions are only applicable to a semi-infinite plate. The effect of this limitation can be seen by considering the variation of local Nusselt number along the plate. If the flow near the leading edge for the forced flow is first considered and the Nusselt number based on the distance from this edge is used, then at this leading edge, the Nusselt number will have the same value as in pure forced flow. With increasing distance, the Nusselt number will, theoretically, fall more and more below the value that would exist with purely forced flow. If, however, the flow near the other end, which is the leading edge for the free flow, is considered and the Nusselt number based on the distance from this edge is used, then since the Grashof number based on this distance is also zero at this edge, the Nusselt number must here again have the forced flow value corresponding to the Reynolds number based on the distance from the other edge. With increasing distance from this edge the Nusselt number will, theoretically, tend to become closer and closer to the purely free flow value. The actual flow is obviously some combination of these two predicted flow patterns. The effect of this two-edge effect on the temperature distribution with uniform heat flux, which was what existed in the experimental work, is illustrated in fig.7. This figure shows how the average temperature of the whole plate can be less than that which would exist if the flow were pure forced or pure free. This means that the average combined convective Nusselt number can be higher than that that would exist in either of the latter two cases, as was found experimentally.

The opposing flow results are plotted in the forms $(N_{\ell} - N_{\ell for} / N_{\ell for}$ and $(N_{\ell} - N_{\ell free})/ N_{\ell free}$ against $(G_{\ell} /R_{\ell}^{\;2})$ and $(R_{\ell} /G_{\ell}^{0.5})$ in figs. 8 and 9. In deriving these results, equation (5) was used to give $N_{\ell free}$ and, as with the assisting flow results, equation (4) was used to give $N_{\ell for}$. Figs. 8 and 9 show that the opposing flow results can be correlated in the same way as the assisting flow results although the functional relation between the variables is different in the two cases.

<u>Comparison with Predicted Results:</u>

The only theoretical result likely to be applicable in the Reynolds - Grashof number range covered in the present tests is the predicted variation of $(N_{\ell} - N_{\ell for})/N_{\ell for}$ with G_{ℓ}/R_{ℓ}^2 given in ref.5. This is plotted, for comparison with the experimental results, in fig.4. It is seen that the experimental results are considerably higher than those predicted. The reason for this discrepancy is not clear. Experimentally it is possibly the result of turbulence in the working stream or, although this seems very unlikely, the occurrence of partial transition to turbulence.

Conclusions:

An experimental study of the heat transfer by combined convection from a vertical flat plate has been carried out under such conditions that neither the forced nor the free flow effects is predominent. The results show that, under such circumstances, the actual heat transfer in both assisting and opposing flows is higher than that which would exist if the flow were considered to be either pure free or pure forced. The results obtained have been shown to correlate satisfactorily using the forms derived in ref.3.

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2. WIEBELT, J.A., "Engineering Radiation Heat Transfer", 1st ed., Holt, Rinehart and Winston, 1966, page 220.

3. Paper 6 of this thesis.

4. SCHLICHTING, H., "Boundary Layer Theory", 4th ed., Mc Graw-Hill, 1960, page 317.

5. Paper 8 of this thesis.

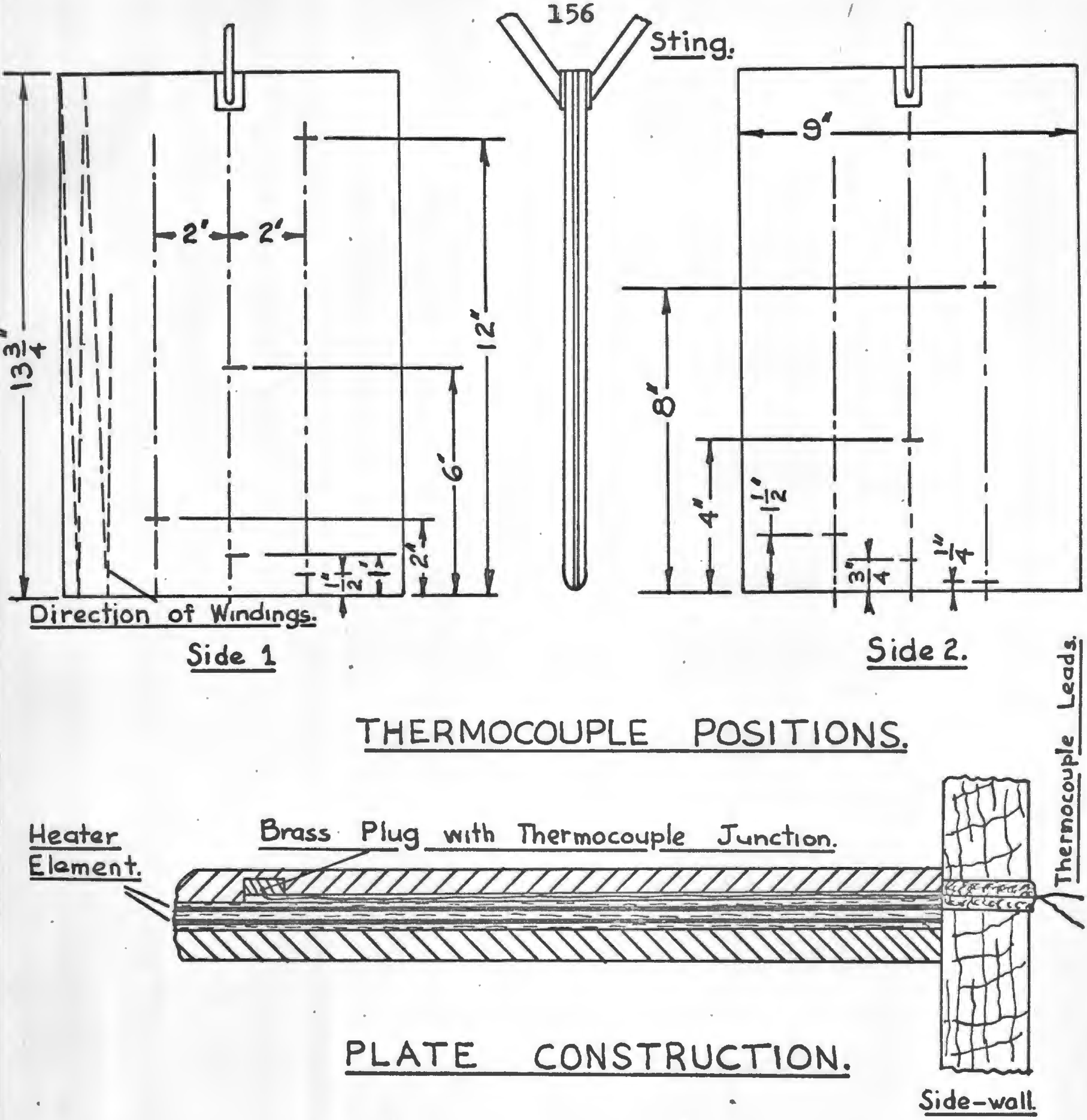

Fig. 1. Plate construction and position of thermocouples.

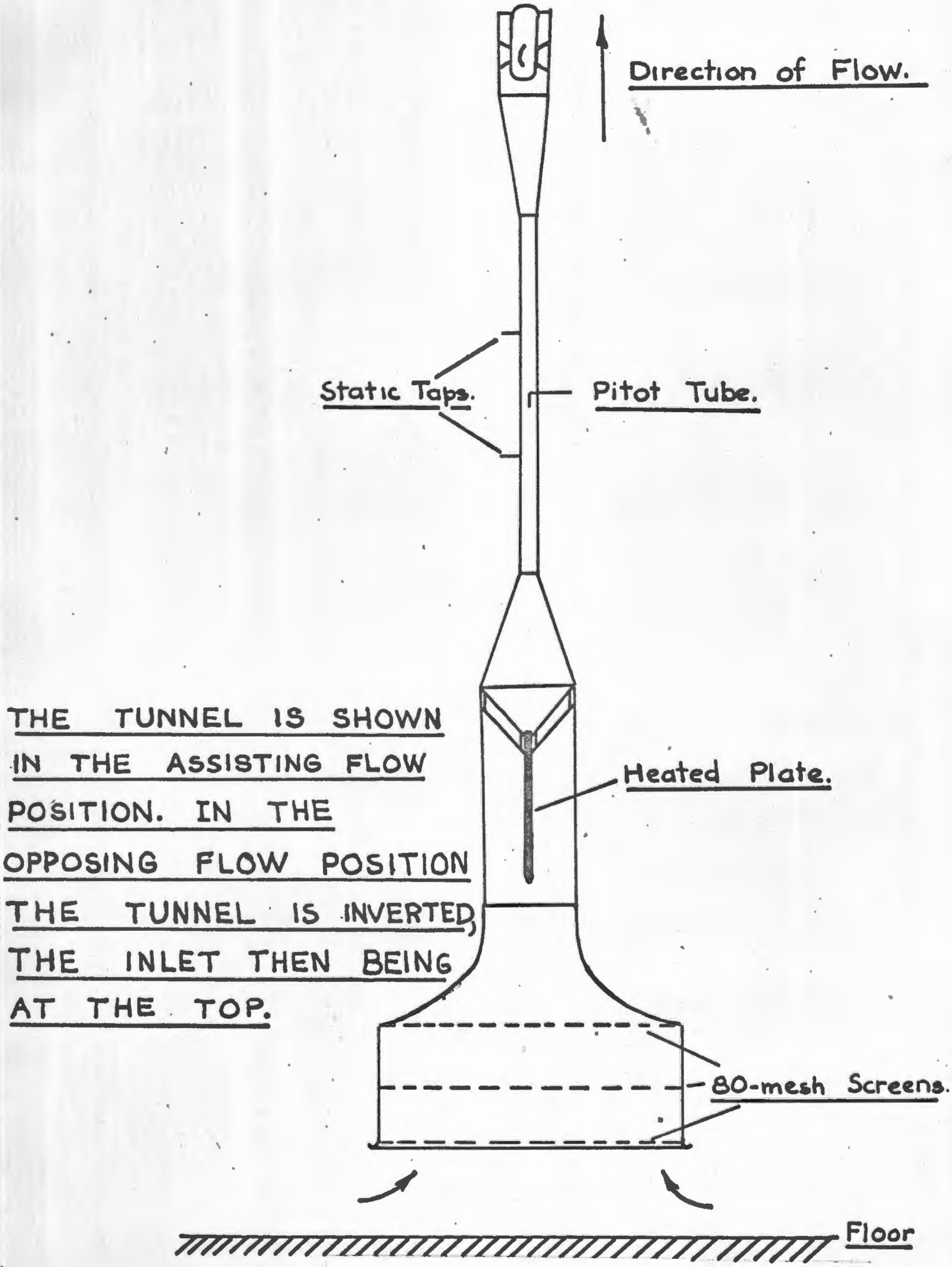

Fig. 2. Layout of tunnel.

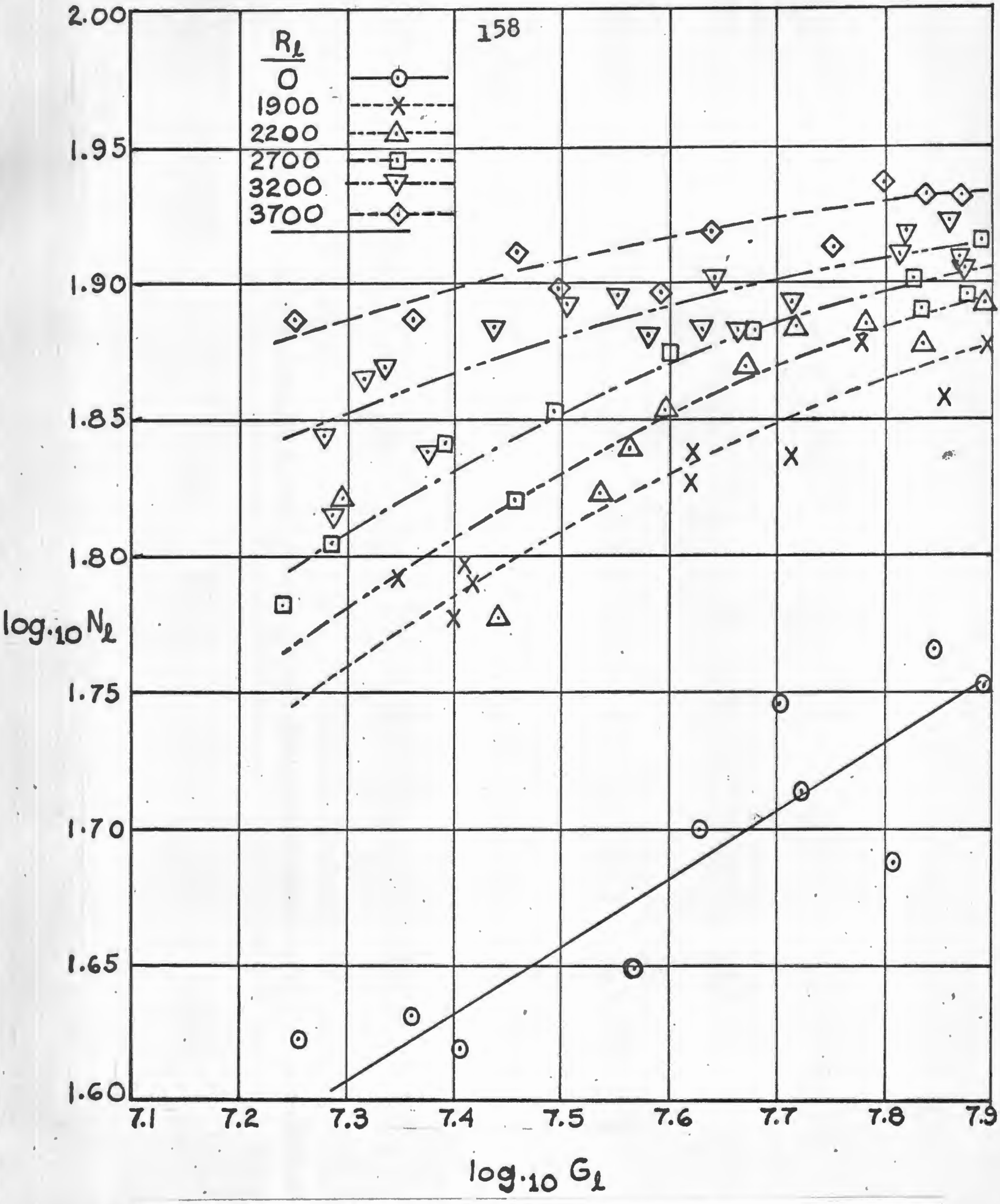

Fig. 3. Basic results for the plate in the assisting flow position.

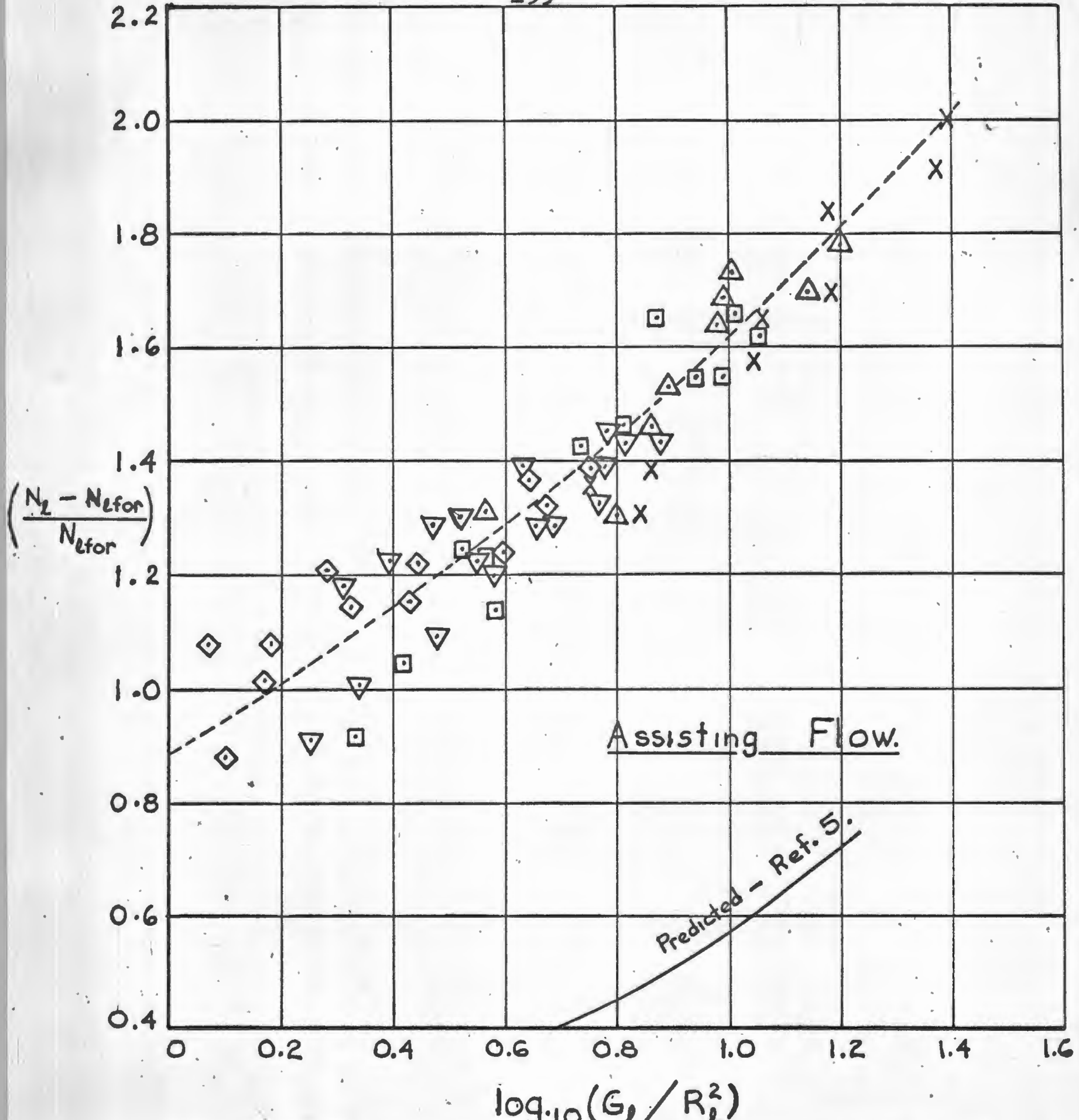

Fig. 4. Correlation of assisting flow results using purely forced flow conditions for reference. Experimental points are marked using the same code as in Fig. 3.

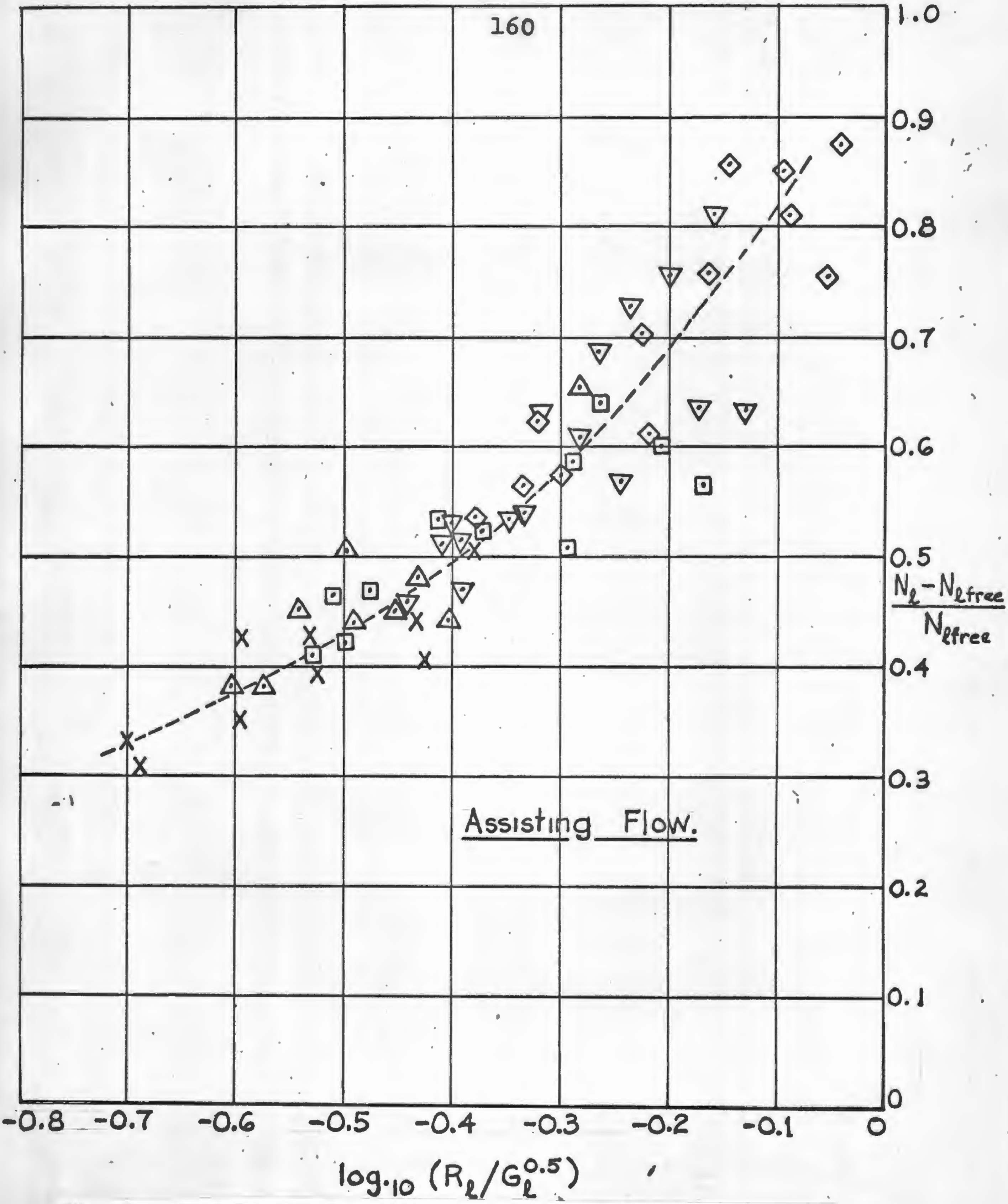

Fig. 5. Correlation of assisting flow results using purely free flow conditions for reference. Experimental points are marked using the same code as in Fig. 3.

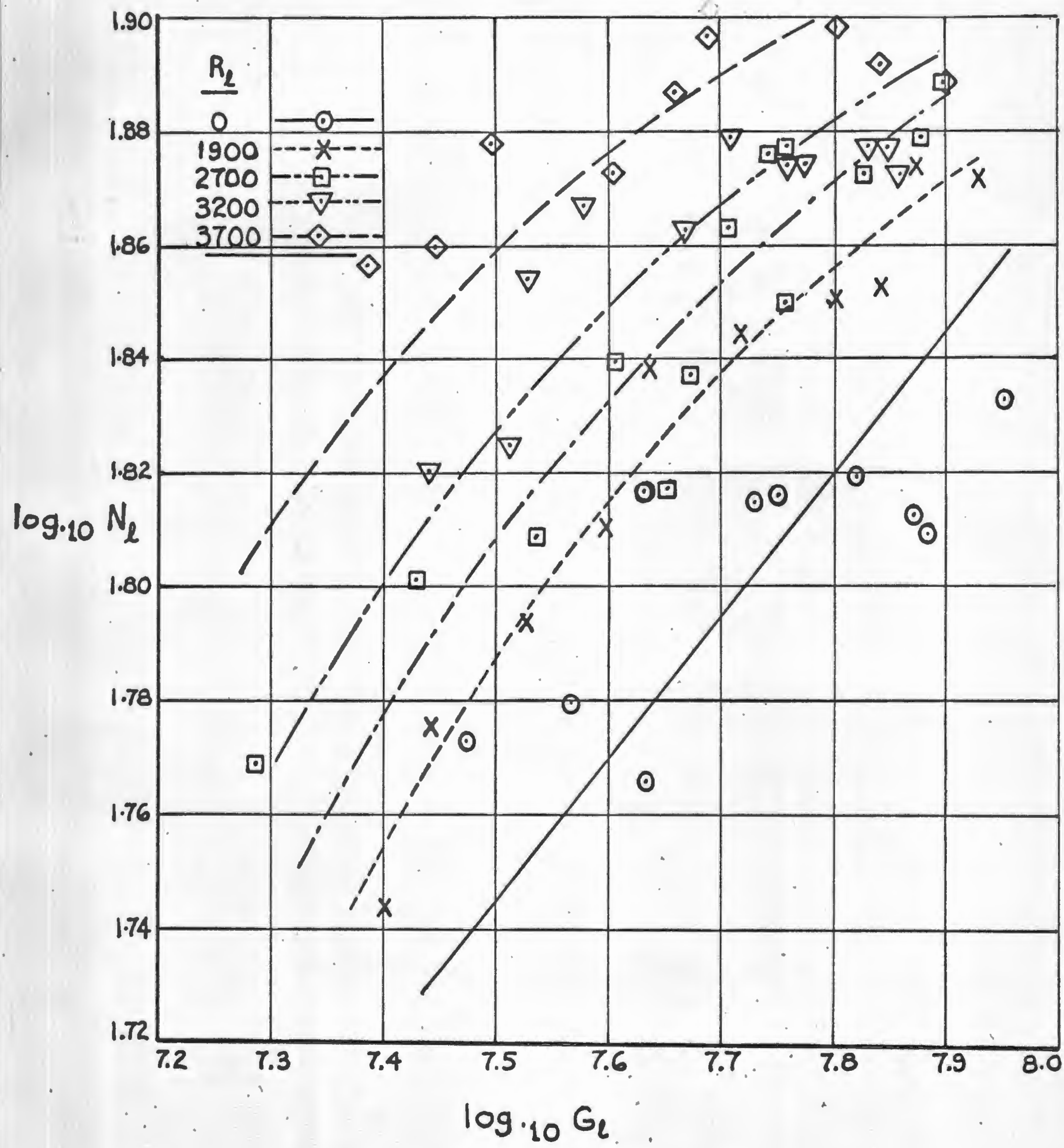

Fig. 6. Basic results for the plate in the opposing flow position.

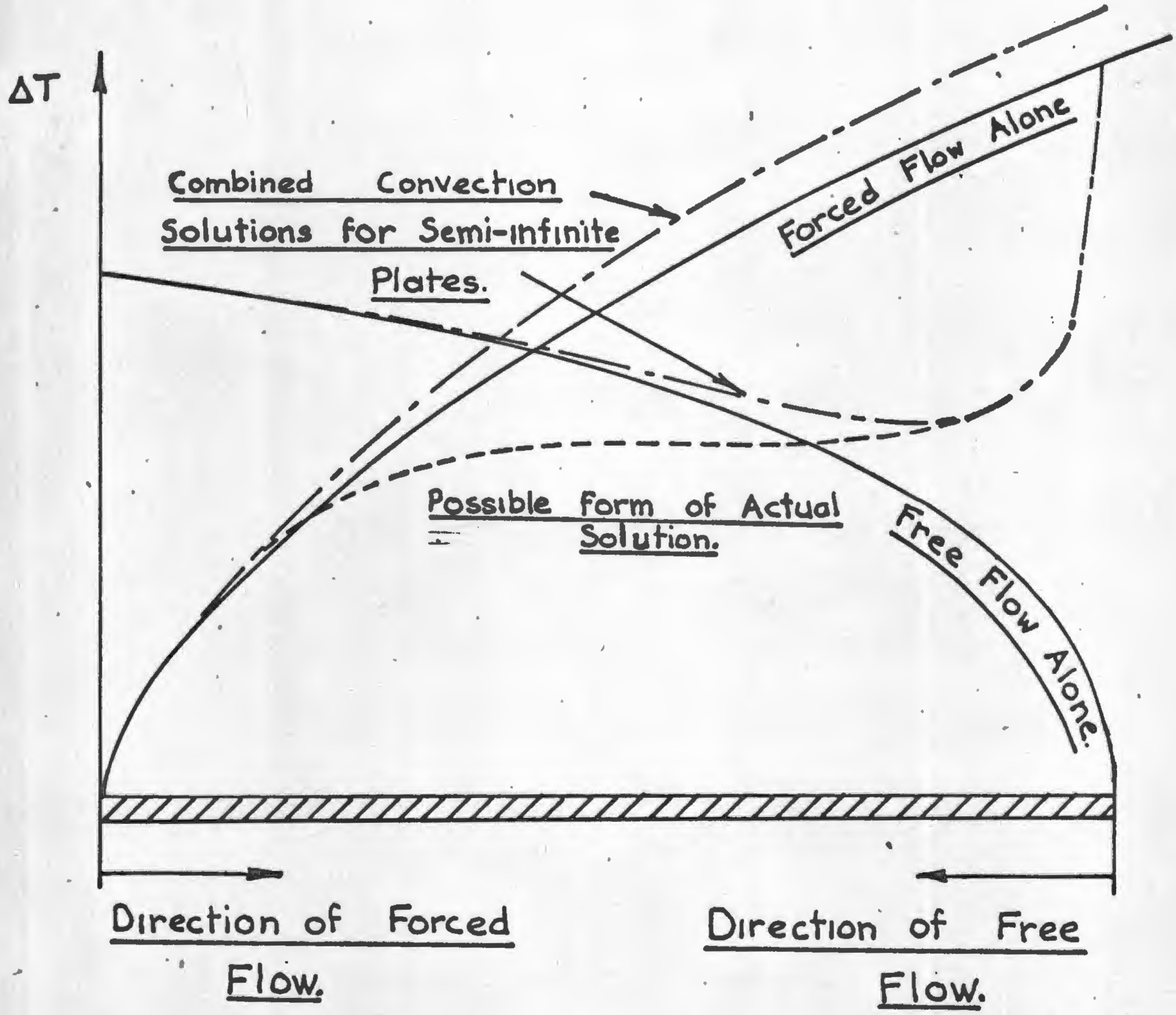

Fig. 7. Possible relation between actual temperature distribution and semi-infinite plate solutions for constant surface heat flux.

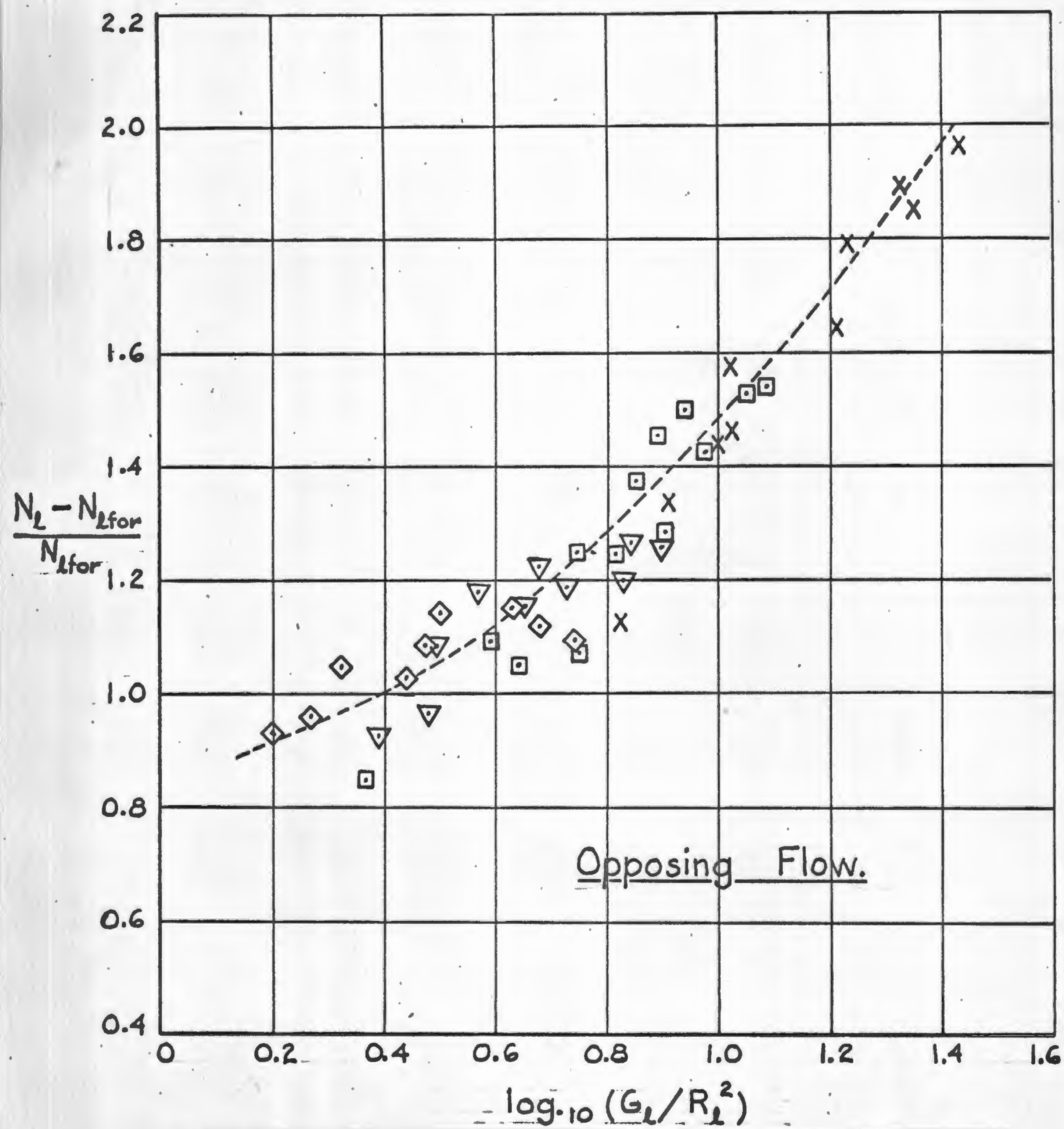

Fig. 8. Correlation of opposing flow results using purely forced flow conditions for reference. Experimental points are marked using the same code as in Fig. 6.

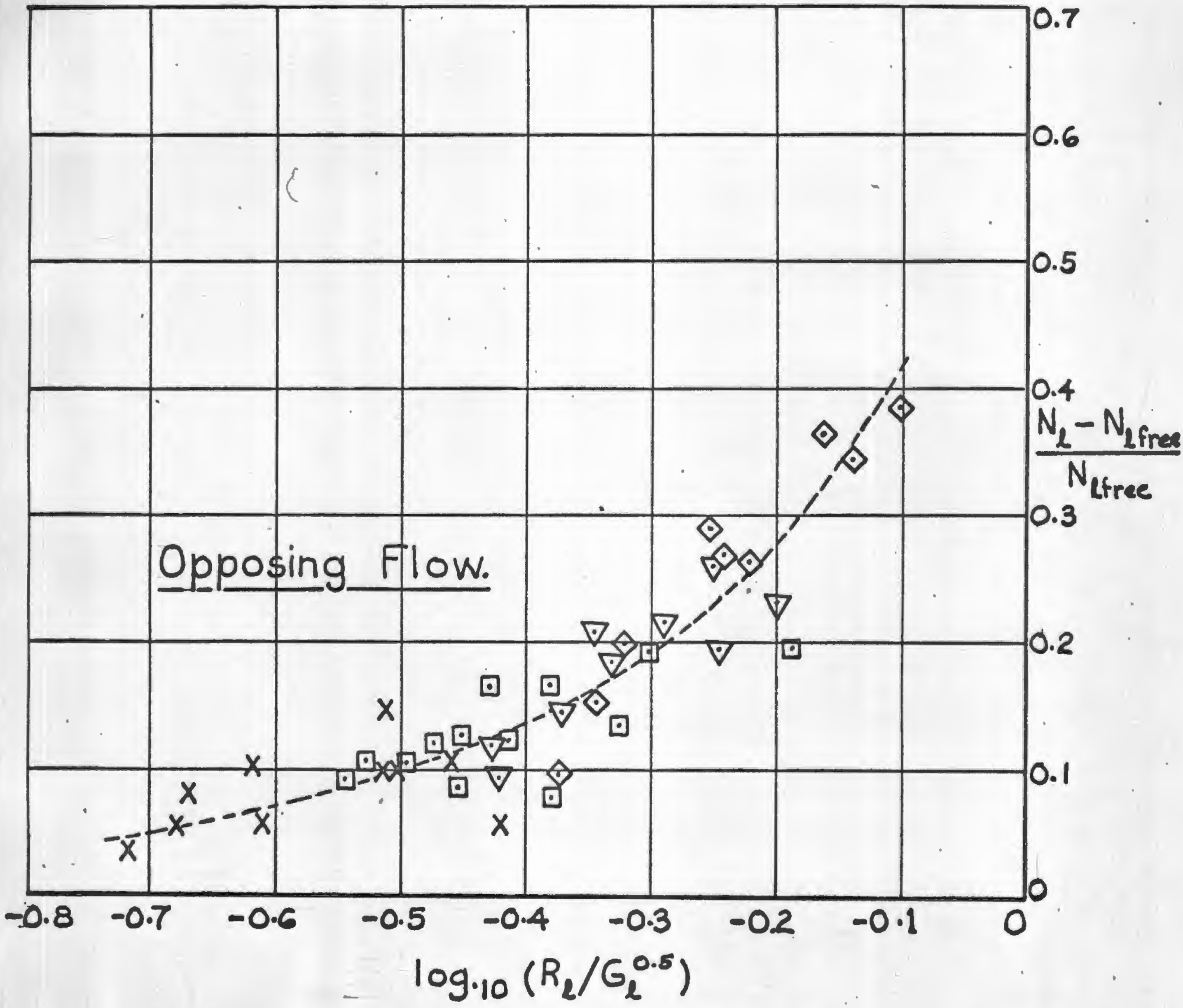

Fig. 9. Correlation of opposing flow results using purely free flow conditions for reference. Experimental points are marked using the same code as in Fig. 6.

Paper 11.

A NOTE ON THE TRANSITION POINT IN A FREE CONVECTIVE BOUNDARY LAYER ON AN ISOTHERMAL VERTICAL PLANE SURFACE

A note on the transition point in a free convective boundary layer on an isothermal vertical plane surface

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The application of a recently published semi-empirical boundary layer transition theory to the free convective flow over a flat plate has been considered and shown to give results which agree fairly well with experiment. A modification to the theory has been introduced which appears to improve this agreement.

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INTRODUCTION

When a heated plate is placed vertically in a fluid a flow over the plate will be caused by the bouyancy forces which arise due to the change in fluid density with temperature. This flow, which carries heat from the plate to the fluid, always has, in practice, a boundary layer character.

The flow in this boundary layer can be either laminar or turbulent. If the plate is long enough the flow will be laminar near the leading edge followed by a transition region, which can be quite long, and the flow over the rest of the plate will be turbulent.

This note presents an attempt to predict the point at which the laminar flow first starts to break down i.e. the beginning of the transition region.

THE VAN DRIEST AND BLUMER THEORY

Van Driest and Blumer[1] have recently suggested that transition from laminar to turbulent flow occurs in a boundary layer when the ratio of the local 'inertial' stress to the local viscous stress reaches some limiting value at some point in the boundary layer. Therefore, taking the local inertial stress as $\rho l^2 (du/dy) \,|(du/dy)|$ and the local viscous stress as $\mu(du/dy)$, this criterion states that transition occurs when:—

$$(\rho l^2/\mu)\,|(du/dy)| = \text{some limiting value} \quad \ldots (1)$$

where l is a length, ρ the density, μ the viscosity, (du/dy) the velocity gradient along the normal to the wall and $|(du/dy)|$ the arithmetic magnitude of this velocity gradient. The latter is introduced to ensure the correct sign for the inertial stress.

Van Driest and Blumer assume that l is proportional to the distance from the wall, y, to the point in the boundary layer considered. With this assumption equation (1) gives that at transition:—

$$(\rho y^2/\mu)\,|(du/dy)| = T \quad \ldots (2)$$

T being the transition constant.

APPLICATION TO THE PRESENT PROBLEM

This note considers the application of the theory, outlined above, to the free convective boundary layer on a vertical isothermal plate.

On such a plate when the fluid properties are assumed to be constant the Pohlhausen analysis[2] gives for laminar flow:—

$$u/2[g\beta x\theta_o]^{0.5} = f\,[(y/x)(G_x/4)^{0.25}] = f(\eta) \quad \ldots (3)$$

where u = velocity at distance y from the surface,
x = distance from leading edge,
β = coefficient of cubical expansion,
θ_o = difference between the wall temperature and the temperature of the still air outside the boundary layer,
G_x = Groshof number based on x
= $[\beta g \rho^2 x^3 \theta_o/\mu^2]$
η = $[(y/x)\,(G_x/4)^{0.25}]$

Differentiation of equation (3) gives:—

$$\frac{du}{dy} = \frac{\sqrt{2}\,(G_x)^{0.75}}{x^2\,(\rho/\mu)}\,\frac{df}{d\eta} \quad \ldots (4)$$

Substituting this into equation (2) gives that at transition:—

$$\sqrt{2}\,(y/x)^2\,(G_{xT})^{0.75}\,|(df/d\eta)| = T \quad \ldots (5)$$

which can be written as:—

$$2\sqrt{2}\,\eta^2\,(G_x)^{0.25}\,|(df/d\eta)| = T \quad \ldots (6)$$

where G_{xT} is the Groshof number based on the distance x_T from the leading edge to the point at which transition commences.

It may be noted, in passing that $2\sqrt{2}\,(G_x)^{0.25}$ is the parameter G introduced by Szewczyk[3] in his analysis of the present problem using hydrodynamic stability theory.

Now Schuh's solution[4] of the Pohlhausen equation gives the velocity function f of equation (3) for Prandtl numbers, P, of 0·73 (air), 10, 100 and 1000. Using these results the variation of $\eta^2\,|(df/d\eta)|$ with η can be determined and is shown in Fig 1.

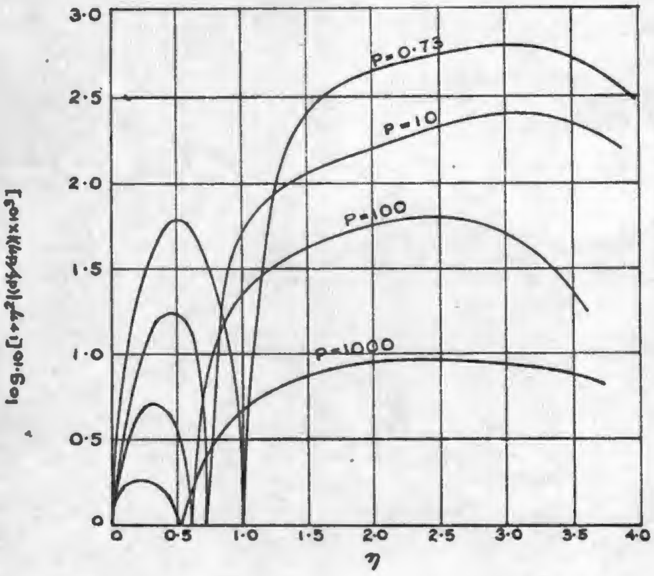

Fig 1 Variation of $\eta^2\,|(df/dn)|$ with η for various values of Prandtl number P

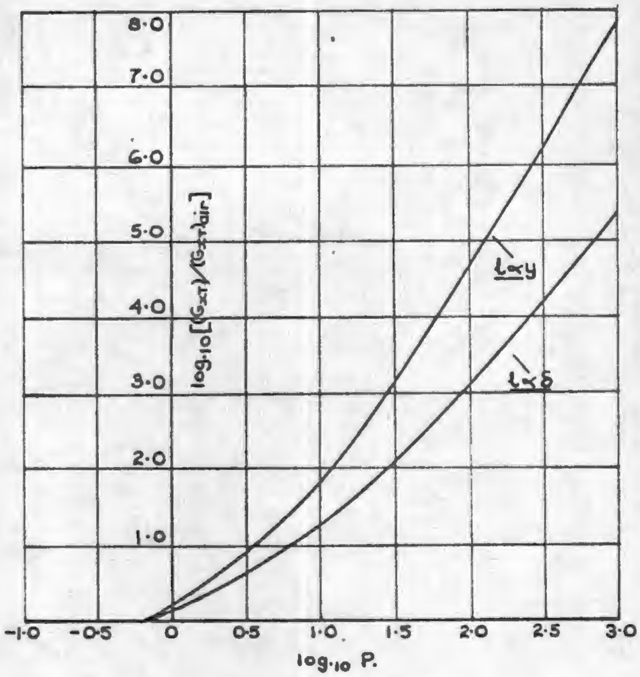

Fig 2 Variation of $(G_{xT})/(G_{xT})_{air}$ with P as predicted with the two assumptions as to l.

Equation (6) gives that at transition:—

$$2\sqrt{2}\ (G_{xT})^{0.25} = T/[\eta^2\,|(df/d\eta)|]_{max} \qquad \ldots\ldots (7)$$

i.e. transition will occur first at the point where $\eta^2\,|(df/d\eta)|$ has a maximum. From Fig 1 it will be seen that the $\eta^2\,|(df/d\eta)|$ against η curve has two maxima, one in the inner portion of the layer and one in the outer portion of the layer, but that in all cases the maximum value of $\eta^2\,|(df/d\eta)|$ occurs in the outer portion i.e. in all cases transition will commence in the outer portion of the layer.

Results

Equation (7) gives, since T is a constant:—

$$\frac{(G_{xT})P_1}{(G_{xT})P_2} = \frac{\{[\eta^2\,|(df/d\eta)|]_{max}\}^4 P_2}{\{[\eta^2\,|(df/d\eta)|]_{max}\}^4 P_1} \qquad \ldots\ldots (8)$$

where subscripts P_1 and P_2 refer to conditions which occur in fluids having Prandtl numbers P_1 and P_2.

With values of $[\eta^2|(df/d\eta)|]_{max}$ taken from Fig 1, equation (8) allows the ratio of G_{xT} for any fluid to G_{xT} for air to be determined. Values of this ratio are given in Table 1 and are plotted for later interpolation in Fig 2.

TABLE 1

Results with l proportional to y

P	$(G_{xT})/(G_{xT})_{air}$	$(G_{xT} \times P)/(G_{xT} \times P)_{air}$	η_m
0·73	1·0	1·0	3·0
10	$5{\cdot}0 \times 10^1$	$6{\cdot}8 \times 10^2$	3·1
100	$1{\cdot}2 \times 10^5$	$1{\cdot}6 \times 10^7$	2·4
1000	$4{\cdot}2 \times 10^7$	$5{\cdot}8 \times 10^{10}$	2·1

Also given in Table 1 are values of $(G_{xT}P)/(G_{xT}\,P)_{air}$. It will be noted that these values differ from unity which means that the present theory indicates that (G_xP) cannot be used as a universal criterion for the point at which transition commences. This factor (G_xP) is, of course, often assumed to have a critical value at transition and although the results from which this criterion was derived, which gave some mean point in the transition region, are not directly comparable with the present theory which gives the beginning of the transition region, it does seem unlikely that (G_xP) can be used over a wide range of Prandtl numbers as a criterion for transition. Indeed Saunders'[5] results in mercury suggest that transition may be occurring at a much lower value of (G_xP) than that obtained for air and water, although there are not sufficient points to make the evidence conclusive, especially as much smaller changes are to be expected at transition with mercury, which has a very low Prandtl number, than with air and water.

The actual point in the boundary layer at which transition commences will be that at which $[\eta^2|(df/d\eta)|]_{max}$ occurs. Therefore in Table 1 are also listed the values of η_m, the value of η at which this occurs.

Comparison with experiment

The theory being considered in the present note can only be expected to indicate the point at which turbulence will first occur i.e. the beginning of the transition region. Therefore, since Tritton's experiments[6] indicate that with a natural convective boundary layer there is a long transition region, many experimental determinations of the transition point are not directly comparable with the present theory. This is because experiments in which the point at which a marked change in boundary layer thickness, as noted by optical means, or the point at which the local heat transfer rate changes, were determined as the transition point will only give some rather arbitrary point in the long transition region.

Experiments in which the approximate point at which turbulence first occurred was determined are:—

(i) In air, from the review given by Tritton[6], by Tritton himself who gives a mean value of G_{xT} of approximately $9{\cdot}2 \times 10^6$, but remarks that his value is probably low due to external disturbances in the room; by Schmidt and Beckman who found intermittent turbulence at values of G_x equal $4{\cdot}3 \times 10^7$ and 14×10^7; by Eckert, Soehngen and Schneider who observed Tollmien-Schlichting waves at $G_x = 4{\cdot}2 \times 10^8$.

(ii) In water by Szewczyk[3] who gives the mean value of G_{xT} as approximately 2×10^9.

Now if the Prandtl numbers for air and water are taken as 0·73 and, due to the low temperature differences generally involved, 7 respectively, then Fig 2 gives $(G_{xT})_{water}/(G_{xT})_{air}$ as approximately 35. The experimental results given above give, if Tritton's result is ignored due to his doubts about it, $(G_{xT})_{water} / (G_{xT})_{air}$ between 4·8 and 46·4. Thus the predicted value is of the same order as the experimental results.

Tritton's experiments also give the value of η at which the first turbulent spots occur for air as between 1·8 and 2·5 while the value of η at which $[\eta^2 \,|(df/d\eta)|]_{max}$ occurs is for air, from Table 1, approximately 3·0. It may be argued that a comparison between these values should not be made as Tritton's value of G_{xT} was considered inaccurate and ignored and therefore his value of η for first turbulence should also be ignored, or at least treated as being approximate. However, the free stream turbulence, which is assumed to cause the low value of G_{xT} observed by Tritton, can cause only small relative changes in the velocity profile, and it is to be expected, therefore, that the value of η at which turbulence first occurs will be little effected by this free-stream turbulence. Thus the difference between Tritton's experimental value of 1·8 to 2·5 and the predicted value of 3·0 would appear to indicate some failing in the theory.

Remarks

Although the evidence is far from conclusive it would appear from the above comparisons that when the van Driest and Blumer theory is applied in the form given by them to a free convective boundary layer it predicts the correct trends but the numerical values are not altogether satisfactory. In an attempt to produce better results a modification to the theory, as originally given, will be considered below.

Alternative expression for l

The assumption that l is proportional to y as used in the original theory is really quite arbitrary. For the type of boundary layer problems treated by van Driest and Blumer it is to be expected that this assumption will give reasonable results. For free convective layers, however, this form of l can only be expected to apply near the wall. In the outer portion of the boundary layer, i.e. beyond the point of maximum velocity, it seems more reasonable to assume that l is proportional to the boundary layer thickness δ which is given by:—

$$\delta/x = \text{constant}/(G_x)^{0.25} \qquad \text{(9)}$$

Therefore in the outer portion of the boundary layer it will be assumed that l is proportional to $x/(G_x)^{0.25}$.

With this assumption the van Driest and Blumer transition criterion, equation (1), gives for the outer part of the layer when equation (4) is used:—

$$(G_{xT})^{0.25} . \,|(df/d\eta)| = T' \qquad \text{(10)}$$

where T' is a new transition constant.

Therefore with this assumption for l, the transition point will depend on $|(df/d\eta)|$ · Values of this derived from Schuh's[4] results are shown in Fig 3 for the outer part of the boundary layer.

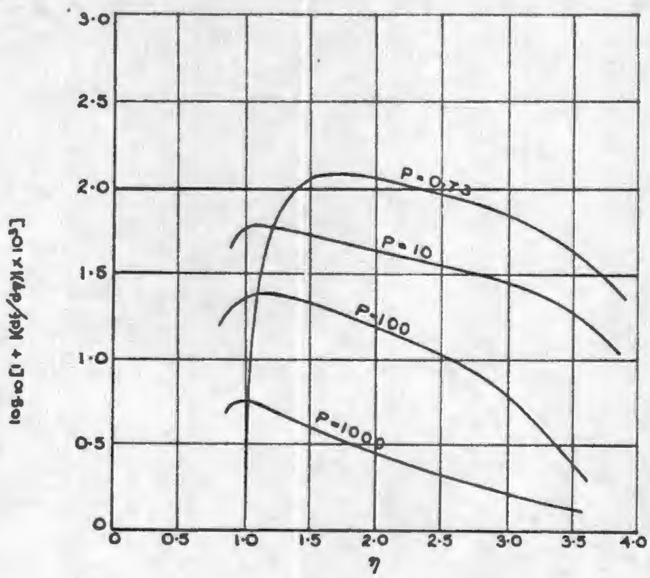

Fig 3 Variation of $|(df/d\eta)|$ with η in the outer portion of the boundary layer for various values of P.

It will be noted by comparing Figs 1 and 3 that in all cases $|(df/d\eta)|_{max}$ in the outer region is greater than the maximum value of $\eta^2 \,|(df/d\eta)|$ in the inner region. Thus if it is assumed that l is proportional to y from the wall out to approximately the maximum velocity point and that l is proportional to δ over the rest of the boundary layer then transition will still occur in the outer region at the point where $(df/d\eta)$ reaches a maximum value.

Results with the alternative expression for l

Equation (10) can be used to give:—

$$\frac{(G_{xT})_{P_1}}{(G_{xT})_{P_2}} = \frac{[\,|(df/d\eta)|_{max}\,]^4_{P_2}}{[\,|(df/d\eta)|_{max}\,]^4_{P_1}} \qquad \text{(11)}$$

where, as before, subscripts P_1 and P_2 refer to conditions in fluids having Prandtl numbers P_1 and P_2. Values of $(G_{xT}) / (G_{xT})_{air}$ derived with the aid of this equation using values of $|(df/d\eta)|_{max}$ obtained from Fig 3, are shown in Table 2 and plotted in Fig 2 where they can be compared with the values of this ratio obtained by assuming l proportional to y.

Also given in Table 2 are the values of $(G_{xT}P)/(G_{xT}P)_{air}$ obtained when l is assumed proportional to δ, and also, the values of η_m i.e. the values of η at which $|(df/d\eta)|_{max}$, and therefore the first turbulence, occurs.

Comparison with experiment

When l is assumed proportional to δ it is seen from Fig 2 that $(G_{xT})_{water}/(G_{xT})_{air}$ is given as approximately 12. This value is also within the experimental range derived previously and cannot be said to be inferior or superior to the value obtained by assuming l proportional to y.

However, the predicted value of η at which the first turbulence will occur in air is seen from Table 2 to be 1·7, which is closer to Tritton's experimental value of 1·8—2·5 than is the value of 3·0 predicted by assuming l proportional to y.

TABLE 2

Results with l proportional to δ

P	$(G_{xT})/(G_{xT})_{air}$	$(G_{xT} \times P)/(G_{xT} \times P)_{air}$	η_m
0·73	1·0	1·0	1·7
10	$2{\cdot}4 \times 10^1$	$2{\cdot}4 \times 10^1$	1·1
100	$5{\cdot}7 \times 10^3$	$5{\cdot}7 \times 10^3$	1·2
1000	$5{\cdot}4 \times 10^6$	$5{\cdot}4 \times 10^5$	1·0

Actual transition point

In order to determine the actual value of G_{xT} for any fluid by means of either theory the transition constant T or T' has to be known from experiment. If Szewczyk's value of G_{xT} for water is used for this purpose and if it is assumed that both theories give the correct value of G_{xT} for water then the values obtained are:—

(i) when l is proportional to y : $T = 160$

(ii) when l is proportional to δ : $T' = 13$

This value of T can be compared with van Driest and Blumer's value of approximately 2500 for the type of boundary layer problem they considered. This comparison demonstrates the strong influence of the type of boundary layer flow on the value of the transition constant. It complies with the well known, loosely expressed, fact that 'a free convective boundary layer is less stable than a normal boundary layer'.

Using the values of T and T' derived above allows the variation of G_{xT} with P as predicted with the two assumptions about l to be determined and is shown in Fig 4.

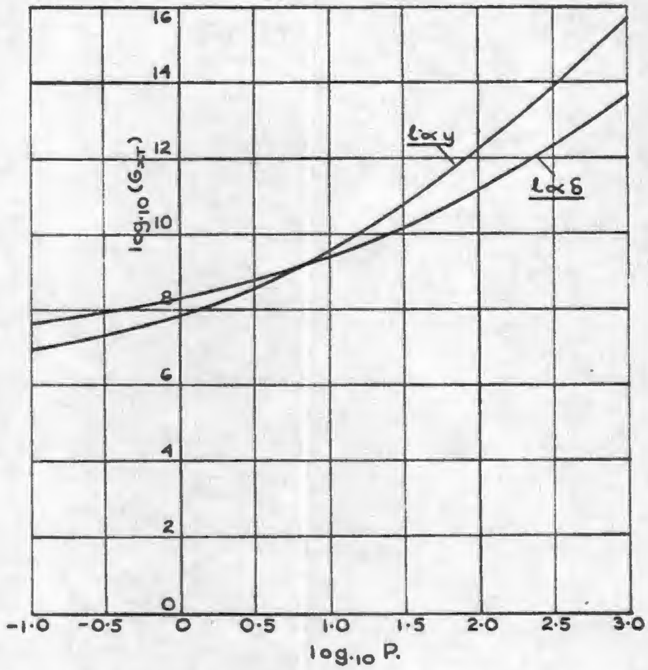

Fig 4 Variation of G_{xT} with P as predicted with the two assumptions as to l

CONCLUSION

The application of the van Driest and Blumer theory to a free convective boundary layer on a vertical plate has been shown to give results which agree reasonably well with the scanty evidence available.

If instead of assuming, as do van Driest and Blumer, that the length factor l is proportional to y, it is assumed that it is proportional to the boundary layer thickness δ, some improvement in the agreement with experiment appears to be obtained.

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Paper 12.

AN APPROXIMATE ANALYSIS OF THE TWO-DIMENSIONAL TURBULENT FREE CONVECTIVE FLOW OVER A VERTICAL FLAT PLATE.

Summary:

A method of analysing turbulent free convective flow over a vertical flat plate is presented. The analysis is similar to that of Eckert and Jackson but different forms are assumed for the boundary layer profiles and for the friction coefficient. An attempt is made to extend the analysis, on an approximate basis, to include the effects of a small superimposed forced velocity.

AN APPROXIMATE ANALYSIS OF THE TWO-DIMENSIONAL TURBULENT FREE CONVECTIVE FLOW OVER A VERTICAL FLAT PLATE

Introduction:

In free convective fluid flow, as in most viscous flows, there are two distinct types of flow which can occur, laminar flow and turbulent flow. In free convective flow over a vertical plate, the flow near the leading edge will be laminar, changing, provided certain conditions are satisfied, to turbulent flow some distance from the leading edge. The actual change from laminar to turbulent is not sharp, there being a fairly long transition region.[1]

In free convective flows, the extent of the laminar flow region is usually much greater than in most viscous flows and the study of laminar free convective flows is probably of more direct engineering importance than most studies of laminar flow. There are, however, numerous practical cases in which turbulent free convective flow also exists, and the study of this type of flow is thus also of considerable importance.

Since the analysis of most turbulent flows still relies to some extent on experimental results, the study of free convective turbulent flows is severely hampered by the lack of such experimental results. In fact, all the analyses of turbulent free convective flow at the moment available, and the present method is no exception, rely to some extent on results obtained from experimental studies of forced flow. While this reliance is to some extent justified by the known universal nature of turbulent flow in the region of the wall, thd ultimate justification stems from the fact that the end results obtained by means of this procedure show reasonable agreement with the limited experimental results available for turbulent free convection.

Previous Analyses:

The present analysis is similar to those of Eckert and Jackson[2], and of Bayley[3], both of which make, basically, the same assumptions. Beside the assumptions common to all analyses relying on the boundary layer integral equations, the following assumptions were made by the above investigators:-

(i) The flow near the wall is similar to that which exists in forced turbulent boundary layer flow and the velocity near the wall is, therefore, given by:-

$$(u/u_1) = (y/\delta)^{1/7} \tag{1}$$

where u_1 is a wall velocity profile parameter, which is to be determined, and δ is the thickness of the actual free convective boundary layer.

(ii) The shearing stress acting on the wall can be expressed in terms of a Reynolds' number based on u_1 and δ by the power law relation that is known to apply approximately in forced flow.

(iii) A power law relation, which is virtually the same as that which applies in forced flow, is assumed for the temperature distribution.

The Present Analysis:

The present analysis, while relying on the same fundamental assumptions as the analyses mentioned in the previous section, differs from them in the following respects:-

(i) It is not assumed that the hypothetical forced boundary layer, characterised by the forced velocity u_1, in terms of which the flow near the wall is specified has the same thickness as the actual boundary layer.

(ii) Different expressions are assumed for the velocity distribution near the wall and for the shearing stress.

(iii) A temperature distribution is assumed which is more closely related to the assumed form of the velocity profile i.e. it is assumed that near the wall the temperature distribution is given by the same expression as applies in the forced flow characterized by u_1.

Assumptions Made in the Present Analysis:

The following assumptions, many of which are the same as those made in the previous analyses mentioned above, will be made in the present analysis:-

(i) The free convective flow is restricted, effectively, to a boundary layer of thickness δ, this being the same for both velocity and temperature.

(ii) As in forced flow, a hypothetical origin at which the turbulent boundary layer thickness is zero can be introduced. This is shown in Fig.1 which also shows the co-ordinate system used.

(iii) The fluid properties can, except for the density variation with temperature which causes the buoyancy forces and hence the flow, be taken as constant. While this is, in most actual flows, not exactly true, experience with similar flows indicates that analyses based on the assumption of constant fluid properties can be applied in general provided that the fluid properties in the actual flow are evaluated at some suitable mean temperature in the boundary layer.

(iv) The flow near the wall is governed by conditions at the wall alone and is, therefore, essentially of the same nature as that which exists near the wall in forced flow. The velocity distribution near the wall in free convective flow can, therefore, be expressed in the form:-

$$(u_1 - u)/u_\tau = f\ (y/\delta_1) \qquad (2)$$

where u_τ = friction velocity

$= \sqrt{\tau_w/\rho}$

τ_w = local shearing stress acting on wall,

ρ = fluid density,

u = local velocity,

u_1 = a hypothetical freestream forced velocity which characterises the flow near the wall,

δ_1 = thickness of hypothetical forced boundary layer,

f = a function determined experimentally in forced flow.

The function f does not, of course, apply very near the wall in forced flow but is sufficient for the present purposes in view of the method of solution adopted.

(v) As a consequence of the previous assumption, it will be assumed that the shearing stress can also be expressed in terms of the hypothetical forced flow parameters, u_1 and δ_1, introduced above, by the same expression as applies in forced turbulent flow. One of the most accurate expressions available is that due to Squire[4] which can be expressed in the following form:-

$$\tau_w / \rho u_1^2 = \left[5.89 \log_{10} (4.075 R_\theta) \right]^{-2} \qquad (3)$$

where R_θ is the Reynolds' number based on the momentum thickness, θ, of the hypothetical forced boundary layer i.e.:-

$$R_\theta = \rho u_1 \theta / \mu \qquad (4)$$

μ being the fluid viscosity and θ being defined by :-

$$\theta / \delta_1 = \int_0^1 \left[(u/u_1) - (u/u_1)^2 \right] d(y/\delta_1) \qquad (5)$$

u being given by equation (2). Substituting equation (2) into equation (5) and the result into equation (3) gives:-

$$1 / \sqrt{K_f} = 5.89 \log_{10} \left[4.075 R_{\delta 1} (P \sqrt{K_f} - Q K_f) \right] \qquad (6)$$

where:-

$$K_f = \tau_w / \rho u_1^2 = (u_\tau / u_1)^2 = C_f/2 \qquad (7)$$

C_f being the conventional coefficient of shear stress,

$$P = \int_0^1 f d(y / \delta_1) \tag{8}$$

$$Q = \int_0^1 f^2 \, d\,(y / \delta_1) \tag{9}$$

$$R_{\delta 1} = \rho u_1 \delta_1 / \mu \tag{10}$$

Equation (5) thus gives K_f in terms of $R_{\delta 1}$, the Reynolds' number based on u_1 and δ_1.

P and Q can be evaluated by numerical integration using the experimentally determined[5] form of the function f. This procedure gives:-

$P = 2.49$

$Q = 12.2$

(vi) That the temperature distribution near the wall, as with the velocity distribution near the wall, is given by the same expression as applies in forced flow. It will further be assumed that this temperature distribution is given in terms of the same function f as was used to describe the velocity distribution near the wall i.e. it will be assumed that the temperature distribution near the wall is given by:-

$$(T - T_1) / T_\tau = (u_1 - u) / u_\tau = f\,(y/\delta_1) \tag{11}$$

where $T_\tau = q_w / \rho C_p u_\tau$,

q_w = heat transfer from the wall to the fluid per unit wall area per unit time,

C_p = specific heat of the fluid,

T_1 = actual free-stream temperature.

It may be argued that some hypothetical free-stream temperature, similar to the hypothetical forced velocity u_1, and not the actual free-stream temperature should be used to describe the temperature distribution near the wall but the form of the overall distribution is such that little error can be expected from the assumption made.

It will be noted that the hypothetical forced velocity and temperature boundary layers introduced to describe the flow near the wall have been assumed to have the same thickness and that the same function f has been used to describe both the velocity and temperature distributions. These related assumptions will, strictly, limit the present analysis to fluids having Prandtl numbers near unity.

(vii) As a consequence of the above assumptions concerning the temperature distribution near the wall it will be assumed that the rate of heat transfer from the wall is related to the shearing stress function K_f by [6]:-

$$1/K_s = \left[(1/K_f) + S(P_r)/\sqrt{K_f} \right] \qquad (12)$$

where K_f is as previously defined, P_r is the Prandtl number of the fluid and where the Stanton number K_s is given by:-

$$K_s = q_w / C_p u_1 (T_w - T_1) = T_\gamma \sqrt{K_f} / \theta_w \qquad (13)$$

where, for convenience, the following has been introduced:-

$$\theta_w = T_w - T_1 \qquad (14)$$

T_w being the local wall temperature.

The function S of equation (12) is assumed to be given by[6]:-

$$S = 5\left\{(P_r - 1) + \log_e\left[(5 P_r + 1) / 6\right]\right\} \qquad (15)$$

It is convenient at this stage to introduce the local Nusselt number, N_x, which can be written in terms of the quantities introduced above as follows:-

$$N_x = q_w x / k\theta_w = K_s P_r R_\delta (x/\delta) \qquad (16)$$

where R_δ is the Reynolds' number based on the actual boundary layer thickness, δ, and the hypothetical forced velocity, u_1, i.e.:-

$$R_\delta = \rho u_1 \delta / \mu$$

(viii) The velocity distribution in the outer region of the boundary layer is independent of the wall shearing stress and is given by a power law relation. A continuous expression for the velocity distribution throughout the boundary layer, which satisfies the correct boundary conditions, obtained by combining this power law relation with equation (2), which gives the velocity distribution near the wall, is:-

$$u = (u_1 - u_\tau f) \left[1 - (y/\delta)^n \right] \qquad (18)$$

i.e.

$$(u/u_1) = \left[1 - (y/\delta)^n \right] - \sqrt{K_f} \quad f\ (y/\delta_1) \left[1 - (y/\delta)^n \right] \qquad (19)$$

The power n remains, for the moment, undetermined.

An important parameter, which will be used later, is the maximum velocity in the free convective boundary layer and the point in the layer at which it occurs. Differentiating equation (19) with respect to (y/δ) and assuming that in the region concerned:-

$$f\ (y/\delta_1) = -2.5 \ \log_e (y/\delta_1) \qquad (20)$$

and equating to zero gives:-

$$(\bar{y}/\delta)^n = 2.5 \sqrt{K_f} \Big/ \left[n + 2.5n \sqrt{K_f} \ \log_e \ (\bar{y}/\delta_1) + 2.5 \sqrt{K_f} \right] \qquad (21)$$

where $\bar{y}$ is the distance from the wall to the point of maximum velocity. This equation shows that for any given values of n and (δ/δ_1) (see later):-

$$\bar{y} / \delta = Y\ (K_f) \qquad (22)$$

The maximum velocity $\bar{u}$ can then be obtained by substituting this value of $\bar{y}$ in equation (19).

(ix) The temperature distribution in the outer portion of the boundary layer is independent of the conditions at the wall and is given by a power law relation.

A continuous law will not be assumed for the temperature distribution, it being assumed, instead, that two distinct relations, the inner law and the outer law, apply. The dividing point between the regions of application of the two laws is assumed, more or less arbitrarily, to be the point, $\bar{y}$, at which the maximum velocity occurs i.e. it is assumed that the temperature distribution is given by:-

For $(y/\delta) < (\bar{y}/\delta)$

$$(T - T_1)/\theta_w = K_s \, f \, (y/\delta_1)/\sqrt{K_f} \qquad (23)$$

For $(y/\delta) > (\bar{y}/\delta)$

$$(T - T_1)/\theta_w = 1 - (y/\delta)^m$$

The exponent m, as with the velocity exponent n, remains at this stage undetermined.

The assumed forms for the velocity and temperature distributions throughout the boundary layer are shown in Fig.2.

(x) The ratio of the thickness δ_1 of the hypothetical forced boundary layer, which characterizes the flow near the wall, to the thickness δ of the actual free convective boundary layer is dependent on the Prandtl number of the fluid alone i.e.:-

$$(\delta_1/\delta) = R(P_r) \tag{24}$$

where the function R remains, also, for the moment, undetermined.

There are thus three undetermined parameters n, m and R. These can be determined from velocity and temperature measurements in the outer region of a turbulent free convective boundary layer. Such measurements are easier to make than measurements very near the wall and for this latter region, therefore, more definite assumptions concerning the velocity and temperature distributions had to be made.

Analysis:

The present analysis is based on the integrated boundary layer momentum and energy equations. For the case of two-dimensional free convective flow over a vertical plate these equations are:-

$$\frac{d}{dx} \int_0^{\delta} u^2 \, dy = \beta g \int_0^{\delta} (T - T_1) \, dy - u_{\tau}^2 \tag{25}$$

and:-

$$\frac{d}{dx} \int_0^{\delta} u(T - T_1) \, dy = u_{\tau} T_{\tau} \tag{26}$$

In view of the assumed forms of the velocity and temperature distributions, these equations are conveniently written as:-

$$\frac{d}{dx}(I\, u_1^{\,2}\delta) = J\beta g\ \delta\theta_w - K_f\, u_1^{\,2} \tag{27}$$

and

$$\frac{d}{dx}(L\, u_1\, \delta\theta_w) = K\theta_w\ u_1 \tag{28}$$

where I, J and L are defined as follows:-

$$I = \int_0^1 (u/u_1)^2\ d\,(y/\delta) \tag{29}$$

$$J = \int_0^1 \left[(T - T_1)\ /\theta_w\right]\ d\,(y/\delta) \tag{30}$$

$$L = \int_0^1 (u/u_1)\left[\ (T - T_1)\ /\theta_w\right] d\,(y/\delta) \tag{31}$$

For any given values of n, m and R, equations (19) and (23) allow I, J and L to be evaluated for any given value of K_f, K_s being given in terms of K_f by equation (12).

For the case of constant wall temperature, which will be considered here, equation (28) reduces to:-

$$\frac{d}{dx}(L\, u_1\, \delta) = K_s\, u_1 \tag{32}$$

Equations (27) and (32) in conjunction with equations (6) and (12) together with the equations defining the velocity and temperature profiles and the coefficients in the above equations constitute a solution to the present problem provided the initial conditions on u_1 and δ are known.

Boundary Conditions:

As noted before, a hypothetical origin has been introduced at which the boundary layer thickness is zero and downstream of which the boundary layer is entirely turbulent. Since the flow is the result of the heating of the fluid, an associated assumption, which is based on the same assumptions as used in deriving the boundary layer equations, is that at the same hypothetical origin u_1 is also zero. The set of equations, mentioned in the preceding section, will, therefore, be solved subject to the following initial conditions:-

$$\text{At } x = o\,; \quad \delta = o; \quad u_1 = o \tag{33}$$

Method of Solution:

In solving equations (27) and (32) subject to boundary conditions (33), it is convenient to introduce some reference length ℓ_R and to then rewrite these equations in terms of the following dimensionless variables:-

$$Y = \rho\, u_1\, \delta / \mu \tag{34}$$

and

$$Z = \rho^2\, u_1{}^2\, \delta\, \ell_R \,/\, \mu^2 \tag{35}$$

Equations (27) and (32) then reduce to:-

$$\frac{d(TZ)}{d(x/\ell_R)} = JG_{\ell_R}\, Y^2/Z - K_f\, Z^2/Y^2 \tag{36}$$

and

$$\frac{d\,(LY)}{d\,(x/\ell_R)} = K_S\, Z/Y \tag{37}$$

where G_{ℓ_R} is the Grashof number based on the reference length-ℓ_R i.e.:-

$$G_{\ell_R} = \beta g \rho^2 \ell_R^3 \theta_w / \mu^2 \tag{38}$$

The variable (x/ℓ_R) can be eliminated between equations (36) and (37) to give:-

$$\frac{d\,(IZ)}{d\,(LY)} = J G_{\ell_R} Y^3 / K_s Z^2 - K_f Z / K_s Y \tag{39}$$

Now equation (6), which gives K_f and from which, therefore, K_s can be deduced, can be written as:-

$$1 / \sqrt{K_f} = 5.89 \log_{10} \left[4.075\, Y \left(P \sqrt{K_f} - Q\, K_f\right)\right] \tag{40}$$

Since the velocity and temperature profiles, and hence I, J and L, depend only on K_f and K_s, equations (36) and (37) can be used to determine Z as a function of Y for any given value of G_{ℓ_R}. Equation (37) can then be used to determine the corresponding values of (x/ℓ_R), while (δ/ℓ_R) can be determined by noting that:-

$$(\delta/\ell_R) = Y^2/Z \tag{41}$$

With these results, local Grashof numbers, G_x, and the corresponding local Nusselt Numbers, N_x, can be determined from the following

$$G_x = G_{\ell_R} (x/\ell_R)^3 \tag{42}$$

$$N_x = K_s P_r Y (x/\ell_R)/(\delta/\ell_R) \tag{43}$$

Before any numerical results can be determined, it is necessary to know the values of n, m and R, which, as stated previously, must be determined from experimental results.

<u>The Values of m, n and R for Air</u>:

As noted before, the values of m, n and R are determined by comparing the velocity and temperature profiles as given by equations (19) and (23), for various values of these parameters, with experimental measurements in the outer portion of the turbulent, free convective boundary layer and choosing those values of the parameters which give the closest fit with experiment. In order to make this comparison accurately it is necessary to know sufficient about the conditions under which the results were obtained to determine the values of K_f and K_s, which are required in calculating the theoretical profiles.

The only experimental results available to the author were those of Griffiths and Davies[2] in air. The information given with these results is, however, insufficient to determine K_f and, hence, to derive the value of K_s. However, since these variables are expected to exert only a secondary influence on the profiles, a mean value of K_f, equal to 0.0025, was used in making the comparison of the predicted profiles with the experimental results.

In making the comparison, the variation of $(u/\bar{u})$ and (θ/θ_w) with $(y/\bar{y})$ were determined, $\bar{u}$ and $\bar{y}$ being, as before, the maximum velocity in the layer and the distance from the wall at which this velocity occurs, respectively.

The values of m, n and R which give the best fit between the predicted profiles and experimental results, written in the above form, are:-

$$m = 0.16 \quad n = 0.47 \quad R = 0.6$$

The predicted profiles, as given by these values of the parameters, together with the experimental results form which they were determined are shown in Fig.3.

Numerical Results for Air and Comparison with Experiment:

Using the values of m, n and R given in the previous section, the predicted variation of N_x with G_x can be determined by means of the procedure previously outlined.

Experimentally, it is far more usual to derive the average Nusselt and Grashof numbers, N_ℓ and G_ℓ respectively, which are based on the overall plate length ℓ, than it is to derive values of N_x and G_x. Now N_ℓ is defined in terms of the mean rate of heat transfer, q_m, which is given by:-

$$q_m = \frac{1}{\ell}\int_0^\ell q\,dx \qquad (44)$$

where q is the local rate of heat transfer at any distance x from the leading edge of the plate.

N_ℓ is, therefore , given in terms of N_x by:-

$$N_\ell = \int_0^\ell (N_x / x)\, dx \qquad (45)$$

and can, therefore, be determined from the values of N_x derived by the method previously explained.

With the variation of N_ℓ with G_ℓ determined in this way, it is found that good agreement with experimental results is obtained if the numerical coefficient in equation (40) is reduced from 5.89 to 4.8. This is not an unexpected requirement since the Reynolds numbers in the forced flow in which the experimental results used in deriving the value 5.89 were obtained were much higher than those appropriate to free convection. The predicted resultx, with adjusted constant, are compared with some experimental results in Fig.4. In all that follows this adjusted constant will be used in obtaining numerical results.

Approximate Method of Solution:

While the method of solution, outlined above, is quite general, its use in the extension of the above analysis to various related problems is rather cumbersome. An approximate method of solving equations (27) and (32) will, therefore, be indicated. This solution is based on the assumption that, at the high Grashof numbers associated with fully developed turbulent free convective flow, the variation of K_s and K_f with x is so small that their variation has a negligible effect on the variation of δ and u_1 with x and that an approximate solution for these quantities can be obtained by assuming K_s and K_f to be constant and equal to some mean value.

Therefore, assuming that the consequent solutions for δ and u_1 are of the simple power-law type i.e. that:-

$$\delta = Px^{p} \tag{46}$$

and

$$u_1 = Q\,x^{q} \tag{47}$$

which, of course, satisfy boundary conditions (33), equations (27) and (32) reduce to:-

$$(2q + p)\, I Q^2 P\, x^{2q + p - 1} = J \beta g \theta_w P\, x^{p} - K_f Q^2\, x^{2q} \tag{48}$$

and

$$(p + q)\, L\, Q\, P\, x^{p + q - 1} = K_s\, Q\, x^{q} \tag{49}$$

respectively, since, with K_f and K_s assumed to be constant, I, J and L must be constant.

Equations (48) and (49) give:-

$$p = 1 \; ; \; q = 1/2 .$$

$$P = 2K_s/3L \; ; \; Q = \left[2J K_s / (4I K_s + 3K_f L)\right]^{1/2} (\beta g \theta_w)^{1/2}$$

and hence, therefore, :-

$$(\delta/x) = (K_x / L) \quad (50)$$

and:-

$$u_1 = \left[2J\ K_s / (4I\ K_s + 3\ K_f\ L) \right]^{1/2} (\beta\ g\ \theta_w\ x)^{1/2} \quad (51)$$

which in turn give:-

$$R_\delta = (2K_s / 3L) \left[2J\ K_s / (4I\ K_s + 3\ K_f\ L) \right]^{1/2} G_x^{1/2} \quad (52)$$

G_x being, as before, the local Grashof number.

Since $R_{\delta_1} = R_\delta / R$ the above equqtion can be substituted into equation (6) to give a second approximation to K_f, from which can be determined a second approximation to K_s, for any value of G_x. These results can then be substituted into equation (16) with corrected coefficient to determined the value of N_x corresponding to this value of G_x.

The variation of N_x with G_x for air as determined by this procedure, using the previously determined values of n, m and R, is shown in Fig.6, the initially assumed constant value of K_f used in determining these results being 0.0040. Also shown in Fig.6 are the results given by the more accurate method of solution ddalt with before. The approximate solution is seen to be in fair agreement with the accurate solution.

Correction for Laminar Region:

It will be noted that the length x used in defining the theoretical values of Grashof and Nusselt numbers is measured from a hypothetical origin at which the fully developed turbulent boundary layer has zero thickness while the length used in defining any experimentally measured values of these numbers will be measured from the actual leading edge of the plate. The assessment of the effects of this difference is rendered difficult by the long transition which usually exists with free convective flow. Some idea of the magnitude of the effects can, however, be obtained by assuming that a sharp transition occurs at some intermediate value of G_x within this transition region and assuming, further, that the energy content of the boundary layer remains constant during this sharp transition. (See Fig.7).

A convenient measure of the energy content of the boundary layer to use in the analysis based on the above assumption is:-

$$H = \int_0^{\delta} \rho u (T - T_1)\, dy \qquad (53)$$

Now introducing x_t, the distance from the leading edge of the plate to any point considered, and x, the distance from the hypothetical turbulent origin to the same point, it is assumed (see Fig.7) that these are related by:-

$$x_t = x + d \qquad (54)$$

Squires[6] approximate analysis of the laminar free convective flow over a vertical flat plate gives:-

$$H = C_L \mu \theta_w G_{x_t}^{0.25} \qquad (55)$$

where the coefficient C_L depends only on the Prandtl number of the fluid considered and G_{x_t} is the Grashof number based on x_t the distance from the leading edge of the plate.

Similarly, the approximate analysis of the turbulent flow problem, given in the previous section, leads to

$$H = (2 K_s/3) \left[2J K_s /(4I K_s + 3K_f L) \right]^{0.5} \mu \, \theta_w \, G_x^{0.5} \qquad (56)$$

$$= C_T \, \mu \, \theta_w \, G_x^{0.5}$$

where, here, C_T depends on the type of fluid and the constant value of K_f assumed in the analysis and where G_x is the Grashof number based on the distance x from the hypothetical origin.

Now at the transition point the above two values of H are, by assumption, equal and G_{x_t} is known and equal, say, to G_s. The value of G_x at the transition point is, therefore, given by :-

$$G_x = \left[(C_L/C_T) G_s^{0.25} \right]^2 \qquad (57)$$

Dividing this by G_s and taking the cube root of both sides then gives:-

$$x/x_s = k^{1/3} \qquad (58)$$

where x_s and x are the distances of the transition point from the leading edge of the plate and from the hypothetical turbulent flow origin respectively, and k is a constant for a given fluid.

This equation can be re-arranged and combined with equation (54) to give:-

$$d = (1 - k^{1/3})\, x_s. \qquad (59)$$

Now, if G_{x_t} is the Grashof number based on the length x_t from any point considered in the turbulent flow from the leading edge of the plate and if G_x is the Grashof number based on the distance of the same point from the hypothetical origin used in analysing the turbulent flow, then, since $G_{x_t} = G_x \left[(x+d)/x \right]^3$, :-

$$G_{x_t} = G_x + 3\, G_x^{2/3} G_d^{1/3} + 3\, G_x^{1/3} G_d^{2/3} + G_d \qquad (60)$$

where G_d is the Grashof number based on the correction length d i.e. equal to $\beta g \theta_w \rho^2 d^3 / \mu^2$. But by equation (59):-

$$G_d = (1 - k^{1/3})^3\, G_s \qquad (61)$$

G_s being, as before, the true Grashof number at transition.

Equations (60) and (61) allow the Grashof number, based on the distance from the leading edge, corresponding to any Grashof number, based on the distance from the hypothetical origin used in analysing the turbulent flow, to be determined for any given fluid. The corresponding correction to the Nusselt number can be obtained by noting that:-

$$N_{x_t} = N_x (G_{x_t} / G_x)^{1/3} \qquad (62)$$

The variation of N_{x_t} with G_{x_t}, obtained from the predicted approximate variation of N_x with G_x of the previous section, by assuming that $G_s = 10^9$, is shown in Fig.8 together with the results on which it is based. The effect of the correction is seen to have only a small effect on the shape of the curve.

Effect of a Small Forced Velocity on the Rate of Heat Transfer:

An approximate assessment of the effect of a small forced velocity on the type of free convective flow being dealt with can be obtained by noting that, since only a small forced velocity is being considered, it can have little effect on the flow near the wall since the temperature differences and, hence, the buoyancy forces in this region are high and the free velocities, also, therefore, comparatively high. Further, in the outer region the presence of a small forced velocity can have little effect on the fundamental nature of the flow pattern, and the expressions for the velocity and temperature distributions should have the same form as with free flow alone.

It will be assumed, therefore, that when a small forced velocity, with a free-stream magnitude of u_f, exists the velocity distribution is given by (see Fig.9):-

$$(u/u_1) = \left[1 - (y/\delta)^n\right]\left[1 - \sqrt{K_f}\; f\,(y/\delta_1)\right] + (u_f/u_1)(y/\delta)^n\left[1 - \sqrt{K_f}\; f(y/\delta_1)\right] \tag{63}$$

It will further be assumed that when a small forced velocity exists, the temperature distribution is given by the same expression as with free flow alone, i.e. equation (23), and further that since the forced velocity can have little effect on the flow near the wall the value of $(\bar{y}/\delta)$ is unaffected by the presence of this forced velocity.

With the above assumptions concerning the velocity distribution, the energy equation (26) becomes:-

$$\frac{d}{dx}\left\{\delta u_1\left[\;L + M(u_f/u_1)\right]\right\} = K_s \; u_1 \tag{64}$$

where:-

$$L = \int_0^1 \left[1 - (y/\delta)^n\right]\left[1 - \sqrt{K_f}\;\; f\,(y/\delta_1)\right]\left[(T - T_1)\;/\theta_w\right] d(y/\delta) \tag{65}$$

and for any given value of K_f and the other flow parameters it will have the same value as with free flow alone i.e. as given by equation (31).

Similarly

$$M = \int_0^1 (y/\delta)^n \left[1 - \sqrt{K_f}\;\; f\,(y/\delta_1)\right]\left[(T - T_1)\;/\theta_w\right] d(y/\delta) \tag{66}$$

Now, since u_1 is introduced to characterize the flow near the wall and since the forced velocity has, by assumption, little effect on the flow near the wall it will be assumed that u_1 has the same value as with free flow alone. If, therefore, the approximate method of solution, introduced above, in which K_f is assumed to be constant, is adopted, u_1 is given by equation (51) as:-

$$u \;\; = C\,(\beta g \theta_w)^{0.5}\; x^{0.5} \tag{67}$$

where:-

$$C = \left[2J\,K_s /(4J\,K_s + 3\,K_f\,L) \right]^{0.5} \tag{68}$$

and is, with the method of solution adopted, constant.

Substituting equation (67) into equation (64) and solving for δ gives:-

$$\delta = \frac{(2\,K_s/3)\,x}{\left[L + (M/C)(R_{f_x}/G_x^{0.5}) \right]} \tag{69}$$

where R_{f_x} is the local Reynolds' number based on the forced velocity u_f i.e.

$$R_{f_x} = \rho\, u_f\, x/\mu \tag{70}$$

Combining this with equation (67) then gives:-

$$R_\delta = \frac{(2\,C\,K_s/3)\,G_x^{0.5}}{\left[L + (M/C)\,(R_{f_x}/G_x^{0.5}) \right]} \tag{71}$$

For any given values of R_{f_x} and G_x and any assumed initial value of K_f, and hence K_s, this equation substituted into equation (6), allows a second approximation to K_f, and hence K_s, to be determined from which can be determined N_x. Equation (6) applies to the present case since it is a function of the flow near the wall which, by assumption, is unaffected by the presence of a small forced velocity.

For the case of air, assuming the same values of the parameters m, n and R as derived for free flow alone and using the same initial value of K_f, the variation of N_x with G_x for various values of R_{f_x}, as derived by the above procedure, is shown in Fig. 10.

Discussion:

At first reading it may appear that the present method of analysing turbulent free convective flow relies more on empirical results than the other similar methods of analysis mentioned before. In these other methods of analysis, however, empirically based assumptions had, in fact, to be made for the same three parameters that were derived empirically in the present analysis. Since the assumed values were introduced at a very early stage in these other analyses, the exact reasons for choosing the values used were not, perhaps, altogether clear, and any later re-adjustment of constants to make derived results fit experimental results was of a somewhat arbitrary nature. From this viewpoint alone, it is hoped that the present analysis is, perhaps, an improvement on these earlier analyses.

Conclusion:

An analysis of turbulent free convective flow has been presented and the derived results have been shown to be in fairly good agreement with experimental results for air. The analysis has been extended, on an approximate basis, to include the effects of a small forced velocity on the turbulent free convective flow and the predicted variation of Nusselt numbers with both Grashof and Reynolds' numbers has been presented.

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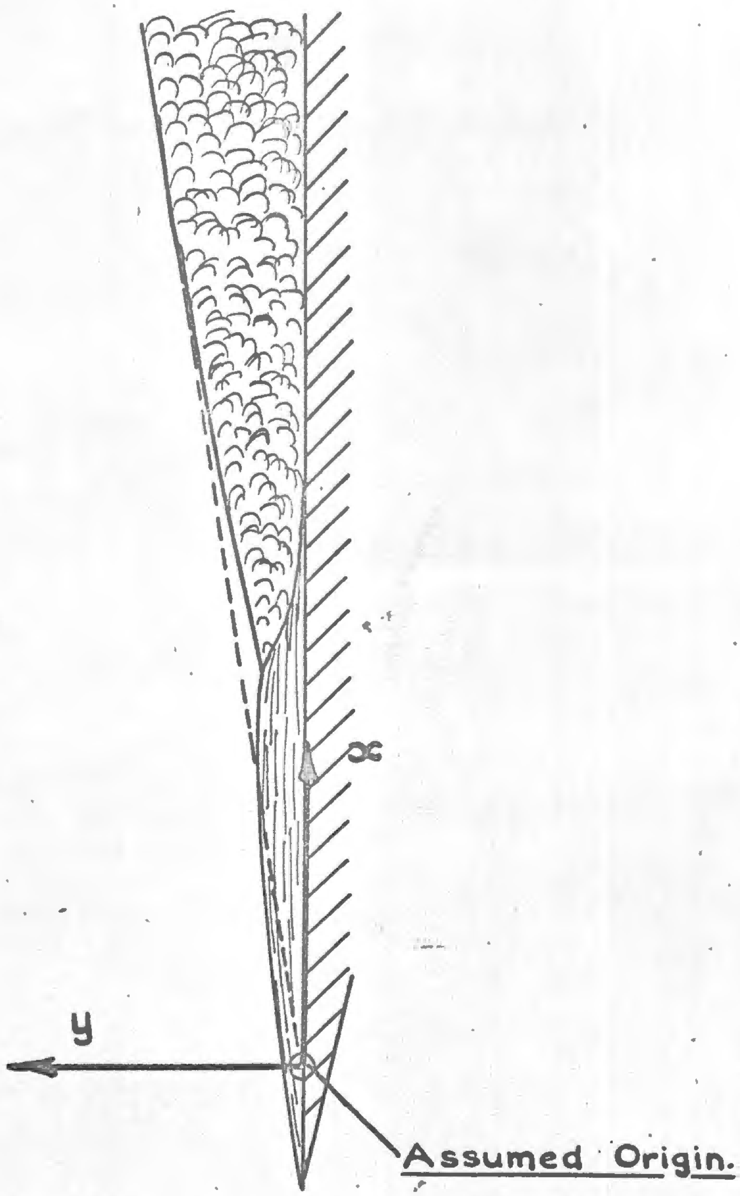

Fig. 1. Co-ordinate systems used in the analysis.

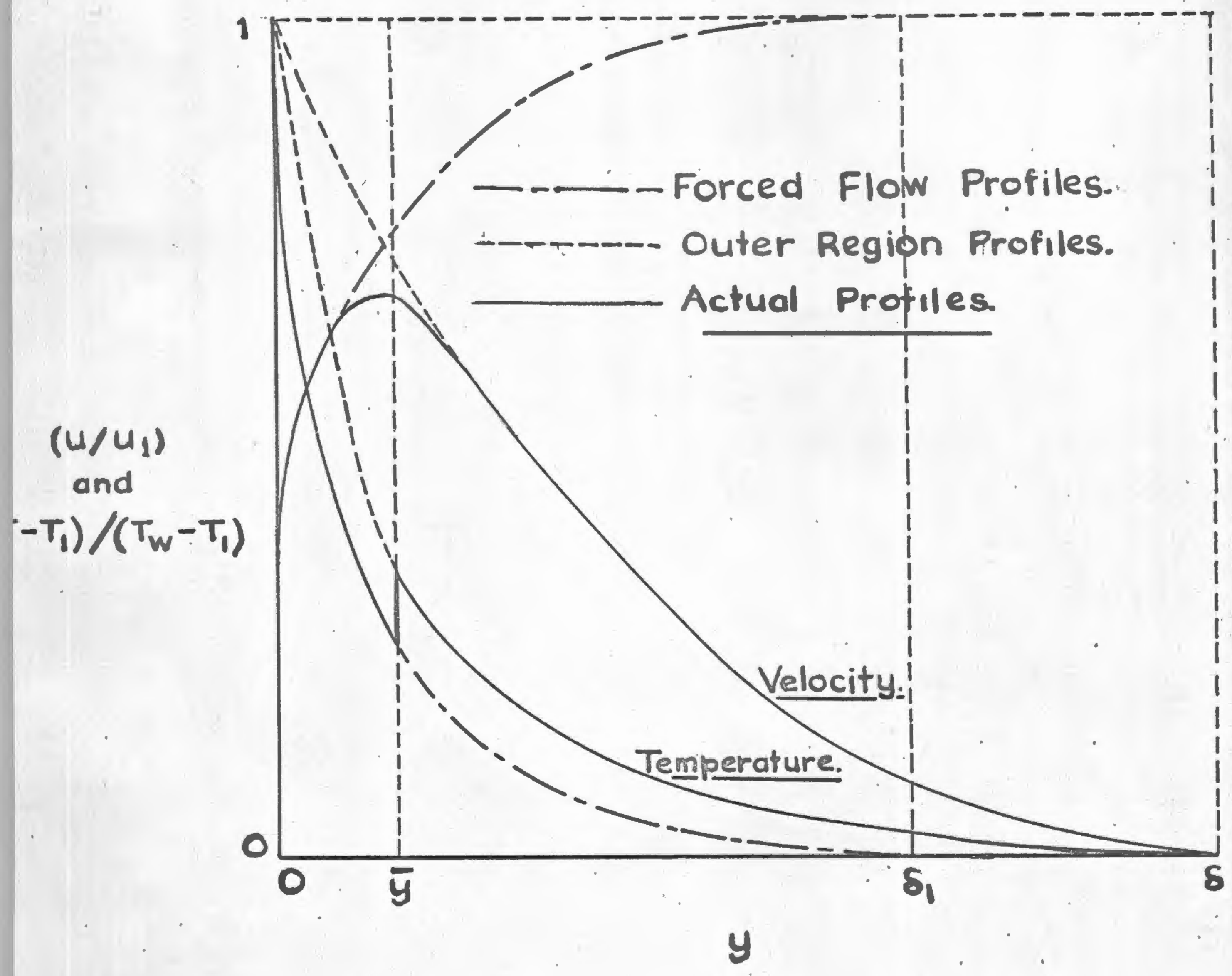

Fig. 2. Assumed velocity and temperature profiles.

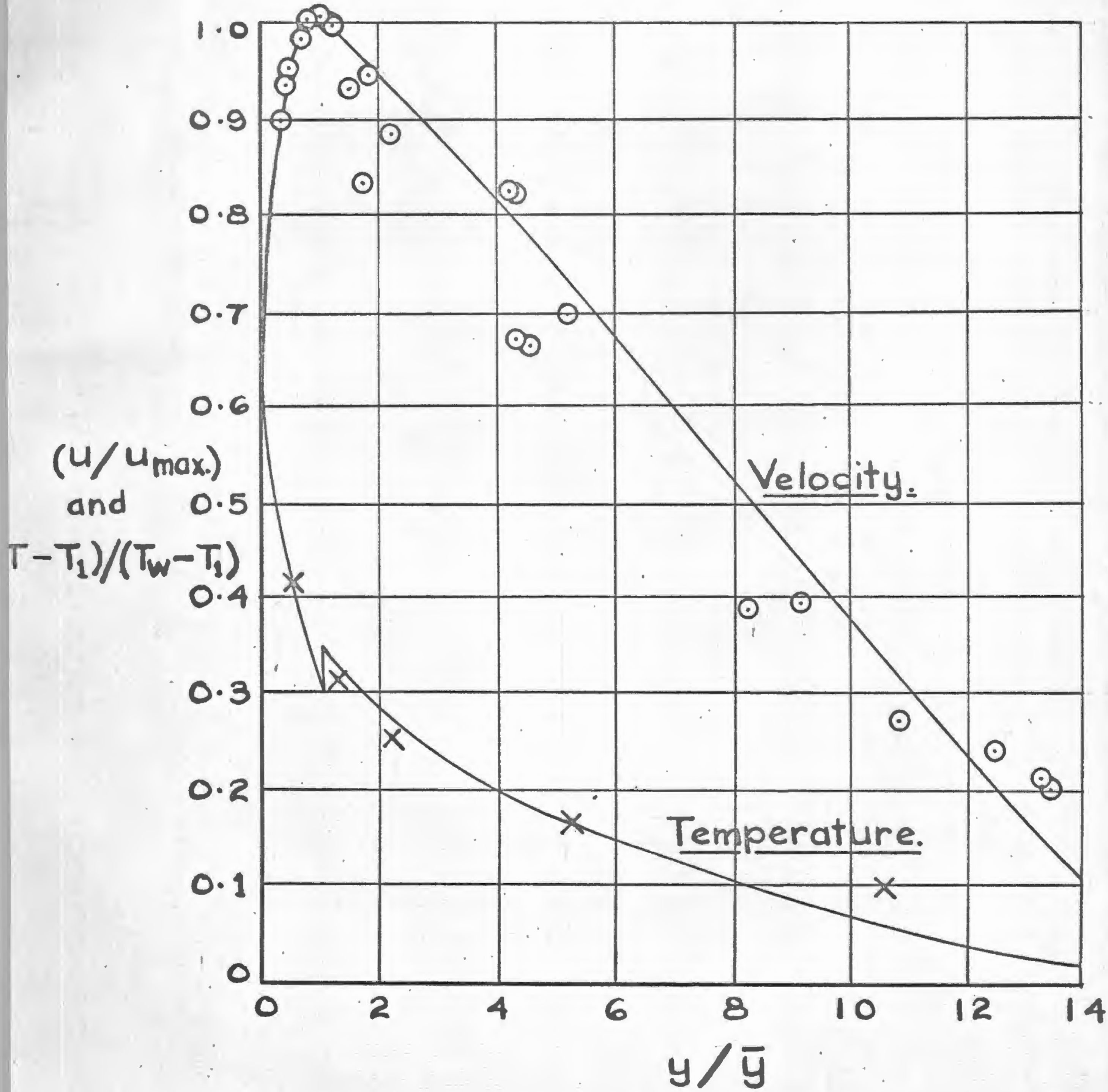

Fig. 3. Comparison of assumed profiles with experimental results.

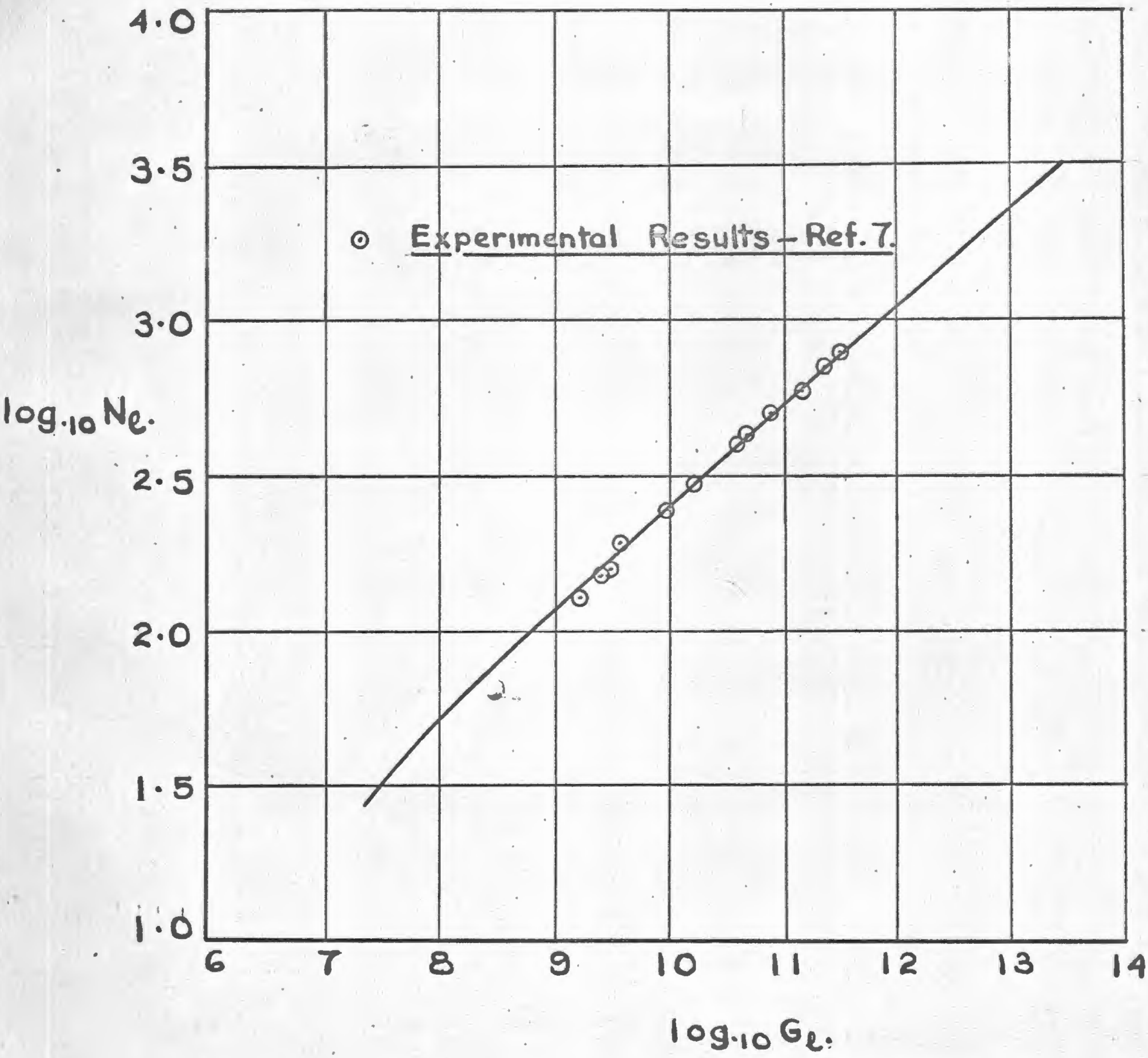

Fig. 4. Comparison of predicted heat transfer results with experimental results.

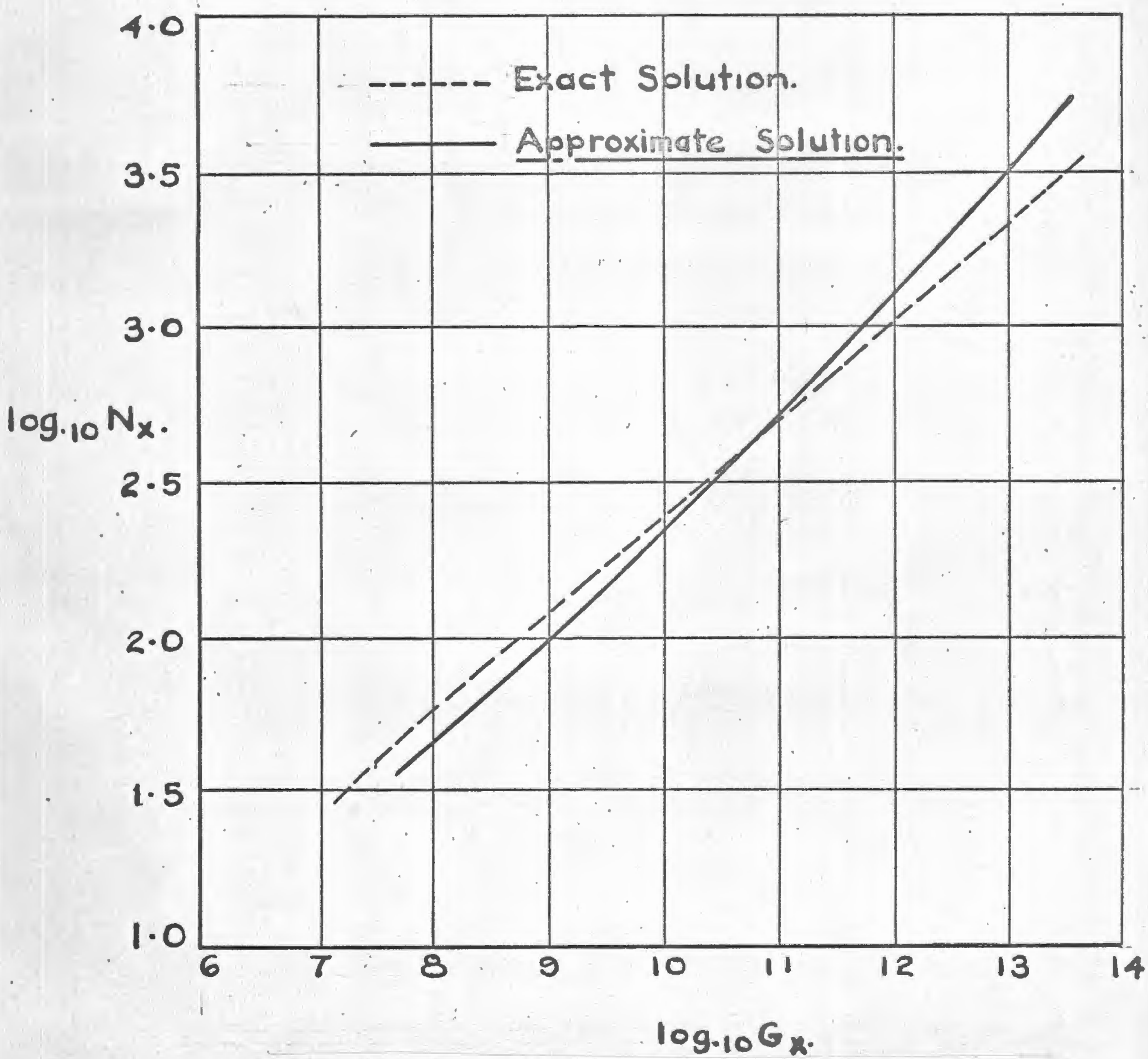

Fig. 5. Comparison of results obtained by approximate method of solution with those obtained by numerical solution.

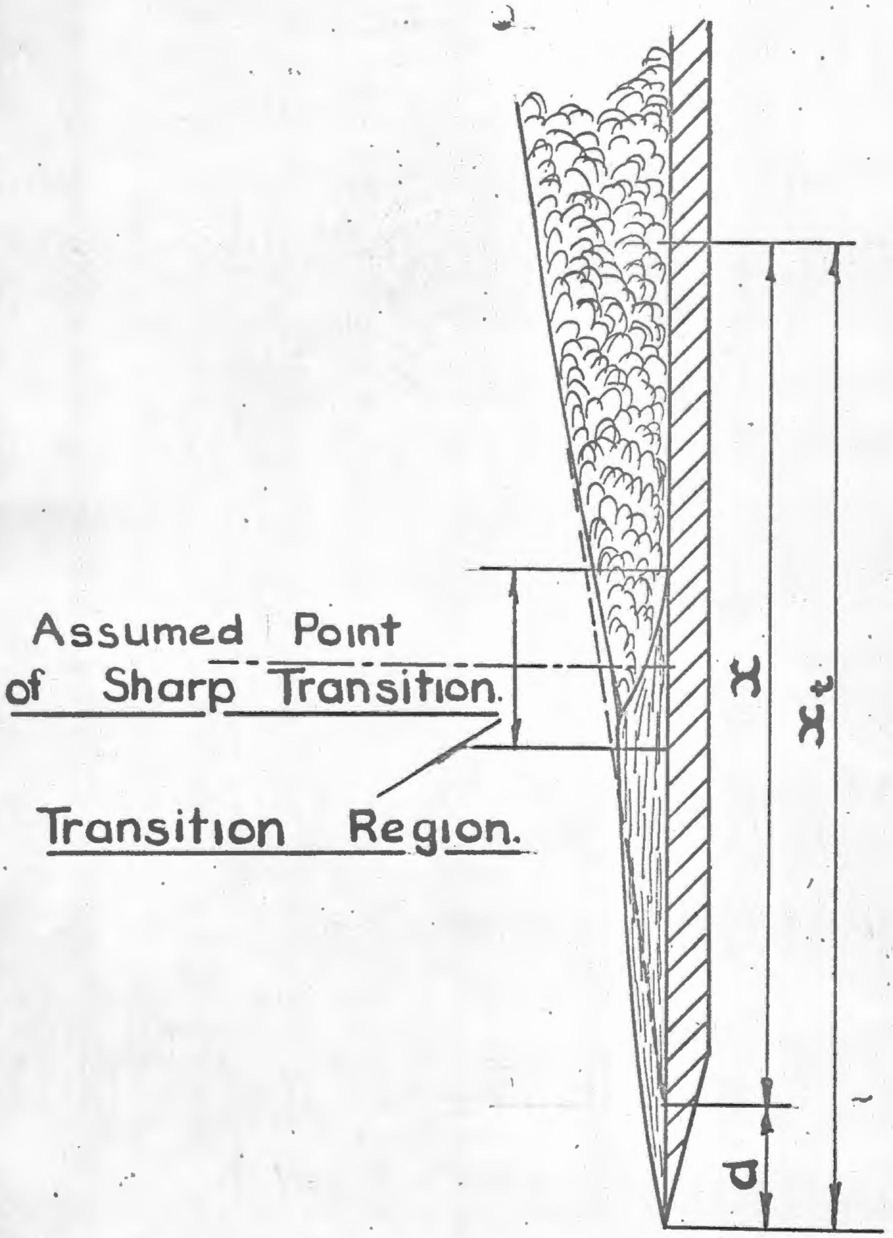

Fig. 6. Assumed flow system.

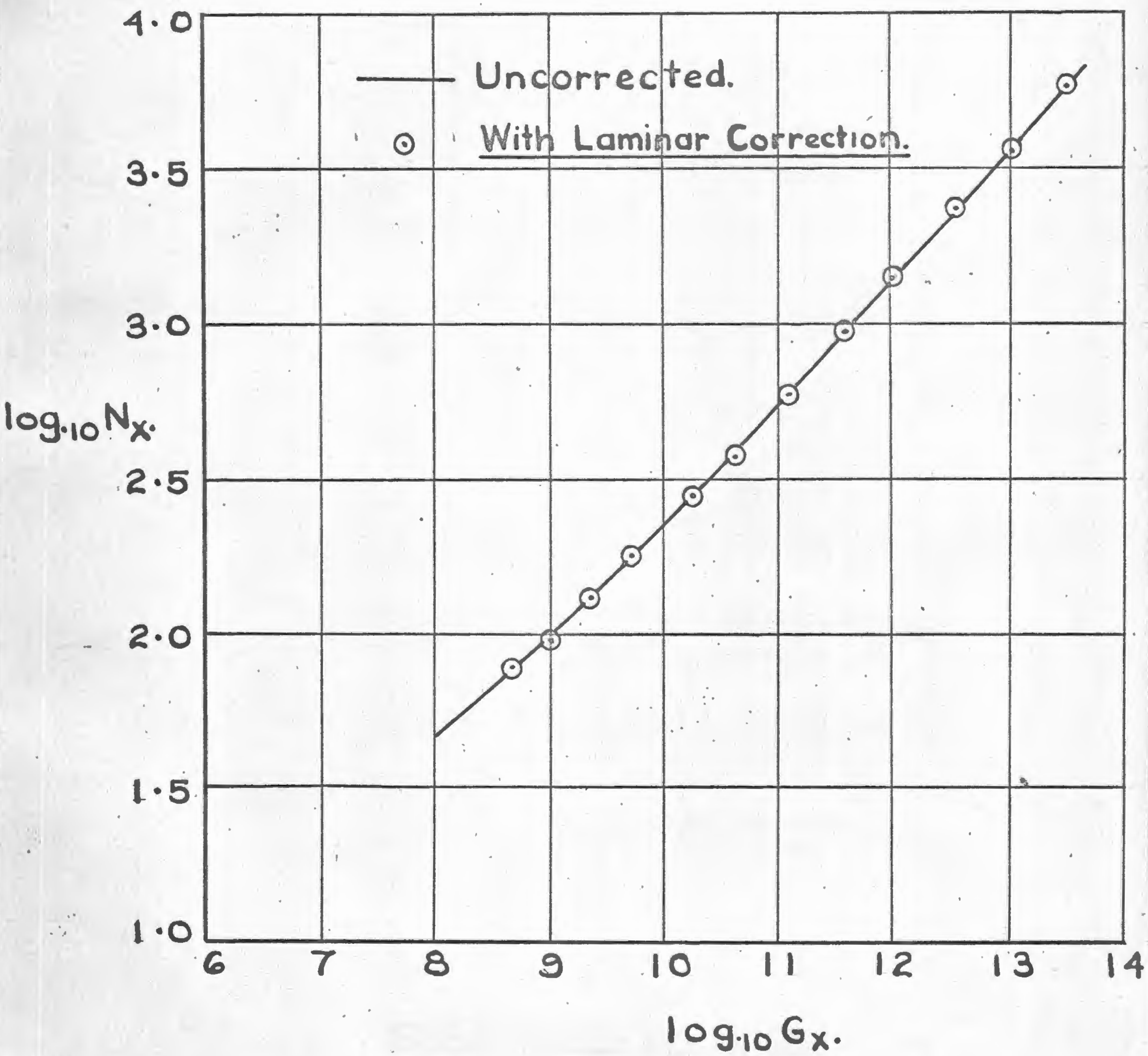

Fig. 7. Effect of laminar region on heat transfer results.

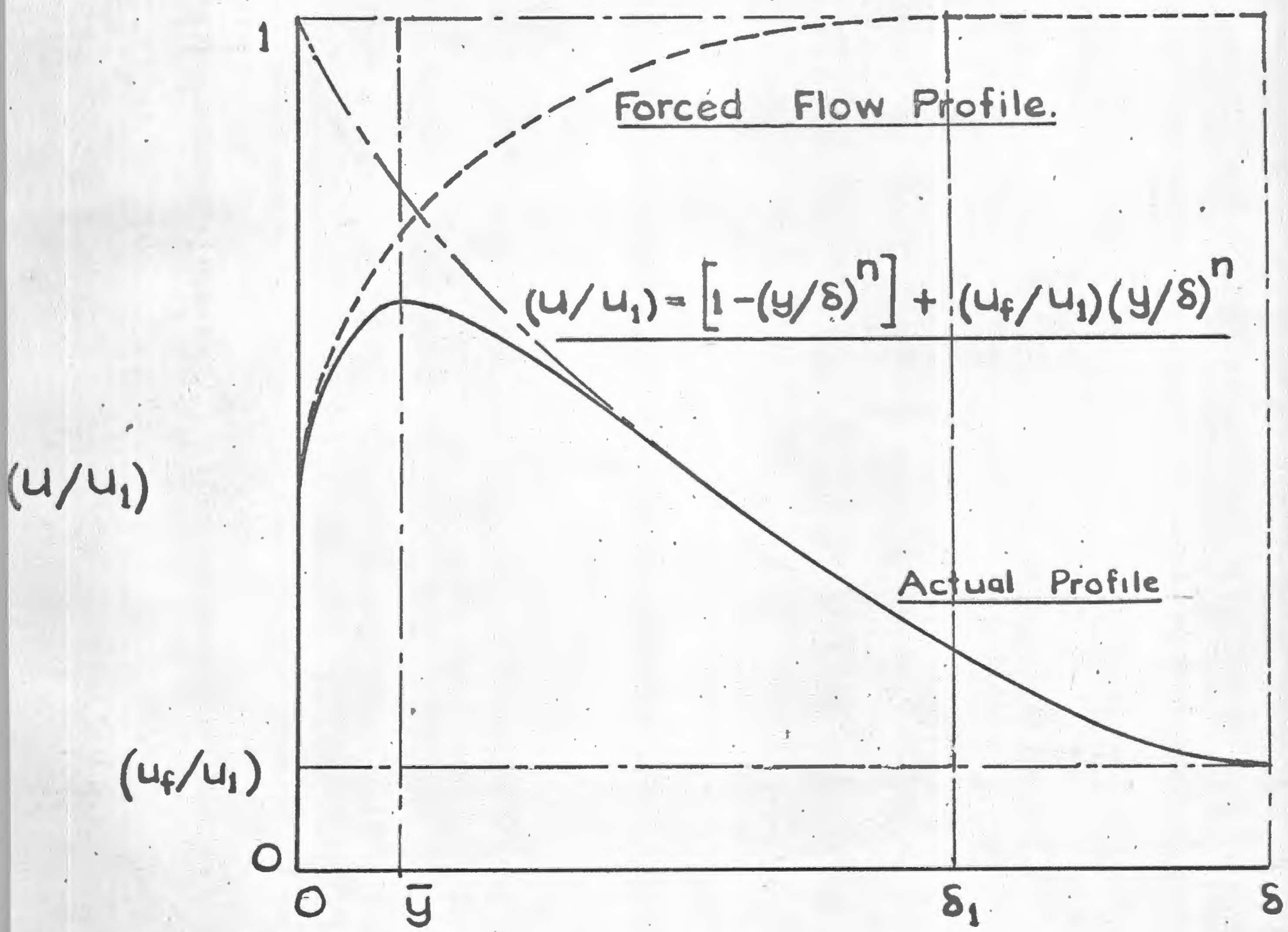

Fig. 8. Assumed boundary layer profiles in combined flow.

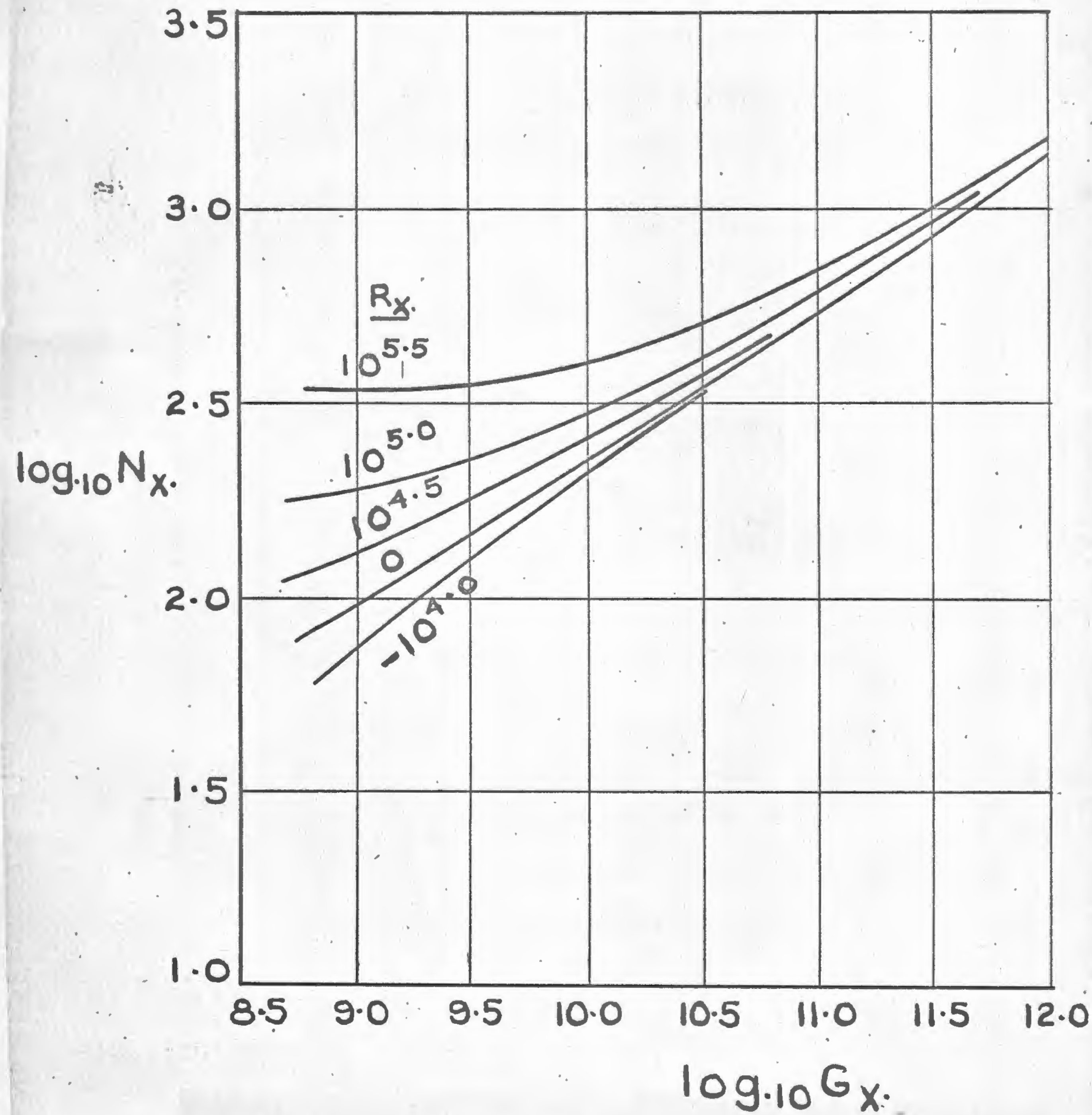

Fig. 9. Predicted effects of a small forced velocity on the heat transfer results.

Paper 13.

HEAT TRANSFER BY TURBULENT FREE CONVECTION FROM A VERTICAL PLATE.

Summary:

In the present analysis, it is assumed that the boundary layer can be split into three distinct regions. In the inner region it is assumed that the velocity and temperature profiles depend only on conditions at the wall, while in the outer region it is assumed that, due to the low buoyancy forces, the flow has the characteristics of free jet flow. The third region, the transition region, is used to join these inner and outer regions. With the boundary layer profiles defined with the aid of the above assumptions, the momentum and energy integral equations are used to derive expressions for the heat transfer from the wall to the fluid in terms of two empirical coefficients. Tentative values for these two coefficients for air flow are suggested and the results derived using these values are shown to agree reasonably well with experiment.

HEAT TRANSFER BY TURBULENT FREE CONVECTION FROM A VERTICAL PLATE

Introduction:

Although problems involving turbulent free convection are fairly common in engineering practice, the theoretical analysis of such problems has not received very much attention. This is largely due to the lack of accurate experimental measurements of the velocity distributions existing in such flows. Indeed, the majority of available analyses of turbulent free convective flow problems have, in some way or another, used the results of measurements in forced convective flows. The present analysis also makes use of such measurements in forced flows to describe the velocity in the universal region near the wall. It is hoped, however, that, taken overall, the present analysis relies less on empirical assumptions than most other available analyses of turbulent free convective flows.

Basic Assumptions:

In analysing the problem under consideration, it will be assumed that:-

(i) The flow is two-dimensional.

(ii) The properties of the fluid are constant except for the density variation with temperature which causes the bouyancy forces and, hence, the flow.

(iii) The velocity and temperature change are restricted to a definite boundary layer whose thickness is small in comparison with the length of the plate. It will be assumed that the boundary layer thickness is the same for velocity and temperature.

(iv) The surface temperature of the plate, over which the flow occurs, is constant.

The Velocity Profile

In describing the velocity and temperature profiles, it will be assumed that the boundary layer can be divided into three distinct regions as shown in fig. 1. The discontinuities in the profiles are assumed to have a negligible effect on the heat transfer parameters due to the method of solution adopted.

The three regions considered and the corr8sponding velocity profiles are:-

(i) The Inner Region - in which the flow is governed solely by the conditions at the wall and in which, therefore, the profile is of the same form as that which exists near the wall in forced flow. Restricting consideration to flow over a smooth wall , the velocity in this inner region is, therefore, assumed to be given by:-

$$(u/u_{\tau}) = F\ (u_{\tau} y/\nu) \qquad (1)$$

where $u_{\tau} = (\tau_w/\rho)^{0.5}$,

τ_w = local shearing stress on wall,
ρ = density
y = distance from wall to point considered
u = local velocity at point considered,
ν = μ/ρ
μ = coefficient of viscosity

In the inner region it will also be assumed that the local shearing stress, τ, at any point is constant and equal, therefore, to the value at the wall, i.e. in the inner region it will be assumed that:-

$$\tau = \tau_w \qquad (2)$$

The inner region is assumed to extend from the wall out to the point at which:-

$$(u_\tau \; y/\nu) = a \qquad (3)$$

a being an, as yet, unknown function of the distance along the plate.

(ii) The Transition Region - in which the velocity is constant and equal, say, to u_A. From equations (1) and (3), u_A is given by:-

$$(u_A/u_\tau) = f\ (a) = A \qquad (4)$$

In the real flow there will, of course, be a smooth transition from the inner to the outer profile but since the transition region is comparatively small, the constant velocity assumption should be adequate for the present purposes.

The transition region is assumed to extend from the outer edge of the inner region, i.e. from $y = a\,\nu/u_\tau$, out to the point at which:-

$$(y/\delta) = b \qquad (5)$$

Where b is a constant and δ is the local boundary layer thickness.

(iii) The Outer Region - in which, since the temperature changes and, hence, buoyancy forces are small, the flow structure is similar to that which exists in free jet flow. Therefore, with the eddy viscosity, defined as usual by:-

$$\tau = \rho \epsilon_\tau \frac{\partial u}{\partial y} \qquad (6)$$

the molecular shearing stress being neglected, it will be assumed that, in this outer region:-

$$\epsilon_\tau = C \quad u_A \delta \qquad (7)$$

Where C is a constant, an equation of this type being known to apply in free jet flow.[2]

In this region, it will also be assumed that the shearing stress distribution is approximately linear and is, therefore, if the stress at the inner edge of this region is equal to τ_T, given by:-

$$(\tau / \tau_W) = \tau_T (1 - \eta)/\tau_W (1 - b) \qquad (8)$$

where:-

$$\eta = (y/\delta) \qquad (9)$$

Combining equations (6), (7) and (8) gives:-

$$C\,(u_A/ u_\tau) \frac{d\,(u/u_\tau)}{d\eta} = \tau_T (1 - \eta) / \tau_W (1 - b) \qquad (10)$$

Integrating this equation inwards across the boundary layer gives the velocity distribution in the outer region as:-

$$(u/u_{\tau}) = -\tau_T (1-\eta)^2 / 2\tau_w C \quad A\,(1-b) \qquad (11)$$

But, at the inner edge of the outer region, where $\eta = b$, equation (4) gives, since the velocity is continuous, $(u/u_{\tau}) = A$. Application of equation (11) to this point, therefore, gives:-

$$\frac{\tau_w \quad C}{\tau_T} = -\ (1-b)\ /2A^2 \qquad (12)$$

i.e. writing, for convenience:-

$$B = (1-b)$$

equation (11) reduces to:-

$$(u/u_{\tau}) = (1-\eta)^2 \ / \ (B^2/A) \qquad (14)$$

which gives the velocity distribution in the outer region.

Temperature Distribution:

As mentioned before, the same three regions that were introduced to define the velocity profile will be used in describing the temperature variation across the boundary layer. It will also be assumed, in this connection, that, throughout the boundary layer:-

$$(q/q_w) = (\tau / \tau_w) \qquad (15)$$

where q = local rate of heat transfer per unit area at any point in the boundary layer,

q_w = local rate of heat transfer per unit area from the wall to the fluid.

With the above assumptions, the temperature distributions in the various regions are:-

(i) Inner Region - because of the assumed structure of the flow in this region, it will be assumed that, as in forced flow, the temperature distribution is given by:-

$$(T_w - T) = (q_w / \rho \; C_p \; u_\tau) \; f \; (u_\tau y/\nu) \qquad (16)$$

where T = local temperature at the point in the boundary layer considered,

T_w = local wall temperature,

C_p = specific heat of the fluid.

The function f of equation (15) is assumed to be the same as that introduced in equation (1) to describe the velocity distribution in this region. Strictly, this assumption will limit the present analysis to fluids having Prandtl numbers near unity.

It follows from equations (2) and (15), that the heat transfer rate in the inner region is constant and equal to the rate of heat transfer from the wall i.e. in the inner region:-

$$q = q_w \qquad (17)$$

(ii) The Transition Region - in which the temperature is constant and equal, say, to T_A . From equations (4) and (16), T_A is given by:-

$$(T_w - T_A) = (q_w / \rho C_p u_\tau) A \qquad (18)$$

(iii) The Outer Region - in which, since the flow is, by previous assumption, similar to that which exists in free jet flow:-

$$\epsilon_q = 2\epsilon_\tau \qquad (19)$$

Where ϵ_q is the eddy conductivity and is defined, as usual, by:-

$$q = -\rho C_p \epsilon_q \frac{\partial T}{\partial y} \qquad (20)$$

Combining equations (19), (20), (15), (8) and (7), and then integrating, as was done with the velocity distribution, then gives, in this outer region:-

$$(\rho C_p u_\tau / q_w)(T - T_1) = (1 - \eta)^2 /(2B^2 /A) \qquad (21)$$

Where T_1 is the constant temperature outside the boundary layer.

Now since the temperature at the outer edge of the transition region and the inner edge of the outer region are, by assumption, equal, equations (18) and (21) give, at this point:-

$$(\rho C_p u_\tau / q_w)(T_w - T_1) = (3A/2) \qquad (22)$$

Therefore, defining, for convenience,:-

$$T_\gamma = q_w/\rho C_p u_\gamma \tag{23}$$

and noting that equation (22) gives:-

$$T_\gamma = 2\,(T_w - T_1)/3A \tag{24}$$

the temperature distribution in the three regions can be expressed in the form:-

(i) Inner Region:-

$$(T - T_1)/T_\gamma = (3A/2) - f \tag{25}$$

(ii) Transition Region:-

$$(T - T_1)/T_\gamma = A/2 = (T_A - T_1)/T_\gamma \tag{26}$$

(iii) Outer Region:-

$$(T - T_1)/T_\gamma = (1 - \eta)^2 / (2B^2/A) \tag{27}$$

<u>Boundary Layer Thickness:</u>

Since the outer region is assumed to have the characteristics of free jet flow and since a free jet boundary spreads linearly, it follows that it is consistent with previous assumptions to assume that:-

$$(\delta - b\delta) = \text{constant}.x \tag{28}$$

Where x is the distance measured in the direction of flow from the hypothetical origin at which the turbulent boundary layer has zero thickness.

But b is, by assumption, a conxtant so that equation (28) gives:-

$$\delta = Kx \tag{29}$$

K being a constant.

<u>Heat Transfer Rate:</u>

With the local Nusselt number N_x, defined as usual as:-

$$N_x = q_w \, x/k \, (T_w - T_1) \tag{30}$$

Where k is the coefficient of conductivity, it is seen, from equation (22), to be given by

$$N_x = (2/3A) \, (\rho u_T \, x/\mu) \, P_r \tag{31}$$

Where P_r is the Prandtl number of the fluid which is equal to $(\mu C_p / k)$.

<u>Analysis:</u>

The momentum and energy integral equations will be used as the basis of the present analysis. These equations can, as a consequence of the general assumptions made above, be written as:-

$$\frac{d}{dx}\left[\rho\int_0^{\delta} u^2 dy\right] = \beta g \rho \int_0^{\delta} (T - T_1)\ dy - \tau_w \qquad (32)$$

and:-

$$\frac{d}{dx}\left[\rho\, C_p \int_0^{\delta} u\,(T - T_1)\, dy\right] = q_w \qquad (33)$$

respectively, β being the coefficient of cubical expansion of the fluid.

Using the forms for the velocity and temperature profiles, introduced above, these equations can be written as:-

$$\frac{d}{dx}\left(M\, u_\tau \nu + N\, u_\tau^2 \delta\right) = (P\,\nu / u_\tau + Q\delta)\,(2/3A)\beta\, g\,(T_w - T_1) - u_\tau^2 \qquad (34)$$

and:-

$$\frac{d}{dx}\left(S\,\nu / A + R\, u_\tau\, \delta / A\right) = (u_\tau / A) \qquad (35)$$

Note having been taken of equations (23) and (24). The coefficients M, N, P. Q, R and S are functions of a and the assumed value of b (and, hence, of A and B) alone, and are defined as follows:-

$$M = \int_0^{a} f^2\, d\,(u_\tau\, y/\nu) - (A^2 a) \qquad (36)$$

$$N = \int_b^{1} \left[(1-\eta)^2 / (B^2/A)\right]^2 d\eta + (A^2 b) \qquad (37)$$

$$P = (Aa) - \int_0^{a} f\ d\,(u_\tau\, y/\nu) \qquad (38)$$

$$Q = \int_b^1 \left[(1-\eta)^2 \,/\, (2B^2/A) \right] d\eta + (Ab/2) \tag{39}$$

$$R = 0.5 \int_b^1 \left[(1-\eta)^2 \,/\, (B^2/A) \right]^2 d\eta + (A^2 b/2) \tag{40}$$

$$S = \int_0^a \left[(3Af/2) - f^2 \right] d\,(u_\tau y/\nu) - (A^2 a/2) \tag{41}$$

For any assumed values of b and K, the above set of equations constitute a solution for the variation of u_τ and A with x. Once these two are known, the values of any of the other required parameters can be determined. The values of b and K finally used will, of course, be those which lead to the best agreement of the end results with experiment.

In the present note, a general dolution of the above equations will not be given. Instead, an approximate solution, which should, however, agree very closely with the exact solution under most circumstances, will be presented.

Approximate Solution:

Although, as shown above, the inner region exerts a governing influence on velocity and temperature in the other regions and, hence, on the integrals in equations (32) and (33), its actual direct contribution to these integrals is very small and can to a good degree of approximation, be neglected. It will be assumed, therefore, that the coefficients M, P and S occurring in equations (34) and (35) can be neglected.

Then, noting that the remaining coefficients can be written as:-

$$N = U A^2 \tag{42}$$

$$Q = V\,A \tag{43}$$

$$R = U\,A^2/2 \tag{44}$$

Where:-

$$U = \int_b^1 \left[(1-\eta)^4/B^4 \right] d\eta + b \tag{45}$$

and:-

$$V = \int_b^1 \left[(1-\eta)^2/2B^2 \right] d\eta + (b/2), \tag{46}$$

Using equation (29) and introducing the following dimensionless variables:-

$$X = (x/l) \tag{47}$$

and:-

$$Z = (\rho\, u_\tau\, l/\mu), \tag{48}$$

Where l can, in general, be any convenient reference length and will, in the present case, be taken as the length of the plate measured from the hypothetical origin used to define x, equations (34) and (35) reduce to:-

$$\frac{dA}{dX} + (A/Z)\frac{dZ}{dX} = (E/AZ^2) - (1 + FA^2)/2FAX \tag{49}$$

and

$$\frac{dA}{dX} + (A/Z)\frac{dZ}{dX} = (2 - FA^2)/FAX \qquad (50)$$

Where:-

$$E = G_l \ V/3U \qquad (51)$$

$$F = K\ U \qquad (52)$$

and G_l is the Grashof number based on the plate length l i.e.:-

$$G_l = \beta\, g\, \rho^2\, l^3\, (T_w - T_l)\, /\mu^2 \qquad (53)$$

Solving for Z between equations (49) and (50) gives:-

$$Z = \left[2\ FEX/(5 - FA^2)\right]^{0.5} \qquad (54)$$

which can be substituted back to give, on integration of the resulting equation:-

$$\int_{A_i}^{A} \left[10\ FA/(5 - FA^2)\ (4 - 3\ FA^2)\right] dA = \log_e\ (X/X_i) \qquad (55)$$

Where A_i and X_i are the values of A and X at some, as yet, undefined, initial point.

Equations (54) and (55) constitute a solution for the variation of A and Z with X for any chosen values of b and K. The, normally used, heat transfer parameters can then be obtained by noting that equation (31) gives the local Nusselt number as:-

$$N_x = (2 P_r Z X/3A) \tag{56}$$

While the local Grashof number, G_x, is, by definition, given by:-

$$G_x = X^3 G_l \tag{57}$$

The average Nusselt number, N_l, for the whole plate, which is defined as:-

$$N_l = \int_{x_t}^{l} \left[q_w x/k \ (T_w - T_l) \right] dx \tag{58}$$

where x_t is the x-co-ordinate defining the true leading edge of the plate, is, similarly, given by:-

$$N_l - N_{li} = \int_{X_i}^{X} (N_x / X)\, d X \tag{59}$$

Where N_{li} is the average Nusselt number for flow up to the initial point, previously introduced.

Initial Values:

The initial values used in the above solution for turbulent free convective flow will be obtained by joining this solution to the known solution for laminar free convective flow. This is rendered difficult by the fact that a long transition length usually exists in free convective flows. Since slight errors in the initial values of the variables cannot, however, be expected to influence the results away from the transition region very greatly, it will be assumed that a sharp transition occurs at some intermediate point within the transition region and that, across this transition, the energy content of the boundary layer remains constant. Now a useful measure of the energy content of the boundary layer is the integral:-

$$E = \int_0^{\delta} u\ (T - T_1)\ dy \qquad (60)$$

and it will, therefore, be assumed that this quantity, E, remains constant during transition.

Squires[3.] approximate solution for laminar flow gives:-

$$E = J \mu\ (T_w - T_1)/\rho \qquad (61)$$

Where the coefficient J depends on the Prandtl number of the fluid and the assumed local Grashof number at transition.

The above analysis of turbulent flow gives, when, as before, the contribution of the inner region to the integral is neglected,:-

$$E = (A\ Z\ XF/3) \left[\mu\ (T_w - T_1)/\rho \right] \qquad (62)$$

Equating these two values for E at the transition point gives:-

$$A_i\ \ Z_i\ X_i\ = 3J/F \qquad (63)$$

If it is assumed, further, that the hypothetical origin, at which the turbulent boundary layer has zero thickness, which is used to define x and, hence, χ, coincides with the actual leading edge of the plate, then:-

$$\chi_i = (G_{xT}/G_l)^{1/3} \tag{64}$$

Where G_{xT} is the local Grashof number, based on the distance from the leading edge of the plate, at which the sharp transition is assumed to occur.

With χ_i obtained in this manner, equation (63) can be written as:-

$$A_i Z_i = (3J/F\chi_i) = D \tag{65}$$

Combining this equation with equation (54) then gives:-

$$Z_i = \left[(2\ FEX_i + F\ D^2)/5 \right]^{0.5} \tag{66}$$

and, therefore,:-

$$A_i = \left[5\ D^2 /(2\ FEX_i + F\ D^2) \right]^{0.5} \tag{67}$$

With the initial values determined in this way and using the value of N_{li} given by Squire's solution, the variation of N_x with G_x and, hence, the variation of N_l with G_l can be determined for any fluid for any assumed values of b and κ.

Numerical Results:

The variation of N_l with G_l for fluids with a Prandtl number of 0.72 has been calculated for various values of b and K on the assumption that $GxT = 10^9$. However, the lack of experimental results, particularly at high Grashof numbers, available to the author, rendered impossible the accurate determination of the values giving the best overall fit of the calculated results with experimental results. However, noting that Griffiths and Davies[4.] velocity profile measurements suggest that b is roughly equal to 0.25, the following values of b and K, for fluids with Prandtl numbers of 0.72, are tentatively suggested:-

$b = 0.25, K = 0.05$

The calculated results, obtained with these values, are compared in fig. 2 with Saunders[5.] experimental measurements in air. Reasonable agreement is seen to be obtained.

Conclusion:

A method of analysing turbulent free convective flow over a vertical flat plate has been presented. The application of this method relies on a knowledge of the values of two coefficients which depend on the type of fluid and which must be determined experimentally. Tentative values for air have been given and the results obtained with these values shown to agree reasonably well with experimental results.

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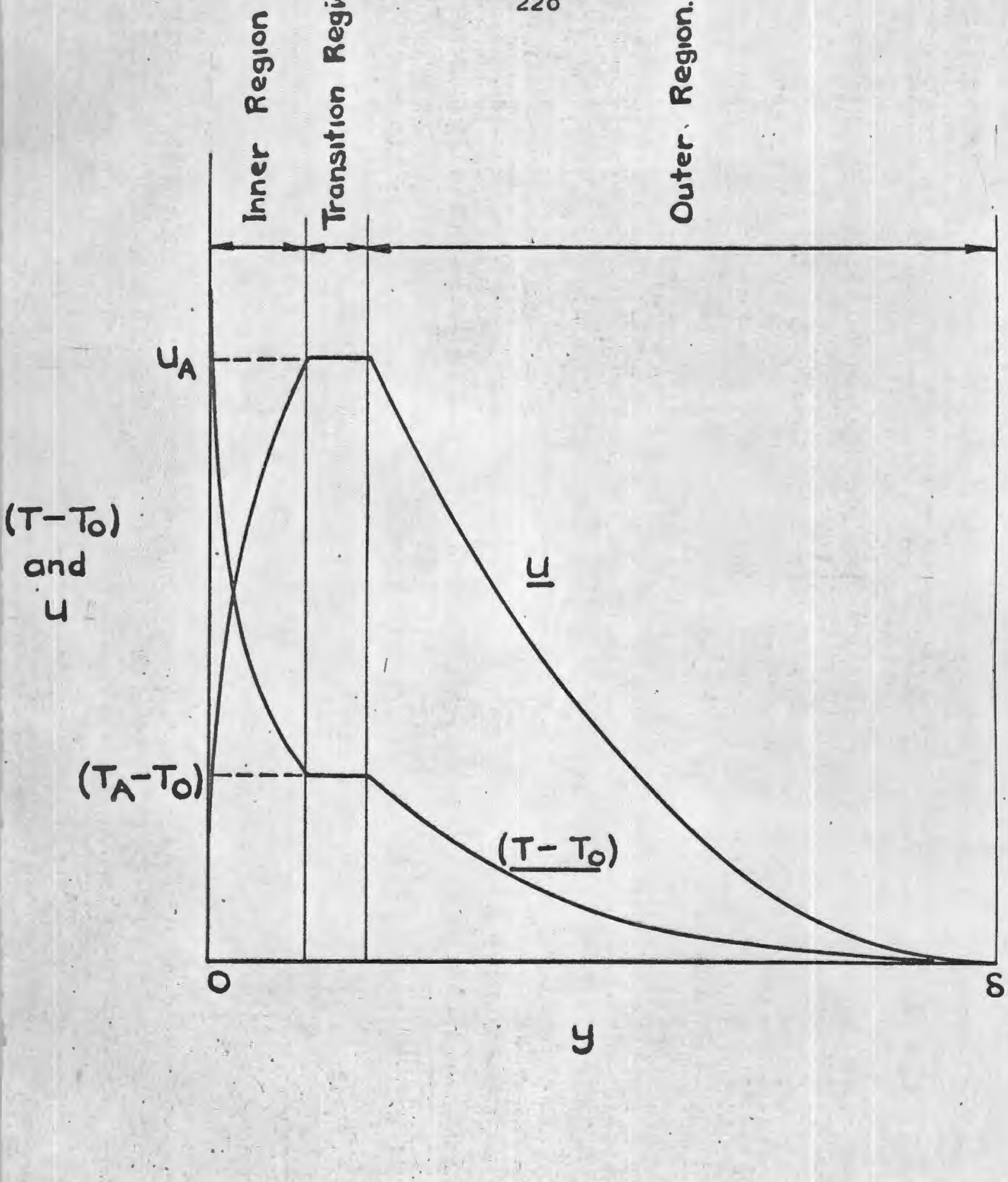

Fig. 1 - Assumed Forms for Boundary Layer Profiles.

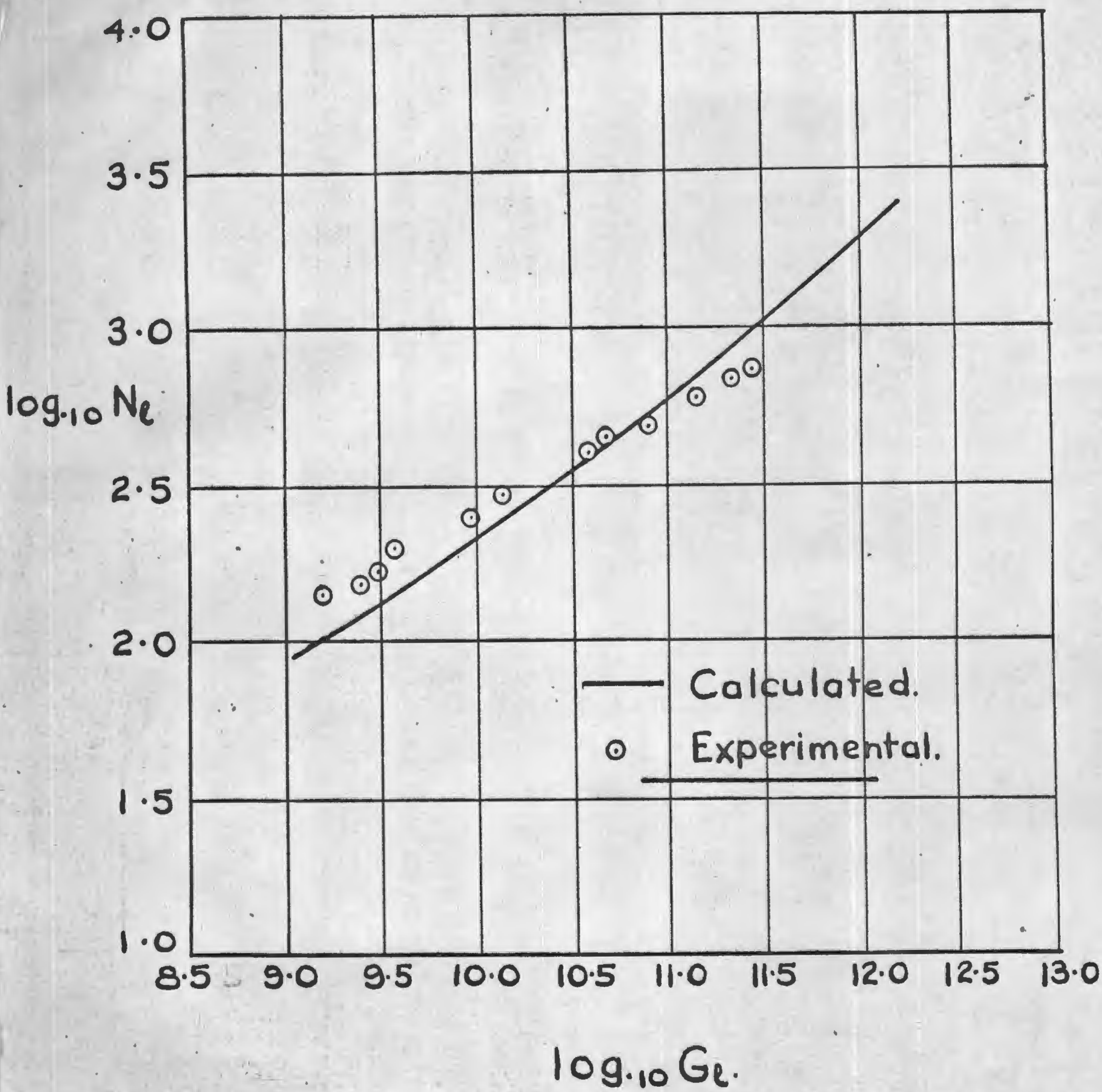

Fig. 2 - Comparison of Predicted Heat Transfer results for Air with Experimental Results.