NEW IDENTITIES FOR LEGENDRE ASSOCIATED FUNCTIONS OF INTEGRAL ORDER AND DEGREE

by

S.R. SCHACH

****

A thesis submitted in fulfilment of the requirements of the Degree of Doctor of Philosophy in the University of Cape Town

August, 1973

****

To my parents

ABSTRACT

In the solution of the boundary value problems of mathematical physics in a separable 3-dimensional coordinate system, the shape of the boundary of the space may be such that the Green's function of the second order differential operator can be expanded as an infinite series of orthogonal functions. In many coordinate systems (such as the spherical, spheroidal and some cyclidal systems) these expansions are given in terms of Legendre associated functions of integral order and degree.

Starting with Dougall's identities for Legendre associated functions of non-integral degree, new identities for infinite series of Legendre associated functions of integral degree are derived. Uniform convergence of each new identity is investigated in detail.

The direct applicability of these identities is demonstrated by using them to verify theorems satisfied by the Dirichlet Green's function of the infinite half-space and of the interior of the prolate hemispheroid.

The results and techniques are then generalized, and a sufficient condition found under which a generalized orthogonal function which satisfies Dougall's identity will also satisfy the new identity. This theorem is applied to the Legendre associated function, the generalized Legendre associated function and to the Jacobi function.

DECLARATION

This is to certify that all ideas and results presented in this thesis are due to the writer, except where due reference is given.

No portion of this thesis has been previously submitted in support of an application for any other degree or qualification in this or any other University.

S.R. Schach

August, 1973

ADDENDA ET CORRIGENDA

Page 12

Last line: for 'Dirac' read 'Kronecker'.

Page 17

Delete last four lines.

Page 18

Delete first two lines.

Page 42

Insert after equation (3.60):

$$(3.61) \quad P'^{m}_{\ell}(0)\Sigma^{\infty}_{n=m}\gamma_{n,m}P^{m}_{n}(x)P^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x)\epsilon(x) + \tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x)$$

$$-1 < x < 1$$

NOTATION

R denotes the set of Real numbers

I denotes the set of Integers

N denotes the set of Non-Negative integers

The lower case Roman letters ℓ, m, n and q all denote elements of the set N, while $s \in I$

The Greek letters λ and μ denote real numbers which may take on integral values, while ν is real and specifically non-integral.

A prime ' is used exclusively to denote differentiation. In particular

$$P'^m_n(x_o) \quad \text{means} \quad \left(\frac{d}{dx} P^m_n(x)\right)_{x=x_o}$$

and similarly for $Q'^m_n(x_o)$

Results taken from (Erdélyi, 1953) are indicated by the letter E followed by the appropriate equation number, e.g. (E 2.9(14)) refers to equation (14) of section 2.9 of (Erdélyi, 1953).

C O N T E N T S

CHAPTER 1

INTRODUCTION

Green's functions play a central rôle in the three dimensional boundary value problems of mathematical physics and applied mathematics. For every differential equation and every space with any given shape, there exists a Green's function, and if the Green's function can be found the corresponding boundary value problem is virtually solved.

Unfortunately, in most cases finding the Green's function is no simpler than solving the original problem. One exception is the case in which the coordinate system is separable. (A coordinate system (u^1,u^2,u^3) is said to be "separable" (Moon and Spencer, 1952) with respect to a differential operator $L(u^1,u^2,u^3)$ if the solution to the equation

$$L(u^1,u^2,u^3)\psi(u^1,u^2,u^3) = 0$$

can be expressed in the form

(1.1) $$\psi(u^1,u^2,u^3) = \psi_1(u^1)\psi_2(u^2)\psi_3(u^3)/R(u^1,u^2,u^3)$$

where $R(u^1,u^2,u^3)$ is independent of the separation constants.) In addition, we require that the boundary of the space be such that the Green's function may be expressed as an infinite series of orthogonal functions. But there is a severe limitation on the number of shapes for which these expansions exist.

Consider, for example, the spherical coordinate system (r,θ,ψ). The Green's function for Laplace's equation $\nabla^2\psi = 0$ can be expressed as an infinite series of terms of the type $r^n P_n^m(\cos\theta)\frac{\sin}{\cos} m\phi$ for the spaces bounded by the following shapes

(i) the sphere $r = a$

(ii) the plane $\theta = \pi/2$

(iii) the planes $\phi = 0, \pi/2, \pi, 3\pi/2$

Since Laplace's equation is separable in only eleven co-ordinate systems plus the six families of cyclidal coordinates (Morse and Feshbach, 1952, pages 519-523) the number of possible shapes to which the technique is applicable is seen to be strictly limited. Permissible shapes include the ellipsoid (with the oblate and prolate spheroids and the sphere as special cases); the infinite plane, cylinder (elliptic, parabolic and circular), cone, paraboloid and hyperboloid; the toroid and the spindle.

As the complexity of the differential equation increases so the number of coordinate systems in which it is separable decreases. For example, unlike Laplace's equation the Helmholtz equation $(\nabla^2+k^2)\psi = 0$ is not separable in any cyclidal coordinate system (of which the toroidal and bi-spherical systems are examples of degenerate forms). The Schroedinger equation

$$(\nabla^2+E)\psi(\underline{u}) = V(\underline{u})\psi(\underline{u}), \quad E \text{ constant}$$

is separable in those eleven coordinate systems in which the Helmholtz equation separates, but only under stringent restrictions as to the allowable form of the potential $V(\underline{u})$.

The solutions of Laplace's equation, the simplest second order differential field equation of mathematical physics (and the one with the greatest number of coordinate systems in which it is separable) play the rôle of a base on which to build Green's functions for other, more complicated equations. For instance, in the Quantum Theory of Scattering the Born expansion (Messiah, 1961, Chapter XIX) enables one to construct a perturbation expansion for the scattering Green's function $G_k(\underline{r}_1,\underline{r}_2)$ from the free field Green's function $G_0(\underline{r}_1,\underline{r}_2)$.

Similarly Kleinman (1965) has developed a technique for solving iteratively the Dirichlet Problem for the Helmholtz equation $(\nabla^2+k^2)\psi = 0$ (valid for sufficiently small values of the wave number k) which employs the Laplace equation Green's function for the relevant space. This has been applied (Ar, 1967) to low frequency accoustical scattering of a plane wave from a soft spindle, for which shape there exists (in bispherical coordinates) a Green's function expansion for Laplace's equation, but not for the Helmholtz equation; we recall that the latter equation is not separable in the bispherical system. This technique has been extended to the Neumann Problem for the Helmholtz equation (Ar and Kleinman, 1966).

Let us now consider the specific orthogonal functions which arise in such separable systems. In the solution of Laplace's equation in spherical, prolate spheroidal and oblate spheroidal coordinates, as well as in the toroidal, bispherical and other cyclidal systems, at least one of the functions $\psi_i(u^i)$ in (1.1) is a Legendre associated function $P_n^m(x_L)$, where x_L is a function of u^i only. In the case of spherical coordinates $x_L = \cos\theta$, and the general solution to Laplace's equation is of the form

(1.2) $\Sigma_{n,m=o}^{\infty} A_{mn} r^n P_n^m(\cos\theta)\cos m\psi$ $\qquad A_{mn}$ constant

The identities which we develop in this thesis are of applicability to bodies bounded by two surfaces, one of which can be expressed in the form $x_L = 0$ (where x_L is the argument of the Legendre associated function), and the second is given in terms of the other two separable co-ordinates only. The simplest example of this is the half-space bounded by the surface $z = 0$ (or in spherical co-ordinates, $\theta = \pi/2$). Here $x_L = \cos\theta = 0$ as required, and our identities can be applied; this example is treated in detail in Chapter 4.

Another example which is dealt with is the interior of the prolate hemispheroid, where the solution to Laplace's equation is given in prolate spheroidal coordinates (ξ,η,ϕ) in the form

(1.3) $\Sigma_{n,m=o}^{\infty} B_{nm} P_n^m(\eta) Q_n^m(\xi)\cos m\phi$ $\qquad B_{nm}$ constant

Here the plane surface of the hemispheroid is given by $x_L = \eta = 0$; the curved surface is defined by $\xi = \alpha > 1$.

When we make use of Green's functions in solving problems of the classes outlined above we encounter integrals of the type

(1.4) $\int_S G(\underline{r}_1(\underline{u}_1),\underline{r}_2(\underline{u}_2))f(\underline{u}_1)dS_1$

and/or

$\int_V G(\underline{r}_1(\underline{u}_1),\underline{r}_2(\underline{u}_2))f(\underline{u}_1)dV_1$

where $\underline{u} = (u^1,u^2,u^3)$.

The Green's function is expressed as an infinite series similar in form to (1.2) or (1.3) (but in the appropriate coordinate system), and the function $f(\underline{u}_1)$ will in general also be expanded as a similar series of orthogonal functions to make best use of the separability of the coordinates in order to perform the indicated integration.

Having established uniform convergence (which is necessary before interchanging the order of infinite summation and integration) (1.4) will give a doubly infinite series of terms if the range of integration is not the one over which the relevant functions are orthogonal. For instance, the Legendre associated functions $P_n^m(\cos\theta)$ are orthogonal over the surface of the sphere $0 \leq \theta \leq \pi$, but not over a hemisphere (or any other portion of a sphere for that matter). Since the majority of the techniques for evaluating $G_k(\underline{r}_1,\underline{r}_2)$ in terms of $G_0(\underline{r}_1,\underline{r}_2)$ as an expansion in k (low frequency expansion) are iterative, to obtain the second term of the expansion the resultant doubly infinite series will be multiplied by some new function (itself expanded as an infinite series) and integrated thereby giving a trebly infinite series, and so on. Any attempt to sum these series numerically will be unsatisfactory because the expansions converge very slowly.

If the coordinate system and bounding surface satisfy our above criteria then the identities established in this thesis are of great value for two reasons.

(a) They sum an infinite series of Legendre associated functions of the type generated by (1.4), expressing the result as a single term. The form of the identities is such that they may be applied directly to the infinite series without any additional analysis being required.

(b) The single term to which the series sums is again a Legendre associated function (or a constant multiple of one). This means that after applying our identities to the doubly infinite series the resulting singly infinite series inserted into the integral of type (1.4) for the second iteration leads to an integral of a very similar type to the first, and in general it is possible to construct a recurrence relation for finding all subsequent terms from the first term.

This technique is especially useful in the field of low frequency scattering of scalar plane waves (accoustical radar). Results published in this field show that even for relatively simple shapes bounded by a single surface such as an ellipsoid (Sleeman, 1967) by the time only the second term is reached the integrals are of frightening complexity. However, a recurrence relation technique such as that of (Asvestas and Kleinman, 1969) for the similarly elementary spheroid and disk gives an explicit algebraic formula for each term of the low frequency expansion in terms of previously calculated ones.

Low frequency expansions of solutions of the radar scattering equation and similar iterative schemes involving Green's functions for Laplace's equation can now be applied to a larger number of shapes other than the "basic" shapes already considered, provided that the coordinate system is separable, that one of the coordinates u^i appears in the Green's function in the form $P_n^m(x_L)$, $x_L = x_L(u^i)$ only, and that one of the two surfaces of the body is given by $x_L = 0$, the other being defined independently of x_L.

The outline of this thesis is as follows:

In Chapter 2 we quote fundamental results and theorems from the theory of Legendre associated functions. For uniformity the majority of these are taken from (Erdélyi, 1953); they are stated without proof. Only Dougall's identities are derived, because we wish to show that they have a wider range of validity than is given by Erdélyi.

Dougall's identities are expansions for Legendre associated functions $P_\nu^m(x)$ of non-integral degree ν and are hence of limited use, because the majority of series expansions of Green's functions like (1.2) or (1.3) are for Legendre associated functions $P_n^m(x)$ of integral degree n.

Starting from Dougall's identities ("D-type identities"), in Chapter 3 we establish a set of new identities for Legendre associated functions of integral degree ("S-type identities"). Uniform convergence of each S-type series is considered in detail, because they will be used in integrals like (1.4) and interchange of summation and integration must be justified before it may be carried out.

In Chapter 4 we apply our new identities to the volumes already mentioned above (i.e. the half-space and the interior of the prolate hemispheroid), as well as to a purely technical problem. This last is a series of Legendre associated functions which the author encountered and attempted to sum, thereby leading to his discovery of the identities derived in Chapter 3.

Now we extend the theorems, methods and results of Chapter 3 leading to new identities for generalized orthogonal polynomials. The first part of Chapter 5 is a summary of basic results from the theory of orthogonal polynomials; using these results we then find a sufficient condition under which a generalized orthogonal polynomial which satisfies a D-type identity (i.e. for a generalized polynomial of non-integral degree) will satisfy an S-type identity (with generalized polynomials of integral degree). We then consider two applications of this theorem; firstly to Legendre associated functions to show that our theorem of Chapter 5 gives the same results as were obtained in Chapter 3, and secondly to generalized Legendre associated functions of (Kuipers and Meulenbeld, 1957) and hence to Jacobi polynomials.

CHAPTER 2

LEGENDRE ASSOCIATED FUNCTIONS

In this chapter we quote standard formulae from the theory of Legendre associated functions $P_n^m(x)$. Only where we disagree with the range of validity of the stated formulae are proofs given. The majority of our results are taken from (Erdélyi, 1953); these are indicated by the letter E followed by the appropriate equation number, e.g. (E 2.7(23)).

Differential Equation

Legendre's differential equation of degree λ and order μ is (E 3.2(1))

$$(2.1) \qquad (1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + (\lambda(\lambda+1)-\mu^2(1-z^2)^{-1})\, w = 0,$$

z,λ,μ unrestricted

The hypergeometric function $u = F(a,b;\, c;\, z)$ satisfies Gauss' differential equation (E 2.1.1(1))

$$(2.2) \qquad z(1-z)\frac{d^2u}{dz^2} + (c-(a+b+1)z)\frac{du}{dz} - abu = 0.$$

F can be expanded in the series (E 2.1.1(2))

$$(2.3) \qquad F(a,b;\, c;\, z) = \Sigma_{q=o}^{\infty} \frac{\Gamma(q+a)\Gamma(q+b)\Gamma(c)z^q}{\Gamma(a)\Gamma(b)\Gamma(q+c)q!} \qquad |z| < 1$$

Substituting $w = (z^2-1)^{\frac{1}{2}\mu}y$, $\zeta = \frac{1}{2}-\frac{1}{2}z$ in (2.1) we obtain Gauss' differential equation (2.2) with

$$a = \mu-\lambda, \quad b = \mu+\lambda+1, \quad c = \mu+1$$

Hence (E 3.2(3))

$$(2.4) \qquad w = P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} F(-\lambda,\lambda+1;\, 1-\mu;\, \tfrac{1}{2}-\tfrac{1}{2}z)$$

$|1-z| < 2$

where we have used the fact (E 2.3(1)) that

$$F(a,b;\ c;\ z) = z^{1-c}F(a-c+1,b-c+1;\ 2-c;\ z)$$

Setting $w = (z^2-1)^{\frac{1}{2}\mu}y$, $\zeta = z^2$ in (2.1) leads to a hypergeometric-type equation with

$$a = \tfrac{1}{2}(\lambda+\mu+1),\ b = \tfrac{1}{2}(\mu-\lambda),\ c = \tfrac{1}{2}$$

Now from Kummer's 24 solutions of the hypergeometric equation (E 2.9(1), (9))

$$F(a,b;\ c;\ z) = (-z)^{-a}F(a,a+1-c;\ a+1-b;\ z^{-1})$$

whence

$$F(\tfrac{1}{2}(\lambda+\mu+1),\ \tfrac{1}{2}(\mu-\lambda);\ \tfrac{1}{2};\ z^2)$$

$$= (-z^2)^{-\frac{1}{2}(\lambda+\mu+1)}F(\tfrac{1}{2}(\lambda+\mu+1),\tfrac{1}{2}(\lambda+\mu+2);\ \lambda+3/2;\ z^{-2})$$

and (2.1) has as second solution (E 3.2(5))

$$(2.5)\qquad w = Q^{\mu}_{\lambda}(z) = e^{i\mu\pi}2^{-\lambda-1}\pi^{\frac{1}{2}}\,\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+3/2)}\,z^{-\lambda-\mu-1}(z^2-1)^{\frac{1}{2}\mu}$$

$$\cdot F(\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu+1,\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu+1;\ \lambda+3/2;\ z^{-2})\qquad |z| > 1$$

We define $P^{\mu}_{\lambda}(z)$ and $Q^{\mu}_{\lambda}(z)$ of (2.4) and (2.5) to be the Legendre associated functions of the first and second kinds respectively.

Wronskian

From (2.1) we can show that

$$(2.6)\qquad W\{P^{\mu}_{\lambda}(z),Q^{\mu}_{\lambda}(z)\} = P^{\mu}_{\lambda}(z)\,\frac{d}{dz}\,Q^{\mu}_{\lambda}(z)-Q^{\mu}_{\lambda}(z)\,\frac{d}{dz}\,P^{\mu}_{\lambda}(z)$$

must be of the form $c/(1-z^2)$. From formulae for $P^{\mu}_{\lambda}(z),Q^{\mu}_{\lambda}(z)$ and their derivatives at $z = 0$ (see later) we can evaluate the constant c and deduce that (E 3.2(13))

$$(2.7)\qquad W\{P^{\mu}_{\lambda}(z),Q^{\mu}_{\lambda}(z)\} = \frac{e^{i\mu\pi}2^{2\mu}\Gamma(1+\frac{1}{2}\lambda+\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\lambda+\frac{1}{2}\mu)}{(1-z^2)\ \Gamma(1+\frac{1}{2}\lambda-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\lambda-\frac{1}{2}\mu)}$$

Now (E 1.3(15))

$$(2.8) \qquad \Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

Define

$$(2.9) \qquad \gamma_{\lambda,\mu} = \frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda+\mu+1)}$$

and hence (2.7) gives

$$(2.10) \qquad W\{P_\lambda^\mu(z),Q_\lambda^\mu(z)\} = \frac{e^{i\mu\pi}}{1-z^2}\,\gamma_{\lambda,\mu}^{-1}$$

<u>Legendre associated functions on the cut</u>

Setting $z = \cos\theta$, $\theta \in R$ in (2.4) defines a Legendre associated function on the real interval $(-1,1)$, the so-called "cut". However, we see from (2.4) that

$$P_\lambda^\mu(x-i0) \neq P_\lambda^\mu(x+i0)$$

unless μ is an even integer. For $x \in (-1,1)$ we therefore define (E 3.4(1) and (2))

$$(2.11) \qquad P_\lambda^\mu(x) = \tfrac{1}{2}(e^{i\frac{1}{2}\mu\pi}P_\lambda^\mu(x+i0) + e^{-i\frac{1}{2}\mu\pi}P_\lambda^\mu(x-i0))$$

and

$$(2.12) \qquad Q_\lambda^\mu(x) = \tfrac{1}{2}e^{-i\mu\pi}(e^{-i\frac{1}{2}\mu\pi}Q_\lambda^\mu(x+i0) + e^{i\frac{1}{2}\mu\pi}Q_\lambda^\mu(x-i0))$$

$$-1 < x < 1$$

whence (E 3.4(6))

$$(2.13) \qquad P_\lambda^\mu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} F(-\lambda,\lambda+1;\ 1-\mu;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

$$-1 < x < 1$$

From this result and the properties of the hypergeometric function we may deduce the following identities (E 3.4(14) and (17))

$$(2.14) \qquad P_\lambda^\mu(-x) = P_\lambda^\mu(x)\cos((\lambda+\mu)\pi) - (2/\pi)Q_\lambda^\mu(x)\sin((\lambda+\mu)\pi)$$

$$0 < x < 1$$

$$(2.15) \qquad P_\lambda^{-\mu}(x) = \gamma_{\lambda,\mu}(P_\lambda^\mu(x)\cos(\mu\pi)-(2/\pi)\sin(\mu\pi)Q_\lambda^\mu(x))$$

$$-1 < x < 1$$

where we have defined $\gamma_{\lambda,\mu}$ in (2.9).

We also have the explicit expressions (E 3.4(20) and (22))

(2.16) $$P_\lambda^\mu(0) = 2^\mu \pi^{-\frac{1}{2}} \cos(\tfrac{1}{2}(\lambda+\mu)\pi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\lambda-\tfrac{1}{2}\mu)$$

(2.17) $$\left(\frac{dP_\lambda^\mu(x)}{dx}\right)_{x=0} = 2^{\mu+1}\pi^{-\frac{1}{2}} \sin(\tfrac{1}{2}\lambda+\mu)\pi)\Gamma(1+\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda-\tfrac{1}{2}\mu)$$

Similar results can be found for $Q_\lambda^\mu(0)$ and $\left(\frac{dQ_\lambda^\mu(x)}{dx}\right)_{x=0}$

For x near 1 (E 3.9.2(8)) the leading term of $P_\lambda^{-\mu}(x)$, $\mu \geq 0$ is

(2.18) $$2^{-\frac{1}{2}\mu}(1-x)^{\frac{1}{2}\mu}/\Gamma(1+\mu)$$

Integral representation

We can show (E 3.7(27)) that

(2.19) $$P_\lambda^\mu(\cos\theta) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \frac{(\sin\theta)^\mu}{\Gamma(\frac{1}{2}-\mu)} \int_0^\theta d\psi(\cos\psi-\cos\theta)^{-\mu-\frac{1}{2}} \cdot \cos((\lambda+\tfrac{1}{2})\psi) \qquad 0 < \theta < \pi, \quad \mathrm{Re}\,\mu < \tfrac{1}{2}$$

Integral of product of two Legendre associated functions

Let $M_\nu^m(z)$, $M_\lambda^m(z)$ be any two solutions of Legendre's differential equation (2.1). Then (E 3.12(1))

(2.20) $$(\nu-\lambda)(\nu+\lambda+1)\int_a^b dz M_\nu^m(z)M_\lambda^m(z) = \Big[z(\nu-\lambda)M_\nu^m(z)M_\lambda^m(z)+(\lambda+m)M_\nu^m(z)M_{\lambda-1}^m(z) - (\nu+m)M_{\nu+1}^m(z)M_\lambda^m(z)\Big]_a^b$$

and (E 3.12(19) and (21)).

(2.21) $$\int_{-1}^1 dx P_n^m(x)P_\ell^m(x) = \frac{2}{2n+1}\gamma_{n,\ell}^{-1}\delta_{n,\ell}$$

(where $\delta_{n,\ell}$ is the Dirac delta function.)

Results for μ a non-negative integer

If μ is a non-negative integer write

$$\mu = m, \qquad m = 0,1,2,\ldots$$

Starting from (2.4) we can show that (E 3.6.1(6))

$$(2.22) \qquad P_\lambda^m(x) = (-1)^m(1-x^2)^{\frac{1}{2}m} \frac{d^m P_\lambda(x)}{dx^m} \qquad -1 < x < 1$$

From (E 3.9.2(8) and (9)) we see that for $m > 0$ and x near 1

$$(2.23) \qquad P_\lambda^m(x) \to 0 \quad \text{as} \quad x \to 1$$

$$(2.24) \qquad P_\lambda^{-m}(x) \to 0 \quad \text{as} \quad x \to 1$$

and for $m > 0$ and x near -1 (E 3.9.2(13) and (14))

$$(2.25) \qquad P_\lambda^m(x) \to \infty \quad \text{as} \quad x \to -1$$

$$(2.26) \qquad P_\lambda^{-m}(x) \to \infty \quad \text{as} \quad x \to -1$$

For $m = 0$ (Hobson, 1931, equation II (15)) or from (2.4))

$$(2.27) \qquad P_n(1) = 1$$
$$P_n(-1) = (-1)^n$$

If λ is a non-negative integer n, $P_n(x)$ is a polynomial of degree n in x, and hence from (2.22)

$$(2.28) \qquad P_n^m(x) = 0 \qquad \text{for} \quad m > n$$

A bound on $P_n^m(\cos\theta)$ is given by (Gradshteyn and Ryzhik, 1965, equation 8.724(3))

$$(2.29) \qquad |P_n^m(\cos\theta)| < \frac{2 \quad \Gamma(n+m+1)}{\sqrt{n\pi}\ \Gamma(n+1)\sin^{m+\frac{1}{2}}\theta} \qquad n \geq 1,\ m \geq 0,\ m \leq n,$$

and by (Hobson, 1931, page 303)

$$(2.30) \qquad n^{-m}P_n^m(\cos\theta) = (\tfrac{1}{2}n\pi\sin\theta)^{-\frac{1}{2}}\cos((n+\tfrac{1}{2})\theta-\pi/4+\tfrac{1}{2}m\pi)$$

$$+O(n^{-3/2}),\quad 0 < \varepsilon < \theta < \pi - \varepsilon,\quad n > 1,\quad n >> m$$

where $O(n^{-3/2})$ depends on ε.

Recurrence relations

Of the many well-known recurrence relations we will be using only (E 3.8(19) and (12))

$$(2.31)\qquad (1-x^2)\frac{dP_n^m(x)}{dx} = (n+1)xP_n^m(x) - (n-m+1)P_{n+1}^m(x)$$

$$= -nxP_n^m(x) + (n+m)P_{n-1}^m(x)$$

and

$$(2.32)\qquad (n-m+1)P_{n+1}^m(x) = (2n+1)xP_n^m(x) - (n+m)P_{n-1}^m(x)$$

Addition theorem

Define

$$(2.33)\qquad \varepsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m > 0 \end{cases}$$

then (E 3.11(2))

$$(2.34)\qquad P_n(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2))$$

$$= \Sigma_{m=o}^{n}\varepsilon_m\gamma_{n,m}P_n^m(\cos\theta_1)P_n^m(\cos\theta_2)\cos(m(\phi_1-\phi_2))$$

$$0 \leq \theta_1, \theta_2 < \pi, \quad \theta_1 + \theta_2 < \pi, \; \phi_1-\phi_2 \in R$$

Laplace series for Legendre associated functions

We will employ the following theorem from the theory of Legendre associated functions, proved in (Hobson, 1931, page 344), which we will label as Theorem 2.1.

Theorem 2.1

The Laplace series

$$\Sigma_{n=o}^{\infty}\frac{2n+1}{4\pi}\int_{-\pi}^{\pi}d\phi_2\int_{o}^{\pi}d\theta_2\,\sin\theta_2 f(\theta_2,\phi_2)P_n(\cos\gamma)$$

(where $\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2))$

in which $f(\theta_2,\phi_2)$ has an absolutely convergent integral (Lebesgue) over the spherical surface, will converge at (θ_1,ϕ_1) to the value $f(\theta_1,\phi_1)$ if (θ_1,ϕ_1) is a point of continuity of the function with respect to (θ_1,ϕ_1), or to the value

$$\tfrac{1}{2}\{f_1(\theta_1,\phi_1) + f_2(\theta_1,\phi_1)\}$$

if the point (θ_1,ϕ_1) is such that there passes through it a line of discontinuity such that $f_1(\theta_1,\phi_1)$ and $f_2(\theta_1,\phi_1)$ are the limits of the function at the point taken from the two sides of the line, provided that the function $\psi(\gamma)$, which is the mean value of the function $f(\theta_1,\phi_1)$, for each fixed value of γ over the small circle for which γ has that value, has bounded variation in the whole interval $(0,\pi)$ of γ.

Dougall's Identities

Dougall's 3 identities (which first appeared in (MacRobert, 1934)) are given in reference (E 3.10(6),(8) and (9)) as

$$(2.35)\qquad P_\nu^{-\mu}(\cos\theta) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n P_n^{-\mu}(\cos\theta)\cdot(1/(\nu-n)-1/(\nu+n+1))\qquad -\pi<\theta<\pi,\quad \mu\geq 0$$

$$(2.36)\qquad P_\nu^{-\mu}(\cos\theta)P_\nu^{-\lambda}(\cos\xi) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n \cdot P_n^{-\mu}(\cos\theta)P_n^{-\lambda}(\cos\xi)(1/(\nu-n)-1/(\nu+n+1))\qquad -\pi<\theta\pm\xi<\pi,\ \mu,\lambda\geq 0$$

and

$$(2.37)\qquad P_\nu^{m}(\cos\theta)P_\nu^{-m}(\cos\xi) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n \cdot P_n^{m}(\cos\theta)P_n^{-m}(\cos\xi)(1/(\nu-n)-1/(\nu+n+1))$$

m a positive integer, $0<\theta,\xi<\pi$, $\theta+\xi<\pi$

For $\nu \in I$ equations (2.35) - (2.37) reduce to trivial identities, because

$$\lim_{\nu \to n} \frac{\sin(\nu\pi)}{\nu - n} = \pi(-1)^n \tag{2.38}$$

We will rederive Dougall's three identities in order to show that the ranges of validity of (2.35) - (2.37) as stated in (E) are overrestrictive; in addition we will prove uniform convergence over their entire ranges of validity.

Lemma 2.2

Let

$$I_\Gamma = \int_\Gamma dz \, \frac{\cos((z+\frac{1}{2})\psi)}{(z-\nu)\sin(\pi z)} \qquad \nu \in R \sim I, \quad -\pi < \psi < \pi \tag{2.39}$$

where Γ is the circle centre the origin and radius $R = N+\frac{1}{2}$, $N \in I$, $N > |\nu|$

Then

$$\lim_{N \to \infty} I_\Gamma = 0 \tag{2.40}$$

Proof

The integrand is finite on Γ, because

$$\sin(\pi z) = 0 \text{ implies } e^{i2\pi z} = 1$$
$$\text{or } z \in I$$

Thus

$$|\sin(\pi z)| > \eta > 0 \quad \text{for all } z \in \Gamma$$

Now

$$\cos((z+\tfrac{1}{2})\psi) = \tfrac{1}{2}\{e^{i(R(\cos\theta + i\sin\theta)+\frac{1}{2})\psi}$$
$$+ e^{-i(R(\cos\theta + i\sin\theta)+\frac{1}{2})\psi}\} \quad \text{where } R = N+\tfrac{1}{2}$$

$$\therefore \quad |\cos((z+\tfrac{1}{2})\psi)| \le \tfrac{1}{2}(e^{R\psi\sin\theta} + e^{-R\psi\sin\theta})$$
$$= \tfrac{1}{2}(e^{R|\psi|\sin\theta} + e^{-R|\psi|\sin\theta}) \text{ by symmetry}$$

$$\sin(\pi z) = -\tfrac{1}{2}i\{e^{i\pi(R(\cos\theta+i\sin\theta))} - e^{-i\pi(R(\cos\theta+i\sin\theta))}\}$$

$$\therefore \qquad |\sin(\pi z)| \geq \tfrac{1}{2}|e^{\pi R\sin\theta} - e^{-\pi R\sin\theta}|$$

$\therefore$ From (2.39)

$$|I_\Gamma| \leq \frac{R}{R-|\nu|}\int_0^{2\pi} d\theta \frac{e^{R|\psi|\sin\theta}+e^{-R|\psi|\sin\theta}}{|e^{\pi R\sin\theta}-e^{-\pi R\sin\theta}|}$$

$$= \frac{2R}{R-|\nu|}\int_0^{\pi} d\theta \frac{e^{R|\psi|\sin\theta}+e^{-R|\psi|\sin\theta}}{e^{\pi R\sin\theta}-e^{-\pi R\sin\theta}}$$

$$= \frac{2R}{R-|\nu|}\int_0^{\pi} d\theta e^{-R(\pi-|\psi|)\sin\theta}\left(\frac{1+e^{-2R|\psi|\sin\theta}}{1-e^{-2\pi R\sin\theta}}\right)$$

For $0 < \theta < \pi$ we have that $e^{-2R|\psi|\sin\theta} < 1$, $e^{-2\pi R\sin\theta} < 1$ and hence

$$|I_\Gamma| < \frac{4R}{R-|\nu|}\int_0^{\pi} d\theta e^{-R(\pi-|\psi|)\sin\theta}$$

$$= \frac{8R}{R-|\nu|}\int_0^{\frac{1}{2}\pi} d\theta e^{-R(\pi-|\psi|)\sin\theta}$$

Now for $0 < \theta < \frac{1}{2}\pi$ we know that

$$1 > (\sin\theta)/\theta > 2/\pi$$

and hence

$$|I_\Gamma| < \frac{8R}{R-|\nu|}\int_0^{\frac{1}{2}\pi} d\theta e^{-R(\pi-|\psi|)2\theta/\pi}$$

$$= \frac{8R}{R-|\nu|}\,\frac{\pi}{2R(\pi-|\psi|)}\,(1-e^{-(\pi-|\psi|)R})$$

$$\to 0 \quad \text{as} \quad R\to\infty \quad \text{as required, since} \quad -\pi < \psi < \pi$$

The restriction on the range of ψ may be demonstrated explicitly as follows:

If $\qquad \psi = \pm\pi$

$$I_\Gamma = \int_\Gamma \frac{\cos((z+\frac{1}{2})\pi)}{(z-\nu)\sin\pi z}\,dz$$

$$= -\int_\Gamma \frac{1}{z-\nu}\,dz$$

which diverges logarithmically.

Corollary 2.3

(2.41) $$\cos((\nu+\tfrac{1}{2})\psi) = (\sin(\nu\pi)/\pi)\Sigma_{n=0}^{\infty}(-1)^n\cos((n+\tfrac{1}{2})\psi)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1)) \qquad -\pi < \psi < \pi$$

Proof

I_Γ as defined by (2.39) has simple poles at

$$z = \nu, 0, \pm 1, \pm 2, \ldots$$

Lemma 2.2 then gives, using Cauchy's Residue Theorem,

$$0 = 2\pi i\{\frac{\cos((\nu+\frac{1}{2})\psi)}{\sin(\pi\nu)} + \Sigma_{s=-\infty}^{\infty}\frac{\cos((s+\frac{1}{2})\psi)}{(s-\nu)\pi(-1)^s}\}$$

or

(2.42) $$\cos((\nu+\tfrac{1}{2})\psi) = -(\sin(\nu\pi)/\pi)\ \{\Sigma_{s=0}^{\infty}(-1)^s\ \frac{\cos((s+\frac{1}{2})\psi)}{s-\nu}$$

$$+ \Sigma_{s=-\infty}^{-1}(-1)^s\ \frac{\cos((s+\frac{1}{2})\psi)}{s-\nu}\}$$

In the first sum set $s = n$; in the second set $s = -n-1$. (2.42) then reads

$$\cos((\nu+\tfrac{1}{2})\psi) = (\sin(\nu\pi)/\pi)\{\Sigma_{n=0}^{\infty}(-1)^n\cos((n+\tfrac{1}{2})\psi)/(\nu-n)$$

$$- \Sigma_{n=0}^{\infty}(-1)^n\cos((n+\tfrac{1}{2})\psi)/(\nu+n+1)\}$$

(2.43) $$= (\sin(\nu\pi)/\pi)\ \Sigma_{n=0}^{\infty}(-1)^n\cos((n+\tfrac{1}{2})\psi)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1)) \qquad -\pi < \psi < \pi$$

Lemma 2.4

The series (2.41) is uniformly convergent for $\psi \in (-\pi,\pi)$.

Proof

Write (2.41) as

$$\cos((\nu+\tfrac{1}{2})\psi) = -(\sin(\nu\pi)/\pi)S_o(\psi)$$

where

(2.44) $$S_o(\psi) = \Sigma_{n=o}^{\infty}(-1)^n\cos((n+\tfrac{1}{2})\psi)(1/(n-\nu) + 1/(n+\nu+1))$$

Put

$$u_n(\psi) = (-1)^n\cos((n+\tfrac{1}{2})\psi)$$

$$v_n = (1/(n-\nu) + 1/(n+\nu+1))$$

then

(i) $$\Sigma_{n=o}^{M-1} u_n(\psi) = \Sigma_{n=o}^{M-1} e^{i\pi n}\tfrac{1}{2}(e^{i(n+\frac{1}{2})\psi}+e^{-i(n+\frac{1}{2})\psi})$$

$$= \tfrac{1}{2}e^{i\frac{1}{2}\psi}\Sigma_{n=o}^{M-1} e^{in(\pi+\psi)}+\tfrac{1}{2}e^{-i\frac{1}{2}\psi}\Sigma_{m=o}^{M-1}e^{in(\pi-\psi)}$$

$$= \tfrac{1}{2}e^{i\frac{1}{2}\psi}\left(\frac{1-e^{i(\pi+\psi)M}}{1-e^{i(\pi+\psi)}}\right)+\tfrac{1}{2}e^{-i\frac{1}{2}\psi}\left(\frac{1-e^{i(\pi-\psi)M}}{1-e^{i(\pi-\psi)}}\right)$$

$$= \frac{\tfrac{1}{2}e^{i\frac{1}{2}\psi}(1-(-1)^M e^{iM\psi})(1+e^{-i\psi})+\tfrac{1}{2}e^{-i\frac{1}{2}\psi}(1-(-1)^M e^{-iM\psi})(1+e^{i\psi})}{(1+e^{-i\psi})(1+e^{i\psi})}$$

$$= \cos(\tfrac{1}{2}\psi)(1-(-1)^M\cos(M\psi))/(1+\cos\psi)$$

$$\therefore \quad |\Sigma_{n=o}^{M-1} u_n(\psi)| < 2/(1+\cos\psi) < K$$

where K is a constant independent of ψ, $\psi \in (-\pi,\pi)$.

We note that this breaks down for $\psi = \pm\pi$, which is of course not entirely unexpected.

(ii) $v_n > 0$ and decreases with n $\forall n > |\nu|$

(iii) $v_n \to 0$ as $n \to \infty$

Then by Dirichlet's test for uniform convergence (Bromwich, 1926, page 125) $S_o(\psi)$ is uniformly convergent, whence (2.41) is uniformly convergent in $(-\pi,\pi)$ as required.

Lemma 2.5

If $S_n(x) = \Sigma_{q=o}^{n} f_q(x)$ is uniformly convergent to $F(x)$ on $(a,x_o) \cup (x_o,b)$ and $S_n(x_o)$ converges to $F(x_o)$, then $S_n(x)$ is uniformly convergent on (a,b).

Proof

$S_n(x)$ is uniformly convergent on $(a,x_o) \cup (x_o,b)$ implies that

$$\forall\varepsilon > 0 \; \exists \; N_1 \; (\forall n > N_1) \; V \; (\forall x \in (a,x_o)) |S_n(x)-F(x)|<\varepsilon$$

$$\forall\varepsilon > 0 \; \exists \; N_2 \; (\forall n > N_2) \; V \; (\forall x \in (x_o,b)) |S_n(x)-F(x)|<\varepsilon$$

$S_n(x_o)$ converges to $F(x_o)$ means that

$$\forall\varepsilon > 0 \; \exists \; N_3 \; (\forall n > N_3) \; |S_n(x_o)-F(x_o)| < \varepsilon$$

Choosing $N = \max(N_1,N_2,N_3)$ we see that

$$(\forall\varepsilon > 0) \; V \; (\forall x \in (a,b)) \; (n > N) \Rightarrow |S_n(x)-F(x)|<\varepsilon$$

i.e. we have uniform convergence on (a,b).

Lemma 2.6

(2.45) $$P_\nu^{-\mu}(\cos\theta) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n P_n^{-\mu}(\cos\theta) \cdot (1/(\nu-n)-1/(\nu+n+1))$$

where $\{\theta,\mu\}$ satisfies

(2.46) $$\begin{cases} \mu > -\tfrac{1}{2}, & 0 < |\theta| < \pi \\ \mu \geq 0, & \theta = 0 \end{cases}$$

and the series is uniformly convergent in $(-\pi,\pi)$.

Proof

From (2.19)

(2.47) $$P_\nu^{-\mu}(\cos\theta) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \frac{(\sin\theta)^{-\mu}}{\Gamma(\frac{1}{2}+\mu)} \int_0^\theta d\psi(\cos\psi-\cos\theta)^{\mu-\frac{1}{2}} \cdot \cos((\nu+\tfrac{1}{2})\psi) \qquad 0 < \theta < \pi, \qquad \mu > -\tfrac{1}{2}$$

Multiply both sides of (2.41) by

(2.48) $$(\tfrac{1}{2}\pi)^{-\frac{1}{2}}\frac{(\sin\theta)^{-\mu}}{\Gamma(\frac{1}{2}+\mu)}(\cos\psi-\cos\theta)^{\mu-\frac{1}{2}}$$

Since (2.48) is a bounded function of ψ for $0 < \psi < \theta$, $0 < \theta < \pi$, the resulting series is uniformly convergent and may be integrated term-by-term to give a uniformly convergent series. Utilizing (2.47) we obtain

(2.49) $$P_\nu^{-\mu}(\cos\theta) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n P_n^{-\mu}(\cos\theta)$$
$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

uniformly convergent in $(0,\pi)$ for $\mu > -\frac{1}{2}$.

Now setting $\theta = -\theta$ in (2.47) we deduce

(2.50) $$P_\nu^{-\mu}(\cos\theta) = -e^{-i\pi\mu}(\tfrac{1}{2}\pi)^{-\frac{1}{2}}\frac{(\sin\theta)^{-\mu}}{\Gamma(\frac{1}{2}+\mu)}$$
$$\cdot\int_o^\theta d\psi(\cos\psi-\cos\theta)^{\mu-\frac{1}{2}}\cos((\nu+\tfrac{1}{2})\psi)$$
$$-\pi < \theta < 0, \quad \mu > -\tfrac{1}{2}$$

The previous argument may now be repeated, thereby proving that (2.47) is uniformly convergent in $(-\pi,0) \cup (0,\pi)$ for $\mu > -\frac{1}{2}$.

The point $\theta = 0$ is included in the range of validity of (2.45) by first considering the case $\mu > 0$, and then $\mu = 0$. (2.45) clearly holds for $\mu > 0$ for from (2.18) it reduces to a trivial identity at $\theta = 0$.

Now from (2.3) and (2.4)

(2.51) $$P_\nu(1) = 1$$

Thus setting $\psi = 0$ in (2.41) we obtain

$$1 = P_\nu(1)$$
$$= (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n(1/(\nu-n)-1/(\nu+n+1))$$
$$= (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n P_n(1)(1/(\nu-n)-1/(\nu+n+1))$$

Thus

(2.52) $$P_\nu^{-\mu}(\cos\theta) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n P_n^{-\mu}(\cos\theta)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

for $-\pi < \theta < \pi$, $\mu \geq 0$, and is uniformly convergent for $\theta \in (-\pi,0) \cup (0,\pi)$. Thus by Lemma 2.5 it is uniformly convergent in $(-\pi,\pi)$ for $\mu \geq 0$.

We see that we have obtained (2.35) under slightly less restrictive conditions than in (E); we have also proved uniform convergence over the entire range of validity.

Lemma 2.7

(2.53) $$P_\nu^{-\mu}(\cos\theta)P_\nu^{-\lambda}(\cos\xi) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n P_n^{-\mu}(\cos\theta)$$

$$\cdot P_n^{-\lambda}(\cos\xi)(1/(\nu-n)-1/(\nu+n+1))$$

where $\{\theta,\mu\}$ and $\{\xi,\lambda\}$ satisfy (2.46), viz.

$$\begin{cases} \mu > -\tfrac{1}{2}, & 0 < |\theta| < \pi \\ \mu \geq 0, & \theta = 0 \end{cases}$$

and

$$\begin{cases} \lambda > -\tfrac{1}{2}, & 0 < |\xi| < \pi \\ \lambda \geq 0, & \xi = 0 \end{cases}$$

and in addition $-\pi < \theta \pm \xi < \pi$. With these restrictions the series (2.53) is uniformly convergent.

Proof

Using integral representation (2.19) twice we find

(2.54) $$P_\nu^{-\mu}(\cos\theta)P_\nu^{-\lambda}(\cos\xi) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}}\frac{(\sin\theta)^{-\mu}}{\Gamma(\frac{1}{2}+\mu)}$$

$$\cdot\int_0^\theta d\psi(\cos\psi-\cos\theta)^{\mu-\frac{1}{2}}\cos((\nu+\tfrac{1}{2})\psi)(\tfrac{1}{2}\pi)^{-\frac{1}{2}}$$

$$\cdot\frac{(\sin\xi)^{-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}\int_0^\xi d\chi(\cos\chi-\cos\xi)^{\lambda-\frac{1}{2}}\cos((\nu+\tfrac{1}{2})\chi)$$

for $0 < \theta < \pi$, $0 < \xi < \pi$, $\mu,\lambda > -\frac{1}{2}$

From (2.41)

$$(2.55) \qquad \cos((\nu+\tfrac{1}{2})(\psi+\chi)) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n$$

$$\cdot\cos((n+\tfrac{1}{2})(\psi+\chi))(1/(\nu-n)-1/(\nu+n+1))$$

$$(2.56) \qquad \cos((\nu+\tfrac{1}{2})(\psi-\chi)) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n$$

$$\cdot\cos((n+\tfrac{1}{2})(\psi-\chi))(1/(\nu-n)-1(\nu+n+1))$$

$$\text{for} \quad -\pi < \psi \pm \chi < \pi$$

Adding (2.55) and (2.56) we obtain

$$(2.57) \qquad \cos((\nu+\tfrac{1}{2})\psi)\cos((\nu+\tfrac{1}{2})\chi) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n$$

$$\cdot\cos((n+\tfrac{1}{2})\psi)\cos((n+\tfrac{1}{2})\chi)(1/(\nu-n)-1/(\nu+n+1))$$

$$\text{with} \quad -\pi < \psi \pm \chi < \pi$$

Multiply both sides of (2.57) by

$$(\tfrac{1}{2}\pi)^{-\frac{1}{2}}\frac{(\sin\theta)^{-\mu}}{\Gamma(\frac{1}{2}+\mu)}(\cos\psi-\cos\theta)^{\mu-\frac{1}{2}}$$

$$\circ(\tfrac{1}{2}\pi)^{-\frac{1}{2}}\frac{(\sin\xi)^{-\lambda}}{\Gamma(\frac{1}{2}+\lambda)}(\cos\chi-\cos\xi)^{\lambda-\frac{1}{2}}$$

which is bounded with respect to both ψ in $(0,\theta)$ and χ in $(0,\xi)$. We integrate the resultant uniformly convergent series termwise with respect to ψ and χ utilizing (2.54) to obtain (2.53) for $0 < \theta,\xi < \pi$. The extension to the range $(-\pi,0) \cup (0,\pi)$ for $\lambda,\mu > -\frac{1}{2}$ and to $(-\pi,\pi)$ for $\lambda,\mu \geq 0$ follows as in Lemma 2.6; uniform convergence does likewise.

We note that we have derived (2.36) under slightly less restrictive conditions than those given by (E), and in addition have proved uniform convergence over the extended range of validity.

Lemma 2.8

(2.58) $$P_\nu^m(\cos\theta)P_\nu^{-m}(\cos\xi) = (\sin(\nu\pi)/\pi)\Sigma_{n=m}^{\infty}(-1)^n$$

$$\cdot P_n^m(\cos\theta)P_n^{-m}(\cos\xi)(1/(\nu-n)-1/(\nu+n+1))$$

m a positive integer, $-\pi < \theta \pm \xi < \pi$,

and this series is uniformly convergent over its range of validity.

Proof

Recall relation (2.22)

$$P_\nu^m(x) = (-1)^m(1-x^2)^{\frac{1}{2}m} \frac{d^m P_\nu(x)}{dx^m} \qquad -1 < x < 1,\ m > 0$$

or

(2.59) $$P_\nu^m(\cos\theta) = (-1)^m \sin^m\theta \frac{d^m P_\nu(\cos\theta)}{d(\cos\theta)^m}$$

$$-\pi < \theta < \pi,\ \theta \neq 0, m > 0$$

Consider equation (2.53). Set $\mu = 0$, $\lambda = m$ (where m is a positive integer) and differentiate m times with respect to $x = \cos\theta$. Utilization of (2.59) gives

(2.60) $$P_\nu^m(\cos\theta)P_\nu^{-m}(\cos\xi) = (\sin(\nu\pi)/\pi)\Sigma_{n=m}^{\infty}(-1)^n$$

$$\cdot P_n^m(\cos\theta)P_n^{-m}(\cos\xi)(1/(\nu-n)-1/(\nu+n+1))$$

for $-\pi < \theta \pm \xi < \pi$, $\theta,\xi \neq 0$, $m > 0$.

The case $m = 0$ is included in (2.53) with $\lambda = \mu = 0$.

The lower limit of summation follows from the fact (2.28) that

$$P_n^m(x) \equiv 0 \quad \text{for} \quad m > n.$$

The term-by-term differentiation of (2.53) is justified by the uniform convergence of (2.60) which is proved as follows:

From (2.30)

(2.61) $$n^{-m}P_n^m(\cos\theta) = (\tfrac{1}{2}n\pi\sin\theta)^{-\frac{1}{2}}\cos((n+\tfrac{1}{2})\theta-\pi/4+\tfrac{1}{2}m\pi)+O(n^{-3/2})$$

$$0 < \varepsilon < \theta < \pi - \varepsilon, \quad n > 1, \quad n \gg m$$

Thus

$$P_n^m(\cos\theta)P_n^{-m}(\cos\xi) = \frac{2}{n\pi\sqrt{\sin\theta\sin\xi}}$$

$$\cdot\cos((n+\tfrac{1}{2})\theta-\pi/4+\tfrac{1}{2}m\pi)\cos((n+\tfrac{1}{2})\xi-\pi/4-\tfrac{1}{2}m\pi)+O(n^{-2})$$

Since

$$(1/(\nu-n)-1/(\nu+n+1)) = O(n^{-1})$$

it is obvious by Weierstrass' M-test that (2.53) is uniformly convergent for θ,ξ in $(0,\pi)$; the extension of range to $(-\pi,0)$ and inclusion of the points $\xi = \theta = 0$, as well as uniform convergence over the entire range of validity follow as in Lemmas 2.6 and 2.7.

Note that in this case the range of validity of the Dougall identity has been extended fourfold.

CHAPTER 3

NEW IDENTITIES FOR INTEGER ν

Starting with equation (2.58) which we renumber (3.1) for convenience, namely

$$(3.1)\qquad P_\nu^m(\cos\theta)P_\nu^{-m}(\cos\xi) = (\sin(\nu\pi)/\pi)\Sigma_{n=m}^{\infty}(-1)^n$$

$$\cdot P_n^m(\cos\theta)P_n^{-m}(\cos\xi)(1/(\nu-n)-1/(\nu+n+1))$$

$$-\pi < \theta \pm \xi < \pi,$$

we derive a set of identities for Legendre polynomials of integral order and degree. Uniform convergence of each one is investigated in detail because these identities are used in situations where one would wish to interchange the order of summation and integration, and uniform convergence is required before this may be performed.

As mentioned in the introductory section on notation, we will write

$$(3.2)\qquad \left(\frac{d}{dx}P_n^m(x)\right)_{x=x_o} \quad \text{as} \quad P'^m_n(x_o)$$

<u>Lemma 3.1</u>

For all $\nu \in R\sim I$, and for all $m \in N$, the series

$$(3.3)\qquad S_1(\theta) = \Sigma_{n=m}^{\infty}(-1)^n\gamma_{n,m}P_n^m(\cos\theta)P'^m_n(0)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

is uniformly convergent for $0 < \theta < \frac{1}{2}\pi$

and

$$(3.4)\qquad S_2(\theta) = \Sigma_{n=m}^{\infty}(-1)^n\gamma_{n,m}P_n^m(\cos\theta)P_n^m(0)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

is uniformly convergent for $0 < \theta < \pi$.

Proof

From (2.15) we deduce that

(3.5) $$P_n^{-m}(x) = (-1)^m \gamma_{n,m} P_n^m(x) \qquad -1 < x < 1$$

where $\gamma_{n,m}$ has been defined in (2.9) as

(3.6) $$\gamma_{n,m} = \Gamma(n-m+1)/\Gamma(n+m+1)$$

Recurrence relation (2.31) gives

(3.7) $$P'^m_n(0) = (m+n)P^m_{n-1}(0)$$

Further, equation (2.30) states that

(3.8) $$n^{-m}P_n^m(\cos\theta) = (\tfrac{1}{2}n\pi\sin\theta)^{-\frac{1}{2}}\cos((n+\tfrac{1}{2})\theta-\pi/4+\tfrac{1}{2}m\pi)+O(n^{-3/2})$$

$$0 < \varepsilon < \theta < \pi - \varepsilon, \quad n > 1, \quad n >> m$$

where $O(n^{-3/2})$ depends on ε, whence

$$\gamma_{n,m}P_n^m(\cos\theta)P'^m_n(0)$$

$$= (-1)^m P_n^{-m}(\cos\theta)(n+m)P^m_{n-1}(0) \quad \text{(by (3.5) and (3.7))}$$

$$= (-1)^m(n+m)((n-1)/n)^{m-\frac{1}{2}}\{(\tfrac{1}{2}n^2\pi\sin\theta)^{-\frac{1}{2}}$$

$$\cdot\cos((n+\tfrac{1}{2})\theta-\pi/4-\tfrac{1}{2}m\pi)(\tfrac{1}{2}\pi)^{-\frac{1}{2}}\sin(\tfrac{1}{2}(n+m)\pi)+O(n^{-2})\}$$

(3.9) $$= K\sin(\tfrac{1}{2}(n+m)\pi)\cos((n+\tfrac{1}{2})\theta-\pi/4-\tfrac{1}{2}m\pi) + O(n^{-1})$$

$$0 < \varepsilon < \theta < \pi - \varepsilon, \quad n > 1, \quad n >> m$$

for large n, where K is a constant independent of ε, and where $O(n^{-1})$ depends on ε.

As in Lemma 2.4 set

$$u_n(\theta) = (-1)^n\sin(\tfrac{1}{2}(n+m)\pi)\cos((n+\tfrac{1}{2})\theta-\pi/4-\tfrac{1}{2}m\pi)$$

(3.10) $$= \tfrac{1}{2}(-1)^n\{\sin(n(\theta+\tfrac{1}{2}\pi)-\pi/4+\tfrac{1}{2}\theta) - \sin(n(\theta-\tfrac{1}{2}\pi)-\pi/4+\tfrac{1}{2}\theta-m\pi)\}$$

and

(3.11) $$v_n = 1/(\nu-n)-1/(\nu+n+1)$$

Consider

$$\Sigma_{n=o}^{M-1}(-1)^n \sin(n\psi+\alpha)$$

$$= -\tfrac{1}{2}i\Sigma_{n=o}^{M-1}(-1)^n(e^{i(n\psi+\alpha)}-e^{-i(n\psi+\alpha)})$$

$$= -\tfrac{1}{2}i(e^{i\alpha}\Sigma_{n=o}^{M-1}\ e^{in(\pi+\psi)}-e^{-i\alpha}\Sigma_{n=o}^{M-1}\ e^{in(\pi-\psi)})$$

$$= -\tfrac{1}{2}i(e^{i\alpha}(\frac{1-e^{iM(\pi+\psi)}}{1+e^{i\psi}})\ -e^{-i\alpha}(\frac{1-e^{iM(\pi-\psi)}}{1+e^{-i\psi}}))$$

$$= -\tfrac{1}{2}i\frac{(e^{i\alpha}(1+e^{-i\psi})(1-(-1)^M e^{iM\psi})-e^{-i\alpha}(1+e^{i\psi})(1-(-1)^M e^{-iM\psi}))}{(1+e^{i\psi})(1+e^{-i\psi})}$$

$$= -\tfrac{1}{2}i(e^{i(\alpha-\frac{1}{2}\psi)}\cos(\tfrac{1}{2}\psi)(1-(-1)^M e^{iM\psi})-e^{-i(\alpha-\frac{1}{2}\psi)}$$

$$\cdot\cos(\tfrac{1}{2}\psi)(1-(-1)^M e^{-iM\psi}))/(1+\cos\ \psi)$$

$$= \cos(\tfrac{1}{2}\psi)(\sin(\alpha-\tfrac{1}{2}\psi)-(-1)^M\sin(\alpha-\tfrac{1}{2}\psi+M\psi))/(1+\cos\ \psi)$$

$$\therefore\quad |\Sigma_{n=o}^{M-1}(-1)^n\sin(n\psi+\alpha)| < 2/(1+\cos\ \psi) < K$$

where K is a constant independent of ψ, provided $-\pi < \psi < \pi$.

Thus from (3.10) we see that

(i) $|\Sigma_{n=o}^{M-1}\ u_n(\theta)| < K$ if $0 < \theta < \frac{1}{2}\pi$.

From definition (3.11) we have

(ii) $v_n > 0$ and decreases with n $\forall n > |\nu|$

(iii) $v_n \to 0$ as $n \to \infty$

Then by Dirichlet's test for uniform convergence (Bromwich, 1926, page 125) $S_1(\theta)$ of equation (3.3) is uniformly convergent for $0 < \theta < \frac{1}{2}\pi$ as required.

Now consider $S_2(\theta)$.

$$|(-1)^{n+m}\gamma_{n,m}P_n^m(\cos\,\theta)P_n^m(0)(1/(\nu-n)-1/(\nu+n+1))|$$

$$= |P_n^{-m}(\cos\,\theta)P_n^m(0)(2n+1)/((\nu-n)(\nu+n+1))| \quad \text{by } (3.5)$$

$$= |K(\tfrac{1}{4}n^2\pi^2\sin\,\theta)^{-\frac{1}{2}}\cos((n+\tfrac{1}{2})\theta-\pi/4-\tfrac{1}{2}m\pi)\cos(\tfrac{1}{2}(n+m)\pi)/n|$$

$$+ O(n^{-3}) \quad \text{by } (3.8)$$

$$\text{for } n >> m, \quad 0 < \varepsilon < \theta < \pi - \varepsilon$$

$$= K/n^2 + O(n^{-3}),$$

where $O(n^{-3})$ depends on ε, but where K is a constant independent of ε.

Hence by Weierstrass' M-test $S_2(\theta)$ is uniformly convergent for $0 < \theta < \pi$ as stated.

Lemma 3.2

For all $\nu \in R\sim I$, $m \in N$ and $x \in (0,1)$

$$(3.15) \qquad P_\nu^m(x) = P_\nu^m(0)\Sigma_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

$$(3.16) \qquad = P'^m_\nu(0)\Sigma_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

and both series are uniformly convergent over their range of validity.

Proof

Substitute for $P_\nu^{-m}(\cos\,\xi)$ and $P_n^{-m}(\cos\,\xi)$ from (3.5) into (3.1) and differentiate with respect to ξ; set $\xi = \pi/2$. We obtain (writing $x = \cos\,\theta$)

$$(3.17) \qquad \gamma_{\nu,m}P_\nu^m(x)P'^m_\nu(0) = (\sin(\nu\pi)/\pi)\Sigma_{n=m}^{\infty}(-1)^n\gamma_{n,m}$$

$$\cdot P_n^m(x)P'^m_n(0)(1/(\nu-n)-1/(\nu+n+1))$$

$$0 < x < 1$$

Validity of the term-by-term differentiation of (3.5) follows from the uniform convergence of (3.17) which was proved in Lemma 3.1.

Now (2.16) and (2.17) give

$$(3.18) \qquad P_\nu^m(0) = 2^m\pi^{-\frac{1}{2}}\cos(\tfrac{1}{2}(\nu+m)\pi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}m)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}m)$$

$$(3.19) \qquad P'^m_\nu(0) = 2^{m+1}\pi^{-\frac{1}{2}}\sin(\tfrac{1}{2}(\nu+m)\pi)\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}m)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}m)$$

From (3.18) and (3.19) and definition (3.6) we find

$$\gamma_{\nu,m}P_\nu^m(0)P'^m_\nu(0)$$

$$= \gamma_{\nu,m}2^{2m+1}\pi^{-1}\tfrac{1}{2}\sin((m+\nu)\pi)(\Gamma(1+\nu+m)/2^{1+\nu+m-1})$$

$$/(\Gamma(1+\nu-m)/2^{1+\nu-m-1})$$

$$(3.20) \qquad = (-1)^m\sin(\nu\pi)/\pi$$

where we have used Legendre's duplication formula (2.8) for gamma functions, namely

$$(3.21) \qquad \Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

Multiplying both sides of (3.17) by $P_\nu^m(0)$ and using (3.20) we obtain (3.15), namely

$$(3.22) \qquad P_\nu^m(x) = P_\nu^m(0)\Sigma_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

Similarly by multiplying both sides of (3.1) by $P'^m_\nu(0)$ and setting $\xi = \pi/2$ we find (using (3.20)) that

$$(3.23) \qquad P_\nu^m(x) = P'^m_\nu(0)\Sigma_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

as required.

Uniform convergence of both (3.22) and (3.23) follows from Lemma 3.1.

We are now in a position to derive our basic new identities, which we do in the following theorem.

Theorem 3.3

For all $m,\ell \in N$, $\ell \geq m$ and for all $x \in (0,1)$,

$$P_\ell^m(0)\Sigma_{\substack{n=m\\n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1)) = \tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x) \tag{3.24}$$

$$P'^m_\ell(0)\Sigma_{\substack{n=m\\n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)(1/(\ell-n)-1/(\ell+n+1)) = \tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x). \tag{3.25}$$

The convergence is uniform.

Proof

Consider equation (3.15). For some $\ell \geq m$ we may rewrite it as

$$P_\nu^m(x) = (-1)^{\ell+m}P_\nu^m(0)\gamma_{\ell,m}P_\ell^m(x)P'^m_\ell(0)(1/\nu-\ell)-1/(\nu+\ell+1)$$
$$+ P_\nu^m(0)\Sigma_{\substack{n=m\\n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)$$
$$\cdot(1/(\nu-n)-1/(\nu+n+1)) \tag{3.26}$$

We now wish to go to the limit $\nu \to \ell \in N$. The only piece of (3.26) for which this limit is not smooth is the first term of the dexter which reads

$$(3.27)\qquad \lim_{\nu\to\ell} (-1)^{\ell+m}P_\nu^m(0)\gamma_{\ell,m}P_\ell^m(x)P'^m_\ell(0)/(\nu-\ell)$$

$$= \lim_{\nu\to\ell} (-1)^{\ell+m}\gamma_{\ell,m}P_\ell^m(x)2^m\pi^{-\frac{1}{2}}\cos(\tfrac{1}{2}(\nu+m)\pi)\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}m)}$$

$$\cdot 2^{m+1}\pi^{-\frac{1}{2}}\sin(\tfrac{1}{2}\ell+m)\pi)\frac{\Gamma(1+\frac{1}{2}\ell+\frac{1}{2}m)}{\Gamma(\frac{1}{2}+\frac{1}{2}\ell-\frac{1}{2}m)}\ \frac{1}{\nu-\ell} \qquad \text{(using (3.18) and (3.19))}$$

$$= -(-1)^{\ell+m}\gamma_{\ell,m}P_\ell^m(x)2^{2m+1}\pi^{-1}(\tfrac{1}{2}\pi)$$

$$\cdot\sin^2(\tfrac{1}{2}(\ell+m)\pi)\frac{\Gamma(1+\ell+m)}{\Gamma(1+\ell-m)}\ 2^{-2m} \qquad \text{(from (3.21))}$$

$$= -(-1)^{\ell+m}\sin^2(\tfrac{1}{2}(\ell+m)\pi)P_\ell^m(x) \qquad \text{(by definition (3.6))}$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)$$

Taking the limit $\nu \to \ell \in N$ of (3.26) and employing (3.27) we see that the limit exists and gives

$$P_\ell^m(x) = \tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x) + P^m(0)\Sigma_{\substack{n=m\\ n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}$$

$$\cdot P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$

or

$$(3.28)\qquad P_\ell^m(0)\Sigma_{\substack{n=m\\ n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)$$

thus deriving (3.24). Proof of its uniform convergence follows similar lines to that of Lemma 3.1; we can allow ν of Lemma 3.1 to be an integer ℓ because the term $n = \ell$ has been excluded from the summation in (3.28).

We turn now to equation (3.16). Again separate out the term for $n = \ell$, and take the limit $\nu \to \ell \in N$. The non-smooth term is

$$(3.29)\quad \lim_{\nu\to\ell} P'^{m}_{\nu}(0)(-1)^{\ell+m}\gamma_{\ell,m}P^m_\ell(x)P^m_\ell(0)/(\nu-\ell)$$

$$= \lim_{\nu\to\ell}(-1)^{\ell+m}\gamma_{\ell,m}2^{m+1}\pi^{-\frac{1}{2}}\sin(\tfrac{1}{2}(\nu+m)\pi)\frac{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}m)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}m)}$$

$$\cdot 2^m\pi^{-\frac{1}{2}}\cos(\tfrac{1}{2}(\ell+m)\pi)\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\ell+\frac{1}{2}m)}{\Gamma(1+\frac{1}{2}\ell-\frac{1}{2}m)}\; P^m_\ell(x)/(\nu-\ell)$$

(from (3.18) and (3.19))

$$= (-1)^{\ell+m}\cos^2(\tfrac{1}{2}(\ell+m)\pi)P^m_\ell(x)$$

$$= \tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x)$$

Substitute this limit into the term for $n = \ell$; (3.16) then gives

$$(3.30)\quad P'^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n=m\\ n\neq\ell}}(-1)^{n+m}\gamma_{n,m}P^m_n(x)P^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x)$$

which proves (3.25). Uniform convergence follows as in Lemma 3.1 like before. Q.E.D.

<u>Corollary 3.4</u>

From (3.18) and (3.19) we deduce that

$$P^m_n(0) = 0 \quad \text{unless} \quad (m+n) \quad \text{is even}$$

and $\quad P'^{m}_{n}(0) = 0 \quad$ unless $\quad (m+n) \quad$ is odd

whence (3.24) and (3.25) become

$$(3.31)\quad P^m_\ell(0)\Sigma^{\infty}_{\substack{n=m\\ n\neq\ell}}\gamma_{n,m}P^m_n(x)P'^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= -\tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x),$$

$$(3.32)\quad P'^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n=m\\ n\neq\ell}}\gamma_{n,m}P^m_n(x)P^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x) \qquad 0 < x < 1$$

Corollary 3.5

Equations (3.24), (3.25), (3.31) and (3.32) hold for $0 < x < 1$. However, if $-1 < x < 0$ then $0 < -x < 1$ and $(-x)$ may be substituted into these four equations. From (2.14) we deduce that

(3.34) $$P_n^m(-x) = (-1)^{m+n}P_n^m(x) \qquad 0 < x < 1$$

We obtain for $-1 < x < 0$

(3.35) $$P_\ell^m(0)\sum_{\substack{n=m\\ n\neq\ell}}^{\infty}\gamma_{n,m}P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$
$$= \tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)$$

(3.36) $$P'^m_\ell(0)\sum_{\substack{n=m\\ n\neq\ell}}^{\infty}\gamma_{n,m}P_n^m(x)P_n^m(0)(1/(\ell-n)-1/(\ell+n+1))$$
$$= -\tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)$$

(3.37) $$P_\ell^m(0)\sum_{\substack{n=m\\ n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$
$$= -\tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)$$

(3.38) $$P'^m_\ell(0)\sum_{\substack{n=m\\ n\neq\ell}}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)(1/\ell-n)-1/(\ell+n+1))$$
$$= -\tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)$$

and the convergence is uniform.

Note that the dexters of (3.35) - (3.38) are opposite in sign to those of the corresponding identities for $0 < x < 1$.

Having derived identities uniformly convergent for $0 < |x| < 1$, we now investigate whether we can extend our identities to include the points $x = 0, \pm 1$.

Corollary 3.6

Equations (3.24), (3.25), (3.31), (3.32) and (3.35) - (3.38) can not be extended to include the points $x = \pm 1$ unless $m = 0$, in which case they reduce to

$$(3.39)\quad \begin{aligned} &P_\ell(0)\Sigma_{\substack{n=0\\ n\neq\ell}}^{\infty}(-1)^n P'_n(0)(1/(\ell-n)-1/(\ell+n+1))=\tfrac{1}{2}(1+(-1)^\ell) \\ &P'_\ell(0)\Sigma_{\substack{n=0\\ n\neq\ell}}^{\infty}(-1)^n P_n(0)(1/(\ell-n)-1/(\ell+n+1))=\tfrac{1}{2}(1-(-1)^\ell) \\ &P_\ell(0)\Sigma_{\substack{n=0\\ n\neq\ell}}^{\infty} P'_n(0)(1/(\ell-n)-1/(\ell+n+1)) = -\tfrac{1}{2}(1+(-1)^\ell) \\ &P'_\ell(0)\Sigma_{\substack{n=0\\ n\neq\ell}}^{\infty} P_n(0)(1/(\ell-n)-1/(\ell+n+1)) = \tfrac{1}{2}(1-(-1)^\ell) \end{aligned}$$

Proof

From (2.23) for $m > 0$ we have

$$(3.40)\qquad P_n^m(x) \to 0 \quad \text{as} \quad x \to 1$$

and our identities become trivial at $x = 1$.

Also

$$(3.41)\qquad (-1)^m \gamma_{n,m} P_n^m(x) = P_n^{-m}(x) \qquad \text{(by (3.5))}$$

$$\to \infty \quad \text{as} \quad x \to -1 \quad \text{(by (2.26))}$$

and we cannot extend our identities to the point $x = -1$ if $m > 0$.

If $m = 0$ we use (2.27)

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

to obtain (3.39); the validity of substituting $x = \pm 1$ in this case comes from the fact that identities (3.39) can be derived starting from (2.45) using the methods of Theorem 3.3.

Corollary 3.7

The sinisters of relations (3.24), (3.31), (3.35) and (3.37) are identically zero if we set $x = 0$.

Proof

From (2.16) and (2.17)

(3.42) $$P_n^m(0)P'^m_n(0) \equiv 0$$

We now demonstrate the power of Theorem 2.1 by rederiving (3.31) assuming (3.32), and then (3.35) assuming (3.36).

Let

(3.43) $$f(\theta_1,\phi_1) = \begin{cases} P_\lambda^m(\cos\ \theta_1)\cos(m\ \phi_1) \\ \qquad 0 \le \theta_1 \le \frac{\pi}{2}, \quad 0 \le \phi_1 \le 2\pi \\ 0 \qquad \frac{\pi}{2} < \theta_1 \le \pi, \quad 0 \le \phi_1 \le 2\pi \end{cases}$$

(a) Choose $\theta_1 \in (0,\pi/2)$ i.e. a point of continuity of $f(\theta_1,\phi_1)$, which function obviously satisfies the conditions of Theorem 2.1 from which we obtain (using addition theorem (2.34))

$$P_\lambda^m(\cos\ \theta_1)\cos(m\ \phi_1)$$

$$= \Sigma_{n=o}^{\infty} \frac{2n+1}{4\pi} \int_{-\pi}^{\pi} d\phi_2 \int_o^{\pi} d\theta_2 \sin\ \theta_2 f(\theta_2,\phi_2)\Sigma_{q=o}^{n}\gamma_{n,q}$$

$$\cdot P_n^q(\cos\ \theta_1)P_n^q(\cos\ \theta_2)\varepsilon_q\cos(q(\phi_1-\phi_2))$$

(3.44) $$=\Sigma_{n=o}^{\infty}\gamma_{n,m}\tfrac{1}{2}(2n+1)\left(\int_0^1 dx P_n^m(x)P_\lambda^m(x)\right)P_n^m(\cos\ \theta_1)\cos(m\ \phi_1)$$

Now (2.20) gives (using (3.7))

$$(n-\lambda)(n+\lambda+1)\int_0^1 dx P_n^m(x)P_\lambda^m(x)$$

$$= \left[x(n-\lambda)P_n^m(x)P_\lambda^m(x) + P_n^m(x)P'^m_\lambda(x) - P'^m_n(x)P_\lambda^m(x)\right]_o^1$$

$$= -P_n^m(0)P'^m_\lambda(0) + P'^m_n(0)P_\lambda^m(0)$$

whence

(3.45) $$\int_0^1 dx P_\lambda^m(x)P_n^m(x) = \frac{1}{2n+1}\left(1/(\lambda-n)-1/(\lambda+n+1)\right)$$

$$\cdot\left(P_n^m(0)P'^m_\lambda(0)-P'^m_n(0)P_\lambda^m(0)\right)$$

Substituting (3.45) into (3.44) gives

$$P_\lambda^m(\cos\ \theta_1) = \tfrac{1}{2}\Sigma_{n=o}^{\infty}\gamma_{n,m}\left(P_n^m(0)P'^m_\lambda(0)\right.$$

$$\left.-P'^m_n(0)P_\lambda^m(0)\right)P_n^m(\cos\ \theta)$$

$$\cdot\left(1/(\lambda-n)-1/(\lambda+n+1)\right)$$

(3.46) $$= \tfrac{1}{2}\gamma_{\ell,m}P_\ell^m(0)P'^m_\lambda(0)\left(1/(\lambda-\ell)-1/(\lambda+\ell+1)\right)P_\ell^m(\cos\ \theta_1)$$

$$- \tfrac{1}{2}\gamma_{\ell,m}P'^m_\ell(0)P_\lambda^m(0)\left(1/(\lambda-\ell)-1/(\lambda+\ell+1)\right)P_\ell^m(\cos\ \theta_1)$$

$$+ \tfrac{1}{2}\Sigma_{\substack{n=o\\ n\neq\ell}}^{\infty}\gamma_{n,m}P_n^m(0)P'^m_\lambda(0)\left(1/(\lambda-n)-1/(\lambda+n+1)\right)P_n^m(\cos\ \theta_1)$$

$$- \tfrac{1}{2}\Sigma_{\substack{n=o\\ n\neq\ell}}^{\infty}\gamma_{n,m}P'^m_n(0)P_\lambda^m(0)\left(1/(\lambda-n)-1/(\lambda+n+1)\right)P_n^m(\cos\ \theta_1)$$

Now take $\lim_{\lambda\to\ell\in N}$ of (3.46) and utilize (3.27) and (3.29) (writing $x = \cos\ \theta_1$) to give

We can now combine the above results into 2 basic identities, valid over the entire range $(-1,1)$. We do this in the following two theorems.

Theorem 3.8

The series

$$P'^{m}_{\ell}(0)\Sigma^{\infty}_{n=m}\gamma_{n,m}P^{m}_{n}(x)P^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

is uniformly convergent on the interval $(-1,1)$ to

$$\begin{cases} \tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x) & x > 0 \\ 0 & x = 0 \\ -\tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x) & x < 0 \end{cases}$$

Proof

Defining $f(\theta_1,\phi_1)$ by (3.43) choose $\theta_1 = 0$, the point of discontinuity of f. Theorem 2.1 then gives

$$\tfrac{1}{2}(0+P^{m}_{\lambda}(0)\cos(m\ \phi_1)) = \Sigma^{\infty}_{n=o}\gamma_{n,m}\tfrac{1}{2}(2n+1)$$

$$\cdot(\int_0^1 dxP^{m}_{\lambda}(x)P^{m}_{n}(x))P^{m}_{n}(0)\cos(m\ \phi_1)$$

whence

(3.51) $$P^{m}_{\lambda}(0) = \gamma_{\ell,m}P^{m}_{\ell}(0)P'^{m}_{\lambda}(0)(1/(\lambda-\ell)-1/(\lambda+\ell+1))P^{m}_{\ell}(0)$$

$$+\ \Sigma^{\infty}_{\substack{n=m\\ n\neq\ell}}\gamma_{n,m}P^{m}_{n}(0)P'^{m}_{\lambda}(0)(1/(\lambda-n)-1/(\lambda+n+1))P^{m}_{n}(0)$$

(where we have used (3.45) and (3.42)).

Taking $\lim\limits_{\lambda\to\ell\in N}$ of (3.51) we obtain (using (3.29))

(3.52) $$P'^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n=m\\ n\neq\ell}}\gamma_{n,m}P^{m}_{n}(0)P^{m}_{n}(0)(1/\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(0)$$

$$= 0 \qquad \text{(by (3.18))}$$

$$(3.47)\qquad 2P_\ell^m(x) = \tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)+\tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)$$

$$+ P'^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P^m_n(0)P^m_n(x)$$

$$\cdot(1/(\ell-n)-1/(\ell+n+1))$$

$$- P^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P'^m_n(0)P^m_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

$$0 < x < 1$$

But by (3.32)

$$(3.48)\qquad P'^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P^m_n(0)P^m_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x) \qquad 0 < x < 1$$

and (3.47) becomes

$$(3.49)\qquad P^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P'^m_n(0)P^m_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

$$= -P^m_\ell(x)+\tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x)$$

$$= -\tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x) \qquad 0 < x < 1$$

which is (3.31) as required.

(b) Similarly choosing $\theta_1 \in (\pi/2,\pi)$ we obtain ($x = \cos\theta_1$ and hence $-1 < x < 0$)

$$0 = \tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x)+\tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x)$$

$$+ P'^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P^m_n(0)P^m_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

$$- P^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P'^m_n(0)P^m_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

whence (using (3.36))

$$(3.50)\qquad P^m_\ell(0)\Sigma^\infty_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P'^m_n(0)P^m_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

$$= P^m_\ell(x)-\tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x)$$

$$= \tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x) \qquad -1 < x < 0$$

which is (3.35) as required.

Define

$$S_{\ell m}(x) \equiv P'^{m}_{\ell}(0)\Sigma^{\infty}_{n=m}\gamma_{n,m}P^{m}_{n}(x)P^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

and

$$F(x) \equiv \begin{cases} \tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x) & x > 0 \\ 0 & x = 0 \\ -\tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x) & x < 0 \end{cases}$$

By Corollary 3.4 $S_{\ell m}(x)$ is uniformly convergent to $F(x)$ for $x \in (0,1)$, and by Corollary 3.5 $S_{\ell m}(x)$ is uniformly convergent to $F(x)$ for $x \in (-1,0)$. Finally from (3.52) we see that $S_{\ell m}(0) = F(0)$. Hence by Lemma 2.5 $S_{\ell m}(x)$ is uniformly convergent to $F(x)$ on $(-1,1)$.

Q.E.D.

Theorem 3.9

The series

$$P^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n=m \\ n\neq\ell}}\gamma_{n,m}P^{m}_{n}(x)P'^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

converges on the interval $(-1,1)$ to

$$\begin{cases} -\tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x) & x > 0 \\ 0 & x = 0 \\ \tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x) & x < 0 \end{cases}$$

The convergence is uniform on any interval which excludes the origin.

Proof

Define

$$T_{\ell m}(x) \equiv P^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n=m \\ n\neq\ell}}\gamma_{n,m}P^{m}_{n}(x)P'^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

and

$$G(x) \equiv \begin{cases} -\tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x) & x > 0 \\ 0 & x = 0 \\ \tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x) & x < 0 \end{cases}$$

Uniform convergence of $T_{\ell m}(x)$ to $G(x)$ on $(-1,0) \cup (0,1)$ was proved in Corollaries 3.4 and 3.5. From Corollary 3.7 we see that

$$T_{\ell m}(0) = 0 = G(0)$$

since each term of $T_{\ell m}(0)$ is zero.

However, for $(\ell+m)$ even

$$\lim_{x\to 0^+} G(x) = -P_\ell^m(0) \neq 0$$

and

$$\lim_{x\to 0^-} G(x) = P_\ell^m(0) \neq 0$$

(for $(\ell+m)$ odd both sides are identically zero)

Hence no uniform convergence is possible in any neighbourhood of the origin. Q.E.D.

We can express the results of Theorems 3.8 and 3.9 in a compact form if we define $\varepsilon(x)$ by

$$\varepsilon(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{3.54}$$

Then Theorems 3.8 and 3.9 become

$$P_\ell^m(0)\sum_{\substack{n=m \\ n\neq\ell}}^{\infty}\gamma_{n,m}P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1)) = -\tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)\varepsilon(x) \qquad -1 < x < 1 \tag{3.55}$$

and

$$P'^m_\ell(0)\sum_{\substack{n=m \\ n\neq\ell}}^{\infty}\gamma_{n,m}P_n^m(x)P_n^m(0)(1/(\ell-n)-1/(\ell+n+1)) = \tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)\varepsilon(x) \qquad -1 < x < 1 \tag{3.56}$$

A third, related, identity is found by differentiating (3.51) with respect to x, giving

(3.57) $$P'^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n\bar{=}m\\ n\neq\ell}}\gamma_{n,m}P^{m}_{n}(x)P'^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= -\tfrac{1}{2}(1+(-1)^{\ell+m})P'^{m}_{\ell}(x)\varepsilon(x) \qquad -1 < x < 1$$

That this step is a valid one follows from the uniform convergence of (3.57) in $(-1,0) \cup (0,1)$ which is proved analogously to Lemma 3.1 and Corollary 3.5. The series vanishes at $x = 0$ as can be seen from Corollary 3.7.

Setting $x = -x$ in equations (3.55), (3.56) gives

(3.58) $$P^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n\bar{=}m\\ n\neq\ell}}(-1)^{n+m}\gamma_{n,m}P^{m}_{n}(x)P'^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x)\varepsilon(x) \qquad -1 < x < 1$$

(3.59) $$P'^{m}_{\ell}(0)\Sigma^{\infty}_{\substack{n\bar{=}m\\ n\neq\ell}}(-1)^{n+m}\gamma_{n,m}P^{m}_{n}(x)P^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x)\varepsilon(x) \qquad -1 < x < 1$$

A similar result can be derived from (3.57).

In many applications (see on) we are required to evaluate $\Sigma^{\infty}_{n=m}$ rather than $\Sigma^{\infty}_{\substack{n\bar{=}m\\ n\neq\ell}}$. To do this we merely re-insert the $n=\ell$ term which we have evaluated using limits (3.27) and (3.29). We obtain

(3.60) $$P^{m}_{\ell}(0)\Sigma^{\infty}_{n=m}\gamma_{n,m}P^{m}_{n}(x)P'^{m}_{n}(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= -\tfrac{1}{2}(1+(-1)^{\ell+m})P^{m}_{\ell}(x)\varepsilon(x)-\tfrac{1}{2}(1-(-1)^{\ell+m})P^{m}_{\ell}(x)$$

$$-1 < x < 1$$

and setting $x = -x$

$$(3.62)\qquad P_\ell^m(0)\Sigma_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P'^m_n(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)\varepsilon(x)+\tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)$$

$$-1 < x < 1$$

$$(3.63)\qquad P'^m_\ell(0)\Sigma_{n=m}^{\infty}(-1)^{n+m}\gamma_{n,m}P_n^m(x)P_n^m(0)(1/(\ell-n)-1/(\ell+n+1))$$

$$= \tfrac{1}{2}(1-(-1)^{\ell+m})P_\ell^m(x)\varepsilon(x)+\tfrac{1}{2}(1+(-1)^{\ell+m})P_\ell^m(x)$$

$$-1 < x < 1$$

Similar identities follow from (3.57).

Uniform convergence for equations (3.55) - (3.63) is the same as that of the respective parent equation from which each is derived.

CHAPTER 4

APPLICATIONS

We begin with the problem which led to the author's interest in Dougall Identities for integral values of ν.

Define $f(x) \equiv \begin{cases} P^m_\ell(x) & 0 \le x \le 1 \\ 0 & -1 \le x < 0 \end{cases}$

and try to expand $f(x)$ as a series of Legendre associated functions

(4.1) $$f(x) = \Sigma^\infty_{s=o} c_s P^m_s(x)$$

In order to find the $\{c_s\}$ multiply both sides of (4.1) by $P^m_n(x)$, and integrate over $(-1,1)$

$$\therefore \int_{-1}^{1} dx f(x) P^m_n(x) = \Sigma^\infty_{s=o} c_s \int_{-1}^{1} dx P^m_s(x) P^m_n(x)$$

$$= c_n \frac{2}{2n+1} \gamma^{-1}_{n,m} \qquad \text{(by (2.21))}$$

(4.2) $$\therefore \quad c_n = \tfrac{1}{2}(2n+1)\gamma_{n,m} \int_0^1 dx P^m_\ell(x) P^m_n(x)$$

$$= \tfrac{1}{2}\gamma_{n,m}(P^m_n(0)P'^m_\ell(0) - P^m_\ell(0)P'^m_n(0))$$

$$\cdot(1/(\ell-n) - 1/(\ell+n+1))$$

(using (3.45))

Using the c_n of equation (4.2), can we now explicitly sum $\Sigma_n c_n P^m_n(x)$; and if so, does it converge to $f(x)$?

$$\Sigma^\infty_{n=o} c_n P^m_n(x) = \tfrac{1}{2}P'^m_\ell(0)\Sigma^\infty_{n=m}\gamma_{n,m} P^m_n(0) P^m_n(x)(1/(\ell-n) - 1/(\ell+n+1))$$

$$- \tfrac{1}{2}P^m_\ell(0)\Sigma^\infty_{n=m}\gamma_{n,m} P'^m_n(0) P^m_n(x)(1/(\ell-n) - 1/(\ell+n+1))$$

$$= \tfrac{1}{2}\{\tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x)\varepsilon(x) + \tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x)\}$$

$$- \tfrac{1}{2}\{-\tfrac{1}{2}(1+(-1)^{\ell+m})P^m_\ell(x)\varepsilon(x) - \tfrac{1}{2}(1-(-1)^{\ell+m})P^m_\ell(x)\}$$

(by (3.61), (3.60))

$$= \tfrac{1}{2}P_\ell^m(x)\varepsilon(x) + \tfrac{1}{2}P_\ell^m(x)$$

$$= \begin{cases} P_\ell^m(x) & x > 0 \\ \tfrac{1}{2}P_\ell^m(x) & x = 0 \\ 0 & x < 0 \end{cases}$$

We see that the series converges to the required value except at the point of discontinuity; the result at $x = 0$ does not surprise us bearing in mind Theorem 2.1.

Dirichlet Green's Function for the Half Space

We require the following basic results from the theory of Green's functions.

Definition 4.1

Let $\underline{r}$ denote a point in R^3.

We define the Dirichlet Green's Function $G(\underline{r}_1,\underline{r}_2)$ for a volume V bounded by a surface S as follows:

Write

$$G(\underline{r}_1,\underline{r}_2) = -\frac{1}{4\pi|\underline{r}_1-\underline{r}_2|} + U(\underline{r}_1,\underline{r}_2) \qquad \underline{r}_1,\underline{r}_2 \in V$$

Then $U(\underline{r}_1,\underline{r}_2)$, the regular part of the Green's function, satisfies

(i) $\nabla^2 U(\underline{r}_1,\underline{r}_2) = 0$ for all $\underline{r}_1,\underline{r}_2 \in V$

(4.3) (ii) $U(\underline{r}_1,\underline{r}_2) = \dfrac{1}{4\pi|\underline{r}_1-\underline{r}_2|}$ for all $\underline{r}_1,\underline{r}_2 \in S$

(iii) $U(\underline{r}_1,\underline{r}_2) \to 0$ as $\underline{r}_1,\underline{r}_2 \to \infty$

Lemma 4.2 (Uniqueness)

Conversely, if any harmonic function $U(\underline{r}_1,\underline{r}_2)$ satisfies (i), (ii) and (iii) of (4.3) then

$$-\frac{1}{4\pi|\underline{r}_1-\underline{r}_2|} + U(\underline{r}_1,\underline{r}_2)$$

is the Green's function for the volume V.

Proof

Let there be two distinct Green's functions, namely

$$G_1(\underline{r}_1,\underline{r}_2) = -\frac{1}{4\pi|\underline{r}_1-\underline{r}_2|} + U_1(\underline{r}_1,\underline{r}_2)$$

$$G_2(\underline{r}_1,\underline{r}_2) = -\frac{1}{4\pi|\underline{r}_1-\underline{r}_2|} + U_2(\underline{r}_1,\underline{r}_2)$$

Then $G_1 - G_2 = U_1 - U_2$ is a harmonic function everywhere in V and vanishes everywhere on S. Hence (Kellogg, 1929, Chapter VIII, Theorem II)

$$G_1 - G_2 = U_1 - U_2 = 0$$

everywhere in V and on S, and the Green's function is unique.

Lemma 4.3

For any function $\phi(\underline{r})$ harmonic in V

(4.4) $$\phi(\underline{r}_2) = \int_S dS_1 \phi(\underline{r}_1) \frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2) \qquad \underline{r}_2 \in V$$

where $\frac{\partial}{\partial n_1}$ denotes differentiation along the normal out of V.

Proof

See (Kellogg, 1927, page 237).

Corollary 4.4

Choose $\phi(\underline{r})$ to be the regular part of the Green's function. (4.4) then gives (using Definition 4.1 and equation (4.3))

(4.5) $$U(\underline{r}_2,\underline{r}_3) = \int_S dS_1 U(\underline{r}_1,\underline{r}_3) \frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2)$$

$$= \int_S dS_1 \frac{1}{4\pi|\underline{r}_1-\underline{r}_3|} \frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2) \qquad \underline{r}_2,\underline{r}_3 \in V$$

Suppose we are working in an orthogonal curvilinear system of coordinates (u^1,u^2,u^3) in which Laplace's equation $\nabla^2\psi(u^1,u^2,u^3) = 0$ is separable (Moon and Spencer, 1952). The solution can be expressed in the form

$$\psi(u^1,u^2,u^3) = \psi_1(u^1)\psi_2(u^2)\psi_3(u^3)$$

If at least one of the functions $\psi_i(u^i)$ is a Legendre associated function $P_n^m(x_L)$, x_L being a function of u^i alone, then, as previously mentioned in the Introduction, our identities are applicable to bodies bounded by two surfaces one of which is given by $x_L = 0$ and the other is defined independently of x_L.

Consider the half space $z \geq 0$. In spherical polar coordinates (where the solution to Laplace's equation takes the form $r^n P_n^m(\cos\theta)\cos(m\phi)$) the bounding surface $z = 0$ is given by $x_L = \cos\theta = \cos(\frac{1}{2}\pi) = 0$. (The other surface is the hemisphere at infinity.)

We now construct the Dirichlet Green's function for the half space $z \geq 0$ or, in spherical coordinates, $0 \leq \theta \leq \frac{1}{2}\pi$.

$G(\underline{r}_1,\underline{r}_2)$ is the potential at $\underline{r}_1 = (r_1,\theta_1,\phi_1)$ due to a unit point charge at $\underline{r}_2 = (r_2,\theta_2,\phi_2)$ in the presence of an earthed conductor at $z = 0$. By the method of images, if a point charge of -1 be placed at $\underline{r}_2^* = (r_2,\pi-\theta_2,\phi_2)$ which is the reflection of $\underline{r}_2$ in the plane $z = 0$, the potential vanishes everywhere on the plane.

(4.6) $$G(\underline{r}_1,\underline{r}_2) = -\frac{1}{4\pi|\underline{r}_1-\underline{r}_2|} + \frac{1}{4\pi|\underline{r}_1-r_2^*|}$$

Since $\underline{r}_2 = \underline{r}_2^*$ on the plane $z_2 = 0$ (or $\theta_2 = \frac{1}{2}\pi$) $G(\underline{r}_1,\underline{r}_2)$ vanishes there and conditions (4.3) are clearly satisfied for $z_1,z_2 \geq 0$, by the uniqueness theorem

(Lemma 4.2) equation (4.6) is the Green's function for the half-space.

Let us now express (4.6) in spherical coordinates (r,θ,ϕ) using the expansion (Morse and Feshbach, 1953, equation 10.3.37)

$$(4.7) \qquad \frac{1}{|\underline{r}_1-\underline{r}_2|} = \Sigma_{n=o}^{\infty} \frac{r_<^n(1,2)}{r_>^{n+1}(1,2)} \Sigma_{m=o}^{n} \gamma_{n,m} P_n^m(\cos\theta_1) \cdot P_n^m(\cos\theta_2)\varepsilon_m \cos(m(\phi_1-\phi_2))$$

where $r_<(1,2) = \min(r_1,r_2)$

and $r_>(1,2) = \max(r_1,r_2)$

ε_m and $\gamma_{n,m}$ are defined by equations (2.33) and (2.9) respectively.

Recall (2.14), namely

$$P_n^m(\cos(\pi-\theta)) = P_n^m(-\cos\theta) = (-1)^{n+m} P_n^m(\cos\theta) \qquad 0 < \theta < \tfrac{1}{2}\pi$$

$$(4.8) \quad \therefore\ U(\underline{r}_1,\underline{r}_2) = \frac{1}{4\pi|\underline{r}_1-\underline{r}_2^*|} = (1/4\pi)\Sigma_{n=o}^{\infty} \frac{r_<^n(1,2)}{r_>^{n+1}(1,2)} \Sigma_{m=o}^{n} (-1)^{n+m}\gamma_{n,m} \cdot P_n^m(\cos\theta_1)P_n^m(\cos\theta_2)\varepsilon_m\cos(m(\phi_1-\phi_2))$$

$$(4.9) \quad \therefore\ G(\underline{r}_1,\underline{r}_2) = -(1/4\pi)\Sigma_{n=o}^{\infty} \frac{r_<^n(1,2)}{r_>^{n+1}(1,2)} \Sigma_{m=o}^{n} (1-(-1)^{n+m})\gamma_{n,m} \cdot P_n^m(\cos\theta_1)P_n^m(\cos\theta_2)\varepsilon_m\cos(m(\phi_1-\phi_2))$$

In order to demonstrate our identities let us verify that $G(\underline{r}_1,\underline{r}_2)$ satisfies (4.5), namely

(4.10) $$U(\underline{r}_2,\underline{r}_3) = \int_S dS_1 \frac{1}{4\pi|\underline{r}_1-\underline{r}_3|} \frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2)$$

$$0 \leq \theta_2,\theta_3 < \tfrac{1}{2}\pi$$

From (4.7)

(4.11) $$\frac{1}{4\pi|\underline{r}_1-\underline{r}_3|} = (1/4\pi)\Sigma_{n=o}^{\infty} \frac{r_<^n(1,3)}{r_>^{n+1}(1,3)} \Sigma_{q=o}^{n}\gamma_{n,q}$$

$$\cdot P_n^q(\cos\theta_1)P_n^q(\cos\theta_3)\varepsilon_q\cos(q(\phi_1-\phi_3))$$

Also

(4.12) $$\left.\frac{\partial}{\partial n_1}\right)_{\underline{r}_1 \text{ on } S} = -\left.\frac{\partial}{\partial z_1}\right)_{\theta_1=\frac{1}{2}\pi}$$

$$= -\hat{\underline{z}}_1\cdot\nabla_1)_{\theta_1=\frac{1}{2}\pi}$$

$$= -(\cos\theta_1,-\sin\theta_1,0).\left(\frac{\partial}{\partial r_1},\frac{1}{r_1}\frac{\partial}{\partial\theta_1},\frac{1}{r_1\sin\theta_1}\frac{\partial}{\partial\theta_1}\right)\Big)_{\theta_1=\frac{1}{2}\pi}$$

$$= \frac{1}{r_1}\left.\frac{\partial}{\partial\theta_1}\right)_{\theta_1=\frac{1}{2}\pi}$$

$$= -\frac{1}{r_1}\left.\frac{\partial}{\partial(\cos\theta_1)}\right)_{\theta_1=\frac{1}{2}\pi}$$

whence using expansion (4.9)

(4.13) $$\left.\frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2)\right)_{\underline{r}_1 \text{ on } S} = (1/4\pi r_1)\Sigma_{\ell=o}^{\infty} \frac{r_<^{\ell}(1,2)}{r_>^{\ell+1}(1,2)}$$

$$\cdot\Sigma_{m=o}^{\ell}(1-(-1)^{\ell+m})\gamma_{\ell,m}P_\ell^m(\cos\theta_2)$$

$$\cdot P'^m_\ell(0)\varepsilon_m\cos(m(\phi_1-\phi_2))$$

Further

(4.14) $$\int_0^{2\pi} d\phi_1\cos(m(\phi_1-\phi_2))\cos(q(\phi_1-\phi_3))=(2\pi/\varepsilon_m)$$

$$\cdot\delta_{m,q}\cos(m(\phi_2-\phi_3))$$

We will also be using the identity

(4.15) $$\Sigma_{n=o}^{\infty}\Sigma_{m=o}^{n}\delta_{m,q} = \Sigma_{m=o}^{\infty}\Sigma_{n=m}^{\infty}\delta_{m,q}$$

$$= \Sigma_{n=q}^{\infty}$$

In spherical coordinates

$$(4.16)\qquad \int_S dS = \int_0^\infty dr_1 r_1 \int_0^{2\pi} d\phi_1$$

Substituting (4.11) and (4.13) into the dexter of (4.10) and employing (4.16) and (4.14) we obtain

$$(4.17)\quad \text{dexter of } (4.10) = (1/8\pi)\Sigma_{n=o}^{\infty}\Sigma_{q=o}^{n}\Sigma_{\ell=o}^{\infty}\Sigma_{m=o}^{\ell}\delta_{m,q}$$

$$\cdot(1-(-1)^{\ell+m})\gamma_{\ell,m}\gamma_{n,q}\varepsilon_m\cos(m(\phi_2-\phi_3))P'^{m}_{\ell}(0)$$

$$\cdot P^m_\ell(\cos\ \theta_2)P^m_n(0)P^m_n(\cos\ \theta_3)$$

$$\cdot\left(\int_0^\infty dr_1\ \frac{r_<^\ell(1,2)r_<^n(1,3)}{r_>^{\ell+1}(1,2)r_>^{n+1}(1,3)}\right)$$

Let us perform the indicated integration. Without any loss of generality we may assume that $r_2 < r_3$. Further, since $P'^{m}_{\ell}(0)P^m_\ell(0) \equiv 0$ (cf. (3.42)) n can never equal ℓ in (4.17). Thus the integral with respect to r_1 is equal to

$$(4.18)\qquad \int_0^{r_2} dr_1\ \frac{r_1^{\ell+n}}{r_2^{\ell+1}r_3^{n+1}} + \int_{r_2}^{r_3} dr_1\ \frac{r_2^\ell r_1^{n-\ell-1}}{r_3^{n+1}} + \int_{r_3}^{\infty} dr_1\ \frac{r_2^\ell\ r_3^n}{r_1^{n+\ell+2}}$$

$$= -(r_2^n/r_3^{n+1})(1/(n-\ell)-1/(n+\ell+1))-(r_2^\ell/r_3^{\ell+1})$$

$$\cdot(1/(\ell-n)-1/(\ell+n+1))$$

Inserting (4.18) into (4.17) and making use of identity (4.15) for ℓ and n in turn we obtain

$$\text{dexter of } (4.10) = -(1/8\pi)\Sigma_{n=o}^{\infty}(r_2^n/r_3^{n+1})\Sigma_{m=o}^{n}\gamma_{n,m}P^m_n(\cos\ \theta_3)$$

$$\cdot\{P^m_n(0)\Sigma^{\infty}_{\substack{\ell=m\\ \ell\neq n}}(1-(-1)^{\ell+m})\gamma_{\ell,m}P'^{m}_{\ell}(0)P^m_\ell(\cos\ \theta_2)$$

$$\cdot(1/(n-\ell)-1/(n+\ell+1))\}\varepsilon_m\cos(m(\phi_2-\phi_3))$$

$$-(1/8\pi)\Sigma_{\ell=o}^{\infty}(r_2^\ell/r_3^{\ell+1})\Sigma_{m=o}^{\ell}\gamma_{\ell,m}(1-(-1)^{\ell+m})P^m_\ell(\cos\ \theta_2)$$

$$\cdot\{P'^m_\ell(0)\Sigma^{\infty}_{\substack{n=m\\ n\neq\ell}}\gamma_{n,m}P^m_n(0)P^m_n(\cos\theta_3)(1/(\ell-n)-1/(\ell+n+1))\}$$

$$\cdot\varepsilon_m\cos(m(\phi_2-\phi_3))$$

We employ identities (3.55) and (3.58) in the first set of braces and (3.56) in the second (interchanging ℓ and n) to obtain (since $0 \leq \theta_2,\theta_3 < \frac{1}{2}\pi$)

$$\text{dexter of (4.10)} = -(1/8\pi)\Sigma^{\infty}_{n=o}(r_2^n/r_3^{n+1})\Sigma^n_{m=o}\gamma_{n,m}$$

$$\cdot P^m_n(\cos\theta_2)P^m_n(\cos\theta_3)\{-\tfrac{1}{2}(1+(-1)^{n+m})$$

$$-\tfrac{1}{2}(1+(-1)^{n+m})+(1-(-1)^{n+m})\tfrac{1}{2}(1-(-1)^{n+m})\}$$

$$\cdot\varepsilon_m\cos(m(\phi_2-\phi_3))$$

$$= (1/4\pi)\Sigma^{\infty}_{n=o}(r_2^n/r_3^{n+1})\Sigma^n_{m=o}(-1)^{n+m}\gamma_{n,m}$$

$$\cdot P^m_n(\cos\theta_2)P^m_n(\cos\theta_3)\varepsilon_m\cos(m(\phi_2-\phi_3))$$

$$= U(\underline{r}_2,\underline{r}_3) \qquad \text{(see (4.8))}$$

$$= \text{sinister of (4.10) as required.}$$

Dirichlet Green's Function for the Prolate Hemispheroid

In order to demonstrate that our identities are useful for solving problems not only in spherical polar coordinates but also in all other coordinate system in which the solution to Laplace's equation is given in terms of Legendre associated functions, we now carry out a similar calculation in prolate spheroidal coordinates to verify explicitly identity (4.5) for a prolate hemispheroid.

In prolate spheroidal coordinates (ξ,η,ϕ) the solution to Laplace's equation has the form $P^m_n(\eta)Q^m_n(\xi)\cos(m\,\phi)$.

The plane face of the prolate hemispheroid is the surface $\eta = 0$; the curved surface is given, independently of η as required, by the equation $\xi = \alpha > 1$. Thus we are once again in a situation in which our identities may be applied.

Prolate spheroidal coordinates are formed by rotating elliptical coordinates about the major axis. Suppose that the foci of the spheroid are at $x = 0$, $y = 0$, $z = \pm\frac{1}{2}a$.

Define the prolate spheroidal coordinate system (ξ,η,ϕ) as follows:

$$x = \tfrac{1}{2}a\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi$$

$$y = \tfrac{1}{2}a\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi$$

$$z = \tfrac{1}{2}a\,\xi\eta$$

with

$$1 \le \xi \le \infty, \quad -1 \le \eta \le 1, \quad 0 \le \phi \le 2\pi$$

then

$$h_\xi = \tfrac{1}{2}a\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}} \tag{4.19}$$

$$h_\eta = \tfrac{1}{2}a\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}$$

$$h_\phi = \tfrac{1}{2}a\sqrt{(\xi^2-1)(1-\eta^2)}$$

In this system (Morse and Feshbach, 1953, equation 10.3.53; but see note)

$$\frac{1}{|\underline{r}_1-\underline{r}_3|} = (2/a)\Sigma_{\ell=0}^{\infty}(2\ell+1)\Sigma_{\ell=0}^{q}(-1)^q\gamma_{\ell,q}^2 P_\ell^q(\eta_1)P_\ell^q(\eta_3) \tag{4.20}$$

$$\cdot P_\ell^q(\xi_<(1,3))Q_\ell^q(\xi_>(1,3))\varepsilon_q\cos(q(\phi_1-\phi_3))$$

where $\xi_<(1,3) = \min(\xi_1,\xi_3)$

$\xi_>(1,3) = \max(\xi_1,\xi_3)$

The prolate hemispheroid is the body bounded by the surfaces $r = a$, $z \geq 0$ and $r \leq a$, $z = 0$, or in prolate spheroidal coordinates

(4.21) $$S_H = \{(\xi,\eta,\phi) | \xi = \alpha,\ 0 \leq \eta \leq 1,\ 0 \leq \phi \leq 2\pi\}$$

and

$$S_F = \{(\xi,\eta,\phi) |\ 1 \leq \xi \leq \alpha,\ \eta = 0,\ 0 \leq \phi \leq 2\pi\}$$

where α is a constant greater than 1.

Hence

(4.22) $$G(\underline{r}_1,\underline{r}_2) = -(1/2\pi a)\Sigma_{n=o}^{\infty}(2n+1)\Sigma_{m=o}^{n}(-1)^m$$

$$\cdot(1-(-1)^{n+m})\gamma_{n,m}^2 P_n^m(\eta_1)P_n^m(\eta_2)\{P_n^m(\xi_<)Q_n^m(\xi_>)$$

$$- P_n^m(\xi_1)P_n^m(\xi_2)(Q_n^m(\alpha)/P_n^m(\alpha))\}\varepsilon_m\cos(m(\phi_1-\phi_2))$$

Clearly $G(\underline{r}_1,\underline{r}_2)$ vanishes for η_1 or $\eta_2 = 0$ and $\xi_> = \alpha$, and has the required $-\frac{1}{4\pi|\underline{r}_1-\underline{r}_2|}$ singularity for $\underline{r}_1$ near $\underline{r}_2$. By Lemma 4.2, (4.22) is the Dirichlet Green's function for the interior of the prolate hemispheroid.

Let us verify that $G(\underline{r}_1,\underline{r}_2)$ satisfies (4.5), namely

(4.23) $$U(\underline{r}_2,\underline{r}_3) = \int_{S_H \cup S_F} dS_1 \frac{1}{4\pi|\underline{r}_1-\underline{r}_3|} \frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2)$$

$$0 < \eta_2,\eta_3 < 1$$

We have that

(4.24) $$\left.\frac{\partial}{\partial n_1}\right)_{\underline{r}_1 \text{ on } S_H} = \frac{1}{h_\xi}\left.\frac{\partial}{\partial \xi_1}\right)_{\xi_1=\alpha}$$

The sign of (4.24) follows from the fact that the outward normal goes from $\xi < \alpha$ to $\xi > \alpha$, and hence the derivative must be taken in the direction of ξ increasing.

Now

$$(4.25)\quad \frac{\partial}{\partial \xi_1}\{P_n^m(\xi_2)Q_n^m(\xi_1)-P_n^m(\xi_1)P_n^m(\xi_2)(Q_n^m(\alpha)/P_n^m(\alpha))\})_{\xi_1=\alpha}$$

$$= P_n^m(\xi_2)\{Q'^m_n(\alpha)P_n^m(\alpha)-Q_n^m(\alpha)P'^m_n(\alpha)\}/P_n^m(\alpha)$$

$$= -(P_n^m(\xi_2)/P_n^m(\alpha))\gamma_{n,m}^{-1}(-1)^m/(\alpha^2-1)$$

since by (2.10) the Wronskian of $P_n^m(z)$ and $Q_n^m(z)$ is given by

$$(4.26)\quad Q'^m_n(\alpha)P_n^m(\alpha)-Q_n^m(\alpha)P'^m_n(\alpha)\} = \gamma_{n,m}^{-1}(-1)^m/(1-\alpha^2)$$

From (4.22) and (4.25) we see that

$$(4.27)\quad \frac{\partial}{\partial \xi_1} G(\underline{r}_1,\underline{r}_2))_{\xi_1=\alpha} = (1/2\pi a)\Sigma_{n=o}^{\infty}(2n+1)\Sigma_{m=o}^{n}$$

$$\cdot(1-(-1)^{n+m})\gamma_{n,m}P_n^m(\eta_1)P_n^m(\eta_2)(P_n^m(\xi_2)/$$

$$\cdot P_n^m(\alpha))\varepsilon_m\cos(m(\phi_1-\phi_2))/(\alpha^2-1)$$

Let us split up the dexter of (4.23) into I_1 and I_2, where

$$(4.28)\quad I_1 = \int_{S_H} dS_1 \frac{1}{4\pi|\underline{r}_1-\underline{r}_3|}\frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2)$$

$$(4.29)\quad I_2 = \int_{S_F} dS_1 \frac{1}{4\pi|\underline{r}_1-r_3|}\frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2)$$

(S_H and S_F were defined in (4.21)).

From (4.28), using (4.19) and (4.24)

$$(4.30)\quad I_1 = \tfrac{1}{2}a\int_0^1 d\eta_1\int_0^{2\pi} d\phi_1(h_\eta h_\phi/h_\xi)(\frac{1}{4\pi|\underline{r}_1-\underline{r}_3|}\frac{\partial}{\partial \xi_1}$$

$$\cdot G(\underline{r}_1\underline{r}_2))_{\xi_1=\alpha}$$

$$= \tfrac{1}{2}a(\alpha^2-1)\int_0^1 d\eta_1\int_0^{2\pi} d\phi_1(\frac{1}{4\pi|\underline{r}_1-\underline{r}_3|}\frac{\partial}{\partial \xi_1}G(\underline{r}_1,\underline{r}_2))_{\xi_1=\alpha}$$

Substitute for $\frac{1}{|\underline{r}_1-\underline{r}_3|}$ from (4.20) and for $\frac{\partial G}{\partial \xi_1}$ from (4.27) to give (using (4.15))

$$(4.31)\qquad I_1 = (1/4\pi a)\Sigma_{\ell=o}^{\infty}\Sigma_{m=o}^{\ell}\Sigma_{n=m}^{\infty}(2\ell+1)(2n+1)(-1)^m(1-(-1)^{n+m})$$

$$\cdot\gamma_{\ell,m}^2\gamma_{n,m}P_\ell^m(\eta_3)P_n^m(\eta_2)(\int_0^1 d\eta_1 P_\ell^m(\eta_1)P_n^m(\eta_1))$$

$$\cdot P_\ell^m(\xi_3)Q_\ell^m(\alpha)(P_n^m(\xi_2)/P_n^m(\alpha))\varepsilon_m\cos(m(\phi_2-\phi_3))$$

$$= (1/4\pi a)\Sigma_{\ell=o}^{\infty}(2\ell+1)\Sigma_{m=o}^{\ell}(-1)^m\gamma_{\ell,m}^2P_\ell^m(\eta_3)P_\ell^m(\xi_3)Q_\ell^m(\alpha)$$

$$\cdot\{\Sigma_{n=m}^{\infty}(1-(-1)^{n+m})\gamma_{n,m}(1/(\ell-n)-1/(\ell+n+1))P_n^m(\eta_2)$$

$$\cdot(P_n^m(\xi_2)/P_n^m(\alpha))(P'^m_\ell(0)P_n^m(0)-P_\ell^m(0)P'^m_n(0)\}$$

$$\cdot\varepsilon_m\cos(\phi_2-\phi_3)$$

where we have used (3.45) to evaluate the integral.

Consider the function $g(\xi_2,\eta_2,\phi_2)$ defined by

$$(4.32)\qquad g(\xi_2,\eta_2,\phi_2) = \{P_\ell^m(0)\Sigma_{n=m}^{\infty}(1-(-1)^{n+m})\gamma_{n,m}P'^m_n(0)P_n^m(\eta_2)$$

$$\cdot(P_n^m(\xi_2)/P_n^m(\alpha))(1/(\ell-n)-1/(\ell+n+1))$$

$$+\ 2(P_\ell^m(\xi_2)/P_\ell^m(\alpha))P_\ell^m(\eta_2)\}\cos(m\ \phi_2)$$

By (3.60) and (3.62) (since from (4.23) $0 < \eta_2 < 1$) $g(\xi_2,\eta_2,\phi_2)$ vanishes on the surface $\xi_2 = \alpha$; g is a harmonic function of (ξ_2,η_2,ϕ_2) and hence

$$(4.33)\qquad g(\xi_2,\eta_2,\phi_2) = 0$$

inside its radius of convergence (i.e. for $\xi_2 \leq \alpha$), or

$$(4.34)\qquad P_\ell^m(0)\Sigma_{n=m}^{\infty}(1-(-1)^{n+m})\gamma_{n,m}P'^m_n(0)P_n^m(\eta_2)(P_n^m(\xi_2)/$$

$$\cdot P_n^m(\alpha))(1/(\ell-n)-1/(\ell+n+1))$$

$$= -2P_\ell^m(\eta_2)(P_\ell^m(\xi_2)/P_\ell^m(\alpha))$$

Similarly (using (3.61) and (3.63))

(4.35) $$P'^{m}_{\ell}(0)\Sigma^{\infty}_{n=m}(1-(-1)^{n+m})\gamma_{n,m}P^{m}_{n}(0)P^{m}_{n}(\eta_2)(P^{m}_{n}(\xi_2)/$$

$$\cdot P^{m}_{n}(\alpha))(1/(\ell-n)-1/(\ell+n+1)) = 0$$

Substitute (4.34) and (4.35) into (4.31) to give

(4.36) $$I_1 = (1/2\pi a)\Sigma^{\infty}_{\ell=o}(2\ell+1)\Sigma^{\ell}_{m=o}(-1)^{m}\gamma^{2}_{\ell,m}P^{m}_{\ell}(\eta_2)P^{m}_{\ell}(\eta_3)$$

$$\cdot(Q^{m}_{\ell}(\alpha)/P^{m}_{\ell}(\alpha))P^{m}_{\ell}(\xi_2)P^{m}_{\ell}(\xi_3)\varepsilon_{m}\cos(m(\phi_2-\phi_3))$$

Now consider I_2 (equation (4.28))

(4.37) $$\frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2))_{\underline{r}_1 \text{ on } S_F} = -\frac{1}{h_\eta}\frac{\partial}{\partial \eta_1} G(\underline{r}_1,\underline{r}_2))_{\eta_1=0}$$

The minus sign follows from the fact that the outward normal is from $z > 0$ to $z < 0$, i.e. from $\eta > 0$ to $\eta < 0$.

Now

(4.38) $$\frac{\partial}{\partial n_1} G(\underline{r}_1,\underline{r}_2) = -(1/2\pi a)\Sigma^{\infty}_{n=o}(2n+1)\Sigma^{n}_{m=o}(-1)^{m}$$

$$\cdot(1-(-1)^{n+m})\gamma^{2}_{n,m}P'^{m}_{n}(0)P^{m}_{n}(\eta_2)\{P^{m}_{n}(\xi_<)Q^{m}_{n}(\xi_>)$$

$$- P^{m}_{n}(\xi_1)P^{m}_{n}(\xi_2)(Q^{m}_{n}(\alpha)/P^{m}_{n}(\alpha))\}\varepsilon_{m}\cos(m(\phi_1-\phi_2))$$

where $G(\underline{r}_1,\underline{r}_2)$ is given by equation (4.22).

From (4.19), for $\eta = 0$

(4.39) $$h_\phi h_\xi/h_\eta = \tfrac{1}{2}a$$

Substitute (4.20), (4.37), (4.38) and (4.39) into (4.29) to give

(4.40) $$I_2 = -\int_1^\alpha d\xi_1\int_0^{2\pi} d\phi_1(h_\phi h_\xi/h_\eta)\left(\frac{1}{4\pi|\underline{r}_1-\underline{r}_3|}\;\frac{\partial}{\partial \eta_1}G(\underline{r}_1,\underline{r}_2)\right)_{\eta_1=0}$$

$$= (1/4\pi a)\Sigma^{\infty}_{\ell=o}\Sigma^{\ell}_{m=o}\Sigma^{\infty}_{n=o}\Sigma^{n}_{q=o}\delta_{q,m}(-1)^{m+q}(2\ell+1)(2n+1)$$

$$\cdot(1-(-1)^{n+m})\gamma^{2}_{\ell,q}\gamma^{2}_{n,m}P^{q}_{\ell}(0)P^{q}_{\ell}(\eta_3)P'^{m}_{n}(0)P^{m}_{n}(\eta_2)$$

$$\cdot\left(\int_1^{\alpha} d\xi_1 \{P_n^m(\xi_<(1,2))Q_n^m(\xi_>(1,2)) - P_n^m(\xi_1)P_n^m(\xi_2)\right.$$

$$\left.\cdot(Q_n^m(\alpha)/P_n^m(\alpha))\}P_\ell^q(\xi_<(1,3))Q_\ell^q(\xi_>(1,3))\varepsilon_m \cos(m(\phi_2-\phi_3)\right)$$

It is clear from the presence of the product $\delta_{q,m}P_\ell^q(0)P'^m_n(0)$ that n cannot equal ℓ. Suppose, without any loss of generality, that $\xi_2 < \xi_3$. Then the integration over ξ_1 can be written as

(4.41) $$I_\xi = \int_1^{\xi_2} d\xi_1 \{P_n^m(\xi_1)Q_n^m(\xi_2) - P_n^m(\xi_1)P_n^m(\xi_2)(Q_n^m(\alpha)/$$

$$\cdot P_n^m(\alpha))\}P_\ell^m(\xi_1)Q_\ell^m(\xi_3)$$

$$+ \int_{\xi_2}^{\xi_3} d\xi_1 \{P_n^m(\xi_2)Q_n^m(\xi_1) - P_n^m(\xi_1)P_n^m(\xi_2)(Q_n^m(\alpha)/$$

$$\cdot P_n^m(\alpha))\}P_\ell^m(\xi_1)Q_\ell^m(\xi_3)$$

$$+ \int_{\xi_3}^{\alpha} d\xi_1 \{P_n^m(\xi_2)Q_n^m(\xi_1) - P_n^m(\xi_1)P_n^m(\xi_2)(Q_n^m(\alpha)/$$

$$\cdot P_n^m(\alpha))\}P_\ell^m(\xi_3)Q_\ell^m(\xi_1)$$

$$= Q_n^m(\xi_2)Q_\ell^m(\xi_3)\int_1^{\xi_2} d\xi_1 P_n^m(\xi_1)P_\ell^m(\xi_1)$$

$$- (Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_2)Q_\ell^m(\xi_3)\int_1^{\xi_3} d\xi_1 P_n^m(\xi_1)P_\ell^m(\xi_1)$$

$$+ P_n^m(\xi_2)Q_\ell^m(\xi_3)\int_{\xi_2}^{\xi_3} d\xi_1 Q_n^m(\xi_1)P_\ell^m(\xi_1)$$

$$+ P_n^m(\xi_2)P_\ell^m(\xi_3)\int_{\xi_3}^{\alpha} d\xi_1 Q_n^m(\xi_1)Q_\ell^m(\xi_1)$$

$$- (Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_2)P_\ell^m(\xi_3)\int_{\xi_3}^{\alpha} d\xi_1 P_n^m(\xi_1)Q_\ell^m(\xi_1)$$

Now by (2.20) if M_n^m, M_ℓ^m are any two solutions of Legendre's differential equation (2.1)

$$(4.42)\qquad (n-\ell)(n+\ell+1)\int_a^b M_n^m(z)M_\ell^m(z)dz$$

$$= \left[z(n-\ell)M_n^m(z)M_\ell^m(z)+(\ell+m)M_n^m(z)M_{\ell-1}^m(z) - (n+m)M_{n-1}^m(z)M_\ell^m(z)\right]_a^b$$

But from (2.31)

$$(4.43)\qquad (\ell+m)M_{\ell-1}^m(z) = \ell z M_\ell^m(z) - (z^2-1)M'^m_\ell(z)$$

and dexter of (4.42) becomes

$$(4.44)\qquad \left[(z^2-1)(M_\ell^m(z)M'^m_n(z) - M_n^m(z)M'^m_\ell(z)\right]_a^b$$

Inserting this result into (4.41) we see that

$$(4.45)\qquad (n-\ell)(n+\ell+1)I_\xi$$

$$= Q_n^m(\xi_2)Q_\ell^m(\xi_3)(\xi_2^2-1)\{P_\ell^m(\xi_2)P'^m_n(\xi_2)-P_n^m(\xi_2)P'^m_\ell(\xi_2)\}$$

$$- (Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_2)Q_\ell^m(\xi_3)(\xi_3^2-1)\{P_\ell^m(\xi_3)P'^m_n(\xi_3) -P_n^m(\xi_3)P'^m_\ell(\xi_3)\}$$

$$+ P_n^m(\xi_2)Q_\ell^m(\xi_3)(\xi_3^2-1)\{P_\ell^m(\xi_3)Q'^m_n(\xi_3)-Q_n^m(\xi_3)P'^m_\ell(\xi_3)\}$$

$$- P_n^m(\xi_2)Q_\ell^m(\xi_3)(\xi_2^2-1)\{P_\ell^m(\xi_2)Q'^m_n(\xi_2)-Q_n^m(\xi_2)P'^m_\ell(\xi_2)\}$$

$$+ P_n^m(\xi_2)P_\ell^m(\xi_3)(\alpha^2-1)\{Q_\ell^m(\alpha)Q'^m_n(\alpha) - Q_n^m(\alpha)Q'^m_\ell(\alpha)\}$$

$$- P_n^m(\xi_2)P_\ell^m(\xi_3)(\xi_3^2-1)\{Q_\ell^m(\xi_3)Q'^m_n(\xi_3)-Q_n^m(\xi_3)Q'^m_\ell(\xi_3)\}$$

$$- (Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_2)P_\ell^m(\xi_3)(\alpha^2-1)\{Q_\ell^m(\alpha)P'^m_n(\alpha) -P_n^m(\alpha)Q'^m_\ell(\alpha)\}$$

$$+ (Q_n^m(\alpha))/P_n^m(\alpha)P_n^m(\xi_2)P_\ell^m(\xi_3)(\xi_3^2-1)\{Q_\ell^m(\xi_3)P'^m_n(\xi_3) -P_n^m(\xi_3)Q'^m_\ell(\xi_3)\}$$

$$= -(\xi_2^2-1)P_\ell^m(\xi_2)Q_\ell^m(\xi_3)\{Q'^m_n(\xi_2)P_n^m(\xi_2)-Q_n^m(\xi_2)P'^m_n(\xi_2)\}$$

$$+ (\alpha^2-1)(Q_\ell^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_2)P_\ell^m(\xi_3)\{Q'^m_n(\alpha)P_n^m(\alpha)-Q_n^m(\alpha)P'^m_n(\alpha)\}$$

$$+ (\xi_3^2-1)P_n^m(\xi_2)Q_n^m(\xi_3)\{Q'^m_\ell(\xi_3)P_\ell^m(\xi_3)-Q_\ell^m(\xi_3)P'^m_\ell(\xi_3)\}$$

$$- (\xi_3^2-1)(Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_2)P_n^m(\xi_3)\{Q'^m_\ell(\xi_3)P_\ell^m(\xi_3) -Q_\ell^m(\xi_3)P'^m_\ell(\xi_3)\}$$

$$= \gamma_{n,m}^{-1}(-1)^m\{Q_\ell^m(\xi_3)P_\ell^m(\xi_2)-(Q_\ell^m(\alpha)/P_n^m(\alpha))P_\ell^m(\xi_3)P_n^m(\xi_2)\}$$

$$- \gamma_{\ell,m}^{-1}(-1)^m P_n^m(\xi_2)\{Q_n^m(\xi_3)-(Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_3)\}$$

where we have used (4.26).

∴ (4.40) gives (using (4.15))

(4.46) $$I_2 = -(1/4\pi a)\Sigma_{\ell=o}^{\infty}(2\ell+1)\Sigma_{m=o}^{\ell}(-1)^m\gamma_{\ell,m}^2 P_\ell^m(\eta_3) \cdot\{Q_\ell^m(\xi_3)P_\ell^m(\xi_2) \cdot\{P_\ell^m(0)\Sigma_{\substack{n\geq m\\ n\neq\ell}}^{\infty}(1-(-1)^{n+m})\gamma_{n,m}P'^m_n(0)P_n^m(\eta_2) \cdot(1/(\ell-n)-1/(\ell+n+1))\}$$
$$- P_\ell^m(\xi_3)Q_\ell^m(\alpha)\{P_\ell^m(0)\Sigma_{\substack{n\geq m\\ n\neq\ell}}^{\infty}(1-(-1)^{n+m})\gamma_{n,m} \cdot(P_n^m(\xi_2)/P_n^m(\alpha))P'^m_n(0) \cdot P_n^m(\eta_2)(1/(\ell-n)-1/(\ell+n+1))\}\}\varepsilon_m\cos(m(\phi_2-\phi_3))$$
$$- (1/4\pi a)\Sigma_{n=o}^{\infty}(2n+1)\Sigma_{m=o}^{n}(-1)^m(1-(-1)^{n+m})\gamma_{n,m}^2 \cdot P_n^m(\eta_2)P_n^m(\xi_2)\{Q_n^m(\xi_3)-(Q_n^m(\alpha)/P_n^m(\alpha))P_n^m(\xi_3)\} \cdot\{P'^m_n(0)\Sigma_{\substack{\ell\geq m\\ \ell\neq n}}^{\infty}\gamma_{\ell,m}P_\ell^m(0)P_\ell^m(\eta_3)(1/(n-\ell)-1/(n+\ell+1))\} \cdot\varepsilon_m\cos(m(\phi_2-\phi_3))$$

The first sum is evaluated by taking the difference between (3.55) and (3.58) to give

(4.47) $$P_\ell^m(0)\Sigma_{\substack{n\geq m\\ n\neq\ell}}^{\infty}(1-(-1)^{n+m})\gamma_{n,m}P'^m_n(0)P_n^m(\eta_2)$$

$$\cdot(1/(\ell-n)-1/(\ell+n+1))$$

$$= -(1+(-1)^{\ell+m})P_\ell^m(\eta_2)$$

(since $\eta_2 > 0$ from (4.23)).

Using a technique similar to that employed in deriving (4.34) we obtain for the second sum (from (3.55) and (3.58))

(4.48) $$P_\ell^m(0)\Sigma_{\substack{n\geq m\\ n\neq\ell}}^{\infty}(1-(-1)^{n+m})\gamma_{n,m}P'^m_n(0)P_n^m(\eta_2)(P_n^m(\xi_2)/P_n^m(\alpha))$$

$$\cdot(1/(\ell-n)-1/(\ell+n+1))$$

$$= -(1+(-1)^{\ell+m})P_\ell^m(\eta_2)(P_\ell^m(\xi_2)/P_\ell^m(\alpha))$$

The third sum of (4.46) is evaluated using (3.56), namely

(4.49) $$P'^m_n(0)\Sigma_{\substack{\ell\geq m\\ \ell\neq n}}^{\infty}\gamma_{\ell,m}P_\ell^m(0)P_\ell^m(\eta_3)(1/(n-\ell)-1/(n+\ell+1))$$

$$= \tfrac{1}{2}(1-(-1)^{n+m})P_n^m(\eta_3)$$

Substituting (4.47) - (4.49) into (4.46) we see that

(4.50) $$I_2 = -(1/4\pi a)\Sigma_{\ell=o}^{\infty}(2\ell+1)\Sigma_{m=o}^{\ell}(-1)^m\gamma_{\ell,m}^2P_\ell^m(\eta_2)P_\ell^m(\eta_3)$$

$$\cdot P_\ell^m(\xi_2)\{Q_\ell^m(\xi_3)-P_\ell^m(\xi_3)(Q_\ell^m(\alpha)/P_\ell^m(\alpha))\}$$

$$\cdot\{-(1+(-1)^{\ell+m})+(1-(-1)^{\ell+m})\tfrac{1}{2}(1-(-1)^{\ell+m})\}$$

$$\cdot\varepsilon_m\cos(m(\phi_2-\phi_3))$$

$$= (1/2\pi a)\Sigma_{\ell=o}^{\infty}(2\ell+1)\Sigma_{m=o}^{\ell}(-1)^m(-1)^{\ell+m}\gamma_{\ell,m}^2P_\ell^m(\eta_2)$$

$$\cdot P_\ell^m(\eta_3)P_\ell^m(\xi_2)\{Q_\ell^m(\xi_3)-P_\ell^m(\xi_3)(Q_\ell^m(\alpha)/P_\ell^m(\alpha))\}$$

$$\cdot\varepsilon_m\cos(m(\phi_2-\phi_3))$$

Adding (4.36) and (4.50) and comparing with (4.22) we see that

$$\text{dexter of } (4.23) = I_1 + I_2$$

$$= U(\underline{r}_2, \underline{r}_3)$$

$$= \text{sinister of } (4.23) \text{ as required.}$$

CHAPTER 5

EXTENSION TO OTHER POLYNOMIALS

In this chapter we study certain functions $p_\nu(x)$ which satisfy Dougall identities analogous to (2.35) - (2.37), namely

$$D_\nu p_\nu(x)p_\nu(y) = (\sin(\nu\pi)/\pi)\Sigma_{n=0}^{\infty}(-1)^n D_n p_n(x)p_n(y)$$
$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

for $\nu \in R\sim I$, D_ν constant, $x,y \in X \subset (a,b)$,

and investigate whether or not these functions also satisfy identities of the type (3.55) and (3.56), viz.

$$p_\ell(x) = \Sigma_{\substack{n=0\\n\neq\ell}}^{\infty} S_n^\ell(x_0)p_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

for $\ell \in N$, for some $x_0 \in X \subset (a,b)$, $S_n^\ell(x_0)$ constant. We will refer to the former as D-type identities, and to the latter as S-type identities.

Meulenbeld and van de Wetering (1967) have derived a D-type identity for the generalized Legendre associated function (or GLAF for short) $P_\nu^{m,n}(x)$. Hence those functions which can be expressed in terms of GLAFs (e.g. the Jacobi function $P_\nu^{(\alpha,\beta)}$) will satisfy a D-type identity for non-integral ν, as will those which can be expressed in terms of Legendre associated functions (such as Gegenbauer or ultraspherical functions $C_\nu^{(\lambda)}$). Thus the set $\{p_\nu(x)\}$ is certainly non-trivial.

In searching for such functions we will consider only those functions which are generalizations of the classical orthogonal polynomials $p_n(x)$ to the case of non-integral n by expressing $p_n(x)$ in terms of the hypergeometric function and replacing n by ν, or by considering the original differential equation defining $p_n(x)$ and making the same replacement. We shall refer to such functions as generalized orthogonal functions. Before following this programme we must recall a few basic properties of orthogonal polynomials in order to generalize them later.

Properties of orthogonal polynomials

(The results of this and the following subsection are taken from (Erdélyi, 1953, Chapter 10) and (Szegö, 1939)).

Let $\{p_n(x) | n = 0,1,\ldots\}$ be a sequence of polynomials of exact degree n, defined on the interval (a,b). Further let $w(x)$, the weight function, be a non-negative function (measurable in the Lebesgue sense) for which

$$\int_a^b dx w(x) > 0$$

Then if $\int_a^b dx w(x) p_i(x) p_j(x)$ exists for all $i,j \in N$ (in Lebesgue's sense) we may define the scalar product

$$(p_i, p_j) = \int_a^b dx w(x) p_i(x) p_j(x)$$

If

$$(p_i, p_j) = 0 \quad \text{for} \quad i \neq j$$

then the sequence $\{p_n(x)\}$ is said to be a system of orthogonal polynomials.

We can show that every orthogonal polynomial system is complete on (a,b) if the interval is finite.

The classical polynomials have the intervals and weight functions given in the table below.

	a	b	w(x)	Name
(i)	0	∞	$e^{-x}x^{\alpha}$ $\alpha>-1$	Laguerre polynomial $L_n^{(\alpha)}(x)$
(ii)	$-\infty$	∞	e^{-x^2}	Hermite polynomial $H_n(x)$
(iii)	-1	1	$(1-x)^{\alpha}(1+x)^{\beta}$ $\alpha,\beta>-1$	Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$

Some special cases of (iii) are

(iv)	$\alpha = \beta = \lambda - \frac{1}{2}$	Ultraspherical (Gegenbauer) polynomial $C_n^{(\lambda)}(x)$
(v)	$\alpha = \beta = -\frac{1}{2}$	Chebychev polynomial of 1st kind $T_n(x)$
(vi)	$\alpha = \beta = \frac{1}{2}$	Chebychev polynomial of 2nd kind $U_n(x)$
(vii)	$\alpha = -\beta = \frac{1}{2}$	Polynomial $U_{2n}(\cos(\frac{1}{2}\theta))=\sin((n+\frac{1}{2})\theta)/\cdot\sin(\frac{1}{2}\theta)$ of $\cos\theta = x$
(viii)	$\alpha = \beta = 0$	Legendre polynomial $P_n(x)$

These polynomials are characterized by three major properties:

(I) $\{p'_n(x)\}$ is a system of orthogonal polynomials

(II) $p_n(x)$ satisfies a differential equation of the form

(5.1) $A(x)y'' + B(x)y' + \lambda_n y$

where $A(x)$ and $B(x)$ are independent of n, and λ_n is independent of x.

(III) There is a generalized Rodrigues formula

(5.2) $$p_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n} (w(x)X^n)$$

where K_n is a constant and X is a polynomial of degree at most 2 in x whose coefficients are independent of n.

Conversely, any one of these properties characterizes the classical orthogonal polynomial in the sense that any system of orthogonal polynomials which has one of these properties can be reduced to a classical system. Thus by considering only the classical orthogonal polynomials we nevertheless are including a wide range of functions.

From (5.2) we can deduce that the differential equation for $y = p_n(x)$ has the form

(5.3) $$X \frac{d^2y}{dx^2} + K_1 p_1(x) \frac{dy}{dx} + \lambda_n y = 0$$

where

(5.4) $$\lambda_n = -n(k_1 K_1 + \tfrac{1}{2}(n-1)X'')$$

with k_1 the coefficient of x in $p_1(x)$, K_1 and X as defined in (5.2).

The self-adjoint form of the differential equation is

(5.5) $$\frac{d}{dx} (Xw(x) \frac{dy}{dx}) + \lambda_n w(x)y = 0$$

Since X is at most quadratic in x, and $p_1(x)$ is a linear function of x, (5.3) can be reduced to the hypergeometric equation or one of its special or limiting cases.

The recurrence and differentiation formulae

Let k_n, $\tilde{k}_n$ be the coefficients of x^n, x^{n-1} respectively in $p_n(x)$; $r_n = \tilde{k}_n/k_n$ and $h_n = (p_n,p_n)$. Then (Szegö, 1939, equation (3.2.1))

(5.6) $$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \quad n = 0,1,2,\ldots$$

with $p_{-1}(x) = 0$

where

$$A_n = k_{n+1}/k_n$$

(5.7) $$B_n = A_n(r_{n+1} - r_n)$$

$$C_n = A_n h_n/(A_{n-1}h_{n-1}) = k_{n+1}k_{n-1}h_n/(k_n^2 h_{n-1})$$

This can be shown as follows:

(5.8) $$p_{n+1}(x) - A_n(x)xp_n(x)$$

$$= (k_{n+1}x^{n+1} + \tilde{k}_{n+1}x^n + \ldots) - (k_{n+1}/k_n)x(k_n x^n + \tilde{k}_n x^{n-1} + \ldots)$$

and is hence a polynomial of degree n or less. By the completeness property of the orthogonal polynomial $p_n(x)$ we may therefore expand (5.8) as

$$p_{n+1}(x) - A_n(x)xp_n(x) = \Sigma_{q=o}^{n} \gamma_q p_q(x)$$

By the orthogonal property of $\{p_n(x)\}$

$$(p_{n+1}, p_q) = 0 \qquad q \leq n$$

and $$(xp_n, p_q) = (p_n, xp_q)$$

$$= 0 \qquad q \leq n - 2$$

Thus

$$\gamma_0 = \gamma_1 = \ldots = \gamma_{n-2} = 0$$

and

$$-A_n(p_n, xp_{n-1}) = \gamma_{n-1}(p_{n-1}, p_{n-1})$$

$$= \gamma_{n-1}h_{n-1}$$

now $xp_{n-1}(x) - (k_{n-1}/k_n)p_n(x)$ is a polynomial of degree $n-1$ or less, and therefore is orthogonal to p_n, whence

$$-A_n h_n (k_{n-1}/k_n) = \gamma_{n-1} h_{n-1}$$

or
$$\gamma_{n-1} = -C_n$$

Compare coefficients of x^n on both sides of (5.6). We obtain

$$\tilde{k}_{n+1} = A_n \tilde{k}_n + B_n k_n$$

or
$$r_{n+1} k_{n+1} = A_n r_n k_n + B_n k_n$$

or
$$r_{n+1} A_n = A_n r_n + B_n$$

whence $B_n = A_n (r_{n+1} - r_n)$ as required by (5.7).

We now come to the differentiation formula (E 10.7(4))

(5.9)
$$X \frac{dp_n(x)}{dx} = (\alpha_n + \tfrac{1}{2} n X'' x) p_n(x) + \beta_n p_{n-1}(x)$$

where

(5.10)
$$\alpha_n = nX'(0) - \tfrac{1}{2} X'' r_n$$

$$A_n \beta_n = -C_n (k_1 K_1 + (n - \tfrac{1}{2}) X'')$$

where X, K_1 are defined in (5.2), r_n and k_1 in (5.6) and A_n, C_n in (5.7).

The proof of (5.9) appears in (Tricomi, 1948, pages 212-215) and is based on the fact that

$$Xp'_n(x) - \tfrac{1}{2} n X'' x p_n(x)$$

is a polynomial of degree at most n and hence is of the form

$$\alpha_n p_n(x) + \beta_n p_{n-1}(x) + \Sigma_{q=o}^{n-2} \gamma_q p_q(x)$$

The coefficients α_n, γ_q are then determined by the orthogonality property; (5.5) is used to determine β_n.

Generalized orthogonal functions

Since $p_n(x)$, the solution of (5.3), is a hypergeometric function or one of its special or limiting cases, let us define the generalized orthogonal function $z = p_\nu(x)$ to be the solution of the equation (cf. (5.3))

$$X \frac{d^2z}{dx^2} + K_1 p_1(x) \frac{dz}{dx} + \lambda_\nu z = 0 \tag{5.11}$$

where

$$\lambda_\nu = -\nu(k_1 K_1 + \tfrac{1}{2}(\nu-1)X'')$$

Further, by replacing n by ν in (5.6) and (5.9) we obtain

$$p_{\nu+1}(x) = (A_\nu x + B_\nu)p_\nu(x) - C_\nu p_{\nu-1}(x), \quad p_{-1}(x) = 0 \tag{5.12}$$

$$X \frac{dp_\nu(x)}{dx} = (\alpha_\nu + \tfrac{1}{2}\nu X'' x)p_\nu(x) + \beta_\nu p_{\nu-1}(x) \tag{5.13}$$

This substitution of n by ν can be justified rigorously by considering the appropriate hypergeometric function for each classical system and using Gauss' 15 relations between contiguous hypergeometric functions (E 2.1.2) to deduce (5.12) and the differentiation formula for $F(a,b;c;z)$ (E 2.8.20) to obtain (5.13)

Armed with the above results we can now find a sufficient condition for a generalised orthogonal function which satisfies a D-type identity to satisfy an S-type identity.

Our main result follows from the following two theorems.

Theorem 5.1

Let $p_\nu(x)$ be a generalized orthogonal function which satisfies a D-type identity, viz.

(5.14) $$D_\nu p_\nu(x)p_\nu(y) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)p_n(y) \cdot(1/\nu-n)-1/(\nu+n+1))$$

with D_ν constant, $\nu \in R\sim I$, for all $x,y \in X \subset (a,b)$

Then a sufficient condition that $p_\nu(x)$ satisfies an S-type identity

(5.15) $$p_\ell(x) = \Sigma_{\substack{n=o\\ n\neq\ell}}^{\infty} S_n^\ell (x_o)p_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

for $\ell \in N$, for some $x_o \in X \subset (a,b)$, $S_n^\ell(x_o)$ constant, is

(5.16) (a) $$D_\nu p_\nu(x)p'_\nu(y) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)p'_n(y) \cdot(1/(\nu-n)-1/(\nu+n+1))$$

is uniformly convergent for all $x,y \in X$

and

(5.17) (b) $$p_\nu(x_o) = M_\nu(x_o)\cos(\tfrac{1}{2}(\nu+m)\pi), \quad m \in N.$$

$S_n^\ell(x_o)$ is given by

$\ell+m$ even

(5.18) $$S_n^\ell(x_o) = -(2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)(D_n/D_\ell)\{\beta_n M_{n-1}(x_o) \cdot\sin(\tfrac{1}{2}(n+m)\pi)+M_n(x_o)((\alpha_n-\alpha_\ell)+\tfrac{1}{2}(n-\ell)X''x_o) \cdot\cos(\tfrac{1}{2}(n+m)\pi)\}/\{M_{\ell-1}(x_o)\beta_\ell+(2/\pi)M_\ell(x_o) \cdot(\alpha'_\ell+\tfrac{1}{2}X''x_o)\}$$

where $\alpha'_\ell = (\frac{d\alpha_\nu}{d\nu})_{\nu=\ell}$

ℓ+m odd

(5.19) $$S_n^{\ell}(x_o) = (2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)\,\frac{D_n M_n(x_o)}{D_\ell M_\ell(x_o)}\cos(\tfrac{1}{2}(n+m)\pi)$$

Proof

Differentiate (5.14) term-by-term with respect to y, and set $y = x_o$; this step is valid by virtue of uniform convergence condition (5.16) (which in turn implies uniform convergence of (5.14)). Multiplying both sides by $X(x_o)$ we obtain

(5.20) $$D_\nu X(x_o)p_\nu(x)p'_\nu(x_o) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n \cdot D_n p_n(x)X(x_o)p'_n(x_o)(1/(\nu-n)-1/(\nu+n+1))$$

Multiply both sides of (5.14) by $(\alpha_\nu+\frac{1}{2}\nu X''y)$ and set $y = x_o$ to give

(5.21) $$D_\nu(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_\nu(x)p_\nu(x_o) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_n(x_o) \cdot(1/(\nu-n)-1/(\nu+n+1))$$

Subtract (5.21) from (5.20). Using (5.13) we obtain

$$D_\nu p_\nu(x)\{X(x_o)p'_\nu(x_o)-(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_\nu(x_o)\}$$

(5.22) $$= D_\nu p_\nu(x)\beta_\nu p_{\nu-1}(x_o)$$

$$= (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)\{X(x_o)p'_n(x_o) -(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_n(x_o)\}(1/(\nu-n)-1/(\nu+n+1))$$

$$= (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)\{\beta_n p_{n-1}(x_o) +(\alpha_n+\tfrac{1}{2}nX''x_o)p_n(x_o) -(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_n(x_o)\}(1/(\nu-n)-1/(\nu+n+1))$$

Now if

$$p_\nu(x_o) = M_\nu(x_o)\cos(\tfrac{1}{2}(\nu+m)\pi)$$

then

(5.23) $$p_\nu(x_o)p_{\nu-1}(x_o) = M_\nu(x_o)M_{\nu-1}(x_o)\cos(\tfrac{1}{2}(\nu+m)\pi)\sin(\tfrac{1}{2}(\nu+m)\pi)$$
$$= \tfrac{1}{2}M_\nu(x_o)M_{\nu-1}(x_o)\sin((\nu+m)\pi)$$
$$= \tfrac{1}{2}M_\nu(x_o)M_{\nu-1}(x_o)\sin(\nu\pi)(-1)^m$$

Thus multiplying both sides of (5.22) by $p_\nu(x_o)$ gives

(5.24) $$(-1)^m D_\nu \beta_\nu M_\nu(x_o)M_{\nu-1}(x_o)p_\nu(x)$$
$$= (2/\pi)p_\nu(x_o)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)\{\beta_n p_{n-1}(x_o)+(\alpha_n+\tfrac{1}{2}nX''x_o)$$
$$\cdot p_n(x_o)-(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_n(x_o)\}(1/(\nu-n)-1/(\nu+n+1))$$

As in Chapter 3 separate out the term for $n = \ell \geq 0$

(5.25) $$\therefore (-1)^m D_\nu \beta_\nu M_\nu(x_o)M_{\nu-1}(x_o)p_\nu(x)$$
$$= (2/\pi)p_\nu(x_o)(-1)^\ell D_\ell p_\ell(x)\{\beta_\ell p_{\ell-1}(x_o)+(\alpha_\ell+\tfrac{1}{2}\ell X''x_o)$$
$$\cdot p_\ell(x_o)-(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_\ell(x_o)\}(1/(\nu-\ell)-1/(\nu+\ell+1))$$
$$+(2/\pi)p_\nu(x_o)\Sigma_{\substack{n=o\\n\neq\ell}}^{\infty}(-1)^n D_n p_n(x)\{\beta_n p_{n-1}(x_o)$$
$$+(\alpha_n+\tfrac{1}{2}nX''x_o)p_n(x_o)-(\alpha_\nu+\tfrac{1}{2}\nu X''x_o)p_n(x_o)\}$$
$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

Now take $\lim_{\nu\to\ell\in N}$ of (5.25)

The only non-smooth term is

(5.26) $$\lim_{\nu\to\ell} p_\nu(x_o)p_{\ell-1}(x_o)/(\nu-\ell)$$
$$= \lim_{\nu\to\ell} M_\nu(x_o)\cos(\tfrac{1}{2}(\nu+m)\pi)M_{\ell-1}(x_o)\sin(\tfrac{1}{2}(\ell+m)\pi)/(\nu-\ell)$$
$$= -(\tfrac{1}{2}\pi)M_\ell(x_o)M_{\ell-1}(x_o)\sin^2(\tfrac{1}{2}(\ell+m)\pi)$$

Substituting this term into $\lim_{\nu\to\ell}$ of (5.25) we obtain

(5.27) $$M_\ell(x_o)M_{\ell-1}(x_o)\beta_\ell p_\ell(x)D_\ell(-1)^m\{1+(-1)^{\ell+m}\sin^2(\tfrac{1}{2}(\ell+m)\pi)\}$$

$$= -(2/\pi)(-1)^\ell M_\ell^2(x_o)p_\ell(x)D_\ell(\alpha'_\ell+\tfrac{1}{2}X''x_o)\cos^2(\tfrac{1}{2}(\ell+m)\pi)$$

$$+(2/\pi)M_\ell(x_o)\cos(\tfrac{1}{2}(\ell+m)\pi)\Sigma_{\substack{n=0\\n\neq\ell}}^{\infty}(-1)^n D_n p_n(x)$$

$$\cdot\{\beta_n M_{n-1}(x_o)\sin(\tfrac{1}{2}(n+m)\pi)$$

$$+ M_n(x_o)((\alpha_n-\alpha_\ell)+\tfrac{1}{2}(n-\ell)X''x_o)\cos(\tfrac{1}{2}(n+m)\pi)\}$$

$$\cdot(1/(\ell-n)-1/(\ell+n+1))$$

The identity becomes trivial unless we assume

(5.28) $M_\ell(x_o) \neq 0$ for all $\ell \in N$

Further

$$\{1+(-1)^{\ell+m}\sin^2(\tfrac{1}{2}(\ell+m)\pi)\} = \begin{cases}1 & \text{for } (\ell+m) \text{ even}\\ 0 & \text{for } (\ell+m) \text{ odd}\end{cases}$$

and

$$\cos(\tfrac{1}{2}(\ell+m)\pi) = \begin{cases}(-1)^{\frac{1}{2}(\ell+m)} & \text{for } (\ell+m) \text{ even}\\ 0 & \text{for } (\ell+m) \text{ odd}\end{cases}$$

We must therefore assume that $(\ell+m)$ is even, and (5.27) then gives

(5.29) $$p_\ell(x) = (2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)\Sigma_{\substack{n=0\\n\neq\ell}}^{\infty}(-1)^{n+m}(D_n/D_\ell)$$

$$\cdot\{\beta_n M_{n-1}(x_o)\sin(\tfrac{1}{2}(n+m)\pi)+M_n(x_o)((\alpha_n-\alpha_\ell)$$

$$+ \tfrac{1}{2}(n-\ell)X''x_o)\cos(\tfrac{1}{2}(n+m)\pi)\}p_n(x)(1/(\ell-n)-1/(\ell+n+1))/$$

$$\{M_{\ell-1}(x_o)\beta_\ell+(2/\pi)M_\ell(x_o)(\alpha'_\ell+\tfrac{1}{2}X''x_o)\}$$

Now, $\sin(\tfrac{1}{2}k\pi) = 0$ unless k is odd

and $\cos(\tfrac{1}{2}k\pi) = 0$ unless k is even

therefore defining (cf (5.18)) for $(\ell+m)$ even

$$(5.30) \quad S_n^{\ell}(x_o) = -(2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)(D_n/D_\ell)\{\beta_n M_{n-1}(x_o)$$
$$\cdot\sin(\tfrac{1}{2}(n+m)\pi)-M_n(x_o)((\alpha_n-\alpha_\ell)+\tfrac{1}{2}(n-\ell)X''x_o)$$
$$\cdot\cos(\tfrac{1}{2}(n+m)\pi)\}/\{M_{\ell-1}(x_o)\beta_\ell+(2/\pi)M_\ell(x_o)(\alpha'_\ell+\tfrac{1}{2}X''x_o)\}$$

we obtain

$$(5.31) \quad p_\ell(x) = \Sigma_{\substack{n=o\\ n\neq\ell}}^{\infty} S_n^{\ell}(x_o)p_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

as required.

The corresponding identity for $(\ell+m)$ odd is found by setting $y = x_o$ in (5.14) and multiplying both sides by $p_{\nu-1}(x_o)$; we obtain (using (5.23))

$$(5.32) \quad D_\nu p_{\nu-1}(x_o)p_\nu(x_o)p_\nu(x)$$
$$= (-1)^m\tfrac{1}{2}D_\nu M_\nu(x_o)M_{\nu-1}(x_o)\sin(\nu\pi)p_\nu(x)$$
$$= (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n D_n p_n(x)p_n(x_o)p_{\nu-1}(x_o)$$
$$\cdot(1/(\nu-n)-1/(\nu+n+1)).$$

Again separating out the $n = \ell \geq 0$ term we obtain

$$(5.33) \quad (-1)^m D_\nu M_\nu(x_o)M_{\nu-1}(x_o)p_\nu(x)$$
$$= (2/\pi)(-1)^\ell D_\ell p_\ell(x)p_\ell(x_o)p_{\nu-1}(x_o)(1/(\nu-\ell)-1/(\nu+\ell+1))$$
$$+ (2/\pi)\Sigma_{\substack{n=o\\ n\neq\ell}}^{\infty}(-1)^n D_n p_n(x)p_n(x_o)p_{\nu-1}(x_o)$$
$$\cdot(1/(\nu-n)-1(\nu+n+1))$$

Now

$$(5.34) \quad \lim_{\nu\to\ell} p_\ell(x_o)p_{\nu-1}(x_o)/(\nu-\ell)$$
$$= \lim_{\nu\to\ell} M_\ell(x_o)\cos(\tfrac{1}{2}(\ell+m)\pi)M_{\nu-1}(x_o)\sin(\tfrac{1}{2}(\nu+m)\pi)/(\nu-\ell)$$
$$= \tfrac{1}{2}\pi M_\ell(x_o)M_{\ell-1}(x_o)\cos^2(\tfrac{1}{2}(\ell+m)\pi)$$

Thus taking $\lim_{\nu\to\ell\in N}$ of (5.33) we find

$$(5.35)\qquad (-1)^m D_\ell M_\ell(x_o)M_{\ell-1}(x_o)p_\ell(x)\{1-(-1)^{\ell+m}\cos^2(\tfrac{1}{2}(\ell+m)\pi)\}$$

$$= (2/\pi)M_{\ell-1}(x_o)\sin(\tfrac{1}{2}(\ell+m)\pi)\Sigma^\infty_{\substack{n=o\\n\neq\ell}}(-1)^n D_n p_n(x)$$

$$\cdot M_n(x_o)\cos(\tfrac{1}{2}(n+m)\pi)(1/(\ell-n)-1/(\ell+n+1))$$

Again (5.35) reduces to a trivial identity unless we assume (5.28); further $(\ell+m)$ must be odd, in which case

$$\{1-(-1)^{\ell+m}\cos^2(\tfrac{1}{2}(\ell+m)\pi)\} = 1$$

and (5.35) reduces to

$$(5.36)\qquad p_\ell(x) = (2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)\Sigma^\infty_{\substack{n=o\\n\neq\ell}}(-1)^{n+m}\frac{D_n M_n(x_o)}{D_\ell M_\ell(x_o)}$$

$$\cdot\cos(\tfrac{1}{2}(n+m)\pi)p_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

or defining $S^\ell_n(x_o)$ by (cf (5.19))

$$(5.37)\qquad S^\ell_n(x_o) = (2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)\ \frac{D_n M_n(x_o)}{D_\ell M_\ell(x_o)}\ \cos(\tfrac{1}{2}(n+m)\pi)$$

where we have noted that $(n+m)$ must be even, we obtain

$$(5.38)\qquad p_\ell(x) = \Sigma^\infty_{\substack{n=o\\n\neq\ell}}S^\ell_n(x_o)p_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

as required. Q.E.D.

Theorem 5.2

Let $p_\nu(x)$ be a generalized orthogonal function which satisfies a D-type identity, viz.

$$(5.39)\qquad D_\nu p_\nu(x)p_\nu(y) = (\sin(\nu\pi)/\pi)\Sigma^\infty_{n=o}(-1)^n D_n p_n(x)p_n(y)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

with D_ν constant, $\nu \in R\sim I$, for all $x,y \in X \subset (a,b)$, where $0 \in X$.

Then a sufficient condition that $p_n(x)$ satisfies an S-type identity

(5.40) $$p_\ell(x) = \Sigma_{\substack{n=0\\ n\neq \ell}}^{\infty} S_n^\ell p_n(x)(1/(\ell-n)-1/(\ell+n+1))$$

for $\ell, m \in N$ and for all $x \in X$,

is

(5.41) (i) $$D_\nu p_\nu(x)p'_\nu(y) = (\sin(\nu\pi)/\pi)\Sigma_{n=0}^{\infty}(-1)^n D_n p_n(x)p'_n(y)$$

$$\cdot(1/(\nu-n)-1/(\nu+n+1))$$

is uniformly convergent for all $x,y \in X$

(5.42) (ii) $p_\nu(0) = M_\nu \cos(\frac{1}{2}(\nu+m)\pi)$, $m \in N$, M_ν constant

and

(5.43) (iii) $r_n = \tilde{k}_n/k_n$

$= r$, constant for all n

S_n^ℓ is given by

<u>(ℓ+m) even</u>

(5.44) $$S_n^\ell = -(2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)(D_n/D_\ell)\{\beta_n M_{n-1}\sin(\tfrac{1}{2}(n+m)\pi)$$

$$-M_n(\alpha_n-\alpha_\ell)\cos(\tfrac{1}{2}(n+m)\pi)\}/\{M_{\ell-1}\beta_\ell+(2/\pi)M_\ell\alpha'_\ell\}$$

<u>(ℓ+m) odd</u>

(5.45) $$S_n^\ell = (2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)\ \frac{D_n M_n}{D_\ell M_\ell}\ \cos(\tfrac{1}{2}(n+m)\pi)$$

<u>Proof</u>

We first show that if $0 \in X$ then condition (b) of Theorem 5.1 is equivalent to conditions (ii) and (iii) of Theorem 5.2.

<u>Case A : m even</u>

<u>Necessity:</u> Recall recurrence relation (5.6) - (5.7)

(5.6) $p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \quad p_{-1}(x) = 0$

with $A_n = k_{n+1}/k_n$

(5.7) $B_n = A_n(r_{n+1} - r_n)$

$C_n = A_n h_n/(A_{n-1}h_{n-1})$

$= k_{n+1}k_{n-1}h_n/(k_n^2 h_{n-1})$

where $k_n, \tilde{k}_n$ are the coefficients of x^n, x^{n-1} in $p_n(x)$ and $r_n = \tilde{k}_n/k_n$.

Assume $p_s(x_o) = M_s(x_o)\cos(\frac{1}{2}(s+m)\pi)$ for $s \leq N$

$= (-1)^{\frac{1}{2}m} M_s(x_o)\cos(\frac{1}{2}s\pi)$

Choose N even. Then by (5.6) since m is even

(5.46) $p_{N+1}(x_o) = (-1)^{\frac{1}{2}m}(A_N x_o + B_N)M_N\cos(\frac{1}{2}N\pi)$

This is zero as required iff

$M_N = 0$ (impossible by (5.28))

or

(5.47) $A_N x_o + B_N = 0$

Since A_N and B_N are independent of x this implies

(5.48) $B_N = 0$ and $A_N x_o = 0$

Now $B_N = A_N(r_{N+1} - r_N)$, and (5.48) is equivalent to

(5.49) $x_o = 0$ and $r_{N+1} - r_N = 0$

or $A_N = 0$

But $A_N = k_{N+1}/k_N$ by (5.7), and $A_N = 0$ means that k_{N+1} (the coefficient of x^{N+1} in p_{N+1}) is zero, which is impossible because $p_{N+1}(x)$ is defined to be a polynomial of exact degree $(N+1)$.

Thus (5.49) gives

(5.50) $\quad x_o = 0$ and r_n = constant for all n.

Sufficiency : follows similar lines

Case B : m odd

Choosing N odd leads to an analogous proof.

Thus if $0 \in X$, Theorem 5.2 follows as a result of Theorem 5.1. Q.E.D.

We now look at particular instances of Theorems 5.1 and 5.2.

Let us apply Theorem 5.2 to Legendre associated functions $P_n^m(x)$ with m fixed. Since $\{P_n^m(x)\}$ satisfy a differential equation of the type (5.1) they can be reduced to a classical system.

From the proof of Lemma 2.8 we know that

$$(5.51) \qquad \gamma_{\nu,m} P_\nu^m(x) P'^m_\nu(y) = (\sin(\nu\pi)/\pi)\Sigma_{n=o}^{\infty}(-1)^n \gamma_{n,m} \cdot P_n^m(x) P'^m_n(y)(1/(\nu-n)-1/(\nu+n+1))$$

is uniformly convergent for

$$-\pi < \theta \pm \xi < \pi, \quad \text{where} \quad x = \cos\theta,\ y = \cos\xi$$

We see that the range of validity of (5.51) is not in the form required by condition (i) of Theorem 5.2, but following the methods of Chapter 3 we can easily extend Theorem 5.2 to include limits of this nature.

We are going to show that S_n^ℓ of equations (5.44) and (5.45) are in agreement with equations (3.55) and (3.56)

(i) By uniform convergence of (5.51) condition (5.41) is satisfied with

(5.52) $$D_\nu = \gamma_{\nu,m}$$

(ii) From (2.16)

(5.53) $$P_\nu^m(0) = 2^m\pi^{-\frac{1}{2}}\cos(\tfrac{1}{2}(\nu+m)\pi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}m)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}m)$$

whence (cf. (5.42))

(5.54) $$M_n = 2^m\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}m)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}m)$$

(iii) Further, from (2.31)

(5.55) $$(1-x^2)\frac{dP_n^m(x)}{dx} = (n+1)xP_n^m(x)-(n-m+1)P_{n+1}^m(x)$$

(5.56) $$= -nxP_n^m(x)+(n+m)P_{n-1}^m(x)$$

Comparing (5.56) with (5.9), namely

$$X\frac{dp_n(x)}{dx} = (\alpha_n+\tfrac{1}{2}nX''x)p_n(x)+\beta_n p_{n-1}(x)$$

we see that

(5.57) $$\alpha_n = 0$$
$$\beta_n = n+m$$
$$X = 1-x^2$$

whence from (5.10), viz

$$\alpha_n = nX'(0) - \tfrac{1}{2}X''r_n$$

we conclude

$$r_n = 0 \quad \text{for all} \quad n.$$

Thus $\{P_n^m(x)\}$ satisfies conditions (i), (ii) and (iii) of Theorem 5.2.

Case A : $(\ell+m)$ even

From definition (5.44), substituting for D_n, α_n, α'_n and β_n from (5.52) and (5.57) we obtain

$$(5.58)\qquad S_n^\ell = -(2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)\,\frac{D_n\beta_n M_{n-1}}{D_\ell\beta_\ell M_{\ell-1}}\,\sin(\tfrac{1}{2}(n+m)\pi)$$

$$= -(2/\pi)(\cos(\tfrac{1}{2}(\ell+m)\pi)/M_{\ell-1})(M_{n-1}\sin(\tfrac{1}{2}(n+m)\pi)$$

$$\cdot\frac{\gamma_{n,m}(n+m)}{\gamma_{\ell,m}(\ell+m)}$$

Now from (5.53), using (5.54)

$$(5.59)\qquad P^m_{n-1}(0) = 2^m\pi^{-\frac{1}{2}}\cos(\tfrac{1}{2}(n+m-1)\pi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}(n-1)+\tfrac{1}{2}m)/$$

$$\cdot\Gamma(1+\tfrac{1}{2}(n-1)-\tfrac{1}{2}m)$$

$$= M_{n-1}\sin(\tfrac{1}{2}(n+m)\pi)$$

and from (2.17)

$$(5.60)\qquad P'^{-m}_{\ell-1}(0) = 2^{-m+1}\pi^{-\frac{1}{2}}\sin(\tfrac{1}{2}(\ell-m-1)\pi)\Gamma(1+\tfrac{1}{2}(\ell-1)-\tfrac{1}{2}m)/$$

$$\cdot\Gamma(\tfrac{1}{2}+\tfrac{1}{2}(\ell-1)+\tfrac{1}{2}m)$$

$$= -2^{-m+1}\pi^{-\frac{1}{2}}\cos(\tfrac{1}{2}(\ell+m-2m)\pi)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\ell-\tfrac{1}{2}m)/\Gamma(\tfrac{1}{2}\ell+\tfrac{1}{2}m)$$

$$= -(-1)^m(2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)(2^m\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}\ell+\tfrac{1}{2}m)/$$

$$\cdot\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\ell-\tfrac{1}{2}m))^{-1}$$

$$= -(-1)^m(2/\pi)\cos(\tfrac{1}{2}(\ell+m)\pi)/M_{\ell-1}$$

Substitute (5.59) and (5.60) into (5.58) to obtain

$$(5.61)\qquad S_n^\ell = (-1)^m\,\frac{\gamma_{n,m}(n+m)}{\gamma_{\ell,m}(\ell+m)}\,P^m_{n-1}(0)P'^{-m}_{\ell-1}(0)$$

Now from (5.56)

$$(5.62)\qquad P'^m_n(0) = (n+m)P^m_{n-1}(0)$$

Employing (5.55) and then (2.15)

(5.63) $$P'^{-m}_{\ell-1}(0) = -(\ell+m)P^{-m}_{\ell}(0)$$

$$= -(-1)^m(\ell+m)P^m_\ell(0)\gamma_{\ell,m}$$

Substituting (5.62) and (5.63) into (5.61) we see that

(5.64) $$S^\ell_n = -\gamma_{n,m}P^m_\ell(0)P'^m_n(0)$$

which agrees with (3.55).

Case B : $(\ell+m)$ odd

From (5.45) using (5.52)

(5.65) $$S^\ell_n = (2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)\ \frac{D_nM_n}{D_\ell M_\ell}\ \cos(\tfrac{1}{2}(n+m)\pi)$$

$$= (\gamma_{n,m}/\gamma_{\ell,m})(2/\pi)(\sin(\tfrac{1}{2}(\ell+m)\pi)/M_\ell)(M_n\cos(\tfrac{1}{2}(n+m)\pi)$$

Now from (5.53) and (5.54)

(5.66) $$P^m_n(0) = M_n\cos(\tfrac{1}{2}(n+m)\pi)$$

and from (2.17)

(5.67) $$P'^{-m}_\ell(0) = 2^{-m+1}\pi^{-\frac{1}{2}}\sin(\tfrac{1}{2}(\ell-m)\pi)\Gamma(1+\tfrac{1}{2}\ell-\tfrac{1}{2}m)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\ell+\tfrac{1}{2}m)$$

$$= 2^{-m+1}\pi^{-\frac{1}{2}}\sin(\tfrac{1}{2}(\ell+m-2m)\pi)\Gamma(1+\tfrac{1}{2}\ell-\tfrac{1}{2}m)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\ell+\tfrac{1}{2}m)$$

$$= (-1)^m(2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)(2^m\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\ell+\tfrac{1}{2}m)/$$

$$\cdot\Gamma(1+\tfrac{1}{2}\ell-\tfrac{1}{2}m))^{-1}$$

$$= (-1)^m(2/\pi)\sin(\tfrac{1}{2}(\ell+m)\pi)/M_\ell$$

Substituting (5.66) and (5.67) into (5.65) we see that

(5.68) $$S^\ell_n = (-1)^m(\gamma_{n,m}/\gamma_{\ell,m})P^m_n(0)P'^{-m}_\ell(0)$$

Now

(5.69) $$P'^{-m}_\ell(0) = (\ell-m)P^{-m}_{\ell-1}\ (0) \qquad \text{by (5.56)}$$

$$= (\ell-m)(-1)^m\gamma_{\ell-1,m}P^m_{\ell-1}(0) \qquad \text{by (2.15)}$$

$$= (\ell-m)(-1)^m\frac{(\ell-1-m)!}{(\ell-1+m)!}\ \frac{P'^m_\ell(0)}{(\ell+m)} \quad \text{by (5.56)}$$

$$= (-1)^m\gamma_{\ell,m}P'^m_\ell(0)$$

and (5.68) becomes

(5.70) $$S_n^{\ell} = \gamma_{n,m} P_n^m(0) P'^m_{\ell}(0)$$

which agrees with (3.56).

We have noted that Meulenbeld and van de Wetering have derived a D-type identity for the generalized Legendre associated function $P_\nu^{m,n}(x)$ (Meulenbeld and van de Wetering, 1967). Starting from this identity and making use of the fact that the Jacobi function $P_\nu^{(\alpha,\beta)}$ can be expressed in terms of $P_\nu^{m,n}$, we derive a D-type identity for the Jacobi function and using Theorems 5.1 and 5.2 investigate whether an S-type identity can be derived for $P_n^{(\alpha,\beta)}$

Generalized Legendre associated functions

Let $P_\nu^{m,n}(z)$ be the solution of the differential equation (Kuipers and Meulenbeld, 1957, equation (1))

(5.71) $$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \{\nu(\nu+1) - \tfrac{1}{2}m^2/(1-z) - \tfrac{1}{2}n^2/(1+z)\}$$

If $m = n$ then (5.71) reduces to

(5.72) $$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \{\nu(\nu+1) - m^2/(1-z^2)\} = 0$$

the differential equation (2.1) defining the Legendre associated function $P_\nu^m(z)$, and we see that for $m \neq n$ $P_\nu^{m,n}$ is indeed a generalization of $P_\nu^m(z)$.

For $-1 < x < 1$ we define (Meulenbeld, 1958, equation (1)) as in the case of the Legendre associated function (see (2.11))

$$(5.73)\qquad P_\nu^{m,n}(x) = \tfrac{1}{2}\{e^{i\frac{1}{2}\pi m}P_\nu^{m,n}(x+i0)+e^{-i\frac{1}{2}\pi m}P_\nu^{m,n}(x-i0)\}$$

For m a non-negative integer (Kuipers and Meulenbeld, 1957, equation (12)) and ν non-integral

$$(5.74)\qquad P_\nu^{m,n}(z) = \frac{(-1)^m\Gamma(\nu+\frac{1}{2}(m+n)+1)\Gamma(-\nu+\frac{1}{2}(m+n))}{2^m\Gamma(1+m)\Gamma(\nu-\frac{1}{2}(m-n)+1)\Gamma(-\nu-\frac{1}{2}(m-n))}$$

$$\cdot(z+1)^{\frac{1}{2}n}(z-1)^{\frac{1}{2}m}$$

$$\cdot F(-\nu+\tfrac{1}{2}(m+n),\nu+\tfrac{1}{2}(m+n)+1;\ m+1;\ \tfrac{1}{2}-\tfrac{1}{2}z)$$

and limiting z to $(-1,1)$ we find (using $\Gamma(z)\Gamma(1-z) = \pi\,\mathrm{cosec}(\pi z)$)

$$(5.75)\qquad P_\nu^{m,n}(x) = \frac{(-1)^m(1+x)^{\frac{1}{2}n}(1-x)^{\frac{1}{2}m}\Gamma(\nu+\frac{1}{2}(m+n)+1)\Gamma(\nu+\frac{1}{2}(m-n)+1)}{2^m\Gamma(1+m)\Gamma(\nu-\frac{1}{2}(m-n)+1)\Gamma(\nu-\frac{1}{2}(m+n)+1)}$$

$$\cdot\frac{\sin(\pi(\nu+\frac{1}{2}(m-n)+1))}{\sin(\pi(\nu-\frac{1}{2}(m+n)+1))}$$

$$\cdot F(-\nu+\tfrac{1}{2}(m+n),\nu+\tfrac{1}{2}(m+n)+1;\ m+1;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

$$= \frac{(1+x)^{\frac{1}{2}n}(1-x)^{\frac{1}{2}m}}{2^m\Gamma(1+m)\gamma_{\nu,\frac{1}{2}(m+n)}\gamma_{\nu,\frac{1}{2}(m-n)}}$$

$$\cdot F(-\nu+\tfrac{1}{2}(m+n),\nu+\tfrac{1}{2}(m+n)+1;\ m+1;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

also (Meulenbeld, 1958, equation (8)).

$$(5.76)\qquad P_\nu^{-m,-n}(x) = 2^{m-n}\gamma_{\nu,\frac{1}{2}(m+n)}\gamma_{\nu,\frac{1}{2}(m-n)}\{\cos(m\pi)P_\nu^{m,n}(x)$$

$$- 2(\sin(m\pi)/\pi)Q_\nu^{m,n}(x)\}$$

(where $Q_\nu^{m,n}$ is the second solution of (5.71)). For integral m this reduces to (writing m for n and vice versa)

$$(5.77)\qquad P_\nu^{-n,-m}(x) = (-1)^n 2^{n-m}\gamma_{\nu,\frac{1}{2}(m+n)}\gamma_{\nu,\frac{1}{2}(n-m)}P_\nu^{n,m}(x)$$

In (Meulenbeld and van de Wetering, 1967, equation (6)) we find the following D-type identity for ν non-integral, for m and n constants such that $m, \frac{1}{2}(n-m)$ and $\frac{1}{2}(n+m)$ are non-negative integers

$$(5.78) \qquad \gamma_{\nu,\frac{1}{2}(m-n)} P_\nu^{-n,-m}(\cos\theta) P_\nu^{m,n}(\cos\xi)$$

$$= (\sin(\nu\pi)/\pi)\Sigma_{q=\frac{1}{2}(m+n)}^{\infty} (-1)^q \gamma_{q,\frac{1}{2}(m-n)} P_q^{-n,-m}(\cos\theta)$$

$$\cdot P_q^{m,n}(\cos\xi)(1/(\nu-q)-1/(\nu+q+1))$$

$$-\pi < \theta \pm \xi < \pi$$

Since m is a non-negative integer we may substitute (5.77) into (5.78) (noting that $\gamma_{\alpha,\beta}\,\gamma_{\alpha,-\beta} = 1$) to obtain the more convenient expression

$$(5.79) \qquad \gamma_{\nu,\frac{1}{2}(n+m)} P_\nu^{n,m}(x) P_\nu^{m,n}(y) = (\sin(\nu\pi)/\pi)\Sigma_{q=\frac{1}{2}(m+n)}^{\infty}(-1)^q$$

$$\cdot \gamma_{q,\frac{1}{2}(n+m)} P_q^{n,m}(x) P_q^{m,n}(y)(1/(\nu-q)-1/(\nu+q+1))$$

Now the Jacobi function $P_\nu^{(\alpha,\beta)}$, the generalization of the Jacobi polynomial, satisfies the differential equation (E 10.8(14))

$$(1-x^2)y'+(\beta-\alpha-(\alpha+\beta+2)x)y'+\nu(\nu+\alpha+\beta+1)y = 0$$

with solution (E 10.8(16))

$$(5.80) \qquad P_\nu^{(\alpha,\beta)}(x) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)} F(-\nu,\nu+\alpha+\beta+1;\ 1+\alpha;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

whence

$$(5.81) \qquad P_{\nu-\frac{1}{2}(m+n)}^{(m,n)}(x) = \frac{\Gamma(\nu+\frac{1}{2}(m-n)+1)}{\Gamma(\nu-\frac{1}{2}(m+n)+1)\Gamma(m+1)}$$

$$\cdot F(-\nu+\tfrac{1}{2}(m+n),\nu+\tfrac{1}{2}(m+n)+1;\ m+1;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

Comparing (5.81) with (5.75) we see that

(5.82) $$P_\nu^{m,n}(x) = 2^{-m}(1+x)^{\frac{1}{2}n}(1-x)^{\frac{1}{2}m}\frac{\Gamma(\nu+\frac{1}{2}(m+n)+1)}{\Gamma(\nu-\frac{1}{2}(m-n)+1)}P^{(m,n)}_{\nu-\frac{1}{2}(m+n)}(x)$$

Substituting (5.82) into (5.79) (remembering that m, $\frac{1}{2}(n\pm m)$ must be non-negative integers) we deduce

(5.83) $$\frac{\Gamma(\nu-\frac{1}{2}(m+n)+1)\Gamma(\nu+\frac{1}{2}(m+n)+1)}{\Gamma(\nu-\frac{1}{2}(n-m)+1)\Gamma(\nu+\frac{1}{2}(n-m)+1)}P^{(n,m)}_{\nu-\frac{1}{2}(m+n)}(x)P^{(m,n)}_{\nu-\frac{1}{2}(m+n)}(y)$$

$$= (\sin(\nu\pi)/\pi)\Sigma^{\infty}_{q=\frac{1}{2}(m+n)}(-1)^q$$

$$\cdot\frac{\Gamma(q-\frac{1}{2}(m+n)+1)\Gamma(q+\frac{1}{2}(m+n)+1)}{\Gamma(q-\frac{1}{2}(n-m)+1)\Gamma(q+\frac{1}{2}(n-m)+1)}P^{(n,m)}_{q-\frac{1}{2}(m+n)}(x)$$

$$\cdot P^{(m,n)}_{q-\frac{1}{2}(m+n)}(y)(1/(\nu-q)-1/(\nu+q+1))$$

which is the required D-type identity.

For the Jacobi polynomial $P_n^{(\alpha,\beta)}$ (E 10.8(5))

(5.84) $$r_n = n(\alpha-\beta)/(2n+\alpha+\beta)$$

Thus r_n is a constant iff $\alpha = \beta$, and for this case an S-type identity exists assuming that the uniform convergence condition (5.41) holds. However, we need not check this point explicitly because a Jacobi polynomial with $\alpha = \beta = m$ is proportional (by (5.82)) to a GLAF with $n = m$. From (5.72) this in turn means that $P_\nu^{(m,m)}(x)$ is proportional to $P_\nu^m(x)$, and we know that (5.41) holds for the Legendre associated function. We must therefore find the explicit relation between $P_\nu^m(x)$ and $P_\nu^{(m,m)}(x)$.

We know that from (2.13)

(5.85) $$P_\nu^{-m}(x) = \frac{1}{\Gamma(1+m)}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}m}F(-\nu,\nu+1;\ 1+m;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$
$$-1 < x < -1$$

while from (5.80)

(5.86) $$P_\nu^{(m,m)}(x) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)\Gamma(m+1)} F(-\nu,\nu+2m+1;\ 1+m;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

whence

(5.87) $$P_{\nu-m}^{(m,m)}(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-m+1)\Gamma(m+1)} F(-\nu+m,\nu+m+1;\ 1+m;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

Now Kummer's second solution of the hypergeometric equation is (E 2.9(5) and (6))

$$u_2 = F(a,b;\ a+b+1-c;\ 1-z)$$

(5.88) $$= z^{1-c}F(a+1-c,b+1-c;\ a+b+1-c;\ 1-z)$$

Set $a = -\nu+m$

$b = \nu+m+1$

$c = 1+m$

$z = \tfrac{1}{2}+\tfrac{1}{2}x$

Then $a+b+1-c = 1+m$, and (5.88) gives

(5.89) $$F(-\nu+m,\nu+m+1;\ 1+m;\ \tfrac{1}{2}-\tfrac{1}{2}x)=(\tfrac{1}{2}(1+x))^{-m} \cdot F(-\nu,\nu+1;\ 1+m;\ \tfrac{1}{2}-\tfrac{1}{2}x)$$

Combining (5.85), (5.87) and (5.89) we obtain

(5.90) $$P_{\nu-m}^{(m,m)}(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu-m+1)} 2^m(1-x^2)^{-\frac{1}{2}m}P_\nu^{-m}(x)$$

$$= 2^m(1-x^2)^{-\frac{1}{2}m} \frac{\Gamma(\nu+1)(-1)^m}{\Gamma(\nu+m+1)} P_\nu^m(x)$$

by (2.15)

Substitute (5.90) into (5.83) with $n=m$, to give

(5.91) $$\gamma_{\nu,m}P_\nu^m(x)P_\nu^m(y) = (\sin(\nu\pi)/\pi)\Sigma_{q=m}^{\infty}(-1)^q\gamma_{q,m}P_q^m(x)P_q^m(y) \cdot(1/(\nu-q)-1/(\nu+q+1))$$

which (from the proof of Lemma 2.8) satisfies condition (i) of Theorem 5.2.

Setting $n=m$ in equation (5.83) gives

$$(5.92)\qquad \frac{\Gamma(\nu-m+1)\Gamma(\nu+m+1)}{\Gamma^2(\nu+1)} P_{\nu-m}^{(m,m)}(x)P_{\nu-m}^{(m,m)}(y)$$

$$= (\sin(\nu\pi)/\pi)\Sigma_{q=m}^{\infty}(-1)^q \frac{\Gamma(q-m+1)\Gamma(q+m+1)}{\Gamma^2(q+1)}$$

$$\cdot P_{q-m}^{(m,m)}(x)P_{q-m}^{(m,m)}(y)(1/(\nu-q)-1/(\nu+q+1))$$

a D-type identity which we have shown to satisfy condition (i) of Theorem 5.2.

From (5.90), using (2.16)

$$(5.93)\qquad P_{n-m}^{(m,m)}(0) = 2^m(-1)^m \frac{\Gamma(n+1)}{\Gamma(n+m+1)} P_n^m(0)$$

$$= 2^{2m}\pi^{-\frac{1}{2}}(-1)^m \frac{\Gamma(n+1)\quad\Gamma(\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}m)}{\Gamma(n+m+1)\Gamma(1+\frac{1}{2}n-\frac{1}{2}m)}$$

$$\cdot\cos(\tfrac{1}{2}(n+m)\pi)$$

thereby satisfying condition (ii) of the same theorem. Condition (iii) follows from (5.84) with $\alpha = \beta = m$.

We may thus apply Theorem 5.2 to the Jacobi polynomial $P_{n-m}^{(m,m)}(x)$. From (5.93)

$$(5.94)\qquad M_n = 2^{2m}\pi^{-\frac{1}{2}}(-1)^m \frac{\Gamma(n+1)\quad\Gamma(\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}m)}{\Gamma(n+m+1)\Gamma(1+\frac{1}{2}n-\frac{1}{2}m)}$$

From (5.92)

$$(5.95)\qquad D_n = \frac{\Gamma(n-m+1)\Gamma(n+m+1)}{\Gamma^2(n+1)}$$

$$= \gamma_{n,m} \frac{\Gamma^2(n+m+1)}{\Gamma^2(n+1)}$$

Also, from (E 10.8(9))

$$(5.96)\qquad \alpha_n = 0$$

and $\qquad \beta_n = n$

Thus the coefficients $\{S_n^{\ell}\}$ for Jacobi polynomials can be obtained from those for Legendre associated functions by replacing (cf. (5.52), (5.54) and (5.57))

(5.97) D_n by $\frac{\Gamma^2(n+m+1)}{\Gamma^2(n+1)} D_n$

(5.98) M_n by $2^m(-1)^m \frac{\Gamma(n+1)}{\Gamma(n+m+1)} M_n$

(5.99) and β_n by $\frac{n}{n+m} \beta_n$

whence from (5.44) and (5.64)

<u>ℓ+m even</u>

$$S_n^\ell = - \frac{\Gamma^2(n+m+1)\Gamma^2(\ell+1)\Gamma(n)\Gamma(\ell+m)n(\ell+m)}{\Gamma^2(n+1)\Gamma^2(\ell+m+1)\Gamma(n+m)\Gamma(\ell)(n+m)\ell} \cdot \gamma_{n,m} P_\ell^m(0) P'^m_n(0) \tag{5.100}$$

$$= - \frac{\Gamma(n+m+1)\Gamma(\ell+1)}{\Gamma(n+1)\Gamma(\ell+m+1)} \gamma_{n,m} P_\ell^m(0) P'^m_n(0)$$

From (E 10.8(15)) we obtain

$$P'^{(m,m)}_{n-m}(0) = nP^{(m,m)}_{n-m-1}(0) \tag{5.101}$$

while by (5.62)

$$P'^m_n(0) = (n+m)P^m_{n-1}(0)$$

whence by (5.93)

$$P'^{(m,m)}_{n-m}(0) = 2^m(-1)^m \frac{\Gamma(n+1)}{\Gamma(n+m+1)} P'^m_n(0) \tag{5.102}$$

Hence using (5.93) and (5.102), equation (5.100) becomes

$$S_n^\ell = -\gamma_{n,m}\left(P^{(m,m)}_{\ell-m}(0)(-1)^m 2^{-m} \frac{\Gamma(\ell+m+1)}{\Gamma(\ell+1)}\right) \tag{5.103}$$

$$\cdot\left(P'^{(m,m)}_{n-m}(0)(-1)^m 2^{-m} \frac{\Gamma(n+m+1)}{\Gamma(n+1)}\right)\left(\frac{\Gamma(n+m+1)\Gamma(\ell+1)}{\Gamma(n+1)\Gamma(\ell+m+1)}\right)$$

$$= -\gamma_{n,m} 2^{-2m} P^{(m,m)}_{\ell-m}(0) P'^{(m,m)}_{n-m}(0) \frac{\Gamma^2(n+m+1)}{\Gamma^2(n+1)}$$

Similarly using (5.97) and (5.98) on (5.45) and (5.70) we deduce (employing once again (5.93) and (5.102))

$\ell+m$ odd

$$(5.104)\quad S_n^{\ell} = \gamma_{n,m} P_n^m(0) P'^m_{\ell}(0) \frac{\Gamma(n+m+1)\Gamma(\ell+1)}{\Gamma(n+1)\Gamma(\ell+m+1)}$$

$$= \gamma_{n,m}\left(P_{n-m}^{(m,m)}(0)(-1)^m 2^{-m} \frac{\Gamma(n+m+1)}{\Gamma(n+1)}\right)$$

$$\cdot\left(P'^{(m,m)}_{\ell-m}(0)(-1)^m 2^{-m} \frac{\Gamma(\ell+m+1)}{\Gamma(\ell+1)}\right)\frac{\Gamma(n+m+1)\Gamma(\ell+1)}{\Gamma(n+1)\Gamma(\ell+m+1)}$$

$$= \gamma_{n,m} 2^{-2m} P_{n-m}^{(m,m)}(0) P'^{(m,m)}_{\ell-m}(0) \frac{\Gamma^2(n+m+1)}{\Gamma^2(n+1)}$$

Equations (5.103) and (5.104) give the coefficients of the S-type identity for $P_{n-m}^{(m,m)}(x)$.

ACKNOWLEDGEMENTS

For the past three years my supervisor, Dr. G.B. Brundrit has been burdened with an overwhelming teaching and research load; despite this he has always found time for advice, helpful criticism and constant encouragement.

I should also like to thank Mrs. M.I. Cousins for her meticulous transcription of my semi-illegible draft.

REFERENCES

Ar, E. (1967) *On the Low Frequency Accoustical Scattering of a Plane Wave from a Soft Spindle at Nose-on Incidence.* Quart. Appl. Math. 25, 285-292.

Ar, E. and Kleinman, R.E. (1966) *The Exterior Neumann Problem for the Three-Dimensional Helmholtz Equation.* Arch. Rational Mech. Analysis 23, 218-236.

Asvestas, J.S. and Kleinman, R.E. (1969) *Low Frequency Scattering by Spheroids and Disks I. Dirichlet Problem for a Prolate Spheroid.* J. Inst. Maths Applics 6, 42-56.

Bromwich, T.J.I'A. (1926) *An Introduction to the Theory of Infinite Series,* MacMillan.

Erdélyi, A. (Ed.) (1953) *Higher Transcendental Functions (Bateman Manuscript Project),* McGraw Hill, New York.

Gradshteyn, I.S. and Ryzhik, I.M. (1965) *Table of Integrals, Series and Products (Translated from the Russian by Alan Jeffrey),* Academic Press, New York.

Hobson, E.W. (1931) *The Theory of Spherical and Elliptical Harmonics,* Cambridge University Press.

Kellogg, O.D. (1929) *Foundations of Potential Theory,* Ungar, New York.

Kleinman, R.E. (1965) *The Dirichlet Problem for the Helmholtz Equation.* Arch. Rational Mech. Analysis 18, 205-229.

Kuipers, L. and Meulenbeld, B. (1957) *On a Generalization of Legendre's Associated Differential Equation I.* Proc. Neder. Akad. Wetensch. *A60, 436-443.*

MacRobert, T.M. (1934) *Some Integrals, with respect to their Degrees, of Associated Legendre Functions.* Proc. Roy. Soc. Edinburgh 54, 135-144.

Messiah, A. (1961) *Quantum Mechanics,* North-Holland, Amsterdam.

Meulenbeld, B. (1958) *Generalized Legendre's Associated Functions for Real Values of Argument Numerically less than Unity.* Proc. Neder. Akad. Wetensch. A61, 557-563.

Meulenbeld, B. and van de Wetering, R.L. (1967) *On Some Series of Dougall.* Proc. Neder. Akad. Wetensch. A70, 530-534.

Moon, P. and Spencer, D.E. (1952) *Separability Conditions for the Laplace and Helmholtz Equations.* J. Franklin Inst. 253, 585-600.

Morse, P.M. and Feshbach, H. (1952) *Methods of Theoretical Physics,* McGraw Hill, New York.

Note that $P_n^m(z)$ and $Q_n^m(z)$ (as defined in Appendix 10 of Morse and Feshbach) differ by a factor of $(-1)^m$ from the normalization which we are using, which is that of (Erdélyi, 1953).

Also, in order to ensure that $P_n^m(z)$ is real for $|z| > 1$, Morse and Feshbach have introduced an additional factor of i^m (see page 1286).

Sleeman, B.D. (1967) *The Low-Frequency Scalar Dirichlet Scattering by a General Ellipsoid.* J. Inst. Maths Applics 3, 291-312.

Szegö, G. (1939) *Orthogonal Polynomials,* American Mathematical Society, New York.

Tricomi, F.G. (1948) *Serie Ortogonali di Funzioni,* S.I.E. Istituto Editoriale Gheroni, Torino.