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ASPECTS OF SPECTRAL THEORY FOR
ALGEBRAS OF MEASURABLE OPERATORS

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Abstract

Title: Aspects of Spectral Theory for Algebras of Measurable Operators.

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The spectral theory for bounded normal operators on a Hilbert space and the various functional calculi for such operators is closely related to the representation theory of commutative C^* - and von Neumann algebras as algebras of bounded continuous or measurable functions. For unbounded operators the corresponding theory leads to algebras of unbounded densely defined operators.

The thesis looks at aspects of spectral theory in the non-commutative generalisations of these algebras. Given a von Neumann algebra \mathcal{M} , there are various notions of measurability for operators affiliated with \mathcal{M} , and the measurable operators of a particular kind form an involutive algebra under the strong sum and product. Algebras of this kind can usually be equipped with a topology modelled on the topology of convergence in measure under which they become topological algebras. The emphasis in this thesis is on a semi-finite von Neumann algebra \mathcal{M} equipped with a semi-finite faithful normal trace τ and the corresponding algebra $\widetilde{\mathcal{M}}$ of τ -measurable operators. The notion of a locally τ -measurable operator is introduced and it is shown that the set $\mathcal{L}(\widetilde{\mathcal{M}})$ of such operators forms an algebra which becomes a topological algebra when equipped with an appropriate topology of local τ -convergence in measure.

Invertibility, continuity of inversion and conditions under which the group of invertible elements is open are studied in these algebras. It is shown that it is possible to give characterizations of the spectrum, essential spectrum and point spectrum of self-adjoint operators in $\widetilde{\mathcal{M}}$ in terms of the spectral family and generalized singular value function of the operator. Similar characterizations are considered for spectra of elements of $\widetilde{\mathcal{M}}_0$, the ideal of τ -compact operators, and for the operator obtained in the Schmidt decomposition. The spectrum relative to a subalgebra is also considered.

In the last part of the thesis the functional calculus for measurable operators is considered. Borel functions which preserve τ -measurability under the functional calculus are studied. Since $\widetilde{\mathcal{M}}$ is a generalized B*-algebra, the functional calculus for such algebras can be compared to the Borel functional calculus. Finally some spectral mapping results for elements of $\widetilde{\mathcal{M}}_0$ are given, and the functional calculus for the operators obtained using the Schmidt spectral decomposition investigated.

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Dedication

To the memory of my little boy Siphe-Uxolo.

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Cape Town, South Africa

Isaac Daniel Tembo

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Introduction

Spectral theory for bounded operators on a Hilbert space is closely related to C^* - and von Neumann algebra theory. The representation theory for commutative algebras of this kind can be used to derive spectral theorems and functional calculi for bounded normal operators, and the spectrum of such an operator can be identified with its spectrum as a member of such an algebra.

More care has to be taken when trying to impose an algebraic structure on sets of unbounded, not necessarily everywhere defined, linear operators. Even when dealing with closed densely defined operators, we have no guarantee that their usual sum and product will yield operators of the same kind. If we start with a von Neumann algebra, it is possible to define various types of unbounded operators related, or affiliated, to the von Neumann algebra. The motivation comes from the commutative situation, in which the von Neumann algebra can be represented as an L_∞ space, and the related unbounded operators can be represented by (unbounded) measurable functions. For this reason various notions of measurability for operators have been introduced. The conditions imposed in these definitions are strong enough to ensure that with appropriate modifications of the definitions of sums and products we obtain *algebras* of measurable operators. These algebras of unbounded operators can often be given a suitable topology which turns them into topological algebras (which need not be normed algebras). The motivation for these topologies comes from the corresponding topologies of convergence in measure in the commutative situation. It is also possible to take a topological approach: if a commutative von Neumann algebra is represented as a

space of continuous functions on a compact Hausdorff space X , the associated algebra of unbounded operators can be represented by functions which need only be defined on open dense subsets of X .

One of the main aims of the thesis is to look at aspects of spectral theory in topological algebras of unbounded operators. Although some general results for topological algebras are available, the intention here is to do a systematic study of these algebras of unbounded operators. Invertibility, continuity of inversion, openness of the group of invertible elements and characterizations of various types of spectra will be considered. A second aim of the thesis is a study of the spectral theory for measurable operators. Since all such operators are unbounded, the general spectral theory and functional calculi for unbounded operators are available. The thesis looks at what can be said more specifically in the case where we consider measurable operators.

We now give a more detailed discussion of the contents of each of the chapters.

In Chapter 1 we give a brief overview of certain well known results in functional analysis, measure theory, operator theory, spectral theory and the functional calculus that we shall need in the course of the thesis.

Chapter 2 deals with algebras of both bounded and unbounded functions and operators. In the first part of the chapter we review known results and in the second we introduce a new notion of local measurability and consider algebras of such locally measurable operators. The first four sections contains an overview of the relevant parts of the theory of von Neumann algebras, traces, unbounded operators affiliated with a von Neumann algebra, topological algebras and generalized B^* -algebras (GB^* -algebras). We then look at various notions of measurability for operators affiliated to a von Neumann algebra. The emphasis is on a semi-finite von Neumann algebra equipped with a semi-finite, faithful normal trace τ and the τ -

measurable operators affiliated to such an algebra. The algebra $\widetilde{\mathcal{M}}$ of τ -measurable operators is a complete metrizable topological $*$ -algebra with respect to the topology τ_m of convergence in measure. We show that it is also a GB $*$ -algebra, and survey some of its topological and order properties. A section is then devoted to algebras of densely defined functions on extremally disconnected compact Hausdorff spaces which can be used to represent the algebra of closed densely defined operators affiliated to a commutative von Neumann algebra. We show that such function algebras are isomorphic to algebras of measurable functions. In the last three sections we define the algebra $\mathcal{L}(\widetilde{\mathcal{M}})$ of locally τ -measurable operators, which contains the algebra of τ -measurable operators, and show that it is a $*$ -algebra with respect to strong sum, strong product and adjunction. Motivated by the way Dixon [Dix71] and Yeadon [Yea73] have defined the topology of local convergence in measure on the algebra of locally measurable operators in terms of the dimension function, we consider two topologies, τ_{lm} and τ_{lmc} of local convergence in measure on the algebra of locally measurable operators, both defined in terms of the trace. We show that under certain conditions we obtain a metrizable Hausdorff vector topology, and investigate continuity of multiplication and adjunction. In the commutative case both topologies turn out to be the well known measure theoretic topology of local convergence in measure. We end the chapter by investigating the local convexity of these topologies.

Chapter 3 deals with invertibility and the characterization of different kinds of spectra in algebras of unbounded functions and operators. We begin by giving results in the commutative case, where we use measure theoretic arguments, as a motivation for similar results in the general case, which are considered next. We then show that inversion is τ_m -continuous on $\widetilde{\mathcal{M}}$, and that under certain conditions we have τ_{lmc} -continuity of inversion on $\mathcal{L}(\widetilde{\mathcal{M}})$. We give a brief outline of the Schmidt spectral decomposition for τ -measurable operators first introduced by Ovchinnikov and then extended by Dodds, Dodds and de Pagter, and discuss its use in generating commutative subalgebras of the underlying von Neumann algebra which are proper in the sense that the trace on the von Neumann algebra restricted to the subalgebra is semi-finite. We show that the spectrum of an operator relative to a proper algebra

coincides with its spectrum in the algebra itself. In the next three sections we characterize various types of spectra for operators in $\widetilde{\mathcal{M}}$ and in the ideal $\widetilde{\mathcal{M}}_0$ of τ -compact operators, making use of the generalized singular value function. We also compare the spectrum of the operator S_0 occurring in the Schmidt spectral decomposition with the spectrum of the operator S itself. We end chapter 3 by showing that in $\widetilde{\mathcal{M}}$, the set of invertible elements is open, with respect to the topology τ_m of convergence in measure, if and only if $\widetilde{\mathcal{M}} = \mathcal{M}$.

Chapter 4 is devoted to the functional calculus for measurable operators. In the first section we look at those functions f which are τ -measurability preserving, that is to say, $f(S)$ is in $\widetilde{\mathcal{M}}$ whenever $S \in \widetilde{\mathcal{M}}$. We show that the set of all such Borel functions is a $*$ -algebra. This is followed by a comparison between the usual Borel functional calculus and the functional calculus introduced by Allan in [All65] in the GB*-algebra setting, both in the context of the algebra $\widetilde{\mathcal{M}}$. The last two sections are devoted to spectral mapping results for elements of $\widetilde{\mathcal{M}}_0$ and the functional calculus for the operators obtained using the Schmidt spectral decomposition.

Index of notation

We give a list of symbols that will be frequently used and a brief indication of their meaning. (For more details see also the Preliminaries).

H	Hilbert space
$\mathcal{B}(H)$	algebra of bounded operators on H
S, T, R	bounded or densely defined unbounded operators in a Hilbert space
I	identity operator on H
$D(S)$	domain of S , only used for unbounded operators
$\ker(S)$	kernel, or null space of S
$\text{ran}(S)$	range of S
$N(S)$	projection onto the kernel of S
$\mathbf{r}(S = I - N(S))$	right support of S
$\mathbf{l}(S) = I - N(S^*)$	left support of S
\mathcal{M}	a von Neumann algebra, understood to be a subalgebra of $\mathcal{B}(H)$
η	affiliation
τ	trace
\mathcal{M}'	commutant of \mathcal{M} in $\mathcal{B}(H)$
\mathcal{M}^p	lattice of self-adjoint projections in \mathcal{M}
\mathcal{M}^{sa}	set of self-adjoint operators in \mathcal{M}
\mathcal{M}^+	positive elements of \mathcal{M}
$\mathcal{N}(\mathcal{M})$	algebra of closed densely defined operators affiliated with \mathcal{M}
$Z(\mathcal{M})$	centre of \mathcal{M} , that is $\mathcal{M} \cap \mathcal{M}'$
\mathcal{A}^p	$\mathcal{A} \cap \mathcal{M}^p$, where it is usually understood that $\mathcal{A} \subset \mathcal{M}$
$P \sim Q$	P and Q are equivalent projections
$\widetilde{\mathcal{M}}$	algebra of τ -measurable operators
$\widetilde{\mathcal{M}}_0$	$\{x \in \widetilde{\mathcal{M}} : \mu_\infty(S) = 0\}$.
$\mathcal{S}(\mathcal{M})$	algebra of measurable operators
$\mathcal{L}(\mathcal{M})$	algebra of locally measurable operators

$\mathcal{L}(\widetilde{\mathcal{M}})$	algebra of locally τ – measurable operators
$d_t(S)$	distribution function of S in $\widetilde{\mathcal{M}}^{sa}$
$\mu_t(S)$	generalised singular function of S in $\widetilde{\mathcal{M}}$
$\mu_\infty(S) = \lambda_0$	$\lim_{t \rightarrow \infty} \mu_t(S)$
α_0	$\inf\{t > 0 : \mu_t(S) = \lambda_0\}$
w_0	weak-operator topology
s_0	strong-operator topology
τ_m	topology of convergence in measure
τ_{lm}	topology of local convergence in measure on $\widetilde{\mathcal{M}}$ defined in terms of the trace
γ_{lm}	Yeadon’s topology of local convergence in measure on $\mathcal{L}(\mathcal{M})$ defined in terms of the dimension function
$L_0(X, \Sigma, \mu)$	space of equivalence classes (a.e) of \mathbb{C} – valued measurable functions on (X, Σ, μ)
$L_\infty(X, \Sigma, \mu)$	space of equivalence classes (a.e) of \mathbb{C} – valued essentially bounded measurable functions on (X, Σ, μ)
$\widetilde{L}_\infty(X, \Sigma, \mu)$	space of equivalence classes (a.e) of \mathbb{C} – valued measurable functions essentially bounded except on a set of finite measure
$\widetilde{\mathcal{M}}(\epsilon, \delta)$	basic neighbourhood of 0 in the topology of convergence in measure on $\widetilde{\mathcal{M}}$
$\widetilde{\mathcal{M}}(Q, \epsilon)$	basic neighbourhood of 0 in the topology of local convergence in measure defined in terms of the trace

Chapter 1

Preliminaries

Throughout this thesis, we shall assume a knowledge of basic functional analysis as can be found in, for example [Con85], [KR83], [RSN53], [Sim63].

1.1 Topological Vector Spaces

Here we present some results about topological vector spaces that we shall use throughout this thesis. For further details the reader can consult [RR64] and [Jar81].

Definition 1.1.1 *A subset A of a vector space E is called*

1. *convex if, for all $x, y \in A$, $\lambda x + \mu y \in A$ whenever $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$.*
2. *balanced if for all $x \in A$, $\lambda x \in A$ whenever $|\lambda| \leq 1$.*
3. *absolutely convex if it is both convex and balanced.*
4. *absorbent if for each $x \in E$ there is some $\lambda > 0$ such that $x \in \mu A$ for all μ with $|\mu| \geq \lambda$.*

Definition 1.1.2 Let E be a vector space over a field \mathbb{F} . A non-negative (finite) real valued function p defined on E is called a seminorm if it satisfies

1. $p(x) \geq 0$
2. $p(\lambda x) = |\lambda|p(x)$
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$ and all $\lambda \in \mathbb{F}$.

Proposition 1.1.3 ([Jar81], 2.2.5, 2.8.1) Suppose E is a vector space and \mathcal{F} is a system of sets satisfying:-

1. \mathcal{F} is a filter base
2. for all $U \in \mathcal{F}$ there exists a $V \in \mathcal{F}$ such that $V + V \subset U$
3. for all $U \in \mathcal{F}$, U is balanced and absorbing.

Then \mathcal{F} induces upon translation a vector topology (that is, topology on E compatible with the algebraic structure of E) with \mathcal{F} a basic system of neighbourhoods of 0. A vector topology is metrizable if and only if it has a countable base of neighbourhoods of 0.

Definition 1.1.4 A vector topology on a vector space is said to be locally convex if there is a base for the neighbourhoods of 0 consisting of convex sets. A vector space with a locally convex topology is called a locally convex space.

Theorem 1.1.5 ([RR64]. I.3) Given any family \mathcal{P} of seminorms on a vector space E , there is a coarsest topology on E in which every seminorm is continuous. Under this topology E is a locally convex space and a base of closed neighbourhoods is formed by sets

$$\{x : \sup_{1 \leq i \leq n} p_i(x) \leq \epsilon\} \quad (\epsilon > 0, p_i \in \mathcal{P}).$$

1.2 Measure Theory

We will assume knowledge of elementary measure theory, as can be found in [Ber65], [Coh80] and [Hal74]. Below we give some terminology, notation and results that will be used frequently in the sequel.

Definition 1.2.1 ([Ber65], [Coh80], [Hal74]) *A measure space (X, Σ, μ) is said to be*

1. *finite if $\mu(X) < \infty$.*
2. *σ -finite if there exists a (disjoint) sequence $(X_n)_{n \in \mathbb{N}}$ in Σ such that $X = \cup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$ for every n .*
3. *localizable if there exists a family $\{X_\alpha : \alpha \in A\}$ in Σ such that $X = \cup_{\alpha \in A} X_\alpha$ and $\mu(X_\alpha) < \infty$ for all α , where A is some indexing set.*
4. *semi-finite if for each $E \in \Sigma$, there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$.*

We mention here that a localizable measure space is semi-finite.

Definition 1.2.2 *Suppose (X, Σ, μ) is a measure space. We denote by $L_0(X, \Sigma, \mu)$ the set of equivalence classes (modulo everywhere equality) of complex valued measurable functions on X , and by $L_\infty(X, \Sigma, \mu)$ the set of equivalence classes (modulo almost everywhere equality) of complex valued essentially bounded measurable functions on X . We will denote by $\tilde{L}_\infty(X, \Sigma, \mu)$ the set of all complex valued measurable functions essentially bounded except on a set of finite measure.*

We now define some topologies on $\tilde{L}_\infty(X, \Sigma, \mu)$ and $L_0(X, \Sigma, \mu)$.

Definition 1.2.3 *For $\epsilon, \delta > 0$, define*

$$N(\epsilon, \delta) = \{f \in \tilde{L}_\infty(X, \Sigma, \mu) : \mu\{x \in X : |f(x)| > \epsilon\} \leq \delta\}.$$

We write $N(\epsilon)$ for $N(\epsilon, \epsilon)$.

Let $\Sigma_f = \{E \in \Sigma : \mu(E) < \infty\}$. For $\epsilon > 0$ and $E \in \Sigma_f$, let

$$N(E, \epsilon) = \{f \in L_0 : \mu\{x \in E : |f(x)| > \epsilon\} \leq \epsilon\}.$$

Theorem 1.2.4 ([Wes90], 1.4)

The set $\{N(\epsilon, \delta) : \epsilon, \delta > 0\}$ forms a basic system of neighbourhoods at 0 for a complete metrizable vector topology on $\tilde{L}_\infty(X, \Sigma, \mu)$, called the topology of convergence in measure, and denoted by τ_m .

A sequence (f_n) converges to f in measure, written $f_n \rightarrow_{\tau_m} f$ if and only if for all $\epsilon > 0$, $\mu\{x \in X : |f_n(x) - f(x)| > \epsilon\} \rightarrow 0$ as $n \rightarrow \infty$.

The set $\{N(E, \epsilon) : E \in \Sigma_f, \epsilon > 0\}$ forms a basic system of neighbourhoods at 0 for a vector topology on L_0 , denoted by τ_{lm} , called the topology of local convergence in measure.

A net (f_α) converges to f with respect to the topology τ_{lm} (written $f_\alpha \rightarrow_{\tau_{lm}} f$) if and only if for all $\epsilon > 0$, for all $E \in \Sigma_f$, $\mu\{x \in E : |(f_\alpha - f)(x)| > \epsilon\} \rightarrow 0$, or equivalently if and only if for all $E \in \Sigma_f$, $f_\alpha \chi_E \rightarrow_{\tau_m} f \chi_E$.

1.3 Operator Theory

We give a summary of results about bounded as well as unbounded operators on Hilbert spaces. For further details the reader can consult [Con85], [KR83], [Zhu87] and [SZ79].

Throughout this thesis H will denote a Hilbert space and $\langle x, y \rangle$ the inner product of x and y in H . Denote by $\mathcal{B}(H)$ the algebra of bounded linear operators from H into itself. For $T \in \mathcal{B}(H)$ we denote by $T^* \in \mathcal{B}(H)$ the adjoint of T . T is *self-adjoint* if $T = T^*$, *normal* if $TT^* = T^*T$ and *unitary* if $TT^* = T^*T = I$, where I denotes the identity operator on H . Each $T \in \mathcal{B}(H)$ can be uniquely expressed as a linear combination of two self-adjoint elements, i.e. $T = S + iR$ where $S = \frac{1}{2}(T + T^*)$ and $R = \frac{1}{2i}(T - T^*)$. Denote by $\mathcal{B}(H)^{sa}$ the set of all self-adjoint elements of $\mathcal{B}(H)$. $T \in \mathcal{B}(H)^{sa}$ is called *positive*, written $T \geq 0$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$.

A partial ordering on $\mathcal{B}(H)^{sa}$ can be defined by saying that, for $T, S \in \mathcal{B}(H)^{sa}$, $T \geq S$ if and only if $T - S \geq 0$, that is $\langle Tx, x \rangle \geq \langle Sx, x \rangle$ for all $x \in H$.

A net $(T_\alpha) \subseteq \mathcal{B}(H)$ increases to $T \in \mathcal{B}(H)$, written $T_\alpha \uparrow_\alpha T$, if (T_α) is increasing with respect to the partial ordering and $\langle Tx, x \rangle = \sup_\alpha \langle T_\alpha x, x \rangle$ for all $x \in H$. We define convergence of a decreasing net similarly. We recall that the *strong-operator topology* is the locally convex topology on $\mathcal{B}(H)$ determined by the family of seminorms: $\{p_x : x \in H\}$ where $p_x(T) = \|Tx\|$. Thus the net (T_α) is *strong-operator convergent* to $T \in \mathcal{B}(H)$, denoted $T_\alpha \rightarrow_{so} T$, if $\|(T - T_\alpha)x\| \rightarrow_\alpha 0$ for all $x \in H$. The *weak-operator topology* is the locally convex topology determined by the family of seminorms: $\{p_{x,y} : x, y \in H\}$, where $p_{x,y}(T) = |\langle Tx, y \rangle|$. A net $(T_\alpha) \subseteq \mathcal{B}(H)$ is *weak-operator convergent* to $T \in \mathcal{B}(H)$, denoted $T_\alpha \rightarrow_{wo} T$, if $\langle (T - T_\alpha)x, y \rangle \rightarrow_\alpha 0$ in H for all $x, y \in H$.

Whilst in general the weak operator topology is weaker than the strong operator topology, the weak and strong-operator closures of a convex subset of $\mathcal{B}(H)$ coincide.

A projection in $\mathcal{B}(H)$ is a self-adjoint operator $P \in \mathcal{B}(H)$ such that $P^2 = P$. A projection is always positive and of norm one unless it is the zero projection. We shall denote the set of all projections in $\mathcal{B}(H)$ by $\mathcal{B}(H)^p$. It is a complete lattice, that is, every family of projections $\{P_\alpha\}$ has a least upper bound and greatest lower bound, with respect to the partial ordering in $\mathcal{B}(H)$, denoted respectively, by $\vee_\alpha P_\alpha$ and $\wedge_\alpha P_\alpha$. If $P, Q \in \mathcal{B}(H)^p$ we shall write $P \vee Q$ for their upper bound and $P \wedge Q$ for their lower bound. We have that $\wedge_\alpha P_\alpha$ is the projection onto $\bigcap_\alpha P_\alpha(H)$ and $\vee_\alpha P_\alpha$ is the projection onto the closure of the linear span of $\bigcup_\alpha P_\alpha(H)$. For a family of projections $\{P_\alpha\} \subseteq \mathcal{B}(H)^p$, $\vee_\alpha (I - P_\alpha) = I - \wedge_\alpha P_\alpha$ and $\wedge_\alpha (I - P_\alpha) = I - \vee_\alpha P_\alpha$ hold. An increasing net of projections (P_α) is strong operator convergent to $\vee_\alpha P_\alpha$. If $P, Q \in \mathcal{B}(H)^p$ then $P \leq Q$ if and only if $QP = PQ = P$. The projections $P, Q \in \mathcal{B}(H)^p$ are called *orthogonal* if $PQ = 0$. If P and Q commute, then $PQ = P \wedge Q$.

In what follows the term *operator* will mean a linear map from a subspace of H into H . The term *unbounded operator* will be used for an operator which is not necessarily bounded.

Let $T : D(T) \rightarrow H$ be a linear operator, where $D(T)$ denotes the domain of T , a linear subspace of H . With T we associate its graph $\mathcal{G}(T) = \{(x, Tx) : x \in D(T)\} \subseteq H \times H$.

An operator T is said to be closed if its graph is closed in the product topology of $H \times H$. This is equivalent to saying that if (x_n) is a sequence in $D(T)$ converging to x and Tx_n converges to y then $x \in D(T)$ and $Tx = y$. The operator T is densely defined if $D(T)$ is dense in H . If T is defined on all of H and its graph is closed, then by the Closed Graph Theorem T is bounded.

We say that S is an *extension* of T , written $T \subseteq S$, whenever $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$. Two operators S and T are said to be equal, and we write $T = S$, if $D(T) = D(S)$ and $Tx = Sx$ for all $x \in D(T)$.

We let $\text{ran}(T)$ denote the range of T and $\text{ker}(T)$ the kernel of T . The kernel of T is closed if T is closed.

The closure $\overline{\mathcal{G}(T)}$ of $\mathcal{G}(T)$ is a linear subspace of $H \times H$. When $\overline{\mathcal{G}(T)}$ is the graph of an operator S , we say that T is *preclosed* and write $S = \overline{T}$. We call \overline{T} the *closure* of T ; it is clearly an extension of T .

If T is a linear operator and D is a subspace of $D(T)$ with the property that the graph of T is contained in the the closure of the graph of the restriction of T to D , the D is called a *core* for T .

The adjoint T^* of a densely defined linear operator T is defined as follows: $D(T^*)$ consists of those y in H such that for some z in H we have that $\langle Tx, y \rangle = \langle x, z \rangle$ for each $x \in D(T)$. For such y we put $T^*y = z$; since T is densely defined, T^* is well-defined. It also follows that T^* is a closed linear operator.

For operators S and T we define $S + T$ and ST by putting

$$D(S + T) = D(S) \cap D(T) \text{ and } (S + T)(x) = Sx + Tx \text{ for } x \in D(S + T)$$

$$D(ST) = \{x \in D(T) : Tx \in D(S)\} \text{ and } (ST)(x) = S(Tx) \text{ for } x \in D(ST).$$

If T is an injective operator, we define the inverse T^{-1} of T by putting $D(T^{-1}) = \text{ran}(T)$ and for $y = Tx \in \text{ran}(T)$, $T^{-1}y = x$. Then we have $TT^{-1} \subseteq I, T^{-1}T \subseteq I$.

Proposition 1.3.1 ([KR83], 5.6), ([RSN53], VIII.114) Suppose S, T, R are densely defined operators.

1. If $S \subseteq T$ and $R \subseteq Q$, then $S + R \subseteq T + Q$.
2. If $S \subseteq T$, then $RS \subseteq RT$ and $SR \subseteq TR$.
3. $(S + T)R = SR + TR$, $RS + RT \subseteq R(S + T)$.
4. If $S \subseteq T$, then $T^* \subseteq S^*$.
5. If $S + T$ is densely defined, then $S^* + T^* \subseteq (S + T)^*$.
6. If ST is densely defined, $T^*S^* \subseteq (ST)^*$.
7. When S is bounded, then $T^* + S^* = (T + S)^*$ and $T^*S^* = (ST)^*$.

If ST and $S + T$ are preclosed, we shall denote by \overline{ST} and $\overline{S + T}$ the strong product and strong sum of the densely defined operators S and T , that is the closures of the ordinary product and sum.

The densely defined operator T is called *self-adjoint* if $T = T^*$ and *normal* if it is closed, densely defined and $TT^* = T^*T$. A self-adjoint operator T is called *positive*, written $T \geq 0$ if

$$\langle Tx, x \rangle \geq 0 \text{ for all } x \in D(T).$$

The projection of H onto $\ker(T)$ is denoted by $N(T)$. The *right support* of T is the projection $\mathbf{r}(T) = I - N(T)$. It is the smallest projection P such that $T = TP$. The left support of T , denoted $\mathbf{l}(T)$, is the projection onto the closure of $\text{ran}(T)$ and is the smallest projection P such that $T = PT$. Furthermore, we have that

$$\mathbf{l}(T) = I - N(T^*), \quad N(T) = I - \mathbf{l}(T^*),$$

$$\mathbf{l}(T^*T) = \mathbf{l}(T^*), \quad N(T^*T) = N(T).$$

When T is self-adjoint then $\mathbf{l}(T) = \mathbf{r}(T)$ and we shall call this the support of T and denote it by $\mathbf{s}(T)$.

A *partial isometry* is an operator $V \in \mathcal{B}(H)$ such that for x in $(\ker(V))^\perp$, $\|Vx\| = \|x\|$. The space $(\ker(V))^\perp$ is called the *initial space* of V and the space $\text{ran}(V)$ (which is closed) is called the *final space* of V . The operator $V \in \mathcal{B}(H)$ is a partial isometry if and only if $P = V^*V$ and $Q = VV^*$ are projections; P is the projection onto $(\ker(V))^\perp$ and called the initial projection of V and Q is the projection onto $\text{ran}(V)$ and called the final projection of V .

1.4 Spectral Theory

We now present most of the spectral theory for unbounded operators that we shall need in this thesis. Details can be found in [KR83], [DS71], [Zhu87] and [RSN53]. In what follows H will be a Hilbert space.

If S is an injective operator, it has an inverse $S^{-1} : \text{ran}(S) \rightarrow H$. If S is a closed, densely defined and bijective, S^{-1} is everywhere defined and it follows from the closed graph theorem that S^{-1} is bounded and maps H onto $D(S)$.

Definition 1.4.1 *Let S be a closed densely defined linear operator. We define the resolvent set of S by*

$$\rho(S) = \{\lambda \in \mathbb{C} : \lambda I - S \text{ is bijective}\},$$

and the spectrum of S by $\sigma(S) = \mathbb{C} \setminus \rho(S)$.

The point spectrum of S , denoted by $\sigma_p(S)$, is defined by

$$\sigma_p(S) = \{\lambda \in \mathbb{C} : \text{there exists an } x \in H, x \neq 0 \text{ such that } Sx = \lambda x\}.$$

The elements of the point spectrum are called eigenvalues.

Definition 1.4.2 A spectral measure is a Boolean algebra homomorphism

$$\mathfrak{B}(\mathbb{C}) \rightarrow \mathcal{B}(H)^p : B \rightarrow E_B$$

such that $E_{\mathbb{C}} = I$, where $\mathfrak{B}(\mathbb{C})$ denotes the σ -algebra of Borel measurable subsets of the complex plane \mathbb{C} . A spectral measure is said to be countably additive if $\sum_{i=1}^{\infty} E_{B_i} = E_{\cup_{i=1}^{\infty} B_i}$ for (B_i) a disjoint sequence in $\mathfrak{B}(\mathbb{C})$.

Definition 1.4.3 A family $\{E_t : t \in \mathbb{R}\}$ of projections on H is called a resolution of the identity if

1. for $t_1 \leq t_2$, $E_{t_1} \leq E_{t_2}$;
2. The family is right continuous, that is, $E_s \downarrow E_t$ as $s \downarrow t$;
3. $E_t \uparrow I$ as $t \uparrow \infty$;
4. $E_t \downarrow 0$ as $t \downarrow -\infty$.

Theorem 1.4.4 ([DS71] Theorem XII 2.3, [KR83] Theorems 5.6.10, 5.2.3)

Suppose S is a self-adjoint operator acting on H . Then its spectrum, $\sigma(S)$, is real, and there exists a countably additive spectral measure $E(S)$ with the following properties:

1. $E(S)$ vanishes off $\sigma(S)$.
2. If for $t \in \mathbb{R}$, we write $E_t(S) = E_{(-\infty, t] \cap \sigma(S)}(S)$, then the family $\{E_t(S) : t \in \mathbb{R}\}$ is a resolution of the identity.
3. $SE_t(S) \leq tE_t(S)$ for all $t \in \mathbb{R}$.
4. $t(I - E_t(S)) \leq S(I - E_t(S))$ for all $t \in \mathbb{R}$.
5. If for each $n \in \mathbb{N}$, $F_n = E_n - E_{-n}$ then $\cup_{n=1}^{\infty} F_n(H)$ is a core for S .
6. For each n , $Sx = \int_{-n}^n t dE_t x$ for $x \in F_n(H)$, in the sense of norm convergence of approximating Riemann sums.

The family $\{E_t(S) : t \in \mathbb{R}\}$ is called the spectral family for S , or the spectral resolution of S . The spectral family $\{E_t(S) : t \in \mathbb{R}\}$ is uniquely determined by (5) and (6): If $\{E'_t : t \in \mathbb{R}\}$ is a resolution of the identity on H such that $Sx = \int_{-n}^n \lambda dE'_\lambda x$ for each $x \in F'_n(H)$ and all n , where $F'_n = E'_n - E'_{-n}$ and $\cup_{n=1}^\infty F'_n(H)$ is a core for S , then $E_t = E'_t$ for all $t \in \mathbb{R}$.

If S is a bounded self adjoint operator, then $S = \int_{-\|S\|}^{\|S\|} t dE_t(S)$.

S is positive if and only if $E_t(S) = 0$ for all $t < 0$.

If S is positive then $N(S) = E_0(S)$.

For $s, t \in \mathbb{R}$, with $s < t$ we have:

$$E_{(t, \infty)}(S) = I - E_t(S)$$

$$E_{(-\infty, t)}(S) = \sup_{\theta < t} E_{(-\infty, \theta]}(S) \text{ and } E_{(-\infty, t)}(S) = E_{[0, t]}(S) \text{ if } S \text{ is positive}$$

$$E_{(s, t]}(S) = E_{(-\infty, t]}(S) - E_{(-\infty, s]}(S)$$

$$E_{[s, t]}(S) = E_{(-\infty, t]}(S) - E_{(-\infty, s)}(S)$$

$$E_{(s, t)}(S) = E_{(-\infty, t)}(S) - E_{(-\infty, s]}(S)$$

$$E_{[s, t)}(S) = E_{(-\infty, t)}(S) - E_{(-\infty, s)}(S).$$

1.5 The Functional calculus

Here we present some results about the functional calculus for self-adjoint operators. There are similar results for normal operators ([KR83] 5.6.26.), but we shall mainly be dealing with self-adjoint operators in what follows.

Theorem 1.5.1 ([DS71] Theorems XII 2.6, 2.7 and 2.9, [KR83] 5.6.26., 5.6.29)

Suppose S is a self-adjoint operator acting on a Hilbert space H with spectral family $\{E_t(S) : t \in \mathbb{R}\}$ and $\mathcal{B}(\sigma(S))$ is the algebra of all complex Borel functions on the spectrum of S . For $f \in \mathcal{B}(\sigma(S))$, there is an operator $f(S)$ with the following properties:

1. $D(f(S)) = \{x \in H : \int_{-\infty}^{\infty} |f(t)|^2 d\|E_t(S)x\|^2 < \infty\}$.

2. For $x \in D(f(S))$ and $y \in H$, $\langle f(S)x, y \rangle = \int_{-\infty}^{\infty} f(t) d\langle E_t(S)x, y \rangle$.

3. $f(S)$ is closed and densely defined.
4. $f(S)$ commutes with $E_B(S)$ for any Borel subset B of $\sigma(S)$.
5. $\sigma(f(S))$ is the intersection of all sets $\overline{f(B)}$ where B varies over the Borel subsets of $\sigma(S)$ with $E_B(S) = I$.
6. If f is real valued then $f(S)$ is self-adjoint and

$$E_B(f(S)) = E_{f^{-1}(B)}(S) \quad (1.1)$$

for every Borel subset B of $\sigma(S)$. This result will be referred to as **the change of measure principle** in what is to follow.

7. The map $f \rightarrow f(S)$ is σ -normal, that is, for every increasing sequence (f_n) of real-valued Borel functions converging point-wise to the real-valued Borel function f , $f_n(S) \uparrow_{so} f(S)$.
8. If $f = \chi_B$, where B is a Borel set, then $f(S) = E_B(S)$
9. $f(S)^* = \bar{f}(S)$
10. If $g, f \in \mathcal{B}(\sigma(S))$ then

$$f(S) + g(S) \subset (f + g)(S)$$

$$f(S)g(S) \subset (fg)(S)$$

$$(f \circ g)(S) = f(g(S)).$$

For every scalar α , $(\alpha f)(S) = \alpha f(S)$, with the exception that if $\alpha = 0$, then $\alpha f(S)$ is the everywhere defined zero operator.

Example 1.5.2 ([SZ79], 9.11, 9.14, 9.26, 9.28), ([RSN53], IX.128) Let S be an operator on H .

1. If S is positive, $\sigma(S) \subseteq [0, \infty)$ and $f(t) = t^{\frac{1}{2}}$ defines a Borel function on $\sigma(S)$. We denote $f(S)$ by $S^{\frac{1}{2}}$.

2. If S is densely defined, S^* exists and S^*S is positive. We put $|S| = (S^*S)^{\frac{1}{2}}$. Similarly, $|S^*| = (SS^*)^{\frac{1}{2}}$.
3. Let S be self-adjoint and injective, and $f(t) = \frac{1}{t}$ if $t \neq 0$ and $f(t) = 0$ for $t = 0$. Then $f(S) = S^{-1}$, as defined earlier, and if $g(t) = t$, $f(S)S = f(S)g(S) \subseteq (fg)(S) = I$.

1.6 Polar Decomposition

Theorem 1.6.1 ([KR86], section 6.1), ([SZ79], sections 9.29, 9.30)

Suppose S is a closed densely defined operator on H and $|S| = (S^*S)^{\frac{1}{2}}$. Then

1. There is a partial isometry V with initial space the closure of $\text{ran}(|S|)$ and final space the closure of $\text{ran}(S)$ such that $S = V|S| = (SS^*)^{1/2}V$ and $S^* = V^*|S^*|$.
2. $D(S) = D(|S|)$, $\ker(S) = \ker(|S|)$
3. $\mathbf{1}(|S|) = \mathbf{1}(S^*) = \mathbf{1}(S^*S)$.
4. $V^*V = \mathbf{1}(|S|) = \mathbf{s}(|S|) = E_{(0,\infty)}(|S|)$.
5. V and $|S|$ are uniquely determined up to $|S|$ being positive and V being a partial isometry with initial space the closure of $\text{ran}(S^*)$.
6. $SS^* = VS^*SV^*$, and so restricted to the closures of $\text{ran}(S^*)$ and $\text{ran}(S)$ respectively, S^*S and SS^* are unitarily equivalent, and the equivalence is implemented by V .
7. $|S^*| = V|S|V^*$.

Chapter 2

Algebras of Unbounded Operators

In this chapter we introduce the algebras that will be used in the rest of the thesis. These are mostly algebras of unbounded functions or unbounded operators and generally not Banach algebras. In the first part of the chapter we review known results that will be needed later. The second part of the chapter introduces a new notion of local measurability and corresponding algebras of locally measurable operators.

The chapter starts with a brief introduction to von Neumann algebras and traces on such algebras, and this is followed by a discussion of the notion of unbounded operators affiliated to a von Neumann algebra. Some basic results from the theory of topological algebras are presented next. The generalised B^* -algebras (GB*-algebras) introduced by Allan and Dixon provide a framework for discussion of algebras of unbounded operators; these are briefly discussed in the next section.

In the rest of the chapter we consider various types of operators affiliated with a given von Neumann algebra. The emphasis is on a semi-finite von Neumann algebra \mathcal{M} equipped with a semi-finite faithful normal trace τ , and the algebra $\widetilde{\mathcal{M}}$ of τ -measurable operators. We introduce the topology of convergence in measure on this algebra and show that it is a GB*-algebra. The generalised singular functional is an important tool in the theory of measurable operators. We list some of its important properties, as well as the ideal of τ -

compact operators in $\widetilde{\mathcal{M}}$. The order properties of $\widetilde{\mathcal{M}}$ and the local convexity of the measure topology are also discussed.

If \mathcal{M} is a commutative von Neumann algebra, it is isometrically isomorphic to $C(X)$, where X is an extremely disconnected compact Hausdorff space. When considering commutative algebras of unbounded operators affiliated with \mathcal{M} , it becomes necessary to introduce algebras of unbounded functions on X , called the algebras of normal functions. This makes it possible to use a topological approach to aspects of spectral theory. Since the extremely disconnected spaces occurring in the representation of von Neumann algebras admit normal measures, it is possible to represent these function algebras as algebras of measurable functions as well. We discuss the algebra of normal functions, which is a topological algebra and show that there is an isomorphism between the algebra of normal functions and the algebra of Borel measurable functions.

Various authors have considered notions of local measurability and topologies of local convergence in measure on algebras of such operators in the context of general von Neumann algebras. In the last part of the chapter we restrict attention to a semi-finite von Neumann algebra equipped with a semifinite normal trace τ . We define the algebra $\mathcal{L}(\widetilde{\mathcal{M}})$ of locally τ -measurable operators, which contains $\widetilde{\mathcal{M}}$, and show that it is a $*$ -algebra with respect to strong sum, strong product and adjunction. We introduce two topologies τ_{lm} and τ_{lmc} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ and show that under certain conditions this topology is a metrizable Hausdorff vector topology. In the commutative case this topology turns out to be the well known topology of local convergence in measure. We also show that under certain conditions multiplication is jointly τ_{lmc} -continuous and adjunction is τ_{lmc} -continuous. We end this chapter by discussing the local convexity of the topology τ_{lmc} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$.

2.1 von Neumann Algebras and traces

We present a summary of some standard results on von Neumann algebras that we shall use throughout this thesis. Details and further results can be found in, for example, [Dix81], [KR83], [Sak79], [SZ79] and [Tak79].

Let H be a Hilbert space, and $B(H)$ the algebra of all bounded linear operators on H . A subalgebra \mathcal{A} of $B(H)$ is self-adjoint, or a $*$ -subalgebra, if $T \in \mathcal{A} \Rightarrow T^* \in \mathcal{A}$. The commutant of a subset \mathcal{W} of $B(H)$, denoted by \mathcal{W}' , is the set of all operators in $B(H)$ that commute with every element of \mathcal{W} . The commutant is weak operator closed. The commutant $(\mathcal{W}')'$ of the commutant \mathcal{W}' is called the bicommutant, or double commutant, of \mathcal{W} and denoted by \mathcal{W}'' .

Throughout this thesis \mathcal{M} will denote a von Neumann algebra, that is, a $*$ -subalgebra of $B(H)$ that is closed in the weak operator topology and contains the identity I . The centre $Z(\mathcal{M})$ of \mathcal{M} is the set $\mathcal{M} \cap \mathcal{M}'$. We say that \mathcal{M} is a factor if $Z(\mathcal{M}) = \{\lambda I : \lambda \in \mathbb{C}\}$.

Theorem 2.1.1 ([KR83], 5.3.1) *If \mathcal{A} is a self-adjoint subalgebra of $B(H)$ containing the identity, then the weak and strong operator closures of \mathcal{A} coincide with \mathcal{A}'' .*

Definition 2.1.2 1. *Two projections E and F in a von Neumann algebra \mathcal{M} are said to be equivalent, written $E \sim F$ when $E = V^*V$ and $F = VV^*$ for some partial isometry V in \mathcal{M} .*

2. *If E and F are projections in \mathcal{M} , we say that E is weaker than F , written $E \preceq F$, when E is equivalent to a subprojection of F .*

3. *A projection E in \mathcal{M} is said to be finite if when $E \sim F \leq E$ then $E = F$; infinite if it is not finite; and purely infinite (or type III) if there is no non-zero finite projection F_1 such that $F_1 < E$. \mathcal{M} is said to be finite, infinite, or purely infinite if the identity I is respectively finite, infinite, or purely infinite.*

4. \mathcal{M} is said to be semi-finite if any non-zero central projection contains a non zero finite projection.

Lemma 2.1.3 ([Nel74], section 1) If P and Q are projections in \mathcal{M} with $P \wedge Q = 0$, then $P \preceq I - Q$.

Definition 2.1.4 A projection $P \in \mathcal{M}$ is said to be countably decomposable if any family of mutually orthogonal non-zero sub-projections of P in \mathcal{M} is at most countable. When I is countably decomposable we say that the von Neumann algebra \mathcal{M} is countably decomposable.

For more details on equivalence of projections and types, the reader can consult ([KR86]6.1, 6.2) or ([SZ79], chapter 4).

Definition 2.1.5 Let \mathcal{M} be a von Neumann algebra and \mathcal{M}^+ denote the set of positive elements of \mathcal{M} . A trace on \mathcal{M} is a function $\tau : \mathcal{M}^+ \rightarrow [0, \infty]$ such that

1. $\tau(T + S) = \tau(T) + \tau(S)$ for all $T, S \in \mathcal{M}^+$,
2. $\tau(\lambda T) = \lambda \tau(T)$ for all $\lambda \in \mathbf{R}^+$ and all $T \in \mathcal{M}^+$
3. $\tau(TT^*) = \tau(T^*T)$ for all $T \in \mathcal{M}$.

A trace τ is said to be

normal if for every net (T_i) in \mathcal{M}^+ such that $T_i \uparrow T \in \mathcal{M}$, $\tau(T_i) \uparrow \tau(T)$;

faithful if $T \in \mathcal{M}^+$, $\tau(T) = 0 \Rightarrow T = 0$;

finite if $\tau(I) < \infty$;

semi-finite if for every $0 < T \in \mathcal{M}^+$ there exists $0 < R \leq T$ such that $\tau(R) < \infty$, (equivalently, for every $0 < S \in \mathcal{M}^+$, there exist a net (S_i) in \mathcal{M}^+ , with $\tau(S_i) < \infty$ for every i , such that $S_i \uparrow_{so} S$).

Example 2.1.6 Consider the von Neumann algebra $\mathcal{M} = \mathcal{B}(H)$ of all bounded linear operators on the Hilbert space H . Let $\{e_i\}_{i \in I}$ be an orthonormal basis for H . We define

$$\tau : \mathcal{B}(H)^+ \rightarrow [0, \infty] : S \rightarrow \sum_{i \in I} \langle Se_i, e_i \rangle.$$

Then it can be shown that τ is a faithful semi-finite normal trace on $\mathcal{B}(H)$, called the diagonal trace, which is independent of the choice of $\{e_i\}_{i \in I}$.

Example 2.1.7 [Seg51], ([Sak79], 1.18), ([Tak79]III 1.18). If \mathcal{M} is a commutative von Neumann algebra, then \mathcal{M} is $*$ -isomorphic to $L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ) . We define a trace $\tau : L_\infty(X, \Sigma, \mu) \rightarrow [0, \infty]$ by $\tau(f) = \int_X f d\mu$. τ is faithful semi-finite and normal (by the Monotone Convergence Theorem), and finite if and only if μ is finite.

A von Neumann algebra is semi-finite if and only if it admits a faithful semi-finite normal trace ([Tak79], V. 2.15).

Denote by \mathcal{M}^p the set of all self-adjoint projections in \mathcal{M} . \mathcal{M}^p is a complete lattice under the usual ordering of self-adjoint operators.

Proposition 2.1.8 ([KR83], 2.5) Let \mathcal{M} be a von Neumann algebra and τ a trace on \mathcal{M} .

1. If $P_1, P_2, \dots, P_n \in \mathcal{M}^p$, then $\tau[\bigvee_{i=1}^n P_i] \leq \sum_{i=1}^n \tau(P_i)$.
2. If $E, F \in \mathcal{M}^p$, then $E \sim F$ implies $\tau(E) = \tau(F)$, and $E \preceq F$ implies $\tau(E) \leq \tau(F)$.
3. If $E, F \in \mathcal{M}^p$, then $E \vee F - E \sim F - E \wedge F$ and hence $\tau(E \vee F - E) = \tau(F - E \wedge F)$.

2.2 Affiliation

Definition 2.2.1 We say that a densely defined operator T is affiliated with a von Neumann algebra \mathcal{M} (and we write $T \eta \mathcal{M}$) when $U^*TU = T$ for each unitary operator U in \mathcal{M} .

Proposition 2.2.2 ([Wes90], 7.2) $S\eta\mathcal{M} \iff$ for all $R \in \mathcal{M}'$, $RS \subseteq SR$.

Denote by $\mathcal{N}(\mathcal{M})$ the set of all closed densely defined operators affiliated with \mathcal{M} .

Proposition 2.2.3 ([Wes90], 7.6), ([SZ79], 9.29) Let \mathcal{M} be a von Neumann algebra and S a densely defined operator on H . Then

1. If S is bounded, then $S\eta\mathcal{M}$ if and only if $S \in \mathcal{M}$.
2. If S is preclosed and $S\eta\mathcal{M}$, then $\overline{S}\eta\mathcal{M}$.
3. If S is closed, with polar decomposition $S = V|S|$, then $S\eta\mathcal{M}$ if and only if $V \in \mathcal{M}$ and $|S|\eta\mathcal{M}$.
4. $S\eta\mathcal{M}$ implies $S^*\eta\mathcal{M}$.
5. If $R, S\eta\mathcal{M}$, then $R + S\eta\mathcal{M}$, $RS\eta\mathcal{M}$.

Suppose S is a self adjoint operator acting on a Hilbert space H . Then its spectrum $\sigma(S)$ is real, and therefore $S + iI$ and $S - iI$ are boundedly invertible. The inverses S_+ and S_- of $S + iI$ and $S - iI$ respectively are bounded everywhere defined operators with norm not exceeding 1. ([KR83], 5.6.7).

Theorem 2.2.4 ([KR83], 5.6), ([MVN36], 4.1)

Let S be a self-adjoint operator on H , with spectral family $\{E_t(S) : t \in \mathbb{R}\}$.

1. The von Neumann algebra \mathcal{A} generated by I, S_+ and S_- is abelian (and referred to as the von Neumann algebra generated by S).
2. $S\eta\mathcal{A}$, and \mathcal{A} is the smallest von Neumann algebra to which S is affiliated.
3. $\{E_t(S) : t \in \mathbb{R}\} \subseteq \mathcal{A}$.
4. If \mathcal{M} is a von Neumann algebra, then $S\eta\mathcal{M}$ if and only if $\{E_t(S) : t \in \mathbb{R}\} \subseteq \mathcal{M}$.

5. If $S \eta \mathcal{M}$ and f is a real-valued measurable function, then $f(S)$ is closed, densely defined and affiliated with \mathcal{M} .

Corollary 2.2.5 *Let S be a closed densely defined operator with polar decomposition $S = V|S|$, and \mathcal{M} a von Neumann algebra. Then $S \eta \mathcal{M} \Leftrightarrow V \in \mathcal{M}$ and $\{E_t(|S|) : t \geq 0\} \subseteq \mathcal{M}$.*

We shall repeatedly use the following lemma in the sequel.

Lemma 2.2.6 *Let S be a closed, densely defined injective normal operator affiliated with a von Neumann algebra \mathcal{M} . Then its inverse $S^{-1} : \text{ran}(S) \rightarrow D(S)$, defined by $S^{-1}(Sx) = x$ for every $x \in D(S)$, is also affiliated with \mathcal{M} .*

Proof: Since S is normal, $SS^* = S^*S$ so that $|S| = |S^*|$. Then we have $\ker(S) = \ker(|S|) = \ker(|S^*|) = \ker(S^*)$. Since S is injective we have that S^* is also injective so that $\text{ran}(S)$ is dense in H , so that S^{-1} is densely defined (see section 1.4). Since S is closed, S^{-1} is closed as well. Since $S \eta \mathcal{M}$, we have $US = SU$ for every $U \in \mathcal{M}'$, that is

$$D(S) = D(SU) \text{ and } USx = SUx \quad \text{for all } x \in D(S). \quad (2.1)$$

We want to show that for every $U \in \mathcal{M}'$, $US^{-1} = S^{-1}U$. Let $x \in \text{ran}(S)$, then $x = Sz$ for some $z \in D(S) = D(SU)$. Then we have

$$Ux = USz = SUz \in \text{ran}(S) = D(S^{-1}), \quad \text{from equation (2.1)}. \quad (2.2)$$

Hence $x \in D(S^{-1}U)$. Thus $\text{ran}(S) \subseteq D(S^{-1}U)$.

Conversely, if $x \in D(S^{-1}U)$, then $Ux \in D(S^{-1}) = \text{ran}(S)$ so that $x = U^*Ux \in U^*(\text{ran}(S))$.

Now

$$\begin{aligned} x &= U^*Ux = U^*Sz = SU^*z, \text{ for some } z \in D(S) \quad \text{from equation (2.1)} \\ &\in \text{ran}(S) \end{aligned}$$

Hence $D(S^{-1}U) \subseteq \text{ran}(S)$, and so, $D(S^{-1}U) = \text{ran}(S) = D(S^{-1})$.

Now if $x \in D(S^{-1}) = \text{ran}(S)$, $x = Sz$, for some $z \in D(S)$. Now

$$\begin{aligned} S^{-1}Ux &= S^{-1}USz \\ &= S^{-1}SUz \\ &= Uz \\ &= US^{-1}x. \end{aligned}$$

Hence we have that $S^{-1}\eta\mathcal{M}$. Δ

Proposition 2.2.7 ([KR83], 5.6.18) *S is normal if and only if it is affiliated with an abelian von Neumann algebra. If S is normal, there is a smallest von Neumann algebra \mathcal{A}_0 such that S is affiliated to \mathcal{A}_0 . The algebra \mathcal{A}_0 is abelian.*

Remark: If S is normal and $S\eta\mathcal{M}$, then $\mathcal{A}_0 \subseteq \mathcal{M}$.

Corollary 2.2.8 *Suppose $S\eta\mathcal{M}$ is normal, and f is a Borel function on $\sigma(S)$. Then*

$$f(S) \in \mathcal{N}(\mathcal{M}).$$

Proof: By Proposition 2.2.7, S is affiliated with an abelian von Neumann subalgebra \mathcal{A}_0 of \mathcal{M} . By ([KR83], Theorem 5.6.26), $f(S) \in \mathcal{N}(\mathcal{A}_0)$. Now $\mathcal{A}_0 \subseteq \mathcal{M}$ implies that $\mathcal{M}' \subseteq \mathcal{A}'_0$, and so $f(S)\eta\mathcal{M}$. Being closed and densely defined, $f(S) \in \mathcal{N}(\mathcal{M})$.

Δ

Definition 2.2.9 *Let $S \in \mathcal{M}$ and $Q \in \mathcal{M}^p$. Denote by S_Q the restriction of QS to $Q(H)$. We define $\mathcal{M}_Q = \{S_Q : S \in \mathcal{M}\}$. \mathcal{M}_Q is called the reduction of \mathcal{M} by Q .*

\mathcal{M}_Q is a von Neumann algebra acting on $Q(H)$, since $(\mathcal{M}')_Q = (\mathcal{M}_Q)'$ ([Dix81], I.2.1).

2.3 Topological Algebras

In this section we present some results on topological algebras that we shall need in the course of this thesis, in particular those related to invertibility. For further results the reader can consult [Mal86], [Zel65] and [Zel71].

Definition 2.3.1 *A topological algebra (\mathcal{A}, γ) , is a (complex) algebra \mathcal{A} with identity together with a Hausdorff topology γ such that \mathcal{A} is a topological vector space and multiplication is separately continuous. If \mathcal{A} is a $*$ -algebra and the involution mapping $w \rightarrow w^*$ is continuous, then \mathcal{A} is called a topological $*$ -algebra.*

Definition 2.3.2 *An element w in a topological algebra \mathcal{A} with unit e is said to be invertible in \mathcal{A} if there exists a $v \in \mathcal{A}$ such that $wv = vw = e$. It can be shown that v is unique; we write $v = w^{-1}$ and call it the inverse of w . We denote the set of invertible elements of \mathcal{A} by \mathcal{Q} . If $\lambda \in \mathbb{C}$ and the element $w - \lambda e$ is not invertible in \mathcal{A} , then we say that λ lies in the spectrum $\sigma_{\mathcal{A}}(w)$ of w in \mathcal{A} . Denote by $\rho_{\mathcal{A}}(w)$ the resolvent set of w , that is, the set of all $\lambda \in \mathbb{C}$ such that $w - \lambda e$ is invertible in \mathcal{A} .*

Definition 2.3.3 *A topological algebra in which the map $w \rightarrow w^{-1}$ is continuous at e , the identity, is called an algebra with continuous inversion.*

In an algebra with continuous inversion the map $w \rightarrow w^{-1}$ is continuous everywhere on \mathcal{Q} .

Definition 2.3.4 *The topological algebra \mathcal{A} is called a \mathcal{Q} -algebra if the set \mathcal{Q} of all invertible elements in \mathcal{A} is open.*

Lemma 2.3.5 ([BS77], 4.8-3.) *For any topological algebra \mathcal{A}*

1. *\mathcal{A} is a \mathcal{Q} -algebra if and only if the set of invertible elements has a non empty interior.*
2. *If \mathcal{A} is a \mathcal{Q} -algebra then the spectrum $\sigma_{\mathcal{A}}(w)$ of each element $w \in \mathcal{A}$ is bounded.*

Lemma 2.3.6 *If the identity $e \in \mathcal{A}$ has no neighbourhood in \mathcal{Q} , then every $y \in \mathcal{Q}$ has no neighbourhood contained in \mathcal{Q} .*

Proof: Suppose there is a $y \in \mathcal{Q}$ and a neighbourhood $N(y)$ of y with $N(y) \subseteq \mathcal{Q}$. Since the map $x \rightarrow y^{-1}x$ is a homeomorphism of \mathcal{Q} to \mathcal{Q} , $y^{-1}N(y)$ is a neighbourhood of e in \mathcal{Q} , a contradiction.

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Every Banach algebra is a \mathcal{Q} -algebra with continuous inversion. The spectrum of an element in a Banach algebra is always compact and non-empty, but this is not necessarily the case in a topological algebra.

2.4 GB*-Algebras

We give a brief summary of the definition and properties of the so called generalized Banach *-algebras (GB*-algebras). For more information on GB*-algebras the reader may consult [All65], [All67], [Bha79], [Dix71], [Dix70]. In [All65], [All67], [Bha79], GB*-algebras are defined to be locally convex, whereas in [Dix71], [Dix70] a more general definition is given. In what follows we will adopt the more general definition.

Definition 2.4.1 *If \mathcal{A} is a topological *-algebra, then \mathcal{B}' will denote the collection of subsets B of \mathcal{A} satisfying:*

1. B is closed and bounded
2. the identity e is in B , $B^2 \subset B$, $B^* = B$, where $B^2 = \{wv : w, v \in B\}$ and $B^* = \{w^* : w \in B\}$.

If $B \in \mathcal{B}'$ is absolutely convex, the linear span of B forms an algebra which is normed by the Minkowski functional of B . This normed algebra is denoted by $\mathcal{A}(B)$.

Definition 2.4.2 An element $w \in \mathcal{A}$ is said to be bounded if, for some non zero complex number λ , the set $\{(\lambda w)^n : n = 1, 2, \dots\}$ is bounded. The set of all bounded elements of \mathcal{A} will be denoted by \mathcal{A}_0 .

\mathcal{A} is said to be symmetric if for every $w \in \mathcal{A}$, $(e + w^*w)^{-1}$ exists and lies in \mathcal{A}_0 .

Definition 2.4.3 ([Dix70] 2.5). A GB^* -algebra is a topological $*$ -algebra \mathcal{A} such that:

1. \mathcal{B}' has a greatest member B_0 , in the partial ordering of inclusion, and B_0 is absolutely convex;
2. \mathcal{A} is symmetric;
3. $\mathcal{A}(B_0)$ is complete.

Theorem 2.4.4 ([Dix71], 7.1)

Let \mathcal{A} be a topological $*$ -algebra and \mathcal{B}' be as in Definition 2.4.1. \mathcal{A} is a GB^* -algebra if and only if there is a subalgebra \mathcal{A}_b of \mathcal{A} , which is a C^* -algebra in some norm, and such that $(e + w * w)^{-1} \in \mathcal{A}_b$ for every $w \in \mathcal{A}$ and such that the unit ball B_0 of \mathcal{A}_b is the greatest member of \mathcal{B}' .

2.5 Measurable and Locally Measurable Operators

In this section we collect some results about measurable and locally measurable operators as defined in [Seg53], [Yea73], [San59] and [Dix71]. We shall need these ideas as a motivation for the other concepts of measurability that we shall introduce later.

Definition 2.5.1 ([Seg53], 2.1) Let \mathcal{M} be a von Neumann algebra on a Hilbert space H . A linear subspace \mathcal{D} of H is said to be strongly dense in H with respect to \mathcal{M} if

1. $U(\mathcal{D}) \subseteq \mathcal{D}$ for every unitary $U \in \mathcal{M}'$ (written $\mathcal{D}\eta\mathcal{M}$);

2. there is a sequence (\mathcal{K}_n) of closed linear subspaces of H such that $\mathcal{K}_n^\perp \downarrow 0$, that is, the projections onto \mathcal{K}_n^\perp decrease to zero, and, for each n , $\mathcal{K}_n \eta \mathcal{M}$, $\mathcal{K}_n \subseteq \mathcal{D}$ and the projection onto \mathcal{K}_n^\perp is finite in \mathcal{M} .

The condition $\mathcal{K}_n^\perp \downarrow 0$ implies that every strongly dense subspace of H is dense in H .

Definition 2.5.2 ([Seg53], 2.1) *An operator T on H is said to be measurable with respect to \mathcal{M} if*

1. $T \eta \mathcal{M}$;
2. T has a strongly dense domain;
3. T is closed.

Theorem 2.5.3 ([Seg53], 2.3)

The collection of all operators on H that are measurable with respect to \mathcal{M} is a $$ -algebra relative to the strong sum and strong product, the usual operations of multiplications by scalars (except that multiplication by 0 gives the everywhere defined zero operator), and the usual adjunction.*

We shall denote this algebra by $\mathcal{S}(\mathcal{M})$.

Example 2.5.4 *When $\mathcal{M} = \mathcal{B}(H)$, $\mathcal{S}(\mathcal{M}) = \mathcal{M}$.*

An alternative characterization of measurability was given in [Yea73] and we present it here as we shall need it when we consider invertibility.

Theorem 2.5.5 ([Yea73], 2.1) . *Let $T \eta \mathcal{M}$ be a closed operator. T is measurable with respect to \mathcal{M} if and only if $D(T)$ is dense in H and $I - E_\lambda(|T|)$ is finite for some $\lambda > 0$, where $\{E_\lambda(|T|) : \lambda > 0\}$ is the spectral family of $|T|$.*

Definition 2.5.6 ([Yea73], Theorem 2.1(2)(ii)) A closed operator $T \in \mathcal{M}$ is said to be locally measurable with respect to \mathcal{M} if there exist projections $Q_n \in Z(\mathcal{M})$, the centre of \mathcal{M} , such that $Q_n \uparrow I$ and TQ_n is measurable with respect to \mathcal{M} for each n .

It is immediate that every measurable operator is locally measurable.

An example of a locally measurable operator that is not measurable has been given in ([San59], section 4) in the case when \mathcal{M} is purely infinite, with a countably decomposable non trivial centre.

Denote by $\mathcal{L}(\mathcal{M})$ the set of all locally measurable operators. It was shown in ([San59], 3.4) and ([Dix71], 6.6) that $\mathcal{L}(\mathcal{M})$ forms a $*$ -algebra with the usual adjunction, strong sum, strong product and scalar multiplication.

2.6 The Algebra $\widetilde{\mathcal{M}}$ of τ -Measurable operators

In this section we introduce the algebra of τ -measurable operators, which algebra will be at the centre of most of our work throughout this thesis. We give some examples of such algebras and also show that this algebra is a GB^* -algebra.

Let \mathcal{M} be a semi-finite von Neumann algebra, with underlying Hilbert space H , equipped with a faithful semi-finite normal trace τ .

Definition 2.6.1 A subset E of H is called τ -dense if for every $\delta > 0$ there exists $P \in \mathcal{M}^p$ such that $P(H) \subseteq E$ and $\tau(I - P) \leq \delta$.

Proposition 2.6.2 ([Ter81] 1.10.)

E is a τ -dense subspace of H if and only if there exists a sequence $(P_n)_{n \in \mathbb{N}}$ in \mathcal{M}^p such that $P_n \uparrow_{so} I$, $\tau(I - P_n) \downarrow 0$, $\bigcup_{n=1}^{\infty} P_n(H) \subset E$

From the above proposition we have that a τ -dense subspace is necessarily norm-dense.

Definition 2.6.3 An operator $S \in \mathcal{M}$ is called τ -premeasurable if for every $\delta > 0$ there exists a $P \in \mathcal{M}^p$ such that $P(H) \subseteq D(S)$, $\|SP\| < \infty$ and $\tau(I - P) \leq \delta$.

Definition 2.6.4 We define $\widetilde{\mathcal{M}}$ to be the set of all $S \in \mathcal{N}(\mathcal{M})$ with a τ -dense domain.

From the definition of $\widetilde{\mathcal{M}}$ and the closed graph theorem we have that an affiliated operator is τ -measurable if and only if it is closed and τ -premeasurable .

Note that every τ -measurable operator is measurable, for if $E \sim F \leq E$ and $\tau(E) < \infty$, then $\tau(E) = \tau(F)$ and hence by faithfulness of the trace $E = F$.

Proposition 2.6.5 ([Ter81], 1.20)

If S and T are premeasurable, then $S + T$ and ST are premeasurable.

Proposition 2.6.6 ([Ter81] 1.21.)

Suppose $S \in \mathcal{N}(\mathcal{M})$ and $S = V|S|$ is the polar decomposition of S . The following are equivalent:

1. $S \in \widetilde{\mathcal{M}}$
2. $|S| \in \widetilde{\mathcal{M}}$
3. there exists a $t > 0$ such that $\tau(E_{(t,\infty)}(|S|)) < \infty$.

Proposition 2.6.7 ([Ter81] 1.15.)

A τ -premeasurable operator admits at most one extension in $\widetilde{\mathcal{M}}$.

Example 2.6.8 Consider the von Neumann algebra $\mathcal{M} = \mathcal{B}(H)$, of all bounded linear operators on the Hilbert space H , with the trace as defined in example 2.1.6. We have that $\mathcal{M} = \widetilde{\mathcal{M}}$ in this case.

Example 2.6.9 Let $\mathcal{M} = L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ) (example 2.1.7). Then $\widetilde{\mathcal{M}} = \widetilde{L}_\infty(X, \Sigma, \mu)$ (Definition 1.2.2).

Example 2.6.10 ([HN87], section 2) Given a von Neumann algebra \mathcal{M} with a faithful normal finite trace τ , we define a faithful normal finite trace $\hat{\tau}$ on the von Neumann algebra

$$M_2(\mathcal{M}) = \left\{ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : T_{ij} \in \mathcal{M}, \quad i, j = 1, 2 \right\}, \text{ by}$$

$$\hat{\tau} \left(\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \right) = \tau(T_{11}) + \tau(T_{22}), \quad \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in M_2(\mathcal{M}).$$

Then $\widetilde{M_2(\mathcal{M})} = M_2(\widetilde{\mathcal{M}})$.

$\widetilde{\mathcal{M}}$ is a *-algebra with respect to strong sum, strong product, adjunction and scalar multiplication, except that for $S \in \widetilde{\mathcal{M}}$ we define $0S$ to be the everywhere defined zero operator.

For $\epsilon > 0, \delta > 0$, define

$$\widetilde{\mathcal{M}}(\epsilon, \delta) = \{S \in \widetilde{\mathcal{M}}; \exists P \in \mathcal{M}^p \text{ such that } P(H) \subseteq D(S), \|SP\| \leq \epsilon, \tau(I - P) \leq \delta\}.$$

We shall write $\widetilde{\mathcal{M}}(\epsilon)$ for $\widetilde{\mathcal{M}}(\epsilon, \epsilon)$.

Proposition 2.6.11 ([Ter81], 1.26)

Suppose $\epsilon, \epsilon_1, \epsilon_2, \delta, \delta_1, \delta_2 > 0$ and $0 \neq \lambda \in \mathbb{C}$. Then

1. $\widetilde{\mathcal{M}}(\epsilon_1, \delta_1) + \widetilde{\mathcal{M}}(\epsilon_2, \delta_2) \subset \widetilde{\mathcal{M}}(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$ (strong sum)
2. $\widetilde{\mathcal{M}}(|\lambda|\epsilon, \delta) = \lambda\widetilde{\mathcal{M}}(\epsilon, \delta)$.
3. $\epsilon_1 < \epsilon_2, \delta_1 < \delta_2 \Rightarrow \widetilde{\mathcal{M}}(\epsilon_1, \delta_1) \subset \widetilde{\mathcal{M}}(\epsilon_2, \delta_2)$.
4. $\widetilde{\mathcal{M}}(\epsilon_1 \wedge \epsilon_2, \delta_1 \wedge \delta_2) \subseteq \widetilde{\mathcal{M}}(\epsilon_1, \delta_1) \cap \widetilde{\mathcal{M}}(\epsilon_2, \delta_2)$.
5. $\widetilde{\mathcal{M}}(\epsilon_1, \delta_1)\widetilde{\mathcal{M}}(\epsilon_2, \delta_2) \subset \widetilde{\mathcal{M}}(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$ (strong product)
6. $\widetilde{\mathcal{M}}(\epsilon, \delta)^* = \widetilde{\mathcal{M}}(\epsilon, \delta)$.

The family $\{\widetilde{\mathcal{M}}(\epsilon, \delta) : \epsilon > 0, \delta > 0\}$ forms a neighbourhood basis at 0 for a metrizable vector topology τ_m on $\widetilde{\mathcal{M}}$, called the *topology of convergence in measure*. Note that this generalizes the topology of convergence in measure for the commutative case. (Theorem 1.2.4).

Theorem 2.6.12 ([Ter81], 1.28, [Wes90], 8.27).

$\widetilde{\mathcal{M}}$, with the topology τ_m , is a complete metrizable topological $*$ -algebra in which \mathcal{M} is dense.

△

Proposition 2.6.13 (Adapted from [Dix71], Theorem 7.3).

$\widetilde{\mathcal{M}}$ is a GB^* -algebra, with $(\widetilde{\mathcal{M}})_b = \mathcal{M}$.

Proof: Suppose $T \in \widetilde{\mathcal{M}}$. By Lemma 2.2.6 and ([RSN53], 118 Theorem), $(I + T^*T)^{-1} \in \mathcal{M}$ so that $\widetilde{\mathcal{M}}$ is symmetric. Since $\widetilde{\mathcal{M}}$ is complete we only need to show that the unit ball B_0 of \mathcal{M} is the greatest member of \mathcal{B}' . We show that $B \subseteq B_0$ for all $B \in \mathcal{B}'$. Let $B \in \mathcal{B}'$ and $T \in B$, and suppose $T \notin B_0$. Then $T^*T \notin B_0$. Writing $T^*T = \int_{\mathbb{R}} \lambda dE_{\lambda}$, where $\{E_{\lambda} : \lambda \in \mathbb{R}\}$ is the spectral resolution of T^*T , we can find, for some $k > 1$ (since $T^*T \notin B_0$), a projection $P = I - E_k$ such that

$$\|(T^*T)^n x\| \geq k^n \|x\|, \text{ for all } x \in \text{ran}(P), \text{ for all } n \in \mathbb{N}.$$

For if $x \in \text{ran}(P) = \text{ran}(I - E_k)$, then $x = (I - E_k)x$ and

$$\begin{aligned} \|(T^*T)^n x\|^2 &= \int_0^{\infty} \lambda^{2n} d\|E_{\lambda}(I - E_k)x\|^2 \\ &= \int_k^{\infty} \lambda^{2n} d\|E_{\lambda}x\|^2 \\ &\geq k^{2n} \int_k^{\infty} d\|E_{\lambda}x\|^2, \quad (\text{since } \lambda \geq k \text{ implies } \lambda^{2n} \geq k^{2n}) \\ &= k^{2n} \int_0^{\infty} d\|E_{\lambda}(I - E_k)x\|^2 \\ &= k^{2n} \int_0^{\infty} d\|E_{\lambda}x\|^2 \\ &= k^{2n} \|x\|^2. \end{aligned}$$

Since τ is faithful and $P \neq 0$, there exists an $r > 0$ such that $\tau(P) > r > 0$. Let $\epsilon = \frac{r}{2}$. Since $T \in B \in \mathcal{B}'$, $\{(T^*T)^n : n = 1, 2, \dots\} \subseteq B$. Since B is τ_m -bounded, there exists an $N > 0$ such that

$$\{(T^*T)^n : n = 1, 2, \dots\} \subset N\widetilde{\mathcal{M}}(\epsilon, \epsilon) = \widetilde{\mathcal{M}}(N\epsilon, \epsilon).$$

Hence for every n , there is a projection P_n such that

$$\|(T^*T)^n P_n\| \leq N\epsilon \text{ and } \tau(I - P_n) \leq \epsilon.$$

We claim that $\text{ran}(P_n) \cap \text{ran}(P) \neq \{0\}$ for all n . Suppose $\text{ran}(P_n) \cap \text{ran}(P) = \{0\}$ for some $n \in \mathbb{N}$. Then by Lemma 2.1.3 and Proposition 2.1.8, $\tau(P) \leq \tau(I - P_n) < \epsilon = \frac{r}{2} < r < \tau(P)$, a contradiction. Thus for every $n \in \mathbb{N}$ there exists a non-zero $x_n \in \text{ran}(P_n) \cap \text{ran}(P)$. Since $x_n \in \text{ran}(P_n)$, $x_n = P_n x_n$ and hence

$$\|(T^*T)^n x_n\| = \|(T^*T)^n P_n x_n\| \leq \|(T^*T)^n P_n\| \|x_n\| \leq N\epsilon \|x_n\|.$$

Since $x_n \in \text{ran}(P)$,

$$\begin{aligned} \|(T^*T)^n x_n\| &\geq k^n \|x_n\| \text{ for all } n \in \mathbb{N} \\ &> N\epsilon \|x_n\| \text{ for } n \text{ large enough, since } k > 1. \end{aligned}$$

This is a contradiction, hence $T \in B_0$, and so $B \subseteq B_0$.

We next show that $B_0 \in \mathcal{B}'$.

The closure of B_0 in the measure topology, $\overline{B_0}^{\tau_m}$, is certainly τ_m -closed. Let $\widetilde{\mathcal{M}}(\epsilon, \delta)$ be a basic neighbourhood of $0 \in \widetilde{\mathcal{M}}$. Then $\widetilde{\mathcal{M}}(\epsilon, \delta) \cap \mathcal{M}$ is a τ_m -neighbourhood of $0 \in \mathcal{M}$. Since $\tau_m|_{\mathcal{M}}$ is weaker than the norm topology on \mathcal{M} , we have that $\widetilde{\mathcal{M}}(\epsilon, \delta) \cap \mathcal{M}$ is a norm neighbourhood of $0 \in \mathcal{M}$. Thus there exists a $\lambda > 0$ such that $B_0 \subseteq \lambda(\widetilde{\mathcal{M}}(\epsilon, \delta) \cap \mathcal{M}) \subseteq \lambda\widetilde{\mathcal{M}}(\epsilon, \delta)$. Therefore, B_0 is τ_m -bounded. It follows that $\overline{B_0}^{\tau_m}$ is τ_m -bounded and so lies in \mathcal{B}' . This implies that $\overline{B_0}^{\tau_m} \subseteq B_0$. It follows that B_0 is τ_m -closed and so is in \mathcal{B}' . Therefore $\widetilde{\mathcal{M}}$ is a GB*-algebra, with $(\widetilde{\mathcal{M}})_b = \widetilde{\mathcal{M}}(B_0) = \mathcal{M}$.

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Definition 2.6.14 *Suppose $Q \in \mathcal{M}^p$. We define τ_Q on \mathcal{M}_Q (definition 2.2.9) by*

$$\tau_Q(S_Q) = \tau(QSQ).$$

We also define $\widetilde{\mathcal{M}}_Q = \{S_Q : S \in \widetilde{\mathcal{M}}\}$.

It can be shown that τ_Q is a faithful semi-finite normal trace on \mathcal{M}_Q . ([Wes90], 3.31). Now let $\widetilde{\mathcal{M}}_Q$ denote the completion of the semi-finite von Neumann algebra \mathcal{M}_Q with respect to the topology of convergence in measure determined by the reduced trace τ_Q on \mathcal{M}_Q . We have the following

Proposition 2.6.15 ([Wes93], 6.2.8) *Suppose $Q \in \mathcal{M}^p$. Then*

$$\widetilde{\mathcal{M}}_Q = \widetilde{\mathcal{M}}_Q.$$

2.7 The Distribution and Generalised Singular Value Functions

In this section we give a brief introduction to the generalised singular function and distribution function of a τ -measurable operator. We present some results that we will need in the sequel. We mention here that the generalised singular function is a generalization of the singular value sequence of a compact operator in $\mathcal{B}(H)$, and also of the decreasing rearrangement of a measurable function.

Definition 2.7.1 [FK86] *Suppose $S \in \widetilde{\mathcal{M}}$ and $S = V|S|$ is its polar decomposition. For $t > 0$, let $d_t(S) = \tau(E_{(t,\infty)}(|S|))$. The function $d_t(S)$ is known as the distribution function of S .*

By Proposition 2.6.6(3), $d_t(S)$ is eventually finite valued, and $d_t(S) \rightarrow 0$ as $t \rightarrow \infty$. Hence for any $t > 0$ there exists an $s > 0$ such that $d_s(S) \leq t$.

Definition 2.7.2 We define the generalized singular value function $\mu_t(S)$ of $S \in \widetilde{\mathcal{M}}$ by

$$\mu_t(S) = \inf\{s \geq 0 : d_s(S) \leq t\} = \inf\{s \geq 0 : \tau(E_{(s,\infty)}(|S|)) \leq t\}$$

It has been shown in ([FK86], 2.2) that

$$\mu_t(S) = \inf\{\|SP\| : P \in \mathcal{M}^p, P(H) \subseteq D(S), \tau(I - P) \leq t\}.$$

The generalized singular function is finite valued, decreasing and right continuous. It is in fact almost everywhere continuous (with respect to Lebesgue measure on the positive real line).

If $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$ then $\mu_t(f) = \inf\{\theta \geq 0 : \mu\{x \in X : |f(x)| > \theta\} \leq t\}$, the decreasing rearrangement of f .

Definition 2.7.3 $S, T \in \widetilde{\mathcal{M}}$ are said to be equimeasurable if $\mu_t(S) = \mu_t(T)$ for all $t > 0$.

Lemma 2.7.4 ([FK86], 2.5, 3.1)

Let $S, R, T \in \widetilde{\mathcal{M}}$, $\alpha \in \mathbb{C}$, $s, t > 0$ and $\epsilon > 0$. Then

1. $\mu_t(\alpha S) = |\alpha| \mu_t(S)$ for all $t > 0$.
2. $\mu_{t+s}(S + T) \leq \mu_t(S) + \mu_s(T)$.
3. $\mu_{t+s}(ST) \leq \mu_t(S) \mu_s(T)$.
4. $\mu_t(S) = 0$ for all $t > 0 \Leftrightarrow S = 0$.
5. $\mu_t(RST) \leq \|R\| \mu_t(S) \|T\|$.
6. $S \in \widetilde{\mathcal{M}}(\epsilon, t) \Leftrightarrow \mu_t(S) \leq \epsilon$.
7. $S_i \rightarrow_{\tau_m} S \Leftrightarrow \mu_t(S_i - S) \rightarrow 0$ for all $t > 0$.
8. $\lim_{t \downarrow 0} \mu_t(S) = \|S\|$, where $\|S\| = \infty$ when S is unbounded.

2.8 The order structure in $\widetilde{\mathcal{M}}$

We shall need some results on the order structure in $\widetilde{\mathcal{M}}$ and here we present a summary of some relevant results. For full details the reader is referred to [DdP93].

Definition 2.8.1 $\widetilde{\mathcal{M}}^{sa} = \{S \in \widetilde{\mathcal{M}} : S = S^*\}$ is an ordered vector space with respect to the partial ordering defined by setting $S \geq T$ if and only if $S - T \geq 0$, where $S - T$ denotes the closure of the algebraic difference of S and T .

Proposition 2.8.2 ([DdP93], 1.1, 1.3, 1.4, 1.7, 1.8)

1. $\widetilde{\mathcal{M}}^{sa}$ is order complete, i.e. if (S_α) is an increasing net in $\widetilde{\mathcal{M}}^+$ bounded above by some $T \in \widetilde{\mathcal{M}}$, then $S = \sup_\alpha S_\alpha$ exists in $\widetilde{\mathcal{M}}$.
2. If $0 \leq S \in \widetilde{\mathcal{M}}$, then $T^*ST \geq 0$ for all $T \in \widetilde{\mathcal{M}}$.
3. If $0 \leq S_\alpha \uparrow_\alpha S$ holds in $\widetilde{\mathcal{M}}$, then $0 \leq T^*S_\alpha T \uparrow_\alpha T^*ST$ holds in $\widetilde{\mathcal{M}}$ for all $T \in \widetilde{\mathcal{M}}$.
4. The positive cone $\widetilde{\mathcal{M}}^+$ is closed for the measure topology.
5. If $0 \leq S_\alpha \uparrow_\alpha S$ in $\widetilde{\mathcal{M}}$ then $\mu_t(S_\alpha) \uparrow_\alpha \mu_t(S)$ holds for all $t > 0$.
6. Let $0 \leq S \in \widetilde{\mathcal{M}}$. Then there exists a net $(S_\alpha) \subseteq \mathcal{M}$ with $\tau(S_\alpha) < \infty$ for each α such that $0 \leq S_\alpha \uparrow_\alpha S$ in $\widetilde{\mathcal{M}}$.

2.9 The subspace $\widetilde{\mathcal{M}}_0$ of $\widetilde{\mathcal{M}}$

When $\mathcal{M} = \mathcal{B}(H)$, it is well known that the compact operators are exactly those operators whose s -number sequence decreases to zero. In this section we introduce a generalisation of compactness to elements of $\widetilde{\mathcal{M}}$ that we shall refer to as τ -compactness. Suppose $S \in \widetilde{\mathcal{M}}$, then we know that $\mu_t(S) < \infty$ for every $t > 0$. Therefore, since $\mu_t(S)$ is decreasing in t , $\lim_{t \rightarrow \infty} \mu_t(S)$ exists and we denote it by $\mu_\infty(S)$.

Proposition 2.9.1 ([SW93] section 2)

If $S, T \in \widetilde{\mathcal{M}}$ and $\lambda \in \mathbb{C}$, then

1. $\mu_\infty(S) \leq \|S\|$.
2. $\mu_\infty(\lambda S) = |\lambda| \mu_\infty(S)$.
3. $\mu_\infty(S + T) \leq \mu_\infty(S) + \mu_\infty(T)$.
4. $\mu_\infty(ST) \leq \mu_\infty(S) \mu_\infty(T)$.
5. $\mu_\infty(S) = \mu_\infty(S^*)$.
6. $\mu_\infty(S) = \mu_\infty(|S|)$

It follows from (2) to (5) that μ_∞ is a *-algebra semi-norm on $\widetilde{\mathcal{M}}$.

If $S_\alpha \rightarrow_{\tau_m} 0$ in $\widetilde{\mathcal{M}}$ then $\mu_t(S_\alpha) \rightarrow_\alpha 0$ for all $t > 0$ and hence $\mu_\infty(S_\alpha) \rightarrow_\alpha 0$. Thus μ_∞ is continuous at 0, and hence on $\widetilde{\mathcal{M}}$.

Definition 2.9.2 We define $\widetilde{\mathcal{M}}_0$ to be the kernel of μ_∞ , i.e.

$$\widetilde{\mathcal{M}}_0 = \{x \in \widetilde{\mathcal{M}} : \mu_\infty(x) = 0\}.$$

By definition of $\widetilde{\mathcal{M}}_0$ it is clear that $S \in \widetilde{\mathcal{M}}_0$ if and only if $\tau(E_{(t,\infty)}(|S|)) < \infty$ for all $t > 0$.

Thus we have that all τ -measurable operators with finite trace lie in $\widetilde{\mathcal{M}}_0$.

If $\mathcal{M} = \mathcal{B}(H)$, with the canonical trace, then $\widetilde{\mathcal{M}}_0 = \mathcal{K}(H)$, the set of all compact operators on H .

Proposition 2.9.3 ([SW93], example 2.2.3.) The following are equivalent:

1. $\widetilde{\mathcal{M}} = \mathcal{M}$
2. $\inf_{0 \neq P \in \mathcal{M}^p} \tau(P) > 0$

3. The topology of convergence in measure coincides with the norm topology.
4. $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0$, where $\widetilde{\mathcal{M}}_0 \cap \mathcal{M} = \mathcal{M}_0$.

The $*$ -algebra semi-norm μ_∞ on $\widetilde{\mathcal{M}}$ induces canonically a $*$ -algebra norm, also denoted by μ_∞ , on the quotient $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$, if we put $\mu_\infty(S + \widetilde{\mathcal{M}}_0) = \mu_\infty(S)$.

Proposition 2.9.4 ([SW93], section 3, [Wes93], 6.3.3, 6.3.4)

1. For any $S \in \widetilde{\mathcal{M}}$, $\mu_\infty(S) = \inf_{T \in \widetilde{\mathcal{M}}_0} \|S - T\|$.
2. $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$, equipped with the norm μ_∞ , is isometrically $*$ -isomorphic to $\mathcal{M}/\mathcal{M}_0$, with its usual quotient norm. Furthermore, $(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0, \mu_\infty)$ is a C^* -algebra, the so called τ -Calkin algebra.

We have the following relationships among the algebras we have looked at so far.

Proposition 2.9.5 ([MC91], Remark 1)

1. If $\tau(I) < \infty$ then $\widetilde{\mathcal{M}}_0 = \widetilde{\mathcal{M}} = \mathcal{S}(\mathcal{M}) = \mathcal{L}(\mathcal{M})$.
2. If $\tau(I) = \infty$, then I does not belong to $\widetilde{\mathcal{M}}_0$, in particular $\widetilde{\mathcal{M}}_0 \neq \widetilde{\mathcal{M}}$.
3. If \mathcal{M} is a factor of type II_∞ , then $\tau(P) < \infty$ if and only if P is a finite projection. In this case $\mathcal{M} \neq \widetilde{\mathcal{M}} = \mathcal{S}(\mathcal{M})$.
4. If \mathcal{M} is a factor of type I then $\mathcal{M} = \widetilde{\mathcal{M}} = \mathcal{S}(\mathcal{M}) = \mathcal{L}(\mathcal{M})$.

2.10 The local convexity of $\widetilde{\mathcal{M}}$

Crowther in [Cro97] has investigated when the topology of convergence in measure is locally convex, and shown that the answer depends on the nature of the projections in the von Neumann algebra. We present a summary of the results here.

Definition 2.10.1 A projection is said to be atomic if it has no nonzero subprojections and is said to be nonatomic (or continuous) if it has no atomic subprojections. \mathcal{M}^p is called atomic if it contains no nonatomic projections and \mathcal{M}^p is nonatomic (or continuous) if it possesses no atomic projections.

Theorem 2.10.2 ([Cro97], 1.5.1.) Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ and suppose that \mathcal{M}^p has a nonatomic projection. Then the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is not locally convex.

Theorem 2.10.3 ([Cro97], 1.5.2., 1.5.3, 1.5.5) Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ and suppose \mathcal{M}^p is atomic.

1. If $\inf\{\tau(P) : P \in \mathcal{M}^p, \tau(P) \neq 0\} > 0$, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is locally convex and equal to the norm topology on \mathcal{M} .
2. If $\inf\{\tau(P) : P \in \mathcal{M}^p, \tau(P) \neq 0\} = 0$ and there exists a constant $K > 0$ such that

$$\sum_{\substack{\tau(P) < K \\ P \text{ atomic}}} \tau(P) < \infty,$$

then the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is locally convex.

3. If

$$\inf\{\tau(P) : P \in \mathcal{M}^p, \tau(P) \neq 0\} = 0$$

and there exists a sequence of mutually orthogonal atomic projections (P_n) in \mathcal{M}^p with $\tau(P_n) \downarrow_n 0$ such that

$$\sum_{n=1}^{\infty} \tau(P_n) = \infty,$$

then the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is not locally convex.

A complete characterisation of local convexity of $\widetilde{\mathcal{M}}$ in the general case has not yet been given, as it is still unknown what happens in the case when

$$\inf\{\tau(P) : P \in \mathcal{M}^p, \tau(P) \neq 0\} = 0$$

and there is no mutually orthogonal sequence of atomic projections (P_n) in \mathcal{M}^p such that $\tau(P_n) \downarrow_n 0$ and $\sum_{n=1}^{\infty} \tau(P_n) = \infty$. However, in the commutative case atomic projections in this setting are mutually orthogonal, so that we have a complete characterisation in this case:

Corollary 2.10.4 ([Cro97], 1.5.7) *Let (X, Σ, μ) be a localizable measure space and μ a semi-finite measure. The topology of convergence in measure on $\widetilde{L}_{\infty}(X, \Sigma, \mu)$ is locally convex if and only if (X, Σ, μ) is atomic and $\inf\{\mu(A) : A \in \Sigma, \mu(A) \neq 0\} > 0$, or (X, Σ, μ) is atomic, $\inf\{\mu(A) : A \in \Sigma, \mu(A) \neq 0\} = 0$ and there exists $K > 0$ such that*

$$\sum_{\substack{\mu(A) < K \\ A \text{ atomic}}} \mu(A) < \infty.$$

2.11 Algebras of Normal Functions: $\mathcal{N}(X)$ and $\mathcal{J}(X)$

If \mathcal{M} is a commutative von Neumann algebra, it is isometrically isomorphic to $C(X)$, where X is an extremely disconnected compact Hausdorff space. When considering representations of commutative algebras of unbounded operators affiliated with a von Neumann algebra \mathcal{M} , it becomes necessary to introduce algebras of unbounded functions on X . This makes it possible to use a topological approach to aspects of spectral theory. Since the extremely disconnected spaces occurring in the representation of von Neumann algebras admit normal measures, it is possible to represent these function algebras as algebras of measurable functions as well, as is shown in this section.

Definition 2.11.1 1. *A topological space X is said to be extremely disconnected if the closure of each open set is open.*

2. *A subset of a topological space X is said to be nowhere dense (in X) if its closure has an empty interior and it is said to be meager (or of the first category) in X if it is a countable union of nowhere dense sets in X .*

Definition 2.11.2 If X is an extremely disconnected compact Hausdorff space, a normal function on X is a continuous complex-valued function f defined on an open dense subset O of X such that $\lim_{q \rightarrow p} |f(q)| = \infty$ for every $p \in X \setminus O$ (where $q \in O$), that is to say for every $p \in X \setminus O$ and for every $k > 0$ there exists a neighbourhood $N(p)$ of p such that $|f(q)| > k$ for $q \in O \cap N(p)$. We denote the set of normal functions by $\mathcal{N}(X)$. A real valued normal function is called a self-adjoint function, and we denote the set of self-adjoint normal functions by $\mathcal{J}(X)$.

Throughout this section X will denote an extremely disconnected compact Hausdorff space.

Theorem 2.11.3 ([LZ83] Theorem 47.1) Let f be a continuous real-valued function defined on the open dense subset O of the extremely disconnected compact Hausdorff space X . Then f can be extended uniquely to a self-adjoint function on X .

Definition 2.11.4 If f and $g \in \mathcal{J}(X)$ are defined on a common open dense subset K of X , denote the extension of their sum $f + g$ on K to X by $f \hat{+} g$ and the extension of their product fg on K to X by $f \hat{\cdot} g$.

Theorem 2.11.5 ([KR83], 5.6) Let X be an extremely disconnected compact Hausdorff space and $f, g \in \mathcal{N}(X)$ be defined on open dense subsets O, O' of X , respectively. Denote their real and imaginary parts by $Re(f), Re(g)$, and $Im(f), Im(g)$ respectively. Then $f + g$ and fg are defined and continuous on $O \cap O'$ and have normal extensions

$$f \hat{+} g = (Re(f) \hat{+} Re(g)) \hat{+} i(Im(f) \hat{+} Im(g)) \text{ and}$$

$$f \hat{\cdot} g = (Re(f) \hat{\cdot} Re(g) \hat{-} Im(f) \hat{\cdot} Im(g)) \hat{+} i(Re(f) \hat{\cdot} Im(g) \hat{+} Im(f) \hat{\cdot} Re(g)).$$

With the operations $(f, g) \rightarrow f \hat{+} g$, $(f, g) \rightarrow f \hat{\cdot} g$, $(\alpha, f) \rightarrow \alpha f$ and $f \rightarrow \bar{f}$, $\mathcal{N}(X)$ becomes an associative, commutative algebra with unit 1.

Proof: $Re(f) + Re(g)$ and $Im(f) + Im(g)$ are defined and continuous on $O \cap O'$, a dense subset of X , and by Theorem 2.11.3 have unique self-adjoint extensions $Re(f) \hat{+} Re(g)$ and

$Im(f)\widehat{+}Im(g)$, respectively. Then $f\widehat{+}g$ as defined above is the unique normal extension for $f + g$. In the same way we have that $f\widehat{\cdot}g$ is a unique normal extension for fg . Repeated use of Theorem 2.11.3 shows that we have an associative, commutative algebra with unit 1 and adjoint operation $f \rightarrow \bar{f}$. Δ

We now want to show that $\mathcal{N}(X)$ is isomorphic to $L_0(X, \Sigma_X, \mu)$ for an appropriate measure space (X, Σ_X, μ) . To do this we consider measures on an extremely disconnected compact Hausdorff space, following the approach of ([KA81], X 2.5).

Definition 2.11.6 *Let X be an extremely disconnected compact Hausdorff space, \mathcal{G}_X the collection of all open-and-closed subsets of X , and \mathcal{F}_X the collection of all sets of first category in X . Define Σ_X by $\Sigma_X = \{G\Delta N : G \in \mathcal{G}_X, N \in \mathcal{F}_X\}$, where $G\Delta N = (G \setminus N) \cup (N \setminus G)$, the symmetric difference of G and N .*

Lemma 2.11.7 ([KA81], Lemma 2.5.6) Σ_X is a σ -algebra containing the Borel σ -algebra of X .

Definition 2.11.8 *A measure μ defined on the σ -algebra Σ_X is said to be normal if $N \in \mathcal{F}_X$ implies $\mu(N) = 0$ and for every $G \in \mathcal{G}_X$ with $\mu(G) = \infty$, there exists $G_1 \in \mathcal{G}_X$ such that $G_1 \subset G$ and $0 < \mu(G_1) < \infty$.*

A normal measure μ is said to be strictly positive if $G \in \mathcal{G}_X, G \neq \emptyset$, implies $\mu(G) > 0$.

Definition 2.11.9 *An extremely disconnected compact Hausdorff space X on which there is a strictly positive normal measure is said to be hyperstonean.*

Note that if \mathcal{M} is a commutative von Neumann algebra, then \mathcal{M} is isometrically isomorphic to $C(X)$, where X is hyperstonean. ([Tak79], III.1.18)

Proposition 2.11.10 ([KA81], 2.5)

Let X be hyperstonean and μ a the strictly positive normal measure on Σ_X . Then we have

1. $\mu(G\Delta N) = \mu(G)$, if $G \in \mathcal{G}_X, N \in \mathcal{F}_X$.

2. The measure μ is complete.

3. The measure μ is semi-finite.

Proof: (1) Since $G \setminus N$ and $N \setminus G$ are disjoint, we have

$$\begin{aligned}\mu(G \Delta N) &= \mu(G \setminus N) + \mu(N \setminus G) \\ &= \mu(G \setminus N), \quad \text{since } N \setminus G \subseteq N \in \mathcal{F}_X \\ &= \mu(G \setminus N) + \mu(N \cap G) \quad \text{since } N \cap G \in \mathcal{F}_X \\ &= \mu(G).\end{aligned}$$

(2) If $A \subset G \Delta N$, $G \in \mathcal{G}_X$, $N \in \mathcal{F}_X$ and $\mu(G \Delta N) = 0$, then since μ is strictly positive and by part (a) we have $G = \emptyset$, so that $A \subset N$ and hence $A \in \mathcal{F}_X$ so that $A = \emptyset \Delta A \in \Sigma_X$.

(3) If $\mu(G \Delta N) = \infty$, $G \in \mathcal{G}_X$, $N \in \mathcal{F}_X$, then since $\mu(N) = 0$, $\mu(G) = \infty$ since $G \setminus N \subset G$. By definition of normality, there exists a $G_1 \in \mathcal{G}_X$ such that $G_1 \subset G$ and $0 < \mu(G_1) < \infty$. Hence $G_1 \Delta N \subset G \Delta N$ and $\mu(G_1 \Delta N) = \mu(G_1) < \infty$, so that μ is semi-finite.

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Theorem 2.11.11 ([KA81] 2.5.4) *Let μ be a strictly positive normal measure on Σ_X , with X extremely disconnected.*

1. *If $f \in \mathcal{J}(X)$, then f is a measurable μ -almost everywhere finite function.*
2. *If $f \neq g$ in $\mathcal{J}(X)$, then $\mu\{x \in X : f(x) \neq g(x)\} \neq 0$.*
3. *If g is measurable and μ -almost everywhere finite on X , then there exists $f \in \mathcal{J}(X)$ such that $f(x) = g(x)$ μ -almost everywhere.*

Proposition 2.11.12 *If X is hyperstonean and μ a strictly positive normal measure on Σ_X , then $\mathcal{N}(X)$ is $*$ -algebra isomorphic to $L_0(X, \Sigma_X, \mu)$*

Proof: If $f \in \mathcal{N}(X)$, $f = f_1 + if_2$ where $f_1 = \text{Re}f(f)$, $f_2 = \text{Im}(f)$. By Theorem 2.11.11(1), f_1 and f_2 are μ -almost everywhere finite real-valued measurable functions. Hence f is an almost everywhere complex-valued measurable function. We define the map $\Pi : \mathcal{N}(X) \rightarrow L_0(X, \Sigma_X, \mu)$ by $\Pi(f) = [f]$, where $[f]$ is the equivalence class of all complex-valued measurable functions equal μ -almost everywhere to f . It follows from Theorem 2.11.11(2) that Π is injective, and from Theorem 2.11.11(3) that Π is surjective.

If $f, g \in \mathcal{N}(X)$,

$$\Pi(f \widehat{+} g) = [f \widehat{+} g] = [f + g] = [f] + [g] = \Pi(f) + \Pi(g).$$

$$\Pi(f \widehat{\cdot} g) = [f \widehat{\cdot} g] = [fg] = [f][g] = \Pi(f)\Pi(g).$$

$$\Pi(\overline{f}) = [\overline{f}] = [\overline{f_1 + if_2}] = [f_1 - if_2] = [f_1] - i[f_2] = \overline{[f_1] + i[f_2]} = \overline{\Pi(f)}.$$

$$\Pi(\lambda f) = [\lambda f] = \lambda[f] = \lambda\Pi(f).$$

Hence Π is a $*$ -isomorphism.

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Theorem 2.11.13 *([KR83], 5.6.19) If \mathcal{A} is an abelian von Neumann algebra and ϕ is an $*$ -isomorphism of \mathcal{A} onto $C(X)$, where X is an extremely disconnected compact Hausdorff space, then ϕ extends to a $*$ -isomorphism from $\mathcal{N}(\mathcal{A})$ onto $\mathcal{N}(X)$.*

Corollary 2.11.14 *If \mathcal{A} is an abelian von Neumann algebra, then there is a localizable measure space (X, Σ, μ) such that $\mathcal{N}(\mathcal{A})$ is $*$ -isomorphic to $L_0(X, \Sigma, \mu)$.*

2.12 The Algebra of Locally τ -Measurable Operators

Various authors have introduced the notion of locally measurable operators, among them are Segal in [Seg53], Yeadon in [Yea73], Sankaran in [San59] and Dixon in [Dix71]. Their definitions of locally measurable operators use algebraic finiteness of projections, as can be seen in section 2.5. In this section we follow a similar path, but here we use the notion of the trace, and replace algebraic finiteness by finiteness of the trace. We show that the set of operators defined in this way forms a $*$ -algebra with respect to strong sum, strong multiplication, adjunction and scalar multiplication.

Motivated by definition 2.5.6, we make the following definition:

Definition 2.12.1 *Suppose \mathcal{M} is a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ . Let S be a closed densely defined operator affiliated with \mathcal{M} . S is said to be locally τ -measurable if there exist projections $Q_n \in Z(\mathcal{M})$ such that $Q_n \uparrow I$ and $SQ_n \in \widetilde{\mathcal{M}}$ for every $n \in \mathbb{N}$.*

We denote the set of locally τ -measurable operators by $\mathcal{L}(\widetilde{\mathcal{M}})$.

It is immediate from the definition and the fact that every τ -measurable operator is measurable that $\widetilde{\mathcal{M}} \subseteq \mathcal{L}(\widetilde{\mathcal{M}}) \subseteq \mathcal{L}(\mathcal{M})$. If \mathcal{M} is a factor, $\widetilde{\mathcal{M}} = \mathcal{L}(\widetilde{\mathcal{M}})$, and in the case $\mathcal{M} = B(H)$, we have $\widetilde{\mathcal{M}} = \mathcal{L}(\widetilde{\mathcal{M}}) = \mathcal{L}(\mathcal{M}) = B(H)$.

Example 2.12.2 *If (X, Σ, μ) is semi-finite, $f \in L_0(X, \Sigma, \mu)$ and $n \in \mathbb{N}$, put $X_n = \{x \in X : |f(x)| \leq n\}$. Then $\chi_{X_n} \uparrow \chi_X$ and $f\chi_{X_n} \in \widetilde{L}_\infty(X, \Sigma, \mu)$ for all $n \in \mathbb{N}$. Hence $L_0(X, \Sigma, \mu) \subseteq \mathcal{L}(\widetilde{L}_\infty)$. Conversely, suppose $f \in \mathcal{L}(\widetilde{L}_\infty)$. Then there exists a sequence (Q_n) of projections in L_∞ such that $Q_n \uparrow I$, and we can write $Q_n = \chi_{X_n}$ for some $X_n \in \Sigma$, and $f\chi_{X_n} \in \widetilde{L}_\infty$. It follows that $\chi_{X_n} \rightarrow \chi_X$, almost everywhere. Thus $f\chi_{X_n} \rightarrow f$ almost everywhere. But an almost everywhere limit of a sequence in L_0 lies in L_0 . Hence $f \in L_0$. Thus $\mathcal{L}(\widetilde{L}_\infty) = L_0(X, \Sigma, \mu)$.*

The following Lemma will be useful in the sequel.

Lemma 2.12.3 *Suppose $Q \in \mathcal{M}^p$ and S is a closed densely defined operator affiliated with \mathcal{M} such that $QS \subseteq SQ$. (This is the case, in particular, when Q is a central projection, by Proposition 2.2.2.) Then $SQ = QSQ$.*

Proof: $D(SQ) = \{x \in H : Qx \in D(S)\} = D(QSQ)$, since Q is everywhere defined. Since $QS \subseteq SQ$, then $QSQ \subseteq SQ$ by Proposition 1.3.1. Hence $QSQ = SQ$ since they have the same domain.

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Definition 2.12.4 *Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ . Let E be a projection in $Z(\mathcal{M})$. A subset D of H is said to be τ -measurable (E) with respect to \mathcal{M} if*

1. $U(D) \subseteq D$ for all unitaries $U \in \mathcal{M}'$.
2. for every $\delta > 0$, there exists a projection P in $E\mathcal{M}$ such that $P(H) \subseteq E(D)$ and $\tau(E - P) \leq \delta$.

Definition 2.12.5 *A subset D of H is said to be locally τ -measurable with respect to \mathcal{M} if there exists a sequence (E_i) of projections in $Z(\mathcal{M})$, with $E_i \uparrow I$ such that D is τ -measurable (E_i) with respect to \mathcal{M} , for each i .*

Lemma 2.12.6 *Let $E \in Z(\mathcal{M})^p$.*

1. ([San59], 2.3) *If D is a subset of H such that $U(D) \subseteq D$ for all unitaries $U \in \mathcal{M}'$, then $E(D) = \text{ran}(E) \cap D$.*
2. *If $S \eta \mathcal{M}$, then $D(SE) = (I - E)(H) + E(D(S))$ and $E(D(S)) \subseteq D(SE)$.*

Proof: (2): From part (1), we have

$$\begin{aligned} D(SE) &= \{x \in H : Ex \in D(S)\} = \{(I - E)x + Ex : Ex \in D(S)\} \\ &\subseteq (I - E)(H) + E(H) \cap D(S) \\ &= (I - E)(H) + E(D(S)). \end{aligned}$$

Conversely, if $x = (I - E)y + z$, with $y \in H$, $z \in E(H) \cap D(S)$, then $Ex = 0 + Ez = z$ since $z \in E(H)$. Hence $Ex = z \in D(S)$, so that $x \in D(SE)$. Hence $D(SE) = (I - E)(H) + E(D(S))$.

From this we obtain $E(D(S)) \subseteq D(SE)$.

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Lemma 2.12.7 *Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ . A closed densely defined operator S affiliated with \mathcal{M} is locally τ -measurable if and only if $D(S)$ is locally τ -measurable with respect to \mathcal{M} .*

Proof: Suppose $D(S)$ is locally τ -measurable with respect to \mathcal{M} . Then there exists a sequence (E_i) in $Z(\mathcal{M})^p$ with $E_i \uparrow I$ such that $UD(S) \subseteq D(S)$ for all unitaries $U \in \mathcal{M}'$ and for every $\delta > 0$ there exists a projection P_i in $E_i\mathcal{M}$ such that $P_i(H) \subseteq E_iD(S)$ and $\tau(E_i - P_i) \leq \delta$. We have straight away that SE_i is closed for each i , and $SE_i\eta\mathcal{M}$ since $S, E_i\eta\mathcal{M}$. Let $Q_i = I - E_i + P_i$. Now $\tau(I - Q_i) = \tau(E_i - P_i) \leq \delta$ and

$$\begin{aligned} Q_i(H) &= (I - E_i + P_i)(H) = (I - E_i)(H) + P_i(H) \\ &\subseteq (I - E_i)(H) + E_i(D(S)) \\ &= D(SE_i), \quad \text{by Lemma 2.12.6(2)}. \end{aligned}$$

Hence $SE_i \in \widetilde{\mathcal{M}}$, and so the sequence of projections (E_i) demonstrates that $S \in \mathcal{L}(\widetilde{\mathcal{M}})$.

Conversely, if $S \in \mathcal{L}(\widetilde{\mathcal{M}})$, then there exists a sequence (E_i) in $Z(\mathcal{M})^p$ with $E_i \uparrow I$ and $SE_i \in \widetilde{\mathcal{M}}$. This implies that for every $\delta > 0$, for each i , there exists a projection $P_i \in \mathcal{M}^p$ such that $P_i(H) \subseteq D(SE_i)$ and $\tau(I - P_i) \leq \delta$. Let $Q_i = E_i \wedge P_i = E_iP_i$. Then,

$$Q_i(H) \subseteq D(SE_i) \cap E_i(H) = [(I - E_i)(H) + E_i(D(S))] \cap E_i(H) = E_iD(S),$$

by Lemma 2.12.6(2), and $\tau(E_i - Q_i) = \tau(E_i - E_i \wedge P_i) = \tau(E_i \vee P_i - P_i) \leq \tau(I - P_i) \leq \delta$. Since $S\eta\mathcal{M}$, $UD(S) \subseteq D(S)$ for all $U \in \mathcal{M}'$. Hence $D(S)$ is locally τ -measurable.

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Corollary 2.12.8 *If $S\eta\mathcal{M}$ and $E \in Z(\mathcal{M})^p$, then $D(SE)$ is τ -dense if and only if $D(S)$ is τ -measurable (E) with respect to \mathcal{M} .*

Proof: Follows straight away from the proof of Lemma 2.12.7.

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Proposition 2.12.9 1. *The set $\mathcal{L}(\widetilde{\mathcal{M}})$ is closed with respect to strong sum, strong product, scalar multiplication and adjunction.*

2. *The set $\mathcal{L}(\widetilde{\mathcal{M}})$ is a $*$ -algebra with respect to strong sum, strong product, scalar multiplication and adjunction.*

Proof: (1) Let $S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$. We show that

(a) $S^* \in \mathcal{L}(\widetilde{\mathcal{M}})$, (b) $\overline{S+T} \in \mathcal{L}(\widetilde{\mathcal{M}})$ and (c) $\overline{ST} \in \mathcal{L}(\widetilde{\mathcal{M}})$:

(a) If $S \in \mathcal{L}(\widetilde{\mathcal{M}})$, then there exist projections $Q_n \in Z(\mathcal{M})$ such that $Q_n \uparrow I$ and $SQ_n \in \widetilde{\mathcal{M}}$ for each n . Since $S\eta\mathcal{M}$ and $Q_n \in \mathcal{M}'$, $Q_n S \subseteq SQ_n$ (Proposition 2.2.2). Hence $(SQ_n)^* \subseteq (Q_n S)^* = S^*Q_n$ since Q_n is bounded, by Proposition 1.3.1. But $SQ_n \in \widetilde{\mathcal{M}} \Rightarrow (SQ_n)^* \in \widetilde{\mathcal{M}}$, and so S^*Q_n is τ -premeasurable. Since S^*Q_n is closed, it is in $\widetilde{\mathcal{M}}$. In fact $S^*Q_n = (SQ_n)^*$, by Proposition 2.6.7. Thus the projections Q_n also demonstrate that $S^* \in \mathcal{L}(\widetilde{\mathcal{M}})$.

(b) Suppose $S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$, then there exist sequences of projections $(Q_n), (P_n)$ in $Z(\mathcal{M})$ such that $Q_n \uparrow I, P_n \uparrow I$ and $SQ_n, TP_n \in \widetilde{\mathcal{M}}$ for each n . Let $H_n = Q_n \wedge P_n = Q_n P_n$ (since $Q_n \in Z(\mathcal{M})^p$). Then $H_n \in Z(\mathcal{M})^p$ and

$$\begin{aligned} H_{n+1} - H_n &= Q_{n+1}P_{n+1} - Q_nP_n \\ &= (Q_{n+1} - Q_n)P_n + Q_{n+1}(P_{n+1} - P_n) \\ &= P_n(Q_{n+1} - Q_n)P_n + Q_{n+1}(P_{n+1} - P_n)Q_{n+1} \\ &\geq 0. \end{aligned}$$

Thus $H_n \uparrow R = \sup H_n$, so that $H_n \rightarrow_{so} R$. But $H_n \rightarrow_{so} I$ since multiplication is strongly continuous on the unit ball of $\mathcal{B}(H)$. Hence $H_n \in Z(\mathcal{M})$ and $H_n \uparrow I$. Now

$$(S + T)H_n = (S + T)Q_nP_nH_n = SQ_nH_n + TP_nH_n.$$

Since $SQ_n, TP_n, H_n \in \widetilde{\mathcal{M}}$, we have that SQ_nH_n and TP_nH_n both lie in $\widetilde{\mathcal{M}}$. Thus

$$\overline{S + T}H_n = \overline{(S + T)H_n} = \overline{SQ_nH_n + TP_nH_n} \in \widetilde{\mathcal{M}},$$

by Propositions 2.6.5 and 2.6.7. Thus the projections H_n demonstrate that $\overline{S + T} \in \mathcal{L}(\widetilde{\mathcal{M}})$.

(c) Just as in part (b), we have $H_n \in Z(\mathcal{M})^p$, $H_n \uparrow I$, $TH_n, SH_n \in \widetilde{\mathcal{M}}$ for all n . Since SH_n and TH_n are in $\widetilde{\mathcal{M}}$, SH_nTH_n is τ -premeasurable, so that $\overline{SH_nTH_n} \in \widetilde{\mathcal{M}}$. Now $\overline{ST}H_n = \overline{STH_n} = \overline{SH_nTH_n}$ using Lemma 2.12.3.

Thus the projections H_n also demonstrate that $\overline{ST} \in \mathcal{L}(\widetilde{\mathcal{M}})$.

(2) Since $\mathcal{L}(\widetilde{\mathcal{M}})$ is closed with respect to strong sum, strong product, scalar multiplication and adjunction, and since $\mathcal{L}(\widetilde{\mathcal{M}})$ is a subset of the $*$ -algebra $\mathcal{L}(\mathcal{M})$, it follows that $\mathcal{L}(\widetilde{\mathcal{M}})$ is a $*$ -algebra itself.

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Lemma 2.12.10 *Suppose $Z(\mathcal{M})^p$ is countably decomposable. An operator $S \in \mathcal{M}$ with a locally τ -measurable domain admits at most one extension in $\mathcal{L}(\widetilde{\mathcal{M}})$.*

Proof: A locally τ -measurable set is locally measurable in the sense of ([San59], 2.1), and so by ([Yea73], Theorem 2.4), if $Z(\mathcal{M})^p$ is countably decomposable, then a locally τ -measurable set is locally measurable in the sense of ([Yea73], 2.3). So suppose $S_1, S_2 \in \mathcal{L}(\widetilde{\mathcal{M}})$ are extensions of S , and that S_1 and S_2 agree on $D(S)$. Let $S_{1r} = S_1|_{D(S)}$, then we show that $\overline{S_{1r}} = S_1$. Since $\mathcal{L}(\widetilde{\mathcal{M}}) \subseteq \mathcal{L}(\mathcal{M})$, by ([Yea73], Corollary 2.8), we have that $\overline{S_{1r}} = S_1$. In the same way, we have that $\overline{S_{2r}} = S_2$. But then $\overline{S_{1r}} = \overline{S_{2r}}$. The result follows.

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Corollary 2.12.11 *Suppose $Q \in \mathcal{M}^p$ and S is a closed densely defined operator affiliated with \mathcal{M} such that $QS \subseteq SQ$ (this is the case, in particular, when Q is a central projection) and $SQ \in \widetilde{\mathcal{M}}$. Then $QS^*Q = (QSQ)^*$.*

Proof: As in the proof of Proposition 2.12.9(a), we have $S^*Q = (SQ)^*$, so that by Lemma 2.12.3, $QS^*Q = (QSQ)^*$.

△

Lemma 2.12.12 *Suppose $S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$. Then $D(ST)$ is locally τ -measurable.*

Proof: Let (H_n) be the sequence of central projections as in Proposition 2.12.9(a); that is $H_n \in Z(\mathcal{M})^p$, $H_n \uparrow I$, $TH_n, SH_n \in \widetilde{\mathcal{M}}$ for all n . That $U(D(ST)) \subseteq D(ST)$ for each unitary U in \mathcal{M}' follows since the product of two affiliated operators is affiliated. By Proposition 2.12.9(c), SH_nTH_n is τ -premeasurable. By Lemma 2.12.3, $STH_n = SH_nTH_n$, so that STH_n is τ -premeasurable, and so $D(STH_n)$ is τ -dense. Hence, by Lemma 2.12.8, $D(ST)$ is τ -measurable (H_n) with respect to \mathcal{M} for each n , and hence $D(ST)$ is locally τ -measurable.

△

Proposition 2.12.13 *If $S \in \mathcal{L}(\widetilde{\mathcal{M}})$ and $Q \in \mathcal{M}^p$ with $\tau(Q) < \infty$ then $QSQ \in \widetilde{\mathcal{M}}$.*

Proof: Since $S \in \mathcal{L}(\widetilde{\mathcal{M}})$, there exists a sequence (Q_n) in $\mathcal{M}^p \cap Z(\mathcal{M})$ such that $Q_n \uparrow I$ and $SQ_n \in \widetilde{\mathcal{M}}$ for all $n \in \mathbb{N}$. Let $\delta > 0$. Since $Q(I - Q_n)Q \downarrow 0$ and $Q(I - Q_n)Q \leq Q$, and $\tau(Q) < \infty$, there exists an $n \in \mathbb{N}$ such that $\tau(Q(I - Q_n)Q) < \frac{\delta}{2}$. Since $Q_nSQ_nQ = Q_nSQ_nQ \in \widetilde{\mathcal{M}}$,

$$QQ_nSQ_nQQ_n \in \widetilde{\mathcal{M}}_{QQ_n} = \widetilde{\mathcal{M}}_{QQ_n}, \text{ by Proposition 2.6.15.}$$

Hence there exists a $P \in (\mathcal{M}_{QQ_n})^p$, i.e. $P \leq QQ_n \leq Q$, such that

$$\|QQ_nSQ_nQP\| < \infty \text{ and } \tau(QQ_n - P) = \tau(QQ_n - PQQ_n) < \frac{\delta}{2}.$$

Now

$$\begin{aligned}
\tau(I - (I - Q + P)) &= \tau(Q - P) = \tau(Q - QQ_n + QQ_n - P) \\
&= \tau(Q(I - Q_n)) + \tau(QQ_n - P) \\
&< \frac{\delta}{2} + \frac{\delta}{2} = \delta
\end{aligned}$$

and

$$\begin{aligned}
\|QSQ(I - Q + P)\| &= \|QSQP\| \\
&= \|QSQQQ_nP\| \quad (P \leq QQ_n) \\
&= \|QQ_nSQ_nQP\|, \quad \text{by Lemma 2.12.3} \\
&< \infty.
\end{aligned}$$

Thus the projection $I - Q + P$ demonstrates that $QSQ \in \widetilde{\mathcal{M}}$.

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Corollary 2.12.14 *If $S \in \mathcal{L}(\widetilde{\mathcal{M}})$ and $P \in \mathcal{M}^p$ with $\tau(P) < \infty$, then $(PSP)^* = PT^*P$.*

Proof: By Proposition 1.3.1, we have that $(PTP)^* = (TP)^*P$ since P is bounded, and also we have $PT^* \subseteq (TP)^*$, so that $PT^*P \subseteq (TP)^*P = (PTP)^*$. By Proposition 2.12.13, we have that $PT^*P, (PTP)^* \in \widetilde{\mathcal{M}}$, so that by uniqueness of extensions in $\widetilde{\mathcal{M}}$, we have $PT^*P = (PTP)^*$.

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Corollary 2.12.15 *If $S \in \mathcal{L}(\widetilde{\mathcal{M}})$ and $P, Q \in \mathcal{M}^p$ with $\tau(Q) < \infty$, $\tau(P) < \infty$, then $PSQ \in \widetilde{\mathcal{M}}$.*

Proof: Let $P_0 = P \vee Q$, then $PSQ = PP_0SP_0Q \in \widetilde{\mathcal{M}}$, since by Proposition 2.12.13, $P_0SP_0 \in \widetilde{\mathcal{M}}$, and since PP_0SP_0Q is closed.

Δ

2.13 The Topology of Local Convergence in Measure on $\mathcal{L}(\widetilde{\mathcal{M}})$

Let (X, Σ, μ) be a localizable measure space. As we have seen in Theorem 1.2.4 it is possible to define a topology (the topology of local convergence in measure) on $L_0(X, \Sigma, \mu)$ such that $L_0(X, \Sigma, \mu)$ becomes a complete topological $*$ -algebra. In [Dix71] and [Yea73] this topology has been generalized to the algebra $\mathcal{L}(\mathcal{M})$ of locally measurable operators, using the notion of a dimension function as introduced by Segal.

We first recall the definition of a dimension function:

Definition 2.13.1 *Let \mathcal{M} be a von Neumann algebra (not necessarily semi-finite) with centre $Z(\mathcal{M})$ and ϕ be an isomorphism of $Z(\mathcal{M})$ onto $L_\infty(X, \Sigma, \mu)$, with (X, Σ, μ) a localizable measure space. A dimension function on \mathcal{M} is a function D on the projections of \mathcal{M} to the non negative extended real-valued measurable functions on (X, Σ, μ) such that*

1. $D(P) < \infty$ a.e if and only if P is finite;
2. if $P \perp Q$, then $D(P + Q) = D(P) + D(Q)$;
3. if $\{P_\lambda\}$ is a chain of projections, then $D(\bigvee_\lambda P_\lambda) = \sup_\lambda D(P_\lambda)$
4. if U is a partial isometry in \mathcal{M} , then $D(U^*U) = D(UU^*)$;
5. if E is a non-zero projection in $Z(\mathcal{M})$, and P is a projection in \mathcal{M} , then $D(E) \neq 0$ and $D(EP) = \phi(E)D(P)$.

We can now define the topology of convergence in measure, as introduced by Yeadon:

Definition 2.13.2 ([Yea73], section 3)

Let (X, Σ, μ) and D be as defined in definition 2.13.1. For K a subset of X with $\mu(K) < \infty$

and $\lambda > 0$, define the set

$$N_{K,\lambda}(0) = \{S \in \mathcal{L}(\mathcal{M}) : \text{for some projection } E \in \mathcal{M}, \|SE\| \leq \lambda$$

$$\text{and } \mu(\{x \in K : D(I - E)(x) > \lambda\}) \leq \lambda\}.$$

Then the system $\{N_{K,\lambda}(0) : \mu(K) < \infty, \lambda > 0\}$ forms a basis for the neighbourhoods of 0 in $\mathcal{L}(\mathcal{M})$ for a vector topology γ_{lm} called the topology of local convergence in measure.

In the case where \mathcal{M} is commutative, it can be identified with $L_\infty(X, \Sigma, \mu)$, for some localisable measure space (X, Σ, μ) , and $\mathcal{L}(\mathcal{M})$ with $L_0(X, \Sigma, \mu)$. In this case γ_{lm} coincides with the usual topology of local convergence in measure with the γ_{lm} neighbourhoods being defined by

$$N_{F,\epsilon} = \{f \in L_0(X, \Sigma, \mu) : \mu(\{x \in F : |f(x)| > \epsilon\}) \leq \epsilon\},$$

for each $\epsilon > 0$ and each $F \in \Sigma$ with $\mu(F) < \infty$ (see, for example, the proof of Theorem 2.2 in [Pat83]).

When $\mathcal{M} = \mathcal{B}(H)$, γ_{lm} coincides with the norm topology on $\mathcal{B}(H)$.

In this section we restrict attention to a semi-finite von Neumann algebra equipped with a semi-finite normal faithful trace τ and define a topology of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ in terms of the trace τ .

Definition 2.13.3 Let $Q \in \mathcal{M}^p \cap \mathcal{M}_0$, and $\epsilon > 0$. Let

$$\widetilde{\mathcal{M}}(Q, \epsilon) = \{S \in \mathcal{L}(\widetilde{\mathcal{M}}) : QSQ \in \widetilde{\mathcal{M}}(\epsilon)\},$$

where $\widetilde{\mathcal{M}}(\epsilon)$ is as defined in section 2.6.

We show that the family $\{\widetilde{\mathcal{M}}(Q, \epsilon) : Q \in \mathcal{M}^p \cap \mathcal{M}_0, \epsilon > 0\}$ forms a neighbourhood base at 0 for a vector topology on $\mathcal{L}(\widetilde{\mathcal{M}})$.

The following is immediate from Definition 2.13.3:

Proposition 2.13.4 For $S \in \mathcal{L}(\widetilde{\mathcal{M}})$, $S \in \widetilde{\mathcal{M}}(Q, \epsilon) \iff \mu_\epsilon(QSQ) < \epsilon$.

Proposition 2.13.5 Let $Q, Q_1, Q_2 \in \mathcal{M}^p \cap \mathcal{M}_0$, and $\epsilon_1, \epsilon_2 > 0$.

1. If $Q_1 \leq Q_2$ and if $\epsilon_1 \leq \epsilon_2$, then $\widetilde{\mathcal{M}}(Q_2, \epsilon_1) \subseteq \widetilde{\mathcal{M}}(Q_1, \epsilon_2)$.
2. If $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, then $\widetilde{\mathcal{M}}(Q_1 \vee Q_2, \epsilon) \subseteq \widetilde{\mathcal{M}}(Q_1, \epsilon) \cap \widetilde{\mathcal{M}}(Q_2, \epsilon)$.
3. Let $\epsilon = \epsilon_1 + \epsilon_2$. Then $\widetilde{\mathcal{M}}(Q, \epsilon_1) + \widetilde{\mathcal{M}}(Q, \epsilon_2) \subseteq \widetilde{\mathcal{M}}(Q, \epsilon)$, where the sum is the strong sum.

Proof:

1. Suppose $S \in \widetilde{\mathcal{M}}(Q_2, \epsilon_1)$. Then $\mu_{\epsilon_1}(Q_2 S Q_2) < \epsilon_1$, so that

$$\begin{aligned} \mu_{\epsilon_1}(Q_1 S Q_1) &= \mu_{\epsilon_1}(Q_1 Q_2 S Q_2 Q_1) \\ &\leq \|Q_1\| \mu_{\epsilon_1}(Q_2 S Q_2) \|Q_1\| < \epsilon_1 \leq \epsilon_2 \end{aligned}$$

Hence $S \in \widetilde{\mathcal{M}}(Q_1, \epsilon_2)$.

2. Follows from (1).

3. Let $S_1 \in \widetilde{\mathcal{M}}(Q, \epsilon_1)$ and $S_2 \in \widetilde{\mathcal{M}}(Q, \epsilon_2)$. Then

$$Q(S_1 + S_2)Q \in \widetilde{\mathcal{M}}(\epsilon_1) + \widetilde{\mathcal{M}}(\epsilon_2) \subseteq \widetilde{\mathcal{M}}(\epsilon_1 + \epsilon_2),$$

by Proposition 2.6.11, so that $S_1 + S_2 \in \widetilde{\mathcal{M}}(Q, \epsilon_1 + \epsilon_2)$.

△

Proposition 2.13.6 Suppose $Q \in \mathcal{M}^p \cap \mathcal{M}_0$ and $\epsilon > 0$.

1. $\widetilde{\mathcal{M}}(Q, \epsilon)$ is absorbing.
2. $\widetilde{\mathcal{M}}(Q, \frac{\epsilon}{2}) + \widetilde{\mathcal{M}}(Q, \frac{\epsilon}{2}) \subseteq \widetilde{\mathcal{M}}(Q, \epsilon)$.
3. $\widetilde{\mathcal{M}}(Q, \epsilon)$ is balanced.

Proof: (1) $S \in \mathcal{L}(\widetilde{\mathcal{M}})$ implies that $QSQ \in \widetilde{\mathcal{M}}$, by Proposition 2.12.13. Since $\widetilde{\mathcal{M}}(\epsilon)$ is absorber, there exists a $\lambda > 0$ such that $QSQ \in \lambda\widetilde{\mathcal{M}}(\epsilon)$. Thus $Q(\frac{1}{\lambda}S)Q \in \widetilde{\mathcal{M}}(\epsilon)$, and so $\frac{1}{\lambda}S \in \widetilde{\mathcal{M}}(Q, \epsilon)$. Thus $S \in \lambda\widetilde{\mathcal{M}}(Q, \epsilon)$.

(2) Follows from Proposition 2.13.5(3).

(3) Let $|\lambda| \leq 1$ and $S \in \widetilde{\mathcal{M}}(Q, \epsilon)$. Then $QSQ \in \widetilde{\mathcal{M}}(\epsilon)$. Since $\widetilde{\mathcal{M}}(\epsilon)$ is balanced, $Q(\lambda S)Q = \lambda(QSQ) \in \widetilde{\mathcal{M}}(\epsilon)$. Hence $\lambda S \in \widetilde{\mathcal{M}}(Q, \epsilon)$.

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Proposition 2.13.7 Suppose $Q \in \mathcal{M}^p \cap \mathcal{M}_0$ and $\epsilon > 0$.

1. The family $\{\widetilde{\mathcal{M}}(Q, \epsilon) : Q \in \mathcal{M}^p \cap \mathcal{M}_0, \epsilon > 0\}$ is a filter base.
2. The family $\{\widetilde{\mathcal{M}}(Q, \epsilon) : Q \in \mathcal{M}^p \cap \mathcal{M}_0, \epsilon > 0\}$ forms a neighbourhood base at 0 for a vector topology τ_{lm} on $\mathcal{L}(\widetilde{\mathcal{M}})$, called the topology of local convergence in measure.

Proof: (1) Follows from Proposition 2.13.5(2).

(2) Follows by Propositions 1.1.3 and 2.13.7.

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For a net (S_α) in $\mathcal{L}(\widetilde{\mathcal{M}})$ it follows that $S_\alpha \rightarrow_{\tau_{lm}} 0 \iff QS_\alpha Q \rightarrow_{\tau_m} 0$ for all $Q \in \mathcal{M}^p \cap \mathcal{M}_0$.

Example 2.13.8 If (X, Σ, μ) is semi-finite, then by example 2.12.2, $\mathcal{L}(\widetilde{\mathcal{M}}) = L_0(X, \Sigma, \mu)$.

We have that τ_{lm} is the usual topology of local convergence in measure (Theorem 1.2.4).

Proposition 2.13.9 The topology τ_{lm} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ is Hausdorff.

Proof: Let $S \in \bigcap \{\widetilde{\mathcal{M}}(Q, \epsilon) : Q \in \mathcal{M}^p \cap \mathcal{M}_0, \epsilon > 0\}$. Then $QSQ \in \widetilde{\mathcal{M}}(\epsilon)$ for all $Q \in \mathcal{M}^p \cap \mathcal{M}_0$ and all $\epsilon > 0$. Since the topology of convergence in measure τ_m on $\widetilde{\mathcal{M}}$ is Hausdorff, $QSQ = 0$ for all $Q \in \mathcal{M}^p \cap \mathcal{M}_0$. This implies that also $QS^*Q = 0$ for all $Q \in \mathcal{M}^p \cap \mathcal{M}_0$. Consequently, it may be assumed, without loss of generality that $S = S^*$.

Define spectral projections E_n , $n \in \mathbb{Z}$, by setting $E_n = E_{(n, n+1]}(S)$ and let $S_n = SE_n$. Let $n \in \mathbb{Z}$ be fixed for a moment. Since the trace is semi-finite, there exists a net $\{Q_\alpha\}$ in \mathcal{M}^p such that $Q_\alpha \uparrow_\alpha E_n$ and $\tau(Q_\alpha) < \infty$ for all α . Since $Q_\alpha = E_n Q_\alpha$, it follows that $Q_\alpha S_n Q_\alpha = Q_\alpha S E_n Q_\alpha = Q_\alpha S Q_\alpha = 0$ for all α . Using that $Q_\alpha \uparrow_\alpha E_n$ implies that $Q_\alpha \rightarrow_\alpha E_n$ strongly, that $S_n \in \mathcal{M}$ and that multiplication is jointly continuous on norm bounded sets, it follows that $Q_\alpha S_n Q_\alpha \rightarrow_\alpha E_n S_n E_n = S_n$ strongly. Consequently, $S_n = 0$. Since this holds for all $n \in \mathbb{Z}$, we conclude that $S = 0$. Thus τ_{lm} is Hausdorff on $\mathcal{L}(\widetilde{\mathcal{M}})$

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The following example shows that multiplication need not be jointly τ_{lm} -continuous.

Example 2.13.10 Consider the von Neumann algebra $\mathcal{M} = \mathcal{B}(H)$, of all bounded linear operators on an infinite dimensional Hilbert space H , with the canonical diagonal trace. Then it is well known that $\widetilde{\mathcal{M}} = \mathcal{M}$ and that the topology τ_m of convergence in measure is equal to the norm topology on $\mathcal{B}(H)$ (Proposition 2.9.3). Then $\mathcal{L}(\widetilde{\mathcal{M}}) = \mathcal{B}(H)$. Suppose (S_α) is a net in $\mathcal{L}(\widetilde{\mathcal{M}})$ and that $S_\alpha \rightarrow_{\tau_m} 0$. Then $Q S_\alpha Q \rightarrow 0$ in norm for every $Q \in \mathcal{M}^p \cap \mathcal{M}_0$. If $x_0 \in H$, $\|x_0\| = 1$, then $Qx = \langle x, x_0 \rangle x_0$ defines a projection Q in \mathcal{M}_0 . Since $\|Q S_\alpha Q\| = \sup_{\|x\|=1} \|Q S_\alpha Q x\| = |\langle S_\alpha x_0, x_0 \rangle|$, $\langle S_\alpha x_0, x_0 \rangle \rightarrow 0$. It follows that $\langle S_\alpha x, x \rangle \rightarrow 0$ for every $x \in H$, and hence $S_\alpha \rightarrow 0$ in the weak operator topology.

Conversely, suppose $S_\alpha \rightarrow_{wo} 0$ and $x_1, x_2, \dots, x_n \in H$ and $\|x_i\| = 1$ for all $i = 1, 2, \dots, n$. Then $\langle S_\alpha x_i, x_j \rangle \rightarrow 0$ for all i, j . Let

$$Px = \sum_{i=1}^n \langle x, x_i \rangle x_i. \quad (2.3)$$

Then $P \in \mathcal{M}^p \cap \mathcal{M}_0$ and $PS_\alpha Px = \sum_{i,j=1}^n \langle S_\alpha x_i, x_j \rangle \langle x, x_i \rangle x_j$ so that

$$\|PS_\alpha P\| \leq \sum_{i,j=1}^n |\langle S_\alpha x_i, x_j \rangle| \rightarrow 0.$$

Since any $P \in \mathcal{M}^p \cap \mathcal{M}_0$ can be written in the form of equation (2.3), this shows that $S_\alpha \rightarrow_{\tau_m} 0$. It follows that the topology τ_{lm} is the weak operator topology on $\mathcal{B}(H)$. Now it is well known that multiplication is separately continuous but not jointly continuous with respect to the weak operator topology, ([KR83], Remark 2.5.10). Thus in this case we have

that multiplication is not jointly continuous for the topology τ_{lm} of local convergence in measure. This also shows that the topology τ_{lm} is different from the topology γ_{lm} of Yeadon, since it is known ([Yea73], Theorem 3.3) that multiplication is γ_{lm} jointly continuous.

Proposition 2.13.11 *The family $\{\widetilde{\mathcal{M}}(Q, \epsilon) : Q \in Z(\mathcal{M})^p \cap \mathcal{M}_0, \epsilon > 0\}$ is a basis for the neighbourhoods of 0 for a vector topology τ_{lmc} on $\mathcal{L}(\widetilde{\mathcal{M}})$.*

Proof: The proof is similar to that for the topology τ_{lm} .

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It is immediate that the topology τ_{lmc} is weaker than τ_{lm} and that $S_\alpha \rightarrow_{\tau_{lmc}} 0 \iff QS_\alpha Q \rightarrow_{\tau_m} 0 \iff S_\alpha Q \rightarrow_{\tau_m} 0$ for all $Q \in Z(\mathcal{M})^p \cap \mathcal{M}_0$.

A similar argument to that in Proposition 2.13.9 shows that τ_{lmc} is Hausdorff if the trace restricted to $Z(\mathcal{M})$ is semi-finite. The following example shows that in general τ_{lmc} is not necessarily Hausdorff.

Example 2.13.12 Consider $\mathcal{M} = \mathcal{B}(H)$, with H infinite dimensional. Since $Z(\mathcal{M})^p \cap \mathcal{M}_0 = \{0\}$, the neighbourhood base at 0 for τ_{lmc} consists of the single element $\mathcal{B}(H)$. Thus τ_{lmc} is the indiscrete topology, and so τ_{lmc} cannot be Hausdorff.

Proposition 2.13.13 *Adjunction on $\mathcal{L}(\widetilde{\mathcal{M}})$ is τ_{lm} -continuous.*

Proof: Suppose $S_\alpha \rightarrow_{\tau_{lm}} S$ in $\mathcal{L}(\widetilde{\mathcal{M}})$. Then $QS_\alpha Q \rightarrow_{\tau_m} QSQ$ for all $Q \in \mathcal{M}^p \cap \mathcal{M}_0$. Then $(QS_\alpha Q)^* \rightarrow_{\tau_m} (QSQ)^*$, by continuity of adjunction in $\widetilde{\mathcal{M}}$, so that $QS_\alpha^* Q \rightarrow_{\tau_m} QS^* Q$ for all $Q \in \mathcal{M}^p \cap \mathcal{M}_0$, by Corollary 2.12.14, and so adjunction is τ_{lm} -continuous.

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In ([Bik04]) the notion of the τ -local convergence in measure on $\widetilde{\mathcal{M}}$ has been discussed, and it is proved there that multiplication is separately continuous. Using a similar approach we have the following.

Proposition 2.13.14 *Let $S_i, S \in \mathcal{L}(\widetilde{\mathcal{M}})$.*

If $S_i \rightarrow_{\tau_{lm}} S$, then $S_i T \rightarrow_{\tau_{lm}} ST$ and $TS_i \rightarrow_{\tau_{lm}} TS$ for every fixed $T \in \mathcal{L}(\widetilde{\mathcal{M}})$.

Proof: Let $S_i, S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$ and $Q \in \mathcal{M}^p \cap \mathcal{M}_0$. Since $\mathbf{l}(TQ) \sim \mathbf{l}(QT^*) \leq Q$, we have that $\mathbf{l}(TQ) \in \mathcal{M}^p \cap \mathcal{M}_0$, where $\mathbf{l}(TQ)$ is the left support of TQ . We have that $P_0 = P \vee Q \in \mathcal{M}^p \cap \mathcal{M}_0$ for every $P, Q \in \mathcal{M}^p \cap \mathcal{M}_0$. Now $S_i \rightarrow_{\tau_{lm}} S$ if and only if $PS_iQ \rightarrow_{\tau_m} PSQ$. For, from $P_0S_iP_0 \rightarrow_{\tau_m} P_0SP_0$, it follows that $PS_iQ = PP_0S_iP_0Q \rightarrow PP_0SP_0Q = PSQ$, by using the τ_m -continuity of multiplication on $\widetilde{\mathcal{M}}$, and since $P_0SP_0 \in \widetilde{\mathcal{M}}$ by Proposition 2.12.13. We have

$$PS_iTP = PS_i\mathbf{l}(TP)TP = PS_i\mathbf{l}(TP)TP \rightarrow_{\tau_m} PS\mathbf{l}(TP)TP = PSTP,$$

by Corollary 2.12.15, and using continuity of multiplication in $\widetilde{\mathcal{M}}$. Hence $S_iT \rightarrow_{\tau_{lm}} ST$.

Similarly, using the right support of PT , $\mathbf{r}(PT)$, we have that $\mathbf{r}(PT) \sim \mathbf{l}(PT) \leq P$, so that $\mathbf{r}(PT) \in \mathcal{M}^p \cap \mathcal{M}_0$. Hence

$$PTS_iP = PT\mathbf{r}(PT)S_iP = PT\mathbf{r}(PT)S_iP \rightarrow_{\tau_m} PT\mathbf{r}(PT)SP = PTSP,$$

again using continuity of multiplication in $\widetilde{\mathcal{M}}$ and using Corollary 2.12.15. Hence we have $TS_i \rightarrow_{\tau_{lm}} TS$.

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Corollary 2.13.15 $\mathcal{L}(\widetilde{\mathcal{M}})$ is a topological $*$ -algebra when equipped with the topology τ_{lm} .

Proof: Follows from Propositions 2.12.9, 2.13.13 and 2.13.14.

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Proposition 2.13.16 Multiplication is jointly continuous, and adjunction is continuous on $\mathcal{L}(\widetilde{\mathcal{M}})$ with respect to the topology τ_{lmc} ; hence $(\mathcal{L}(\widetilde{\mathcal{M}}), \tau_{lmc})$ is a topological $*$ -algebra.

Proof: Suppose $S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$, and $(S_\alpha), (T_\alpha)$ are nets in $\mathcal{L}(\widetilde{\mathcal{M}})$ such that $S_\alpha \rightarrow_{\tau_{lmc}} S$ and $T_\alpha \rightarrow_{\tau_{lmc}} T$. Then for all $Q \in Z(\mathcal{M})^p \cap \mathcal{M}_0$, $QS_\alpha Q \rightarrow_{\tau_m} QSQ$ and $QT_\alpha Q \rightarrow_{\tau_m} QTQ$. By continuity of multiplication with respect to the topology τ_m of convergence in measure,

$$QS_\alpha T_\alpha Q = QS_\alpha QQT_\alpha Q \rightarrow_{\tau_m} QSQQTQ = QSTQ, \quad \text{using Lemma 2.12.3.}$$

Thus multiplication is τ_{lmc} -continuous.

Suppose $S_\alpha \rightarrow_{\tau_{lmc}} S$ in $\mathcal{L}(\widetilde{\mathcal{M}})$. Then $QS_\alpha Q \rightarrow_{\tau_m} QSQ$ for all $Q \in Z(\mathcal{M})^p \cap \mathcal{M}_0$. Then $(QS_\alpha Q)^* \rightarrow_{\tau_m} (QSQ)^*$, so that $QS_\alpha^* Q \rightarrow_{\tau_m} QS^* Q$ for all $Q \in Z(\mathcal{M})^p \cap \mathcal{M}_0$, by Corollary 2.12.11, and so adjunction is τ_{lmc} -continuous.

The continuity of addition and scalar multiplication follows as for the topology τ_{lm} .

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Proposition 2.13.17 *If $Z(\mathcal{M})$, the centre of \mathcal{M} , is countably decomposable and $\tau|_{Z(\mathcal{M})}$ is semi-finite then the topology τ_{lmc} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ is metrizable.*

Proof: By example 2.1.7, there exists a $*$ -isomorphism

$$\phi : Z(\mathcal{M}) \rightarrow L_\infty(X, \Sigma, \mu),$$

with (X, Σ, μ) a localizable measure space. Now if $Z(\mathcal{M})$ is countably decomposable (definition 2.1.4), then the measure μ is σ -finite. To see this, note that we have $X = \cup_\lambda B_\lambda$ where B_λ is a mutually disjoint family in Σ , with $\mu(B_\lambda) < \infty$ for all λ . So the corresponding orthogonal family of projections $(P_\lambda) \in Z(\mathcal{M})^p$, with $\phi(P_\lambda) = \chi_{B_\lambda}$, must be countable, by countable decomposability of $Z(\mathcal{M})$, thus making the family (B_λ) countable. We therefore have $\phi(I) = \chi_X$, and $X = \cup_{n=1}^\infty B_n$ where the $B_n \in \Sigma$ are mutually disjoint and $\mu(B_n) < \infty$, for all n . For each n , there is a $P_n \in Z(\mathcal{M})^p$ such that $\chi_{B_n} = \phi(P_n)$. We have

$$\sum_{n=1}^\infty \phi(P_n) = \phi\left(\sum_{n=1}^\infty P_n\right), \quad (2.4)$$

since ϕ is bicontinuous with the weak operator topology on $Z(\mathcal{M})$ and the weak*-topology on $L_\infty(X, \Sigma, \mu)$, ([Seg53], Remark 1.2). Let D be the restriction of ϕ to the projections in $Z(\mathcal{M})$. Then D is a dimension function on $Z(\mathcal{M})$.

By ([Yea73], section 1), since $Z(\mathcal{M})$ is semi-finite, τ , D and μ are related by

$$\tau(P) = \int_X D(P) d\mu,$$

for $P \in Z(\mathcal{M})^p$. Thus $\tau(P_n) = \int_X D(P_n) d\mu = \int_X \phi(P_n) d\mu = \int_X \chi_{B_n} d\mu = \mu(B_n) < \infty$. Also we have

$$I = \phi^{-1}(\chi_X) = \phi^{-1}(\chi_{\cup_{n=1}^{\infty} B_n}) = \phi^{-1}\left(\sum_{n=1}^{\infty} \chi_{B_n}\right) = \phi^{-1}\left(\sum_{n=1}^{\infty} \phi(P_n)\right) = \phi^{-1}\left(\phi\left(\sum_{n=1}^{\infty} (P_n)\right)\right) = \vee_{n=1}^{\infty} P_n,$$

using equation (2.4), and the fact that the P_n are mutually orthogonal.

Now suppose $P \in Z(\mathcal{M})^p$, $\tau(P) < \infty$ and $\epsilon > 0$. Choose an $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. Since $\vee_{n=1}^k P_n \uparrow I$ implies $\vee_{n=1}^k P_n \wedge P \uparrow P$, and since $\tau(P) < \infty$, it follows from the normality of the trace that we can choose an $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. Then choose a $k \in \mathbb{N}$ such that $\tau(P - \vee_{n=1}^k P_n \wedge P) \leq \frac{1}{2m}$. Since $P - \vee_{n=1}^k P_n \leq P - \vee_{n=1}^k P_n \wedge P$, we have that $\tau(P - \vee_{n=1}^k P_n) \leq \frac{1}{2m}$.

Set $P^k = \vee_{n=1}^k P_n$, then $P^k \in Z(\mathcal{M})^p$.

Suppose now that $S \in \widetilde{\mathcal{M}}(P^k, \frac{1}{2m})$. Then there exists a $Q \in \mathcal{M}^p$ such that

$$\|P^k S P^k Q\| \leq \frac{1}{2m}, \quad \tau(I - Q) \leq \frac{1}{2m} \quad \text{and} \quad Q(H) \subseteq D(P^k S P^k) = D(S P^k) \text{ by Lemma 2.12.3.}$$

Now let $Q_1 = (I - P) + P^k$, then $Q_1 \in Z(\mathcal{M})^p$ and $\tau(I - Q_1) = \tau(P - P^k) < \frac{1}{2m}$.

Set $Q_2 = Q_1 \wedge Q$. Then $\tau(I - Q_2) = \tau(I - Q_1 \wedge Q) \leq \tau(I - Q_1) + \tau(I - Q) < \frac{1}{m} < \epsilon$, by Proposition 2.1.8.

We have that $Q(H) \subseteq D(P^k S P^k) = D(S P^k)$, so that $P^k Q y \in D(S)$ for all $y \in H$. Now

$$P Q_2 = P Q_1 Q_2 = P P^k Q Q_2 = P^k Q P Q_2.$$

If $x \in H$, $y = P Q_2 x \in H$, then $P Q_2 x = P^k Q y \in D(S)$. Hence $Q_2(H) \subseteq D(S P) = D(P S P)$.

Also

$$\begin{aligned}
\|PSPQ_2\| &= \|PSPQ_2Q_1Q\|, \quad (Q_1 \text{ is central, } Q_2 \leq Q, Q_2 \leq Q_1) \\
&= \|PSPQ_1QQ_2\| \\
&\leq \|PSPQ_1Q\| \\
&\leq \|SPQ_1Q\| \\
&= \|SP P^k Q\|, \quad (PQ_1 = PP^k) \\
&= \|SP^k QP\| \quad \text{since } P \text{ is central} \\
&\leq \|P^k SP^k Q\|, \quad \text{using Lemma 2.12.3} \\
&\leq \frac{1}{2m} < \frac{1}{m} < \epsilon.
\end{aligned}$$

Hence $S \in \widetilde{\mathcal{M}}(P, \epsilon)$. Hence the family $\{\widetilde{\mathcal{M}}(P^k, \frac{1}{2m}) : k, m \in \mathbb{N}\}$ forms a countable neighbourhood basis at 0, so that the topology τ_{lmc} of local convergence in measure is metrizable.

Δ

Corollary 2.13.18 *The system $\{\widetilde{\mathcal{M}}(P_n, \frac{1}{m}) : n, m \in \mathbb{N}\}$ is a countable subbase at 0 for the topology τ_{lmc} of local convergence in measure.*

Proof: Suppose $\widetilde{\mathcal{M}}(P^k, \frac{1}{m})$ is one of the basic sets already considered in Proposition 2.13.17. Suppose $S \in \cap_{n=1}^k \widetilde{\mathcal{M}}(P_n, \frac{1}{mk})$. Then there exist projections $Q_n \in \mathcal{M}^p$, $n = 1, 2, 3, \dots, k$, such that $\|P_n S P_n Q_n\| \leq \frac{1}{mk}$, $\tau(I - Q_n) \leq \frac{1}{mk}$ and $Q_n(H) \subseteq D(P_n S P_n)$. Now $P^k S P^k = \left(\sum_{n=1}^k P_n\right) S \left(\sum_{n=1}^k P_n\right) = \sum_{n=1}^k P_n S P_n$ since the P_n are orthogonal, and using Lemma 2.12.3. Let $H_k = \wedge_{n=1}^k Q_n$, then $\tau(I - H_k) \leq \frac{1}{m}$ by Proposition 2.1.8, and for all $n = 1, 2, 3, \dots, k$, $H_k(H) \subseteq D(P_n S P_n)$ so that

$$H_k(H) \subseteq \cap_{n=1}^k D(P_n S P_n) = D(P^k S P^k)$$

since the P_n are mutually orthogonal. Thus by the triangle inequality,

$$\begin{aligned} \|P^k S P^k H_k\| &\leq \sum_{n=1}^k \|P_n S P_n Q_n H_k\|, & \text{since } H_k \leq Q_n \\ &\leq \sum_{n=1}^k \|P_n S P_n Q_n\| \\ &\leq \sum_{n=1}^k \frac{1}{mk} = \frac{1}{m}. \end{aligned}$$

Therefore, $\bigcap_{n=1}^k \widetilde{\mathcal{M}}(P_n, \frac{1}{mk}) \subseteq \widetilde{\mathcal{M}}(P^k, \frac{1}{m})$. Hence $\{\widetilde{\mathcal{M}}(P_n, \frac{1}{m}) : n, m \in \mathbb{N}\}$ is a countable subbase at 0 for the topology τ_{lm} of local convergence in measure.

Δ

2.14 The local convexity of $\mathcal{L}(\widetilde{\mathcal{M}})$

It was shown in [Cro97] that if \mathcal{M} is a semi-finite von Neumann algebra, the local convexity of the topology τ_m of convergence in measure on $\widetilde{\mathcal{M}}$ depends on the nature of projections in \mathcal{M} . In the case where $\mathcal{M} = L_\infty(X, \Sigma, \mu)$, with (X, Σ, μ) a localizable measure space, it is possible to characterize the semi-finite measure spaces for which the topology τ_{lm} of local convergence in measure is locally convex.

Theorem 2.14.1 ([Wes90], 1.21) *Suppose (X, Σ, μ) is a semi-finite measure space. Then τ_{lmc} is locally convex if and only if Σ does not contain a non-atomic set of positive measure.*

In this section we obtain some partial results for the non-commutative case. The proof of the first result is an adaption of the proof of Theorem 2.10.2, where the local convexity of τ_m on $\widetilde{\mathcal{M}}$ is considered.

Theorem 2.14.2 *Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ and suppose that \mathcal{M}^p has a non-atomic central projection. Suppose also that $\tau_{Z(\mathcal{M})}$ is semi-finite. Then the topology τ_{lmc} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ is not locally convex.*

Proof: Suppose \mathcal{M}^p has a non-atomic central projection P . Since $\tau_{|Z(\mathcal{M})}$ is semi-finite, we can choose a $Q \in Z(\mathcal{M})^p \cap \mathcal{M}_0$ such that $0 < Q \leq P$. Then Q is non-atomic. We show that for each $Q_1 \in Z(\mathcal{M})^p \cap \mathcal{M}_0$ and each $\delta > 0$ with $0 < \delta < \frac{1}{2}\tau(Q_1Q)$, if $0 < \epsilon < \delta$ then, $\text{conv}\widetilde{\mathcal{M}}(Q_1, \epsilon) \not\subseteq \widetilde{\mathcal{M}}(Q, \delta)$. It follows that $\widetilde{\mathcal{M}}(Q, \delta)$ does not contain any convex neighbourhood of 0.

Let $Q_0 = Q \wedge Q_1 = Q_1Q$; then Q_0 is non-atomic. Suppose $Q_0 \neq 0$ and let $0 < \epsilon < \delta < \frac{1}{2}\tau(Q_0)$ be given. Choose $n \in \mathbb{N}$ such that $\frac{2\delta}{n} < \epsilon$. Since Q_0 is non-atomic, we can find a $P_0 \leq Q_0$ such that $\tau(P_0) = 2\delta$. Since P_0 is also non-atomic, there are disjoint projections P_1, \dots, P_n such that $\tau(P_k) < \epsilon$ for $k = 1, \dots, n$ and $\sum_{k=1}^n P_k = P_0$. For each $k = 1, \dots, n$ define $R_k = 2n\delta P_k$. Then each $R_k \in \widetilde{\mathcal{M}}(Q_1, \epsilon)$ since for each $k = 1, \dots, n$, $d_\epsilon(R_k) = \tau(P_k) < \epsilon$ shows that $R_k \in \widetilde{\mathcal{M}}(\epsilon)$. Since $R_k = Q_1 R_k Q_1$, we also have that $R_k \in \widetilde{\mathcal{M}}(Q_1, \epsilon)$. Define

$$\begin{aligned} R &= \sum_{k=1}^n \frac{1}{n} R_k \\ &= 2\delta \vee_{k=1}^n P_k \\ &= 2\delta P_0. \end{aligned}$$

Then $R \in \text{conv}\widetilde{\mathcal{M}}(Q_1, \epsilon)$ but $d_\delta(R) = \tau(P_0) = 2\delta > \delta$ so that $R = QRQ \notin \widetilde{\mathcal{M}}(Q, \delta)$. Hence $\text{conv}\widetilde{\mathcal{M}}(Q_1, \epsilon) \not\subseteq \widetilde{\mathcal{M}}(Q, \delta)$.

Now suppose $Q_0 = Q_1 \wedge Q = Q_1Q = 0$. For each $\delta > 0$ with $0 < \delta < \frac{1}{2}\tau(Q)$, if $0 < \epsilon < \delta$ then let $R = 2\delta Q$. Then $Q_1 R Q_1 = 2\delta Q_1 Q Q_1 = 0$. Hence $R \in \widetilde{\mathcal{M}}(Q_1, \epsilon)$. However, $d_\delta(R) = \tau(Q) > 2\delta > \delta$, so that $R = QRQ \notin \widetilde{\mathcal{M}}(Q, \delta)$. Hence $\widetilde{\mathcal{M}}(Q_1, \epsilon) \not\subseteq \widetilde{\mathcal{M}}(Q, \delta)$ so that $\text{conv}\widetilde{\mathcal{M}}(Q_1, \epsilon) \not\subseteq \widetilde{\mathcal{M}}(Q, \delta)$. Hence the topology τ_{lmc} of local convergence in measure is not locally convex.

△

Theorem 2.14.3 *Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ and suppose \mathcal{M}^p is atomic. If $\inf\{\tau(P) : P \in \mathcal{M}^p, \tau(P) \neq 0\} > 0$, then the topology τ_{lm} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ is locally convex.*

Proof: It follows from Proposition 2.9.3 that under the stated condition, $\widetilde{\mathcal{M}} = \mathcal{M}$ and the topology τ_m is the norm topology. For every $Q \in \mathcal{M}^p \cap \mathcal{M}_0$ and $S \in \mathcal{L}(\widetilde{\mathcal{M}})$, we have that $QSQ \in \widetilde{\mathcal{M}} = \mathcal{M}$ (Proposition 2.12.13) and

$$\begin{aligned} S \in \widetilde{\mathcal{M}}(Q, \epsilon) &\iff QSQ \in \widetilde{\mathcal{M}}(\epsilon) = \{T \in \mathcal{M} : \|T\| < \epsilon\} \\ &\iff \|QSQ\| < \epsilon. \end{aligned}$$

For every $Q \in \mathcal{M}^p \cap \mathcal{M}_0$, define p_Q by $p_Q(S) = \|QSQ\|$ for all $S \in \mathcal{L}(\widetilde{\mathcal{M}})$. Then p_Q is clearly a seminorm on $\mathcal{L}(\widetilde{\mathcal{M}})$, and the family $\{p_Q : Q \in \mathcal{M}^p \cap \mathcal{M}_0\}$ of seminorms determines the topology τ_m . By Theorem 1.1.5, the topology τ_m is locally convex.

Δ

We note that the locally convex topology τ_m defined in the above theorem is Hausdorff if and only if for each $0 \neq S \in \mathcal{L}(\widetilde{\mathcal{M}})$, there exists a $Q \in \mathcal{M}^p \cap \mathcal{M}_0$ such that $QSQ \neq 0$. This is the case, for example, when $\mathcal{M} = B(H)$ and the trace τ is the usual diagonal trace.

Chapter 3

Spectra and Invertibility for Algebras of Measurable Operators

In this chapter we present characterizations of various types of spectra for elements of algebras of measurable functions and measurable operators. We also investigate the continuity of inversion and conditions under which the group of invertible elements is open.

We start in section 3.1 by characterizing different types of spectra for elements of $L_0(X, \Sigma, \mu)$, for a localizable measure space (X, Σ, μ) . The results obtained provide a motivation generalizations to the non-commutative case in later sections. In section 3.2 we characterize invertibility for self adjoint elements of $\widetilde{\mathcal{M}}$. We also give invertibility conditions for normal elements of $\widetilde{\mathcal{M}}$. This enables us to show in the next section that inversion is τ_m -continuous on the set of invertible elements in $\widetilde{\mathcal{M}}$. We start by giving a measure theoretic argument in the commutative case and then generalising to the non-commutative case. Similar arguments allow us to show in section 3.4 that inversion is τ_{lmc} -continuous on $\mathcal{L}(\widetilde{\mathcal{M}})$.

In section 3.5 we introduce the notion of Schmidt spectral decomposition that we shall need in subsequent sections. The following section is devoted to subalgebras of $\widetilde{\mathcal{M}}$ which are proper in the sense that when the trace τ is restricted to the subalgebra, it is still semi-finite. We also look at spectral permanence for such subalgebras.

In the next three sections we characterize various types of spectra for operators in $\widetilde{\mathcal{M}}$ and in the ideal $\widetilde{\mathcal{M}}_0$ of τ -compact operators, making use of the generalized singular value function. We also compare the spectrum of the operator S_0 occurring in the Schmidt spectral decomposition with the spectrum of the operator S itself. We end the chapter by showing that in $\widetilde{\mathcal{M}}$, the set of invertible elements is open, with respect to the topology τ_m of convergence in measure, if and only if $\widetilde{\mathcal{M}} = \mathcal{M}$.

3.1 Spectra in Commutative Algebras

In this section we consider spectra of elements in the function algebras $L_0(X, \Sigma, \mu)$, $\widetilde{L}_\infty(X, \Sigma, \mu)$ and $\mathcal{N}(X)$.

Proposition 3.1.1 *If $f \in L_0(X, \Sigma, \mu)$, for some localizable measure space (X, Σ, μ) , then $\sigma_{L_0}(f) = \{\lambda \in \mathbb{C} : \mu(\{x \in X : f(x) = \lambda\}) > 0\}$.*

Proof:

$$\begin{aligned} \lambda \notin \sigma_{L_0}(f) &\iff \text{there exists } g \in L_0 \text{ such that } (f - \lambda 1)g = 1 \text{ in } L_0 \\ &\iff g, \text{ defined by } g(x) = \frac{1}{f(x) - \lambda}, \text{ is finite } \mu - a.e \\ &\iff \mu\{x \in X : f(x) = \lambda\} = 0. \end{aligned}$$

Δ

For $f \in L_0(X, \Sigma, \mu)$, we define the operator M_f on $L_2(X, \Sigma, \mu)$ by $M_f g = fg$. Then M_f is a closed densely defined operator.

Corollary 3.1.2 *If $f \in L_0(X, \Sigma, \mu)$, for some localizable measure space (X, Σ, μ) , then*

$$\sigma_p(M_f) = \sigma_{L_0}(f).$$

Proof:

$$\begin{aligned}
\lambda \in \sigma_p(M_f) &\iff \text{there exists a } g \in L_2(X, \Sigma, \mu), \text{ such that } g \neq 0 \text{ and } M_f g = \lambda g \\
&\iff \text{there exists a } g \in L_0(X, \Sigma, \mu) \text{ such that } \mu\{x \in X : g(x) \neq 0\} > 0 \text{ and} \\
&\quad f(x)g(x) = \lambda g(x) \quad \mu\text{-a.e} \\
&\iff \mu\{x \in X : f(x) = \lambda\} > 0 \\
&\iff \lambda \in \sigma_{L_0}(f), \text{ by Proposition 3.1.1.}
\end{aligned}$$

Δ We present below an alternative proof to that which appears in ([KR83], page 357).

Corollary 3.1.3 *If S is a normal operator, then $\sigma_p(S) = \sigma_{\mathcal{N}(\mathcal{A})}(S)$, where \mathcal{A} is the smallest abelian von Neumann algebra to which S is affiliated.*

Proof: By Proposition 2.2.7, there exists a commutative von Neumann algebra \mathcal{A} to which S is affiliated. Since \mathcal{A} is commutative, there exists a hyperstonean X such that \mathcal{A} is isometrically isomorphic to $C(X)$ and $\mathcal{N}(\mathcal{A}) \cong \mathcal{N}(X)$. Since X is hyperstonean, there is a strictly positive normal measure μ on X such that $C(X) \cong L_\infty(X, \Sigma, \mu)$ and $\mathcal{N}(\mathcal{A}) \cong \mathcal{N}(X) \cong L_0(X, \Sigma, \mu)$, by Corollary 2.11.14. If we consider \mathcal{A} as the algebra of multiplication operators on $L_2(X, \Sigma, \mu)$, we have that $S = M_f$ for some $f \in L_0(X, \Sigma, \mu)$. The rest follows from Corollary 3.1.2. Δ

Corollary 3.1.4 *Suppose $f \in \mathcal{N}(X)$ for X an extremely disconnected compact Hausdorff space. Then*

$$\sigma_{\mathcal{N}(X)}(f) = \{\lambda \in \mathbb{C} : \{x \in X : f(x) = \lambda\} \text{ contains a non-zero clopen subset of } X\}$$

Proof: Let $\lambda \in \mathbb{C}$. Let N be a clopen subset of X such that $f(x) = \lambda$ for every $x \in N$. Suppose $\lambda \notin \sigma_{\mathcal{N}(X)}(f)$. Then there is a $g \in \mathcal{N}(X)$ and an open dense subset O of X such that $(f(x) - \lambda)g(x) = 1$ for every $x \in O$. Since N is open and O is dense, there is an $x \in N \cap O$, and so $(f(x) - \lambda)g(x) = 0$, a contradiction. Hence $\lambda \in \sigma_{\mathcal{N}(X)}(f)$.

Conversely, suppose the closed set $F = \{x \in X : f(x) = \lambda\}$ contains no non-empty clopen subset. Then it follows that F has empty interior, and therefore the open set $X \setminus F$ is dense in X . Let O be an open dense set on which f is defined, and define g on $U = O \cap (X \setminus F)$ by $g(x) = (f(x) - \lambda)^{-1}$. Then $g \in \mathcal{N}(X)$ and $(f(x) - \lambda)g(x) = 1$ for all $x \in U$, so $\lambda \notin \sigma_{\mathcal{N}(X)}(f)$. Δ

Definition 3.1.5 If $f \in L_0(X, \Sigma, \mu)$, for some localizable measure space (X, Σ, μ) , we define

1. $\rho(f) = \{\lambda \in \mathbb{C} : \text{there is a } g \in L_\infty(X, \Sigma, \mu) \text{ such that } g(x)(f(x) - \lambda) = 1 \text{ a.e.}\}$
2. $\sigma(f) = \mathbb{C} \setminus \rho(f)$
3. the essential range of f , $\mathcal{E}(f)$, by

$$\mathcal{E}(f) = \{\lambda \in \mathbb{C} : \mu(\{x \in X : |f(x) - \lambda| < \epsilon\}) > 0, \text{ for all } \epsilon > 0\}.$$

The following appears in [Pat83]. We present the proof for completeness.

Proposition 3.1.6 ([Pat83], 1.9) Suppose $f \in L_0(X, \Sigma, \mu)$, for some localizable measure space (X, Σ, μ) . Then $\mathcal{E}(f) = \sigma(f)$.

Proof: Let $f \in L_0(X, \Sigma, \mu)$. Suppose $\lambda \notin \sigma(f)$. Then there is a $g \in L_\infty(X, \Sigma, \mu)$ such that $(f(x) - \lambda)g(x) = 1$ a.e. Then $\mu(\{x \in X : |f(x) - \lambda| < \frac{1}{2\|g\|_\infty}\}) = 0$. Hence $\lambda \notin \mathcal{E}(f)$. Conversely, if $\lambda \notin \mathcal{E}(f)$ then there is an $\epsilon > 0$ such that the set $D = \{x \in X : |f(x) - \lambda| < \epsilon\}$ has μ -measure zero. Define g on X by

$$g(x) = \begin{cases} \frac{1}{f(x) - \lambda} & x \in X \setminus D \\ 0 & x \in D. \end{cases}$$

Then $g \in L_\infty(X, \Sigma, \mu)$, and $g(x)(f(x) - \lambda) = 1$ a.e. Hence $\lambda \notin \sigma(f)$.

Δ

Proposition 3.1.7 If $f \in \tilde{L}_\infty(X, \Sigma, \mu)$, then

$$\sigma_{\tilde{L}_\infty}(f) = \{\lambda \in \mathbb{C} : \mu\{x \in X : |f(x) - \lambda| < \epsilon\} = \infty, \forall \epsilon > 0 \text{ or } \mu\{x \in X : f(x) = \lambda\} > 0\}.$$

Proof:

$$\begin{aligned}
\lambda \notin \sigma_{\tilde{L}_\infty}(f) &\iff \text{there exists a } g \in \tilde{L}_\infty \text{ such that } (f(x) - \lambda)g(x) = 1 \text{ } \mu - a.e \\
&\iff \text{the function } g \text{ defined by } g(x) = \frac{1}{f(x) - \lambda} \text{ is in } \tilde{L}_\infty \\
&\iff \mu\{x \in X : f(x) = \lambda\} = 0 \text{ and there exists a } C > 0 \text{ such that} \\
&\quad \mu\{x \in X : \frac{1}{|f(x) - \lambda|} > C\} < \infty \\
&\iff \mu\{x \in X : f(x) = \lambda\} = 0 \text{ and there exists an } \epsilon > 0 \text{ such that} \\
&\quad \mu\{x \in X : |f(x) - \lambda| < \epsilon\} < \infty.
\end{aligned}$$

Hence

$$\lambda \in \sigma_{\tilde{L}_\infty}(f) \iff \mu\{x \in X : f(x) = \lambda\} > 0 \text{ or } \mu\{x \in X : |f(x) - \lambda| < \epsilon\} = \infty \text{ for all } \epsilon > 0.$$

Δ

It follows from Proposition 3.1.1 and Proposition 3.1.6 that $\sigma_{L_0}(f) \subseteq \sigma(f)$. Note that if $\mu(X) < \infty$, then for $f \in \tilde{L}_\infty(X, \Sigma, \mu) = L_0(X, \Sigma, \mu)$, $\sigma_{\tilde{L}_\infty}(f) = \sigma_{L_0}(f) \subseteq \sigma(f)$.

The following example shows that $\sigma_{\tilde{L}_\infty}(S)$ is not necessarily closed.

Example 3.1.8 Let m be Lebesgue measure on \mathbb{R}^+ . Consider the function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, defined by

$$f(x) = \begin{cases} 1 + \frac{1}{x-1} & x > 1 \\ \frac{1}{n} & \frac{1}{n+1} < x \leq \frac{1}{n}, n = 1, 2, 3, \dots \end{cases}$$

Then $f \in \tilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m)$, where \mathcal{B} is the Borel σ -algebra on \mathbb{R}^+ . Now for $0 < \epsilon < 1$,

$$m\{x \in \mathbb{R}^+ : |f(x)| < \epsilon\} < \infty$$

and

$$m\{x \in \mathbb{R}^+ : f(x) = 0\} = m\{\emptyset\} = 0.$$

Hence $0 \notin \sigma_{\tilde{L}_\infty}(f)$. However, $m\{x \in \mathbb{R}^+ : f(x) = \frac{1}{n}\} > 0 \quad \forall n \in \mathbb{N}$ so that $\frac{1}{n} \in \sigma_{\tilde{L}_\infty}(f)$. Now $\frac{1}{n} \rightarrow 0$ shows that $\sigma_{\tilde{L}_\infty}(f)$ is not closed. In fact we have that $\sigma_{\tilde{L}_\infty}(f) = \{\frac{1}{n} : n \in \mathbb{N}^+\}$. This example also demonstrates that if the spectrum of $f \in \tilde{L}_\infty$ is bounded, f need not be bounded.

Example 3.1.9 The spectrum of $f \in \tilde{L}_\infty(X, \Sigma, \mu)$ may be unbounded.

Consider $f \in \tilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m)$ defined by

$$f(x) = \begin{cases} \sum_{n=1}^{\infty} n \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(x) & 0 < x \leq 1 \\ 1 & x > 1. \end{cases}$$

Clearly, we have that $f \in \tilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m)$ and $m\{x \in \mathbb{R}^+ : f(x) = n\} > 0$ so that $n \in \sigma_{\tilde{L}_\infty}(f)$ for all $n \in \mathbb{N}$. Thus the spectrum is unbounded.

Example 3.1.10 The following well-known example shows that spectrum of an element in $\tilde{L}_\infty(X, \Sigma, \mu)$ may be empty. Consider $L_\infty([0, 1], \mathcal{B}, m)$. Then $\tilde{L}_\infty = L_0([0, 1], \mathcal{B}, m)$. Let $f(t) = t$ for all $t \in [0, 1]$. Since we are on a finite measure space, Propositions 3.1.6 and 3.1.7 imply that

$$\sigma_{\tilde{L}_\infty}(f) = \sigma_{L_0}(f) = \emptyset$$

and $\sigma(f) = [0, 1]$.

We assume for the rest of this section that $\mu(X) = \infty$.

Definition 3.1.11 Suppose that $f \in \tilde{L}_\infty(X, \Sigma, \mu)$. Denote by π the canonical quotient map

$$\pi : \tilde{L}_\infty \rightarrow \tilde{L}_\infty / (\tilde{L}_\infty)_0,$$

where $(\tilde{L}_\infty)_0 = \{f \in \tilde{L}_\infty : \mu_\infty(f) = 0\}$. Then we define the essential spectrum of f , denoted by $\sigma_e(f)$, to be $\sigma_e(f) = \sigma_{\tilde{L}_\infty / (\tilde{L}_\infty)_0}(\pi(f))$.

Proposition 3.1.12 Suppose $f \in \tilde{L}_\infty(X, \Sigma, \mu)$. Then

$$\sigma_e(f) = \sigma(\pi(f)) = \{\lambda \in \mathbb{C} : \mu\{x \in X : |f(x) - \lambda| < \epsilon\} = \infty \text{ for all } \epsilon > 0\}.$$

Proof: Suppose $\mu\{x \in X : |f(x) - \lambda| < \delta\} < \infty$ for some $\delta > 0$.

Put $A = \{x \in X : f(x) = \lambda\}$. Then $\mu(A) < \infty$, so $\chi_A \in (\tilde{L}_\infty)_0$. Put

$$g(x) = \begin{cases} \frac{1}{f(x)-\lambda} & x \notin A \\ 0 & x \in A. \end{cases}$$

Then $g \in \tilde{L}_\infty$. Then we have $(f(x) - \lambda)g(x) = 1 - \chi_A(x)$ for all $x \in X$, so $(f - \lambda)g - 1 \in (\tilde{L}_\infty)_0$, and so $\pi(f - \lambda)$ is invertible in $\tilde{L}_\infty/(\tilde{L}_\infty)_0$.

Conversely, suppose $\pi(f) - \lambda 1$ is invertible in $\tilde{L}_\infty/(\tilde{L}_\infty)_0$, then there exist $g \in \tilde{L}_\infty$, $h \in (\tilde{L}_\infty)_0$, such that $(f(x) - \lambda)g(x) = 1 + h(x)$ for all μ - a.e. $x \in X$. For $\epsilon > 0$, let $k_\epsilon(x) = \chi_{B_\epsilon}(x)$, where $B_\epsilon = \{x \in X : |f(x) - \lambda| < \epsilon\}$. Then

$$\begin{aligned} \mu_\infty(k_\epsilon) &= \mu_\infty(k_\epsilon[(f - \lambda)g - h]) \\ &\leq \mu_\infty(k_\epsilon(f - \lambda))\mu_\infty(g) + \mu_\infty(k_\epsilon h) \\ &= \mu_\infty(k_\epsilon(f - \lambda))\mu_\infty(g), \text{ since } k_\epsilon h \in (\tilde{L}_\infty)_0 \\ &\leq \|k_\epsilon(f - \lambda)\|\mu_\infty(g) \\ &\leq \epsilon\mu_\infty(g). \end{aligned}$$

If $\mu_\infty(g) = 0$, then $\mu_\infty(k_\epsilon) = 0$ for every $\epsilon > 0$, so $\mu(B_\epsilon) < \infty$ for all $\epsilon > 0$. If $\mu_\infty(g) > 0$, choose $\epsilon < \frac{1}{\mu_\infty(g)}$. Then $\mu_\infty(k_\epsilon) < 1$. Since $\mu_\infty(k_\epsilon)$ is either 1 or 0, we must have $\mu_\infty(k_\epsilon) = 0$, that is to say $\mu(B_\epsilon) < \infty$. Thus

$$\{\lambda \in \mathbb{C} : \mu\{x \in X : |f(x) - \lambda| < \epsilon\} = \infty \text{ for all } \epsilon > 0\} \subset \sigma_\epsilon(S).$$

Δ

Consequently, we have that

Corollary 3.1.13 *Suppose $f \in \tilde{L}_\infty(X, \Sigma, \mu)$. Then*

$$\sigma_{\tilde{L}_\infty}(f) = \sigma_\epsilon(f) \cup \sigma_{L_0}(f).$$

Proof: This follows from Propositions 3.1.2, 3.1.7 and 3.1.12.

Δ

Proposition 3.1.14 *Suppose $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$. Then*

$$\sup\{|\lambda| : \lambda \in \sigma_\epsilon(f)\} = \mu_\infty(f).$$

Proof: Since $\mu\{x \in X : |f(x) - \mu_\infty(f)| < \epsilon\} = \infty$ for all $\epsilon > 0$, we have that $\mu_\infty(f) \in \sigma_\epsilon(f)$. Thus $\sup\{|\lambda| : \lambda \in \sigma_\epsilon(f)\} \geq \mu_\infty(f)$. Now suppose that there exists a $\lambda_1 \in \sigma_\epsilon(f)$ such that $|\lambda_1| > \mu_\infty(f)$. Let $\epsilon_1 = |\lambda_1| - \mu_\infty(f)$. Then $\epsilon_1 > 0$, and $\mu\{x \in X : |f(x) - \lambda_1| < \epsilon_1\} < \infty$, contradicting the assumption that $\lambda_1 \in \sigma_\epsilon(f)$. Hence $\sup\{|\lambda| : \lambda \in \sigma_\epsilon(f)\} \leq \mu_\infty(f)$, from where the result follows.

△

3.2 Invertibility in $\widetilde{\mathcal{M}}$ and $\mathcal{S}(\mathcal{M})$

In this section we characterize invertibility for self-adjoint τ -measurable operators in terms of the trace of the spectral projections. We also give some invertibility results for normal τ -measurable operators. We finish with an invertibility criterion for Segal's measurable operators.

The following lemma is adapted from [Kre78], 9.11–1 but here we extend it to the unbounded case.

Lemma 3.2.1 *Let $S : D(S) \rightarrow H$ be a self-adjoint (possibly unbounded) linear operator and $(E_\lambda(S))_{\lambda \in \mathbb{R}}$ the corresponding spectral family. Then for every $\lambda \in \mathbb{R}$,*

$$\ker(S - \lambda I) = (E_\lambda(S) - E_{\lambda-0}(S))(H)$$

where $E_{\lambda-0}(S) = \lim_{\mu \uparrow \lambda} E_\mu(S)$.

Proof: For each $n \in \mathbb{N}$, let $E(\Delta_n) = E_\lambda(S) - E_{\lambda - \frac{1}{n}}(S)$. Then we have (see equation 7 in the proof in ([KR83], Theorem(5.6.7))),

$$\left(\lambda - \frac{1}{n}\right)E(\Delta_n) \leq SE(\Delta_n) \leq \lambda E(\Delta_n). \quad (3.1)$$

Letting $n \rightarrow \infty$ and putting

$$F_0 = E_\lambda(S) - E_{\lambda-0}(S),$$

we get $\lambda F_0 \leq SF_0 \leq \lambda F_0$. Hence $SF_0 = \lambda F_0$, that is, $(S - \lambda I)F_0 = 0$. Thus

$$F_0(H) \subseteq \ker(S - \lambda I).$$

Conversely, we show that $F_0(H) \supseteq \ker(S - \lambda I)$. Suppose $\xi \in \ker(S - \lambda I)$. If $\lambda \in \rho(S)$, then $S - \lambda I$ is injective so that in this case $\ker(S - \lambda I) = \{0\} \subset F_0(H)$. Let $\lambda \notin \rho(S)$. By assumption $(S - \lambda I)\xi = 0$. This implies $(S - \lambda I)^2\xi = 0$, so that by the functional calculus

$$\int_{-\infty}^{\infty} (\mu - \lambda)^2 d\langle E_\mu(S)\xi, \xi \rangle = 0$$

Here $(\mu - \lambda)^2 \geq 0$ and $\mu \rightarrow \langle E_\mu(S)\xi, \xi \rangle$ is monotone increasing. Hence the integral over any subinterval of positive length must be zero. In particular, for every $\epsilon > 0$ we must have

$$\begin{aligned} 0 &= \int_{-\infty}^{\lambda-\epsilon} (\mu - \lambda)^2 d\langle E_\mu(S)\xi, \xi \rangle \\ &\geq \epsilon^2 \int_{-\infty}^{\lambda-\epsilon} d\langle E_\mu(S)\xi, \xi \rangle, \quad (\mu < \lambda - \epsilon) \\ &= \epsilon^2 \langle E_{\lambda-\epsilon}(S)\xi, \xi \rangle. \end{aligned}$$

Also

$$\begin{aligned} 0 &= \int_{\lambda+\epsilon}^{\infty} (\mu - \lambda)^2 d\langle E_\mu(S)\xi, \xi \rangle \\ &\geq \epsilon^2 \int_{\lambda+\epsilon}^{\infty} d\langle E_\mu(S)\xi, \xi \rangle, \quad (\mu - \lambda > \epsilon) \\ &= \epsilon^2 \langle I\xi, \xi \rangle - \epsilon^2 \langle E_{\lambda+\epsilon}(S)\xi, \xi \rangle. \end{aligned}$$

Since $\epsilon > 0$, we obtain $0 = \langle E_{\lambda-\epsilon}(S)\xi, \xi \rangle$ hence $E_{\lambda-\epsilon}(S)\xi = 0$ and $\langle \xi - E_{\lambda+\epsilon}(S)\xi, \xi \rangle = 0$, so that $\xi - E_{\lambda+\epsilon}(S)\xi = 0$. Thus we may write

$$\xi = (E_{\lambda+\epsilon}(S) - E_{\lambda-\epsilon}(S))\xi.$$

Letting $\epsilon \rightarrow 0$, we obtain $\xi = F_0\xi$ since $\mu \rightarrow E_\mu(S)$ is right continuous. Thus $\xi \in F_0(H)$. Thus we have shown that $\ker(S - \lambda I) = (E_\lambda(S) - E_{\lambda-0}(S))(H)$. Δ

We also need the following lemma whose proof can also be adapted from [Kre78], Theorem 9.11-2 only that in its proof we replace Theorem 9.1-2 in [Kre78] by its unbounded analogue in Theorem 10.4-1 in [Kre78]. We therefore omit its proof.

Lemma 3.2.2 *Let $S : D(S) \rightarrow H$ be a self-adjoint (possibly unbounded) linear operator and $(E_\lambda(S))_{\lambda \in \mathbb{R}}$ the corresponding spectral family. Then $\lambda \in \mathbb{R}$ belongs to the resolvent set $\rho(S)$ of S if and only if there is a $\gamma > 0$ such that $(E_\lambda(S))_{\lambda \in \mathbb{R}}$ is constant on the interval $[\lambda - \gamma, \lambda + \gamma]$.*

We shall need the following Lemma in what follows.

Lemma 3.2.3 *Suppose $S, T \in \widetilde{\mathcal{M}}$ are such that $\overline{ST} = \overline{TS} = I$. Then $\text{ran}(S)$ is τ -dense in H and $T = S^{-1}$ where $S^{-1} : \text{ran}(S) \rightarrow D(S)$ is defined by $S^{-1}y = x$ where $y = Sx \in \text{ran}(S)$, for some $x \in D(S)$.*

Proof: We have $STx = x$ for all $x \in D(ST)$. Hence if $x \in D(ST)$, $x = STx \in \text{ran}(S)$ and therefore $D(ST) \subseteq \text{ran}(S)$. But by Proposition 2.6.5, ST is τ -premeasurable so that $D(ST)$ is τ -dense. Hence $\text{ran}(S)$ is τ -dense.

Also we have $TSx = x$ for all $x \in D(TS)$, where $D(TS) = \{x \in D(S) : Sx \in D(T)\}$. Hence $Ty = x$ if $y = Sx \in \text{ran}(S)$ and $y \in D(T)$. Let $D = \text{ran}(S) \cap D(T)$. Then D is τ -dense, since for all $\delta > 0$, there exist projections $P_1, P_2 \in \mathcal{M}^p$ such that $P_1(H) \subseteq \text{ran}(S)$ and $\tau(I - P_1) \leq \frac{\delta}{2}$, and $P_2(H) \subseteq D(T)$ and $\tau(I - P_2) \leq \frac{\delta}{2}$. Let $P = P_1 \wedge P_2$. Then $P(H) \subseteq D$ and $\tau(I - P) = \tau(I - P_1 \wedge P_2) = \tau((I - P_1) \vee (I - P_2)) \leq \tau(I - P_1) + \tau(I - P_2) \leq \delta$, showing that D is τ -dense. We also have that $T|_D \subseteq S^{-1}$. Now S^{-1} is closed and by Lemma 2.2.6, $S^{-1} \in \widetilde{\eta\mathcal{M}}$. Since from what we have shown above, $\text{ran}(S)$ is τ -dense, we have that $S^{-1} \in \widetilde{\mathcal{M}}$. By the uniqueness of extensions in $\widetilde{\mathcal{M}}$ (Proposition 2.6.7), $T = S^{-1}$.

Δ

Proposition 3.2.4 *Suppose $S \in \widetilde{\mathcal{M}}^{\text{sa}}$ and $\{E_\lambda(S)\}_{\lambda \in \mathbb{R}}$ is the spectral family for S . S is invertible in $\widetilde{\mathcal{M}}$ if and only if S is injective and there exists a $t > 0$ such that*

$$\tau(E_{(-t, t)}(S)) < \infty.$$

In this case $S^{-1} = h(S)$, where $h : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$h(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}$$

Proof: Suppose S is injective and there exists a $t > 0$ such that $\tau(E_{(-t,t)}(S)) < \infty$. Since S is injective, it has an inverse $S^{-1} : \text{ran}(S) \rightarrow D(S)$, and since S is self-adjoint, $\text{ran}(S)$ is dense in H since $R(S) = I - N(S^*) = I - N(S) = I$. By Lemma 2.2.6, $S^{-1}\eta\mathcal{M}$ and $S^{-1} = h(S)$. Let h be as above and $g(\lambda) = |\lambda|$. It follows from Theorem 1.5.1(6) that, for $\lambda > 0$,

$$\begin{aligned} E_\lambda(|h(S)|) &= E_{((g \circ h)^{-1}(-\infty, \lambda])}(S) \\ &= E_{(-\infty, -\frac{1}{\lambda}]}(S) + E_{[\frac{1}{\lambda}, \infty)}(S). \end{aligned}$$

Thus we have, for $\lambda > 0$,

$$\begin{aligned} I - E_\lambda(|h(S)|) &= I - E_{(-\infty, -\frac{1}{\lambda}]}(S) - E_{[\frac{1}{\lambda}, \infty)}(S) \\ &= E_{(-\infty, \frac{1}{\lambda})}(S) - E_{(-\infty, -\frac{1}{\lambda}]}(S) \\ &= E_{(-\frac{1}{\lambda}, \frac{1}{\lambda})}(S). \end{aligned}$$

Now $h(S)$ is a closed densely defined operator affiliated with \mathcal{M} , by Theorem 2.2.4. And by assumption there exists a $t > 0$ such that

$$\tau(I - E_{\frac{1}{t}}(|h(S)|)) = \tau(E_{(-t,t)}(S)) < \infty.$$

Thus $h(S) \in \widetilde{\mathcal{M}}$. Since S and $h(S)$ are τ -measurable, their product is τ -premeasurable, by Proposition 2.6.5, and by Corollary 2.6.7, τ -premeasurable operators have at most one extension in $\widetilde{\mathcal{M}}$. By Theorem 1.5.1(i), we have

$$\begin{aligned} \overline{Sh(S)} &= \overline{h(S)S} = \int_{-\infty}^{\infty} \lambda h(\lambda) dE_\lambda(S) \\ &= \int_{-\infty}^{0^-} dE_\lambda(S) + \int_{0^+}^{\infty} dE_\lambda(S), \text{ since } \lambda h(\lambda) = 0 \text{ if } \lambda = 0, \text{ and } \lambda h(\lambda) = 1 \text{ otherwise} \\ &= I - (E_0(S) - E_{0^-}(S)), \text{ since } \{E_\lambda(S)\} \text{ is right continuous} \\ &= I, \text{ by Lemma 3.2.1 since } S \text{ is injective.} \end{aligned}$$

Thus S is invertible in $\widetilde{\mathcal{M}}$ with inverse $h(S)$.

Conversely, suppose S is invertible in $\widetilde{\mathcal{M}}$. Then there is a $T \in \widetilde{\mathcal{M}}$ such that $\overline{ST} = \overline{TS} = I$. Suppose $Sx = 0$. Then $Sx \in D(T)$. Now since $TS \subseteq \overline{TS}$, we have $0 = TSx = \overline{TS}x = Ix = x$, and so S is injective. By Lemma 2.2.6, S has an inverse S^{-1} which is affiliated with \mathcal{M} , and $S^{-1} = h(S)$. By Lemma 3.2.3 $S^{-1} = T \in \widetilde{\mathcal{M}}$. Thus there exists a $t > 0$ such that

$$\infty > \tau(I - E_{\frac{1}{t}}(|h(S)|)) = \tau(E_{(-t, t)}(S)).$$

Δ

Proposition 3.2.5 *Suppose $S \in \widetilde{\mathcal{M}}$, $S = U|S|$ the polar decomposition for S . Then*

1. *if S is invertible in $\widetilde{\mathcal{M}}$ then $|S|$ is invertible in $\widetilde{\mathcal{M}}$ and $|S|^{-1} = S^{-1}U$.*
2. *if S is normal and $|S|$ is invertible in $\widetilde{\mathcal{M}}$ then S is invertible in $\widetilde{\mathcal{M}}$.*

Proof: (1) Suppose S is invertible in $\widetilde{\mathcal{M}}$. Then there exists a $T \in \widetilde{\mathcal{M}}$ such that

$$\overline{ST} = \overline{TS} = I.$$

Also S invertible implies that S is injective and so $\text{ran}(S^*)$ is dense in H . Furthermore $\text{ran}(S)$ is τ -dense by Lemma 3.2.3. Hence U is a unitary, since it is a partial isometry with initial space the closure of $\text{ran}(S^*)$ and final space the closure of $\text{ran}(S)$. Thus $U^*U = UU^* = I$.

We then have

$$\begin{aligned} & \overline{ST} = \overline{TS} = I \\ \Rightarrow & \overline{U|S|T} = \overline{TU|S|} = I \\ \Rightarrow & I = \overline{U^*U|S|TU} = \overline{U^*U|S|TU} \\ & = \overline{U^*U|S|TU}, \text{ by Proposition 2.6.7, since } \overline{U^*U|S|TU} \text{ and} \\ & \overline{U^*U|S|TU} \text{ are both extensions of } U^*U|S|TU. \\ & = \overline{|S|TU} \\ \Rightarrow & \overline{|S|TU} = \overline{TU|S|} = I. \end{aligned}$$

Since T is closed and U is bounded, $TU = \overline{TU} \in \widetilde{\mathcal{M}}$ is the inverse of $|S|$ in $\widetilde{\mathcal{M}}$.

(2) Suppose $|S|$ is invertible in $\widetilde{\mathcal{M}}$ and S is normal, then there is a $T \in \widetilde{\mathcal{M}}$ such that $\overline{T|S|} = I = \overline{|S|T}$. Since S is normal, $SS^* = S^*S$ so that $|S| = |S^*|$. Then we have $\ker(S) = \ker(|S|) = \ker(|S^*|) = \ker(S^*)$. Since S is injective then we have that S^* is also injective so that $\text{ran}(S)$ and $\text{ran}(S^*)$ are dense in H (section 1.3). Hence U is a unitary since U is a partial isometry with initial space the closure of $\text{ran}(S^*)$ and final space the closure of $\text{ran}(S)$, (section 1.6). Since S is normal, it is affiliated with some abelian von Neumann subalgebra \mathcal{A} of \mathcal{M} , by Proposition 2.2.7. From this we have that $U \in \mathcal{A}$ and $|S| \eta \mathcal{A}$, by Proposition 2.2.3. Now $|S|$ affiliated with \mathcal{A} implies its inverse T is affiliated with \mathcal{A} , by Lemma 2.2.6. Thus $UTU^* = T$ since $U \in \mathcal{A} \subset \mathcal{A}'$. Now

$$\begin{aligned} \overline{TU^*S} &= \overline{TU^*U|S|} \\ &= \overline{T|S|} \\ &= I. \end{aligned}$$

Also

$$\begin{aligned} \overline{STU^*} &= \overline{|S^*|UTU^*} \\ &= \overline{|S^*|T} \quad \text{since } UTU^* = T \\ &= \overline{|S|T} \quad \text{since } |S| = |S^*| \text{ when } S \text{ is normal} \\ &= I. \end{aligned}$$

Thus we have that $TU^* \in \widetilde{\mathcal{M}}$ is the inverse of S in $\widetilde{\mathcal{M}}$.

△

Proposition 3.2.6 *Suppose $S \in \mathcal{S}(\mathcal{M})$ is self-adjoint and $\{E_\lambda(S)\}_{\lambda \in \mathbb{R}}$ is the spectral family for S . S is invertible in $\mathcal{S}(\mathcal{M})$ if and only if S is injective and there exists a $t > 0$ such that $E_{(-t,t)}(S)$ is a finite projection in \mathcal{M} .*

Proof: The proof is almost the same as that for Proposition 3.2.4 only that here we use [Seg53], corollary 5.1 instead of Corollary 2.6.7 and the characterisation of measurability of

Theorem 2.5.5. Hence we omit the proof.

△

3.3 Continuity of Inversion in $\widetilde{\mathcal{M}}$

In this section we show that inversion is continuous on the set \mathcal{Q} of invertible elements in $\widetilde{\mathcal{M}}$ with respect to the topology of convergence in measure.

We first give an elementary measure theoretic proof to show that inversion is continuous in the commutative case.

Proposition 3.3.1 *Let (X, Σ, μ) be a localizable measure space and μ a semi-finite measure. Then inversion is continuous in $\widetilde{L}_\infty(X, \Sigma, \mu)$.*

Proof: First we show that we have continuity of inversion for positive functions, and then use the polar decomposition to extend to the complex case. Let \mathcal{Q} be the set of invertible elements in $\widetilde{L}_\infty(X, \Sigma, \mu)$. Suppose (f_n) is a sequence of positive functions in \mathcal{Q} and $f_n \rightarrow_{\tau_m} 1$. Then, by Theorem 1.2.4, for all $t > 0$, $\mu\{x \in X : |f_n(x) - 1| > t\} \rightarrow 0$ as $n \rightarrow \infty$. We first show that if $y > 0$ and $|\frac{1}{y} - 1| > t$, then $|y - 1| > \min\{1, \frac{t}{1+t}\}$. We consider two cases.

Let $y > 0$ and $0 < t < 1$.

Then

$$\begin{aligned} |\frac{1}{y} - 1| > t &\Rightarrow \frac{1}{y} - 1 > t \text{ or } 1 - \frac{1}{y} > t \\ &\Rightarrow y - 1 < \frac{-t}{t+1} \text{ or } y - 1 > \frac{t}{1-t} > \frac{t}{t+1} \\ &\Rightarrow -(y - 1) > \frac{t}{t+1} \text{ or } y - 1 > \frac{t}{t+1} \end{aligned}$$

Hence $|y - 1| > \frac{t}{t+1}$.

Now let $y > 0$ and $t \geq 1$. Suppose $|\frac{1}{y} - 1| > t$. Since $t \geq 1$ and $y > 0$, we cannot have

$1 - \frac{1}{y} > t$. So we have

$$\begin{aligned} \left| \frac{1}{y} - 1 \right| > t &\Rightarrow \frac{1}{y} - 1 > t \\ &\Rightarrow 1 - y > \frac{t}{t+1} \\ &\Rightarrow |y - 1| > \frac{t}{1+t}. \end{aligned}$$

Let $s_t = \min\{1, \frac{t}{1+t}\}$. Then, for all $t > 0$,

$$\{y \in \mathbb{R} : y > 0, \left| \frac{1}{y} - 1 \right| > t\} \subseteq \{y \in \mathbb{R} : |f(x) - 1| > s_t\}.$$

Hence we have that for every $t > 0$ and every $n \in \mathbb{N}$, there exists $s_t > 0$ (depending on t only) such that

$$\{x \in X : f_n(x) > 0, \left| \frac{1}{f_n(x)} - 1 \right| > t\} \subseteq \{x \in X : |f_n(x) - 1| > s_t\}.$$

Thus

$$\mu\{x \in X : f_n(x) > 0, \left| \frac{1}{f_n(x)} - 1 \right| > t\} \leq \mu\{x \in X : |f_n(x) - 1| > s_t\} \rightarrow_n 0.$$

Thus $\frac{1}{f_n} \rightarrow_{\tau_m} 1$ so that inversion is continuous on the positive part of \mathcal{Q} .

Now suppose that for each n , $f_n \in \mathcal{Q}$, f_n is complex-valued, $f_n(x) = |f_n(x)|e^{i\theta_n(x)}$ (polar form) and $f_n \rightarrow_{\tau_m} 1$. As a consequence of the inequality $||f_n(x)| - 1| \leq |f_n(x) - 1|$, we have that $|f_n| \rightarrow_{\tau_m} 1$. By what we have shown above, $|f_n| \rightarrow_{\tau_m} 1$ implies that $\frac{1}{|f_n|} \rightarrow_{\tau_m} 1$. By continuity of multiplication in $\widetilde{L}_\infty(X, \Sigma, \mu)$, we have that $\frac{|f_n|e^{i\theta_n}}{|f_n|} \rightarrow_{\tau_m} 1$ so that $e^{i\theta_n} \rightarrow_{\tau_m} 1$. By continuity of conjugation we have $e^{-i\theta_n} \rightarrow_{\tau_m} 1$. Hence, by continuity of multiplication in $\widetilde{L}_\infty(X, \Sigma, \mu)$ again, we have that $\frac{1}{f_n} = \frac{e^{-i\theta_n}}{|f_n|} \rightarrow_{\tau_m} 1$. Hence inversion is continuous in $\widetilde{L}_\infty(X, \Sigma, \mu)$.

△

To prove continuity of inversion in the general case, we shall need the following result:

Theorem 3.3.2 [DdP97], [Tik87] *Let $S \in \widetilde{\mathcal{M}}$, (S_n) be a sequence in $\widetilde{\mathcal{M}}$, and $S_n \rightarrow_{\tau_m} S$. Then*

$$|S_n| \rightarrow_{\tau_m} |S|.$$

Proposition 3.3.3 *Let \mathcal{Q} be the set of invertible elements in $\widetilde{\mathcal{M}}$, and (S_n) a sequence in \mathcal{Q} such that $S_n \rightarrow_{\tau_m} I$. Then*

$$S_n^{-1} \rightarrow_{\tau_m} I,$$

that is to say inversion is τ_m -continuous on \mathcal{Q} .

Proof: We first prove the result for the case when the S_n are positive. So suppose $S_n > 0$ for all $n \geq 1$. From the equality $I - S_n^{-1} = (S_n - I)S_n^{-1}$ we have, for $t > 0$,

$$\begin{aligned} \mu_t(I - S_n^{-1}) &= \mu_t((S_n - I)S_n^{-1}) \\ &\leq \mu_{\frac{t}{2}}(S_n - I)\mu_{\frac{t}{2}}(S_n^{-1}), \text{ by Lemma 2.7.4.} \end{aligned}$$

Now since $S_n \rightarrow_{\tau_m} I$, it follows that $\mu_t(S_n - I) \rightarrow 0$ as $n \rightarrow \infty$ for all $t > 0$. It will suffice to show that for fixed $t > 0$, the $\mu_t(S_n^{-1})$ have a common upper bound for all $n \in \mathbb{N}$. By Proposition 3.2.4, we have that $S_n^{-1} = h(S_n)$ where

$$h(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}$$

and that for $\beta > 0$,

$$\tau(E_{(\beta, \infty)}(|h(S_n)|)) = \tau(E_{(0, \frac{1}{\beta})}(S_n)).$$

Hence

$$\begin{aligned} \mu_t(S_n^{-1}) &= \inf\{\beta > 0 : \tau(E_{(0, \frac{1}{\beta})}(S_n)) < t\} \\ &= \inf\left\{\frac{1}{\alpha} : \tau(E_{(0, \alpha)}(S_n)) < t\right\} \\ &= \frac{1}{\sup\{\alpha > 0 : \tau(E_{(0, \alpha)}(S_n)) < t\}} \end{aligned}$$

We have that

$$\mu_t(S_n - I) = \inf\{\theta \geq 0 : \tau(E_{(\theta, \infty)}(|S_n - I|)) < t\}$$

and $\mu_t(S_n - I) \rightarrow_n 0$ for each $t > 0$, by assumption, since $S_n \rightarrow_{\tau_m} I$. Let $t > 0$. There exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\inf\{\theta \geq 0 : \tau(E_{(\theta, \infty)}(|S_n - I|)) < t\} = \mu_t(S_n - I) < \frac{1}{2}$. Hence $\tau(E_{(\frac{1}{2}, \infty)}(|S_n - I|)) < t$ for all $n \geq n_0$. Now, by the functional calculus, $|S_n - I| = k(S_n)$ where $k(\lambda) = |\lambda - 1|$. Thus

$$E_{(\theta, \infty)}(|S_n - I|) = E_{(\theta, \infty)}(k(S_n)) = E_{k^{-1}(\theta, \infty)}(S_n),$$

the last equality following by the change of measure principle. Now

$$\begin{aligned} k^{-1}(\theta, \infty) &= \{\lambda \in \mathbb{R}^+ : \theta < k(\lambda) < \infty\} \\ &= (-\infty, 1 - \theta) \cup (\theta + 1, \infty) \end{aligned}$$

With $\theta = \frac{1}{2}$, and for all $n \geq n_0$, we have

$$t > \tau(E_{(\frac{1}{2}, \infty)}(|S_n - I|)) = \tau[E_{(0, \frac{1}{2})}(S_n) + E_{(\frac{3}{2}, \infty)}(S_n)] \geq \tau(E_{(0, \frac{1}{2})}(S_n)).$$

Thus $\sup\{\alpha > 0 : \tau(E_{(0, \alpha)}(S_n)) < t\} \geq \frac{1}{2}$ so that for all $n \geq n_0$, $\mu_t(S_n^{-1}) \leq 2$. Now let $q = \max\{\mu_t(S_n^{-1}) : n = 1, 2, \dots, n_0 - 1\}$ and $K = \max\{q, 2\}$. Then for all $n \in \mathbb{N}$, $\mu_t(S_n^{-1}) \leq K$. Thus

$$\mu_t(S_n^{-1} - I) \leq K \mu_{\frac{t}{2}}(S_n - I) \rightarrow_n 0.$$

Since this holds for all $t > 0$, $S_n^{-1} \rightarrow_{\tau_m} I$.

Now let (S_n) be an arbitrary sequence in \mathcal{Q} with $S_n \rightarrow_{\tau_m} I$. Let $S_n = U_n|S_n|$ be the polar decompositions for S_n . By Proposition 3.2.5(1), $|S_n| \in \mathcal{Q}$, $|S_n|^{-1} = S_n^{-1}U_n$ and $S_n^{-1} = |S_n|^{-1}U_n^*$. By Theorem 3.3.2, we have that $|S_n| \rightarrow_{\tau_m} I$. By the first part of the proof, $|S_n|^{-1} \rightarrow_{\tau_m} I$. Now we have

$$\begin{aligned} S_n \rightarrow I &\Rightarrow U_n|S_n| \rightarrow_{\tau_m} I \\ &\Rightarrow U_n|S_n||S_n|^{-1} \rightarrow_{\tau_m} I, \text{ by continuity of multiplication} \\ &\Rightarrow U_n \rightarrow_{\tau_m} I \\ &\Rightarrow U_n^* \rightarrow_{\tau_m} I^* = I, \text{ by continuity of adjunction} \\ &\Rightarrow S_n^{-1} = |S_n|^{-1}U_n^* \rightarrow_{\tau_m} I.I = I, \text{ by continuity of multiplication.} \end{aligned}$$

Hence inversion is τ_m -continuous.

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3.4 Continuity of Inversion in $(\mathcal{L}(\widetilde{\mathcal{M}}), \tau_{lmc})$

We have seen in example 2.12.2 that if $\mathcal{M} = L_\infty(X, \Sigma, \mu)$ for a semi-finite measure space (X, Σ, μ) , then $\mathcal{L}(\widetilde{\mathcal{M}}) = L_0(X, \Sigma, \mu)$ and the topology τ_{lmc} in this case is the usual topology of local convergence in measure.

Proposition 3.4.1 *Suppose (X, Σ, μ) is a σ -finite measure space. Then inversion is continuous in $L_0(X, \Sigma, \mu)$, with respect to the topology τ_{lmc} of local convergence in measure.*

Proof: An element f of $L_0(X, \Sigma, \mu)$ is invertible in $L_0(X, \Sigma, \mu)$ if and only if $\mu\{x \in X : f(x) = 0\} = 0$, and in this case the inverse of f is the function g such that $g(x) = (f(x))^{-1}$ if $f(x) \neq 0$ and $g(x) = 0$ if $f(x) = 0$ (or any function equal a.e. to g). Let \mathcal{Q} be the set of invertible elements in $L_0(X, \Sigma, \mu)$ and (f_n) a sequence in \mathcal{Q} such that $f_n \rightarrow_{\tau_{lmc}} f \in \mathcal{Q}$. By ([Wes90], 1.22), every subsequence of (f_n) has a subsequence, call it (f_{n_k}) , such that $f_{n_k}(x) \rightarrow f(x)$ almost everywhere. For each n , let g_n be the inverse of f_n , and let g be the inverse of f . Given a subsequence of (g_n) , consider the corresponding subsequence of (f_n) . By the above, this subsequence has a subsequence, call it (f_{n_k}) , such that $f_{n_k}(x) \rightarrow f(x)$ almost everywhere. Let $A_n = \{x \in X : f_n(x) = 0\}$, $A_0 = \{x \in X : f(x) = 0\}$ and $A = \{x \in X : f_n(x) \rightarrow f(x)\}$. Then $B = \cup_{n=1}^{\infty} A_n \cup A_0 \cup A$ has measure zero and

$$g_{n_k}(x) - g(x) = \frac{1}{f_{n_k}(x)} - \frac{1}{f(x)} \rightarrow 0 \text{ for all } x \notin B.$$

This implies that every subsequence of (g_n) has a subsequence converging to g a.e. By ([Wes90], 1.22) again, we have that $g_n \rightarrow_{\tau_{lmc}} g$. Thus inversion is continuous in $L_0(X, \Sigma, \mu)$, with respect to the topology of local convergence in measure.

△

Lemma 3.4.2 *Suppose $P \in \mathcal{M}^p$ and S is a closed densely defined operator affiliated with \mathcal{M} such that $SP \in \widetilde{\mathcal{M}}$, $PS \subseteq SP$ and $P|S| \subseteq |S|P$. Let $\{E_\lambda(|S|) : \lambda \in \mathbb{R}\}$ be the spectral family of $|S|$. Then*

1. $|SP| = |S|P$.

2. The spectral family of $|S|P$ is given by

$$E_\lambda(|S|P) = \begin{cases} I - P + E_\lambda(|S|)P, & \lambda \geq 0 \\ 0, & \lambda < 0 \end{cases}.$$

Proof: (1). We have that $D(|S|P) = \{x \in H : Px \in D(|S|)\} = \{x \in H : Px \in D(S)\} = D(SP) = D(|SP|)$ and if $x \in D(|SP|) = D(SP)$,

$$\begin{aligned} |SP|^2x &= (SP)^*SPx = (PSP)^*SPx, \text{ by Lemma 2.12.3} \\ &= PS^*PSPx, \text{ by Corollary 2.12.11} \\ &= PS^*SPx, \text{ by Lemma 2.12.3} \\ &= P|S|^2Px \\ &= P|S|P|S|Px, \text{ by Lemma 2.12.3} \\ &= |S|P|S|Px, \text{ by Lemma 2.12.3} \\ &= (|S|P)^2x. \end{aligned}$$

For (2), we note first that the family given in the statement of part (2) is a spectral family; in particular, $E_\lambda(|S|P) \rightarrow I$ as $\lambda \rightarrow \infty$ since $E_\lambda(|S|)P \rightarrow P$ as $\lambda \rightarrow \infty$. For each n , let $F_n = E_n(|S|) - E_0(|S|)$. We have that $\cup_{n=1}^\infty F_n(H)$ is a core for $|S|$ and for each n and $x \in F_n(H)$ that $|S|x = \int_0^n \lambda dE_\lambda(|S|x)$. It follows that $\cup_{n=1}^\infty F_nP(H)$ is a core for $|S|P$ and for $x \in F_nP(H)$,

$$\begin{aligned} |S|Px &= \int_0^n \lambda dE_\lambda(|S|)Px \\ &= \int_0^n \lambda d(E_\lambda(|S|)P)x \\ &= \int_0^n \lambda d(I - P + E_\lambda(|S|)P)x. \end{aligned}$$

It follows from Theorem 1.4.4 that the family given is the spectral family for $|S|P$. Δ

Lemma 3.4.3 Suppose $Z(\mathcal{M})$ is countably decomposable and $S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$ are such that $\overline{ST} = \overline{TS} = I$. Then $\text{ran}(S)$ is locally τ -measurable with respect to \mathcal{M} and $T = S^{-1}$ where $S^{-1} : \text{ran}(S) \rightarrow D(S)$ is defined by $S^{-1}y = x$ where $y = Sx$ for some $x \in D(S)$.

Proof: Just like in Lemma 3.2.3, we have that $D(ST) \subseteq \text{ran}(S)$. Since $S, T \in \mathcal{L}(\widetilde{\mathcal{M}})$, $D(ST)$ is locally τ -measurable with respect to \mathcal{M} , by Lemma 2.12.12, and so $\text{ran}(S)$ is also locally τ -measurable. Also, for all $U \in \mathcal{M}'$, and $x \in \text{ran}(S)$, $x = Sy$ for some $y \in D(S)$. Thus $Ux = USy = SUy \in \text{ran}(S)$, since $S\eta\mathcal{M}$. Thus $U(\text{ran}(S)) \subseteq \text{ran}(S)$. It follows that $\text{ran}(S) = D(S^{-1})$ is locally τ -measurable. Since S is closed, S^{-1} is closed, and by Lemma 2.2.6, S^{-1} is affiliated with \mathcal{M} . Hence $S^{-1} \in \mathcal{L}(\widetilde{\mathcal{M}})$, by Lemma 2.12.7. Let $D = \text{ran}(S) \cap D(T)$. Let $x \in D$ and $U \in \mathcal{M}'$. Then $Ux \in \text{ran}(S)$ since $U(\text{ran}(S)) \subseteq \text{ran}(S)$ as shown above. Also $Ux \in D(T)$ since $T\eta\mathcal{M}$. Hence $Ux \in D$ so that $U(D) \subseteq D$. Now since $\text{ran}(S)$ and $D(T)$ are locally τ -measurable, there exist, for $n = 1, 2$, sequences $(E_{i,n})$ of projections in $Z(\mathcal{M})$, with $E_{i,n} \uparrow I$ such that $\text{ran}(S)$ is τ -measurable $(E_{i,1})$ with respect to \mathcal{M} , for each i , and $D(T)$ is τ -measurable $(E_{i,2})$ with respect to \mathcal{M} , for each i . Let $G_i = E_{i,1} \wedge E_{i,2}$. Then $G_i \in Z(\mathcal{M})^p$ and $G_i \uparrow I$ as $i \uparrow \infty$. Then the sequence (G_i) demonstrates that D is locally τ -measurable (Definition 2.12.5). We have that $T|_D \subseteq S^{-1}$. By uniqueness of extensions in $\mathcal{L}(\widetilde{\mathcal{M}})$ (Lemma 2.12.10) it follows that $S^{-1} = T$.

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Lemma 3.4.4 *Suppose $Z(\mathcal{M})$ is countably decomposable and $S \in \mathcal{L}(\widetilde{\mathcal{M}})$ is invertible in $\mathcal{L}(\widetilde{\mathcal{M}})$ and $S = U|S|$ is the polar decomposition of S . Then $|S|$ is also invertible in $\mathcal{L}(\widetilde{\mathcal{M}})$ with $|S|^{-1} = S^{-1}U$.*

Proof: Follows just like in Proposition 3.2.5(1) by using the uniqueness of extensions in $\mathcal{L}(\widetilde{\mathcal{M}})$ (Lemma 2.12.10).

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Proposition 3.4.5 *Suppose $Z(\mathcal{M})$ is countably decomposable and $\tau|_{Z(\mathcal{M})}$ is semi-finite. Suppose that (S_n) is a sequence of invertible elements in $\mathcal{L}(\widetilde{\mathcal{M}})$ such that $S_n \rightarrow_{\tau_{lmc}} I$, then $S_n^{-1} \rightarrow_{\tau_{lmc}} I$. Hence inversion is continuous in $(\mathcal{L}(\widetilde{\mathcal{M}}), \tau_{lmc})$.*

Proof: Since $Z(\mathcal{M})^p$ is countably decomposable and $\tau|_{Z(\mathcal{M})}$ is semi-finite, the topology τ_{lmc} of local convergence in measure on $\mathcal{L}(\widetilde{\mathcal{M}})$ is metrizable, by Proposition 2.13.17. In fact there

exists a sequence (Q_m) in $Z(\mathcal{M})^p$ such that $\{\widetilde{\mathcal{M}}(Q_m, \frac{1}{m'}) : m, m' \in \mathbb{N}\}$ is a neighbourhood basis of 0. Just as in Proposition 3.3.3, we first prove the result for the $S_n > 0$ and then use continuity of multiplication and adjunction on $\mathcal{L}(\widetilde{\mathcal{M}})$ with respect to τ_{lmc} (Lemma 2.13.16) to derive the general case. Now by Lemma 2.13.4

$$S_n - I \in \widetilde{\mathcal{M}}(Q_m, \frac{1}{m'}) \iff Q_m(S_n - I)Q_m \in \widetilde{\mathcal{M}}(\frac{1}{m'}) \iff \mu_{\frac{1}{m'}}(Q_m(S_n - I)Q_m) \leq \frac{1}{m'}.$$

Now if we first assume $S_n > 0$ and fix $m \in \mathbb{N}$, then we have that

$$\begin{aligned} \mu_{\frac{1}{m'}}(Q_m(I - S_n^{-1})Q_m) &= \mu_{\frac{1}{m'}}(Q_m(S_n - I)S_n^{-1}Q_m) \\ &= \mu_{\frac{1}{m'}}(Q_m(S_n - I)Q_m S_n^{-1}Q_m), \quad \text{by Lemma 2.12.3, since } S_n^{-1}\eta\mathcal{M} \\ &\quad \text{by Lemma 2.2.6} \\ &\leq \mu_{\frac{1}{2m'}}(Q_m(S_n - I)Q_m)\mu_{\frac{1}{2m'}}(Q_m S_n^{-1}Q_m) \end{aligned}$$

Now using Lemma 3.4.3, we have that $S_n^{-1} = h(S_n)$ where

$$h(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}$$

and a similar argument to that in the proof of Proposition 3.2.4 shows that for $\beta > 0$,

$$E_\beta(|h(S_n)|) = E_{(\frac{1}{\beta}, \infty)}(S_n).$$

By Lemma 3.4.2, $E_\beta(|h(S_n)|Q_m) = I - Q_m + Q_m E_{(\frac{1}{\beta}, \infty)}(S_n)$. Thus

$$\begin{aligned} E_{(\beta, \infty)}(|h(S_n)|Q_m) &= Q_m - Q_m E_{[\frac{1}{\beta}, \infty)}(S_n) \\ &= Q_m(I - E_{[\frac{1}{\beta}, \infty)}(S_n)) \\ &= Q_m E_{(0, \frac{1}{\beta})}(S_n). \end{aligned}$$

We have that

$$\begin{aligned} \mu_t(Q_m(S_n - I)Q_m) &= \mu_t((S_n - I)Q_m), \quad \text{by Lemma 2.12.3} \\ &= \inf\{\theta \geq 0 : \tau(E_{(\theta, \infty)}(|S_n - I|Q_m)) < t\}, \quad \text{by Lemma 3.4.2,} \end{aligned}$$

and $\mu_t(Q_m(S_n - I)Q_m) \rightarrow_n 0$ for each $t > 0$, since $S_n \rightarrow_{\tau_{lmc}} I$. Now, just as in Proposition 3.3.3, there exists an $n_0 \in \mathbb{N}$ such that

$$\inf\{\theta \geq 0 : \tau(E_{(\theta, \infty)}(|S_n - I|Q_m)) < t\} = \mu_t((S_n - I)Q_m) < \frac{1}{2} \quad \text{for all } n \geq n_0.$$

Hence $\tau(E_{(\frac{1}{2}, \infty)}(|S_n - I|)Q_m) < t$ for all $n \geq n_0$. This yields

$$t > \tau(E_{(\frac{1}{2}, \infty)}(|S_n - I|)Q_m) = \tau(E_{(0, \frac{1}{2})}(S_n)Q_m + E_{(\frac{3}{2}, \infty)}(S_n)Q_m) \geq \tau(E_{(0, \frac{1}{2})}(S_n)Q_m),$$

by the same techniques as in Proposition 3.3.3. Thus

$$\mu_t(Q_m S_n^{-1} Q_m) = \frac{1}{\sup\{\alpha > 0 : \tau(E_{(0, \alpha)}(S_n)Q_m) < t\}} \leq 2$$

for all $n \geq n_0$. Let $q = \max\{\mu_t(Q_m S_n^{-1} Q_m) : n = 1, 2, \dots, n_0 - 1\}$ and $K = \max\{q, 2\}$.

Then $\mu_t(Q_m(S_n^{-1} - I)Q_m) \leq K\mu_{\frac{t}{2}}(Q_m(S_n - I)Q_m) \rightarrow_n 0$. Since this holds for all $t > 0$ and all $m \in \mathbb{N}$, we conclude that $S_n^{-1} \rightarrow_{\tau_{lmc}} I$.

For the case when (S_n) is an arbitrary sequence in \mathcal{Q} with $S_n \rightarrow_{\tau_{lmc}} I$, a similar argument to that in Proposition 3.3.3, using Lemma 3.4.4 and the τ_{lmc} -continuity of multiplication and adjunction (Proposition 2.13.16) shows that inversion is τ_{lmc} -continuous.

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3.5 The Schmidt Spectral Decomposition

We discuss a type of spectral representation first introduced by Ovchinnikov in [Ovc70] and extended in [DdP92]. We present a brief discussion of this representation as an understanding of the techniques involved will be crucial for subsequent results. This is a generalisation of the Schmidt expansion of a bounded compact operator in ([GK69], II.2).

Proposition 3.5.1 [DdP92] *Let $0 \leq S \in \widetilde{\mathcal{M}}$, with spectral family $\{E_\lambda(S) : \lambda \geq 0\}$. Let $\mu_t(S)$ be the decreasing rearrangement of S . Let*

$$\lambda_0 = \lim_{t \rightarrow \infty} \mu_t(S) \text{ and } \alpha_0 = \inf\{t > 0 : \mu_t(S) = \lambda_0\}.$$

and $S_0 = (I - E_{\lambda_0}(S))S$. Then

$$S_0 = \int_0^\infty \mu_t(S_0) d\tilde{E}_t(S),$$

where

$$\tilde{E}_t(S) = \begin{cases} 0 & t < 0 \\ (I - E_{\lambda_0}(S))(I - E_{\mu_t(S)_-}(S)) & 0 \leq t < \alpha_0 \\ I - E_{\lambda_0}(S) & t \geq \alpha_0 \end{cases}$$

This representation of S is called the Schmidt spectral decomposition.

Proof: S_0 is self-adjoint and since the projection $I - E_{\lambda_0}(S)$ satisfies the conditions of Lemma 3.4.2, the spectral family $\{E_\lambda(S_0) : \lambda \in \mathbb{R}\}$ is given by

$$\begin{aligned} E_\lambda(S_0) &= \begin{cases} 0, & \text{if } \lambda < 0 \\ E_\lambda(S)(I - E_{\lambda_0}(S)) + E_{\lambda_0}(S), & \text{if } \lambda \geq 0 \end{cases} \\ &= \begin{cases} 0, & \text{if } \lambda < 0 \\ E_{\lambda_0}(S), & \text{if } 0 \leq \lambda < \lambda_0 \\ E_\lambda(S), & \text{if } \lambda \geq \lambda_0 \end{cases} . \end{aligned}$$

It follows from the spectral theorem that $S_0 = \int_{\mathbb{R}} \lambda dE_\lambda(S_0)$.

We also have that $\mu_t(S_0) = \mu_t(S)\chi_{[0, \alpha_0)}(t)$. To see this, note that

$$\tau(I - E_\lambda(S_0)) = \begin{cases} \tau(I - E_{\lambda_0}(S)), & 0 \leq \lambda < \lambda_0 \\ \tau(I - E_\lambda(S)), & \lambda_0 \leq \lambda \end{cases}$$

so that

$$\mu_t(S_0) = \inf\{0 < \lambda < \lambda_0 : \tau(I - E_\lambda(S_0)) \leq t\} \wedge \inf\{\lambda \geq \lambda_0 : \tau(I - E_\lambda(S)) \leq t\}.$$

If $\tau(I - E_{\lambda_0}(S)) \leq t$, $\mu_t(S_0) = \inf_{0 < \lambda < \lambda_0} \lambda = 0$ and if $\tau(I - E_{\lambda_0}(S)) > t$,

$$\mu_t(S_0) = \inf\{\lambda \geq \lambda_0 : \tau(I - E_\lambda(S)) \leq t\} = \mu_t(S).$$

But

$$\tau(I - E_{\lambda_0}(S)) \leq t \Leftrightarrow d_{\lambda_0}(S) \leq t \Leftrightarrow \mu_t(S) \leq \lambda_0 \Leftrightarrow \mu_t(S) = \lambda_0 \Leftrightarrow t \geq \alpha_0.$$

Hence

$$\mu_t(S_0) = \mu_t(S)\chi_{[0,\alpha_0)}(t). \quad (3.2)$$

If we write $d_\lambda(S_0) = \tau(I - E_\lambda(S_0))$, $\lambda > 0$, then the set

$$A = \{\lambda > 0 : \mu_{d_\lambda(S_0)}(S_0) \neq \lambda\}$$

is of measure zero for the spectral measure induced on $(0, \infty)$ by the spectral family $\{E_\lambda(S_0) : \lambda \in \mathbb{R}\}$. Let $f(\lambda) = \mu_\lambda(S_0)$, $g(\lambda) = d_\lambda(S_0)$. Then $f(g(S_0)) = S_0$ by the functional calculus. If $\tilde{E}_t(S) = E_t(g(S_0))$ is the spectral family for $g(S_0)$, then $g(S_0) = \int_0^\infty t d\tilde{E}_t(S)$. Hence

$$S_0 = f(g(S_0)) = \int_0^\infty f(t) d\tilde{E}_t(S) = \int_0^\infty \mu_t(S_0) d\tilde{E}_t(S).$$

This yields in particular that

$$S_0 = \int_0^{\alpha_0} \mu_t(S) d\tilde{E}_t(S) \quad (3.3)$$

by equation (3.2).

We observe that $A_t = \{\lambda > 0 : d_\lambda(S_0) \leq t\} = [\mu_t(S_0), \infty)$.

For $\lambda \geq \mu_t(S_0) \Rightarrow d_\lambda(S_0) \leq d_{\mu_t(S_0)}(S_0) \leq t$, and conversely, if $d_\lambda(S_0) \leq t$, then $\mu_t(S_0) \leq \lambda$.

It follows that if $0 \leq t < \alpha_0$, then $A_t = [\mu_t(S), \infty)$, and if $t \geq \alpha_0$, then $A_t = [\lambda_0, \infty)$.

We have

$$\begin{aligned} \tilde{E}_t(S) &= E_t(g(S_0)) \\ &= E_{(-\infty, t]}(g(S_0)) \\ &= E_{g^{-1}((-\infty, t])}(S_0) \\ &= E_{A_t}(S_0). \end{aligned}$$

For $t \geq \alpha_0$,

$$\tilde{E}_t(S) = E_{A_t}(S_0) = E_{[\lambda_0, \infty)}(S_0) = I - \sup_{\lambda < \lambda_0} E_\lambda(S_0) = I - E_{\lambda_0}(S).$$

For $0 \leq t < \alpha_0$,

$$\begin{aligned}
\tilde{E}_t(S) &= E_{A_t} = E_{[\mu_t(S), \infty)}(S_0) \\
&= I - E_{(-\infty, \mu_t(S))}(S_0) \\
&= I - \sup_{r < \mu_t(S)} E_r(S_0) \\
&= I - \sup_{r \leq \lambda_0} E_r(S_0) \vee \sup_{\lambda_0 < r < \mu_t(S)} E_r(S_0) \\
&= I - E_{\lambda_0}(S) \vee E_{\mu_t(S)_-}(S) \\
&= (I - E_{\lambda_0}(S)) \wedge (I - E_{\mu_t(S)_-}(S)) \\
&= (I - E_{\lambda_0}(S))(I - E_{\mu_t(S)_-}(S)),
\end{aligned}$$

where $E_{\mu_t(S)_-}(S) = \lim_{\lambda \uparrow \mu_t(S)} E_\lambda(S)$.

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We note that if $0 \leq S \in \widetilde{\mathcal{M}}_0$, then $S = S_0$. This follows from the fact that in this case $\lambda_0 = 0$, so that $S_0 = (I - E_0(S))S = \mathbf{s}(S)S = S$, where $\mathbf{s}(S)$ is the support of S .

The following examples show that in the case that $S \notin \widetilde{\mathcal{M}}_0$, both $S = S_0$ and $S_0 \neq S$ can occur.

Example 3.5.2 Define f by $f(t) = 2 + \frac{1}{t}$, $t > 0$. Then clearly $f \in \widetilde{L}_\infty(\mathbb{R}^+, \Sigma, m)$ where m is Lebesgue measure on \mathbb{R}^+ , $\lambda_0 = 2$ and $\alpha_0 = \infty$. Let $S = M_f$ be the operator of multiplication by f . The spectral projections of M_f correspond to characteristic functions χ_{X_λ} where $X_\lambda = \{s \in \mathbb{R}^+ : f(s) \leq \lambda\}$. Since $\lambda_0 \neq 0$, $f \notin (\widetilde{L}_\infty)_0$, but $(1 - \chi_{X_2})f = f$. It follows that $S_0 = S$ in this case.

If we define

$$g(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 2 + \frac{1}{t-1} & t > 1, \end{cases}$$

then again $g \in \widetilde{L}_\infty(\mathbb{R}^+, \Sigma, m)$, $\lambda_0 = 2$, $\alpha_0 = \infty$ and $g \notin (\widetilde{L}_\infty)_0$, but

$X_2 = \{t \in \mathbb{R}^+ : |g(t)| \leq 2\} = (0, 1]$, so that $(1 - \chi_{X_2})g \neq g$. Thus if $S = M_g$, $S_0 \neq S$.

3.6 Proper Subalgebras

In this section we consider abelian subalgebras of the underlying semi-finite von Neumann algebra \mathcal{M} which are such that the trace on \mathcal{M} restricted to the abelian subalgebra is semi-finite. We shall call such subalgebras *proper*. We show that the spectrum of an element in a proper subalgebra coincides with the spectrum of the element in the algebra itself. We also show that for $S \in \widetilde{\mathcal{M}}^{sa}$, the von Neumann algebra generated by the spectral family for S need not be proper. We shall need proper subalgebras in the sequel when we consider \mathcal{Q} -algebras.

Definition 3.6.1 *A subalgebra \mathcal{A} of a semi-finite von Neumann algebra \mathcal{M} with a faithful semi-finite normal trace τ is said to be proper if the restriction of τ to \mathcal{A} is a semi-finite trace on \mathcal{A} .*

We shall need the following elementary lemma in what follows:

Lemma 3.6.2 *Suppose \mathcal{A} is a subalgebra of a semi-finite von Neumann algebra \mathcal{M} with a faithful semi-finite normal trace τ such that for each $P \in \mathcal{A}^p$, there exists a $0 < Q \in \mathcal{A}^p$ such that $Q \leq P$ and $\tau(Q) < \infty$. Then the restriction of the trace τ on \mathcal{M} to \mathcal{A} is semi-finite.*

Proof: Suppose $0 < S \in \mathcal{A}$, then by the spectral theorem there exists a $\delta > 0$ and a spectral projection P of S such that $P \in \mathcal{A}^p$ and $0 < \delta P \leq S$. Now by assumption there exists a $0 < Q \in \mathcal{A}^p$ such that $Q \leq P$ and $\tau(Q) < \infty$. Thus $0 < \delta Q \leq \delta P \leq S$ and $\tau(\delta Q) = \delta\tau(Q) < \infty$. Hence τ restricted to \mathcal{A} is semi-finite.

Δ The following result is analogous to the well-known C^* -algebra result that the spectrum of an element in a C^* -subalgebra of the C^* -algebra is the same when taken with respect to the subalgebra as when taken with respect to the algebra itself ([KR83], 4.1.5).

Proposition 3.6.3 *Suppose \mathcal{M} is a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ , $T \in \widetilde{\mathcal{M}}$ and \mathcal{A} is a proper von Neumann subalgebra (that is a proper*

subalgebra which contains the identity of \mathcal{M} and is closed in the weak operator topology) of \mathcal{M} such that $T\eta\mathcal{A}$. Let $\tau_{\mathcal{A}}$ be the restriction of τ to \mathcal{A} and $\tilde{\mathcal{A}}$ be the algebra of $\tau_{\mathcal{A}}$ -measurable operators. Then $T \in \tilde{\mathcal{A}}$ and

$$\sigma_{\tilde{\mathcal{M}}}(T) = \sigma_{\tilde{\mathcal{A}}}(T).$$

Proof: The trace $\tau_{\mathcal{A}}$ is faithful, normal and semi-finite on \mathcal{A}^+ since \mathcal{A} is proper. We first show that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{M}}$. Since $\mathcal{A} \subseteq \mathcal{M}$, we have that $\mathcal{M}' \subseteq \mathcal{A}'$. Thus for any $R \in \tilde{\mathcal{A}}$ we have that $R\eta\mathcal{A}$ so that $VR = RV$ for all unitaries V in \mathcal{A}' . Since $\mathcal{M}' \subseteq \mathcal{A}'$, we have that $VR = RV$ for all unitaries V in \mathcal{M}' . Thus $R\eta\mathcal{M}$. Since $R \in \tilde{\mathcal{A}}$, there is a $t_0 > 0$ such that $\tau_{\mathcal{A}}(I - E_{t_0}(|R|)) < \infty$. But $I - E_{t_0}(|R|) \in \mathcal{M}$, so

$$\tau(I - E_{t_0}(|R|)) = \tau_{\mathcal{A}}(I - E_{t_0}(|R|)) < \infty.$$

Hence we have that $R \in \tilde{\mathcal{M}}$, and so $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{M}}$. From this we have that

$$\sigma_{\tilde{\mathcal{M}}}(T) \subseteq \sigma_{\tilde{\mathcal{A}}}(T).$$

For the converse it will suffice to prove that if $T \in \tilde{\mathcal{A}}$ has an inverse S in $\tilde{\mathcal{M}}$, then $S \in \tilde{\mathcal{A}}$. We first assume that T is self-adjoint. By Proposition 3.2.4, the inverse is given by $S = h(T)$ where

$$h(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}.$$

Now $T\eta\mathcal{A}$ implies that $S\eta\mathcal{A}$, by Lemma 2.2.6. Since $S \in \tilde{\mathcal{M}}$, there exists a $t_0 > 0$ such that $\tau(E_{(t_0, \infty)}(|h(T)|)) < \infty$. $T\eta\mathcal{A}$ implies that $E_t(T) \in \mathcal{A}$ for all $t > 0$. Since, by the change of measure principle, the spectral family $\{E_t(|h(T)|) : t > 0\}$ is generated by $\{E_t(T) : t > 0\}$, we have that $E_t(|h(T)|) \in \mathcal{A}$ for all $t > 0$. Thus we have

$$\tau_{\mathcal{A}}(E_{(t_0, \infty)}(|h(T)|)) = \tau(E_{(t_0, \infty)}(|h(T)|)) < \infty.$$

Hence $S = h(T) \in \tilde{\mathcal{A}}$. Thus $\sigma_{\tilde{\mathcal{M}}}(T) \supseteq \sigma_{\tilde{\mathcal{A}}}(T)$, and hence we have

$$\sigma_{\tilde{\mathcal{M}}}(T) = \sigma_{\tilde{\mathcal{A}}}(T).$$

Again, suppose $T \in \widetilde{\mathcal{M}}$, without being necessarily self adjoint. Just like in the first part, we have that

$$\sigma_{\widetilde{\mathcal{M}}}(T) \subseteq \sigma_{\widetilde{\mathcal{A}}}(T).$$

Now suppose, without loss of generality, that T has an inverse S in $\widetilde{\mathcal{M}}$, then by Proposition 3.2.5(a), we have that $|T|$ has an inverse SU in $\widetilde{\mathcal{M}}$, where $T = U|T|$ is the polar decomposition of T . Furthermore, we note that U is a unitary in this case since its initial space $\text{ran}(T^*)$ and its final space $\text{ran}(T)$ are dense in H . Now by what we have just proved for the self adjoint elements of $\widetilde{\mathcal{M}}$ above, we have that $SU \in \widetilde{\mathcal{A}}$. Note that $T\eta\mathcal{A}$ implies that $|T|\eta\mathcal{A}$ and $U \in \mathcal{A}$. Hence $U^* \in \mathcal{A}$ and so $S = (SU)U^* \in \widetilde{\mathcal{A}}$. Thus $\sigma_{\widetilde{\mathcal{M}}}(T) \supseteq \sigma_{\widetilde{\mathcal{A}}}(T)$, and hence we have

$$\sigma_{\widetilde{\mathcal{M}}}(T) = \sigma_{\widetilde{\mathcal{A}}}(T).$$

Δ

Remark: If τ is finite, then trivially every von Neumann subalgebra \mathcal{A} of \mathcal{M} is a proper algebra. Consequently we have that in this case

$$\sigma_{\widetilde{\mathcal{M}}}(S) = \sigma_{\widetilde{\mathcal{A}}}(S)$$

for each $S\eta\mathcal{A}$. Therefore, in what follows when we consider proper algebras, we shall assume that $\tau(I) = \infty$.

Let $0 \leq S \in \widetilde{\mathcal{M}}$. Then by Theorem 2.2.4, $S\eta\mathcal{M}$ if and only if $\{E_t(S) : t \geq 0\} \subseteq \mathcal{M}$. Hence the von Neumann algebra \mathcal{A}_S generated by $\{E_t(S) : t \geq 0\}$ and I is the smallest abelian von Neumann subalgebra of \mathcal{M} such that $S\eta\mathcal{M}$. The subalgebra \mathcal{A}_S could be, but need not be proper, as is illustrated in Example 3.6.5 below.

We now specialise to the case where $S \in \widetilde{\mathcal{M}}_0$. Then $d_\lambda(S) = \tau(I - E_\lambda(S)) < \infty$ for all $\lambda > 0$. We also have that $I - E_\lambda(S) \in \mathcal{A}_S$ for all $\lambda > 0$. For $0 < \lambda \leq \mu$, $E_\mu(S) - E_\lambda(S) = (I - E_\lambda(S)) - (I - E_\mu(S))$ shows that $E_\mu(S) - E_\lambda(S)$ has finite trace. Let $\lambda > 0$ and

suppose there exists $0 < \mu < \lambda$ such that $E_\mu(S) < E_\lambda(S)$. Then $I - E_\mu(S) > I - E_\lambda(S)$ and $(I - E_\mu(S)) - (I - E_\lambda(S)) \leq I - (I - E_\lambda(S)) = E_\lambda(S)$. This shows that $E_\lambda(S)$ has a non-zero subprojection of finite trace. If $\lambda > 0$ and $E_\lambda(S) = E_\mu(S)$ for all $0 < \mu < \lambda$, then $E_\lambda(S) = E_0(S)$, by the right continuity of the spectral family. Since \mathcal{A}_S does not contain any proper non-zero subprojection of $E_0(S)$, we have proved:

Proposition 3.6.4 *Let $0 \leq S \in \widetilde{\mathcal{M}}_0$ and let \mathcal{A}_S be the von Neumann subalgebra generated by the projections $\{E_t(S) : t \geq 0\}$ and I . Then \mathcal{A}_S is proper if and only if $\tau(E_0(S)) < \infty$.*

Example 3.6.5 Define $0 < f \in (\widetilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m))_0$ by

$$f(t) = \begin{cases} \frac{1}{t} & 0 < t < 1 \\ 0 & t \geq 1. \end{cases}$$

Let S be the multiplication operator M_f . Now the projections in the von Neumann algebra \mathcal{A}_S generated by S correspond to characteristic functions e_λ of sets $X_\lambda = \{t \in \mathbb{R}^+ : f(t) \leq \lambda\}$. Now $e_0 = \chi_{X_0} = \chi_{[1, \infty)}$. It follows from the proposition above that \mathcal{A}_S is not proper. However, if $g(t) = \frac{1}{t}$ for all $t > 0$ and $S = M_g$, then $e_0 = 0$, so that by the above proposition again, the von Neumann algebra \mathcal{A}_S is proper.

We shall need the following result when we consider \mathcal{Q} -algebras in the sequel.

Proposition 3.6.6 *Suppose $0 < S \in \widetilde{\mathcal{M}}_0$ and let the projections $\widetilde{E}_t(S)$, $t \in \mathbb{R}$ be defined as in Proposition 3.5.1. Then the commutative subalgebra \mathcal{M}_S of \mathcal{M} generated by the projections $\{\widetilde{E}_t(S) : t \in \mathbb{R}\}$ is proper.*

Proof: This follows immediately from the fact that $\tau(I - E_\lambda(S)) < \infty$ for all $\lambda > 0$, since $S \in \widetilde{\mathcal{M}}_0$.

Δ

Some conditions for the existence of proper subalgebras in terms of conditional expectations can be found in ([Sun87], Lemma 4.3.12.) and ([Tak79], V2.36.)

3.7 Characterisations of Spectra in \mathcal{M}

In this section we generalise the results of Section 3.1 to the non-commutative setting, using the characterisation of invertibility obtained in Proposition 3.1.13.

Proposition 3.7.1 *Suppose $S \in \widetilde{\mathcal{M}}$. Then $\sigma_{\widetilde{\mathcal{M}}}(S) \subseteq \sigma(S)$*

Proof: Suppose $\lambda \notin \sigma(S)$, then $S - \lambda I$ has a bounded inverse B . Since $S - \lambda I \in \mathcal{M}$, it follows from Lemma 2.2.6, that B is affiliated to \mathcal{M} and so $B \in \mathcal{M}$. This implies that $B \in \mathcal{M} \subseteq \widetilde{\mathcal{M}}$. Thus $\lambda \notin \sigma_{\widetilde{\mathcal{M}}}(S)$. Consequently, $\sigma_{\widetilde{\mathcal{M}}}(S) \subseteq \sigma(S)$.

Definition 3.7.2 *Suppose that $S \in \widetilde{\mathcal{M}}$. Denote by π the canonical quotient map*

$$\pi : \widetilde{\mathcal{M}} \rightarrow \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0.$$

Then we define the essential spectrum of S , denoted by $\sigma_e(S)$ by

$$\sigma_e(S) = \sigma_{\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0}(\pi(S)).$$

We have the following:

Proposition 3.7.3 *([SW93], 3.6)*

Suppose $S \in \widetilde{\mathcal{M}}^{sa}$. Then

$$\sigma_e(S) = \{\lambda \in \mathbb{R} : \tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) = \infty \forall \epsilon > 0\}.$$

We observe here that if $S \in \widetilde{\mathcal{M}}_0$ then $\sigma_e(S) = \{0\}$.

Proposition 3.7.4 *Suppose $S \in \widetilde{\mathcal{M}}^{sa}$, then $\sigma_{\widetilde{\mathcal{M}}}(S) = \sigma_p(S) \cup \sigma_e(S)$, where $\sigma_p(S)$ is the point spectrum of S .*

Proof: For $\mu \in \mathbb{R}$, $\mu \notin \sigma_{\widetilde{\mathcal{M}}}(S)$ if and only if $S - \mu I$ is invertible in $\widetilde{\mathcal{M}}$. Let $E_\lambda(S)$ be the spectral family for S . Set $k(\lambda) = \lambda - \mu$. Then $k(S) = S - \mu I$. By the change of measure principle, $E_\lambda(k(S)) = E_{k^{-1}(-\infty, \lambda]}(S) = E_{(-\infty, \lambda + \mu]}(S)$ and so $\{E_{\lambda + \mu}(S) : \lambda \in \mathbb{R}\}$ is the spectral family for $S - \mu I$. Now by Proposition 3.1.13

$\lambda \notin \sigma_{\widetilde{\mathcal{M}}}(S) \Leftrightarrow S - \lambda I$ is injective and there exists a $t > 0$ such that

$$\tau(E_{(\lambda-t, \lambda+t)}(S)) = \tau(E_{(-t, t)}(S - \lambda I)) < \infty$$

$\lambda \in \sigma_{\widetilde{\mathcal{M}}}(S) \Leftrightarrow S - \lambda I$ is not injective or $\tau(E_{(\lambda-t, \lambda+t)}(S)) = \infty$ for all $t > 0$.

$$\Leftrightarrow \lambda \in \sigma_p(S) \text{ or } \lambda \in \sigma_e(S).$$

Thus we have that $\sigma_{\widetilde{\mathcal{M}}}(S) = \sigma_p(S) \cup \sigma_e(S)$.

△

Proposition 3.7.5 *Let $S \in \widetilde{\mathcal{M}}$. Then $\mu_\infty(S) = \sup\{|\lambda| : \lambda \in \sigma_e(S)\}$.*

Proof: By Proposition 2.9.4, $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ equipped with the norm $\|\pi(S)\| = \mu_\infty(S)$ is a C^* -algebra. The result follows from the fact that the spectral radius of an element in a C^* -algebra equals its norm.

△

Corollary 3.7.6 *Suppose $S \in \widetilde{\mathcal{M}}$. Then $\sigma_e(S) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \mu_\infty(S)\}$.*

Proof: Follows by Proposition 3.7.5.

△

Proposition 3.7.7 *Let $0 \leq S \in \widetilde{\mathcal{M}}$ and $\lambda_0 = \lim_{t \rightarrow \infty} \mu_t(S)$. Then*

$$\sigma(S) \cap [\lambda_0, \infty) = \overline{\{\mu_t(S) : t > 0\}}.$$

Proof: Suppose $t > 0$. We first show that $\mu_t(S) \in \sigma(S)$. Since $\sigma(S)$ is always closed, it suffices to show that

$$\sup[\sigma(S) \cap [0, \mu_t(S)]] = \mu_t(S).$$

So assume for a contradiction that there exists some $\alpha > 0$ such that

$$\sigma(S) \cap (\mu_t(S) - \alpha, \mu_t(S)] = \emptyset.$$

By Theorem 3.2.2, it follows that $E_{(\mu_t(S) - \alpha, \mu_t(S)]}(S) = 0$, that is

$$E_{\mu_t(S)}(S) = E_{\mu_t(S) - \alpha}(S)$$

and so

$$t \geq d_{\mu_t(S)}(S) = d_{\mu_t(S) - \alpha}(S)$$

which implies that

$$\mu_t(S) = \inf\{\theta > 0 : d_\theta(S) \leq t\} \leq \mu_t(S) - \alpha$$

which is the desired contradiction. Thus $\{\mu_t(S) : t > 0\} \subseteq \sigma(S)$ from which it follows that

$$\overline{\{\mu_t(S) : t > 0\}} \subseteq \sigma(S)$$

since $\sigma(S)$ is closed. Since $\mu_t(S) \geq \lambda_0$ for every $t > 0$, $\overline{\{\mu_t(S) : t > 0\}} \subseteq \sigma(S) \cap [\lambda_0, \infty)$.

Conversely, suppose $\lambda \notin \overline{\{\mu_t(S) : t > 0\}}$. We show that $\lambda \notin [\lambda_0, \infty) \cap \sigma(S)$. If $\lambda < \lambda_0$, this is clear. Since $\lambda_0 \in \overline{\{\mu_t(S) : t > 0\}}$, it follows that $\lambda \neq \lambda_0$. If $\lambda > \lambda_0$, then there exists an $\epsilon > 0$ such that $\lambda - \epsilon > \lambda_0$ and $(\lambda - \epsilon, \lambda + \epsilon] \cap \{\mu_t(S) : t > 0\} = \emptyset$. This implies that $(\lambda - \epsilon, \lambda + \epsilon]$ is an interval of constancy for $d_\lambda(S)$, so that

$$d_{\lambda - \epsilon}(S) = d_{\lambda + \epsilon}(S) < \infty,$$

since $\lambda - \epsilon > \lambda_0$ and $d_\lambda(S)$ is a decreasing function. It follows via the faithfulness of τ that

$$E_{(\lambda - \epsilon, \infty)}(S) = E_{(\lambda + \epsilon, \infty)}(S)$$

so that $\lambda \in \rho(S)$, since by Lemma 3.2.2, $\lambda \in \rho(S) \cap \mathbb{R}$ if and only if λ lies in an interval of constancy for $\{E_t(S)\}$. Hence $\lambda \notin \sigma(S)$. This implies that for all $\lambda > 0$, if $\lambda \notin \overline{\{\mu_t(S) : t > 0\}}$,

then $\lambda \notin \sigma(S) \cap [\lambda_0, \infty)$. This completes the proof.

△

The following result generalizes, for the self-adjoint case, Proposition 3.1.6, and characterizes the spectrum in terms of the trace.

Proposition 3.7.8 *Suppose $S \in \widetilde{\mathcal{M}}^{sa}$. Then*

$$\sigma(S) = \{\lambda \in \mathbb{R} : \tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) > 0, \text{ for all } \epsilon > 0\}.$$

Proof: $\sigma(S) \subseteq \mathbb{R}$ since S is self-adjoint. Suppose $\lambda \in \mathbb{R}$ and $\tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) = 0$ for some $\epsilon > 0$. Then $E_{(\lambda-\epsilon, \lambda+\epsilon)}(S) = 0$, by the faithfulness of the trace. Thus $0 = E_{(\lambda-\epsilon, \lambda+\epsilon)}(S) = E_{(-\infty, \lambda+\epsilon)}(S) - E_{(-\infty, \lambda-\epsilon]}(S)$, so that $E_{(-\infty, \lambda+\epsilon)}(S) = E_{(-\infty, \lambda-\epsilon]}(S)$. Thus λ lies in an interval of constancy for $\{E_\mu(S)\}$. By Lemma 3.2.2, $\lambda \in \rho(S)$. Hence $\sigma(S) \subseteq \{\lambda \in \mathbb{R} : \tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) > 0 \text{ for all } \epsilon > 0\}$.

Conversely, suppose $\lambda \notin \sigma(S)$, then either $\lambda \notin \mathbb{R}$ or $\lambda \in \rho(S) \cap \mathbb{R}$. If $\lambda \in \rho(S) \cap \mathbb{R}$, then there exists an $\epsilon > 0$ such that $E_{(\lambda-\epsilon, \infty)}(S) = E_{(\lambda+\epsilon, \infty)}(S)$, by Lemma 3.2.2. This implies that $E_{(\lambda-\epsilon, \lambda+\epsilon]}(S) = 0$ and so $E_{(\lambda-\epsilon, \lambda+\epsilon)}(S) = 0$. Thus $\tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) = 0$. Thus in either case $\lambda \notin \{\lambda \in \mathbb{R} : \tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) > 0 \text{ for all } \epsilon > 0\}$. Thus

$$\{\lambda \in \mathbb{R} : \tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S)) > 0 \text{ for all } \epsilon > 0\} \subseteq \sigma(S),$$

completing the proof.

△

3.8 Spectra of the operator S_0

Recall that if $0 \leq S \in \widetilde{\mathcal{M}}$, we put $S_0 = (I - E_{\lambda_0}(S))S$. In this section we investigate the relationship between the spectra of S and S_0 .

If $S \in \widetilde{\mathcal{M}}_0$, then we have seen that $\sigma_e(S) = \{0\}$, and $S_0 = (I - E_0(S))S = S$, so that $\sigma_e(S_0) = \{0\}$. When considering the essential spectrum it is therefore enough to restrict attention to the case where $\lambda_0 = \lim_{t \rightarrow \infty} \mu_t(S) > 0$.

Proposition 3.8.1 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}$ and $\lambda_0 > 0$. Then*

1. $(0, \lambda_0) \cap \sigma_e(S_0) = \emptyset$.
2. $0 \in \sigma_e(S_0) \iff \tau(E_{\lambda_0}(S)) = \infty$.
3. *We have $\sigma_e(S_0) \subseteq \{0, 1\}$ and $\sigma_e(S_0)$ contains at least one of 0 and λ_0 . When $\alpha_0 < \infty$, 0 is the only element of $\sigma_e(S_0)$.*

Proof: We have that

$$\sigma_e(S_0) = \{\lambda \in \mathbb{R}^+ : \tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S_0)) = \infty \text{ for all } \epsilon > 0\}.$$

By the construction of S_0 in Proposition 3.5.1, we have that

$$E_\lambda(S_0) = \begin{cases} 0, & \text{if } \lambda < 0 \\ E_{\lambda_0}(S), & \text{if } 0 \leq \lambda < \lambda_0 \\ E_\lambda(S), & \text{if } \lambda_0 \leq \lambda \end{cases}$$

We have by definition (section 1.4) that

$$E_{(\lambda-\epsilon, \lambda+\epsilon)}(S_0) = \sup_{\theta < \lambda+\epsilon} E_{[0, \theta]}(S_0) - E_{\lambda-\epsilon}(S_0).$$

Now

$$\sup_{\theta < \lambda+\epsilon} E_{[0, \theta]}(S_0) = \begin{cases} \sup_{\theta < \lambda+\epsilon} E_{[0, \theta]}(S), & \text{if } \lambda + \epsilon > \lambda_0 \\ E_{\lambda_0}(S), & \text{if } 0 < \lambda + \epsilon \leq \lambda_0 \\ 0, & \text{if } \lambda + \epsilon \leq 0 \end{cases}$$

and

$$E_{\lambda-\epsilon}(S_0) = \begin{cases} E_{\lambda-\epsilon}(S), & \text{if } \lambda - \epsilon \geq \lambda_0 \\ E_{\lambda_0}(S), & \text{if } 0 \leq \lambda - \epsilon < \lambda_0 \\ 0, & \text{if } \lambda - \epsilon < 0 \end{cases}$$

Therefore we have

$$E_{(\lambda-\epsilon, \lambda+\epsilon)}(S_0) = \begin{cases} 0, & \text{if } 0 \leq \lambda - \epsilon < \lambda + \epsilon \leq \lambda_0 \\ E_{\lambda_0}(S), & \text{if } \lambda - \epsilon < 0 \leq \lambda + \epsilon \leq \lambda_0 \end{cases}$$

Now for part (1), if $0 < \lambda < \lambda_0$, there exists an $\epsilon > 0$ such that $0 \leq \lambda - \epsilon < \lambda + \epsilon \leq \lambda_0$. Hence we have that $\tau(E_{(\lambda-\epsilon, \lambda+\epsilon)}(S_0)) = 0$ so that $\lambda \notin \sigma_e(S_0)$.

For part (2), we have that $E_{(-\epsilon, \epsilon)}(S_0) = E_{\lambda_0}(S)$ so that by Proposition 3.7.3, $0 \in \sigma_e(S_0)$ if and only if $\tau(E_{\lambda_0}(S)) = \infty$.

(3) We have that $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ is a C*-algebra when given the norm $\|[T]\| = \lim_{t \rightarrow \infty} \mu_t(T) = \lambda_0$, where $[T] = \pi(T)$ and π is the canonical quotient map. The essential spectrum of T is the spectrum of $[T]$ in $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$. Now from known C*-algebra results it follows that the essential spectrum is non-empty and contained in $\{\lambda \in \mathbb{C} : |\lambda| \leq \lambda_0\}$. If $T \geq 0$, this becomes the interval $[0, \lambda_0]$. With $T = S_0$, and since we have already shown in part (1) that $\sigma_e(S_0) \cap (0, \lambda_0) = \emptyset$, we have that $\sigma_e(S_0) \subseteq \{0, \lambda_0\}$. This shows that at least one of 0 or λ_0 must be in $\sigma_e(S_0)$. When $\alpha_0 < \infty$, then $\lambda_0 = \lim_{t \rightarrow \infty} \mu_t(S_0) = \lim_{t \rightarrow \infty} \mu_t(S)\chi_{[0, \alpha_0)}(t) = 0$ so that 0 is the only member of $\sigma_e(S_0)$.

△

The following examples show that 0 could be, but need not be, in $\sigma_e(S_0)$.

Example 3.8.2 Let S be the multiplication operator M_f , where $f \in \widetilde{L}_\infty(\mathbb{R}, \mathcal{B}, m)$ is defined by

$$f(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ 1 + \frac{1}{t} & \text{if } t > 0 \end{cases}$$

Then $\lambda_0 = 1$ and $m\{t \in \mathbb{R} : |f(t)| \leq 1\} = \infty$ and therefore $0 \in \sigma_e(S_0)$. Also, we have $\mu\{t \in \mathbb{R} : |f(t) - 1| < \epsilon\} = \infty$ for all $\epsilon > 0$, so that $\lambda_0 = 1$ is also in $\sigma_e(S_0)$. However, if we define $f \in \widetilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m)$ by

$$f(t) = \begin{cases} 1, & \text{if } 0 < t \leq 1 \\ \frac{1}{t} + 1 & \text{if } t > 1 \end{cases}$$

then $\lambda_0 = 1$ and $m\{t \in \mathbb{R}^+ : |f(t)| \leq 1\} < \infty$, and therefore $0 \notin \sigma_e(S_0)$. However, $\mu\{t \in \mathbb{R}^+ : |f(t) - 1| < \epsilon\} = \infty$ for all $\epsilon > 0$, so that $\lambda_0 = 1$ is in $\sigma_e(S_0)$.

The following example shows that in contrast with the above results for S_0 , $(0, \lambda_0) \cap \sigma_e(S)$ could be non-empty. Furthermore, we show that $\lambda = 0$ is not always in $\sigma_e(S)$ when $S \notin \widetilde{\mathcal{M}}_0$, that is when $\lambda_0 \neq 0$.

Example 3.8.3 Consider the multiplication operator $S = M_f$ where f is the function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ \frac{1}{x} + 2 & \text{if } x > 0 \end{cases}$$

Then $f \in \widetilde{L}_\infty(\mathbb{R}, \mathcal{B}, m)$ and $\lambda_0 = 2$. The spectral projections of M_f are the characteristic functions χ_{X_λ} of the sets $X_\lambda = \{x \in \mathbb{R} : f(x) \leq \lambda\}$. Then $S_0 = M_{f_0}$ where $f_0(x) = \frac{1}{x} + 2$, $x > 0$, and $f_0(x) = 0$ if $x \leq 0$. Now $\mu\{x \in \mathbb{R} : |f(x) - 1| < \epsilon\} = \infty$ for all $\epsilon > 0$, so that $1 \in \sigma_e(S)$. But $\mu\{x \in \mathbb{R} : |f(x)| < \epsilon\} = 0$ for all $\epsilon < 1$, so that $0 \notin \sigma_e(S)$.

Δ

In the same way we compared the essential spectra for S and S_0 , we now look at the point spectra of S and S_0 . We have the following:

Proposition 3.8.4 Suppose $0 \leq S \in \widetilde{\mathcal{M}}$, $S_0 = (I - E_{\lambda_0}(S))S$ and $\lambda_0 > 0$. Then

1. $(\lambda_0, \infty) \cap \sigma_p(S) = (\lambda_0, \infty) \cap \sigma_p(S_0)$.
2. $(0, \lambda_0] \cap \sigma_p(S_0) = \emptyset$.

Proof: (1) Let $\lambda > \lambda_0$, then $E_\lambda(S_0) = E_\lambda(S)$ and $E_{\lambda-0}(S_0) = E_{\lambda-0}(S)$. Using Proposition 3.2.1 we have

$$\begin{aligned} \ker(S_0 - \lambda I) &= (E_\lambda(S_0) - E_{\lambda-0}(S_0))(H) \\ &= (E_\lambda(S) - E_{\lambda-0}(S))(H) \\ &= \ker(S - \lambda I). \end{aligned}$$

and hence $\lambda \in \sigma_p(S_0) \iff \lambda \in \sigma_p(S)$.

(2) For $0 < \lambda \leq \lambda_0$ we have by Proposition 3.2.1 again that

$$\begin{aligned} \ker(S_0 - \lambda I) &= (E_\lambda(S_0) - E_{\lambda-0}(S_0))(H) \\ &= (E_{\lambda_0}(S) - E_{\lambda_0}(S))(H) \\ &= \{0\}. \end{aligned} \tag{3.4}$$

Hence for $0 < \lambda \leq \lambda_0$, $\lambda \notin \sigma_p(S_0)$, completing the proof.

Δ

The following example shows that $\lambda = 0$ may or may not lie in $\sigma_p(S_0)$.

Example 3.8.5 Since

$$E_\lambda(S_0) = \begin{cases} 0, & \text{if } \lambda < 0 \\ E_{\lambda_0}(S), & \text{if } 0 \leq \lambda < \lambda_0 \\ E_\lambda(S), & \text{if } \lambda_0 \leq \lambda, \end{cases}$$

we have $\ker(S_0) = E_{\lambda_0}(S_0)(H)$. Now let S be the multiplication operator M_f , where $f \in \widetilde{L}_\infty(\mathbb{R}, \mathcal{B}, m)$ is defined by

$$f(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1 \\ 1 + \frac{1}{t} & \text{if } t > 1. \end{cases}$$

Then $S = S_0$, $\lambda_0 = 1$ and $m\{t \in \mathbb{R} : f(t) \leq 1\} > 0$ and therefore $0 \in \sigma_p(S_0)$. However, if we define $g \in \widetilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m)$ by $g(t) = \frac{1}{t} + 1$ for $t > 0$ and $S = M_g$, then $S \neq S_0$ again, $\lambda_0 = 1$ and $m\{t \in \mathbb{R}^+ : g(t) \leq 1\} = 0$, and therefore $0 \notin \sigma_p(S_0)$.

3.9 Spectra in $\widetilde{\mathcal{M}}_0$

The point spectrum of a positive compact operator on a Hilbert space H is countable (perhaps finite), and the eigenvalues can be arranged in decreasing order, with the only possible point of accumulation being zero. In this case the spectrum is just the closure of the point spectrum. If we let $\mathcal{M} = B(H)$, with the canonical diagonal trace, then $\widetilde{\mathcal{M}} = B(H)$ and $\widetilde{\mathcal{M}}_0$ is the ideal of compact operators. The singular value function of a compact operator is a piecewise constant function, with range equal to the set of singular values of the operator. The following is an analogous result for the positive τ -compact operators, and is a special case of Proposition 3.7.7.

Proposition 3.9.1 [SW93], 4.10. Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$. Then

$$\sigma(S) = \overline{\{\mu_t(S) : t > 0\}}.$$

Proof: Follows from Proposition 3.7.7, since $\lambda_0 = 0$ and $\sigma(S) \subseteq [0, \infty)$.

Δ

Lemma 3.9.2 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$, $\alpha_0 < \infty$ and $\tau(I) = \infty$. Then $0 \in \sigma_p(S)$.*

Proof: If $S \in \widetilde{\mathcal{M}}_0$, then $\lambda_0 = 0$. If $\alpha_0 < \infty$ then $\tau(I - E_0(S)) = \alpha_0 < \infty$. If $\tau(I) = \infty$, then $E_0(S) \neq 0$ so that $\ker(S) = E_0(S)(H) \neq \{0\}$. Hence $0 \in \sigma_p(S)$.

Δ

In the case $\mathcal{M} = B(H)$, every non-zero element of the spectrum of a compact operator is in the point spectrum, the singular value function is piecewise constant and the non-zero elements of the spectrum coincide with the range of this function. The next result generalizes these well-known facts.

Proposition 3.9.3 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$ and let m be Lebesgue measure on the positive real line. Then*

$$\{\lambda > 0 : m\{t > 0 : \mu_t(S) = \lambda\} > 0\} \subseteq \sigma_p(S),$$

and

$$\sigma_p(S) \subseteq \{\lambda > 0 : m\{t > 0 : \mu_t(S) = \lambda\} > 0\} \cup \{0\}.$$

If furthermore $\tau(I) = \infty$ and $\alpha_0 < \infty$, then

$$\sigma_p(S) = \{\lambda \geq 0 : m\{t > 0 : \mu_t(S) = \lambda\} > 0\}.$$

Proof: Suppose $\lambda_1 > 0$ and $m\{t > 0 : \mu_t(S) = \lambda_1\} > 0$, then the set $\{t > 0 : \mu_t(S) = \lambda_1\}$ is an interval of constancy for $\mu_t(S)$. Thus λ_1 is a point of discontinuity for $d_\lambda(S)$. This implies that

$$\lim_{\lambda \uparrow \lambda_1} \tau(E_{(\lambda, \infty)}(S)) = d_{\lambda_1-0}(S) \neq d_{\lambda_1+0}(S) = d_{\lambda_1}(S), \quad (3.5)$$

using the right continuity of $d_\lambda(S)$. Now we show that $E_{\lambda_1-0}(S) = \lim_{\lambda \uparrow \lambda_1} E_\lambda(S) \neq E_{\lambda_1}(S)$. For each $0 < \lambda < \lambda_1$, $d_\lambda(S) < \infty$ since $S \in \widetilde{\mathcal{M}}_0$. Now suppose, to the contrary that

$E_{\lambda_1-0}(S) = E_{\lambda_1}(S)$, then $E_\lambda(S) \uparrow E_{\lambda_1}(S)$ as $\lambda \uparrow \lambda_1$, so that $E_{(\lambda,\infty)}(S) \downarrow E_{(\lambda_1,\infty)}$ as $\lambda \uparrow \lambda_1$. By the normality of the trace and the fact that $d_\lambda(S) < \infty$, we have that

$$d_\lambda(S) = \tau(E_{(\lambda,\infty)}(S)) \downarrow \tau(E_{(\lambda_1,\infty)}(S)) = d_{\lambda_1}(S), \text{ as } \lambda \uparrow \lambda_1,$$

a contradiction to equation (3.5). Hence $E_{\lambda_1-0}(S) \neq E_{\lambda_1}(S)$, and by Lemma 3.2.1 we have

$$\ker(S - \lambda_1 I) = (E_{\lambda_1}(S) - E_{\lambda_1-0}(S))(H) \neq \{0\}.$$

Hence $\lambda_1 \in \sigma_p(S)$.

Now suppose $\lambda_1 > 0$ and $m\{t > 0 : \mu_t(S) = \lambda_1\} = 0$. Then $d_\lambda(S)$ is continuous at λ_1 . Thus we have

$$\lim_{\lambda \uparrow \lambda_1} \tau(E_{(\lambda,\infty)}(S)) = \tau(E_{(\lambda_1,\infty)}(S)) < \infty, \text{ since } S \in \widetilde{\mathcal{M}}_0. \quad (3.6)$$

Suppose to the contrary that $I - E_{\lambda_1-0}(S) \neq E_{(\lambda_1,\infty)}(S)$, then there exists a projection Q such that

$$E_{(\lambda,\infty)}(S) \downarrow Q > E_{(\lambda_1,\infty)}(S) \text{ as } \lambda \uparrow \lambda_1.$$

Now since $S \in \widetilde{\mathcal{M}}_0$, just as in the first part of the proof, the normality and faithfulness of the trace implies that

$$\tau(E_{(\lambda,\infty)}(S)) \downarrow \tau(Q) > \tau(E_{(\lambda_1,\infty)}(S)) \text{ as } \lambda \uparrow \lambda_1.$$

From this it follows that $\lim_{\lambda \uparrow \lambda_1} \tau(E_{(\lambda,\infty)}(S)) \neq \tau(E_{(\lambda_1,\infty)}(S))$, a contradiction to equation (3.6). Thus $I - E_{\lambda_1-0}(S) = E_{(\lambda_1,\infty)}(S)$, from which we have that $E_{\lambda_1-0}(S) = E_{\lambda_1}(S)$. Hence, again by Lemma 3.2.1

$$\ker(S - \lambda_1 I) = (E_{\lambda_1}(S) - E_{\lambda_1-0}(S))(H) = \{0\}.$$

Hence $\lambda_1 \notin \sigma_p(S)$, and therefore,

$$\sigma_p(S) \subseteq \{\lambda > 0 : m\{t > 0 : \mu_t(S) = \lambda\} > 0\} \cup \{0\}.$$

Now if $\alpha_0 < \infty$, then $m\{t > 0 : \mu_t(S) = 0\} > 0$ and by Lemma 3.9.2, $0 \in \sigma_p(S)$. Thus, together with what we have already proved above, we have that

$$\sigma_p(S) = \{\lambda \geq 0 : m\{t > 0 : \mu_t(S) = \lambda\} > 0\}.$$

△

Remark The inclusions in the above proposition can be proper, depending on whether S is injective or not. For, when S is injective then $\ker(S) = \{0\}$, and so $0 \notin \sigma_p(S)$ so that the second inclusion in the statement of Proposition 3.9.3 is proper. If S is not injective then $0 \in \sigma_p(S)$, and so the first inclusion of Proposition 3.9.3 is proper.

The above proposition does not hold in general when $S \notin \widetilde{\mathcal{M}}_0$, that is when $\lambda_0 > 0$ as the following example in the commutative case shows. This is due to the suppressing effect of the decreasing rearrangement.

Example 3.9.4 Define f by

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & x \geq 1 \end{cases}$$

Then clearly $f \in \widetilde{L}_\infty(\mathbb{R}^+, \Sigma, m)$, where m is Lebesgue measure on \mathbb{R}^+ , $\lambda_0 = 2$ and by Proposition 3.1.2, $\sigma_p(M_f) = \{1, 2\}$ but $\{\lambda \in \mathbb{R}^+ : m\{t > 0 : \mu_t(f) = \lambda\} > 0\} = \{2\}$

The following two corollaries show that positive equimeasurable operators in $\widetilde{\mathcal{M}}_0$ have the same spectrum.

Corollary 3.9.5 Suppose $0 \leq S, T \in \widetilde{\mathcal{M}}_0$ and $\mu_t(S) = \mu_t(T)$. Then

$$\sigma(S) = \sigma(T).$$

Proof: Follows easily from Corollary 3.9.1.

△

Corollary 3.9.6 Suppose $0 \leq S, T \in \widetilde{\mathcal{M}}_0$ and $\mu_t(S) = \mu_t(T)$. Then

$$\sigma_{\widetilde{\mathcal{M}}}(S) = \sigma_{\widetilde{\mathcal{M}}}(T).$$

Proof: From Propositions 3.7.4 and 3.9.3 we have that

$$\begin{aligned}
\sigma_{\widetilde{\mathcal{M}}}(S) &= \sigma_p(S) \cup \{0\} \\
&= \{\lambda > 0 : \mu\{t > 0 : \mu_t(S) = \lambda\} > 0\} \cup \{0\} \\
&= \{\lambda > 0 : \mu\{t > 0 : \mu_t(T) = \lambda\} > 0\} \cup \{0\} \\
&= \sigma_{\widetilde{\mathcal{M}}}(T)
\end{aligned}$$

△

Another immediate consequence of Propositions 3.9.3 and 3.7.4 is the following:

Corollary 3.9.7 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$. Then*

$$\sigma_{\widetilde{L}_\infty(\mathbb{R}^+)}(\mu_t(S)) = \sigma_{\widetilde{\mathcal{M}}}(S)$$

Proof: We have that

$$\begin{aligned}
\sigma_{\widetilde{\mathcal{M}}}(S) &= \sigma_p(S) \cup \{0\} \\
&= \{\lambda \in \mathbb{R}^+ : \mu\{t > 0 : \mu_t(S) = \lambda\} > 0\} \cup \{0\} \text{ from Propositions 3.9.3 and 3.7.4} \\
&= \sigma_{\widetilde{L}_\infty(\mathbb{R}^+)}(\mu_t(S)) \text{ from Propositions 3.1.2 and 3.1.13.}
\end{aligned}$$

△

Proposition 3.9.8 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}$, and let $S_0 = (I - E_{\lambda_0}(S))S$, with $\alpha_0 < \infty$. Then*

$$\sigma(S_0) = \overline{\{\mu_t(S) : t \in [0, \alpha_0]\}}.$$

Proof: $S_0 = (I - E_{\lambda_0}(S))S \in \widetilde{\mathcal{M}}$. Furthermore, since $\mu_t(S_0) = \mu_t(S)\chi_{[0, \alpha_0]}(t)$, by what we have shown in the proof of Proposition 3.5.1, we have that $\lim_{t \rightarrow \infty} \mu_t(S_0) = 0$ since $\alpha_0 < \infty$, so that $S_0 \in \widetilde{\mathcal{M}}_0$. Since $S_0 = (I - E_{\lambda_0}(S))S = (I - E_{\lambda_0}(S))S(I - E_{\lambda_0}(S))$ and $S \geq 0$, we have that $S_0 \geq 0$. Thus by Corollary 3.9.1, we have

$$\begin{aligned}
\sigma(S_0) &= \overline{\{\mu_t(S_0) : t > 0\}} \\
&= \overline{\{\mu_t(S)\chi_{[0, \alpha_0]} : t > 0\}} \\
&= \overline{\{\mu_t(S) : t \in [0, \alpha_0]\}}.
\end{aligned}$$

△

We note here that when $\alpha_0 = \infty$, S_0 may not be in $\widetilde{\mathcal{M}}_0$, unless $\lambda_0 = \lim_{t \rightarrow \infty} \mu_t(S) = 0$.

We observe also that Proposition 3.9.8 above, and Propositions 3.8.1 and 3.8.4 we considered earlier, are generalisations of the result in the bounded case mentioned in ([GK69], II.7).

3.10 $\widetilde{\mathcal{M}}$ as a \mathcal{Q} -algebra

A topological algebra in which the set of invertible elements is open is called a \mathcal{Q} -algebra. Every Banach algebra is a \mathcal{Q} -algebra, and so are some commutative locally m -convex algebras, ([Zel76], Corollary 3). In this section we show that $\widetilde{\mathcal{M}}$ is a \mathcal{Q} -algebra if and only if $\widetilde{\mathcal{M}} = \mathcal{M}$. We start with the commutative case to illustrate the elementary measure theoretic arguments involved in the proofs. We then give the general result which depends on some more technical results.

Proposition 3.10.1 *Suppose (X, Σ, μ) is a localizable measure space and μ a semi-finite measure. Then $\widetilde{L}_\infty(X, \Sigma, \mu)$ is a \mathcal{Q} -algebra if and only if $\widetilde{L}_\infty(X, \Sigma, \mu) = L_\infty(X, \Sigma, \mu)$.*

Proof: Let \mathcal{Q} be the set of invertible elements of $\widetilde{L}_\infty = \widetilde{L}_\infty(X, \Sigma, \mu)$. It follows from Proposition 3.1.7 that

$$\mathcal{Q} = \{f \in \widetilde{L}_\infty : \mu\{x \in X : |f(x)| < \delta\} < \infty, \text{ some } \delta > 0 \text{ and } \mu\{x \in X : f(x) = 0\} = 0\}.$$

Let us suppose that $\widetilde{L}_\infty(X, \Sigma, \mu) \neq L_\infty(X, \Sigma, \mu)$. It follows from Proposition 2.9.3 that $\inf\{\mu(A) : A \in \Sigma, \mu(A) \neq 0\} = 0$. We show that $\widetilde{L}_\infty(X, \Sigma, \mu)$ is not a \mathcal{Q} -algebra. By Lemma 2.3.6, it suffices to show that there is no neighbourhood of e contained in \mathcal{Q} , where e is the constant 1 function. For any $\epsilon > 0$, $f_0 \in e + \widetilde{\mathcal{N}}(\epsilon)$ if and only if $\mu\{x \in X : |f_0(x) - 1| > \epsilon\} \leq \epsilon$. Since we have that $\inf\{\mu(A) : A \in \Sigma, \mu(A) \neq 0\} = 0$, then for each $n \in \mathbb{N}$, there exists A_n , $A_n \in \Sigma$, $\mu(A_n) \neq 0$ and $\mu(A_n) \leq \frac{1}{n}$. Define

$$f_n(x) = \begin{cases} 0, & x \in A_n \\ 1, & x \notin A_n \end{cases}$$

Now $\mu\{x \in X : |f_n(x) - 1| > \frac{1}{n}\} = \mu(A_n) \leq \frac{1}{n}$. Hence $f_n \in e + \tilde{\mathcal{N}}(\frac{1}{n})$. However, $\mu\{x \in X : f_n(x) = 0\} = \mu(A_n) \neq 0$ so that $f_n \notin Q$. Since $\{\tilde{\mathcal{N}}(\frac{1}{n}), n \in \mathbb{N}\}$ constitutes a base of neighbourhoods at 0 for τ_m on $\tilde{L}_\infty(X, \Sigma, \mu)$, every neighbourhood of e contains a non-invertible element. Hence $\tilde{L}_\infty(X, \Sigma, \mu)$ is not a Q -algebra.

Conversely, if $\inf\{\mu(A) : A \in \Sigma, \mu(A) \neq 0\} > 0$ then

$$\tilde{L}_\infty(X, \Sigma, \mu) = L_\infty(X, \Sigma, \mu) \quad (\text{Proposition 2.9.3}).$$

so that it is a Q -algebra since every Banach algebra is a Q -algebra. Hence $\tilde{L}_\infty(X, \Sigma, \mu)$ is a Q -algebra if and only if $\tilde{L}_\infty(X, \Sigma, \mu) = L_\infty(X, \Sigma, \mu)$.

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To extend the previous result to the non-commutative case, we will need the following lemma.

Lemma 3.10.2 (*[Wes90], 9.25.*)

Suppose $0 \leq S \in \tilde{\mathcal{M}}$. Then

1. the function $\nu_S : \mathcal{B}([0, \infty)) \rightarrow [0, \infty] : B \rightarrow \tau(E_B(S))$ is a measure, where $\mathcal{B}([0, \infty))$ is the Borel σ -algebra on the positive real line.

2. $\tau(E_B(S)) = \int_0^\infty \chi_B(t) d\tau(E_t(S))$

The following is a special case of a result which appears in the proof of ([DdP92], Theorem 3.5). We present its proof for completeness.

Lemma 3.10.3 Let $0 \leq S \in \tilde{\mathcal{M}}_0$ and denote by \mathcal{K} the collection of the intervals of constancy of $\mu_t(S)$. Put $C = \cup\{F : F \in \mathcal{K}\}$. Let M_S be the von Neumann subalgebra of $L_\infty(\mathbb{R}^+)$ generated by the characteristic functions of members of \mathcal{K} and the Lebesgue measurable subsets of $\mathbb{R}^+ \setminus C$, and let \mathcal{M}_S be the proper abelian subalgebra of \mathcal{M} defined in Proposition 3.6.6. For each $f \in \tilde{M}_S$, let $J_S(f) = \int_0^\infty f(t) d\tilde{E}_t(S)$, where $\tilde{E}_t(S)$ is as defined in Proposition 3.5.1. Then $J_S(f) \in \tilde{M}_S$ and $\mu_t(J_S(f)) = \mu_t(f)$.

Proof: By the spectral theorem, $J_S(f)$ is closed densely defined and affiliated with \mathcal{M}_S , since its spectral family $\{\tilde{E}_{f^{-1}(0,t]}(S)\}$ lies in \mathcal{M}_S , as $\{\tilde{E}_t(S)\}$ lies in \mathcal{M}_S . If $f \in \widetilde{\mathcal{M}}_S$ and if $\lambda > 0$, we set $F_\lambda = \{s \geq 0 : |f|(s) > \lambda\}$. It follows from the definition of $J_S(f)$ and the functional calculus that

$$\begin{aligned} |J_S(f)| &= \int_0^\infty |f|(t) d\tilde{E}_t(S) \\ &= \int_0^\infty t d\tilde{E}_{(|f|^{-1}(0,t])}(S), \end{aligned}$$

so that

$$\begin{aligned} \tau(E_{(\lambda,\infty)}(|J_S(f)|)) &= \tau(\tilde{E}_{(|f|^{-1}(\lambda,\infty))}(S)) \\ &= \tau(\tilde{E}_{F_\lambda}(S)) \\ &= \tau\left(\int_0^\infty \chi_{F_\lambda}(t) d\tilde{E}_t(S)\right) \text{ by the functional calculus} \\ &= \int_0^\infty \chi_{F_\lambda}(t) d\tau(\tilde{E}_t(S)) \text{ by Lemma 3.10.2} \\ &= \int_0^\infty \chi_{F_\lambda}(t) dt \\ &= m(F_\lambda) \downarrow 0 \text{ as } \lambda \uparrow \infty \text{ since } f \in \widetilde{\mathcal{M}}_S. \end{aligned}$$

We observe here that $\tau(\tilde{E}_t(S)) = \tau(I - E_{\mu_t(S)}(S)) = d_{\mu_t(S)}(S) = t$ at all points of continuity for $\mu_t(S)$, so we have that $\tau(\tilde{E}_t(S)) = t$, m -almost everywhere, since $\mu_t(S)$ is continuous m -almost everywhere. And redefining $\chi_{F_\lambda}(t)$ to be zero at points of discontinuity of $\tau(\tilde{E}_t(S))$, we have the equality in the second but last equality sign above. Hence $J_S(f) \in \widetilde{\mathcal{M}}_S$. Also from what we have above, $d_\lambda(J_S(f)) = m(F_\lambda) = d_\lambda(f)$, so that $\mu_t(J_S(f)) = \mu_t(f)$.

Δ

Now we construct an unbounded function in $\widetilde{\mathcal{M}}_S$ that we shall use in what follows to characterize those $\widetilde{\mathcal{M}}$ which are \mathcal{Q} -algebras.

Proposition 3.10.4 *Suppose $0 < S \in \widetilde{\mathcal{M}}_0 \setminus \mathcal{M}$ and let $\widetilde{\mathcal{M}}_S$ be defined as in Proposition 3.6.6. Then there exists an unbounded operator $T \in \widetilde{\mathcal{M}}_S \cap \widetilde{\mathcal{M}}_0$ such that $\sigma_{\widetilde{\mathcal{M}}}(T)$ is unbounded.*

Proof: We show that we can define $T = J_S(f)$ for an appropriate $f \in \widetilde{M}_S$, where M_S is as in Lemma 3.10.3. Let C be defined as in Lemma 3.10.3. We consider two cases.

Case 1: There exists a $t_0 > 0$ such that $(0, t_0) \cap C = \emptyset$. This implies that there exists an $n_0 \in \mathbb{N}^+$, say, such that $(0, \frac{1}{n_0}) \cap C = \emptyset$. Then in this case define

$$f(t) = \begin{cases} \sum_{n=n_0}^{\infty} n \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) & t < \frac{1}{n_0} \\ 0 & t \geq \frac{1}{n_0} \end{cases}$$

Case 2: For all $t_0 > 0$, $(0, t_0) \cap C \neq \emptyset$. This implies that for each $t_0 > 0$ there exists an interval $F_{t_0} \in \mathcal{K}$ such that $(0, t_0) \cap F_{t_0} \neq \emptyset$, with $(0, t_0) \cap F_{t_0}$ an interval of constancy of $\mu_t(S)$. Now since S is unbounded, and so $\mu_t(S) \uparrow \infty$ as $t \downarrow 0$, we can find a_1, b_1 such that $0 < a_1 < b_1 \leq 1$ and $(0, 1) \cap F_1 = (a_1, b_1)$. Put $K_1 = (a_1, b_1)$. Now by assumption there exists an $F_{a_1} \in \mathcal{K}$ such that $(0, a_1) \cap F_{a_1} \neq \emptyset$, and as before we can find $0 < a_2 < b_2 \leq a_1$ such that $(0, a_1) \cap F_{a_1} = (a_2, b_2)$. Put $K_2 = (a_2, b_2)$.

Continuing in this way, we get a sequence of intervals $\{K_k\}_{k=1}^{\infty}$ such that $m(K_k) > 0$ for all k , the K_k are mutually disjoint, and $m(K_k) \downarrow 0$ as $k \uparrow \infty$, where

$$K_k = (0, a_{k-1}) \cap F_{a_{k-1}} = (a_k, b_k), \text{ with } 0 < a_k < b_k \leq a_{k-1}.$$

Let $G_k = [b_{k+1}, a_k]$.

Now in this case, define the function

$$f(t) = \begin{cases} \sum_{k=1}^{\infty} [k \chi_{K_k}(t) + k \chi_{G_k}(t)] & t < b_1 \\ 0 & t \geq b_1 \end{cases}$$

Then in both cases 1 and 2, we have that $f \in \widetilde{M}_S$, since by construction, f is bounded except on a set of m -measure less than 1, and f is affiliated with M_S since it is generated by characteristic functions from M_S . Now with the function f the one constructed above and using Lemma 3.10.3, define the operator

$$T = J_S(f) = \int_0^{\infty} f(t) d\tilde{E}_t(S).$$

We have that $T \in \widetilde{\mathcal{M}}_S \subseteq \widetilde{\mathcal{M}}$, $\mu_t(T) = \mu_t(f) \rightarrow 0$ as $t \rightarrow \infty$, by the construction of f , so that in fact $0 < T \in \widetilde{\mathcal{M}}_0$. Then we have

$$\begin{aligned}
\sigma_{\widetilde{\mathcal{M}}}(T) &= \sigma_p(T) \cup \sigma_e(T) \text{ from Proposition 3.7.4} \\
&= \{\lambda \in \mathbb{R}^+ : m\{t > 0 : \mu_t(T) = \lambda\} > 0\} \cup \{0\} \text{ from Proposition 3.9.3} \\
&= \{\lambda \in \mathbb{R}^+ : m\{t > 0 : \mu_t(f) = \lambda\} > 0\} \cup \{0\} \\
&= \mathbb{N},
\end{aligned}$$

since by the construction of f , for each $n \in \mathbb{N}^+$ there is an interval of constancy for $\mu_t(f)$ on which $\mu_t(f) = n$. This shows that $\sigma_{\widetilde{\mathcal{M}}}(T)$ is unbounded.

Δ

Proposition 3.10.5 $\widetilde{\mathcal{M}}$ is a \mathcal{Q} -algebra if and only if $\widetilde{\mathcal{M}} = \mathcal{M}$.

Proof: If $\widetilde{\mathcal{M}} = \mathcal{M}$, then $\widetilde{\mathcal{M}}$ is a Banach algebra and it follows that $\widetilde{\mathcal{M}}$ is a \mathcal{Q} -algebra. To show that $\widetilde{\mathcal{M}}$ a \mathcal{Q} -algebra implies that $\widetilde{\mathcal{M}} = \mathcal{M}$, we suppose that $\widetilde{\mathcal{M}} \neq \mathcal{M}$ and then show that $\widetilde{\mathcal{M}}$ is not a \mathcal{Q} -algebra. Suppose $\widetilde{\mathcal{M}} \neq \mathcal{M}$, then by Proposition 2.9.3, we have that $\widetilde{\mathcal{M}}_0 \neq \mathcal{M}_0$. So let $0 < S \in \widetilde{\mathcal{M}}_0 \setminus \mathcal{M}_0$, that is S is a positive unbounded operator in $\widetilde{\mathcal{M}}_0$. With this S define $\widetilde{\mathcal{M}}_S$ as before, as in Proposition 3.6.6. Now by Proposition 3.10.4, there is a $T \in \widetilde{\mathcal{M}}$ such that $\sigma_{\widetilde{\mathcal{M}}}(T)$ is unbounded. By Lemma 2.3.5, $\widetilde{\mathcal{M}}$ is not a \mathcal{Q} -algebra. Thus we have that if $\widetilde{\mathcal{M}}$ is a \mathcal{Q} -algebra, then $\widetilde{\mathcal{M}} = \mathcal{M}$. This completes the proof.

Δ

Corollary 3.10.6 $\widetilde{\mathcal{M}}$ is a \mathcal{Q} -algebra if and only if $\inf\{\tau(P) : P \in \mathcal{M}^p : \tau(P) \neq 0\} > 0$.

Proof: Follows from Proposition 2.9.3

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Chapter 4

A Functional Calculus for τ -Measurable Operators

In this chapter we investigate the functional calculus for those functions whose action on a τ -measurable operator preserve τ -measurability. Let \mathcal{M} be a semi-finite von Neumann algebra with a faithful semi-finite normal trace τ . For every normal operator $T \in \widetilde{\mathcal{M}}$ and every Borel function f on $\sigma(T)$, $f(T)$ can be defined using the Borel functional calculus. However, $f(T)$ need not be in $\widetilde{\mathcal{M}}$. In section 4.1, we start by identifying classes of functions f with the property that $f(T) \in \widetilde{\mathcal{M}}$ whenever $T \in \widetilde{\mathcal{M}}$.

In ([Dix70]) Dixon developed a functional calculus for normal elements of a GB*-algebra and a class of continuous functions defined on the spectrum of such an element. Since $\widetilde{\mathcal{M}}$ is a GB*-algebra, this construction is applicable. In section 4.2 we compare the Borel and the Dixon functional calculus for $\widetilde{\mathcal{M}}$.

In section 4.3 we present some spectral mapping results for elements of $\widetilde{\mathcal{M}}_0$ and in the last section we look at the functional calculus for the operators obtained using the Schmidt spectral decomposition.

4.1 τ -Measurability Preserving Functions

We start by presenting the following result from [FK86] which we shall use later.

Proposition 4.1.1 ([FK86], 2.5(iv)) *Suppose $S \in \widetilde{\mathcal{M}}$ and let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function. Then for every $t > 0$,*

$$\mu_t(f(|S|)) = f(\mu_t(S)).$$

Proof: We present an alternative proof to the one in [FK86] for the case where f is strictly increasing. Note that for $\alpha \geq 0$ and $\theta = f(\alpha)$, $(\alpha, \infty) = f^{-1}(\theta, \infty)$ since f is strictly increasing.

$$\begin{aligned} f(\mu_t(S)) &= f(\inf\{\alpha \geq 0 : \tau(E_{(\alpha, \infty)}(|S|)) \leq t\}) \\ &= \inf\{f(\alpha) \geq 0 : \tau(E_{(\alpha, \infty)}(|S|)) \leq t\} \text{ since } f \text{ is increasing and continuous} \\ &= \inf\{\theta \geq 0 : \tau(E_{(f^{-1}(\theta, \infty))}(|S|)) \leq t\} \\ &= \inf\{\theta \geq 0 : \tau(E_{(\theta, \infty)}(f(|S|))) \leq t\} \text{ by the change of measure principle.} \\ &= \mu_t(f(|S|)). \end{aligned}$$

Δ

Our main aim in this section is to study Borel functions which, under the functional calculus, preserve measurability of an operator. We start with an example showing that it is possible to find a Borel function f and an $S \in \widetilde{\mathcal{M}}$ such that $f(S) \notin \widetilde{\mathcal{M}}$.

Example 4.1.2 Suppose $S \in \widetilde{\mathcal{M}}_0^{sa}$. Let

$$h(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}$$

Now since $S \in \widetilde{\mathcal{M}}_0$, we have that $\sigma_e(S) = \{0\}$, so that, by Proposition 3.7.3,

$$\tau(e_{(t, \infty)}(|h(S)|)) = \tau(e_{(-\frac{1}{t}, \frac{1}{t})}(S)) = \infty$$

for all $t > 0$. Hence $h(S) \notin \widetilde{\mathcal{M}}$.

The following recent result identifies a large class of Borel functions which preserve the measurability of an operator. We give the proof for the sake of completeness.

Proposition 4.1.3 ([dPS07], 3.1) *Suppose $S \in \widetilde{\mathcal{M}}^{sa}$. Let $\mathcal{B}_b(\sigma(S))$ be the $*$ -algebra of all complex valued Borel functions on $\sigma(S)$ which are bounded on all bounded subsets of $\sigma(S)$. Then $f(S) \in \widetilde{\mathcal{M}}$ for all $f \in \mathcal{B}_b(\sigma(S))$.*

Proof: Since $S \in \widetilde{\mathcal{M}}$, there exists an $\alpha > 0$ such that

$$\tau(E_{(\alpha, \infty)}(|S|)) < \infty.$$

If $f \in \mathcal{B}_b(\sigma(S))$, then there is a $K > 0$ such that $|f(t)| \leq K$ for all $t \in [-\alpha, \alpha]$, and therefore

$$\{t \in \mathbb{R} : |f(t)| > K\} \subseteq \{t \in \mathbb{R} : |t| > \alpha\}.$$

Hence

$$\tau(E_{(K, \infty)}(|f(S)|)) = \tau(E_{f^{-1}(\{t: |t| > K\})}(S)) \leq \tau(E_{\{t: |t| > \alpha\}}(S)) = \tau(E_{(\alpha, \infty)}(|S|)) < \infty.$$

Hence $f(S) \in \widetilde{\mathcal{M}}$.

△

The condition given to ensure preservation of measurability is sufficient, but not necessary, as the following example shows.

Example 4.1.4 Let h be the function defined in Example 4.1.2 and let $S \in \widetilde{\mathcal{M}}^{sa}$ be such that $0 \notin \sigma_e(S)$ (as, for example, in Example 3.8.2). Then it follows from Proposition 3.7.3 that for some $t > 0$, $\tau(E_{(-t, t)}(S)) < \infty$, and hence $\tau(E_{(\frac{1}{t}, \infty)}(|h(S)|)) = \tau(E_{(-t, t)}(S)) < \infty$, so that $h(S) \in \widetilde{\mathcal{M}}$. But clearly h is not bounded on all bounded subsets of \mathbb{R} .

Proposition 4.1.5 *Let $S \in \widetilde{\mathcal{M}}$ be normal and $\mathcal{B}_{\mathcal{M}}(S)$ be the collection of all Borel functions $f : \sigma(S) \rightarrow \mathbb{C}$ such that $f(S) \in \widetilde{\mathcal{M}}$. Then $\mathcal{B}_{\mathcal{M}}(S)$ is a $*$ -subalgebra of the algebra of all Borel functions on $\sigma(S)$.*

Proof: Suppose $f, g \in \mathcal{B}_{\mathcal{M}}(S)$ and $\alpha \in \mathbb{C}$. Then by Theorem 1.5.1, we have $(f + g)(S) \supseteq f(S) + g(S)$ and $(fg)(S) \supseteq f(S)g(S)$. Furthermore, $(\alpha f)(S) = \alpha f(S) \in \widetilde{\mathcal{M}}$, so that $\alpha f \in \mathcal{B}_{\mathcal{M}}(S)$. By Lemma 2.6.7 we have that a τ -pre-measurable operator has at most one extension in $\widetilde{\mathcal{M}}$. Now since $f(S) + g(S)$ and $f(S)g(S)$ are τ -pre-measurable, by Proposition 2.6.5, and their closures are τ -measurable, and since $(f + g)(S)$ and $(fg)(S)$ are closed, by Theorem 2.2.4, we have that $(f + g)(S) = \overline{f(S) + g(S)} \in \widetilde{\mathcal{M}}$ and $(fg)(S) = \overline{f(S)g(S)} \in \widetilde{\mathcal{M}}$. Thus we have $f + g, fg \in \mathcal{B}_{\mathcal{M}}(S)$. Also, we have $\bar{f}(S) = f(S)^* \in \widetilde{\mathcal{M}}$ whenever $f(S) \in \widetilde{\mathcal{M}}$ so that $\mathcal{B}_{\mathcal{M}}(S)$ is closed under adjunction. Therefore, $\mathcal{B}_{\mathcal{M}}(S)$ is a $*$ -algebra.

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If S is normal, the algebra $\mathcal{B}_{\mathcal{M}}(S)$ contains the polynomials. For by Proposition 2.2.7, S is affiliated with some abelian von Neumann algebra. If $p(t) = \sum_{k=0}^n a_k t^k$, then by ([KR83], 5.6.35), $p(S) = \sum_{k=0}^n a_k S^k$ and so $\sum_{k=0}^n a_k S^k$ is closed and hence in $\widetilde{\mathcal{M}}$. For $S \in \widetilde{\mathcal{M}}^{sa}$, $\mathcal{B}_{\mathcal{M}}(S)$ also contains $\mathcal{B}_b(\sigma(S))$, by Proposition 4.1.3.

Definition 4.1.6 *Suppose $S \in \widetilde{\mathcal{M}}^{sa}$. Let \mathfrak{F} be a set of functions $h : \sigma(S) \rightarrow (0, \infty)$ such that*

1. $h \in \mathfrak{F}$ implies $h(S) \in \widetilde{\mathcal{M}}$
2. $h \in \mathfrak{F}$ implies $h^2 \in \mathfrak{F}$
3. $h_1, h_2 \in \mathfrak{F}$ implies there exists an $h \in \mathfrak{F}$ such that

$$h_1 \leq h \text{ and } h_2 \leq h.$$

Let $\mathcal{D}_{\mathfrak{F}}(S)$ be the set of all complex-valued Borel measurable functions f defined on $\sigma(S)$ for which there is an $h \in \mathfrak{F}$ such that

$$\sup_{\lambda \in \sigma(S)} \frac{|f(\lambda)|}{h(\lambda)} = M < \infty \quad (4.1)$$

Example 4.1.7 Let $S \in \widetilde{\mathcal{M}}^{sa}$. An example of a class \mathfrak{F} of functions which satisfy the conditions of Definition 4.1.6 is the set of all positive Borel functions bounded on bounded subsets of $\sigma(S)$. From Proposition 4.1.3 we have that $f(S) \in \widetilde{\mathcal{M}}$, so that (1) is satisfied. For (2), we have that if f is bounded on bounded subsets of $\sigma(S)$, then so is f^2 . Lastly for (3) if f_1 and f_2 are both bounded on bounded subsets of $\sigma(S)$, then the same is true of $f_1 + f_2$, so we can take $h = f_1 + f_2$.

We note that the set of all positive increasing functions on $\sigma(S)$ is a subset of the above class of functions, and forms a class satisfying the conditions of Definition 4.1.6 itself, as can easily be checked.

Example 4.1.8 Another example of the class of functions \mathfrak{F} satisfying the conditions of Definition 4.1.6 is the class

$$\mathfrak{F} = \{(1 + \lambda^2)^m : m \in \mathbb{N}\}.$$

For, if $S \in \widetilde{\mathcal{M}}^{sa}$, then if $f(\lambda) = (1 + \lambda^2)^m$, then f is a polynomial, and as noted before $f(S) = (I + S^2)^m$ is closed, and so is in $\widetilde{\mathcal{M}}$. Furthermore, $f^2 = ((1 + \lambda^2)^m)^2 = (1 + \lambda^2)^{2m} \in \mathfrak{F}$, and if $f_1, f_2 \in \mathfrak{F}$, we take $h = \max\{f_1, f_2\}$, which is in \mathfrak{F} . We note that this class is also contained in the class of positive Borel functions bounded on bounded subsets of \mathbb{R} .

Proposition 4.1.9 Suppose $S \in \widetilde{\mathcal{M}}^{sa}$ and let \mathfrak{F} be any class of functions satisfying the conditions of definition 4.1.6. Then $\mathcal{D}_{\mathfrak{F}}(S)$ is a $*$ -subalgebra of $\mathcal{B}_{\mathcal{M}}(S)$.

Proof: Suppose $f \in \mathcal{D}_{\mathfrak{F}}(S)$, then there exists an $h \in \mathfrak{F}$ satisfying equation (4.1) of definition 4.1.6. We first show that $f(S) \in \widetilde{\mathcal{M}}$. We have that $h(S) \in \widetilde{\mathcal{M}}$ so that $h(S)$ has a τ -dense domain in \mathcal{H} . Now let $x \in D(h(S))$, then

$$\begin{aligned} \int_{\mathbb{R}} |f(\lambda)|^2 d\|E_{\lambda}(S)x\|^2 &= \int_{\mathbb{R}} |f(\lambda)|^2 h(\lambda)^{-2} h(\lambda)^2 d\|E_{\lambda}(S)x\|^2 \\ &\leq M^2 \int_{\mathbb{R}} h(\lambda)^2 d\|E_{\lambda}(S)x\|^2 \\ &< \infty \quad \text{since } x \in D(h(S)). \end{aligned}$$

Hence $x \in D(f(S))$ and so $D(h(S)) \subseteq D(f(S))$ so that $f(S)$ has a τ -dense domain. Since $f(S)$ is closed and affiliated with \mathcal{M} , we have that $f(S) \in \widetilde{\mathcal{M}}$ so that $\mathcal{D}_{\mathfrak{F}}(S) \subset \mathcal{B}_{\mathcal{M}}(S)$.

Now for $f, g \in \mathcal{D}_{\mathfrak{F}}(S)$ there exist $h, h_1, h_2 \in \mathfrak{F}$ such that

$$M = \sup_{\lambda \in \sigma(S)} \frac{|f(\lambda)|}{h_1(\lambda)} < \infty, \quad N = \sup_{\lambda \in \sigma(S)} \frac{|g(\lambda)|}{h_2(\lambda)} < \infty$$

and $h_1 \leq h, h_2 \leq h$. Now we have

$$\begin{aligned} \sup_{\lambda \in \sigma(S)} \frac{|f(\lambda)g(\lambda)|}{h^2(\lambda)} &\leq \sup_{\lambda \in \sigma(S)} \frac{|f(\lambda)|}{h_1(\lambda)} \cdot \sup_{\lambda \in \sigma(S)} \frac{|g(\lambda)|}{h_2(\lambda)} \\ &= MN < \infty \end{aligned}$$

Since $h^2 \in \mathfrak{F}$ we have that $fg \in \mathcal{D}_{\mathfrak{F}}(S)$. Also

$$\begin{aligned} \sup_{\lambda \in \sigma(S)} \frac{|f(\lambda) + g(\lambda)|}{h(\lambda)} &\leq \sup_{\lambda \in \sigma(S)} \left[\frac{|f(\lambda)|}{h_1(\lambda)} + \frac{|g(\lambda)|}{h_2(\lambda)} \right] \\ &= M + N < \infty \end{aligned}$$

so that $f + g \in \mathcal{D}_{\mathfrak{F}}(S)$. In a similar way for each $\alpha \in \mathbb{C}, f \in \mathcal{D}_{\mathfrak{F}}(S)$, we have that $\bar{f}, \alpha f \in \mathcal{D}_{\mathfrak{F}}(S)$. Hence $\mathcal{D}_{\mathfrak{F}}(S)$ is a $*$ -subalgebra of $\mathcal{B}_{\mathcal{M}}(S)$.

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With $\mathfrak{F} = \{(1 + \lambda^2)^m : m \in \mathbb{N}\}$ denote by $\mathcal{D}_u(S)$ the corresponding $\mathcal{D}_{\mathfrak{F}}(S)$. We will use this particular $*$ -subalgebra in the sequel.

4.2 A Functional Calculus for τ -measurable operators

In this section we present a functional calculus for τ -measurable operators and for a class of Borel measurable functions such that the operators obtained by the functional calculus are τ -measurable. In [Dix70] Dixon developed a functional calculus for normal elements of a GB*-algebra. Since $\widetilde{\mathcal{M}}$ is a GB*-algebra, this functional calculus can be applied to

normal elements of $\widetilde{\mathcal{M}}$, and will yield an operator in $\widetilde{\mathcal{M}}$. The usual Borel functional calculus for normal operators ([KR83] 5.6.26) can be applied to operators in $\widetilde{\mathcal{M}}$, but will in general not yield an operator in $\widetilde{\mathcal{M}}$ again. In this section we show that the two functional calculi coincide for the class $C_1(\sigma(S))$ of functions considered in the Dixon functional calculus, and then extend the calculus to the larger class $\mathcal{D}_u(S)$ of functions, for operators $S \in \widetilde{\mathcal{M}}^{sa}$.

Definition 4.2.1 *Let S be a normal element of the GB*-algebra $\widetilde{\mathcal{M}}$. Let $C_1(\sigma(S))$ be the set of all continuous complex valued functions f on $\sigma(S)$ (see Definition 1.4.1) such that for some non negative integer n (depending on f), the function*

$$\lambda \rightarrow \frac{f(\lambda)}{(1 + |\lambda|^2)^n}$$

extends to a bounded, continuous function on the whole of $\sigma(S) \cup \{\infty\}$.

We note from Definition 4.1.6 that in the case where $S \in \widetilde{\mathcal{M}}^{sa}$, $C_1(\sigma(S)) \subseteq \mathcal{D}_u(S)$.

Let $S \in \widetilde{\mathcal{M}}^{sa}$. Let \mathfrak{F} be a class of Borel measurable functions as in Definition 4.1.6, and $D_{\mathfrak{F}}(S)$ be the corresponding *-subalgebra of $\mathcal{B}_{\mathcal{M}}(S)$. For $f \in D_{\mathfrak{F}}(S)$, there is an $h \in \mathfrak{F}$ such that $\sup_{\lambda \in \sigma(S)} \frac{|f(\lambda)|}{h(\lambda)} < \infty$ and $h(S) \in \widetilde{\mathcal{M}}$. Let $g(\lambda) = \frac{f(\lambda)}{h(\lambda)}$. Then g is a bounded Borel function, and we can define $g(S)$ using the bounded Borel functional calculus, and $g(S)$ is bounded and affiliated with \mathcal{M} , hence $g(S) \in \mathcal{M}$. Since $D_{\mathfrak{F}}(S) \subseteq \mathcal{B}_{\mathcal{M}}(S)$, $f(S) \in \widetilde{\mathcal{M}}$. Since the Borel functional calculus is an algebra homomorphism,

$$f(S) = (hg)(S) = \overline{h(S)g(S)} = h(S)g(S) \quad (4.2)$$

since $h(S)$ is closed and $g(S)$ is bounded.

In the case where $\mathfrak{F} = \{(1 + \lambda^2)^m : m \in \mathbb{N}\}$, and $h \in \mathfrak{F}$, with $h(\lambda) = (1 + \lambda^2)^m$, $h(S)$ coincides with $(I + S^2)^m$, obtained using ordinary sums and products, by [KR83], Proposition 5.6.35. Using the functional calculus developed by Allan and Dixon in [All67], [All65] and [Dix70] for GB*-algebras, we can define $f(S)$ for a normal operator S in the GB*-algebra $\widetilde{\mathcal{M}}$ and a function $f \in C_1(\sigma(S))$. For such functions, $f(S)$ is defined by

$$f(S) = g(S)(1 + S^*S)^m, \quad (4.3)$$

where $g(\lambda) = \frac{f(\lambda)}{(1+|\lambda|^2)^m}$ and $g(S)$ is defined using a functional calculus for bounded continuous functions. Since every $f \in C_1(\sigma(S))$ is a Borel function, we have two ways to define $f(S)$: using the Borel functional calculus, or using the Allan-Dixon functional calculus.

For $S \in \widetilde{\mathcal{M}}^{sa}$, equation 4.3 becomes

$$f(S) = g(S)(I + S^2)^n. \quad (4.4)$$

By comparing equation 4.2, with $h(S) = (I + S^2)^n$, and equation 4.4, we see that it will follow that the two methods give the same result if $g(S)$ defined using the bounded Borel functional calculus coincides with $g(S)$ defined using Allan-Dixon functional calculus for bounded continuous functions.

We show below that indeed, the two functional calculi coincide on $C_1(\sigma(S))$.

Lemma 4.2.2 (*[Dix71], Proposition 2.4*) *Let \mathcal{A} be a set of closed, densely defined operators on a Hilbert space H which is a $*$ -algebra under strong sum and strong multiplication, and let $\mathcal{A}_b = B(H) \cap \mathcal{A}$ be a von Neumann algebra. Then $(I + T * T)^{-1} \in \mathcal{A}$ for every $T \in \mathcal{A}$ if and only if $T\eta\mathcal{A}_b$ for every $T \in \mathcal{A}$.*

Lemma 4.2.3 (*adapted from [Dix71], Lemma 2.1 and [Dix70], Lemma 4.10*) *Let \mathcal{C} be a maximal commutative $*$ -subalgebra of $\widetilde{\mathcal{M}}$. Then $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{M}$ is closed in the weak operator topology, and so is a von Neumann subalgebra of \mathcal{M} . Furthermore, \mathcal{C} is a GB^* -algebra with bounded part $\mathcal{C}_b = \mathcal{C}_0$.*

Theorem 4.2.4 (*[Dix70], Theorem 4.12, specialized to $\widetilde{\mathcal{M}}$.*)

Let S be a normal element of $\widetilde{\mathcal{M}}$. Then there is a unique $$ -isomorphism F of $C_1(\sigma(S))$ into $\widetilde{\mathcal{M}}$ such that:*

1. *if $u_0(\lambda) \equiv 1$, then $F(u_0) = I$.*
2. *if $u_1(\lambda) \equiv \lambda$, then $F(u_1) = S$*

3. for every maximal abelian $*$ -subalgebra \mathcal{C} of $\widetilde{\mathcal{M}}$ containing S , and every $f \in C_1(\sigma(S))$, $F(f) \in \mathcal{C}$; also $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{M}$ is a von Neumann algebra and if ϕ is the isomorphism of \mathcal{C}_0 onto $C(X)$, where X is an extremely disconnected compact Hausdorff space, then ϕ can be extended to an isomorphism from \mathcal{C} to $\mathcal{N}(X)$, and $\phi(F(f))(x) = f(\phi(S))(x)$ for every $x \in X$.

The isomorphism F is a functional calculus for S , and we write, as usual, $f(S)$ for $F(f)$.

Proposition 4.2.5 *Suppose \mathcal{M} is a semi-finite von Neumann algebra with a semi-finite faithful normal trace τ and let $S \in \widetilde{\mathcal{M}}$ be normal. Then the Borel functional calculus $f \rightarrow f(S)$ restricted to $\mathcal{B}_{\mathcal{M}}(S)$ is a $*$ -isomorphism of $\mathcal{B}_{\mathcal{M}}(S)$ into $\widetilde{\mathcal{M}}$ satisfying:*

1. $u_0(\lambda) \equiv 1$ implies $u_0(S) = I$;
2. $u_1(\lambda) \equiv \lambda$ implies $u_1(S) = S$;
3. for every maximal abelian subalgebra \mathcal{C} of $\widetilde{\mathcal{M}}$ such that $S \in \mathcal{C}$, and every $f \in \mathcal{B}_{\mathcal{M}}(S)$, $f(S) \in \mathcal{C}$;
4. for every proper von Neumann subalgebra \mathcal{A} of \mathcal{M} such that $S \in \widetilde{\mathcal{A}}$, and every $f \in \mathcal{B}_{\mathcal{A}}(S)$, $f(S) \in \widetilde{\mathcal{A}}$.

Proof: The definition of $\mathcal{B}_{\mathcal{M}}(S)$ ensures that $f(S) \in \widetilde{\mathcal{M}}$ if $f \in \mathcal{B}_{\mathcal{M}}(S)$. It follows from ([KR83], 5.6.26) that the Borel functional calculus is a $*$ -isomorphism with the first two properties listed above.

Now let \mathcal{C} be a maximal abelian subalgebra of $\widetilde{\mathcal{M}}$ such that $S \in \mathcal{C}$. Then by Lemma 4.2.2 and Lemma 4.2.3 we have that $S\eta\mathcal{C}_0$ and hence $f(S)\eta\mathcal{C}_0$. It follows that $f(S) \in \mathcal{N}(\mathcal{C}_0)$. We have, similarly, that for every $T \in \mathcal{C}$, $T \in \mathcal{N}(\mathcal{C}_0)$. Hence $\mathcal{C} \subseteq \mathcal{N}(\mathcal{C}_0) \cap \widetilde{\mathcal{M}}$. But $\mathcal{N}(\mathcal{C}_0) \cap \widetilde{\mathcal{M}}$ is an abelian $*$ -subalgebra of $\widetilde{\mathcal{M}}$, so by maximality of \mathcal{C} , $\mathcal{C} = \mathcal{N}(\mathcal{C}_0) \cap \widetilde{\mathcal{M}}$. It follows that $f(S) \in \mathcal{C}$.

The last statement is immediate from the definition of $\mathcal{B}_{\mathcal{A}}(S)$.

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We note that if $S \in \widetilde{\mathcal{M}}^{sa}$, we may replace $\mathcal{B}_{\mathcal{M}}(S)$ and $\mathcal{B}_{\mathcal{A}}(S)$ in the proposition above by $\mathcal{B}_b(\sigma(S))$. If $0 \leq S \in \widetilde{\mathcal{M}}_0$ and $\tau(E_0(S)) < \infty$, then it follows from Proposition 3.6.4 that in (4) in the above proposition we can take \mathcal{A} to be \mathcal{A}_S , the von Neumann subalgebra generated by I and the spectral projections of S .

Corollary 4.2.6 *Let $S \in \widetilde{\mathcal{M}}$ be normal. Then the Allan-Dixon functional calculus and the Borel functional calculus coincide for functions in $C_1(\sigma(S))$.*

Proof: It follows from the definition of the Borel functional calculus (see ([KR83], 5.6.25, 26)) and (3) above that it satisfies the conditions of Theorem 4.2.4. The result follows by uniqueness.

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4.3 Spectral Mapping Theorems for elements of $\widetilde{\mathcal{M}}_0$

In this section we consider spectral mapping theorems for positive operators in $\widetilde{\mathcal{M}}_0$.

Let S be an (unbounded) self adjoint operator and f is a real valued Borel function on $\sigma(S)$. Then

$$\sigma(f(S)) = \overline{f(\sigma(S))}.$$

([DS71], XII.2.9, [DdP87], Proposition 6.11). When $0 \leq S \in \widetilde{\mathcal{M}}_0$ and f is a positive increasing continuous function which vanishes at the origin, we can improve on the above spectral mapping theorem. We also give a spectral mapping theorems for the spectrum $\sigma_{\widetilde{\mathcal{M}}}(S)$ in $\widetilde{\mathcal{M}}$ of such an operator.

Lemma 4.3.1 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$ and let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function such that $f(0) = 0$. Then $0 \leq f(S) \in \widetilde{\mathcal{M}}_0$.*

Proof: $\mu_\infty(S) = 0$ since $S \in \widetilde{\mathcal{M}}_0$. By Proposition 4.1.1, we have that $f(\mu_t(S)) = \mu_t(f(S))$ so that $\mu_\infty(f(S)) = f(\mu_\infty(S)) = f(0) = 0$ by continuity of f . Δ

The proof of the following special case of the spectral mapping theorem depends on the properties of decreasing rearrangements.

Proposition 4.3.2 *Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$ and let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function such that $f(0) = 0$. Then*

$$\sigma(f(S)) = f(\sigma(S)).$$

Proof: It follows from Proposition 4.1.3 that $f(S) \in \widetilde{\mathcal{M}}$. By Lemma 4.3.1, $0 \leq f(S) \in \widetilde{\mathcal{M}}_0$. By Proposition 3.9.1 we have that $\sigma(S) = \overline{\{\mu_t(S) : t > 0\}}$ so that

$$\begin{aligned} f(\sigma(S)) &= f(\overline{\{\mu_t(S) : t > 0\}}) \\ &\subseteq \overline{\{f(\mu_t(S)) : t > 0\}} \text{ since } f \text{ is continuous} \\ &= \overline{\{\mu_t(f(S)) : t > 0\}} \text{ by Proposition 4.1.1} \\ &= \sigma(f(S)). \end{aligned}$$

Thus $f(\sigma(S)) \subseteq \sigma(f(S))$.

For the other inclusion we have first that

$$\begin{aligned} \{\mu_t(f(S)) : t > 0\} &= \{f(\mu_t(S)) : t > 0\}, \quad \text{by Proposition 4.1.1} \\ &= f(\{\mu_t(S) : t > 0\}) \\ &\subseteq f(\overline{\{\mu_t(S) : t > 0\}}), \end{aligned}$$

so that

$$\{\mu_t(f(S)) : t > 0\} \subseteq \overline{f(\{\mu_t(S) : t > 0\})} = f(\sigma(S)). \quad (4.5)$$

We want to show that $\overline{\{\mu_t(f(S)) : t > 0\}} \subseteq \overline{f(\{\mu_t(S) : t > 0\})}$. Now suppose

$$0 \neq \lambda \in \overline{\{\mu_t(f(S)) : t > 0\}} \setminus \{\mu_t(f(S)) : t > 0\},$$

then $\lambda = \lim_{t \rightarrow t_0^-} \mu_t(f(S))$ where t_0 corresponds to a jump discontinuity for $\mu_t(f(S))$. We are taking the left limit here since the decreasing rearrangement is right continuous and a λ not in the range of $\mu_t(f(S))$ but in the closure $\overline{\{\mu_t(f(S)) : t > 0\}}$ can only correspond to a

left limit. Now

$$\begin{aligned}
\lambda &= \lim_{t \rightarrow t_0^-} \mu_t(f(S)) \\
&= \lim_{t \rightarrow t_0^-} f(\mu_t(S)) \text{ by Proposition 4.1.1} \\
&= f(\lim_{t \rightarrow t_0^-} \mu_t(S)) \text{ by continuity of } f \\
&= f(\alpha), \text{ for some } \alpha \in \overline{\{\mu_t(S) : t > 0\}}.
\end{aligned}$$

Hence $\lambda \in f(\overline{\{\mu_t(S) : t > 0\}})$ and consequently, together with the inclusion (4.5), we have

$$\sigma(f(S)) = \overline{\{\mu_t(f(S)) : t > 0\}} \subseteq f(\overline{\{\mu_t(S) : t > 0\}}) = f(\sigma(S)).$$

For the case $\lambda = 0$, we observe that the zero element lies in $\sigma(f(S))$ since the decreasing rearrangements of elements of $\widetilde{\mathcal{M}}_0$ decrease to zero and $\sigma(f(S))$ is closed. Also since $f(0) = 0$, we have that $0 \in f(\sigma(S))$. Hence $\sigma(f(S)) = f(\sigma(S))$. This completes the proof.

△

Proposition 4.3.3 *Let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous increasing function such that $f(0) = 0$. Suppose $0 \leq S \in \widetilde{\mathcal{M}}_0$. Then*

$$f(\sigma_{\widetilde{\mathcal{M}}}(S)) \subset \sigma_{\widetilde{\mathcal{M}}}(f(S)).$$

Furthermore, if f is strictly increasing, then $f(\sigma_{\widetilde{\mathcal{M}}}(S)) = \sigma_{\widetilde{\mathcal{M}}}(f(S))$.

Proof: We have by Propositions 3.9.3 and 3.7.4 that

$$\sigma_{\widetilde{\mathcal{M}}}(S) = \sigma_p(S) \cup \sigma_e(S) = \{\lambda \in \mathbb{R}^+ : \mu\{t > 0 : \mu_t(S) = \lambda\} > 0\} \cup \{0\}.$$

If $0 \leq S \in \widetilde{\mathcal{M}}_0$ then $f(S) \in \widetilde{\mathcal{M}}_0$ by Lemma 4.3.1. If $\mu_t(S)$ is constant on some interval, then it follows that $\mu_t(f(S)) = f(\mu_t(S))$ is constant on that interval. Now we have

$$\begin{aligned}
f(\sigma_{\widetilde{\mathcal{M}}}(S)) &= \{f(\delta) \in \mathbb{R}^+ : \mu\{t > 0 : \mu_t(S) = \delta\} > 0\} \cup \{0\} \\
&\subseteq \{\lambda \in \mathbb{R}^+ : \mu\{t > 0 : f(\mu_t(S)) = \lambda\} > 0\} \cup \{0\} \\
&= \{\lambda \in \mathbb{R}^+ : \mu\{t > 0 : \mu_t(f(S)) = \lambda\} > 0\} \cup \{0\} \\
&= \sigma_{\widetilde{\mathcal{M}}}(f(S)).
\end{aligned}$$

If further f is strictly increasing, then we have

$$\begin{aligned}
\sigma_{\widetilde{\mathcal{M}}}(f(S)) &= \{\lambda \in \mathbb{R}^+ : \mu\{t > 0 : \mu_t(f(S)) = \lambda\} > 0\} \cup \{0\} \\
&= \{\lambda \in \mathbb{R}^+ : \mu\{t > 0 : f(\mu_t(S)) = \lambda\} > 0\} \cup \{0\} \\
&= \{f(\delta) \in \mathbb{R}^+ : \mu\{t > 0 : f(\mu_t(S)) = f(\delta)\} > 0\} \cup \{0\} \\
&= \{f(\delta) \in \mathbb{R}^+ : \mu\{t > 0 : \mu_t(S) = \delta\} > 0\} \cup \{0\}, \text{ since } f \text{ is injective} \\
&= f(\sigma_{\widetilde{\mathcal{M}}}(S)) \text{ since also } f(0) = 0.
\end{aligned}$$

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The above inclusion can be proper when f is not strictly increasing, as can be seen in the following example:

Example 4.3.4 Let

$$g(t) = \begin{cases} \frac{1}{t}, & 0 < t \leq 1 \\ 1, & 1 \leq t \leq 2 \\ 0, & t > 2 \end{cases}$$

Then $0 \leq g \in (\widetilde{L}_\infty(\mathbb{R}^+, \mathcal{B}, m))_0$. Let M_g be the operator of multiplication by g . Let

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t \leq 2 \\ t - 1, & 2 < t \leq 3 \\ 2, & 3 < t \leq 4 \\ t - 2, & t > 4 \end{cases} .$$

Then f is a positive continuous increasing function, which is not strictly increasing, and

$$(f \circ g)(t) = \begin{cases} \frac{1}{t} - 2, & 0 < t \leq \frac{1}{4} \\ 2, & \frac{1}{4} \leq t \leq \frac{1}{3} \\ \frac{1}{t} - 1, & \frac{1}{3} \leq t \leq \frac{1}{2} \\ 1, & \frac{1}{2} \leq t \leq 2 \\ 0, & t > 2 \end{cases}$$

so that $f \circ g \in (\widetilde{L_\infty})_0$. Now since $f(M_g) = M_{f \circ g}$, we can show that $\sigma_{\widetilde{L_\infty}}(M_{f \circ g}) = \{0, 1, 2\}$, where we have used Propositions 3.1.13 and 3.1.2. However, $f(\sigma_{\widetilde{L_\infty}}(M_g)) = \{0, 1\}$, so that we have a proper inclusion.

△

4.4 Functional Calculus for operators arising in the Schmidt decomposition

In this section we consider the functional calculus for the operator S_0 arising in the Schmidt decomposition. In what follows S will always denote a positive element of $\widetilde{\mathcal{M}}$ with spectral family $\{E_\lambda(S) : \lambda \in \mathbb{R}\}$ and $S_0 = (I - E_{\lambda_0}(S))S$. We recall from section 3.5 that the spectral resolution of S_0 is given by $E_\lambda(S_0) = E_\lambda(S)(I - E_{\lambda_0}(S)) + E_{\lambda_0}(S)$ for $\lambda \geq 0$ and $E_\lambda(S_0) = 0$ for $\lambda < 0$. The Schmidt decomposition is given by

$$S_0 = (I - E_{\lambda_0}(S))S = \int_{\mathbb{R}} \mu_t(S) d\tilde{E}_t(S),$$

where

$$\tilde{E}_t(S) = \begin{cases} 0 & t < 0 \\ (I - E_{\lambda_0}(S))(I - E_{\mu_t(S)_-}(S)) & 0 \leq t < \alpha_0 \\ I - E_{\lambda_0}(S) & t \geq \alpha_0, \end{cases}$$

where $\lambda_0 = \lim_{t \rightarrow \infty} \mu_t(S)$ and $\alpha_0 = \inf\{t > 0 : \mu_t(S) = \lambda_0\}$.

Lemma 4.4.1 *Let $f : [0, \infty) \rightarrow [0, \infty)$ be a Borel function. Then*

$$((f(S))_0) = (I - E_{f(\lambda_0)}(f(S)))f(S) \text{ and } f(S_0) = (I - E_{\lambda_0}(S))f(S).$$

Proof: The first statement follows directly from the definition of $(f(S))_0$. For the second, we have

$$\begin{aligned}
f(S_0) &= \int_0^\infty f(t) dE_t(S_0) \\
&= \int_0^\infty f(t) d(E_t(S)(I - E_{\lambda_0}(S))) \\
&= \left(\int_0^\infty f(t) dE_t(S) \right) (I - E_{\lambda_0}(S)) \\
&= f(S)(I - E_{\lambda_0}(S)) \\
&= (I - E_{\lambda_0}(S))f(S).
\end{aligned}$$

△

Lemma 4.4.2 *Let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous strictly increasing function such that $f(0) = 0$. Then*

1. $\lim_{t \rightarrow \infty} \mu_t(f(S)) = f(\lambda_0)$;
2. $\inf\{t > 0 : \mu_t(f(S)) = f(\lambda_0)\} = \alpha_0$;
3. $E_{f(\lambda_0)}(f(S)) = E_{\lambda_0}(S)$;
4. $\tilde{E}_t(f(S)) = \tilde{E}_t(S)$ for all $t \geq 0$.

Proof: (1)

$$\begin{aligned}
\lim_{t \rightarrow \infty} \mu_t(f(S)) &= \lim_{t \rightarrow \infty} f(\mu_t(S)), \text{ by Proposition 4.1.1} \\
&= f(\lim_{t \rightarrow \infty} \mu_t(S)), \text{ by continuity of } f \\
&= f(\lambda_0).
\end{aligned}$$

(2) $\inf\{t > 0 : \mu_t(f(S)) = f(\lambda_0)\} = \inf\{t > 0 : f(\mu_t(S)) = f(\lambda_0)\} = \alpha_0$,

since f is strictly increasing.

(3) By the change of measure principle and the fact that f is strictly increasing and $f(0) = 0$, we have that $E_{f(\lambda_0)}(f(S)) = E_{(0, f(\lambda_0)]}(f(S)) = E_{f^{-1}(0, f(\lambda_0)]}(S) = E_{(0, \lambda_0]}(S) = E_{\lambda_0}(S)$.

(4) We have that

$$\tilde{E}_t(f(S)) = \begin{cases} I - E_{f(\lambda_0)}(f(S)) & t \geq \alpha_0 \\ (I - E_{f(\lambda_0)}(f(S)))(I - E_{\mu_t(f(S))_-}(f(S))) & 0 < t < \alpha_0 \\ 0 & t \leq 0 \end{cases}$$

Using the fact that $\mu_t(f(S)) = f(\mu_t(S))$, we show first that $E_{\mu_t(f(S))_-}(f(S)) = E_{\mu_t(S)_-}(S)$.

$$\begin{aligned} E_{\mu_t(f(S))_-}(f(S)) &= \lim_{q \uparrow \mu_t(f(S))} E_q(f(S)) \\ &= \lim_{q \uparrow f(\mu_t(S))} E_{f^{-1}([0,q])}(S) \text{ (change of measure principle)} \\ &= \lim_{q \uparrow f(\mu_t(S))} E_{([0, f^{-1}(q)])}(S) \text{ since } f \text{ is strictly increasing} \\ &= \lim_{\delta \uparrow \mu_t(S)} E_{[0,\delta]}(S) \\ &= E_{\mu_t(S)_-}(S) \end{aligned}$$

It now follows from (1), (2) and (3) that $\tilde{E}_t(f(S)) = \tilde{E}_t(S)$ for every $t \geq 0$. Δ

Proposition 4.4.3 *Let $f : [0, \infty) \rightarrow [0, \infty)$ be a continuous strictly increasing function such that $f(0) = 0$. Then*

$$f(S_0) = (f(S))_0 = \int_0^{\alpha_0} f(\mu_t(S_0)) d\tilde{E}_t(S).$$

Proof: We have that

$$\begin{aligned} (f(S))_0 &= (I - E_{f(\lambda_0)}(f(S)))f(S) \quad (\text{by Lemma 4.4.1}) \\ &= (I - E_{\lambda_0}(S))f(S), \quad (\text{by Lemma 4.4.2 (3)}) \\ &= f(S_0) \quad (\text{by Lemma 4.4.1}). \end{aligned}$$

Hence, by Lemma 4.4.2 (4) and Proposition 4.1.1

$$\begin{aligned} f(S_0) &= (f(S))_0 \\ &= \int_0^\infty \mu_t(f(S)) d\tilde{E}_t(f(S)) \\ &= \int_0^\infty \mu_t(f(S)) d\tilde{E}_t(S) \\ &= \int_0^\infty f(\mu_t(S)) d\tilde{E}_t(S) \\ &= \int_0^{\alpha_0} f(\mu_t(S)) d\tilde{E}_t(S) \\ &= \int_0^{\alpha_0} f(\mu_t(S_0)) d\tilde{E}_t(S). \end{aligned}$$

△

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