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Norm-determining Subspaces

by

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INTRODUCTION

This thesis will centre on the concept of norming subspaces in the dual of a Banach space. We shall consider a weakly-dense subspace V of E', and the natural norm it generates on E. If this new norm is equivalent to the original norm on E, then V is called norming or norm-determining.

In 1948 Dixmier published his results [1 3] on norming subspaces i.e. subspaces of characteristic greater than zero. Since that time writers have given several alternative proofs [19, 31, 34, 44] for many of his theorems and gone on to prove new and varied results.

In this thesis, no new results are stated, although Corollary 9.1 is a slight generalization of Theorem 9.3 . The new proofs given are : Theorems 2.2, 2.4, 4.1, 4.2, 4.6, 7.1, and 9.3. In example 2 of Section 2 and Theorem 4.4 the proofs have been completed rather than changed.

The aim is to revise the subject of norming subspaces, unify it and point out its relevance to other areas of analysis.

In Section 2 we follow Dixmier's reasoning to arrive at the concept of the characteristic of a subspace. This is done by enlarging its weak sequential closure. Discussion in Section 3 hinges on both the theory of analytic functions and the theory of two-norm spaces. These topics are connected by an important characterization of the characteristic. Section 4 contains further investigation of this concept and leads up to a characterization of reflexive Banach spaces. So far, the setting has been that of a Banach space though many of the results are valid in normed linear spaces. In Section 5 the characteristic is discussed in a locally convex setting. This generalization is due to Krishnamurthy [31,32,33] . In Section 6 minimal subspaces are introduced in a locally convex setting, with special reference to Banach spaces. With all the necessary tools established, Section 7 provides an application of these to the question of spaces equivalent to the dual or bidual of a Banach space. The "dual" question was originally investigated independently by Dixmier [13] and Ruston [44]. In a manner reminiscent of Dixmier, Civin and Yood [7] investigated the "bidual" case. These two authors are also responsible for the definition of quasi - reflexivity discussed in Section 8. So many papers have appeared on this topic, e.g. [8, 10, 15, 19, 21, 33, 35, 40, 48 and 49] that we are obliged to restrict our discussion to those results which relate directly to results obtained within this thesis. Section 9 contains a completeness theorem [52] which serves to connect the characteristic with James's work on compactness, and so relates the former once more with a large section of analysis. The bibliography is in Section 10.

Notation : In general, we employ the Bourbaki notation. However, we draw attention to the following : Given a dual pair $<E,F>$ of linear spaces,

$\sigma(E,F)$: the topology of uniform convergence on the finite subsets of F, or the weak topology on E.

$\mu(E,F)$: the topology of uniform convergence on the absolutely convex weakly-compact subsets of F, or the Mackey topology on E.

$\beta(E,F)$: the topology of uniform convergence on the weakly-bounded subsets of F, or the strong topology on E.

$\gamma(E,F)$: the topology of uniform convergence on the strongly-bounded subsets of F.

Clearly, $\gamma(E,E') = \beta(E'',E')|_E$ i.e. the strong topology on E'' restricted to E. These topologies are all locally convex Hausdorff topologies. A linear space E with such a topology is called a "locally convex Hausdorff topological linear space". We abbreviate this to "l.c. space". We shall also use "iff" for "if, and only if".

The dual of a l.c.space E will be denoted by E', and the bidual $[E',\beta(E',E)]'$ by E''. For a subset V of E', V^o and V_o will denote the polars in E'' and E respectively. Other spaces in which polars are taken will be specified as they appear. Unless alternative symbols are explicitly named, S_E and $S_{E''}$ will denote the unit balls in E and E'' and S_i will denote the i'th multiple of the unit ball S_1 in E'.

The scalar field in question can be either the real or complex number field.

I should like to thank my supervisor, Dr. J.H.Webb, for his constant help in all aspects of this project, and for his permission to quote results from a paper of his which has not yet appeared, see [52] .

THE DIXMIER CHARACTERISTIC

Dixmier's concept [13] of the characteristic of a subspace developed naturally from the desire to generalise the following theorem [3;213] to the case of a non-separable Banach space.

<u>Theorem 2.1</u> If E is a separable Banach space, V a subspace of E', then every element of E' is the limit of a $\sigma(E',E)$-convergent sequence of elements from V iff $\exists$ a constant $k > 0$ such that for every x in E,

$$\sup_{\substack{f \in V \\ \|f\|=1}} |f(x)| \geq k.\|x\|$$

<u>Definition 2.1</u> If A is a subset of E', the dual of the Banach space E, we denote by $\underline{A}$ the set of limits of weakly convergent sequences from A, and call it the weak sequential closure of A.

The condition "$\underline{V} = E'$" in Theorem 2.1 is stronger than the condition "V is $\sigma(E',E)$-dense in E'". Mazurkiewicz [36] gives an example for which only the latter condition holds. In some topologies these two concepts do coincide e.g. metrizable topologies. For further results in this connection the reader is referred to papers by Webb [51] and Franklin [17].

To generalise Theorem 2.1 we expand the weak sequential closure of V :

<u>Definition 2.2</u> If A is a subset of E', let

$$A^* = \bigcup_{r \geq 0} \overline{A \cap S_r}$$

where $\overline{}$ denotes the $\sigma(E',E)$ closure. Now, if F is a bounded set in E', then $\exists\ r : F \subset S_r$ and then

$$\overline{A \cap F} \subset \overline{A \cap S_r} \subset A^*$$

thus $\underline{A} = \bigcup_F \overline{A \cap F}$ where the union is taken over all the bounded sets in E'.

<u>Theorem 2.2</u> (1) $\underline{A} \subset A^*$

(2) If E is separable, then $\underline{A} = A^*$.

<u>Proof</u>. (1) If $f \in \underline{A}$, then $\exists$ a sequence $\{f_i\}$ in A such that the f_i are $\sigma(E',E)$-convergent to f. So by [3;73], $\sup_i \| f_i \| = m < \infty$, and $f \in \overline{A \cap S_m} \subset A^*$

(2) Let $f \in A^*$. Then $\exists\ n : f \in \overline{A \cap S_n} = n\,\overline{A \cap S_1}$ Since E is separable, the unit ball S_1 in E' is metrisable [9;43]. So $\exists$ a sequence $\{f_i\}$ in $n(A \cap S_1)$ which is $\sigma(E',E)$-convergent to f. Hence $f \in \underline{A}$.

<u>Theorem 2.3</u> If V is a subspace of E', then

V is $\sigma(E',E)$-closed iff $V = V^*$.

<u>Proof</u>. If V is $\sigma(E',E)$-closed, then $V \cap S_1$ is too. $V \cap S_r = V \cap rS_1 = r\,(V \cap S_1)$ is also $\sigma(E',E)$ - closed. Hence $V^* = \bigcup \overline{V \cap S_r} = \bigcup V \cap S_r = V \cap \bigcup S_r$.

Since $E' = \bigcup Sr$, it follows that $V = V^*$. Conversely, if $V = V^*$, then $\overline{V \cap S_1} \subset V$. But $V \cap S_1 \subset \overline{V \cap S_1}$, thus $V \cap S_1 = \overline{V \cap S_1}$. In a Banach space this is equivalent to V being $\sigma(E',E)$-closed [4;74].

<u>Definition 2.3</u> A set $B \subset E'$ is said to be bw*- closed or almost closed if $B \cap S_1$ is $\sigma(E',E)$ - closed in E'.

The terms bw*- closed and almost closed appear in [9] and [22] respectively; in general the bw*- and weak - closures of B do not coincide. However, for convex sets B in a Banach space, they do. So an alternative statement of Theorem 2.3 is :

$$V = V^* \quad \text{iff} \quad V \text{ is bw*- closed}$$

<u>Theorem 2.4</u> For a subspace V of E', $V^* = E'$ iff $\overline{V \cap S_1}$ contains an S_r, for $r > 0$.

<u>Proof</u>. Suppose $\exists\, r > 0 : \overline{V \cap S_1}$ contains S_r. Then $\overline{V \cap S_r} \supset S_k$, for some k. Thus $V^* = \bigcup \overline{V \cap S_r} \supset S_k$. Since V^* is an absorbent subspace, $V^* = E'$. Conversely, if $V^* = E' = \bigcup_r \overline{V \cap S_r}$, then $\overline{V \cap S_1}$ is absorbent. Since E is normed, its strong dual $E', \beta(E',E)$ is complete and so is barrelled. Now $\overline{V \cap S_1}$ is a barrel, and it follows that $\exists\, r>0 : \overline{V \cap S_1} \supset S_r$.

The above is the generalization of Theorem 2.1 which suggests the following :

<u>Definition 2.4</u> If V is a subspace of E', we define the characteristic of V, denoted by $D(V)$, as

$$D(V) = \sup\{ r \geq 0 : \overline{V \cap S_1} \text{ contains } S_r \}$$

If the weak closure of $V \cap S_1$ contains a sequence of balls $\{S_{r_i}\}$, where r_i approaches r from below, then it contains S_r. So $D(V)$ is well-defined. In fact, since $\overline{V \cap S_1}$ is contained in S_1, we have that $0 \leq D(V) \leq 1$. Furthermore, if $\overline{V \cap S_1} = S_r$ for some r, then $r = 1$ as $\overline{V \cap S_1}$ contains elements of norm one.

The concept of the characteristic of a subspace V can be readily generalised : Consider $\bar{V}$ the weak closure of V. The characteristic of V relative to $\bar{V}$ can be defined as the largest number r such that $V \cap S_1$ is weakly dense in $\bar{V} \cap S_r$. Without difficulty, many of the characterizations of $D(V)$ which appear in Sections 4 and 5 can be extended to this case [13] .

To obtain Theorem 2.4 we extended $\underline{V}$ to V^*. However, M^c Williams [35] defined a number which he called $\mu(V)$ which depended on $\underline{V}$ only. Fleming investigated the relationship between $D(V)$ and $\mu(V)$ and proved that [15] If E is separable and $\underline{V} = E'$, then $D(V) = \frac{1}{\mu(V)}$. For weaker conditions Fleming obtained correspondingly weaker relations.

AN ALTERNATIVE APPROACH

In an investigation of extremum properties of analytic functions, Rogosinski and Shapiro [42] generalised the problem to normed linear spaces:

Let W be a subspace of the normed linear space E. For $x \in E$ and f a continuous linear functional on W, we define

$$\| f \|_W = \sup_x | f(x) | \qquad \ldots.(1)$$

where the sup is taken over all x in $W \cap S_E$. Dually, if V is a subspace of E', define for $x \in E$

$$\| x \|_V = \sup_f | f(x) | \qquad \ldots.(2)$$

where the sup is taken over all f in $V \cap S_1$. Now (1) and (2) define seminorms weaker than the original norms, which are norms iff W and V are weakly dense in E and E', respectively.

Theorem 3.1 Let W be a subspace of E. If $f \in E'$, then

$$\| f \|_W = \inf_h \| f+h \|$$

where the inf is taken over all h in W° and is in fact attained.

Proof. By the Hahn-Banach theorem, every continuous linear functional f on W can be extended to a continuous linear functional g on E. Then $f - g$ is in W°. Moreover, there is at least one such g for which $\|f\|_W = \|g\|$. For fixed f in E' $\{g: g=h+f, h\epsilon W^\circ\} = \{g: g(w)=f(w), w\epsilon W\}$. So the statement of Theorem 3.1 [6] is equivalent to the original theorem in [42].

Using again the Hahn - Banach theorem, a dual version of Theorem 3.1, in which the sup instead of the inf is attained, was proved by Bonsall [6;Theorem 2].

<u>Theorem 3.2</u> If $x_\circ \epsilon E$, then

$$\sup_f |f(x_\circ)| = \inf_{y\epsilon W} \|x_\circ + y\|$$

where the sup is taken over all f in $W^\circ \cap S_1$ and is in fact attained.

However, Theorem 3.2 is not a completely dualized version of Theorem 3.1, as Bonsall himself observed. With this in mind, it is natural to ask for what subspaces V of E' it is true that for all x in E,

$$\|x\|_V = \sup_{\substack{f\epsilon V \\ \|f\|\leqslant 1}} |f(x)| = \inf_{y\epsilon V_\circ} \|x+y\|$$

Now, if V is $\sigma(E',E)$ - dense in E', then $V_\circ = (0)$ and we ask for which V is $\|x\|_V = \|x\|$. The initial problem can always be reduced to the latter by considering the quotient space $E/V_\circ$. Bonsall's question had been answered in 1948 by Dixmier who proved the next theorem.

<u>Theorem 3.3</u> Let V be a subspace with $D(V) = r$. If

$$s = \inf_{\substack{x\in E \\ x\neq 0}} \|x\|_V . \|x\|^{-1}$$

then $r = s$.

<u>Proof</u>. Since $D(V)=r$, $\overline{V \cap S_1}$ contains S_r. Let $x\in E$, $x\neq 0$ then $\exists g\in E'$, $\|g\| = r$ and $g(x) = r.\|x\|$. Consider the weak neighbourhood of g defined by $|f(x) - g(x)| \leq \varepsilon$. Call it U. Then $\exists$ $f\in V \cap S_1$ such that $f\in U$. Thus $|f(x) - r\|x\|| \leq \varepsilon$. So $\sup_{\substack{x\in E \\ x\neq 0}} |f(x)| . \|x\|^{-1} \geq r$.
Hence $s \geq r$.

Conversely, let g be such that $\|g\| < s$, and $|g(x)| < s.\|x\|$ for all x, then $|g(x)| < \sup_{\substack{f\in V \\ \|f\|=1}} |f(x)|$
i.e. for every x $\exists$ $f\in V\cap S_1$ such that
$|g(x)| \leq |f(x)|$. It follows that $g \in \overline{V \cap S_1}$, for if not, since $\overline{V \cap S_1}$ is weakly - closed and absolutely convex,

$$\exists\ x \in E \qquad |g(x)| > \sup_{\substack{f\in V \\ \|f\|\leq 1}} |f(x)| \qquad \text{by [41;30]}$$

Hence $\overline{V \cap S_1} \supset \{g: \|g\| < s\}$. So $\overline{V \cap S_1} \supset S_s$ and $r \geq s$.

Now $\|x\|_V = \|x\|$ for all x in E iff $D(V) = 1$, by the above theorem. The terms "duxial" [43,44] and "absolutely total" [56] are also used to describe such subspaces. If V is such that for every x in E, $\exists$ $f \in V$: $|f| = 1$ and $f(x) = \|x\|$; then V is called a "d-manifold" [56]. Obviously, every d - manifold V has $D(V) = 1$.

For a full discussion of d - manifolds the reader is referred to a paper by Wilder [56] .

Again by Theorem 3.3 $D(V) > 0$ iff the norms $\|x\|_V$ and $\|x\|$ are equivalent for all x in E. Such subspaces are also called norming [54,55] , norm - determining [50,56] , or fundamental [1] .

Definition 3.1 A subspace V of E', the dual of the Banach space E, is called strictly norming [2,54,55] if every $\sigma(E,V)$ - bounded subset of E is $\sigma(E,E')$ - bounded.

Theorem 3.4 A strongly - closed norming subspace is strictly norming, [54,55]

Proof. Let A be a $\sigma(E,V)$ - bounded subset of E, i.e. for all f in V $\sup_{x\in A} |f(x)| < \infty$. Since V is a Banach space $\sup \|x\|_V = k < \infty$. Now $\exists\ c > 0 : c.\|x\| \leq \|x\|_V$ for all x in A. So for arbitrary x_1 and x_2 in A $\|x_1 - x_2\| \leq \|x_1\| + \|x_2\| \leq \frac{2k}{c}$. Hence A is bounded.

Although this result is not true for arbitrary norming subspaces [54], we do have

Theorem 3.5 Any strictly norming subspace is norming, [54,55] .

Proof. Suppose $V \subset E'$ is strictly norming but not norming. For every integer k, $\exists\ x_k$ in E : $\|x_k\| > k$, $\|x_k\|_V < \frac{1}{k}$

It follows that the sequence $\{x_k\}$ is $\sigma(E,V)$ - bounded but not norm bounded.

For further results on norming subspaces and their role in the theory of two - norm spaces see for example [2, 38, 55]. We mention the following question posed by Orlicz and Pták: let $X, \|\ \|^*, \|\ \|$ be a two - norm space.

Let Z : dual of $X, \|\ \|$

Z^* : dual of $X, \|\ \|^*$

Z' : $\{ f \in Z : f$ is $\|\ \|^*$-continuous on the $\|\ \|$ - bounded sets.$\}$

Given a Banach space $X, \|\ \|$ and a closed subspace Y of the dual of X, can we find $\|\ \|^*$ ON X weaker than the original norm such that $Z' = Y$?

The answer, provided in [38] is negative.

We would allude briefly to the following two norms :

(1) That derived by Goldberg [18] with properties resembling that of $\|\ \|_V$: If E and F are Banach spaces and T a densely defined closed operator $T : E \to F$, then for $y \in F$ we define

$$\|y\|_1 = \sup_{y'} \frac{|\langle y, y' \rangle|}{\|y'\| + \|T'y'\|}$$

where the sup is taken over all y', $y' \neq 0$ in the domain of T', the adjoint of T. It can be shown that $\|y\|_1 \leq \|y\|$ and $(F, \|\ \|_1)' = D(T')$. (See Theorem 4.1)

(2) The norm "$|x(M)|$" which occurs in the general theory of Banach algebras [37;193] . This norm is again weaker than the original norm of the algebra.

FURTHER PROPERTIES.

We proceed with our investigation of the norm $\| \ \|_V$ and use our results to establish properties of $D(V)$.

Theorem 4.1 If E is a normed linear space with unit ball B, and V a subspace of E', then

$$V^* = (E, \| \ \|_V)'$$

Proof.

$$\begin{aligned}
(E, \| \ \|_V)' &= \{ f \in E' : \sup_{\|x\|_V \leq 1} |f(x)| \leq k, \text{ some } k \} \\
&= \bigcup_{n=1}^{\infty} n\{ f \in E' : \sup_{\|x\|_V \leq 1} |f(x)| \leq 1 \} \\
&= \bigcup_{n=1}^{\infty} n\, B_V{}^\circ \quad \text{for } B_V = \{ x : \|x\|_V \leq 1 \} \\
&= \bigcup_{n=1}^{\infty} n\, (B^\circ)_\circ{}^\circ \quad \text{(first polar in } V) \\
&= \bigcup_{n=1}^{\infty} n\overline{B}^\circ \quad (\sigma(E',E)\text{-closure}) \\
&= \bigcup_{n=1}^{\infty} n\, \overline{B^\circ \cap V} \quad (\text{polar in } E',\ \sigma(E',E)\text{-closure.}) \\
&= \bigcup_{n=1}^{\infty} n\, \overline{V \cap S_1} \\
&= V^*
\end{aligned}$$

Corollary 4.1 For a Banach space E,

(1) $(E, \| \ \|_V)' = E'$ iff $D(V) > 0$.

(2) $(E, \| \ \|_V)' = V$ iff V is $\sigma(E',E)$-closed

Hence, if V is $\sigma(E',E)$-dense in E',

$$(E, \| \ \|_V)' = V \quad \text{iff} \quad V = E'.$$

Proof. (1) Follows immediately from Theorems 2.4 and 4.1, and (2) follows from Theorem 2.3 .

Theorem 4.1 and its consequences are essentially due to Kerr [28] who proved his results in a seminormed space setting.

<u>Theorem 4.2</u> Given a Banach space E with unit ball B, let $B^{\sim}$ denote the $\sigma(E,V)$ - closure of B. Then,

(1) $\{x: \|x\|_V \leqslant 1\} = B^{\sim}$

(2) $1/D(V) = \sup_{x \in B^{\sim}} \|x\|$

<u>Proof.</u> (1)

$$\{x: \|x\|_V \leqslant 1\} = \bigcap_{\substack{f \in V \\ \|f\|=1}} \{x: |f(x)| \leqslant 1\}$$

$$= \bigcap_{S_1 \cap V} \{x: |f(x)| \leqslant 1\}$$

$$= \bigcap_{B^{\circ} \cap V} \{x: |f(x)| \leqslant 1\}$$

$$= B^{\circ}{}_{\circ} \text{ (polar in } V\text{, bipolar in } E)$$

$$= B^{\sim}$$

(2)

$$1/D(V) = \sup_{x \neq 0} \|x\| . \|x\|_V^{-1} \quad \text{by Theorem 3.3}$$

$$= \sup_{\|x\|_V = 1} \|x\|$$

$$\leqslant \sup_{\|x\|_V \leqslant 1} \|x\|$$

$$\leqslant \sup_{0 < \|x\|_V \leqslant 1} \|x\|$$

$$\leqslant \sup_{x \neq 0} \|x\| . \|x\|_V^{-1}$$

$$= 1/D(V)$$

thus $1/D(V) = \sup_{\|x\|_V \leqslant 1} \|x\| = \sup_{x \in B^{\sim}} \|x\|$

Corollary 4.2 For a subspace V of E',

(1) $D(V) > 0$ iff $\tilde{B}$ is bounded.

(2) $D(V) = 1$ iff B is $\sigma(E,V)$-closed.

Proof. $D(V) = 1$ iff $\|x\|_V = \|x\|$. Also , the latter condition implies that $\tilde{B} = B$. So B is $\sigma(E,V)$-closed. Conversely, if B is $\sigma(E,V)$-closed, then the two norms define the same unit balls, and so are equal.

The proof of Theorem 4.2 is essentially Dixmier's [13] , but in our proof we have replaced the use of the Hahn-Banach Theorem by that of polars. Corollary 4.2(2) was first noted by Petunin [39;Lemma1] who used it to prove a reflexivity criterion - see Theorem 4.7 .

Corollary 4.3 If E is embedded in its bidual, then $E \cap S_{E''}$ is $\sigma(E'',E')$-dense in $S_{E''}$.

Corollary 4.4 Let J_V denote the canonical map, $J_V : E \to V'$ where V' is the dual of the subspace V of E'. Then $J_V(S_E)$ is $\sigma(V',V)$-dense in $S_{V'}$, the unit ball of V'.

Proof. Follows from Corollary 4.3 by application of the Hahn-Banach Theorem.

Singer [46] proved Corollary 4.4, from which 4.3 follows trivially. In a later paper [47] he noted that these two results were equivalent. Fleming [16] has given a different proof of Corollary 4.4 using Smulian's compactness criterion [27;16.6].

We now give a characterization of $D(V)$ in terms of E'' :

<u>Theorem 4.3</u> For a subspace V of E',

$$D(V) = \inf_{\substack{x\in E, x\neq 0\\ z\in V^\circ}} \|x+z\| . \|x\|^{-1}$$

<u>Proof.</u> Let $x \in E$, $x \neq 0$ and $z \in V^\circ$, the functionals x and $x + z$ coincide on V and

$$\begin{aligned} \|x+z\| &= \sup_{f\in S_1} |f(x+z)| \\ &\geqslant \sup_{f\in V\cap S_1} |f(x+z)| \\ &= \sup_{f\in V\cap S_1} |f(x)| \\ &\geqslant \sup_{f\in S_r} |f(x)| \\ &= r . \|x\| \qquad \text{where } r = D(V). \end{aligned}$$

On the other hand, given $\varepsilon > 0$, $\exists\, x \in E$, $x \neq 0$ such that

$$\sup_{f\in V\cap S_1} |f(x)| . \|x\|^{-1} \leqslant D(V) + \epsilon$$

Now, $\exists\, x''$ in E'' $f(x'') = f(x)$ for all x in V

and
$$\begin{aligned} \|x''\| &= \sup_{f\in V\cap S_1} |f(x'')| \\ &= \sup_{f\in V\cap S_1} |f(x)| \\ &\leqslant \|x\| . (r+\epsilon) \end{aligned}$$

We have $x'' = x + z$, $z \in V^\circ$ and as ϵ is arbitrary, the result follows.

Theorem 4.3 is due to Dixmier [13]

<u>Theorem 4.4</u> For a $\sigma(E',E)$-dense subspace V of E', with $V^\circ \subset E''$, $V^* = E'$ iff $V^\circ \oplus E$ is strongly - closed.

<u>Proof.</u> Since V is $\sigma(E',E)$-dense in E', $V^\circ \cap E = (0)$. The map $V^\circ \oplus E \to E$ is onto, continuous (being a projection) and $V^\circ \oplus E$ is closed, hence complete. So, by the Open Map Theorem $\exists\ k < \infty$ $\|x\| \leq k.\|x+z\|$ for all x in E, z in V°. But by Theorem 4.3, $D(V) = \inf_{\substack{x\in E, x\neq 0\\ z\in V^\circ}} \|x+z\|.\|x\|^{-1} \geq 1/_k > 0$ and by Theorem2.4 $V^* = E'$.

Conversely, $V^* = E'$ iff $\inf_{\substack{x\in E, x\neq 0\\ z\in V^\circ}} \|x+z\|.\|x\|^{-1} \geq k > 0.$ So, $\|x+z\| \geq k.\|x\| > 0.$ Consider the map of $V^\circ \times E \to V^\circ \oplus E \subset E''$, defined by $(z,x) \to z+x$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in E and V° respectively. If $\|x_n+z_n\| \to 0$ then $x_n \to 0$. Thus $z_n \to 0$ i.e. $(z_n,x_n) \to 0$. So the map from E'' into $V^\circ \times E$ is continuous. As E is a Banach space, V° and E are strongly closed. Hence result.

If E is a Banach space, we can denote the natural norm on E' by $\|\ \|'$. More generally, if $|\ |$ is a norm on E, equivalent to the given norm of E, $\|\ \|$, then for f in E',

$$|f|' = \sup_{\substack{x\in E\\ x\neq 0}} |f(x)| \,.\, |x|^{-1}$$

is called the dual norm of $|\ |$. Clearly, this norm is equivalent to the natural norm on E'. However, every norm on E'

which is equivalent to $\| \|'$ is not a dual norm. Spaces for which this is true are reflexive. In fact, Williams [57] has shown that this property characterizes reflexive spaces. In this connection we have the following result which is due to Dieudonné [13].

Theorem 4.5 If V is a subspace of E' with $D(V) > 0$, then there exists a norm $| \, |$ on E such that $D(V) = 1$ in the new system of norms.

Proof. $B^{\sim}$, the $\sigma(E,V)$-closure of the unit ball B of E is closed, absolutely convex, bounded and absorbent by Corollary 4.2. So it defines a norm $| \, |$ on E, equivalent to $\| \|$ such that $B^{\sim} = \{x: |x| \leq 1\}$. Then by Theorem 4.2,

$$D(V) = \sup_{x \in B^{\sim}} |x| = 1.$$

So, if $D(V) > 0$ for $\| \|$, then $D(V) > 0$ for all norms equivalent to $\| \|$. Similarly, if $D(V) = 0$ for $\| \|$, then $D(V) = 0$ for all norms equivalent to $\| \|$. Hence, V takes on only two fundamental values : 0 and 1.

Examples. (1) If E is a normed space, then $D(E) = 1$ in E'':

$$D(E) = \inf_{\substack{x' \in E' \\ x' \neq 0}} \{ \sup_{\substack{x'' \in i(E) \\ \|x''\| \neq 0}} |\langle x', x'' \rangle| \, . \, \|x'\|^{-1} \} \quad \text{where } i: E \to E''$$

$$= \inf_{\substack{x' \in E' \\ x' \neq 0}} \{ \sup_{\substack{x \in E \\ \|x\| \leq 1}} |\langle x, x' \rangle| \, . \, \|x'\|^{-1} \}$$

but this $\sup |\langle x, x' \rangle| = \|x'\|$, and so $D(E) = 1$.

In particular, the sequence space $\varphi \subset l^\infty$ has $D(\varphi) = 1$.

(2) Clearly, if V has $D(V) > 0$, then V is $\sigma(E',E)$ - dense in E'. We now give Bonsall's example [6] of a $\sigma(E',E)$ dense subspace V with $D(V) \neq 1$.

Consider the space C_0, with dual l^1. Let

$$V = \{ x \in l^1 : x=(x_i) \text{ and } \sum_{i=1}^{\infty} x_i = 0 \}$$

and $z = (2,0,0,0,\ldots)$. Then $z \in C_0$ and $\| z \| = 2$.

If $y \in V$, $\| y \| \leqslant 1$, then

$$|y(z)| = |2y_1| = |y_1 - \sum_{i=2}^{\infty} y_i| \leqslant \sum_{i=1}^{\infty} |y_i| \leqslant 1$$

So $\sup\limits_{\substack{y \in V \\ \|y\| \leqslant 1}} |y(z)| \leqslant 1$ and $\sup\limits_{\substack{y \in V \\ \|y\| \leqslant 1}} |y(z)| \,.\, \|z\|^{-1} \leqslant 1/2$

Thus $D(V) \leqslant 1/2$.

We now show that V is $\sigma(l^1,C_0)$ - dense in l^1 :

Let $e^i = (0,0,0\ldots0,1,0,\ldots)$. Then $e^i \in \bar{V}$, the $\sigma(l^1,C_0)$-closure of V. Let $y \in C_0$. Given $\varepsilon > 0$, choose k_0 such that $|y_k| < \varepsilon$ for $k > k_0$. Then $x = (1,0,0,\ldots,-1/2, -1/4, -1/8, \ldots) \in V$, where $-1/2$ is in the $(k+1)$th position. Now $|y(e^1 - x)| = |\sum_{i=1}^{\infty} 1/2^i \, y_{i+k_0}| < \varepsilon$.

So $e^i \in V$, for all i. Hence $\varphi \subset \bar{V}$ and the result follows.

For a discussion of those weakly-dense subspaces V with $D(V) = 0$, the reader is referred to Petunin [40]. The question of the existence of such subspaces in the dual of an arbitrary Banach space remains open ; the question for reflexive Banach spaces is settled by the next theorem [56; page1015] .

<u>Theorem 4.6</u> For a subspace V of E', the dual of a reflexive Banach space, the following conditions are equivalent :

(1) V is weakly-dense in E'

(2) $D(V) = 1$

(3) V is strongly-dense in E'

<u>Proof.</u> (2) $\rightarrow$ (1) is obviously true in any Banach space. For a reflexive space E the $\sigma(E',E)$, $\sigma(E',E'')$ and $\beta(E',E)$ closures of V coincide. Hence (1) $\rightarrow$ (3). We show that (3) $\rightarrow$ (2) :

$$\|x\|_V = \sup_{\substack{f \in V \\ f \neq 0}} |f(x)| \cdot \|f\|^{-1}$$

Now x is $\beta(E',E)$-continuous on E' for $x \in E \subset E''$, so $f \rightarrow |f(x)|$ is strongly continuous. The map of $f \rightarrow \|f\|$ is also strongly continuous. Hence, since $\|f\| \neq 0$ for $f \neq 0$, the map of $f \rightarrow |f(x)| \cdot \|f\|^{-1}$ is strongly continuous on V, and $\sup_{f \in V} |f(x)| \cdot \|f\|^{-1} = \sup_{f \in \bar{V}} |f(x)| \cdot \|f\|^{-1}$ where $\bar{V}$ denotes the strong closure. Since $\bar{V} = E'$, the result follows.

In connection with Theorem 4.6, Ruston [44; Theorem 3] proved that if (1) - (3) coincide for every subspace V of E', then E is reflexive - See Corollary 6.3 .

<u>Corollary 4.5.</u> If E is a Banach space, then E is reflexive iff every subspace V of E'' with $D(V) = 1$ is norm-dense in E''.

<u>Proof.</u> If E is reflexive, then E' is also reflexive, and the result follows by Theorem 4.6. Conversely, E is a subspace of E'' with $D(E) = 1$. So E is norm-dense in E''. Since E is complete, it is closed in E'', hence $E = E''$.

We conclude this section with another characterization of reflexivity [39; Theorem1].

<u>Theorem 4.7</u> A Banach space E is reflexive iff S_E is closed in any l.c. topology comparable with the initial topology on E.

<u>Proof.</u> If E is reflexive, then the result follows by Theorem 4.6 and Corollary 4.2(2). Suppose E is not reflexive. Then $\exists$ elements $x \in E$, $\|x\| = 1$ and $z'' \notin E$ such that x is not orthogonal to the subspace $\{\lambda z''\}$. Let V denote the polar of z'' in E'. Then $D(V) < 1$. But $z'' \notin E$ so V is $\sigma(E',E)$-dense in E' and the $\sigma(E,V)$ topology provides the necessary contradiction.

Petunin [39;Theorem 2] shows that it is possible to characterize reflexive Banach spaces by the condition that S_E be closed in any normable topology, comparable with the initial topology on E, using the above theorem.

THE LOCALLY CONVEX SETTING.

We turn to a discussion of Krishnamurthy's [31,32] generalization of the characteristic which leads to his claim: " that a proper setting for the discussion of the characteristic is that of a l.c. space." [32; page 526].

Notation: Given a l.c. space E,τ , $\mathfrak{U}$ will denote its fundamental system of closed absolutely convex neighbourhoods of zero. $\mathfrak{M}$ and $\mathfrak{B}$ will denote the classes of absolutely convex, closed and bounded sets in E,τ and $E',\beta(E',E)$ respectively. If p_U denotes the gauge of $U \in \mathfrak{U}$. For $M \in \mathfrak{M}$ let $p_U(M) = \sup\{p_U(x) : x \in M\}$. Then ,

$$p_U(M) = \inf\{\lambda \geq 0 : M \subset \lambda U\}.$$

Definition 5.1 A subspace V of E' is said to be duxial if every absolutely convex $\sigma(E',E)$-compact subset of E' is contained in the $\sigma(E',E)$-closure of some $V \cap F$, where F is a $\beta(E',E)$-closed and bounded subset of E'.

In the case of a normed linear space " V duxial " re - duces to " $D(V) > 0$ " by Theorem 2.4 . As remarked earlier, the term "duxial " was used by Ruston [43, 44] to describe subspaces V with $D(V) = 1$.

Definition 5.2 A l.c. space E is called semidistinguished if every absolutely convex $\sigma(E'',E')$-compact subset of E'' is contained in the $\sigma(E'',E')$-closure of some absolutely convex $\sigma(E,E')$-closed and bounded subset of E ; or, equivalently, if $\beta(E',E) = \mu(E',E'')$.

Examples.

(1) All distinguished spaces i.e. spaces for which the strong dual is barrelled [30;page 309] are semidistinguished. In particular, all normed linear spaces are semidistinguished since their strong duals are complete [27; 19.5] .

(2) All semireflexive spaces are semidistinguished [29;page 189]

(3) For an example of a non-semidistinguished space - a very pathological situation - the reader is referred to an example of Komura [29; page 157].

Theorem 5.1 Embed E in E''. Then, E is duxial in E'' iff E is semidistinguished.

Proof. In general, $\sigma(E',E) \leq \tau \leq \gamma(E,E')$. Let $K \subset E''$ be an absolutely convex $\sigma(E'',E')$-compact. Now E is duxial in E'' iff $\exists$ a γ-bounded set A such that $K \subset \bar{A}$, the $\sigma(E'',E')$-closure of A. But the $\gamma(E,E')$-bounded and $\sigma(E,E')$-bounded sets coincide, so E is duxial in E'' iff $\exists$ a τ-bounded set A such that $K \subset \bar{A}$. This, by definition, is equivalent to saying that E is semidistinguished.

Theorem 5.1 is a partial analogue to Example1 in Section 2.

For an example of a space E which is not duxial in E'', the reader is referred to the same example of Komura [29] which was previously quoted. Further results will show that the role played by duxial subspaces is analogous to that played by subspaces V with $D(V) = 1$ in Banach spaces. However, whereas $D(V)$ was defined as a number, $0 \leq D(V) \leq 1$, there is no number associated with duxial subspaces. Thus motivated we define the following two numbers :

<u>Definition 5.3</u> Given a subspace V of E', we define

$\alpha(V,U,M) = \max\{\lambda \geq 0 : \overline{V \cap M^\circ} \supset \lambda U^\circ\}$ for every $U \in \mathfrak{U}$ and $M \in \mathfrak{M}$, such that $p_U(M) \neq 0$.

$\alpha(V,U) = \inf\{\alpha(V,U,M).p_U(M): M \in \mathfrak{M}, p_U(M) \neq 0\}$ for every $U \in \mathfrak{U}$.

$\alpha(V) = \inf\{\alpha(V,U): U \in \mathfrak{U}\}$

$\beta(V,U,B) = \max\{\lambda \geq 0 : \overline{V \cap U^\circ} \supset \lambda B\}$ for every $U \in \mathfrak{U}$, $B \in \mathfrak{B}$.

$\beta(V,U) = \beta(V,U,U^\circ)$

$\beta(V) = \inf\{\beta(V,U) : U \in \mathfrak{U}\}$

then $\beta(V)$ and $\alpha(V)$ are called the β- and α-characteristics of V, respectively.

Whenever $p_U(M) = \lambda > 0$, we have $(1/\lambda)U^\circ \subset M^\circ$, in fact $1/\lambda$ is the largest multiple of U° for which this is true. Now, if $\overline{V \cap M^\circ} \supset \rho U^\circ$, then $\rho \leq 1/\lambda$ i.e. we have

$0 \leq \alpha(V,U,M).p_U(M) \leq 1$. So $0 \leq \alpha(V) \leq 1$. Similarly, it can be shown that $0 \leq \beta(V) \leq 1$. Now, suppose E is a normed linear space with unit ball S_E. Then $\alpha(V,S_E,S_E) = D(V)$. Take $U = rS_E$, $M = nS_E$, then $\alpha(V,U,M) = r/n \cdot \alpha(V,S_E,S_E)$. But $p_U(M) = n/r$, so $\alpha(V,U,M).p_U(M) = \alpha(V,S_E,S_E)$ and $\alpha(V,U,M).p_U(M)$ is constant for all U,M and equals $D(V)$. Hence $\alpha(V,U) = \alpha(V) = D(V)$. Similarly $\beta(V,U)$ is constant for all U and equals $D(V)$. So, in normed linear spaces, $\alpha(V) = \beta(V) = D(V)$. In fact if $\beta(V) = 1$, then V is duxial.

<u>Theorem 5.2</u> Let $M^{\sim}$ denote the $\sigma(E,V)$-closure of M in E. Suppose $p_U(M) \neq 0$. Consider the following statements : (a) $p_U(M^{\sim}) < \infty$

(b) $\alpha(V,U,M) = \alpha > 0$

(c) $\alpha = 1/p_U(M^{\sim})$

then (a) and (b) are equivalent, and each implies (c).

<u>Proof.</u> We show that (1) If $p_U(M) \neq 0$ and $p_U(M^{\sim}) < \infty$, then $\alpha \geq 1/p_U(M^{\sim})$ and (2) If $p_U(M) \neq 0$ and $\alpha > 0$, then $\alpha \leq 1/p_U(M^{\sim})$. The result follows from these statements.

Now $M^{\sim} = (V \cap M^{\circ})_{\circ}$, so $(M^{\sim})^{\circ} = (V \cap M^{\circ})_{\circ}{}^{\circ} = \overline{V \cap M^{\circ}}$

If $M^{\sim} \subset \lambda U$, for $\lambda > 0$, then $\overline{V \cap M^{\circ}} \supset 1/\lambda . U^{\circ}$ i.e. $1/\lambda \leq \alpha$. In particular, $1/p_U(M^{\sim}) \leq \alpha$. Hence (1) is true.

If $\alpha > 0$, then $M^{\sim} \subset 1/\alpha . U$ i.e. $p_U(M^{\sim}) \leq 1/\alpha$. Hence (2).

<u>Theorem 5.3</u> Suppose $\beta(V,U,B) = \beta > 0$ and $p_U(M) \neq 0$.

Then $\beta \leqslant p_U(M) / p_{M^\circ}(B)$

$\beta \leqslant p_U(M) / p_{B^\circ}(M^\sim)$

In particular, $\beta(V,U) \leqslant p_U(M) / p_U(M^\sim)$

<u>Proof.</u> Put $p_U(M) = \lambda$. Then

$$\lambda M^\circ \supset \overline{V \cap \lambda M^\circ} \supset \overline{V \cap U^\circ} \supset \beta B$$

So $B \subset \lambda/\beta\ M^\circ$, i.e. $p_{M^\circ}(B) \leqslant \lambda/\beta$.

Also $\overline{V \cap \lambda M^\circ} \supset \beta B$, so $(V \cap \lambda M^\circ)_\circ \subset 1/\beta\ B_\circ$. Hence $1/\lambda M^\sim \subset 1/\beta\ B_\circ$ and $p_{B_\circ}(M^\sim) \leqslant \lambda/\beta$.

<u>Theorem 5.4</u> $\beta(V) \leqslant \alpha(V)$

<u>Proof.</u> It suffices to prove $\beta = \beta(V,U) \leqslant \alpha(V,U)$. Now $\overline{V \cap U^\circ} \supset \beta U^\circ$. Suppose $p_U(M) = \lambda \neq 0$. Then

$$\lambda\ \overline{V \cap M^\circ} = \overline{V \cap \lambda M^\circ} \supset \beta U^\circ$$

so $\overline{V \cap M^\circ} \supset \beta/\lambda . U^\circ$ and $\alpha(V,U,M) \geqslant \beta/\lambda$ for all M for which $p_U(M) \neq 0$. Hence result.

<u>Theorem 5.5</u> Embed E in E'', then $\alpha(E) = \beta(E) = 1$.

<u>Proof.</u> Now $E \cap M^{\circ\circ}$ is $\sigma(E'',E')$-dense in $M^{\circ\circ}$. Since $\{M^\circ : M \in \mathfrak{M}\}$ forms a fundamental system of neighbourhoods for $E', \beta(E',E)$ we have $\beta(E,M^\circ) = 1$ for every $M \in \mathfrak{M}$ i.e. $\beta(E) = 1$. The result follows by Theorem 5.3 .

This theorem provides a tidy analogue to Example 1 of Section 2. For further generalizations, e.g. of Theorem 3.3 see [32]. No such generalization of Theorem 4.2 exists.

MINIMAL SUBSPACES.

In Section 7 we shall answer the question :
When is a Banach space E equivalent to the strong dual of some other Banach space F ?
Now, if E is equivalent to F', then their duals E' and F'' are also equivalent, and the embedding of F in F'' makes F a norm-closed, weakly-dense subspace of E'. For this reason we shall investigate the norm-closed, weakly-dense subspaces of E'. Before doing so, it is necessary to define our terms precisely [3] because of the large number of alternative terms in existence - see for example [4, 13, 19, 30, 45, 50.].

Definition 6.1 Given Banach spaces E and F, E is isomorphic to F if there is a one-one continuous linear map of E onto F. Note that its inverse is necessarily continuous. E and F are equivalent if there is a norm preserving linear map of E onto F. If F is a closed subspace of E', then there is a natural map of E into F', defined by $\langle f, Jx \rangle = \langle x, f \rangle$ called the canonical map of E into F'. The image of E under this map is called the canonical image of E in F'. If this map sets up an isomorphism or an equivalence, then the spaces are said to be canonically isomorphic or canonically equivalent.

<u>Definition 6.2</u> A subspace V of E', the dual of a l.c. space E is minimal if

(1) V is $\beta(E',E)$-closed in E'

(2) V is $\sigma(E',E)$-dense in E'

(3) No proper subspace of V has both the above properties.

In our investigation of minimal subspaces we shall make use of the next result [34]. Recall that E,τ denotes a l.c. space and $\mathfrak{M}$ the class of all absolutely convex $\sigma(E,E')$-closed and bounded subsets of E. We use the standard dual pair notation of [27].

<u>Theorem 6.1</u> If V_1 and V_2 are $\sigma(E',E)$-dense subspaces of E', their strong closures are identical iff the topologies $\sigma(E,V_1)$ and $\sigma(E,V_2)$ coincide on each $M \in \mathfrak{M}$.

<u>Proof.</u> Suppose $\overline{V}_1 = \overline{V}_2$, where $\overline{V}_1$, $\overline{V}_2$ denote the strong closures of V_1 and V_2 respectively. Now $\sigma(E,V_1)$ is weaker than $\sigma(E,\overline{V}_1)$ and $\sigma(E,\overline{V}_1)$ induces on each M a topology finer than that induced by $\sigma(E,V_1)$. If $x \in \overline{V}_1$ and $M \in \mathfrak{M}$, then $\exists$ a sequence $\{x_n\}$ in V_1 which converges uniformly to x on M. Thus x is $\sigma(E,V_1)$-continuous on M. Conversely, we show that $\overline{V}_1 \subset \overline{V}_2$. Let U be a $\beta(E',E)$ neighbourhood of zero, and $x \in \overline{V}_1$. We show that $\exists\ y \in V_2$

such that $x - y \in U$. Then the result will follow by symmetry. Now $\exists\ M \in \mathfrak{M}$: $M^\circ \subset U$. By the first part of the proof $\sigma(E,\bar{V}_1) = \sigma(E,V_1) = \sigma(E,V_2)$ on M. So $\exists$ $y_1,\dots,y_n$ in V_2 and x' in M such that $\sup_i |\langle x',y_i\rangle| \leq 1$ implies that $|\langle x',x\rangle| \leq 1$. Let $V = \{x \in E : \langle x,y_i\rangle = 0$ for $1 \leq i \leq n\}$. If p denotes the gauge of M, then for all x' in V, $|\langle x',x\rangle| \leq p(x)$. By the Hahn-Banach Theorem $\exists z \in E^*$: $\langle x',x\rangle = \langle x',z\rangle$ for all x' in V and $|\langle x',z\rangle| \leq p(x)$ for all x' in E. Now $\langle x',x-z\rangle = 0$ for all x' in V and $x-z = y$ where y is a linear combination of $\{y_i\}_{1\leq i\leq n}$. But $z = x-y \in E'$, so $z \in M^\circ \subset U$. Combining these results $\exists\ y \in V_2$: $x-y \in U$.

The next related result was proved somewhat earlier by Dixmier [13; Lemma 1].

<u>Theorem 6.1'</u> If E is a normed linear space and V_1, V_2 are two $\sigma(E',E)$-dense subspaces of E', then their strong closures are identical iff $\sigma(E,V_1)$ and $\sigma(E,V_2)$ coincide on S_E, the unit ball of E.

<u>Proof.</u> Similar to Theorem 6.1.

<u>Corollary 6.1</u> If E is a normed linear space and V a subspace of E', then V is norm-dense in E' iff the $\sigma(E,V)$ and $\sigma(E,E')$ topologies coincide on S_E.

Proof. Since the topology of E' is the topology of uniform convergence on the bounded subsets of E, the result follows immediately from Theorem 6.1 .

For further results on dense subspaces and their uses the reader is referred to the work of Kasahara, for example [26] .

Theorem 6.2 Let V be a $\beta(E',E)$-closed, $\sigma(E',E)$-dense subspace of E', the dual of a l.c. space. Then the following statements are equivalent :

(1) V is minimal in E'

(2) $E'' = E \oplus V^\circ$

(3) Each $M \in \mathfrak{M}$ is relatively $\sigma(E,V)$-compact

Proof. (1) → (2) : Since V is $\sigma(E',E)$-dense in E', $V^\circ \cap E = \{0\}$. Let $z \in E''$ and $z \notin V^\circ$. Consider

$$W = \{ y \in V : < z,y > = 0 \} = Z_\circ \cap V$$

where Z is the subspace in E'' spanned by z . Now $Z_\circ$ and V are $\beta(E',E)$-closed, hence W is $\beta(E',E)$-closed and $W \neq V$. So W is not $\sigma(E',E)$-dense in E' i.e. $\exists\, x \in E,\ x \neq 0 : < x,w > = 0$ for all $w \in W$. Now z and x are linear functionals on V, vanishing on W. Thus

$$z'' = z - \frac{< z,y_\circ >}{< x,y_\circ >} \qquad y_\circ \in V,\ \ y_\circ \notin W$$

is a linear functional which vanishes on V, i.e. $z'' \in V^\circ$. So $z = \lambda x + z''$ for some λ such that $\lambda x \in E$, $z'' \in V^\circ$.

By our initial remark this representation is unique.

(2) → (3) : Since $E''/\mathcal{N}^\circ$ is algebraically isomorphic to E and $E''/\mathcal{N}^\circ = [V,\beta(E',E)/\mathcal{N}]'$, we have that $E,\sigma(E,V)$ is isomorphic to $E''/\mathcal{N}^\circ, \sigma(E''/\mathcal{N}^\circ,V)$. For every $M \in \mathfrak{M}$, $M^\circ \cap V$ is a $\beta(E',E)$ neighbourhood of zero in V, thus $(M^\circ \cap V)^\circ \subset E''/\mathcal{N}^\circ$ is $\sigma(E''/\mathcal{N}^\circ,V)$-compact. But, for $\tilde{M}$ the $\sigma(E,V)$-closure of M, we have $\tilde{M} = (M^\circ \cap V)_\circ = (M^\circ \cap V)^\circ$. So M is relatively $\sigma(E,V)$-compact.

(3) → (1) : Let $W \subset V$ be $\beta(E',E)$-closed in E'. If W were $\sigma(E',E)$-dense then $\sigma(E,W)$ would be Hausdorff on E, and so on $\tilde{M}$. Since $\tilde{M}$ is $\sigma(E,V)$-compact, $\sigma(E,V)$ and $\sigma(E,W)$ would coincide on $\tilde{M}$, so on M. By Theorem 6.1 this implies that V and W have the same $\beta(E',E)$ closures. This contradicts our initial assumption.

The above proof [31;Theorem 1] is essentially that of Dixmier [13; Theorems 11, 13] who proved the result for Banach spaces making use of the presence of a norm. However, his proof translates easily to the more general setting because the concept of minimality is expressible independent of a norm.

<u>Corollary 6.2</u> For any l.c. space, the following are equivalent

(1) E' is minimal

(2) $E = E''$ or E is semireflexive.

Corollary 6.3 A quasi-barrelled space E is reflexive iff its strong dual does not contain any proper minimal subspaces.

Since every Banach space is quasi-barrelled, Corollary 6.3 contains Ruston's result [44; Theorem 3] as a special case. So, finally, we have the promised characterization of reflexivity - see the remark after Theorem 4.6 .

Corollary 6.4 If E is a Banach space and V a minimal subspace of E', then $D(V) > 0$.

Proof. Follows immediately from Theorems 6.2 and 4.4

Corollary 6.5 If E is a Banach space and V a $\beta(E',E)$-closed, $\sigma(E',E)$-dense subspace of E', then the following statements are equivalent :

(1) V is minimal

(2) S_E, the unit ball of E, is relatively $\sigma(E,V)$-compact

Proof. As in Theorem 6.2, using Dixmier's variation of Theorem 6.1

Corollary 6.6 If V is a $\beta(E',E)$-closed and $\sigma(E',E)$-dense subspace of E', then V is minimal and $D(V)$ is one iff S_E is relatively $\sigma(E,V)$-compact.

Proof. Apply Corollary 6.5 and Theorem 4.2 to the Banach space E.

<u>Definition 6.3</u> A l.c. space E is said to be quasi- M - barrelled if, in E', every absolutely convex $\beta(E',E)$ - bounded set is relatively $\sigma(E',E)$ - compact.

Clearly, each of the following statements could equally well be used to define quasi - M-barrelled spaces :

(1) The strong bidual E'', $\beta(E'',E')$ induces the Mackey topology $\mu(E,E')$ on E.

(2) E, $\mu(E,E')$ is quasi - barrelled.

(3) Every bornivorous barrel in E is a $\mu(E,E')$ neighbourhood of zero.

Every quasi - barrelled space is quasi - M - barrelled. In particular, every metrisable space is quasi - M - barrelled.

<u>Example.</u> A quasi - M - barrelled space which is not quasi - barrelled:

Suppose E is an infinite dimensional reflexive Banach space. Then $\sigma(E,E') < \mu(E,E') = \gamma(E,E') = \beta(E,E')$.

If $\sigma(E,E') = \mu(E,E')$ then every bounded set in E is weakly compact, and hence norm compact i.e. E is finite dimensional.

<u>Theorem 6.3</u> If E is a l.c. space, then E is minimal in E'' iff E is a quasi-M-barrelled space which is $\beta(E'',E')$-closed in E''.

Proof. If E is minimal in E'', then E is $\beta(E'',E')$ - closed in E''. Since $E'' = [E', \beta(E',E)]'$ every absolutely convex, closed and bounded set B in E' is relatively $\sigma(E',E)$-compact, by Theorem 6.2 . Hence $\bar{B}$, the $\beta(E',E)$-closure of B is relatively $\sigma(E',E)$-compact. But B and $\bar{B}$ have the same $\sigma(E',E)$ closures. Hence E is quasi- M - barrelled. If B is an absolutely convex, $\beta(E',E)$-closed and bounded subset of E' then, by hypothesis, B is relatively $\sigma(E',E)$-compact. Since E is $\beta(E'',E')$-closed and clearly $\sigma(E'',E')$-dense, the result follows from Theorem 6.2 .

Corollary 6.7 If E is a complete quasi-M-barrelled space, then E is minimal in E''. In particular, if E is a Banach space, then E is minimal in E''.

Proof. Since $\mu(E,E') = \gamma(E,E')$ is a complete topology, E is $\beta(E'',E')$-closed in E''.

Theorem 6.4 The following two conditions are necessary and sufficient for a $\beta(E',E)$-closed and $\sigma(E',E)$-dense subspace V of E' to be minimal and duxial in E' :

(1) Every $M \in \mathfrak{M}$ is relatively $\sigma(E,V)$-compact

(2) $\mu(E,E') < \beta(E,V)$

Proof. By Theorem 6.2, V is minimal iff every M is relatively $\sigma(E,V)$-compact. If V is minimal, the $\sigma(E,V)$-bounded and closed absolutely convex sets of V are the $\beta(E',E)/_V$ bounded and closed sets of V. Result follows from the definition of duxial subspaces.

Theorem 6.5 In order that E be minimal and duxial in E'' the following three conditions are necessary and sufficient :

(1) E is $\beta(E'',E')$-closed in E''

(2) E is quasi-M-barrelled

(3) E is semidistinguished

Proof. Property (2) in Theorem 6.4, applied to $E \subset E''$ states that $\mu(E',E'') \leqslant \beta(E',E)$. The result then follows from Theorems 6.2 and 6.3 .

Theorem 6.5 is a generalization of Dixmier's result [13 ; Theorem 15] and shows that this generalization is the best possible.

Unless explicit reference to other papers was given, all results of this Section can be found in [31] .

CONCERNING DUALS AND BIDUALS.

Having established all the necessary results on minimal subspaces in Section 6, we can now provide an answer to the question, " When is a Banach space isomorphic or equivalent to the dual of another Banach space ? " or, more generally, " When is a l.c. space isomorphic or equivalent to the strong dual of another l.c. space ? "

A Banach space E was said to be canonically equiv - alent to F', the dual of the subspace F of E', if the canonical map of E to F' was an equivalence. Singer [46] uses the term "F-reflexive". This terminology has the advantage that it suggests that F-reflexivity is a generalization of the usual reflexivity. This is true, since the E'-reflexivity of E is in fact its reflexivity in the usual sense. The following definition [31;340] extends this term to arbitrary l.c. spaces.

Definition 7.1 Let V be a subspace of E', the dual of the l.c. space E,τ . If the canonical embedding of E in E'' is such that

$$[\, V,\ \beta(E',E)_{|\,V}]' = E$$

and $$E,\ \beta(E,V) = E,\tau$$

then E,τ is said to be V-reflexive.

<u>Theorem 7.1</u> Embed the l.c. space E in its bidual E''. E', $\beta(E',E)$ is E-reflexive iff E is quasi-M-barrelled.

<u>Proof.</u> E', $\beta(E',E)$ is E-reflexive

iff E, $\gamma(E,E') = E$

iff $\gamma(E,E') \leqslant \mu(E,E')$

iff E is quasi-M-barrelled.

We have remarked that a Banach space E is always E'-reflexive. Theorem 7.1 shows that the class of quasi-M-barrelled spaces is the most general class of spaces for which this is true [31;340].

<u>Theorem 7.2</u> Let E,τ be a l.c. space with its Mackey topology. The following statements are equivalent :

(1) E is isomorphic to the strong dual of a l.c. space F which has the properties :

(a) F is $\beta(F'',F')$-closed in F''.

(b) F is quasi-M-barrelled.

(c) F is semidistinguished.

(2) There exists a minimal duxial subspace V in E' such that on E, $\beta(E,V) \leqslant \mu(E,E')$.

(3) E,τ is V-reflexive for a $\beta(E',E)$-closed and $\sigma(E',E)$-dense subspace V of E'.

Proof. (1) → (2) : Suppose (1) is true, and consider the identity map $T : E \to F'$. Its adjoint $T' : F'' \to E'$ is also an identity whose restriction to F embeds F as a subspace of E'. Call this subspace V. Now, F'', $\beta(F'',F')$ is isomorphic to $E',\beta(E',E)$. Also, by Theorem 6.5, (a) – (c) means that F is minimal and duxial in E'. Since, $\beta(F',F)$ is weaker than $\mu(F',F'')$, it follows that $\beta(E,V) \leq \mu(E,E')$ and (2) is satisfied.

(2) → (3) : Suppose (2) is true. Since V is minimal, $[V, \beta(E',E)|V]' = E''/V^{\circ}$, and by Theorem 6.2 $E'' = E \oplus V^{\circ}$. Since V is duxial, $\mu(E,E') < \beta(E,V)$ by Theorem 6.4. But by hypothesis, $\beta(E,V) \leq \mu(E,E')$. Hence $\beta(E,V) = \mu(E,E') = \tau$, as E has the Mackey topology. So E is V-reflexive and (3) is satisfied.

(3) → (1) : Suppose E is V-reflexive. Since E has the Mackey topology it follows that $\mu(E,E') = \beta(E,V)$. Since $E'' = E \oplus V^{\circ}$ for V $\beta(E',E)$-closed and $\sigma(E',E)$-dense, every $M \in \mathfrak{M}$ is relatively $\sigma(E,V)$-compact by Theorem 6.2. So, by Theorem 7.4, V is minimal and duxial in E', and it can be shown that V is minimal and duxial in V''. In fact, by Theorem 6.5,

a. V is $\beta(V'',V')$-closed in V''

b. V is quasi-M-barrelled

c. V is semidistinguished

If we replace V, $\beta(E',E)|_V$ by F, then the result follows.

We now use Theorem 7.2 [31;340] to obtain Dixmier's earlier results for Banach spaces [13; Theorems 16, 17, 18, 19, 16', 17'].

<u>Corollary 7.1</u> A Banach space E is equivalent to the dual of another Banach space iff there exists in E' a minimal subspace V with $D(V) = 1$.

<u>Proof.</u> All Banach spaces satisfy (1)a – c of Theorem 7.2 and any $\beta(E',E)$-closed, $\sigma(E',E)$-dense subspace V of E' satisfies (2), i.e. every $\sigma(V,E)$-bounded subset A of V has the form $V \cap B$ where B is a subset of a ball in E'. Hence the $\sigma(V,E)$-closure of A is $\sigma(V,E)$-compact. The isometry follows since S_E is the polar of the unit ball in V i.e. $\beta(E,V)$ and τ define the same norm in E.

By our equivalent statements for a subspace V to be minimal and have $D(V) = 1$, we have

<u>Corollary 7.2</u> (1) If there exists a subspace V of E' with $D(V) = 1$, then E is V-reflexive.
(2) E is equivalent to F', the dual of a Banach space F iff either of the following conditions hold : (a) $\exists$ in E'' a weakly closed subspace W such that $E'' = E \oplus W$ and $\|x\| \leq \|x+z\|$ for $x \in E$, $z \in W$. (b) $\exists$ in E' a subspace V such that S_E is $\sigma(E,V)$-compact.

Corollary 7.3 A Banach space E is isomorphic to F', the dual of some Banach space F iff there exists in E' a minimal subspace V.

Proof. If E is isomorphic to F', then F can be identified with a minimal subspace of E', by the first implication in Theorem 7.2 . Conversely, if V is a minimal subspace of E', then E is algebraically isomorphic to $V' = E''/V^\circ$, and for such V, $\beta(E,V) \leq \mu(E,E')$. Now, the space $V', \beta(E,V) = [V, \beta(E',E)|_V]'$ is a Banach space. Thus on E we have two normable topologies, each of which is complete, and the result follows.

Again by our characterizations of minimal subspaces :

Corollary 7.4 If there is a minimal subspace V of E' with $D(V) > 0$, then there exists an isomorphism $J_V : V' \to E$ such that $\| J_V \| = \dfrac{1}{D(V)}$.

Corollary 7.5 For a Banach space E the following are equivalent :

1. E is isomorphic to a dual.
2. E' contains a minimal subspace.
3. $\exists$ in E' a subspace V with S_E relatively $\sigma(E,V)$-compact.
4. $\exists$ in E'' a subspace W which is weakly closed and $E'' = E \oplus W$.

Corollaries 7.1 – 7.5 include all Dixmier's results as well as the main results of Ruston's paper [57]. Though Ruston's work appeared almost a decade later than Dixmier's, he appears to have been unaware of the earlier paper. His proofs are substantially different. Another series of proofs is found in Goldberg's paper [19]. These proofs are interesting in that they use operator theory, but are lengthier than those we have given as they do not apply the available characterizations of minimal subspaces. In Theorem 7.2 it was proved that if a Banach space E is equivalent to a dual, it is V-reflexive for a suitable subspace V of E'. In fact, in the following theorem which is due to Goldberg [19;244] , V is explicitly determined.

<u>Theorem 7.3</u> If E is a Banach space, let T be an isomorphism, or an equivalence, between E and F', the dual of some Banach space F. Then E is isomorphic, or equivalent, to V' under the map $J_V : E \to V'$ defined by

$$\langle v, J_V x \rangle = \langle x, v \rangle , \quad \text{all } v \in V$$

where $V = T'JF$, and J is the canonical map of F into F''.

<u>Proof.</u> Since V is weakly dense, J_V is one – one. To show that J_V has the required properties, it is sufficient to show that $J_V E = V'$. The result then follows by the Open

Map Theorem. Now $T^{-1}(T'J)'$ is a map from V' to E. Given $v \varepsilon V'$, let $x = T^{-1}(T'J)'v'$. We show that $J_V x = v'$. If $v \in V$,

$$\begin{aligned} < v, v' > &= < v, [(T'J)']^{-1}(Tx) > \\ &= < v, [(T'J)^{-1}]'(Tx) > \\ &= < Tx, (T^{-1})'v > \\ &= < v, J_V x > \end{aligned}$$

Furthermore, if T is an isometry, then $T^{-1}(T'J)'$ is an isometry, hence J_V is too.

Goldberg, who uses Corollary 4.4 and Theorem 7.3, attacks the question from this direction. As previously mentioned, Corollary 4.4 was first proved by Singer [46] who used it to establish the following generalization of a well-known reflexivity criterion [50; 229] . Further generalizations of this type can be found in [45, 47, 49].

<u>Theorem 7.4</u> A necessary and sufficient condition for a Banach space E to be V-reflexive is that S_E be $\sigma(E,V)$-compact and Hausdorff.

<u>Proof.</u> If V is strongly closed and S_E is $\sigma(E,V)$-compact then E is V-reflexive by Corollaries 6.6 and 7.2 . It suffices to prove that we may omit the hypothesis that V is strongly closed. Suppose V is an arbitrary subspace of E', and S_E is $\sigma(E,V)$-compact and Hausdorff. Then $E, \sigma(E,V)$ is Hausdorff. Thus V and $\overline{V}$, the strong closure of V,

are weakly dense in E'. By Theorem 6.1' $\sigma(E,V)$ coincides with $\sigma(E,\overline{V})$ on S_E. Thus S_E is $\sigma(E,\overline{V})$-compact, hence E is V-reflexive. Conversely, if $S_{V'}$ is the unit ball in V', then S_E and $S_{V'}$ are isomorphic. But S_E is $\sigma(V',V)$-compact, hence also $\sigma(E,V)$-compact.

This theorem can be proved without employing any of Dixmier's results, see [46]. In the particular case $V = E'$, Eberlein's theorem [14] shows that we can replace the $\sigma(E,V)$-compactness of S_E by sequential compactness. However, this is not true in general [46]. Still, if V is separable and weakly dense, then it can be replaced by strict sequential compactness [47]. For further results on compactness, see James [23, 24, 25]. A considerable bibliography on this topic is found in [9].

<u>Corollary 7.6</u> The dual E' of a Banach space E is $i(E)$-reflexive, where $i(E)$ denotes the image of E in E'' under the identity map $i : E \to E''$.

<u>Proof</u>. $S_{E'}$ is $\sigma(E',E)$-compact, hence also $\sigma(E',i(E))$ - compact. Apply Theorem 7.4 for $E = E'$, $V = i(E)$.

The sequence space C_0 is a well - known example of a Banach space which is not V-reflexive for any subspace V of its dual l^1 [44; 578].

Obviously, any Banach space E which is not equivalent to the dual of some Banach space is not reflexive with respect to any subspace of its dual. Hence by Corollary 7.6 and Theorem 7.3 the class of all V-reflexive Banach spaces coincides with the class of all Banach spaces which are equivalent to the dual of a Banach space.

Now, using the results of this section, we proceed to characterize those Banach spaces which are isomorphic to the bidual of a Banach space.

<u>Theorem 7.5</u> The following statements are equivalent for a Banach space E :

1. E is isomorphic to a bidual.
2. There exists an equivalent norm for E such that $E' = V \oplus R$, where V is a minimal subspace of E' with $D(V) = 1$ and R is a $\sigma(E',E)$-closed subspace of E'.

<u>Proof.</u> Suppose E is isomorphic to F''. Then $F''' = F' \oplus F^{\circ}$ and F' is $\sigma(F''',F'')$-dense, F° is $\sigma(F''',F'')$-closed. Give E the norm induced by the isomorphism and identify E with F''. The unit ball of F'' is $\sigma(F''', F'')$-compact, and so is compact when F' is regarded as a subspace of F'''.
Conversely, consider E with this equivalent norm. Let $X = R_{\circ} \subset E$ and denote the restriction of $x' \in E'$ to X by

α. The kernel of α is $X^{\circ} = R \subset E'$. Let $z \in X'$. Extend z to $x' \in E'$ where $x' = x_1' + x_2'$, x_1' and x_2' are elements of V and R respectively. Then $z = \alpha(x') = \alpha(x_1')$ Hence α is a one - one bicontinuous map of V onto X'. By Corollary 7.2 E is equivalent to V' and E is isomorphic to X''.

The above theorem was proved by Civin and Yood [7;907] while the following more precise version is due to Goldberg [19;247] .

<u>Theorem 7.5'</u> The following statements are equivalent :

1. T is an isomorphism from E onto the bidual F'' of the Banach space F.

2. $E' = V \oplus X^{\circ}$ where $V = T'jF'$ is minimal, $X = T^{-1}iF$ and i, j are injection maps of F' and F into F''' and F'' respectively.

<u>Theorem 7.6</u> The following statements are equivalent for a Mackey space E,τ :

(1) E,τ is isomorphic to the strong bidual of a l.c. space F with the properties :

(a) F is quasi-M-barrelled.

(b) F is $\beta(F'',F')$-closed in F''.

(c) $F',\beta(F',F)$ is quasi-M-barrelled.

(d) F' is $\beta(F''',F'')$-closed in F'''.

(e) $F'', \beta(F'', F')$ is a Mackey space.

(2) There exists a minimal and duxial subspace V of E' with the properties :

(a) $E' = V \oplus L$

(b) L is $\sigma(E', E)$-closed.

(c) $\beta(E, V) \leq \mu(E, E')$

Proof. See [33; 79].

QUASI - REFLEXIVITY

In 1950 James [23] published an example of a space E for which E''/E was finite dimensional. An account of this example is found in Day's book [9;72]. Civin and Yood, in 1957, called Banach spaces with this property quasi-reflexive.

Definition 8.1 A Banach space E is called weakly complete if every weakly convergent sequence in E is weakly convergent to an element of E.

This definition, by Banach [3;240], proves useful in an investigation of separable Banach spaces. Goldstine [20] used a generalization of this concept to obtain analogous results for non-separable Banach spaces. Obviously, any reflexive Banach space is quasi-reflexive. Conversely, it can be shown that every weakly complete quasi-reflexive Banach space is reflexive [7;909].

We restrict our discussion of quasi-reflexivity to an account of Singer's result [48] which is a generalization of Corollary 6.3.

Definition 8.2 A Banach space E is quasi-reflexive (of order n) if $E''/\iota(E)$ has finite dimension (is n-dimensional) where $\iota : E \to E''$ is the identity map.

<u>Theorem 8.1</u> Given a Banach space E, E is quasi-reflexive of order n iff the following conditions hold :

1. Every $\sigma(E',E)$-dense, $\beta(E',E)$-closed sub-space V of E' has $\dim E'/V \leq n$.

2. There exists a $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspace W of E' with $\dim E'/W = n$.

To prove this theorem, we need two lemmas.

<u>Lemma 1.</u> Given a Banach space E and a non-negative integer k. (1) A closed subspace V of E has codim $\leq k$ iff for every $(k+1)$-dimensional subspace F_{k+1} of E, $F_{k+1} \cap V \supseteq \{x\}$, $x \neq 0$.

(2) $\dim E/V \geq k$ iff $\exists$ a $(k+1)$-dimensional subspace F'_{k+1} of E such that $\exists\ X' \in E$, $X' \neq 0$: $F'_{k+1} \cap V$ contains X', and X' is unique up to a scalar multiple.

<u>Proof.</u> Suppose $\dim E/V \leq k$. $\exists$ h linearly independent functionals $f_1, \ldots\ldots, f_h$ where $h = \dim E/V \leq k$, such that $V = \{ x \in E : f_j(x) = 0,\ 1 \leq j \leq h \}$. Let F_{k+1} be an arbitrary $(k+1)$-dimensional subspace of E and $y_1, \ldots, y_{k+1}$ a Hamel base for F_{k+1}. Then $x = \sum_{i=1}^{k+1} \alpha_i y_i \in V$, i.e. $\sum_{i=1}^{k+1} \alpha_i f_j(y_i) = 0$ and this system has a non-zero solution $\{\alpha_1, \ldots, \alpha_{k+1}\}$. So (1) is true. But $\dim E/V \geq k$ iff $\dim E/V \nleq k-1$. Thus apply (1) to obtain (2).

<u>Lemma 2.</u> Let E be a Banach space and V a $\beta(E',E)$-closed and $\sigma(E',E)$-dense subspace of E'. If $\dim V^\circ$ or $\dim E'/V$ is finite, then $\dim V^\circ = \dim E'/V$. Furthermore, if E is quasi-reflexive of order n then

1. $0 \leq \dim E'/V = \dim V^\circ \leq n$
2. $0 \leq \dim E''/(\iota(E) \oplus V^\circ) \leq n$
3. $\dim E''/(\iota(E) \oplus V^\circ) = n - \dim E'/V$

<u>Proof.</u> Since V° is isometric to $(E'/V)'$,

$$\dim V^\circ = \dim (E'/V)' = \dim E'/V$$

1. If $\dim V^\circ \geq n + 1$, then by Lemma 1 $V^\circ \cap \iota(E) \neq \{0\}$ Contradiction. Thus $\dim V^\circ \leq n$ and $0 \leq \dim E'/V = \dim V^\circ$.

2. Since $\dim E''/\iota(E) = n$ and $V^\circ \cap E = \{0\}$, the result follows.

3. $\dim E''/\iota(E) = n$ and

$$\dim E''/\iota(E) \oplus V^\circ = \dim E''/\iota(E) - \dim V^\circ = n - \dim E'/V.$$

<u>Proof of Theorem.</u> Suppose conditions 1 and 2 are satisfied. Then W is a minimal subspace of E'. Thus $j(E) = W'$ where $j : E \to W'$ is the canonical map. Let $f \in E''$, $f|W \in j(E)$ i.e. $f|W = j(x)$ for $x \in E$. Now $\iota(x)|W = j(x)$ and hence $f - \iota(x) \in W^\circ$. Thus $E'' = \iota(E) \oplus W^\circ$. By Lemma 2 $\dim W^\circ = n$, and so $\dim E''/\iota(E) = n$. Conversely, if 2 is not satisfied, then by 1 $\exists$ an integer n_o, $0 \leq n_o < n$ for which 1 and 2 are true. By the sufficiency part of

this proof, it follows that E is quasi-reflexive of order $n_0 < n$. Contradiction.

<u>Corollary 8.1</u> (1) A Banach space E is quasi-reflexive of order n iff it satisfies Condition 1 of Theorem 8.1.

(2) A Banach space E is quasi-reflexive iff $\sup \{ \dim E'/V \} < \infty$, where the sup is taken over all $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspaces V of E'.

<u>Corollary 8.2</u> A $\sigma(E',E)$-dense subspace V of the dual of a quasi-reflexive Banach space E has $D(V) > 0$.

Corollary 8.2 shows that the question of the existence of weakly dense subspaces in the dual of reflexive and quasi-reflexive Banach spaces is limited to those with characteristic greater than zero. Corollary 8.2 was proved independently by Petunin [40] in an investigation of various spaces which do permit weakly dense subspaces with characteristic zero.

<u>Theorem 8.2</u> A Banach space E is quasi-reflexive of order n iff there exists a $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspace V of E' : $\dim E'/V = n$ and $\dim E''/(\iota(E) \oplus V^\circ) = 0$.

<u>Proof.</u> If E is quasi-reflexive of order n, then by Theorem 8.1 $\exists$ a $\sigma(E',E)$-dense and $\beta(E',E)$-closed subspace V such that $\dim E'/V = n$. For such V, $\dim E''/(\iota(E) \oplus V^{\circ})$ is zero. Conversely, if $\dim E''/(\iota(E) \oplus V^{\circ}) = 0$ then $E'' = \iota(E) \oplus V^{\circ}$, and since $\dim E'/V = n$, $\dim V^{\circ} = n$ and $\dim E''/\iota(E) = n$.

As Singer [48;209] remarked, Theorem 8.2 is equivalent to an earlier theorem proved by Civin and Yood [7 ; Theorem 3.3] :

<u>Theorem 8.3</u> A Banach space E is quasi-reflexive of order n iff there exists an equivalent norm for E such that $E' = V \oplus R$, where V is a $\sigma(E',E)$-dense, $\beta(E',E)$-closed subspace of E' such that S_E is $\sigma(E,V)$-compact and R is n-dimensional.

<u>Proof.</u> Suppose $E'' = E \oplus L$. Let $V = L_{\circ}$. Then V is weakly dense. Now $L \cap E = \{0\}$ and $L_{\circ}$ is $\sigma(E',E'')$-closed, hence also $\beta(E',E)$-closed. Since L is finite dimensional, $L = L_{\circ\circ}$. Furthermore, $\overline{L_{\circ}} = E'$ where $\overline{}$ denotes the $\sigma(E',E)$-closure, i.e. $(L_{\circ})_{\circ}{}^{\circ} = (L_{\circ\circ})^{\circ} = E'$ since $L_{\circ\circ} = L_{\circ\circ} \cap E = \{0\}$. Now, let $x_1'',\ldots,x_n''$ be a base for L. Select $x_1',\ldots,x_n'$ in E' : $x_i''(x_j') = \delta_{ij}$ for $i,j = 1,\ldots,n$. Let $R \subset E'$ be the subspace generated

by the x_j', $1 \leq j \leq n$. Then $E' \supseteq L_\circ \oplus R$. Let $f \in E'$, then $< x_i'', f - \sum_{i=1}^{n} \lambda_i x_i' > = 0$. Put $\lambda_i = f(x_i'')$. Then $f - \sum \lambda_i x_i' \in L_\circ$ and

$$f = f - \sum \lambda_i f(x_i'')x_i' + \sum \lambda_i f(x_i'')x_i'$$

$$\in \quad L_\circ \quad\quad \oplus \quad\quad R$$

So $E' \subseteq L_\circ \oplus R$. Hence $E' = L_\circ \oplus R$. The result follows from Corollary 6.6 .

If E is quasi-reflexive of order n and $E' = V \oplus R$ where V is a $\sigma(E',E)$-dense subspace and R is an n-dimensional subspace of E', is there an equivalent norm for E in which the unit ball is $\sigma(E,V)$-compact ? Hunter [21] , using the above methods, shows that all decompositions of E' of the above type arise from the consideration of the polars of the n - dimensional subspaces of E'.

ON A THEOREM BY DEVITO

We begin with a result by James [24; 139].

Theorem 9.1 A weakly closed subset W of a Banach space E is weakly compact iff each continuous linear functional on E attains its sup on W.

Corollary 9.1 A Banach space E is reflexive iff every continuous linear functional on E attains its sup over S_E.

James obtained several more results on compactness and reflexivity, e.g. [23, 24, 25].

Definition 9.1 A l.c. space E is quasi-complete iff every closed and bounded subset of E is complete [9;44].

Clearly, a complete l.c. space is quasi-complete.

In 1968 DeVito proved a sufficient condition for a l.c. space to be quasi-complete for its Mackey topology [10] and used it to show the connection between the work of James and Dixmier [13]. In 1969 Webb [52] improved the result by obtaining completeness rather than the weaker condition of quasi-completeness.

We now state the latter theorem.

<u>Theorem 9.2</u> If $< E,E' >$ is a dual pair such that $E,\gamma(E,E')$ is complete and separable, and $E',\beta(E',E)$ is complete, then $E,\mu(E,E')$ is complete.

To prove this theorem we use three lemmas which depend on the concept of a sequentially - barrelled space [51].

<u>Definition 9.2</u> A l.c. space E is called sequentially - barrelled if the closed absolutely convex hull of a sequence which is $\sigma(E,E')$ - convergent to zero is equicontinuous.

Analogous to the definition of a quasi-M-barrelled space would be the following definition : A l.c. space E is sequentially - M - barrelled iff $E, \mu(E,E')$ is sequentially - barrelled.

<u>Lemma 1</u> Sequentially - barrelled spaces satisfy the Banach - Steinhaus condition i.e. the weakly and strongly bounded subsets of E, or E', coincide [51].

<u>Proof.</u> Let A and B be weakly bounded subsets of E and E' respectively. Suppose $\sup_{x\in A, y\in B} |< x,y >| = \infty$. Then $\exists$ sequence $\{y_n\}$ in B, such that $\sup_{x\in A} |< x,y_n >| > n^2$, for each n.

Now, $\{(\frac{1}{n})y_n\}$ is weakly convergent to zero, hence is equicontinuous, hence strongly bounded. But,

$\sup_{x\in A, n} |<x, (\frac{1}{n})y_n >| = \infty$. Contradiction.

<u>Lemma 2</u>. Let $< E,E' >$ be a dual pair such that $E,\gamma(E,E')$ is complete. Then $E',\mu(E',E)$ is sequentially - barrelled.

<u>Proof.</u> Let $\{ x_n \}$ be a sequence in E which is $\sigma(E,E')$-convergent to zero. We must show that $\overline{\Gamma}\{x_n\}$, the weak closure of the absolutely convex hull of $\{ x_n \}$, is $\sigma(E,E')$ compact. Let F be the $\gamma(E,E')$-closed linear span of $\{x_n\}$ and give F the topology induced by $\gamma(E,E')$. Then F is complete, separable and $\{x_n\}$ is bounded in F. Construct a map $T : l^1 \to F$. Let $\xi = (\xi_i) \in l^1$. For $f' \in F'$, the sum $\sum_{i=1}^{\infty} \xi_i < x_i, f' >$ is defined. Write $< f_\xi, f' >$ for this sum. Then $f_\xi \in F'^*$, the algebraic dual of F'. To prove that $f_\xi \in F$, it is sufficient to prove that f_ξ is $\sigma(F',F)$ - sequentially continuous on F' [30; Section 5] Let $\{f'_n\}$ be a sequence in F' which is $\sigma(F',F)$-convergent to zero. Let $\varepsilon > 0$. Since F is complete, the Banach-Steinhaus condition holds for $< F,F' >$. So,

$$M = \sup \{ |< x_i , f'_n >| : 1 \leq i, n < \infty \}$$

is finite. Choose $k : \sum_{i=k+1}^{\infty} |\xi_i| < \frac{\varepsilon}{2M}$

and

$$\text{Choose } N : | \xi_i < x_i , f'_n > | < \frac{\varepsilon}{2k}$$

for all $n > N$, and each $i : 1 \leq i \leq k$. Then, for $n > N$, $| < f_\xi, f'_n > | < \varepsilon$. So we have constructed a map from l^1 into F. Denote this map by $T : T\xi = f_\xi$.

Then T is linear and continuous with respect to the topol - ogies $\sigma(l^1, \omega)$ and $\sigma(F,E')$. Let B be the unit ball of l^1. Then B is $\sigma(l^1,C_0)$-compact, so $T(B)$ is $\sigma(F,E')$-compact in F. If e_n denotes the n'th unit vector in l^1 then $Te_n = x_n$, hence $\overline{\Gamma}\{x_n\} \subset T(B)$.

Now, combining Lemmas 1 and 2, we have : If E is $\gamma(E,E')$-complete, then $< E,E' >$ satisfies the Banach - Steinhaus condition.

<u>Lemma 3.</u> Let $< E,E' >$ be a dual pair such that $E,\gamma(E,E')$ is separable and $E,\mu(E,E')$ is sequentially - barrelled. Then $E,\mu(E,E')$ and $E,\gamma(E,E')$ have the same completions.

<u>Proof.</u> Let f be in the completion of $E,\mu(E,E')$. Then f is $\sigma(E',E)$-continuous on every absolutely convex $\sigma(E',E)$ compact subset of E'. Since the $\sigma(E',E)$-convergent sequences in E' are $\mu(E,E')$-equicontinuous, f is $\sigma(E',E)$-sequentially continuous on E'. Since $E, \gamma(E,E')$ is separable, the $\beta(E',E)$-bounded subsets of E' are $\sigma(E',E)$ metrizable. Thus f is $\sigma(E',E)$-continuous on every $\beta(E',E)$-bounded subset of E', hence f is in the comple - tion of $E,\gamma(E,E')$. Conversely, let f be in the comple- tion of $E,\gamma(E,E')$. Then f is $\sigma(E',E)$-continuous on every $\beta(E',E)$-bounded subset of E'; in particular, f is

$\sigma(E',E)$-continuous on every absolutely convex $\sigma(E',E)$ - compact subset of E' i.e. f is in the completion of $E, \mu(E,E')$.

Proof of Theorem 9.2 Since $E,\gamma(E,E')$ is complete, $\gamma(E',E) = \beta(E',E)$, by the remark after Lemma 2. Apply Lemma 2 to $E', \gamma(E',E)$. Then $E, \mu(E,E')$ is sequentially barrelled. The result follows from Lemma 3.

Throughout the remainder of this section, unless otherwise stated, E will denote a real separable Banach space and V a $\beta(E',E)$-closed, $\sigma(E',E)$-dense subspace of E'.

Theorem 9.3 If $D(V) > 0$ (or, V is duxial), then $E, \mu(E,V)$ is complete.

Proof. Since V is $\beta(E',E)$-closed and $D(V) > 0$, V is strictly - norming by Theorem 3.4. Thus the $\sigma(E,E')$- and the $\sigma(E,V)$-bounded sets coincide. Hence the $\beta(E',E)|_V$ and the $\beta(V,E)$ topologies coincide. Thus V is $\beta(V,E)$-complete, since it is closed. Now $\beta(E,E')$ is generated by the polars of $\sigma(E',E)$-bounded sets in E', and since $D(V) > 0$, it is in fact generated by the $\sigma(E',E)|_V$ - bounded sets in V i.e. the $\sigma(V,E)$-bounded sets. Thus $\beta(E,E') = \beta(E,V)$. We show that $\gamma(E,V) = \beta(E,E')$ i.e.

in V the classes of $\sigma(E,V)$ - and $\beta(V,E)$ - bounded sets coincide. Since E is a Banach space, we have for subsets of E', hence of V, that the $\sigma(E',E)$-bounded sets are norm bounded. Thus also the $\sigma(V,E)$-bounded sets are $\beta(V,E)$ bounded. Now $\gamma(E,V)$ is the norm topology on E, so E, $\gamma(E,V)$ is complete and separable, thus E,$\mu(E,V)$ is complete.

The proof of Theorem 9.3 is essentially that of DeVito, but by making use of Webb's improved result, and recalling the definition of a duxial subspace, it is shortened.

<u>Corollary 9.1</u> If E is a separable, barrelled l.c. space such that E', $\beta(E',E)$ is complete, and if V is a strictly norming subspace of E', then E, $\mu(E,V)$ is complete.

<u>Proof.</u> As for Theorem 9.3 .

<u>Corollary 9.2</u> Let V have $D(V) > 0$. If A is a subset of E such that every element of V attains its sup on the $\sigma(E,V)$ closure of A, then A is relatively $\sigma(E,V)$-compact.

<u>Proof.</u> James [24] proved that a subset A of a of a complete l.c. space E is relatively weakly compact if every element of the dual space attains its sup on the weak closure of A. The result then follows by Theorem 9.3

since $E, \mu(E,V)$ is complete.

<u>Corollary 9.3</u> If $D(V) > 0$, then E is isomorphic to V' iff every element of V attains its sup on the $\sigma(E,V)$ closure of S_E.

<u>Proof.</u> This follows immediately from Corollary 7.5 and Corollary 9.2 .

<u>Corollary 9.4</u> If $D(V) = 1$, then E is V-reflexive iff every element of V attains its sup on S_E.

<u>Proof.</u> Follows from Theorem 4.2 and Corollaries 6.2(2) and 9.2 .

<u>Theorem 9.4</u> Let V be a subspace of E' with $D(V) = 1$ such that E is V-reflexive. If P is any norm closed subspace of E' which properly contains V, then P contains an element which does not attain its sup on S_E .

<u>Proof.</u> Assume the result is false. Then, clearly $D(P)=1$ and by Corollary 9.4, E is P-reflexive. Thus by Corollary 9.2, S_E is $\sigma(E,P)$-compact, and also $\sigma(E,V)$-compact. Since $P \supset V$ these topologies agree on S_E. But by Theorem 6.1' this implies that P and V have the same norm closures. This is a contradiction.

We have shown, in Corollary 7.1 , that for subspaces V with $D(V) = 1$, E is canonically equivalent to V' iff V is minimal. Consider now the family of all norm closed subspaces of E' which have the property that each of their elements attains its sup on S_E . If this family contains an element W with $D(W) = 1$, then W is both minimal and maximal with respect to inclusion by Theorem 9.4 .

Theorem 9.5 E is quasi-reflexive of order n iff there are two strongly closed subspaces V and W of E' such that

(1) $D(V) = 1$ and every element of V attains its sup on S_E .

(2) $\dim W = n$, and W has a basis of functionals which do not attain their sups on S_E.

(3) $E' = V \oplus W$

By Theorem 4.5 if $D(V) > 0$ then we can find an equivalent norm for E such that $D(V) = 1$. In this theorem we consider E with this equivalent norm.

Proof. By Corollaries 9.2 and 9.4, S_E is $\sigma(E,V)$ - compact. Then, by Theorem 8.3 , E is quasi-reflexive. Conversely, suppose E is quasi-reflexive of order n. Again by Theorem 8.3, there exists a closed subspace V of

E' which satisfies (1) and such that $\dim E'/V = n$. Let $f_1 \in E'$, $f_1 \notin V$. By Corollary 9.4, E is canonically equivalent to V'. Applying Theorem 9.4, we conclude that $V \oplus f_1$ contains an element g_1 which does not attain its sup on S_E. Choose $f_2 \in E'$, $f_2 \notin V \oplus f_1$. Then $V \oplus f_2$ contains an element g_2 which does not attain its sup on S_E. Suppose $\alpha g_1 + \beta g_2 \in V$ for some scalars α and β. If $\beta \neq 0$, then $g_2 \in V \oplus g_1 = V \oplus f_1$, and so $f_2 \in V \oplus f_1$ Thus $\beta = 0$ and $\alpha g_1 \in V$, i.e. $\alpha = 0$. So V and (f_1, f_2) have only the zero element of E' in common. Continue with this process. We obtain n linearly independent elements $g_1, \ldots, g_n$. The space $(g_1, \ldots, g_n)$ has the properties demanded of W.

Corollary 9.1 is a new generalization of Theorem 9.3. Excluding this, and Webb's completeness theorem [52], all other results of this section are found in [10].

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