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# A STUDY OF HOLOGRAPHIC SUPERCONDUCTORS

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Thesis submitted to the Department of Mathematics and Applied  
Mathematics  
University of Cape Town  
in partial fulfillment of the requirements for the  
Degree of Master of Science  
July 2009

# Abstract

The proposal that the physics of quantum critical phase transition in strongly coupled condensed matter systems can be described by a gravitational theory within the framework of gauge/gravity correspondence is investigated more extensively for s-wave superconductors. We consider a gravitational theory with a black hole solution in anti de Sitter spacetime, coupled to an Abelian-Higgs system in  $(d + 1)$ -dimensions. A wide range of negative mass squared for the scalar field that satisfied the Breitenlohner-Freedman stability bound and the unitarity bound are considered in the probe limit. The dependence of the some of the physical quantities on the scaling dimensions of the dual condensates were thoroughly investigated. We observe that the holographic superconductors can be consistently classified into two, based on the scaling dimensions and the charge of the dual condensates. Holographic superconductors of dimension  $\lambda_-$  exhibit features of type II superconductors while those of dimension  $\lambda_+$  show features of type I. The validity of this classification was confirmed by solving the bulk equations of motion perturbatively near the quantum critical point in order to calculate the superconducting characteristic lengths at a fixed charge  $q$ . The results show that there is a critical scaling dimension beyond which a holographic superconductor behave as type I and below this value it is a type II. The properties of holographic superconductors presented in this report are in qualitative agreement with the Ginzburg-Landau theory.

# Acknowledgement

I hereby acknowledge the financial assistance of the National Astrophysics and Space Science Programme, (NASSP), South Africa, in the form of M.Sc. scholarship, towards this research. It is a great pleasure to thank the present NASSP administrator, Nicky Walker and the former administrator, Penny Middlekoop for their support. I am highly indebted to the director of NASSP, Prof. Peter K. S. Dunsby and other sterling committee members for finding me worthy of the scholarship award. I strongly believe that the results reported in this thesis justify their trusty in me. I thank my thesis advisor, Dr. Jeff Murugan, for encouragement and support. I highly appreciate the support from all the NASSP students especially NASSP Masters/NASSP Honours 2008. I thank Amare Abebe, Daniel Opoloto, Anne marie Nzioki, Ando Ratsimbazafy. and Patrick Oyoriworth for being there for me when the journey was extremely tough. My fellow country men Daniel Oko and Chijioke Nwosa, I thank you all.

I benefited immensely from the following collaborators: Dr. Sean Hartnoll (Harvard University), Dr. Julian Sonner (Cambridge University), Dr. Pallesh Basu (University of British Columbia), Dr. Christopher Herzog (Princeton University, New Jersey) and Mathew Roberts (UCSB). I appreciate the key role played by Dr. Amanda Weltman in facilitating the establishment of some of the international collaborations.

I appreciate the support from all the Post-docs of Cosmology and Gravity Group of the university of Cape Town, they include Dr. Petja Salmi, Dr. Alex Hamilton, Dr. Julian Larena, Dr. Sante Carloni, Dr. Kishore N. Ananda and Dr. Ritu Goswami. To the NASSP IT department, Sebastine and Sephilo, I am very grateful, without your support this work would not have been possible. I lack words to express my gratitude to Asadig Mohammed and Andrea Princloo, for sacrificing their time to ensure that I understand some aspects of string theory/M-theory. I thank Andy and Alex for proof-reading some parts of this work.

# Declaration

The work presented in this thesis is partly based on collaborations with Dr. Jeff Murugan (Department of Mathematics and Applied Mathematics, University of Cape Town) and Dr. Sean Hartnoll (Jefferson Physical Laboratory, Harvard University). Larger part of chapter 3 is based on the following publication,

Obinna C. Umeh, “Scanning the parameter space of holographic superconductors”, **JHEP 0908: 062, 2009**, *arXiv:0907.3136[hep-th]*.

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work...

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Obinna Cosmas Umeh

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# Chapter 1

## GENERAL OVERVIEW

### 1.1 Introduction

The gauge/gravity correspondence is a conjectured duality in string theory that connects two different kinds of theories. On one side is a quantum field theory similar to the Yang-Mills gauge theory of the standard model of particle physics and on the other side is string theory whose low energy limit is a supergravity theory in ten dimensions. The quantum field theory is defined on the non-dynamical boundary of the anti-de Sitter AdS spacetime with its spatial dimension lower by one. An example of this duality is the correspondence between type IIB string theory on  $AdS_5 \times S^5$  spacetime<sup>1</sup> and a  $\mathcal{N} = 4$  supersymmetric (SUSY) Yang-Mills gauge theory on the 4-dimensional boundary of  $AdS_5$  [1]. This correspondence has also been generalized to many AdS/non-AdS backgrounds in  $(d + 1)$ -dimensions and their dual conformal quantum field theories in  $d$ -dimensions [2, 3, 4, 5, 6, 7].

In the low energy limit, the duality connects the strongly coupled sector of Yang-Mills gauge theory where calculations are almost close to impossible to a weakly coupled gravitational theory, where calculations are possible and simple. This impressive connection is the major motivation behind the activities of mapping out the properties of strongly coupled SUSY gauge theories with their gravitational duals. For example it is becoming increasingly clear that this approach may be able to provide an access to the strongly coupled sector of quantum chromodynamics (QCD)<sup>2</sup>. This claim is supported by recent

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<sup>1</sup>A product of five dimensional anti-de Sitter (AdS) spacetime with five dimensional sphere.

<sup>2</sup>Quantum chromodynamics is a section of the standard model of particle physics that describes the

experimental results from the relativistic heavy ion collider (RHIC) which suggest that the viscosity of quark-gluon plasma is very low [9]. The calculation of viscosity to entropy density ratio using gauge/gravity approach for  $\mathcal{N} = 4$  SUSY Yang-Mills gauge theory yields a very low value  $\eta/s = \hbar/4\pi k_B$  [10, 11, 12, 13]. In contrast, the perturbative QCD technique yields a very large viscosity [14, 15], and lattice gauge theory requires a very difficult analytic continuation from the Euclidean theory to extract such a transport coefficient [16].

This remarkable result inspired the application of AdS/CFT technique to certain condensed matter systems. Phenomena such as Hall effect [17], Nernst effect [18, 19] and superconductivity [20] appear to have their dual gravitational descriptions. The gravitational dual of superconductivity is anchored on the understanding that close to a quantum critical point (QCP), the relevant zero temperature, scale invariant theories are analogous to field theories describing second-order phase transitions, such as Ginzburg-Landau theory of superconductivity [21, 22, 23, 24]. In the vicinity of QCP, spin systems which have a dynamical critical exponent  $z = 1$ , becomes invariant under re-scalings of time and distance. This scale invariant symmetry forms part of the larger conformal symmetry group  $SO(d+1, 2)$  of the quantum field theory [24, 25], where  $d$  is the number of spatial dimensions. The emergence of this symmetry implies that its dual gravity description resides in anti-de Sitter spacetime, where this symmetry can be generated as isometries of the background geometry [26, 27]. It is worthy of note that the gauge/gravity duality is well understood when the gravity theory is in anti-de Sitter spacetime and this makes its application to superconductivity very robust.

According to the model of holographic superconductivity proposed in [20], one can study a strongly coupled s-wave superconductors at a finite temperature and chemical potential, by considering a gravitational theory with an action which has a black hole solution<sup>3</sup>. The black hole in this example is charged under a  $U(1)$  gauge field with a minimally coupled complex scalar field. The ‘no hair theorem’<sup>4</sup> [30, 31] does not apply if the scalar field has a non-trivial coupling to the gauge field [32, 33, 34]. In this set

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interaction between quarks. The understanding of this regime of QCD is one of the major problems of theoretical physics [8].

<sup>3</sup>The presence of a black hole breaks all supersymmetry associated with the theory and also ensures that the transport properties are computed at finite temperature and chemical potential [28].

<sup>4</sup>No hair theorem states that "All stationary, asymptotically flat, four dimensional black hole equilibrium solutions of Einstein equations in vacuum or with an electromagnetic field are characterized by their mass, angular momentum and electric/magnetic charge or both" [29]

up, the symmetry breaking in the bulk theory which correspond to a phase transition to superconducting phase in the boundary theory is triggered by a position dependent negative mass squared formed from the gauge covariant derivative [33]. Its contribution becomes significant near the horizon of the black hole, thereby forcing the scalar field to condense. From the gravitational theory perspective, the resulting black hole is often called a superconducting black hole<sup>5</sup> [33, 38].

This model of holographic superconductivity has been studied in various limits by several authors. For example, the authors of [39, 40, 41, 42, 43, 44] mapped the phase diagram of the holographic superconductors in the presence of an external magnetic field. They also found and analyzed the physical properties of the vortex and droplets solutions for the scalar field of mass squared  $m^2 l^2 = -2$  ( $l$  will be defined shortly). The hydrodynamics of holographic superconductors was studied in detail in [45]. The effect of vector current on the order of the phase transition was explored in [46]. The authors of [47] showed that superconductivity is possible for scalar fields of various masses in  $d = 3$  and  $d = 4$  bulk dimensions. A proposal on how to calculate the superconducting characteristic length analytically in the vicinity of quantum critical point was suggested in [48] and the effects of gravitational backreaction was considered for  $m^2 l^2 = -2$  in [49, 50]. The results of these studies show a qualitative agreement with Ginzburg-Landau theory of superconductivity [26] and real superconductors [51].

The rest of the chapter is organized as follows: In section 1.2 we review some aspects of gauge/gravity correspondence proposed by Maldacena [1] and its generalizations in  $d$ -dimensions. A brief introduction to anti-de Sitter spacetime is presented in section 1.3. In section 1.4, we review a method of calculating the correlation functions from the gravitational theory. The description of quantum phase transition is provided in section 1.5. We narrowed down on quantum phase transition to continuous phase transition in superconductors in section 1.6. The aim and structure of the thesis is presented in section 1.7.

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<sup>5</sup>Superconducting black holes are a subset of stable black holes, that support matter fields as hair under certain condition. They belong to a more general class of Reissner-Nordstrom black holes. They have mass  $M$ , which is nearly equal to their charge  $Q$ , and hence they are near extremal black holes [35, 36, 37]

## 1.2 Gauge/Gravity Correspondence

The connection between type IIB string theory compactified on  $AdS_5 \times S^5$  and a  $\mathcal{N} = 4$  SUSY Yang-Mills gauge theory was motivated by the study of D-branes and black holes in string theory<sup>6</sup>. The breakthrough came from the study of excitations from  $N$  coincident  $D3$  branes. The excitations include the closed strings excitations and the open strings excitations. Closed strings excitations are the excitations of the spacetime, while the open strings excitations describe the excitations of the D-branes [54]. At energies lower than the string scale, where perturbative treatment is reliable, only the massless string states can be excited. The closed string massless states give a gravity supermultiplet in ten dimensions and its low energy lagrangian is that of type IIB supergravity. The open string massless states give a  $\mathcal{N} = 4$  vector supermultiplet in  $(3 + 1)$ -dimensions and the low energy effective lagrangian is that of  $\mathcal{N} = 4$  U(N) gauge theory [54]. The complete effective action of the massless modes have the form

$$I = I_{bulk} + I_{brane} + I_{int}. \quad (1.1)$$

$I_{bulk}$  is the action of the ten dimensional type IIB supergravity plus some higher derivative corrections, it is schematically given by

$$I_{bulk} \sim \frac{1}{2\kappa} \int \sqrt{gR} + F_5^2 \sim \int (\partial h)^2 + \kappa (\partial h)^2 h + F_5^2 + \dots \quad (1.2)$$

plus the self-duality constraint for the five form field strength,  $F_5 = *F_5$ . The five form flux is quantized

$$\int_{S_5} F_5 = N, \quad (1.3)$$

where  $N$  is the number of D-branes.  $I_{brane}$  is the action defined on the  $(3 + 1)$ -dimensional brane worldvolume. It contains the  $\mathcal{N} = 4$   $SU(N)$  SUSY-Yang-Mills lagrangian, which

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<sup>6</sup> $D_p$ -branes are solitons that come in various dimensions in string theory.  $D_0$ -brane are like ordinary localized particle, they have zero spatial dimensions. D-branes which have one extended dimension, are called  $D_1$  branes or D-strings. D-strings are much heavier than ordinary fundamental strings when the string coupling is small. The tension of D-branes is proportional to  $1/g_s$ , where  $g_s$  is the string coupling. D-branes serve as the surfaces where open strings can end.  $D_p$ -branes are charged under  $p + 1$ -form gauge potential [52]. The  $p + 1$  gauge potentials have an associated  $p + 2$  form field strength. The massless mode from the perturbation of the open strings describe the oscillations of the branes, a gauge field living on the brane and their fermionic partners since they are supersymmetric [53].

is schematically given by

$$\mathcal{L} = \frac{1}{g^2} \mathbf{Tr} \left[ F^2 + (D\phi)^2 \chi + \bar{\chi} D\chi + \sum_{IJ} [\phi^I, \phi^J]^2 + \bar{\chi} \Gamma^I \phi^I \chi \right] + \theta \mathbf{Tr} [F \wedge F], \quad (1.4)$$

where  $F$  is a non-abelian gauge field strength,  $\phi$  is a scalar field and  $\chi$  is one of the chiral fermions. The lagrangian contains two parameters, the coupling constant  $g$  and a  $\theta$  angle. The theory has zero beta function, i.e it is quantum mechanically scale invariant in all orders.  $I_{int}$  describes the interaction between the brane modes and the bulk modes. In the low energy limit, the interaction terms relating the bulk and the brane vanishes, leaving just the pure  $\mathcal{N} = 4$  U(N) gauge theory in  $(3 + 1)$ -dimensions [1]. This implies that in the low energy limit,  $I_{bulk}$  and the  $I_{brane}$  are two decoupled systems. The D-brane solution from the supergravity action,  $I_{bulk}$ , is given by [54],

$$\begin{aligned} ds^2 &= f^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2} (dr^2 + r^2 d^2\Omega_5), \\ F_5 &= (1 + \star) dt dx_1 dx_2 dx_3 df^{-1}, \\ f &= 1 + \frac{l^4}{r^4}, \\ l^4 &= 4\pi g_s \alpha^{1/2} N, \end{aligned} \quad (1.5)$$

where  $F_5$  is a five form field strength and  $l$  is a curvature radius. The difference in energy as measured by an observer at position  $r$  and an observer at infinity is given by

$$E_\infty = f^{-1/4} E_r \quad (1.6)$$

where the subscripts denote the location where the energy,  $E$ , is measured. For an observer at infinity (1.6), the excitations of the massless particle propagating in the bulk region with wavelength bigger than the gravitational radius of the brane decouples from the excitations that live very close to  $r = 0$ , i.e. the excitations near  $r = 0$  will find it very difficult to climb the gravitational potential and escape to infinity [55], thereby giving two decoupled systems, the free bulk supergravity and the near horizon region of the

geometry. In the near horizon region  $r \ll l$ , the geometry (1.5) becomes

$$ds^2 = \frac{r^2}{l^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{l^2}{r^2} (dr^2 + l^2 d^2\Omega_5). \quad (1.7)$$

Equation (1.7) is the geometry of  $AdS_5 \times S^5$ . From the perspective of the field theory living on the brane and that of the supergravity, there are two decoupled theories describing the same physics in this portion of the geometry. This identification is the conjecture that  $\mathcal{N} = 4, U(N)$  SUSY Yang-Mills theory in 3 + 1-dimensions is the same as type IIB superstring theory on  $AdS_5 \times S^5$ . Since Yang-Mills coupling is the same the string coupling<sup>7</sup> a perturbative analysis of Yang-Mills theory is possible when

$$g_{YM}^2 N \sim g_s N \sim \frac{l^4}{l_s^4} \ll 1 \quad (1.8)$$

and classical gravity description becomes reliable when the radius of curvature of  $AdS_5 \times S^5$ ,  $l$ , becomes large compared to the string length  $l_s$

$$\frac{l^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1 \quad (1.9)$$

Symmetry considerations indicate that  $AdS_5$  has the isometry  $SO(4, 2)$  and the  $S^5$  geometry has the isometry  $SO(6)$ . The theory has fermions that belong to the spinor representations, which have the covering group  $SU(2, 2)$  and  $SU(4)$  respectively. The bosonic subgroup  $SU(2, 2) \times SU(4)$  of the supergroup  $PSU(2, 2|4)$  is also realized by  $AdS_5$  geometry. All the 32 supersymmetries of the type IIB superstring theory is realized as the vacuum symmetries of the background geometry. On the field theory side, the  $\mathcal{N} = 4U(N)$  SUSY Yang-Mills theory has  $SU(4)$  symmetry which arises as the global  $SU(4) = SO(6)$  of the R-symmetry. This symmetry rotates the six scalar fields in the lagrangian (1.4) and it does not commute with any of the supercharges. The four chiral fermions transforms as a **4** and those of the opposite chirality transforms as a  $\bar{\mathbf{4}}$ , and the six scalar fields forms a **6**. The linearly realized supersymmetries account for 16 fermionic symmetries. Also there are additional 16 non-linearly realized symmetries. In general it yields the desired superconformal algebra  $PSU(2, 2|4)$  [1].

AdS/CFT correspondence may be realized in various  $d$ -dimensions, provided that the

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<sup>7</sup>The string coupling constant  $g_s$  is related to the gauge theory coupling constant  $g_{YM}$  via  $4\pi g_{YM}^2 = g_s$ .

symmetry of the conformal field theory<sup>8</sup> can be generated as isometries of the background geometry of the gravitational theory. The symmetry algebra of any field theory is most conveniently defined, at an ultraviolet fixed point, to ensure that the theory remains valid at all scales [26, 56]. The generators of CFT symmetry algebra include the rotations  $M_{ij}$ , translations  $P_j$ , time translations  $H$  and dilatation  $D$ . These generators satisfy the standard commutation relations for  $M_{ij}, P_k, H$  and the action of dilatation

$$\begin{aligned} [D, M_{ij}] &= 0, \\ [D, P_i] &= iP_i, \\ [D, H] &= izH, \end{aligned} \tag{1.11}$$

where  $z$  is the dynamical critical exponent. The commutation relations (1.11) can be realized from the gravitational theory for  $z = 1$ , by considering the general background geometry of the form

$$ds^2 = \frac{l^2}{r^2} (-dt^2 + dx^i dx^i + dr^2), \tag{1.12}$$

where  $i$  is an index which runs from 1 to  $d-2$ . The symmetry algebra of this background is generated by the following killing vectors

$$\begin{aligned} M_{ij} &= -ix^i \partial_j - x^j \partial_i, \\ P_i &= -i \partial_i, \\ H &= -\partial_t, \\ D &= -i(t \partial_t + x^i \partial_i + r \partial_r). \end{aligned} \tag{1.13}$$

The dynamical critical exponent  $z$  can take any positive value, for example  $z = 1$ , gives the anti-de Sitter spacetime and  $z = 2$  corresponds to theories with Galilean symmetry

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<sup>8</sup>Conformally invariant quantum field theories describe the critical behaviour of systems at second order phase transitions. The canonical example is the Ising model in two dimensions, with spins  $z = \pm 1$  on sites of a square lattice. At the phase transition, typical configurations have fluctuations on all length scales, so that field theory describing the model at its critical point should be expected to be invariant at least under changes of scale. Critical theories are generally invariant under the full conformal group. The conformal group is the group of transformation which preserve the form of the metric up to an arbitrary scale factor,  $g_{\mu\nu} \rightarrow \Omega^2 x g_{\mu\nu}(x)$  ( $\mu\nu = 0, \dots, d-1$ ). The conformal group of Minkowski space is generated by Poincare transformations, the scale transformation  $x^\mu \rightarrow \lambda x^\mu$  and the special conformal transformations.

$$x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2} \tag{1.10}$$

algebra<sup>9</sup>.

### 1.3 Anti de Sitter Space

The  $(p+2)$ -dimensional anti de Sitter space ( $AdS_{p+2}$ ) can be represented as a hyperboloid

$$X_0^2 + X_{p+2}^2 - \sum_{l=1}^{p+1} X_l^2 = R^2, \quad (1.14)$$

in the flat  $(p+3)$ -dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{l=1}^{p+1} dX_l^2 \quad (1.15)$$

The ambient space has two time directions, but the surface contains only one time direction, the other time direction is orthogonal to the surface. It has an isometry  $SO(2, p+1)$  and it is homogeneous and isotropic. By setting

$$\begin{aligned} X_0 &= R \cosh \rho \cos \tau, \\ X_{p+2} &= R \cosh \rho \sin \tau, \\ X_i &= R \sinh \rho \Omega_i, \\ &(i = 1, \dots, P+1; \sum_i \Omega_i^2 = 1) \end{aligned} \quad (1.16)$$

in equation (1.15) gives the metric on  $AdS_{p+2}$  as

$$ds^2 = R^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2) \quad (1.17)$$

For  $0 \leq \rho$  and  $0 \leq \tau < 2\pi$ , the solution (1.17) covers the entire hyperboloid once. The coordinates  $(\tau, \rho, \Omega_i)$  are called the global coordinates of AdS space. Near the boundary ( $\rho = 0$ ) the metric (1.17) behaves as

$$ds^2 \approx R^2(-d\tau^2 + d\rho^2 + \rho^2 d\Omega^2) \quad (1.18)$$

---

<sup>9</sup>This thesis is concerned with the case of  $z = 1$ . Discussion on the case of  $z = 2$  can be found in [6, 57, 7, 2, 58]

The hyperboloid has the topology of  $S^1 \times \mathbb{R}^{p+1}$ , with the  $S^1$  representing closed timelike curves in the  $\tau$  direction. The isometry group  $SO(2, p+1)$  of  $AdS_{p+2}$  has the maximal compact subgroup  $SO(2) \times SO(p+1)$ . The  $SO(2)$  part represents the constant translation in the  $\tau$  direction, and the  $SO(p+1)$  gives rotations of  $S^p$ . By introducing a coordinate  $\theta$ , which is related to  $\rho$  by  $\tan \theta = \sinh \rho$  ( $0 \leq \theta < \pi/2$ ), the asymptotic structure of the metric (1.17) takes the form

$$ds^2 = \frac{R^2}{\cos^2 \theta} (-d^2\tau + d^2\theta + \sin^2 \theta d^2\Omega). \quad (1.19)$$

The metric (1.19) is invariant under conformal rescaling. If a spacetime can be conformally compactified into a region which has the same boundary structure as the one half of the Einstein static universe, the spacetime is called asymptotically AdS. This form of spacetime (1.17) satisfies Einstein field equations with negative cosmological constant

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \Lambda g_{\mu\nu}, \quad (1.20)$$

where the cosmological constant is defined in terms of the radius of curvature of anti-de Sitter spacetime  $l$ ,  $R_{\mu\nu}$  is the Ricci tensor,  $R$  is the Ricci scalar and  $g_{\mu\nu}$  is the metric.

$$\Lambda = \frac{-d(d-1)}{2l^2}. \quad (1.21)$$

A spacetime  $(\mathcal{M}, g)$  is asymptotically AdS if there exists a spacetime  $(\hat{\mathcal{M}}, \hat{g})$ , where  $(\mathcal{M})$  is a manifold with boundary  $\partial\hat{\mathcal{M}}$  such that [59, 60];

- The manifold  $\mathcal{M}$  can be identified with the interior of  $\hat{\mathcal{M}}$  by a diffeomorphism from  $\mathcal{M}$  onto  $\hat{\mathcal{M}} - \partial\hat{\mathcal{M}}$
- There exist a positive function  $\Omega$  on  $\hat{\mathcal{M}}$ , known as a defining function, such that  $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ .  $\Omega$  has a first order pole and a non-vanishing gradient on  $\partial\hat{\mathcal{M}}$ .
- The Weyl tensor  $\hat{C}_{\lambda\mu\sigma\nu}$  on  $\hat{\mathcal{M}}$ , constructed from  $\hat{g}$ , is such that  $\Omega^{3-d}\hat{C}_{\lambda\mu\sigma\nu}$  is smooth on  $\hat{\mathcal{M}}$  and  $\hat{C}_{\lambda\mu\sigma\nu}$  vanishes on  $\partial\hat{\mathcal{M}}$ .
- The boundary  $\partial\hat{\mathcal{M}}$  is topologically  $\mathcal{R} \times S^{d-1}$

## 1.4 Correlation Functions

In any field theory, correlation functions are the most important physical observable [61, 62]. For example, in a theory of phase transitions information about the critical exponent can be extracted from the correlation functions [22, 48].

A universal/global operator in any conformal field theory is a symmetric stress-energy tensor  $T_{\mu\nu}$ . The correlation function of  $T_{\mu\nu}$  in the boundary theory can be calculated by considering the propagation of the gravitons in the AdS spacetime [63, 64]. Inserting  $T_{\mu\nu}$  in a correlation function corresponds to a graviton emitted from point  $x$  on the boundary of the AdS space. Thus computation of the correlations of the stress-energy components at different points in the field theory is translated to a problem of scattering of gravitons in the interior of the curved spacetime [63, 65]. In general the connection is made at the level of the generating functional for the correlations which is defined at the boundary of the AdS spacetime as

$$Z_{Gravity}[\phi_{(0)}] = \int_{\phi \sim \phi_{(0)}} D\phi \exp(-S[\phi]) = \left\langle \exp\left(-\int_{\partial AdS} \phi \mathcal{O}\right) \right\rangle_{CFT}. \quad (1.22)$$

The information encoded in equation (1.22) is that, the partition function of the string theory compactified on the AdS spacetime with prescribed boundary conditions for the bulk fields is equal to the generating functional of the conformal field theory correlation functions. The expectation value of an operator  $\langle \mathcal{O} \rangle$  is over the conformal field theory path integral and  $\partial AdS$  denotes the boundary of asymptotically AdS spacetime. The boundary value of the fields are now interpreted as sources for the operators of the dual conformal field theory [65, 66]. Correlation functions of the operator  $\mathcal{O}$  are computed by functional differentiation with respect to the source [67, 68, 69]

$$\langle \mathcal{O} \rangle = \left. \frac{\delta S_{onshell}}{\delta \phi_{(0)}(x)} \right|_{\phi_{(0)}=0}. \quad (1.23)$$

where  $S_{onshell}$  is the on-shell value of the supergravity action, in this regime,  $\hbar$  of the gravity theory has been taken to zero so that the path integral may be approximated by evaluating the measure of the classical gravity action. Higher-point functions can also be calculated by higher order functional differentiation. Calculation of correlation functions can be generalized to any number of fields that couple to the classical gravity

action  $S[\phi, A_\mu, g_{\mu\nu}\dots]$ . The dictionary is that there exist exact renormalizable one-point function, one for each of the bulk fields [67, 69, 68]

$$\begin{aligned}\Phi &\longrightarrow \langle \mathcal{O}(x) \rangle_s = \frac{1}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{ren}}{\delta \phi_{(0)}(x)}, \\ A_\mu &\longrightarrow \langle J_i(x) \rangle_s = \frac{1}{g_{(0)}(x)} \frac{\delta S_{ren}}{\delta A_{i(0)}(x)}, \\ g_{\mu\nu} &\longrightarrow \langle T_{ij}(x) \rangle_s = \frac{2}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{ren}}{\delta g_{ij}(x)},\end{aligned}\tag{1.24}$$

where  $J_i$  is the boundary symmetry current that couples to the bulk gauge field  $A_\mu$ .  $T_{ij}$  is the boundary stress energy tensor that couples to the boundary metric  $g_{(0)ij}$ ,  $\lambda$  is the scaling dimension of the boundary operator, it satisfies the relation  $\lambda(\lambda - d) = m^2 l^2$ . The fields which appeared on the right hand side of equation (1.24)  $\phi_{(2\lambda-d)}$ ,  $A_i(x)$ ,  $g_{(d)ij}$  result from the asymptotic expansion of the solutions of the bulk field equation. The operators are evaluated at the boundary of the AdS spacetime. The subscript ‘s’ indicates that the expectation values are computed in the presence of the sources. The subscript ‘ren’ stands for renormalized on-shell action. The on-shell action requires renormalization because the choice of background metric needed to define the field theory is related by an infinite conformal transformation to the induced metric on the boundary of the AdS spacetime [69].

## 1.5 Quantum Phase Transitions

Phase transitions occur when a material changes its properties in a spectacular way. It is common in everyday life; for example, the boiling of water or the melting of ice. Cosmologically, it is believed that the universe underwent a series of phase transitions, which resulted in the formation of structures. [35]. As common as phase transition might be, its understanding at zero temperature remains an unsolved problem in theoretical condensed matter physics [8]. It is believed that presence of such zero temperature quantum critical points (QCP) holds the key to so far unsolved puzzles in many condensed matter systems [23]. Examples include rare-earth magnetic insulators, heavy fermion compounds, high- $T_c$  superconductors [70].

In the vicinity of zero temperature phase transition, a non-thermal control parameter such as pressure, magnetic field or chemical potential is varied to access the critical point [70]. In this case the order is destroyed solely by quantum fluctuations which are the

manifestations of the Heisenberg uncertainty principle. For a larger class of these systems, it is believed that the dynamics of the phase near the QCP requires novel theories that have no analogue in the traditional framework of phase transitions [23, 71].

Phase transitions are classified into first-order and second-order or continuous phase transitions. At first order transitions the two phases co-exist at the critical temperature, examples include ice and water at  $0^\circ C$ , water and steam at  $100^\circ C$ . In contrast, at second order phase transition, the two phases do not co-exist. An important example is the ferromagnetic transition at  $770^\circ C$  above which the magnetic moment vanishes [23]. The transition point of the second-order phase transition is also called the critical point [21, 72]. Second-order phase transition is characterized by a thermodynamic quantity called order parameter. An order parameter is zero in one phase (disordered phase) and non-zero and non-unique in the other (the ordered) phase [73]. In the vicinity of the critical point, the spatial correlation of the order parameter fluctuations become long-ranged, hence the typical scale, i.e the correlation length,  $\xi$ , diverges as

$$\xi \propto |t|^{-\nu}, \quad (1.25)$$

where  $\nu$  is the correlation length critical exponent and  $t$  is dimensionless the measure of the distance from the critical point. It is defined as  $t = |T - T_c|/T_c$ , where  $T_c$  is the critical temperature. Close to the quantum critical point, the system also has an energy scale  $\Delta$  associated with the energy difference between the ground and excited state.

$$\Delta \propto |t|^{-\nu z} \quad (1.26)$$

The energy scale and the correlation length are related through the dynamical critical exponent  $z$ , as  $\Delta \propto \xi^z$ . Close to the critical point there is no characteristic length scale other than  $\xi$  and no other characteristic energy scale other than  $\Delta$ . This implies that the system is invariant under the rescalings of time and distance [25],  $t \rightarrow \lambda^z t$ , and  $x \rightarrow \lambda x$ . Different values of  $z$  occur in various condensed matter systems, for example  $z = 1$  is common for spin systems. The case of  $z = 1$  is very special because the quantum critical system has a Lorentz symmetry and the scaling becomes part of a larger conformal symmetry group  $SO(d+1, 2)$  for a system in  $d$  spatial dimensions. The system with  $z = 2$ , is invariant under free Schrödinger group (see for example [5] for details).

## 1.6 The Ginzburg-Landau Theory of Superconductivity

Superconductivity is an example of a quantum phase transition [73], it is characterized by a transition at a critical temperature from a normal phase to a phase with no resistance. The microscopic theory of superconductivity was proposed by Bardeen Cooper and Schrieffer [21], and now known as the BCS theory of superconductivity. This effective theory describes superconductivity as pairing between particles of spin one-half close to the Fermi level to form a spin zero particle which minimizes the energy of the system. In electronic systems, this interaction is mediated by acoustic phonons, which screens electron-electron repulsion.

An effective theory due to Ginzburg and Landau [21], provides a very instructive description of superconductivity in the vicinity of the critical point. The approach is based on the earlier work by Landau on the theory of second order phase transitions [73], in which the free energy is expanded in terms of the order parameter (the order parameter is small close to the QCP). For a system of fermions coupled to the  $U(1)$  gauge field  $A_\mu$ , the order parameter is  $\psi = \langle c^{\alpha\beta} \psi_\alpha \psi_\beta \rangle$ . It breaks the  $U(1)$  gauge invariance spontaneously. Assuming the system is time independent and the order parameter and its gradient are small, the Lagrangian of the system can be expanded as

$$\mathcal{L} = \int d^3x \left( -\frac{1}{2m} |(\nabla + 2ie\vec{A})\psi|^2 + \frac{1}{2} m_H^2 (\psi^* \psi) + \frac{1}{2} g (\psi^* \psi)^2 \right), \quad (1.27)$$

where  $m_H$  and  $g(T)$  are unknown temperature dependent parameters of the theory. The scalar field can be decomposed as follows

$$\psi = \rho e^{2ie\eta}. \quad (1.28)$$

The potential is then defined as

$$V(\rho) = -\frac{1}{2} m_H^2 \rho^2 + \frac{1}{4} g \rho^4. \quad (1.29)$$

The minimum of the free energy is achieved when  $\rho^2 = m_H^2/g$ . In the presence of a magnetic field the Ginzburg-Landau lagrangian in the vicinity of phase transition  $m_H(T_c) = 0$

is given by

$$\mathcal{E} = \int d^3x \left\{ \frac{\hbar^2}{2m} \left| \left( \nabla - \frac{iqA(x)}{\hbar c} \right) \psi(x) \right|^2 - m_H^2 |\psi(x)|^2 + \frac{g}{2} |\psi(x)|^4 + \frac{1}{8\pi} (\nabla \times A(x))^2 \right\}. \quad (1.30)$$

The equations of motion obtained by minimizing with respect to the vector potential  $A$  gives,

$$\frac{1}{4\pi} (\nabla^2 A - \nabla(\nabla \cdot A)) = \frac{\hbar q}{4mi} \left( \psi^\dagger \left( \nabla - \frac{iq}{\hbar c} A \right) \psi - \left( \nabla - \frac{iq}{\hbar c} A \right) \psi^\dagger \psi \right). \quad (1.31)$$

Minimization of the free energy with respect to the order parameter  $\psi$  yields,

$$\frac{\hbar^2}{2m} \left( \nabla^2 - \frac{iq}{\hbar c} A \right)^2 \psi = -m_H^2 \psi + g |\psi|^2 \psi \quad (1.32)$$

Equation (1.31) can be written as

$$\nabla \times \nabla \times B(x) = -\frac{4\pi q^2 m_H^2}{m_H c^2 g^2} A(x) \quad (1.33)$$

This implies that an external magnetic field  $\vec{B}$  decays over a length scale, given by

$$\lambda_m = \left[ \frac{c^2}{4\pi q^2 \frac{m_H}{g^3}} \right]^{1/2}. \quad (1.34)$$

The solution is  $B(x) = e^{-x/\lambda}$  which indicates that the magnetic field penetrates  $\lambda$  past the surface of the superconductor. Similarly equation (1.32) in the neighbourhood of phase transition gives

$$\nabla^2 \psi = -m_H^2 \psi \quad (1.35)$$

which implies that the variation of the order parameter relaxes over a characteristic length scale

$$\xi = \frac{1}{m_H}. \quad (1.36)$$

These length scales are inversely proportional to the order parameter, hence as  $T \rightarrow T_c$ , both of them diverge. The two length scales, the magnetic penetration depth and the

superconducting coherence length constitute the Ginzburg-Landau parameter

$$\kappa = \frac{\lambda_m}{\xi}. \quad (1.37)$$

The relative size of  $\lambda_m$  and  $\xi$  distinguishes between type II and type I superconductors. i.e  $\kappa < 1/\sqrt{2}$  for type I superconductors and  $\kappa > 1/\sqrt{2}$  for type II superconductors. In type a II superconductor, magnetic flux can penetrate the sample in the form of vortices. At the core of a vortex, the order parameter vanishes ( $\rho = 0$ ). The core is much smaller than the region over which the magnetic field goes to zero. The magnetic flux can be computed as

$$\int_{\Lambda} \vec{B} \cdot \vec{S} = \oint_{\partial A} \vec{A} \cdot d\vec{l} = \oint_{\partial A} \vec{\nabla} \psi \cdot d\vec{l} = \frac{n\pi\hbar}{e}. \quad (1.38)$$

It is quantized in units of  $\pi\hbar/e$ . In a type II superconductor, magnetic vortices repel each other and form a regular lattice known as the Abrikosov lattice [73]. However in a type I material  $\xi > \lambda_B$ , vortices are not stable and the magnetic field can only penetrate the sample if the superconductivity is destroyed.

## 1.7 Aim and Structure

The proposal that quantum phase transitions in condensed matter systems may be described using the gauge/gravity duality approach is still at its infancy, for example the studies so far are limited to the case of  $m^2 l^2 = -2$ , with the exception of [47, 51], that studied the value of  $m^2 l^2$  that saturate the Brietenloner-Freedmen stability bound. There has not been any study that show how most of the physical quantities associated with the model depend on the scaling dimensions of the dual condensates. The possible classification scheme for holographic superconductors had not been deduced and it is still not clear if the holographic superconductor is a type I or type II. The study by [50] suggests that it is a type II, while the result of [48] says it is type I. It is against this backdrop that this thesis aims:

- To extend the holographic model of superconductivity proposed in [20], to include scalar fields with various mass squared that satisfy the unitarity bound in  $d$ -dimensions. Then study the dependence of most of the physical quantities associated the model on the scaling dimensions of the dual condensates.

- To compute the superconducting coherence length, the magnetic penetration depth, and the Ginzburg-Landau parameter for all the condensates using the perturbative approach proposed in [48]. Then identify a possible classification scheme for holographic superconductors based on the dependence of the physical quantities on the scaling dimensions of the dual condensates.

This thesis is organized as follows: in chapter 2 we present a review of the mechanism driving superconducting instability in the bulk theory and show that a marginally stable mode exists. Latter part of chapter 2 is devoted to the holographic description of superconductivity in  $(2 + 1)$ -dimensional conformal field theory for  $m^2 l^2 = -2$ . This chapter is essentially a review of most of the works on s-wave holographic superconductors. Chapter 3 is where the new results of this work is presented. It has five sections that cover phase transition, conductivity and superconducting characteristic length scale in  $(2 + 1)$ -dimensional and  $(3 + 1)$ -dimensional boundary theory for all the condensates. Discussions and conclusion are presented in Chapter 4. Some of the results are presented at the appendices.

# Chapter 2

## A REVIEW OF HOLOGRAPHIC SUPERCONDUCTORS

### 2.1 Introduction

Holographic superconductors are those strongly coupled superconductors that have a dual gravitational description [50]. Strongly interacting condensed matter systems which preserve Lorentz invariance include all spin systems with dynamical critical exponent  $z = 1$ . The proposal in [20] is that the physics in the vicinity of QCP of these systems may be described by an Abelian-Higgs system coupled to a black hole in anti-de Sitter space-time. The event horizon of the black hole and the coupling between the gauge field and scalar field through the gauge covariant derivative are responsible for the superconducting instability in the gravitational theory.

Prior to the proposal that the event horizon of a charged black hole could trigger a superconducting instability which breaks the  $U(1)$  gauge symmetry of an Abelian-Higgs system, geometry considerations [74, 75, 76] also indicate that a charged black hole exhibits some properties of a superconductor, e.g. diamagnetism. The argument, due to Gubser [33] is that the coupling of the charged scalar field to the  $U(1)$  gauge field through a gauge covariant derivative, contributes an extra negative mass squared to the mass squared of the scalar field. The extra negative mass squared<sup>1</sup> becomes significant near

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<sup>1</sup>The position dependent mass squared is negative because of the signature of the time component of the black hole metric.

the horizon of the black hole, i.e where  $g_{tt} \rightarrow 0$ , forcing the scalar field to condense near the horizon. This mechanism of symmetry breaking is sometimes referred to as position dependent Higgs mechanism [20, 39, 46].

However, as  $g_{tt} \rightarrow 0$ , the position dependent mass squared diverges. This divergence requires that a boundary condition must be imposed to ensure that the scalar potential is well-behaved at the horizon. This constraint does not cancel the entire effect of the position dependent mass, if the charge of the black hole is large enough and that the scalar field is strongly coupled [33]. If these conditions are satisfied, then its contribution will still be significant near the horizon. In this large charge limit, the stability of the charged condensate is almost guaranteed, if the black hole is in anti de Sitter spacetime, since the charge of the black hole cannot expel the massive charged condensate to infinity in a finite proper time [77].

In line with the gauge/gravity correspondence, the symmetry breaking in the gravitational theory corresponds to a breaking of symmetry in the boundary theory. Although there is a technical subtlety in this interpretation, since the broken symmetry in the bulk is a local  $U(1)$  gauge field while the boundary theory is a global  $U(1)$  theory. The major difference is that the current in the global  $U(1)$  gauge theory does not generate its own magnetic field. The work of [50, 49, 33] suggests that one can weakly gauge the boundary theory.

This chapter is organized as follows: in section 2.2 a discussion on the mechanism for generating a holographic superconducting instability is presented. In section 2.3, the bulk equations of motion is solved numerically in order to calculate the non-trivial solution describing the superconducting phase. A review of conductivity in the dual  $(2+1)$ dimensional boundary theory is presented in section 2.4. In section 2.5, a summary of other works on holographic s-wave superconductors is presented. This chapter ends with a conclusion in section 2.6. This chapter is largely based on the works of [20] and [33]

## 2.2 Criterion for Superconductivity

The action for the dual gravitational theory of superconductivity is given by

$$I = I_{EH} + I_{matter}. \quad (2.1)$$

$I_{EH}$  is the Einstein-Hilbert action with negative cosmological constant  $\Lambda$ , it is given by

$$I_{EH} = \frac{1}{2\kappa_4} \int d^4x \sqrt{-g} \left\{ R + \frac{6}{l^2} \right\}, \quad (2.2)$$

where  $\kappa_4$  is related to Newton gravitational constant in four dimensions  $\kappa_4 = 8\pi G_N$  and the cosmological constant can be defined in terms of the radius of curvature of AdS spacetime,  $l$ , by  $\Lambda = \frac{6}{l^2}$ . The matter action  $I_{matter}$  is the Abelian-Higgs system action given by

$$I_{matter} = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |\partial\Psi - iqA\Psi|^2 - V(\Psi) \right\}. \quad (2.3)$$

The gauge field and scalar field are coupled through the gauge covariant derivative,  $D_\mu = \partial_\mu + iqA_\mu$ . Here  $\partial_\mu$  is the space time covariant derivative,  $A_\mu$  is the gauge field, with associated field strength  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ .  $\Psi$  is a complex scalar field with the charge  $q$ . We consider the following ansatz in the probe limit:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d^2\Omega_{2,k} \\ \Psi &= \Psi(r) \\ A_\mu dx^\mu &= \Phi dt \\ f(r) &= \frac{r^2}{l^2} + k - \frac{M}{r}, \end{aligned} \quad (2.4)$$

where  $\Phi$  is the scalar potential and  $k$  defines the curvature of the spacetime. For  $k = 0$ ,  $d^2\Omega_{2,k}$  defines the line element of a flat space  $\mathbb{R}^2$ . If  $k > 0$ , then  $d^2\Omega_{2,k}$  is the metric of a 2-sphere. whereas, if  $k < 0$ , then  $d^2\Omega_{2,k}$  is the metric of the hyperbolic plane with radius of curvature  $1/\sqrt{-k^2}$  [33]. The case of  $k = 0$  which corresponds to having a conformal field theory in flat space at the AdS boundary is considered. The spacetime is incomplete as a result of the horizon at  $r = r_+$ , which is the largest solution to the equation

$$\frac{r^2}{l^2} + k - \frac{M}{r} = 0. \quad (2.5)$$

The Hawking temperature of the black hole is obtained by performing a Wick rotation ( $t = it$ ) to the Euclidean sector of the black hole ansatz and requiring regularity at the

horizon, the period is identified with the inverse temperature ( $\beta = 4\pi/f'(r)$ ), which gives

$$T = \frac{3M^{1/3}}{4\pi l^{4/3}}. \quad (2.6)$$

With these ansatze. the lagrangian of a decoupled Abelian-Higgs system becomes,

$$\mathcal{L} = \frac{1}{16\pi G_N} (-g^{rr}g^{tt}|\partial_r A_t|^2 - g^{rr}|\partial_r \Psi|^2 - g^{tt}q^2\Phi^2|\Psi|^2 - m^2|\Psi|^2). \quad (2.7)$$

where a quadratic potential studied in [33] is adopted as well and  $m$  is the mass of the scalar field. The effective mass squared of the scalar field in (2.7) becomes

$$m_{eff}^2 = m^2 + g^{tt}q^2\Phi^2. \quad (2.8)$$

The mass of the scalar field gets an additional piece due to the coupling of the scalar field to the charge of the black hole. The equations of motion from (2.7) becomes

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \frac{\Phi^2\Psi}{f^2} - \frac{m^2\Psi}{f} = 0, \quad (2.9)$$

$$\Phi'' + \frac{2}{r}\Phi - \frac{2\Psi^2}{f}\Phi = 0. \quad (2.10)$$

The regularity condition at the horizon implies that  $\Phi$  must vanish there. This leads to the following constraint for the scalar field equation at the horizon

$$\Psi' = -\frac{2\Psi}{3r_+}. \quad (2.11)$$

To ensure stability of AdS vacuum in 4-dimensions, the negative mass squared must satisfy the Breitenlohner and Freedman bound [78]  $m^2 \geq -9/4$  and in general the unitarity bound<sup>2</sup>  $\lambda \geq 1/2$  [79]. With the choice of  $m^2 = -2$  in equation (2.9) and (2.10), and integrating the equations out to the AdS boundary, the scalar field and the scalar potential falls off as

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \dots \quad (2.12)$$

---

<sup>2</sup> $m^2$  and  $\lambda$  are related through the constraint  $\lambda(\lambda - d) = m^2 l^2$

and

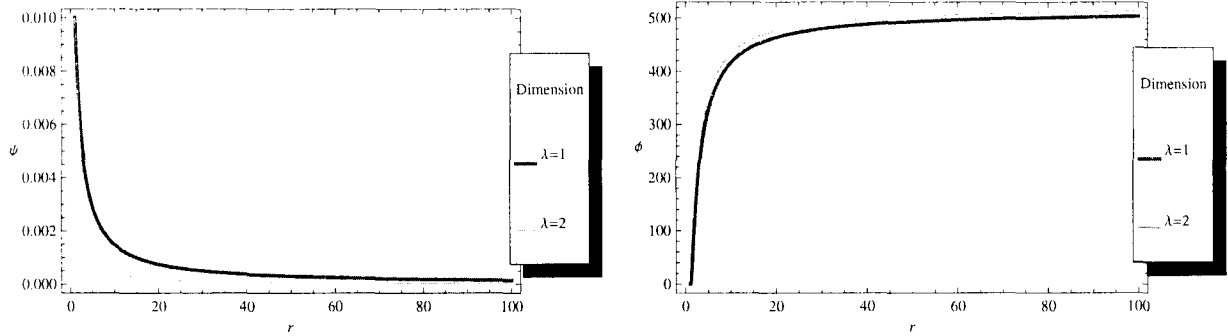
$$\Phi = \mu - \frac{\rho}{r} + \dots \quad (2.13)$$

respectively. Generally scalar field equation normally has two solutions—one normalizable and one that diverges at the boundary. The solution that diverges at the boundary is the source for the dual field. The presence of the divergent solution means that the (AdS invariant) quantization of the field is unique. In the case at hand, the (AdS-invariant) quantization of the field is not unique, since both solutions are normalizable. At the AdS boundary, these values have various re-interpretations,  $\mu$  is interpreted as the chemical potential while  $\rho$  is the charge density. There exists an ambiguity in the case of scalar field since both  $\Psi_1$  and  $\Psi_2$  are normalizable. Scalar operator  $\mathcal{O}_1$  of conformal dimension one, is identified as  $\Psi^{(1)} = \langle \mathcal{O}_1 \rangle$ , with  $\Psi^{(2)}$  as the source. Then the operator  $\mathcal{O}_2$  of conformal dimension two is identified as  $\Psi^{(2)} = \langle \mathcal{O}_2 \rangle$ , while  $\Psi^{(1)}$  is the source. The critical temperature in the boundary theory is defined by imposing a Neumann boundary condition on one of the condensates at the AdS boundary. A phase transition occurs when the vacuum expectation value does not have a source.

Analytical solutions to equations (2.9) and (2.10) are the trivial solutions  $\Psi = 0$  and  $\Phi = \mu - \rho/r$ . Other solutions are obtained numerically with suitable boundary conditions. In anti de Sitter spacetime, one of the boundary conditions is that the fields must fall off as described in equation (2.12) and also must respect the regularity conditions at the horizon. Since  $r = r_+$  is a singular point of the equations of motion, and hence is not a good place to set up initial data, equations (2.9) and (2.10) are first solved by a series expansion to fifth order near the horizon. With the series approximation, an initial data may be set up at a very small distance away from the horizon. The marginally stable mode obtained by solving equations (2.9) and (2.10) numerically is shown in Figure 2.1. The graphs are labelled by the dimensions of the dual operator.

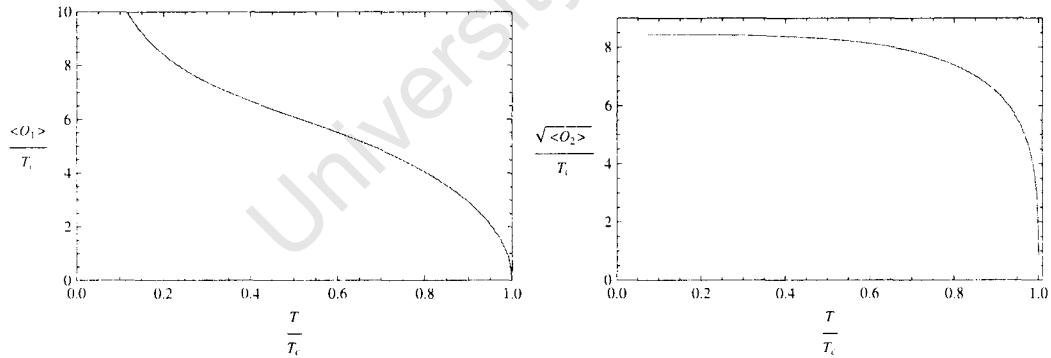
## 2.3 Phase Transition

In this section, equations (2.9) and (2.10) will be solved numerically to calculate the relevant physical quantities in the boundary theory. These quantities may be calculated by first, finding an approximate series solutions which have the required fall-offs in the neighborhood of the AdS boundary. Then the initial data at the horizon, are evolved to



**Figure 2.1:** Marginally stable modes for the scalar field together with plots of the scalar potential as function of the radial coordinate

match with the approximate series solution near the boundary. The solution describing the superconducting phase is obtained by imposing a Neumann boundary condition for  $\Psi^{(1)}$  or  $\Psi^{(2)}$ , hence we have two condensates for  $m^2 = -2$ . Figure 2.2 is the solutions obtained by solving equations (2.9) and (2.10) numerically using the shooting method. The condensate appears below the critical temperature defined in terms of the charge density or chemical potential of the dual field theory. The critical temperature is related



**Figure 2.2:** The condensates, as a function of temperature for the two operators  $\mathcal{O}_1$  and  $\mathcal{O}_2$ .

to the charge as  $T_c \sim \rho^{1/2}$ , which is obtained from dimensional analysis, this ensures that only dimensionless quantities are plotted. A curve-fitting analysis applied to figure 2.2 close to the critical temperature ( $T \sim T_c$ ), indicates a square root type behavior typical

of a second order phase transition. For the condensate of dimension one

$$\langle \mathcal{O} \rangle_1 \approx 9.3T_c (1 - T/T_c)^{1/2}, \quad (2.14)$$

where  $T_c \approx 0.226\rho^{1/2}$ , and for the condensate of dimension two

$$\langle \mathcal{O} \rangle \approx 144T_c^2(1 - T/T_c)^{1/2} \quad (2.15)$$

with  $T_c \approx 0.118\rho^{1/2}$ . Observe that the condensates/order parameters vanish at the critical temperature in a square-root type transition, typical of Ginzburg-Landau theory.

## 2.4 Conductivity

The unique property of a superconducting sample is the infinite DC conductivity. Conductivity in the dual quantum field theory is related to retarded current-current two-point function of the global  $U(1)$  symmetry. Under the frame work of AdS/ CFT correspondence, this may be calculated from the electromagnetic fluctuations of the bulk Maxwell field. The electromagnetic perturbation of the hairy black hole at zero spatial momentum and with a time dependence of the form  $\delta A = A_x(r) e^{-i\omega t} dx$  results in a decoupled equation of motion,

$$A_x'' + \frac{f'}{f} A_x' + \left( \frac{\omega}{f^2} - \frac{2\Psi^2}{f} \right) A_x = 0. \quad (2.16)$$

The prescription [80, 81] for computing the retarded Greens function from the bulk theory, requires equation (2.16) to be solved with an ingoing wave boundary condition at the horizon. This is to ensure that near horizon oscillations are removed from the Maxwell field  $A_x$ . The boundary condition is given by

$$A_x(r) = f(r)^{-4\pi i\omega/T} A_x(r), \quad (2.17)$$

where  $T$  is the Hawking temperature of the black hole background. Near the AdS boundary, the solution to equation (2.16) falls off as

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r} + \dots \quad (2.18)$$

From the AdS/CFT dictionary (see chapter 1), the slower fall off is the dual source, while the faster fall off is the expectation value of the current.

$$\begin{aligned} A_x &= A_x^{(0)}, \\ \langle J_x \rangle &= A_x^{(1)}. \end{aligned} \quad (2.19)$$

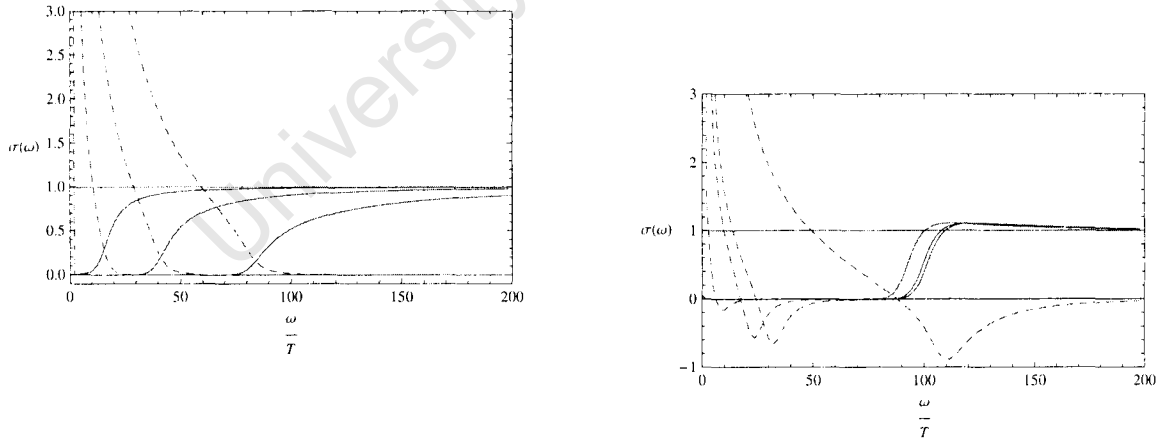
The conductivity can easily be computed using Ohm's law,

$$\sigma(\omega) = \frac{\langle J_x \rangle}{E_x}, \quad (2.20)$$

where  $E = -\dot{A}_x = -i\omega A_x$ . Upon substitution

$$\sigma(\omega) = \frac{A_x^{(1)}}{i\omega A_x^{(0)}}. \quad (2.21)$$

The real part of the frequency dependent conductivity, computed by solving equation (2.16) numerically, is shown in the figure 2.3 for the superconducting condensates of dimension one and two. The conductivity has a delta function at  $\omega = 0$ , but it is difficult



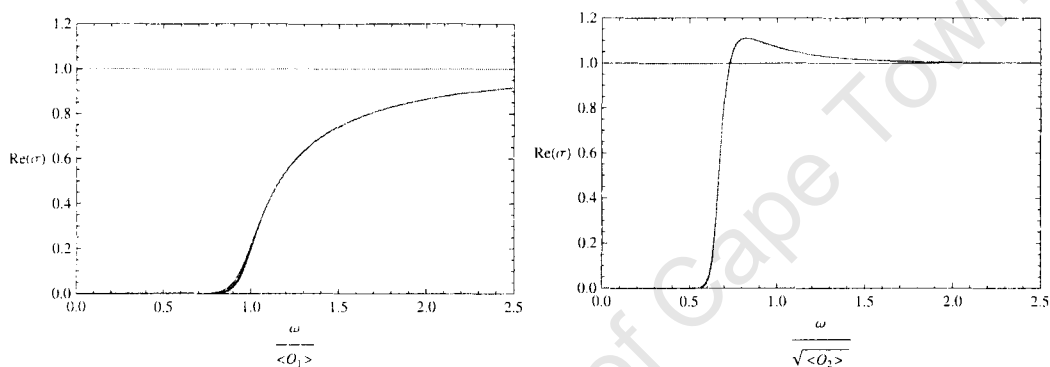
**Figure 2.3:** The real part of the frequency dependent conductivity for the superconducting condensate of dimension one  $\mathcal{O}_1$ (right) and the condensate of dimension two  $\mathcal{O}_2$ (left). The gap opens up below the critical temperature. The subsequent curves describe successive lower temperatures. The conductivity is plotted at temperature less  $T/T_c = 0.2$

to resolve it numerically. The delta function appears below the critical temperature. At

the critical temperature, the conductivity is independent of the frequency. The gap opens up at a critical frequency  $\omega_g$ , defined as the frequency at which the imaginary part  $Im[\sigma]$  of the conductivity is minimum. The value of  $\omega_g$  calculated from the numerics for the two operators is approximately

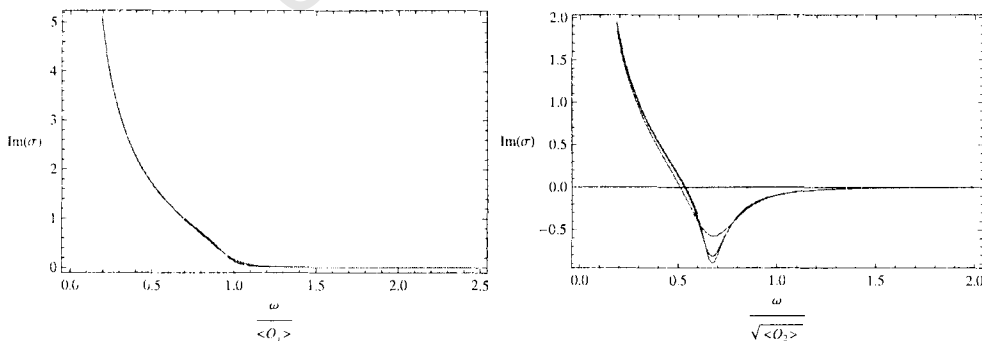
$$\frac{\omega_g}{T_c} \approx 8. \quad (2.22)$$

Figure 2.4, shows the real part of the conductivity versus frequency normalized by the condensate at low temperature. According to [20], the slower rise of conductivity for



**Figure 2.4:** The real part of the conductivity with the frequency normalized by the condensate.

the dimension one condensate and the suppression at the peak of conductivity in the dimension two condensate are features expected from superconductors of type I and type II coherence factors respectively. The imaginary part of conductivity at small temperature shows the pole at  $\omega = 0$ , as can be seen in the figure 2.5 below. The superfluid density



**Figure 2.5:** The imaginary part of the conductivity at very low temperature  $T/T_c = 0.1$

$n_n$  in the normal phase can be calculated from the real part of conductivity in the limit

of  $\omega \rightarrow 0$ . From the numerics, we observe that this quantity is exponentially suppressed as

$$n_n = \text{Re}[\sigma(\omega)] \approx e^{-\Delta/T_c}, \quad (2.23)$$

where  $\Delta$  is defined as the energy gap. This form of an exponential suppression is interpreted as an evidence that an energy gap exist in any superconducting sample [73].

## 2.5 Summary of Other Related Works

In this section, we present a brief account of other related works on the holographic  $s$ -wave superconductors. This will help to sharpen our focus on the objectives of this thesis. The works include:

- A study of the effects of a constant external magnetic field on the superconducting condensate. It was shown that a family of condensates can form and, for a finite magnetic field, the condensates were found to be localized in dimension with a profile that is exactly solvable. It was also shown that, as the magnetic field increases, the condensate shrinks in size which is a feature related to the Meissner effect [39]. More recent studies including the effects of gravitational back-reaction provided more evidence for the existence of stable vortex and droplets solutions [40, 41, 42, 43, 44]
- The effect of gravitational backreaction was considered in [50], where it was shown that the qualitative features of the dual superconductor are similar for all charge limits. Signatures indicating an existence of a new type of instability were observed, which the Authors suggested that it might be due to emergence of an  $AdS_2$  symmetry in the zero charge limit. Their argument is largely based on the differences in the Breitenlohner-Freedman stability bound [78] for scalar fields of negative mass squared in  $AdS_4$  and  $AdS_2$ . They also suggested, based on the results obtained from weakly gauging the global  $U(1)$  gauge field at the boundary, that the holographic superconductors may be of Type II.
- Studies beyond probe limit by [49], shows that superconducting black hole has a lower free energy than a bare Reissner-Nordstrom black hole. A domain wall-like structure was observed in the small charge limit.

- It was shown in [46] that black holes in anti de Sitter spacetime can also support vector hair. Since an additional vector field decreases the amount of position dependent mass, the authors argued that the presence of magnetic field will make the condensation of the scalar field less likely if the black hole is more magnetically charged.
- The extension of the model to include various condensates in  $2 + 1$  and  $3 + 1$ -dimensional boundary theory was done in [47], The authors observed a gap in the frequency dependent conductivity for all the condensates.
- A method of calculating the superconducting characteristic lengths perturbatively was suggested in [48] for  $m^2 = -2$  in  $3 + 1$ -dimensional boundary theory. It was shown analytically that the superconducting coherence length near the critical temperature behaves as in the case of the Ginzburg-Landau theory. By adding an external homogeneous magnetic field, it was also found that a stationary diamagnetic current is proportional to the square-root of the order parameter, and that Ginzburg-Landau parameter ( $\kappa = \lambda_m/\xi$ ) is proportional to the critical temperature.
- Signatures of the butterfly effect in the holographic superconductors were demonstrated in [51]. The authors calculated the specific heat capacity for  $m^2 = -2$  and  $m^2 = 1.36$  in  $2 + 1$ -dimensional boundary theory and observed a significant correlation with the experimentally measured values.
- A superconducting instability was also observed in [82], when the black hole spacetime is Kerr-Newman.

## 2.6 Conclusion

In this chapter a discussion of the mechanism and conditions for a superconducting instability in the bulk theory was presented. It was shown that a marginally stable mode exists for the scalar field in the bulk theory. Under the frame work of gauge/gravity correspondence it was demonstrated that the superconducting instability in the bulk corresponds to a second order phase transition (to the superconducting phase) in the boundary theory. The conductivity calculated for these condensates have features, which are in agreement

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with the Ginzburg-Landau theory. A brief discussion on several other related works were also presented.

University of Cape Town

# Chapter 3

## PROPERTIES OF HOLOGRAPHIC SUPERCONDUCTORS

### 3.1 Introduction

This chapter is devoted to the study various physical quantities associated with holographic s-wave superconductors as functions of the scaling dimensions of the dual condensates. A bulk scalar field with negative mass squared  $m^2$ , satisfying the Breitenlohner-Freedman stability bound and the unitarity bound, and allowed to vary in 0.5 unit intervals, were considered. We observe that all the physical quantities investigated are sensitive to the scaling dimensions of the dual condensates. For all the  $m^2$ , the characteristic lengths diverge at the critical temperature in agreement with the Ginzburg-Landau theory. The Ginzburg-Landau parameter, obtained from these length scales indicates that the holographic superconductors can be type I or type II depending on the charge and the scaling dimensions of the dual condensates. For a fixed charge, there exists a critical scaling dimension, above which a holographic superconductor is type I, below which it becomes a type II

This chapter is organized as follows: In section 3.2, we define our conventions and derive the equations of motion. In section 3.3, we show that the superconducting phase of holographic superconductors of the class  $\Psi_{\lambda_-}$  is very different from that of the class  $\Psi_{\lambda_+}$ . We present a discussion of the conductivity in section 3.4 and show that in the limit in which the frequency  $\omega$  approaches zero ( $\omega \approx 0$ ), the superfluid density can be obtained

from the frequency dependent conductivity. In section 3.5, we solve the equations of motion perturbatively in order to calculate the characteristic lengths and the Ginzburg-Landau parameter. The conclusion is provided in section 3.6, while various results relating to the conductivity in the boundary theory are presented in the appendices.

## 3.2 Background Equations of Motion

The action of a gravitational theory with a  $d + 1$  black hole solution in anti de Sitter spacetime  $AdS_{d+1}$  coupled to a matter field is given by

$$I = I_{EH} + I_{matter}, \quad (3.1)$$

where  $I_{EH}$  is the Einstein-Hilbert action with a negative cosmological constant  $\Lambda$

$$I_{EH} = \frac{1}{2\kappa_d^2} \int d^{d+1}x \sqrt{-g} \left\{ R + \frac{d(d-1)}{2l^2} \right\}, \quad (3.2)$$

with  $\kappa_d$  related to Newton's gravitational constant in  $d$ -dimensions  $\kappa_d = 8\pi G_N$ . The cosmological constant  $\Lambda$  depends on the radius of curvature of the anti de Sitter spacetime,  $l$ ,  $\Lambda = d(d-1)/2l^2$ .  $I_{matter}$  is the action for the Abelian Higgs system expanded to quadratic order in the scalar field

$$I_{matter} = \frac{1}{2\kappa_d^2} \int d^{d+1}x \sqrt{-g} \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |\partial\Psi - iqA\Psi|^2 - m^2|\Psi|^2 \right\}, \quad (3.3)$$

where the gauge field and the scalar field are coupled through the gauge covariant derivative,  $D_\mu = \partial_\mu + iqA_\mu$ . Here  $\partial_\mu$  is the spacetime covariant derivative,  $A_\mu$  is the gauge field, with associated field strength  $F_{\mu\nu}$ , and  $\Psi$  is a complex scalar field. In the probe limit, the matter field can be re-scaled as

$$\begin{aligned} A_\mu &\rightarrow A_\mu/q \\ \Psi &\rightarrow \Psi/q, \end{aligned} \quad (3.4)$$

which ensures that the quadratic potential scales as  $V(|\Psi|^2) \rightarrow V(|\Psi|^2)/q^2$  and the entire matter action as  $I_{matter} \rightarrow I_{matter}/q^2$ . In the limit  $q \rightarrow \infty$ , the action for Abelian-Higgs system  $I_{matter}$  decouples from the Einstein-Hilbert action  $I_{EH}$ . As noted in [20], the

probe approximation remains valid as long as  $\Psi$  and scalar potential  $\Phi$  are not large in the Planck limit. Another way to implement the probe approximation suggested in [83], is to consider a formal expansion of the full backreacted geometry in inverse powers of  $q$ . Then the leading order matter solutions will depend on  $q$  as  $\mathcal{O}(q^{-1})$ , while the leading order metric  $\mathcal{O}(q^2)$  receives  $\mathcal{O}(q^{-2})$  corrections.

The equations of motion for the scalar field and Maxwell fields reads

$$\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} g^{\mu\nu} D_\nu \Psi) = m^2 \Psi, \quad (3.5)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\nu\lambda} g^{\mu\sigma} F_{\lambda\sigma}) = g^{\mu\nu} J_\mu, \quad (3.6)$$

where the current  $J_\mu$  is given by

$$J_\mu = (i (\Psi \partial_\mu \Psi^\dagger - \partial_\mu \Psi \Psi^\dagger) + 2A_\mu \Psi \Psi^\dagger). \quad (3.7)$$

We consider the  $d + 1$  planar black hole ansatz

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 dx_i dx^i, \quad (3.8)$$

where  $f(r) = \frac{r^2}{l^2} (1 - \frac{r_0^d}{r^d})$  and  $i$  runs from 1 to  $(d - 2)$ . Here  $r = r_0$  is the event horizon and the Hawking temperature of the black hole is given by

$$T = \frac{r_0 d}{4\pi l^2}. \quad (3.9)$$

It is more convenient to make a change of coordinates  $z = r_0/r$ , so that the metric (3.8) becomes

$$ds^2 = \frac{l^2 \alpha(T)}{z^2} (-h(z) dt^2 + dx_i dx^i) + \frac{l^2 dz^2}{z^2 h(z)}, \quad (3.10)$$

where  $\alpha(T) \equiv 4\pi T = r_0 d/l^2$  and  $h(z) = (1 - z^d)$ . Here  $z = 1$  and  $z = 0$  is the event horizon and AdS boundary respectively. We consider the following ansatz<sup>1</sup> for the matter fields  $A_\mu dx^\mu = \Phi(z) dt$  and  $\Psi = \Psi(z)$ . Using the ansatz in the equations of motion, the

<sup>1</sup>From these ansatz we can see that the phase of the scalar field is fixed.

scalar and gauge fields yield respectively

$$\Psi'' + \left( \frac{h'}{h} + \frac{d-1}{z} \right) \Psi' + \frac{\tilde{\Phi}^2 \Psi}{h^2} - \frac{m^2}{hz^2} \Psi = 0, \quad (3.11)$$

and

$$\tilde{\Phi}'' - \frac{d-3}{z} \tilde{\Phi}' - \frac{2\Psi^2}{hz^2} \tilde{\Phi} = 0, \quad (3.12)$$

where  $\tilde{\Phi} \equiv \Phi/\alpha(T)$  and  $l = 1$ . Regularity at the horizon requires

$$\begin{aligned} \Psi' \Big|_{z=1} &= \frac{m^2 \Psi}{d} \Big|_{z=1}, \\ \tilde{\Phi} \Big|_{z=1} &= 0. \end{aligned} \quad (3.13)$$

Near the *AdS* boundary the scalar field and the scalar potential behave as

$$\begin{aligned} \Psi &= \Psi_{\lambda_-} z^{\lambda_-} + \Psi_{\lambda_+} z^{\lambda_+} + \dots \\ \tilde{\Phi} &= \mu - \rho z^{d-2} + \dots, \end{aligned} \quad (3.14)$$

where  $\lambda$  is the dimension of the dual operator, which satisfies the relation

$$\lambda(\lambda - d) = m^2, \quad (3.15)$$

with solutions  $\lambda_{\pm} = \frac{1}{2} (d \pm \sqrt{d^2 + 4m^2})$ . The stability of AdS vacuum, requires that the scalar field of negative mass squared must satisfy the BF bound [78],  $m^2 \geq -d^2/4$ , and in general the unitarity bound [79],  $\lambda \geq (d-2)/2$ . In the analysis that follows, we consider the values of  $m^2$  within the range  $-d^2/4 \leq m^2 < -d^2/4 + 1$ . Both modes of the asymptotic values of the scalar fields whose  $m^2$  are within this range are normalizable, except at the saturation of the BF bound. For  $m^2 \geq d^2/4 + 1$ , only the  $\lambda_+$  is normalizable, since  $\lambda_-$  is below the unitarity bound. As mentioned in the introduction our primary focus is on the scalar fields<sup>2</sup> with  $m^2$  within this range ( $-d^2/4 \leq m^2 < -d^2/4 + 1$ ), which we can achieve by considering  $m^2$  in 0.5 unit interval. The fixed interval makes the analysis and interpretation of the results less challenging.

The AdS/CFT dictionary [65, 63] relates the constant coefficients of the asymptotic solutions (equation (3.14)) to physical quantities in the boundary theory. The coefficients

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<sup>2</sup>The marginally stable mode for the scalar field with mass squared within this is presented in appendix A figure A.1 and figure A.2

$\Psi_{\lambda_i}$  are coefficients of the normalizable modes of the scalar field equation, they both correspond to expectation values in the dual field theory  $\Psi_{\lambda_i} = \langle \mathcal{O}_{\lambda_i} \rangle$ .  $\mu$  and  $\rho$  correspond to the chemical potential and charge density in the dual field theory, respectively.

### 3.3 Phase Transitions for Various Condensates

Apart from the trivial solutions  $\Psi = 0$  and  $\tilde{\Phi} = \mu - \rho z^{d-2}$ , a non-trivial solution to equations (3.11) and (3.12) which describe the superconducting phase in the dual field theory, exist below a critical temperature<sup>3</sup>. The critical temperature is defined, for  $\Psi_{\lambda_-}$ , when  $\Psi_{\lambda_+}$  vanishes and for  $\Psi_{\lambda_+}$ , when  $\Psi_{\lambda_-}$  vanishes. We present the solutions to equations (3.11) and (3.12) obtained numerically in figure 3.1.

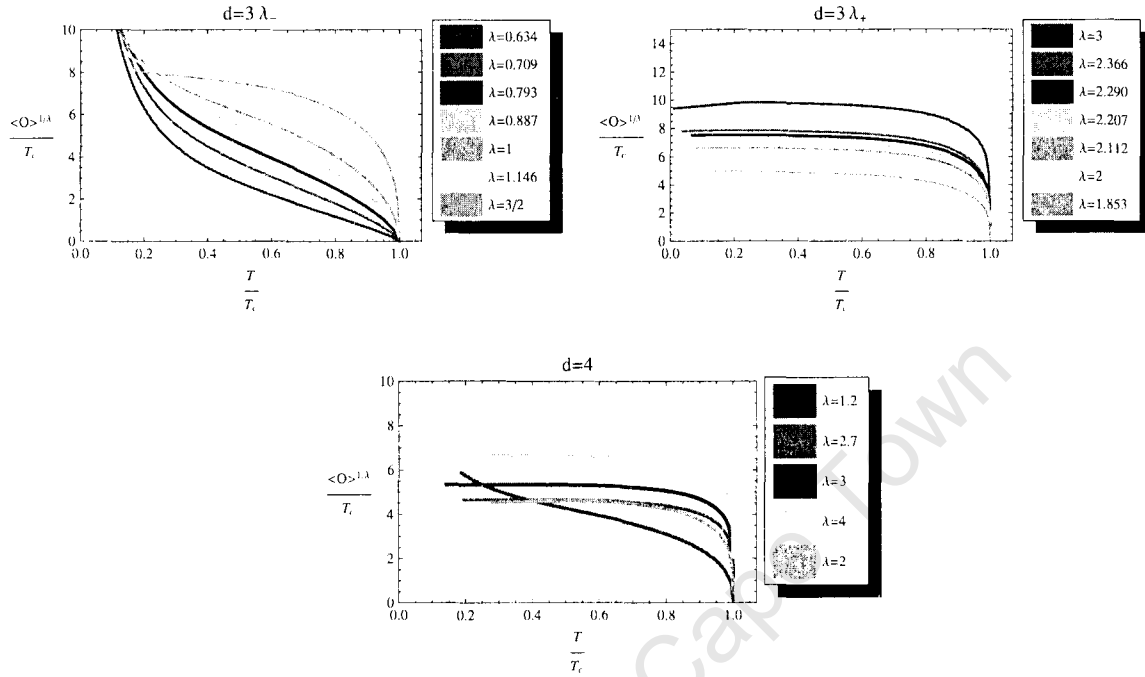
The temperature scales as  $T \sim \rho^{1/2}$  and  $T \sim \rho^{1/3}$  in the 2+1 and 3+1 boundary theory respectively. Notice that the condensates of the class  $\Psi_{\lambda_-}$  converge at  $\langle \mathcal{O} \rangle / T_c \approx 10$  before they collectively diverge. The signatures of the divergence near zero temperature become more pronounced as  $\lambda$  approaches the unitarity bound. A similar divergence was observed in [20] for  $\lambda = 1$  and was attributed to the probe approximation. But recent study [50] which considered gravitational backreaction, also show some signatures of divergence for  $\lambda = 1$  when the charge  $q$  becomes large. This divergence might be an artifact of large  $N$ . There are obvious differences between the superconducting phase of  $\Psi_{\lambda_-}$  and that of  $\Psi_{\lambda_+}$ . The condensates of the class  $\Psi_{\lambda_-}$  show a gradual transition to the superconducting phase<sup>4</sup>.

The amount of condensate in each case can be calculated from the numerical solutions to equations (3.11) and (3.12) at a fixed temperature  $T/T_c$ , in the vicinity of QCP. The results are shown in figure 3.1 (right). There appears to be a discontinuity in the amount of condensates between holographic superconductors of the class  $\Psi_{\lambda_-}$  and that of class  $\Psi_{\lambda_+}$  at  $\lambda_{crit} = \lambda_{BF}$  in both 2+1 and 3+1 boundary theories. This might be an indication that the two classes have different superconducting coherence factors [20]. The height of the discontinuous gap increases as the temperature decreases.

The dependence of the critical temperatures for various condensates on the dimension

<sup>3</sup>That is the field has a non-zero expectation value even after the source is turned off. This is a classic test for symmetry breaking

<sup>4</sup>In  $d = 4$  bulk dimensions the range of permissible values of  $m^2$  is small hence we did not distinguish between the two classes in the graphical representation. All the features as explained for the 2 + 1 boundary theory are also present.

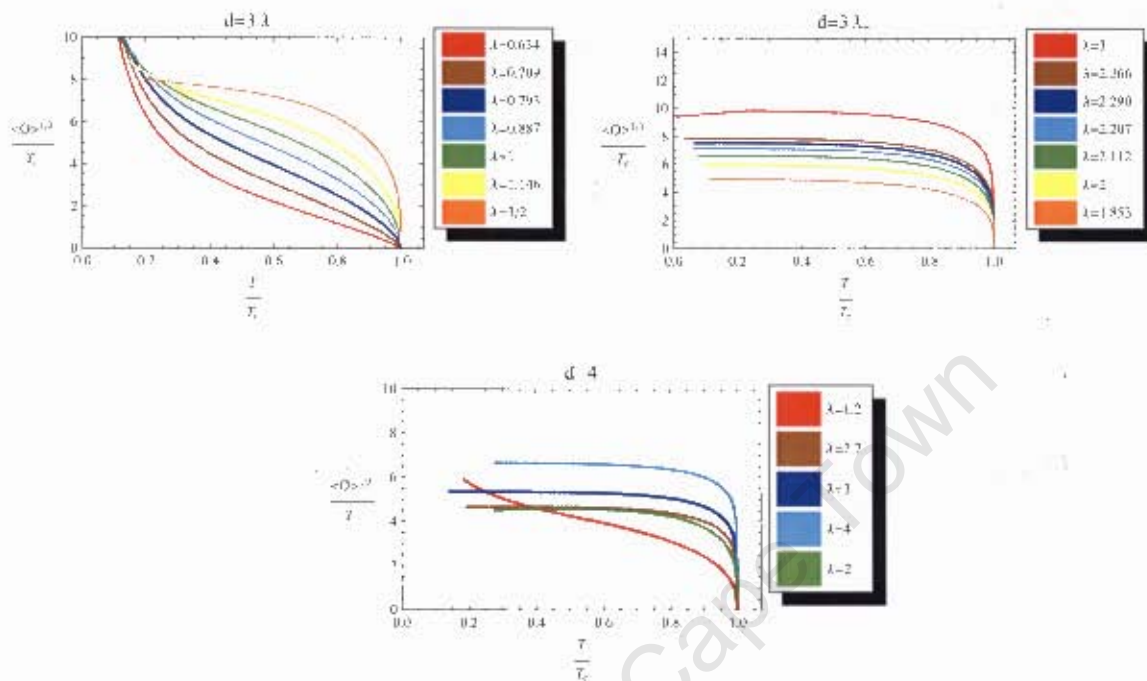


**Figure 3.1:** The condensates as a function of temperature for various condensates  $\mathcal{O}_\lambda$  in the boundary theory. The figures are labelled by the scaling dimensions of the dual condensates. The upper left graphs are condensates dual to the modes of the scalar field with slower fall-off  $\Psi_{\lambda_-}$ , while the upper right figure shows the condensates dual to the modes of scalar fields with faster fall-off  $\Psi_{\lambda_+}$ . The graphs are labelled by  $\lambda_-$  and  $\lambda_+$  to distinguish between the two classes of condensates in the 2 + 1 boundary theory. Below the two graphs is the condensates as a function of temperature for various condensate  $\mathcal{O}_\lambda$  in 3 + 1 dual field theory

of the dual operator is shown in figure 3.3. The condensates with high scaling dimensions have relatively very low critical temperature. In general as the critical temperature decreases as the dimension of dual condensate increases in both the 3 + 1 and 2 + 1 boundary theories.

### 3.4 Conductivity

Within the frame work of the AdS/CFT correspondence, the conductivity in the boundary theory can be calculated from the Maxwell field in the bulk theory. This can be done in the

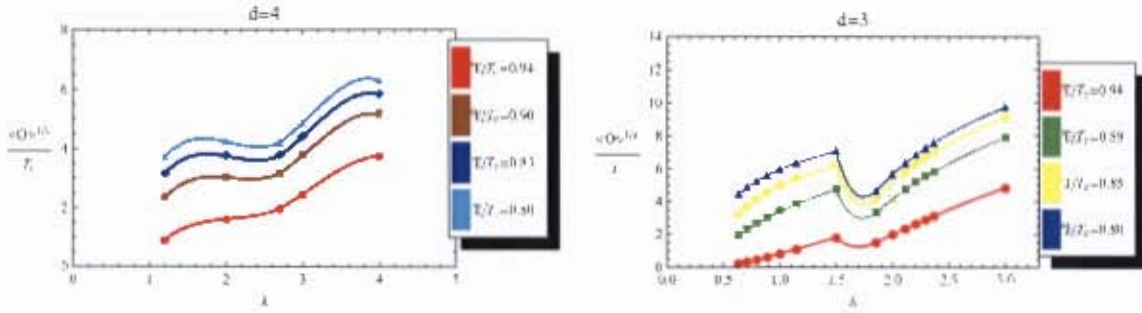


**Figure 3.1:** The condensates as a function of temperature for various condensates  $\mathcal{O}_\lambda$  in the boundary theory. The figures are labelled by the scaling dimensions of the dual condensates. The upper left graphs are condensates dual to the modes of the scalar field with slower fall-off  $\Psi_{\lambda_-}$ , while the upper right figure shows the condensates dual to the modes of scalar fields with faster fall-off  $\Psi_{\lambda_+}$ . The graphs are labelled by  $\lambda_-$  and  $\lambda_+$  to distinguish between the two classes of condensates in the  $2+1$  boundary theory. Below the two graphs is the condensates as a function of temperature for various condensate  $\mathcal{O}_\lambda$  in  $3+1$  dual field theory

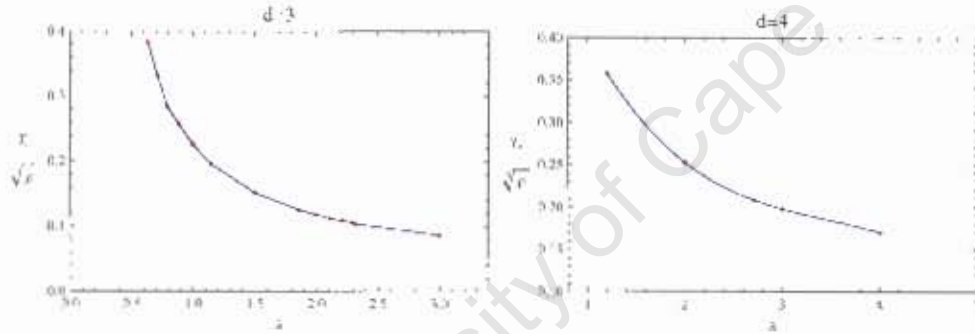
of the dual operator is shown in figure 3.3. The condensates with high scaling dimensions have relatively very low critical temperature. In general as the critical temperature decreases as the dimension of dual condensate increases in both the  $3+1$  and  $2+1$  boundary theories.

### 3.4 Conductivity

Within the frame work of the AdS/CFT correspondence, the conductivity in the boundary theory can be calculated from the Maxwell field in the bulk theory. This can be done in the



**Figure 3.2:** The amount of condensates as a function of the dimension of the dual operator (right) computed at different fixed temperatures. The dots represent the actual value and the continuous line is an interpolation between the actual values. This form of representation is used in the rest of the report.



**Figure 3.3:** The dependence of the critical temperature on the dimensions of the dual condensates in 2 + 1 and 3 + 1 boundary theories

probe limit by perturbing the Maxwell field at zero spatial momentum on the fixed black hole background: With the ansatz for the perturbed Maxwell field,  $\delta A_x = A_x(z)e^{i\omega t}dx$ , a linearized equation of motion results

$$A_x'' - \left( \frac{h'}{h} - \frac{d-3}{z} \right) A_x' + \left( \frac{\omega}{h^2} - \frac{2\Psi^2}{z^2 h} \right) A_x = 0. \quad (3.16)$$

Equation (3.16) is solved with an ingoing wave boundary condition [80] near the horizon of the black hole in order to suppress near horizon oscillations:

$$A_x(z) = h(z)^{-4\pi i \omega / T} \tilde{A}_x(z), \quad (3.17)$$

### 3.4.1 Conductivity in the $(2 + 1)$ -dimensional dual field theory

In an odd number of dimensions (e.g.  $d = 3$ ) the solution to the Maxwell's equation (3.16) behaves near the boundary as

$$A_x = A^{(0)} + A^{(1)}z + \dots \quad (3.18)$$

From Ohm's law and the dictionary of AdS/CFT correspondence, the conductivity becomes

$$\sigma(\omega) = \frac{A^{(1)}}{i\omega A^{(0)}}. \quad (3.19)$$

The plots of the real and imaginary part of the conductivity against the frequency normalized by individual condensate are shown in appendix B, figure C.1 and appendix C, figure C.2 for the two classes of holographic superconductors in the  $2 + 1$  boundary theory.

### 3.4.2 Conductivity in the $(3 + 1)$ -dimensional dual field theory

When the bulk dimension is even (e.g.  $d = 4$ ), there exists a logarithmic divergence of the Maxwell's field in the action 3.1:

$$A_x = A^{(0)} + A^{(2)}z^2 + A^{(0)}\omega^2 z^2 \log \frac{\Lambda}{z}. \quad (3.20)$$

A boundary counter term may be added to remove the divergence [84], so that the conductivity becomes [47]

$$\sigma(\omega) = \frac{2A^{(2)}}{i\omega A^{(0)}} + \frac{i\omega}{2} \quad (3.21)$$

The numerical solutions to equation (3.16) in the  $3 + 1$  boundary theory, is shown in appendix D, figure D.2 and for the frequency normalized by the individual superconducting condensate. We could not resolve the delta function at  $\omega = 0$  numerically. However, it can be seen from the Kramers-Kronig relation

$$Im[\sigma(\omega)] = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{Re[\sigma(\omega')]d\omega'}{\omega' - \omega}, \quad (3.22)$$

that there is a delta function at  $\omega = 0$  for all the condensates, since  $\omega = 0$  is a pole in the imaginary part of the conductivity. The gap frequency  $\omega_g$  remain approximately the

same for all the condensates  $\omega_g/T_c \approx 8$ , irrespective of the number of bulk dimensions.

### 3.4.3 Superfluid density and magnetic penetration depth

In the limit  $\omega \rightarrow 0$ , the superfluid density  $n_s$  is defined as the coefficient of the pole in the imaginary part of conductivity  $\text{Im}[\sigma] = n_s/\omega$ , where  $n_s$  is the superfluid density. The results of the superfluid density computed by solving equation (3.16) in this limit, is shown in figure 3.4. The vanishing of  $n_s$  at the critical temperature is in agreement with

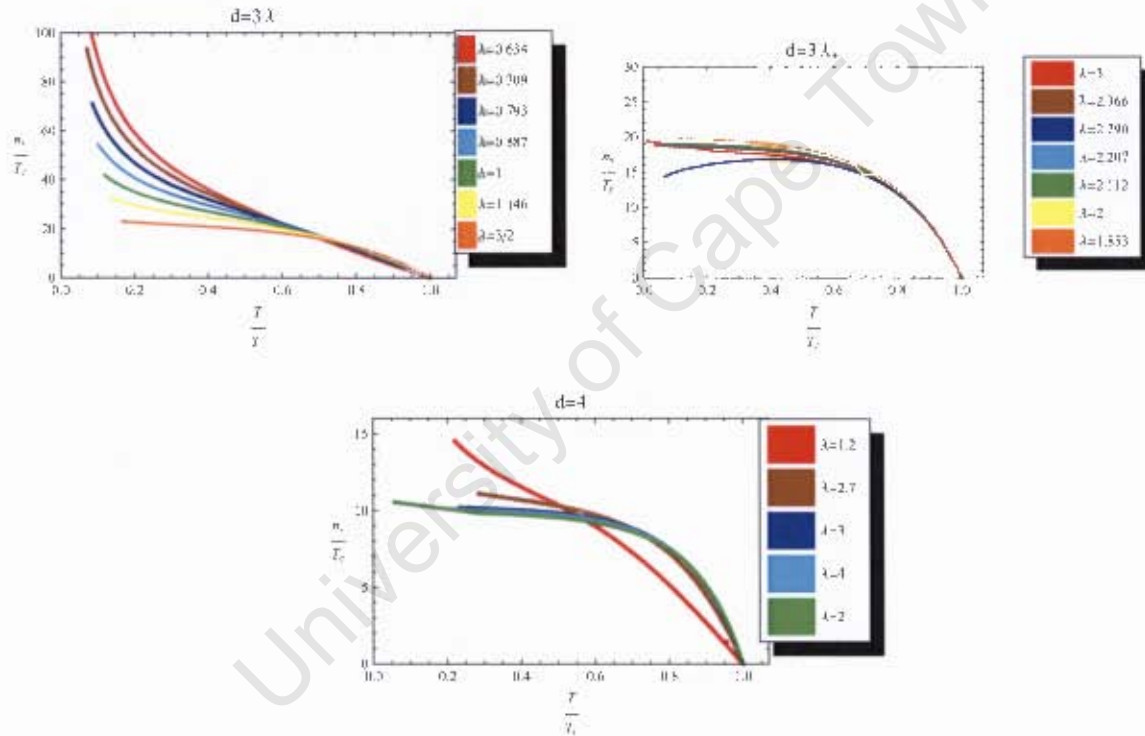
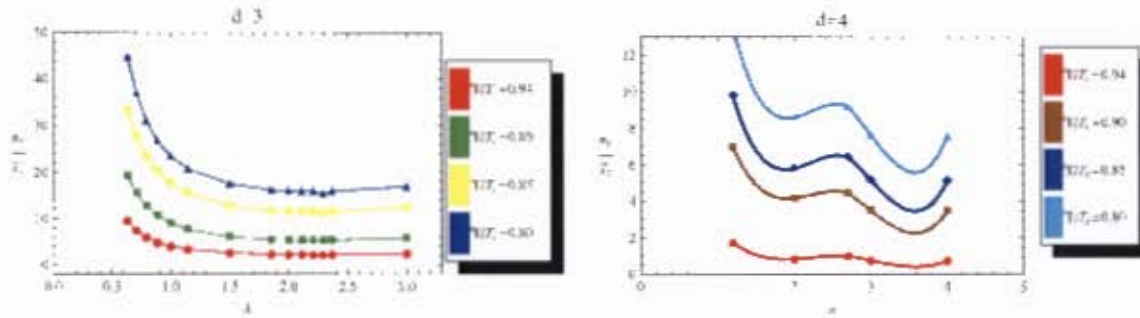


Figure 3.4: Superfluid density below the critical temperature in the boundary theory.

the Ginzburg-Landau theory.

The dependence of the  $n_s$  on the scaling dimension calculated at various fixed temperatures below  $T_c$  is shown in figure 3.5. Observe that for  $\lambda \geq \lambda_{BF}$ ,  $n_s$  is not sensitive to changes in  $\lambda$ , suggesting that this class of holographic superconductors may not be stable against perturbations by external magnetic field, since  $n_s$  is related to the current which generate the electromagnetic field if the boundary theory was gauged.



**Figure 3.5:** Superfluid density as function of the dimension of the dual operator, at different fixed temperatures

Thus the superfluid density is related to the magnetic penetration depth  $\lambda_m$  through the first London equation

$$J = -e_s n_s A, \quad (3.23)$$

where  $e_s$  is the charge of the order parameter. Using the Maxwell's equation for the curl of the magnetic field and assuming that the current at the boundary can generate its own magnetic field<sup>5</sup>, the relation between the superfluid density and the magnetic penetration depth appear more explicitly

$$\begin{aligned} \nabla^2 B &= \nabla \times (\nabla \times B) = 4\pi \nabla \times J = -4\pi n_s \nabla \times A = 4\pi n_s B \\ \nabla^2 B &= \frac{1}{\lambda_m^2} B, \end{aligned} \quad (3.24)$$

where  $\lambda_m^2 = \frac{1}{4\pi n_s}$ . The magnetic penetration depth obtained using this relation for both classes of holographic superconductors is shown in figure 3.6. Notice that the magnetic penetration depth diverges at  $T_c$  which is an expected behavior. Its dependence on the dimension of the dual operator is presented in figure 3.7

### 3.5 Perturbative Solution

At the quantum critical point, equations (3.11) and (3.12) can be solved exactly:

$$\tilde{\Phi} = q_c \left( 1 - \frac{z^d - 1}{d-2} \right). \quad (3.25)$$

<sup>5</sup>i.e weakly gauging the boundary theory as suggested in [50].

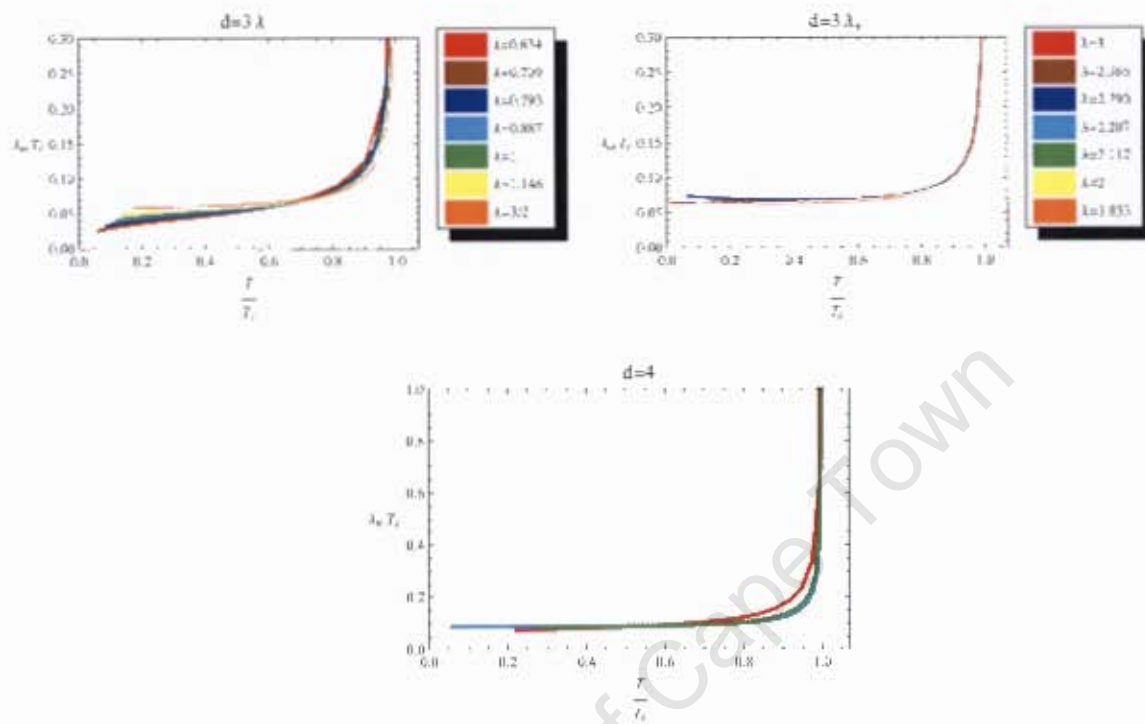


Figure 3.6: The magnetic penetration depth below the critical temperature in the dual field theory.

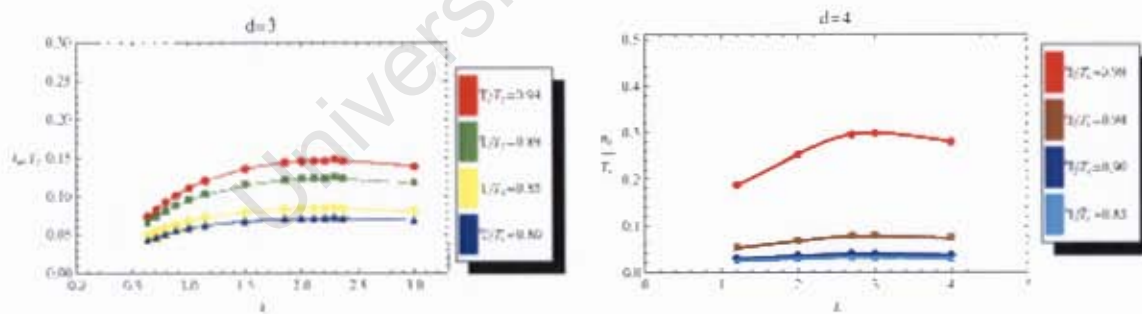


Figure 3.7: Magnetic penetration depth as function of the dimension of the dual operator, at temperatures below  $T_c$ .

Other solutions to equations (3.11) and (3.12) can be found in the vicinity of quantum critical point, by a perturbative expansion, since the superconducting condensate behaves as (see section 3.3)

$$\langle O \rangle \approx T_c (1 - T/T_c)^{1/2} \quad (3.26)$$

and vanishes at  $T_c$ . The results of the numerical calculations in section 3.3 show that at the critical temperature  $\mu = \rho = q_c$ <sup>6</sup>.

Other solutions to equations (3.11) and (3.12) may be obtained to higher order in  $\epsilon = (1 - T/T_c)$  and in the manner which still yield the expected fall offs at the AdS boundary.

$$\begin{aligned}\Psi(z) &= \epsilon^{1/2}\Psi_1(z) + \epsilon^{3/2}\Psi_2(z) + \epsilon^{5/2}\Psi_3(z) + \dots \\ \tilde{\Phi}(z) &= \tilde{\Phi}_c(z) + \epsilon\tilde{\Phi}_1(z) + \epsilon^2\tilde{\Phi}_2(z) + \dots\end{aligned}\tag{3.27}$$

Using equation (3.27) in equations (3.11) and (3.12) gives

$$\left[ z^{d-1} \frac{d}{dz} \frac{h(z)}{z^{d-1}} \frac{d}{dz} - \frac{m^2}{z^2} + \frac{\tilde{\Phi}_c^2}{h(z)} \right] \Psi_1 = 0,\tag{3.28}$$

$$\left[ z^{d-3} h(z) \frac{d}{dz} \frac{1}{z^{d-3}} \frac{d}{dz} \right] \Phi_1 - \frac{2\tilde{\Phi}_c \Psi_1^2}{z^2} = 0.\tag{3.29}$$

The equations are written in a form most convenient for use in the following analysis. Equation (3.28) decouples from equation (3.29) to first order in the perturbative expansion. We make the following definitions for clearer presentation:

$$\begin{aligned}\mathcal{L}_\psi &:= \left[ z^{d-1} \frac{d}{dz} \frac{h(z)}{z^{d-1}} \frac{d}{dz} - \frac{m^2}{z^2} + \frac{\tilde{\Phi}_c^2}{h(z)} \right] \\ \mathcal{L}_\phi &:= \left[ z^{d-3} h(z) \frac{d}{dz} \frac{1}{z^{d-3}} \frac{d}{dz} \right].\end{aligned}\tag{3.30}$$

### 3.5.1 Superconducting coherence length

The correlation length of the order parameter is related to the superconducting coherence length  $\xi$ , which appears as a complex pole of the static correlation function of the order parameter fluctuation in Fourier space [48]:

$$\langle \mathcal{O}(\vec{k}) \mathcal{O}(-\vec{k}) \rangle \sim \frac{1}{|\vec{k}|^2 + 1/\xi^2}.\tag{3.31}$$

Following the technique of AdS/CFT correspondence, this correlation length may be calculated within the probe approximation by perturbing the Maxwell and scalar fields on

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<sup>6</sup>We use the conventions of [48].

a fixed black hole background. We consider only the linear perturbation, with fluctuations of the fields in the  $x$ -direction, in the form

$$\begin{aligned}\delta A_\mu(z, x) dx^\mu &= [A_x(z, k) dx + A_y(z, k) dy + \phi(z, k) dt] e^{ikx}, \\ \delta\psi(z, x) &= \frac{1}{\alpha(T)} [\psi(z, k) + i\tilde{\psi}(z, k)] e^{ikx}.\end{aligned}\quad (3.32)$$

Using (3.32) on the perturbed Maxwell and scalar fields give the following eigenvalue equations

$$\psi'' + \left(\frac{h'}{h} + \frac{d-1}{z}\right) \psi' - \frac{\tilde{k}^2 \psi}{h} + \frac{\tilde{\Phi}^2 \psi}{h^2} + \frac{2\tilde{\Phi}\Psi}{h^2} \phi - \frac{m^2}{z^2 h} \psi = 0, \quad (3.33)$$

$$\phi'' - \frac{d-3}{z} \phi' - \tilde{k}^2 \phi - \frac{2\Psi}{z^2} \phi - \frac{4\tilde{\Phi}\Psi}{hz^2} \psi = 0, \quad (3.34)$$

$$A_y'' + \left[\frac{h'}{h} - \frac{d-3}{z}\right] A_y' - \frac{\tilde{k}^2}{h} A_y - \frac{2\Psi^2}{z^2 h} A_y = 0, \quad (3.35)$$

where  $\tilde{k} = k/\alpha(T)$ . Regularity at the horizon implies that

$$\begin{aligned}\phi &= 0 \\ \psi' &= -\frac{k^2 \psi}{dz_+^{d-1}} - \frac{m^2 \psi}{dz_+^{d+1}} \\ A_y' &= -\frac{k^2 A_y}{dz_+^{d-1}} - \frac{2|\Psi|^2 A_y}{dz_+^{d+1}}.\end{aligned}\quad (3.36)$$

Analytical treatment is possible for the eigenvalue equations in the limit  $T \rightarrow T_c$  [48].

Using the series expansion (3.27) in equations (3.33), (3.34) and (3.35) yield

$$\begin{aligned}\mathcal{L}_\psi \psi &= \tilde{k}^2 \psi - \frac{2\epsilon \tilde{\Phi}_c \tilde{\Phi}_1}{h(z)} \psi - \frac{2\epsilon^{1/2} \tilde{\Phi}_c \Psi_1}{h(z)} \phi \\ \mathcal{L}_\phi \phi &= \tilde{k}^2 \phi + \frac{2\epsilon \Psi_1^2}{z^2} \phi + \frac{4\epsilon^{1/2} \tilde{\Phi}_c \Psi_1}{z^2} \psi.\end{aligned}\quad (3.37)$$

The solution to equation (3.37) of interest are those that satisfy the regularity condition at the horizon (3.36) and have an expected fall off at the AdS boundary (equation 3.14). One trivial solution is the zeroth order solution  $\phi_0$  and  $\psi_0$ :

$$\begin{aligned}\psi_0 &= \Psi_1 \\ \phi_0 &= 0\end{aligned}\quad (3.38)$$

Non-trivial solutions can be found by a series expansion around the zeroth order solution in powers of  $\epsilon$

$$\begin{aligned}\psi &= \Psi_1 + \epsilon\psi_1 + \epsilon^2\psi_2 + \dots \\ \phi &= \epsilon^{1/2}\phi_1 + \epsilon^{3/2}\phi_2 + \dots \\ \tilde{k}^2 &= \epsilon\tilde{k}_1^2 + \epsilon^2\tilde{k}_2^2 + \dots\end{aligned}\tag{3.39}$$

Substituting the expansion (3.39) into equation (3.37) yields

$$\mathcal{L}_\psi\psi = k_1^2\Psi_1 - \frac{2\tilde{\Phi}_c\Psi_1}{h(z)}\left(\tilde{\Phi}_1 + \phi_1\right)\tag{3.40}$$

$$\mathcal{L}_\phi\phi_1 = \frac{4\tilde{\Phi}_c\Psi_1^2}{z^2}.\tag{3.41}$$

In this approximation it is easy to see that the equations of motion for  $\Phi_1$  and  $\phi_1$  only differ by a factor of two. Equation (3.40) can be solved for  $k$  by defining an inner product for the states  $\psi_1$  and  $\psi_2$  which satisfy the boundary condition at the AdS boundary and is well behaved at the horizon (3.36).

$$\langle\psi_I|\psi_{II}\rangle = \int_0^1 \frac{dz}{z^{d-1}}\psi_I^*\psi_{II}.\tag{3.42}$$

Because  $\mathcal{L}_\psi$  is hermitian for non-zero negative mass squared, taking the inner product of equation (3.40) gives

$$\langle\Psi_1|\mathcal{L}_\psi|\psi_1\rangle = k_1^2\langle\Psi_1|\Psi_1\rangle - \left\langle\Psi_1\frac{2\tilde{\Phi}_c\Psi_1}{h(z)}\left(\tilde{\Phi}_1 + \phi_1\right)\right\rangle\tag{3.43}$$

Using the inner product (3.42) and the constraint  $\mathcal{L}_\psi\Psi_1 = 0$  in equation (3.43) we obtain

$$k_1^2\langle\Psi_1|\Psi_1\rangle = \left\langle\Psi_1\frac{2\tilde{\Phi}_c\Psi_1}{h(z)}\tilde{\Phi}_1\right\rangle + 2\int_0^1 dz\frac{\tilde{\Phi}_c\Psi_1^2}{z^{d-1}h(z)}\phi_1.\tag{3.44}$$

Equation (3.44) may be simplified by considering the equation of motion for the mode  $\Psi_2$ :

$$\mathcal{L}_\psi\Psi_2 = \frac{2\tilde{\Phi}_c\Psi_1}{h(z)}\tilde{\Phi}_1.\tag{3.45}$$

Since equation (3.42) is well defined for  $\Psi_1$  and it is hermitian, the first term in the right hand side of equation (3.44) is zero. Using equation (3.42) and  $\tilde{k}^2 = \epsilon \tilde{k}_1^2$ , the eigenvalue  $\tilde{k}$  in a first order approximation may be written as.

$$\tilde{k}^2 = \epsilon \frac{N}{D} + \mathcal{O}(\epsilon^2) \quad (3.46)$$

where

$$N = 2 \int_0^1 dz \frac{\tilde{\Phi}_c \Psi_1^2}{z^{d-1} h(z)} \phi_1$$

$$D = \int_0^1 \frac{\Psi_1^2}{z^{d-1}}$$

This result was first derived in [48] for  $m^2 = -2$ , and it is shown to hold for all the masses that satisfy the unitarity bound in  $d$ -dimensions, except for  $d = 2$  where the scalar potential diverges. Now the superconducting coherence length is given by

$$\xi = \frac{\epsilon^{-1/2}}{\alpha(T)} \sqrt{\frac{D}{N}} + \mathcal{O}(\epsilon^2) \quad (3.47)$$

Figure 3.8 shows the results obtained from calculating the  $\xi$  using equation (3.47) for various condensates. We have used the boundary conditions obtained in section 3.3 to solve for  $\Psi_1$  and  $\phi_1$ .

The numerical accuracy becomes very unsatisfactory for  $m^2 = 0$ . As a result, we did not include it in the figure 3.8. The dependence of the superconducting correlation length on the scaling dimensions of the dual condensates is shown in figure 3.9.

### 3.5.2 Magnetic penetration depth

As stated in section 3.4, the magnetic penetration depth may be calculated from the London current. This can also be calculated by solving equation 3.35 perturbatively in the limit  $T \rightarrow T_c$  at zero frequency and momentum. The relevant portion of the Maxwell's equation is given by

$$z^{d-3} \frac{d}{dz} \frac{h}{z^{d-3}} \frac{dA_y}{dz} - \frac{2\Psi^2}{z^2} A_y = 0. \quad (3.48)$$

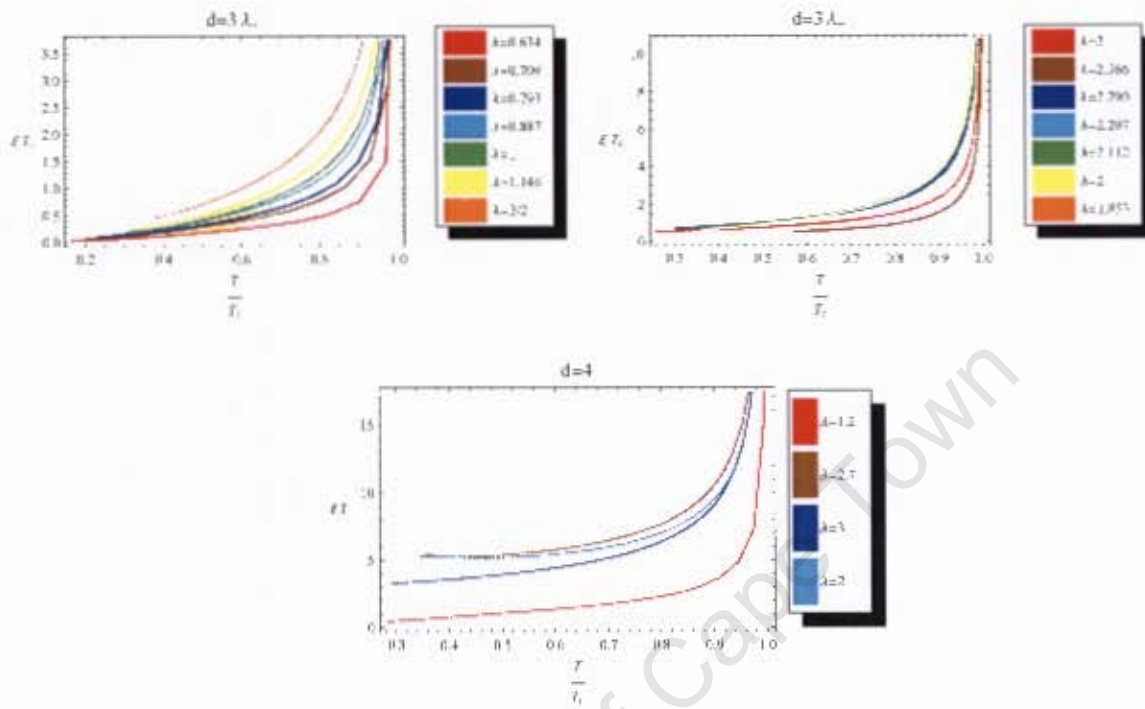


Figure 3.8: Superconducting coherence length of holographic superconductors plotted as a function of temperature.

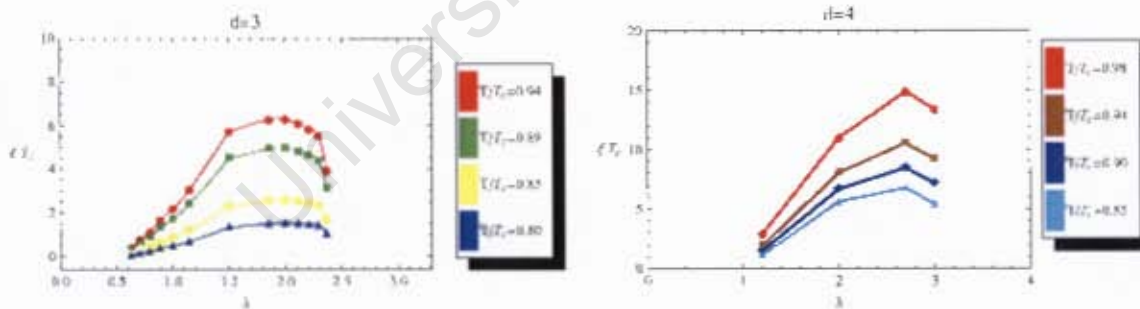


Figure 3.9: Superconducting correlation length as a function \$\lambda\$

The Maxwell field can be expanded as  $A = A_0 - \epsilon A_1$  in the neighborhood of the QCP, which leads to the following equations

$$\frac{d}{dz} \frac{h}{z^d} \frac{d}{dz} A_0 = 0 \tag{3.49}$$

$$\frac{d}{dz} \frac{h}{z^{d-3}} \frac{d}{dz} A_1 - \frac{2\Psi_1}{z^{d-1}} A_0 = 0, \quad (3.50)$$

where the subscript,  $y$ , has been dropped for clarity. One of the solutions to equation (3.49), which satisfies the required boundary conditions is

$$A_0 = C, \quad (3.51)$$

where  $C$  is a constant. Hence the first order mode becomes

$$\frac{dA_1}{dz} = -\frac{2A_0 z^{d-3}}{h(z)} \int_{z_0^1} dz_0 \frac{|\Psi_1(z_0)|^2}{z_0^{d-1}} \quad (3.52)$$

Integrating this expression (3.52) yields

$$A_1(z) = A_0 - 2A_0 \int_0^1 dz \frac{z^{d-3}}{h(z)} \int_{z_0}^1 dz_0 \frac{|\Psi_1(z_0)|^2}{z_0^{d-1}} + \mathcal{O}(\epsilon^2) \quad (3.53)$$

Here  $A_0$  is the constant of integration. Using  $A = A_0 + \epsilon A_1$

$$A(z) = A_0 - 2\epsilon A_0 \int_z^1 dz \frac{z^{d-3}}{h(z)} \int_{z_0}^1 dz_0 \frac{|\Psi_1(z_0)|^2}{z_0^{d-1}} + \mathcal{O}(\epsilon^2). \quad (3.54)$$

Near the boundary  $h(z) \approx 1$

$$A(z) = A_0 - 2\epsilon A_0 \int_z^1 dz z^{d-3} \int_{z_0}^1 dz_0 \frac{|\Psi_1(z_0)|^2}{z_0^{d-1}} + \mathcal{O}(\epsilon^2). \quad (3.55)$$

From the dictionary of AdS/CFT correspondence, the current is identified as

$$\langle j \rangle = -\frac{1}{\kappa_d^2} \left( \frac{4\pi T_c}{d(d-2)} \right) \epsilon \int_0^1 dz \frac{\Psi_1^2}{z^{d-1}} A_0(x) + \mathcal{O}(\epsilon^2), \quad (3.56)$$

and, for  $\epsilon = (1 - T/T_c)$ , the current becomes

$$\langle j \rangle = -\frac{1}{\kappa_d^2} \left( \frac{4\pi T_c}{d(d-2)} \right) (1 - T/T_c) \int_0^1 dz \frac{\Psi_1^2}{z^{d-1}} A_0 + \mathcal{O}(\epsilon^2) \quad (3.57)$$

The magnetic penetration depth is then defined (see equation 3.24) as

$$\lambda_m = \sqrt{\left[ \frac{1}{\kappa_d^2} \left( \frac{4\pi T_c}{d(d-2)} \right) (1 - T/T_c) \int_0^1 dz \frac{\Psi_1^2}{z^{d-1}} \right]^{-1}} \quad (3.58)$$

Using equation (3.47) and (3.58), the Ginzburg-Landau parameter becomes

$$\kappa = \frac{\lambda_m}{\xi} \quad (3.59)$$

To solve for  $\lambda_m$  we use the relation  $\Psi = \epsilon^{1/2}\Psi_1 + \mathcal{O}(\epsilon)$  to compute  $\Psi$  instead of  $\Psi_1$ . This offers some numerical simplification. The results of the numerical computations are presented in figure 3.10. The dependence of the magnetic penetration depth  $\lambda_m$  on the

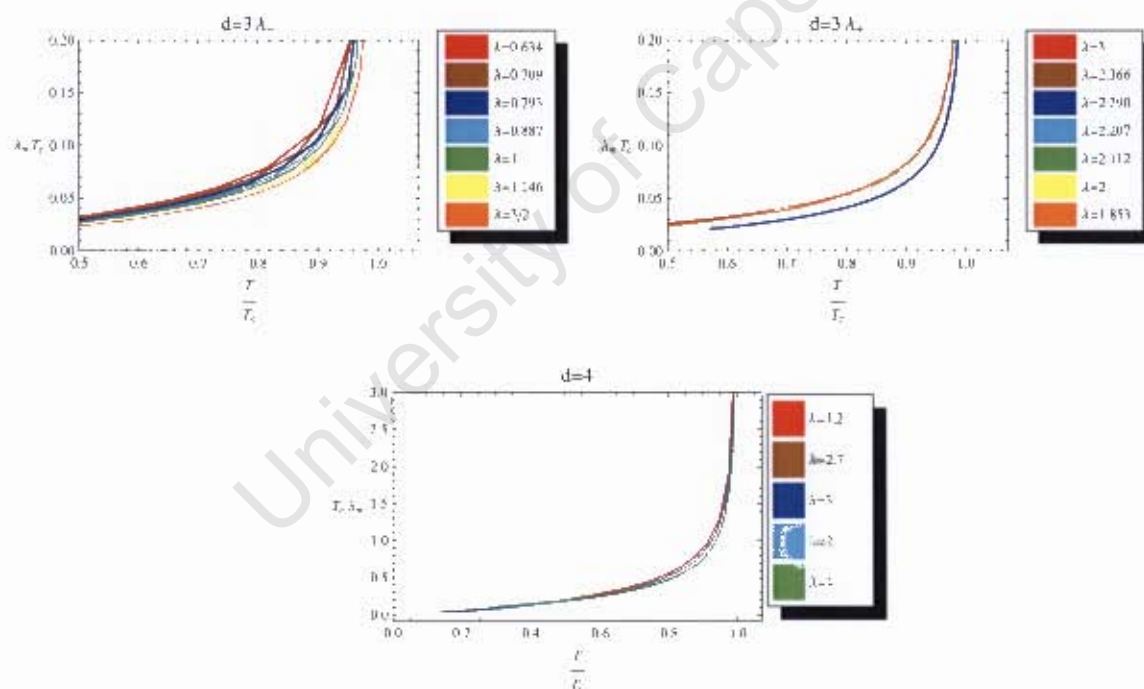


Figure 3.10: Magnetic penetration depth below the critical temperature in the superconducting phase.

scaling dimensions of the dual condensates is shown in figure 3.11.

Observe that the results of the magnetic penetration depth, calculated using a perturbative approach and the one calculated from superfluid density are in agreement. This

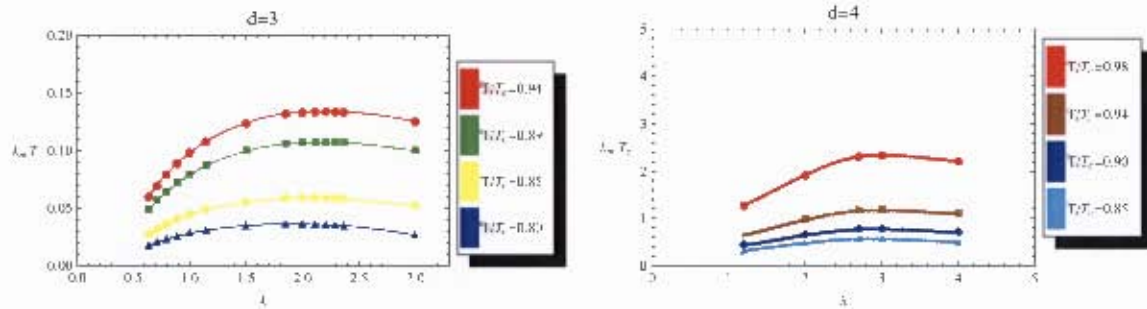


Figure 3.11: Magnetic penetration depth as a function  $\lambda$ .

agreement show that the perturbative treatment captures the physics of interest in the vicinity of the QCP.

The Ginzburg-Landau parameter  $\kappa = \lambda_m/\xi$  can be calculated from equations (3.47) and (3.58). The results obtained are plotted in figure 3.12 against the dimension of the dual condensate.

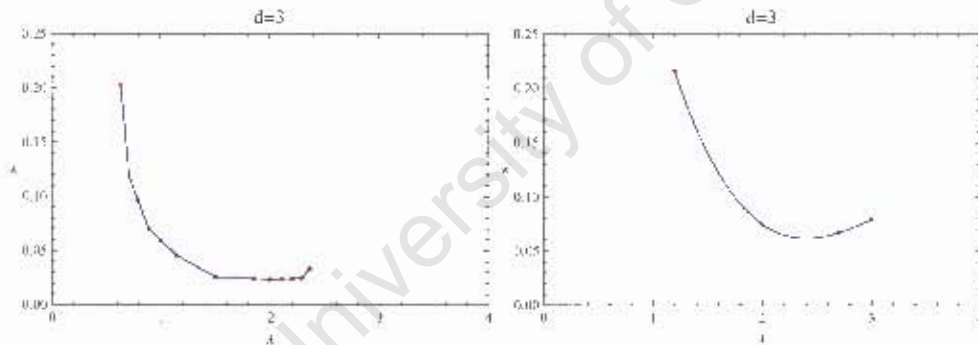


Figure 3.12: Ginzburg-Landau parameter against  $\lambda$ .

In Ginzburg-Landau theory, the coefficient  $\kappa$  classifies superconductors into two types, i.e.  $\kappa < 1/\sqrt{2}$  for type I superconductors and  $\kappa > 1/\sqrt{2}$  for type II superconductors. If our boundary theory was gauged, the results in figure 3.12 show that at  $\lambda = \lambda_{BF}$ , there is a change in the relative size of  $\kappa$ . An obvious interpretation is that for  $\lambda < \lambda_{BF}$  superconducting condensates are of type II, while for  $\lambda > \lambda_{BF}$  they are of type I. It is interesting to see that similar clear distinction also exist for holographic superconductors. Although, we should note that the London current also depends on  $q$ , which was scaled away in the probe limit. The effect of large but finite  $q$  is to ensure that  $\lambda_m$  is greater than  $\xi$ , i.e. the condensate must be type II. Despite being large, there are still indications

that a holographic superconductor can be type I. This result is in agreement with Maeda et. al. [48], who suggested that holographic superconductors which have low critical temperature are type I. But Hartnoll et. al. [50] showed that holographic superconductor corresponding to dimension one operator, which they studied with high accuracy is a type II. These results are not in any way contradicting, as we have seen that both deductions are correct limits of the larger class of condensates considered here.

### 3.6 Conclusion

We have studied the dependence of various physical quantities associated with the holographic model of superconductivity on the scaling dimensions of the dual condensates in the  $(2 + 1)$  and  $(3 + 1)$ -dimensional boundary theories. Each of these physical quantities was calculated at a fixed temperature, but for different values of mass squared  $m^2$  (varied in 0.5 unit intervals) in  $d = 3$  and  $d = 4$  bulk spacetime dimensions. We considered mainly bulk scalar fields which have normalizable fall-offs at the AdS boundary. The results of this indicate that, there are two distinct superconducting condensates dual to the two modes of scalar field, which have different fall-off behaviors at the AdS boundary. The amount of the condensate dual to the bulk scalar field with slower fall-off  $\Psi_{\lambda_-}$  converges, before diverging collectively near zero temperature. Its superconducting phase is different from that of the scalar fields with a faster fall-off  $\Psi_{\lambda_+}$ . Certain features indicating a discontinuity in the amount of condensates were observed between condensates of the class  $\Psi_{\lambda_-}$  and those of the class  $\Psi_{\lambda_+}$  at  $\lambda = \lambda_{BF}$ . This discontinuity distinguishes between the two classes. The Ginzburg-Landau parameter  $\kappa$ , obtained from the superconducting coherence length  $\xi$  and magnetic penetration depth  $\lambda_m$ , indicates that there is a critical scaling dimension  $\lambda_{crit}$  at which the holographic superconductors change from type II to type I. Type I holographic superconductors have very low critical temperatures, unlike those of type II, which have relatively high critical temperatures.

# Chapter 4

## DISCUSSION AND CONCLUSION

The major objective of this study has been to study the physical quantities associated with the holographic superconductors as a function of the scaling dimensions of the dual condensates in various dimensions. We focussed attention mainly on  $(3 + 1)$  and  $(2 + 1)$ -dimensional boundary theories, since the physics of most condensed matter systems are tractable in these dimensions. We considered an Abelian-Higgs system with a quadratic potential on a fixed anti-de Sitter black hole background. The negative mass squared of the scalar field that satisfy the BF stability bound and the unitarity bound, and allowed to vary in intervals of 0.5 units were considered. The results of the study indicate that:

- The holographic superconductors can be consistently classified into two based on the normalizations of the bulk scalar field at the AdS boundary. The properties of the condensates dual to the mode of the scalar field with a slower fall-off  $\Psi_{\lambda_-}$  are different from those dual to the mode with a faster fall-off  $\Psi_{\lambda_+}$ . Both classes of condensates have distinct superconducting phase, for example near zero temperature, the superconducting phase of the condensates of the class  $\Psi_{\lambda_-}$  are diverging while those of the class  $\Psi_{\lambda_+}$  are well behaved. There appears to be a discontinuity in the amount of condensates of the class  $\Psi_{\lambda_-}$  and that of the class  $\Psi_{\lambda_+}$  at  $\lambda = \lambda_{BF}$ . This discontinuity at  $\lambda_{BF}$  provides a clear distinction between the properties of the two classes. The holographic superconductors of the class  $\Psi_{\lambda_-}$  have higher critical temperatures than those of class  $\Psi_{\lambda_+}$ . These features are invariant in both  $(2 + 1)$  and  $(3 + 1)$ -dimensional boundary theories.
- The Ginzburg-Landau parameter calculated in the vicinity of the QCP at fixed

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charge indicate the holographic superconductors of the class  $\Psi_{\lambda_-}$  and  $\Psi_{\lambda_+}$  correspond to type II and type I superconductors respectively. In Ginzburg-Landau theory, classification of superconductors into type I and type II is based on the critical value of the Ginzburg-Landau parameter  $\kappa$ , i.e  $\kappa < 1/\sqrt{2}$  for type I superconductors and  $\kappa > 1/\sqrt{2}$  for type II superconductors. For holographic superconductors, the critical value is given by  $\lambda_{BF}$ , i.e condensate with the scaling dimension  $\lambda < \lambda_{BF}$  are type II while those with the scaling dimension  $\lambda > \lambda_{BF}$  are type I.

Loosely speaking, these results are similar to the differences in the superconducting phase between the conventional superconductors and the unconventional superconductors. Conventional superconductors are low temperature superconductors that can be described by the weak coupling approximation of BCS theory, they are mainly type I superconductors. In contrast unconventional superconductors are mostly high- $T_c$  superconductors, which have different superconducting phase that cannot be described by BCS theory. A list of all known type I and type II superconductors can be found [85] and [86] respectively. The superconducting condensate with dual dimension  $\lambda_{BF}$  has features of both type I and type II holographic superconductors. The physical superconductor that has mixed features of type I and type II is Magnesium diboride ( $MgB_2$ ), it is often classified as type 1.5 [87].

Based on the qualitative remarkable agreement of results of the holographic model of superconductivity considered here and that of Ginzburg-Landau theory, one can conclude that the holographic approach provides a good qualitative insight into the nature of physical superconductors, atleast for s-wave superconductors.

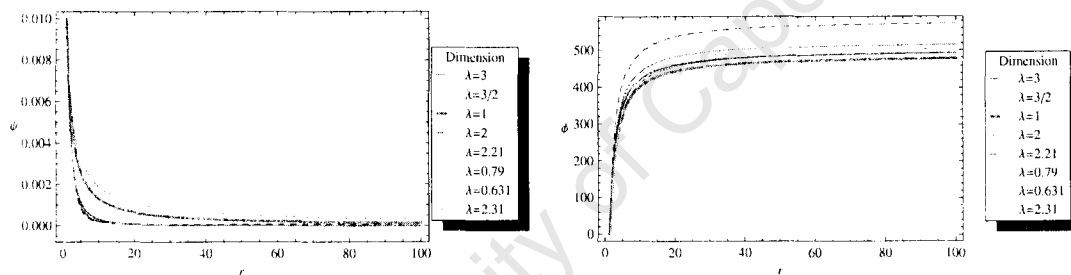
It would be very interesting to extend the computations presented in here to include the effects of the backreaction of the scalar field on the gravitational background. This would enable us to understand the source of the divergence for the condensates of the class  $\Psi_{\lambda_-}$ . A treatment involving a complete backreacted geometry would shed some light on the class of condensate that would be associated with the vortex and droplets solutions found in [40, 41]. Based on an understanding of real superconductors, one would not expect a type I holographic superconductor to support a stable vortex solution. One might also repeat the analysis presented here for the action, involving a matter field considered in [88]. This would indicate whether the features observed here are general, and might apply to an entire class of theories with gravity duals.

Note added: After this thesis was completed, two papers [89, 90] appeared on the arxiv, which presented models of holographic superconductivity based on the M-theory. Their constructions capture the physics of quantum phase transition at zero temperature. Recall that the understanding of this limit in the model studied here is marred by the numerical instability. Hence it will be very interesting to ask whether the results presented here will hold in M-theory based model of holographic superconductivity.

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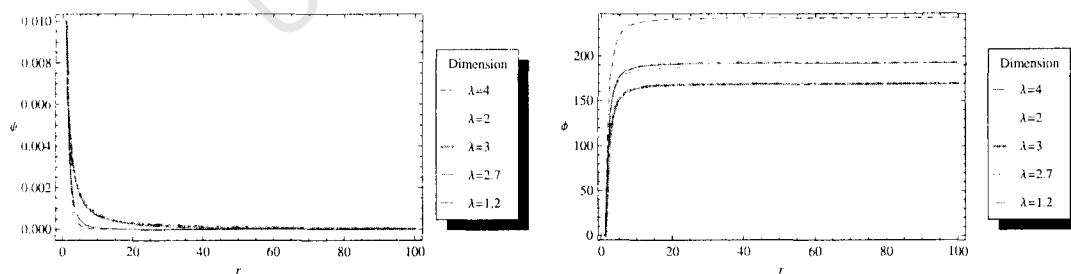
# Appendix A

## Marginally Stable Mode



**Figure A.1:** Marginally stable modes for the scalar field (Right) and Scalar potential as function of  $r$  (Left) in  $d = 3$  bulk spacetime dimensions

The field is everywhere positive, up to large  $r$ .

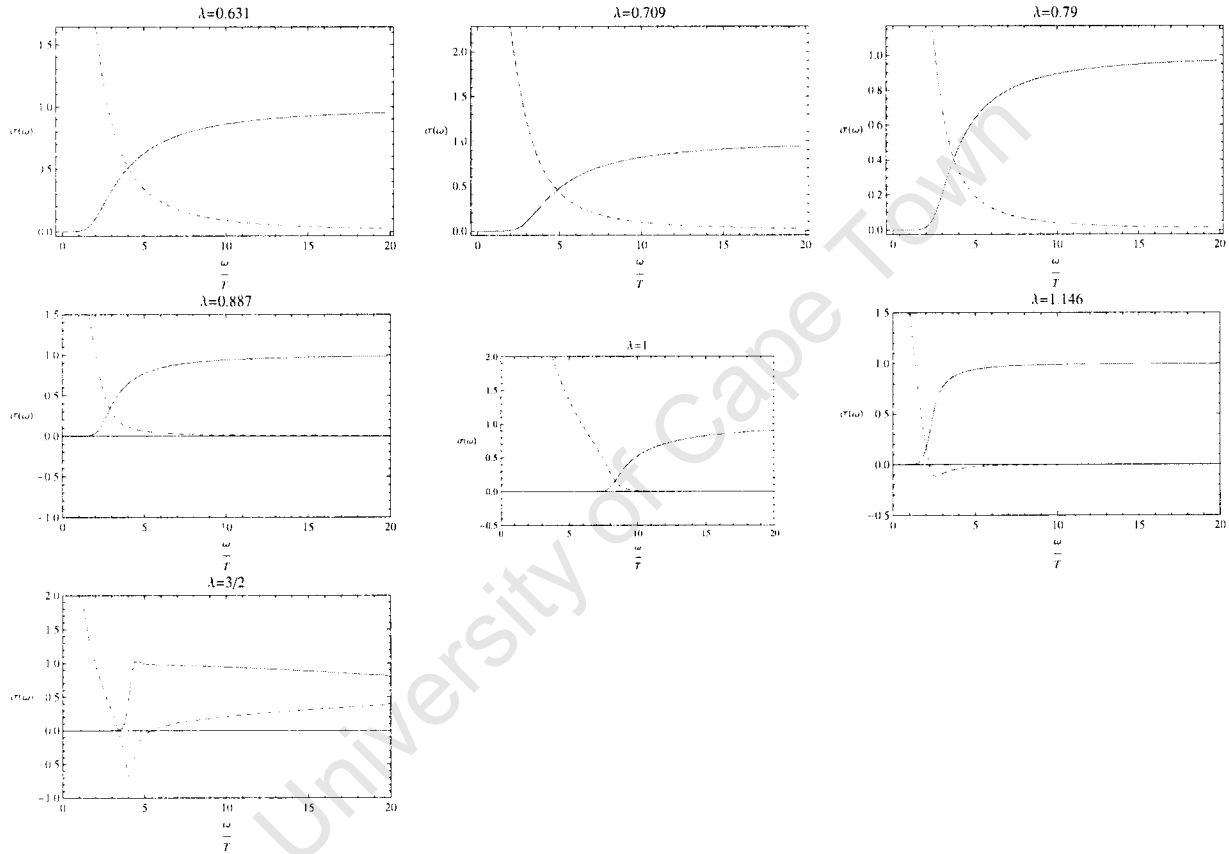


**Figure A.2:** Marginally stable modes for the scalar field (Right) and Scalar potential as function of  $r$  (Left) in  $d = 4$  bulk spacetime dimensions.

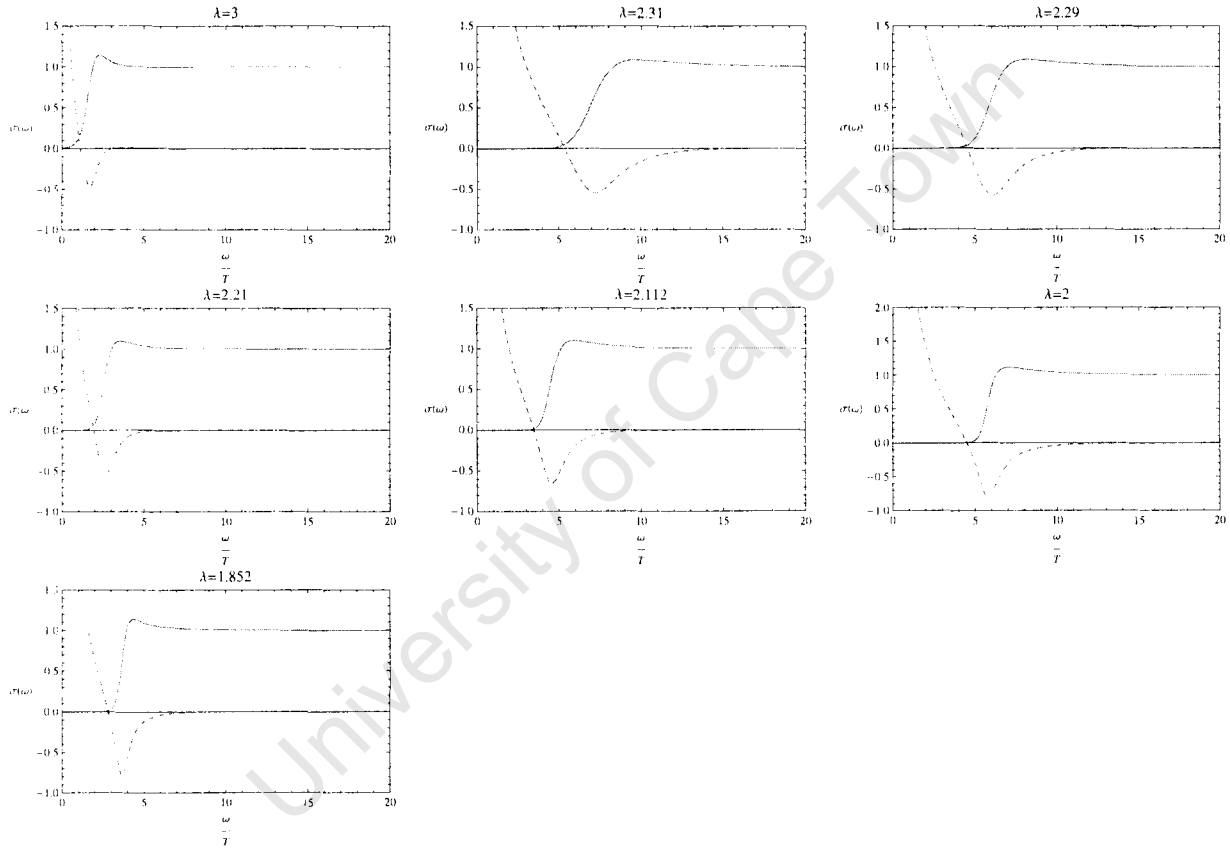
## Appendix B

# Conductivity in $(2 + 1)$ -dimensional Boundary Theory

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**Figure B.1:** Plots of conductivity in the superconducting phase for superconductors of class  $\lambda_-$  in  $(2+1)$ -dimensional boundary field theory. Each plot were captured at  $T/T_c \approx 0.3$ . The plots are labelled by the dimension of the operator in the dual field theory. Observe that for  $\lambda = 3/2$ , no pole is found in the real part. Dashed lines indicates the imaginary part of conductivity, while the thick line represents the real part.

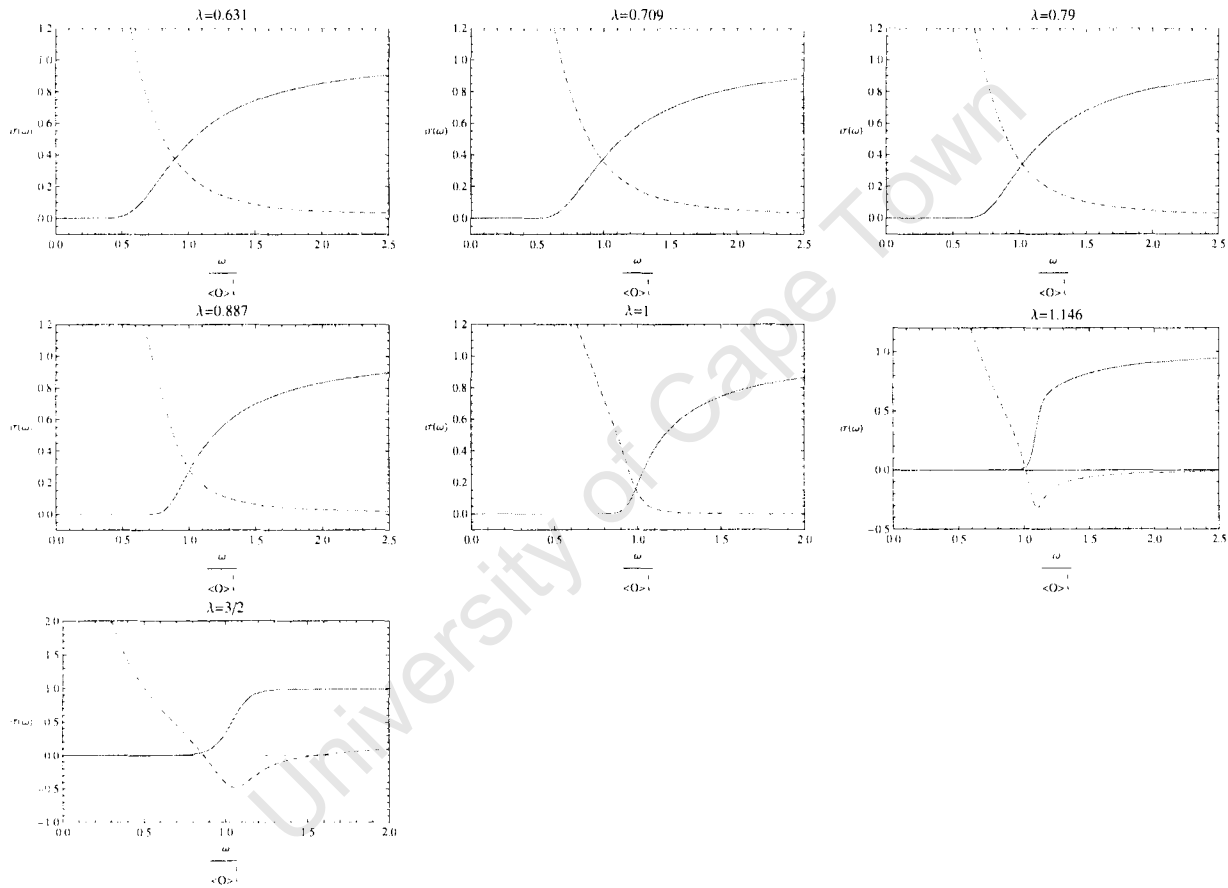


**Figure B.2:** Plots of conductivity in the superconducting phase for condensates of class  $\lambda_+$  in  $(2+1)$ -dimensional boundary field theory. Each plot were captured at  $T/T_c \approx 0.3$ . The plots are labelled by the dimension of the operator in the dual field theory.

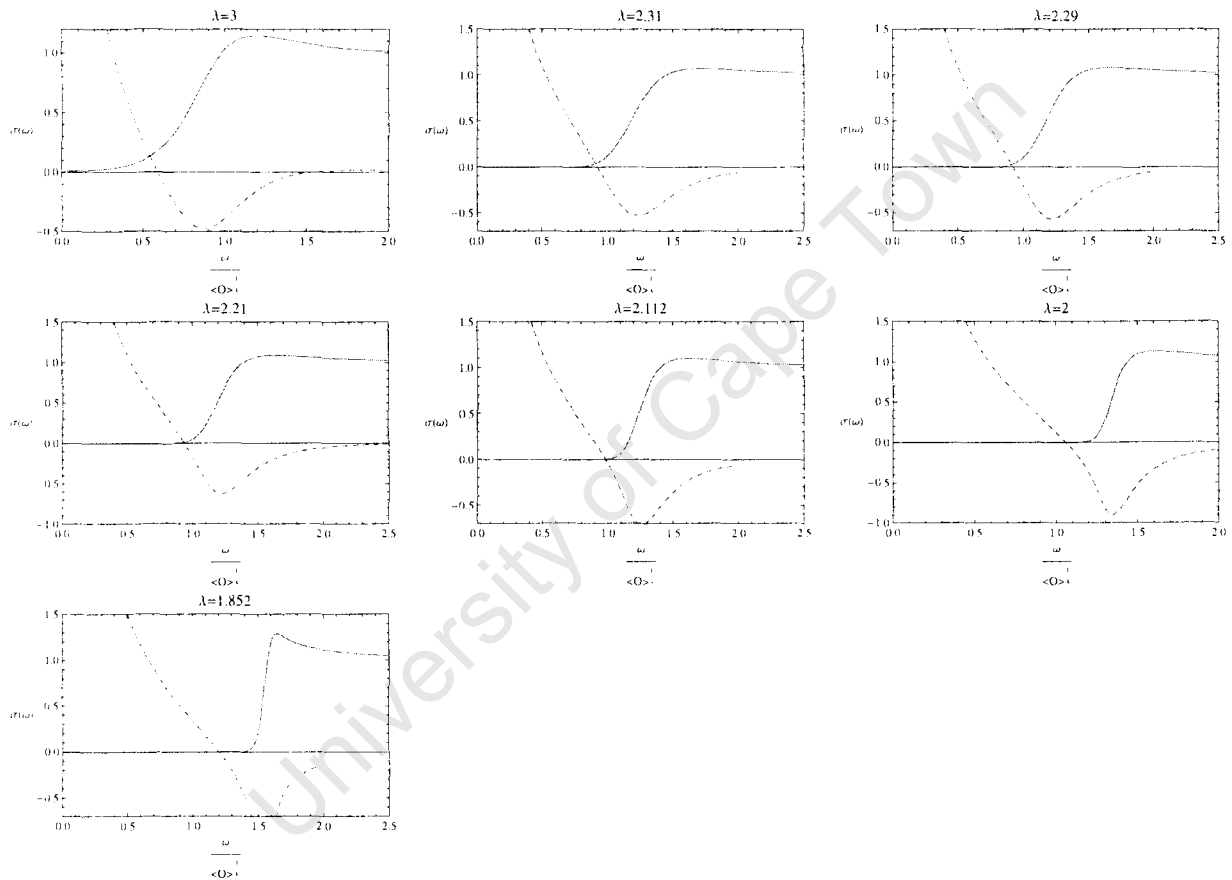
## Appendix C

Frequency dependent conductivity  
for condensates of both classes

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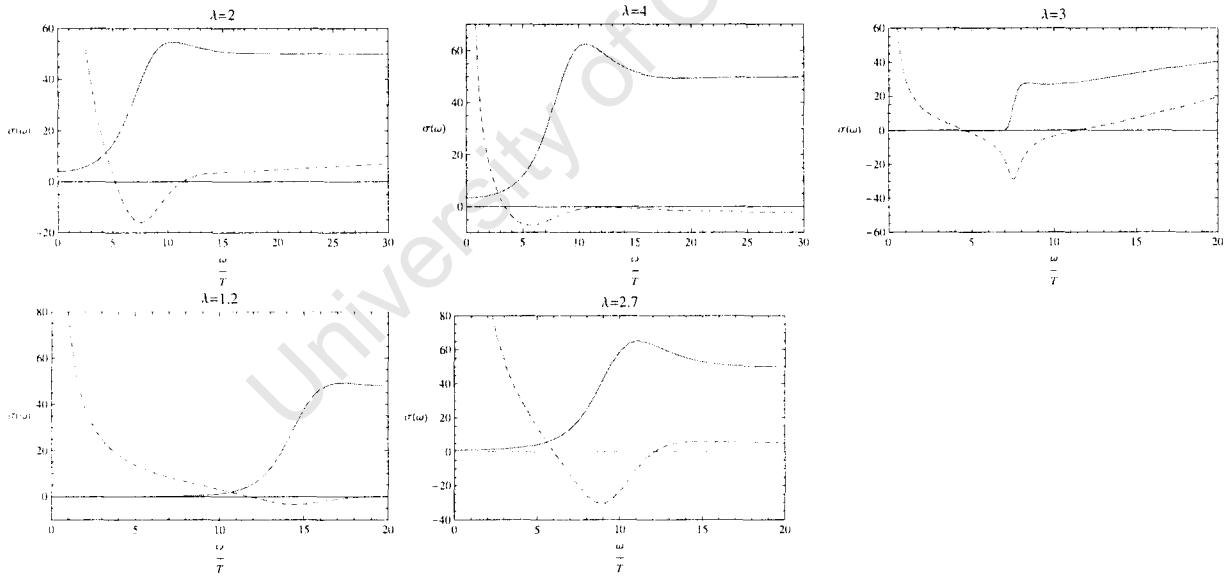
**Figure C.1:** Plots of frequency dependent conductivity for condensates of class  $\lambda_-$ . The frequency is normalized by the condensate in the superconducting phase. The plots are labelled by the dimension of the condensates in the dual field theory.



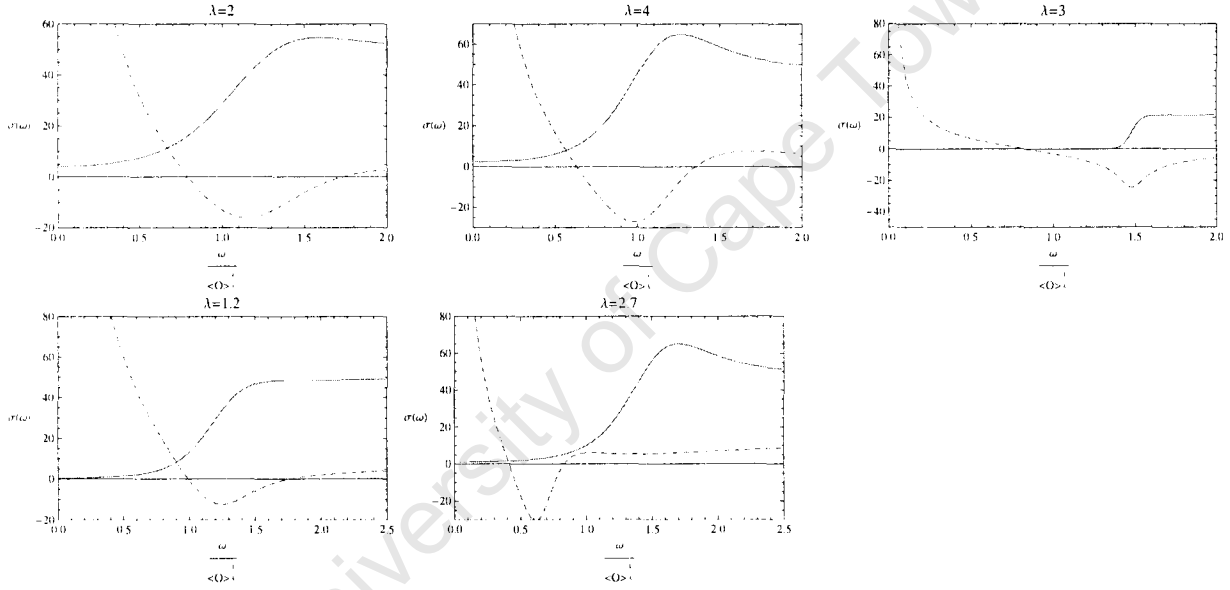
**Figure C.2:** Plots of frequency dependent conductivity for condensates of class  $\lambda_+$ . The frequency is normalized by the condensate in the superconducting phase. The plots are labelled by the dimension of the operator in the dual field theory

# Appendix D

## Conductivity in $(3 + 1)$ -dimensional Boundary Theory



**Figure D.1:** Plots of conductivity in the superconducting phase for  $(3 + 1)$ -dimensional boundary field theory. Each plot were captured at  $T/T_c = 0.3$ . The plots are labelled by the dimension of the operator in the dual field theory. Observe that for  $\lambda = 2$ , no pole is found in the real part.



**Figure D.2:** Plots of conductivity versus frequency normalized by the condensate in the superconducting phase in  $(3 + 1)$ -dimensional boundary field theory. Each plot was captured at  $T/T_c = 0.3$ . The plots are labelled by the dimension of the corresponding operator.

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