The Differential Space Concept: A Generalization of The Manifold Concept.

Patrice Pungu Ntumba

Supervised by Professor P.F. Cherenack.

February 1997

A dissertation submitted to the University of Cape Town in fulfillment of the requirement for the degree of Masters of Science in Mathematics.

Acknowledgements

I wish to thank my supervisor, Professor P.F. Cherenack, for his help and encouragement during the course of this thesis. His ability to pick a flaw in an argument is both an inspiration and a frustration! I would also like to express my deepest gratitude for his patience to bear with me because of my very limited English.

I acknowledge the part-time employement offered by the Department of Mathematics, and the financial support received from Professor G. Brummer, head of the Category and Topology research group.

I wish to thank the Department of Mathematics for the use of their facilities, G. Oliver and K. Colville for their advice in the typing of this work, and also Dr. I. Ntantu for his moral support.

Contents

Introduction

It is well known that the theory of smooth manifolds is incapable in dealing with the classical singularity problem in relativistic cosmology and relativistic astrophysics (see, for instance, [8, 29]). To overcome this problem, attempts have been made to obtain a more general and geometrically manageable concept than the traditional manifold concept. It is to this end that Aronszajn [1] and Marshall [16] developed the theory of the so-called sub-cartesian spaces that essentially are manifolds with "singularities" such as piecewise manifolds and quasianalytic sets of $\mathbb{R}^n$. In the spirit of this generalization, Sikorski [27] proposed the so-called differential space (or d-space, for short) by dropping the axiom forcing the manifold to be locally diffeomorphic to the Euclidean space of some dimension.

Any subset of $\mathbb{R}^n$ is a d-space, and there are many d-spaces which cannot be embedded in any Euclidean differential manifold. This makes the differential space concept a suitable tool to deal with the classical singularity problem . One should notice that space-time with its singular boundary is no longer a differentiable manifold, but it can be viewed as a differential space. Singularities (at least regular and some quasi-regular singularities) need not be considered as belonging to "singular boundaries" of space-time, but can be regarded as "internal domains" of a corresponding differential space (see [5]).

The aim of this work is to show the applicability of differential space theory in relativistic cosmology. As we will see, the notion of differential space enables us to investigate problems in differential geometry where differentiable manifolds do not suffice; for instance a geometrical analysis of quasi-regular singularities is possible within this framework. In this respect, we devote some time to the quasi-regular singularities of both the cosmic string and closed Friedman world. According to the classification scheme developed by Ellis and Schmidt [3], quasi-regular singularities are defined as those points of space-time through which no space-time extension is posssible altough the local geometry is well behaved as one approaches the singularity point. An observer approaching such a singularity has no warning until his history abruptly comes to an end (see [6]).

We organize our material in the following way. In Chapter 1, we give, for the reader's convenience, a short account of the theory of differential spaces. The book by Sikorski presents differential geometry in terms of d-spaces. Here we give only necessary definitions and theorems, providing a background for the following chapters. In Chapter 2, we define smooth forms on differential spaces. We begin with the cartesian products of differential spaces, which help us introduce bundles in the category of differential spaces. Chapter 2 ends up with the construction of a graded algebra where exterior differentiation and pull-back operations are allowed. Chapter 3 deals with linear connections and Riemannian geometry. It is in this chapter that we discuss the Riemannian and Lorentzian structures carried by a differential space. In Chapter 4, we effectively construct two space-time models (those of a cosmic string and the closed Friedman world) in terms of differential spaces.

Chapter 1

BASIC THEORY OF DIFFERENTIAL SPACES.

A smooth manifold is a set which is locally diffeomorphic to a smooth Euclidean space. This local resemblance of a differentiable manifold to a Euclidean space doesn't play any role when formulating many concepts of differential geometry. By dropping this property in the definition of the manifold concept, one obtains the so-called differential space. Many definitions and theorems related to differentiable manifolds preserve their contents almost untouched in the context of differential spaces (see [27]). Of course, every differentiable manifold is a differential space, but not vice-versa.

In this chapter, we define the concept of differential spaces and show how it generalizes the time-honoured manifold concept. In section 1.2, we speak of differential bases on differential spaces. It is easy to see that differential bases on d-spaces are linearly independent, i.e. if $\mathcal{F}_0$ is a differential basis and $f \in \mathcal{F}_0$, then f is linearly independent of the other functions of $\mathcal{F}_0$. Chapter 1 closes with vector fields and tensor fields on differential spaces, which are more general than vector fields and tensor fields on manifolds.

1.1 Preliminary Definitions.

We recall necessary definitions and theorems from the theory of differential spaces.

Let M be a non-empty set and $\mathcal{F}$ a non-empty subset of $\mathbb{R}^M$ ($\mathbb{R}^M$ is the collection of all set maps $f : M \mapsto \mathbb{R}$). The initial topology on M inherited from $\mathbb{R}$ using the functions in $\mathcal{F}$ will be denoted by $\tau_{\mathcal{F}}$.

We will assume, throughout this section, that the space M is endowed with the topology $\tau_{\mathcal{F}}$, defined by a family $\mathcal{F}$ of real-valued functions on M.

Definition 1.1. [26] *A real-valued function $f : A \mapsto \mathbb{R}$, where $A \subset M$, is said to be a local $\mathcal{F}$-function on A provided, for every point $p \in A$, there exist a neighbourhood U of p in the subspace A and a function $g \in \mathcal{F}$ such that $f|_U = g|_U$.*

The set of all local $\mathcal{F}$-functions on A will be denoted by $\mathcal{F}_A$. One can easily see that $\mathcal{F}|_A \subset \mathcal{F}_A$; in particular, $\mathcal{F} \subset \mathcal{F}_M$.

Note that we can restate Definition 1.1. as follows: a function f defined on a set $A \subset M$ is a local $\mathcal{F}$-function provided there exists an open covering $\mathcal{U}$ of the space A, such that for every set $U \in \mathcal{U}$ there exists a function $g_U \in \mathcal{F}$ with $f|_U = g_U|_U$.

Lemma 1.1. *If $A \subset B \subset M$, then $(\mathcal{F}_B)_A = \mathcal{F}_A$. In particular, $(\mathcal{F}_A)_A = \mathcal{F}_A$.*

Proof. Let's take $f \in (\mathcal{F}_B)_A$. For any point $p \in A$, there is a neighbourhood U of p in A such that $f|_U = g|_U$, for some $g \in \mathcal{F}_B$. Since $g \in \mathcal{F}_B$, there exist a neigbourhood V of p in B and a function $h \in \mathcal{F}$ such that $g|_V = h|_V$. But $W := U \cap V$ is a neighbourhood of p in A and $f|_W = h|_W$, which means that $f \in \mathcal{F}_A$. Therefore $(\mathcal{F}_B)_A \subset \mathcal{F}_A$.
Now let's show that $\mathcal{F}_A \subset (\mathcal{F}_B)_A$. Given $f \in \mathcal{F}_A$ and p a point in A, there exist a neighbourhood U of p in A and a function $g \in \mathcal{F}$ such that $f|_U = g|_U$.

But since $U \subset A \subset B$, and $h := g|_B \in \mathcal{F}|_B \subset \mathcal{F}_B$ is such that $f|_U = h|_U$, it follows that $f \in (\mathcal{F}_B)_A$, i.e. $\mathcal{F}_A \subset (\mathcal{F}_B)_A$. □

Proposition 1.1. *If $\mathcal{V}$ is an open covering of M, f is a function defined on M, and $f|_V \in \mathcal{F}_V$ for every $V \in \mathcal{V}$, then $f \in \mathcal{F}_M$.*

Proof. Let p be a point in M. There exists a neighbourhood $V \in \mathcal{V}$ such that $p \in V$. Since $f|_V \in \mathcal{F}_V$, it follows that there is a neighbourhood U of p in V such that $f|_U = g|_U$, for some $g \in \mathcal{F}$; therefore $f \in \mathcal{F}_M$. □

Proposition 1.1. asserts that if locally a function comes from $\mathcal{F}$, it must be in $\mathcal{F}$. This is a sheaf property.

Definition 1.2. [26] *The pair $(M, \mathcal{F})$ is said to be a differential space (d-space, for short) provided that*

(i) $\mathcal{F}$ is closed with respect to localization, i.e. $\mathcal{F} = \mathcal{F}_M$,

(ii) $\mathcal{F}$ is closed with respect to composition with smooth functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, i.e., if $f_1, \dots, f_n \in \mathcal{F}$ and ω is a smooth real-valued function ($= C^\infty$- function) defined on $\mathbb{R}^n$, then the composition

$$\omega \circ (f_1(p), \dots, f_n(p)),$$

for any $p \in M$, is a function in $\mathcal{F}$.

The set $\mathcal{F}$ is called the differential structure on M.

Let

$$sc\,\mathcal{F} := \{\omega \circ (f_1, \dots, f_n) \in \mathcal{F} | f_1, \dots, f_n \in \mathcal{F},\ \omega \in \varepsilon_n := C^\infty(\mathbb{R}^n, \mathbb{R});\ n \in \mathbb{N}\}.$$

The axiom (ii) becomes simply

$$sc\,\mathcal{F} = \mathcal{F}.$$

Example 1.1. *The pair* $(\mathbb{R}^n, \varepsilon_n)$, $n \in \mathbb{N}$, *where* $\varepsilon_n = C^\infty(\mathbb{R}^n, \mathbb{R})$, *i.e., the set of all infinitely differentiable functions on* $\mathbb{R}^n$, *is a differential space. The d-space* $(\mathbb{R}^n, \varepsilon_n)$ *is called the Euclidean differential space.*

Proof. Any function which is locally C^∞ on $\mathbb{R}^n$, must be C^∞ on all $\mathbb{R}^n$, i.e., $\varepsilon_n = (\varepsilon_n)_{\mathbb{R}^n}$. It is also clear that any composition of C^∞-functions is also C^∞. Therefore $sc\ \varepsilon_n = \varepsilon_n$. □

Example 1.2. *Let* $(M, \mathcal{A})$ *be a* C^∞*-n-dimensional differentiable manifold, where* $\mathcal{A}$ *is an atlas on* M. *The ring* $\mathcal{F}$ *of all smooth functions on* M *defined by*

$$\mathcal{F} := \{f : M \mapsto \mathbb{R} | f \circ \varphi^{-1} \in \varepsilon_n, \textit{ for all chart } (U, \varphi) \in \mathcal{A}\}$$

determines a differential structure on M. *The differential structure* $\mathcal{F}$ *is said to be determined by the atlas* $\mathcal{A}$.

Proof. First we need to show that the necessary and sufficient condition for every function $f : M \mapsto \mathbb{R}$ to be smooth, is that for every chart $(U, \varphi) \in \mathcal{A}$, $f \circ \varphi^{-1}$ be smooth, $f \in \mathcal{F}$. Indeed, if f is smooth, $f \circ \varphi^{-1}$ is smooth since φ is a diffeomorphism. On the other hand, if we assume that $f \circ \varphi^{-1}$ is smooth , then $f|_U = (f \circ \varphi^{-1}) \circ \varphi \in \mathcal{F}_U$; and since the U's form an open covering of M, $f \in \mathcal{F}$.
Now let's prove that $\mathcal{F}$ is a differential structure on M. Take $f \in \mathcal{F}_M$. For every $p \in M$, there is an open neighbourhood V of p such that $f|_V = g|_V$, for some $g \in \mathcal{F}$. If (U, φ) is a chart, then $(V \cap U, \varphi)$ is also a chart and $f \circ \varphi^{-1}|_{V \cap U} = g \circ \varphi^{-1}|_{V \cap U} \in \varepsilon_n$. It follows that $f \in \mathcal{F}$. Therefore $\mathcal{F}_M \subset \mathcal{F}$.
The inclusion $sc\ \mathcal{F} \subset \mathcal{F}$ is easy. Indeed, take $\omega \in \varepsilon_n$ and $f_1, \ldots, f_n \in \mathcal{F}$, $n \in \mathbb{N}$. Since $f_i \circ \varphi^{-1} \in \varepsilon_n$ for every chart (U, φ), $(\omega \circ (f_1, \ldots, f_n)) \circ \varphi^{-1} = \omega \circ (f_1 \circ \varphi^{-1}, \ldots, f_n \circ \varphi^{-1}) \in \varepsilon_n$. Therefore $\omega \circ (f_1, \ldots, f_n) \in \mathcal{F}$ and $sc\ \mathcal{F} \subset \mathcal{F}$.
□

As it is well known from the theory of manifolds, subsets of differentiable manifols are not, generally speaking, differentiable manifolds. But in the differential spaces context, differential structures can be induced from a base space to a subset.

Proposition 1.2. [26, 9] *If $(M, \mathcal{F})$ is a differential space and A an arbitrary subset of M, then the pair $(A, \mathcal{F}_A)$ is a d-space. It will be called a differential subspace of $(M, \mathcal{F})$ (or d-subspace, for short).*

Proof. By Lemma 1.1., $\mathcal{F}_A$ is closed with respect to localization. To prove the second axiom of d-spaces, let's consider the set

$$sc\, \mathcal{F}_A := \{\omega \circ (f_1, \ldots, f_n) | f_1, \ldots, f_n \in \mathcal{F}_A, \omega \in \varepsilon_n, n \in \mathbb{N}\}.$$

For all $i = 1, \ldots, n$, if $f_i \in \mathcal{F}_A$ and $p \in A$, there exists a $\tau_{\mathcal{F}}|_A$-open neighbourhood U_i of p such that $f_i|_{U_i} = g_i|_{U_i}$, for some $g_i \in \mathcal{F}$. Since $\bigcap_{i=1}^n U_i$ is a neighbourhood of $p \in A$ and $\omega \circ (f_1, \ldots, f_n)|_{\bigcap_{i=1}^n U_i} = \omega \circ (g_1, \ldots, g_n)|_{\bigcap_{i=1}^n U_i}$, with $\omega \circ (g_1, \ldots, g_n) \in \mathcal{F}$ (remember $(M, \mathcal{F})$ is a d-space), it follows that $\omega \circ (f_1, \ldots, f_n) \in \mathcal{F}_A$; which implies that $sc\, \mathcal{F}_A \subset \mathcal{F}_A$. □

From Proposition 1.2. it follows that every subset of a Euclidean space is a differential space. This property of Euclidean spaces is enough to show one the scope of the differential space concept. Indeed, differential spaces are a generalization of differentiable manifolds since manifolds are locally diffeomorphic to Euclidean spaces. Later we will discuss differential spaces which can not be embedded into any Euclidean space.

Proposition 1.3. [5] *Let N be any set. For any set $\mathcal{G}_0$ of real-valued functions on N, there is a smallest differential structure $\mathcal{G}$ such that $\mathcal{G}_0 \subset \mathcal{G}$, and the topology $\tau_{\mathcal{G}}$ coincides with the topology $\tau_{\mathcal{G}_0}$. The set $\mathcal{G}_0$ is said to generate the differential structure $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be the family of all functions $f : N \mapsto \mathbb{R}$ such that $f \in \mathcal{G}$ if and only if, for every $p \in N$ there is a $\tau_{\mathcal{G}_0}$-neighbourhood U of p in N such that

$$f|_U = \omega \circ (f_1, \ldots, f_n)|_U$$

where $f_1, \ldots, f_n \in \mathcal{G}_0$ and $\omega \in \varepsilon_n$, $n \in \mathbb{N}$. Clearly, $\mathcal{G}_0 \subset \mathcal{G}$.
The family $\mathcal{G}$ is closed with respect to localization, i.e., $\mathcal{G}_N = \mathcal{G}$. Indeed, let's take $f \in \mathcal{G}_N$. For all point $p \in N$, there exist a $\tau_{\mathcal{G}}$-neighbourhood V of p and a function $g \in \mathcal{G}$ such that $f|_V = g|_V$. Since $g \in \mathcal{G}$, one can

find a $\tau_{\mathcal{G}_0}$-neighbourhood U of p such that $g|_U = \omega \circ (g_1, \ldots, g_n)|_U$, where $g_1, \ldots, g_n \in \mathcal{G}_0$ and $\omega \in \varepsilon_n$, $n \in \mathbb{N}$. But $U \cap V$ is a $\tau_{\mathcal{G}_0}$-neighbourhood of p and $f|_{U\cap V} = \omega \circ (g_1, \ldots, g_n)|_{U\cap V}$. Therefore $\mathcal{G}_N \subset \mathcal{G}$. The next axiom requires that $sc\ \mathcal{F} \subset \mathcal{F}$. Consider $f_1, \ldots, f_n \in \mathcal{F}$, $n \in \mathbb{N}$, and $\omega \in \varepsilon_n$. For any point $p \in N$, $f_i \in \mathcal{F}$, $i = 1, \ldots, n$, implies that there exists a $\tau_{\mathcal{G}_0}$-open neighbourhood U_i of p such that $f_i|_{U_i} = \omega_i \circ (f_{i1}, \ldots, f_{im_i})|_{U_i}$, where $f_{i1}, \ldots, f_{im_i} \in \mathcal{G}_0$ and $\omega_i \in \varepsilon_{m_i}$, $m_i \in \mathbb{N}$. It follows that

$$\omega \circ (f_1, \ldots, f_n)|_{\bigcap_{i=1}^n U_i} = \\ \omega \circ (\omega_1, \ldots, \omega_n) \circ (f_{11}, \ldots, f_{1m_1}, \ldots, f_{n1}, \ldots, f_{nm_n})|_{\bigcap_{i=1}^n U_i}.$$

Since $f_{11}, \ldots, f_{nm_n} \in \mathcal{G}_0$ and $\omega \circ (\omega_1, \ldots, \omega_n) \in \varepsilon_{n'}$, $n' := \sum_{i=1}^n m_i$, then $\omega \circ (f_1, \ldots, f_n) \in \mathcal{G}$; i.e. $sc\ \mathcal{G} \subset \mathcal{G}$.
Now, suppose that there exists a differential structure $\mathcal{G}_1$ such that $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}$. Clearly we have $\mathcal{G} = (sc\ \mathcal{G}_0)_N \subset (sc\ \mathcal{G}_1)_N = (\mathcal{G}_1)_N = \mathcal{G}_1$, which proves that $\mathcal{G}_1 = \mathcal{G}$. The equality $\tau_{\mathcal{G}} = \tau_{\mathcal{G}_0}$ is easy to check. □

Then we have the following:

Definition 1.3. [5] *Let $\mathcal{F}_0$ be a set of real-valued functions on a set M. The pair $(M, \mathcal{F})$ is said to be finitely generated by $\mathcal{F}_0$ if $\mathcal{F}_0$ is finite and generates $\mathcal{F}$.*

Example 1.3. *The differential space $(\mathbb{R}^n, \varepsilon_n)$ is finitely generated: $\varepsilon_n = sc\ \{\pi_1, \ldots, \pi_n\}$, where $\pi_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i = 1, \ldots, n$, is the projection onto the i-th coordinate.*

It follows from Example 1.3. that if N is a subset of $(\mathbb{R}^n, \varepsilon_n)$, the differential structure induced by ε_n on N is $\mathcal{G} := (sc\ \{\pi_1|_N, \ldots, \pi_n|_N\})_N$.

Definition 1.4. [6] *A differential space $(M, \mathcal{F})$ is said to be Hausdorff if the topological space $(M, \tau_{\mathcal{F}})$ is Hausdorff.*

For example, the differential space $(\mathbb{R}^n, \varepsilon_n)$ is Hausdorff.

One can show easily that if $(M, \mathcal{F})$ is a differential space such that $\mathcal{F} = \text{Gen}\ \mathcal{F}_0$, then it is Hausdorff if and only if, for any $p, q \in M$, $p \neq q$, there exists a function $f \in \mathcal{F}_0$ such that $f(p) \neq f(q)$.

Let's now devote some place to the definition of mappings between differential spaces.

Definition 1.5. [23] *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be differential spaces. A mapping $\varphi : M \mapsto N$ is said to be a smooth mapping of $(M, \mathcal{F})$ into $(N, \mathcal{G})$ if $\varphi^*\mathcal{G} := \{g \circ \varphi | g \in \mathcal{G}\} \subset \mathcal{F}$. The set of smooth maps make the collection of differential spaces into a category, denoted $\mathcal{DSP}$.*

This definition also gives some evidence about the wider scope of the theory of differential spaces. In fact, let $f : X \mapsto Y$ be a map from a m-dimensional differentiable manifold to a n-dimensional differentiable manifold and let $\{x^\mu\}$ and $\{y^\alpha\}$ be charts on X and Y respectively. The map f is said to be smooth at the point (x^μ) if $y^\alpha = y^\alpha(x^\mu)$ is C^∞ with respect to each x^μ. The smoothness in the sense of differential spaces does not require the above-mentioned concept of differentiation in $\mathbb{R}^n$, it relies on the primary smoothness of functions which form a differential structure of a differential space.

In Definition 1.5., if φ is one-to-one and onto, φ is said to be a diffeomorphism provided both mappings $\varphi : M \mapsto N$ and $\varphi^{-1} : N \mapsto M$ are smooth. And the differential spaces $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are said to be diffeomorphic.

Definition 1.6. [26] *A differential space $(M, \mathcal{F})$ is a n-dimensional differential manifold provided every point $p \in M$ has a neighbourhood A such that $(A, \mathcal{F}_A)$ is diffeomorphic, in the sense of d-spaces, to a pair (O, ε_O), where O is an open subset of $\mathbb{R}^n$, and $\varepsilon_O = (\varepsilon_n)_O$.*

The following definition makes a contact with the time-honoured manifold concept in terms of maps and atlases.

Definition 1.7. [9] *Let $(M, \mathcal{F})$ be a n-dimensional differential manifold. Any pair (U, φ), where U is an open subset of M and φ is a diffeomorphism*

$U \mapsto \varphi(U) \subset \mathbb{R}^n$, is called chart, or coordinate system on U.
A set of charts $(U_i, \varphi_i), i = 1, 2, \ldots$, is called atlas if the sets U_i form an open covering of M.

Definition 1.7. motivates the following:

Theorem 1.1. [9] *Let $\mathcal{A}$ be an atlas of a differential manifold $(M, \mathcal{F})$. $\mathcal{A}$ has the following properties :*

(i) If $(U, \varphi) \in \mathcal{A}$, $\varphi : U \mapsto \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

(ii) $M = \bigcup_i U_i$, $(U_i, \varphi_i) \in \mathcal{A}$ for all $i \in I \subset \mathbb{N}$.

(iii) Given (U, φ) and (V, ψ) such that $U \cap V \neq \emptyset$, the maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \mapsto \psi(U \cap V)$$

and

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \mapsto \varphi(U \cap V)$$

are diffeomorphisms.

Proof. (i) and (ii) follow from Definition 1.7.. The maps in (iii) are compositions of diffeomorphisms. □

From Example 1.2. and Theorem 1.1., it is clear that the differential manifold concept is equivalent to the traditional differentiable manifold concept. For this reason, we will prefer to use the term"differential" (space, manifold).

The definition of differential manifolds in terms of the family $\mathcal{F}$ turns out to be more natural than the traditional one. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two atlases on M and $\mathcal{F}_1$ and $\mathcal{F}_2$ two families of all smooth functions determined by the atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We have the following result.

Proposition 1.4. [5] *$\mathcal{F}_1 = \mathcal{F}_2$ if and only if, for any $(U_1, \varphi_1) \in \mathcal{A}_1$ and $(U_2, \varphi_2) \in \mathcal{A}_2$, $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism.*

Proof. Suppose that $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism. Let's consider $f \in \mathcal{F}_1$. By Example 1.2., $f \circ \varphi_1^{-1}$ is smooth. Therefore $f \circ \varphi_2^{-1} = f \circ \varphi_1^{-1} \circ \varphi_1 \circ \varphi_2^{-1}$ is smooth. Thus $f \in \mathcal{F}_2$. In the same way, one can show that $\mathcal{F}_2 \subset \mathcal{F}_1$. The reverse implication is always true since φ_1 and φ_2 are diffeomorphisms.□

This proposition establishes an equivalence relation between differential manifolds: $(M, \mathcal{A}_1) \sim (M, \mathcal{A}_2)$ if and only if $\mathcal{F}_1 = \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are differential structures determined by atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. It follows that every given equivalence class is uniquely determined by a family $\mathcal{F}$ of functions on M. It should be noticed, by defining a d-space as a pair $(M, \mathcal{F})$, that M can be assumed to be any set and there is no need to ascribe to it, from the beginning, the structure of a topological space.

1.2 Differential Basis on Differential Spaces

The notion of differential basis on differential spaces is important when one is dealing with the dimensionality of the differential space of a cosmic string. We discuss this matter in Chapter 4.

But first of all let's define the notion of tangent vector to a differential space.

Definition 1.8. [26] *Any linear mapping $v : \mathcal{F} \mapsto \mathbb{R}$, satisfying the Leibnitz condition*

$$v(f.g) = v(f).g(p) + f(p).v(g), \tag{1.1}$$

for any $f, g \in \mathcal{F}$, is said to be a tangent vector to a d-space $(M, \mathcal{F})$ at $p \in M$.

The real number $v(f)$ is called the directional derivative of the function $f \in \mathcal{F}$ in the direction v and is often denoted by the symbol $\partial_v f$. Thus (1.1) can be written as follows

$$\partial_v(f.g) = \partial_v f.g(p) + f(p).\partial_v g$$

for all $f, g \in \mathcal{F}$.
The set of all tangent vectors to $(M, \mathcal{F})$ at $p \in M$ is a linear space, called the tangent space to $(M, \mathcal{F})$ at p, and will be denoted by T_pM.

Proposition 1.5. [26] *Let $(M, \mathcal{F})$ be a differential space, $f \in \mathcal{F}$ and $f|_A = 0$ for a neighbourhood A of a point $p \in M$. Then $\partial_v f = 0$ for every $v \in T_pM$. Consequently, if functions $f, g \in \mathcal{F}$ are equal on a neighbourhood A of a point $p \in M$, then $\partial_v f = \partial_v g$ for every $v \in T_pM$.*

Proof. Reasonably clear. □

Let $(A, \mathcal{F}_A)$ be a differential subspace of a differential space $(M, \mathcal{F})$ and let $p \in A$. If $v \in T_pA$, i.e., if v is a vector tangent to A at p, then the formula

$$\bar{v}(f) = v(f|_A) \tag{1.2}$$

for all $f \in \mathcal{F}$, defines a vector $\bar{v} \in T_pM$. Indeed, $\bar{v}$ is linear and

$$\begin{aligned}\bar{v}(f.g) &= v(f|_A)g(p) + f(p)v(g|_A) \\ &= \bar{v}(f)g(p) + f(p)\bar{v}(g)\end{aligned}$$

for all $f, g \in \mathcal{F}$ and $p \in A$. Clearly the map $T_pA \mapsto T_pM$ which assigns $\bar{v} \in T_pM$ to $v \in T_pA$ is a linear monomorphism. We shall identify v with $\bar{v}$.

The following proposition is true.

Proposition 1.6. [26] *The tangent space T_pA at $p \in A$ to a subspace $(A, \mathcal{F}_A)$ of a differential space $(M, \mathcal{F})$ is a linear subspace of the tangent space T_pM. If A is an open subset of M, then $T_pA = T_pM$ for every $p \in A$.*

Proof. It is easy to see that T_pA is a linear subspace of T_pM by virtue of (1.2). Now, let's assume that A is open in M. If $f, g \in \mathcal{F}_A$, there exists a $\tau_{\mathcal{F}}$-open subset U in A such that

$$\begin{aligned}f|_U &= f'|_U, \qquad & f' \in \mathcal{F} \\ g|_U &= g'|_U, \qquad & g' \in \mathcal{F}.\end{aligned}$$

Therefore, for all $v \in T_pM$, we have using Proposition 1.5.

$$\begin{aligned}v(f.g) &= v(f'.g') \\ &= v(f')g'(p) + f'(p)v(g') \\ &= v(f)g(p) + f(p)v(g),\end{aligned}$$

which proves that $v \in T_p A$.□

Let $(M, \mathcal{F})$ be a differential space.

Definition 1.9. [10] *A function $f \in \mathcal{F}$ is said to be differentially dependent (briefly, d-dependent) on functions $g_1, \ldots, g_n \in \mathcal{F}$ at a point $p \in M$ if there exist a neighbourhood $U \in \tau_{\mathcal{F}}$ of the point p and a function $\omega \in \varepsilon_n$ such that*

$$f|_U = \omega \circ (g_1, \ldots, g_n)|_U.$$

Example 1.4. . *Any function $f \in \varepsilon_n$ differentially depends on projections $\pi_1, \pi_2, \ldots, \pi_n \in \varepsilon_n$ at any point $p \in \mathbb{R}^n$.*

Proof. Since $\varepsilon_n = \mathrm{Gen}\{\pi_1, \ldots, \pi_n\}$, $f \in \varepsilon_n$ if and only if for any point $p \in \mathbb{R}^n$ there exist a neighbourhood $U \in \tau_{\varepsilon_n}$ and a smooth function $\omega : \mathbb{R}^n \to \mathbb{R}$ such that

$$f|_U = \omega \circ (\pi_1, \ldots, \pi_n)|_U.$$

□

Definition 1.10. [10] *A set $\{f_1, \ldots, f_n\} \subset \mathcal{F}$ is said to be differentially independent at a point $p \in M$ if no function f_i, $i = 1, \ldots, n$ differentially depends on other functions of this set at p. Any set $\mathcal{F}_0 \subset \mathcal{F}$ is said to be differentially independent at $p \in M$ if every finite subset of $\mathcal{F}_0$ is differentially independent at p.*

Example 1.5. . *The set $\{\pi_1, \ldots, \pi_n\} \subset \varepsilon_n$ is differentially independent at any point $p \in \mathbb{R}^n$.*

Evidently, from Definitions 1.9. and 1.10. it follows that both d-dependence and d-independence of a set $\mathcal{F}_0 \subset \mathcal{F}$ are local properties of $\mathcal{F}_0$.

Let's prove

Lemma 1.2. [18] *Let $M \subset \mathbb{R}^n$ be a k-dimensional algebraic variety. Then the set of the projections $\{\pi_1|_M, \ldots, \pi_n|_M\} \subset C^\infty(M)$ is d-independent at an arbitrary point $p \in M$ if and only if $k = n$.*

Proof. ($\Longrightarrow$) Suppose that the set $\{\pi_1|_M, \ldots, \pi_n|_M\}$ is d-independent. It follows that the family $\{\frac{\partial}{\partial x_1}|_M, \ldots, \frac{\partial}{\partial x_n}|_M\}$ is linearly independent ($\frac{\partial}{\partial x_i}$ is the tangent vector along the i-th coordinate axis, $i = 1, \ldots, n$). Therefore, at any arbitrary point $p \in M$, one can define n linearly independent tangent vectors; which implies that M is n-dimensional.
($\Longleftarrow$) Clear, by using Example 1.3.

By using Lemma 1.2., one can prove the following:

Proposition 1.7. [10] *Let $M \subset \mathbb{R}^n$ be a non-empty subset and $\mathcal{F} = (\varepsilon_n)_M$. The set of the projections $\{\pi_1|_M, \ldots, \pi_n|_M\}$ is d-independent at $p \in M$ if and only if $\dim T_pM = n$.*

More generally, we also prove

Proposition 1.8. *Let $(M, \mathcal{F})$ be a d-space with $\mathcal{F} = \mathrm{Gen}\{f_1, \ldots, f_n\}$, where the set $\{f_1, \ldots, f_n\}$ is d-independent at any point $p \in M$. Then, $\dim T_pM = n$ for any $p \in M$.*

Proof. Since $\mathcal{F} = \mathrm{Gen}\{f_1, \ldots, f_n\}$ and the set $\{f_1, \ldots, f_n\}$ is d-independent, the function $\varphi := (f_1, \ldots, f_n)$ is a diffeomorphism from $(M, \mathcal{F})$ onto $(\varphi(M), (\varepsilon_n)_{\varphi(M)})$. It is easy to see that φ is one-to-one and onto. On the other hand, φ is smooth since $\omega \circ (\pi_1|_{\varphi(M)}, \ldots, \pi_n|_{\varphi(M)}) \circ (f_1, \ldots, f_n) = \omega(f_1, \ldots, f_n) \in \mathcal{F}$, for any $\omega \in \varepsilon_n$. φ^{-1} is also smooth; indeed, for any $\sigma \in \varepsilon_n$, $\sigma \circ (f_1, \ldots, f_n) \circ \varphi^{-1} = \sigma$. Hence, for any $p \in (M, \mathcal{F})$, dim $T_pM =$ dim $T_x\varphi(M) = n$, where $x = (f_1(p), \ldots, f_n(p))$. □

It is easy to check the following:

Corollary 1.1. *Let $(M, \mathcal{F})$ be a differential space finitely generated by $\mathcal{F}_0 := \{f_1, \ldots, f_n\}$, and $p \in M$. The following conditions are equivalent:*

(i) $\mathcal{F}_0$ *is differentially independent at* p.

(ii) $\dim T_pM = n$.

The immediate proof to this corollary is omitted.

Another useful characterization of the d-independence of a set of real-valued functions belonging to $\mathcal{F}$ is given by the following:

Theorem 1.2. [6] *A subset* $\{f_1, \ldots, f_n\} \subset \mathcal{F}$ *is differentially independent at* $p \in M$ *if and only if for any function* $\omega \in \varepsilon_n$ *and any neighbourhood* $U \in \tau_{\mathcal{F}}$ *of* p*, the following condition is satisfied*

$$\omega \circ (f_1, \ldots, f_n) = 0 \Longrightarrow \text{for all } 1 \leq i \leq n, \ \partial_i\omega(f_1(p), \ldots, f_n(p)) = 0 \quad (1.3)$$

Proof. The implication ($\Rightarrow$) is immediate.
($\Leftarrow$) Let's assume that for any function $\omega \in \varepsilon_n$ and any neighbourhood $U \in \tau_{\mathcal{F}}$ of p, (1.3) is true. Let's suppose that one of the f_i's differentially depends on the other functions of the set, i.e., on $f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n$. Without loss of generality, suppose that there exists a function $\sigma \in \varepsilon_{n-1}$ such that $f_1 = \sigma \circ (f_2, \ldots, f_n)$ on some neighbourhood of p, i.e., f differentially depends on $f_2, \ldots, f_n$. Let ω be a function in ε_n such that $\omega \circ (f_1, \ldots, f_n) = f_1 - \sigma \circ (f_2, \ldots, f_n) = 0$, and then $\partial_1\omega = 1$. This contradicts (1.3). Thus the set $\{f_1, \ldots, f_n\}$ is d-independent. □

To look deeper into a local structure of a differential space we introduce the following:

Definition 1.11. [10] *A subset* $\mathcal{G} \subset \mathcal{F}$ *reproduces* $\mathcal{F}$ *at* $p \in M$ *if, for any function* $f \in \mathcal{F}$*, there exist a neighbourhood* $U \in \tau_{\mathcal{F}}$ *of* p *and functions* $g_1, \ldots, g_n \in \mathcal{G}$, $\omega \in \varepsilon_n$ *such that* $f|_U = \omega \circ (g_1, \ldots, g_n)|_U$.

Remark 1.1. [10] *Let* $(M, \mathcal{F})$ *be a locally finitely generated differential space. One can easily see that a subset* $\mathcal{G} \subset \mathcal{F}$ *reproduces* $\mathcal{F}$ *at* $p \in M$ *if and only if* $\mathcal{G}$ *locally generates the structure* $\mathcal{F}$ *in a certain neighbourhood of the point* p.

We can now introduce the notion of differential bases on differential spaces.

Definition 1.12. [*10*] *A set $\mathcal{G} \subset \mathcal{F}$ is a differential basis of the differential structure $\mathcal{F}$ at $p \in M$ if $\mathcal{G}$ is differentially independent at p and $\mathcal{G}$ reproduces $\mathcal{F}$ at p.*

Example 1.6. . *Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences, i.e.*

$$\mathbb{R}^{\mathbb{N}} = \{x := (x_i)_{i \in \mathbb{N}} : x_i \in \mathbb{R} \textit{for all } i \in \mathbb{N}\}.$$

The projection of $\mathbb{R}^{\mathbb{N}}$ onto the i-th coordinate is given by

$$\pi_i(x) = x_i$$

for $x = (x_i) \in \mathbb{R}^{\mathbb{N}}$.
Let $\varepsilon_{\mathbb{N}}$ be the differential structure on $\mathbb{R}^{\mathbb{N}}$ generated by the set $\{\pi_i : i \in \mathbb{N}\}$. The set $\{\pi_i : i \in \mathbb{N}\}$ turns out to be a differential basis of the differential structure $\varepsilon_{\mathbb{N}}$ at a point $x \in \mathbb{R}^{\mathbb{N}}$.

The following lemma is proved in [18].

Lemma 1.3. *Let $(M, \mathcal{F})$ be a differential space and $\mathcal{B}$ a differential basis of the differential structure $\mathcal{F}$ at $p \in M$. Then, for each function $u_0 : \mathcal{B} \to \mathbb{R}$, there exists exactly one tangent vector u at p such that $u|_{\mathcal{B}} = u_0$.*

Suppose we are dealing with differential manifolds, where differential bases are finite. Then, for a function $u_0 : \mathcal{B} \to \mathbb{R}$, with $\mathcal{B}$ a differential basis, the tangent vector u at a point p such that $u|_{\mathcal{B}} = u_0$ is given by

$$u = u_0(f_1)\frac{\partial}{\partial f_1} + \ldots + u_0(f_n)\frac{\partial}{\partial f_n},$$

$f_1, \ldots, f_n \in \mathcal{B}$, $n = \text{Card } \mathcal{B}$.

By using Lemma (1.3.), one can prove

Proposition 1.9. [*10*] *Let $(M, \mathcal{F})$ be a differential space, and $\mathcal{B} \subset \mathcal{F}$ a differential basis of $\mathcal{F}$ at $p \in M$. The mapping $\lambda : T_pM \to \mathbb{R}^{\mathcal{B}}$, given by*

$$\lambda(u) = u|_{\mathcal{B}}$$

for $u \in T_pM$, is an isomorphism of linear spaces.

We omit the simple proof to this proposition.

From Proposition 1.9., it follows that

Corollary 1.2. *Let $(M, \mathcal{F})$ and $\mathcal{B}$ be as in Proposition 1.9..*

(i) If $\text{Card } \mathcal{B} < \infty$, *then* $\dim T_pM = \text{Card } \mathcal{B}$.

(ii) If $\text{Card } \mathcal{B} = \infty$, *then* $\dim T_pM = 2^{\text{Card } \mathcal{B}}$.

Proof. [18].

The following corollary gives a characterization of a differential space of constant differential dimension.

Corollary 1.3. [*10*] *Let $(M, \mathcal{F})$ be a d-space and $n \in \mathbb{N}$, fixed. If, for every point $p \in M$, there exists a neighbourhood $U \in \tau_{\mathcal{F}}$ of p and a set of functions $\{f_1, \ldots, f_n\} \subset \mathcal{F}$ which is a differential basis of $\mathcal{F}$ at any point $q \in U$, then the d-space $(M, \mathcal{F})$ is of constant differential dimension, i.e.* $\dim T_pM = n$, $p \in M$.

Proof. [18].

Example 1.7. [*14*] *Let $\mathcal{F}$ be the differential structure on $\mathbb{R}$ generated by the set*

$$\{f_n(x) = \sin \frac{x}{n} : n \in \mathbb{N}, x \in \mathbb{R}\} \cup$$
$$\{g_n(x) = \cos \frac{x}{n} : n \in \mathbb{N}, x \in \mathbb{R}\}.$$

For any $n \in \mathbb{N}$, the function f_n is a differential basis of $\mathcal{F}$ at these points $x \in \mathbb{R}$ for which $f_n'(x) \neq 0$ and analogously g_n forms a differential basis of $\mathcal{F}$ at points $x \in \mathbb{R}$ for which $g_n'(x) \neq 0$ (See Heller et al 1991). For every point $x \in \mathbb{R}$, there exist a neighbourhood $U \in \tau_{\mathcal{F}}$ of x and a subset of $\mathcal{F}$ consisting of one element which is a differential basis of $\mathcal{F}$ at every point $q \in U$. Therefore, by Corollary 1.3., the d-space $(\mathbb{R}, \mathcal{F})$ considered here has differential dimension equal to 1.

1.3 Vector Fields and Tensor Fields on Differential Spaces

Vector fields and tensor fields on a differential space are defined in a natural way by the differential structure of the d-space.

Before we go over to defining vector fields and tensor fields on a d-space, we need the following:

Definition 1.13. [9] *Let $(M, \mathcal{F})$ be a d-space, and let us consider a function $\phi : p \mapsto \phi(p)$ assigning to every $p \in M$ a vector space $\phi(p)$. Any function W on M such that $W(p) \in \phi(p)$ is called a ϕ-vector field (or simply ϕ-field).*

Let $(M, \mathcal{F})$ be a d-space. For all ϕ-fields V, W and $f, g \in \mathcal{F}$, one assumes that

$$\begin{aligned} f(p)(V(p) + W(p)) &= f(p)V(p) + f(p)W(p) \\ (f(p) + g(p))V(p) &= f(p)V(p) + g(p)V(p) \\ (f(p)g(p))V(p) &= f(p)(g(p)V(p)), \end{aligned}$$

$p \in M$. Consequently, the set of all ϕ-fields on the d-space $(M, \mathcal{F})$ is an $\mathcal{F}$-module. In the following we will not be interested in examining all ϕ-fields on $(M, \mathcal{F})$ but only an $\mathcal{F}$-module of certain ϕ-fields which via subsequent definitions should be considered to be smooth. The set of all such smooth ϕ-fields will be denoted by $\mathcal{W}$, and we shall additionally assume it to be closed with respect to localization, i.e., $\mathcal{W} = \mathcal{W}_M$.

Definition 1.14. [9] *A finite sequence*

$$W_1, \ldots, W_m \tag{1.4}$$

is said to be a vector basis of an $\mathcal{F}$-module $\mathcal{W}$ of ϕ-fields provided that

(i) for every point $p \in M$, the sequence $W_1(p), \ldots, W_m(p)$ is a vector basis of the vector space $\phi(p)$

(ii) every ϕ-field $W \in \mathcal{W}$ is a linear combination of $W_1, \ldots, W_m$ with coefficients from $\mathcal{F}$.

The following definition will help us later define the notion of differential dimension of a d-space.

Definition 1.15. [26] *A $\mathcal{F}$-module $\mathcal{W}$ of ϕ-fields on a d-space $(M, \mathcal{F})$ is said to be a differential module provided that*

(i) it is closed with respect to localization,

(ii) it has locally a vector basis composed of m ϕ-fields, i.e. for every $p \in M$ there is a neighbourhood A of p such that $W_1, \ldots, W_m$ is a vector basis of $\mathcal{W}$ on A.

The number m is called the dimension of $\mathcal{W}$, which is denoted by $\dim \mathcal{W}$. Note that $\dim \mathcal{W} = \dim \phi(p)$, for $p \in M$.

Example 1.8. *Let $(M, \mathcal{F})$ be a d-space. $\mathcal{F}$ is a differential module. In fact, for every $p \in M$, $\phi(p) \equiv \mathbb{R}$. And the vector basis for $\mathcal{F}$ is just the constant function one.*

Note the following

Theorem 1.3. [26] *Let $(M, \mathcal{F})$ be a differential space and $\emptyset \neq A \subset M$. If $\mathcal{W}$ is a differential module of ϕ-fields on $(M, \mathcal{F})$, then $\mathcal{W}_A$ is a differential module on $(A, \mathcal{F}_A)$ and* $\dim \mathcal{W} = \dim \mathcal{W}_A$.

Proof. First, let's prove that $\mathcal{W}_A$ is a $\mathcal{F}_A$-module. Indeed, if $p \in A$, $f \in \mathcal{F}_A$ and $V \in \mathcal{W}_A$, there exist a neighbourhood U of p in A, a function $g \in \mathcal{F}$ and a ϕ-field $W \in \mathcal{W}$ such that $fV|_U = gW|_U$. Therefore, $f(p)V(p) = g(p)W(p) \in \phi(p)$. And by a simple extension of Lemma 1.1., $(\mathcal{W}_A)_A = \mathcal{W}_A$; i.e. $\mathcal{W}_A$ is closed with respect to localization.
Now, suppose that (1.4) is a vector basis of the differential module $\mathcal{W}$. Let's prove that

$$W_1|_A, \ldots, W_m|_A \tag{1.5}$$

is a vector basis of $\mathcal{W}$ on $(A, \mathcal{F}_A)$. It's evident that for all $p \in A$, $W_1(p), \ldots, W_m(p)$ is a vector basis of $\phi(p)$. It suffices to prove that (1.5) has the property (ii) of Definition 1.14.. Let $V \in \mathcal{W}_A$. By the property (i) of Definition 1.14., there exists a unique sequence of real functions $\alpha^1, \ldots, \alpha^m$ on A such that $V = \alpha^i W_i|_A$. We shall prove that the functions α^i are in $\mathcal{F}_A$. In fact, for all point $p \in A$ there exist a neighbourhood U of p in A and a ϕ-field $W \in \mathcal{W}$ such that $V|_U = W|_U$. But $W = \beta^i W_i$ for certain functions $\beta^i \in \mathcal{F}$. Hence, it follows that $\beta^i|_U = \alpha^i|_U$, which proves that $\alpha^i \in \mathcal{F}_A$. Thus (1.5) is a vector basis of $\mathcal{W}$ on $(A, \mathcal{F}_A)$. □

Corollary 1.4. *If $\mathcal{W}$ is a $\mathcal{F}$-differential module of ϕ-fields on the d-space $(M, \mathcal{F})$, then for every $p \in M$ and for every $w \in \phi(p)$ there exists a $W \in \mathcal{W}$ such that $w = W(p)$.*

Proof. We assume that (1.4) is a vector basis of $\mathcal{W}$ and A a neighbourhood of p. Thus $w = a^i W_i(p)$ for certain real numbers a^i. Since $a^i W_i|_A \in \mathcal{W}_A$, there exists a $W \in \mathcal{W}$ such that $W|_V = a^i W_i|_V$ for a neighbourhood V of p in A. Hence, it follows that $W(p) = a^i W_i(p) = w$. □

Let's now introduce the notion of tangent vector fields on a differential space.

Definition 1.16. [9] *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be two differential spaces and let $f : M \mapsto N$. Let ϕ be a function which assigns a linear space $\phi(p)$ to any* *point $p \in M$. If we assume $\phi(p) = T_{f(p)}N$, for all $p \in M$, then ϕ-fields are called f- vector fields on M, tangent to N. In other words, an f-vector field on M tangent to N is a mapping*

$$V : M \mapsto \bigcup_{q \in N} T_q N \tag{1.6}$$

ascribing to every $p \in M$ a vector $V(p) \in T_q N$, $q = f(p)$.

In the case where $M = N, \mathcal{F} = \mathcal{G}$ and $f = id_M, V$ is called a tangent vector field on M.

Let V be an f-vector field on M, tangent to N, and $\alpha \in \mathcal{G}$. The symbol $\partial_V \alpha$ will denote the real function defined by

$$(\partial_V \alpha)(p) = V(p)\alpha = \partial_{V(p)}\alpha$$

for all $p \in M$. The function $\partial_V \alpha$ is the directional derivative of α with respect to V.

Definition 1.17. *[26] Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be differential spaces and let $f : M \mapsto N$. An f-vector field V is said to be smooth provided*

(i) f is smooth, i.e. $\mathcal{G} \circ f \subset \mathcal{F}$,

(ii) $\partial_V \mathcal{G} \subset \mathcal{F}$.

It is evident that if V is a smooth vector field on M, tangent to N, then $\partial_V : \mathcal{G} \mapsto \mathcal{F}$ is an $\mathbb{R}$-linear mapping satisfying the Leibnitz condition

$$\partial_V(\alpha\beta) = \partial_V \alpha(\beta \circ f) + (\alpha \circ f)\partial_V \beta \tag{1.7}$$

for $\alpha, \beta \in \mathcal{G}$.
Conversely, if $f : M \mapsto N$ is smooth, then every $\mathbb{R}$- linear mapping from $\mathcal{G}$ into $\mathcal{F}$, satisfying (1.7) is of the form ∂_V for exactly one smooth f-vector field V. We shall identify V with ∂_V. After this identification, we ascribe to the notion of vector field two interpretations :

(i) in the pointwise interpretation it is a mapping V as in (1.6), such that $\partial_V : \mathcal{G} \mapsto \mathcal{F}$

(ii) in the global interpretation it is an $\mathbb{R}$-linear mapping $\partial_V : \mathcal{G} \mapsto \mathcal{F}$ satisfying (1.7).

Notation 1.1. *For any smooth mapping $f : M \mapsto N$ the symbol $\mathcal{X}_f(M, N)$ will denote the set of all smooth f-vector fields on M tangent to N. If $M = N, \mathcal{G} = \mathcal{F}$ and $f = id_M$, we write $\mathcal{X}(M)$ to denote the set of tangent vector fields on M. It is easy to verify that $\mathcal{X}_f(M, N)$ is an $\mathcal{F}$-module.*

Definition 1.18. *[26] A sequence*

$$W_1, \ldots, W_n \tag{1.8}$$

is said to be a vector basis of the $\mathcal{F}$-module $\mathcal{X}_f(M, N)$ provided that

(i) for every $p \in M$, the sequence $W_1(p), \ldots, W_n(p)$ is a basis of the linear space $T_{f(p)}N$,

(ii) every f-vector field $W \in \mathcal{X}_f(M, N)$ is an $\mathcal{F}$-linear combination of $W_1, \ldots, W_n$.

In the case where $N = M$, the sequence $W_1, \ldots, W_n$ is said to be a vector basis of the $\mathcal{F}$-module $\mathcal{X}(M)$. Most often, we simply say that the sequence $W_1, \ldots, W_n$ is a vector basis on the differential space $(M, \mathcal{F})$. Furthermore, if

- $W_1(p), \ldots, W_n(p)$ are linearly independent, for every $p \in M$, i.e. $\dim T_pM = n$,
- for every $p \in M$ and every $v \in T_pM$, there is locally a smooth tangent vector field V on $(M, \mathcal{F})$ such that $V(p) = v$,

we define the large or global dimension of $(M, \mathcal{F})$ to be

$$\operatorname{Dim} M = n.$$

The dimension of T_pM is called the small or local dimension of the differential space $(M, \mathcal{F})$ at the point p. We denote the local dimension at a point p by $\dim_p M$.
In the case where the small dimension is constant on M both dimensions coincide, i.e.

$$\dim_p M = \operatorname{Dim} M,$$

for all $p \in M$.

Example 1.9. . *Let us consider the pair $(A, (\varepsilon_2)_A)$, where $A = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$. For all $p \neq (0,0)$, $\dim T_pA = 1$. At $p = (0,0)$, there are linear mappings $V_1, V_2 : (\varepsilon_2)_A \mapsto \mathbb{R}$ defined by*

$$V_1(f) = \frac{\partial f(0,0)}{\partial x}, \qquad V_2(f) = \frac{\partial f(0,0)}{\partial y}$$

where $f \in (\varepsilon_2)_A$. Of course, the mappings V_1, V_2 satisfy the Leibnitz condition and are linearly independent. Therefore, $\dim T_pA = 2$ at $p = (0,0)$. Hence, the d-space $(A, (\varepsilon_2)_A)$ has no global dimension.

Example 1.10. . *Let $(A, \mathcal{F})$ be a d-space, where $A = \mathbb{R}$ and $\mathcal{F} = \{f : \mathbb{R} \mapsto \mathbb{R};\ f$ is a continous function and there exist both left and right derivatives at each point $p \in \mathbb{R}\}$. For all $p \in \mathbb{R}$, $\dim_p A = \operatorname{Dim} A = 2$.*

Let $(M, \mathcal{F})$ be a differential space and let ϕ be a function, which assigns a linear space $\phi(p)$ to any $p \in M$. Since $\phi(p)$ is a linear space, there exists a dual vector space to $\phi(p)$, whose members are real- valued linear functions on $\phi(p)$. If $\phi(p)$ is the tangent space at p, the dual space is called the cotangent space at p and it is denoted by $\phi^*(p)$. An element $\omega : \phi(p) \mapsto \mathbb{R}$ of $\phi^*(p)$ is called a one-form or a covariant vector on M.

If v is a vector at p, i.e. $v \in T_pM$, the number into which ω maps v will be written $< \omega, v >$; in that case linearity implies that

$$< \omega, \alpha v + \beta v_1 >= \alpha < \omega, v > + \beta < \omega, v_1 >$$

holds for all $\alpha, \beta \in \mathbb{R}$ and $v, v_1 \in \phi(p)$.

Example 1.11. *The simplest example of a one-form is the differential df of a map $f \in \mathcal{F}$. The action of $df : \bigcup_{p\in M} T_pM \mapsto \mathbb{R}$ on a vector $v \in T_pM$ is defined by*

$$< df, v >= v(f) \in \mathbb{R}.$$

Let's now introduce the notion of tensor field on a differential space. We shall consider $n+1$ fixed functions $\phi_j(p),\ j = 1,\dots,n+1$, which assign linear spaces $\phi_j(p)$ to any point p of a differential space $(M, \mathcal{F})$. To every $\phi_j,\ j = 1,\dots,n+1$ corresponds an $\mathcal{F}$- module $\mathcal{W}_j$ of ϕ_j-fields.

Definition 1.19. *A tensor field on a differential space $(M, \mathcal{F})$ is a function T which assigns to every point $p \in M$, an n- linear mapping*

$$T(p) : \phi_1(p) \times \ldots \times \phi_n(p) \mapsto \phi_{n+1}(p). \tag{1.9}$$

The n-linear mapping (1.9) is called an n-tensor. The set of all n-tensors defined at $p \in M$ is a vector space over $\mathbb{R}$ and shall be denoted by $\varphi(p)$, that is

$$\varphi(p) = \mathcal{L}_{\mathbb{R}}(\phi_1(p),\dots,\phi_n(p); \phi_{n+1}(p)).$$

It follows that every tensor field T is a φ-field.

Similarly to the notion of vector field, the notion of tensor field has two interpretations [26]:

- in the pointwise interpretation, T is a function which assigns the n-linear tensor (1.9) to any $p \in M$.
- in the global interpretation, T is an $\mathcal{F}$-tensor with following properties:

$$\begin{aligned} T(W_1,\dots,W_n) &\in \mathcal{W}_{n+1} \\ T(W_1,\dots,W_n)(p) &= T(p)(W_1(p),\dots,W_n(p)), \end{aligned}$$

  for $p \in M$ and $W_i \in \mathcal{W}_i,\ i = 1,\dots,n$.

Chapter 2

SMOOTH FORMS ON DIFFERENTIAL SPACES

Changing from the manifold definition in terms of atlases to that in terms of ring of functions leads to a generalization of the manifold concept where one drops the axiom ensuring the manifold to be locally diffeomorphic to the Euclidean space $\mathbb{R}^n$. Within the framework of this new concept, the so-called differential space, we want to look at tangent bundles and smooth forms. The class of all bundles together with all bundle morphisms is a category. In this category, we will single out the Whitney sum bundle so as to define the Whitney sum of k copies of the tangent bundle TM of a differential space $(M, \mathcal{F})$ and then look at differential forms.
Differential forms are classified into two groups: the graded algebra $A(M)$ of pointwise differential forms and the graded algebra $\Omega(M)$ of global forms, which may be not isomorphic if the differential space $(M, \mathcal{F})$ is not of a constant differential dimension. To circumvent this difficulty we define adequately a special graded algebra $\hat{\mathcal{A}}(M)$, where both pull-back and exterior differentiation operators are possible. But prior to all this is a focus on cartesian products of differential spaces.

2.1 Cartesian Products of Differential Spaces

Definition 2.1. [9] *Let* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$ *be non-empty d-spaces. Let* $\mathcal{F} \times \mathcal{G}$ *be the differential structure on the cartesian product* $M \times N$*, generated by the set of real-valued functions*

$$\{f \circ \pi_1 : f \in \mathcal{F}\} \cup \{g \circ \pi_2 : g \in \mathcal{G}\},$$

where $\pi_1(p,q) = p, \quad \pi_2(p,q) = q$ *for all* $(p,q) \in M \times N$.
The d-space $(M \times N, \mathcal{F} \times \mathcal{G})$ *is called the cartesian product of the d-spaces* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$.

Proposition 2.1. *The natural projections*

$$\pi_1 : (M \times N, \mathcal{F} \times \mathcal{G}) \longrightarrow (M, \mathcal{F})$$

and

$$\pi_2 : (M \times N, \mathcal{F} \times \mathcal{G}) \longrightarrow (M, \mathcal{G})$$

are smooth.

Proof. Clearly, for every $f \in \mathcal{F}$, $f \circ \pi_1 \in \mathcal{F} \times \mathcal{G}$ and, for every $g \in \mathcal{G}$, $g \circ \pi_2 \in \mathcal{F} \times \mathcal{G}$. □

Let $(M \times N, \mathcal{F} \times \mathcal{G})$ be the cartesian product of differential spaces $(M, \mathcal{F})$ and $(N, \mathcal{G})$. For an arbitrary point $p \in M$, let $j_p : N \longrightarrow M \times N$ be the embedding defined by

$$j_p(q) = (p,q) \qquad \text{for } q \in N.$$

In the same way, let $j_q : M \longrightarrow M \times N$, $q \in N$, be the embedding defined by

$$j_q(p) = (p,q) \qquad \text{for } p \in M.$$

It is easy to verify the following equalities:

$$\pi_1 \circ j_q = id_M,$$

$$\pi_2 \circ j_p = id_N,$$

for $p \in M$ and $q \in N$.

Proposition 2.2. [22] *Let's consider the cartesian product* $(M \times N, \mathcal{F} \times \mathcal{G})$ *of d-spaces* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$. *For any tangent vector* $w \in T_{(p,q)}(M \times N)$, *let's put*

$$w_M = (j_q \circ \pi_1)_{*(p,q)} w,$$

$$w_N = (j_p \circ \pi_2)_{*(p,q)} w.$$

Then, we have the following :

$$w = w_M + w_N, \tag{2.1}$$

$$w_M(g \circ \pi_2) = 0 \qquad \text{for any } g \in \mathcal{G}, \tag{2.2}$$

$$w_N(f \circ \pi_1) = 0 \qquad \text{for any } f \in \mathcal{F}. \tag{2.3}$$

Proof. For any $u \in \mathcal{F} \times \mathcal{G}$, we have

$$\begin{aligned} w_M(u) + w_N(u) &= w(u \circ j_q \circ \pi_1(p,q)) + w(u \circ j_p \circ \pi_2(p,q)) \\ &= w(u). \end{aligned}$$

Equations 2.2 and 2.3 are straightforward.
In fact, for any $g \in \mathcal{G}$, we have $w_M(g \circ \pi_2)(p,q) = w(g(q)) = 0$, since $g(q)$ is constant. □

More generally, we have

Proposition 2.3. *Let* $U \in \mathcal{X}(M)$ *and* $V \in \mathcal{X}(N)$ *be vector fields tangent to the differential spaces* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$ *respectively. Then, we have*

$$(j_q)_{*p} U_p(g \circ \pi_2) = 0, \tag{2.4}$$

$$(j_p)_{*q} V_q(f \circ \pi_1) = 0, \tag{2.5}$$

$$T_{(p,q)}(M \times N) = (j_q)_{*p}(T_p M) \oplus (j_p)_{*q}(T_q N). \tag{2.6}$$

for $(p,q) \in M \times N, f \in \mathcal{F}, g \in \mathcal{G}$. *(Note that we have used, in the equations* (2.4) *and* (2.5), *the identifications* $U(p) \equiv U_p$ *and* $V(p) \equiv V_p$. *)*

Proof. One checks easily that

$$\begin{aligned}(j_q)_{*p}U_p(g\circ\pi_2) &= U_p(g\circ\pi_2\circ j_q(p))\\ &= U_p(g(q))\\ &= 0, \quad \text{since } g(q) \text{ is constant}\end{aligned}$$

In an analogous way, one proves (2.5).
From (2.4) and (2.5), it follows that $(j_q)_{*p}(T_pM)\cap(j_p)_{*q}(T_qN)=\{0\}$. On the other hand, for all $X\in\mathcal{X}(M)$ and $Y\in\mathcal{X}(N)$, it is easy to see that $(j_q)_{*p}X_p\in T_{(p,q)}(M\times N)$, $(j_p)_{*q}Y_q\in T_{(p,q)}(M\times N)$. Therefore, $(j_q)_{*p}(T_pM)\oplus(j_p)_{*q}(T_qN)$ is contained in $T_{(p,q)}(M\times N)$.
Now, let's take $z\in T_{(p,q)}(M\times N)$. By Proposition 2.2., $z=z_M+z_N$, where $z_M=(j_q\circ\pi_1)_{*(p,q)}z$, $z_N=(j_p\circ\pi_2)_{*(p,q)}z$. But $z_M\in(j_q)_{*p}(T_pM)$ and $z_N\in(j_p)_{*q}(T_qN)$ since $(\pi_1)_{*(p,q)}z\in T_pM$ and $(\pi_2)_{*(p,q)}z\in T_qN$. Therefore, $z\in(j_q)_{*p}(T_pM)\oplus(j_p)_{*q}(T_qN)$. □

Definition 2.2. [22] *Let $(M,\mathcal{F})$ and $(N,\mathcal{G})$ be two differential spaces. A vector $w\in T_{(p,q)}(M\times N)$ is said to be parallel to $(M,\mathcal{F})$ if $w(g\circ\pi_2)=0$ for any $g\in\mathcal{G}$. A vector $w\in T_{(p,q)}(M\times N)$ is said to be parallel to $(N,\mathcal{G})$ if $w(f\circ\pi_1)=0$ for any $f\in\mathcal{F}$.*

More generally, we have

Definition 2.3. *A vector field $Z\in\mathcal{X}(M\times N)$ is said to be parallel to $(M,\mathcal{F})$ if $(\pi_2)_*Z(p,q)=0$ for every $(p,q)\in M\times N$.
A vector field $Z\in\mathcal{X}(M\times N)$ is said to be parallel to $(N,\mathcal{G})$ if $(\pi_1)_*Z(p,q)=0$ for every $(p,q)\in M\times N$.*

We denote by $\mathcal{X}_M(M\times N)$ the set of all smooth vector fields tangent to $(M\times N,\mathcal{F}\times\mathcal{G})$ which are parallel to $(M,\mathcal{F})$. Similarly, the set of all smooth vector fields tangent to $(M\times N,\mathcal{F}\times\mathcal{G})$ which are parallel to $(N,\mathcal{G})$ will be denoted by $\mathcal{X}_N(M\times N)$.

As in Proposition 2.2., let's put

$$Z_M(p,q) = (j_q\circ\pi_1)_{*(p,q)}Z(p,q), \tag{2.7}$$

$$Z_N(p,q) = (j_p \circ \pi_2)_{*(p,q)} Z(p,q), \tag{2.8}$$

for $(p,q) \in M \times N$. It is easy to see that $Z_M \in \mathcal{X}_M(M \times N)$ and $Z_N \in \mathcal{X}_N(M \times N)$. Moreover, $Z = Z_M + Z_N$.

One can prove [27]:

Proposition 2.4. *The $\mathcal{F} \times \mathcal{G}$-module $\mathcal{X}(M \times N)$ is a direct sum of $\mathcal{F} \times \mathcal{G}$-modules $\mathcal{X}_M(M \times N)$ and $\mathcal{X}_N(M \times N)$.*

Proof. Clearly $\mathcal{X}_M(M \times N) \oplus \mathcal{X}_N(M \times N) \subset \mathcal{X}(M \times N)$. On the other hand, for any $Z \in \mathcal{X}(M \times N)$, we have, by using Proposition 2.3., $Z = Z_M + Z_N$, where $Z_M \in \mathcal{X}_M(M \times N)$ and $Z_N \in \mathcal{X}_N(M \times N)$. Thus $\mathcal{X}(M \times N) \subset \mathcal{X}_M(M \times N) \oplus \mathcal{X}_N(M \times N)$. □

Now, let $X \in \mathcal{X}_M(M \times N)$ and $Y \in \mathcal{X}_N(M \times N)$. For any $q \in N$, let $X^q : M \longrightarrow TM$ be defined by

$$X^q(p) = (\pi_1)_{*(p,q)} X_{(p,q)} \qquad \text{for } p \in M; \tag{2.9}$$

and analogously for any $p \in M$, let $Y^p : N \longrightarrow TN$ be defined by

$$Y^p(q) = (\pi_2)_{*(p,q)} Y_{(p,q)} \qquad \text{for } q \in N. \tag{2.10}$$

One can easily prove from (2.9) and (2.10) that $X^q \in \mathcal{X}(M)$, for every $q \in N$, and $Y^p \in \mathcal{X}(N)$, for every $p \in M$, correspondingly.

The following lemma can be proved [23]:

Lemma 2.1. *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be differential spaces. $(M, \mathcal{F})$ is a differential space of differential dimension m if and only if the $\mathcal{F} \times \mathcal{G}$-module $\mathcal{X}_M(M \times N)$ is a m-dimensional differential module. Similarly, $(N, \mathcal{G})$ has a differential dimension n if and only if the $\mathcal{F} \times \mathcal{G}$-module $\mathcal{X}_N(M \times N)$ is a n-dimensional differential module.*

Proof.($\Longrightarrow$) Assume that $(M, \mathcal{F})$ has a differential dimension m. Let (p,q) be an arbitrary point of $M \times N$. Let $V \in \tau_{\mathcal{F}}$ be an open neighbourhood of p such that on V there is a local vector basis $X_1, \ldots, X_m \in \mathcal{X}(V)$ of the $\mathcal{F}$-module $\mathcal{X}(M)$. Let us define

$$\bar{X}_i(p,q) = (j_q)_{*p} X_i(p) \qquad \text{for } (p,q) \in M \times N, \; i = 1, \ldots, m. \tag{2.11}$$

For any $f \in \mathcal{F}$ and $g \in \mathcal{G}$, one has

$$\bar{X}_i(p,q)(f \circ \pi_1) = X_i(p)(f(p)) \tag{2.12}$$

and

$$\bar{X}_i(p,q)(g \circ \pi_2) = 0. \tag{2.13}$$

Equation (2.13) asserts that the vector field $\bar{X}_i$ is parallel to the d-space $(M, \mathcal{F})$ (See Definition 2.3.), i.e. $\bar{X}_i \in \mathcal{X}_M(M \times N)$. Equation (2.12) proves that the sequence of vector fields (2.11) is a local vector basis of the $\mathcal{F} \times \mathcal{G}$-module $\mathcal{X}_M(M \times N)$ on $V \times N \in \tau_{\mathcal{F} \times \mathcal{G}}$.
($\Longleftarrow$) Assume that $\mathcal{X}_M(M \times N)$ is a m-dimensional differential module. Every $Z \in \mathcal{X}_M(M \times N)$ is such that $Z(p,q) \in (j_q)_{*p}(T_pM)$ for $(p,q) \in M \times N$. Therefore, $\mathcal{X}_M(M \times N)$ is a $\mathcal{F} \times \mathcal{G}$-module of ϕ- fields, where $\phi(p,q) = (j_q)_{*p}(T_pM)$ for $(p,q) \in M \times N$.
Since, by virtue of (2.11), $(j_q)_{*p} : T_pM \longrightarrow \phi(p,q)$ is an isomorphism for every $(p,q) \in M \times N$, $\dim T_pM = \dim \phi(p,q) = m$ for any $p \in M$. It is enough to show that for an arbitrary vector $u \in TM$ there exist a vector field $X \in \mathcal{X}(M)$ such that $u = X(\pi_M(u))$, where $\pi_M : TM \longrightarrow M$ is the projection. Indeed, for the vector $\bar{u} = (j_q)_{*p} u \in \phi(p,q)$, where $u \in T_pM$, there exists a vector field $Z \in \mathcal{X}_M(M \times N)$ such that $\bar{u} = Z(p,q)$.
Hence, we have

$$(\pi_1)_{*(p,q)} \bar{u} = (\pi_1)_{*(p,q)} Z(p,q)$$

or equivalently

$$u = Z^q(p),$$

where $Z^q \in \mathcal{X}(M)$ is defined by (2.9).
The second part of Lemma 2.1. can be proved analogously.□

Lemma 2.2. [23] *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be differential spaces. Then, $\dim T_{(p,q)}(M \times N)$ is constant for any $(p,q) \in M \times N$ if and only if $\dim T_pM$ is constant for any $p \in M$ and $\dim T_qN$ is constant for any $q \in N$.*

Proof. ($\Longrightarrow$) Assume that $\dim T_{(p,q)}(M \times N)$ is constant for any $(p,q) \in M \times N$. Since

$$(j_q)_{*p} : T_pM \rightarrow (j_q)_{*p}(T_pM),$$
$$(j_p)_{*q} : T_qN \rightarrow (j_p)_{*q}(T_qN),$$

are isomorphisms for all $(p,q) \in M \times N$, and since (2.6) holds it is true that

$$\dim T_{(p,q)}(M \times N) = \dim T_pM + \dim T_qN \tag{2.14}$$

for all $(p,q) \in M \times N$.
Let $q_0, q_1 \in N$. Then,

$$\dim T_{(p,q_0)}(M \times N) = \dim T_pM + \dim T_{q_0}N$$

and

$$\dim T_{(p,q_1)}(M \times N) = \dim T_pM + \dim T_{q_1}N.$$

But $\dim T_{(p,q_0)}(M \times N) = \dim T_{(p,q_1)}(M \times N)$, therefore

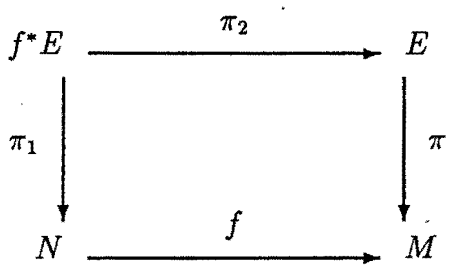
$$\dim T_{q_0}N = \dim T_{q_1}N.$$

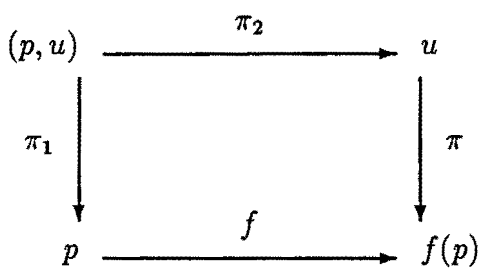
Thus, $\dim T_qN$ is constant for $q \in N$. In the same way, one proves that $\dim T_pM$ is constant for any $p \in M$.
($\Longleftarrow$) holds because of (2.14).□

By using Lemmas 2.1. and 2.2., we prove

Proposition 2.5. [23] *Let $(M, \mathcal{F})$ and $(N, \mathcal{G})$ be differential spaces. The cartesian product $(M \times N, \mathcal{F} \times \mathcal{G})$ is a differential space of constant differential dimension if and only if $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are differential spaces of constant differential dimension.*

Proof. ($\Longrightarrow$) Assume that the cartesian product $(M \times N, \mathcal{F} \times \mathcal{G})$ is a differential space of constant differential dimension. Assume that $\dim T_{p_0}M = m$ and $\dim T_{q_0} = n$ for certain points $p_0 \in M$ and $q_0 \in N$. In view of Lemma 2.2., $\dim T_pM = m$ for any $p \in M$ and $\dim T_qN = n$ for $q \in N$. It is enough to prove that every vector tangent to $(M, \mathcal{F})$ or $(N, \mathcal{G})$ is extendible to a smooth vector field tangent to $(M, \mathcal{F})$ or $(N, \mathcal{G})$, respectively.





Let's now define the product bundle and the Whitney sum bundle in the category $\mathcal{DSP}$.

Definition 2.6. *Given two bundles $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$. The product bundle is the bundle*

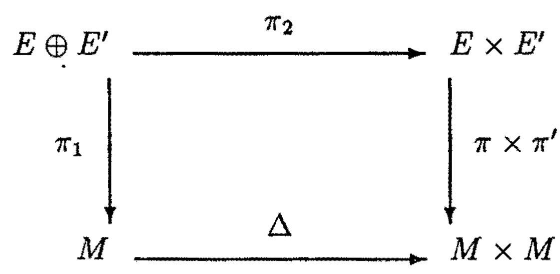
$$E \times E' \xrightarrow{\pi \times \pi'} M \times M',$$

where the mapping $\pi \times \pi'$ is defined by

$$\pi \times \pi'(u, u') = (\pi(u), \pi'(u'))$$

for $u \in E$ and $u' \in E'$.

Definition 2.7. *[19] Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ be two bundles. The Whitney sum bundle $(E \oplus E', \hat{\pi}, M)$ is a pullback bundle of $E \times E'$ by Δ in the diagram*



where Δ is defined by $\Delta(p) = (p, p)$.
Note that

$$E \oplus E' = \{(u, u') \in E \times E' : \pi(u) = \pi'(u')\}.$$

Example 2.3. *Let* $(M,\mathcal{F})$ *be a differential space. The triple* $((T^kM, T^k\mathcal{F}), \pi_0, (M,\mathcal{F}))$, $k \in \mathbb{N}$, *where*

$$T^kM := \{(v_1,\ldots,v_k) \in \prod_{i=1}^{k} TM : \pi(v_1) = \ldots = \pi(v_k)\}$$

(*the symbol* $\prod_{i=1}^{k} TM$ *denotes the cartesian product of* k *copies of* TM),

$$T^k\mathcal{F} := (\prod_{i=1}^{k} T\mathcal{F})_{T^kM},$$

(*the notation* $(\ldots)_{T^kM}$ *means localisation to* T^kM), *and*

$$\pi_0 : T^kM \to M$$

is given by the formula $\pi_0(v_1,\ldots,v_k) := \pi(v_1) = \ldots = \pi(v_k) = p$, *for every* $(v_1,\ldots,v_k) \in T^k_pM := \prod_{i=1}^{k} T_pM$, $p \in M$, *is a Whitney sum of* k *copies of* TM.

Proposition 2.6. [14] *Let the differential structure of* $(M,\mathcal{F})$ *be generated by a set* $\mathcal{F}_0 \subset \mathcal{F}$. *Then the differential structure* $T^k\mathcal{F}$, $k \in \mathbb{N}$, *is generated by the set*

$$\mathcal{F}_0^k := \{f \circ \pi_0 : f \in \mathcal{F}_0\} \cup (\cup_{i=1}^{k}\{df \circ \pi_i : f \in \mathcal{F}_0\}),$$

where $\pi_i(v_1,\ldots,v_k) = v_i$ *for any* $v_1,\ldots,v_k \in T_pM$ $(p = \pi_0(v_1,\ldots,v_k))$ *and* $i = 1,\ldots,k$.

Proof. By Example 2.2.,

$$T\mathcal{F} = \text{Gen}(\{f \circ \pi : f \in \mathcal{F}\} \cup \{df : f \in \mathcal{F}\}),$$

where $\pi(v) = p$, for any $v \in T_pM$, and $df(v) = v(f)$, for any $v \in TM$. Since $\mathcal{F} = \text{Gen}\ \mathcal{F}_0$, it follows that

$$T\mathcal{F} = \text{Gen}(\{f \circ \pi : f \in \mathcal{F}_0\} \cup \{df : f \in \mathcal{F}_0\}).$$

Therefore, the differential structure of the cartesian product T^kM is

$$T^k\mathcal{F} = \text{Gen}(\cup_{i=1}^{k}\{(f \circ \pi) \circ \pi_i : f \in \mathcal{F}_0\} \cup (\cup_{i=1}^{k}\{df \circ \pi_i : f \in \mathcal{F}_0\}))$$

or

$$T^k\mathcal{F} = \mathrm{Gen}(\{f \circ \pi_0 : f \in \mathcal{F}_0\} \cup (\cup_{i=1}^k \{df \circ \pi_i : f \in \mathcal{F}_0\}))$$

where $\pi_0(v_1, \ldots, v_k) = \pi(v_1) = \ldots = \pi(v_k) = p$, for every $(v_1, \ldots, v_k) \in T_p^k M$.□

Let $\varphi : (M, \mathcal{F}) \to (N, \mathcal{G})$ be a smooth map. One can easily prove, by means of Proposition 2.6. above, that the map

$$T^k\varphi : (T^kM, T^k\mathcal{F}) \to (T^kN, T^k\mathcal{G})$$

defined by $T^k\varphi(v_1, \ldots, v_k) = (d\varphi(v_1), \ldots, d\varphi(v_k))$, where if $(v_1, \ldots, v_k) \in T_p^k M$ $(p \in M)$, then $(d\varphi(v_1), \ldots, d\varphi(v_k)) \in T_{\varphi(p)}^k N$, is a smooth map of T^kM into T^kN.

Proposition 2.7. *The operator T^k, $k \in \mathbb{N}$, is a covariant functor in the category $\mathcal{DSP}$ of differential spaces.*

Let $(A, \mathcal{F}_A)$ be a differential subspace of a d-space $(M, \mathcal{F})$ and $i_A : (A, \mathcal{F}_A) \to (M, \mathcal{F})$ the inclusion map. Since T^k, $k \in \mathbb{N}$, is a functor, the following diagram

$$\begin{array}{ccc} (A, \mathcal{F}_A) & \xrightarrow{T^k} & (T^kA, T^k\mathcal{F}_A) \\ \downarrow i_A & & \downarrow T^k i_A \\ (M, \mathcal{F}) & \xrightarrow{T^k} & (T^kM, T^k\mathcal{F}) \end{array}$$

is commutative. $T^k i_A$ is the natural injection $(T^kA, T^k\mathcal{F}_A) \to (T^kM, T^k\mathcal{F})$. From now on, we shall identify T^kA and $T^k i_A(T^kA)$. This is justified by the following theorem.

Theorem 2.1. *[14] Let $(A, \mathcal{F}_A)$ be a subspace of a d-space $(M, \mathcal{F})$ and let i_A stand for the inclusion map: $(A, \mathcal{F}_A) \to (M, \mathcal{F})$. Then the induced map $T^k i_A : (T^kA, T^k\mathcal{F}_A) \to (T^kA, (T^k\mathcal{F})_{T^kA})$ is a diffeomorphism.*

Proof. By Proposition 2.6., $T^k\mathcal{F}$ is generated by

$$\mathcal{F}^k := \{f \circ \pi_0 : f \in \mathcal{F}\} \cup (\cup_{i=1}^k \{df \circ \pi_i : f \in \mathcal{F}\}).$$

Let $\pi_{0A} := \pi_0|_{T^kA} : T^kA \to A$ and $\pi_{iA} := \pi_i|_{T^kA} : T^kA \to TA$. According to the equality

$$df|_{TA} = d(f|_A), \qquad \text{for } \ f \in \mathcal{F},$$

the differential structure $(T^k\mathcal{F})_{T^kA}$ is generated by

$$\begin{aligned} G &:= \{\beta|_{T^kA} : \beta \in \mathcal{F}^k\} \\ &= \{(f|_A) \circ \pi_{0A} : f \in \mathcal{F}\} \cup (\cup_{i=1}^k \{d(f|_A) \circ \pi_{iA} : f \in \mathcal{F}\}). \qquad (2.16) \end{aligned}$$

Since the set $\{f|_A : f \in \mathcal{F}\}$ generates the differential structure $\mathcal{F}_A$, the equality (2.16) implies that G generates $T^k\mathcal{F}_A$. Hence $T^k\mathcal{F}_A = (T^k\mathcal{F})_{T^kA}$. □

2.3 Smooth Forms

Let us consider a differential space $(M, \mathcal{F})$ and the $\mathcal{F}$-module $\mathcal{X}(M)$ of all smooth vector fields tangent to $(M, \mathcal{F})$.

Definition 2.8. [9] *Any mapping $\omega : T^kM \to \mathbb{R}$ such that the mapping $\omega|_{T_pM\times\ldots\times T_pM}$ is skew-symmetric k-linear, for each point $p \in M$, is called a pointwise differential form on $(M, \mathcal{F})$. A k-form ω is said to be smooth on $(M, \mathcal{F})$ if $\omega \in T^k\mathcal{F}$.*

If $\varphi : (M, \mathcal{F}) \to (N, \mathcal{G})$ is a smooth mapping between differential spaces, then for any pointwise k-form ω on $(N, \mathcal{G})$ one defines its pull-back $\varphi^*\omega$ on $(M, \mathcal{F})$ by

$$(\varphi^*\omega)(v_1, \ldots, v_k) := \omega(\varphi_* v_1, \ldots, \varphi_* v_k)$$

for any $(v_1, \ldots, v_k) \in T^kM$.
The set of all pointwise differential k-forms on $(M, \mathcal{F})$ will be denoted by $A^k(M)$. For any $\omega, \omega_1, \omega_2 \in A^k(M)$ and $f \in \mathcal{F}$ we put

$$\begin{aligned} (\omega_1 + \omega_2)(v_1, \ldots, v_k) &:= \omega_1(v_1, \ldots, v_k) + \omega_2(v_1, \ldots, v_k), \\ (f.\omega)(v_1, \ldots, v_k) &:= f.\omega(v_1, \ldots, v_k) \end{aligned}$$

where $(v_1, \ldots, v_k) \in T^kM$.
The set $A^k(M)$ with the above operations is a $\mathcal{F}$- module.

For the purpose of the next theorem, we need the following definitions.

Definition 2.9. [23] *Let the differential structure of $(M, \mathcal{F})$ be generated by a set $\mathcal{F}_0 \subset \mathcal{F}$. A real function $f : M \to \mathbb{R}$ is said to be of class C^k if , for any point $p \in M$, there exist a neighbourhood $V \in \tau_{\mathcal{F}}$ of p, functions $f_1, \ldots, f_n \in \mathcal{F}_0$ and a function $\sigma : \mathbb{R}^n \to \mathbb{R}$ of class C^k such that*

$$f|_V = \sigma \circ (f_1, \ldots, f_n)|_V.$$

Definition 2.10. [6] *A k-form $\omega : T^kM \to \mathbb{R}$ is said to be smooth of class C^r on $(M, \mathcal{F})$ (shortly C^r – k-form) if ω is a C^r-function on the differential space $(T^kM, T^k\mathcal{F})$.*

Lemma 2.3. [23] *Let $\sigma : \mathbb{R}^n \to \mathbb{R}$ a C^k-function. If there exists a point $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ such that*

$$\sigma(ku) = k\sigma(u) \qquad \text{for any } k \in \mathbb{R}, \tag{2.17}$$

then

$$\sigma(u) = \sum \frac{\partial \sigma(0, \ldots, 0)}{\partial x_i} u_i.$$

Proof. Indeed, $\sigma'|_u(0) = \lim_{t \to 0} \frac{\sigma(tu) - \sigma(0)}{t} = \lim_{t \to 0} \frac{t\sigma(u)}{t} = \sigma(u)$.
Hence, $\sigma(u) = \sigma'|_u(0) = \sum_{i=1}^n \frac{\partial \sigma(0)}{\partial x_i} u_i$. □

Now we shall prove an important result.

Theorem 2.2. [14, 23] *Let $(M, \mathcal{F})$ be a differential space with $\mathcal{F} = Gen\, \mathcal{F}_0$, $p \in M$ an arbitrary point, $\omega : T^kM \to \mathbb{R}$ a smooth k-form of class C^r on $(M, \mathcal{F})$ and $k \leq r$.*
Then there exist a smooth mapping $F : (M, \mathcal{F}) \to (\mathbb{R}^n, \varepsilon_n)$ with the coordinates $F_1, \ldots, F_n \in \mathcal{F}_0$, a k-form $\theta : T^k\mathbb{R}^n \to \mathbb{R}$ of class C^r on $(\mathbb{R}^n, \varepsilon_n)$ and an open neighbourhood $V \in \tau_{\mathcal{F}}$ of p such that

$$\omega|_{\pi_0^{-1}(V)} = F^*\theta|_{\pi_0^{-1}(V)},$$

where $\pi_0 : T^kM \to M$ *is the projection* $(v_1, \ldots, v_k) \mapsto p = \pi(v_1) = \ldots = \pi(v_k)$*, i.e. the following diagram*

$$\begin{array}{ccccc}
(M, \mathcal{F}) & & (T^kM, T^k\mathcal{F}) & & \\
\Big\downarrow F & & \Big\downarrow F_* & \searrow^{\omega} & \\
(\mathbb{R}^n, \varepsilon_n) & & (T^k\mathbb{R}_n, T^k\varepsilon_n) & \xrightarrow[\theta]{} & \mathbb{R}
\end{array}$$

is commutative.

First we make the following statement

Remark 2.1. : *The differential space* $(T^k\mathbb{R}^n, T^k\varepsilon_n)$ *is usually identified with* $(\mathbb{R}^{(k+1)n}, \varepsilon_{(k+1)n})$ *by the identification which maps* $(v_1, \ldots, v_k)$ *to*

$$(\pi^1(p), \ldots, \pi^n(p), d\pi^1(v_1), \ldots, d\pi^n(v_1), \ldots, d\pi^1(v_k), \ldots, d\pi^n(v_k)),$$

where $v_1, \ldots, v_k \in T_p\mathbb{R}^n$, $p \in \mathbb{R}^n$, *and the* π^i*'s,* $i = 1, \ldots, n$, *are coordinate functions on* $\mathbb{R}^n$. *Under this identification,* ω *is a smooth* k*-form on* $(\mathbb{R}^n, \varepsilon_n)$ *if and only if* $\omega : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$ *belongs to* $\varepsilon_{(k+1)n}$ *and if it is totally skew-symmetric and* k*- linear with respect to the last* k *arguments. If a mapping* $F : (M, \mathcal{F}) \to (\mathbb{R}^n, \varepsilon_n)$ *has the coordinates* $F_1, \ldots, F_n$*, then*
$T^kF : (T^kM, T^k\mathcal{F}) \to (T^k\mathbb{R}^n, T^k\varepsilon_n) = (\mathbb{R}^{(k+1)n}, \varepsilon_{(k+1)n})$ *sends* $(v_1, \ldots, v_k)$ *to* $(F_1(p), \ldots, F_n(p), dF_1 \circ v_1, \ldots, dF_n \circ v_1, \ldots, dF_1 \circ v_k, \ldots, dF_n \circ v_k)$ *for any* $v_1, \ldots, v_n \in T_pM$, $p \in M$.

Proof {Theorem 2.2.}. Since $\omega : T^kM \to \mathbb{R}$ is a smooth k-form of class C^r on $(M, \mathcal{F})$, by Definition 2.9. and the preceding remark, there exist a neighbourhood $V \in \tau_{\mathcal{F}}$ of p and functions $F_1, \ldots, F_n \in \mathcal{F}_0$, $n \in \mathbb{N}$, and a C^r-function $\sigma : \mathbb{R}^{(k+1)n} \to \mathbb{R}$ such that

$$\begin{aligned}
&\omega|_{\pi_0^{-1}(V)} = \\
&\quad \sigma \circ (F_1 \circ \pi_0, \ldots, F_n \circ \pi_0, dF_1 \circ \pi_1, \ldots, dF_n \circ \pi_1, \ldots, dF_1 \circ \pi_k, \ldots \\
&\quad \ldots, dF_n \circ \pi_k)|_{\pi_0^{-1}(V)}.
\end{aligned}$$

Let $\theta : T^k\mathbb{R}^n \to \mathbb{R}$ be the k-form of class C^r defined by

$$\theta = \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k \sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\ldots\partial x_{kn+i_k}} \circ \iota_{n,(k+1)n} dx_{i_1} \otimes \ldots \otimes dx_{i_k}$$

where $\iota_{n,(k+1)_n} : \mathbb{R}^n \to \mathbb{R}^{(k+1)n}$ is given by

$$\iota_{n,(k+1)n}(x_1,\ldots,x_n) = (x_1,\ldots,x_n,0,\ldots,0)$$

for $(x_1,\ldots,x_n) \in \mathbb{R}^n$.
Let's show that $\omega|_{\pi_0^{-1}(V)} = F^*\theta|_{\pi_0^{-1}(V)}$, where F is the map $(M,\mathcal{F}) \to (\mathbb{R}^n, \varepsilon_n)$ with coordinates $F_1,\ldots,F_n \in \mathcal{F}_0$.

Consider the C^r-function $\alpha : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\alpha(x_1,\ldots,x_n) = \sigma(F_1(p),\ldots,F_n(p),x_1,\ldots,x_n,v_2(F_1),\ldots,v_2(F_n),\ldots,v_k(F_1),\ldots,v_k(F_n))$$

for $(x_1,\ldots,x_n) \in \mathbb{R}^n$ and $v_1,\ldots,v_k \in T_p\mathbb{R}^n$.
For the point $u = (v_1(F_1),\ldots,v_1(F_n))$, we have $\alpha(u) = \omega(v_1,\ldots,v_k)$. This implies that, for any $t \in \mathbb{R}$ we have

$$\begin{aligned} \alpha(tu) &= \alpha(tv_1(F_1),\ldots,tv_1(F_n)) \\ &= \omega(tv_1,v_2,\ldots,v_k) \\ &= t\alpha(u), \end{aligned}$$

i.e., the function α satisfies (2.17). Thus from Lemma 2.3. it follows that

$$\begin{aligned} \alpha(u) &= \sigma(F_1(p),\ldots,F_n(p),v_1(F_1)\ldots,v_1(F_n),\ldots,v_k(F_1),\ldots,v_k(F_n)) \\ &= \sum_{i_1=1}^{n} \frac{\partial\sigma}{\partial x_{n+i_1}}(F_1(p),\ldots,F_n(p),0,\ldots,0,v_2(F_1),\ldots \\ &\quad \ldots,v_2(F_n),\ldots,v_k(F_1),\ldots,v_k(F_n)).v_1(F_{i_1}). \end{aligned}$$

Now using the same Lemma 2.3. $(k-1)$ times, in the similar way, one checks that

$$\begin{aligned} &\omega(v_1,\ldots,v_k) = \\ &\quad \sigma(F_1(p),\ldots,F_n(p),v_1(F_1),\ldots,v_1(F_n),\ldots,v_k(F_1),\ldots,v_k(F_n)) \\ &\quad = \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k\sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\ldots\partial x_{kn+i_k}}(F_1(p),\ldots,F_n(p),0,\ldots,0). \\ &\quad v_1(F_{i_1})\ldots v_k(F_{i_k}) \end{aligned} \tag{2.18}$$

But,

$$
\begin{aligned}
F^*\theta(v_1,\ldots,v_k) &= \theta(F_*v_1,\ldots,F_*v_k) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k\sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\ldots\partial x_{kn+i_k}}(F_1(p),\ldots,F_n(p),0,\ldots,0). \\
&\quad dx_{i_1}(F_*v_1)\ldots dx_{i_k}(F_*v_k) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k\sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\ldots\partial x_{kn+i_k}}(F_1(p),\ldots,F_n(p),0,\ldots,0). \\
&\quad v_1(x_{i_1}\circ F)\ldots v_k(x_{i_k}\circ F) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} \frac{\partial^k\sigma}{\partial x_{n+i_1}\partial x_{2n+i_2}\ldots\partial x_{kn+i_k}}(F_1(p),\ldots,F_n(p),0,\ldots,0) \\
&\quad v_1(F_{i_1})\ldots v_k(F_{i_k}) \\
&= \omega(v_1,\ldots,v_k), \qquad \text{by using(2.18).}\square
\end{aligned}
$$

Corollary 2.1. [23] *Let $(M,\mathcal{F})$ be a differential space and $p \in M$ any of its points. If there exists a non-degenerate k-form ω of class C^r $(k \leq r)$ on $(M,\mathcal{F})$, then there is an open neighbourhood $V \in \tau_{\mathcal{F}}$ of p such that $(V,\mathcal{F}_V)$ can be immersed in the Euclidean space. Moreover,* $\dim T_q(M,\mathcal{F}) < +\infty$ *for any $q \in M$.*

Proof. We define F and θ as in Theorem 2.2.. The mapping F is a smooth immersion. Indeed, since ω is non-degenerate and $\omega(v_1,\ldots,v_k) = \theta(F_*v_1,\ldots,F_*v_k)$ for all $(v_1,\ldots,v_k) \in T^kV$, where V is an open set containing p, F_{*q} is injective for every $q \in V$. Thus

$$\dim T_qM = \dim F_{*q}(T_qM) \leq \dim T_{F(q)}\mathbb{R}^n = n. \ \square$$

From the above theorem it follows that, for an arbitrary $\omega \in A^k(M)$ and each point $p \in M$, there exists an open neighbourhood $V \in \tau_{\mathcal{F}}$ of the point p, and smooth functions $F_{i_1\ldots i_k}, F_{i_1},\ldots,F_{i_k} \in \mathcal{F}_V, (i_1,\ldots,i_k) \in I \subset \mathbb{N}^k$, such that

$$\omega|_{\pi_0^{-1}(V)} = \sum_I F_{i_1\ldots i_k} dF_{i_1} \wedge \ldots \wedge dF_{i_k}, \tag{2.19}$$

where I is a certain finite set of indices.

Another particular result deduced from Theorem 2.2. is :

Proposition 2.8. *[12] Let $(M, \mathcal{F})$ be a differential space with the differential structure $\mathcal{F}$ generated by a set $\mathcal{F}_0 = \{f_1, \ldots, f_n\}$, and let $p \in M$ be a point such that $\dim T_pM = n$. If $g : T^2M \to \mathbb{R}$ is a symmetric, nondegenerate 2-form of class C^k $(k \geq 2)$ of signature (r, s) at p, then there exists an open neighbourhood $U \in \tau_{\mathcal{F}}$ of p and a pseudo-Riemannian metric η of class C^k with the signature (r, s) on a certain open subspace of $(\mathbb{R}^n, \varepsilon_n)$ such that*

$$g|_{\pi_0^{-1}(U)} = F^*\eta|_{\pi_0^{-1}(U)},$$

where $F = (f_1, \ldots, f_n)$ and $\pi_0 : T^2M \to M$ is the projection defined by $(v_1, v_2) \to p$, for $(v_1, v_2) \in T_p^2M$.

Proof. There exist a neighbourhood $V \in \tau_{\mathcal{F}}$ of p and a function $\sigma : \mathbb{R}^{3n} \to \mathbb{R}$ of class C^k such that

$$\begin{aligned} g|_{\pi_0^{-1}(V)} &= \sigma \circ (f_1 \circ \pi_0, \ldots, f_n \circ \pi_0, df_1 \circ \pi_1, \ldots, df_n \circ \pi_1, \\ &\quad df_1 \circ \pi_2, \ldots, df_n \circ \pi_2)|_{\pi_0^{-1}(V)} \end{aligned}$$

Let $\eta : T^2\mathbb{R}^n \to \mathbb{R}$ be the 2-form defined by

$$\eta = \sum_{i,j=1}^{n} \frac{\partial^2 \sigma}{\partial x_{n+i} \partial x_{2n+j}} \circ \iota_{n,3n} dx_i \otimes dx_j,$$

where $\iota_{n,3n} : \mathbb{R}^n \to \mathbb{R}^{3n}$ is the mapping $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$. As in the proof of Theorem 2.3., one can check that $g|_{\pi_0^{-1}(V)} = F^*\eta|_{\pi_0^{-1}(V)}$. Since F is smooth, there exists a connected open neighbourhood A of the point $F(p) \in \mathbb{R}^n$ such that $F^{-1}(A) \subset V$, and

$$\det\left(\frac{\partial^2 \sigma}{\partial x_{n+i} \partial x_{2n+j}}(\iota_{n,3n}(q))\right) \neq 0$$

for $q \in A$. Since F_{*p} is an isomorphism, the signature of η at $F(p)$ is (r, s), and, consequently, η is of signature (r, s) on A. Of course, η is a symmetric and nondegenerate 2-form on A. If we put $U = F^{-1}(A)$, we have $g|_{\pi_0^{-1}(U)} = F^*\eta|_{\pi_0^{-1}(U)}$, which ends the proof. □

One can also define global k-forms on a differential space $(M, \mathcal{F})$.

Definition 2.11. [9] *Any skew-symmetric k-linear mapping*

$$\omega : \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \to \mathcal{F}$$

is called a global k-form on the d-space $(M, \mathcal{F})$.

Let us denote by $\Omega^k(M)$ the set of all global k-forms on the d- space $(M, \mathcal{F})$. For any pointwise k-form $\omega \in A^k(M)$ let $\tilde{\omega} : \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \to \mathcal{F}$ be the mapping defined by

$$\tilde{\omega}(X_1, \ldots, X_n)(p) := \omega(X_1(p), \ldots, X_n(p)),$$

where $X_1, \ldots, X_n \in \mathcal{X}(M)$, and $p \in M$.
It can be easily seen that $\tilde{\omega}$ is a global smooth k-form on $(M, \mathcal{F})$.

Given $\omega, \omega_1, \omega_2 \in \Omega^k(M)$ and $f \in \mathcal{F}$, we put

$$(\omega_1 + \omega_2)(X_1, \ldots, X_k) := \omega_1(X_1, \ldots, X_k) + \omega_2(X_1, \ldots, X_k),$$

$$(f.\omega)(X_1, \ldots, X_k) := f.\omega(X_1, \ldots, X_k),$$

for $X_1, \ldots, X_k \in \mathcal{X}(M)$. The triple $(\Omega^k(M), +, .)$ is an $\mathcal{F}$-module.

Definition 2.12. *Let $(M, \mathcal{F})$ be a differential space and $k, l \in \mathbb{N}$. For $\omega \in \Omega^k(M)$ and $\xi \in \Omega^l(M)$, we define the exterior product $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$ by the formula*

$$\begin{aligned}(\omega \wedge \xi)(X_1, \ldots, X_k, X_{k+1}, \ldots, X_{k+l}) = \\ \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn\sigma\ \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\xi(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}),\end{aligned} \tag{2.20}$$

where $X_i \in \mathcal{X}(M)$, $i = 1, \ldots, k + l$.

Then, we have

Proposition 2.9. *Let $(M, \mathcal{F})$ be a differential space of constant differential dimension n and two forms $\omega \in \Omega^k(M), \xi \in \Omega^l(M)$ $(k, l \in \mathbb{N})$. If $k + l > n$, $\omega \wedge \xi$ vanishes identically.*

Proof. Since $k + l > n$, there exists an $X_i \in X_1, \ldots, X_{k+l}$ which is a linear combination of the rest. Thus, $(\omega \wedge \xi)(X_1, \ldots, X_{k+l}) = 0$. But $X_1, \ldots, X_{k+l}$ are arbitrary in $\mathcal{X}(M)$, and therefore $\omega \wedge \xi = 0$. □

The direct sum $\Omega(M) = \oplus_{k \geq 0} \Omega^k(M)$, where $\Omega^0(M) := \mathcal{F}$, is the space of all differential global k-forms on the d-space $(M, \mathcal{F})$ and is closed under the exterior product. Thus, $\Omega(M)$ is a graded algebra over $\mathbb{R}$. Similarly, one shows that the triple $(A(M), +, \wedge)$, where $A(M) := \oplus_{k \geq 0} A^k(M)$ and $\wedge$ is defined at each point by (2.20), is a graded algebra over $\mathbb{R}$.

Definition 2.13. *Let $(M, \mathcal{F})$ be a differential space. In the algebra $\Omega(M)$, the exterior derivative d_k is a map $\Omega^k(M) \to \Omega^{k+1}(M)$ whose action is defined by the formulas:*

(i) if $f \in \mathcal{F}$, then $(df)(X) = X(f)$, for $X \in \mathcal{X}(M)$.

(ii) if $k \geq 1$ and $\omega \in \Omega^k(M)$, then

$$(d_k\omega)(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

where $X_i \in \mathcal{X}(M)$, $i = 1, \ldots, k+1$.

It is common to drop the subscript k and write simply d if there is no danger of misunderstanding.

Proposition 2.10. *Let $(M,\mathcal{F})$ be a d-space. The operator d satisfies the following well known conditions :*

(i) d *is* $\mathbb{R}$*-linear*

(ii) $d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^{deg\omega}\omega \wedge d\xi$

(iii) $d \circ d = 0$

for any $\omega, \xi \in \Omega(M)$.

Proof. [9] □

Next, we show :

Proposition 2.11. *[12] Let $(M,\mathcal{F})$ be a differential space of a constant differential dimension. The mapping (for $k \in \mathbb{N}$) $h : A^k(M) \to \Omega^k(M)$ defined by*

$$h(\omega)(X_1,\dots,X_k)(p) = \omega(X_1(p),\dots,X_k(p)), \tag{2.21}$$

for any $\omega \in A^k(M)$, $X_1,\dots,X_k \in \mathcal{X}(M)$, $p \in M$ and $h = id_{\mathcal{F}}$, for $k = 0$, is an isomorphism of $\mathcal{F}$-modules.

Proof. As it can be easily seen, h is a monomorphism. Indeed, let's assume that $h(\omega) = 0$. Then, for any $X_1,\dots,X_k \in \mathcal{X}(M)$ and $p \in M$,

$$\omega(X_1(p),\dots,X_k(p)) = 0.$$

Therefore, $\omega = 0$.
Let $g : \Omega^k(M) \to A^k(M)$ be given by

$$g(\theta)(v_1,\dots,v_k) = \theta(X_1,\dots,X_k)(p)$$

for any $v_1,\dots,v_k \in T_pM$, $p \in M$, $X_1,\dots,X_k \in \mathcal{X}(M)$, $X_i(p) = v_i$, $i = 1,\dots,k$. We can easily see that for any $X_1,\dots,X_k \in \mathcal{X}(M)$ and $p \in M$,

$$\begin{aligned} h(g(\theta))(X_1,\dots,X_k)(p) &= g(\theta)(X_1(p),\dots,X_k(p)) \\ &= (\theta)(X_1,\dots,X_k)(p), \end{aligned}$$

which demonstrates that h is an isomorphism. □

Remark 2.2. *In view of formula (2.21), if the differential space is not of a constant differential dimension, the graded algebras $A(M)$ and $\Omega(M)$ will not necessarily be isomorphic. As we can see, in the graded algebra $\Omega(M)$ there is an exterior differentiation, but many difficulties appear when one wants to introduce the operator of pull-back for global forms. On the other hand, in the graded algebra $A(M)$ the pull-back operation is well defined, but difficulties arise in defining exterior differentiation. In order to get rid of these defects we introduce the following construction [9, 12].*

Let $\mathcal{M}^k(M)$, for $k \geq 1$, denote the set of all elements $\omega \in A^k(M)$ such that, for every point $p \in M$, there exists an open neighbourhood $U \in \tau_{\mathcal{F}}$ of p and a family of smooth functions

$$f_{i_1}, \ldots, f_{i_{k-1}}, f_{i_1 \ldots i_{k-1}} \in \mathcal{F}_U$$

for $(i_1, \ldots, i_{k-1}) \in I \subset \mathbb{N}^{k-1}$, where I is a finite set of indices, such that

$$\omega|\pi^{-1}(U) = \sum_I df_{i_1 \ldots i_{k-1}} \wedge df_{i_1} \wedge \ldots \wedge df_{i_{k-1}}$$

but

$$\sum_I f_{i_1 \ldots i_{k-1}} df_{i_1} \wedge \ldots \wedge df_{i_{k-1}} = 0.$$

Moreover, for $k = 0$, we assume $\mathcal{M}^0(M) = \{0\}$, where 0 denotes the real function identically equal to zero on M.

Example 2.4. [9] *Consider the set $M = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$. The element $\omega = dx \wedge dy \in \mathcal{M}^2(M)$ since it is a differential of the 1-form $\frac{1}{2}xdy - \frac{1}{2}ydx = 0$. Thus, $\mathcal{M}^2(M) \neq 0$.*

This example shows us that, in general, the set

$$\mathcal{M}(M) = \oplus_{k \geq 0} \mathcal{M}^k(M) \neq \{0\}.$$

Now, we will prove

Lemma 2.4. *[20] $\mathcal{M}(M)$ is a homogeneous ideal in the graded algebra $A(M)$.*

Proof. Let $\mu \in \mathcal{M}^k(M)$ and $\omega \in A^l(M)$ be arbitrary differential forms. We will show that $\mu \wedge \omega \in \mathcal{M}^{k+l}(M)$.
In fact, let p be an arbitrary point of M. There exist an open neighbourhood $U \in \tau_{\mathcal{F}}$ of p and smooth functions $f_{i_1 \dots i_{k-1}}, f_{i_1}, \dots, f_{i_{k-1}} \in \mathcal{F}_U, (i_1, \dots, i_{k-1}) \in I \subset \mathbb{N}^{k-1}$, I is finite, such that

$$\mu|_{\pi^{-1}(U)} = \sum_I df_{i_1 \dots i_{k-1}} \wedge df_{i_1} \wedge \dots \wedge df_{i_{k-1}}$$

as well as

$$\sum_I f_{i_1 \dots i_{k-1}} df_{i_1} \wedge \dots \wedge df_{i_{k-1}} = 0,$$

and at the same time, according to (2.19), $\omega \in A^l(M)$ has the following form

$$\omega|_{\pi^{-1}(U)} = \sum_J \omega_{j_1 \dots j_l} d\omega_{j_1} \wedge \dots \wedge d\omega_{j_l},$$

where $\omega_{j_1 \dots j_l}, \omega_{j_1}, \dots, \omega_{j_l} \in \mathcal{F}_U$ are smooth functions on U and $(j_1, \dots, j_l) \in J \subset \mathbb{N}^l$, J a certain finite set of indices.
Observe that

$$\sum f_{i_1 \dots i_{k-1}} \omega_{j_1 \dots j_l} df_{i_1} \wedge \dots \wedge df_{i_{k-1}} \wedge d\omega_{j_1} \wedge \dots \wedge d\omega_{j_l} = 0 \tag{2.22}$$

and

$$\begin{aligned}
&\sum d(f_{i_1 \dots i_{k-1}} \omega_{j_1 \dots j_l}) \wedge df_{i_1} \wedge \dots \wedge df_{i_{k-1}} \wedge d\omega_{j_1} \wedge \dots \wedge d\omega_{j_l} \\
&= \sum (df_{i_1 \dots i_{k-1}} \omega_{j_1 \dots j_l} + f_{i_1 \dots i_{k-1}} d\omega_{j_1 \dots j_l}) \wedge df_{i_1} \wedge \dots \\
&\dots \wedge df_{i_{k-1}} \wedge d\omega_{j_1} \wedge \dots \wedge d\omega_{j_l} \\
&= \mu \wedge \omega|_{\pi^{-1}(U)}
\end{aligned} \tag{2.23}$$

From (2.22) and (2.23) it follows that $\mu \wedge \omega \in \mathcal{M}^{k+l}(M)$. □

Lemma 2.5. . *[20] If $(M, \mathcal{F})$ is a differential space of constant differential dimension, then the homogenous ideal $\mathcal{M}(M) = \oplus_{k \geq 0} \mathcal{M}^k(M)$ is the zero ideal.*

Proof. If $(M, \mathcal{F})$ is a differential space of constant differential dimension then the mapping: $h : A(M) \to \Omega(M)$, defined by 2.21, is an isomorphism. Hence Ker$h = \{0\}$. So it suffices to show the inclusion

$$\mathcal{M}(M) \subset \mathrm{Ker}h.$$

In fact, let $\mu \in \mathcal{M}^{k+1}(M)$ and p be an arbitrary point of M. There exist an open neighbourhood $U \in \tau_{\mathcal{F}}$ of the point p as well as a family of smooth functions $f_{i_1 \dots i_k}, f_{i_1}, \dots, f_{i_k} \in \mathcal{F}_U, (i_1, \dots, i_k) \in I \subset \mathbb{N}^k$ such that the following equalities are fulfilled

$$\mu|_{\pi^{-1}(U)} = \sum df_{i_1 \dots i_k} \wedge df_{i_1} \wedge \dots \wedge df_{i_k}.$$

$$\sum f_{i_1 \dots i_k} df_{i_1} \wedge \dots \wedge df_{i_k} = 0.$$

Taking the images of these two equations, by means of h , we obtain

$$h_U(\mu|_{\pi^{-1}(U)}) = \sum h_U(df_{i_1 \dots i_k}) \wedge h_U(df_{i_1}) \wedge \dots \wedge h_U(df_{i_k}) \tag{2.24}$$

and

$$\sum h_U(f_{i_1 \dots i_k}) \wedge h_U(df_{i_1}) \wedge \dots \wedge h_U(df_{i_k}) = 0. \tag{2.25}$$

Since h_U is an isomorphism, it is evident that $h_U(df_{i_1 \dots i_k}) = \underline{d}f_{i_1 \dots i_k}, h_U(df_{i_1}) = \underline{d}f_{i_1}, \dots, h_U(df_{i_k}) = \underline{d}f_{i_k} \in \Omega(U)$, where $\underline{d}$ is a derivation on U. Therefore, from (2.24) and (2.25), we obtain

$$h(\mu)|_U = \sum \underline{d}f_{i_1 \dots i_k} \wedge \underline{d}f_{i_1} \wedge \dots \wedge \underline{d}f_{i_k}, \tag{2.26}$$

and

$$\sum f_{i_1 \dots i_k} \underline{d}f_{i_1} \wedge \dots \wedge \underline{d}f_{i_k} = 0. \tag{2.27}$$

From (2.26) and (2.27) as well as from properties of the exterior derivation $\underline{d}$ it follows

$$h(\mu)|_U = \underline{d}(\sum f_{i_1 \dots i_k} \underline{d}f_{i_1} \wedge \dots \wedge \underline{d}f_{i_k}) = \underline{d}0 = 0.$$

Since p was an arbitrary point of M then there exists an open covering $\mathcal{U}$ of M such that for each $U \in \mathcal{U}$, $h(\mu)|_U = 0$. Consequently, $h(\mu) = 0$, i.e. $\mu \in \mathrm{Ker}h$. Thus, $\mathcal{M}(M) = \{0\}$. □

Lemma 2.4. and Lemma 2.5. prelude an important theorem, proved in details in [20]. We don't reproduce the proof here. However, for the sake of the theorem, one needs the following.

Let $(M, \mathcal{F})$ be a d-space, $\mathcal{A}^k$ the sheaf $U \to A^k(U)$ and $\mathcal{M}^k$ the sheaf $U \to \mathcal{M}^k(U)$, where $U \in \tau_{\mathcal{F}}$ and $k = 1, 2, \ldots$. The sheaf $\mathcal{M}^k$ is evidently a subsheaf of $\mathcal{F}$-modules of the sheaf $\mathcal{A}^k$, for $k \geq 1$. Furthermore, let $\mathcal{A} = \oplus_{k \geq 0} \mathcal{A}^k$ and $\mathcal{M} = \oplus_{k \geq 0} \mathcal{M}^k$ be direct sums of the corresponding sheaves. One can see that both sheaves $\mathcal{A}$ and $\mathcal{M}$ are sheaves of graded algebras, and $\mathcal{M}$ is a subsheaf of homogenous ideals in the sheaf $\mathcal{A}$.

Let $\bigwedge = \mathcal{A}/\mathcal{M}$ be a quotient presheaf. Let us denote by $\hat{\mathcal{A}}$ the quotient sheaf associated with the presheaf $\bigwedge$.

With this notation, we state the following

Theorem 2.3. *Let $(M, \mathcal{F})$ be a d-space. In the graded algebra $\hat{\mathcal{A}}(M)$, there exists exactly one operator $\hat{d} : \hat{\mathcal{A}}^k(M) \to \hat{\mathcal{A}}^{k+1}(M)$, for $k = 0, 1, \ldots$ satisfying the following conditions:*

(i) $\hat{d}$ *is an* $\mathbb{R}$*-linear mapping,*

(ii) $(\hat{d}f)(p) := [df]_p$, *for any point* $p \in M$ *and any function* $f \in \mathcal{F}$,

(iii) $\hat{d}(\omega \wedge \eta) = \hat{d}\omega \wedge +(-1)^{deg\omega} \omega \wedge \hat{d}\eta$, *for any* $\omega, \eta \in \hat{\mathcal{A}}(M)$,

(iv) $\hat{d} \circ \hat{d} = 0$.

Proof. [20]. □

Chapter 3

LINEAR CONNECTIONS AND RIEMANNIAN GEOMETRY

Calculus on a differential space is assured by the existence of smooth vector fields, and the notion of connection on a differential space is not so much different from that on a manifold. However, the Christoffel symbols of a covariant derivative on a differential space $(M, \mathcal{F})$ are defined with respect to a basis of $\mathcal{X}(M)$ and a basis of a differential module $\mathcal{W}$.
A differential space is Riemannian or Lorentzian if it is endowed with a Riemannian or Lorentz metric tensor accordingly. To any smooth metric tensor one can assign a covariant derivative, called the Levi-Civita connection.
The chapter closes considering contravariant derivatives as a means to generalize the Levi-Civita connection of a degenerate metric.

3.1 Linear Connections, Curvature, and Torsion

Throughout the present section, we shall assume that the pair $(M, \mathcal{F})$ is a differential space and $\mathcal{W}$ is a $\mathcal{F}$-differential module of ϕ-linear fields on $(M, \mathcal{F})$. $\mathcal{L}_{\mathcal{F}}(A, B)$ will denote the set of all $\mathcal{F}$-module maps from A to B.

Definition 3.1. [9] *The covariant derivative or linear connection in the $\mathcal{F}$-differential module $\mathcal{W}$ is any mapping*

$$\nabla \in \mathcal{L}_{\mathcal{F}}(\mathcal{X}(M); \mathcal{L}_{\mathbb{R}}(\mathcal{W}; \mathcal{W}))$$

satisfying the condition

$$\nabla_V(fW) = \partial_V f.W + f.\nabla_V W, \tag{3.1}$$

where $V \in \mathcal{X}(M)$, $W \in \mathcal{W}$, $f \in \mathcal{F}$, and $\nabla_V : \mathcal{W} \to \mathcal{W}$ is a function assigning to every linear field $W \in \mathcal{W}$ a linear field $\nabla_V W$, called the directional derivative of W in the direction V.

Let ∇ be a covariant derivative in $\mathcal{W}$. By definition, ∇ is a mapping which assigns to every $V \in \mathcal{X}(M)$ another mapping, namely

$$\nabla : \mathcal{X}(M) \to \mathcal{L}_{\mathbb{R}}(\mathcal{W}; \mathcal{W}), \tag{3.2}$$

which is a module map over $\mathcal{F}$. It can be easily seen that ∇ has the following properties:

$$\begin{aligned} \nabla_{fV} W &= f\nabla_V W \\ \nabla_{V_1+V_2} W &= \nabla_{V_1} W + \nabla_{V_2} W \\ \nabla_V \alpha W &= \alpha \nabla_V W \\ \nabla_V (W_1 + W_2) &= \nabla_V W_1 + \nabla_V W_2 \end{aligned}$$

where $f \in \mathcal{F}$, $V, V_1, V_2 \in \mathcal{X}(M)$, $\alpha \in \mathbb{R}$, $W, W_1, W_2 \in \mathcal{W}$.

Let $V_1, \ldots, V_m$ be a vector basis of the $\mathcal{F}$-module $\mathcal{X}(M)$, and $W_1, \ldots, W_n$ a vector basis of the $\mathcal{F}$-module $\mathcal{W}$. On account of (3.2) , one has

$$\nabla_{V_i} W_j = \Gamma^k_{ij} W_k$$

where $\Gamma^k_{ij} \in \mathcal{F}$, $i = 1, \ldots, m; j, k = 1, \ldots, n$. The functions Γ^k_{ij} are called Christoffel symbols or coordinates or coefficients of the covariant derivative ∇ with respect to the bases $V_1, \ldots, V_m$ and $W_1, \ldots, W_n$.

Proposition 3.1. *Let $V_1, \ldots, V_m$ be a vector basis of $\mathcal{X}(M)$ and $W_1, \ldots, W_n$ a vector basis of $\mathcal{W}$. If $V = f^i V_i \in \mathcal{X}(M), f^i \in \mathcal{F}$, and $W = g^i W_i \in \mathcal{W}, g^i \in \mathcal{F}$, then*

$$\nabla_V W = (f^i \partial_i g^k + f^i g^j \Gamma^k_{ij}) W_k. \tag{3.3}$$

Proof. The proof utilizes formula (3.1) and properties of ∇ stated above. □

It follows that the functions $\gamma^k := f^i \partial_i g^k + f^i g^j \Gamma^k_{ij}$ are coordinates of $\nabla_V W$ in the considered bases. Since $\{f^i\}$, $i = 1, \ldots, m$, and $\{g^j\}$, $j = 1, \ldots, n$, uniquely determine V and W respectively, the functions γ^k uinquely determine the covariant derivative $\nabla_V W$.

Definition 3.2. *Let X, Y be two vector fields on the differential space $(M, \mathcal{F})$. By the Lie bracket $[X, Y]$, we mean the vector field defined by*

$$[X, Y](f) = X[Y(f)] - Y[X(f)]$$

where $f \in \mathcal{F}$.

Let's note that neither XY nor YX is a vector field since they involve second-order derivatives. The Lie bracket $[X, Y]$, however, involves first order derivatives and is indeed a vector field.
It's easy to see that the Lie bracket satisfies

(i) bilinearity

$$[X, a_1 Y_1 + a_2 Y_2] = a_1 [X, Y_1] + a_2 [X, Y_2],$$

$$[a_1 X_1 + a_2 X_2, Y] = a_1 [X_1, Y] + a_2 [X_2, Y],$$

with $a, a_1, a_2 \in \mathbb{R}$.

(ii) skew-symmetry

$$[X,Y] = -[Y,X],$$

(iii) the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Let ∇ be a covariant derivative in the differential module $\mathcal{W}$. ∇ doesn't, in general, preserve Lie brackets, that is to say

$$R_{XY} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{3.4}$$

may not vanish for all $X, Y \in \mathcal{X}(M)$.

Proposition 3.2. [9] *Let $X, Y \in \mathcal{X}(M)$ and $W \in \mathcal{W}$. $R_{XY}W$, viewed as a function of the three variables X, Y, W is a tensor.*

Proof. Let us prove, for instance:

$$R_{XY}(fW) = R_{XY}f.W + f.R_{XY}W,$$

$f \in \mathcal{F}$. Indeed,

$$\begin{aligned} R_{XY}(fW) &= [\nabla_X, \nabla_Y](fW) - \nabla_{[X,Y]}(fW) \\ &= \nabla_X\nabla_Y(fW) - \nabla_Y\nabla_X(fW) - \partial_{[X,Y]}f.W - f.\nabla_{[X,Y]}W \\ &= \partial_X\partial_Y f.W + \partial_Y f\nabla_X W + \partial_X f\nabla_Y W + f\nabla_X\nabla_Y W - \\ &\quad \partial_Y\partial_X f.W - \partial_X f.\nabla_Y W - \partial_Y f.\nabla_X W - f\nabla_Y\nabla_X W - \\ &\quad \partial_{[X,Y]}f.W - f\nabla_{[X,Y]}W \\ &= (\partial_X\partial_Y f - \partial_Y\partial_X f)W - \partial_{[X,Y]}f.W + f(\nabla_X\nabla_Y W - \\ &\quad \nabla_Y\nabla_X W - \nabla_{[X,Y]}W) \\ &= R_{XY}f.W + f.R_{XY}W \end{aligned}$$

It can also be shown easily that $R_{(fX)Y}W = fR_{XY}W$ and $R_{X(fY)}W = fR_{XY}W, f \in \mathcal{F}$. □

Definition 3.3. [12] *The curvature tensor of the covariant derivative* ∇ *is the mapping* R *which assigns to any two vectors* $X, Y \in \mathcal{X}(M)$ *the mapping* R_{XY}. *Thus*

$$R \in \mathcal{L}_{\mathcal{F}}(\mathcal{X}(M) \times \mathcal{X}(M); \mathcal{L}_{\mathcal{F}}(\mathcal{W}, \mathcal{W})).$$

The curvature tensor is also called the Riemann tensor.

The following proposition gives two basic properties of the curvature tensor.

Proposition 3.3. *Let* $X, Y, Z \in \mathcal{X}(M)$. *Then*

(i) $R_{XY} = -R_{YX}$,

(ii) $\nabla_X R_{YZ} + R_{X[Y,Z]} + \nabla_Y R_{ZX} + R_{Y[Z,X]} + \nabla_Z R_{XY} + R_{Z[X,Y]} = 0.$

Proof.[9]. □

Definition 3.4. [9] *The multilinear mapping*

$$\hat{R} : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{W} \times \mathcal{W}^* \to \mathcal{F},$$

defined by $\hat{R}(X, Y, W, U) = U(R_{XY}W)$, $X, Y \in \mathcal{X}(M), W \in \mathcal{W}, U \in \mathcal{W}^*$, *is called the scalar curvature tensor.*

Proposition 3.4. [9] *If* $\mathcal{W}^{**} = \mathcal{W}$, *i.e.* $\mathcal{W}$ *is reflexive , the scalar curvature tensor* $\hat{R}$ *uniquely determines the curvature tensor* R.

Proof. $\mathcal{W}^{**} = \mathcal{W}$ implies that for any linear function $L : \mathcal{W}^* \to \mathcal{F}$, there is exactly one element $V \in \mathcal{W}$ such that $L(U) = U(V)$, for every $U \in \mathcal{W}^*$. For any fixed elements: $X, Y \in \mathcal{X}(M)$ and $W \in \mathcal{W}$, $R_{XY}W$ is the only element of $\mathcal{W}$ such that $U(R_{XY}W) = \hat{R}(X, Y, W, U)$, for any $U \in \mathcal{W}^*$. Therefore, in the case of a reflexive module $\mathcal{W}$, $\hat{R}$ uniquely determines R. □

Let's assume that the differential space $(M, \mathcal{F})$ is of a constant differential dimension and $V_1, \ldots, V_m$ a vector basis of $\mathcal{X}(M)$, $m = \dim(M, \mathcal{F})$. If

$W_1, \ldots, W_n$ is a vector basis of $\mathcal{W}$ ($n = \dim \mathcal{W}$), and $W^1, \ldots, W^n$ its dual in $\mathcal{W}^*$, clearly one has

$$R_{V_i V_j} W_k = R^l_{ijk} W_l.$$

The functions $R^l_{ijk} \in \mathcal{F}$ are uniquely determined by the equation

$$R^l_{ijk} = W^l(R_{V_i V_j} W_k). \tag{3.5}$$

Equation (3.5) shows that R^l_{ijk} are components of the scalar curvature tensor. Direct computation from (3.5) gives

$$R^l_{ijk} = (\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik}) - (\Gamma^r_{ik}\Gamma^l_{jr} - \Gamma^r_{jk}\Gamma^l_{ir}) - \gamma^s_{ij}\Gamma^l_{sk}$$

where $\Gamma^k_{ij} = W^k(\nabla_{V_i} W_j)$, $[V_i, V_j] = \gamma^s_{ij} V_s$, $\partial_i = \nabla_{V_i}$.

We next look at the concept of the Ricci tensor. But we first need the following digression on the concept of the trace of a (multi)linear mapping.

Definition 3.5. *[9] Let $W_1, \ldots, W_n$ be a vector basis of the $\mathcal{F}$- differential module $\mathcal{W}$ and let $W^1, \ldots, W^n$ be the dual of $W_1, \ldots, W_n$ in the $\mathcal{F}$-module $\mathcal{W}^*$. Let $L : \mathcal{W} \to \mathcal{W}$ be a linear mapping. The trace of L, trL, is defined to be*

$$trL = W^i(LW_i) \in \mathcal{F},$$

$i = 1, \ldots, n$.

Definition 3.5. does not depend on the choice of a particular basis.

Remark 3.1. *Let $L : \mathcal{W} \to \mathcal{W}$ be a linear mapping. For any $f \in \mathcal{F}$, we have*

$$tr(fL) = W^i((fL)W_i) = fW^i(LW_i) = ftrL, \tag{3.6}$$

where $W_1, \ldots, W_n$ and $W^1, \ldots, W^n$ are as in Definition 3.5..
From (3.6), it follows that, since additivity for tr is clear,

$$tr \in \mathcal{L}_{\mathcal{F}}(\mathcal{L}_{\mathcal{F}}(\mathcal{W}; \mathcal{W});\ \mathcal{F}).$$

Example 3.1. *Let $\mathcal{V}$ be a n-dimensional $\mathbb{R}$-vector space and $\mathcal{V}^*$ its dual. Let $V_1, \ldots, V_n$ and $V^1, \ldots, V^n$ be vector bases in $\mathcal{V}$ and $\mathcal{V}^*$ respectively. If a^i_j are coordinates of a linear mapping $L \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, i.e. $L(V_i) = a^j_i V_j$, one has $trL = a^i_i$.*

Now, let ∇ be a covariant derivative in the differential module $\mathcal{X}(M)$ and let R be the curvature tensor of ∇. We introduce the tensor

$$trR \in \mathcal{L}_{\mathcal{F}}(\mathcal{X}(M) \times \mathcal{X}(M);\ \mathcal{F}), \tag{3.7}$$

defined by

$$(trR)_{XY} = tr(R_{XY}), \qquad X, Y \in \mathcal{X}(M). \tag{3.8}$$

Definition 3.6. *[12] The tensor*

$$Ric \in \mathcal{L}_{\mathcal{F}}(\mathcal{X}(M), \mathcal{X}(M); \mathcal{F})$$

defined by

$$Ric(Y, Z) = tr_X(R_{XY}Z), \qquad Y, Z \in \mathcal{X}(M), \tag{3.9}$$

for any $X \in \mathcal{X}(M)$, is called the Ricci tensor.

$R_{XY}Z$, with Y and Z fixed, should be thought of as a linear function of the variable X transforming the module $\mathcal{X}(M)$ into itself, i.e., $R_{\_Y}Z \in \mathcal{L}_{\mathcal{F}}(\mathcal{X}(M); \mathcal{X}(M))$. $Ric(Y, Z)$ is the trace of this function (Heller 1991).

We now wish to define another object: the torsion tensor. For this purpose we assume that $\mathcal{X}(M) \subset \mathcal{W}$.

Definition 3.7. *[9] Let ∇ be a covariant derivative taking values in the differential module $\mathcal{W}$. The torsion tensor of the covariant derivative ∇ is a mapping $T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{W}$, given by*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

$X, Y \in \mathcal{X}(M)$.

It can be easily checked that T is a tensor.

Definition 3.8. *Let T is the torsion tensor of the covariant derivative ∇. ∇ is said to be symmetric if $T(X,Y) = 0$ for all $X, Y \in \mathcal{X}(M)$.*

The relationship between the curvature and torsion tensors is given by the following

Proposition 3.5. [9] *For any $X, Y, Z \in \mathcal{X}(M)$*

$$R_{XY}Z - \nabla_X(T(Y,Z)) - T(X,[Y,Z]) + R_{YZ}X - \nabla_Y(T(Z,X)) - \\ -T(Y,[Z,X]) + R_{ZX}Y - \nabla_Z(T(X,Y)) - T(Z,[X,Y]) = 0 \quad (3.10)$$

The proof is straightforward. One uses Lie brackets Jacobi identities.

Proposition 3.5. leads to the following

Corollary 3.1. *If the covariant derivative ∇ is symmetric, then (3.10) becomes*

$$R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0 \quad (3.11)$$

Equation (3.11) is called the first Bianchi identity. The second Bianchi identity will be deduced from

Proposition 3.6. *If $\mathcal{X}(M) = \mathcal{W}$, then*

$$(\nabla_X R)_{YZ} - R_{X,T(Y,Z)} + (\nabla_Y R)_{ZX} - R_{Y,T(Z,X)} + \\ +(\nabla_Z R)_{XY} - R_{Z,T(X,Y)} = 0. \quad (3.12)$$

(the coma in the subscript of the second term introduces a "useful inconsistency" in the notation).

Proof. By using the Leibnitz rule:

$$\nabla_Z R_{XY} = (\nabla_Z R)_{XY} + R_{\nabla_Z X, Y} + R_{X, \nabla_Z Y}.$$

□

Corollary 3.2. *If the covariant derivative is symmetric (3.12) becomes*

$$(\nabla_X R)_{YZ} + (\nabla_Y R)_{ZX} + (\nabla_Z R)_{XY} = 0 \tag{3.13}$$

Equation (3.13) is called the second Bianchi identity.

If the sequences $V_1, \ldots, V_m; W_1, \ldots, W_n$ and $W^1, \ldots, W^n$ are bases in $\mathcal{X}(M)$, $\mathcal{W}$ and $\mathcal{W}^*$ respectively, then the coordinates of the torsion tensor T in these bases are given by

$$T(V_i, V_j) = T_{ij}^k W_k$$

or

$$T_{ij}^k = W^k(T(V_i, V_j)).$$

Let $[V_i, V_j] = \gamma_{ij}^s V_s$ and let

$$\Gamma_{ij}^k = W^k(\nabla_{V_i} W_j) \tag{3.14}$$

$$\hat{\Gamma}_{ij}^k = W^k(\nabla_{V_i} V_j) \tag{3.15}$$

be the coordinates of the covariant derivative in the corresponding bases. By direct computation, it is easy to show that

$$T_{ij}^k = \hat{\Gamma}_{ij}^k - \hat{\Gamma}_{ji}^k - \alpha_s^k \gamma_{ij}^s \tag{3.16}$$

where α_s^k is defined by

$$V_s = \alpha_s^k W_k.$$

Lemma 3.1. *If $\mathcal{X}(M) = \mathcal{W}$, the basis $V_1, \ldots, V_m$, in $\mathcal{X}(M)$, coincides with the basis $W_1, \ldots, W_n$, in $\mathcal{W}$, and the coordinates (3.14) of the covariant derivative in the respective bases are equal, i.e. $\hat{\Gamma}_{ij}^k = \Gamma_{ij}^k$. In such a case the torsion coordinates (3.16) become*

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k - \gamma_{ij}^k.$$

The simple proof of this lemma is omitted.

If, in addition, the basis $V_1, \ldots, V_n$ is abelian, i.e. if $[V_i, V_j] = 0$, one has $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$. Therefore we see

Proposition 3.7. [9] *If $\mathcal{X}(M) = \mathcal{W}$ and the basis of $\mathcal{X}(M)$ is abelian, then the covariant derivative is symmetric if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$.*

Under the assumption $\mathcal{X}(M) = \mathcal{W}$, let's prove

Proposition 3.8. [9] *If ∇ in $\mathcal{X}(M)$ is symmetric and the basis $V_1, \ldots, V_n$ of $\mathcal{X}(M)$ is abelian, one has*

$$Ric(X,Y) - Ric(Y,X) + trR_{XY} = 0, \tag{3.17}$$

$X, Y \in \mathcal{X}(M)$.

Proof. Let's put $X = \alpha^i V_i, Y = \beta^j V_j$. Then, Equation (3.17) simply be- comes

$$Ric(V_i, V_j) - Ric(V_j, V_i) - trR_{V_i V_j} = 0.$$

But by (3.9), we have

$$\begin{aligned} &Ric(V_i, V_j) - Ric(V_j, V_i) - trR_{V_i V_j} \\ =\ &tr_{V_k}(R_{V_k V_i} V_j) - tr_{V_k}(R_{V_k V_j} V_i) - trR_{V_i V_j} \\ =\ &V^k(R_{V_k V_i} V_j) - V^k(R_{V_k V_j} V_i) - trR_{V_i V_j}. \end{aligned}$$

Using the fact that $[V_i, V_j] = 0$, for all $i, j = 1, \ldots, n$ the last equality becomes

$$\begin{aligned} &V^k([\nabla_{V_k}, \nabla_{V_i}]V_j) - V^k([\nabla_{V_k}, \nabla_{V_j}]V_i) - V^k([\nabla_{V_i}, \nabla_{V_j}]V_k) \\ =\ &\partial_k(\Gamma_{ij}^k - \Gamma_{ji}^k) + (\Gamma_{ij}^s - \Gamma_{ji}^s)\Gamma_{ks}^k - \partial_i(\Gamma_{kj}^k - \Gamma_{jk}^k) \\ &-(\Gamma_{kj}^s - \Gamma_{jk}^s)\Gamma_{is}^k + \partial_j(\Gamma_{ki}^k - \Gamma_{ik}) + (\Gamma_{ki}^s - \Gamma_{ik}^s)\Gamma_{js}^k \\ =\ &0, \qquad \text{since } \Gamma_{ji}^k = \Gamma_{ji}^k. \square \end{aligned}$$

Corollary 3.3. *The Ricci tensor Ric is symmetric if and only if* $trR = 0$.

3.2 Riemannian Geometry

We now discuss the existence of further structures, namely Riemannian and Lorentzian structures, carried by a differential space when it is endowed with a metric tensor, which is a natural generalisation of the inner product between two vectors in $\mathbb{R}^n$ to an arbitrary differential space.

As in section 3.1, we assume that $(M, \mathcal{F})$ is a differential space and $\mathcal{W}$ is a $\mathcal{F}$-differential module of ϕ-linear fields on $(M, \mathcal{F})$.

Definition 3.9. [5] *Let* $g(p)$ *denote a scalar product in the tangent sapce* T_pM, $p \in M$.
A metric on a differential space $(M, \mathcal{F})$ *is a two-covariant, symmetric and non-degenerate* $\mathcal{F}$*-tensor* g *such that*

$$g(V, W)(p) = g(p)(V(p), W(p)),$$

$V, W \in \mathcal{W}$, $p \in M$.
If, for any $V, W \in \mathcal{W}$, $g(V, W) \in \mathcal{F}$, g *is said to be smooth, i.e., if* $g \in \mathcal{L}_{\mathcal{F}}(\mathcal{W}, \mathcal{W}; \mathcal{F})$, *where the last symbol denotes the set of all bilinear module mappings with values in* $\mathcal{F}$.

Theorem 3.1. [5] *If* g *is a smooth metric in the differential module* $\mathcal{W}$ *on* $(M, \mathcal{F})$, *every point* p *of* M *has a neighbourhood* U *on which there is a* g*-orthonormal vector basis* $V_1, \ldots, V_n$ *of the module* $\mathcal{W}$, *i.e.,*

$$g(V_i, V_j) = \delta^i_j \varepsilon_i,$$

where $\varepsilon_i = g(V_i, V_i) = \pm 1, i, j = 1, \ldots, n = \dim \mathcal{W}$.

Proof. By construction through the standard Gramm-Schmidt g- orthogonalisation.

The g-orthogonal basis may be ordered so that negative signs, if any, come first. The number I of minus signs is called the index of the module $\mathcal{W}$. The Gramm-Schmidt g-orthogonalisation ensures that the index I of the module $\mathcal{W}$ is basis independent.

Definition 3.10. [5] *The pair $(\mathcal{W}, g)$, where g is a smooth metric in $\mathcal{W}$, is called the pseudo-Riemannian differential module. If $I = 0$ or $n = \dim \mathcal{W}$, $(\mathcal{W}, g)$ is said to be Riemannian differential module; if $I = 1$ or $n - 1$, it is said to be Lorentz differential module.*

We shall show that a smooth metric g induces a certain covariant derivative in $\mathcal{W}$. Let's first prove

Lemma 3.2. [12] *Let g be a smooth metric on the differential space $(M, \mathcal{F})$. For any $\mathcal{F}$-linear mapping $\varphi : \mathcal{X}(M) \to \mathcal{F}$, there exists exactly one vector field $V \in \mathcal{X}(M)$ such that*

$$\varphi(W) = g(V, W)$$

for any $W \in \mathcal{X}(M)$.

Proof. Let $V_1, \ldots, V_n$ be a vector basis of the $\mathcal{F}$-module $\mathcal{X}(M)$. With respect to this basis, we solve for $V = \alpha^i V_i$, where $\alpha^i \in \mathcal{F}$, such that

$$\varphi(W) = g(V, W)$$

for any $W \in \mathcal{X}(M)$. Substituting for W successively $V_1, \ldots, V_n$ one obtains

$$\varphi(V_i) = g(V, V_i), \qquad i = 1, \ldots, n.$$

Hence

$$\varphi(V_i) = \alpha^j g(V_j, V_i), \qquad i, j = 1, \ldots, n,$$

i.e.,

$$\begin{pmatrix} \varphi(V_1) \\ \vdots \\ \varphi(V_n) \end{pmatrix} = \begin{pmatrix} g(V_1, V_1) & \cdots & g(V_n, V_1) \\ \vdots & \ddots & \vdots \\ g(V_1, V_n) & \cdots & g(V_n, V_n) \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix},$$

where the matrix $(g(V_j, V_i))_{i,j=1,\dots,n}$ is non-degenerate. By using Cramer formulas, one can easily compute $\alpha^1, \dots, \alpha^n$. The functions α^i are unique. Therefore V is uniquely defined.

Now, we prove

Theorem 3.2. *[12] For any smooth metric g in the $\mathcal{F}$-module $\mathcal{W} = \mathcal{X}(M)$, there is one and only one covariant derivative ∇ in $\mathcal{X}(M)$ such that*

$$\begin{aligned} Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y), & (3.18)\\ \nabla_X Y &= \nabla_Y X + [X,Y], & (3.19)\end{aligned}$$

for any $X, Y, Z \in \mathcal{X}(M)$.

Proof. For any $X, Y \in \mathcal{X}(M)$ let $\varphi_{X,Y} : \mathcal{X}(M) \to \mathcal{F}$ be the $\mathcal{F}$-linear mapping given by

$$\begin{aligned} \varphi_{X,Y}(Z) &= 1/2(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + \\ &\quad g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X)), \qquad (3.20)\end{aligned}$$

for $Z \in \mathcal{X}(M)$.
Using (3.18) and (3.19) in (3.20), one obtains

$$\begin{aligned} 2\varphi_{X,Y}(Z) &= g(\nabla_X Y + \nabla_Y X + [X,Y], Z) + g(Y, \nabla_X Z - \nabla_Z X - [X,Z]) + \\ &\quad g(X, \nabla_Y Z - \nabla_Z Y - [Y,Z]) \\ &= 2g(\nabla_X Y, Z)\end{aligned}$$

or

$$\varphi_{X,Y}(Z) = g(\nabla_X Y, Z).$$

From Lemma 3.2. it follows that $\nabla_X Y$ is unique.
Let $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \mapsto \mathcal{X}(M)$ be the mapping given by

$$\nabla(X,Y) = \nabla_X Y$$

for $X, Y \in \mathcal{X}(M)$. One can check easily that ∇ is a covariant derivative. Moreover, it satisfies (3.18)and (3.19). In fact, for any $X, Y, Z \in \mathcal{X}(M)$, one has

$$\begin{aligned} 2g(\nabla_Z X, Y) + 2g(X, \nabla_Z Y) &= 2\varphi_{Z,X}(Y) + 2\varphi_{Z,Y}(X) \\ &= 2Zg(X,Y), \qquad \text{by virtue of (3.20).} \end{aligned}$$

For the equation (3.19), let's put $Z = X$ in (3.20). Then we have,

$$2\varphi_{X,Y}(X) = Yg(X,X) + 2g([X,Y],X). \tag{3.21}$$

Applying Equation (3.18) to $Yg(X,X)$, (3.21) becomes

$$2\varphi_{X,Y}(X) = 2g(\nabla_Y X, X) + 2g([X,Y],X). \tag{3.22}$$

Since $2\varphi_{X,Y}(X) = 2g(\nabla_X Y, X)$, (3.22) can be written as

$$g(\nabla_X Y, X) = g(\nabla_Y X, X) + g([X,Y],X),$$

which ends the proof. □

The covariant derivative of Theorem 3.2. is said to be compatible with the metric g in $\mathcal{X}(M)$, and it is called the natural covariant derivative, or the Levi-Civita connection, in $\mathcal{X}(M)$.

In opposition to covariant derivatives in the differential module $\mathcal{X}(M)$, we define the notion of contravariant derivative to help generalize the Levi-Civita connection of a degenerate metric. For this purpose, let $\mathcal{X}^*(M) = \mathcal{L}_{\mathcal{F}}(\mathcal{X}(M), \mathcal{F})$ be the dual $\mathcal{F}$-module with respect to the symmetric 2-form g, i.e., for any $X \in \mathcal{X}(M)$ let $X^* \in \mathcal{X}^*(M)$ be a linear form defined by

$$X^*(Z) = g(X,Z), \qquad \text{for } Z \in \mathcal{X}(M). \tag{3.23}$$

Definition 3.11. *[25] A mapping $\nabla^* : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}^*(M)$ is called the contravariant derivative in the $\mathcal{F}$-module $\mathcal{X}(M)$ if*

(i) $\nabla^*_{X_1+X_2} Y = \nabla^*_{X_1} Y + \nabla^*_{X_2} Y$

(ii) $\nabla^*_{fX} Y = f\nabla^*_X Y$

(iii) $\nabla^*_X(Y_1+Y_2) = \nabla^*_X Y_1 + \nabla^*_X Y_2$

(iv) $\nabla^*_X(fY) = (Xf)Y^* + f\nabla^*_X Y$

for all $X_1, X_2, Y, Y_1, Y_2 \in \mathcal{X}(M)$, *and* $f \in \mathcal{F}$.

If $V_1, V_2, \ldots, V_n$ is a $\mathcal{F}$- vector basis of the differential module $\mathcal{X}(M)$ then one can define the components of ∇^* with respect to the basis in the following way:

$$\Gamma_{ijk} = (\nabla^*_{V_i} V_j)(V_k), \tag{3.24}$$

for $i, j, k = 1, \ldots, n$. On the other hand, for any $W \in \mathcal{X}(M)$, we have

$$\begin{aligned} \nabla^*_{V_i} V_j(W) &= g(\nabla_{V_i} V_j, W) \\ &= g(\Gamma^k_{ij} V_k, W) \\ &= \Gamma^k_{ij} V^*_k(W) \end{aligned}$$

for $i, j, k = 1, \ldots, n$. Therefore it is immediate that, for arbitrary $X = \sum_{i=1}^n \alpha^i V_i$ and $Y = \sum_{j=1}^n \beta^j V_j$,

$$\nabla^*_X Y = \sum_{i,j=1}^n \alpha^i V_i(\beta^j) V^*_j + \sum_{i,j,k=1}^n \alpha^i \beta^j \Gamma^k_{ij} V^*_k. \tag{3.25}$$

Given the elements $\Gamma_{ijk} \in \mathcal{F},\ i, j, k = 1, \ldots, n$, one can define the contravariant derivative ∇^* with components Γ_{ijk} using formula (3.24).

Theorem 3.3. [25] *For an arbitrary symmetric 2-form* $g : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}$ *there exists exactly one contravariant derivative* $\nabla^* : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}^*(M)$ *satisfying the following conditions:*

$$\nabla^*_X Y - \nabla^*_Y X = [X, Y]^* \tag{3.26}$$

$$Zg(X, Y) = (\nabla^*_Z X)(Y) + (\nabla^*_Z Y)(X) \tag{3.27}$$

for any $X, Y, Z \in \mathcal{X}(M)$.

The proof is analogous to that of Theorem 3.2..

The contravariant derivative, defined by the Koszul formula

$$(\nabla^*_X Y)(Z) = \frac{1}{2}\{Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + \\ +g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z])\}, \qquad (3.28)$$

will be called the contravariant Levi-Civita connection of the symmetric 2-form g. If, in addition, the basis $V_1, \ldots, V_n$ of the module $\mathcal{X}(M)$ is abelian, i.e. $[V_i, V_j] = 0$ for all $i, j = 1, \ldots, n$, the components of ∇^* have the form:

$$\Gamma_{ijk} = \frac{1}{2}[V_i(g_{ik}) + V_j(g_{ki}) - V_k(g_{ij})],$$

where $g_{ij} = g(V_i, V_j)$.

Now, let $\psi : \mathcal{X}(M) \to \mathcal{X}^*(M)$ be a mapping defined by

$$\psi(X) = X^*, \qquad (3.29)$$

for $X \in \mathcal{X}(M)$.
Let us denote by Ker $g := \{X \in \mathcal{X}(M) : g(X, \_) = 0\}$, the kernel of the 2-form g. Of course, by virtue of (3.23), Ker g = Ker ψ. It is easy to show that the quotient map $\hat{\psi} : \mathcal{X}(M)/\text{Ker } g \to \mathcal{X}^*(M)$ is a monomorphism, and Im ψ = Im $\hat{\psi}$. Let us observe that for all $X, Y \in \mathcal{X}(M)$,

$$\begin{aligned} (X+Y)^* &= X^* + Y^* \\ (-X)^* &= -X^*. \end{aligned}$$

One can prove that (Im $\psi, +, .$) is a $\mathcal{F}$-module. Let us denote by $\mathcal{M}$ the set

$$\mathcal{M} = \{(X,Y) \in \mathcal{X}(M) \times \mathcal{X}(M) : \nabla^*_X Y \in \text{Im } \psi\}.$$

Since (Im $\psi, +, .$) is a $\mathcal{F}$-module, it is easy to see that the set $\mathcal{M}$ has the following properties

(i) If $(X,Y) \in \mathcal{M}$ then $(Y,X) \in \mathcal{M}$.

(ii) If $(X_1,Y) \in \mathcal{M}$ and $(X_2,Y) \in \mathcal{M}$ then $(X_1 + X_2, Y) \in \mathcal{M}$.

(iii) If $(X,Y) \in \mathcal{M}$ and $f \in \mathcal{F}$ then $(fX,Y) \in \mathcal{M}$.

Proposition 3.9. [25] *For an arbitrary* 2*-linear symmetric mapping* $g : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}$*, there exists exactly one mapping* $\nabla : \mathcal{M} \to \mathcal{X}(M)/\mathrm{Ker}g$ *satisfying the conditions:*

(1) $\nabla_{X_1}Y + \nabla_{X_2}Y = \nabla_{X_1+X_2}Y$,

(2) $\nabla_{fX}Y = f\nabla_X Y$,

(3) $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$,

(4) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$,

(5) $\nabla_X Y - \nabla_Y X = [X, Y]$,

(6) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

for $(X_1, Y), (X_2, Y), (X, Y), (X, Y_1), (X, Y_2), (X, Z), (Y, Z) \in \mathcal{M}$ *and* $f \in \mathcal{F}$*, where* $[X, Y]$ *is an equivalence class of* $[X, Y]$ *in the quotient module* $\mathcal{X}(M)/\mathrm{Ker}g$*.*

Proof Let us define $\nabla : \mathcal{M} \to \mathcal{X}(M)/\mathrm{Ker}g$ by the formula

$$\nabla_X Y = \hat{\psi}^{-1}(\nabla^*_X Y). \tag{3.30}$$

It is evident, in view of Theorem 3.3. and the linearity of $\hat{\psi}^{-1}$, that ∇ satisfies (1) – (6). □

Remark 3.2. *If* g *is nondegenerate and* $\psi : \mathcal{X}(M) \to \mathcal{X}^*(M)$ *is an isomorphism of* $\mathcal{F}$*-modules then* $\mathcal{M} = \mathcal{X}(M) \times \mathcal{X}(M)$*. In this case, we obtain* ∇ *which is the Levi-Civita connection of the tensor* g*.*

Let's emphasize the fact that Theorem 3.3. holds even if $(M, \mathcal{F})$ has singularities and g is degenerate. In the case when $(M, \mathcal{F})$ has constant differential dimension and g is non-degenerate, one can prove that the mapping ψ in (3.29) is an isomorphism and evidently

$$\mathcal{M} = \mathcal{X}(M) \times \mathcal{X}(M), \qquad \mathrm{Ker}g = \{0\}.$$

The notion of the contravariant Levi-Civita connection of g gives us a new light on the existence of the Levi-Civita connection of an arbitrary symmetric tensor on $(M, \mathcal{F})$ (which may be degenerate). This is very important in the singular semi-Riemannian geometry (See [15]).

Chapter 4

SOME APPLICATIONS OF DIFFERENTIAL SPACES TO COSMOLOGY

In this chapter, we present a brief exposition of singular points on a differential space, and succinctly describe their classification. Fundamental differential spaces, which are differential spaces with some sort of boundary, are likely to model space-time with singularities. With such special differential space, one can prolong a metric tensor to the boundary.
The description of the differential spaces of the cosmic string and of the Friedman universe is the last step of this chapter. Note that the classifiaction of singularities presented here is that of Ellis and Schmidt [3].

4.1 Singularities of the Fundamental Differential Space

In this section we deal with some applications of the theory of differential spaces to relativistic physics. The first question that arises is [12]: if we employ differential spaces instead of smooth manifolds to model space-time, do we obtain a substantial generalization of general relativity? The answer is "yes", provided we use to this end differential spaces with a special type of boundary.

Definition 4.1. [23] *The pair* $((\bar{M},\bar{\mathcal{F}}),(M,\bar{\mathcal{F}}_M))$ *is a fundamental differential space (shortly f-d-space) if*

(i) $(\bar{M},\bar{\mathcal{F}})$ *is a differential space*

(ii) M *is dense in* $\bar{M}$

(iii) $(M,\bar{\mathcal{F}}_M)$ *is an n-dimensional differential manifold.*

Condition (i) implies that $\bar{M}$ contains the boundary and all the limit points of M. The set $\partial M = \bar{M} - M$ is called the boundary of the f-d-space $((\bar{M},\bar{\mathcal{F}}),(M,\bar{\mathcal{F}}_M))$.
Let $((\bar{M},\bar{\mathcal{F}}),(M,\bar{\mathcal{F}}_M))$ and $((\bar{N},\bar{G}),(N,\bar{\mathcal{G}}_N))$ be two fundamental differential spaces. $((\bar{M}\times\bar{N},\bar{\mathcal{F}}\times\bar{G}),(M\times N,(\bar{\mathcal{F}}\times\bar{\mathcal{G}}_{M\times N}))$ is a fundamental differential space with the boundary $\partial(M\times N) = \partial M\times\bar{N}\cup\bar{M}\times\partial N$.

Definition 4.2. [23] *Let* $((\bar{M},\bar{\mathcal{F}}),(M,\bar{\mathcal{F}}_M))$ *be a fundamental differential space.*
A boundary point $p\in\partial M$ *is called regular if there exists a neighbourhood* $U\in\tau_{\bar{\mathcal{F}}}$ *of* p *such that the differential subspace* $(U,\bar{\mathcal{F}}_U)$ *has constant differential dimension* $n(=\dim M)$*.*
The fundamental differential space $((\bar{M},\bar{\mathcal{F}}),(M,\bar{\mathcal{F}}_M))$ *is said to be regular if all the boundary points are regular, i.e. the differential space* $(\bar{M},\bar{\mathcal{F}})$ *is of constant differential dimension.*

Example 4.1. *Let $\bar{M} = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$, i.e., $\bar{M}$ is the upper half-plane, and $\bar{\mathcal{F}} = (\varepsilon_2)_{\bar{M}}$. M is considered to be the set $\{(x,y) \in \mathbb{R}^2 : y > 0\}$. The boundary points $\partial M = \{(x,y) \in \bar{M} : y = 0\}$ are regular.*

A boundary point is called singular if it is not regular.

Example 4.2. *[23] Let $\bar{M} = \mathbb{R}^2$ and $\bar{\mathcal{F}}$ be the differential structure on $\bar{M}$ generated by the set $\{\pi_1, \pi_2, z\}$, where $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are natural projections and $z : \mathbb{R}^2 \to \mathbb{R}$ is the function defined by*

$$z(x,y) = \sqrt{x^2 + y^2},$$

for $(x,y) \in \mathbb{R}^2$.
It is clear that $\bar{\mathcal{F}}$ defines a 2-dimensional cone. Let $M = \mathbb{R}^2 - \{(0,0)\}$. One can easily see that $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ is a fundamental differential space and $\partial M = \{(0,0)\}$. The boundary point $(0,0)$ is singular. In fact, $\dim T_{(0,0)}\bar{M} = 3$ and $\dim T_p\bar{M} = 2$ for $p \neq (0,0)$.

Some differential spaces can be viewed as "parts" of a differentiable manifold; they are called differential spaces of class D_0. More precisely, a differential space $(M, \mathcal{F})$ is said to be of class D_0 if, for every $p \in M$, there exist an open neighbourhood U of p and a differentiable manifold N such that U is diffeomorphic, in the sense of differential spaces, to some open subset $V \subset N$, $\dim N = \dim T_pM$. See [7].

Then, we define:

Definition 4.3. *[23] A boundary point p of a fundamental differential space $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ is said to be of class D_0 (shortly D_0-point) if there exists a neighbourhood $U \in \tau_{\bar{\mathcal{F}}}$ of p such that $(U, \bar{\mathcal{F}}_U)$ is a differential space of class D_0.*
A boundary point is called a non-D_0-point if it is not of class D_0.

Example 4.3. *The boundary points in Examples 4.1. and 4.2. are D_0-points. More precisely, the boundary points in Example 4.1. are D_0-regular, and the boundary point of Example 4.2. is D_0-singular.*

Example 4.4. [23] *Let $\bar{\mathcal{F}}$ be the differential structure on $\bar{M} = \mathbb{R}^2$ generated by the set $\{\pi_1, \pi_2\} \cup \{f_n : n \in \mathbb{N}\}$, where $f_n : \mathbb{R}^2 \to \mathbb{R}$ is the function defined by*

$$f_n(x,y) = \sqrt[3^n]{x^2 + y^2},$$

for $(x,y) \in \mathbb{R}^2$.
Let's take $M = \mathbb{R}^2 - \{(0,0)\}$. Clearly, M is dense in $\mathbb{R}^2$, and a neighbourhood $(U, \bar{\mathcal{F}}_U)$ of a point $p \in M$ is diffeomorphic to a differential space $(U, (\varepsilon_2)_U)$ since $f_n(x,y) = \sqrt[3^n]{\pi_1^2(x,y) + \pi_2^2(x,y)} \in \varepsilon_2$. Therefore the pair $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ is a fundamental differential space, whose boundary is $\partial M = \{(0,0)\}$. The boundary point $(0,0)$ is non-D_0-singular since locally at $(0,0)$ the space $((\bar{M}, \bar{\mathcal{F}}_M))$ is not a differential subspace of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

In the following example, we provide a case of non-D_0-regular boundary points.

Example 4.5. [23] *Let $N := \{\frac{1}{n} \in \mathbb{R} : n \in \mathbb{N}\} \cup \{0\}$. Let $\mathcal{G}$ be the differential structure on N generated by the set $\{id_N\} \cup \{f_n : n \in \mathbb{N}\}$, where $f_n : N \to \mathbb{R}$, for $n \in \mathbb{N}$, is defined by*

$$f_n(x) = \sqrt[3^n]{x}$$

for $x \in N$. The space $N - \{0\}$ is discrete, therefore $\dim T_x(N, \mathcal{G}) = 0$, for $x \neq 0$. Furthemore, since $\frac{\partial \sqrt[3^n]{x}}{\partial x}|_0$ doesn't exist, $\dim T_x(N, \mathcal{G}) = 0$ for all $x \in N$.
Let us take the cartesian product $(\bar{M}, \bar{\mathcal{F}}) = (N \times \mathbb{R}^2, \mathcal{G} \times \varepsilon_2)$ and let $M = \{\frac{1}{n} \in \mathbb{R} : n \in \mathbb{N}\} \times \mathbb{R}^2$. Clearly, $\bar{M}$ is a bundle of parallel Euclidean planes. It is a 2-dimensional differential manifold. Therefore the pair $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ is a fundamental differential space. For any point $p \in \bar{M}$, we have $\dim T_p\bar{M} = \dim T_pN + \dim T_p\mathbb{R}^2 = 2$. Thus $(\bar{M}, \bar{\mathcal{F}})$ is a differential space of constant differential dimension 2, and the boundary points are regular. And since any neighbourhood of any boundary point, in $\bar{M}$, is disconnected the boundary points are non-D_0. Hence, the boundary points are non-D_0-regular.

Now, let us employ fundamental differential spaces to model space-time. We will assume in the following definition that the pair $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ is a fundamental differential space.

Definition 4.4. [23] *The pair* $((\bar{M}, \bar{\mathcal{F}}), (M, g))$ *is said to be a* C^k*-differential space-time if* (M, g) *is an* n*-dimensional* C^k*- Lorentz submanifold.*
The set $\partial M = \bar{M} - M$ *is called the boundary of the* C^k*-differential space-time. The* C^k*-Lorentz metric* g *is said to be extendible on the boundary* ∂M *if there exists a* C^k *Lorentz metric* $\bar{g}$ *on* $(\bar{M}, \bar{\mathcal{F}})$ *such that* $g = \iota_M^* \bar{g}$*, where* $\iota_M : M \to \bar{M}$ *is the inclusion mapping.*

Example 4.6. [23] *Let* $\bar{M} = \{(x, y, z) \in \mathbb{R}^3 : x = 0 \text{ or } y = 0\}$ *and let* $\bar{\mathcal{F}}$ *be the differential structure on* $\bar{M}$*, induced from* ε_3*, i.e.,* $\bar{\mathcal{F}} = (\varepsilon_3)_{\bar{M}}$*. Let us consider the axis* OZ *to be the time-like axis. Then the metric*

$$\bar{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is an extension of the Lorentz metric from the space-time (M, g)*, which is*

$$(\{(x, y, z) \in \bar{M} : y \neq 0\}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}) \uplus$$

$$(\{(x, y, z) \in \bar{M} : x \neq 0\}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}),$$

($\uplus$ *denotes the disjoint union between these two 2-dimensional Minkwoski spacetimes*). *The axis* OZ *is the boundary of the differential space-time* $((\bar{M}, \bar{\mathcal{F}}), (M, g))$.

An important question arises as to whether it is possible to uniquely prolong geometric objects, i.e. vector fields, k-forms, covariant derivatives from M to $\bar{M}$. The answer to this question turns out to be positive [24]. Let's show it.

Proposition 4.1. *Let* $((\bar{M}, \bar{\mathcal{F}}), (M, g))$ *be a* C^k*-differential space-time. Let* $X_1, X_2 \in \mathcal{X}(\bar{M}), \omega_1, \omega_2 \in \Omega(\bar{M})$ *and* ∇_1, ∇_2 *be covariant derivatives on* $(\bar{M}, \bar{\mathcal{F}})$*. If*

$$X_1|_M = X_2|_M, \quad \omega_1|_M = \omega_2|_M, \quad \nabla_1|_M = \nabla_2|_M,$$

then

$$X_1 = X_2, \quad \omega_1 = \omega_2, \quad \nabla_1 = \nabla_2.$$

Proof. For any function $f \in \bar{\mathcal{F}}$, we have

$$X_1(f)|_M = X_1|_M(f|_M) = X_2|_M(f|_M) = X_2(f)|_M.$$

Since M is dense in $\bar{M}$, $X_1(f) = X_2(f)$ for any $f \in \bar{\mathcal{F}}$. Thus $X_1 = X_2$.
Now, let $\omega_1, \omega_2 \in \Omega^k(\bar{M})$, and $X_1, \ldots, X_k \in \mathcal{X}(\bar{M})$. We have

$$\begin{aligned} \omega_1(X_1, \ldots, X_k)|_M &= \omega_1|_M(X_1|_M, \ldots, X_k|_M) \\ &= \omega_2|_M(X_1|_M, \ldots, X_k|_M) \\ &= \omega_2(X_1, \ldots, X_k)|_M. \end{aligned}$$

Since M is dense $\bar{M}$, $\omega_1(X_1, \ldots, X_k) = \omega_2(X_1, \ldots, X_k)$ for $X_1, \ldots, X_k \in \mathcal{X}(\bar{M})$. Therefore, $\omega_1 = \omega_2$.
In analogous way, one proves that $\nabla_1 = \nabla_2$. □
The following corollary is immediate.

Corollary 4.1. *If geometric objects on M have prolongations to $\bar{M}$, they are unique.*

Whether it is possible to prolong the Lorentz metric g from the manifold M to the base space $\bar{M}$ of the C^k-differential space-time $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ motivates the following definition.

Definition 4.5. *[24] A boundary point $p \in \partial M$ is said to be g-metric if there exist a neighbourhood $U \in \tau_{\bar{\mathcal{F}}}$ of p and a C^k Lorentz metric $\bar{g}$ on $(U, \mathcal{F}_U)$ such that*

$$\bar{g}|_{U \cap M} = g|_{U \cap M}.$$

A point $p \in \partial M$ is a non-metric point if it is not a g-metric point.

Example 4.7. *Let $g = \iota_M^* \eta$ be the metric on the submanifold M from Example 4.1., where η is the Minkowski metric on $(\mathbb{R}^2, \varepsilon_2)$. The metric $\bar{g} = \iota_{\bar{M}}^* \eta$ is an extension of g onto $\bar{M}$. Every point $p \in \partial M$ is D_0-g-metric regular.*

Proposition 4.2. [23] *Let $((\bar{M}, \bar{\mathcal{F}}), (M, g))$ be a C^k-differential space-time. If a point $p \in \partial M$ is g-metric and $k \geq 2$, then there exist a neighbourhood $V \in \tau_{\bar{\mathcal{F}}}$ of p and the integer $m \in \mathbb{N}$ such that*

$$\dim T_q(\bar{M}, \bar{\mathcal{F}}) \leq m$$

for $q \in V$.

Proof. From Corollary 2.1. it follows that there exist an open neighbourhood $V \in \tau_{\bar{\mathcal{F}}}$ of p and a mapping $F : (V, \bar{\mathcal{F}}_V) \to (\mathbb{R}^m, \varepsilon_m)$ such that F_{*q} is injective for $q \in V$. Therefore

$$\dim T_q(\bar{M}, \bar{\mathcal{F}}) = \dim T_q(V, \bar{\mathcal{F}}_V) \leq m$$

for any $q \in V$.

Example 4.8. [27] *Let $\mathbb{R}^{\mathbb{N}}$ be the set of all real sequences, i.e.,*

$$\mathbb{R}^{\mathbb{N}} = \{x := (x_i)_{i \in \mathbb{N}} : x_i \in \mathbb{R} \text{ for all } i \in \mathbb{N}\}.$$

The projection of $\mathbb{R}^{\mathbb{N}}$ onto the i-th coordinate is given by

$$\pi_i(x) = x_i$$

for $x = (x_i) \in \mathbb{R}^{\mathbb{N}}$.
Let $\varepsilon_{\mathbb{N}}$ be the differential structure on $\mathbb{R}^{\mathbb{N}}$ generated by the set $\{\pi_i : i \in \mathbb{N}\}$. Let us put

$$M_i = \{x \in \mathbb{R}^{\mathbb{N}} : x_j = 0 \text{ for } j \neq i\}, \qquad i \in \mathbb{N}.$$

The set M_i is the i-th coordinate line of the infinite dimensional space $\mathbb{R}^{\mathbb{N}}$. Let $\bar{M} := \cup_{i \in \mathbb{N}} M_i$ and $\bar{\mathcal{F}} := (\varepsilon_{\mathbb{N}})_{\bar{M}}, M = \bar{M} - \{0\}$. The differential space $(\bar{M}, \bar{\mathcal{F}})$ is such that $\dim T_0 \bar{M} = \infty$ and $\dim T_x \bar{M} = 1$, for any $x \neq 0$. Since M is one-dimensional manifold, dense in $\bar{M}$, the pair $((\bar{M}, \bar{\mathcal{F}}), (M, \bar{\mathcal{F}}_M))$ is a fundmental differential space. And since $\dim T_0(\bar{M}, \bar{\mathcal{F}}) = \infty$, there is no non-degenerate 2-form of class $C^k (k \geq 2)$ in a neighbourhood of the singular point $(0) \in \mathbb{R}^{\mathbb{N}}$.

Now, let's see how from a Lorentz metric, on a differential space $(M, \mathcal{F})$, and a Riemannian metric, on a differential space $(N, \mathcal{G})$, one can construct a Lorentz metric on the product space $(M \times N, \mathcal{F} \times \mathcal{G})$.
For this purpose we need the following:

Lemma 4.1. *Let $(p,q) \in M \times N$ be an arbitrary point. Let $v_1, \ldots, v_m \in T_pM$ be a vector basis of T_pM, and let $u_1, \ldots, u_n \in T_qN$ be a vector basis of T_qN. The sequence*

$$\begin{array}{lll} w_1 = (j_q)_{*p}v_1 & ,\ldots, & w_m = (j_q)_{*p}v_m, \\ w_{m+1} = (j_p)_{*q}u_1 & ,\ldots, & w_{m+n} = (j_p)_{*q}u_n, \end{array}$$

where $j_q : M \to M \times N, j_q(p) = (p,q)$, and $j_p : N \to M \times N, j_p(q) = (p,q)$, is a vector basis of $T_{(p,q)}(M \times N)$.

Proof. We know that the vectors $w_1, \ldots, w_m$ and $w_{m+1}, \ldots, w_{m+n}$ are linearly independent, because j_p and j_q are embeddings.
Now, suppose there exist $\alpha_1, \ldots, \alpha_m \in \mathcal{F} \times \mathcal{G}$ such that

$$w_{m+i} = \alpha_1(p,q)w_1 + \ldots + \alpha_m(p,q)w_m, \qquad i = 1, \ldots, n.$$

Then

$$w_{m+i}(f \circ \pi_1) = \alpha_1(p,q)w_1(f \circ \pi_1) + \ldots + \alpha_m(p,q)w_m(f \circ \pi_1),$$

for $f \in \mathcal{F}$. More explicitly, we have

$$(j_p)_{*q}u_i(f \circ \pi_1) = \alpha_1(p,q)(j_q)_{*p}v_1(f \circ \pi_1) + \ldots + \alpha_m(p,q)(j_q)_{*p}v_m(f \circ \pi_1),$$

or

$$u_i(f(p)) = \alpha_1(p,q)v_1(f(p)) + \ldots + \alpha_m(p,q)v_m(f(p)),$$

which will infer that $u_i \in T_pM$ if there is a non-zero function $\alpha_j \in \mathcal{F} \times \mathcal{G}, j = 1, \ldots, m$. Therefore, $w_{m+i}, i = 1, \ldots, n$, can not be a linear combination of $w_1, \ldots, w_m$. Thus, extending this argument, the sequence $w_1, \ldots, w_{m+n}$ is a vector basis of $T_{(p,q)}(M \times N)$. □

Proposition 4.3. *[23] Let $g : T^2M \to \mathbb{R}$ be a C^k Lorentz metric on $(M, \mathcal{F})$, $h : T^2N \to \mathbb{R}$ a C^k Riemannian metric on $(N, \mathcal{G})$ and $f : M \to (0, +\infty)$ a* *smooth function of class C^k on $(M, \mathcal{F})$. Then, the 2-form $\bar{g} : T^2(M \times N) \to \mathbb{R}$, defined by*

$$\bar{g}(w_1, w_2) = (\pi_1^* g)(w_1, w_2) + f(\pi_1(\pi(w_1))).(\pi_2^* h)(w_1, w_2), \tag{4.1}$$

for $(w_1, w_2) \in T^2(M \times N)$, is a C^k Lorentz metric on $(M \times N, \mathcal{F} \times \mathcal{G})$, where $\pi : T(M \times N) \to M \times N$ is the natural projection.

Proof. Let $(p, q) \in M \times N$ be an arbitrary point. Let $v_1, \ldots, v_m \in T_pM$ be a vector basis of T_pM such that

$$(g(v_i, v_j)) = \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & -1 \end{pmatrix}$$

and let $u_1, \ldots, u_n \in T_qN$ be a vector basis of T_qN such that

$$(h(u_i, u_j)) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Since $(\pi_2)_* \circ (j_q)_{*p}(T_pM) = 0$ and $(\pi_1)_* \circ (j_p)_{*q}(T_qN) = 0$, the vectors

$$\begin{array}{lll} w_1 = (j_q)_{*p}v_1, & \ldots, & w_m = (j_q)_{*p}v_m, \\ w_{m+1} = (j_p)_{*q}u_1, & \ldots, & w_{m+n} = (j_p)_{*q}u_n \end{array}$$

form a vector basis of $T_{(p,q)}(M \times N)$ such that

$$(\bar{g}(w_i, w_j)) = \left(\begin{array}{c|c} (g(v_i, v_j)) & 0 \\ \hline 0 & f(p)(h(u_i, u_j)) \end{array}\right)$$

$$= \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & f(p) & & \\ & 0 & & & & \ddots & \\ & & & & & & f(p) \end{pmatrix}$$

Thus $\bar{g}$ is a C^k Lorentz metric on $(M \times N, \mathcal{F} \times \mathcal{G})$. □

Example 4.9. *Let $(N, \mathcal{G})$ be the differential space from Example 4.5. and $(\bar{M}, \bar{\mathcal{F}})$ be the differential space from Example 4.6.. Then the 2-form $\pi_1^*\bar{g}$, where $\bar{g} = d\pi_1^2 + d\pi_2^2 - d\pi_3^2$, is a Lorentz metric on $(\bar{M} \times N, \bar{\mathcal{F}} \times \mathcal{G})$. The pair $((\bar{M} \times N, \mathcal{F} \times \mathcal{G}), (\bar{M} \times N_0, (\pi_1^* g))$ is a differential space-time, where $N_0 := \{\frac{1}{n} \in \mathbb{R} : n \in \mathbb{N}\}$. It is clear that $\dim T_{(p,0)}(\bar{M} \times N) = 3, p \in \bar{M}$, and $\dim T_{(p,x)}(\bar{M} \times N) = 2, x \in N_0$. It follows that the boundary points are singular. The boundary points are also non-D_0 since any neighbourhood of a boundary point is disconnected. One can also check that the boundary points are g-metric. Therefore, the boundary points are non-D_0-g-metric singular.*

4.2 Differential Space of a Cosmic String

In the present section, we define a differential space which describes the external gravitational field of the cosmic string as a manifold with singularity.

Let us consider the Levi-Civita metric [17]

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \tag{4.2}$$

where, $t, z \in (-\infty, \infty), r \in (0, \infty), \theta \in [0, 2\pi - \Delta), \Delta \in (0, 2\pi)$. The metric (4.2) is interpreted as describing the external gravitational field of a straight cosmic string ([28, 13, 4]).

Let $C^{(4)} = \{p \in \mathbb{R}^5 : [(z^1)^2 + (z^2)^2]^{\frac{1}{2}} = az^4\}$, where $p = (z^0, z^1, z^2, z^3, z^4)$, $a \in \mathbb{R}$, be a 4-dimensional hypersurface immersed in $\mathbb{R}^5$. The set of singular points of this hypersurface has the explicit form

$$S = \{p \in \mathbb{R}^5 : p = (z^0, 0, 0, z^3, 0), z^0, z^3 \in \mathbb{R}\}.$$

Now , we will describe $C^{(4)}$ with its singular points as a finitely generated differential space.

Let $\tilde{P} := \mathbb{R}^2 \times [0, \infty) \times [0, 2\pi]$ be a "parameter space", and $\alpha_i : \tilde{P} \to \mathbb{R}, i = 0, 1, \ldots, 4$, real-valued functions parametrizing the hypersurface $C^{(4)}$ as follows

$$\begin{array}{rcccl} z^0 & = & \alpha_0(q) & := & t, \\ z^1 & = & \alpha_1(q) & := & \rho\cos\phi, \\ z^2 & = & \alpha_2(q) & := & \rho\sin\phi, \\ z^3 & = & \alpha_3(q) & := & z, \\ z^4 & = & \alpha_4(q) & := & a\rho, \end{array} \tag{4.3}$$

for $q = (t, z, \rho, \phi) \in \tilde{P}$. With respect to this parametrization of the hypersurface C^4, the metric (4.2) becomes

$$ds^2 = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2$$

or

$$ds^2 = -dt^2 + (a^2 + 1)d\rho^2 + \rho^2 d\phi^2 + dz^2 \tag{4.4}$$

By identifying (4.2) and (4.3), one obtains

$$r = \rho(a^2 + 1)^{\frac{1}{2}}, \theta = \phi(a^2 + 1)^{\frac{-1}{2}}.$$

Proposition 4.4. *Let $\tilde{\mathcal{P}}$ be the differential structure, on the parameter space $\tilde{P} = \mathbb{R}^2 \times [0, \infty) \times [0, 2\pi]$, generated by $\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$, i.e., $\tilde{\mathcal{P}} = \mathrm{Gen}\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$. The differential space $(\tilde{P}, \tilde{\mathcal{P}})$ is not Hausdorff.*

Proof. Indeed, none of the functions $\alpha_i, i = 0, 1, \ldots, 4$, distinguishes the points $(t, z, \rho, 0)$ and $(t, z, \rho, 2\pi)$. □

Let ρ_H be a Hausdorff equivalence relation on the space $\tilde{P}$ such that for any $q_1, q_2 \in \tilde{P}$, $q_1 \rho_H q_2 \Leftrightarrow \alpha_i(q_1) = \alpha_i(q_2)$, $i = 0, 1, \ldots, 4$. Let $\mathcal{P} =$

$\text{Gen}\{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4\}$ be the differential structure on $P := \tilde{P}/\rho_H$. The functions $\hat{\alpha}_i$, $i = 0, 1, \ldots, 4$, are defined by:

$$\hat{\alpha}_i([p]) = \alpha_i(p) \tag{4.5}$$

for $p \in \tilde{P}$.

For the sake of the following theorem, we need the following

Lemma 4.2. [21] *Let $(M, \mathcal{F})$ be a Hausdorff differential space with $\mathcal{F} = \text{Gen}\{f_1, f_2, \ldots, f_n\}$. The mapping $F : M \to \mathbb{R}^n$, $F := (f_1, f_2, \ldots, f_n)$, is a diffeomorphism onto the image $(F(M), (\varepsilon_n)_{F(M)})$.*

Proof. Since $(M, \mathcal{F})$ is Hausdorff, the mapping F is one-to-one. Therefore $F : M \to F(M)$ is a bijection.
The mapping F is smooth. In fact, for any $\omega \in \varepsilon_n$, since $\mathcal{F} = \text{Gen}\{f_1, \ldots, f_n\}$,

$$\omega(f_1, f_2, \ldots, f_n) \in \mathcal{F}.$$

Now, let $\sigma \in \varepsilon_n$. Then

$$\sigma \circ (f_1, \ldots, f_n) \circ F^{-1} = \sigma \in \varepsilon_n.$$

Thus F^{-1} is also smooth. □

Theorem 4.1. [6] *The differential space $(P, \mathcal{P})$ is diffeomorphic to the differential space $(C^4, (\varepsilon_5)_{C^{(4)}})$ which is a differential space of $(\mathbb{R}^5, \varepsilon_5)$.*

Proof. On the strength of Lemma 4.2., the mapping $\hat{F} : P \to \mathbb{R}^5$, $\hat{F} := (\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4)$ is a diffeomorphism of $(P, \mathcal{P})$ onto the image $(\hat{F}(P), (\varepsilon_5)_{\hat{F}(P)})$. In view of formulae (4.3) and (4.5), $\hat{F}(P) = C^{(4)}$. Therefore $\hat{F} : (P, \mathcal{P}) \to (C^{(4)}, (\varepsilon_5)_{C^{(4)}})$ is the diffeomorphism onto a differential subspace of $(\mathbb{R}^5, \varepsilon_5)$.
□

From Theorem 4.1., it follows that the pair $(P, \mathcal{P})$ represents the differential space of a cosmic string with singularity. Singular points of the space P are points of the form $[(t, z, 0, \phi)]$, where $t, z \in \mathbb{R}$, $\phi \in [0, 2\pi]$, since

$\hat{F}[(t,z,0,\phi)] = (t,0,0,z,0) \in S$. Let us denote the set of all singular points of P by P^0, and let $\hat{P} = P - P^0$. It follows that the mapping $\hat{F}|_{\hat{P}} : (\hat{P}, \mathcal{P}_{\hat{P}}) \to (C^{(4)} - S, (\varepsilon_5)_{C^{(4)}-S})$ is a diffeomorphism. But $(C^{(4)} - S, (\varepsilon_5)_{C^{(4)}-S})$ is the space-time manifold of the cosmic string, consequently the d-space $(P, \mathcal{P})$ can regarded as an extension of the space-time manifold $(C^{(4)} - S, (\varepsilon_5)_{C^{(4)}-S})$ of the cosmic string onto the singularity (see [6]).

Now, let's discuss the dimensionality of the differential space of a cosmic string with singularity, the existence of smooth vector fields on it, and the possibility to extend Lorentz metric onto singular points.

Proposition 4.5. [6] *Let $(P, \mathcal{P})$ be a differential space of a cosmic string with singularity. For any singular point $p \in P$,* $\dim T_pP = 5$.

Proof. The set of generators $\{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4\} \subset \mathcal{P}$ satisfies condition (1.3) of Theorem 1.2. at any singular point $p \in P^0$. In fact, for any $p \in P^0$,

$$\frac{\partial}{\partial z^i}|_p(a^2(z^4)^2 - (z^1)^2 - (z^2)^2) = 0,$$

$i = 0, 1, \ldots, 4$. Therefore, by using Theorem 1.2., the set $\{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4\}$ is differentially independent at $p \in P^0$. Hence, on the strength of Corollary 1.1., $\dim T_pP = 5$. □

Remark 4.1. *For any $p \in \hat{P} := P - P^0$,* $\dim T_pP = 4$. *Indeed, we showed earlier that the mapping $\hat{F}|_{\hat{P}}$ is a diffeomorphism of $\hat{P}$ onto the 4-dimensional manifold $C^{(4)} - S$.*

It is worthwhile to notice that smooth vector fields on the d-space $(P, \mathcal{P})$ exist. For example, the vector fields $V_0, V_1, V_2, V_3 : \mathcal{P} \to \mathcal{P}$, given by

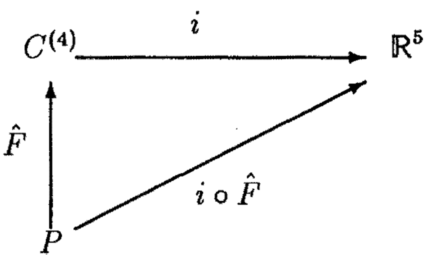
$$\begin{array}{lll} V_0(\hat{\alpha}_0) = 1, & V_0(\hat{\alpha}_i) = 0, & \text{for } i = 1,2,3,4 \\ V_1(\hat{\alpha}_1) = 1, & V_1(\hat{\alpha}_i) = 0, & \text{for } i = 0,2,3,4 \\ V_2(\hat{\alpha}_2) = 1, & V_2(\hat{\alpha}_i) = 0, & \text{for } i = 0,1,3,4 \\ V_3(\hat{\alpha}_3) = 1, & V_3(\hat{\alpha}_i) = 0, & \text{for } i = 0,1,2,4 \end{array} \tag{4.6}$$

are smooth. Let's note that we can not have a vector field V_4 such that $V_4(\hat{\alpha}_4) = 1$ and $V_4(\hat{\alpha}_i) = 0$, for $i = 0, 1, 2, 3$ at any point $p \in \hat{P}$ because

$$\cos\phi\ V_4(\hat{\alpha}_4) + \sin\phi\ V_4(\hat{\alpha}_2) = \frac{2}{a}$$

at any $p \in \hat{P}$.

Another striking property of the differential space $(P, \mathcal{P})$ representing the cosmic string with singularity is that the Lorentz metric can be "extended" to singular points. See [6]. Let $i : C^{(4)} \to \mathbb{R}^5$ be the immersion mapping, and $\hat{F} : P \to C^{(4)}$ the diffeomorphism appearing in the proof of Theorem 4.1.. It is easy to see that the mapping $i \circ \hat{F}$, defined in the following commutative diagram



is a diffeomorphism of P onto $i \circ \hat{F}(P)$. Therefore if we pull-back the Lorentz metric $\eta^{(5)} = (-1, 1, 1, 1, 1)$ from $\mathbb{R}^5$ to P, $\hat{\eta} := (i \circ \hat{F})^* \eta^{(5)}$, we obtain the Lorentz differential space of a cosmic string with singularity. In view of formula (4.3) and (4.5), one obtains

$$\hat{\eta} = -(d\hat{\alpha}_0)^2 + (d\hat{\alpha}_1)^2 + (d\hat{\alpha}_2)^2 + (d\hat{\alpha}_3)^2 + (d\hat{\alpha}_4)^2. \tag{4.7}$$

The metric $\hat{\eta}$ is smooth in spite of change in dimensionality. At regular points, (4.7) coincides with (4.2).

Another interesting example is the differential space of the Closed Friedman Universe, filled with radiation. Its metric has the form

$$ds^2 = a^2(\eta)(-d\eta^2 + d\chi^2 + sin^2\chi(d\theta^2 + sin^2\theta d\phi^2)), \tag{4.8}$$

where $a(\eta) = a_1 sin\eta$, $\eta, \chi, \theta \in (0, \pi)$, $\phi \in [0, 2\pi]$, and a_1 is a constant. Let

$$\tilde{P} := \{p : p = (\eta, \chi, \theta, \phi) \in (0, \pi) \times [0, \pi] \times [0, \pi] \times [0, 2\pi]\}$$

be a "parameter space", and let $\alpha_i : \tilde{P} \to \mathbb{R},\ i = 0, 1, \ldots, 4$, be the parametrizing functions, given by:

$$\begin{array}{lll} z_0 = & a_1 \eta & =: \alpha_0(p) \\ z_1 = & a_1 \sin\eta \cos\chi & =: \alpha_1(p) \\ z_2 = & a_1 \sin\eta \sin\chi \cos\theta & =: \alpha_2(p) \\ z_3 = & a_1 \sin\eta \sin\chi \sin\theta \cos\phi & =: \alpha_3(p) \\ z_4 = & a_1 \sin\eta \sin\chi \sin\theta \sin\phi & =: \alpha_4(p) \end{array} \tag{4.9}$$

with $p \in \tilde{P}$.

The d-space $(\tilde{P}, \tilde{\mathcal{P}})$, $\tilde{\mathcal{P}} = \text{Gen}\{\alpha_0, \alpha_1, \ldots, \alpha_4\}$, is not Hausdorff; indeed , $\alpha_i(\eta, \chi, \theta, 0) = \alpha_i(\eta, \chi, \theta, 2\pi),\ i = 0, 1, \ldots, 4$.

Let us introduce a Hausdorff equivalence relation ρ_H on the space $\tilde{P}$ such that for any $p_1, p_2 \in \tilde{P},\ p_1\ \rho_H\ p_2 \Leftrightarrow \alpha_i(p_1) = \alpha_i(p_2),\ i = 0, 1, \ldots, 4$. Let $\mathcal{P} := \text{Gen}\{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_4\}, \hat{\alpha}_i([p]) := \alpha_i(p)$, for $p \in \tilde{P}, [p] \in P$. As in the previous example, we formulate the following result.

Theorem 4.2. [7] *The d-space $(P, \mathcal{P})$ is diffeomorphic to the background manifold of the Closed Friedman world model with radiation.*

Proof. The proof is similar to that of Theorem 4.1..

Bibliography

[1] Aronszajn N., *Subcartesian and Subriemannain Spaces. Notices Amer. Math. Soc.* **14** *(1967), 111*

[2] Choquet-Bruhat Y., DeWitt-Morette C., *Analysis, Manifolds and Physics* (revised edition) (North-Holland, Amsterdam, 1982).

[3] Ellis G.F.R., Schmidt B.G., *Singular Space-Times, General Relativity and Gravitation,* **8** *(11) (1977), 915-953.*

[4] Gott III, J.R. *Astrophys. J. 288(1985), 422-427*

[5] Gruszczak et al *A Generalization of Manifolds as Space- Time models, J. Math. Phys.* **29** *(12), December 1988*

[6] Gruszczak et al *Quasi-regular Singularity of A Cosmic String, Acta Cosmologica-Fasciculus* **18** *1992*

[7] Gruszczak J., Heller M., *Differential Structures of Space-Time and Its Prolomgations to Singular Boundaries, International Journal of Theoretical Physics (1993),* **324**.

[8] Hawking S.W. and Ellis G.F.R., *The Large-Scale Structure of Space-Time.* Cambrige University Press, Cambridge, 1973.

[9] Heller M. et al *The Algebraic Approach of Space-Time Geometry, Acta Cosmologica- Fasciculus* **16** *1989*

[10] Heller M. et al *Local Differential Dimension of Space-Time, Acta Cosmologica-Fasciculus* **17**, *1991*

[11] Heller M. et al *Differential Spaces and New Aspects of Schmidt's b-Boundary of Space-Time, Acta Cosmologica-Fasciculus* **18**, *1992*

[12] Heller M., and Sasin W. *Structured Spaces and Their Application to Relativistic Physics, J.Math. Phys.* **36***(7), July 1995*

[13] Hiscock, W.A. *Phys. Rev.* **D23***(1985), 852-857.*

[14] Kowalczyk A. *Tangent Differential Spaces and Smooth Forms, Demonstratio Mathematica* **13***(4), 1980.*

[15] Larsen J. Chr. *Singular Semi-Riemannian Gemoetry, Journal of Geometry and Physics (1992)* **9**, *3-23*

[16] Marshall C.D., *Calculus on Subcartesian Spaces. J. Diff. Geo.* **10** *(1975), 551-573*

[17] Levi-Civita T., *Rend. Accad. Lincei* **26***(1917), 308*

[18] Multarzynski P, Sasin W. *Demonstratio Math.* **23**, *405-415.*

[19] Nakahara M., *Geometry, Topology and Physics*, Adam Hilger, Bristol and New York, 1990.

[20] Sasin W. *On Exterior Algebra of Differential Forms over A Differential Space, Demonstratio Mathematica* **19***(4), 1986*

[21] Sasin W., Zekanowski Z. *Demonstratio Mathematica* **20***(1987), 477-487*

[22] Sasin W. *The de Rham Cohomology of Differential Spaces, Demonstratio Mathematica* **22***(1989), 249-270*

[23] Sasin W., *Differential Spaces and Singularities in Differential Space-Times, Demonstratio Mathematica* **24***(1991), 601-634*

[24] Sasin W., Heller M., *Space-Time with b-Boundary as A Generalized Differential Space, Acta Cosmologica Fasciculus* **19**, *1993*

[25] Sasin W., Eledrisi Y., *On A Generalization of The Levi-Civita Connections of A Degenerate Metric, Acta Cosmologica Fasciculus* **21***2 1995*

[26] Sikorski R., *Differential Modules, Colloquium Mathematica* **24***(1971), 46-79*

[27] Sikorski R., *Wstep do geometrii rozniczkowej (Introduction to diffeen-rtial geometry).* In Polish. Panstwowe Wydawnictwo Naukowe (Polish Scientific Publishers). Warszawa 1972.

[28] Vilenkin A., *Gravitational Field of Vacuum Domain Walls and Strings, Physical Review D* **23***(4), 1981*

[29] Woodhouse N.M.J., *the Differential and Causal Structures of Space-Time, J. Math. Phys.,* **14***(1973), 495-501.*