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UNIVERSITY OF CAPE TOWN

Department of Statistical Sciences

Faculty of Commerce

**Options and Volatility Effects
in South Africa**

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A thesis prepared under the supervision of Associate Professor
D.J. Bradfield in fulfilment of the requirements for the Degree of
Doctor of Philosophy in Finance.

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To my parents

Abstract

This thesis examines and extends research into option price modeling in the South African market with a particular focus on its most important parameter, namely the volatility of the underlying. The primary objective of the thesis therefore is to offer an option price model that takes account of the conditions of the environment prevailing in South Africa.

The initial aim of the thesis is to describe the behaviour of the volatility in the South African market. This is achieved by conducting three empirical examinations using data from the South African Futures Exchange (SAFEX). The empirical examinations are partly based on standard methodologies (that have been modified in the thesis) and partly based on original methodologies adapted for the South African environment. The analysis establishes that the assumption of constant volatility is certainly inappropriate for the South African data over the period 1992 to 1996. Furthermore, the results reveal that systematic deviations are found in the implied volatility across strike prices and across expirations.

The incorporation of the systematic deviations of the implied volatility in the form of striking price biases and expiration biases are taken account of in a newly developed option price model for the South African options on futures market in the thesis. This proposed option price model is based upon the characteristics of the implied volatility option price model by Derman and Kani (1994). The primary departures from the Derman and Kani (1994) model take account of the problem of negative transition probabilities using an algorithm proposed in the thesis. Additionally, novel methods are developed in the thesis for the required inter- and extrapolation of option prices to ensure that the proposed option price model reflects the market information with greater realism.

The new proposed option price model with its implemented extrapolation and interpolation methods is assessed empirically and it is evident that the proposed option price model stands up well in the South African environment. Thereafter, the proposed option price model is used to establish return distributions which capture the market information in option prices. The findings of these computations give new insights into the applicability of the lognormal distribution assumed in many option price models (such as the Black and Scholes (1973) model). The established distributions show clear evidence that the lognormal

distribution underestimates the probability of large price declines (as occurred in October 1987).

In the light of the evidence regarding the deviations in the calculated return distributions and the finding of volatility biases, the effects of non-constant volatility are examined in detail in the thesis. The examination focuses on the incorporation of the full volatility surface and its effects on option price sensitivities and on portfolio management strategies. The effects on the option price sensitivities and on portfolio management strategies reveal substantial and impressive evidence that the incorporation of the full volatility surface is a necessary requirement as reflected in the proposed option price model for the South African environment.

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Chapter 1

1 Introduction

1.1 Options and Volatility Effects

Although options on commodities have been in existence as trading instruments since the middle ages or even before, the year 1973 signified the turning point for option markets world-wide. In 1973, for the first time, standardized option contracts were traded on the new Chicago Board Options Exchange (CBOE). The contracts traded on the CBOE were structured so that option contracts were standardized in their strike prices and expirations. The South African Futures Exchange (SAFEX) was established in 1990 with futures contracts initially being traded only, subsequently however options on futures were introduced on the 16 October 1992.

The year 1973 is also significant to the contribution of modern option pricing in that the first option price model with a closed form solution appeared in the literature. This option price model published by Fischer Black and Myron Scholes has subsequently become the standard pricing model for financial options. Further extensions and derivatives of the model have been plentiful with one of the most notable extensions being published by Robert C. Merton (1973) (who extended the model to options on stocks paying dividends). The importance of the model was honoured by the Nobel Price committee with the Nobel Price in Economics for Myron Scholes and Robert C. Merton (Fischer Black died in 1995) in October 1997.

Subsequently, the Black and Scholes (1973) model and its extensions have come under further critical scrutiny as the underlying assumptions have been questioned regarding their realism. The questions mainly draw attention to the assumption of a constant volatility across strike prices (striking price bias¹) and across expirations (expiration bias²). The

¹ The striking price bias is also known as volatility smile or volatility skew.

² The expiration bias or time-to-expiration bias is also known as term structure or maturity bias.

assumption of constant volatility is required for the assumed lognormal distribution of the Black and Scholes (1973) model. Hence, use of the lognormal distribution also becomes questionable because it assumes that the probability of a rise in market prices is the same as the probability of a fall in market prices and that large rises or falls in market prices are very remote. However, the substantial fall in market prices in October 1987 highlighted the shortcomings of the lognormal distribution theory. It is not surprising therefore that a wealth of international literature reports on significant departures from the constant volatility assumption (in a series of markets, for example US indices and stocks, British stocks, and a Dutch index)³.

The evidence of non-constant volatility is also recognized in many different option price models that attempt to either model the volatility process (e.g. stochastic volatility) or take the information of the volatility process directly out of the market. However, the attempt to model volatility produces additional problems related to the ability to hedge options with the underlying asset⁴. These problems are not evident in the latter type of the volatility models that estimate the volatility process directly from option prices. Therefore in the thesis the focus is on the option price models that incorporate the estimation of the volatility process directly from the market (which were initially published by Dupire (1994), Derman and Kani (1994), and Rubinstein (1994)).

The more recent option price models are not only able to price options according to the volatility process of the market but they are also independent of fixed assumptions about volatility and return distributions. This flexibility of the more recent option price models is particularly useful especially because of their ability to combine a variety of option types. Nowadays, exchange-traded options such as calls and puts are only one important part of options markets. Additionally, the so-called “second generation options” or “exotic options” are gaining more and more market share although they are mainly traded in over-the-counter (OTC) markets. Moreover, exotic options are priced according to the underlying conditions prevailing in the exchange-traded options and are tailored to investor specific needs. For example, the exotic options are tailored for the use in speculation strategies on the direction of the market (with high leverage), for hedging strategies against

³ See for example the studies by Rubinstein (1985a), Duque and Paxson (1994), and Heynen (1994) respectively.

market risk, or for strategies on specified parameters such as the volatility. However, many strategies can also be produced by exchange-traded options. Hence, the focus of this thesis is on exchange-traded options because the exchange-traded options and their underlying conditions (e.g. volatility) set the standards for the valuation of all other kinds of options.

Today, highly sophisticated option products are available on the market which allow many interesting considerations for portfolio construction. For example, it is possible with the advantages of options to build guaranteed portfolio products. These products guarantee the invested capital and a participation of the capital gains in the stock market. Other examples of advantages of options in portfolio management are portfolio insurance, the alteration of portfolio distributions, or the improved facilities for a fast and simple restructuring of portfolios. Although the advantages are numerous, the use of options has been relatively small in South Africa especially in the investment industry⁵. Consequently, the option market in stock index contracts in South Africa has been relatively illiquid in comparison to the major international markets of the world.

The main focus of this thesis therefore is on the understanding of the emerging derivatives market in South Africa. Developed option models and theories are mainly based on empirical results from the large US markets which do not suffer from anomalies, which may influence the option market in an emerging market. The motivation for this thesis also arises from the fact that implied volatility structures and their effects on option pricing have not been researched in a small market context like South Africa. Furthermore, nor have the effects of non-constant volatility been examined on portfolio management strategies with options. These topics are the focus of the thesis.

Apart from the literature review in Chapter 2 and the conclusion and the outlook outlined in Chapter 6, the investigations into options and volatility effects can be categorized into three main parts. The first part (Chapter 3) consists of the examination of the assumption of a constant volatility for the South African option on futures market. The assessment of the

⁴ Dupire (1994) defines the ability to hedge options with the underlying as “completeness”. He values this “completeness” of the highest value.

constant volatility assumption is emphasized in three different research methods that are based on methods in the international literature but have been modified for the South African environment. The second part (Chapter 4) includes some novel modifications to option price models as well as a completely new approach to option pricing in South Africa. Additionally (in Chapter 4), the focus is on the theoretical foundation of a new option price model by incorporating market information. Finally, the third part (Chapter 5) consists of two components to demonstrate the effect of non-constant volatility and the use of the proposed option price model. Firstly in Chapter 5, the effects of non-constant volatility are analysed for the option price sensitivities (i.e. delta, gamma, theta, and vega). Secondly in Chapter 5, the effect of non-constant volatility and the use of the proposed option price model is examined for portfolio management strategies with options. Hence, it is felt that the approaches adopted in this thesis are not only applicable to the South African environment but they will also give useful guidance to a more general environment worldwide. A more detailed structure of the contents of this thesis is given in the next section.

1.2 Organization of the Thesis

Throughout the thesis, a style has been adopted of concentrating on the main results and relegating some of the technical and analytical developments to appendices (approximately a quarter of the thesis). Moreover, empirical findings that merely confirm results in the Chapter are also relegated to the appendices. However, these appendices still contain very important results and they are therefore a very substantial part of the thesis. The organization of the thesis is presented in more detail below.

Chapter 2 contains three important literature reviews on the main subjects of the thesis. Firstly, a literature review is conducted on options on American and European options on stock indices as well as on options on futures. In order to achieve a better understanding of the subsequent research in the thesis and the special problem of American options on futures at SAFEX, the review focuses on the valuation process and the differences between

⁵ Firer and Israelsohn (1991) report on the relatively small use of options in the portfolio management. In addition, a more recent report by Bruce (1996) indicates little change.

American and European options. Secondly, an introductory literature review on volatility is presented. In this review, the main differences between historical and implied volatility are considered and computational concepts are reviewed. Moreover, the importance of the use of the implied volatility for recently developed option price models is discussed. Thirdly and finally, a background discussion on strategies in the portfolio management with options is included to assess the importance of options in modern portfolio management. This part concludes Chapter 2.

In Chapter 3, a description of the environment in South Africa is presented beginning with a historical introduction. Secondly, a description of the data used in the ensuing empirical research follows. The emphasis in Chapter 3 is on the analyses of the constant volatility assumption from Black and Scholes (1973) for the three indices, All Share Index (ALSI), Gold Share Index (GLDI), and Industrial Index (INDI). However, this analysis of the volatility requires an option price model for options on futures at SAFEX which is an appropriate modification of the Black and Scholes (1973) model for the South African environment. This so-called modified Black model is thereafter used to compute the implied volatility required for the tests of the constant volatility assumption. The first test results reveal substantial differences between historical and implied volatility and between implied volatilities with different times to expiration. The second descriptive approach attempts to detect a strike price bias and an expiration bias in the implied volatility. The results of the implementation of both methods (in the second test) is that a strike price bias and an expiration bias is evident for all three indices.

In addition to the two tests, a third nonparametric test based on Rubinstein's (1985a) methodology is conducted. This nonparametric test however requires modifications to take account of the available data in South Africa. The third test confirmed the results of the prior two tests statistically. Hence, in this latter test the realism of the constant volatility assumption is tackled more formally by conducting tests of significance. The rejection of the constant volatility assumption consequently also leads to a rejection of the modified Black model used for the computation of the implied volatilities because the assumption of constant volatility across strike prices and expirations is no longer valid as one of the underlying assumptions of the modified Black model.

In Chapter 4, an innovative option price model for options on futures is developed. The proposed model incorporates the market information in a flexible way (and is based on the

option price model by Derman and Kani (1994)). The innovative option price model solves the problem (and the influence) of malfunctions of the Derman and Kani (1994) model by using modified and new algorithms. Additionally, novel approaches to the inter- and extrapolation methods of implied volatilities are examined and tested. The innovative option price model for the South African option market is furthermore tested under different scenarios (of constant volatility and non-constant volatility) and is shown to perform well. Moreover, a method for speeding up the new option price model is proposed and applied with a substantial gain in speed, without loss of accuracy. Finally, the proposed option price model is implemented to establish return distributions ("implied distributions") with representative data for the South African market. The representative data is constructed in the form of a volatility surface from the results of prior research of the implied volatility in Chapter 3. Hence, the effect of non-constant volatility is presented graphically confirming that the implied distribution shows a higher probability of large market falls than expected from the lognormal distribution in the Black and Scholes (1973) model or the modified Black model respectively. In particular, it is important to note that the implementation of the whole volatility surface is essential and that different results would be obtained if, for example, only the implied volatilities across strike prices for one fixed expiration are incorporated.

Chapter 5 focuses on the effects of non-constant volatility on option price sensitivities and on portfolio management strategies with options. The examination of the effects of non-constant volatility on option price sensitivities show that the option price sensitivities differ substantially compared to constant volatility or even compared to constant volatility across expirations and non-constant volatility across strike prices. Moreover, the effects of non-constant volatility on portfolio management strategies also display considerable differences between option price models that are able to incorporate the whole volatility surface and that are not. Hence, the results support the need for an option price model that implements the whole volatility surface (such as the proposed model in Chapter 4).

Finally, Chapter 6 contains a summary of the conclusions obtained from the research in this thesis as well as giving direction for further research in this area. In particular, some final thoughts and new ideas are presented for the direction of further research.

Chapter 2

2 A Review of Options on Futures, Volatility, and Options in Portfolio Management

2.1 Introduction

This chapter reviews the pertinent theories necessary for the developments in the thesis. In particular, the concepts of options on futures, historical and implied volatility, and options in portfolio management will be reviewed.

The first aim of the review is to impart a clear understanding of American options on index futures. As American options on index futures are the only options which have been traded with sufficient volume⁶ and history since October 1992 at the South African Futures Exchange (SAFEX), a clear understanding of American options on index futures is therefore essential for the ensuing developments in the thesis.

Since the development of the Black and Scholes (1973) model, much research has been done on refining the accuracy of option valuation; in particular on American options. In this depth, the special characteristics of American options will be discussed and contrasted to European options. Furthermore, the differences between options on indices and options on index futures are reviewed. The review in section 2.2 is consequently structured as follows:

- European options on indices,
- European options on index futures,
- American options on indices, and
- American options on index futures.

Comparisons of the different types of options are also discussed.

The second aim of the review is to distinguish between historical and implied volatility. The conceptual differences between historical and implied volatility are discussed

⁶ American options on stocks for Anglo American, De Beers, Richemont, Sasol, SA Breweries, and Liberty were only introduced in September 1997 by SAFEX.

particularly with the view of the incorporation of volatility in option price models. Recent research in implied volatility option pricing methods conclude section 2.3.

Finally, a discussion on the background of options in portfolio management is found in section 2.4. Here, different strategies are proposed for the application of options in portfolio management. These discussions are divided into two main categories:

1. the time horizon of the investment (i.e. options are implemented in asset allocation strategies)
2. the risk of the investment (i.e. options are used as protection or speculation utility)

Additionally, the different uses of traded options and synthetic options as well as the advantages and disadvantages of the use of options in portfolio management are discussed. Section 2.4 ends with a concluding summary.

The first part of the review begins with an introduction to options on stock indices as well as to options on futures.

2.2 Options on Stock Indices and Stock Index Futures

The first standardized stock index options were established in the United States in 1983. Subsequently, they have become established around the world with trading of options on futures commencing in South Africa in October 1992 on SAFEX. As the South African market is somewhat unique, it is important to lay the foundation for the valuation of future options (which are appropriate for the South African case) prior to the empirical and theoretical research. American options on futures are based on a framework that is relatively complex however. The review therefore considers the development of the framework for American options on futures from the simpler European option and stock index option forms. Additionally, for the purposes of this review, it is assumed that the indices pay dividends (through the stocks in them) on which no taxes are paid (as in South Africa, for example Alexander (1996)).

First, an analysis of European options on stock indices is presented.

2.2.1 European Options on Stock Indices

European options on stock indices pave the way for American options on futures because their construction is simpler than American options on futures in two aspects. First, the European options can only be exercised at expiration. Second, index options are based on the stock index instead of more complex instruments such as futures. The aim of the brief analysis of European options on stock indices below is to discuss an appropriate valuation method for these options in the South African context.

European options on stock indices cannot be priced using the Black and Scholes (1973) model because this option price model is not able to price an option on an underlying (i.e. stock index) that pays dividends (as assumed here). Merton (1973) however proposes an extension to the Black and Scholes (1973) model for options on an underlying that pays dividends. He extends the Black and Scholes (1973) model by discounting the price of the underlying with the continuous compounded dividend yield, q , over the life of the option, $T-t$. Hence, the put-call parity of the Black and Scholes (1973) model, i.e.

$$c + Xe^{-r(T-t)} = p + S$$

changes to the Merton (1973) model, i.e.

$$c + Xe^{-r(T-t)} = p + Se^{-q(T-t)}$$

where the European call price, c , is compared with the European put price, p , under the consideration of the continuously compounded interest rate, r , and the strike price, X . Assuming a positive continuous dividend yield, the put-call parity reveals that the values of the call and the put option are different when compared to options without dividends.

Moreover, the valuation of European options on stock indices is the same as the valuation of options on stocks paying dividends. The effect of this valuation method is that the call option on a dividend paying index is worth less than an equal call option on an index that does not pay a dividend. The reason for the different prices is that the holder of a call option does not have the right to receive the dividend, instead the holder of the index receives the dividend. Under the assumption of an arbitrage-free condition⁷, the gain or the

⁷ The arbitrage-free condition means that neither call nor index holder have an advantage.

loss for the holder of the call has to be the same as for the holder of the index. However, the gain or the loss can only be equal for both if the call is worth less than a call option on an index that does not pay a dividend. This is also valid, *vice versa*, for European put options.

The assumption of a continuous dividend yield however is not appropriate for the majority of indices in the world. For example, the Japanese stock index (Nikkei), the German stock index (DAX) and the stock indices in South Africa⁸ only pay dividends in particular seasons. The application of the Merton (1973) model on such indices would lead to inaccurate pricing because of an incorrect assumption concerning the dividend distribution.

Hence, an alternative valuation approach to the continuous dividend yield for European options on stock indices is proposed by Hull (1993). This alternative approach estimates and calculates every dividend daily for each stock in the stock index until expiration of the option. Hull (1993) then deducts the sum of the estimated dividends from the spot index⁹ price, S , to obtain a new spot index price, S^* . Hence, the put-call parity of the Merton (1973) model changes to the alternative approach by Hull (1993), i.e.

$$c + Xe^{-r(T-t)} = p + S^*$$

This alternative approach would result in the same put-call parity as Merton's (1973) model if the dividends are distributed as a continuous dividend yield on the stock index. Hence

$$Se^{-q(T-t)} = S^*$$

If S^* is then substituted in the put-call parity by Hull (1993), the result is the same as the put-call parity by Merton (1973). However, a continuous dividend yield can only become evident if the stocks in a stock index pay the same amount of dividends through infinite intervals with the same length throughout the year. As noted, dividends are not paid in this manner in South Africa.

Consequently, the valuation of European options with the Merton (1973) model is unlikely to be sufficiently accurate enough for South African indices. Hence, the alternative approach by Hull (1993) would be preferable if options on stock indices were available.

⁸ Dividends on stocks included in the indices in South Africa are normally paid twice a year for each stock but with different dates of the dividend payment for each stock.

⁹ The spot index price or spot is interchangeably used with the index price.

The second type of options that pave the way to modelling American options on futures are the European options on stock index futures. They are discussed below.

2.2.2 European Options on Stock Index Futures

European options on stock index futures assist in the model development for American options on futures because they are simpler in construction to American options on futures from one significant aspect. The European options on stock index futures are only exercisable at expiration. The aim of the brief analysis of European options on stock index futures is to promote an appropriate valuation method for these options which could be used in South Africa. Moreover, the influence of the futures as underlying for the options is discussed to gain a clear understanding of the problems of modelling American options on futures in South Africa.

Options on stock index futures and options on stock indices do not only differ in their underlyings (index futures vs. index) but also in the required delivery when exercised. Options on stock index futures normally require the delivery of an underlying future whilst options on stock indices are normally settled in cash. Hence, if a call option on a future is exercised, the call holder gets a long position in the underlying future contract plus a cash amount equal to the difference between the current future price and the strike price. If a put option on a future is exercised, the put holder gets a short position in the underlying future contract plus a cash amount equal to the difference between the strike price and the current future price.

Besides the differences in delivery, the valuation of European options on index futures may also differ from the valuation of European options on indices because of the different underlyings. Black (1976) was the first to show the valuation differences using an analytical approach by modifying the Black and Scholes (1973) model (based on Merton's (1973) findings). As discussed in section 2.2.1, Merton (1973) extended the Black and Scholes (1973) model to accommodate dividends by assuming a continuous dividend yield, q . Black (1976) also assumes a continuous dividend yield, q , and additionally assumes a risk-free rate

of interest, r , to adjust the underlying price, S , in the Black and Scholes (1973) model. Through the adjustment, Black (1976) obtains a formula for the price of the future F :

$$F = Se^{(r-q)(T-t)}$$

Hence, Black (1976) replaces the price of the underlying, S , with the price of the future, F , in the Black and Scholes (1973) model. The put-call parity for the Black (1976) model differs slightly from the Merton (1973) model because of the inclusion of the risk-free interest-rate. That is

$$c + Xe^{-r(T-t)} = p + Se^{-q(T-t)} \quad \text{(Merton model)}$$

$$c + Xe^{-r(T-t)} = p + Fe^{-r(T-t)} \quad \text{(Black model)}$$

However, the difference in the valuation of both models disappears for European options on futures and European options on stock indices. The value of both European options will be identical for options on the same index, with the same strike price and with the same expiration date. The reason is that both options can only be exercised at the expiration date because of their European style. At expiration, the future and the index will be identical in value.

The equality of European future and stock index options can also be calculated analytically. If the definition of the future price is substituted into the put-call parity of the Black (1976) model, the put-call parity will become identical to the index price put-call parity of the Merton (1973) model.

However, the assumption of a continuous dividend yield is not appropriate for the indices in South Africa (as discussed in section 2.2.1). Hence, Hull's (1993) alternative approach to accommodate discrete dividends (instead of a continuous dividend yield) can also be applied for the valuation of European options on stock index futures. The only new parameter is the risk-free rate of interest so that the new future price¹⁰, F^* , can be derived from the new spot index price, S^* , (as defined in section 2.2.1):

$$F^* = S^* e^{r(T-t)}$$

This new future price, F^* , is also the same as the new spot index price, S^* , at *expiration* of the option. Consequently, no valuation difference appears for European future options

and European index options for Hull's (1993) alternative approach. The put-call parity in Black (1976) is however adjusted through the exchange of the futures price, F , with the new future price, F^* , to take account of the dividends appropriately.

The only difference between Hull's (1993) alternative approaches (in section 2.2.1 and 2.2.2) and the models by Merton (1973) and by Black (1976) is the assumption concerning the dividend distribution. The advantage of the alternative approach by Hull (1993) is the accurate valuation of options for every index. The disadvantage of the alternative approaches is however that all dividends have to be calculated for every stock in the index, which can be very time consuming.

In summary, the sections 2.2.1 and 2.2.2 argued that the differences between European options on index futures and European options on stock indices have no consequences for their valuation. These European options are valued using the same methods and have the same value. The discussion concerning the dividend distribution reveals that for indices with no continuous dividend payout only an option valuation model with discrete dividends is appropriate.

However, only options on futures with American exercise style are traded in the South African environment at SAFEX. Hence, an introduction to American options on stock indices is given below to lay the final foundation for the later review of American options on stock index futures.

2.2.3 American Options on Stock Indices

A review on American options on *stock indices* constitutes the final step towards modelling American options on *stock index futures* because only their underlying differs, but are identical in their exercise style. The American exercise style introduces problems for modelling options on *stock indices* as well as for options on *futures* (for example the analytical valuation). The aim of the analysis of American options on *stock indices* is

¹⁰ This can also be written as $F^* = (S - \sum_{i=1}^N D_i e^{-r(t_i-t)})e^{r(T-t)}$ or as $F^* = Se^{r(T-t)} - \sum_{i=1}^N D_i e^{r(T-t_i)}$ where the latter is proposed by Brenner, Courtadon, and Subrahmanyam (1985).

therefore to propose an appropriate valuation method that solves the early exercise problem of the American options. Moreover, a comparison between American and European options is presented to highlight their fundamental differences.

The difference between American and European *stock index* options is their exercise feature. Hence, a difference in their values only exists if one exercise feature is more valuable than the other. Consequently, the American option could only be worth more than an European option if it is optimal to exercise the American option early. The conditions for an early exercise are discussed below for an American *stock index* call and an American *stock index* put.

American Index Call

In general, early exercise of American index calls could be optimal if specific conditions are met. These specific conditions are discussed below:

Under the assumption of a continuous dividend yield, q , for the stock index, early exercise is only optimal if the continuous dividend yield, q , is greater than the risk-free interest rate, r (i.e. $q > r$). Simultaneously, the option has to be deep *in-the-money*¹¹ to be exercised optimally. However, the condition of a greater risk-free interest rate, r , than the continuous dividend yield, q , (i.e. $r > q$) prevails for almost every stock index world-wide, so that an early exercise of American index calls would be very rare¹².

The assumption of a continuous dividend yield however is not appropriate in South Africa (as discussed in the sections 2.2.1 and 2.2.2). Hence, it might be plausible to expect that on a few occasions during the year the value of the dividends is higher than the interest for the period up to expiration of the option contract (i.e. $q > r$). In this instance, early exercise of American index calls would become optimal if the option contract is simultaneously deep *in-the-money*.

¹¹ If the American option is deep *in-the-money*, $N(d_1)$ and $N(d_2)$ tend to the value of one in the Black and Scholes formula. The Black and Scholes formula therefore becomes $C = Se^{-q(T-t)} - Xe^{-r(T-t)}$ or $P = Xe^{-r(T-t)} - Se^{-q(T-t)}$. Additionally, it is assumed that $r > q$, so that the option price is higher than $S-X$ for the call and $X-S$ is higher than the option price for the put. Hence, it would be optimal to exercise the American put and not to exercise the American call early.

¹² One exemption could be Japan where very low interest rates may result in a profitable early exercise opportunity for American index calls.

In sum, American index calls could be exercised optimally under the right set of conditions. However, their early exercise is unlikely to be optimal in the usual market situation (i.e. $r > q$).

American Index Puts

In general, early exercise of American index puts can be optimal under the specific conditions discussed below:

The early exercise of American index puts is only optimal if American index puts are deep *in-the-money* and the risk-free interest rate, r , is greater than the continuous dividend yield, q , ($r > q$). The reason is analogous to American index calls. Moreover, the assumption of $r > q$ reflects the usual situation in financial markets so that an early exercise of American put options is more likely than that of American call options.

The assumption of a continuous dividend yield could also lead to a false valuation as in the case of American call index options. Hence, the exact dividend dates and dividend amounts must also be incorporated in the valuation model of American index puts.

In sum, American index puts could be exercised optimally under the right set of conditions. Furthermore, their early exercise is likely to be optimal in the usual market situation (i.e. $r > q$).

The discussion on the American index call and put options suggest that early exercise could be optimal. The optimal early exercise means that a profit can be gained by exercising the American options in contrast to non-exercisable European options. Consequently, the early exercise right adds a value to European call and put options. The value of American index options can therefore be calculated by summing up the value of European index options and the value of the early exercise right. Consequently, American options have a higher value than their European counterparts because profitable early exercise might be likely under the right conditions¹³.

The price of American index options can therefore not be obtained with formulas designed for European index options. In fact, no analytical solution has been produced for American options so far, but a few analytical approximations and numerical solutions exist.

¹³ An analytic illustration of the differences between European and American index options is given in Appendix 2.A.

A history of analytical approximations is presented in Appendix 2.B followed by an overview of the most important numerical methods in Appendix 2.C.

The number of analytical approximations are plentiful but none of these are without approximation errors. The most acceptable results for the American option valuation are obtained using numerical methods, but they are mostly more complex (or at least more time consuming computationally) than the analytical approximations. The improved accuracy of the numerical methods is however their main advantage compared to analytical approximations (whilst the disadvantage of a higher computer time consumption diminishes with faster and faster computers and new faster techniques for their computation¹⁴).

In conclusion, American options have a higher value than European options because of the early exercise right. The valuation of American options consequently has to be different to European options. However, no analytical solutions exist for the valuation of American options so that analytical approximations or numerical solutions have to be implemented. Due to the higher accuracy, American options should be valued with a numerical method. In particular, numerical methods such as the binomial and trinomial tree or even finite difference methods are recommended in this thesis (according to Appendix 2.B and 2.C) for the valuation of American options. However, the binomial tree is the preferred numerical method because it is more stable than the implicit finite difference method and simpler to construct than the explicit finite difference method or the trinomial tree.

The review of American options on stock indices as well as the prior reviews of European options on stock indices and on futures laid the foundation to review the American options on stock index futures below.

¹⁴ The area of faster computation techniques for numerical methods consists of a wealth of literature, for example Dempster and Hutton (1997) and Leisen and Reimer (1996).

2.2.4 American Options on Stock Index Futures

American options on stock index futures are based on a framework that is relatively complex. The review therefore considers the framework for American options on stock index futures from simpler forms of European *stock index* and *stock index future options* as well as of American stock index options. Consequently, the three main characteristics of American options on stock index futures are only briefly reviewed with reference to the prior discussions. These characteristics are:

1. the difference between *American* and *European* options on stock index futures
2. the difference between American *stock index options* and American *options on stock index futures*
3. the valuation of American options on stock index futures

Firstly, *American* and *European* options on stock index futures only differ in their exercise right. As already discussed in section 2.2.3, the opportunity of optimal early exercise during the lifetime of *American* options raises its option price above that of *European* options. Consequently, *American* options on stock index futures are worth more than *European* options on stock index futures (Ramaswamy and Sundaresan (1985) confirm this assertion).

Secondly, it is evident that American *options on stock index futures* differ not only notionally but also in value from American *stock index options*. The difference in value between them is a consequence of the different trading prices for *stock index futures* and *stock indices*. Hence, the *stock index future* price is normally higher than the *stock index* price assuming that the risk-free interest rate, r , is greater than the continuous dividend yield of the index, q (i.e. $r > q$). Consequently, the difference between *stock index future* price and strike price is greater than the difference between *stock index* price and strike price. The price of American calls on *stock index futures*, C_F , is therefore higher than the price of American calls on *stock indices*, C_S , whilst the price of American puts on *stock index futures*, P_F , is lower than the price of American puts on *stock indices*, P_S . The opposite is true, if the continuous dividend yield on the stock index, q , is greater than the risk-free interest rate, r (i.e. $q > r$).

Brenner, Courtadon, and Subrahmanyam (1985, 1989) show that the price difference between American *stock index options* and American *options on stock index futures* depend on the relationship between the risk-free interest rate and the continuous dividend yield of the stock index. Another important relationship is derived by Ball and Torous (1986) who present an arbitrage relationship for American *options on stock index futures*. This arbitrage relationship (confirmed by Whaley (1986b)) replaces the non-existent put-call parity for American *options on stock index futures*:

$$C - F + Xe^{-r(T-t)} \leq P \leq C - Fe^{-r(T-t)} + X$$

On the left side, the arbitrage relationship shows a lower bound and on the right side an upper bound for the American put. Ball and Torous (1986) refer to the left side as lower put-call parity and to the right side as upper put-call parity for American *options on stock index futures*.

Thirdly and finally, the valuation of American *options on stock index futures* is discussed. Similar to the valuation of American *stock index options* in section 2.2.3, no analytic solutions exist for American *options on stock index futures*. However, the valuation methods of American *stock index options* reviewed in section 2.2.3 are also applicable to American *options on stock index futures* with the exception that the underlying variable changes from the *stock index* to the *stock index future*.

Some approximation methods for American *options on stock index futures* have been proposed in the literature. For example, Barone-Adesi and Whaley (1987) develop a valuation method for American options on futures on commodities, similar to a method by Whaley (1986a). Shastri and Tandon (1986) apply the Geske and Johnson (1987) approach adjusted for the valuation of American options on S&P 500 Index futures and on German Mark futures in their research. Whaley (1986b) also implements a compound valuation approach that is related to the Geske and Johnson (1984) method.

However, these approximation methods (mentioned above) also suffer from the weaknesses of the approximation methods discussed in section 2.2.3. Consequently, numerical methods are recommended in the thesis for the pricing of American *options on stock index futures* because they are more appropriate. Examples of the application of numerical methods are found in Brenner, Courtadon, and Subrahmanyam (1985, 1989) who value American future options with finite difference methods like Schwartz (1977) and

Ramaswamy and Sundaresan (1985). Amongst the methods considered in the thesis the binomial tree has gained the most favour (as explained in section 2.2.3 and Appendix 2.C as well as in Geske and Shastri (1985)).

The primary result of the above discussion is that *American options on stock index futures* have no closed-form solutions that can be applied to price *American options on stock index futures*. Here, a numerical method like the binomial tree is recommended for the valuation of American options on stock index futures in this thesis.

In the ensuing review, the application of options in the portfolio management is discussed, but first a very important parameter in the pricing of options, namely the volatility, is reviewed below.

2.3 Volatility¹⁵

Volatility is one of the most important parameters in the option pricing process as the valuation of an option is very sensitive to the volatility estimate used in the option price model (Gemmill (1993)). The influence of volatility on the option price raises the question of how to compute volatility and which aspect of volatility to consider. In the ensuing discussion on volatility four aspects are considered:

1. future volatility
2. actual volatility
3. historical volatility
4. implied volatility

The discussion concerning volatility commences with the first category i.e. the future volatility.

¹⁵ Volatility and standard deviation are used interchangeably.

2.3.1 Future Volatility

The *future volatility* is the appropriate parameter for the option valuation as option prices should reflect future expectations. Nevertheless, for obvious reasons the *future volatility* cannot be computed directly. Different authors (e.g. Whaley (1982), Canina and Figlewski (1993)) discuss for example the appropriateness of historical volatility or implied volatility as a proxy for *future volatility*. The objective of the thesis is to value options directly on the available data in an arbitrage-free equilibrium. Consequently, the attempt to model *future volatility* with statistical methods (e.g. neural networks) is not the objective of this thesis and is therefore not considered in this thesis.

The review of volatility continues with the second category i.e. the *actual volatility*.

2.3.2 Actual Volatility

The *actual volatility* (as defined by Tompkins (1994)) is intended to capture the instantaneous *actual* price changes in the market. However, the measurement of the price change in practise can only be determined immediately after it has taken place. Hence, the measured volatility can no longer be referred to as *actual volatility*. Consequently, the *actual volatility* although theoretically appropriate cannot practically be measured.

The above discussed two aspects of future and actual volatility are theoretically appealing but have little practical use. Hence, they are not considered further in the ensuing analysis. However, the third and fourth categories, historical and implied volatility, are reviewed in depth in the two following sections 2.3.3 and 2.3.4. Here, computational methods will be discussed for the two remaining volatility considerations. Additionally under the heading of implied volatility, extensions of option valuation methods will be discussed that incorporate implied volatility as input estimates in option price models.

The historical volatility is considered below.



2.3.3 Historical Volatility

The review of *historical volatility* is differentiated into two main categories. First, the general computation method of *historical volatility* is discussed. This discussion includes the technical requirements of the computation, for example, the length of computation periods, the use of trading or calendar days, and the type of trading prices implemented (e.g. closing prices). Second, the specific computation problems of *historical volatility* for options on futures are addressed. Much emphasis lies on the consideration of which underlying should be used to compute *historical volatility* for options on futures (the future or the underling of the future).

Traditionally, *historical volatility* is measured as the annualized standard deviation of continuously compounded returns from the past. The continuously compounded returns are determined by the natural logarithm of the ratio of two consecutive asset prices ($\ln S_t/S_{t-1}$). The differencing interval between consecutive asset prices should be fixed intervals of time, for example, every day or every month. Additionally, the overall length of the computation period for the volatility has to be considered. Hull (1993), for example, asserts that the longer the computation period the higher the accuracy of the volatility estimate. However simultaneously, the longer the computational period the less representative is the time period for the lifetime of the option as argued by Tompkins (1994). In practice, periods as long as the time to expiration (or alternatively 20-trading-day periods) are implemented. However, Tompkins (1994) recommends that the return differencing intervals should consist of at least daily prices.

As indicated above a further consideration for the computation of *historical volatility* is the choice of whether trading or calendar days should be utilized. Fama (1965) finds that non-trading days as weekends exhibit nearly no volatility¹⁶. This result is confirmed by

¹⁶ Fama (1965) finds that the difference in variance between consecutive trading days and non-consecutive trading days is only 22 percent. This difference would have been just less than 300 percent if calendar days have the same variance as trading days. The mix of weekends with a three day difference and also holidays, which have only a two day difference, result in an expected value of a little less than 300 percent variance.

French (1980) in a similar study¹⁷. The conclusion from the results of both tests suggest that the use of trading days is preferable.

A further dimension concerning the computation of the *historical volatility* is the consideration of which particular trading prices should be used (the majority of research work uses closing prices for the calculation of the *historical volatility*). The impact of differing intra-day trading prices on the volatility computation was considered by Parkinson (1980). He computes the *historical volatility* based on both the highest and lowest prices in the estimation procedure (instead of using closing prices). Parkinson (1980) finds that his method needs 80 percent less data for the same accuracy of the *historical volatility* compared to methods which use only closing prices. Extensions to his method have subsequently been proposed by Beckers (1983) who employs high, low and closing prices, and by Garman and Klass (1980) who work with opening, closing, high, and low prices.

The computation of the *historical volatility* is complicated further by the consideration of which underlying, the future or the index, the *historical volatility* should be calculated on. Bearing in mind that with the future as underlying of the option and the index as underlying of the future it is uncertain which underlying to use for the computation of the *historical volatility*. Whaley (1986a) and Stoll and Whaley (1990) assert that the standard deviation of both index and future price changes is the same if the risk-free interest rate and the dividend payments on indices are considered. Their assertion would however only be correct if the market between the *index* and the *future* is always in equilibrium. However, the equilibrium condition between *future* and *index* is not always met; it failed, for example, in the market crash at the 19 October 1987. On this day, the *future* was traded considerably lower than its calculated fair value¹⁸ (for example, the S&P 500 *future* declined 29 percent whilst the S&P 500 *index* weakened only 20 percent). A further problem of the equilibrium condition is caused if future trading is stopped or future selling is prohibited for exchange regulation reasons¹⁹. These arguments lead to the conclusion that only the volatility computed from the

¹⁷ French (1980) confirms the direction of Fama's results with an observed difference of only 19 percent between consecutive trading days and non-consecutive trading days.

¹⁸ The calculation of the fair value can be found in section 2.2.2.

¹⁹ For example, the control commission of the Chicago Board Options Exchange is able to stop the selling of futures if a defined daily loss limit is reached.

future is appropriate for the *historical volatility* computation because only this *historical volatility* reflects the changes in the underlying of the options on futures precisely.

This conclusion concerning the use of futures for the *historical volatility* computation is supported by research on the lead-lag relationship between *future* and *index* markets. Stoll and Whaley (1990) present results indicating that the *future* leads the *index* on an intraday basis. For example, they find that the S&P 500 *future* leads the S&P 500 *index* by about five minutes on average. Herbst, McCormack, and West (1987) and others confirm this result in their research. Hence for intraday volatility computations, the volatility of *options on futures* should not be determined using the volatility of the *index* because this spot index lags behind all index derivatives (i.e. the future).

Moreover, various empirical studies show that the volatility of the *future* is higher than the *index* volatility (see for example Cornell (1985) and Brenner, Subrahmanyam and Uno (1989) for daily data, MacKinlay and Ramaswamy (1988) for minute-to-minute data). In these studies, the basis effect²⁰ is already taken account of. The results from these studies also support the conclusion that only the volatility of *futures*, instead of the volatility of the *index*, should be used as *historical volatility* for options on futures.

Finally, the implications for the use of futures for *historical volatility* computations are considered. The volatility of *futures* is more difficult to compute than volatility of the *index* because *futures* are simultaneously traded with different expirations. For example, often only *futures* are traded close to expiration, and *futures* only have a limited time series of trading prices. Hence, *futures* with different expirations may differ in their volatilities. However, the problem of the limited time series of trading prices may be solved by constructing a time series of only the *futures* closest to expiration. This means that the expiring *future* would be rolled over to the next *future*. However, this new time series of trading prices causes an additional problem because a jump between the two consecutive *futures* prices appears at every roll over date. The jump appears because the expiring *future* is identical²¹ to the *index* at expiration, whilst the *future* with the longer time to expiration is

²⁰ The basis effect requires that the *future* value must change daily if the *index* does not change and the interest value is not the same as the dividend sum. Hence, the basis is the difference between *future* and *index*. It is calculated using the risk-free rate of interest and dividend payments for the time to expiration. The basis declines when time proceeds so that the basis and the *future* must consequently change. The basis becomes zero and the future value equals the *index* value at expiration.

²¹ The basis is zero at expiration so that *future* and *index* are worth the same.

different in value from the *index*²². Consequently, each *future* and its *historical volatility* must be considered separately.

In summary, the *historical volatility* for options on futures should be computed from the underlying *future*. Furthermore, the calculation of the *historical volatility* from the *index* for options on futures could lead to severe errors (as mentioned above). Moreover, use of historical data alone is questionable because the past is not always representative of the future.

The methodology of implied volatility (reviewed in the next section) considers the use of the information in the market to price options.

2.3.4 Implied Volatility

This section discusses the characteristics of *implied volatility*. The structure of this discussion begins with a brief introduction on the *implied volatility*. Thereafter, different techniques for the calculation of *implied volatility* are discussed. A discussion contrasting the assumed *implied volatility* behaviour in option price models (e.g. Black and Scholes (1973)) and the *implied volatility* behaviour found in a wealth of international literature follows. Thereafter, implications concerning the *implied volatility* behaviour are discussed.

The volatility implicit in the option price is termed the “implied volatility”. The *implied volatility* represents the assessment of the average volatility over the time to expiration of the option by the market participants. In theory, an option price model such as the Black and Scholes (1973) model should be used to solve for the *implied volatility*, however an explicit analytical solution is not possible²³. Instead, the analytical problem of computing the *implied volatility* from the option prices can be handled with iterative procedures. Two possible iterative procedures are analysed in depth below.

²² The basis is above zero. In addition to this, to roll over a future to another future involves the risk of a roll over bias as Ma, Mercer, and Walker (1992) report.

²³ Approximations of analytical inversions are made by Brenner and Subrahmanyam (1988), Bharadia, Christofides and Salkin (1996), and Corrado and Miller (1996). These attempts generally suffer from

The *Newton-Raphson* method is a technique widely used for the computation of accurate *implied volatilities* (e.g. Gemmill (1993)). Another is the method of *Bisection*. It is essential for the further developments in the thesis to explain how *implied volatility* is computed. Hence, the *Newton-Raphson* technique and the method of *Bisection* are reviewed.

The input parameters required to calculate the *implied volatility* in both procedures are listed below:

- Traded Option Price
- Option Price Model
- Strike Price
- Price of the Underlying
- Time to Expiration
- Risk-free Interest Rate
- Dividend Payments until Expiration

Newton-Raphson

The *Newton-Raphson* method requires a reasonable volatility estimate to start the iterative process for the computation of the *implied volatility*. The accuracy of the *Newton-Raphson* procedure is normally very good with only very few iterations. However, Tompkins (1994) notes that the *Newton-Raphson* procedure is only applicable for European options because the procedure depends on the linearity of the “option price/volatility relationship” (i.e. option price with respect to the volatility). This option price/volatility relationship is linear for European options but it is non-linear for American options because of their early exercise right.

Bisection

The method of *Bisection*²⁴ yields accurate results for American and European options because it does not depend on the linearity of the option price/volatility relationship. An

approximation errors compared to the results of iterative procedures and they are consequently not considered here!

²⁴ The method of *Bisection* works with two estimates of volatility (as described by Tompkins (1994)). One volatility estimate is chosen as the high estimate, σ_{high} , corresponding to an option price, P_{high} , above the market price. The other volatility estimate is chosen as the low estimate, σ_{low} , corresponding to an

additional advantage of the method of *Bisection* over and above the *Newton-Raphson* method is that it is less sensitive to the initial volatility estimate. However, the method of *Bisection* is slightly slower computationally than the *Newton-Raphson* method but it has similar accuracy (if compared for European options) according to Tompkins (1994).

The slightly slower computational speed of the *Bisection* method by contrast to the *Newton-Raphson* method is rather insignificant when one trades off the advantage that it is applicable to European and American options. As a consequence, the method of *Bisection* is adopted for the determination of *implied volatility* in this thesis.

The computation of the *implied volatility* is required to perform tests on the behaviour of *implied volatility* in the South African option market. Currently, evidence in the international literature reveals that the Black and Scholes (1973) model and its assumption of a constant *implied volatility* across strike prices and across expirations have come under critical scrutiny (mainly because the constant volatility assumption has been questioned regarding its realism). In particular, substantial international literature shows evidence that *implied volatility* differs for options with different strike prices (see for example Latané and Rendleman (1976), Chiras and Manaster (1978), and Schmalensee and Trippi (1978)) and is therefore not constant. Additionally, MacBeth and Merville (1979) argue that the Black and Scholes (1973) model overprices *out-of-the-money* options and underprices *in-the-money* options which is found *vice versa* by Black (1975) (using a different period).

More recently, a rigorous paper by Rubinstein (1985a) not only confirms these findings of the striking price bias in the US but additionally differentiates between the striking price bias and the expiration bias. Rubinstein (1985a) finds that the observed striking price bias and the expiration bias change from period to period. Additionally, Rubinstein (1985a) finds that both biases are statistically significant.

Shastri and Tandon (1986) also find a striking price bias and an expiration bias for S&P 500 index future options in the period February 1983 to September 1984. They present a similar result for options on the German mark future in the period February 1984 to

option value, P_{low} , below the market price. The *implied volatility* is then estimated through a comparison with the traded market option price, P_{Market} . That is:

$$\text{Implied Volatility} = \sigma_{low} + (\sigma_{high} - \sigma_{low}) (P_{Market} - P_{low}) / (P_{high} - P_{low})$$

December 1984. However, Shastri and Tandon (1986) argue that these biases, used in a hedging strategy, cannot cover the transaction costs. Rubinstein (1994) confirms Shastri and Tandon's (1986) results for the time before 1987 but he simultaneously refers to the radical changes of the biases after 1987 where the transactions costs are covered. He finds that the striking price biases have grown over the period 1987 to 1992. A clear indication of the grown striking price biases can be seen in the development of the *implied volatility* range²⁵. The *implied volatility* range has grown from around 1.5 percent in 1986 to around 6.5 percent for *implied volatilities* in 1992. Consequently, Rubinstein (1994) assumes that the transaction costs covering striking price bias and the expiration bias have only appeared since 1987 as a result of the stock market crash in 1987.

The implications and consequences of biases in the *implied volatility* are very important for the pricing of options as the violation of the constant volatility assumption underpinning the Black and Scholes (1973) model casts doubt on the accuracy of the model. Clearly, if computed option prices are inaccurate, investors could lose substantial sums of money. Adaptations that attempt to overcome both striking price biases and expiration biases through weighting schemes²⁶ still do not adequately address the problems of the constant volatility assumption. Hence, the accuracy of these adaptations is still questionable.

Evidence of non-constant volatility has not been restricted to the US markets alone. To date, patterns of striking price biases and expiration biases have also been established in markets other than the US, for example, the studies by Heynen (1994) for the Dutch European Options Exchange Index and by Duque and Paxson (1994) for British stocks. Duque and Paxson (1994) examine the striking price bias for equity options and index call options at the London International Financial Futures and Options Exchange (LIFFE). They discover that the striking price bias is primarily caused by a relatively high *implied volatility*

²⁵ For example, the *implied volatility* range is calculated as 1.5 percent when the highest *implied volatility* is 18 percent and the lowest is 16.5 percent. The *implied volatility* range is calculated over a -9 percent to +9 percent striking price range. The striking price range means that only the *implied volatilities* that are below the *at-the-money* striking price - 9 percent or above the *at-the-money* striking price + 9 percent are considered. For example, if the *at-the-money* striking price is 100, then only striking prices between 91 and 109 are considered.

²⁶ Different weighting methods are described in Latané and Rendleman (1976), Chiras and Manaster (1978), Schmalensee and Trippi (1978) or Whaley (1982). Their results reveal that weighted *implied volatility* is superior to historical volatility for volatility forecasting whereas Canina and Figlewski (1993) find evidence contrary to this hypothesis.

for *in-the-money* call options. Further research by Taylor and Xu (1994) find significant expiration biases for currency options and Heynen, Kemna, and Vorst (1994) conclude that EGARCH gives the best description of expiration biases. To the author's knowledge, no such studies have been conducted in the South African environment (bearing in mind that the standardized option market has only been in existence since 1992²⁷).

In general, all presented evidence of biases in *implied volatilities* clearly contradicts the assumptions of the Black and Scholes (1973) model. Their assumption of a lognormal distribution of prices over time suggest that *implied volatilities* must be identical for options across strike prices and across expirations. Hence, other option price models are developed that are able to incorporate such biases, for example, stochastic volatility models. Firstly, Hull and White (1987) derive a solution for a call option on an asset with stochastic volatility. Besides their model, a model approach for discontinuous jumps in the asset price has already been presented by Merton (1976). However, both models suffer from a large disadvantage compared to the Black and Scholes (1973) model because the option valuation in both models is no longer preference-free. Consequently, the replication and the hedging of options in both models are currently impossible with known securities and derivatives (Dupire (1994)).

In particular, substantial evidence of biases in *implied volatilities* in the literature (e.g. Rubinstein (1994), Shimko (1993)) has lead to new option price models attempting to take account of the biases. As a consequence, Rubinstein (1994) proposes a new model, the "Implied Binomial Tree". He constructs an implied volatility process into the implied binomial tree with the help of the binomial probabilities derived from the option market. This process allows to create a volatility "smile" (i.e. non-constant volatility across strike prices) for a given expiration by first inferring the probability process and then building the binomial tree.

While Rubinstein's approach focuses only on the incorporation of the striking price bias, Derman and Kani (1994) use a different tree method, the "Implied Volatility Tree", to

²⁷ A study on the valuation of calls on gilts and warrants on stocks was conducted by Plastrier, Thomas, and Affleck-Graves (1986) who apply the Black and Scholes (1973) model. Their conclusions suggest that

incorporate the striking price bias and the expiration bias. Their method is different to Rubinstein (1994) because they use the option prices from the market to construct the binomial tree. Thereafter, they are able to obtain the probability process directly from the tree. Dupire (1994) follows the same idea as Derman and Kani (1994) and works with a trinomial lattice approach to incorporate the implied volatility process for the striking price bias and the expiration bias.

All three approaches assume that risk-neutral probabilities can only be derived from the prices of European options because European options are well-defined with their ending nodal probabilities. Consequently, Rubinstein (1994) states that his approach does not fully cover the stochastic process contained in the price of American options because of their different interior and ending nodal probabilities. However, the information derived from European option prices can be used to infer a probability process which can then be used for the valuation of American options and path-dependent options.

More recently, Jackwerth (1996) found a solution to the problem that the probability process can only be inferred from European option prices. He expands the "Implied Binomial Trees" approach from Rubinstein (1994) with a new approach referred to as "Generalized Binomial Trees". The "Generalized Binomial Trees" approach is able to recover the probability process from all kind of options (American, European, and Exotic options). However, this model uses a technique similar to Rubinstein (1994) where the probability process is inferred from the option prices and then the binomial tree is calculated.

A further discussion of the implied volatility models and their characteristics follows in Chapter 4 where an implied volatility model is proposed for the South African environment. A review on the prospects for options in portfolio management follows in the third and final part of this chapter below.

the Black and Scholes (1973) model gives good results on average but they allude to the fact that major questions revolve around the volatility input.

2.4 Background of Portfolio Management with Options

The academic literature has been fairly silent on the topic of options in portfolio management. This section does not therefore constitute a review in the usual sense, but rather takes the form of a background discussion. However, the discussion attempts to structure the common practical strategies of options in portfolio management in an academic framework.

This background discussion is considered here to establish the foundation for Chapter 5. It is within this context that the background discussion on strategies of portfolio management with options is placed. The field of option strategies in portfolio management is nevertheless vast hence this background discussion is unlikely to be fully comprehensive but it gives the necessary background for the thesis.

Firer and Israelsohn (1991) argue that the usefulness of options has not been fully recognized by fund managers for the purpose of portfolio management in South Africa. This disclosure appears to have changed little over the subsequent years as articles in the finance press indicate (for instance by Bruce (1996)). Hence, the incorporation of option strategies and their uses in the modern portfolio management will be reviewed here (and considered in the South African context in Chapter 5). In the ensuing review, portfolio strategies with options are discussed in order to lay the foundations for the further developments in the thesis.

Options in portfolio management, particularly index options, are based on the foundations of the portfolio theory. Markowitz (1952, 1959), with his ideas on portfolio selection, puts the main emphasis on the risk-return relationship of assets and the diversification of risk to reduce the overall portfolio risk. Sharpe (1963, 1970) simplifies the computation of the portfolio selection problem by developing a model where the assets are dependent on a market index. The use of options on a stock market index assumes that the stock market index is a good proxy for the true market portfolio, especially when the stock market index is the benchmark for the performance of a portfolio.

Further development of portfolio theory produced the Capital Asset Pricing Model (CAPM) by Sharpe (1964), Lintner (1965), and Mossin (1966). The CAPM differentiates

risk into asset specific risk (unsystematic risk) and market risk (systematic risk). The unsystematic risk can be diversified away with the diversification process described by Markowitz (1959), so that only the systematic risk is relevant. Whilst the theories of capital markets suggest systematic risk cannot be avoided, the systematic risk can however be reduced by strategies which involve options (or as Beighley (1994) writes, options can be used to alter return distributions of portfolios). Hence, an individual relationship between systematic risk and return can be constructed very easily with the help of options.

A brief discussion of options in different aspects of portfolio management follows. The review differentiates between *strategic* asset allocation (in section 2.4.1) and *tactical* asset allocation (in section 2.4.2). In particular, options utilized for speculation (in section 2.4.3), portfolio hedging (in section 2.4.4), and portfolio insurance (in section 2.4.5) with purchased options (in section 2.4.5.1) and synthetic options (in section 2.4.5.2) are discussed in the ensuing sections.

2.4.1 Options in Strategic Asset Allocation

Strategic asset allocation is defined as a strategy to attain investment targets in the long run. The simple investment strategy of buying and holding an equity portfolio (whose benchmark is a stock market index) can be replicated with the combination of options on the index and a cash deposit. In addition, a portfolio manager can pre-determine a minimum value for the portfolio at the end of her/his investment horizons. Depending on the chosen minimum value, the portfolio manager may determine the participation ratio²⁸ of the option strategy compared to a fully invested portfolio in stocks (if the stock market index rises).

The above mentioned asset allocation approach²⁹ has differing advantages. One advantage is the ability to guarantee a minimum price for a portfolio; another is that

²⁸ For example, a participation ratio of one means that the option holder participates in the same way in a rising market as the holder of the equity portfolio.

²⁹ A second asset allocation strategy is the capped return strategy where a call option with a very low strike price (i.e. well below the actual underlying price) is bought and simultaneously a call option with a high strike price (i.e. above the actual underlying price) is sold as cap. The best return for this strategy is obtained when the underlying asset price is at the strike price of the cap option at expiration. However, the capped return strategy has one main disadvantage compared to the guaranteed portfolio strategy. The return is capped so that the capped return strategy only has a very small return in bull markets while other strategies can profit fully from the rise in prices.

participation rates could be higher than that of a “normal” equity portfolio. In general, additional advantages of options in strategic asset allocation are lower transaction costs and simplified market access because a whole index can be bought with one transaction.

A variety of disadvantages related to the use of options in strategic asset allocation do however exist. The availability of only short-term options is one problem. Another is that only option expirations below one year are normally traded with sufficient liquidity at option exchanges. Further disadvantages³⁰ of options in strategic asset allocation are related to rolling-over effects from one to another option³¹, margin requirements, margin management, and sometimes position-limits in option contracts (imposed by exchanges). For example, position-limits in option contracts are sometimes set to prevent the domination of one market participant. Hence, large trades of option contracts are nearly impossible at exchanges, especially in illiquid markets like South Africa.

Some of the above mentioned problems of options in the strategic asset allocation can however be solved by synthetically replicated options. The synthetic replication of options with an underlying asset and cash position is discussed in depth later in the section 2.4.5.2 under the heading “Synthetic Options”.

However, the main disadvantage of options are their costs (for exchange-traded as well as synthetically replicated options). The option premium reflects the cost of options in the strategic asset allocation. These costs of the use of options can be reduced, but not without cutting the profit at an upper level. For example, the costs for the above-mentioned guaranteed portfolio strategy with a purchased call can be reduced with the writing (i.e. selling) of a put. Nevertheless, the portfolio manager takes an additional risk for the put premium received because the minimum level of the portfolio is no longer guaranteed with this strategy.

³⁰ The above mentioned guaranteed portfolio strategy is selected to make an important disadvantage of the exercise style of options clear. The cash deposit in a guaranteed portfolio is normally fixed until expiration of the option so that it would be reasonable to buy European style call options on futures (with the same expiration). However, only American style options on futures are traded at SAFEX. These American style call options on futures have the disadvantage of being more expensive than European options on futures (as described in section 2.2). Hence, the portfolio manager is forced to buy the more expensive options with the advantage of early exercise, which he/she does not require. Consequently, the outcome of the option strategy is not as advantageous as it would be with European call options on futures.

³¹ Rubinstein (1985b) finds for a sequel of options, which are rolled over from one to another, that their return is below the return of one long-term option.

In general, written option strategies can be used further in combination with an existing equity portfolio, in particular in covered option-writing strategies (e.g. a put is normally covered by a cash deposit). Covered written options are normally used to reduce the risk of the portfolio and to profit from the option premium received. The return of the different strategies for written calls or puts depend on the relation of strike price to the price of the underlying asset. Gladstein, Merton, and Scholes (1978) show for a covered call option writing strategy that the higher the premiums received, the higher the returns will be. Gladstein, Merton, and Scholes (1982), find in a similar study, that written put options display the same results as written call options. The covered call and covered put option strategies described by Gladstein, Merton, and Scholes (1978,1982) do not have an overall return as high as from holding the underlying. Nevertheless, the return is significantly higher than on low risk securities (i.e. fixed income securities). In summary, strategies with written options have a significantly lower risk exposure for portfolios than for the equity portfolio alone. Simultaneously, the return on written options is below the return of a portfolio without options.

In sum, every portfolio manager should consider whether an option strategy (for example strategic asset allocation) holds more advantage than a simple market investment in equity.

2.4.2 Tactical Asset Allocation

Tactical asset allocation strategies take heed of the market timing of the investment process. Brenner (1990) asserts that the objective of tactical asset allocation is to enhance returns by varying the allocation of funds among asset classes. Options can for example be used in tactical asset allocation by influencing the beta of a portfolio. For instance, a strategy with purchased call options (or written put options) is able to increase the portfolio beta whereas a strategy with bought put options (or written call options) can decrease the portfolio beta. Hence, it is possible to construct a portfolio with a beta of zero and a rate of return, that is equal to the risk-free rate of interest, with such strategies³².

³² It is assumed in the ensuing discussion that taxes and transaction costs do not exist.

Cox and Rubinstein (1985) propose that betas of options can be calculated using the underlying asset beta:

$$\beta_{\text{Option}} = \beta_{\text{Asset}} \times \Omega_{\text{Option}}$$

where Ω_{Option} is the elasticity of the option. The elasticity is

$$\Omega_{\text{Option}} = \frac{\partial \text{Option}}{\partial \text{Asset}} \times \frac{\text{Asset}}{\text{Option}}$$

where $\frac{\partial \text{Option}}{\partial \text{Asset}}$ is the delta of the option, "Option" is the price of the option and "Asset" is the price of the asset (i.e. the underlying). The option beta is normally several times larger than the asset beta. It is also more volatile than the asset beta, because of its dependency on the delta of the option. However, as Cox and Rubinstein (1985) argue the beta of the option can simply be added to the beta of a portfolio because the portfolio beta is the value-weighted average of betas of the individual assets in the portfolio.

An important parameter in the calculation of the beta of the option is the delta. The delta of the option depends on the relationship between asset price and strike price. The more the call option is *in-the-money* the more the delta moves closer to its maximum of one (for exchange-traded options). Consequently, the elasticity of the option would become larger and the beta of the option would consequently become larger. The option price of a call option however rises simultaneously the more the call option becomes *in-the-money*. In addition, the effect of becoming more *in-the-money* is greater on the option price than on the delta. Consequently, the option beta decreases the more the call option becomes *in-the-money* and it increases the more the call option becomes *out-of-the-money*³³. Hence, the option beta is normally above one and very sensitive to changes in the delta.

The sensitivity of deltas and therefore the option betas vary substantially for *at-the-money* options³⁴. Hence, the portfolio manager is required to recalculate the beta of the managed portfolio every time large fluctuations occur in the underlying asset of the option (because of the strong effects on the option delta and on the option beta). Additionally, the beta of

³³ For example, under the assumption of an asset beta equal to one, a beta value equal to one for a call option can therefore only be obtained if the strike price is zero, because then the delta would be equal to one and the asset price and the option price would be the same (and *vice versa*, when the option is a put option).

the portfolio requires recalculation daily because of the diminishing time to expiration effect for the option delta and the beta of the option.

The effects of the delta and beta changes require additional transactions to rebalance the portfolio. Hence, the more the changes in the option delta and consequently in the beta of the option occur respectively, the more transactions are necessary so that transaction costs for option strategies will consequently increase rapidly.

Tactical asset allocation with options requires a continuous rebalancing of the option portfolio to prevent substantial sources of errors. This approach is very elaborate and tedious for the portfolio management. A simpler use of options in the portfolio management is presented below.

2.4.3 Options as Instruments for Speculation

Speculation with options is conceivable in different ways. One way would be to invest the whole portfolio value in one option strategy. For example, purchasing as many call options as possible so that the option premium for the calls equals the portfolio value. The risk exposure is very high for such a strategy; i.e. all the money invested will be lost if the underlying asset is not *in-the-money* at expiration. On the other hand, if the option rises above the striking price (including the option premium paid), the profit can be substantially larger compared to a fully invested equity portfolio. This strategy would be profitable if the market rises but it has no insurance for market declines. Alternatively, a full investment in puts would be profitable if the market declines (below the strike price minus the paid option premium) but the risk exposure again remains very high.

It is also possible to write options, calls if the market is not expected to rise or puts if the market is not expected to fall. Both writing strategies also contain a very high risk exposure, in particular when the options are written without cover. Characteristic of uncovered option writing or buying strategies is their unlimited loss potential. Such positions should therefore be handled very cautiously.

³⁴ *In-the-money* and *out-of-the-money* option deltas only vary substantially when the time to expiration diminishes. Instead, *at-the-money* options deltas scarcely change when the time to expiration diminishes.

These above-mentioned strategies are by no means comprehensive, but are just examples of how it is possible to speculate on predicted trends in the market. Options offer a variety of profit potentials. For example, portfolio managers can not only bet on the direction of the underlying market with options but they can also bet on the trend of volatility for example.

All speculative option strategies however include risk. The assumption concerning the market is that the market is efficient and that a free lunch or windfall profit is not conceivable. Nevertheless, the advantage of options lies in their flexibility to profit from the market in a variety of ways in contrast to an investment in the underlying asset. It is also quite feasible to avoid an exposure to a total loss (as described above) by mixing the exposure and returns of a portfolio investment. Positions can also be created that have a risk and return relationship suitable for every single portfolio manager.

The speculation with options is a high risk way of using options in portfolios. An approach to reduce the risk in portfolios would be to use options for hedging (as reviewed below).

2.4.4 Options as Hedging Instruments

Hedging is defined as the process of reducing the risk of the portfolio against any kind of market movement by simultaneously forgoing a potential profit. The tactical asset allocation discussed above can also be viewed to some extent as a hedging method because the reducing of the beta has a similar consequence as hedging.

However, the definition of hedging is not always used correctly. A differentiation must be made between hedging and the purchase of a put option to protect the portfolio. The purchase of the put protects the risk of the portfolio only at expiration of the put option. Nevertheless, the portfolio participates fully in a rise of the market. Hedging by contrast requires a continuously reduced risk but also forgoes all profits if the market rises. The objective of reducing the risk continuously can become very important for a portfolio manager if, for example, the fund liquidity diminishes. Hence, the portfolio manager would have to sell parts of the portfolio contents instantly. In such a situation, a purchased put

may not fully cover a market decline whilst a hedge covers fully the losses. Hence, the “so-called hedge” related to a purchase of a put option is not a hedge in the sense of the hedge definition because the risk of selling the underlying is not fully covered by the put option.

After the differentiation between put option and hedge is explained, two hedging techniques are reviewed. First, the “delta hedging” technique is reviewed and its advantages and disadvantages explained. Second, the “delta-gamma hedging” technique is reviewed and compared to the delta hedging technique.

The delta hedging technique is based on above mentioned delta (see section 2.4.2). Hence, the delta of an option, Δ , is the rate of change of the option price with respect to the price of the underlying asset

$$\Delta = \frac{\partial \text{Option}}{\partial \text{Asset}}$$

Options gaining from a rising market have a positive delta (e.g. purchased call and sold put options) whilst options gaining from a declining market have a negative delta (e.g. purchased put and sold call options). An additional definition is that the delta of the underlying asset is always equal to one (if it is no option). The delta hedging technique therefore simply sums the deltas of the underlying assets (e.g. equities) and the deltas of the options together and attempts to balance the overall delta to zero.

Hence, it is necessary to calculate the required number of options to hedge an existing portfolio. For example, the required number of puts, N_{Puts} , can be calculated as

$$N_{\text{Puts}} = \frac{P_v}{\Delta \times N_{\text{Contract}}}$$

where P_v is the portfolio value and N_{Contract} is the number of shares or indices in every contract of the put option.

The number of puts is however not constant because the delta of the option changes over time as well as in relation to the underlying asset price. Hence, the hedge (i.e. the portfolio of options and underlying assets) has to be rebalanced with diminishing time to expiration of the option as well as with every price change of the underlying asset. However, the rebalancing of the portfolio after every delta change causes transaction costs that rises with the frequency of the portfolio rebalancing. Besides the problem of transaction costs, a second problem arises in the form of the possible failure of delta hedging. For example, the

crash from 1987 showed that delta hedging is very risky if the delta changes erratically in short periods.

The delta hedging technique can be improved if the sensitivity of the delta to the underlying asset price is taken account of. Hence, the knowledge of this sensitivity helps to prevent delta hedges having a large changing risk with respect to the underlying asset price. The risk of such large delta-changes can be identified using the gamma, Γ (for example Hull (1993)). Gamma is the change of the delta with respect to the price of the underlying asset

$$\Gamma = \frac{\partial^2 \text{Option}}{\partial \text{Asset}^2}$$

New hedging techniques therefore include gamma as, for example, the delta-gamma hedging technique. The aim of the delta-gamma hedging technique is to obtain a zero delta and a small gamma, which also tends to zero in order to be protected against large price changes in the underlying asset. The advantage of the incorporation of gamma into the delta hedging strategy is to reduce the number of rebalances for the portfolio hedging strategy. Hence, the transaction costs of a delta-gamma hedge should decline compared to the delta hedge. Nevertheless, erratic changes, for example the market crash in October 1987, cannot be absorbed fully either by the delta-gamma hedging technique but this hedging method works better in the long run than delta hedging.

The hedging mechanisms mentioned thus far, are only able to hedge the overall-risk of a portfolio. However, the overall-risk is not the only risk that can be hedged with options. Appropriate option strategies can also hedge, for example, the influence of the volatility and the influence of the interest rate on portfolios.

Hence, a portfolio manager can choose different strategies to reduce the risk of portfolios. Some of the risk reducing strategies are only possible because of the existence of options. Portfolio management without options would therefore drastically reduce the variety of strategies. In conclusion, options open new horizons for the hedging of risks in portfolios.

A further use of options for portfolio insurance in portfolio management is reviewed in the section below.

2.4.5 Options as Instruments for Portfolio Insurance

Portfolio insurance can be thought of as a technique to insure portfolios against a drop in their value. The essential difference compared to a hedge is that the portfolio insurance insures the portfolio against a fall below a specific level and does not eliminate all potential profits of a rising market. Portfolio insurance techniques are also possible without options, for example, with stop-loss strategies, with the constant-proportion portfolio insurance (proposed by Perold and Sharpe (1988)), or with futures instead of options. For the purpose of this thesis, only option strategies will be discussed for portfolio insurance techniques.

For all portfolio insurance strategies, portfolio manager need to decide how long the investment horizon is and what the insured level of the portfolio should be (i.e. how much loss can be accepted). Leland (1980) and Leland and Rubinstein (1981) argue that investors whose risk tolerance increases with wealth more rapidly than that of an average investor, would benefit more from portfolio insurance. Leland (1980) classifies such an investor as a “safety first” investor. Additionally, Leland (1980) argues that an investor who is more optimistic than the average investor would also benefit from portfolio insurance. This investor class includes for example institutions such as investment funds managed with the expectation of above-average returns compared to the average investor.

Portfolio insurance with options is further subdivided into two categories:

1. portfolio insurance with purchased options
2. portfolio insurance with synthetic options

Both portfolio insurance strategies will be discussed below and finally summarized in a conclusion.

2.4.5.1 Purchased Options

One very well known strategy of purchased options is the “protective put”³⁵. For the strategy of the protective put, puts are bought with expiration equal to the investment

³⁵ Another portfolio insurance strategy with purchased options is the portfolio insurance with purchased calls (fiduciary calls). The portfolio insurance with purchased calls is also a special case of strategic asset

horizon and with a strike price equal to the insured minimum level of the portfolio. The return of the protective put strategy depends on the chosen relation between the strike price of the option and the price of the underlying asset (portfolio) as described by Gladstein, Merton, and Scholes (1982). Hence, the protective put strategy depends on the insured level of the portfolio. The result of such a strategy is that the more the portfolio is insured (i.e. the higher the strike price) the less is the return of the portfolio. The number of contracts required for a protective put strategy is very simple to calculate. The portfolio value is consequently divided by the value of the index or the future and the multiplier³⁶.

Benninga and Blume (1985) assume that an investor purchases a protective put only in "less complete" markets. A "less complete" market is, for example, a market where continuous portfolio rebalancing is not possible because of transaction costs. Leland (1985) argues that the higher the transaction costs the less often an adjustment of the portfolio is appropriate. This means that it could be more attractive for the portfolio manager to buy a protective put than, for example, to replicate an option³⁷ because of the transaction costs.

A durable protection of the portfolio is normally achieved by rolling over options. The rolling-over is necessary because options for long-term protection are very rarely traded at option exchanges requiring short-term options to be combined with a long-term option. The problem of the rolling-over is the insecurity concerning the future option price and the future strike level that will be available for the insurance strategy. Besides that, Choie and Novometsky (1989) find that the rolling-over of options at expiration into another option, replicated a long-term option, but only under the condition of a constant volatility and a constant interest rate over time. They find a difference of up to circa 32 percent between the return of a strategy with synthetic short-term options and the return of a synthetic long-term option caused by a changing volatility. Additionally, Rubinstein (1985b) finds that a sequel of short-term options has a lower expected rate of return than one long-term option where both options cover the same time for the same amount of insurance. Moreover, transaction

allocation. The creation of the call strategy requires a combination of call options with a cash deposit. Hence, the minimum value of the portfolio is determined through the investment in the cash deposit plus its interest over the investment horizon. The remaining money from the initial investment in the cash deposit is then used to invest in call options in order to participate in a rising market.

³⁶ The multiplier is the number that describes the relation between one option contract and the index or the future as underlying. For example, one index option contract consists of ten options at SAFEX where each option is on one future. Hence, the multiplier is ten at SAFEX.

costs for a short-term option strategy, that replicates a long-term option, increase with a longer time horizon for the insurance because more transactions and roll-overs have to be made (in contrast to a long-term option).

However, the advantages of a protective put strategy are firstly the certainty about the minimum level of the portfolio and secondly the certainty about the transaction costs if the option is bought once for the entire investment horizon. Nevertheless, the protective put buyer would prefer not to use the put but to participate from a rising asset market.

2.4.5.2 Synthetic Options

Options can be replicated to solve problems like the non-existence of particular expirations or specific strike prices. Moreover, synthetic options can be used if position limits are in existence at option exchanges or if the market lacks liquidity for large trades. For example, European puts can be replicated with a short sale of the asset and a cash deposit without changing the insured portfolio³⁸.

The replication of the protective put requires a position in the underlying asset of the option in proportion to the delta of the option. The delta of the synthetic put will always change with a changing price of the underlying asset and a diminishing time to expiration. Hence, the strategy is not as static as the purchase of the protective put but dynamic because of the regular readjustments that have to be made. The advantages of the synthetic protective put are that the strike price and the expiration can be selected independently from available options at an exchange.

The accurate valuation of synthetic options is also necessary for the consideration of the portfolio valuation. However, whichever option price model is implemented for the option valuation, all require some essential parameters for the valuation of the option. Most of the parameters such as the interest rate and the underlying asset price are easy to obtain. However, the volatility is not as simple to obtain but is essential to price the option accurately (and to compute the correct delta of the option). If the volatility estimate is

³⁷ For the replication, see the section 2.4.5.2 "Synthetic Options".

inaccurate, the delta will also be inaccurate and therefore the whole portfolio insurance computation would consequently be inaccurate. O'Brien and Rendleman (1990) show that the misestimation of volatility results in an inaccurate mix of asset and cash, consequently, the accurate volatility estimation is essential for portfolio insurance strategies with synthetic options.

One of two (already discussed) volatility considerations could be used

- Historical Volatility
- Implied Volatility

However, criticism of the historical volatility aspect is evident from the discussion above (in section 2.3.3), hence, the historical volatility is not considered further in the thesis. Consequently, the implied volatility is viewed as a superior measure in the framework of the portfolio insurance with replicated options.

Hence, the implied volatility is used to determine the value of a synthetic option and additionally to determine the required delta of the option for this portfolio insurance strategy. An accurate estimation of the implied volatility can be achieved by the novel option price models proposed by Dupire (1994), Derman and Kani (1994), Rubinstein (1994), and Jackwerth (1996). These option price models can then be used replicate synthetic options for portfolio insurance by computing an appropriate delta and consequently obtaining an accurate replication for the synthetic option.

A dynamic hedging strategy with synthetic options may nevertheless suffer from the above mentioned delta problems (discussed in the section 2.4.4). Additionally, a portfolio insurance strategy with synthetic options may suffer from rising transaction costs caused by frequent changes in the delta value. Hence, the more the transaction costs rise, the fewer readjustments are optimal as Leland (1985) argues.

Apart from the discussed problems that may damage the synthetic option replication, a legislative regulation may limit the use of synthetic option replication in portfolio insurance strategies in South Africa. For example, unit trusts in South Africa have the legislative requirement of holding a minimum of five percent in cash. This is problematic because the

³⁸ Ordinarily, the position for the option replication is not sold short for another account, instead, it is balanced with the insured portfolio. Finally, both positions have to be viewed separately, as with the purchased protective put and the insured portfolio.

delta is zero or one at the expiration of the synthetic option. This means that if the market declined below the strike price, the delta is zero for call options and consequently the portfolio only consists of cash. If the market rises above the strike price, the delta is one for call options and consequently the portfolio only consists of the assets. Hence, a portfolio consisting of only assets violates the legislative regulations whilst a portfolio consisting of only cash may violate contract conditions of the portfolio manager.

The reviewed portfolio strategies with options are fairly elaborate and are therefore concluded below.

2.4.6 Conclusion

The ways in which options can be used in portfolios is diverse (as discussed). For example, options are able to alter return distributions of portfolios in the most appropriate way. In addition, guaranteed portfolio strategies or high speculative strategies are imaginable. Furthermore, options are used to reduce general or specific portfolio risks. Consequently, the reduction of the general portfolio risk is one of the most important roles of options in portfolio management.

Some common problems of portfolio strategies with options are the non-existence of particular expirations or specific strike prices. However, these two problems can be solved by replicating options synthetically. Nevertheless, the synthetic replication also inherits some problems, for example the use of inaccurate implied volatility. This problem can be overcome with the development of option price models incorporating the implied volatility. However, large declines of the markets (as in October 1987) with unexpected and sudden increases in transaction costs and large discontinuities in market prices, may also endanger synthetic option replication strategies as Rubinstein (1988) argues. To prevent substantial losses by such large declines put options must be purchased. The problem of purchased put options however lies in the availability of strike prices and of expirations, the market liquidity, and the position limits per market participant at exchanges. In general, it cannot be expected that risk reducing strategies pay off over every period but it can be expected to pay off in the periods when protection is really needed.

The background discussion of the portfolio strategies have highlighted the importance of the implied volatility. Consequently, the behaviour of the implied volatility is examined at SAFEX in the next chapter.

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Appendix 2.A

Analytical Illustration of Differences between European and American Index Options

The analytical illustration of the difference between American and European index options is presented for the upper and lower bounds for both exercise styles. The bounds of the option price are portrayed in simplified terms with the assumption of a continuous dividend yield for the index. However, this simplification has no influence on the meaningfulness of the discussion. American options are noted in capitals, C and P, European options are noted in small letters, c and p.

First, the identical upper bound for American and European options is presented:

$$c \leq S \text{ and } p \leq X \quad \text{European options}$$

$$C \leq S \text{ and } P \leq X \quad \text{American options}$$

where X is the strike price and S is the price of the index.

Second, the lower bound is displayed which differs between American and European options. The lower bound for European index options is

$$c > Se^{-q(T-t)} - Xe^{-r(T-t)} \quad \text{European call}$$

$$p > Xe^{-r(T-t)} - Se^{-q(T-t)} \quad \text{European put}$$

where r is the interest rate and T-t is the lifetime of the option contract.

The lower bound for American index options is

$$C > S - X \quad \text{American call}$$

$$P > X - S \quad \text{American put}$$

Additionally, it is differentiated between two cases for American call and put options. In the first case, the risk-free interest rate, r, is greater than the continuous dividend yield, q ($r > q$). In the second case, the continuous dividend yield is greater than the risk-free interest rate ($q > r$).

1. $r > q$: The lower bound for European options is always greater than the lower bound for American options:

$$Se^{-q(T-t)} - Xe^{-r(T-t)} > S - X$$

Hence, it is never optimal to exercise an American call before expiration under the assumption of $r > q$. American calls are consequently worth the same as European calls, so that $C = c$.

However, American put options could be exercised optimally before expiration under the assumption of $r > q$. The equation below proves that an early exercise could be optimal because the lower bound for American options is greater than the lower bound for European options:

$$X - S > Xe^{-r(T-t)} - Se^{-q(T-t)}$$

American puts are therefore worth more than European puts, so that $P > p$.

A put-call relationship for American call and put options is derived in a similar way to the put-call parity for European options. Under the assumption of $r > q$, the put-call relationship is

$$P + Se^{-q(T-t)} > c + Xe^{-r(T-t)}$$

2. $q > r$: The lower bound for American options is greater than the lower bound for European options:

$$S - X > Se^{-q(T-t)} - Xe^{-r(T-t)}$$

Hence, it could be optimal to exercise an American call before expiration under the assumption of $q > r$. American calls are therefore worth more than European calls, so that $C > c$.

However, the lower bound of European put options is greater than the lower bound of American put options:

$$Xe^{-r(T-t)} - Se^{-q(T-t)} > X - S$$

Early exercise of American put options is never optimal under the assumption of $q > r$. Hence, American put options are worth the same as European put options, so that $P = p$.

Under the assumption of $q > r$, the put-call relationship is

$$p + Se^{-q(T-t)} < C + Xe^{-r(T-t)}$$

The analytical results of the two cases above confirm the prior analysis result regarding the relationship between American and European index options. Additionally, the lower and upper bounds are defined for American and European index options. A put-call relationship for the American index options is also derived from the upper and lower bounds. Nevertheless, the put-call relationship for American index options cannot derive an exact valuation.

Appendix 2.B

Analytical Approximations

The wealth of analytical approximations in the international literature is summarized here. The summary is required to obtain an overview about the most important analytical approximations and their advantages as well as their disadvantages. Hence, the review briefly describes each method and its usefulness for the valuation of American options. The review further tries to list the most important analytical approximations in a historical order. A list of the discussed methods is given below:

- Black's (1975) pseudo-American approximation
- the Roll-Geske-Whaley model
- Johnson's (1983) model
- the Geske and Johnson (1984) model
- the Barone-Adesi and Whaley (1987) model

The review of the analytical approximations starts with the pseudo-American approximation approach by Black (1975). The pseudo-American approach develops a technique where the American option on a dividend paying underlying is divided into two different options. The first option values the American option like a European option. The second option prices the American option like a European option with an expiration in the second before the ex-dividend date. Both of these created options are valued separately with the Black and Scholes (1973) model. The highest value of both created options will be the value of the American option. However, the pseudo-American model is only an approximation that restricts the exercise dates to the second before the ex-dividend date and to the expiration. Hence, the pseudo-American approach undervalues the American call option. This undervaluation is confirmed empirically by Whaley (1982) showing an undervaluation of 1.48 percent.

The second analytical approximation method reviewed is the in stages developed model by Roll (1977), Geske (1979b), and Whaley (1981). The Roll-Geske-Whaley model implements a technique that divides the American call option into a portfolio of three options (discussed below).

The first option of the above mentioned option portfolio is a European call option characterized by the same expiration and strike price as the American call option. The second option is also a European option but with the expiration at a second before the ex-dividend date. In addition, the strike price of the second option is altered by the dividend. The third and last part is a compound option³⁹ which is written on the first European option in the option portfolio. Consequently, the value of the American call option in the Roll-Geske-Whaley model is the sum of the three options in the created option portfolio. The valuation of the American put on a underlying without dividend payment is analogous.

The Roll-Geske-Whaley model is however limited to handle only American calls on a dividend paying underlying and American puts on a non-dividend paying underlying. This limitation is the main disadvantage of the Roll-Geske-Whaley model. Another disadvantage of the model is the problematic handling of more

³⁹ Compound options are options on options. The compound option displays the cost of exercising the option and it is priced by the compound option model by Geske (1979a).

than one dividend before expiration, which leads to a trivariate or higher-order multivariate cumulative density function (as argued by Whaley (1982)). Furthermore, Whaley (1982) notes a prediction error of the Roll-Geske-Whaley model of 1.08 percent with a standard deviation of 23.82 percent. This approximation error may be better than for the pseudo-American approach by Black (1975) but it is still inappropriate for the use of option pricing.

Hence, further approximation methods are discussed. One of these approximations is a method for American puts on stocks without dividend payment proposed by Johnson (1983). His approach is based on an interpolation between two European puts with different strike prices. The main problems of Johnson's (1983) method are however the assumption of only two discrete exercise dates and the assumption of no dividend payment. Barone-Adesi and Whaley (1987) pointed out that Johnson's (1983) method misprices long-term options. Hence, the Johnson (1983) method is also not useful in the appropriate pricing of options.

However, Blomeyer (1986) extends Johnson's (1983) method to a stock with one dividend payment to avoid some of the prior criticism. Nevertheless, Blomeyer (1986) still faces the problems of only up to five discrete exercise dates and only one allowed dividend payment. The method proposed by Blomeyer (1986) is therefore still inappropriate for the pricing of options.

So far, no appropriate approximation methods for the pricing of American options are found in the international literature. The discussed approximation methods are partly complex and inaccurate. Hence, their use in the pricing of American options is limited. Nevertheless, two more promising approximation methods (Geske and Johnson (1984) and Barone-Adesi and Whaley (1987)) are discussed in detail.

Geske and Johnson (1984) propose another approach for the valuation of the American put based on Johnson's (1983) method. The Geske and Johnson (1984) approach constructs a sequence of compound options (whose valuation is based on the technique by Geske (1979a)) for the valuation of the American put. Their result is very accurate.

The high accuracy of the Geske and Johnson (1984) results is confirmed in the research by Shastri and Tandon (1986) who also price American call options with this technique. However, the Geske and Johnson (1984) approximation method has some important disadvantages as listed below.

The Geske and Johnson (1984) method still suffers from the inherited problem by the Johnson (1983) method that exercise is only possible at a few discrete points in time. Moreover, the Geske and Johnson (1984) method requires at least the evaluation of cumulative univariate, bivariate and trivariate normal density functions⁴⁰. According to Shastri and Tandon (1986) is the disadvantage of a cumulative multivariate normal density functions, in the case of a large number of options, that very large amounts of computer time are required. Furthermore, Barone-Adesi and Whaley (1987) demonstrate that the pricing error of Geske and Johnson option prices becomes larger for long-term options. In spite of the noted problems, the Geske and Johnson (1984) method is better than the prior described approximation methods and it works well for short-term American options (which have the highest liquidity).

An attempt by Ohlberg (1987) to improve the Geske and Johnson (1984) method however leads to a problem that diminishes the successful properties of both the Geske and Johnson (1984) method as well as

⁴⁰ For example, Geske and Johnson (1984) show results that are based on a cumulative quadivariate normal density function.

the Ohmberg (1987) method. Ohmberg (1987) describes the interpolation policy of the put values with different dividends by Geske and Johnson (1984) as a source of "substantial (positive) interpolation errors". Ohmberg's (1987) own approximation technique derives by contrast a solution for the American put under the assumption that the critical strike price⁴¹ is an exponential function of time. The advantage of Ohmberg's technique is the speed and the accuracy, which is far greater than the Geske and Johnson (1984) approximation can perform for stocks without dividend payment.

However, Ohmberg (1987) reveals that the approximation technique by Geske and Johnson (1984) as well as his own method have potential errors if dividends are included. In particular, the potential errors appear for dividends that are below the critical amount of dividend payment⁴². The put values for dividend amounts below the critical amount of dividend payment are interpolated in the Ohmberg as well as in the Geske and Johnson formulas. Ohmberg (1987) gives empirical evidence that this interpolation of the put values causes an average error of 3.6 cent and an maximum error of 27.6 cent compared to numerical solutions. Hence, the Ohmberg (1987) technique as well as the Geske and Johnson (1984) method should only be implemented for American put options on underlyings that do not pay dividends.

Still, none of the above discussed approximation method is appropriate for American options on underlyings that pay dividends. Hence, a final approximation method is introduced that solves this problem. This method is the Barone-Adesi and Whaley (1987) approximation method that gives the best approximation results of all discussed techniques under examination.

The Barone-Adesi and Whaley (1987) method is one of the most applied approximation techniques because of its accuracy. Historically, the Barone-Adesi and Whaley (1987) approximation technique is based on MacMillan's (1986) quadratic approximation approach on stocks without dividend payment. However, the Barone-Adesi and Whaley (1987) method also prices American options on underlying assets that pay dividends under the assumption of a continuous dividend yield.

The approximation by Barone-Adesi and Whaley (1987) estimates the difference between the premium for American options and European options. The estimated difference is referred to as "early exercise premium" of American options. This early exercise premium is treated as an option in the Barone-Adesi and Whaley (1987) model. Hence, the early exercise premium option can be used for the valuation of American options on indices and index futures under the assumption of a continuous dividend yield for indices.

The accuracy of the Barone-Adesi and Whaley (1987) approximation technique is very good compared to the prior approximation methods above. The approximation error is only up to 0.4 percent error for short-term options (as revealed by Barone-Adesi and Whaley (1987)). However, long-term options with times to expiration from one year up to three years have considerably greater pricing errors, as the research by Barone-Adesi and Whaley (1987) shows. Consequently, Barone-Adesi and Whaley (1987) recommend the implementation of their method or of the Geske and Johnson (1984) model for options with less than one year to expiration. However, for options with expirations beyond one year Barone-Adesi and Whaley (1987) advise instead numerical methods, for example binomial or finite-difference models because of their higher accuracy.

⁴¹ The critical strike price is the strike price above which the option will be exercised.

In sum, no approximation method is without weakness although the Barone-Adesi and Whaley (1987) technique is very accurate for short-term American options. It is furthermore superior in its accuracy to the other discussed methods. Hence, the Barone-Adesi and Whaley (1987) method is the only method to recommend if small approximation errors are less important. Otherwise, none of the approximation methods could be recommended and other solutions have therefore to be developed for the problem of the American option valuation.

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⁴² The critical amount of dividend payment is defined as the amount that is needed for exercising the option contract.

Appendix 2.C

Numerical Methods

An analytical solution of the valuation of American options does not exist. However, the number of analytical approximations to value American options is plentiful but they are not without problems. The problems of analytical approximations are their inaccuracy and their limitation to specific tasks (e.g. underlyings without dividends). In this light of the unsatisfactory results by analytical approximations, new ways have to be found to value American options accurately. Such new ways are combined in numerical methods that comply with the aim of accuracy. However, the number of existing numerical methods is too extensive to be reviewed here, hence only the most important and representative methods are considered. A list of the discussed numerical methods is given below:

- multinomial trees (e.g. binomial tree)
- finite difference
- Monte Carlo simulation

The first numerical method to be reviewed is the binomial model derived by Cox, Ross, and Rubinstein (1979) and independently by Barter and Rendleman (1979). The binomial model is based on the assumption that price movements of a stock are binomial for each time step (i.e. two outcomes are possible at every time step). The lifetime of an option is therefore subdivided into a determined number of time steps so that a binomial tree is created with its size depending on the number of time steps. The valuation process of the binomial tree model first calculates a complete tree of stock prices. Second, the options are evaluated by starting at the end of the tree and computing backward to the start of the tree. If the number of time steps becomes greater, the model moves towards a continuous stochastic process, which is one of the assumptions of the Black and Scholes (1973) model. Hence, the Black and Scholes (1973) model is a limiting case of the binomial model.

The advantage of the binomial tree model is the flexibility of the application for different options styles. The binomial tree method is able to value American and European options with or without dividend payment. However, the disadvantage of the numerical binomial tree model is the high computational cost compared to the closed-form Black and Scholes (1973) model. The more accuracy required from the binomial tree model, the greater the number of time steps necessary. For example, Cox, Ross, and Rubinstein (1979) split the lifetime of an European call into 150 time steps to obtain approximate values with a maximum absolute error of one "cent" compared to the option price of the Black and Scholes (1973) model. In sum, the binomial tree method is an appropriate technique to price American options accurately and it can therefore be recommended.

The accuracy of the binomial model is also evident in other tree methods. For example, Parkinson (1977) priced the American put option with a numerical integration that is a combined binomial and trinomial method. Boyle (1986) picked up the idea of the trinomial method to gain a greater efficiency from the trinomial tree compared to the binomial tree. The greater efficiency of the trinomial tree is achieved because three outcomes are computed at each time step instead of two outcomes per time step in the binomial tree. Hence, the trinomial tree is able to obtain the same results as the binomial tree with only half the number of

time steps or improves the result with the same number of time steps respectively. In spite of the gain of efficiency, the trinomial tree is slightly more complex than the binomial tree. However, the trinomial tree also values the prices of American options very accurately.

The accuracy of both tree methods is appropriate for the pricing of American options. Although the computation of the option price is more efficient with the trinomial tree, the trinomial tree has a higher complexity than the binomial tree. Hence, both models can be recommended with an own preference to the less complex binomial tree.

The second point in the list of reviewed numerical methods are finite difference techniques (i.e. the explicit and the implicit finite difference). In fact, the finite difference techniques are quite similar to the discussed tree methods. Brennan and Schwartz (1978) demonstrate that binomial and trinomial trees with an infinite number of time steps are the same as the explicit finite difference method.

The explicit finite difference method is one of two finite difference methods. The other finite difference method is referred to as the implicit finite difference method. Hull and White (1990) comment that the implicit finite difference method is equivalent to a multinomial tree with an infinite number of time steps. The technique of finite difference methods is more complex than tree methods and is briefly described below.

Finite difference models approximate the differential equation on which the option price formula is based. Geske and Shastri (1985) demonstrate that the explicit finite difference method is capable of solving the differential equation for the unknown option price with the previous option prices whilst the implicit finite difference method has to solve a set of equations simultaneously. The application of the implicit finite difference method is therefore more complex than the explicit finite difference method, especially because the simultaneous solving process of a set of equations requires the inversion of matrices. Furthermore, Geske and Shastri (1985) find (in a comparison between binomial, explicit, and implicit methods) that the explicit finite difference method with a logarithmic transformation⁴³ is more efficient than the binomial method for a larger number of options. They find also that the explicit finite difference method with logarithmic transformation is more efficient than implicit finite difference methods.

The result of the finite difference techniques is very accurate and faster than tree methods if the difference method achieves a solution. However, the weakness of the finite difference methods is a small probability that no solution is found for the option price of the American option. Hence, the finite difference methods are not superior to the tree methods and therefore the tree methods are recommended for the use in the valuation of American options.

The recommendation to implement tree methods, in particular the binomial tree, is not altered by the final numerical method on the list, the Monte Carlo simulation. This last numerical method is briefly discussed below.

The Monte Carlo method simulates an outcome a large number of times and attempts to obtain the price of a derivative, for example an option, from the large number of simulated outcomes. Boyle (1977) demonstrates the use of the Monte Carlo method for the valuation of European options. The main advantage of the Monte Carlo method is the flexibility of the valuation of different options, especially when no other pricing procedures exist. However, the Monte Carlo method has two main disadvantages. The first disadvantage is that large number of simulations are required. The second disadvantage and the more

⁴³ The finite difference method uses $\ln S$ instead S as underlying variable.

important one is that the valuation of American style options is difficult or impossible (as discussed in Deynne, Howison, and Wilmott (1995)). Hence, the Monte Carlo method is not appropriate for the valuation of American options and the recommendation to use binomial trees is maintained.

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Chapter 3

3 Descriptive Empirical Examination of the Environment in South Africa

3.1 Introduction

This chapter's central aim is to provide a descriptive empirical assessment of the South African option market. The descriptive empirical assessment focuses in particular on the behaviour of the implied and the historical volatility and serves as a foundation for the remainder of the thesis.

Before the empirical assessment starts, three different topics require introduction to the framework for the empirical examination that follows. First, the historical development of the South African option market is considered in section 3.2. Thereafter, a detailed analysis of the options data follows in section 3.3. Thirdly in section 3.4, an option price model is introduced to compute the required implied volatility for the empirical research. This option price model is required for the computation of the implied volatility in this chapter but it is discarded later because of its shortfalls revealed by the empirical research. Hence, a new option price model for South Africa is proposed in Chapter 4.

Based upon the option price model introduced in section 3.4, an empirical examination of volatility in South Africa follows in section 3.5. In particular, the validity of the constant volatility assumption (assumed by the option price model implemented here as well as by the Black and Scholes (1973) model) is questioned. Hence in a first empirical examination in section 3.5.1, the implied volatility is compared to the historical volatility to establish if there are significant departures from the constant volatility assumption (as is evident in the international literature⁴⁴). Moreover in the sections 3.5.2 and 3.5.3, two methods are considered to analyse the behaviour of implied volatility across strike prices and across expirations. The first method constitutes a novel approach to the construction of volatility indices for options on futures (using daily data) for descriptive purposes. The second nonparametric methodology is based upon Rubinstein (1985a) but modified in order to

examine implied volatility in South Africa. Consequently, this second method has the advantage over the first method by enabling tests of statistical significance. Both methods, that is, the implied volatility index method, as well as the nonparametric test method, yield consistent results.

The historical development of the South African derivatives market follows (in section 3.2).

3.2 History

The trading of equity options has been documented in South Africa as early as the end of the last century. Futures trading began in April 1987 with the Rand Merchant Bank acting as informal clearing house for futures on the Actuaries⁴⁵ All Shares, All Gold, and All Industrial indices. In 1990, the South African Futures Exchange (SAFEX) was established, trading standardized equity future contracts on the All Share Index (ALSI), the Gold Index (GLDI), and the Industrial Index (INDI) in the financial division⁴⁶ of SAFEX. The Financial and Industrial Index (FNDI) was added to the existing future and option contracts in October 1995, however the trading in this future and its options has been highly illiquid⁴⁷. The FNDI is consequently not considered further in the forthcoming investigations in this thesis.

On 16 October 1992, the first options on futures (available on all equity futures) were launched by SAFEX⁴⁸. The most liquid option contracts exist on the ALSI futures, followed by the INDI futures, and lastly by the option contracts for the GLDI futures.

⁴⁴ References can be found in Chapter 2 in section 2.3.4.

⁴⁵ The Actuaries indices were 80 percent market capitalisation indices and were abolished by SAFEX as underlying for futures with expiration in March 1996. Instead, SAFEX has implemented stock indices with a standardized number of shares as underlying for futures. Further details are given in section 3.3.

⁴⁶ Seven other non-equity contracts are currently traded in the financial division of SAFEX: the Krugerrand future, the Rand/Dollar future, four different Republic of South Africa Bond futures, and the short-term Interest Rate future. In addition, the agricultural division at SAFEX actually trades six futures on agricultural products: the beef index future, the beef future, the white and the yellow maize futures, the potatoes future and the wheat future.

⁴⁷ Only 218 transactions with an amount of 12,230 future contracts with the value of R 1,205,604,800 were traded in the first year of existence between 6 October 1995 and 30 September 1996.

⁴⁸ The first trade for the INDI option contract dated from the 7 January 1993 and the first trade for the GLDI option contract dated from the 19 January 1993.

On 1 September 1997, the first options on stocks were traded. SAFEX started the American style stock options only on the more liquid shares namely Anglo American, DeBeers, Richemont, Sasol, SA Breweries, and Liberty. Currently, SAFEX considers to expand the stock options to all component shares of the indices. Naturally, no comprehensive empirical research is reasonable for the stock options because of the current paucity of available data. Hence, the research in this thesis focuses only on the options on the futures.

3.3 Data

The data for this empirical research consists of reported trades for high, low, first, and last trading prices for options and futures on the ALSI, the GLDI, and the INDI. All the data examined was downloaded from the SAFEX website⁴⁹. The captured data consists of prices traded at SAFEX from the 16 October 1992 to the 31 December 1996. In this period, the underlying indices changed from a so-called 80 percent market capitalisation index to indices with 40 shares for the ALSI, 10 shares for the GLDI, and 25 shares for the INDI respectively. The new indices (with typically smaller component holdings) have been in existence since 18 June 1995⁵⁰. Figure 3.3-1 displays the chart of all three old and new indices in the period 1 October 1992 to 31 December 1996.

⁴⁹ The address of SAFEX homepage is <http://www.safex.co.za>.

⁵⁰ Options and futures with expirations up until March 1996 were based on the 80 percent market capitalisation indices. Subsequent to March 1996, options and futures have been based on the new indices. A separation between the old and the new indices is deemed unnecessary for the purpose of this research because the behaviour of the old and new indices is nearly identical during the overlapping time between 15 June 1995 to 29 March 1996 with a correlation of 0.9963 for the ALSI, 0.9979 for the GLDI, and 0.9915 for the INDI.

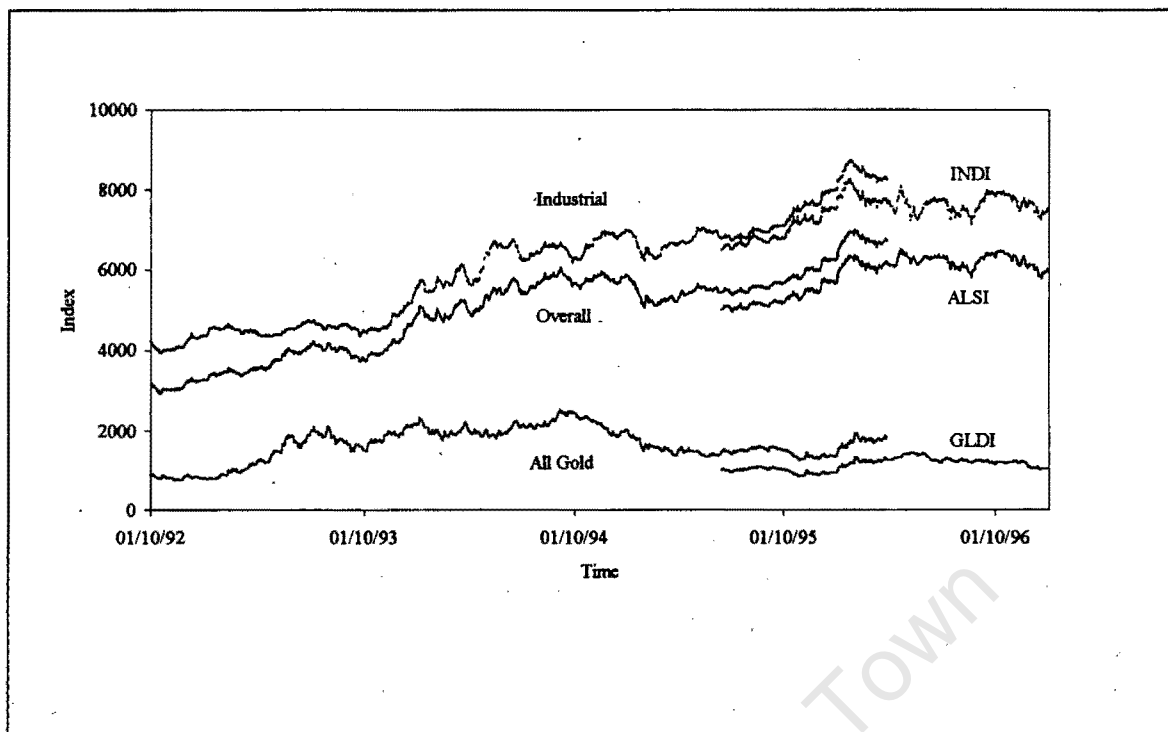


Figure 3.3-1. Daily Closing Prices of the 80 percent Market Capitalisation Indices and the Fixed Number Share Indices

The 80 percent indices are displayed for the period 1 October 1992 to 29 March 1996 whilst the new fixed number share indices are reported for the period 15 June 1995 to 31 December 1996.

The selection of the trading data follows Rubinstein's (1985a) approach. A data "cleaning" procedure is necessary for the prevention of arbitrage violations and mistake trades. These are briefly discussed below:

Arbitrage Violation

One kind of arbitrage violation results when implied volatilities become zero or negative. The existence of negative or zero implied volatilities are not feasible by the definition of the underlying option price models because such an implied volatility value suggests that a risk-free profit is possible. The option is valued below the intrinsic value if the volatility is below zero so that an American option can immediately be exercised with a profit, because the exercise value will be higher than the option premium. Such arbitrage violations should be excluded from the data to prevent a distortion of the data.

Mistake Trades

A trade is classified as a mistake trade when the implied volatility is far beyond the regular volatility. A standard deviation in excess of 150 percent per annum typically constitutes a

mistake trade for the period from 1992 to 1996. The second condition for mistake trades is that the trades at one day are followed by the same trade (the same option price, nearly the same high irregular volatility, and the same volume of options recorded) on the next day. The impression is that these trades were a mistake on the first trading date and that these trades were neutralized with compensating trades on the next trading day. Such trades also have to be excluded from the data set to prevent a distortion of the data.

For example, the ALSI data over the 1992-1996 period consisted of 35844 trading prices (see Table 3.3-1). This excludes 1020 trading prices which were identified as arbitrage violations and mistake trades and were consequently omitted from the data. Hence, only 2.77 percent of all ALSI trading prices were identified as arbitrage violations and mistake trades. In addition, the conditions are similar to the GLDI and the INDI with 2.20 and 1.49 percent of all trading prices respectively identified as arbitrage violations and mistake trades.

Further selection of the data set with the view of searching for other arbitrage violations or mistake trades cannot be tested⁵¹ because of the type of the data. Hence, the possibility of other violations (not discovered) cannot be excluded from the data and derived results from the data might therefore be questionable. Nevertheless, it is possible to obtain significant results that enhance the understanding of the prevailing conditions in the South African option market.

Table 3.3-1 provides a summary⁵² of the data in this research. The number of traded option series⁵³ is divided into the following categories of trading prices: first, last, high, and low price. If only one trade was made in an option series, this trade is recorded as the first, last, high, and low price of the option series for this day. Additionally, the sum of all numbers of trading prices of the categories first, last, high, and low prices are calculated in

⁵¹ It is not known for example if the last quote of one particular option was traded at the same time as the last quote for another option or if some hours of trading separate them. The possible differences in the trading time in particular are a problem in a market where often only one or two trades in one contract are traded a trading day. It is assumed that only the trading time differences during the trading day causes arbitrage violations. Furthermore, it is assumed that the arbitrage violations result only from the way in which the trades are reported. Therefore, the recognized arbitrage violations are not handled as arbitrage violations but as additional information for the particular trading day.

⁵² A further very detailed analysis of the data is relegated to Appendix 3.A.

⁵³ An option series is defined here as the traded prices (first, last, high, and low) of one trading day for one particular strike and one particular expiration.

the last column of Table 3.3-1 so that the research is based upon 35844 trading prices of the ALSI, 12925 trading prices of the INDI, and 6943 trading prices of the GLDI.

Table 3.3-1
Data Categorized into Trading Classes

The data is reported for the ALSI, INDI, and GLDI after screening for arbitrage violations and mistake trades in the period 16 October 1992 to 31 December 1996. The column "option series" is the sum of option series for each trading day over all trading days (1066). Furthermore, the trades are differentiated in first, last, high, and low prices and summed up to the total number of available trading prices for each index in the last column. If only one trade took place in an option series for a particular trading day, the price of this trade will be first, last, high and low price for the particular day.

Index	Option Series	Trades				Total of (1) to (4)
		First (1)	Last (2)	High (3)	Low (4)	
ALSI	9216	8981	8999	8820	9044	35844
INDI	3304	3246	3243	3201	3235	12925
GLDI	1762	1743	1736	1730	1734	6943

In conclusion, the data description gives a perspective of the trading environment. On a point of terminology it must also be noted that strike ratios⁵⁴ below one will be referred to as *out-of-the-money* options and strike ratios above one will be referred to *in-the-money* options. Nevertheless, the so called *out-of-the-money* options consist of *out-of-the-money* puts and *in-the-money* calls and the so-called *in-the-money* options consist of *in-the-money* puts and *out-of-the-money* calls. However, it is assumed⁵⁵ that the *out-of-the-money* puts (*in-the-money* puts) and the *in-the-money* calls (*out-of-the-money* calls) have the same characteristics (because of the put-call-parity), so that they can be combined.

The empirical description of the environment in South Africa and in particular at SAFEX is continued by the consideration of a widely in South Africa used option price model for options on futures (e.g. from SAFEX).

3.4 An Option Price Model with Constant Volatility Assumption

The focus of this section lies on the specification of an option price model that has been used for example by SAFEX for the computation of the mark-to-market prices since 1992.

⁵⁴ A strike ratio is defined as the strike price divided by the underlying asset price.

⁵⁵ The assumption is based on the explanation given in Appendix 3.A.

This option price model implemented by SAFEX⁵⁶ constitutes a modification of the Black (1976) model for American options on futures with mark-to-market procedure. The modification of the Black (1976) model is presented below for the South African environment. Consequently, this modified option price model is henceforth referred to as the modified Black model.

In addition, an understanding of the ensuing research requires the basic knowledge of the specifications of the option contracts traded at SAFEX. Hence, a brief introduction of the option contract specifications follows below.

3.4.1 Contract Specifications

The contract specifications for option contracts at SAFEX are described below:

Traded option contracts at SAFEX are American style options on futures. Their underlying futures and the option contracts themselves expire at the same date and the same time that is the 15th day (or next business day) of the cycle March, June, September, and December. *In-the-money* options are automatically exercised in the underlying futures contract at expiration and exercised options are randomly assigned to short positions. The exercised option contracts are settled in the corresponding future contract. In addition, the relation between option contract and future contract is one to one, which means that each option contains the right on one future.

Moreover, the price of the option (i.e. the premium) is not paid upfront; instead, the option premium is mark-to-market like a future⁵⁷. Mark-to-market is performed at mid-market price of the option daily⁵⁸.

Nevertheless, a margin⁵⁹ has to be paid on the option premium, but a market-related short-term interest-rate is paid on the deposited margin from SAFEX. Finally, no restrictions on the price movement of options or futures are imposed.

⁵⁶ Kindly pointed out by Khonaye Penxa, Risk Management Department, SAFEX Clearing Company (SAFCOM) and furthermore described in the option premium specification of the contract specifications for futures and options contracts published by SAFEX on <http://www.safex.co.za>. SAFEX references the Black (1976) model but actually uses a modification of this model for the mark-to-market of the option price characterised by Asay (1982) and analytically developed by Lieu (1990).

⁵⁷ The future is mark-to-market at mid-market price at close of trading daily.

⁵⁸ SAFEX is able to perform intraday mark-to-market under extreme volatile conditions.

⁵⁹ SAFEX uses the SPAN (Standard Portfolio Analysis of Risk) method to find the correct margin amount to be paid to SAFEX. No margin has to be paid on positions that will offset each other.

3.4.2 The Option Price Formula for American Options on Futures with a Futures-Style Mark-to-Market Procedure in the South African Environment

As discussed in the review, American options on futures cannot be valued with accuracy using the Black and Scholes (1973) model or any other analytical model. Hence, numerical methods such as binomial models or finite difference models have to be used to calculate option values accurately. However, this view is only reasonable for option contracts which require an upfront payment (i.e. at the completion of the trade) of the option premium.

In South Africa, a peculiarity of the traded option contracts is that at SAFEX American options on futures with a future-style mark-to-market procedure⁶⁰ for the option premium are traded. This future-style mark-to-market procedure for the option premium means that no premium has to be paid until the expiration of the option. The option premium therefore does not require financing over the lifetime of the option (in contrast to an upfront premium) in South Africa.

However, a margin has to be paid for the purchased options. The margin has to be deposited at SAFEX to meet potential losses from the position in options. However, SAFEX pays interest at a market-related interest rate for the deposited margin. Consequently, no opportunity costs are produced (neither from the option premium nor from the margin) so that the purchase of an option can be done without initial investment costs at SAFEX.

This feature of no initial investment costs is one of the most important characteristics that has to be taken account of in an option price formula for American options on futures with the mark-to-market procedure. A first approach to the implementation of the future style mark-to-market procedure was proposed by Asay (1982). Asay's (1982) solution is confirmed by Lieu (1990) who additionally develops the theoretical background for an option price model for futures-style options. An in-depth theoretical derivation of the option price model for American options on futures with mark-to-market procedure is given in Appendix 3.B.

⁶⁰ The option premium is not paid upfront and only the changes in the daily mark-to-market prices of the option price are debited or credited to the margin account.

The theoretical derivation of the option price formula in Appendix 3.B as well as by Asay (1982) and by Lieu (1990) produces the following modified Black model which differs to the Black (1976) model by the absence of the interest rate term:

- a call option is valued as

$$C_F = FN(d_1) - XN(d_2)$$

- a put option as

$$P_F = XN(-d_2) - FN(-d_1)$$

where

$$d_1 = \frac{\ln \frac{F}{X} + (\sigma^2 / 2)(T - t)}{\sigma \sqrt{(T - t)}}$$

$$d_2 = d_1 - \sigma \sqrt{(T - t)}$$

with $N(\cdot)$ as the cumulative standard normal distribution and σ as the volatility.

In addition, the put-call parity of this option price model reveals only one difference to the Black (1976) model. This difference is again caused by the absence of the interest rate term in the put-call parity formula. Hence, the put-call parity for the option price model (as in Lieu (1990)) is

$$C_F - P_F = F - X$$

The aim of this section was to discuss and to arrive at a suitable option price model for the use in the following empirical examinations. The proposed option price model turns out to be a modification of the Black (1976) model (that can be applied to European options as well as to American options). Because the early exercise of the American call or put options is never optimal, the modified Black model can be applied to American options on futures with mark-to-market procedure in the South African environment.

However, the modified Black model is based on the assumption of constant volatility. This assumption will be examined in the remainder of this chapter using two different research methodologies. First, the constant volatility assumption is examined descriptively in section 3.5.2 and secondly with nonparametric tests in section 3.5.3. In addition, historical volatility will be compared to implied volatility in section 3.5.1 in order to assess their differences.

3.5 Volatility in South Africa

The aim of this section is to examine the behaviour of volatility and particularly the constant volatility assumption required in all models of the Black and Scholes family (which includes the modified Black model). This volatility assumption is assessed using the South African data described earlier (in section 3.3).

Volatility can be defined in different ways. Different forms of volatility were discussed in Chapter 2 under the heading “Volatility” (in section 2.3) where the difference between historical volatility and implied volatility was emphasized. In the ensuing section, problems of implementing the historical volatility will be considered briefly but the primary emphasis is on the examination of the behaviour of the implied volatility in the South African environment. Strike price biases and expiration biases are evident in markets other than South Africa (see for example, Rubinstein (1985a) for US stocks, Sheikh (1991) for the S&P 100, Duque and Paxson (1994) for British stocks, and Heynen (1994) for the Dutch EOE index). The specific aim of Chapter 3 (section 3.5.2 and 3.5.3) however is to test the South African market for strike price biases and expiration biases.

The existence of strike price biases and expiration biases in any markets suggest that the assumption of a constant implied volatility is inappropriate. As a consequence, the appropriateness of the modified Black option price model is brought into question.

The following empirical investigation into the behaviour of volatility in South Africa has three stages. First, a brief examination on the differences between historical and implied volatility is presented. Second, a “Volatility Smile Index” (VSI) similar to Tompkins (1994) and a proposed “Volatility Term Structure Index” (VTSI) is constructed. The purpose of the VSI and the VTSI is to assist in the observation of volatility patterns such as strike price biases and expiration biases. The final and third stage contains an examination of the non-constant volatility assumption using nonparametric tests (based on Rubinstein (1985a)). This third stage verifies the prior results from the VSI and the VTSI analysis and it establishes the foundation for further implied volatility developments in the thesis.

The first stage consists of a brief comparison between historical volatility and implied volatility.

3.5.1 A Comparison between Historical Volatility and Implied Volatility

In this section, the historical volatility is compared to the implied volatility of the ALSI, the GLDI, and the INDI. In particular, the comparison between historical and implied volatility for the ALSI is discussed here and for the sake of brevity the results for the GLDI and INDI are relegated to Appendix 3.C.

In the first test of volatility, historical volatility over different periods is investigated and compared to *at-the-money* implied volatility with around one month to expiration. In addition, this comparison between historical volatility and implied volatility is established for similar dates for each year in the period 1993 to 1996. The historical volatilities are only calculated on the basis of the closing price for periods when trading took place at least once a day for each trading day, otherwise they are noted as not available, NA. The results⁶¹ for the ALSI are displayed in Table 3.5.1-1.

Table 3.5.1-1

A Comparison between Historical Volatility and Implied Volatility for the ALSI

The *at-the-money* implied volatility is calculated for options on the ALSI with an expiration of around one calendar month later. The historical volatilities are based on the future of the ALSI with one month to expiration and they are calculated on the basis of the closing price over 21, 42, and 63 trading days prior to the referenced date. The nearest date to one month to expiration is chosen for the recorded date.

Date	Implied Volatility At-the-money	Historical Volatility over prior days			Expiration
		21 Days	42 Days	63 Days	
13/08/93	0.1914	0.2072	0.2059	NA	September
15/02/94	0.3074	0.2559	0.2244	0.2102	March
15/08/94	0.1862	0.1237	0.2102	NA	September
15/02/95	0.1740	0.2535	0.2223	0.2098	March
15/08/95	0.1402	0.1360	NA	NA	September
15/02/96	0.1485	0.1054	0.1178	0.1308	March
15/08/96	0.1850	0.1534	0.1667	NA	September

*NA indicates that trading did not take place at least once a day for each trading day.

The results of the comparison between the historical volatility and the implied volatility for the ALSI in Table 3.5.1-1 are similar to evidence in international markets⁶². The historical volatility seems to be dependent on the period over which it is calculated. For example, the 15th February 1994 presents results for the ALSI that reveal historical volatilities of 0.2559, 0.2244, and 0.2102 respectively for periods of 21, 42, and 63 days

⁶¹ The results for the INDI and the GLDI are presented in Table 3.C-1 and Table 3.C-2 respectively in Appendix 3.C.

⁶² For example, see Jackwerth and Rubinstein (1995).

prior to the 15th February 1994 (i.e. declining volatilities with increasing period for computation).

However, the direction of the trend of the historical volatility is not always consistent for all dates. For example the 15th February 1994, the trend in the historical volatility is downward (0.2559, 0.2244, and 0.2102) for periods 21, 42, and 63 days prior to the 15th February 1994 whereas the trend is upward (0.1054, 0.1178, and 0.1308) for the defined periods prior the 15th February 1996. Nevertheless, the results of the analysis suggest that the historical volatility is dependent on the selected period, but that the direction of the trend of the historical volatility is not consistent across different dates.

Moreover, the historical volatility appears to be a poor indicator of the implied volatility. For example, the differences between implied volatility and historical volatility, calculated over a period of 21 days prior the 15th August 1994, 1995 and 1996, are 0.0625, 0.0042, and 0.0316 respectively for the ALSI, instead of being zero to be a perfect indicator.

The results of the INDI and the GLDI are quite similar but it appears that the GLDI (with the highest level of volatility) has the highest differences between the historical volatility and the implied volatility. Hence, the absolute level of volatility seems to influence the differences between historical volatility and implied volatility.

Consequently, the results suggest that the historical volatility is a poor indicator of the implied volatility and that the historical volatility is not constant across different periods.

Although the differences between historical and implied volatility are observed for selected dates, it cannot be concluded that one of both volatility types is correct or both false. Nevertheless, the results of the analysis of the historical volatility suggest the null hypothesis of constant volatility is questionable because historical volatilities (of the same date) differ with interval length. However, the above analysis does not constitute a direct statistical test on implied volatility. A more direct statistical test is conducted in section 3.5.3, however in section 3.5.2 below further descriptive evidence mounts against the constant volatility assumption.

3.5.2 The Volatility Smile Index and the Volatility Term Structure Index⁶³

The objective of the following empirical research is to assess the realism of the constant volatility assumption by relying on both quantitative and graphical analysis of the implied volatility. The purpose of the graphical analysis is to display the implied volatility across different strike prices and across expirations for the South African option market. It is assumed (as null hypothesis) initially that such implied volatilities (across strike prices and across expirations) are constant, as required by the modified Black model.

This research therefore implements two procedures to establish a summary across strike prices and across expirations. The first procedure involves the computation of the "Volatility Smile Index" (VSI) which enables the assessment of the constant volatility assumption across strike prices. The VSI, when depicted graphically, shows the relationship between implied volatility and strike prices. The second procedure involves the computation of the "Volatility Term Structure Index" (VTSI) which enables the assessment of the constant volatility assumption through time, i.e. across expirations. The VTSI, when depicted graphically, shows the relationship between implied volatility and expirations.

Both procedures, the VSI and the VTSI, are based on a standardization technique by Tompkins (1994). We have amended his procedure to take account of the South African environment. Because the procedures are fairly elaborate and tedious, details of them are relegated to Appendix 3.D.

The analysis of Tompkins (1994) is further expanded with a distinction between call options and put options for all three indices. While Tompkins (1994) analyses only calls and puts combined, the examination of the calls and the puts separately constitute an attempt to gain new insights into the behaviour of implied volatility.

Often, only the last trading price is used for empirical research which has the disadvantage that the last trading price can be strongly influenced by tactical or manipulative orders. For example, traders often try to manipulate the last trading price to change their settlement price. Hence, a criticism of Tompkins (1994) is his use of only one

⁶³ The methodology and parts of the results have been published in Wandmacher and Bradfield (1998a).

trading price per day. The approach here chosen constructs a data set from four trading prices (first, last, high, and low) for each particular strike price and for each particular expiration per trading day. A similar approach is employed by Garman and Klass (1980) who use the high, low, opening, and closing prices for the estimation of historical volatility. They claim that they obtain a higher accuracy with their method compared to the use of only the closing price.

The entire VSI and VTSI analysis is based on the assumption that the implied volatility can be computed from the option prices. As a start, it is necessary to assume that the modified Black model is suitable for the computation of the implied volatility in the South African context. As an explicit solution is not possible using the modified Black model (or any other option price model), the iterative method of Bisection⁶⁴ is selected to assist in the estimation of the implied volatility.

The results of the implied volatility examination in the South African option market are presented below for the ALSI (the results of the INDI and the GLDI are relegated to Appendix 3.E). Although similar, some differences between the three indices are highlighted in a conclusion of the VSI and VTSI results (in section 3.5.2.2).

3.5.2.1 Empirical Evidence of Volatility Smiles and of Volatility Term Structures

The empirical evidence is quantitatively summarized and thereafter the results are graphically analysed. The analysis focuses on the VSI and VTSI separately.

Firstly, the results of the implied volatility analysis concerning the VSI are shown in Table 3.5.2.1-1. The VSI is produced using only the nearest expiration per trading day to avoid noise from the term structure. For computation purposes, the data is divided into

⁶⁴ The method of Bisection is explained in detail in the section "Implied Volatility" in Chapter 2 in section 2.3.4.

three categories: firstly the call and put data combined, secondly the call data separately, and thirdly the put data separately⁶⁵.

Table 3.5.2.1-1

The Analysis of the Smile Pattern through the VSI for the ALSI

The index volatility values are sorted in strike classes and in expiration classes for the defined expiration classes. Only index volatility values are used from trading days that have at least one trading price.

Data Category	Expiration Class	Strike Classes				
		deep out-of-the-money 0.60 to 0.85	out-of-the-money 0.85 to 0.95	at-the-money 0.95 to 1.05	in-the-money 1.05 to 1.15	deep in-the-money 1.15 to 1.40
Call/Put	1 to 10	NA	172.34	115.92	126.77	108.11
	11 to 20	494.49	145.50	109.61	126.68	115.56
	21 to 30	305.91	134.76	106.40	113.88	128.93
	31 to 60	223.31	115.60	106.26	112.55	110.09
	61 to 90	185.64	112.79	104.97	110.55	110.16
	91 to 180	139.58	107.86	102.66	103.43	112.88
	181 to 270	106.32	103.69	102.71	106.77	106.11
	271+	103.13	102.15	100.31	104.48	102.10
Call	1 to 10	NA	152.19	102.35	122.06	99.79
	11 to 20	324.21	157.69	103.94	107.20	109.35
	21 to 30	274.89	158.24	100.94	112.74	139.33
	31 to 60	227.28	124.34	104.78	106.67	108.43
	61 to 90	154.40	111.48	102.55	104.54	109.07
	91 to 180	129.05	101.97	100.49	100.66	106.38
	181 to 270	100.85	100.04	100.38	102.51	100.81
	271+	121.22	97.18	100.66	103.10	106.10
Put	1 to 10	93.98	137.46	125.17	101.71	NA
	11 to 20	101.56	134.16	112.46	158.34	NA
	21 to 30	162.69	123.98	108.54	128.03	NA
	31 to 60	188.06	118.97	110.70	114.37	113.71
	61 to 90	153.37	113.82	108.44	106.25	159.15
	91 to 180	129.50	108.37	102.81	127.82	NA
	181 to 270	105.68	106.77	106.71	99.65	NA
	271+	107.11	103.40	106.14	115.49	101.45

*NA indicates that trading did not take place at least once.

a. The strike class headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent to the volatility.

From Table 3.5.2.1-1, which summarizes the VSI results, it is evident that the index volatility in all strike classes *decreases* with increasing time to expiration (i.e. higher expiration class), see for example the combined call and put data (i.e. call/put). Here, the *out-of-the-money*, the *at-the-money* and the *in-the-money* index volatility of the first expiration class (172.34, 115.92, and 126.77 respectively) *decrease* towards the last expiration class (102.15, 100.31, and 104.48 respectively).

⁶⁵ The three data formats are required to examine potential peculiarities and differences between calls and puts.

The classical “smile” discussed in international literature has the lowest index volatility *at-the-money* and higher index volatilities *in-the-money* and *out-of-the-money*, however the data does not conform exactly to this pattern. From Table 3.5.2.1-1 it is evident that the “smile” for the ALSI seems to have the lowest index volatility *at-the-money* and higher index volatilities *in-the-money* and *out-of-the-money* but a higher *out-of-the-money* index volatility than *in-the-money* index volatility. See for example the call data category in the expiration class 21 to 30 days to expiration in Table 3.5.2.1-1 where the *at-the-money* index volatility (of 100.94) is below the *in-the-money* index volatility (of 112.74) which is in turn below the *out-of-the-money* index volatility (of 158.24).

Another common pattern can be found in the expiration class 31 to 60 days to expiration in the combined call and put data. Here, the deep *in-the-money* index volatility (110.09) is below the *in-the-money* index volatility (112.55) so that no full “smile” appears. The appearance is more like a skewed “grin”.

In Figure 3.5.2.1-1, it is evident that the “smiles” and the “grins” are more pronounced the nearer the time to expiration. The closeness to expiration and probable distortions (e.g. roll-overs) seem to support the pronunciation of the “smiles” and the “grins” especially in the first and second expiration class.

In sum, it is concluded that “smiles” and “grins” do exist for the ALSI. The “smiles” sometimes appear more like “grins” in reality, which is consistent with a study of British stocks by Duque and Paxson (1994). Their finding that the “smile effect” increases as the time to expiration decreases matches the findings here closely. Hence, it is concluded that the null hypothesis of a constant volatility is again questionable based on the findings.

A graphical analysis illustrating the “smiles”, the “grins”, and their characteristics follows in Figure 3.5.2.1-1. The graphical analysis consists of the eight curves (for each expiration class) of the combined call and put data in Table 3.5.2.1-1. It can be observed from Figure 3.5.2.1-1 that the first “smiles” or “grins” are more pronounced and that they are nearly flat for high expiration classes. The “smiles” or “grins” are also skewed to the *out-of-the-money* strike class. Furthermore, it can be observed that the *at-the-money* index volatility levels grow from near to far to expiration and that the far *out-of-the-money* index volatility levels decrease from near to far to expiration indicating a flatter index volatility pattern for high expiration classes.

The analytical and graphical analysis is also consistent with the findings of the implied volatility smiles by Derman and Kani (1994). They also find a volatility skew (grin) for the volatility across different strikes having high implied volatility levels for *out-of-the-money* strikes and a flat or only slightly rising volatility for *in-the-money* strikes.

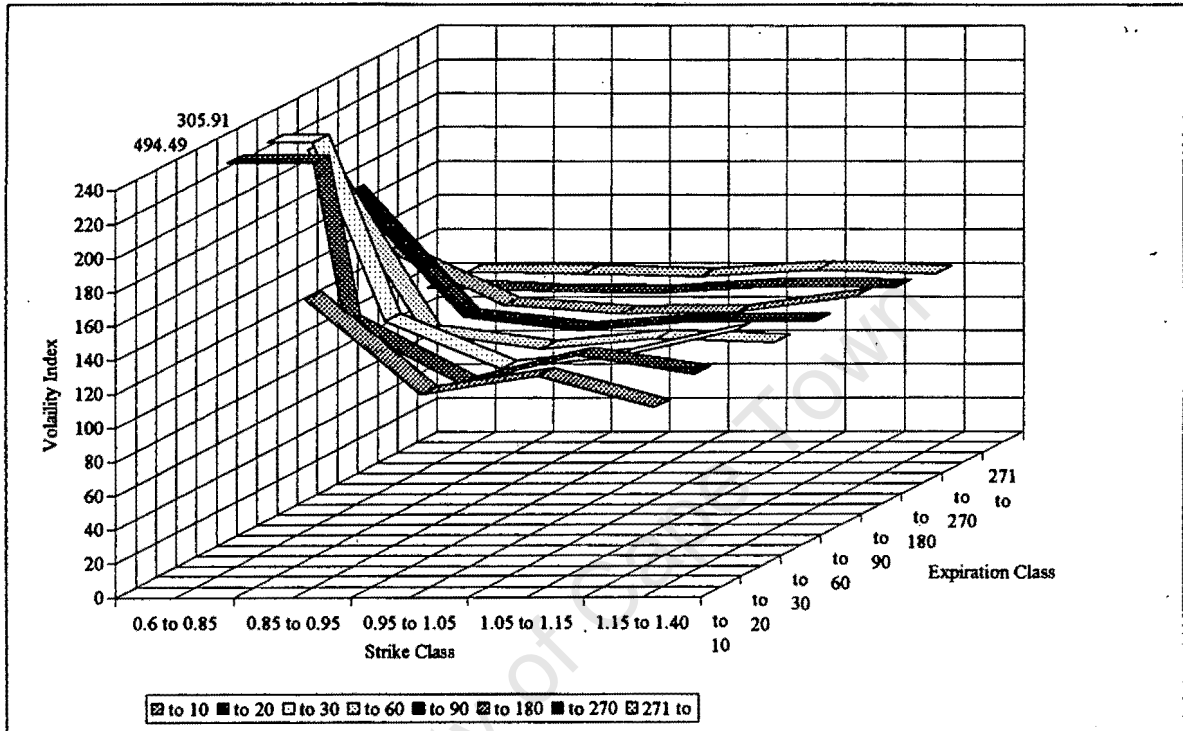


Figure 3.5.2.1-1. Eight Smile and Grin Patterns for the Combined Call and Put Data of the ALSI

The index volatility values are sorted into the strike classes and expiration classes. The volatility values (494.49 and 305.91) for the second and third expiration classes are not plotted exactly to prevent a distortion of the graphic.

Both the quantitative and graphical analysis of the VSI for the ALSI suggest the rejection of the null hypothesis of constant volatility, however the appearance of implied volatility across different expirations is continued to be examined in the VTSI analysis below. The results of the implied volatilities concerning the VTSI are shown in Table 3.5.2.1-2.

Table 3.5.2.1-2

The Analysis of the Term Structure with the VTSI for the ALSI

The index volatility values are sorted in strike classes and in expiration intervals. Index volatility values are only used from trading days that have three different expirations equivalent to the expiration intervals.

Data Category	Expiration Interval (days to expiration)	Strike Class				
		deep out-of-the-money 0.6 to 0.85	out-of-the-money 0.85 to 0.95	at-the-money 0.95 to 1.05	in-the-money 1.05 to 1.15	deep in-the-money 1.15 to 1.4
Call/Put	11 to 90	207.46	102.93	91.51	109.33	NA
	101 to 180	132.74	102.38	98.43	96.69	104.85
	191 to 540	119.72	107.88	104.48	99.59	97.87
Call	11 to 90	228.80	NA	94.25	117.68	NA
	101 to 180	NA	107.05	103.82	100.90	97.69
	191 to 540	155.75	141.00	112.79	108.27	116.70
Put	11 to 90	NA	105.57	98.35	98.11	NA
	101 to 180	111.74	107.89	103.20	105.42	NA
	191 to 540	106.78	109.29	104.44	97.87	NA

*NA indicates that trading did not take place at least once.

a. The strike class headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent to the volatility.

In Table 3.5.2.1-2, however a significant trend in the volatility index for the *at-the-money* options can be observed (e.g. the call/put category has an index volatility of 91.51 for the first expiration interval but an index volatility of 104.48 for the third expiration interval). All three data categories indicate that the volatility index grows from near to expiration to far to expiration for the *at-the-money* strike class. This rising volatility index pattern can also be observed for the *out-of-the-money* strike class for all data types, whereas the deep *out-of-the-money* strike class displays the opposite effect (the volatility index decreases with rising time to expiration). The *in-the-money* strike classes by contrast present no clear picture of the direction of the volatility index for time to expiration.

A second trend is noted in the growing volatility index from *at-the-money* options to deep *out-of-the-money* options in all data types. This increasing trend in the volatility index can be seen for example by the rise from 91.51 to 207.46 in the first expiration interval in the combined call and put category. The *in-the-money* index volatility is more distorted but two directions can be identified. The index volatility decreases from *at-the-money* to *in-the-money* strike classes, or it increases. The increase of the index volatility from the *at-the-money* to *in-the-money* strike classes in combination with the increase from the *at-the-money* to *out-of-the-money* strike classes can be described as “smile”, whilst the decrease of the index volatility from the *at-the-money* to *in-the-money* strike classes in combination with the increase from the *at-the-money* to *out-of-the-money* strike classes can be described

as “grin”. A good example of the “smile” effect is the first expiration interval of the call/put category (from 91.51 to 109.33), whereas the “grin” is aptly described by the third expiration interval of the combined call and put category (from 104.48 to 97.87).

It is perhaps a little bit surprising to find no deep *in-the-money* puts in Table 3.5.2.1-2. Normally, deep *in-the-money* options are not bought; instead, the underlying is sold because the delta⁶⁶ of the options is already equal to one. Hence, a deep *in-the-money* put only occurs if existing put options become deep *in-the-money* (as a consequence of a large fall in the market). Deep *in-the-money* puts cannot be found in the data because no large fall in the ALSI is reported for the period between October 1992 and December 1996. Consequently, it is less surprising that nearly no deep *in-the-money* put options can be found.

In sum, the evidence suggests that the null hypothesis of a constant volatility through time is questionable as clear trends are evident in the term structure of implied volatility. This is especially notable for the *at-the-money* volatility index (that grows from near to expiration to far to expiration).

A further graphical analysis of the VTSI displays the properties discussed above. The graphical analysis consists of a three-dimensional layer (for the three levels of expiration intervals) of the combined call and put data in Table 3.5.2.1-2 and is depicted in Figure 3.5.2.1-2.

It can be observed from Figure 3.5.2.1-2 that the layers across expiration intervals become less skewed for the far to expiration range. In addition, the *at-the-money* volatility grows from near to far to expiration and that the deep *out-of-the-money* volatility decreases from near to far to expiration.

Figure 3.5.2.1-2 shows explicitly that the *at-the-money* volatility grows from near to far to expiration. This result is consistent with evidence in the S&P 500 option market by Derman and Kani (1994). They present the term structure for only one day but also find an increasing volatility with the time to expiration for *at-the-money* options.

⁶⁶ The delta of an option measures the rate of change of its price with respect to the underlying price. A delta of one means that the option changes in the same way as the underlying.

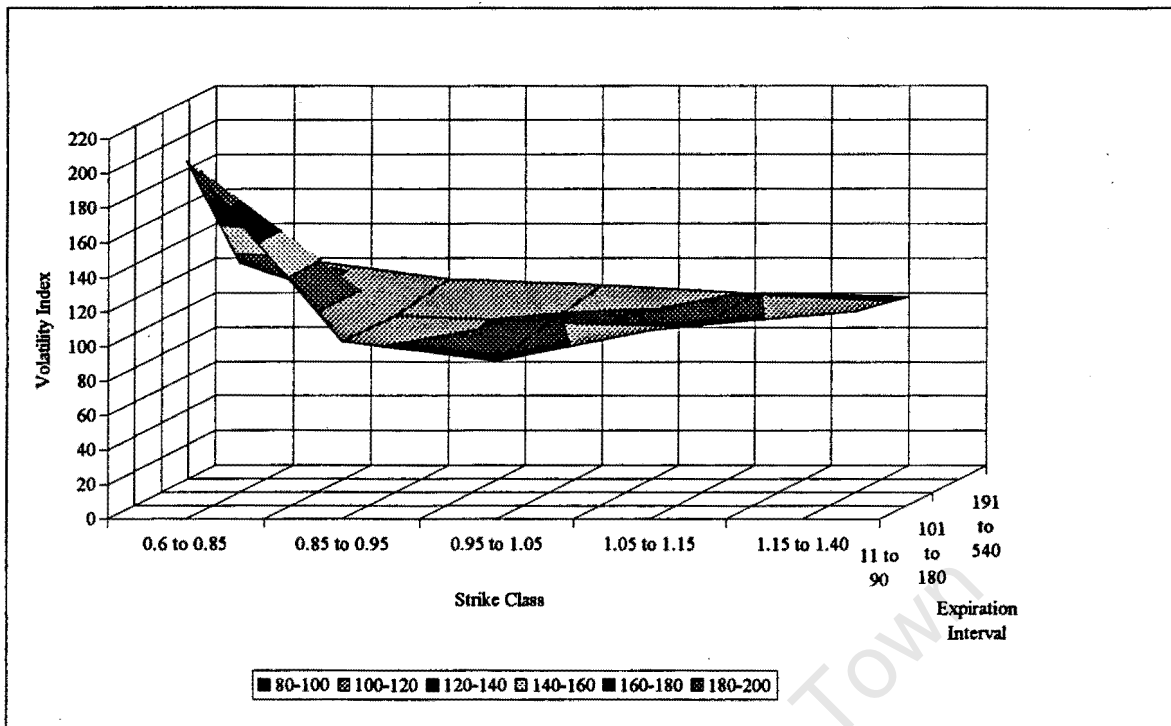


Figure 3.5.2.1-2. Three-dimensional Layer of the Term Structure for the Combined Call and Put Data of the ALSI

The index volatility values are sorted into the strike classes and expiration intervals. Missing data points are interpolated to produce a layer of the term structure over the expiration intervals.

3.5.2.2 Conclusion and Summary

The purpose of the VSI and the VTSI analysis was the assessment of the null hypothesis that the implied volatility is constant across different strike prices and through time. All the results suggest that the null hypothesis of constant volatility is not appropriate because the patterns found for implied volatility differ from the constant volatility assumption. The shapes of the patterns were also found to differ systematically across the various expiration classes and can be seen follow the “smile” patterns found in the North-American, UK, and Dutch markets. Moreover, no significant difference between the three data sets, the call, the put, and their combination (i.e. call and put) was evident. The most important results are briefly summarized below:

It was found that the implied volatility increases with time to expiration for the ALSI and the INDI whilst the implied volatility decreases with time to expiration for the GLDI. The term structure of the implied volatility was most significant for the *at-the-money*

options whereas the term structure effect becomes smaller or is reversed the more the options depart from the *at-the-money* range. In addition, it was observed that *out-of-the-money* options tend to have a higher implied volatility the more they are *out-of-the-money*. *At-the-money* and *in-the-money* options tend to have constantly lower implied volatilities than *out-of-the-money* options except for some expiration classes in the GLDI (where unrepresentative values of the small database might have caused the exceptions). This described pattern of a low *at-the-money* implied volatility combined with a higher *in-the-money* and *out-of-the-money* implied volatility is then referred to as “smile”. Otherwise, *in-the-money* implied volatility is often flatter or lower compared to *at-the-money* implied volatility so that “grins” arise with higher *out-of-the-money* implied volatilities.

The “smiles” or “grins” seem to diminish with the higher the time to expiration, and they are most significant the shorter the time to expiration. It can therefore be concluded that long-term options with more than 180 days to expiration are significantly less influenced by their “moneyness”. In addition, the fact that the “smiles” or “grins” become more significant the shorter the time to expiration can be caused by distortions near to expiration. For example, distortions can be caused by rollovers from the near to expiration options to options with a longer expiration.

In sum, the results confirm that the assumption of constant volatility required by the modified Black model (and others) is unrealistic in the South African environment. In the pursuit of greater accuracy in the pricing of options in South Africa, it is recommended that models that do not rely on the constant volatility assumption may therefore be more suitable.

3.5.3 Nonparametric Tests of Implied Volatility⁶⁷

The advantage of nonparametric tests by comparison to the VSI and VTSI analysis is that not only do they represent direct statistical tests but that they are “distribution-free” (in the sense that they assume nothing about the underlying population from which the sample is drawn). The nonparametric methodology is implemented in several papers (see for

⁶⁷ The methodology and parts of the results have been published in Wandmacher and Bradfield (1998b).

example Sheikh (1991) and Heynen (1994)) and has become a standard methodology for tests of implied volatility. To the author's knowledge this nonparametric methodology has not been implemented on American options on futures in the South African environment. In addition, the conclusions of the VSI and VTSI analysis are confirmed using the more rigorous nonparametric tests.

The advantage of the nonparametric tests is to assess **statistically** the null hypothesis of constant volatility across strike prices and across expirations (as required by the modified Black model). The results of the nonparametric tests are then presented in the form of tables for the *time-to-expiration bias* test as well as for the *striking price bias* test.

This research therefore applies the nonparametric approach of Rubinstein (1985a) to analyse the implied volatility across strike prices and across expirations. However, due to the different environment in South Africa, the nonparametric tests require modification. The two main differences are summarized below:

1. The nonparametric research of implied volatilities by Rubinstein (1985a), Sheikh (1991), and Heynen (1994) is only performed for options where the underlying asset is a non-derivative (Rubinstein (1985a) examines options on shares, Sheikh (1991) investigates options on the S&P 100 index, and Heynen (1994) analyses options on the Dutch European Options Exchange Index). In the South African environment, only options on futures (i.e. a derivative underlying) have a sufficient history of trading to conduct such an analysis. The influence of the derivative underlying on the analysis is explained in Appendix 3.F under the heading "South African Modifications".
2. The studies by Rubinstein (1985a), Sheikh (1991), and Heynen (1994) only implement data from call options (that cannot be exercised prematurely). Instead, a data set is utilized where the call and put data are combined because early exercise is not profitable in the South African environment due to the applied mark-to-market procedure for the option price at SAFEX⁶⁸. In addition, the results for two additional data sets having call and put data partitioned separately are also presented.

⁶⁸ Margin yields market-related short-term interest rates at SAFEX.

Two more differences to Rubinstein's (1985a) methodology arise through the implementation of the nonparametric methodology in South Africa. The first difference concerns the option price model used to compute the required implied volatility estimates. In the South African environment, the modified Black model is applied. This option price model represents a significant simplification (to the Black and Scholes (1973) model used by Rubinstein (1985a)) because the modified Black model does not require the interest rate for the computation of the implied volatility. Hence, the results are immune from the biases of incorrectly approximated interest rates, whilst the results of several of the above mentioned studies may well be influenced by such a problem.

The second difference to Rubinstein's (1985a) methodology is the use of futures as underlying assets in the South African situation. The use of futures has one important advantage compared to non-derivative underlyings (i.e. the index) in that the dividend calculation and estimation are not required. The dividend estimation is very costly with large data sets and is a source of many approximation errors that are avoided by the use of derivative underlyings (as in the ensuing analysis).

Having discussed the major departures from the prior research above, the focus moves onto the construction of the nonparametric tests below.

A first step in the construction of nonparametric tests is the categorisation of each option price as a function of the strike ratio and time to expiration. Details of the definition of the categories are similar to Rubinstein (1985a), but are modified for the South African environment. These details are relegated to Appendix 3.F and are discussed under the heading "Category Definitions".

The categorised option prices are required to identify "pairs" (as described in Rubinstein (1985a)) for the nonparametric tests. A pair for the *time-to-expiration bias* test is defined as two option prices that are observed at the same date in the same constant price interval of the same underlying and that have the same strike price, but that have different expirations. A pair for the *striking price bias* test is defined as two option prices that are observed at the same date in the same constant price interval of the same underlying and that have the same expiration, but that have different strike prices. Each option price is only used once for each nonparametric test.

Both definitions of pairs for the *time-to-expiration bias* test and for the *striking price bias* test require modifications for the South African environment. These modifications are also explained in detail in Appendix 3.F under the heading “South African Modifications”.

The number of pairs in each data set is shown in Table 3.5.3-1, where the pairs of the ALSI are calculated for the call and put data combined, the call data separately, and the put data separately. It can be observed that 3128 pairs are defined for the *time-to-expiration bias* test compared to 6439 pairs for the *striking price bias* test in the combined call and put data set.

Table 3.5.3-1
Number of Pairs in Different Data Sets

The numbers of pairs are shown for each index and their subsequent data sets. Both, the pairs for the *time-to-expiration bias* test and the pairs for the *striking price bias* test are displayed.

Data Set	ALSI		INDI		GLDI	
	Striking Price Bias Test	Time-to-Expiration Bias Test	Striking Price Bias Test	Time-to-Expiration Bias Test	Striking Price Bias Test	Time-to-Expiration Bias Test
Call and Put	6439	3128	1493	846	1282	181
Call	3851	1212	642	364	954	63
Put	2840	863	723	238	325	8

Henceforth, the techniques⁶⁹ by Rubinstein (1985a) are implemented for the calculation of the nonparametric test results of the *time-to-expiration bias* test [alternately the *striking price bias* test]. The test methodology contains two components. First, a probability, presented by p , is calculated assuming the null hypothesis of a constant volatility is true. According to Rubinstein (1985a) the probability, p , is computed from the equation:

$$p = 1 - N\left(\frac{(Sh + 0.5) - (Tn / 2)}{\sqrt{Tn / 2}}\right) \quad (1)$$

where Sh is the number of pairs that have a higher implied volatility with a shorter expiration [alternately with a lower strike ratio]. The total number of pairs is denoted with Tn while $N(\cdot)$ represents the standard normal distribution function.

The resulting probability, p , should be equal to 0.5 according to the null hypothesis of constant volatility. A probability, p , close to 0 gives evidence that the shorter expiration [alternately the lower strike ratio] has a higher implied volatility. A probability, p , close to

⁶⁹ These techniques are also applied by Heynen (1994) and Sheikh (1991), although Sheikh (1991) gives an incorrect formula for the statistical significance test.

1 by contrast, implies that the longer expiration [alternately the higher strike ratio] has a higher implied volatility. Under a standard 5% significance level test⁷⁰, the null hypothesis would be rejected if the probability, p , is between 0 and 0.025, or when it is between 0.975 and 1.

The statistical significance test is supplemented by a second nonparametric test with a measure of economic importance⁷¹. According to Rubinstein (1985a), the test for economic importance requires the median difference in implied volatilities to be zero in each comparison⁷² (under the null hypothesis of a constant volatility). Nevertheless, the median difference in implied volatilities has nothing to do with a bias from the modified Black option values. Hence, the percentage difference between the traded option prices and their theoretical modified Black values is equated for the two option prices in each pair. This is done by using the implied volatility as input variable. According to Rubinstein (1985a), the solution of the equation (2) below results in a measure of economic importance, α , (discussed further below).

$$\frac{B_1^{\text{theo.}}(\sigma) - P_1}{P_1} = \frac{P_2 - B_2^{\text{theo.}}(\sigma)}{P_2} \equiv 0.01\alpha \quad (2)$$

where P_i is the traded price for each option in the pair ($i = 1, 2$) and $B_i^{\text{theo.}}$ is the theoretical option price of the modified Black model for the implied volatility, σ . The measure of economic importance is presented by α .

The economic importance, α , is the value that displays the minimum difference between the market price and the theoretical modified Black value for the option prices in a pair. The α is therefore the lower bound of economic bias in the modified Black model. In accordance with Rubinstein (1985a), the median of the α 's is used for each comparison of pairs.

The null hypothesis implies that the probability, p (measure of statistical significance) should be 0.5 and the α (measure of economic importance) should be zero. Although the

⁷⁰ With a 10% significance level test, the null hypothesis would be rejected for probabilities, p , between 0 and 0.05 or 0.95 and 1.

⁷¹ The measure of economic importance is originally referred to as economic significance in Rubinstein (1985a). The potential ambiguity between statistical significance and economic significance leads to the more cautious reference of economic importance.

⁷² For example, a comparison between shorter and longer expiration with 10 pairs may reveal in each pair a difference in the implied volatility. The result for the comparison is the median of the differences in the implied volatility for the 10 pairs.

rejection of the null hypothesis can only be based upon the statistical significance, the economic importance nevertheless provides additional information to assess the implications of the size of the statistical significance for the practical use. This distinction is expanded below.

Statistical Significance and Economic Importance

A result may be statistically significant but may still not be pronounced enough to be practically useful. Hence, a statistically significant result is perhaps not always sufficient for practical purposes. In such cases, the computed measure of economic importance (similar to the jargon of economic significance in Rubinstein (1985a)) will be low. By contrast, if the computed measure of economic importance is high, it would suggest that the statistically significant result is practically useful⁷³.

The results of the examination of biases for implied volatility in the South African option market are presented for the ALSI below. For brevity, the results of the INDI and the GLDI are relegated to Appendix 3.G and 3.H (these results are very similar to the findings of the ALSI). However, differences between the three indices are highlighted in the conclusion and summary of the *striking price biases* and *time-to-expiration biases* (in section 3.5.3.2).

3.5.3.1 Empirical Evidence of Striking Price Biases and Time-to-Expiration Biases

The empirical evidence for the combined call and put data of the ALSI is presented in Table 3.5.3.1-1 for the *time-to-expiration bias* test and in Table 3.5.3.1-2 for the *striking price bias* test. Further evidence is presented in Appendix 3.H for the call data separately (Table 3.H-1 and Table 3.H-2 respectively) and the put data separately (Table 3.H-3 and Table 3.H-4 respectively).

Recognising that the presentation of our results in tabulation form are unavoidably complex, a similar table structure to those found in Rubinstein (1985a), Sheikh (1991), and

⁷³ Furthermore, statistically *insignificant* results with either a low measure of economic importance or a high measure of economic importance are possible. Such results are defined as unimportant.

Heynen (1994) is implemented. Due to this unavoidable complexity of the tables, a brief explanation of their structure begins this section. The results for the INDI and the GLDI are presented using the same table structure and are found in Appendix 3.G (for the combined call and put data) and in Appendix 3.H (for the separate call and the separate put data).

Table 3.5.3.1-1 shows the results for the *time-to-expiration bias* test on the ALSI (Table 3.H-1 for the separate call data and Table 3.H-3 for the separate put data respectively) for three panels (10-90 vs. 100-180, 10-90 vs. 190-540, and 100-180 vs. 190-540). The results for each panel are presented for each of the strike ratios (0.60-0.85, 0.85-0.95, 0.95-1.05, 1.05-1.15, and 1.15-1.40) in Table 3.5.3.1-1. The first column of each strike ratio displays the total number of pairs, T_n , for a panel. The second column of each strike ratio shows the value of the economic importance, α , on the first line and the value of the probability, p , (or statistical significance) on the second line. Hence, the third column of each strike ratio represents the number of pairs for each panel where the shorter expiration, Sh , has a higher implied volatility than the longer expiration. The first column value T_n and the third column value Sh are required in equation (1) to calculate the probability, p . This computed probability, p , is given as the second value in the second column.

Table 3.5.3.1-2 summarizing the results of the *striking price bias* test for the ALSI (and Table 3.H-2 and Table 3.H-4 in Appendix 3.H for the separate call data and the separate put data respectively) is constructed in similar manner to Table 3.5.3.1-1. The only differences are that ten panels exist (0.60-0.85 vs. 0.85-0.95, 0.60-0.85 vs. 0.95-1.05, ..., and 1.05-1.15 vs. 1.15-1.40) and that the results for each panel are presented across days to expiration (10-30, 30-60, 60-90, 90-180, 180-270, >270). Hence, column one and two across days to expiration in Table 3.5.3.1-2 are defined analogous to Table 3.5.3.1-1, whilst column three across days to expiration represents the number of pairs for each panel where the lower strike ratio, Sh , has a higher implied volatility than the higher strike ratio.

Having focused on the structure of the tables, the focus moves to the results within the tables below.

Table 3.5.3.1-1

Nonparametric Time-to-Expiration Bias Test for the Combined Call and Put Data of the ALSI

The test is carried out for nearly identical calls and puts only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the- money 0.60-0.85			out-of-the- money 0.85-0.95			at-the- money 0.95-1.05			in-the- money 1.05-1.15			deep in-the- money 1.15-1.40		
	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh
10-90 vs. 100-180	0	NA	0	29	2.19	11	609	5.41	137	16	NA	10	0	NA	0
		NA			0.87			1.00**			NA			NA	
10-90 vs. 190-540	16	NA	16	112	1.94	60	972	6.47	461	86	2.71	30	5	NA	0
		NA			0.20			1.00**			1.00**			NA	
100-180 vs. 190-540	0	NA	0	34	4.61	12	216	5.44	28	30	2.93	10	3	NA	3
		NA			0.94			1.00**			0.95*			NA	

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

A discussion of the results of the *time-to-expiration bias* test for the ALSI begins the analysis. The results of the *time-to-expiration bias* test for the ALSI are shown in Table 3.5.3.1-1. From Table 3.5.3.1-1, it is evident that the *at-the-money* ratios result in the rejection of the null hypothesis at the 5 percent level because their probabilities are 1.00 (i.e. the longer time to expiration has the higher implied volatility). Additional to the statistical significance, the measure of economic importance is also large having values between 5.41 and 6.47. Similar results can be observed in Appendix 3.H for the call data separately in Table 3.H-1 and for the put data separately in Table 3.H-3. This pattern of a higher implied volatility for the longer time to maturity was also documented by Rubinstein (1985a) for *at-the-money* options (in the first period⁷⁴ of his research). Sheikh (1991) finds a similar pattern in one of three investigated subperiods of his research.

⁷⁴ Rubinstein (1985a) differentiates his research into two periods.

Moreover, it can be observed in Table 3.5.3.1-1 that the statistical significance decreases from the *at-the-money* ratio to the *in-the-money* ratio (also in Table 3.H-1 and Table 3.H-3 for the separate call data and the separate put data respectively). See for example Table 3.5.3.1-1 where the statistical significance in the third panel (i.e. 100-180 vs. 190-540) for the *in-the-money* ratio (i.e. 1.05-1.15) is only statistically significant at the 10 percent level ($p = 0.95$). The economic importance also declines simultaneously with the decrease of the statistical significance and therefore indicates that the results of *in-the-money* options are less significant than the results of *at-the-money* options. The decrease of the statistical significance from the *at-the-money* to the *in-the-money* ratio is also observable in Rubinstein (1985a) and Sheikh (1991).

Additionally, the statistical significance for the *out-of-the-money* ratio diminishes to insignificant results in Table 3.5.3.1-1 (and in Table 3.H-3 for the separate put data respectively). However, Table 3.H-1 for the separate call data shows one significant result at the 10 percent level for the *out-of-the-money* ratio with a probability of 0.03 (i.e. the shorter time to expiration has the higher implied volatility). Although this result is based upon a small sample ($T_n = 22$), it indicates a substantial change in the implied volatility direction from the *at-the-money* ratio to the *out-of-the-money* ratio. This substantial change of the implied volatility implies that the *at-the-money* ratio and the *out-of-the-money* ratio have differing directions in time-to-expiration biases. Nevertheless, both ratios suggest the violation of the non-constant volatility assumption.

Concluding the discussion on the *time-to-expiration bias* test for the ALSI, the null hypothesis is rejected at a 5 percent significance level for the *at-the-money* ratio. The results of the *in-the-money* ratio suggest the rejection of the null hypothesis at the 10 percent significance level with exception of the third panel in Table 3.H-1 for the separate call data. The results of the *out-of-the-money* ratio also suggest the rejection of the null hypothesis at the 10 percent significance level. Furthermore, the levels of the economic importance are consistent with the statistical significance results. From Table 3.5.3.1-1, it is evident that the economic importance declines across the strike ratio (strike \div future) from having the highest value for the *at-the-money* ratio to lower values for the *out-of-the-money* ratio and *in-the-money* ratio.

The above discussion dealt with the *time-to-expiration bias* test for the ALSI (in Table 3.5.3.1-1, Table 3.H-1, and Table 3.H-3 for the combined call and put data as well as for the separate call and the separate put data respectively). The examination is continued with the analysis of the *striking price bias* test below.

The results of the *striking price bias* test for the combined call and put data set are shown in Table 3.5.3.1-2, whilst the results for the separate call data and put data are shown in Table 3.H-2 and Table 3.H-4 in Appendix 3.H respectively.

It is evident in the three tables that the probabilities are mostly 0 or close to 0 (i.e. the lower strike ratio has the higher implied volatility) and therefore highly statistically significant. The exceptions of these probabilities of 0 or close to 0 are observed across more than “180 days” to expiration (i.e. 180-270 and >270) and in the *at-the-money/in-the-money* panel (i.e. 0.95-1.05 vs. 1.05-1.15).

For the combined call and put data in Table 3.5.3.1-2, the results of the *at-the-money/in-the-money* panel (i.e. 0.95-1.05 vs. 1.05-1.15) show a decreasing probability as the time to expiration increases. For example, the probability decreases from 1.00 to 0 across the “10-30” to the “>270 days” to expiration in Table 3.5.3.1-2. The results (1.00 and 0) are statistically significant at the 5 percent level and suggest to reject the null hypothesis of a constant volatility.

The “more than 180 days” to expiration categories (i.e. 180-270 and >270) show several statistically significant results at the 5 percent level but also a few insignificant results. It seems that with increasing time to expiration the results become less significant. This conclusion is supported by very low values for the economic importance for longer time to expirations. However, the economic importance⁷⁵ also shows relatively small values for statistical significant results. Furthermore, our results are similar to the findings by Rubinstein (1985a) (for his first period investigated).

⁷⁵ The economic importance may be questionable because of its frequent low value in the tables for the striking price bias test. Heynen (1994) compares the α with the bid-ask spread and only views α 's higher than 3 percent as significant. Rubinstein (1985a) argues that the economic importance depends on the market participant and the purpose of the option. He also explains that α is only the minimum deviation from the modified Black value for both options in the pair, and that the economic importance is only designed to show the weakness of the assumption of constant volatility. We think that some market participants make riskless arbitrage profits with small deviations from zero in the option markets if almost all market participants calculate the option prices according to the modified Black model. Riskless arbitrage profits conflict with the assumption of constant volatility so that low values of the economic importance can be interpreted as an indicator of inefficiency of the modified Black model.

Table 3.5.3.1-2

Nonparametric Striking Price Bias Test for the Combined Call and Put Data of the ALSI

The test is carried out for nearly identical calls and puts only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike ÷ future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration																						
	10-30		30-60		60-90		90-180		180-270		>270		Sh										
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh											
0.60-0.85 vs. 0.85-0.95	2	NA	2	28	-1.53	0**	25	39	-2.40	0**	34	166	-0.33	0.01**	98	215	-0.37	0.01**	124	331	0.15	0.81	157
0.60-0.85 vs. 0.95-1.05	3	NA	2	9	NA	NA	9	0	NA	NA	0	23	-2.64	0**	20	47	-0.21	0.12	27	110	-0.52	0**	69
0.60-0.85 vs. 1.05-1.15	0	NA	0	0	NA	NA	0	0	NA	NA	0	0	NA	NA	0	0	NA	NA	0	8	NA	NA	8
0.60-0.85 vs. 1.15-1.40	0	NA	0	0	NA	NA	0	0	NA	NA	0	0	NA	NA	0	0	NA	NA	0	16	NA	NA	16
0.85-0.95 vs. 0.95-1.05	106	-4.74	71	250	-3.35	0**	17	274	-1.21	0**	166	641	-0.41	0**	353	466	-0.33	0**	272	921	-0.25	0.01*	498
0.85-0.95 vs. 1.05-1.15	0	NA	0	7	NA	NA	6	1	NA	NA	1	32	-1.38	0**	25	8	NA	NA	8	95	-1.16	0**	65
0.85-0.95 vs. 1.15-1.40	0	NA	0	0	NA	NA	0	0	NA	NA	0	0	NA	NA	0	0	NA	NA	0	8	NA	NA	8
0.95-1.05 vs. 1.05-1.15	76	1.96	26	157	0.78	0.79	73	182	0.03	0.47	91	533	-0.38	0**	300	359	-0.23	0.05*	195	750	-1.33	0**	524
0.95-1.05 vs. 1.15-1.40	4	NA	0	0	NA	NA	0	0	NA	NA	0	23	-1.34	0.05*	15	20	-0.44	0.25	11	71	-1.56	0	52
1.05-1.15 vs. 1.15-1.40	2	NA	1	9	NA	NA	3	19	NA	NA	8	121	-0.72	0.05*	69	118	0.30	0.68	56	189	-0.09	0.19	100

- a. The ratio of strike price and future price headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

The empirical results suggest that the lower strike ratio has predominantly higher implied volatilities. These results become more pronounced in terms of the economic importance with declining strike ratios and with decreasing time to expiration. Rubinstein's (1985a) results also correspond favourably to the obtained results.

In sum, it is concluded that the lower strike ratio generally has the higher implied volatility for the ALSI in South Africa, in particular for *out-of-the-money* comparisons (panel 1 to 7 in Table 3.5.3.1-2). Nevertheless, it is also observed that the pronunciation decreases for increasing time to expiration because the results become less significant (statistically and economically) with increasing time to expiration. Furthermore, the exception of a higher implied volatility for a higher strike ratio for the *at-the-money/in-the-money* panel (i.e. 0.95-1.05 vs. 1.05-1.15) with a short time to expiration seems significant and should be taken account of in option-pricing decisions. Finally, the null hypothesis of constant volatilities across strike prices for options on ALSI futures in South Africa is rejected.

A conclusion and a summary follow below where the results for the time-to-expiration bias test and the striking price bias test are summarized and where the results of the INDI and the GLDI are included.

3.5.3.2 Conclusion and Summary

The purpose of the nonparametric tests was to test the null hypothesis that the implied volatility is constant across strike prices and across expirations. The results of the tests, based upon statistical significance and economic importance, suggest that the null hypothesis of constant volatility should be rejected. Furthermore, a distinction between the three data sets, the call, the put, and their combination was found to be unnecessary because no significant difference was found between them. The most important results are briefly summarized below:

The nonparametric *time-to-expiration bias* test revealed that the implied volatility generally increased with time to expiration in the *at-the-money* range for the ALSI, the

INDI, and the GLDI. Similar results are found by Rubinstein (1985a) and Sheikh (1991). Furthermore, it was observed that the implied volatility is generally higher for the lower strike ratio for the ALSI and the INDI. Evidence in Heynen (1994) and Rubinstein (1985a) also show similar results. However, the results of the GLDI showed that the implied volatility is sometimes higher for the higher strike ratio but based on only a very small data set for the GLDI. Finally, it was found that the pronounced effect of non-constant implied volatility decreases (i.e. becomes more constant) with increasing time to expiration for all three indices.

A plausible explanation for why the higher implied volatility for lower strike ratios is evident is because such options increased in value as a consequence of the market crash in October 1987. The expectation of large profits due to a large fall in market prices is consequently priced in the options by higher implied volatilities. Additionally, one might expect that the effect of higher implied volatilities would be more pronounced for a short time to expiration than for a long time to expiration. This is because a large fall in market prices tend to take place in a very short period of time (e.g. the market crash in 1987) so that an option with long time to expiration has a higher probability of a market recovery. Hence, the option with a long time to expiration has a less pronounced effect of volatilities across expirations. Finally, a plausible explanation for the finding that the implied volatilities for *at-the-money* strikes increase using the *time-to-expiration bias* test may be explained by the additional risk of significant price movements the longer the time to expiration.

To the author's knowledge the methodological adaptations to derivative underlyings as opposed to traditional implementation on non-derivative underlying assets is the first of its kind. The advantage of implementing the nonparametric test methodology on options on *futures* is twofold. Firstly, no dividends have to be calculated or estimated. The widely used approximation of dividend yields instead of the discrete dividends is one of the problems that we avoid. Secondly, options on *futures* are mark-to-market and therefore are not influenced by any interest rate. Consequently, the use of approximate interest rates like the Treasury bills or the Bank bills are not required and approximation errors do not therefore occur. Furthermore, early exercise of the American options on *futures* is not

optimal. Hence, all data can be used and does not have to be filtered because of the early exercise problems experienced by Rubinstein (1985a).

Nevertheless, some shortcomings are alluded to the analysis: the main weakness of the nonparametric analysis is the unavoidable non-simultaneous price problem. This non-simultaneous price problem arises because the price of the options and the price of the futures might not be traded at the same time. Although the same trading classes for option and future prices are implemented, this problem cannot totally be avoided. Sheikh (1991) assumes that artificial prices are a further problem, especially for the daily mark-to-market products at the end of the day because market makers may try to influence their margin requirements by manipulating the closing bid and ask prices⁷⁶.

In sum, the results obtained reject the assumption of constant volatility required by the modified Black model (among others) for the South African environment. Hence, the pricing of options in South Africa requires models that do not rely on the constant volatility assumption.

3.6 Conclusion of Chapter 3

The results of the descriptive assessment concerning the comparison between historical volatility and implied volatility (section 3.5.1) as well as the results of the VSI and VTSI analysis (section 3.5.2) suggest the rejection of the assumption of constant volatility across strike prices and across expirations for the South African environment. These results were confirmed using the more rigorous nonparametric test approach (section 3.5.3). All the results are consistent across all approaches with the partly exception of the GLDI. However, the GLDI analysis is based on a very small data set so that distortions are thinkable caused by the lack of data.

Hence, the suitability of the modified Black model must also be questioned for option pricing in South Africa (as its underlying assumption of constant volatility is found unrealistic). These results found for the South African market are consistent with the

⁷⁶ The daily mark-to-market price is calculated as the mid-market price at the closing of trading at SAFEX.

mentioned findings in various international markets. In particular, the finding that *out-of-the-money* puts have higher implied volatilities than expected (by the modified Black model and the Black and Scholes (1973) model) is consistent with results in the literature on other markets. These higher implied volatilities especially for *out-of-the-money* puts may well be a consequence of the market crash in October 1987. On this occasion, the actual probability of large and sudden losses was substantial higher than expected by the lognormal distribution implemented in the Black and Scholes (1973) model, or the modified Black model.

The results obtained of volatility biases, or “smiles”, are the basis on which a new option price model with non-constant volatility should be constructed for the South African environment. This aim forms a major focus of the subsequent work in this thesis. The viewpoint is adopted that the option price model should price options according to the information contained in the market. Hence, it is proposed that volatility inputs be implemented in the “proposed” option price model by incorporating the implied volatilities from all traded options across strike prices and expirations.

The development of this “proposed” option price model is presented in the following chapter.

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Appendix 3.A

A Detailed Analysis of the Data

The aim of this appendix is a more detailed analysis of the data, especially the examination of dependencies on the option type or the strike price. A description of the trading data is essential and interesting in the case of a new exchange like SAFEX. The trades are listed for each trading year separately for calls and puts for the ALSI, the INDI, and the GLDI in Table 3.A-1. In Table 3.A-1, it can be observed that the trading volume was zero for the GLDI and the INDI in 1992 and also very small for the ALSI with 164 trades. It is also evident from Table 3.A-1 that the trading volume has increased substantially for each of the following years since then.

Table 3.A-1
Data Categorized in Trading Years

The data is reported for the ALSI, INDI, and GLDI after screening for arbitrage violations and mistake trades in the period 16 October 1992 to 31 December 1996. The number of trades separated for calls and puts are shown for each of the years in the data.

Year	ALSI		INDI		GLDI	
	Calls	Puts	Calls	Puts	Calls	Puts
1992	108	56	0	0	0	0
1993	3454	2462	900	830	1288	578
1994	4839	4383	1302	1324	752	370
1995	5177	4064	1797	1534	1493	296
1996	6661	4640	2446	2792	1591	575
Total	20239	15605	6445	6480	5124	1819

Additionally, the option series are further divided and their distribution is examined across strike ratios⁷⁷ in Table 3.A-2. Hence, the total number of options is differentiated into five strike classes⁷⁸. Beginning with a strike class of "below 0.85", four more strike classes are constructed: "0.85 to 0.95", "0.95 to 1.05", "1.05 to 1.15", and "above 1.15". Finally, the option series are additionally divided into "Call" and "Put" trades.

A strike class of "0.95 to 1.05" is interpreted as an *at-the-money* option for either call option or put option. However, the interpretation is not that simple for other strike classes. All strike classes below the *at-the-money* class are interpreted for calls as *in-the-money* or for puts as *out-of-the-money* respectively. *Vice versa*, all strike classes above the *at-the-money* class are interpreted as *out-of-the-money* calls or *in-the-money* puts respectively.

This interpretation is correct if the validity of the put-call parity by Black and Scholes (1973) is assumed. The put-call parity implies that an *out-of-the-money* put has the same parameters (e.g. volatility, interest-rate, and time) as an *in-the-money* call and *vice versa*. Hence, call and put options in each strike class can be interpreted as the same concerning their parameters.

⁷⁷ A strike ratio is defined as the division of the strike price by the underlying asset price.

⁷⁸ A strike class defines all strike ratios that are included in a particular range of strike ratios.

Table 3.A-2

Distribution of the Number of Traded Options to their Strike Class

The three indices are divided in five strike classes and additionally in call and put trades to assess the distribution of the traded options in them. The last column shows the sum of trades depending on its type for each index. The numbers in bold are the percentage numbers in percent calculated with respect to the sum of call and put in the last column. The other numbers are the absolute values of traded options.

Index	Type	Strike Classes					Sum
		below 0.85	0.85 to 0.95	0.95 to 1.05	1.05 to 1.15	above 1.15	
ALSI	Call	703	2090	12109	4128	1209	20239
		1.96	5.83	33.78	11.52	3.37	56.46
	Put	1093	4937	8813	743	19	15605
		3.05	13.77	24.59	2.07	0.05	43.54
INDI	Call	25	342	5253	759	66	6445
		0.19	2.65	40.64	5.87	0.51	49.86
	Put	98	1225	5054	103	0	6480
		0.76	9.48	39.10	0.80	0.00	50.14
GLDI	Call	45	302	2906	1293	578	5124
		0.65	4.35	41.86	18.62	8.32	73.80
	Put	176	453	1087	87	16	1819
		2.53	6.52	15.66	1.25	0.23	26.20

The distribution of the traded option prices with respect to the class is presented in Table 3.A-2. Table 3.A-2 shows that most frequently traded options are *at-the-money* options (ALSI with 58.37 percent *at-the-money* options, GLDI with 57.51 percent *at-the-money* options, and INDI with 79.74 percent *at-the-money* options). Additionally, the distribution of trading prices is skewed towards the strike classes below *at-the-money* for put options and above *at-the-money* for call options for each of the indices in Table 3.A-2. These options are therefore *out-of-the-money* options and are traded more often than *in-the-money* options in the referenced period. Table 3.A-2 displays that the ALSI data exhibited 31.71 percent *out-of-the-money* options, the GLDI contains 36.01 percent *out-of-the-money* options, and the INDI contains 16.62 percent *out-of-the-money* options. The remainder⁷⁹ of the number of traded option prices is the *in-the-money* options with 9.92 percent of the ALSI, 6.48 percent of the GLDI, and 3.64 percent of the INDI.

The analysis of the differentiation between calls and puts produces interesting results for the *out-of-the-money* options as well. The portion of *out-of-the-money* calls from the overall *out-of-the-money* options is 46.95 percent for the ALSI, 74.84 percent for the GLDI, and 38.41 percent for the INDI. The GLDI with 74.84 percent in *out-of-the-money* calls with respect to all *out-of-the-money* options is extremely high. However, if the overall number of trading prices is considered, the results of the INDI and GLDI should be interpreted more cautiously because of the sparse data in comparison to the ALSI results (which is based on a larger data set).

⁷⁹ Inaccuracies are possible because of rounding.

Table 3.A-3

Distribution of the Number of Traded Options around the At-the-money Strike

The three indices are divided in three strike classes and further separated in call and put trades to assess the distribution of the traded option prices around the *at-the-money* strike. The last column shows the sum of trades depending of its type for each index. The numbers in bold are the percentage numbers in percent calculated with respect to the sum of call and put in the last column. The other numbers are the absolute values of traded options.

Index	Type	Strike Classes			Sum
		0.95 to 0.99	0.99 to 1.01	1.01 to 1.05	
ALSI	Call	2849	5065	4195	12109
		13.62	24.21	20.05	57.88
	Put	4013	3382	1418	8813
		19.18	16.16	6.78	42.12
INDI	Call	833	2513	1907	5253
		8.08	24.38	18.50	50.97
	Put	2487	2011	556	5054
		24.13	19.51	5.39	49.03
GLDI	Call	763	890	1253	2906
		19.11	22.29	31.38	72.78
	Put	539	307	241	1087
		13.50	7.69	6.04	27.22

Finally, the *at-the-money* strike class (0.95 to 1.05) is further partitioned because of its high percentage of all trading prices. Hence, the *at-the-money* strike class is further split in three intervals in Table 3.A-3 displaying very similar results to the results obtained by the prior analysis of all strike classes. However, the number of trades of near *out-of-the-money* put options (0.95 to 0.99) is larger than the number of trades for *at-the-money* options (0.99 to 1.01) in all three indices. Another difference is found in the results of the GLDI in Table 3.A-3. The distribution of the trading prices for the GLDI exhibits a larger number of near *out-of-the-money* calls than *at-the-money* calls.

Appendix 3.B

Theoretical Model Derivation

In this section of the appendix a detailed derivation of an option price model for options on futures in the South African environment follows. Black's (1976) model for European options on futures gives a first idea for the valuation of options on futures in South Africa. However, it cannot be applied to the options on futures in South Africa in its current form because Black's (1976) model does not handle the future-style mark-to-market procedure of the option premium at SAFEX. Nevertheless, it is imaginable to modify the Black (1976) model to incorporate the future style mark-to-market procedure. Hence, the development of the option price model is based upon the Black (1976) model that is again based upon the Black and Scholes (1973) model. To understand what modifications are required in the Black (1976) model in order to be applicable to the South African environment, the proposed modifications to the Black (1976) model need to be derived.

Before the start of an technical in-depth analysis of the option price model, the assumptions by Black and Scholes (1973) for an option price model are discussed in the context of the South African environment.

Assumptions

Following the "ideal conditions" from the Black and Scholes (1973) model for the option and the underlying, it is assumed that

1. the option is European style (i.e. that the option can only be exercised at expiration)
2. the underlying pays no dividends or other distributions
3. no transaction costs in buying or selling the underlying or the option are imposed and no taxes exist. Short selling is allowed and borrowing any fraction of a security is possible. The borrowing rate equals the lending rate.
4. the short-term interest rate is constant and known.
5. the underlying follows a random walk in continuous time. This implies that the underlying has a constant volatility and that the price path of the underlying is continuous. The distribution of possible underlying prices is lognormal at the end of any finite interval.

The assumption of Black and Scholes (1973) that the option is European style is obviously inappropriate for the South African environment. Nevertheless, it is initially assumed to be appropriate for the purpose of the derivation of the model. Later, this assumption is relaxed and the early exercise feature of American options is included into the option price model. Additionally, the assumption about no dividend payments from the underlying is assumed to be appropriate in the South African context with futures as underlying.

The third assumption of short selling being allowed is also appropriate in South Africa. In addition, the assumption that no transaction costs and taxes occur is considered as a simplified approach to the option modelling problem. Although transaction costs and taxes exist in reality, both are particularly dependent on the personal situation of the market participant. Moreover, the possibility of borrowing fractions and of borrowing or lending at the same interest rate depends again on the status of each market participant but is assumed

appropriate in South Africa. For example, large market participants such as investment banks or life insurer have nearly identical borrowing and lending rates.

The fourth assumption concerning the short-term interest rate is examined in the course of the model development.

The fifth and last assumption of the Black and Scholes (1973) model that the underlying follows a random walk will be considered as appropriate for the purposes here.

The discussed assumptions are the framework on which the technical development of the option price model is based below.

After the discussion of the assumptions by Black and Scholes (1973) underpinning an option price model, a technical in-depth analysis of an option price model for the South African environment follows. It is initially assumed by Black (1976) that the future follows a random walk much like the spot underlying assumed by Black and Scholes (1973) first. This process can be described by the following model of future price behavior where the instantaneous future price change relative is

$$dF/F = \mu dt + \sigma dz \quad (3.B-1)$$

with σ as the instantaneous standard deviation, μ as the expected instantaneous price change of the future contract⁸⁰, and dz as a Wiener process⁸¹.

Hence, a riskless hedge portfolio is created with the future, F , and the option on the future, f , as contents. The option, f , is in this process, a function of the future, F , and of the time, t . The portfolio is defined as risk-free if it consists of one option short, $(-f)$, and $\frac{\partial f}{\partial F}$ futures long (or *vice versa*) as the hedge condition in a infinitesimal time interval dt . This hedged portfolio is risk-free under the assumptions of a continuous adjustment of the hedge and a market without any restrictions⁸². Hence, the value of the portfolio, Π , is the combined value of both positions:

$$\Pi = -f + \frac{\partial f}{\partial F} F$$

Arbitrage would normally provide an expected rate of return of this portfolio equal to the short-term interest rate as described by Black and Scholes (1973). However, under the assumption of a portfolio consisting of a future and an option on a future with future-style mark-to-market procedure, the expected rate of return differs from the result in Black and Scholes (1973).

⁸⁰ Whaley (1986a) defines the expected futures price change relative, μ , as equal to the expected spot price change relative, α , less the difference between the riskless rate of interest, r , and the assumed continuous rate of receipt, d , [hence $\alpha - (r - d)$]. He assumes additionally that the spot price, S , follows the stochastic differential equation

$$dS/S = \alpha dt + \sigma dz$$

with the standard deviation, σ , that is the same for both the underlying spot and futures price changes.

⁸¹ A Wiener process is a particular type of a Markov stochastic process. The Markov process considers only the present price as relevant for the future. Hence, the Wiener Process contains the randomness that is required for the random walk.

⁸² Restrictions are transaction costs, short selling limitations or differences in the borrowing or lending interest rate.

The value of a future contract is zero at inception of the trade and so is the value of an option on futures with future-style mark-to-market procedure. Asay (1982) argues that risk-free hedges between the future contract and the option can therefore be established without any investment. This means for the portfolio value at inception that its value is zero.

$$\Pi = 0$$

The change of the portfolio value, $d\Pi$, with the change of time, dt , depends on the change in the option position, $(-df)$, and on the change of the future position, dF , where the dt , df , and dF are the symbols for an infinitesimal change of the respective parameter t , f , and F . The change of the portfolio value is

$$d\Pi = -df + \frac{\partial f}{\partial F} dF \quad (3.B-2)$$

The change of the portfolio value (i.e. the value of the hedge) is not zero in dt , so that the solution is not trivial. The solution of the problem gives an insight into the correct option price model. Hence, the change of the portfolio value has to be analysed. The equation of the change in portfolio value above contains two parts, df and dF , that must be defined.

The model for the future price behavior (in equation 3.B-1) delivers the definition for dF that is required to solve the first definition problem. One still has to define df to obtain a solution for the option price model. However, the definition of df needs more attention to the theoretical analysis. Hence, the findings of Ito (see for example, Merton (1992) or Hull (1993)), Ito's process and Ito's lemma⁸³ are introduced. The equation of future price behavior (equation 3.B-1) is an Ito process on which Ito's lemma can be applied.

Hence, Ito's lemma is (in consistent notation of the thesis):

$$df = \frac{\partial f}{\partial F} dF + \frac{\partial f}{\partial t} dt + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 f}{\partial F^2}$$

The lemma is then applied to the Ito process (in equation 3.B-1). This is done by substituting dF (the change of the future position) with equation 3.B-1 into Ito's lemma. Hence, the result of the utilized lemma is

$$df = \left(\mu F \frac{\partial f}{\partial F} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 f}{\partial F^2} \right) dt + \sigma F \frac{\partial f}{\partial F} dz$$

After df (the change of the option position) is defined, the equation can be solved with respect to the change of the portfolio value (equation 3.B-2). Substituting for $-df$ and $\frac{\partial f}{\partial F} dF$ in $d\Pi$ (equation 3.B-2), the change of the portfolio value is

⁸³ Ito's lemma can be applied on functions with a stochastic process that is known as Ito process. The Ito process is a generalized Wiener process where A , the expected drift rate, and B , the expected variance rate, are functions of the value of any random variable G and time t :

$$dG = A(G, t)dt + B(G, t)dz$$

where dz is a Wiener process. Ito's lemma is

$$df = \frac{\partial f}{\partial G} dG + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial G^2} B^2 dt$$

By substituting dG in Ito's lemma by the function of the Ito process, which depends on G and t , Ito's lemma shows that the Ito process follows

$$df = \left(A \frac{\partial f}{\partial G} + \frac{\partial f}{\partial t} + \frac{1}{2} B^2 \frac{\partial^2 f}{\partial G^2} \right) dt + B \frac{\partial f}{\partial G} dz$$

$$d\Pi = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 f}{\partial F^2}\right)dt$$

The change of the portfolio value, $d\Pi$, is risk-free because of the constructed hedge. Additionally, it must be shown that the portfolio is arbitrage-free. Otherwise arbitrageurs could profit from incorrectly priced options. Hence, the change in the portfolio value must be equal to short-term risk-free securities that can earn the short-term rate of interest, r , in the same time. It is derived that

$$d\Pi = r\Pi dt$$

If the change of the portfolio value, $d\Pi$, is substituted by $r\Pi dt$ (as an arbitrage condition), it follows that

$$-\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 f}{\partial F^2}\right)dt = r\Pi dt$$

However, the value of the portfolio is zero ($\Pi = 0$), because neither the future nor the option has to be paid for. Hence

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 f}{\partial F^2} = 0$$

This result can also be achieved in another manner (see also Lieu(1990)). As noted, the change of the portfolio value is risk-free. Hence, the initial investment is zero for the hedge. These conditions mean that arbitrage is only prevented if the return is zero.

$$d\Pi = 0$$

Hence

$$-\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 f}{\partial F^2}\right)dt = 0$$

This solution is identical to the first result by simply multiplying with (-1)

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 f}{\partial F^2} = 0$$

The intention at the start of the theoretical model derivation was to modify the Black (1976) model for the conditions prevailing in South Africa. Consequently, the Black partial differentiation is compared briefly to the derived one above. The Black partial differentiation is

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 f}{\partial F^2} - rF = 0$$

The difference between the Black partial differentiation and the derived partial differentiation is obvious. The component of $(-rF)$ is absent in the derived result for the South African option market. The difference between both partial differentiations can only be explained by the mark-to-market feature of options on futures because all other assumptions and conditions are the same for both models.

The consequences of the absence of $(-rF)$ are important when one considers the assumptions for the South African environment. Hence, the assumption relating to a known and constant short-term interest rate is no longer relevant because the interest rate is irrelevant in the option price model for South Africa. In addition, the assumption about borrowing any fraction of the price of a security to buy or to hold it at the short-term interest rate is also no longer of importance for the option price model because of two reasons:

1. no initial investment has to be made
2. the interest rate has no function in the South African option price model

All consequences of the derived partial differentiation on the option price model for South Africa will be seen in the option price formula.

The option price formula for options on futures with future-style mark-to-market procedure can only be obtained if the boundary conditions are applied to the derived partial differentiation. These boundary conditions however require definition for European options. However, the boundary conditions are different for calls and puts, hence they are taken account of separately. The boundary conditions are valid for the time to expiration, $T-t$, and are for calls on futures, C_F , with strike price, X :

$$\begin{aligned} C_F(F, t) &= 0 && \text{for } F = 0 \\ C_F(F, t) &\sim F && \text{for } F \rightarrow \infty \\ C_F(F, T) &= \max(F - X, 0) \end{aligned}$$

The boundary conditions for puts on futures, P_F , are

$$\begin{aligned} P_F(F, t) &= F && \text{for } F = 0 \\ P_F(F, t) &\sim 0 && \text{for } F \rightarrow \infty \\ P_F(F, T) &= \max(X - F, 0) \end{aligned}$$

The boundary conditions for the derived option price model are equivalent to the boundary conditions of the Black (1976) model. Using the Black and Scholes (1973) method to solve the partial differentiation, the option price formula is consequently produced for the option price model in the South African environment.

This solution is however only appropriate for European options on futures (as assumed above). Consequently, a further step in the development of the option price model must be the incorporation of options with the American exercise style. The solution of the early exercise is directly related to the result of European options. Hence, only the boundary conditions require extension. The new boundary conditions (for either a call or a put) incorporate the early exercise right. Hence, the fourth boundary conditions are

$$\begin{aligned} C_F(F, t_C) &= \text{Max}\{F-X, C_F(F, t_C)\} && \text{for } t \leq t_C < T \\ P_F(F, t_C) &= \text{Max}\{X-F, P_F(F, t_C)\} && \text{for } t \leq t_C < T \end{aligned}$$

where $C_F(F, t_C)$ or $P_F(F, t_C)$ is the value of the call or the put option respectively if not exercised at time t_C (where t_C is between the current time, t , and the time of expiration, T).

However, the early exercise of an option is only a problem for the valuation if the early exercise brings an advantage compared to holding the option. One condition for the early exercise is that the option must be *in-the-money* otherwise the early exercised option would lose the time value of the option.

Consequently, the probability of the profitable early exercise of an *in-the-money* call or an *in-the-money* put option is assessed in the South African environment. If it is not profitable to exercise early, no difference would exist between American and European options. Hence, the same option price model could be used for both expiration types. The proposed problem of the early exercise value and its application to the options on futures in South Africa is assessed in a technical analysis below.

Early Exercise Value

The problem of the early exercise value is first established for a call option. The condition for an optimal early exercise is that the call option is *in-the-money* (i.e. the future price is higher than strike price of the option), so that

$$F > X$$

where F is the price of the future and X is the strike price.

Moreover, the formula for a call option from the Black (1976) model with absent interest rate (as in South Africa) is

$$C_F = FN(d_1) - XN(d_2)$$

where

$$d_1 = \frac{\ln \frac{F}{X} + (\sigma^2 / 2)(T - t)}{\sigma \sqrt{(T - t)}}$$

$$d_2 = d_1 - \sigma \sqrt{(T - t)}$$

with $N(\cdot)$ as the cumulative standard normal distribution and σ as the volatility.

In addition, the fourth boundary condition requires (in the case of an early exercise) that the value of the call option should be lower than the difference between future and strike price, $(F - X)$. It follows

$$C_F < (F - X)$$

Hence, C_F is substituted by the option price formula of the call. Hence

$$FN(d_1) - XN(d_2) < F - X$$

This condition establishes consequently whether an early exercise is profitable.

The formulas for d_1 and d_2 from above imply that d_1 is higher than d_2 ($d_1 > d_2$) for $F > X$, $\sigma > 0$ and $T > t$. Hence

$$N(d_1) > N(d_2)$$

and

$$FN(d_1) - XN(d_2) > F - X$$

This result suggest that early exercise of call options is not profitable because at all times is the price of the call option higher than the early exercise value i.e.

$$C_F > F - X$$

The solution for the *in-the-money* put option can be derived in the same way⁸⁴. The result is the same as for the call option, an early exercise is never profitable.

In summary, the discussion of the early exercise value of American options on futures with the future-style mark-to-market procedure reveals that either the call option value or the put option value always exceeds the early exercise value. Consequently, in South Africa it is never optimal under the condition of a mark-to-market of the option premium to exercise a American call or a put option on futures before expiration⁸⁵. Hence, the American call options on futures with the mark-to-market procedure are valued like European options on futures with the mark-to-market procedure.

⁸⁴ The *in-the-money* condition for the put option is shown as $X > F$. The early exercise of the *in-the-money* put option would be only profitable if $P_F < X - F$. If P_F is substituted using the formula for the put price (analogous to the price of the call option), the condition for a profitable early exercise becomes

$$XN(-d_2) - FN(-d_1) < X - F$$

Moreover, it is shown that $N(-d_2) > N(-d_1)$ and consequently that $P_F > X - F$. Hence, the put value exceeds the early exercise value always. Consequently, an early exercise of the put option is never profitable.

⁸⁵ Lieu (1990) comes to the same conclusion for this type of option.

Appendix 3.C

Comparisons between Historical Volatility and Implied Volatility for the INDI and the GLDI

Table 3.C-1

A Comparison between Historical Volatility and Implied Volatility for the INDI

The *at-the-money* implied volatility is calculated for options on the INDI with an expiration of around one calendar month later. The historical volatilities are based on the future of the INDI with one month to expiration and they are calculated on the basis of the closing price over 21, 42, and 63 trading days prior to the referenced date. The nearest date to one month to expiration is chosen for the recorded date.

Date	Implied	Historical Volatility over prior days			Expiration
	Volatility	21 Days	42 Days	63 Days	
	At-the-money				
13/08/93	0.1797	0.1155	0.1091	0.1119	September
15/02/94	0.2738	0.2382	0.2238	0.2097	March
15/08/94	0.1866	0.1291	0.1830	NA	September
15/02/95	0.1736	0.2297	0.1776	0.1670	March
15/08/95	0.1645	0.1449	0.1203	NA	September
15/02/96	0.1187	0.0998	0.1146	0.1177	March
15/08/96	0.1900	0.1812	0.1785	NA	September

*NA indicates that trading takes place at least once a day for each trading day.

Table 3.C-2

A Comparison between Historical Volatility and Implied Volatility for the GLDI

The *at-the-money* implied volatility is calculated for options on the GLDI with an expiration of around one calendar month later. The historical volatilities are based on the future of the GLDI with one month to expiration and they are calculated on the basis of the closing price over 21, 42, and 63 trading days prior to the referenced date. The nearest date to one month to expiration is chosen for the recorded date.

Date	Implied	Historical Volatility over prior days			Expiration
	Volatility	21 Days	42 Days	63 Days	
	At-the-money				
13/08/93	0.4505	0.4632	0.4560	0.5159	September
15/02/94	0.5914	0.5005	0.5247	NA	March
15/08/94	0.3499	0.2599	0.2909	0.4691	September
15/02/95	0.4815	0.5014	0.4347	0.3944	March
15/08/95	0.3737	0.2810	0.2755	NA	September
15/02/96	0.6223	0.4583	NA	NA	March
15/08/96	0.3475	0.1740	0.2090	NA	September

*NA indicates that trading takes place at least once a day for each trading day.

Appendix 3.D

Construction of the VSI and VTSI⁸⁶

The construction of the VSI follows along similar lines to that of Tompkins (1994). The VTSI is an additional extension to Tompkins (1994) where the implied volatility is examined across expirations. Both, the VSI and VTSI use standardization techniques where the levels of the underlying as well as the levels of the implied volatility are standardized. These standardisation techniques implemented further in the research have the advantage that historical patterns of implied volatility can be compared through time.

The analysis of the VSI and the VTSI requires some fairly technical manipulations, we briefly outline the procedures used together with some definitions below:

Strike Ratio

A strike ratio is produced by dividing the strike price of the option by the price of the underlying.

Volatility Index

The volatility index uses all options per trading day. The option with the strike ratio nearest to 1.0 for a trading day is defined as the *at-the-money* option of the trading day. The volatility of the *at-the-money* option is then employed as denominator for the volatility of all options for the trading day. The result of the division is finally multiplied by 100 to produce the index volatility (e.g. the *at-the-money* option for each trading day has an index volatility of 100).

Each volatility index value is accompanied by its strike ratio and the time to expiration measured in days. They thus form a three-dimensional data unit for each option price. The next step requires the construction of the VSI and VTSI for each trading day.

The construction of the daily VSI and VTSI again requires some definitions that we will provide briefly:

Strike Class

The index volatility is sorted according to the value of the strike ratio and partitioned into one of the following five intervals of strike classes:

1. Deep *out-of-the-money* (0.60 to 0.85)
2. *Out-of-the-money* (0.85 to 0.95)
3. *At-the-money* (0.95 to 1.05)
4. *In-the-money* (1.05 to 1.15)
5. Deep *in-the-money* (1.15 to 1.40)

The five selected strike classes are sufficient for the examination of the data concerning the constant volatility assumption across strike prices. However, a finer separation of the strike classes is required if the implied volatility shall be forecasted as in Tompkins (1994).

The expressions like *in-the-money* or *out-of-the-money* are defined in section 3.3. For example, the *out-of-the-money* strike class describes *out-of-the-money* puts and *in-the-money* calls. The definition is based on the put-

call parity so that the *in-the-money* volatility of a call is the same as the *out-of-the-money* volatility of a put. Hence, *in-the-money* is used for *in-the-money* puts and *out-of-the-money* calls.

Expiration Class (only applicable to the VSI)

Only options with the nearest expiration per trading day are used. Each index volatility with its particular days to expiration is then sorted according to the number of days to expiration into one of the following expiration classes:

1. 1 to 10 days to expiration
2. 11 to 20 days to expiration
3. 21 to 30 days to expiration
4. 31 to 60 days to expiration
5. 61 to 90 days to expiration
6. 91 to 180 days to expiration
7. 181 to 270 days to expiration
8. 271+ days to expiration

The first three expiration classes are chosen in this form to give insights into implied volatility near the expiration⁸⁷. The remaining expiration classes show the appearance of implied volatility over short-time to expiration until long-time to expiration.

The selected concept of expiration classes differs to Tompkins (1994) who instead implements the exact number of days to expiration. However, the difference between expiration classes and the exact number of days to expiration does not affect the results. Instead, the implementation of expiration classes might better explain the effects of time to expiration.

The daily VSI is constructed by averaging the index volatilities in each strike class per trading day. The averaged index volatilities in the strike classes are further accompanied by the corresponding expiration class. Only the nearest expiration class is used per trading day to prevent noise by the term structure (of different expirations per trading day).

The VSI is then constructed by sorting the index volatilities of the strike classes according to their expiration class over all trading days. Index volatilities with the same strike class and expiration class are averaged again. The VSI emerges with one value in each combination of strike class and expiration class.

The VTSI requires a further definition of the expiration intervals because more than one expiration per trading day has to be employed to calculate the term structure of implied volatilities. The implementation of more than one expiration per trading day is added to the concept of the VSI because the VSI incorporates only the nearest expiration per trading day. However, the VTSI requires at least two different expirations per trading day to be different to the VSI. As a consequence, the concept of expiration intervals is further explained below:

Expiration Interval (only applicable to the VTSI)

Only trading days are used where two or three different expirations respectively were traded. Each index volatility with its particular days to expiration is then sorted according to the number of days to expiration into one of the following expiration intervals:

⁸⁶ The methodology has been published in Wandmacher and Bradfield (1998a).

⁸⁷ Implied volatilities near expiration may be distorted because of rollover effects.

1. 11 to 90 days to expiration
2. 101 to 180 days to expiration
3. 191 to 540 days to expiration

Trading in the ALSI options typically takes place for all three different expirations per trading day while the GLDI and the INDI only trade in the first two different expirations per trading day. The fixed expiration intervals are chosen to account for these facts. Distorted results of the VTSI for days near to expiration are avoided by omitting the first ten days of trading similar to Rubinstein (1985a).

Moreover, the calculation of the volatility index is similar to the above method described for the VSI. However, one difference exist for the volatility index calculation of the VTSI because the *at-the-money* volatility is defined as the strike ratio that is nearest to one (independent from its expiration interval).

The daily VTSI is then calculated by averaging the index volatilities in each strike class for each expiration interval per trading day that meets the condition of three different expirations or two different expirations respectively.

Moreover, the VTSI over all trading days averages the index volatilities of the daily VTSI so that we finally obtain a layer of expiration intervals and strike classes.

Hence, a rather critical point of the calculation process emerges. The averaging of the expiration classes may destroy a change in the direction of the term structure or the smile. It can therefore be concluded from examination of the results that a different volatility structure in the market (from the assumed constant volatility by the modified Black model) exists. However, it cannot be concluded that the volatility structure is constant even if a constant volatility structure is observed because the averaging process could have averaged two opposite volatility structures to one constant volatility structure. Hence, the purpose of the VSI and VTSI methods is only to test the null hypothesis of constant volatility across strike prices and across expirations.

Appendix 3.E

Empirical Evidence of Volatility Smiles and Volatility Term Structures for the INDI and the GLDI

The INDI and the GLDI do not have the same data history as the ALSI. While the ALSI normally trades three expirations per trading day, the INDI and the GLDI traded only two expirations per trading day up until 1995. Thereafter, a third expiration has been traded sometimes but the number of days with three traded expirations is so small for the INDI and the GLDI that the definition of the VTSI requires modification. This modification of the VTSI allows us to use trading prices from days where only two different expirations are traded.

The INDI

The implied volatility analysis of the INDI begins with the VSI shown in Table 3.E-1. It is evident from Table 3.E-1 that the results of the INDI are very similar to the findings for the ALSI. For example, the index volatility tends to decrease with increasing time to expiration for the *out-of-the-money* strike class (from 199.31 down to 100.51). However, the results for expiration classes far from expiration differ substantially from the trend in the ALSI but these results of the INDI are only founded on a very small amount of data.

In sum, the results for the INDI match the findings of the ALSI very well. There is also evidence of "grins" and that the index volatilities reflect more pronounced patterns in expiration classes near to expiration than in expiration classes with more time to expiration.

Hence, it can be concluded that "grins" do exist for the INDI as for the ALSI. Consequently, the null hypothesis of constant volatility seems inappropriate on the basis of the results for the INDI.

Table 3.E-1

The Analysis of the Smile Pattern through the VSI for the INDI

The index volatility values are sorted in strike classes and in expiration classes for the defined expiration classes. Only index volatility values are used from trading days that have at least one trading price.

Data Category	Expiration Class	Strike Classes				
		deep out-of-the-money 0.60 to 0.85	out-of-the-money 0.85 to 0.95	at-the-money 0.95 to 1.05	in-the-money 1.05 to 1.15	deep in-the-money 1.15 to 1.40
Call/Put	1 to 10	NA	199.31	115.54	140.98	97.76
	11 to 20	91.97	147.04	108.39	90.79	93.63
	21 to 30	316.32	136.35	107.96	110.63	90.18
	31 to 60	258.40	120.59	105.94	105.45	106.64
	61 to 90	163.81	110.79	103.41	103.51	104.16
	91 to 180	148.98	110.35	106.28	102.52	NA
	181 to 270	99.68	94.53	100.05	109.61	NA
	271+	NA	100.51	107.22	126.83	NA
Call	1 to 10	166.48	218.07	113.44	110.17	75.66
	11 to 20	NA	218.15	103.93	102.67	102.56
	21 to 30	360.70	146.84	98.90	108.85	75.31
	31 to 60	155.32	97.49	101.83	102.28	104.73
	61 to 90	103.90	108.94	98.76	104.39	96.55
	91 to 180	NA	102.37	99.31	94.00	NA
	181 to 270	99.05	92.62	98.20	101.10	NA
	271+	NA	94.91	100.06	146.72	NA
Put	1 to 10	NA	167.59	116.94	167.72	NA
	11 to 20	103.39	137.30	111.18	NA	NA
	21 to 30	104.57	136.84	111.84	76.88	NA
	31 to 60	237.75	128.93	107.97	89.92	NA
	61 to 90	154.22	110.16	106.01	95.11	NA
	91 to 180	118.71	107.33	110.35	NA	NA
	181 to 270	96.58	100.59	102.56	100.94	NA
	271+	84.74	102.39	103.00	NA	NA

*NA indicates that trading did not take place at least once.

- a. The strike class headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent to the volatility.

A graphical analysis (similar to that of the ALSI) of the VSI for the INDI is constructed for the combined call and put data in Figure 3.E-1. The graphical analysis shows a more distorted picture for the INDI but nevertheless the "grins" and "smiles" are more pronounced for expiration classes near to the expiration. As already mentioned, the departures can be put down to the lack of data for the INDI.

In sum, the results of the VSI analysis for the INDI are very similar to the results of the ALSI. Hence, the conclusion of both VSI analyses is the same that the null hypothesis of constant volatility can be doubted for both indices.

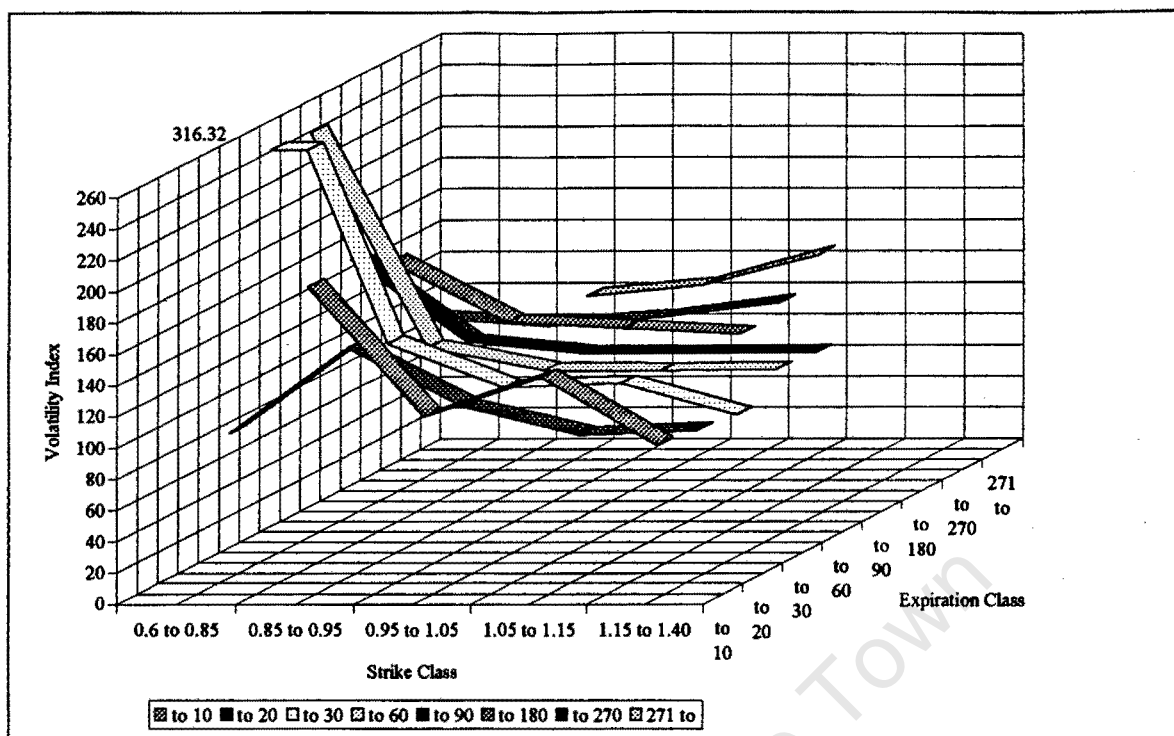


Figure 3.E-1. Eight Smile and Grin Patterns for the Combined Call and Put Data of the INDI
 The index volatility values are sorted into strike classes and expiration classes. The volatility value (316.32) for the third expiration class is not plotted exactly to prevent a distortion of the content.

The analysis of the VSI suggests that the null hypothesis is inappropriate for the behaviour of volatility across strike prices. Hence, the examination of volatility is extended with the VTSI across expirations. The summarized results of the VTSI are presented in Table 3.E-2.

Table 3.E-2
The Analysis of the Term Structure with the VTSI for the INDI

The index volatility values are sorted in strike classes and in expiration classes. Index volatility values are only used from trading days that trade simultaneously trading prices with two different expirations equivalent to the expiration intervals.

Data Category	Expiration Interval (days to expiration)	Strike Classes				
		deep out-of-the-money 0.6 to 0.85	out-of-the-money 0.85 to 0.95	at-the-money 0.95 to 1.05	in-the-money 1.05 to 1.15	deep in-the-money 1.15 to 1.4
Call/Put	11 to 90	358.48	114.30	99.10	100.07	108.32
	101 to 180	125.55	106.50	106.24	100.02	102.45
Call	11 to 90	NA	111.58	99.97	115.84	148.23
	101 to 180	NA	111.73	107.30	107.96	108.46
Put	11 to 90	NA	122.56	99.44	61.86	NA
	101 to 180	164.24	110.02	105.32	85.04	NA

*NA indicates that trading did not take place at least once.

a. The strike class headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent to the volatility.

All three data categories in Table 3.E-2 show similar results to the ALSI analysis. For example, the *at-the-money* index volatility grows from the first expiration interval to the second expiration interval. Moreover, the volatility index for the *out-of-the-money* and the *in-the-money* behaves similarly to the ALSI. In addition, the index volatility tends to increase from the *at-the-money* strike class to the *out-of-the-money* strike class. For

example, the *at-the-money* index volatility is 97.51 for the combined call and put data whereas the deep *out-of-the-money* index volatility is 350.75.

As with the ALSI, the INDI also shows evidence of problems with the availability of deep *out-of-the-money* and deep *in-the-money* strike classes (denoted by NA).

In conclusion, the null hypothesis of constant volatility is also questionable on the basis of the results of the VTSI analysis for the INDI. An additional graphical summary of the VTSI is displayed in Figure 3.E-2.

The presentation in Figure 3.E-2 is only based on two expiration intervals, hence only two lines are constructed. However, it can be observed that the index volatility function becomes less pronounced the higher the time to expiration. From Figure 3.E-2 it is also evident that the *at-the-money* index volatility grows from the near to expiration interval to the far to expiration interval.

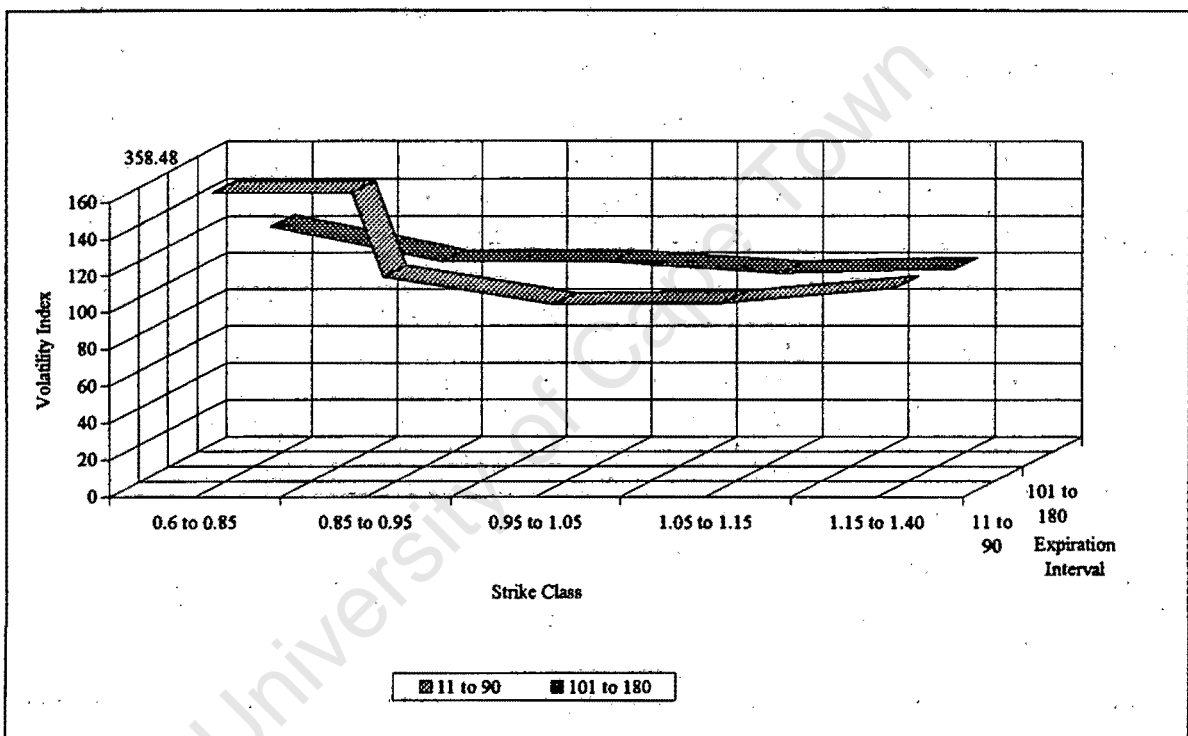


Figure 3.E-2. Two Lines of the Term Structure for the Combined Call and Put Data of the INDI

The index volatility values are sorted into strike classes and expiration intervals. The index volatility value (358.48) for the first expiration interval is not plotted exactly to prevent a distortion of the content.

The GLDI

The implied volatility analysis of the GLDI starts also with the VSI shown in Table 3.E-3. However, as mentioned in the data description the data for the GLDI is very sparse (it is approximately half of the INDI data and around a fifth of the ALSI data). Nevertheless, it is evident from Table 3.E-3 that the pattern in the expiration classes near to expiration are more pronounced than in the expiration classes far from expiration. For example, it is evident that the volatility indices of the combined call and put data are nearly constant after the fourth expiration.

These results correspond to the results of the ALSI and the INDI. A different result of the GLDI is the increasing index volatility up to the third expiration class (however, the sparse data may distort this result).

Finally, the analysis of the term structure ends with a graphical summary of the VTSI for the GLDI using the combined call and put data from Table 3.E-4. Figure 3.E-4 shows the patterns of the first and second expiration in form of two lines. It is evident from Figure 3.E-4 that the first expiration line grows from the *out-of-the-money* to the *in-the-money* strike classes while the second expiration line is nearly flat.

Although the results of the term structure analysis are different to the ALSI and the INDI results, the null hypothesis of constant volatility for the GLDI also appears inappropriate based on the findings.

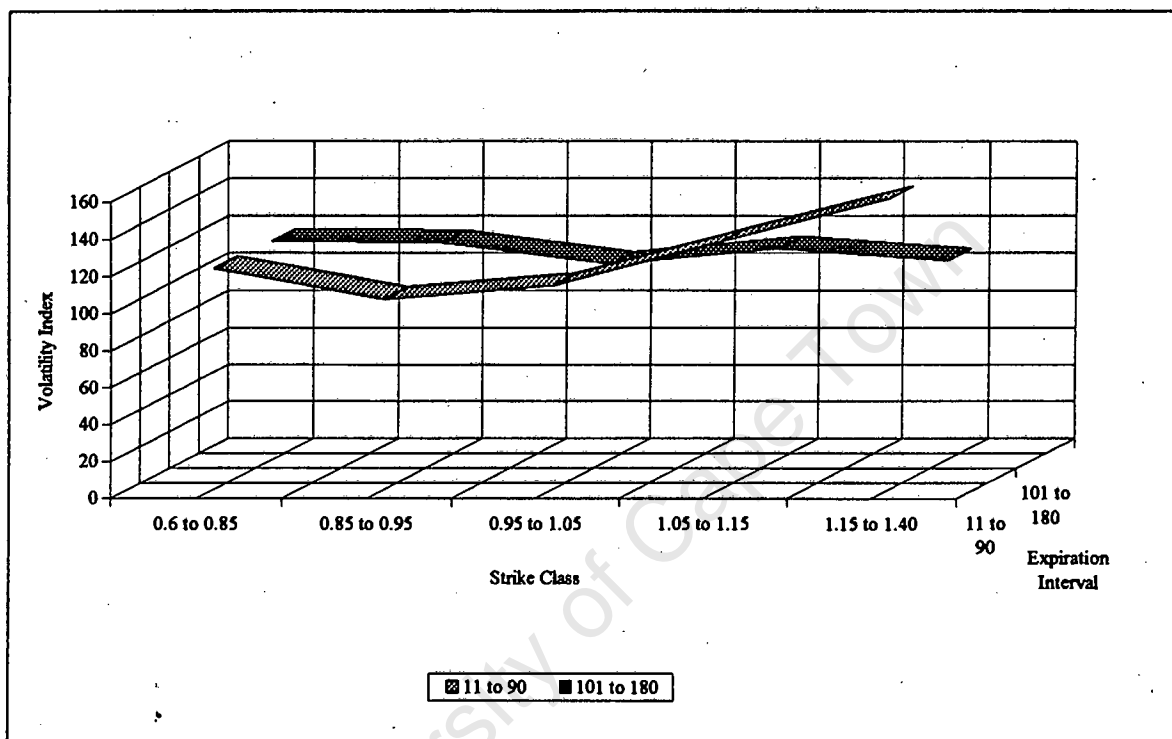


Figure 3.E-4. Two Lines of the Term Structure for the Combined Call and Put Data of the GLDI
The index volatility values are sorted into strike classes and expiration intervals.

Appendix 3.F

Definitions and Modifications for the Nonparametric Test Methodology⁸⁸

The aim of this appendix is to outline and modify Rubinstein's (1985a) methodology of the nonparametric test for the application in the South African environment. Hence, this appendix consists of two main parts:

1. category definitions
2. South African modifications

Both parts follow below.

Category Definitions

Details of the definition of the categories are modified for the South African environment for the *time-to-expiration bias* test and the *striking price bias* test. Both tests require strike classes that are defined below:

1. Deep out-of-the-money (0.60 to 0.85)
2. Out-of-the-money (0.85 to 0.95)
3. At-the-money (0.95 to 1.05)
4. In-the-money (1.05 to 1.15)
5. Deep in-the-money (1.15 to 1.40)

The definition of the time to expiration interval is different from Rubinstein (1985a) because options at SAFEX have quarterly expirations (compared to monthly expiration expirations of the stock options in Rubinstein's research). Additionally, only up to three different expirations were traded at SAFEX at the same time. As a consequence, three intervals of time to expiration are chosen for the nonparametric *time-to-expiration bias* test⁸⁹:

1. nearest expiration (10 to 90 days)
2. middle expiration (100 to 180 days)
3. far expiration (190 to 540 days)

Additionally, we differentiate between the nonparametric *time-to-expiration bias* tests and *striking price bias* tests in regard to the selection of the time to expiration intervals. Hence, more time to expiration intervals are calculated for the nonparametric *striking price bias* tests than for the *time-to-expiration bias* tests (which differs to Rubinstein's (1985a) research). The higher number of time to expiration intervals for the *striking price bias* tests is however essential to obtain a clearer impression of the market conditions across the strike classes. Nevertheless, similar to Rubinstein (1985a) the last ten trading days before expiration are omitted to prevent distortions of the near expiration. The time to expiration intervals for the *striking price bias* tests are

1. nearest expiration (10 to 30 days)
2. very near to expiration (30 to 60 days)

⁸⁸ The methodology is published in Wandmacher and Bradfield (1998b).

⁸⁹ The potential for distortions in the last trading days before expiration is taken account of by the omission of the last ten trading days. Heynen (1994) omits the last 15 trading days whereas Rubinstein (1985a) and Sheikh (1991) omit the last 21 trading days before expiration.

3. near to expiration (60 to 90 days)
4. middle expiration (90 to 180 days)
5. far expiration (180 to 270 days)
6. very far expiration (more than 270 days)

In total, the strike classes and the time to expiration intervals produce 15 categories for the nonparametric *time-to-expiration bias* test and 30 categories for the nonparametric *striking price bias* test. Hence, each option price is sorted into one of the 15 categories for nonparametric *time-to-expiration bias* tests and into one of the 30 categories for nonparametric *striking price bias* tests.

South African Modifications

Both the pair definition for the *time-to-expiration bias* test and for the *striking price bias* test must be modified for the South African environment. The first modification applies to both tests.

Rubinstein (1985a) implements rigorous criteria for both tests to prevent different trading times for options and underlying. Similarly, we define four trading classes with the first, the last, the high, and the low trading price for each option and future. Nevertheless, we must bear in mind that the use of trading classes is a relaxation of Rubinstein's (1985a) rigorous criteria.

A second modification in this thesis only concerns the *time-to-expiration bias* test. As mentioned earlier, prior research has used the nonparametric tests only on non-derivative underlyings while a derivative underlying (i.e. *future*) is the topic of interest examined here. The difference between non-derivative underlyings (e.g. stock indices, stocks) and derivative underlyings (e.g. *futures*) is particularly important because of the definition of the pairs in the *time-to-expiration bias* test. The definition of the pairs in the *time-to-expiration bias* test requires that the expirations should be different but that the strike prices are the same. Rubinstein (1985a) concludes implicitly that the *strike ratios* (strike price \div underlying price) are the same. However, such a conclusion is only valid for non-derivative underlyings.

Derivative underlyings, in particular *futures*, normally differ in their prices for different expirations. The problem with the *time-to-expiration bias* test is that the strike ratios may differ from each other because of the price differences between the *futures*. Hence, an additional condition for the *time-to-expiration bias* test is that only pairs are used that have the same strike ratio.

Appendix 3.G

Empirical Evidence of Time-to-Expiration Biases and Striking Price Biases for the INDI and the GLDI

The structure of the *time-to-expiration bias* tests and *striking price bias* tests for the INDI and the GLDI is the same as for the ALSI. The analysis will be brief and focuses primarily on the results of the INDI and the GLDI. In addition, differences to the ALSI results will be highlighted.

The INDI

The results of the *time-to-expiration bias* test of the INDI are very similar to the results of the ALSI so that only the main aspects are briefly reported. The empirical evidence of the *time-to-expiration bias* test is displayed for the combined call and put data in Table 3.G-1. The results for the separate call and put data sets are presented in Appendix 3.H in the Table 3.H-5 and Table 3.H-7 respectively.

The results in Table 3.G-1 display that the null hypothesis of constant volatility is rejected at the 5 percent level with probabilities of 1.00 for the *at-the-money* strike ratios. These probability values suggest that the longer the time to expiration, the higher the implied volatility for the *at-the-money* range. In addition to the statistical significance, the economic importance is also reflected by large values (of 2.33 and 5.44) for the *at-the-money* options in Table 3.G-1. Further evidence, particularly for other strike ratios, cannot be provided because of the deficiency of sufficient pairs for the analysis.

However, the results of the INDI are similar to that of the ALSI for the *at-the-money* ratio. Hence, it is also sufficient to reject the null hypothesis of constant volatility across expirations for the INDI.

Table 3.G-1

Nonparametric Time-to-Expiration Bias Test for the Combined Call and Put Data of the INDI

The test is carried out for nearly identical calls and puts only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the-money 0.60-0.85		out-of-the-money 0.85-0.95		at-the-money 0.95-1.05		in-the-money 1.05-1.15		deep in-the-money 1.15-1.40						
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh					
10-90 vs. 100-180	0	α p NA NA	0	4	α p NA NA	3	434	2.33 1.00**	164	3	α p NA NA	3	0	α p NA NA	0
10-90 vs. 190-540	0	α p NA NA	0	6	α p NA NA	6	360	1.92 0.94	165	16	α p NA NA	13	0	α p NA NA	0
100-180 vs. 190-540	0	α p NA NA	0	0	α p NA NA	0	21	5.44 1.00**	4	2	α p NA NA	0	0	α p NA NA	0

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

The analysis of the INDI continues with the *striking price bias* test. The results of the *striking price bias* test are shown in Table 3.G-2 for the combined call and put data (and in Appendix 3.H in Table 3.H-6 and Table 3.H-8 respectively for the separately analysed call and put data).

It is evident from all three tables that the probability value is mostly 0 or close to 0 which means that the results are therefore statistically significant at the 5 percent level. Hence, the lower strike ratio generally has the higher implied volatility (as was found for the ALSI). However, the only exceptions are probability values close to 1 for expirations of more than "180 days" to expiration in the *out-of-the-money/at-the-money* panel (i.e. 0.85-0.95 vs. 0.95-1.05) in all three tables. The exceptions are furthermore only statistically significant at the 10 percent level.

The results of the INDI also suggest to reject the null hypothesis of constant volatility across strike prices (as for the ALSI).

Table 3.G-2

Nonparametric Striking Price Bias Test for the Combined Call and Put Data of the INDI

The test is carried out for nearly identical calls and puts only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration																		
	10-30		30-60		60-90		90-180		180-270		>270								
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh							
0.60-0.85 vs. 0.85-0.95	0	NA	0	8	NA	7	10	NA	10	26	-1.90	0.08	NA	NA	0	8	NA	NA	8
0.60-0.85 vs. 0.95-1.05	3	NA	3	5	NA	5	5	NA	5	6	NA	NA	NA	NA	4	10	NA	NA	0
0.60-0.85 vs. 1.05-1.15	0	NA	0	0	NA	0	0	NA	0	0	NA	NA	NA	NA	0	0	NA	NA	0
0.60-0.85 vs. 1.15-1.40	0	NA	0	0	NA	0	0	NA	0	0	NA	NA	NA	NA	0	0	NA	NA	0
0.85-0.95 vs. 0.95-1.05	33	-0.45	19	194	-0.97	11	181	-2.07	119	293	-0.45	0.04*	0.08	0.74	42	72	0.23	0.96*	28
0.85-0.95 vs. 1.05-1.15	0	NA	0	0	NA	0	0	NA	0	6	NA	NA	NA	NA	6	12	NA	NA	0
0.85-0.95 vs. 1.15-1.40	0	NA	0	0	NA	0	0	NA	0	0	NA	NA	NA	NA	0	0	NA	NA	8
0.95-1.05 vs. 1.05-1.15	20	-3.36	7	95	-1.23	58	103	-0.89	62	158	-1.94	0**	-0.08	0.16	27	60	-0.56	0**	41
0.95-1.05 vs. 1.15-1.40	0	NA	0	0	NA	0	0	NA	0	4	NA	NA	NA	NA	2	0	NA	NA	8
1.05-1.15 vs. 1.15-1.40	0	NA	0	1	NA	1	5	NA	1	0	NA	NA	NA	NA	0	8	NA	NA	0

- a. The ratio of strike price and future price headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

The GLDI

The results of the *time-to-expiration bias* test for the GLDI are strongly distorted by the scarcity of data. The results for the combined call and put data are displayed in Table 3.G-3 (whilst the results for the separately calculated call⁹⁰ data are presented in Appendix 3.H in Table 3.H-9).

However, the computed results in Table 3.G-3 are very similar to the results of the ALSI and the INDI for the *time-to-expiration bias* test. The longer time to expiration intervals seem to have the higher implied volatilities but are statistically significant at the 5 percent level on only one occasion. Nevertheless, it seems that the results for the *time-to-expiration bias* test of the GLDI suggest a rejection of the null hypothesis of constant volatility across expirations (as for the ALSI and the INDI).

Table 3.G-3

Nonparametric Time-to-Expiration Bias Test for the Combined Call and Put Data of the GLDI

The test is carried out for nearly identical calls and puts only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the-money 0.60-0.85		out-of-the-money 0.85-0.95		at-the-money 0.95-1.05		in-the-money 1.05-1.15		deep in-the-money 1.15-1.40						
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh					
10-90 vs. 100-180	8	α NA p NA	3	14	α NA p NA	2	118	α 4.08 p 1.00**	35	12	α NA p NA	0	6	α NA p NA	2
10-90 vs. 190-540	0	α NA p NA	0	2	α NA p NA	0	8	α NA p NA	3	1	α NA p NA	1	0	α NA p NA	0
100-180 vs. 190-540	0	α NA p NA	0	0	α NA p NA	0	8	α NA p NA	0	4	α NA p NA	0	0	α NA p NA	0

a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.

b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.

* Significantly different from 0.5 at the 10 percent level.

** Significantly different from 0.5 at the 5 percent level.

Moreover, the analysis of the GLDI continues with a *striking price bias* test. The results of the *striking price bias* test are presented for the combined call and put data in Table 3.G-4 (and for the separately calculated call data and put data in Table 3.H-10 and Table 3.H-11 respectively in Appendix 3.H).

The number of findings for the *striking price bias* test for the GLDI are influenced once more by the sparse data. However for the GLDI, one important difference is evident in the call data results shown in Table 3.H-10 (to the results in the ALSI and the INDI). Most probabilities are not zero or close to zero but instead close to

⁹⁰ The lack of data becomes very obvious in the results for the separately calculated put data because no comparison in the *time-to-expiration bias* test has more than 20 observations (i.e. pairs).

one. For example, the *out-of-the-money/at-the-money* panel (0.85-0.95 vs. 0.95-1.05) in Table 3.H-10 shows one statistically significant result at the 5 percent level and another one at the 10 percent level that have a probability of close to 1. Hence, the higher strike ratio has the higher implied volatility. It is also a pattern that can be detected in an "*out-of-the-money* and *in-the-money*" (original terms) comparison in the second subperiod in Sheikh's (1991) research.

In addition, the economic importance of the two statistical significance results in Table 3.H-10 tends to decrease with increasing time to expiration. This effect is also evident for the GLDI in Table 3.G-4 (and is similar to the results of the ALSI and the INDI).

In sum, the results of the *striking price bias* test for the GLDI give evidence to reject the null hypothesis of constant volatility across strike prices (as in the ALSI and the INDI).

University of Cape Town

Table 3.G-4

Nonparametric Striking Price Bias Test for the Combined Call and Put Data of the GLDI

The test is carried out for nearly identical calls and puts only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration																	
	10-30		30-60		60-90		90-180		180-270		>270		Sh					
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh						
0.60-0.85 vs. 0.85-0.95	2		2	34	-2.32 0.01*	23	34	-1.20 0.20	19	25	3.58 0.66	11	0	NA NA	0	0	NA NA	0
0.60-0.85 vs. 0.95-1.05	4		4	19	NA NA	8	19	NA NA	4	15	NA NA	2	7	NA NA	4	4	NA NA	0
0.60-0.85 vs. 1.05-1.15	0		0	0	NA NA	0	2	NA NA	0	0	NA NA	0	0	NA NA	0	0	NA NA	0
0.60-0.85 vs. 1.15-1.40	0		0	0	NA NA	0	4	NA NA	3	0	NA NA	0	0	NA NA	0	0	NA NA	0
0.85-0.95 vs. 0.95-1.05	58		31	114	-0.90 0.26	59	143	0.20 0.63	69	50	0.31 0.94	19	1	NA NA	0	0	NA NA	0
0.85-0.95 vs. 1.05-1.15	8		2	5	NA NA	4	13	NA NA	6	3	NA NA	2	2	NA NA	2	0	NA NA	0
0.85-0.95 vs. 1.15-1.40	0		0	0	NA NA	0	3	NA NA	1	4	NA NA	0	0	NA NA	0	0	NA NA	0
0.95-1.05 vs. 1.05-1.15	104		38	172	4.21 1.00**	77	162	-0.38 0.35	83	94	-1.14 0**	60	6	NA NA	6	4	NA NA	4
0.95-1.05 vs. 1.15-1.40	11		5	27	NA NA	13	40	0.62 0.50	26	45	1.07 0.93	17	0	NA NA	0	4	NA NA	4
1.05-1.15 vs. 1.15-1.40	2		0	7	NA NA	5	22	-0.21 0.07	14	9	NA NA	5	0	NA NA	0	0	NA NA	0

- a. The ratio of strike price and future price headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Appendix 3.H

Separate Call and Put Data for the ALSI, the INDI, and the GLDI

We include the summarized tables for the nonparametric *time-to-expiration bias* test and for the nonparametric *striking price bias* test of the separately computed call data set and the separately calculated put data set. For the sake of brevity we do not discuss them in any detail, but make the point that the primary results are similar to the results of combined call and put data set.

ALSI

Table 3.H-1

Nonparametric Time-to-Expiration Bias Test for the Call Data of the ALSI

The test is carried out for nearly identical calls only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the-money 0.60-0.85			out-of-the-money 0.85-0.95			at-the-money 0.95-1.05			in-the-money 1.05-1.15			deep in-the-money 1.15-1.40		
	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh
10-90 vs. 100-180	0	NA	0	0	NA	0	209	6.02	42	6	NA	6	0	NA	0
		NA			NA			1.00**			NA			NA	
10-90 vs. 190-540	3	NA	3	22	-1.69	15	755	6.01	168	68	4.33	25	5	NA	0
		NA			0.03*			1.00**			0.98**			NA	
100-180 vs. 190-540	0	NA	0	0	NA	0	112	6.27	8	29	2.24	10	3	NA	3
		NA			NA			1.00**			0.93			NA	

- The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
 - NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
 ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-2
Nonparametric Striking Price Bias Test for the Call Data of the ALSI

The test is carried out for nearly identical calls only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration																				
	10-30		30-60		60-90		90-180		180-270		>270		Tn	Sh							
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh									
0.60-0.85 vs. 0.85-0.95	1	NA	1	26	-1.53	0**	24	19	NA	NA	-0.12	0.09	22	48	-0.08	0.24	26	53	0.03	0.61	25
0.60-0.85 vs. 0.95-1.05	4	NA	4	10	NA	NA	9	3	NA	NA	-0.73	0**	21	67	-0.08	0.23	36	45	-0.23	0**	31
0.60-0.85 vs. 1.05-1.15	0	NA	0	0	NA	NA	0	0	NA	NA	NA	NA	0	12	NA	NA	6	0	NA	NA	0
0.60-0.85 vs. 1.15-1.40	0	NA	0	0	NA	NA	0	0	NA	NA	NA	NA	0	0	NA	NA	0	0	NA	NA	0
0.85-0.95 vs. 0.95-1.05	20	-2.19	16	70	-0.47	0.01*	44	93	-0.37	0**	-0.20	0.01**	168	175	-0.25	0**	111	378	-0.24	0.01**	213
0.85-0.95 vs. 1.05-1.15	0	NA	0	3	NA	NA	2	0	NA	NA	NA	NA	2	4	NA	NA	4	47	-2.23	0**	41
0.85-0.95 vs. 1.15-1.40	0	NA	0	0	NA	NA	0	0	NA	NA	NA	NA	0	2	NA	NA	0	8	NA	NA	8
0.95-1.05 vs. 1.05-1.15	59	0.07	28	135	-0.93	0.01*	81	147	-0.57	0.02**	-0.81	0**	300	329	-0.52	0**	200	698	-1.30	0**	512
0.95-1.05 vs. 1.15-1.40	4	NA	0	1	NA	NA	0	0	NA	NA	NA	NA	15	18	NA	NA	12	105	-1.88	0**	75
1.05-1.15 vs. 1.15-1.40	0	NA	0	8	NA	NA	0	14	NA	NA	-0.81	0.03*	74	88	0.92	0.77	40	178	-0.19	0.04*	100

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-3

Nonparametric Time-to-Expiration Bias Test for the Put Data of the ALSI

The test is carried out for nearly identical puts only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the-money 0.60-0.85			out-of-the-money 0.85-0.95			at-the-money 0.95-1.05			in-the-money 1.05-1.15			deep in-the-money 1.15-1.40		
	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh
10-90 vs. 100-180	0	NA NA	0	22	6.86 0.86	8	224	4.42 1.00**	60	0	NA NA	0	0	NA NA	0
10-90 vs. 190-540	1	NA NA	1	55	6.89 0.29	29	496	6.65 1.00**	113	4	NA NA	2	0	NA NA	0
100-180 vs. 190-540	0	NA NA	0	21	-1.94 0.33	11	40	3.67 1.00**	7	0	NA NA	0	0	NA NA	0

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-4

Nonparametric Striking Price Bias Test for the Put Data of the ALSI

The test is carried out for nearly identical puts only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration															
	10-30		30-60		60-90		90-180		180-270		>270					
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh				
0.60-0.85 vs. 0.85-0.95	0		5		22		132		119		246		121			
		NA		NA		-40.94		-0.33		-0.67		0.07		0.58		
		0	5	5	22	40.94	132	0.33	119	0.67	246	0.07	121	0.58		
0.60-0.85 vs. 0.95-1.05	4		0		0		13		33		47		31			
		NA		NA		NA		NA		-0.15		-1.08		0.01**		
		4	0	0	0	NA	13	NA	33	0.15	47	1.08	31	0.01**		
0.60-0.85 vs. 1.05-1.15	0		0		0		0		0		0		0			
		NA		NA		NA		NA		NA		NA		NA		
		0	0	0	0	NA	0	NA	0	NA	0	NA	0	NA		
0.60-0.85 vs. 1.15-1.40	0		0		0		0		0		0		0			
		NA		NA		NA		NA		NA		NA		NA		
		0	0	0	0	NA	0	NA	0	NA	0	NA	0	NA		
0.85-0.95 vs. 0.95-1.05	78		199		208		470		288		544		257			
		-6.29		-3.31		-1.73		-0.12		-0.35		0.14		0.89		
		0**		0**		0**		0.06		0**		0.89		0.14		
0.85-0.95 vs. 1.05-1.15	0		0		1		7		4		61		57			
		NA		NA		NA		NA		NA		-1.38		0		
		0	0	0	1	NA	7	NA	4	NA	61	1.38	57	0		
0.85-0.95 vs. 1.15-1.40	0		0		0		0		0		0		0			
		NA		NA		NA		NA		NA		NA		NA		
		0	0	0	0	NA	0	NA	0	NA	0	NA	0	NA		
0.95-1.05 vs. 1.05-1.15	25		47		40		82		88		66		45			
		1.45		2.71		-0.13		0.19		0.32		-0.63		0**		
		0.98**		0.99**		0.32		0.89		0.98**		0**		0**		
0.95-1.05 vs. 1.15-1.40	0		0		0		0		0		4		0			
		NA		NA		NA		NA		NA		NA		NA		
		0	0	0	0	NA	0	NA	0	NA	4	NA	0	NA		
1.05-1.15 vs. 1.15-1.40	2		1		4		0		0		0		0			
		NA		NA		NA		NA		NA		NA		NA		
		2	1	1	4	NA	0	NA	0	NA	0	NA	0	NA		

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-5

Nonparametric Time-to-Expiration Bias Test for the Call Data of the INDI

The test is carried out for nearly identical calls only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the-money 0.60-0.85			out-of-the-money 0.85-0.95			at-the-money 0.95-1.05			in-the-money 1.05-1.15			deep in-the-money 1.15-1.40		
	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh
10-90 vs. 100-180	0	NA	0	0	NA	0	182	3.80	44	0	NA	0	0	NA	0
		NA			NA			1.00**			NA			NA	
10-90 vs. 190-540	0	NA	0	0	NA	0	159	1.23	76	11	NA	10	0	NA	0
		NA			NA			0.68			NA			NA	
100-180 vs. 190-540	0	NA	0	0	NA	0	12	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA	

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and *vice versa*). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-6

Nonparametric Striking Price Bias Test for the Call Data of the INDI

The test is carried out for nearly identical calls only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration														
	10-30		30-60		60-90		90-180		180-270		>270		Tn	Sh	
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh			
	α		α		α		α		α		α		α		
	p		p		p		p		p		p		p		
0.60-0.85 vs. 0.85-0.95	0	NA	0 7	NA	7 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0
0.60-0.85 vs. 0.95-1.05	3	NA	3 0	NA	0 1	NA	1 0	NA	0 2	NA	0 0	NA	0 0	NA	0
0.60-0.85 vs. 1.05-1.15	0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0
0.60-0.85 vs. 1.15-1.40	0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0
0.85-0.95 vs. 0.95-1.05	10	NA	7 20	-1.07	12 15	NA	12 99	-0.56	68 25	0.63	8 11	NA	11	NA	0
		NA		0.13		NA		0**		0.95*		NA		NA	
0.85-0.95 vs. 1.05-1.15	0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0
0.85-0.95 vs. 1.15-1.40	0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0 0	NA	0
0.95-1.05 vs. 1.05-1.15	15	NA	10 85	-1.68	57 82	-0.69	50 124	-2.07	95 52	-2.75	42 52	-1.27	52	NA	41
		NA		0**		0.02**		0**		0**		0**		NA	
0.95-1.05 vs. 1.15-1.40	0	NA	0 0	NA	0 0	NA	0 4	NA	2 0	NA	0 12	NA	12	NA	12
1.05-1.15 vs. 1.15-1.40	1	NA	1 1	NA	1 5	NA	1 4	NA	4 8	NA	0 4	NA	4	NA	4

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-7

Nonparametric Time-to-Expiration Bias Test for the Put Data of the INDI

The test is carried out for nearly identical puts only differing in their expirations for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the shorter expiration has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value given is the statistical significance or probability, p.

Days to Expiration	Strike Ratio														
	deep out-of-the-money 0.60-0.85			out-of-the-money 0.85-0.95			at-the-money 0.95-1.05			in-the-money 1.05-1.15			deep in-the-money 1.15-1.40		
	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh	Tn	α	Sh
10-90 vs. 100-180	0	NA NA	0	3	NA NA	2	131	1.50 0.99**	52	0	NA NA	0	0	NA NA	0
10-90 vs. 190-540	0	NA NA	0	6	NA NA	6	98	0.69 0.91	42	0	NA NA	0	0	NA NA	0
100-180 vs. 190-540	0	NA NA	0	0	NA NA	0	40	NA NA	7	0	NA NA	0	0	NA NA	0

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-10

Nonparametric Striking Price Bias Test for the Call Data of the GLDI

The test is carried out for nearly identical calls only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration													
	10-30		30-60		60-90		90-180		180-270		>270			
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh		
0.60-0.85 vs. 0.85-0.95	2	NA	2	1	1	7	4	0	0	0	0	0	0	0
		NA		NA		NA		NA		NA		NA		NA
0.60-0.85 vs. 0.95-1.05	0	NA	0	7	1	4	2	0	0	0	0	0	0	0
		NA		NA		NA		NA		NA		NA		NA
0.60-0.85 vs. 1.05-1.15	0	NA	0	0	0	2	0	0	0	0	0	0	0	0
		NA		NA		NA		NA		NA		NA		NA
0.60-0.85 vs. 1.15-1.40	0	NA	0	0	0	4	3	0	0	0	0	0	0	0
		NA		NA		NA		NA		NA		NA		NA
0.85-0.95 vs. 0.95-1.05	29	-0.44	16	46	17	71	0.84	25	26	0.31	9	0	0	0
		0.23		1.37		0.95*		0.99**		0.92		NA		NA
0.85-0.95 vs. 1.05-1.15	0	NA	0	3	3	12	10	5	1	2	2	0	0	0
		NA		NA		NA		NA		NA		NA		NA
0.85-0.95 vs. 1.15-1.40	0	NA	0	0	0	2	0	4	0	0	0	0	0	0
		NA		NA		NA		NA		NA		NA		NA
0.95-1.05 vs. 1.05-1.15	82	-0.14	38	185	0.45	85	154	-0.54	88	89	-1.35	59	6	6
		0.71		0.85		0.03*		0**		0**		NA		NA
0.95-1.05 vs. 1.15-1.40	10	NA	10	33	0.85	15	39	-3.71	28	45	-1.93	27	4	4
		NA		0.64		0**		0.07		0.07		NA		NA
1.05-1.15 vs. 1.15-1.40	11	NA	8	20	-0.16	11	24	-0.12	14	13	NA	5	0	0
		NA		0.25		0.15		0.15		NA		NA		NA

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

Table 3.H-11

Nonparametric Striking Price Bias Test for the Put Data of the GLDI

The test is carried out for nearly identical puts only differing in the strike prices for the period 16 October 92 to 31 December 1996. The Tn columns give the total number of pairs and the Sh columns represent the number of pairs for which the lower strike ratio (strike + future) has a higher implied volatility. The first value in the middle column of each comparison displays the economic importance, α , in percent and the second value is the statistical significance or probability, p.

Strike Ratio	Days to Expiration																	
	10-30		30-60		60-90		90-180		180-270		>270							
	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh	Tn	Sh						
0.60-0.85 vs. 0.85-0.95	0	NA	0	30	-2.88	21	28	-2.60	23	25	3.57	11	0	NA	0	0	NA	0
		NA			0.01**			0**			0.66			NA			NA	
0.60-0.85 vs. 0.95-1.05	4	NA	4	11	NA	8	16	NA	5	13	NA	1	4	NA	4	0	NA	0
		NA			NA			NA			NA			NA			NA	
0.60-0.85 vs. 1.05-1.15	0	NA	0	0	NA	0	0	NA	0	0	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	
0.60-0.85 vs. 1.15-1.40	0	NA	0	0	NA	0	0	NA	0	0	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	
0.85-0.95 vs. 0.95-1.05	7	NA	1	41	0.90	16	77	0.97	35	14	NA	7	0	NA	0	0	NA	0
		NA			0.89			0.75			NA			NA			NA	
0.85-0.95 vs. 1.05-1.15	3	NA	2	8	NA	5	6	NA	1	0	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	
0.85-0.95 vs. 1.15-1.40	0	NA	0	0	NA	0	0	NA	0	0	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	
0.95-1.05 vs. 1.05-1.15	11	NA	6	14	NA	9	3	NA	1	0	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	
0.95-1.05 vs. 1.15-1.40	3	NA	0	1	NA	1	0	NA	0	4	NA	1	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	
1.05-1.15 vs. 1.15-1.40	0	NA	0	0	NA	0	2	NA	2	0	NA	0	0	NA	0	0	NA	0
		NA			NA			NA			NA			NA			NA	

- a. The strike ratio headings would revert for a call (i.e. *out-of-the-money* becomes *in-the-money* and vice versa). However, the *in-the-money* put (call) volatility is the same as the *out-of-the-money* call (put) volatility because of the put-call parity convention. Hence, the headings are kept consistent.
- b. NA indicates a comparison with less than 20 pairs so that the economic and statistical significance tests are unreliable.
- * Significantly different from 0.5 at the 10 percent level.
- ** Significantly different from 0.5 at the 5 percent level.

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Chapter 4

4 A Non-Constant Volatility Model

4.1 Introduction

The empirical evidence of the volatility tests (VSI/VTSI and nonparametric) in Chapter 3 suggest that it is not appropriate to assume constant volatility across strike prices and across expirations in South Africa. Similar evidence in international markets led to the development of a range of option price models, for example, option price models incorporating non-constant implied volatility, stochastic volatility, or even jumps of the underlying prices. The research in this chapter focuses primarily on the development of an option price model for the South African environment that incorporates non-constant implied volatility in section 4.2. As it turns out this option price model relies on binomial trees. Although option price models such as stochastic volatility models also take account of non-constant volatility, we comment on them only briefly so as to place our development in this thesis in a more general context.

Stochastic volatility models incorporate the volatility as a random process so that, for example, the pattern of volatility smiles can be accommodated. The research on stochastic volatility models is extensive (see for example Hull and White (1987), Scott (1987), Stein and Stein (1991), Heston (1993), or Ball and Roma (1994)). However, stochastic volatility is only one assumption on the state of volatility and other volatility states may well be more appropriate. Nevertheless, the assumption of the volatility state is not the only potential shortcoming of the Black and Scholes (1973) model in practice. Another potential problem is that markets tend empirically to exhibit "jumps"⁹¹ that cannot be accommodated by the Black and Scholes (1973) model. However, the jump diffusion model proposed by Merton (1976) based on the Brownian motion process (with additional implemented occasional jumps independent of the market risk) accommodates the problem of "jumps". Further models attempting to incorporate non-constant volatility or "jumps" for example are the "Constant Elasticity of Variance Model" by Cox and Ross (1976), the

“Compound Option Model” by Geske (1979a), the “Pure Jump Model” by Cox, Ross and Rubinstein (1979), and the “Displaced Diffusion Model” by Rubinstein (1983).

All of these option price models attempt to address the empirical shortcomings of the Black and Scholes (1973) model but most suffer new shortcomings themselves. Dupire (1994) considers the combination of these shortcomings and introduces the jargon “losing the completeness” to summarize them. The “completeness” (as discussed by Dupire (1994)) constitutes the highest value of an option price model because “it [the completeness] allows for arbitrage pricing and hedging” of options. For example, stochastic volatility models are not able to hedge options with the underlying asset so that they are “losing their completeness”. However, recent option price models (for example by Derman and Kani (1994) or Dupire (1994)) incorporate non-constant implied volatility through binomial or trinomial trees without “losing the completeness”. The option price model by Derman and Kani (1994) is based on the binomial tree, the preferred numerical method in this thesis (see section 2.2.3 and Appendix 2.C) and it is therefore selected as foundation for the development of an option price model in the South African environment.

The following brief discussion is intended to give a better insight on these recent option price models.

The first tree-models were presented by Dupire (1994), Derman and Kani (1994), and Rubinstein (1994). These models assume that the volatility of the underlying asset’s return is a “deterministic function” (Dumas, Fleming, and Whaley (1996)) of both the underlying asset price and the time to expiration (however, Rubinstein (1994) does not account for the time to expiration). In addition, these option price models assume that this “deterministic volatility function” can be deduced numerically from the option data. Hence, the widely used (Hull (1993)) model of stock price behaviour⁹² with μ as expected rate of return and σ as stock price volatility (as used by Black and Scholes (1973)) is extended by Dupire (1994) and Derman and Kani (1994) to

$$\frac{dS}{S} = \mu(t) dt + \sigma(S, t) dz$$

⁹¹ “Jumps” mean the sudden increase or decrease in the price of the underlying asset of the option.

⁹² The Black and Scholes stochastic differential equation is $\frac{dS}{S} = \mu dt + \sigma dz$.

where S is the underlying asset, dz a Wiener process, and $\mu(t)$ is the risk-neutral return of the underlying asset depending only on time. Moreover, Derman and Kani (1994) argue that $\sigma(S, t)$ is the local volatility function that is dependent on both the underlying asset price and time. However, Rubinstein (1994) only accounts for the dependence of the volatility from the underlying asset price and ignores the dependence on the time to expiration. Hence, the model of stock price behaviour changes to

$$\frac{dS}{S} = \mu dt + \sigma(S) dz \quad (\text{Rubinstein})$$

The absence of the volatility dependence on the time in the option price model by Rubinstein (1994) shifts the research focus towards the option price models by Derman and Kani (1994) and Dupire (1994) for the South African environment. Nevertheless, the methodological differences between the Rubinstein (1994) model and the models by Derman and Kani (1994) and Dupire (1994) are briefly discussed in the next paragraph to place this shift of emphasis into perspective.

These two main models of implied trees with their different methodologies are discussed in their foundations in the literature. First, the implied *volatility* tree model by Derman and Kani (1994) and Dupire (1994) is constructed using a *forward* process to the end of the tree based upon option prices from the market across strike prices and across expirations. The second model, the implied *binomial* tree by Rubinstein (1994) by contrast is constructed *backwards* from the expiration. Rubinstein's (1994) model therefore infers risk-neutral probabilities⁹³ from the option prices for a chosen expiration. Hence, the Rubinstein (1994) method is limited⁹⁴ because it incorporates only the information on option prices across strike prices.

The models proposed by Derman and Kani (1994) and Dupire (1994) have the advantage of incorporating both the information across strike prices and across expirations. Dupire (1994) uses a trinomial tree while Derman and Kani (1994) prefer a binomial tree. Here, in section 4.2, the focus lies on the implied *volatility* tree by Derman and Kani (1994). The binomial tree by Derman and Kani (1994) is preferred to the trinomial tree by Dupire (1994) because it is less complex (see Appendix 2.C).

⁹³ Jackwerth and Rubinstein (1996) propose a further method to infer the risk-neutral probabilities from option prices.

The proposed option price model for the South African environment in this thesis therefore is based on the Derman and Kani (1994) model. In addition to the construction of the implied volatility tree, the required interpolation and extrapolation method for this option price model are examined in section 4.3. Moreover in section 4.4, the proposed option price model will be tested by considering the implied volatility biases established in Chapter 3. The tests of the proposed option price model for the South African environment aim to assess and compare its accuracy against conventional models such as the modified Black model and the binomial model. Finally in section 4.5, implied return distributions are established by implementing the most appropriate option price model for the use in South Africa.

The chapter proceeds with the construction of an implied volatility tree on options on futures in the environment of South Africa and concludes with the establishment of implied return distributions from evidence in the South African option market.

4.2 Implied Volatility Tree in South Africa

The aim of the ensuing discussion and development in this chapter is to establish an option price model that values American options on futures by taking account of the market conditions in South Africa (i.e. non-constant volatility). The Derman and Kani (1994) model which seems the most appropriate for the South African context is able to incorporate a variety of implied volatilities across strike prices and across expirations. The non-constant implied volatility is therefore accommodated in an implied volatility tree by assuming that the implied volatility depends on the price of the underlying, S , and on the time to expiration, t , in a functional form $(\sigma(S, t))$.

Derman and Kani (1994) assume that the volatility function can be deduced numerically from the option prices. Their framework is based on a binomial tree where the option prices are used to compute the probabilities of the paths to the nodes and the position of the nodes in the tree. However, Derman and Kani's (1994) technique requires a complete set of European option prices across strike prices and across expirations. This complete set of

⁹⁴ Recently, Jackwerth (1997) has presented a method based on Rubinstein (1994) that incorporates different

options can only be obtained by interpolation between existing option prices (a new interpolation technique is proposed in section 4.3). Nevertheless, the focus of this section will be on the Derman and Kani (1994) model and the required extensions and modifications for the South African environment.

The South African environment requires a different approach to implied volatility trees to that of Derman and Kani (1994) because the available option data only consists of American options on futures instead of European options on a non-derivative asset (e.g. equity index).

One consideration for modifications is the different exercise style of options in the Derman and Kani (1994) model and in South Africa (European vs. American⁹⁵). However, modifications are only required if the option prices differ for these two exercise styles. A difference in option prices between European and American options cannot however be observed in South Africa (as already discussed in section 3.4 because of the mark-to-market of the option premium). Hence, the American option on futures should be priced according to European option on futures with mark-to-market procedure. The Derman and Kani (1994) model for European options therefore seems appropriate but still requires modifications for options on futures.

The use of options on futures with a mark-to-market procedure has three consequences. First, the underlying is no longer an asset with a potential dividend payment. Second, no interest rate is required because the premium of the option is mark-to-market. These two modifications lead to a third important consequence, the forward price of the future ($For_{n,i}$) is equal to the future price ($F_{n,i}$) for all nodes (n, i) in the implied volatility tree:

$$For_{n,i} = F_{n,i} \quad (4.2-1)$$

where n is the time step and i is the level in each time step ($i = 1, \dots, n+1$) in the tree.

The equation 4.2-1 is required to satisfy the following risk neutrality condition according to the Derman and Kani (1994) model:

$$For_{n,i} = F_{n,i} = p_{n,i}F_{n+1,i+1} + (1 - p_{n,i})F_{n+1,i} \quad (4.2-2)$$

expirations and different types of options.

⁹⁵ The first method to use American options is presented by Chriss (1997) but his method assumes that the option premium is paid upfront.

where $F_{n+1,i}$ and $F_{n+1,i+1}$ are the prices of the future corresponding to the levels of the nodes (n, i) in the tree and $p_{n,i}$ is the transition probability⁹⁶. The risk neutrality condition in equation 4.2-2 is rearranged for the transition probability, $p_{n,i}$, so that

$$p_{n,i} = \frac{F_{n,i} - F_{n+1,i}}{F_{n+1,i+1} - F_{n+1,i}} \quad (4.2-3)$$

As in Derman and Kani (1994), the implied volatility tree is constructed by forward induction from the level (n) of the tree to the next level $(n+1)$ of the tree. An overview of the construction process is given in Figure 4.2-1 below.

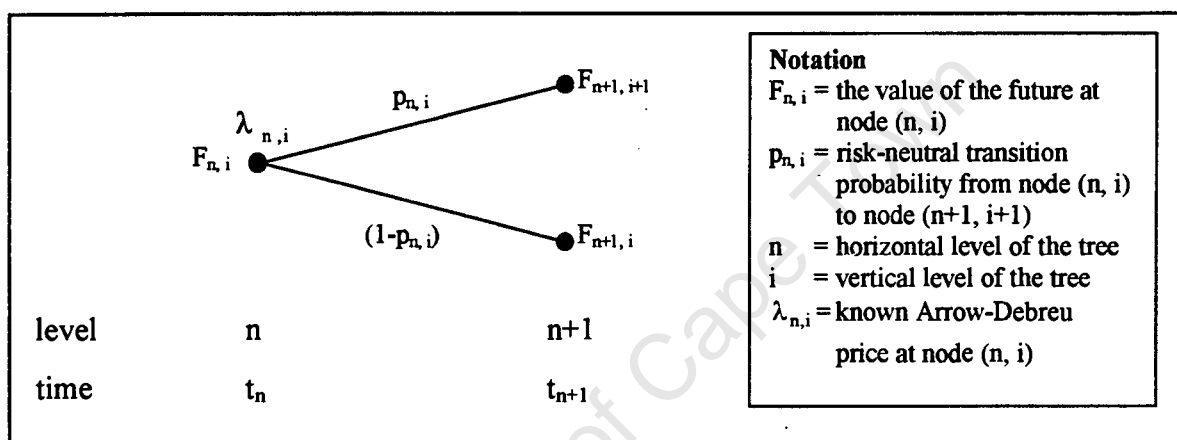


Figure 4.2-1. Construction of the Implied Volatility Tree

The figure shows how the implied volatility tree is constructed from level (n) to the next level $(n+1)$.

As in Derman and Kani (1994), uniformly⁹⁷ spaced levels (n) are used in this thesis to construct the implied volatility tree⁹⁸ for the South African environment. The different idea of non-uniformly spaced levels (n) is discussed by Derman, Kani, and Chriss (1996). They propose non-uniformly spaced levels (n) in an implied volatility tree that is based on a trinomial tree, instead of a binomial tree in the Derman and Kani (1994) model. Derman, Kani, and Chriss (1996) argue that the advantage of the non-uniformly spaced levels (n) is the prevention of negative transition probabilities. These negative transition probabilities may result in negative option prices computed by the implied volatility tree. Negative transition probabilities however emerge only on two occasions. Firstly, the forward price of the nodes at level $(n+1)$ lies above the upward node or below the downward node.

negative transition probabilities

⁹⁶ The transition probability is the probability of the path to the next node.

⁹⁷ Uniformly spaced levels (n) mean that the nodes of each level (n) are the same time differential apart.

⁹⁸ The implied volatility tree is also referred to as implied tree.

Secondly, excessive option prices are implemented in the implied volatility tree (i.e. very large or very small option prices).

The first source of negative transition probabilities is however not applicable to the South African environment because the forward price is equal to the future price (in equation 4.2-1) at the nodes of level (n+1). The second source of negative transition probabilities is applicable to the environment in South Africa but it cannot be solved by the non-uniformly spaced levels (n) as Derman, Kani, and Chriss (1996) acknowledge. Hence, uniformly spaced levels (n) are implemented in the implied volatility tree for the South African environment.

Furthermore, the option price model proposed for the South African environment in this thesis is based on forward induction as in Derman and Kani (1994). The forward induction process requires the corresponding value of the American *option on futures* for every node at level (n) to compute the values of the future for all nodes at the level (n+1). These computed values of the future at level (n+1) are subsequently used to compute the transition probability between the nodes at level (n) and level (n+1) (as in equation 4.2-3).

However, the requirement of options prices of American *options on futures* for each node at level (n) can be problematic because options are sometimes illiquid (i.e. no prices exist) and they are only traded at specific discrete strike steps. Nevertheless, the implied volatility tree is based on the assumption that a continuous function of option prices or of implied volatility respectively across strike prices and across expirations (i.e. for each node is an option value specified) can be found through extrapolation and interpolation⁹⁹ of the discrete trading data. An in-depth discussion of appropriate extrapolation and interpolation techniques can be found in section 4.3.

The novel technical approach of constructing an implied volatility tree model for the South African environment commences in the ensuing section.

⁹⁹ The implemented extrapolation and interpolation methods are discussed in section 4.3.

4.2.1 Modelling of an Implied Volatility Tree Model in South Africa

The modelling of an implied volatility tree model for the South African environment begins with the introduction of “Arrow-Debreu¹⁰⁰ prices”. “Arrow-Debreu prices” are required to compute the values of the future for the nodes at level (n+1). The first Arrow-Debreu price is $\lambda_{1,1}$ at the starting node (1,1) of the tree. All further Arrow-Debreu prices are then calculated as follows:

$$\lambda_{n+1,i} = \begin{cases} p_{n,i-1} \lambda_{n,i-1} & \text{for } i = n + 1 \\ p_{n,i-1} \lambda_{n,i-1} + (1 - p_{n,i}) \lambda_{n,i} & \text{for } 2 \leq i \leq n \\ (1 - p_{n,i}) \lambda_{n,i} & \text{for } i = 1 \end{cases} \quad (4.2.1-1)$$

After the introduction of the “Arrow-Debreu prices”, the construction of the implied volatility tree continues with the definition of its base (i.e. its starting node (1,1)) that is the current value of the future, $F_{1,1}$. Additionally, the prices of the call option, $C(K, t_{n+1})$, and the put option, $P(K, t_{n+1})$, respectively (which expire at time t_{n+1} with a strike price K) are required for the computation of the implied volatility tree. The call option values as well as the put option values are known through interpolation and extrapolation. However, the strike price, K , at time t_{n+1} is equal to $F_{n,i}$ at time t_n for a call option and a put option respectively. Hence, the price of a call option on futures is given in the binomial model by the sum of the probabilities of reaching each node (n+1, i) multiplied by the call payoff over all nodes i at the time t_{n+1} . Consequently, the value of a call¹⁰¹ with expiration t_{n+1} and a current underlying asset price equal to the strike price, K , is

$$C(K, t_{n+1}) = \sum_{i=1}^{n+1} \lambda_{n,i} \max(F_{n,i} - K, 0) \quad (4.2.1-2)$$

Thus far, the Derman and Kani (1994) model has been modified in accordance with the conditions appropriate to the South African environment. In the ensuing discussion on the use of the implied volatility tree for the South African environment, modifications of the

¹⁰⁰ Merton (1992) defines an Arrow-Debreu state contingent security as a security that pays its holder \$1 if a particular state of the world occurs at a particular point in time, and otherwise pays nothing. Here, the Arrow-Debreu price of a node is the value of this security. It pays \$1 if the future price reaches the node and zero otherwise.

¹⁰¹ The research refers to call options while the result for put options is analogous. If the result for put options is different, the put conditions and their results are noted.

Derman and Kani (1994) model by other authors will also be considered. For example, Barle and Cakici (1995) modify the Derman and Kani (1994) model to avoid mispricing of options through the implied volatility tree. Their modifications are discussed within the South African context below.

The first modification by Barle and Cakici (1995) concerns the change of the strike price, K , for which each call option in the tree should be evaluated. Derman and Kani (1994) establish the condition for the strike price that the price of the upward node at time t_{n+1} shall be *greater than or equal* to the spot price of node (i) at time t_n and *less than or equal* to the spot price of node (i+1) at time t_n . The result of this condition is that the strike price is *equal* to the *spot price* of node (i) at time t_n . Barle and Cakici (1995) however argue that this condition of Derman and Kani (1994) is not necessarily true. Hence, Barle and Cakici (1995) modify the condition by Derman and Kani (1994) so that the price of the upward node at time t_{n+1} lies *between* the forward prices of the nodes (i) and (i+1) at time t_n . Consequently, the strike price now *equals* the *forward price* of node (i) at time t_n .

In this manner, Barle and Cakici (1995) account for one of the problems of the Derman and Kani (1994) model (because otherwise the Derman and Kani (1994) model is able to produce upward node prices that fall between the spot price of node (i) at time t_n and its forward, or between the spot price of node (i+1) at time t_n and its forward). As Barle and Cakici (1995) argue these errors can accumulate and cause serious discrepancies. Here, we implement the idea of Barle and Cakici (1995) for the options on futures in the South African implied volatility tree model

$$\left(\text{For}_{n,i} = \right) F_{n,i} \leq F_{n+1,i+1} \leq F_{n,i+1} \left(= \text{For}_{n,i+1} \right) \quad (4.2.1-3)$$

Nevertheless, the result in equation 4.2.1-3 for the South African implied volatility tree model presents an identical result for the Derman and Kani (1994) as well as the Barle and Cakici (1995) approach because the price of the forward is the same as the price of the future (i.e. the spot) in the South African environment. Hence, the strike price is equal to the future (= forward) price of node (i) at time t_n so that

$$C(F_{n,i}, t_{n+1}) = \sum_{j=i+1}^{n+1} \lambda_{n,j} (F_{n,j} - F_{n,i}) \quad (4.2.1-4)$$

where the strike price, K , equals $F_{n,i}$ for $1 \leq i \leq n$. Hence, as in Derman and Kani (1994) the theoretical binomial value of a call option with the strike price $F_{n,i}$ and expiration t_{n+1} is

rearranged for the South African environment. The rearrangement shows the value of the call option in terms of known Arrow-Debreu prices and forward prices when the strike price equals the value of the node at level (n). By implementing the risk neutrality condition in equation 4.2-2, the equation for the value of the call option becomes

$$C(F_{n,i}, t_{n+1}) = p_{n,i} \lambda_{n,i} (F_{n+1,i+1} - F_{n,i}) + \sum_{j=i+1}^n \lambda_{n,j} (F_{n,j} - F_{n,i}) \quad (4.2.1-5)$$

The first term on the right side of equation 4.2.1-5 represents the contribution of the first upward node to the call option price. Derman and Kani (1994) argue that this first term depends on the unknown transition probability, $p_{n,i}$, towards the upward node ($n+1, i+1$) and on the unknown price of the upward node, $F_{n+1,i+1}$. However, the second term of the right hand side of equation 4.2.1-5 is only a sum of known parameters (as in Derman and Kani (1994)). Hence, the values of the two left unknowns, $F_{n+1,i+1}$ and $p_{n,i}$, have to be found.

The unknown price of the upward node, $F_{n+1,i+1}$, and the unknown transition probability, $p_{n,i}$, towards the upward node at level t_{n+1} can be found with the known value of $C(F_{n,i}, t_{n+1})$ and the known value of the future (= forward) at time t_n . Both known values are taken from the market data. Hence, the risk neutrality condition (equation 4.2-2) and the equation 4.2.1-5 are simultaneously solved for $F_{n+1,i+1}$ so that

$$F_{n+1,i+1} = \frac{F_{n+1,i} [C(F_{n,i}, t_{n+1}) - \Sigma] - \lambda_{n,i} F_{n,i} (F_{n,i} - F_{n+1,i})}{[C(F_{n,i}, t_{n+1}) - \Sigma] - \lambda_{n,i} (F_{n,i} - F_{n+1,i})} \quad (4.2.1-6)$$

where

$$\Sigma = \sum_{j=i+1}^n \lambda_{n,j} (F_{n,j} - F_{n,i})$$

and with $p_{n,i}$ is the same as equation 4.2-3.

The implied tree is then constructed for all nodes above the centre of the tree if $F_{n+1,i}$ is known at one node. The centre of the tree and the procedure to obtain $F_{n+1,i}$ is discussed below and constitutes the second modification of the Derman and Kani (1994) model by Barle and Cakici (1995).

The second modification of Barle and Cakici (1995) affects the centre condition of the tree. While Derman and Kani (1994) choose the current spot price as the centre of the tree,

Barle and Cakici (1995) propose to centre the tree at the most probable movement of the underlying, at the forward price. As discussed, the forward price and the future (i.e. the spot price here) are the same in the South African environment so that the second modification plays no role in the model for options on futures. Hence, for the South African environment the following centring conditions are implemented

$$\left. \begin{array}{l} \underline{F_{n+1, n/2+1} = F_{1,1}} \\ \underline{F_{n+1, i} \times F_{n+1, i+1} = F_{1,1}^2} \end{array} \right\} \begin{array}{l} \text{if } (n+1) \text{ is odd} \\ \text{if } (n+1) \text{ is even} \end{array} \quad \left. \begin{array}{l} \Rightarrow S_{n+1, \frac{n+1}{2}} = S_{0,0} \\ \Rightarrow S_{n+1, \frac{n}{2}} = S^2 \end{array} \right\} \text{same as}$$

where $F_{1,1}$ is the price of the underlying at the base of the implied volatility tree (i.e. initial price at the begin of the tree). Hence, the “(n+1) is even” condition is substituted into the formula for $F_{n+1, i+1}$ to obtain the upper node price of the two central nodes [i.e. lower node (n+1, i) and upper node (n+1, i+1) with $i = (n+1)/2$]. If n is odd, then the upper central node is

$$F_{n+1, i+1} = \frac{F_{1,1} [C(F_{1,1}, t_{n+1}) + \lambda_{n,i} F_{1,1} - \Sigma]}{\lambda_{n,i} F_{1,1} - C(F_{1,1}, t_{n+1}) + \Sigma} \quad \text{for } i = (n+1)/2 \quad (4.2.1-7)$$

where

$$\Sigma = \sum_{j=i+1}^n \lambda_{n,j} (F_{n,j} - F_{n,i})$$

All further nodes above the value of the node (n+1, i+1) for $i = (n+1)/2$ can be found using the equation established already for $F_{n+1, i+1}$ (in equation 4.2.1-6).

The calculation of the nodes below the central node are based upon extrapolated and interpolated put options, denoted by $P(F_{n,i}, t_{n+1})$ with the strike price $F_{n,i}$ and the expiration time t_{n+1} . The formula for the lower nodes depends on the knowledge of the price of $F_{n+1, i+1}$ as it can be observed in the equation

$$F_{n+1, i} = \frac{F_{n+1, i+1} [P(F_{n,i}, t_{n+1}) - \Sigma] + \lambda_{n,i} F_{n,i} (F_{n,i} - F_{n+1, i+1})}{[P(F_{n,i}, t_{n+1}) - \Sigma] + \lambda_{n,i} (F_{n,i} - F_{n+1, i+1})} \quad (4.2.1-8)$$

where

$$\Sigma = \sum_{j=i}^{i-1} \lambda_{n,j} (F_{n,i} - F_{n,j})$$

However, this proposed implied volatility tree model for the South African environment still has one important shortcoming. That is the introduction of the market option prices

into the implied volatility tree model to recover the price of the future at the time step t_{n+1} may result in a future price $F_{n+1, i+1}$ that lies outside of the “tree building condition” of equation 4.2.1-3. Consequently, a transition probability, $p_{n,i}$, below zero or above one is produced. Derman and Kani (1994) solve the problem rather inefficiently by manually overriding the future price with a value that corresponds to the previous levels and that smoothes the implied volatility function. Barle and Cakici (1995) propose to set the future price (or the forward price in the South African environment), $F_{n+1, i+1}$, equal to the average of $F_{n,i}$ and $F_{n,i+1}$. They propose a further approach that would use the settings from previous levels ($n < n+1$) to set $F_{n+1, i+1}$ nearer to $F_{n,i}$ or $F_{n,i+1}$. However, Barle and Cakici (1995) argue that calculations other than averaging yield little change to their results. Their proposal is more easily understood if their averaging calculation is considered in a practical context. We elaborate on this calculation below.

A price of $F_{n+1, i+1}$ lying outside the $[F_{n,i}, F_{n,i+1}]$ interval only becomes existent if the market price of the required option is *significantly* low or high (i.e. the implied volatility tends toward zero or infinite). Neither of both option prices (i.e. implied volatilities near zero or infinite) would be long in existence because arbitrage would bring the option prices (i.e. implied volatilities) back to appropriate levels. Hence, *arbitrage violating option prices* with extreme implied volatility are not considered as a basis for the option pricing process in the practical context. Moreover, the forward price computed from an *arbitrage violating option price* is also not considered. Instead, this forward price can only be obtained by averaging the forwards of option prices with the strike price below the *arbitrage violating option price* and with the strike price above the *arbitrage violating option price*. Hence, the appropriate forward price for the *arbitrage violating option price* is obtained in the same way as to the method proposed by Barle and Cakici (1995).

Nevertheless, the method of Barle and Cakici (1995) does neglect the spacing of the nodes between different levels (for example, level (n) and level (n+1)). This omission of the node spacing can lead to unrealistic values of nodes in the implied volatility tree (as

observed in a number of tests¹⁰²). Hence, a solution for the spacing problem based on Derman and Kani (1994), Chriss (1997), and Barle and Cakici (1995) is considered in this thesis. A description of the expanded and novel algorithms for the South African environment is however somewhat elaborate and tedious and is therefore relegated to Appendix 4.A.

The solution for the inappropriate transition probabilities problem combines the strengths of the methods by Derman and Kani (1994), Chriss (1997), and Barle and Cakici (1995) and is presented in Appendix 4.A.

For example, Chriss (1997) proposes a solution to the problem of “bad probabilities” (i.e. transition probabilities below zero and above one) that shall ensure a similar spacing between the nodes at level (n+1) and the nodes at level (n) in the tree. In particular, Chriss (1997) bases his solution on the logarithmic spacing method introduced by Derman and Kani (1994). Nevertheless, Chriss’s (1997) logarithmic spacing method fails on some occasions, consequently, he introduces a second method that prevents bad probabilities in his tree, although logarithmic spacing is abandoned by this second method¹⁰³.

In conclusion, inappropriate transition probabilities shall be avoided using the methods discussed (e.g. the logarithmic spacing method). Derman and Kani (1994) and Chriss (1997) implement methods to preserve the logarithmic spacing in the implied volatility tree while Barle and Cakici (1995) use a method that averages the neighbouring forward prices of the miscalculated node value. Barle and Cakici (1995) argue that the implementation of the logarithmic spacing does not always lead to a solution (supported by Chriss (1997)).

¹⁰² The unrealistic values of the nodes normally originate in the highest and lowest node levels of the implied volatility tree. As Chriss (1997) argues these unrealistic values have no impact on the calculation of the option price because these very high or very low values of the nodes are accompanied by low probabilities. However, our research found that these unrealistic values can distort the implied probability distribution and affect the computed option price sensitivities.

¹⁰³ Chriss (1997) differentiates in the implementation of the logarithmic spacing method between up (1.) and down (2.) moves in the tree. His notation is translated according to the notation of the thesis and the application for options on futures, i.e.

$$1. \quad F_{n+1,i+1} = \frac{F_{n+1,i} \times F_{n,i+1}}{F_{n,i}}$$

$$2. \quad F_{n+1,i+1} = \frac{F_{n+1,i+2} \times F_{n,i}}{F_{n,i+1}}$$

If the new calculated $F_{n+1,i+1}$ still violates the tree building condition and produces bad probabilities, Chriss (1997) implements a second method (3.) without the requirement of logarithmic spacing:

$$3. \quad F_{n+1,i+1} = F_{n,i} + \varepsilon$$

where ε is a small number that puts the value of the node (n+1, i+1) just above the value of the node (n, i).

Chriss (1997) attempts to solve this problem in a manner that is both simple and superior to the manual overriding of node values by Derman and Kani (1994). However, his method has the disadvantage that the consistency of the tree can be disrupted significantly by his second method. Chriss (1997) further neglects the upper and lower bounds of the tree (i.e. the highest and lowest step for each level in the tree) with his solution of logarithmic spacing. However, a combination of the solutions by Derman and Kani (1994, Chriss (1997), and Barle and Cakici (1995) and a newly proposed logarithmic spacing method for the South African environment provides an adequate solution to the problems discussed above. The elaborate technical approach and its algorithms are found in Appendix 4.A.

4.2.2 Conclusion

A newly proposed implied volatility tree model for options on futures appropriate for the South African environment has been presented. This implied volatility tree model is based on the Derman and Kani (1994) model but modified and expanded for use in South Africa. The first modification to the Derman and Kani (1994) model concerning the implied volatility tree is the assumption that the forward price and the future price (as underlying) are identical. The consequence of this assumption is a simpler computation because no distinction must be made between forward price and underlying price (i.e. the future price) as in Barle and Cakici (1995). Secondly, the problem of “bad probabilities” is solved with a new extensive algorithm that is based on the methods by Derman and Kani (1994), Chriss (1997), and Barle and Cakici (1995).

In sum, the proposed implied volatility tree for the South African environment is the first of its kind to our knowledge. The implied volatility tree enables the option pricing process to become more consistent to the market environment. In addition, the proposed algorithm to avoid “bad probabilities” can be extended to other market environments.

Finally, the proposed implied volatility tree model (considered in section 4.2) must be tested to assess its pricing abilities. However, the test is only meaningful if reasonable extrapolation and interpolation methods are implemented to obtain an accurate volatility surface (across strike prices and across expirations). Hence, a variety of extrapolation and interpolation methods are discussed below. Afterwards, the proposed implied volatility tree

model for options on futures and the embedded extrapolation and interpolation methods are tested in section 4.4.

4.3 Extrapolation and Interpolation Methods

The aim of the following research is to develop a method for establishing a continuous volatility surface across strike prices and across expirations. As it turns out, extrapolation and interpolation methods are required to construct an appropriate volatility surface across strike prices and across expirations. The aim of this section therefore is to find a technique that is stable, simple, and practical in the South African environment as well as in the international markets.

Thus the focus here is on a continuous *volatility* surface rather than a continuous *option price* function. Shimko (1993) for example, argues that implied volatilities are smoother than option prices suggesting that superior interpolation results can be achieved using implied volatilities.

The development of a suitable extrapolation and interpolation technique depends on the quality of the data in the first instance. If trading data with bid and ask quotes is available, two volatility surfaces should in theory be constructed to price options. In the context of the data in South Africa, only traded option prices are available (i.e. without a bid and an ask quote). Hence, only one volatility surface can be constructed (which can be thought of as a mixture between bid and ask prices).

The quality of the data is also measured by its “smoothness” which is an important characteristic. For example, an implied volatility function across strike prices is not “smooth” if the implied volatility changes direction frequently. In this instance, it would not be possible to find an appropriate function of implied volatility across strike prices. Additionally, the “smoothness” of a volatility function (across strike prices or across expirations respectively) as well as of a volatility surface might be impaired by very small or very large implied volatilities in the data. As discussed in section 4.2.1 earlier, very small or very large volatilities may have to be overridden anyway because they contain no useful information value for the implied volatility tree. Such effects of mispriced options may therefore distort the input data. Hence, the trade off between the information value of

each option price (and its implied volatility) and the smoothness of the implied volatility surface needs to be considered.

In sum, a volatility function or a volatility surface can perhaps never be adequately constructed using one inflexible extrapolation and interpolation method because:

1. the implied volatility might change in every instant
2. the implied volatility obtained from the data is too scattered
3. *out-of-the-money*, *at-the-money*, and *in-the-money* volatilities tend to behave differently during the time to expiration¹⁰⁴

Hence, the aim here is to find a methodology for the extrapolation and interpolation to accommodate the changes in the data. The research in this thesis therefore views the extrapolation and interpolation across *strike prices* and across *expirations* separately. The examination of the interpolation across strike prices below, begins the investigation. The examination across expirations follows thereafter.

A linear interpolation across strike prices is not realistic because it can result in an implied volatility function which is very jagged (i.e. not smooth). Shimko (1993) uses a best-fit least squares parabola to estimate the implied volatility function across strike prices which avoids the jaggedness of the implied volatility function. Moreover, Dumas, Fleming, and Whaley (1996) implement a variety of approaches to model the implied volatility function across strike prices and across time to expiration. They find that among the variety, a quadratic function (across strike prices as well as across time to expiration) yields the best solution. This result is also consistent with Shimko's (1993) suggestion. However, as Jackwerth and Rubinstein (1995) point out the quadratic function is inaccurate for strike prices far away from the money.

Two additional methods for the construction of a volatility surface were proposed recently. Firstly, Chriss (1997) proposes a simple and stable grid interpolation technique for a continuous volatility surface. However, the grid interpolation technique has as disadvantage that it requires four known option prices to compute the missing option price. In addition, the four known option prices have to surround the missing option price. These

¹⁰⁴ The results of the analysis of the implied volatility across expirations show that the *out-of-the-money* and the *in-the-money* implied volatility decrease with increasing time to expiration whilst the *at-the-money* implied volatility increases.

requirements are not realistic in practice, hence the grid interpolation is somewhat impractical (especially in illiquid markets). This technique therefore is deemed inappropriate for extrapolation purposes here.

Secondly, another approach for computing the volatility surface has been proposed by Avellaneda *et al.* (1997) who implement a “relative-entropy minimization” approach. Although they conclude that their results are accurate, their procedure is neither simple to implement nor without shortcomings. One of the shortcomings occurs because Avellaneda *et al.* (1997) use a prior volatility surface that is arbitrarily chosen to calculate the volatility surface. Hence, the volatility surface obtained depends on the prior volatility surface selected which in turn depends on the number of input option prices. Clearly, if the number of input option prices is small, the prior volatility surface does not change much, hence the results depend on the prior volatility surface.

Although a quadratic form may not be most suitable, it is still believed that the general regression method with ordinary least squares should be used for an appropriate approach of building a volatility function across strike prices. Its advantage is that this approach is stable, simple, and practical by contrast to the methods discussed above. Consequently, regression models have been adopted as a suitable approach in this thesis to handle the extrapolation and interpolation.

Finding a suitable volatility function across strike prices also seems difficult using the ordinary least squares regression model because implied volatility can be different for *out-of-the-money*, *at-the-money*, and *in-the-money* strike prices. The recognition of these potential differences in the pattern of implied volatility across strike prices leads to the consideration of a piecewise volatility function across strike prices. Accordingly, the regression will be calculated *piecewise* in the thesis.

The structure of section 4.3 therefore consists of three comparisons of regression models for the extrapolation and interpolation across *strike prices* (in sections 4.3.1, 4.3.2, and 4.3.3). Additionally, a proposed method for extrapolation and interpolation across *expirations* is discussed in section 4.3.4. Finally, a summarizing conclusion can be found in section 4.3.5.

Each comparison begins with a graph of the two implied volatility functions estimated by the regression models. Thereafter a statistical analysis of the results of the two

compared regression models follows. The research in this section considers a quadratic regression model and a trinomial regression model¹⁰⁵:

$$\text{Quadratic Regression Model: } y = \alpha + \beta_1 x + \beta_2 x^2 + \varepsilon$$

$$\text{Trinomial Regression Model: } y = \alpha + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \varepsilon$$

where x is the strike price, y the estimated volatility, α and β_i are the estimated regressors, and ε is the residual parameter. The quadratic regression model is selected to be consistent with prior research (see for example Shimko (1993)) whilst the trinomial regression model is chosen to improve the implied volatility function by avoiding the far away from the money problem (as discussed in Jackwerth and Rubinstein (1995)). The quadratic and the trinomial regression are considered as piecewise functions of the implied volatility function across strike prices. Section 4.3.1 therefore focuses on the comparison between the quadratic and the trinomial regression model whilst section 4.3.2 focuses on the comparison between a piecewise quadratic regression model and a trinomial regression model. The last section 4.3.3 compares a piecewise quadratic regression model to a piecewise trinomial regression model. This comparison is included to investigate the interesting issue of a trade off between more sophisticated modelling approaches and their gain in accuracy. The adopted method for extrapolation and interpolation across expirations is discussed theoretically in section 4.3.4. Initially however, an appropriate data set must be considered for the analysis in this section.

The appropriate data set used in the comparison of quadratic and trinomial regression models should have the following characteristics based on the prior discussions:

1. a significant “smile” or “grin” distribution (i.e. a curved implied volatility function across strike prices)
2. a similar form to the implied volatility function across strike prices as used in Jackwerth and Rubinstein (1995) (i.e. nearly constant implied volatilities far away from the money)¹⁰⁶
3. be representative (i.e. not suffering from the non-simultaneous pricing problem)

¹⁰⁵ The trinomial regression model is also known as cubic regression model.

¹⁰⁶ This second requirement is introduced to prevent the criticism concerning the implied volatility estimation for far from the money strike prices with quadratic volatility functions (as in Jackwerth and Rubinstein (1995)).

In considering the requirements of an appropriate data set, we conclude that South African data is not appropriate in its current form. Hence we consider existing approaches for the modelling of implied volatility functions in the international literature. The data proposed by Jackwerth and Rubinstein (1995) (for the S&P 500 option prices on the 4/12/1990) meets all the necessary requirements¹⁰⁷, hence their data is used in the ensuing comparison of the regression models, instead.

The comparisons of the regression models starts with the comparison between the quadratic and the trinomial regression model below.

4.3.1 Comparison of the Quadratic Regression with the Trinomial Regression Model

The objective of this section is to establish the most suitable regression model for the implementation in the implied volatility tree model. Under consideration in this section is the quadratic and the trinomial regression model. Jackwerth and Rubinstein (1995) argue that the quadratic regression model cannot appropriately fit an implied volatility function (found for the S&P 500 options). Hence, a more suitable solution to establish an appropriate implied volatility function across strike prices might well be obtainable using a trinomial regression model¹⁰⁸.

It can be observed in Figure 4.3.1-1 that the quadratic regression model has similar problems of fit (described in section 4.3 earlier) when the strike prices are far away from the money. The quadratic regression overestimates the implied volatility for strike ratios below 0.79 considerably and underestimates the implied volatility for strike ratios above 1.15 considerably. Further extrapolations above 1.15 and below 0.79 become more inaccurate using the volatility function modelled with the quadratic regression approach. The problems discussed do not however occur for the trinomial regression model in Figure 4.3.1-1. The trinomial regression by contrast seems to fit the “far away” from the money regions fairly well.

¹⁰⁷ Jackwerth and Rubinstein's (1995) data set additionally matches the South African characteristics very well, hence it can be used in this examination without losing the connection to the South African market.

¹⁰⁸ As yet a trinomial model has not been considered in the literature for this purpose.

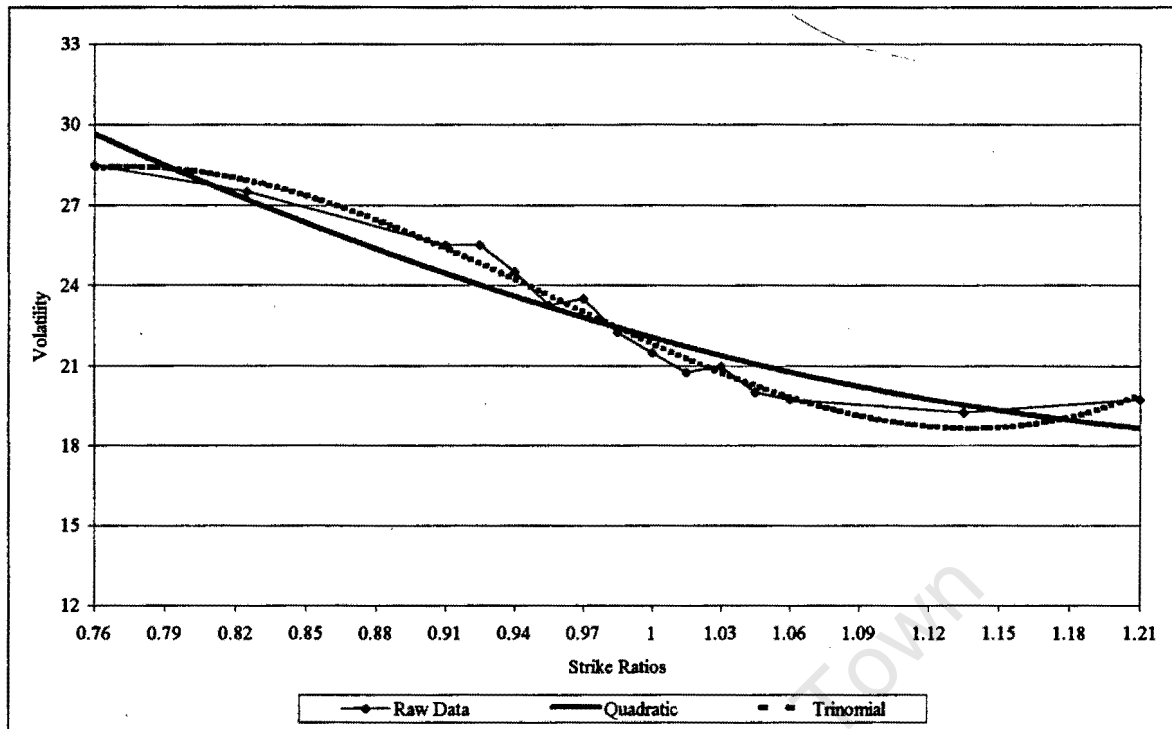


Figure 4.3.1-1. Comparison of the Quadratic Regression with the Trinomial Regression

The data in Jackwerth and Rubinstein (1995) (i.e. implied volatilities of the S&P 500 option prices on December 4, 1990) is fitted using quadratic regression as well as trinomial regression. The initial raw data is displayed with linear interpolation between the data points.

The statistical comparison of the quadratic and trinomial regression models in Table 4.3.1-1 lends support to the graphical evidence that the trinomial model is superior. The “goodness-of-fit” estimator¹⁰⁹ (R^2) increases substantially from 0.8941 for the quadratic model to 0.9782 for the trinomial model. Moreover, the residual sum of squares (RSS) and the variance of the residuals (s^2) decrease substantially for the trinomial regression in Table 4.3.1-1.

¹⁰⁹ The “goodness-of-fit” estimator (R^2) is adjusted to take account of small number of input volatilities in this and in the following sections. The adjustment computation follows Anderson, Sweeney, and Williams (1993) i.e.

$$\text{adjusted } R^2 = 1 - (1 - R^2)(N - 1) / DF$$

where R^2 is the unadjusted “goodness-of-estimator”, N is the number of input volatilities, and DF are the degrees of freedom.

Table 4.3.1-1

Comparison of Quadratic and Trinomial Regression Statistics

The data from Jackwerth and Rubinstein (1995) is examined using the residual sum of squares, RSS, to the variance of the residuals, s^2 , and to the "goodness-of-fit" estimator, R^2 . The value in brackets under the heading of R^2 is the unadjusted value of the "goodness-of-fit" estimator.

Regression Models	Degrees of Freedom DF	Residual Sum of Squares RSS	Variance of Residuals s^2	Goodness- of-Fit R^2
Quadratic Regression ¹¹⁰	12	11.0059	0.9172	0.8941 (0.9092)
Trinomial Regression	11	2.0713	0.1883	0.9782 (0.9829)

The quadratic and the trinomial models are also assessed using an F-test¹¹¹ (under the null hypothesis that the simpler model is more appropriate). The F-Test is especially designed to establish whether R^2 is *significantly* larger for a regression model with higher order. The value of F obtained for the quadratic and trinomial regression test is 47.45 and lies above the critical value of 9.65 for the F-test at a significance level of one percent. The null hypothesis is therefore rejected implying that the trinomial model is the more appropriate regression model.

Although, the trinomial regression model is a reasonable model for the data in *statistical terms*. Figure 4.3.1-1 does however highlight some deviations from the original data that may become *economically* important. See for example the departures in the regions around strike ratios of 0.85 and 1.12. A potential approach to model these deviations more accurately could be the use of higher polynomial regressions, but it should be noted that higher polynomial regressions tend to become more unstable. A novel proposal of two piecewise quadratic regressions instead of one trinomial regression across strike ratios is considered further below.

¹¹⁰ The statistical results presented are only a partial selection. In addition, serial dependence of the residuals appears problematic for the quadratic regression.

¹¹¹ The value of F is defined in the notation of the thesis (similar to Anderson, Sweeney, and Williams (1993)) as

$$F = \frac{(\text{RSS}(\text{quad.}) - \text{RSS}(\text{trinom.})) / (\text{DF}(\text{quad.}) - \text{DF}(\text{trinom.}))}{\text{RSS}(\text{trinom.}) / \text{DF}(\text{trinom.})}$$

where quad. and trinom. are the abbreviations for quadratic and trinomial respectively.

4.3.2 Comparison of the Trinomial Regression with the Piecewise Quadratic Regression Model

The aim of this section is to compare the trinomial regression model (the result of section 4.3.1) with the piecewise quadratic regression model to find the most suitable regression model for implementation in the implied volatility tree model. Hence, the trinomial regression model is compared directly to a quadratic regression model that contains two regressions (as pieces). The first piece of the quadratic regression model is computed for the strike ratio less than or equal to one whereas the second piece of the quadratic regression model is computed for the strike ratio greater than or equal to one. It seems in Figure 4.3.2-1 that the piecewise quadratic regression model is superior to the trinomial regression model. For example, the fit in the regions around strike ratios of 0.85 and 1.12 is better for the piecewise quadratic regression model by contrast to the trinomial regression model in Figure 4.3.2-1. The piecewise quadratic regression model also seems superior for the extrapolation of far away from the money strike ratios by comparison to the trinomial regression model. Consequently, the problem of an inaccurate implied volatility for “far away” from the money strike ratios (i.e. above 1.18 and below 0.79) does not seem to occur for the piecewise quadratic regions. The statistical summary of the comparison between the piecewise quadratic regression and the trinomial regression model in Table 4.3.2-1 below gives quantitative support for this assertion.

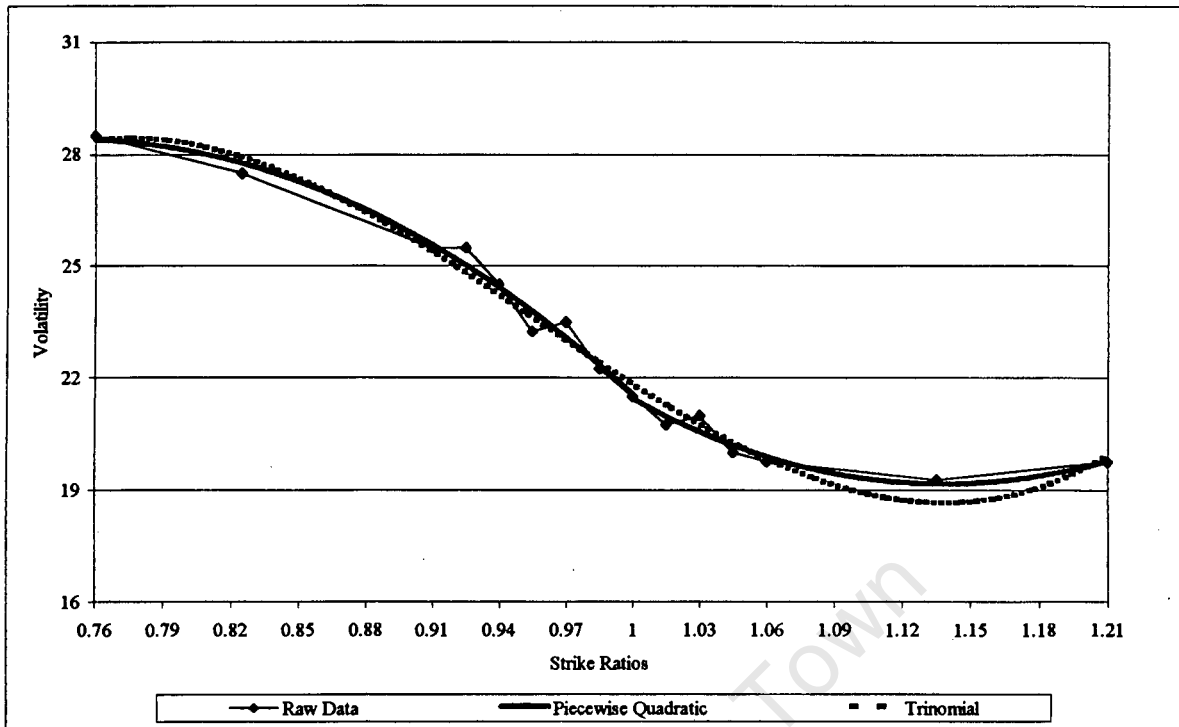


Figure 4.3.2-1. Comparison of the Trinomial Regression with the Piecewise Quadratic Regression
 The data in Jackwerth and Rubinstein (1995) (i.e. implied volatilities of the S&P 500 option prices on December 4, 1990) is fitted using a piecewise quadratic regression as well as trinomial regression. The first piece of the quadratic regression is defined for strike ratios less than or equal to one whereas the second piece of the quadratic regression is defined for strike ratios greater than or equal to one. The initial raw data is displayed with linear interpolation between the data points.

The statistical analysis is however limited in Table 4.3.2-1 because the two regression models (i.e. piecewise quadratic and trinomial regression model) are based on different data sets. The piecewise quadratic regression uses the implied volatilities of the *at-the-money* strike ratio¹¹² (i.e. $x = 1$) for each interval. Consequently, a comparison with the trinomial regression model is not strictly equitable because the piecewise quadratic regression approach has one implied volatility more than the trinomial regression model. Hence, a second trinomial regression model is artificially estimated using the trinomial regression model with two *at-the-money* strike ratios (i.e. both at $x = 1$) in Table 4.3.2-1. However, the artificially computed trinomial regression model and the trinomial regression model do not exhibit substantial differences. Nevertheless, a comparison between the artificial computed trinomial regression model and the piecewise quadratic regression seems appropriate because of their similarities in their data sets.

Although it can be argued that a superior fit is evident for the piecewise quadratic regression model in Figure 4.3.2-1, the “goodness-of-fit” estimator (R^2) reveals a different

¹¹² The strike ratio lies on the x-axis.

picture. The “goodness-of-fit” estimator R^2 is particularly higher for the artificial trinomial regression model in the comparison with the strike ratios above and equal to one. However, it is problematic to compare the two different regression models directly because R^2 is based on the whole data set for the artificial trinomial regression whilst the piecewise quadratic regression model only computes R^2 for each piece of the data set. Nevertheless, the residual sum of squares, RSS, can be used to give some insight into the comparison of the two regression models. If the two RSS values of the piecewise quadratic regression model are summed to 1.0985, the result is still substantially lower than the RSS value of 2.1897 of the trinomial regression model. The lower result of the added RSS suggests that the piecewise quadratic regression is the model that has a superior fit confirming the result of the graphical analysis.

Table 4.3.2-1

Comparison of Piecewise Quadratic and Trinomial Regression Statistics

The data from Jackwerth and Rubinstein (1995) is examined using the residual sum of squares, RSS, the variance of the residuals, s^2 , and the “goodness-of-fit” estimator, R^2 . Moreover, the results are presented for a trinomial regression, an artificial trinomial regression (with two strike ratios of $x=1$), and a piecewise quadratic regression (with strike ratios below and equal to one and above and equal to one). The value in brackets under the heading of R^2 is the unadjusted value of the “goodness-of-fit” estimator.

Regression Models	Degrees of Freedom DF	Residual Sum of Squares RSS	Variance of Residuals s^2	Goodness-of-Fit R^2
Trinomial Regression	11	2.0713	0.1883	0.9782 (0.9829)
Artificial Trinomial Regression	12	2.1897	0.1825	0.9775 (0.9823)
Quadratic Regression for $x \leq 1$	6	0.7832	0.1305	0.9747 (0.9819)
Quadratic Regression for $x \geq 1$	4	0.3153	0.0788	0.8796 (0.9197)

Although the fit of the piecewise quadratic regression deemed satisfactory here, a final attempt to improve the performance of the interpolation method is conducted below.

4.3.3 Comparison of the Piecewise Trinomial Regression with the Piecewise Quadratic Regression Model

A further extension of the interpolation method focuses on a piecewise trinomial regression model in a similar manner to the piecewise quadratic regression model (in the prior section). This last comparison between a piecewise quadratic and a piecewise trinomial regression is conducted to investigate the interesting issue of a trade off between

sophisticated modelling approaches and their gain in accuracy (i.e. quadratic and trinomial regression). The effects may be statistically small but they are perhaps economically important. The graphical analysis in Figure 4.3.3-1 only shows the strike ratios less than or equal to one¹¹³ (because the small differences between the regression models could not be successfully portrayed graphically otherwise).

The advantage of the piecewise trinomial regression is only evident for strike ratios below 0.88 and above 0.79 in Figure 4.3.3-1. The piecewise trinomial regression reveals a superior fit for the strike ratios between 0.79 and 0.88 but shows no superiority for the strike ratios below 0.76. Hence, the piecewise trinomial regression reveals a marginal improvement but the price paid for the added complexity is probably higher than its potential gain. The statistical survey concerning the comparison of the piecewise trinomial regression model and the piecewise quadratic regression model follows.

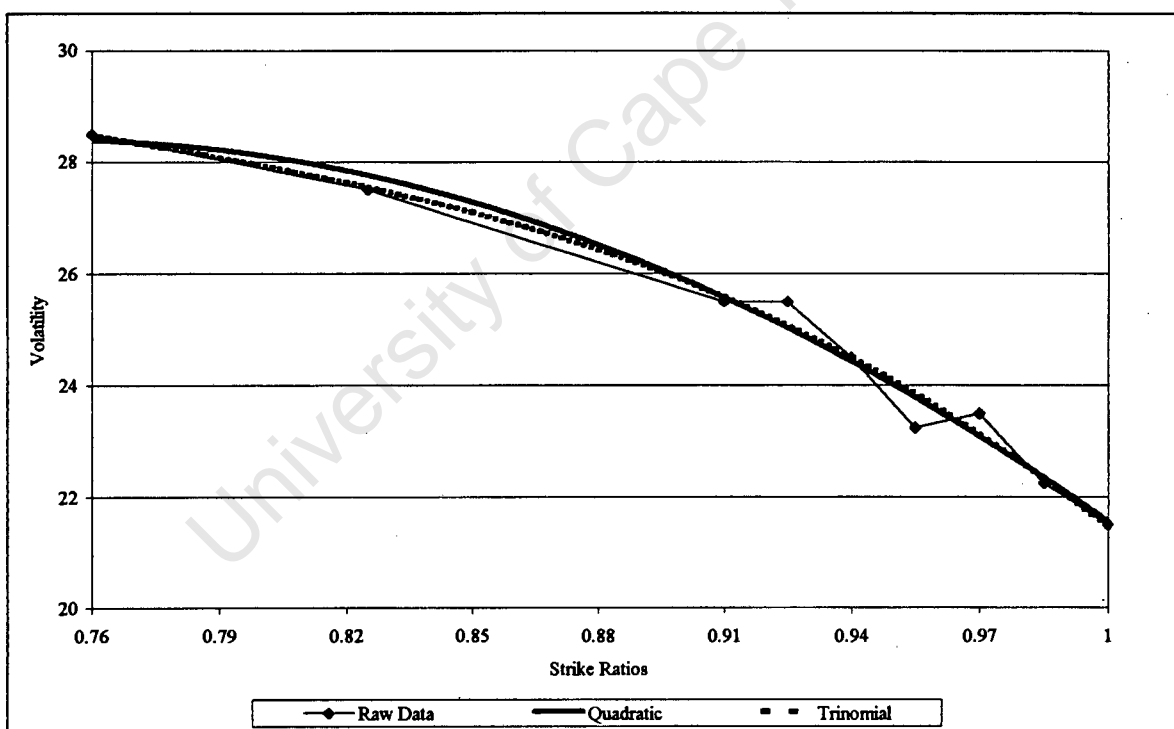


Figure 4.3.3-1. Comparison of the Piecewise Trinomial Regression with the Piecewise Quadratic Regression

The data in Jackwerth and Rubinstein (1995) (i.e. implied volatilities of the S&P 500 option prices on December 4, 1990) is fitted using a piecewise quadratic regression as well as a piecewise trinomial regression for strike ratios less than or equal to one. The initial raw data is displayed with linear interpolation between the data points.

¹¹³ The graphical analysis of strike ratios greater than or equal to one is shown in Figure 4.B-1 in Appendix 4.B.

The statistical analysis in Table 4.3.3-1 is also based on separated data sets for each regression model. The “goodness-of-fit” estimator (R^2) does not confirm the results of the graphical analysis in Figure 4.3.3-1 that the piecewise trinomial regression fits the data slightly better than the piecewise quadratic regression model (0.9717 vs. 0.9747 for $x \leq 1$ and 0.8436 vs. 0.8796 for $x \geq 1$). The residual sums of squares, RSS, is however larger for the piecewise quadratic regression model than for the piecewise trinomial regression model whilst the variance of the residuals, s^2 , is smaller. The decrease in the RSS however is seen to be very small for the piecewise trinomial regression model.

Both regression models are also assessed using the F-test (under the null hypothesis that the simpler is more appropriate). The F-test is carried out separately for the comparison of the quadratic and the trinomial regression model for $x \leq 1$ and $x \geq 1$ respectively. The value of F obtained for the $x \leq 1$ comparison is 0.58 and lies below the critical value of 6.61 for the F-test at a significance level of five percent. The value of F obtained for the $x \geq 1$ comparison is 0.08 and also lies well below the critical value of 10.13 for the F-test at a significance level of five percent. Consequently, the null hypothesis cannot be rejected implying that the piecewise quadratic regression model can still be viewed as the more appropriate regression model.

Table 4.3.3-1

Comparison of Separated Quadratic and Separated Trinomial Regression Statistics

The data from Jackwerth and Rubinstein (1995) is examined using the residual sum of squares, RSS, the variance of the residuals, s^2 , and the “goodness-of-fit” estimator, R^2 . Moreover, the results are presented for a piecewise trinomial regression and a piecewise quadratic regression (with strike ratios below and equal to one and above and equal to one). The value in brackets under the heading of R^2 is the unadjusted value of the “goodness-of-fit” estimator.

Regression Models	Degrees of Freedom DF	Residual Sum of Squares RSS	Variance of Residuals s^2	Goodness- of-Fit R^2
Quadratic Regression for $x \leq 1$	6	0.7832	0.1305	0.9747 (0.9819)
Quadratic Regression for $x \geq 1$	4	0.3153	0.0788	0.8796 (0.9197)
Trinomial Regression for $x \leq 1$	5	0.7016	0.1403	0.9717 (0.9838)
Trinomial Regression for $x \geq 1$	3	0.3071	0.1024	0.8436 (0.9218)

The superior piecewise quadratic regression model will consequently be implemented in the proposed implied volatility tree model in section 4.4. However, this piecewise quadratic regression model is only suitable for extrapolation and interpolation of implied volatilities across *strike prices*. A theoretical discussion about the extrapolation and interpolation method across *expirations* follows.

4.3.4 Extrapolation and Interpolation Method across Expirations

The focus of this section is on a theoretical discussion of a suitable extrapolation and interpolation method for implied volatilities across expirations. The search for a method to extrapolate and interpolate implied volatilities across expirations is however limited to the interpolation of volatilities between available expirations. An extrapolation of implied volatilities across expirations is not required because detailed research (in Chapter 3) has revealed that options with a long time to expiration (i.e. over 180 days) and the same strike price are approximately constant in their implied volatility across different expirations. In addition, options with a short time to expiration (i.e. less than 10 days) also have approximately constant implied volatilities across different expirations. Hence, it is assumed that the implied volatility function across strike prices of the shortest traded expiration captures all implied volatility functions across strike prices with shorter expirations. Consequently, it is also assumed that the implied volatility function across strike prices of the longest traded expiration captures all implied volatility functions across strike prices with longer expirations. Substantial empirical support for these assumptions is given in the analysis of the “smile pattern” in Chapter 3. Hence, the remaining expirations between the shortest and longest expiration available are interpolated by a method introduced below.

The simplest way for the interpolation is certainly a linear interpolation¹¹⁴. Here, the linear interpolation is considered to estimate the implied volatilities between available expirations. The linear interpolation although it was found to be unsuitable for the interpolation across *strike prices* has the advantage of simplicity for estimating the implied volatilities across *expirations*¹¹⁵. Hence, the linear interpolation meets the proposed conditions of a simple, stable, and practical method and is implemented together with the proposed piecewise quadratic regression model in the implied volatility tree model.

¹¹⁴ The linear interpolation is calculated using the following form

$$\sigma(t_2) = \sigma(t_1) + (\sigma(t_3) - \sigma(t_1)) \frac{t_1 - t_2}{t_1 - t_3}$$

where the expirations t are $t_1 < t_2 < t_3$ and $\sigma(t_i)$ is the implied volatility at t_i .

¹¹⁵ The danger of the linear interpolation across expirations is that non-linear functions are incorrectly estimated and an estimation error results. This danger becomes less important the more expirations are available. In addition, the trade off between simplicity and a more sophisticated model suggest the linear interpolation as the more suitable solution.

A further idea for an extrapolation and interpolation method of implied volatilities across *expirations* again is a regression model (as for the extrapolation and interpolation across *strike prices*). However, a regression model for the implied volatilities across expirations is very complicated to implement because an infinite¹¹⁶ number of regressions would be required. A piecewise separation into *at-the-money*, *out-of-the-money*, and *in-the-money* strike ratios does not help either because the implied volatility function can also be different in each of these three “money” regions. In addition, the fewer the number of regressions models the higher is the estimation error of the computed regression models. Hence, a regression model for implied volatilities across *expirations* is not appropriate for the implementation in the implied volatility tree model because it is neither stable, simple, nor practicable. As a consequence, a linear interpolation is used for implied volatilities across *expirations*.

The conclusion regarding the assessment of the extrapolation and interpolation method to be implemented in the implied volatility tree follows.

4.3.5 Conclusion

The aim of the research concerning extrapolation and interpolation methods was to find a suitable method to implement in the proposed implied volatility tree for the South African environment. This aim was successfully achieved. A piecewise quadratic regression model was found in a number of comparisons to be the most suitable model for the extrapolation and interpolation of implied volatilities across *strike prices*. The extrapolation and interpolation method of implied volatilities across *expirations* was limited to an interpolation method because the detailed results of Chapter 3 revealed constant implied volatility for options near to expiration and far from expiration respectively. The established interpolation method for implied volatilities across *expirations* was a linear interpolation. Both the linear interpolation across *expirations* and the piecewise quadratic

¹¹⁶ The infinite number of regressions is required because the implied volatility function across expirations increases with a rising time to expiration for *at-the-money* strike ratios but decreases for *out-of-the-money* and *in-the-money* strike ratios in South Africa (for example, see the ALSI in Chapter 3). This leads to the requirement of a regression model for each strike ratio. Hence, the large number of regression models to

regression model across *strike prices* will be implemented in the proposed implied volatility tree for the South African environment.

This implied volatility tree model will be rigorously tested (in section 4.4) to assess the model performance.

4.4 Tests of the Implied Volatility Tree Model

The aim of the tests of the implied volatility tree for options on futures is to assess the accuracy of the proposed implied volatility tree model for the South African environment. To date, tests of accuracy for implied volatility trees are mainly carried out by establishing whether the option price model has the ability to re-estimate a hypothetical input accurately. These tests have recently been conducted on the assumption of non-constant implied volatility (e.g. Barle and Cakici (1995)). This test methodology is discussed in section 4.4.2. For comparison purposes we conduct equivalent tests on both non-constant and constant implied volatilities. Initially, a test methodology which assumes a constant implied volatility is introduced in section 4.4.1. The constant implied volatility test is required to assess the robustness of the proposed implied volatility tree under this assumption. The conclusion in section 4.4.3 summarizes the results.

Both test methodologies require the consideration of different option price models to compare their results with each other. For example, Barle and Cakici (1995) compare the option prices computed by their implied tree model with option prices based on the Black and Scholes (1973) model. In this thesis, the modified Black model is implemented instead of the Black and Scholes (1973) model and the resulting option prices are compared to the option prices obtained from the implied volatility tree model proposed for the South African environment. It should be noted however that the results of this comparison are dependent on the size of the tree model. The option prices of a binomial tree model only approximate the modified Black option prices with a decreasing approximation error the bigger the binomial tree becomes (i.e. increasing number of steps in the tree). Consequently, the value of the comparison between the modified Black option prices and

compute suggest to reject a regression model as extrapolation and interpolation method for implied volatilities across expirations.

the option prices computed by the implied volatility tree model can be affected by this approximation error. A binomial tree model with constant volatility is included therefore to compare its results with that of the implied volatility tree model (i.e. without approximation error). Besides the modified Black model and the binomial tree model, a third option price model proposed by Chriss (1997) is also included. This third option price model is included because it combines the implied volatility tree model with the Black and Scholes (1973) model (hence the modified Black model in the South African environment). Henceforth in the thesis this model by Chriss (1997) with the embedded modified Black model inside of the implied volatility tree model is referred to as “Implied Black Tree”. A more detailed discussion of this “Implied Black Tree” model is found below.

Implied Black Tree

Chriss (1997) combines an implied volatility tree model with the Black and Scholes (1973) model¹¹⁷. His proposal aims at greater computational efficiency¹¹⁸ of the implied volatility tree by implementing the Black and Scholes (1973) model. The proposed model by Chriss (1997) (i.e. the implied Black tree) is further introduced to study the consequences of the embedded lognormal model (i.e. the modified Black model in South Africa) in this kind of an implied volatility tree. Assuming that a lognormal model prices *out-of-the-money* options incorrect (because the tails of the market distribution are much bigger than in the lognormal distribution), the implied Black tree is expected to perform poorly by comparison to the implied volatility tree model.

One approach comparing the performance of the different option price models is the comparison of their mean absolute errors, MAE's, and their mean squared errors, MSE's (used for example by Gwilym and Buckle (1997)). The values of the MAE and the MSE are calculated as follows

$$\text{MAE} = \frac{\sum_{i=1}^n |\text{option1} - \text{option2}|}{n}$$

¹¹⁷ Here, the modified Black formula for options on futures is used.

¹¹⁸ For example, our tests yielded a computation time of 86 seconds for a 50-step implied volatility tree whilst the implied Black tree required only 30 seconds for the computation.

$$\text{MSE} = \frac{\sum_{i=1}^n (\text{option1} - \text{option2})^2}{n}$$

where option1 is the option price of the first model and option2 is the option price of the second model respectively in the option price model comparison.

Finally, it is assumed that all implied volatilities outside the [0.6,1.4] interval are constant¹¹⁹ as indicated in Chapter 3 because no other strike ratios can be found in the South African data. In addition, implied volatilities near to these extreme strike ratios are nearly constant (as presented in Chapter 3 for the ALSI, the GLDI, and the INDI).

The tests start with the assumption of constant volatility for all chosen option price models below.

4.4.1 Test with Constant Volatility

The first test is based upon the assumption of constant volatility across strike prices and across expirations. Here, a hypothetical call¹²⁰ is calculated with 180 days to expiration and a constant volatility of 20 percent based upon a future with the value of 100 at the beginning of the contract (i.e. $F_{1,1}$). Moreover, the option prices for the tree models are computed using 50 steps¹²¹ in the trees. The results of the first test for call¹²² options can be observed in Table 4.4.1-1 and in Table 4.4.1-2.

Table 4.4.1-1 shows the strike prices in the first column for the computed call option. The prices of the call options are presented in the columns two to five (in Table 4.4.1-1) for the selected option price models (i.e. modified Black model, binomial model, implied volatility tree, and implied Black tree). Table 4.4.1-2 displays supplementary results in the

¹¹⁹ The implied volatility value of the 0.6 strike ratio is valid for the range of strike ratios below 0.6 and the implied volatility value of the 1.4 strike ratio is valid for the range of strike ratios above 1.4.

¹²⁰ In the South African environment for options on futures, the expiration style does not matter as discussed earlier (in section 3.4.2).

¹²¹ A rigorous analysis was conducted to establish the number of steps required but is not included for the sake of brevity (available from the author). In this section, it was found that trees with 50 steps are sufficient by consideration of the trade off between accuracy and computational efficiency. Although 50 step trees are computational expensive, they are necessary for accuracy. Later in the thesis, the requirement of accuracy is relaxed allowing less computational expensive trees (for example 20 steps).

¹²² Similar results for put options are displayed in Table 4.C-1 in Appendix 4.C.

form of a matrix of the MAE and MSE results for the comparisons between the implemented option price models.

Table 4.4.1-1

Comparison of Option Price Models for Call Options with Constant Volatility

The call option prices are based on a volatility of 20 percent with 180 days to expiration and a future price of 100. The tree models (binomial tree, implied tree, and implied Black tree) are calculated using 50 steps. The range of strike prices (60 to 140) proxies the South Africa environment (examined in this thesis) with strike ratios between 0.6 to 1.4.

Strike Prices (1)	Option Prices for Different Option Price Models			
	Modified Black (2)	Binomial Tree (3)	Implied Tree (4)	Implied Black Tree (5)
60	40.000	40.000	40.000	40.000
70	30.021	30.018	30.018	30.018
80	20.299	20.300	20.300	20.284
85	15.791	15.780	15.780	15.751
90	11.745	11.758	11.758	11.701
95	8.318	8.333	8.333	8.260
100	5.599	5.571	5.571	5.488
105	3.581	3.590	3.590	3.513
110	2.179	2.196	2.196	2.132
115	1.265	1.277	1.277	1.229
120	0.702	0.708	0.708	0.675
130	0.192	0.191	0.191	0.181
140	0.046	0.044	0.044	0.042

Note that the option prices are rounded to three decimal places because the traded option contracts consist of ten options for the ALSI, GLDI, and INDI at SAFEX examined.

Table 4.4.1-1 reveals small differences in the option prices between the modified Black model in column (2) and the binomial model in column (3). These small differences reflect the approximation error of the binomial model. Shastri and Tandon (1986) argue that the approximation error between a Black and Scholes (1973) model (i.e. in South Africa the modified Black model) and a binomial tree model becomes smaller the more steps are used for the binomial tree but simultaneously the binomial tree becomes computationally more expensive (i.e. it takes longer for the calculation). Consequently, a comparison between the modified Black model and the implied volatility tree or the implied Black tree is also affected by an approximation error. However, the comparison between the binomial tree model and the implied volatility tree or the implied Black tree is not affected by an approximation error because the approximation error offsets its influence in the comparison between these models.

The comparison between the option prices of the binomial tree in column (3) and the implied tree in column (4) reveals no differences in their option prices for the same strike price. Consequently, this result of no price differences suggests that the implied tree is as

accurate as a binomial tree under the assumption of constant volatility. The price differences in the option prices of the implied tree in column (4) and the modified Black model in column (2) can therefore be put down to the approximation error discussed earlier. Finally, the comparison between the option prices of the binomial tree model in column (3) (as well as the implied tree in column (4)) and the implied Black tree in column (5) reveals option price differences for nearly all strike prices. In particular, the results suggest that the implied Black tree is not as accurate as the binomial tree model (or the implied tree model).

A matrix with the MAE and the MSE results for the different model comparisons with call¹²³ options is introduced in Table 4.4.1-2.

Table 4.4.1-2

Matrix of MAE and MSE Results for Call Options in Comparisons of Different Option Price Models
The results of the comparisons between different option price models in Table 4.4.1-1 yield the mean absolute error, MAE, in the first row of each comparison and as the mean squared error, MSE, in the second row of each comparison in **bold**.

Option Price Models	Option Price Models			
	Modified Black	Binomial Tree	Implied Tree	Implied Black Tree
	MAE MSE	MAE MSE	MAE MSE	MAE MSE
Modified Black	0 0	0.009096 0.000145	0.009091 0.000145	0.035711 0.002188
Binomial Tree	0.009096 0.000145	0 0	0.000020 0	0.037896 0.002321
Implied Tree	0.009091 0.000145	0.000020 0	0 0	0.037906 0.002321
Implied Black Tree	0.035711 0.002188	0.037896 0.002321	0.037906 0.002321	0 0

Note that the results are rounded to six decimal places.

For the comparison between the binomial tree model and the implied tree model Table 4.4.1-2 reveals a MSE value of zero and a MAE value of 0.000020 (caused by differences in their option prices beyond the third decimal place). This result of similar option prices is consistent with the findings in Table 4.4.1-1. In addition in Table 4.4.1-2, it is seen that the MSE values for the comparisons between the modified Black model and the binomial tree model or the implied tree model respectively are identical (0.000145) and that the MAE values only differ in the sixth decimal place (0.009096 vs. 0.009091). These results suggest that the approximation errors between the modified Black model and the binomial tree

¹²³ The equivalent matrix for put options can be found in Table 4.C-2 in Appendix 4.C.

model or the implied tree model respectively are similar whilst the approximation error between the binomial tree model and the implied tree model is nearly zero.

The results for the comparison of the implied Black tree model with the binomial model or the implied tree respectively in Table 4.4.1-2 also confirm the findings of Table 4.4.1-1. The MAE and MSE results for these comparisons reveal an MAE of 0.037896 or 0.037906 respectively and an MSE of 0.002321 twice.

The differences in the option prices between the implied Black tree and the binomial tree or the implied tree respectively can be considered to be of practical relevance. For example, if 1000 option contracts¹²⁴ are traded at a strike price of 100 with the option price of the binomial tree (identical to the option price of the implied tree), the price of these 1000 contracts would be ZAR 55,710 whilst the price of the contract using the implied Black tree price is only ZAR 54,880. The difference between the models is ZAR 830 or 1.5 percent which is practically relevant. Consequently, this practical relevant result may suggest the rejection of the implied Black tree. However, the computational efficiency of the implied Black model (as discussed in section 4.4) is superior to the implied volatility tree. In conclusion, the trade off between computational efficiency and accuracy needs to be considered. The inaccurate and practical relevant results of the implied Black tree with their advantage of computational efficiency do not however seem worth the substantial loss of accuracy. Hence the implied Black tree is not considered as appropriate in the South African environment under the assumption of constant volatility.

The test conducted under the assumption of constant implied volatility reveals that the implied volatility tree model is superior to the implied Black tree. Moreover, this test also reveals that the option prices of the implied tree model are in essence similar to the binomial model. In addition, it should be noted that it is essential to assess the accuracy of the implied tree model comparatively to a model that is based on the same conditions¹²⁵. As a next step in the assessment of the implied tree model, a second test which now assumes non-constant volatility follows.

¹²⁴ The number of 1000 option contracts corresponds to 10,000 options at SAFEX because one option contract consists of 10 options.

4.4.2 Test with Non-Constant Volatility

The objective of the second test is to assess the ability of the models to re-estimate accurately a hypothetical input under the assumption of non-constant volatility. Hence, the computed option prices of five option price models are contrasted to the hypothetical input option prices. The computation of the hypothetical input option prices as well as the computation of the option prices by the implied tree model, the implied Black tree model, and the “implied tree with approximation error” model are based on a non-constant volatility assumption. This non-constant volatility assumption is established by a technique similar to that used by Chriss (1997). Chriss (1997) implements a linear volatility function¹²⁶ by simplifying non-linear “smiles” (as evident, for example, in the ALSI in Chapter 3). This linear simplification is however deemed sufficient for the assessment of the objective of this section (i.e. to re-estimate accurately a hypothetical input under the assumption of non-constant volatility). This implemented linear volatility function is subsequently used as the volatility input for the computation of all required option prices here.

The linear volatility function is defined for two intervals. The first interval is defined over the range of strike prices between 60 and 100 whilst the second interval is defined over the range of strike prices between 100 and 140. The piecewise definition of the volatility function in the two intervals is consistent with prior research (in section 4.3). It is evident in the empirical examination of the ALSI, the GLDI, and the INDI in Chapter 3 that the implied volatility function differs between the *out-of-the-money* interval (i.e. above and equal to the strike price of 100 for call options¹²⁷) and the *in-the-money* interval (i.e. below and equal to the strike price of 100 for call options). Consequently, the volatility function is defined as

¹²⁵ This means that the approximation error between the modified Black model and the binomial model is not relevant for the comparison between the binomial model and the implied tree model.

¹²⁶ The motivation for the use of a linear volatility function for the computation of the option prices in this thesis instead of traded option prices is twofold. First, the use of a volatility function instead of single data points prevents the distortion of the results by scattered data points. Second, the linear volatility function is consistent with similar studies by Barle and Cakici (1995) and Chriss (1997).

¹²⁷ The definition of the piecewise volatility function is reversed for put options.

$$y_1 = 0.20 + \left(\frac{\text{Future} - \text{Strike}}{100} \right) \times 0.15 \quad \text{for} \quad 60 \leq \text{Strike} \leq 100$$

$$y_2 = 0.20 - \left(\frac{\text{Future} - \text{Strike}}{100} \right) \times 0.05 \quad \text{for} \quad 100 \leq \text{Strike} \leq 140$$

where “future” is the price of the underlying asset (which is in South Africa the future) and “strike” is the strike price of the option. The future price is assumed to be 100 and the time to expiration 60 days.

The test results are presented in two tables (Table 4.4.2-1 and Table 4.4.2-2). Table 4.4.2-1 compares the hypothetical input option prices of calls¹²⁸ with call option prices computed using the modified Black model, the binomial tree model, the implied tree model, the implied Black tree model, and an “implied tree model with approximation error”. The comparison of the option prices is required to assess the quality of each option price model. This assessment is supported by the computation of the MAE and the MSE between the hypothetical input option prices and the option prices for the five option price models in Table 4.4.2-2.

In particular, the focus of the test is on the quality of the option prices computed by the proposed implied tree model for the South African environment (shown in column (5) of Table 4.4.2-1). In addition, a second implied tree model, i.e. the “implied tree model with approximation error”, is proposed in this thesis to enhance the computational efficiency of the implied tree model. The computed option prices of this “implied tree model with approximation error” are exhibited in column (7) of Table 4.4.2-1. The motivation and explanation of this option price model is elaborate and is therefore relegated to Appendix 4.E.

The comparison between the hypothetical input option prices and the option prices of the five option price models in Table 4.4.2-1 begins the assessment.

¹²⁸ Similar results to the results of the call option in Table 4.4.2-1 and in Table 4.4.2-2 are observed for the results of the put option in Table 4.D-1 and Table 4.D-2 respectively in Appendix 4.D.

Table 4.4.2-1

Comparison between Input and Output Option Prices for Call Options with Non-Constant Volatility
 The call option prices are based on 60 days to expiration and a future price of 100. The option prices of the implied tree models (i.e. the implied tree, the implied Black tree, and the implied tree with approximation error) are computed using 50 steps under the assumption of non-constant volatility. The option prices of the modified Black model and of the binomial tree model are computed with a constant volatility assumption (i.e. 20 percent across the strike prices). The range of strike prices (60 to 140) is chosen to reflect the strike prices available in the South Africa environment.

Strike Prices (1)	Input Option Price (2)	Output Option Prices for Different Option Price Model				
		Modified Black (3)	Binomial Tree (4)	Implied Tree (5)	Implied Black Tree (6)	Implied Tree with Approximation Error (3) - (4) + (5) = (7)
60	40.000	40.000	40.000	40.000	40.000	40.000
70	30.000	30.000	30.000	30.000	30.000	30.000
80	20.023	20.006	20.006	20.022	20.020	20.022
85	15.118	15.063	15.059	15.117	15.108	15.121
90	10.455	10.351	10.353	10.455	10.427	10.453
95	6.361	6.263	6.267	6.358	6.315	6.354
100	3.234	3.234	3.218	3.218	3.170	3.234
105	1.432	1.397	1.393	1.426	1.380	1.430
110	0.545	0.501	0.497	0.535	0.523	0.539
115	0.181	0.150	0.145	0.177	0.163	0.182
120	0.054	0.038	0.035	0.053	0.032	0.056
130	0.004	0.001	0.001	0.003	0.000	0.003
140	0.000	0.000	0.000	0.000	0.000	0.000

Note that the option prices are rounded to three decimal places because the traded option contracts consist of ten options for the ALSI, GLDI, and INDI at SAFEX examined.

Firstly, all option prices are compared for the *at-the-money* strike price (i.e. 100) in Table 4.4.2-1. For the *at-the-money* strike price the identical option prices of the modified Black model in column (3) and the hypothetical input option price in column (2) are a consequence of the same implied volatility input and the use of the same option price model (i.e. all hypothetical input option prices are computed under the assumption of non-constant volatility across strike prices with the modified Black model). In addition, the hypothetical input option price and the option price computed by the “implied tree with approximation error” in column (7) are also identical. This result is however based on the identical option prices of the binomial model tree in column (4) and the implied tree model in column (5) as well as on the consequently unchanged option price of the modified Black model in column (3). The differences between the hypothetical input option price (as well as the option prices computed by the modified Black model) and the option prices using the binomial tree model in column (4) and the implied tree model in column (5) are identical. These identical differences are a consequence of an approximation error (because of the implementation of 50-step trees). The higher difference between the hypothetical input option price and the option price computed by the implied Black tree model in

column (6) consequently suggests that the implied Black tree model is an inappropriate option price model.

After the introductory discussion of the option price differences for the *at-the-money* strike price, the focus shifts to the differences between the hypothetical input option prices and the option prices computed by the five option price models for the *in-the-money* and *out-of-the-money* strike prices in Table 4.4.2-1.

Differences between the hypothetical input option prices and the option prices of the five option price models are observed for the *in-the-money* strike prices (strike < 100) as well as for the *out-of-the-money* strike prices (strike > 100). For example, the *out-of-the-money* option with strike 110 has an input price of 0.545. The result closest to this option price is found for the “implied tree with approximation error” (in column (7) in Table 4.4.2-1) with a price of 0.539. The second closest result to the hypothetical input option price is found for the implied tree model (in column (5)) with a price of 0.535 (although its price is negatively influenced by the approximation error of the 50-step tree¹²⁹). The option price computed by the implied Black tree model (i.e. 0.523) in column (6) reveals the highest difference from the hypothetical input option price of all option prices computed under the assumption of non-constant volatility. However, the result of the implied Black tree is still better than the results for the modified Black model and the binomial tree model with their option prices computed under the assumption of constant volatility.

The option price models with the constant volatility assumption (i.e. the modified Black model and the binomial model) yield option prices of 0.501 and 0.497 in column (3) and (4) respectively, which differ from the hypothetical input option price by 8.1 percent and 9.1 percent respectively. These price differences reveal a substantial source of pricing error when constant volatility is assumed.

The further assessment of the ability of the option price models to replicate the input prices is given by the results in Table 4.4.2-2.

¹²⁹ The approximation error between the modified Black price and the binomial price is 0.004. A 100-step tree has a price of 0.545.

Table 4.4.2-2

Matrix of MAE and MSE Results for Call Options for Input Replication

The results of the comparisons between input option prices and output option prices of the different models in Table 4.4.2-1 yield the mean absolute error, MAE, in the first row and the mean squared error, MSE, in the second row in bold.

Option Model		Option Price Models				
		Modified Black (1)	Binomial Tree (2)	Implied Tree (3)	Implied Black Tree (4)	Implied Tree with Approximation Error (5)
Input Prices	MAE	0.030918	0.033306	0.003362	0.020550	0.001873
	MSE	0.002164	0.002211	0.000032	0.000847	0.000008

Note that the results are rounded on six decimal places.

The results in Table 4.4.2-2 for call options¹³⁰ reveal that the MAE and MSE values for the “implied tree with approximation error” in column (5) are the smallest of all option price models. The second best results (i.e. the second smallest) for the MAE and the MSE are found for the implied tree model in column (3). Hence, the “implied tree with approximation error” reveals once again its superiority to the implied tree. The MAE and MSE results of the implied Black tree in column (4) however suggest that this option price model is not appropriate (as in section 4.4.1) for the computation of the option prices under the assumption of non-constant volatility. It is suggested that the large MAE and MSE values of the implied Black tree are caused by the embedded modified Black model inside the implied tree as this modified Black model assumes an inappropriate lognormal return distribution.

The results of the option price models with *constant volatility* reveal the largest MAE and MSE values for all option price models. The binomial tree model in column (2) exhibits the largest MAE and MSE values of all the option price models. A plausible suggestion for these poor results can be based on the combination of the approximation error for the binomial tree and the assumption of constant volatility. The second largest MAE and MSE are found for the modified Black model in column (1). These MAE's and MSE's are smaller than the results of the binomial model because no approximation error is accounted for in the modified Black model. However, the results of the modified Black model suggest that the constant volatility assumption is responsible for the differences between the computed option prices and the hypothetical option prices. Consequently, the best result is obtained using the “implied tree with approximation error”.

Finally, the results of the constant and non-constant volatility test are summarized and concluded below.

4.4.3 Conclusion

In summary, the tests have presented evidence that the implied tree model appears to be the best option price model (of the option price models considered) to price options accurately in the South African environment. This finding is upheld under both the assumption of constant volatility as well as of non-constant volatility. Furthermore, it was established that option price models with constant volatility like the binomial tree model or the modified Black model do not have the ability to price the options accurately in the presence of non-constant volatility. Further assessments were conducted on the implied tree model in its original form (with a binomial pricing method for the required options in the implied tree) and in a modified form proposed by Chriss (1997) (with the modified Black pricing method for the required options in the implied tree). The results of the assessments for the constant and the non-constant volatility assumption clearly indicate that the implied Black tree model by Chriss (1997) is not as accurate as the implied tree model. Hence, the implied Black tree model is not considered as an useful model in the South African context. Finally, a new methodology was considered to improve the pricing of options via the implied tree model. This proposed method takes account of the difference between the option price of the modified Black model and the option price of the binomial tree (i.e. the approximation error) by summing up this difference and the option price computed by the implied tree model. This proposed method has the advantage of yielding more accurate option prices by using an “implied tree model with approximation error” by contrast to an implied tree model (without the approximation error adjustment) under the assumption of the same number of steps for both tree models.

The final investigation in this chapter (in section 4.5) considers return distributions implicit in the implied volatility tree (henceforth referred to as implied distributions).

¹³⁰ Similar results for put options are displayed in Table 4.D-2 in Appendix 4.D.

4.5 Implied Distributions

The implied return distribution is *implicit* in all option prices and can be inferred from these option prices. This *implicit* return distribution is henceforth referred to as the “implied distribution”. Currently, two main categories of methodologies are available to establish this “implied distribution”. The first category consists of all methodologies that establish the implied distribution directly from the option prices, whilst the second category requires the construction of an implied volatility tree. Chriss (1997) argues that the first category of the methodologies (in particular the implied binomial tree by Rubinstein (1994)) has the weakness of constructing only a *terminal* distribution without controlling the distribution on other dates. Both categories are briefly explained in more detail in section 4.5.2 for the first category and section 4.5.3 for the second category of methodologies.

This section 4.5 contains two more important segments. Section 4.5.4 describes the construction of a *hypothetical* “volatility surface” across strike prices and across expirations constructed in accordance with the findings on implied volatility in Chapter 3. The motivation for the *hypothetical* construction of this volatility surface is based on the poor quality of the data. The relatively illiquid option market in South Africa for the period between 1992 to 1996 does not provide a sufficient basis of data for one single day to analyse the characteristics of the non-constant volatility on the implied distribution. In particular, the availability of data with only four option prices per contract for each trading day causes substantial problems in the computation (for example the non-simultaneous pricing problem between the option price and the future price). An acceptable data set can however be found in practice when bid and ask quotes are used to construct the volatility surface. In this thesis, bid and ask quotes are however not available in the data set (because they are not available in the data provided by SAFEX). Nevertheless, the effects of non-constant volatility on implied distributions are evident in the international literature (e.g. Shimko (1993)). Hence, a data set is constructed for the volatility surface in section 4.5.4. This data set is constructed in accordance with the results of Chapter 3 (i.e. the volatility smile and the time to expiration bias for the ALSI, the INDI, and the GLDI). Furthermore, this data set is used to analyse that is sufficient to analyse in section 4.5.5 the effects of non-constant volatility on the implied distribution for the South African environment.

Finally, a summarizing conclusion in section 4.5.6 ends the analyses of the implied distribution.

Before the analyses of the implied distribution starts, the background of return distributions in option price models and their connection to volatility assumptions (e.g. the constant volatility assumption in the Black and Scholes (1973) model) is briefly discussed (in section 4.5.1). This discussion is introduced because it explains why implied distribution should be used instead of any pre-selected return distributions (e.g. lognormal distribution).

The discussion continues with the background of return distributions below.

4.5.1 Background of Return Distributions

The assumption of a constant volatility is rejected by the empirical evidence in Chapter 3. Moreover, it is evident in South Africa that implied volatilities differ systematically across strike prices and across expirations (e.g. in the form of “smiles”). Mayhew (1995) argues that volatility smiles can be caused by systematic market imperfections or by a return distribution of the underlying asset that differs from the assumed lognormal distribution in the Black and Scholes (1973) model. In addition, it is suggested that these reasons are also responsible for the non-constant implied volatility across expirations.

Systematic market imperfections are however unlikely to be the cause of volatility smile pattern or volatility term structure pattern because these patterns are evident in a series of substantially different markets (e.g. USA, UK, Netherlands, and South Africa). It seems unreasonable that systematic market imperfections can cause the volatility patterns through all the different markets. Hence, the focus of the ensuing discussion in this section lies on the assumed lognormal distribution (as assumed in option price models like the Black and Scholes (1973) model or the modified Black model).

Consequently, it is assumed that only a different distribution (than the assumed lognormal distribution) can reflect the systematic differences in the volatility patterns for an option price model. However, the international literature contains a wealth of research on return distributions. For example, Fama (1965) and Mandelbrot's (1963) find that the daily returns of underlyings follow Stable Paretian distributions. Black and Scholes (1973) assume that returns follow a specific case of the family of Stable Paretian distributions,

that is, a lognormal distribution (to establish the Black and Scholes (1973) option price model). Sherrick, Irwin, and Forster (1996) prefer the Burr-XII distribution for S&P 500 future prices whereas Klerck and du Toit (1986) find neither a normal distribution nor a Stable Paretian distribution for share returns at the Johannesburg Stock Exchange (JSE). The different distributions show a diffuse picture that could be explained by non-stationary distributions. Fama (1965) assumes that the distribution is normal at any point in time, but the parameter of the distribution can change across time. The only real result of the research about return distribution seems to be that no particular distribution can be applied or as Bates (1995) puts it, that there is no single alternative distributional hypothesis that can eliminate the Black and Scholes strike price biases.

The volatility biases can however be eliminated by a flexible return distribution that is implicitly computed from the option prices and changes with every change in the option prices. This implied distribution can be computed by two categories of methodologies (as discussed in section 4.5). The first category that aims at establishing the implied distribution from the option prices follows below, and the second category follows immediately thereafter in section 4.5.3.

4.5.2 Direct Establishment of Implied Distributions

The direct establishment of implied distributions from option prices has been done for some 20 years. Breeden and Litzenberger (1978) established how implied distributions can be inferred from option prices. They demonstrate that the implied distribution is established using the second derivative of the call option price with respect to the strike price. However, their approach can only be used for options with the same expiration. Recently, Shimko (1993) has implemented the method of Breeden and Litzenberger (1978) to recover an implied distribution for the S&P 500.

Shimko (1993) finds that the implied distribution for the S&P 500 is negatively skewed and more leptokurtic than the lognormal distribution. Rubinstein (1994) also establishes the implied distribution directly from option prices but he uses a quadratic optimization program instead of the Breeden and Litzenberger (1978) method. Rubinstein's (1994) method is improved by Jackwerth and Rubinstein (1996) who find platykurtic implied distributions before the crash in 1987, and leptokurtic implied distributions after the crash

for the S&P 500 (which confirms the result of Shimko (1993)). Rubinstein's (1994) or Jackwerth and Rubinstein's (1996) implied distributions are implemented in the implied *binomial* tree method by Rubinstein (1994) to compute option prices based on the found implied distribution.

Chriss (1997) argues that the problem of this implied *binomial* tree lies with the terminal distribution implemented because the implied *binomial* tree cannot control the distribution on dates other than the date of the terminal distribution. Hence, the implied *binomial* tree method assumes one implied distribution for all dates which does not accurately reflect the environment in South Africa. Consequently, the focus changes to the second category of methodologies that use an implied *volatility* tree to establish the implied distribution.

4.5.3 Indirect Establishment of Implied Distributions

The advantage of the second category of techniques that establish the implied distribution through an implied volatility tree, is that implied distributions can be established for European options as well as for American options and for several expirations. Chriss (1997) argues that the implied distribution for a given future time can be estimated from the Arrow-Debreu prices and from the stock price nodes at the given time. The Arrow-Debreu prices are required to compute the transition probabilities (i.e. the probability of reaching the node (n, i)) by discounting the Arrow-Debreu prices with a discount factor. However, the Arrow-Debreu prices do not need to be discounted in the South African environment because the transition probability is equal to the Arrow-Debreu prices for options on futures with mark-to-market procedure. Hence, the implied distribution can be estimated directly from the Arrow-Debreu prices and the future prices in the implied volatility tree.

The most important advantage of establishing the implied distribution using the implied volatility tree is that the implied volatility tree accommodates the structure of the implied volatility across strike prices and across expirations. Instead, Rubinstein (1994) assumes that only the option prices for the selected expiration are relevant for the establishment of the *terminal* implied distribution in the implied *binomial* tree. Rubinstein (1994) argues that the prices of the options expiring at the end of the tree (i.e. at the expiration) can be

used to infer consistent values for options expiring earlier in the tree. His argument would however lead to option prices that are only consistent with one specified expiration. Instead, the implied volatility changes realistically in each considered expiration as displayed in the analysis of the implied volatility in Chapter 3.

Rubinstein's (1994) argument for the use of only one expiration becomes particularly questionable when compared to the market of interest rate bearing securities (because of their similarity to option prices with implied volatility). In the interest rate market the whole structure of interest rates across different expirations is required to infer the correct price for an interest rate bearing security. The use of the interest rate structure as well as the volatility term structure (i.e. all expirations) makes sense especially if all possible paths for the interest rate or for the implied volatility are considered. The information on these paths cannot be obtained by one expiration. Consequently, from here on the research in this thesis focuses on the implied volatility tree model to estimate implied distributions.

The estimation of the implied distributions require a volatility surface (as discussed in section 4.5). The construction of this required volatility surface is considered in the following section.

4.5.4 The Volatility Surface

The knowledge gained from research in this thesis thus far suggests that the analysis of implied distributions requires a volatility surface (that reflects non-constant volatilities across strike prices and across expirations). Hence, a volatility surface is constructed that to be consistent with the established results (in Chapter 3) having the characteristic volatility smiles and term structures of the ALSI, the GLDI, and the INDI. The results of the implied volatility research for these three indices are very similar hence only one volatility surface is considered for the demonstration of the effects of non-constant volatility on implied distributions here. Because the implied volatility (and consequently the volatility surface) can change every instant, the implied distribution changes too. Hence, it is important to note that the volatility surface obtained and consequently the implied distribution are non-stationary.

The implied volatility surface is constructed by incorporating the results¹³¹ on the implied volatility (from Chapter 3) in a subjective way. As discussed in section 4.5, the available data for the period from 1992 to 1996 for the South African environment is not sufficient for the ensuing analysis in section 4.5.5. Hence, a volatility surface is subjectively constructed¹³² and is displayed by Figure 4.5.4-1.

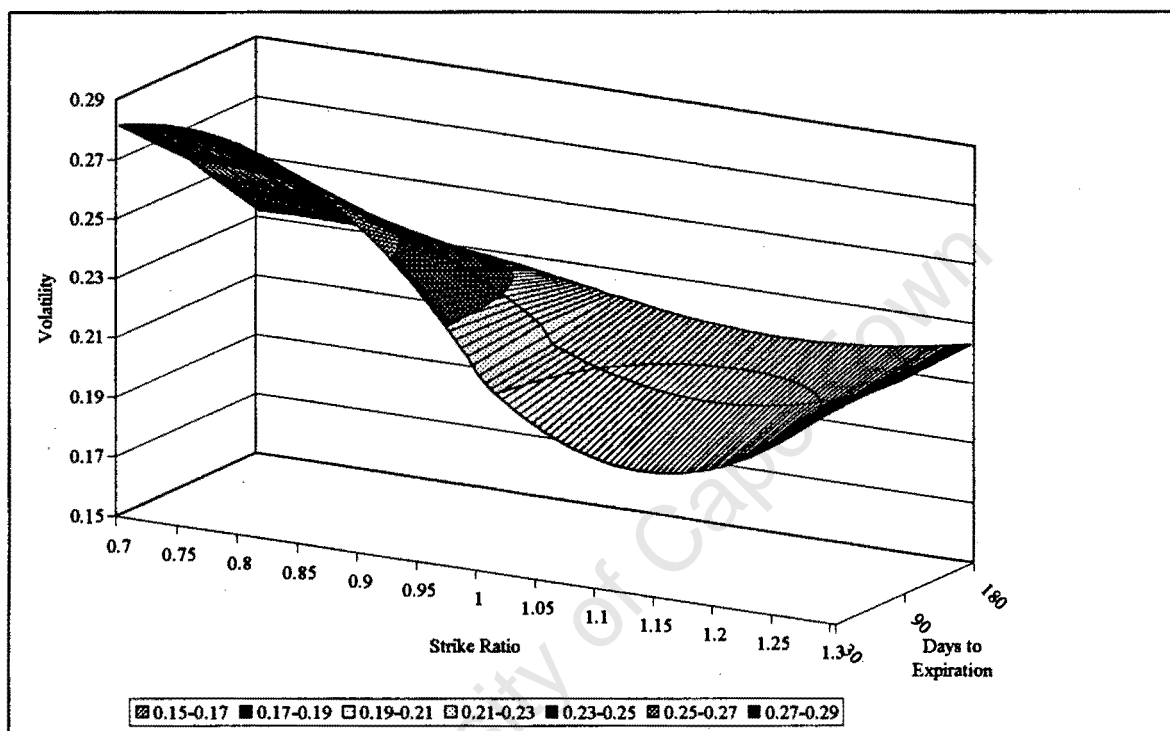


Figure 4.5.4-1. Volatility Surface Consistent with the Environment in South Africa

A volatility surface is constructed for expirations between 30 days and 180 days across strike ratios of 0.7 to 1.3. The volatility smile is most pronounced for a short time to expiration while the pronunciation tends to diminish for an increasing time to expiration.

¹³¹ The research in Chapter 3 reveals very similar results for the ALSI, the INDI, and the GLDI. The implied volatility is higher for *out-of-the-money* and *in-the-money* strike ratios compared to *at-the-money* strike ratios. Moreover, the *out-of-the-money* implied volatilities are higher than the *in-the-money* implied volatilities. Finally, the implied volatilities increases slightly for *at-the-money* options with increasing time to expiration whilst the *out-of-the-money* and the *in-the-money* implied volatilities decrease resulting in a nearly flat volatility smile is produced for far away expirations.

¹³² The volatility surface is constructed by computing implied volatilities for strike ratios from 0.7 to 1.3 for 30, 90, and 180 days to expiration. The initial implied volatilities used for this computation are estimated from the available ALSI, GLDI, and INDI data in Chapter 3 and are averaged. Consequently, each data point is subjectively selected in order to fit into the research results of Chapter 3.

4.5.5 The Effect of Non-Constant Volatility on Implied Distributions

The volatility surface in Figure 4.5.4-1 allows an analysis of implied distributions for several expirations. Here, for the sake of clarity and brevity only the implied distributions for 30, 90 and 180 days to expiration are shown graphically (in Figure 4.5.5-1, Figure 4.5.5-2, and Figure 4.5.5-3 respectively) to focus on the main differences between them.

The assessment of implied distributions is normally conducted by comparing the established implied distribution with the lognormal distribution in the literature (e.g. Derman and Kani (1994)). However, this sort of comparison between an implied distribution (established through a binomial tree, i.e. implied volatility tree with a discrete number of steps) and the lognormal distribution (equivalent to establishing an implied distribution by a tree with an infinite number of steps) can become very problematic.

The binomial tree can only yield an approximation to the lognormal distribution because of its use of a discrete number of steps. This approximation causes an approximation error that becomes larger as the number of steps for the binomial tree are reduced. The approximation error can however impair the analysis. Hence this thesis proposes to compare the implied distribution established by the implied volatility tree *under the assumption of non-constant volatility* with an implied distribution established by a similar implied volatility tree but with *the assumption of constant volatility*¹³³ (across strike prices and across expirations). This proposal avoids the approximation error discussed and consequently its influence on the results of the analysis.

The analysis starts with the implied distribution for 30 days to expiration.

30 Days to Expiration

The first implied distribution for the volatility surface is established for 30 days to expiration in Figure 4.5.5-1 (where the differences from the constant volatility assumption and its implied distribution can be clearly observed). Both implied distributions are based on a 20-step implied volatility tree and a future price of 100. Moreover, the implied distribution of the implied volatility tree with the constant volatility assumption is based on

¹³³ The required constant implied volatility input is assumed to be the *at-the-money* volatility of the expiration for which the implied distribution is established.

the *at-the-money* volatility of the volatility surface for 30 days to expiration (i.e. 21.5 percent).

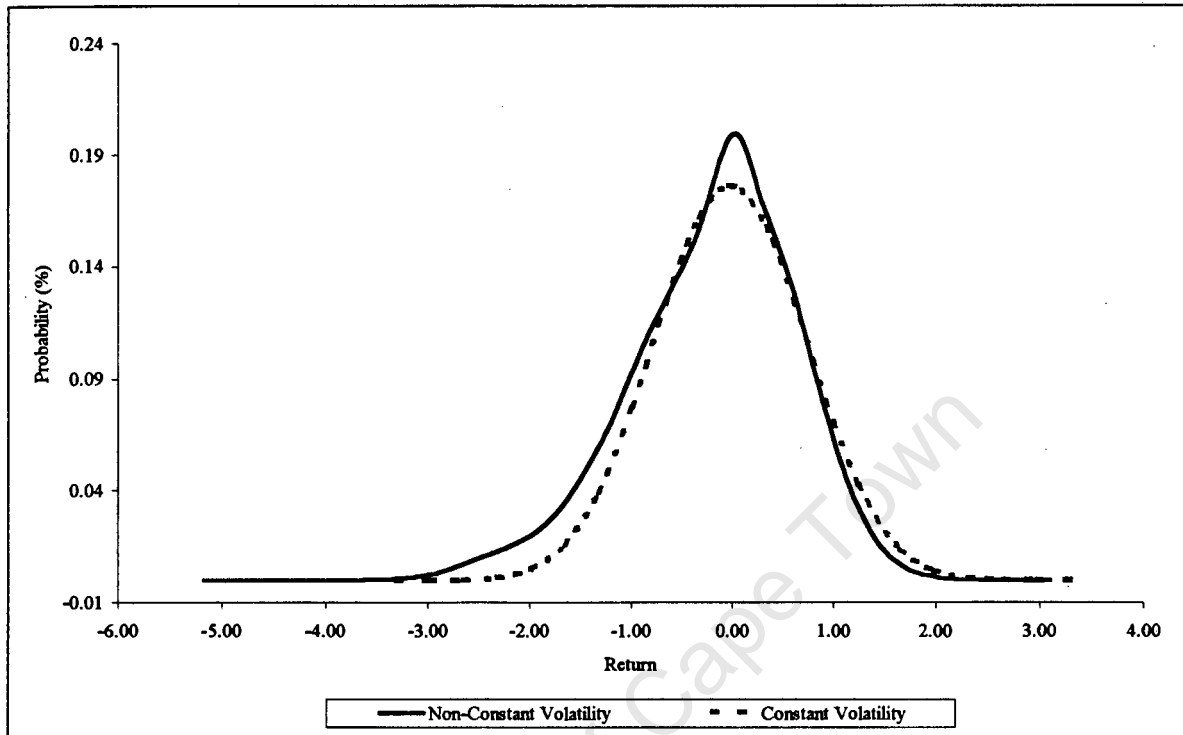


Figure 4.5.5-1. Implied Distributions for 30 Days to Expiration

The implied distributions for constant and non-constant volatility across strike prices are computed for 30 days to expiration. The implied distribution with constant volatility assumes an implied volatility of 21.5 percent, that is, the *at-the-money* volatility of the implied distribution with non-constant volatility.

Figure 4.5.5-1 shows clearly that the implied distribution with non-constant volatility differs from the implied distribution with constant volatility. The implied distribution with non-constant volatility is negatively skewed (i.e. more dispersed with a long tail on the left) and more leptokurtic compared to the implied distribution with constant volatility. Hence, the non-constant implied distribution displays a larger risk of negative returns (i.e. decreasing future prices) than expected by the lognormal theory¹³⁴. These results are similar to the findings of Shimko (1993).

In conclusion, the implied distribution with non-constant volatility presents a higher expected probability of large price falls in the underlying market (i.e. future) by comparison to the implied distribution with constant volatility. The results reveal that the higher expected probability of large price declines (in South Africa) is priced into the options through the pattern of the implied volatility. Consequently, options which profit

¹³⁴ The lognormal theory is here presented by the implied distribution with constant volatility.

from large price drops have a higher implied volatility (i.e. they are more expensive) than options which do not profit from these price drops.

The analysis continues with the implied distribution for 90 days to expiration.

90 Days to Expiration

The second implied distribution examined for 90 days to expiration is again estimated from the volatility surface in Figure 4.5.4-1. The slope of the volatility surface for 90 days to expiration is lower by contrast to the slope of the volatility surface for the 30 days to expiration (as observable in Figure 4.5.4-1). The consequence of the lower slope is that a smaller left tail is expected for the implied distribution for 90 days to expiration than for the implied distribution with 30 days to expiration. However, the left tail of the implied distribution with non-constant volatility is expected to be bigger than the left tail of the implied distribution with constant volatility (similar to 30 days to expiration).

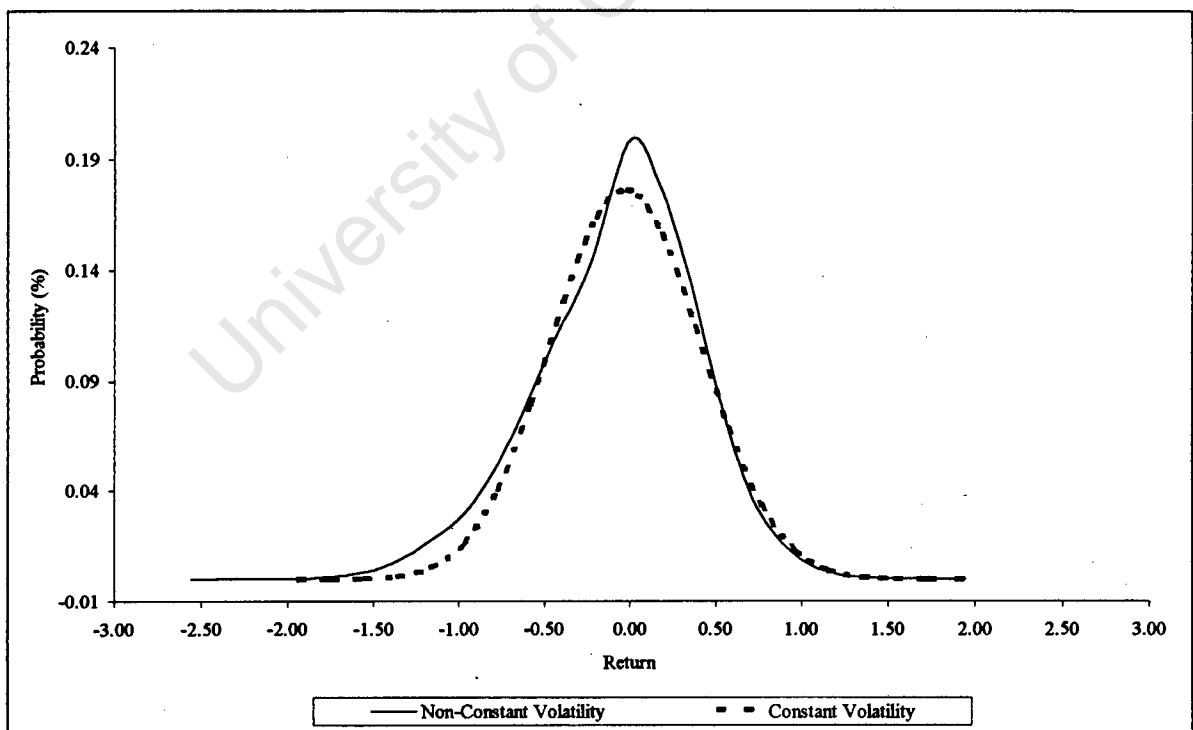


Figure 4.5.5-2. Implied Distributions for 90 Days to Expiration

The implied distributions for constant and non-constant volatility across strike prices are computed for 90 days to expiration. The implied distribution with constant volatility assumes an implied volatility of 21.85 percent, that is, the *at-the-money* volatility of the non-constant volatility distribution.

In Figure 4.5.5-2 it is evident that the left tail of the implied distribution with non-constant volatility is as expected, indeed bigger than the left tail of the implied distribution with constant volatility (and it is smaller than the left tail of the implied distribution with non-constant volatility with 30 days to expiration in Figure 4.5.5-1). In addition, the implied distribution with non-constant volatility is more leptokurtic than the implied distribution with constant volatility. Consequently, the left tail of the implied distribution with non-constant volatility again contradicts the lognormal theory (requiring a smaller left tail). Moreover, the right tails of the implied distribution with non-constant and with constant volatility are nearly identical. These nearly identical tails are caused by a volatility surface that decreases from *at-the-money* to *in-the-money* but increases from *in-the-money* to deep *in-the-money*. This kind of volatility surface shown in Figure 4.5.4-1 seems to be unique to the South African market because other markets (for example, US markets in Derman and Kani (1994) or Shimko (1993)) reflect a monotonically decreasing volatility function in their results.

The implied distribution with non-constant volatility, again, reflects a higher risk of larger price decreases for the underlying asset than expected by the lognormal theory. The higher risk of larger price drops is displayed by the left tail of the implied distribution with non-constant volatility in Figure 4.5.5-2. This bigger left tail indicates higher implied volatilities for options that would profit from price drops. Nevertheless, the implied volatility for these profiting options must be smaller for 90 days to expiration than for 30 days to expiration because the left tail of the implied distribution with non-constant volatility is smaller for 90 days to expiration than for 30 days to expiration.

Finally, the analysis ends with the implied distribution for 180 days to expiration.

180 Days to Expiration

Finally, the third implied distribution for 180 days to expiration is presented in Figure 4.5.5-3. The implied distribution for 180 days to expiration is estimated from the volatility surface (in Figure 4.5.4-1) with a lower slope compared to the volatility surface for implied distributions with 30 or 90 days to expiration. In fact, an almost flat line of implied volatilities is observed for 180 days to expiration in Figure 4.5.4-1. Hence, an implied distribution with non-constant volatility that is nearly identical to the implied distribution

with constant volatility is expected. However, the left tail of the implied distribution with non-constant volatility is expected to be slightly bigger than the left tail of the implied distribution with constant volatility for two different reasons.

The first reason is that the implied volatility for deep *out-of-the-money* strike ratios is still slightly higher than the implemented *at-the-money* implied volatility for an implied distribution with constant volatility. Second and more importantly, the entire volatility surface is required to produce the implied distribution. Hence, the higher slopes of the volatility surface for expirations less than 180 days are also implemented in the implied distribution with non-constant volatility for 180 days to expiration. A brief discussion of the differentiation between non-constant volatility and expirations, and constant volatility across expirations for implied distributions follows below. Thereafter the implied distribution for 180 days to expiration is analysed more detailed.

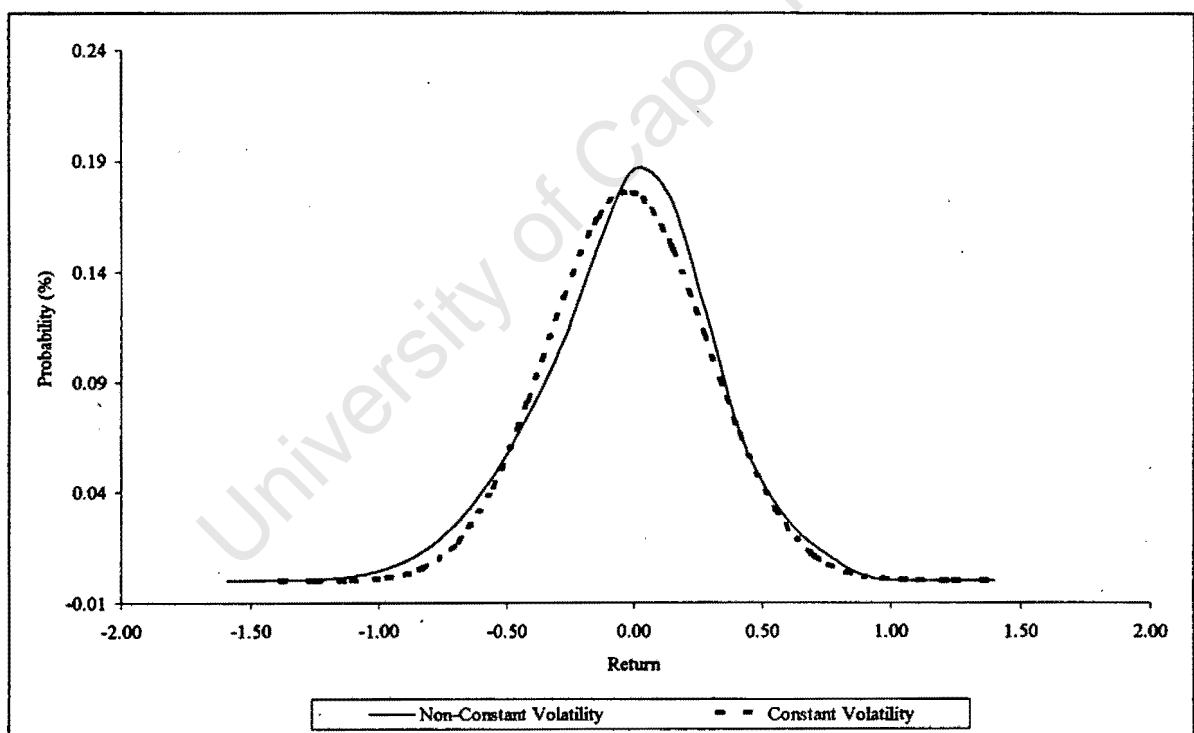


Figure 4.5.5-3. Implied Distributions for 180 Days to Expiration

The implied distributions for constant and non-constant volatility across strike prices are computed for 180 days to expiration. The implied distribution with constant volatility assumes an implied volatility of 22.10 percent, that is, the *at-the-money* volatility of the implied distribution with non-constant volatility.

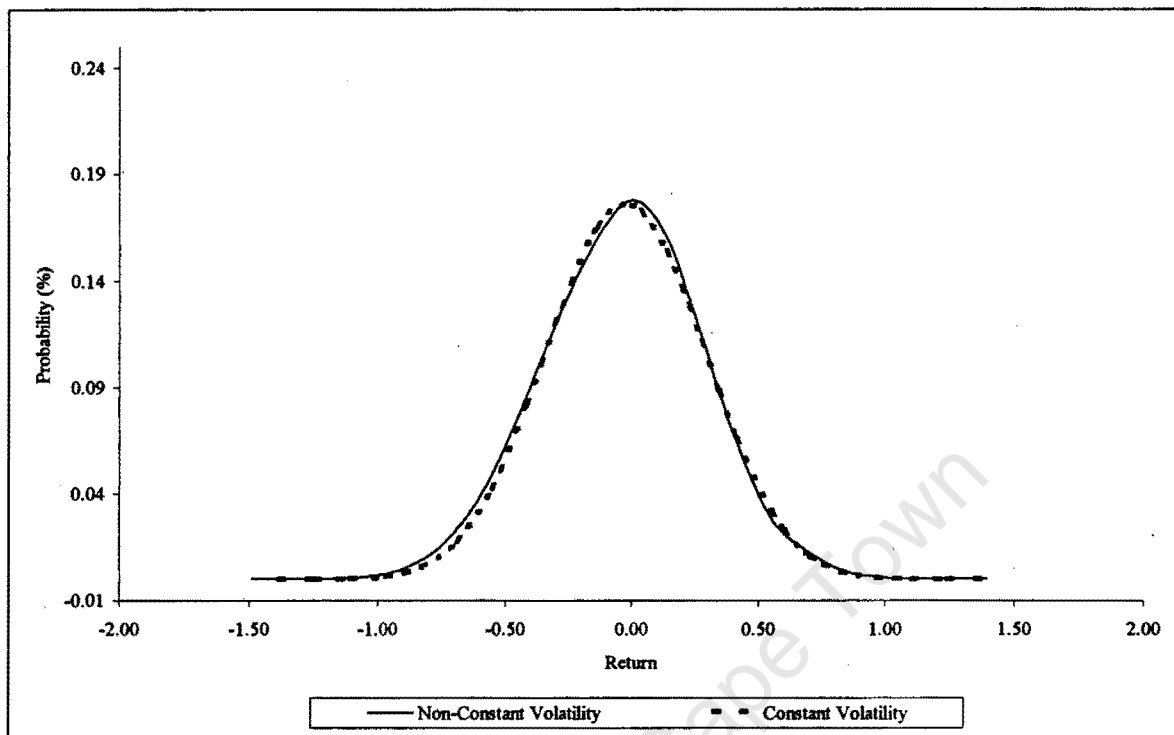
A comparison of Non-Constant Volatility and Constant Volatility across Expirations

Figure 4.5.5-4. Implied Distributions for 180 Days to Expiration with Constant Volatilities across Expirations

The implied distributions for constant and non-constant volatility across strike prices are computed for 180 days to expiration. The implied distribution with constant volatility assumes an implied volatility of 22.10 percent, that is, the *at-the-money* volatility of the implied distribution with non-constant volatility. The implied volatilities across expirations are constant.

The effect of the incorporation of the entire volatility surface is more evident for the implied distribution with non-constant volatility for 180 days to expiration because the differences in implied volatilities are higher between 180 days to expiration and earlier expirations than, for example, between 90 days to expiration and earlier expirations. The effect of the incorporated volatility surface for 180 days to expiration can be assessed by comparing Figure 4.5.5-3 with Figure 4.5.5-4. In Figure 4.5.5-4 (by contrast to Figure 4.5.5-3) the implied distribution is based on the assumption of *constant volatility across expirations* whilst similarly to Figure 4.5.5-3 the volatility is non-constant across strike prices.

It is evident from examining Figure 4.5.5-3 and Figure 4.5.5-4 that the implied distribution with *constant volatility across expirations* in Figure 4.5.5-4 is a superior

match¹³⁵ to the implied distribution with *non-constant volatility across expirations* in Figure 4.5.5-3. This superior match is expected because the implied volatility function for 180 days to expiration has a slope of almost zero and it assumes the volatilities of this expiration as constant across all other expirations. Hence, the assumed implied volatility surface in Figure 4.5.5-4 is almost constant across strike prices and across expirations. Consequently, the implied distribution with *non-constant volatility across strike prices* and *constant volatility across expirations* is very similar to the implied distribution with constant volatility in Figure 4.5.5-4.

However, the comparison between Figure 4.5.5-3 and Figure 4.5.5-4 suggests that the implied volatility surface in its entire structure is required to estimate an implied distribution (and not only across strike prices as in Figure 4.5.5-4). This suggestion is especially important if a substantially pronounced implied volatility surface across expirations is evident. For example, the volatility surface in Figure 4.5.4-1 is responsible for the slightly bigger left tail of the implied distribution with non-constant volatility across expirations in Figure 4.5.5-3 compared to the implied distribution with constant volatility across expirations in Figure 4.5.5-4. The assumption, that the implied volatilities for 180 days to expiration are constant across all expirations, results in biased results especially for the left tail of the implied distribution. The size of the bias can be estimated by comparing the left tails of the implied distribution with non-constant volatility in Figure 4.5.5-3 and the implied distribution with constant volatility across expirations in Figure 4.5.5-4.

Having discussed the impact of constant volatility across expirations above, the focus shifts back to the analysis of the implied distribution with non-constant volatility across strike prices and across expirations for 180 days to expiration below.

Although both the implied distributions with constant and non-constant volatility across expirations in Figure 4.5.5-3 do match better than in Figure 4.5.5-1 and Figure 4.5.5-2, the implied distribution with non-constant volatility still has a slightly larger left tail and is more leptokurtic than the implied distribution with constant volatility (across strike prices and across expirations). This bigger left tail of the implied distribution with non-constant volatility again indicates a higher risk of large price declines for the underlying asset than

¹³⁵ Superior match in the sense that the non-constant volatility distribution across strike prices matches the constant volatility distribution across strike prices.

expected by the lognormal theory. The right tail however seems to be very similar to the lognormal theory.

The bigger left tail of the implied distribution with non-constant volatility implies that options profiting from large price declines (in the underlying asset) have higher implied volatilities. Nevertheless, this profit opportunity is smaller for 180 days to expiration than for 30 or 90 days to expiration (because the left tail of the implied distribution with non-constant volatility is smaller for 180 days to expiration than for 30 or 90 days to expiration).

Finally, the results of the implied distributions are concluded below.

4.5.6 Conclusion

The analysis of implied distributions for options on futures was conducted here for the first time (to the authors knowledge). The advantage of the analysis here is that the implied distributions of three different expirations are directly estimated through the implied volatility tree with its Arrow-Debreu prices and underlying asset prices. This technique of estimating the implied distributions consequently has the advantage that no specific expiration (as in Rubinstein (1994)) has to be selected. The analysis of the implied distributions for 30 days, 90 days, and 180 days to expiration has revealed that the implied distributions change from a fat left tailed distribution to an implied distribution with only a slight left tail for an increasing time to expiration (under the assumption of the volatility surface in Figure 4.5.4-1). In addition, the analysis of the implied distribution with non-constant volatility has revealed more leptokurtic implied distributions by contrast to the implied distributions with constant volatility. In sum, the implied distribution with non-constant volatility suggests a larger risk of price declines than expected by the lognormal theory.

Additionally, it was revealed that the small left tail of the non-constant volatility implied distribution for 180 days to expiration with non-constant volatility arises mainly from the volatility surface (i.e. from implied volatilities of earlier expirations). Hence, it was concluded that without the incorporation of the entire volatility surface (i.e. without taking account of the implied volatility across strike prices *and* across expirations) a bias results

in the estimation of the implied distribution. Consequently, the entire volatility surface is required to infer implied distributions accurately.

An overall summary and conclusion for the entire chapter follows below.

4.6 Conclusion and Summary of Chapter 4

The development of an implied volatility tree model for options on futures in the South African environment was the aim of this fourth Chapter. This aim was successfully achieved using the implied volatility tree model discussed in section 4.2. New approaches for the implementation of the implied volatility tree model, in particular the prevention of negative transition probabilities, were developed and successfully tested. Moreover, a simple, stable, and practical solution was presented and developed by modifying and extending existing methods for the interpolation and extrapolation of implied volatilities. The recognition and the solution of the problems underlying the implied volatility tree model have resulted in a option price model that is not only applicable to South Africa but to international markets as well.

Moreover, the implied volatility tree was assessed by examining its accuracy and efficiency. The assessments display very good results regarding the accuracy of the implied volatility tree (under both the assumption of constant volatility or non-constant volatility). Hence, the implied volatility tree is deemed a suitable modelling approach for the South African environment with non-constant implied volatility.

In addition, a technique is proposed to enhance the computational efficiency of the implied volatility tree and to improve its accuracy simultaneously. This technique sums the difference between the option price of the modified Black model and the option price of the binomial model with the option price of the implied volatility tree model. This approach yields a much greater accuracy (which is achieved with fewer steps) compared to a "standard" implied volatility tree model.

Another attempt by Chriss (1997) to enhance the computational efficiency of the implied volatility tree by introducing the modified Black model into the implied volatility tree model has failed significantly. This implied Black tree model revealed less accurate results

that can be ascribed to the lognormal distribution implemented in the modified Black model.

Finally, the proposed implied volatility tree estimated the implied probability distributions from a hypothetical volatility surface. This hypothetical volatility surface reflects the findings of the research on implied volatility for the South African environment (in Chapter 3). The estimated implied distributions display larger left tails than expected from the lognormal theory assumed by the modified Black model or by the Black and Scholes (1973) model. These larger left tails suggest that the expected probability of large market price falls is higher than in the option price models that assume a lognormal distribution. Moreover, it was found that the entire volatility surface must be implemented to infer implied distributions because implied distributions do not only depend on the pattern of the implied volatilities across strike prices, but depend on the pattern of implied volatilities across expirations as well.

The effects of non-constant volatility on option price sensitivities and on portfolio management strategies with options are analysed further in the following chapter.

Appendix 4.A

Algorithms to Prevent “Bad Probabilities”

The combination of the techniques by Barle and Cakici (1995) and Chriss (1997) as well as some proposed modifications (i.e. for the spacing between nodes) are differentiated into three different cases:

1. Centre of the Implied Tree
2. All Nodes above the Centre of the Implied Tree
3. All Nodes below the Centre of the Implied Tree

The first case outlines the conditions and calculations for the centre of the tree with one centre node, if n is even, and two centre nodes if n is odd. In the second case, the conditions and formulas are outlined for all nodes above the centre of the tree. The third and last case shows the conditions and equations for all nodes below the centre of the tree.

1. Centre of the Implied Tree

1.1. n is odd

1.1.1. up node, $i = (n+1)/2$

Conditions (for $n \geq 1$):

$$F_{n,i} \leq F_{n+1,i+1} \leq F_{n,i+1}$$

if the conditions are violated:

1. $F_{n+1,i+1} = (F_{n,i+1} + F_{n,i})/2$

1.1.2. down node, $i = ((n+1)/2)-1$

Conditions (for $n > 1$):

$$F_{n,i-1} \leq F_{n+1,i} \leq F_{n,i}$$

if the conditions are violated:

1. $F_{n+1,i} = \frac{F_{n,i-1} \times F_{n+1,i+1}}{F_{n,i}}$

if the conditions are still violated after step 1:

2. $F_{n+1,i} = (F_{n,i} + F_{n,i-1})/2$

1.2. n is even

1.2.1. centre node, $i = (n/2)$

Equation (for $n \geq 1$):

$$F_{n+1,i+1} = F_{1,1}$$

2. All Nodes above the Centre of the Implied Tree

2.1. for $n > 1$ and $i < n$

Conditions:

$$F_{n,i} \leq F_{n+1,i+1} \leq F_{n,i+1} \text{ and}$$

$$F_{n+1,i} \leq F_{n+1,i+1}$$

if the conditions are violated:

$$1. \quad F_{n+1,i+1} = \frac{F_{n+1,i} \times F_{n,i}}{F_{n,i-1}}$$

if the conditions are still violated after step 1:

$$2. \quad F_{n+1,i+1} = F_{n+1,i} + (F_{n,i+1} - F_{n,i})$$

if the conditions are still violated after step 1 and step 2:

$$3. \quad F_{n+1,i+1} = F_{n,i} + (F_{n,i+1} - F_{n,i})/2$$

2.2. for $n > 1$ and $i = n$

Conditions:

$$F_{n,i} \leq F_{n+1,i+1} \text{ and}$$

$$F_{n+1,i} \leq F_{n+1,i+1}$$

if the conditions are violated:

$$1. \quad F_{n+1,i+1} = \frac{F_{n+1,i} \times F_{n,i}}{F_{n,i-1}}$$

if the conditions are still violated after step 1:

$$2. \quad F_{n+1,i+1} = F_{n+1,i} + (F_{n,i} - F_{n,i-1})$$

if the conditions are still violated after step 1 and step 2:

$$3. \quad F_{n+1,i+1} = F_{n,i} + (F_{n,i} - F_{n,i-1})/2$$

3. All Nodes below the Centre of the Implied Tree

3.1. for $n > 1$ and $i > 1$

Conditions:

$$F_{n+1,i} \leq \frac{F_{n+1,i+1} \times F_{n,i}}{F_{n,i+1}} \text{ and}$$

$$F_{n,i-1} \leq F_{n+1,i} \leq F_{n,i} \text{ and}$$

$$F_{n+1,i} \leq F_{n+1,i+1}$$

if the conditions are violated:

$$1. \quad F_{n+1,i} = \frac{F_{n+1,i+1} \times F_{n,i}}{F_{n,i+1}}$$

if the conditions are still violated after step 1:

$$2. \quad F_{n+1,i} = F_{n+1,i+1} - (F_{n,i} - F_{n,i-1})$$

if the conditions are still violated after step 1 and step 2:

$$3. \quad F_{n+1,i} = F_{n,i} - (F_{n,i} - F_{n,i-1})/2$$

3.2. for $n > 1$ and $i = 1$

Conditions:

$$F_{n+1,i} \leq \frac{F_{n+1,i+1} \times F_{n,i}}{F_{n,i+1}} \text{ and}$$

$$F_{n+1,i} \leq F_{n,i} \text{ and}$$

$$F_{n+1,i} \leq F_{n+1,i+1}$$

if the conditions are violated:

$$1. \quad F_{n+1,i} = \frac{F_{n+1,i+1} \times F_{n,i}}{F_{n,i+1}}$$

if the conditions are still violated after step 1:

$$2. \quad F_{n+1,i} = F_{n+1,i+1} - (F_{n,i+1} - F_{n,i})$$

if the conditions are still violated after step 1 and step 2:

$$3. \quad F_{n+1,i} = F_{n,i} - (F_{n,i+1} - F_{n,i})/2$$

The first case (i.e. the centre of the implied tree) for the calculation of the nodes in the implied tree is distinct to the other cases because only one or respectively two nodes are calculated. Nevertheless, the first case gives insights for above mentioned further nodes. If n is odd and the computed up centre node violates the established conditions, the averaging method of Barle and Cakici (1995) is implemented. Moreover, if the computed down centre node violates the established conditions, a method similar to the proposed logarithmic spacing method by Chriss (1997) is implemented in a first step. If the logarithmic spacing method also fails (i.e. the node still violates the established conditions), the averaging method by Barle and Cakici (1995) is implemented as second step. The logarithmic spacing method is preferred to the averaging method because the logarithmic spacing preserves the characteristics of the binomial tree.

The second and third case require the same methods for each calculated range of nodes (i.e. all nodes above and below the centre of the tree). The first stage uses the logarithmic spacing method if a node fails the conditions above. Here, a method similar to the logarithmic spacing method by Chriss (1997) is adapted to incorporate the upper boundaries of the tree for the second case as well as the lower boundaries for the third case. However, if the logarithmic spacing fails, a new method is implemented to establish the next node for the second and third case. This new method takes account of the distance between two consecutive nodes from the previous level (n) to the level ($n+1$). This "distance method" reflects an attempt to approximate the logarithmic spacing whilst using the same distance between two nodes at level ($n+1$) as at level (n). However, the "distance method" may also fail so that the averaging method by Barle and Cakici (1995) is implemented as third step. The averaging method has the advantage that it will be successful because the node at level ($n+1$) lies between the two previous nodes at level (n) for $i < n$ and $i > 1$ respectively or it will lay above or below the previous node at level (n) for $i = n$ and $i = 1$ respectively. The disadvantage of the averaging method is that the logarithmic spacing is not preserved. However, the averaging method approximates the logarithmic spacing better than the method proposed by Chriss (1997) if the logarithmic spacing method fails.

Although the three stage method described above yields a satisfactory solution to preventing bad probabilities, the implied tree remains an approximation of the binomial tree. The approximation error of the implied tree model to the binomial tree model is however relatively small (in fact, no difference in the option prices between the two option price models is found under the assumption of constant volatility in Table 4.4.1-1 if rounded to three decimal places).

Appendix 4.B

Comparison of the Piecewise Trinomial Regression with the Piecewise Quadratic Regression for Strike Ratios greater than or equal to One

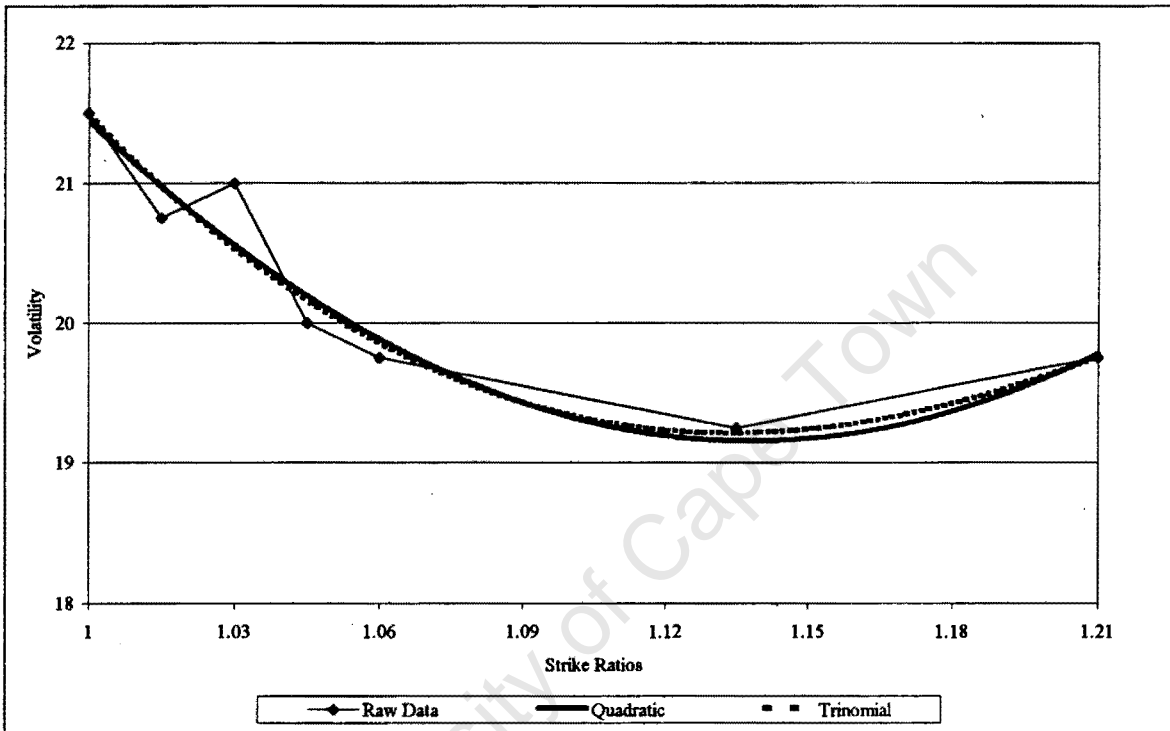


Figure 4.B-1. Comparison of the Piecewise Trinomial Regression with the Piecewise Quadratic Regression

The data in Jackwerth and Rubinstein (1995) for ask volatilities of the S&P 500 option prices on December 4, 1990 is fitted using trinomial regression as well as quadratic regression for strike ratios greater than or equal to one. The initial raw data is displayed with linear interpolation between the data points.

Appendix 4.C

Test with Constant Volatility for Put Options

Table 4.C-1

Comparison of Option Price Models for Put Options with Constant Volatility

The put option prices are based on a volatility of 20 percent with 180 days to expiration and a future price of 100. The tree models (binomial tree, implied tree, and implied Black tree) are calculated using 50 steps. The range of strike prices (60 to 140) proxies the South Africa environment (examined in this thesis) with strike ratios between 0.6 to 1.4.

Strike Prices	Option Prices for Different Option Price Models			
	Modified Black	Binomial Tree	Implied Tree	Implied Black Tree
(1)	(2)	(3)	(4)	(5)
60	0.000	0.000	0.000	0.000
70	0.021	0.018	0.018	0.018
80	0.299	0.300	0.300	0.284
85	0.791	0.780	0.780	0.751
90	1.745	1.758	1.758	1.701
95	3.318	3.333	3.333	3.259
100	5.599	5.571	5.571	5.488
105	8.581	8.590	8.590	8.513
110	12.179	12.196	12.196	12.132
115	16.265	16.277	16.277	16.229
120	20.702	20.707	20.707	20.675
130	30.192	30.191	30.191	30.180
140	40.046	40.044	40.044	40.042

Note that the option prices are rounded to three decimal places because the traded option contracts consist of ten options for the ALSI, GLDI, and INDI at SAFEX examined.

Table 4.C-2

Matrix of MAE and MSE Results for Put Options in Comparisons of Different Option Price Models

The results of the comparisons between different option price models in Table 4.4.1-1 yield the mean absolute error, MAE, in the first row of each comparison and as the mean squared error, MSE, in the second row of each comparison in bold.

Option Price Models	Option Price Models			
	Modified Black	Binomial Tree	Implied Tree	Implied Black Tree
	MAE	MAE	MAE	MAE
	MSE	MSE	MSE	MSE
Modified Black	0	0.009097	0.009091	0.035711
	0	0.000145	0.000145	0.002188
Binomial Tree	0.009097	0	0.000010	0.037901
	0.000145	0	0	0.002321
Implied Tree	0.009091	0.000010	0	0.037906
	0.000145	0	0	0.002321
Implied Black Tree	0.035711	0.037901	0.037906	0
	0.002188	0.002321	0.002321	0

Note that the results are rounded to six decimal places.

Appendix 4.D

Test with Non-Constant Volatility for Put Options

Table 4.D-1

Comparison between Input and Output Option Prices for Put Options with Non-Constant Volatility

The put option prices are based on 60 days to expiration and a future price of 100. The option prices of the implied tree models (i.e. the implied tree, the implied Black tree, and the implied tree with approximation error) are computed using 50 steps under the assumption of non-constant volatility. The option prices of the modified Black model and of the binomial tree model are computed with a constant volatility assumption (i.e. 20 percent across the strike prices). The range of strike prices (60 to 140) is chosen to reflect the strike prices available in the South Africa environment.

Strike Prices (1)	Input Option Price (2)	Output Option Prices for Different Option Price Model				
		Modified Black (3)	Binomial Tree (4)	Implied Tree (5)	Implied Black Tree (6)	Implied Tree with Approximation Error (3) - (4) + (5) = (7)
60	0.000	0.000	0.000	0.000	0.000	0.000
70	0.000	0.000	0.000	0.000	0.000	0.000
80	0.023	0.006	0.006	0.022	0.020	0.022
85	0.118	0.063	0.059	0.117	0.108	0.121
90	0.455	0.351	0.353	0.455	0.427	0.453
95	1.361	1.263	1.267	1.358	1.315	1.354
100	3.234	3.234	3.218	3.218	3.170	3.234
105	6.432	6.397	6.393	6.426	6.380	6.430
110	10.545	10.501	10.497	10.535	10.523	10.539
115	15.181	15.150	15.144	15.176	15.163	15.183
120	20.054	20.038	20.035	20.053	20.032	20.056
130	30.004	30.001	30.001	30.003	30.000	30.003
140	40.000	40.000	40.000	40.000	40.000	40.000

Note that the option prices are rounded to three decimal places because the traded option contracts consist of ten options for the ALSI, the GLDI, and the INDI at SAFEX examined.

Table 4.D-2

Matrix of MAE and MSE Results for Put Options for Input Replication

The results of the comparisons between input option prices and output option prices of the different models in Table 4.D-1 yield the mean absolute error, MAE in the first row and as the mean squared error, MSE, in the second row in bold.

Option Model		Option Price Models				
		Modified Black (1)	Binomial Tree (2)	Implied Tree (3)	Implied Black Tree (4)	Implied Tree with Approximation Error (5)
Input	MAE	0.030951	0.033338	0.003398	0.020586	0.001921
Prices	MSE	0.002166	0.002213	0.000032	0.000849	0.000008

Note that the results are rounded to six decimal places.

Appendix 4.E

The Explanation of the Implied Tree Model with Approximation Error

The aim of this appendix is to explain the "implied tree with approximation error". The "implied tree with approximation error" takes account of the approximation error between the option prices of the modified Black model and the option prices of the binomial tree model and adds this approximation error to the option price computed by the implied tree model. A more detailed explanation is given below.

An N-step implied tree requires $\sum_{i=1}^N i$ binomial calculations because an option price for each step at each

level is calculated. This means that a 50-step implied tree requires the computation of 1275 options (where each option is calculated with a 1 to 50 steps binomial tree).

In a test, the results for a 50-step implied tree and a 25-step "implied tree with approximation error" are computed with otherwise identical parameters. The 50-step implied tree model displays an MAE of 0.003362 and an MSE of 0.00032 respectively (see Table 4.4.2-2), whilst the 25-step "implied tree with approximation error" only has an MAE of 0.002507 and an MSE of 0.000019 respectively. Moreover, the lower values of MAE and MSE for the "implied tree with approximation error" comes at a significantly lower computational cost. The implied tree with 25-steps requires only 325 calculations of options where each option is calculated with a 1 to 25 steps binomial tree. Additionally, only one computation is required for each option price of the binomial tree and the option price of the modified Black model. In sum, the 327 calculations required are far less than the 1275 calculations required for the implied tree model and the "implied tree with approximation error" is therefore clearly the computationally more efficient method.

Nevertheless, the inclusion of the approximation error cannot provide perfect results in the presence of non-constant volatility. The option prices of the modified Black model and of the binomial tree model are based upon a constant volatility assumption so that the approximation error between them is only the approximation error under the assumption of constant volatility. If this approximation error is added to the implied tree, the accuracy of the implied tree is improved but approximation errors especially for *out-of-the-money* options remain.

In sum, the proposed option price model of the "implied tree with approximation error" gains accuracy or computational efficiency respectively by contrast to the "standard" implied tree.

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Chapter 5

5 Options in Portfolio Management – Effects of Non-Constant Volatility

5.1 Introduction

The effects of non-constant implied volatility on options in portfolio management are discussed in this chapter. The analysis starts with a discussion (in section 5.2) on the effects of non-constant implied volatility on the sensitivities of option prices. Shimko (1993) argues that the evidence of non-lognormal implied distributions implies that the Black and Scholes delta of call options overestimates the true market delta. A similar result is noted by Chriss (1997). However, the effects of non-constant implied volatility on option price sensitivities have not been examined extensively in the international literature. Hence, the focus of the discussion in section 5.2 is on the delta (section 5.2.1), the gamma (section 5.2.2), the theta (section 5.2.3), and the vega (section 5.2.4) to reveal the impact of non-constant implied volatility on the sensitivities of option prices. A summarizing conclusion is presented in section 5.2.5.

These four selected option price sensitivities play a primary role for the consideration of options in portfolio management (besides the price of the option). The results of the research concerning the effects of non-constant volatility on the option price sensitivities can thus be expected to influence portfolio management strategies. Results concerning the effect of non-constant implied volatility on the different portfolio strategies will be presented and discussed in section 5.3. As discussed in section 2.4, options in portfolio management play different roles (here it is argued that the effect of non-constant implied volatility differs for the different portfolio management strategies).

Section 5.3 comprises four portfolio strategies and a concluding summary. The first portfolio strategy discussed concerns strategic asset allocation (in section 5.3.1). The effect of non-constant implied volatilities on strategic asset allocation strategies are primarily demonstrated by the influence of the option price change.

By contrast, the tactical asset allocation (discussed in section 5.3.2) reveals the influences of non-constant implied volatilities through the change in the delta sensitivity on this portfolio strategy. The change in delta of the option has important consequences for tactical asset allocation. The change in the delta of the option also implies a change in the beta and the elasticity of an option. The beta of the option is subsequently required to compute the beta of a portfolio. For example, to hedge a portfolio against a market decline so that the portfolio value does not change while the market changes (i.e. to a beta of zero). Option betas are typically several times larger than an asset beta. Consequently, an inaccurately computed option beta has a large influence to the overall beta of the portfolio. Hence, an intended tactical asset allocation strategy can fail by the neglect of the non-constant volatility influence. In order to prevent an inaccurate estimation of the option beta, the delta of the option price should be computed accurately with an option price model that takes account of the non-constant implied volatility.

The discussion on the third portfolio strategy (in section 5.3.3) involves the effects of non-constant implied volatility on a hedging technique. In this hedging strategy, the required option price sensitivities, delta and gamma, are investigated for different hedging strategies. Additionally, the effect of non-constant implied volatility on options used as instrument for speculation is briefly discussed in this section. Finally, the portfolio strategy in section 5.3.4 highlights the effect of non-constant implied volatility on portfolio insurance. This portfolio strategy consists of the synthetic replication of a put option that is also substantially affected by non-constant implied volatility.

As discussed in Chapter 2, the accurate volatility estimation is the basis for a successful portfolio insurance strategy with synthetic options as well as for the strategic asset allocation with synthetic options. Hence, we argue that an option price model like the proposed implied volatility tree (discussed in Chapter 4) should be used to take account of the non-constant volatility.

The examination of the sensitivities of the option prices under the assumption of non-constant implied volatility follows.

5.2 Option Price Sensitivities in the Presence of Non-Constant Implied Volatility

This investigation has the objective of assessing the impact of the assumption of non-constant implied volatility on option price sensitivities (i.e. the non-constant implied volatility surface in section 4.5.4 is implemented). All option price sensitivities examined are computed for 90 days to expiration. The required inputs for the option price computation are a future with the value of 100 and a selected strike price range from 60 to 140. Call options with strike prices below 100 are referred to as *in-the-money* and call options with strike prices above 100 are referred to as *out-of-the-money* options¹³⁶. The objective of this section is an assessment of the differences between

- constant implied volatility across strike prices and across expirations
- constant implied volatility across expirations but non-constant volatility across strike prices and
- non-constant volatilities across strike prices and across expirations

Consequently, option price sensitivities are computed for the modified Black model and the binomial model for each of the following two cases:

1. with constant implied volatility across strike prices and across expirations and
2. with non-constant implied volatilities across strike prices but constant implied volatilities across expirations.

These four computed results for each option price sensitivity are subsequently compared to a fifth option price sensitivity that is computed with an implied volatility tree (i.e. the implied volatilities are non-constant across strike prices and across expirations). The option price sensitivities for the modified Black model are computed according to the methodology outlined in Appendix 5.A. The option price sensitivities for the binomial model and for the implied volatility tree model, by contrast, are computed according to the methodology outlined in Appendix 5.B.

The option price sensitivities for the binomial model are computed in order to compare its option price sensitivities with the option price sensitivities of the implied volatility tree by

¹³⁶ The reverse notation is valid for put options.

avoiding the approximation error to the results of the modified Black model¹³⁷. Due to the construction of the tree models (i.e. binomial tree and implied volatility tree with 50-steps), a small approximation error is produced between the results of the modified Black model and the results of the tree models. This approximation error is however very small (as presented for the comparison between binomial model and modified Black model in Appendix 5.C), especially for the delta, the gamma, and the theta. The approximation error of the vega by contrast is slightly higher for the comparison between binomial model and modified Black model in Appendix 5.C. Consequently, the comparison of the results between the tree models (i.e. binomial tree and implied volatility tree) is preferred.

The comparison of the five computed option price sensitivities is presented in the form of a table. This table structure is similar for all of the option price sensitivities considered. Hence, the structure of the tables is briefly introduced and explained. A range of strike prices is displayed in the first column of Table 5.2.1-1. In addition, the second column of Table 5.2.1-1 displays the option prices with 90 days to expiration computed from the volatility surface (in section 4.5.4). Columns three through to seven give a comparison of option price sensitivities for the option price models (noted in the header of each column) and their implied volatility input (i.e. constant volatility, non-constant implied volatility across strike prices, and the implied volatility surface) for each of the strike prices in column (1).

The examination of the delta follows below.

5.2.1 The Effects of Non-Constant Implied Volatility on the Delta

The effects of the non-constant implied volatility on the delta are presented in the form of a comparison between the deltas of five option price models in the columns (3) to (7) in Table 5.2.1-1.

¹³⁷ The graphical comparison between the results of the modified Black model and the results of the 50-step binomial tree for a constant volatility of 21.85 percent with 90 days to expiration are presented in Appendix 5.C

Table 5.2.1-1
Comparison of Deltas for Call Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The delta values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The deltas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The deltas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the deltas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations from the volatility surface.

Deltas for Different Option Price Models						
Strike Prices	Option Prices	Modified Black constant volatility	Binomial constant volatility	Modified Black strike price volatility	Binomial strike price volatility	Implied Volatility Tree volatility surface
(1)	(2)	(3)	(4)	(5)	(6)	(7)
60	40.000	1.0000	1.0000	1.0000	1.0000	0.9999
70	30.009	0.9996	0.9997	0.9977	0.9979	0.9954
80	20.180	0.9826	0.9833	0.9656	0.9666	0.9501
85	15.513	0.9397	0.9404	0.9146	0.9149	0.8861
90	11.202	0.8474	0.8475	0.8250	0.8264	0.7769
95	7.427	0.7009	0.7010	0.6921	0.6920	0.6290
100	4.326	0.5216	0.5215	0.5216	0.5215	0.4577
105	2.207	0.3462	0.3465	0.3382	0.3382	0.2908
110	1.006	0.2049	0.2034	0.1883	0.1881	0.1611
115	0.420	0.1086	0.1083	0.0925	0.0909	0.0785
120	0.171	0.0519	0.0500	0.0424	0.0414	0.0365
130	0.035	0.0090	0.0086	0.0098	0.0093	0.0069
140	0.004	0.0012	0.0009	0.0013	0.0011	0.0010

A comparison between the delta values of the modified Black model and of the binomial model in column (3) and (4) or column (5) and (6) respectively in Table 5.2.1-1 reveals that small approximation errors are produced between the deltas of the modified Black model and the deltas of the binomial tree model (with 50-steps). The assessment of these small approximation errors is supported by MSE values of 0.000001 in both comparisons (i.e. columns 3 and 4 or columns 5 and 6 respectively). These findings of approximation errors suggests that the binomial tree model should be preferred to the modified Black model for the comparisons with the implied volatility tree model. Consequently, only the comparison between the binomial tree model and the implied volatility tree is examined further (instead of examining the comparison between the modified Black model and the implied volatility tree model). The analysis of the comparison between the binomial tree and the modified Black models however continues in more detail.

The deltas in the comparison of column (3) and (5) or (4) and (6) respectively in Table 5.2.1-1 are very similar for the *at-the-money* strike (i.e. strike at 100) because the modified

Black model as well as the binomial model have the same implied volatility input of 21.85 percent (the small observed difference in the fourth decimal place is caused by rounding during the computation). However, the deltas of the models with constant volatility (i.e. column (3) and (4)) differ considerably to the deltas of the models with non-constant volatility (i.e. column (5) and (6)) for all other strike prices. For example, the deltas for the strike prices of 85 and 115 are 0.9404 and 0.1083 for the binomial model with constant volatility in column (4) whilst the deltas for the same strikes are only 0.9149 and 0.0909 respectively for the binomial model with non-constant volatility across strike prices in column (6). Moreover, the deltas between the option price models with constant volatility and the option price models with non-constant volatility across strike prices differ only slightly for far away from the money strike prices. For example, the deltas for the strike prices of 60 and 140 are 1.000 and 0.0009 respectively for the binomial model with constant volatility in column (4) whilst the deltas for the same strikes are nearly identical with values of 1.000 and 0.0011 respectively for the binomial model with non-constant volatility across strike prices in column (6).

In addition, the difference in the delta values between the models with constant volatility and the models with non-constant volatility across strike prices have two maximums, one between the *at-the-money* (i.e. 100) and the deep *in-the-money* strike prices (i.e. 60) at the strike price of 85 and the other between the *at-the-money* and the deep *out-of-the-money* strike prices (i.e. 140) at the strike price of 115. Hence, the comparison of the results of differences in the deltas between the constant volatility models and the option price models with non-constant volatility across strike prices reveals that an inappropriate implied volatility input (i.e. constant implied volatility) is an important source of an inaccurate delta estimation. The effect of the implementation of the implied volatility (i.e. non-constant implied volatility) is analysed further in a comparison between the delta values of the implied volatility tree (i.e. column (7)) and the binomial tree model (i.e. column 4 and column 6 respectively) in Table 5.2.1-1.

In the comparison between the deltas of the implied volatility tree in column (7) and the binomial tree model with non-constant volatility across strike prices in column (6) in Table 5.2.1-1, it is evident that the deep *in-the-money* and deep *out-of-the-money* strike prices have similar delta values. More importantly, the deltas of the strike prices between the deep

out-of-the-money and deep *in-the-money* strike prices differ substantially. Especially deltas near the *at-the-money* strike price show the largest differences. For example, the delta of the *at-the-money* strike price for the implied volatility tree (7) is only 0.4577 whilst the delta of the same strike price for the binomial tree with non-constant volatility across strike prices (6) is 0.5215 in Table 5.2.1-1. In addition, one maximum (of delta differences) is observed in the comparison between the implied volatility tree model and the binomial tree model with non-constant volatility across strike prices. This maximum is at the *at-the-money* strike price (i.e. 100).

The comparison between deltas of the implied volatility tree in column (7) and the binomial tree with constant volatility in column (4) in Table 5.2.1-1 reveals similar results (as the comparison between the implied volatility tree and the binomial tree with non-constant volatility across strike prices). Again only one maximum between delta values is found at the strike price of 95. However, this maximum is skewed to the *in-the-money* strike prices. The skew to the *in-the-money* strike prices is expected because an *in-the-money* call (like an *out-of-the-money* put) has a higher implied volatility in the volatility surface implemented for the implied volatility tree. Hence, the maximum difference of delta values for a put option is skewed to the *out-of-the-money* strike prices which is presented in Appendix 5.D.

Finally, the implementation of the volatility surface for options with 90 days to expiration (i.e. non-constant volatility across strike prices and expirations) in the implied volatility tree reveals substantially different delta values by comparison to constant volatility models (or even in the comparison with models with non-constant volatility across strike prices). The accurate computation of the delta requires a model that incorporates all implied volatilities across strike prices and across expirations. Option price models with different implied volatility assumptions estimate the deltas inaccurately and therefore can result in incorrect portfolio management strategies using options. The effects on options in portfolio management will be discussed in section 5.3.

The analysis of the gamma option price sensitivity follows below.

5.2.2 The Effects of Non-Constant Implied Volatility on the Gamma

Five different option price models are compared with regard to their computed gammas in the columns (3) to (7) of Table 5.2.2-1. The gammas are computed for 90 days to expiration for

- constant volatility
- non-constant volatility across strike prices
- and non-constant volatility across strike prices and across expirations

In addition, the computed gammas are multiplied by the factor 1000 to be consistent with their use in practical applications. The computed gammas are presented in Table 5.2.2-1.

Table 5.2.2-1
Comparison of Gammas for Call Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The gamma values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The gammas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The gammas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the gammas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations. Additionally, all gammas are multiplied by the factor 1000.

		Gammas for Different Option Price Models				
Strike Prices	Option Prices	Modified Black constant volatility	Binomial constant volatility	Modified Black strike price volatility	Binomial strike price volatility	Implied Volatility Tree volatility surface
(1)	(2)	(3)	(4)	(5)	(6)	(7)
60	40.000	0.000	0.000	0.009	0.005	0.053
70	30.009	0.138	0.112	0.566	0.504	1.316
80	20.180	3.961	3.829	5.992	5.880	9.945
85	15.513	11.023	10.964	12.565	12.554	18.683
90	11.202	21.737	21.818	21.392	21.521	28.702
95	7.427	32.004	32.177	30.444	30.590	36.805
100	4.326	36.719	37.278	36.719	37.278	40.559
105	2.207	34.007	34.173	35.182	35.368	35.087
110	1.006	26.181	26.336	26.478	26.596	24.793
115	0.420	17.173	17.174	16.317	16.208	14.803
120	0.171	9.799	9.569	8.793	8.648	8.070
130	0.035	2.248	2.129	2.385	2.262	1.937
140	0.004	0.354	0.278	0.393	0.327	0.311

The approximation error between gamma values of the modified Black model in column (3) and the binomial tree in column (4) (i.e. both with constant volatility) seems to be more

substantial than for delta values (in section 5.2.1). However, the gammas are multiplied by the factor 1000 so that the approximation error is in fact very small. Without the multiplication by the factor 1000, the MSE value between the two models with constant volatility would be recorded as zero when rounded to six decimal places. Consequently, the binomial tree produces very good estimates for the gamma values. Hence, only the binomial tree is compared to the implied volatility tree in the analysis of this section. Firstly however, the values of the modified Black models in column (3) and (5) are compared to the values of the binomial tree model in column (4) and (6).

In the Table 5.2.2-1, the gammas for the modified Black models shown in columns (3) and (5) as well as the gammas for the binomial models shown in columns (4) and (6) are the same for the *at-the-money* strike price (i.e. 100) because the gammas are computed with the same volatility input (i.e. 21.85 percent) across the option price models. However, the gammas of the option price models with constant volatility (in the columns 3 and 4) differ considerably to the gammas of the option price models with non-constant volatility across strike prices (in the columns 5 and 6). For example, the gammas of the binomial model with constant volatility in column (4) are 10.964 and 17.174 for the strike prices of 85 and 115 respectively compared to 12.554 and 16.208 respectively as gammas of the binomial model with non-constant volatility across strike prices in column (6) in Table 5.2.2-1.

Substantially different gamma differences between the option price models with constant volatility (i.e. columns 3 and 4) and option price models with non-constant volatility across strike prices (i.e. columns 5 and 6) are observed with regard to the strike prices between *at-the-money* (i.e. 100) and far away from the money (i.e. 60 and 140). The gamma differences between *at-the-money* and *in-the-money* strike prices have a maximum at the strike price of 95 and a negative¹³⁸ maximum at the strike price of 80. Similarly, the gamma differences between *at-the-money* and *out-of-the-money* strike prices have a negative maximum at the strike price of 105 and a maximum in the strike price interval [115,120]. From these strike prices on, the gamma differences are close to zero for deep *in-the-money* and deep *out-of-the-money* strike prices.

¹³⁸ A negative difference can result because the second option price sensitivity in the selected comparison is subtracted from the first option price sensitivity.

The comparison between the option price models with constant volatility (i.e. columns 3 and 4) and the option price models with non-constant volatility across strike prices (i.e. columns 5 and 6) reveals a dependency on the volatility input. It seems that *at-the-money* and far away from the money options could be computed independently from the volatility structure. However, the considerable changes in the gamma values near *at-the-money* as well as further *in-the-money* and *out-of-the-money* highlights the problem of an inaccurate gamma estimation caused by an inaccurate volatility input. The full effects of the incorporation of the non-constant volatility across strike prices and across expirations is analysed in comparisons between the implied volatility tree (in column 7) and the binomial models in the columns (4) and (6).

The comparison between the implied volatility tree model in column (7) and the binomial model with non-constant volatility across strike prices in column (6) displays similar gammas for far away from the money strike prices. However, the gamma differences between the far away from the money strike prices differ considerably. The highest gamma difference is found in the near *in-the-money* strike prices at a strike price of 90 whilst a second maximum (in the gamma difference) is observed in the near *out-of-the-money* strike prices at the strike price of 110. The global maximum gamma difference for *in-the-money* strike prices can be explained by the implementation of the volatility surface (i.e. non-constant volatility across strike prices and expirations). Additionally, *out-of-the-money* puts have, as expected from the largest implied volatility input, the largest absolute gamma difference in the *out-of-the-money* strike prices as displayed in Table 5.D-2 in Appendix 5.D.

The comparison between the implied volatility tree in column (7) and the binomial tree with constant volatility in column (4) reveals similar results to the comparison between the implied volatility tree and the binomial tree with non-constant volatility across strike prices. However, the differences in gamma values are larger than before. This shift in the magnitude of the differences is expected because the difference in the volatility input between the implied volatility tree and the binomial tree with constant volatility exacerbates the difference between the implied volatility tree and the binomial tree with non-constant volatility across strike prices.

In sum, only the implementation of the whole volatility surface ((i.e. non-constant volatility across strike prices and expirations) leads to accurate gamma values. Option price models that assume different volatility inputs (than the entire volatility surface) do therefore not only price the option prices inaccurately but also miscalculate the gamma values that are essential, for example, in delta-gamma hedging. The effects of the implemented volatility surface on hedging as well as other portfolio management strategies are analysed in section 5.3.

The analysis of the option price sensitivities continues with the theta below.

5.2.3 The Effects of Non-Constant Implied Volatility on the Theta

The analysis of the effects of non-constant implied volatility on the theta is also based on the comparison of five different option price models with different assumptions about volatility in the columns (3) to (7) in Table 5.2.3-1. In this thesis, the theta is presented in its original form although it is sometimes used in a different form in the options trading environment. There, the theta is divided by the number of days per year¹³⁹ in order to compute the effect of diminishing time with respect to the option premium. As this adjusted theta conveys no additional information to the theta in its original form, the theta in its original form is implemented in the ensuing analysis.

¹³⁹ The number of days per year can be based upon two different assumptions: firstly, the number of calendar days is used, or secondly, the number of trading days (e.g. 250 days) is used.

Table 5.2.3-1
Comparison of Thetas for Call Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The theta values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The thetas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The thetas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the thetas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations from the volatility surface.

Thetas for Different Option Price Models						
Strike Prices	Option Prices	Modified Black constant volatility	Binomial constant volatility	Modified Black strike price volatility	Binomial strike price volatility	Implied Volatility Tree volatility surface
(1)	(2)	(3)	(4)	(5)	(6)	(7)
60	40.000	0.000	0.000	-0.003	-0.002	-0.012
70	30.009	-0.033	-0.027	-0.191	-0.170	-0.310
80	20.180	-0.945	-0.914	-1.960	-1.924	-2.340
85	15.513	-2.631	-2.617	-3.938	-3.936	-4.395
90	11.202	-5.188	-5.208	-6.298	-6.337	-6.752
95	7.427	-7.638	-7.680	-8.242	-8.283	-8.658
100	4.326	-8.763	-8.898	-8.763	-8.898	-9.542
105	2.207	-8.116	-8.157	-7.706	-7.748	-8.254
110	1.006	-6.248	-6.286	-5.579	-5.605	-5.833
115	0.420	-4.099	-4.099	-3.411	-3.389	-3.483
120	0.171	-2.339	-2.284	-1.881	-1.850	-1.899
130	0.035	-0.537	-0.508	-0.583	-0.553	-0.456
140	0.004	-0.085	-0.066	-0.096	-0.080	-0.073

The approximation error between the thetas of the modified Black models (in the columns 3 and 5) and the binomial models (in the columns 4 and 6) with the constant volatility as well as the non-constant volatility assumption is very small in Table 5.2.3-1 (with a MSE value of 0.002211 for option price models with constant volatility and a MSE value of 0.002162 for non-constant volatility across strike prices). Hence, the binomial tree approximates the thetas very well. The binomial tree is therefore used exclusively for the comparison of the thetas of the implied volatility tree in column (7) of Table 5.2.3-1 (to compare option price models on the same basis as discussed in section 4.4).

In addition, the thetas are similar for the *at-the-money* strike price (i.e. 100) in the columns (3) and (5) or in the columns (4) and (6) respectively, due to the same input volatility of 21.85 percent. However, the thetas differ considerably between models with constant volatility and models with non-constant volatility across other strike prices. For example, the thetas of the binomial model with constant volatility for the strike prices 85

and 115 are -2.617 and -4.099 respectively, whilst the thetas of the binomial model with non-constant volatility across strike prices are -3.936 and -3.389 respectively.

These differing theta values are however expected. The value of theta is higher for *in-the-money* call options due to a higher volatility input from the non-constant volatility across strike prices (i.e. the “smile”) than the constant volatility input. Consequently, the value of theta is lower for the *out-of-the-money* call options due to a lower volatility input from the non-constant volatility across strike prices (i.e. the “smile”) than the constant volatility input. However, one exception is evident for deep *out-of-the-money* strike prices (i.e. 130 and 140) because the non-constant volatility across strike prices (i.e. the “smile”) provides a higher volatility input than the constant volatility¹⁴⁰.

In addition, two maximums are evident in the theta difference between the option price models with constant volatility (in the columns 3 and 4) and the option price models with non-constant volatility across strike prices (in the columns 5 and 6). The largest negative maximum in theta differences is found between the *at-the-money* and the deep *in-the-money* strike prices at the strike price of 85 in Table 5.2.3-1. The second maximum of the theta differences is observed between the *at-the-money* and the deep *out-of-the-money* strike prices at the strike price of 115.

The differences in thetas between option price models with constant volatility and models with non-constant volatility across strike prices demand the implementation of an option price model that accurately accounts for non-constant volatility across strike prices. Finally, the analysis of this section focuses on the incorporation of non-constant volatility across strike prices and across expirations (i.e. the volatility surface with 90 days to expiration from section 4.5.3) and its effects on the theta.

The non-constant volatility across strike prices and expirations (i.e. the volatility surface) is implemented in the implied volatility tree in column (7) of Table 5.2.3-1. The thetas computed by the implied volatility tree are firstly compared to the thetas of the binomial tree with non-constant volatility across strike prices in column (6). The comparison between the two option price models reveals that the thetas for far away from the money strike prices are only slightly different. More importantly, the thetas of both option price models differ

¹⁴⁰ The results are analogous for put options and can be found in Table 5.D-3 in Appendix 5.D.

considerably between the far away from the money strike prices with the largest theta difference occurring at the *at-the-money* strike price (i.e. 100). The theta at the *at-the-money* strike price is -9.542 for the implied volatility tree in column (7), whilst the theta is only -8.898 for the binomial tree in column (6) in Table 5.2.3-1. Hence, it is concluded that the incorporation of the volatility surface (i.e. non-constant volatility across strike prices and across expirations) has a considerable effect on the theta. This effect becomes more obvious in the ensuing comparison between the implied volatility tree in column (7) and the binomial tree with constant volatility in column (4) of Table 5.2.3-1.

In Table 5.2.3-1, the maximum in theta differences is placed between the *at-the-money* and the deep *in-the-money* strike prices at the strike price of 85 whilst a lower maximum is found between the *at-the-money* and the deep *out-of-the-money* strike prices at the strike price of 115. In sum, the differences between the thetas of both option price models are skewed to the *in-the-money* strike prices because of the higher volatility input from the volatility surface used (from section 4.5.4).

In conclusion, the implementation of non-constant volatility across strike prices and across expirations (i.e. the volatility surface) in the implied volatility tree model produces substantially different thetas to option price models with constant volatility or non-constant volatility across strike prices. Consequently, the implied volatility tree seems to be the only option price model that is able to incorporate the entire non-constant volatility structure across strike prices and across expirations (in the form of the volatility surface). Hence, the implied volatility tree is the only option price model (of the option price models considered) that is able to compute accurately option prices and thetas. Moreover, the effect of non-constant volatility across strike prices and across expirations (incorporated in the implied volatility tree) on the theta and the two option price sensitivities (analysed in sections 5.2.1 and 5.2.2) is discussed further in the research concerning options in portfolio management summarized in section 5.3.

The last option price sensitivity to be analysed is the vega. This analysis follows in section 5.2.4 below.

5.2.4 The Effects of Non-Constant Implied Volatility on the Vega

The analysis of the vega is achieved by comparing five option price models and their computed vegas as shown in the columns (3) to (7) of Table 5.2.4-1. Additionally, the option prices from the input volatility for 90 days to expiration are also noted in column (2) in Table 5.2.4-1.

Table 5.2.4-1
Comparison of Vegas for Call Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The vega values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The vegas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The vegas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the vegas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations from the volatility surface.

Strike Prices (1)	Option Prices (2)	Vegas for Different Option Price Models				
		Modified Black constant volatility (3)	Binomial constant volatility (4)	Modified Black strike price volatility (5)	Binomial strike price volatility (6)	Implied Volatility Tree volatility surface (7)
60	40.000	0.000	-0.001	0.006	0.014	0.054
70	30.009	0.074	0.065	0.363	0.417	0.872
80	20.180	2.134	2.018	3.779	3.420	4.629
85	15.513	5.938	5.408	7.757	8.115	8.172
90	11.202	11.710	11.532	12.800	12.351	13.660
95	7.427	17.241	17.705	17.467	17.705	17.118
100	4.326	19.781	19.680	19.781	19.680	19.681
105	2.207	18.320	18.615	18.157	18.619	18.959
110	1.006	14.104	13.544	13.403	12.755	12.869
115	0.420	9.251	9.442	8.227	9.444	9.461
120	0.171	5.279	6.430	4.485	4.065	4.176
130	0.035	1.211	1.189	1.301	1.189	1.682
140	0.004	0.191	0.249	0.215	0.249	0.245

From Table 5.2.4-1 it is evident that the approximation error between the vegas of the modified Black models in column (3) and (5) and the binomial tree models in column (4) and (6) is bigger than for any of the prior option price sensitivities analysed. The MSE value for models with constant volatility is 0.178317 whilst the MSE value for models with non-constant volatility is 0.217810. A graphical comparison between the vega values of a modified Black model and a binomial model under the assumption of constant volatility (in Figure 5.C-4 in Appendix 5.C) reveals that the variance of the vegas computed by the

binomial model is higher than the variance of the vegas computed by the modified Black model. However, the vegas computed by the binomial model follow a reverting process that brings the vegas of the binomial model back to the vega values of the modified Black model. Hence, tree models (i.e. binomial tree and implied volatility tree) are exclusively compared with each other to avoid the influence of the approximation error in the following analysis. The focus of the research is however primarily on the comparison of the results between constant volatility and non-constant volatility across strike prices.

Due to the same input of volatility (of 21.85 percent), the vegas are similar for the *at-the-money* strike price of the modified Black models (shown in column 3 and 5) and of the binomial models (shown in column 4 and 6). However, the vegas for the *in-the-money* and the *out-of-the-money* strike prices differ considerably between the option price models with constant volatility (i.e. columns 3 and 4) and non-constant volatility across strike prices (i.e. columns 5 and 6). The vegas are less different between the option price models in the far away from the money strike prices but can differ substantially between the *at-the-money* (i.e. 100) and the far away from the money strike prices. The highest absolute vega difference between the option price models is therefore found in the *out-of-the-money* strike prices at the strike price of 120 (in the comparison of the binomial models). A maximum of a negative difference in vega values is found for the *in-the-money* strike prices at the strike price of 85.

In sum, the comparison between the option price models with constant volatility and with non-constant volatility across strike prices reveals that the differences in the values of the vegas are strongly influenced by the selection of the volatility input. An inappropriate assumption concerning the volatility input (e.g. constant volatility) therefore does not only result in a miscalculation of the option price, but also inaccurately estimates the vega. The ensuing analysis therefore focuses on the incorporation of non-constant volatility across strike prices and across expirations in the implied volatility tree and its effect on the vega.

The vegas in column (7) computed by the implied volatility tree are therefore compared to the vegas in column (6) computed by the binomial tree with non-constant volatility across strike prices. Deep *out-of-the-money* strike prices have similar vegas in Table 5.2.4-1, for example, a vega of 0.249 for the strike price of 140 is found in column (6) whilst the vega for the same strike price is 0.245 in column (7). More importantly, the vegas differ more

considerably for strike prices between far away from the money strike prices (i.e. between deep *out-of-the-money* and deep *in-the-money*). The exception is however the *at-the-money* strike price (i.e. 100) with similar vegas (19.680 and 19.681 respectively) in the columns (6) and (7). Higher differences between the vega values of the two option price models are computed in the *in-the-money* strike prices (e.g. the strike prices of 80 and 90).

More pronounced differences in vegas between option price models are found in the comparison between the vegas of the implied volatility tree in column (7) and the binomial tree with constant volatility in column (4). The differences between the vega values of the different option price models in Table 5.2.4-1 have a higher magnitude than for the comparison between the vegas of the columns (6) and (7). This higher magnitude of the differences between the vega values of the implied volatility tree and the binomial model with constant volatility is expected. This expectation results from the effect of the inappropriate assumption of constant volatility that exacerbates the effect of the assumption of non-constant volatility across strike prices (which is also inappropriate by contrast to the non-constant volatility across strike prices and across expirations). The analogous results are observed for put options in Table 5.D-4 in Appendix 5.D with the highest differences in the vega values between the option price models occurring at the *out-of-the-money* strike price of 80.

Concluding the analysis of the vega, the implementation of non-constant volatility across strike prices and across expirations (i.e. the volatility surface) reveals substantially different vegas when compared to the assumptions of constant volatility or non-constant volatility across strike prices. The necessity to compute vegas accurately therefore requires an option price model that implements non-constant volatility across strike prices and across expirations fully (as the proposed implied volatility tree does). Inaccurate vegas could however have devastating effects on portfolio strategies (discussed further in section 5.3).

Finally, a summarizing conclusion of the effect of non-constant volatility on option price sensitivities follows before the effects of the non-constant volatility are considered in the contrast of portfolio management strategies.

5.2.5 Conclusion

The effect of non-constant volatility on option price sensitivities is substantial. Each of the analysed option price sensitivities changes under the three input assumptions considered:

1. constant volatility
2. non-constant volatility across strike prices
3. non-constant volatility across strike prices and across expirations

All option price sensitivities change substantially under these differing volatility inputs. The least changed option price sensitivities normally occur for far away from the money strike prices. However, even in this case (and in all the other cases of substantial changes for other strike prices) the computation of the option price sensitivities should be conducted with an option price model that is able to incorporate non-constant volatility across strike prices and across expirations. The proposed implied volatility tree model for the South African environment (from Chapter 4) is indeed such an option price model.

Inaccurate and inappropriate computed option price sensitivities have devastating effects on option strategies as well as on portfolio management strategies with options. The requirement of the full incorporation of non-constant volatility across strike prices and across expirations can be compared to the use of the full term structure of interest rates (for the pricing of interest rate bearing securities). Here only a partial interest rate term structure could cause large mispricing effects for such securities especially if the term structure is very well pronounced. Our conclusions here have similar implications. Hence, the implementation of non-constant volatility across strike prices and across expirations together with an appropriate option price model is an inevitable consequence of the results of the above research.

The ensuing analyses examines the effects of non-constant volatility on portfolio management strategies that implement options to enhance or alter their profile.

5.3 The Effects of Non-Constant Volatility on Portfolio Management Strategies

The effects of non-constant volatility on portfolio management strategies with options are analysed using four portfolio strategies (incorporating some of the different strategies discussed in Chapter 2). In addition, the portfolio strategies and their results are based on the option price sensitivities computed in section 5.2.

In this thesis, the focus is on the portfolio management strategies with exchange traded options (to remain consistent with the prior research). However, it is suggested that “exotic options” (i.e. not exchange traded options), for example barrier options, are also substantially influenced by non-constant volatility effects. Barrier and other “exotic” options are not considered further (for the sake of brevity) but additional research is needed in this field. The focus on exchange traded options provides insights on the impact of the non-constant volatility assumption on portfolio management strategies. Exchange traded options are sometimes the only way to incorporate options in portfolio management, for example, legislation prohibits the use of non-exchange traded options for unit trusts in South Africa¹⁴¹.

The structure of section 5.3 therefore consists of four portfolio management strategies. The first portfolio management strategy is the strategic asset allocation with options discussed in section 5.3.1. The second portfolio management strategy is the tactical asset allocation with options presented in section 5.3.2. The portfolio management strategy of a hedging technique and the effects of non-constant volatility on the hedging results is considered in section 5.3.3. The fourth and the last portfolio management strategy proposes a synthetic put replication as an example of portfolio insurance in section 5.3.4.

In each of these four portfolio management strategies, a table is constructed to summarize the results of the respective portfolio strategies with options. The selected portfolio strategy is briefly introduced in each section and the demonstration of the results follows directly after. Finally, a summarizing conclusion is given in section 5.3.5.

¹⁴¹ The legislative regulations for unit trusts are summarized in Alexander (1996).

The first portfolio management strategy analyses the effect of non-constant volatility on the strategic asset allocation below.

5.3.1 Strategic Asset Allocation

The objective of this section is to analyse the effect of non-constant volatility on the strategic asset allocation. This portfolio management strategy is based on a guaranteed investment portfolio (as discussed in section 2.4.1). It is assumed for this strategic asset allocation that a stock index portfolio can be replicated with options and a cash deposit. In addition, the replicated stock index portfolio guarantees the initial investment at the expiration of the option and it participates in an increase of the stock index. Furthermore, the conditions of the South African environment are assumed in this strategic asset allocation strategy (i.e. options on futures with mark-to-market procedure).

The portfolio value, P_V , is assumed to be 1,000,000, the investment horizon is 90 days, and the risk-free interest rate is 14 percent¹⁴². Additionally, it is assumed that the value of the future with one year to expiration is 100 at the start of the option contract, t_0 , and no dividends are paid during the time to expiration of the option. Finally, a call option with a strike price, X , of 95, with a volatility of 23.27 percent¹⁴³, and with 90 days to expiration is computed.

The price of the option is computed using a binomial tree as well as using an implied volatility tree (both with 50 steps). The binomial tree assumes the input volatility to be constant whilst the implied volatility tree uses the volatility surface from section 4.5.4. The binomial tree yields a price of 7.445 for the option whilst the implied volatility yields a price of 7.493 for the option.

This strategic asset allocation strategy aims at the protection of the initial investment. Hence, the initial investment is guaranteed at the expiration of the option. The portfolio value of 1,000,000 is therefore discounted by the interest rate over the time to expiration to obtain a portfolio value of 966,068.49. This discounted amount of 966,068.49 is then

¹⁴² The interest rate is 15.03 percent if continuously compounded.

¹⁴³ The volatility of 23.27 percent is read off the volatility surface in section 4.5.3.

placed into a cash deposit for 90 days to receive 1,000,000 at expiration of the option. The remaining sum of 33,931.51 would normally be invested in the premium of the purchased call options. However, the option premium is not paid upfront at SAFEX. Nevertheless, the amount of 33,931.51 for the purchase of call options is placed into a margin account at SAFEX. This margin account allows the subtraction and the addition of daily losses and gains from the option (i.e. mark-to-market). Moreover, SAFEX pays market-related interest rates on the margin so that the amount of 33,931.51 grows to 35,123.30 over a time of 90 days (assuming a continuous interest rate of 15.03 percent¹⁴⁴). For computational simplification, it is assumed that on average the margin remains constant. Additionally, it is assumed that the invested margin of 33,931.51 is enough security deposit to buy further call options without depositing further margin. Finally, the amount of 35,123.30 is invested in call options. This money can purchase¹⁴⁵ a total of 4,717 options for the option price of the binomial tree whilst only 4,687 options can be bought for the option price of the implied volatility tree.

Table 5.3.1-1

Returns on Portfolio Management Strategies

This table compares four different investment decisions and their returns. A full investment in the underlying is presented in column (2), a full investment in an interest bearing security can be observed in column (3). Column (4) and (5) reveal the returns of a guaranteed portfolio investment strategy with different option prices for the binomial tree (i.e. constant volatility) and the implied volatility tree (i.e. non-constant volatility).

Market Prices at Expiration	Investment Decisions			
	Underlying Index	Interest Rate	Guaranteed Portfolio with Different Models	
			Binomial Tree	Implied Volatility Tree
(1)	(2)	(3)	(4)	(5)
95	950,000	1,035,123.30	1,000,005.42	1,000,003.74
100	1,000,000	1,035,123.30	1,023,590.42	1,023,438.74
105	1,050,000	1,035,123.30	1,047,175.42	1,046,873.74

Table 5.3.1-1 presents the results of a full investment in the underlying in column (2), a full investment in an interest bearing security in column (3), and the investment in the

¹⁴⁴ The interest is calculated under the assumption of a daily interest payment on the notional amount of 33,931.51. Hence

$$35,123.30 = 33,931.51 e^{0.14 \left(\frac{90}{365} \right)}$$

¹⁴⁵ After buying the options, a small amount of money remains unused (because only integer number of options are available). This money is additionally put in the cash deposit as well (5.24 for the binomial tree price and 3.61 for the implied volatility tree price).

guaranteed product with the two different option prices in column (4) and (5) respectively. The results of the guaranteed portfolio strategy reveal minimum results of 1,000,005.42 and 1,000,003.74 for both option price models respectively at expiration of the option. However, the advantage of the downward protection dampens the potential upward performance. The costs for the guaranteed portfolio strategy of 2,824.58 or 3,126.26 respectively if the market price goes up to 105 is far less than the loss of 50,000 that is avoided if the market falls to 95 at expiration of the option. Nevertheless, the guaranteed portfolio strategy loses more in performance if the market rises above the 105 price because the purchased number of options (i.e. 4717 or 4687 respectively) do not present a full participation in the market performance by this guaranteed portfolio strategy.

The difference in the two guaranteed portfolio strategies for the binomial tree in column (4), and the implied volatility tree model in column (5) in Table 5.3.1-1 is caused by the difference in the volatility input (i.e. constant volatility or non-constant volatility respectively). Although both the binomial tree and the implied volatility tree use the same input volatility (i.e. 23.27 percent) for the strike price of 95, a difference in their option prices is obvious. This difference is exhibited because the binomial tree assumes that the volatility is the same at all nodes in the binomial tree whilst the implied volatility tree incorporates the volatility surface at its nodes (i.e. each node has a different volatility). The consequence of the different volatility assumptions for the two option price models results in a difference of only 1.68 for a strike price of 95. This difference rises to 301.68 for a strike price of 105 (at expiration of the option between the guaranteed portfolios of the two option price models). The difference increases further the higher the market rises because the participation rates of the two guaranteed portfolio strategies differ by 30 options.

The effect of non-constant volatility across strike prices and across expirations is small but important for the guaranteed portfolio strategy. Different returns are computed for the same guaranteed portfolio strategy in the columns (4) and (5) in Table 5.3.1-1. The difference in their returns suggests the use of an option price model that is able to incorporate non-constant volatility. The implementation of the proposed implied volatility tree for the South African environment incorporates the effect of non-constant volatility accurately across strike prices and across expirations. Hence, the implied volatility tree

model diminishes the miscalculation of strategic asset allocation strategies as demonstrated for the case of the guaranteed portfolio strategy.

The effects of non-constant volatility on the second portfolio management strategy, i.e. the tactical asset allocation, is considered below.

5.3.2 Tactical Asset Allocation

The effect of non-constant volatility as well as a constant volatility assumption on the beta¹⁴⁶ of an option is considered in the context of tactical asset allocation. The beta computation of the option requires the definition of an asset beta, it is therefore assumed to be one here. Furthermore, it is assumed that the underlying asset price is 100 and that all calculations are based on call options. The results of the option beta are compared in the columns (4) and (7) for the binomial tree and the implied volatility tree respectively in Table 5.3.2-1.

Table 5.3.2-1
Option Betas for Different Volatility Assumptions

The table presents the option price, the delta, and the beta for two different option price models across three strike prices. The calculations are based on call options. The asset beta is assumed to be one and the underlying asset price is assumed to be 100.

Strike Price	Option Price Models					
	Binomial Tree			Implied Volatility Tree		
	Option Price	Delta	Beta	Option Price	Delta	Beta
(1)	(2)	(3)	(4)	(5)	(6)	(7)
85	15.517	0.9404	6.061	15.542	0.8861	5.701
100	4.304	0.5215	12.117	4.303	0.4577	10.637
115	0.415	0.1083	26.096	0.414	0.0785	18.961

The deltas of the implied volatility tree in column (6) and the deltas of the binomial tree in column (3) reveal substantial differences due to their different volatility assumptions. These substantial differences of the deltas consequently have considerable effects on the beta of the computed call option. The option beta for *in-the-money* options differs only slightly between the binomial tree and the implied volatility tree (6.061 in column (4) vs. 5.701 in column (7) for the strike price of 85) whilst the beta differences increase considerably for a

rising strike price. For example, the comparison of the option betas between the binomial tree model (in column 4 with an option beta of 12.117) and the implied volatility tree (in column 7 with an option beta of 10.637) already exhibits a difference of 1.48 in option betas for the *at-the-money* strike price (i.e. 100) in Table 5.3.2-1.

In sum, the effect of non-constant volatility across strike prices and across expirations, as implemented by the volatility surface (from section 4.5.4 and incorporated in the implied volatility tree), reveals substantial differences in the results of the option beta. These substantial differences have a significant effect on tactical asset allocation strategies because beta neutral portfolios (i.e. beta is zero) computed by techniques that do not incorporate non-constant volatility might no longer be beta neutral. In particular, the more the strike prices depart from being *in-the-money* in beta portfolio strategies, the higher is the deviation of the overall portfolio beta in the comparison between constant volatility and non-constant volatility. For example, from inspecting Table 5.3.2-1 it is evident that a beta neutrality strategy¹⁴⁷ with written call options would be beta positive because the betas of the implied volatility tree are lower than the betas of the binomial tree. Hence, a portfolio with such a beta neutrality strategy is no longer protected in the case of a decline in prices. It is therefore important for tactical asset allocation strategies to incorporate the non-constant volatility assumption with techniques such as the proposed implied volatility tree to avoid substantial errors.

The third portfolio management strategy examines the effect of non-constant volatility (across strike prices and across expirations) on hedging strategies. This hedging strategies continue the analysis below.

¹⁴⁶ The calculation of the beta of an option is explained in depth in Chapter 2 in section 2.4.2.

¹⁴⁷ For example, the beta neutrality strategy with written call options has the aim of equalizing the positive portfolio beta with the negative option beta (i.e. sold calls or purchased puts) or *vice versa*, so that the total portfolio is beta neutral (i.e. beta is zero).

5.3.3 Hedging

The effect of the non-constant volatility input on the delta-gamma hedging strategy¹⁴⁸ is analysed in comparison with constant volatility. A portfolio with a delta of zero (i.e. delta neutral) and a gamma¹⁴⁹ of -1500 is assumed for the purpose of this portfolio strategy. The target of a zero gamma is only obtainable by using options because the underlying asset has a gamma of zero and therefore cannot be used to change the total gamma of the portfolio. Instead, delta hedging is possible with the underlying asset because its delta value is equal to one at all times.

The effect of non-constant volatility on the hedging results is studied in a comparison between the results of the implied volatility tree and the binomial tree. Consequently, two hedging processes are analysed. First, the portfolio is hedged under the assumption of constant volatility (i.e. the binomial tree model) by a call option with a strike price of 95. Second, the delta and gamma values are computed under the assumption of non-constant volatility from the implied volatility tree. These results are then implemented in the same hedging strategy as discussed earlier. The results for both option price models are presented in Table 5.3.3-1.

Table 5.3.3-1
Delta-Gamma Hedging

Two option price models (i.e. the binomial tree and the implied volatility tree) are compared for delta-gamma hedging. The binomial tree assumes constant volatility whilst the implied volatility tree uses non-constant volatility (i.e. from the volatility surface in section 4.5.4). The implemented call option has a strike price of 95.

	Option Price Models			
	Binomial Tree		Implied Volatility Tree	
	Portfolio-Delta	Portfolio-Gamma	Portfolio-Delta	Portfolio-Gamma
Portfolio Values	0	-1500	0	-1500
+ 49 call options (buy)	+ 50 * 0.7010	+ 50 * 32.177	+ 50 * 0.6290	+ 50 * 36.805
New Portfolio Values	35.05	108.85	31.45	340.25
- 34 underlying assets (sell)	-35	0	-35	0
Delta-Gamma Hedge	0.05	108.85	-3.55	+340.25

¹⁴⁸ The aim of the delta-gamma hedging is to achieve a zero delta and gamma. However, small deviations from zero are also considered as sufficient.

¹⁴⁹ The gamma values are multiplied by the factor 1000.

The hedging strategy with the delta and gamma values obtained from the binomial tree achieves a relatively¹⁵⁰ small gamma value of 108.85 as shown in Table 5.3.3-1 through the purchase of 50 call options with a strike price of 95. Additionally, the delta changes to 35.05. Hence, 35 underlying assets (each with a delta of one and a gamma of zero) are sold to reduce the delta to 0.05. These hedging results are considered sufficient for practical purposes (because only whole options and underlying assets are available). Hence, the target of the delta-gamma hedging is achieved.

In the second delta-gamma hedging strategy, the delta and gamma values from the implied volatility tree (under the assumption of non-constant volatility) are used. The strategy of buying 50 call options with a strike price of 95 and selling 35 underlying assets is applied to the delta and gamma values of the implied volatility tree. The result of this application is a severe hedging error. First, the portfolio is mishedged by -3.55 underlying assets which means that the portfolio suffers losses if the market rises. Second, the gamma position has changed from -1500 to +340.25 which means that the delta is still sensitive to the price of the underlying. Both these mishedges however could be avoided by implementing an option price model that is able to incorporate non-constant volatility.

In conclusion, it can be noted that the effect of the non-constant volatility is substantial on the result of the delta-gamma hedging. If a portfolio is hedged using an option price model based on the assumption of a constant volatility (at all nodes of the binomial tree), the hedge can result in substantial errors when the volatility is not constant. Hence, it is important to implement an option price model that is able to incorporate non-constant volatility (at all nodes of the implied volatility tree) as the proposed implied volatility tree does. Consequently, this example has revealed that the incorporation of non-constant volatility is not only important for portfolio management but also for the risk management of traded options.

Finally, the last portfolio management strategy considers the effect of non-constant volatility on portfolio insurance.

¹⁵⁰ Gamma is multiplied with 1000 so that the real gamma is only 0.1089.

5.3.4 Portfolio Insurance

The effect of non-constant volatility across strike prices and across expirations on portfolio insurance is demonstrated using the synthetic replication of a put option. The other portfolio insurance strategies of fiduciary calls or bought protective puts (discussed in Chapter 2) are in essence similar to the strategic asset allocation example in section 5.3.1 and are therefore not examined further for the consideration of portfolio insurance.

The synthetic replication of a put option requires a position to be held which consists of the underlying equal to the delta of the synthetic position¹⁵¹. The delta of a put option is negative and therefore a position equal to the delta of the underlying asset has to be sold to replicate the put synthetically. The revenue from the sale of the underlying asset is consequently invested into a cash deposit. However, the synthetic replication of options on futures (as in South Africa) does not produce any revenue because the futures are mark-to-market. In addition, the delta changes continuously with time and movements of the future. Consequently, the sold position (i.e. negative delta) has to be adjusted continuously as well.

A comparison between a binomial tree model (i.e. constant volatility) and an implied volatility tree model (i.e. non-constant volatility) is selected for the demonstration in this section. The results of the demonstration are computed for a put option with 90 days to expiration and a strike price of 100. Moreover, the underlying price of the option is assumed to be 100 and the value of the insured portfolio is 1,000,000. The results of the synthetic put replication are displayed in Table 5.3.4-1.

Table 5.3.4-1
Synthetic Put Option Replication

The table presents a comparison between the binomial tree model and the implied volatility tree model with regard to the portfolio insurance of a synthetic put. A synthetic put is produced by selling a position in the underlying that is equivalent to the delta of the synthetic put option. The synthetic put option is based on a strike price of 100 and has 90 days to expiration.

Underlying Price	Option Price Models					
	Option Price	Binomial Tree		Implied Volatility Tree		
		Delta	Asset Value in Portfolio	Option Price	Delta	Asset Value in Portfolio
100	4.304	-0.4785	521,500	4.302	-0.5423	457,700

The synthetic replication of the put reveals substantial differences in the results for the delta values and asset values in the portfolio between the binomial tree model and the implied volatility tree model. The differences in their computed deltas, -0.4785 for the binomial tree model and -0.5423 for the implied volatility tree model in Table 5.3.4-1, affect the synthetic put replication substantially. The delta of -0.4785 for the binomial tree model means that 478,500 of the underlying asset value must be sold to replicate the put so that the invested portfolio value diminishes to 521,500. Instead, the delta of -0.5423 for the implied volatility tree (i.e. non-constant volatility) means that the invested portfolio value is only 457,700. This substantial difference of 63,800 (between the portfolio values with constant volatility and non-constant volatility) reveals the effect of non-constant volatility on this portfolio insurance strategy of the synthetic put replication.

The difference in assets sold of 6.38 percent between the two option price models used displays a huge source of errors in a portfolio insurance strategy with synthetic option replication. The portfolio insurance strategy with synthetic option replication could lose substantial amounts if the delta of the synthetic option is computed assuming constant volatility (in the presence of non-constant volatility). Hence, the example for portfolio insurance with synthetic option replication reveals the importance of the incorporation of non-constant volatility in option price models.

A summarizing conclusion ends this section below.

5.3.5 Conclusion

The effects of non-constant volatility on portfolio management strategies with options are substantial as highlighted by the different portfolio management strategies. The strategic asset allocation is probably least affected by the assumption of non-constant volatility. Nevertheless important insights (e.g. an option price model should be implemented that takes account of non-constant volatility) were revealed from the performance of the

¹⁵¹ The calculation of the delta is presented for the modified Black model in Appendix 5.A and for the binomial models as well as for the implied volatility tree models in Appendix 5.B.

guaranteed portfolio strategy. Furthermore, the tactical asset allocation revealed that the implementation of non-constant volatility had a substantial influence on the calculation of option betas. Thirdly, the delta-gamma hedging revealed substantial differences in the hedging performance between the constant volatility assumption and the non-constant volatility assumption. Especially, the combined effect of the differences in the delta and the gamma values between the two option price models produced substantial differences in the hedging results. Finally, the portfolio insurance example with the synthetic put option replication also revealed substantial differences between the constant volatility assumption and the non-constant volatility assumption. Hence, portfolio management strategies with options in South Africa should take account of the effect of non-constant volatility to avoid potential losses caused by unrealistic assumptions underpinning option price models used.

Ultimately, the substantial effects of non-constant volatility on the portfolio management strategies suggest that an option price model should be used that is able to incorporate non-constant volatility. Thus far, the proposed implied volatility tree has met these conditions.

Finally, Chapter 5 ends with the ensuing conclusion.

5.4 Conclusion of Chapter 5

The consequences of the non-constant volatility effects on the option price sensitivities (as discussed in section 5.2) were observed in the examples of portfolio management strategies in section 5.3. The effect of non-constant volatility on option price sensitivities and consequently on the portfolio management strategies are considered substantial. Hence, it is essential to incorporate the non-constant volatility into option price models in order to avoid a variety of errors. Such an option price model which is able to implement the non-constant volatility across strike prices and across expirations is the implied volatility tree model proposed in Chapter 4.

This proposed implied volatility tree is therefore not only deemed necessary for the pricing of options but also for the risk management of options and for portfolio strategies. This proposed option price model is also open to further extensions. The proposed model

takes account of the non-constant volatility assumption in a manner similar to the term structure models for interest rates (which incorporate the term structure of interest rates). The implied volatility tree is therefore found to be practically as well as theoretically an intuitively appealing model for the pricing and managing of options in South Africa.

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Appendix 5.A

Option Price Sensitivities Estimated from the Modified Black Model in the South African Environment

The modified Black model for the South African environment has been extensively discussed in Chapter 3. Here, only the calculations for the option price sensitivities are outlined. The outline starts with the delta, Δ :

$$\Delta = N(d_1) \quad \text{for call options}$$

$$\Delta = N(d_1) - 1 \quad \text{for put options}$$

with

$$d_1 = \frac{\ln(F/X) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

where $N(x)$ is the cumulative normal distribution function, F is the future price, X is the strike price, σ is the volatility, and $T-t$ is the time to expiration.

The second computed option price sensitivity is the gamma, Γ :

$$\Gamma = \frac{N'(d_1)}{F\sigma\sqrt{T-t}} \quad \text{for call and put options}$$

where

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and d_1 is defined as above.

As third option price sensitivity, the theta, Θ , is computed as

$$\Theta = -\frac{FN'(d_1)\sigma}{2\sqrt{T-t}} \quad \text{for call and put options}$$

where $N'(x)$ and d_1 are defined as above.

Finally, the vega, K ¹⁵², is computed:

$$K = F\sqrt{T-t}N'(d_1) \quad \text{for call and put options}$$

where $N'(x)$ and d_1 are defined as above.

¹⁵² Here, the vega is indexed by the Greek symbol kappa that is often used as the symbol for the rate of change of the value of the option with respect to the underlying asset (in the literature).

Appendix 5.B

Option Price Sensitivities Estimated from Binomial Trees and Implied Volatility Trees

The option price sensitivities are computed directly from the binomial tree or from the implied volatility tree respectively. All methods presented are valid for call and put option. The calculation method is implemented according to Hull (1993) and adapted into the context and notation of the thesis. The first option price sensitivity to be computed is the delta, Δ :

$$\Delta = \frac{v_{2,2} - v_{2,1}}{F_{2,2} - F_{2,1}}$$

where $v_{n,i}$ is the option price at node (n, i) and $F_{n,i}$ is the price of the future at node (n, i) .

The second computed option price sensitivity is the gamma, Γ , of the option price. The Γ is computed as

$$\Gamma = \frac{[(v_{3,3} - v_{3,2})/(F_{3,3} - F_{3,2})] - [(v_{3,2} - v_{3,1})/(F_{3,2} - F_{3,1})]}{0.5 \times (F_{3,3} - F_{3,1})}$$

Hull (1993) proposes to start two time periods, Δt , before the start of the tree to obtain slightly more accurate delta and gamma estimations. This proposal is however not implemented because it provides no further advantage in the comparison of deltas and gammas between the binomial tree and the implied volatility tree.

Moreover, the calculation method of the theta, Θ , and the vega, K , is presented. The theta, Θ , is computed as follows:

$$\Theta = \frac{(v_{3,2} - v_{1,1})}{2\Delta t}$$

The vega, K , is computed instead from the two option prices, v^* and v , where all conditions are the same. The exception is that the volatility, σ , for v^* is $\Delta\sigma$ percent higher than the volatility, σ , of v . Here, a volatility difference, $\Delta\sigma$, of 1 percent is used. Hence

$$K = \frac{v^* - v}{\Delta\sigma}$$

Appendix 5.C

Graphical Comparison of Option Price Sensitivities between the Modified Black Model and a Binomial Tree

The graphical comparison between option price sensitivities for the modified Black model and the binomial tree model (with 50 steps) are presented for a constant volatility (of 21.85 percent across strike prices and across expirations) for 90 days to expiration. The option price sensitivities are displayed with respect to the strike price in Figure 5.C-1. The strike price range is chosen according to the definition of the volatility surface (in section 4.5.4). Moreover, the graphical analysis is based upon a call option.

The graphical comparison between the delta of the modified Black model and the delta of the binomial tree is presented in Figure 5.C-1.

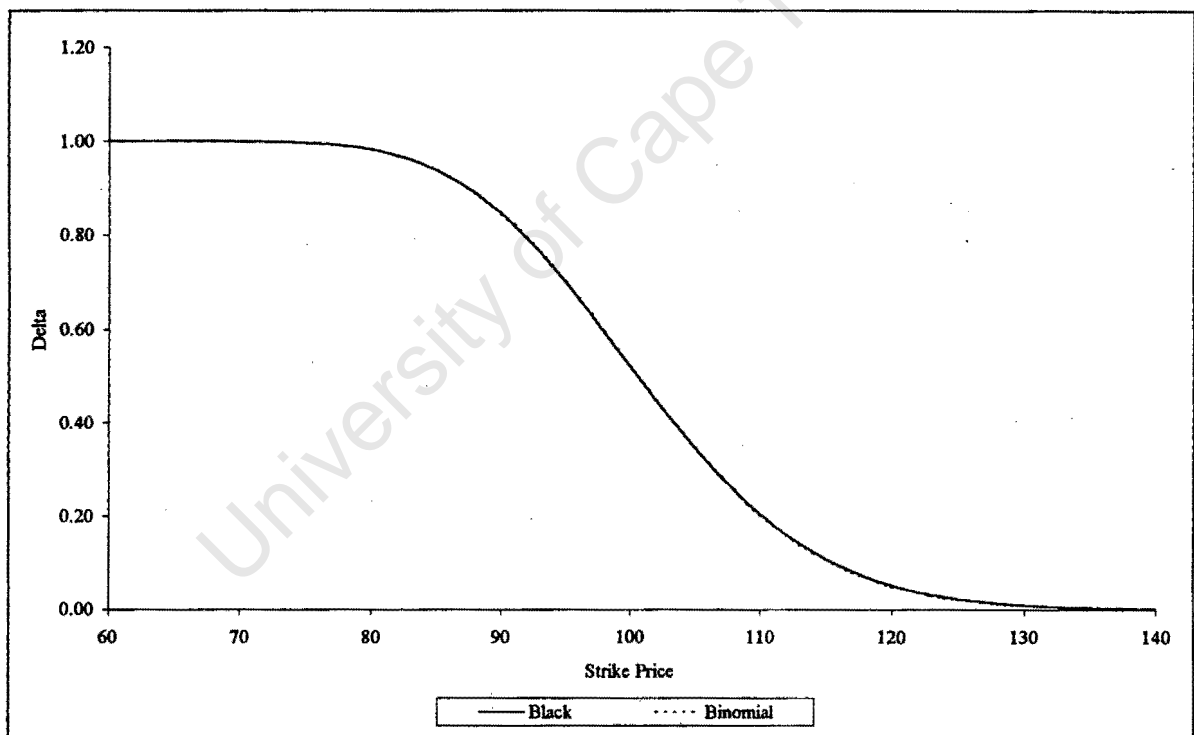


Figure 5.C-1. Delta Comparison between the Modified Black Model and the Binomial Tree

The comparison is computed for the modified Black model and a binomial tree. Both models are based on a constant volatility of 21.85 percent for a call with 90 days to expiration and a future as underlying asset with a value of 100.

Nearly no difference can be observed between the deltas of the modified Black model and the deltas of the binomial tree in Figure 5.C-1. The binomial tree seems to yield a very good approximation that can be deemed sufficient for option pricing in practice (see also Chapter 4). The next comparison for the gamma shows similar results in Figure 5.C-2.

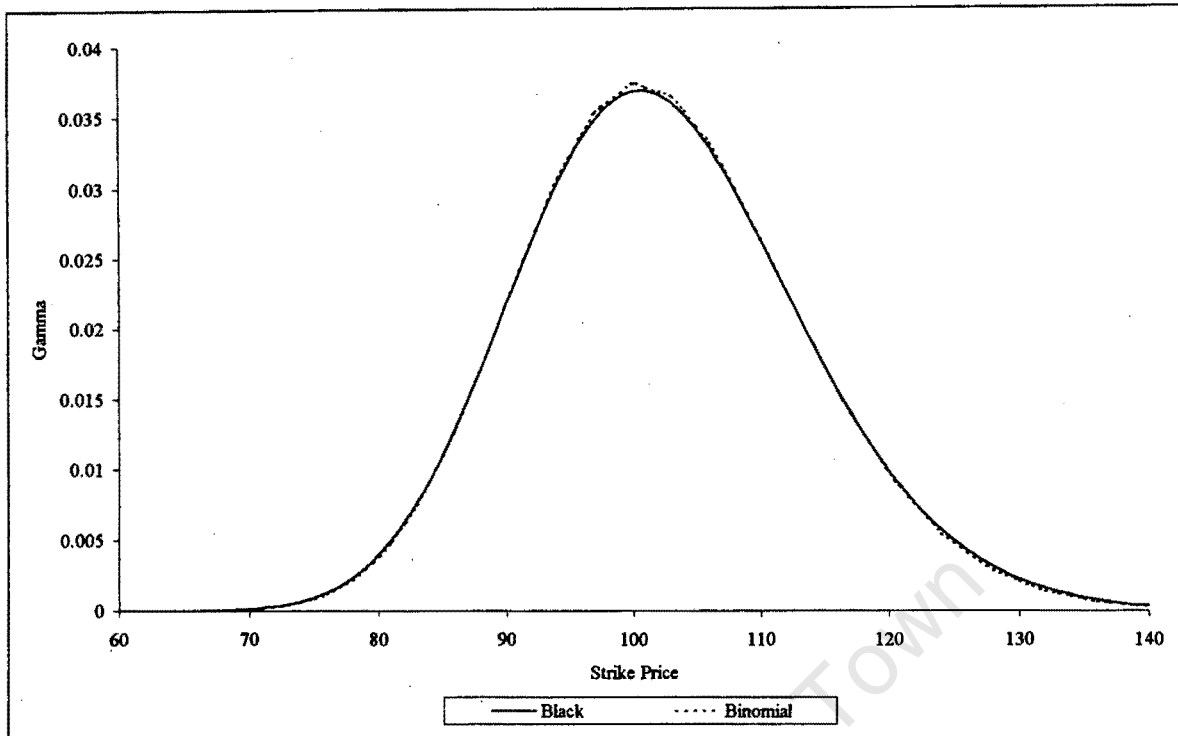


Figure 5.C-2. Gamma Comparison between the Modified Black Model and the Binomial Tree

The comparison is computed for the modified Black model and a binomial tree. Both models are based on a constant volatility of 21.85 percent for a call with 90 days to expiration and a future as underlying asset with a value of 100.

Only a very slight difference for the *at-the-money* range (i.e. strike price is around 100) is observed. However, the difference is so small that the binomial tree is an acceptable approximation for the purpose of option pricing in practice.

The third comparison is presented for the thetas between the modified Black model and the binomial tree in Figure 5.C-3.

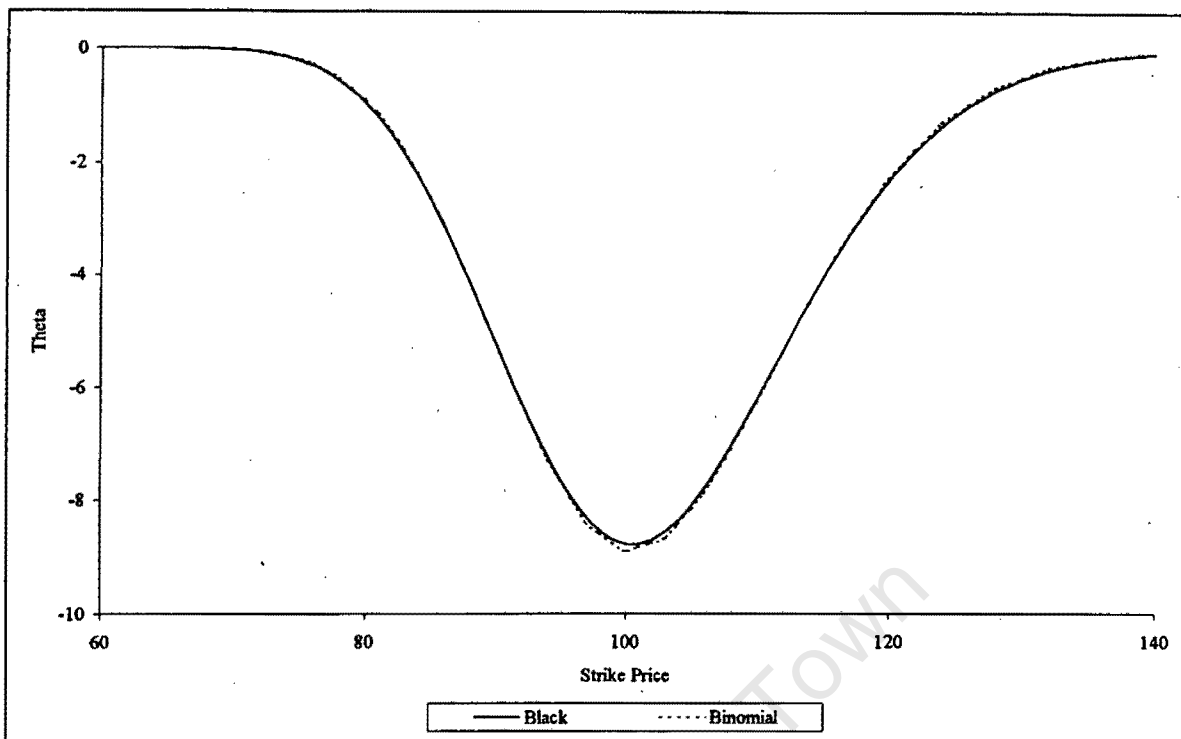


Figure 5.C-3. Theta Comparison between the Modified Black Model and the Binomial Tree

The comparison is computed for the modified Black model and a binomial tree. Both models are based on a constant volatility of 21.85 percent for a call with 90 days to expiration and a future as underlying asset with a value of 100.

The result of the theta comparison is very similar to the gamma comparison. Again, only a slight difference between the both models can be observed for the *at-the-money* range. However, the approximation of the theta is also acceptable for the purpose of option pricing in practice.

Finally, the approximation quality of the binomial tree is analysed graphically for the vega in Figure 5.C-4. The approximation error between the modified Black model and the binomial tree found makes it obvious that a direct comparison between the modified Black model and the implied volatility tree model is influenced by an approximation error. Hence, a comparison between the modified Black model and the implied volatility tree model is only practicable if the approximation error between the modified Black model and the binomial tree model is taken into account. Moreover, the observed mean reverting approximation error between the modified Black model and the binomial tree makes the approximation of the binomial tree still acceptable for the purpose of option pricing in practice and for the purpose of the further research in this thesis.

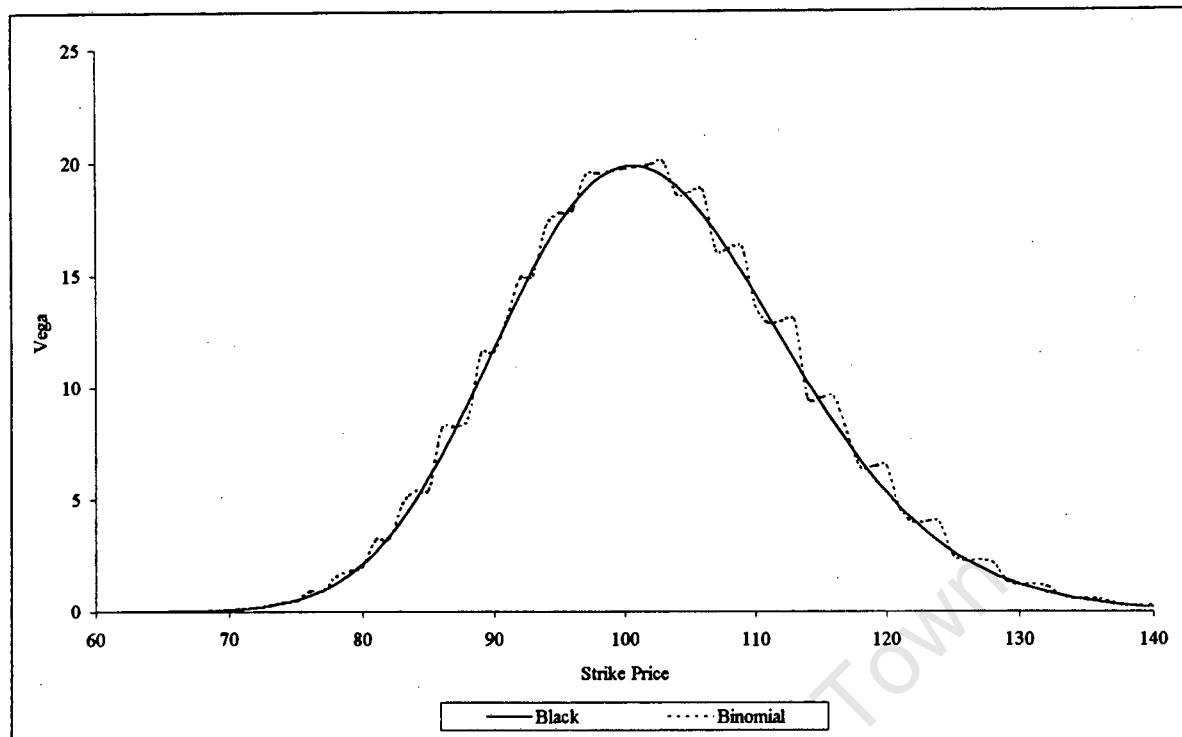


Figure 5.C-4. Vega Comparison between the Modified Black Model and the Binomial Tree

The comparison is computed for the modified Black model and a binomial tree. Both models are based on a constant volatility of 21.85 percent for a call with 90 days to expiration and a future as underlying asset with a value of 100.

Appendix 5.D

The Analysis of Option Price Sensitivities for Put Options

The delta values of different option price models are compared for put options in Table 5.D-1. The analysis of the comparison of delta values for the put options is analogue to the analysis of the call options and is therefore not discussed in detail for the sake of brevity.

Table 5.D-1
Comparison of Deltas for Put Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The delta values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The deltas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The deltas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the deltas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations from the volatility surface.

Strike Prices (1)	Option Prices (2)	Deltas for Different Option Price Models				
		Modified Black constant volatility (3)	Binomial constant volatility (4)	Modified Black strike price volatility (5)	Binomial strike price volatility (6)	Implied Volatility Tree volatility surface (7)
60	0.000	0.0000	0.0000	0.0000	0.0000	-0.0001
70	0.009	-0.0004	-0.0003	-0.0023	-0.0021	-0.0046
80	0.180	-0.0174	-0.0167	-0.0344	-0.0334	-0.0499
85	0.513	-0.0603	-0.0596	-0.0854	-0.0851	-0.1139
90	1.202	-0.1526	-0.1525	-0.1750	-0.1736	-0.2231
95	2.427	-0.2991	-0.2990	-0.3079	-0.3080	-0.3710
100	4.326	-0.4784	-0.4785	-0.4784	-0.4785	-0.5423
105	7.207	-0.6538	-0.6535	-0.6618	-0.6618	-0.7092
110	11.006	-0.7951	-0.7966	-0.8117	-0.8119	-0.8389
115	15.420	-0.8914	-0.8917	-0.9075	-0.9091	-0.9215
120	20.171	-0.9481	-0.9500	-0.9576	-0.9586	-0.9635
130	30.035	-0.9910	-0.9914	-0.9902	-0.9907	-0.9931
140	40.004	-0.9988	-0.9991	-0.9987	-0.9989	-0.9990

The results of the computation of the gamma values for put options are displayed in Table 5.D-2. Again, the results of the analysis for the put options are analogous to the results of the call options.

Table 5.D-2
Comparison of Gammas for Call Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The gamma values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The gammas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The gammas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the gammas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations. Additionally, all gammas are multiplied by the factor 1000.

		Gammas for Different Option Price Models				
Strike Prices	Option Prices	Modified Black constant volatility	Binomial constant volatility	Modified Black strike price volatility	Binomial strike price volatility	Implied Volatility Tree volatility surface
(1)	(2)	(3)	(4)	(5)	(6)	(7)
60	0.000	0.000	0.000	0.009	0.005	0.053
70	0.009	0.138	0.112	0.566	0.505	1.316
80	0.180	3.961	3.829	5.992	5.880	9.946
85	0.513	11.023	10.964	12.565	12.555	18.683
90	1.202	21.737	21.818	21.392	21.521	28.702
95	2.427	32.004	32.177	30.444	30.591	36.805
100	4.326	36.719	37.278	36.719	37.278	40.559
105	7.207	34.007	34.173	35.182	35.369	35.087
110	11.006	26.181	26.336	26.478	26.597	24.793
115	15.420	17.173	17.174	16.317	16.208	14.803
120	20.171	9.799	9.569	8.793	8.647	8.070
130	30.035	2.248	2.129	2.385	2.262	1.937
140	40.004	0.354	0.278	0.393	0.328	0.311

The analysis of the delta and gamma values for put options is similar to the results of the computation of the theta values for put options which is displayed in Table 5.D-3. Again, the results of the analysis for the put options are analogous to the results of the call options.

Table 5.D-3
Comparison of Thetas for Put Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The theta values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The thetas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The thetas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the thetas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations from the volatility surface.

Thetas for Different Option Price Models						
Strike Prices	Option Prices	Modified Black constant volatility	Binomial constant volatility	Modified Black strike price volatility	Binomial strike price volatility	Implied Volatility Tree volatility surface
(1)	(2)	(3)	(4)	(5)	(6)	(7)
60	0.000	0.000	0.000	-0.003	-0.002	-0.012
70	0.009	-0.033	-0.027	-0.191	-0.170	-0.310
80	0.180	-0.945	-0.914	-1.960	-1.924	-2.340
85	0.513	-2.631	-2.617	-3.938	-3.935	-4.395
90	1.202	-5.188	-5.208	-6.298	-6.337	-6.752
95	2.427	-7.638	-7.680	-8.242	-8.283	-8.658
100	4.326	-8.763	-8.898	-8.763	-8.898	-9.542
105	7.207	-8.116	-8.157	-7.706	-7.747	-8.254
110	11.006	-6.248	-6.286	-5.579	-5.605	-5.833
115	15.420	-4.099	-4.099	-3.411	-3.388	-3.483
120	20.171	-2.339	-2.284	-1.881	-1.850	-1.898
130	30.035	-0.537	-0.508	-0.583	-0.553	-0.456
140	40.004	-0.085	-0.067	-0.096	-0.080	-0.073

Similar to the analysis of the three other option price sensitivities for put options, the results of the computation of the vega values for put options are displayed in Table 5.D-4. Again, the results of the analysis for the put options are analogous to the results of the call options.

Table 5.D-4
Comparison of Vegas for Put Options

The option prices in column (2) are computed using the volatility surface (in section 4.5.4) for 90 days to expiration as input. The vega values for the binomial tree and the implied volatility tree are computed based on a 50-step tree. The vegas for the modified Black model and the binomial model in column (3) and (4) respectively are computed using a constant volatility of 21.85 percent (i.e. the *at-the-money* volatility of the volatility surface for 90 days to expiration). The vegas in column (5) and (6) for the modified Black model and the binomial model respectively are computed according to the differing implied volatilities from the volatility surface for 90 days to expiration but with constant implied volatilities across expirations. Finally, the vegas in column (7) for the implied volatility tree model accounts for the non-constant implied volatilities across strike prices and across expirations from the volatility surface.

Vegas for Different Option Price Models						
Strike Prices (1)	Option Prices (2)	Modified Black constant volatility (3)	Binomial constant volatility (4)	Modified Black strike price volatility (5)	Binomial strike price volatility (6)	Implied Volatility Tree volatility surface (7)
60	0.000	0.000	0.000	0.006	0.007	0.055
70	0.009	0.074	0.068	0.363	0.410	0.872
80	0.180	2.134	2.023	3.779	3.419	4.629
85	0.513	5.938	5.412	7.757	8.113	8.172
90	1.202	11.710	11.537	12.800	12.352	13.660
95	2.427	17.241	17.709	17.467	17.703	17.118
100	4.326	19.781	19.684	19.781	19.683	19.681
105	7.207	18.320	18.619	18.157	18.619	18.959
110	11.006	14.104	13.548	13.403	12.757	12.869
115	15.420	9.251	9.446	8.227	9.444	9.460
120	20.171	5.279	6.435	4.485	4.065	4.176
130	30.035	1.211	1.192	1.301	1.188	1.682
140	40.004	0.191	0.251	0.215	0.248	0.245

Chapter 6

6 Conclusion and Summary

Volatility biases across strike prices and across expirations have been documented in the international literature for a number of different option markets. As a consequence, different option price models have been developed to incorporate the biases in the option price models. International literature has however been scant on the effect of volatility biases on option prices and on implied distributions. In addition, the international literature has been almost silent on the effects of volatility biases on option price sensitivities or on portfolio management strategies with options. Furthermore, the strike price bias and the expiration bias have been analysed separately in the most studies (in particular for the effects of the biases on option price sensitivities).

This thesis has been particularly aimed at the South African environment. The peculiarity of the South African market has been its characterisation as an emerging market. Consequently, the established research from the highly developed markets in the US and Europe cannot be directly translated to the South African environment. This thesis firstly aimed at the behaviour of volatility for options on index futures in South Africa. Secondly, an option price model has been proposed for the South African peculiarities of options on futures with the mark-to-market of the option premium at SAFEX. Thirdly, an implied distribution has been identified for the South African environment. Finally, the effects of the volatility behaviour in South Africa have been analysed for option price sensitivities and consequently for portfolio management strategies with options.

The results of the volatility tests in Chapter 3 have revealed empirical evidence that historical volatility and implied volatility are different in South Africa. In addition, historical volatility is not constant for options on futures with different expirations for the same trading day. The result of this test indicated that the Black and Scholes (1973) assumption of constant volatility across expirations is not appropriate. Consequently, the proposed modified Black model for the South African market is also not appropriate because it is based on the assumption of constant volatility. Hence, a new option price model was

considered. Further tests were therefore conducted in Chapter 3 in which the implied volatility was assessed across strike prices and across expirations.

The descriptive test of implied volatility with the two volatility index methods, “Volatility Smile Index” (VSI) and “Volatility Term Structure Index” (VTSI) in Chapter 3, revealed that implied volatility in South Africa is neither constant across strike prices or across expirations as assumed by the Black and Scholes (1973) theory. Systematic volatility biases are evident from the VSI and the VTSI analysis for the ALSI, the GLDI, and the INDI. These volatility biases are similar to evidence in the international literature. The strike price biases have revealed evidence of “grins” (the *out-of-the-money* volatility is higher than the *at-the-money* or the *in-the-money* volatility). Moreover, the empirical results revealed that the implied volatility “grins” across strike prices has been more pronounced the shorter the remaining time to expiration. In addition, the implied volatility across expirations has displayed evidence that the implied volatility increases for *at-the-money* strike prices the longer the remaining time to expiration. The results of the volatility index tests in Chapter 3 resulted in the rejection of the constant volatility assumption, as required by the modified Black model, across strike prices and across expirations. Consequently, the modified Black model which is based on the constant volatility assumption has also been rejected.

The last implied volatility test conducted in Chapter 3 was the nonparametric test across strike prices and across expirations. This test confirmed the results obtained in the volatility index tests above. This confirmation obtained from the nonparametric test was important because the nonparametric test was the only test that constructed tests of statistical significance. In sum, it was concluded in Chapter 3 that the modified Black model for the trading in the South African environment had to be rejected.

The rejection of the modified Black model led to the proposal of a novel option price model for the option trading and pricing in South Africa in Chapter 4. The proposed model in this thesis is based on the implied volatility tree model by Derman and Kani (1994) but it has been improved and modified for the conditions in South Africa. The new implied volatility tree model for South Africa prices options flexibly according to the volatility structure. One of the most important improvements is the solution of the negative transition probability with an algorithm proposed in this thesis (in Chapter 4). Moreover, the essential extrapolation and interpolation methods for the implied volatility tree model were tested and

new methods were introduced in Chapter 4. The interpolation and extrapolation across strike prices resulted in a regression method with a separation between *in-the-money* and *out-of-the-money* strike ranges.

Ultimately, the proposed implied volatility tree model was tested in Chapter 4 and achieved very good results for tests with constant as well as non-constant volatility. The tests resulted in a further modification of the implied tree volatility model to enhance its computational efficiency. The modification considers the inclusion of the approximation error between the modified Black model and the binomial tree model in the implied volatility tree model. It was found in Chapter 4 that the inclusion of the approximation error substantially enhanced the computational efficiency of the modified implied volatility tree.

The successful tests of the implied volatility tree model led to its first application in Chapter 4. The new implied volatility tree model was used to establish implied distributions from a representative data set for the South African environment. The established implied distributions revealed a substantial deviation from distributions with the assumption of constant volatility. The left tail of the implied distributions with non-constant implied volatility was substantially larger than the left tail of implied distributions with constant implied volatility. Hence, the implied distributions reflected a higher probability of large market declines than reflected by the assumption of constant volatility (incorporated in the modified Black model). A second important consequence was the need to incorporate non-constant volatility across strike prices and across expirations (instead of implementing only the volatility process across strike prices or across expirations). It was revealed in Chapter 4 that the incorporation of non-constant volatility is fundamental to establish the appropriate implied distribution accurately.

Finally, the effects of non-constant volatility on option price sensitivities and on portfolio management strategies with options were analysed in Chapter 5. The results of the option price sensitivities presented substantial differences between

- constant implied volatility across strike prices and across expirations
- constant implied volatility across expirations but non-constant implied volatility across strike prices
- non-constant implied volatility across strike prices and across expirations.

All four analysed option price sensitivities (delta, gamma, theta, and vega) displayed evidence in favour of incorporating, fully, non-constant volatility. In particular, it was evident in Chapter 5 that the incorporation of only non-constant implied volatility across strike prices did not reflect the full structure of the implied volatility. Hence, the usual application (in practice) of the modified Black model with non-constant volatility across strike prices may achieve a reasonably accurate option price but nevertheless an incorrect estimate of the option price sensitivities. Consequently, it is important to implement an option price model that incorporates the implied volatility structure fully (as the proposed implied volatility tree model did in this thesis).

Moreover, the effects of non-constant volatility were discussed for four examples of portfolio management strategies with options in Chapter 5. The effects of non-constant volatility in comparison to constant volatility on the portfolio management strategies were substantially different. The results of the comparison between the constant and the non-constant volatility assumption revealed the severe effects of neglecting non-constant volatility (for example, in the delta-gamma hedge and in the synthetic put replication). The importance of an appropriate option price model for portfolio management strategies was highlighted in Chapter 5.

This thesis revealed that option price models based on the inappropriate assumption of constant volatility across strike prices and across expirations did not only perform worse in the option pricing but also in the estimation of the option price sensitivities. The inaccurate estimation of option price sensitivities led to damaging results for two of the most important applications of options today that are hedging techniques and portfolio insurance methods. Hence, new models like the proposed implied volatility tree in this thesis should be implemented to avoid the potential shortcomings in practical applications.

Outlook

The scope for further development of a new model for option pricing and the analysis of non-constant volatility effects does still exist in the South African context. Further research which investigates comparisons between the performance of the proposed implied volatility tree and other option price models, for example, in performance tests for different hedging techniques is a consideration. Moreover, the proposed implied tree technique can still be enhanced computationally. Furthermore, exchange-traded options are not the only options available today. A substantial part of the trading volume consists of exotic options, for example, barrier options. The effects of non-constant volatility on exotic options however have had little coverage in the literature¹⁵³. As noted in section 5.3, further research is needed in this important field, particularly because of the substantial growth in these option contracts (i.e. exotic options).

Finally, the “implied methodology” presented for the option pricing in this thesis can be extended further. For example, Tompkins (1994) displays a method in which he implements the implied volatilities of different underlyings to compute the “implied correlation” between two different assets. A suggested area of further research concerns the calculation of “implied betas” therefore.

The “implied beta” computation however requires the existence of an option market for one or more market indices and an option market for shares so that the beta of the share can be calculated relative to the market index. Nevertheless, some problems have to be addressed before the “implied betas” can be computed. For example, which implied volatility should be used because the implied volatility from the market index as well as from the share differs across strike prices and across expirations. This question also has to be addressed for the computation of the “implied correlation”. A potential solution to this problem may involve the selection of implied volatilities across expirations according to the estimated holding time of the underlying asset in the portfolio. In addition, the implied volatilities across strike prices can be chosen according to the purchase price of the asset. Consequently, further research in this area is needed.

¹⁵³ One notable exemption for barrier options is Chriss (1997).

The implied correlation and implied beta computation as well as the implied distribution and the implied volatility tree give some direction on how to extend the implied methodology towards a fully market driven (i.e. implied) valuation for financial transactions. The advantage of the implied methodology is the independent estimation and valuation of important financial ratios from human biases. Such an independent procedure is important for arbitrage transactions as well as risk estimation. However, the incorporated biases of the human participants in the markets should not be underestimated because the financial world is not based upon fixed parameters as required for the construction of machines or houses for example. Hence, the world of financial engineering might be an illusion and consequently, the computed values have to be judged by a "human rationalist" who is a contradiction in himself.

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