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EXACT POWERS OF SOME MULTIVARIATE TEST CRITERIA

by

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To Genevieve

CONTENTS

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CHAPTER 1

INTRODUCTION AND SUMMARY

In order to test various hypotheses regarding characteristics of multivariate populations, a number of test criteria have been proposed and investigated. These are well summarised by Anderson (1958). For many years the exact distributions of these criteria remained unknown except for the central distributions with low parameters which were obtainable by the classical method of integration. The problem was however met to a certain extent by the development of more easily calculable asymptotic distributions, which had varying orders of accuracy dependent on large sample sizes.

In 1964 a breakthrough was achieved by Schatzoff, who derived an algorithm for obtaining the exact central distribution of Wilks' (1932) likelihood ratio criterion for MANOVA by using the method of convolutions. With this method he was able to calculate exact percentiles for a range of parameters limited mainly by the precision of the computer used, and these results have since been extended by Pillai and Gupta (1969) and Lee (1972).

The major problem remains the calculation of powers (and therefore noncentral distributions) of the proposed criteria against general alternatives. The development of theory around the zonal polynomial, due originally to James, and different approaches by authors such as Consul and

Mathai have not provided a general solution to this problem. So far all results have been confined to distributions with restrictions on the size or type of certain parameters.

Gupta (1971) extended the method of convolutions to the noncentral linear case (when the alternative hypothesis is of unit rank), and was able to represent the distributions of Wilks' criterion by explicit infinite series expressions when the number of variates was low. Gupta also suggested that it might be possible to derive an algorithm which could calculate values of the density and cumulative distribution functions for general values of the parameters.

In this thesis an algorithm for the noncentral linear density and cumulative distribution function of Wilks' likelihood ratio criterion in MANOVA is derived and it is shown how this algorithm, with modifications, can be used to find the distributions of a number of test criteria for different hypotheses. At the same time previous results regarding percentiles and powers of these criteria are examined and discussed.

Chapter 2 contains a summary of the important theory in connection with hypergeometric functions, zonal polynomials and G- and H-functions, for real and complex variables. This provides a broad perspective for the later restriction to the noncentral linear case.

The test criteria examined will be shown to depend on Wishart, univariate and multivariate Beta and Dirichlet distributions, A comprehensive collection of results for these distributions in the central and noncentral, real and complex cases is given in Chapter 3. This is drawn upon in the chapters that follow and is necessary for further extension to the general noncentral case.

In Chapter 4 we show how the test criteria for various hypotheses may be expressed in a general form as a product of a number of independent central and noncentral Beta variates. This applies to tests with both real and complex variables.

The background to the use of the method of convolutions is discussed in Chapter 5 and here it is shown how the density and cumulative distribution functions of a product of independent central and noncentral Beta variables may be expressed in a general form by using this method. A computer-based algorithm is developed which for the first time provides exact distributional results for any number of variables. The limiting factors of computer precision and convergence are also discussed.

Chapter 6 deals with hypotheses regarding the equality of mean vectors, regression coefficients and MANOVA. The algorithm is used to derive exact noncentral linear percentiles and powers of two test criteria proposed by Wilks (1932), the likelihood ratio criterion, $|L|$, and another statistic, $|I-L|$. Power comparisons of these criteria are given and distinct regions of optimality for different parameter values are noted. From these results and from intuitive reasoning on the behaviour of the second statistic, $|I-L|$ is firmly rejected as a practical alternative to $|L|$. The accuracy of previously published results on powers and percentiles (usually asymptotic) of $|L|$ is also commented on. Finally, the power of the statistic $|I-\Sigma L_j|$, for testing for the equality of a number of normal populations, is considered against different alternatives.

The algorithm is extended to cover the test of independence between two sets of variates in Chapter 7. Here again two test criteria are considered, the likelihood ratio criterion, $|I-R|$, and $|R|$. It is shown by means of power comparisons and intuitive discussion that $|R|$ is an inferior criterion for this test except for the case where $|R|$ is the square of the multiple correlation coefficient. A theorem is proved which enables a restriction on the applicable parameter range to be partially overcome. Published results for $|I-R|$ are commented on, and it is shown that the set of parameter values for which powers were previously available can be considerably enlarged.

In Chapter 8 the test for equality of two covariance matrices is discussed. The likelihood ratio criterion is generally biased and the behaviour of two alternative test criteria, $|C|$ and $|I-C|$, is examined. These are both shown to be statistically suitable and calculable by the algorithm, but differ from $|L|$ and $|I-R|$ in that their rejection regions are two-tailed and the convergence of $|I-C|$ is limited. It is seen that two-sided tests with both $|C|$ and $|I-C|$ may be carried out by using only the lower and upper percentiles of $|C|$ and $|I-C|$ respectively. Published results for this test are commented on, and power comparisons of $|C|$ and $|I-C|$ show that the power of each test is dominant over a different range of parameter values.

Chapter 9 shows how the algorithm could be modified to include the general form for tests of hypotheses with complex variables given in Section 4.9 and outlines the differences from the real case in approach and parameter restriction. Finally, the method and implications of using the algorithm to derive the central distribution of the test for reality of a covariance matrix are discussed.

Because of the many test criteria and associated parameters considered, a reasonably complete list of tables would be prohibitively long. It is felt that the selection in Appendices 1 to 12 will adequately illustrate the general trends.

Reference to the more frequently quoted authors will be simplified if the following correspondence is observed in notation used from Chapter 5 onwards.

	number of	degrees of freedom for	
author	variables	hypothesis	error
Anderson	p	q_1	n
de Waal	p	n	m
Gupta	p	f_2	f_1
Hart	p^*	m^*	n^*
Lee	p	q	n
Money	p	n	m
Pillai and Gupta	p	f_2	f_1
Schatzoff	p	q	n
Troskie	p	n	m

CHAPTER 2

FUNCTIONS AND RELATED THEORY

In order to determine the power of any statistical test we need to know both the central and noncentral distributions of the test criterion. In deriving these distributions and in manipulating them into computable form, various functions and techniques are used, some of which are given in this chapter.

2.1 Hypergeometric Functions

Definition 2.1.1 (Erdélyi (1953))

The generalised Gauss function or hypergeometric function of a single variable is defined by

$$(2.1.1)\quad {}_pF_q\,(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ z) = \sum_{j=o}^{\infty} \frac{(a_1)_j\ \ldots\ (a_p)_j}{(b_1)_j\ \ldots\ (b_q)_j}\ \frac{z^j}{j!}$$

where

$$(2.1.2)\quad (a)_j = a(a+1)\ \ldots\ (a+j-1) = \Gamma(a+j)/\Gamma(a) \qquad j = 1,2,\ldots$$

$$(a)_o = 1$$

and z may be real or complex.

Definition 2.1.2 (Constantine (1963), James (1964))

Analogous to 2.1.1, in the case of a real matrix variable S, are the hypergeometric functions of matrix and double matrix argument

(2.1.3) $\quad {}_pF_q\,(a_1,\ldots,a_p;\,b_1,\ldots,b_q;\,S)$

$$= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\,\ldots\,(a_p)_\kappa}{(b_1)_\kappa\,\ldots\,(b_q)_\kappa}\,\frac{C_\kappa(S)}{k!}$$

and

(2.1.4) $\quad {}_pF_q\,(a_1,\ldots,a_p;\,b_1,\ldots,b_q;\,S,T)$

$$= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\,\ldots\,(a_p)_\kappa}{(b_1)_\kappa\,\ldots\,(b_q)_\kappa}\,\frac{C_\kappa(S)C_\kappa(T)}{k!\;\;C_\kappa(I_m)}$$

where S and T are real symmetric matrices of order m (say), $\kappa = (k_1,\ldots,k_m)$ is a partition of k into not more than m components such that $\Sigma k_i = k$, $C_\kappa(S)$ is a homogeneous symmetric polynomial called a zonal polynomial (see Definition 2.2.1) and

(2.1.5) $\quad (a)_\kappa = \Pi_{i=1}^{m}(a+(1-i)/2)_{k_i}$

We will also write the hypergeometric function in abbreviated form e.g. ${}_pF_q(a_i;\,b_j;\,S,T)$. Conditions for convergence of the hypergeometric series are:

(2.1.6) (i) $p \leq q + 1$, otherwise the series may only converge for $S = 0$

(ii) If $p = q + 1$ the series will converge for $|\lambda_1| < 1$ where λ_1 is the largest latent root of S in (2.1.3) or of ST in (2.1.4).

(iii) If $p < q + 1$ the series will always converge.

(iv) The a_i and b_j are arbitrary numbers but none of the b_j may be integers or half-integers less than or equal to $\frac{1}{2}(m-1)$ or else some denominators will vanish.

(v) If an a_i is a negative integer, say $-n$, then for $k > mn + 1$ all the coefficients will vanish, reducing the function to a finite polynomial of degree mn.

Definition 2.1.3 (James (1964))

In the case of a complex Hermitian matrix S, the hypergeometric functions of Hermitian and double Hermitian matrix argument are defined as

$$(2.1.7)\quad {}_p\overline{F}_q\,(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ S) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa}\,\frac{\overline{C}_\kappa(S)}{k!}$$

and

$$(2.1.8)\quad {}_p\overline{F}_q\,(a_1,\ldots,a_p;\ b_1,\ldots,b_q;\ S,T) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa}\,\frac{\overline{C}_\kappa(S)\overline{C}_\kappa(T)}{\overline{C}_\kappa(I_m)\ k!}$$

where S and T are Hermitian matrices, $\overline{C}_\kappa(S)$ is a zonal polynomial of S (see Definition 2.2.1) and

$$(2.1.9)\quad (a)_\kappa = \Pi_{i=1}^{m}(a+1-i)_{k_i}$$

Note that $S = S_R + iS_I$ (R and I denoting real and imaginary parts respectively) and $S = \overline{S}'$ (the Hermitian conjugate of S).

If the gamma function exists, we have

$$(2.1.10)\quad (a)_\kappa = \Gamma_m(a,\kappa)/\Gamma_m(a)$$

where

$$(2.1.11)\quad \Gamma_m(a,\kappa) = \Pi^{m(m-1)/4}\ \Pi_{i=1}^{m}\ \Gamma(a+k_i+(1-i)/2)$$

(2.1.12) $\Gamma_m(a,-\kappa) = \pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma(a-k_i-(m-i)/2)$

$$= (-1)^k\Gamma_m(a)/(-a+\tfrac{1}{2}(m-1))_\kappa$$

for $a > \frac{1}{2}(m-1) + k_i$

and

(2.1.13) $\Gamma_m(a) = \Gamma_m(a,0)$.

Also

(2.1.14) $(a)_\kappa = \overline{\Gamma}_m(a,\kappa)/\overline{\Gamma}_m(a)$

where

(2.1.15) $\overline{\Gamma}_m(a,\kappa) = \pi^{m(m-1)/2}\ \prod_{i=1}^{m}\Gamma(a+k_i+1-i)$

and

(2.1.16) $\overline{\Gamma}_m(a) = \overline{\Gamma}_m(a,0)$

2.2 Zonal Polynomials

Definition 2.2.1 (James (1960))

The zonal polynomial $C_\kappa(S)$ is defined as the component of $(\mathrm{tr}\ S)^k$ in the subspace V_k which is the vector space of homogeneous polynomials $\phi(S)$ of degree k in the $n = \frac{1}{2}m(m+1)$ different elements of the $m\times m$ positive definite symmetric matrix S. In the complex case, $\overline{C}_\kappa(S)$ relates similarly to the $m\times m$ Hermitian matrix S.

James (1960, 1961) and Constantine (1963) were mainly responsible for showing the uses to which the zonal polynomial could be put in solving distribution problems.

We now list some important results.

For S a real symmetric matrix

Property 2.2.1

$(\mathrm{tr}\ S)^k$ has a unique decomposition in terms of zonal polynomials:

(2.2.1) $(\mathrm{tr}\ S)^k = \Sigma_\kappa C_\kappa(S)$

but if $m = 1$

(2.2.2) $C_\kappa(S) = C_\kappa(x) = x^k$

Property 2.2.2

$C_\kappa(S)$ is invariant under the orthogonal group $(O(m))$, i.e.

(2.2.3) $C_\kappa(HSH') = C_\kappa(S) \qquad H \in O(m)$

and is a symmetric homogeneous polynomial of degree k in the latent roots of S.

Definition 2.2.2 (Constantine (1963))

For S symmetric, R positive definite symmetric, we define

(2.2.4) $C_\kappa(RS) = C_\kappa(R^{\frac{1}{2}}SR^{\frac{1}{2}})$

where $R^{\frac{1}{2}}$ is the unique positive definite square root of R. This definition is justified by the fact that RS and $R^{\frac{1}{2}}SR^{\frac{1}{2}}$ have the same latent roots.

Property 2.2.3

(2.2.5) $C_\kappa(bS) = B^kC_\kappa(S)$ for b a scalar.

Theorem 2.2.1 (James (1960))

(2.2.6) $\int_{O(m)} C_\kappa(PHSH')dH = C_\kappa(P)C_\kappa(S)/C_\kappa(I_m)$

where H is an orthogonal matrix and dH is the Haar measure over the orthogonal group $O(m)$, normalised so that the measure over the whole group is one. The normalising constant is given by

$$(2.2.7) \quad (1/2^m) \int_{O(m)} dH = \Pi^{\frac{1}{2}m^2} / \Gamma_m(m/2)$$

This leads to

Property 2.2.4 (James (1964))

$$(2.2.8) \quad \int_{O(m)} {}_pF_q(a_i;\, b_j;\, PHSH')dH = {}_pF_q(a_i;\, b_j;\, P,S)$$

Property 2.2.5

If S is of rank $r \leq m$ then there exists an orthogonal matrix C such that

$$(2.2.9) \quad CSC' = \begin{vmatrix} S_r & 0 \\ 0 & 0 \end{vmatrix}$$

where S_r $(r\times r)$ is of rank r. The zonal polynomial is now given by (James (1964))

$$(2.2.10) \quad C_\kappa(S) = \begin{cases} 0 & k_{i>r} \neq 0 \\ C_\kappa(S_r) & k_{i>r} = 0 \end{cases}$$

where κ is a partition of k into not more than r components.

Property 2.2.6 (James (1964))

$$(2.2.11) \quad C_\kappa(I-S) = \sum_{\tau \leq \kappa} \alpha_\tau C_\tau(S)$$

where τ is a partition of k into not more than m parts. The ordering $\leq$ is lexicographic.

Property 2.2.7 (Constantine (1966))

$$(2.2.12) \quad C_\kappa(I-S) = C_\kappa(I) \sum_{n=o}^{k} \sum_\nu (-1)^n a_{\kappa,\nu} \frac{C_\nu(S)}{C_\nu(I_m)}$$

where ν is a partition of n into not more than m parts.

Constantine has tabulated the coefficients $a_{\kappa,\nu}$ up to order $k = 4$.

Property 2.2.8 (Constantine (1966))

(2.2.13) $C_\kappa(S)C_\tau(S) = \Sigma_\delta g^\delta_{\kappa,\tau} C_\delta(S)$

where κ,τ and δ are partitions of k,t and $d = k+t$ respectively, into not more than m parts. Khatri and Pillai (1968) have tabulated the coefficients $g^\delta_{\kappa,\tau}$ for all partitions of $k+t$ up to order 7.

Property 2.2.9 (Constantine (1963))

(2.2.14) ${}_0F_0(S) = \text{etr}(S) = \sum_{k=0}^{\infty}\sum_\kappa C_\kappa(S)/k!$

Property 2.2.10 (Herz (1955), James (1964))

(2.2.15) ${}_1F_0(a;\ S) = |I-S|^{-a}$

(2.2.16) ${}_1F_0(a;\ -S(I-S)^{-1}) = |I-S|^a$

and

(2.2.17) ${}_1F_0(a;\ S,T) = \int_{O(m)} |I-SHTH'|^{-a}\, dH \qquad H \in O(m)$

Property 2.2.11 (James (1961))

(2.2.18) ${}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}XX') = \int_{O(m)} \text{etr}(XH)dH \qquad H \in O(m)$

where X is a $r\times m$ matrix, $r \leq m$.

For S a Hermitian matrix

Property 2.2.12 (James (1964))

$\overline{C}_\kappa(S)$ is invariant under the Unitary group $U(m)$, i.e.

(2.2.19) $\overline{C}_\kappa(US\overline{U}') = \overline{C}_\kappa(S) \qquad U \in U(m)$

and is a symmetric function of the latent roots of S.

U is the complex equivalent of the real orthogonal matrix and is defined by

(2.2.20) $U\bar{U}' = I_m$

Property 2.2.13

(2.2.21) $\bar{C}_\kappa(ST) = \bar{C}_\kappa(TS)$

are symmetric functions of the latent roots of the product ST where S and T are Hermitian matrices.

Property 2.2.14

(2.2.22) $\bar{C}_\kappa(bS) = b^k\bar{C}_\kappa(S)$, b a scalar

Property 2.2.15

(2.2.23) $\int_{U(m)} \bar{C}_\kappa(PUSU')\, dU = \bar{C}_\kappa(P)\bar{C}_\kappa(S)/\bar{C}_\kappa(I_m)$

where U is a unitary matrix and dU is the invariant Haar measure over the unitary group $U(m)$, normalised so that the measure over the whole group is one. The normalising constant is given by

(2.2.24) $\Pi^{m(m-1)}/\bar{\Gamma}_m(m)$

Properties (2.2.4), (2.2.5), (2.2.9) and (2.2.10) extend similarly to the complex case if H, dH, $O(m)$, $C_\kappa(S)$ and ${}_pF_q$ are replaced respectively by U, dU, $U(m)$, $\bar{C}_\kappa(S)$ and ${}_p\bar{F}_q$.

Property 2.2.16

(2.2.25) ${}_0\bar{F}_1(m;\, X\bar{X}') = \int_{U(m)} \mathrm{etr}(XU+\overline{XU})\, dU$

where X is a $r\times m$ complex matrix, $r \leq m$.

2.3 G-Functions

Many exact distributions of multivariate test criteria may be expressed in terms of Meijer's G-functions or of the more general H-functions.

<u>Definition 2.3.1</u> (Mathai (1971))

The H-function of a complex variable z is defined as

$$(2.3.1)\quad H(z) = H^{mn}_{pq}\left\{ z \left| \begin{matrix} (a_1,\alpha_1),\ldots,(a_p,\alpha_p) \\ (b_1,\beta_1),\ldots,(b_q,\beta_q) \end{matrix} \right. \right\}$$

$$= \frac{1}{2\pi i} \oint_C \frac{\Pi_{j=1}^{m}\Gamma(b_j+\beta_j s)\Pi_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\Pi_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\Pi_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}\, z^{-s} ds$$

where $\oint_C$ represents an integral along the closed contour C taken anticlockwise by convention. Further conditions regarding a_j, b_j, α_j, β_j, n, m, p and q are given by Mathai.

We will state these for the special case of the H-function when $\alpha_1 = \alpha_2 = \ldots = \alpha_p = 1$, $\beta_1 = \beta_2 = \ldots \beta_q = 1$, i.e. the G-function.

<u>Definition 2.3.2</u>

The G-function of a complex variable z is defined as

$$(2.3.2)\quad G(z) = G^{mn}_{pq}\left\{ z \left| \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \right. \right\}$$

$$= \frac{1}{2\pi i} \oint_C \frac{\Pi_{j=1}^{m}\Gamma(b_j+s)\Pi_{j=1}^{n}\Gamma(1-a_j-s)}{\Pi_{j=m+1}^{q}\Gamma(1-b_j-s)\Pi_{j=n+1}^{p}\Gamma(a_j+s)}\, z^{-s}\, ds$$

where p,q,n and m are integers such that $0 \leq n \leq p$ and $1 \leq m \leq q$, $i = (-1)^{\frac{1}{2}}$, z is non-zero, and the a_j and b_h are complex numbers such that

$$b_h + v \neq a_j - 1 - r \quad \text{for} \quad v,r = 0,1,\ldots$$
$$h = 1,\ldots,m$$
$$j = 1,\ldots,n$$

The contour C is such that the points

$$-s = b_h + v, \quad h = 1,\ldots,m; \quad v = 0,1,\ldots$$

are enclosed within it and are separated from the points

$$-s = a_j - 1 - r, \; j = 1,\ldots,n; \quad r = 0,1,\ldots$$

An empty product is interpreted as unity.

In most applications we use a special form of the G-function where $m = q = p$ and $n = 0$, i.e.

$$(2.3.3) \quad G_p\left(z\left|\begin{matrix} a_j \\ b_j \end{matrix}\right.\right) = G^{po}_{pp}\left(z\left|\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_p \end{matrix}\right.\right)$$

$$= \frac{1}{2\pi i} \oint_C \frac{\Pi^p_{j=1}\Gamma(b_j+s)}{\Pi^p_{j=1}\Gamma(a_j+s)} z^{-s} \, ds$$

where C is a limiting contour enclosing all the poles of $\Pi^p_{j=1}\Gamma(b_j+s)$.

It is possible to derive the density of a random variable in terms of G-functions. By Greenacre (1972), if we are given a moment sequence $E(X^h)$ of a random variable X, we may find a corresponding density function by using the inverse Mellin transform:

$$(2.3.4) \quad f(x) = \frac{1}{2\pi i} \oint E(x^h)x^{-h-1} \, dh$$

By Carleman's theorem the density function will be uniquely determined if the moment sequence is such that the series $\sum^{\infty}_{h=1}\{E(X^{2h})\}^{-1/2h}$ is divergent. This will be the case if $0 \leq X \leq 1$.

This technique was originally used by Nair (1938) and later revived by Consul (1966).

Meijer (1946) has shown that for special values of a_j and b_j the G-function assumes the form of known functions such as the gamma function, beta function, Gaussian hypergeometric function and Bessel function. We now state some useful theorems.

Theorem 2.3.1 (Meijer (1946), Pillai, Al-Ani and Jouris (1969))

The G-function (2.3.2) can be expressed as a finite number of generalised hypergeometric functions as follows:

$$G^{mn}_{pq}\left(x\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)$$

$$= \sum_{h=1}^{m}\left(\Pi_{j=1,j\neq h}^{m}\Gamma(b_j-b_h)\Pi_{j=1}^{n}\Gamma(1+b_h-a_j)\right)/$$

$$(2.3.5)\quad \left(\Pi_{j=m+1}^{q}\Gamma(1+b_h-b_j)\Pi_{j=n+1}^{p}\Gamma(a_j-b_h)\right)x^{b_h}$$

$$\cdot {}_pF_{q-1}(1+b_h-a_1,\dots,1+b_h-a_p;\ 1+b_h-b_1,\dots,$$

$$1+b_h-b_{h-1},\ 1+b_h-b_{h+1},\dots,1+b_h-b_q;\ (-1)^{p-m-n}x)$$

Theorem 2.3.2 (Consul (1969), Pillai, Al-Ani and Jouris (1969))

$$G^{2\ 0}_{2\ 2}\left(x\middle|\begin{matrix}a_1,a_2\\ b_1,b_2\end{matrix}\right)$$

$$(2.3.6) = x^{b_1}(1-x)^{a_1+a_2-b_1-b_2-1}/\Gamma(a_1+a_2-b_1-b_2)$$

$$\cdot {}_2F_1(a_2-b_2,a_1-b_2;\ a_1+a_2-b_1-b_2;\ 1-x),\qquad 0 < x < 1$$

The following four theorems hold for $0 < X < 1$ and K independent of h.

Theorem 2.3.3 (Greenacre (1972))

If $E(X^h) = KT^h \dfrac{\Pi_{j=1}^{p}\Gamma(a_j+h)}{\Pi_{j=1}^{p}\Gamma(b_j+h)}$

then

(2.3.7) $$f(x) = Kx^{-1}G_p\left(\frac{x}{T}\middle|\begin{matrix}b_j\\a_j\end{matrix}\right)$$

Theorem 2.3.4 (Greenacre (1972))

If $E(X^h) = KT^h\Gamma_p(a+h,\kappa)/\Gamma_p(b+h,\tau)$

then

(2.3.8) $$f(x) = Kx^{-1}G_p\left(\frac{x}{T}\middle|\begin{matrix}b+t_j-\frac{1}{2}(j+1)\\a+k_j-\frac{1}{2}(j+1)\end{matrix}\right)$$

Theorem 2.3.5 (Greenacre (1972))

If $E(X^h) = KT^h\{\Gamma_p(a_r+h)/\Gamma_p(b_s+h)\}$

$$\cdot{}_mF_n(a_1,\ldots,a_r+h,\ldots,a_m;\ b_1,\ldots,b_s+h,\ldots,b_n;\ S)$$

then

(2.3.9) $$f(x) = Kx^{-1}\sum_{k=o}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_{r-1})_\kappa(a_{r+1})_\kappa\cdots(a_m)_\kappa}{(b_1)_\kappa\cdots(b_{s-1})_\kappa(b_{s+1})_\kappa\cdots(b_n)_\kappa}$$

$$\cdot\frac{C_\kappa(S)}{k!}\ G_p\left(\frac{x}{T}\middle|\begin{matrix}b_s+k_j-\frac{1}{2}(j-1)\\a_r+k_j-\frac{1}{2}(j-1)\end{matrix}\right)$$

Theorem 2.3.6 (Greenacre (1972), Underhill (1973))

If $E(X^h) = KT^h\{\Gamma_p(a_r+h)/\Gamma_p(b_s+h)\}$

$$\cdot{}_mF_n(a_1,\ldots,a_r+h,\ldots,a_m;\ b_1,\ldots,b_s+h,\ldots,b_n;\ S,R)$$

then

(2.3.10) $$f(x) = \sum_{k=o}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_{r-1})_\kappa(a_{r+1})_\kappa\cdots(a_m)_\kappa}{(b_1)_\kappa\cdots(b_{s-1})_\kappa(b_{s+1})_\kappa\cdots(b_n)_\kappa}$$

$$\cdot\frac{C_\kappa(S)C_\kappa(R)}{C_\kappa(I)k!}\; G_p\left(\frac{X}{T}\middle| \begin{matrix} b_s+k_j-\frac{1}{2}(j-1) \\ a_r+k_j-\frac{1}{2}(j-1) \end{matrix}\right)$$

2.4 Some Theorems

We list a few theorems which incorporate the functions of sections 2.1 and 2.2.

Theorem 2.4.1 (Constantine (1963))

For R a $p \times p$ positive definite symmetric matrix and T an arbitrary $p \times p$ symmetric matrix

$$(2.4.1) \quad \int_{S>0} \operatorname{etr}(-RS)|S|^{t-\frac{1}{2}(p+1)} C_\kappa(ST)\, dS$$

$$= \Gamma_p(t,\kappa)|R|^{-t} C_\kappa(TR^{-1})$$

where integration is over the space of all positive definite $p \times p$ matrices and is valid for all $t > \frac{1}{2}(p-1)$.

The left hand side of equation (2.4.1) is essentially the Laplace transform of $|S|^{t-\frac{1}{2}(p+1)}C_\kappa(ST)$. Because the Laplace transform satisfies the convolution theorem it follows that

Theorem 2.4.2 (Constantine (1963))

For R a $p \times p$ positive definite matrix

$$(2.4.2) \quad \int_0^I |S|^{t-\frac{1}{2}(p+1)}|I-S|^{u-\frac{1}{2}(p+1)} C_\kappa(RS)\, dS$$

$$= \{\Gamma_p(t,\kappa)\Gamma_p(u)/\Gamma_p(t+u,\kappa)\} C_\kappa(R)$$

By making suitable transformations we have:

Theorem 2.4.3 (Greenacre (1972))

For R a $p \times p$ positive definite matrix

(2.4.3) $$\int_0^I |S|^{t-\frac{1}{2}(p+1)}|I-S|^{u-\frac{1}{2}(p+1)}C_\kappa(R(I-S))\,dS$$

$$= \{\Gamma_p(t)\Gamma_p(u,\kappa)/\Gamma_p(t+u,\kappa)\}C_\kappa(R)$$

(2.4.4) $$\int_{W>0}|W|^{t-\frac{1}{2}(p+1)}|I+W|^{-(t+u)}C_\kappa(R(I+W)^{-1})dW$$

$$= \{\Gamma_p(t)\Gamma_p(u,\kappa)/\Gamma_p(t+u,\kappa)\}C_\kappa(R)$$

(2.4.5) $$\int_{W>0}|W|^{t-\frac{1}{2}(p+1)}|I+W|^{-(t+u)}C_\kappa(RW(I+W)^{-1})dW$$

$$= \{\Gamma_p(t,\kappa)\Gamma_p(u)/\Gamma_p(t+u,\kappa)\}C_\kappa(R)$$

Theorems 2.4.1, 2.4.2 and 2.4.3 may be expressed in terms of hypergeometric functions as follows:

<u>Theorem 2.4.4</u> (Greenacre (1972))

(2.4.6) $$\int_{S>0}\mathrm{etr}(-RS)|S|^{t-\frac{1}{2}(p+1)}{}_mF_n(a_i;\ b_j;\ ST)\,dS$$

$$= \Gamma_p(t)|R|^{-t}{}_{m+1}F_n(a_i,t;\ b_j;\ TR^{-1})$$

(2.4.7) $$\int_0^I |S|^{t-\frac{1}{2}(p+1)}|I-S|^{u-\frac{1}{2}(p+1)}{}_mF_n(a_i;\ b_j;\ RS)\,dS$$

$$= \frac{\Gamma_p(t)\Gamma_p(u)}{\Gamma_p(t+u)}\ {}_{m+1}F_{n+1}(a_i,t;\ b_j,t+u;\ R)$$

(2.4.8) $$\int_0^I |S|^{t-\frac{1}{2}(p+1)}|I-S|^{u-\frac{1}{2}(p+1)}{}_mF_n(a_i;\ b_j;\ R(I-S))\,dS$$

$$= \frac{\Gamma_p(t)\Gamma_p(u)}{\Gamma_p(t+u)}\ {}_{m+1}F_{n+1}(a_i,u;\ b_j,t+u;\ R)$$

(2.4.9) $$\int_{W>0}|W|^{t-\frac{1}{2}(p+1)}|I+W|^{-(t+u)}{}_mF_n(a_i;\ b_j;\ R(I+W)^{-1})dW$$

$$= \frac{\Gamma_p(t)\Gamma_p(u)}{\Gamma_p(t+u)}\ {}_{m+1}F_{n+1}(a_i,u;\ b_j.t+u;\ R)$$

(2.4.10) $$\int_{W>0} |W|^{t-\frac{1}{2}(p+1)} |I+W|^{-(t+u)} {}_mF_n(a_i;\ b_j;\ RW(I+W)^{-1})dW$$

$$= \frac{\Gamma_p(t)\Gamma_p(u)}{\Gamma_p(t+u)} \ {}_{m+1}F_{n+1}(a_i,u;\ b_j,t+u;\ R)$$

These results are extended by Underhill (1973) to the case of hypergeometric functions with double matrix argument.

CHAPTER 3

SOME MULTIVARIATE DISTRIBUTIONS

This chapter summarises the important results concerning the distributions associated with the multivariate test criteria to be considered. As these are well known, proofs will be omitted, but their sources acknowledged.

3.1 Wishart distributions

Real variable

Let X be a $p \times n$ matrix variable with columns independently and normally distributed with mean vector M and covariance matrix Σ, i.e. $\sim N(M,\Sigma)$. Then $A = XX'$ has the noncentral Wishart distribution with noncentrality parameter $\Lambda = \Sigma^{-1}MM'$, i.e. $A \sim W(\Sigma,n,\Lambda)$ (Constantine (1963)), where

$$(3.1.1)\quad f(A) = (\Gamma_p(\tfrac{1}{2}n))^{-1}|2\Sigma|^{-\frac{1}{2}n}|A|^{\frac{1}{2}(n-p-1)}\operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}A)$$

$$\cdot\operatorname{etr}(-\tfrac{1}{2}\Lambda)\,{}_0F_1(\tfrac{1}{2}n;\ \tfrac{1}{4}\Lambda\Sigma^{-1}A)$$

$$= w(A;\ \Sigma,n,\Lambda) \qquad A > 0.$$

When M(and Λ) $= 0$ this reduces to the central Wishart distribution, i.e. $A \sim W(\Sigma,n)$, where

$$(3.1.2)\quad f(A) = (\Gamma_p(\tfrac{1}{2}n))^{-1}|2\Sigma|^{-\frac{1}{2}n}|A|^{\frac{1}{2}(n-p-1)}\operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}A)$$

$$= w(A;\ \Sigma,n) \qquad A > 0.$$

The "linear case" for general covariance matrix Σ (Anderson and Girshick (1944), Anderson (1946)), which holds when Λ has rank one, is given by

$$(3.1.3)\quad f(A) = w(A;\ \Sigma,n).\mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}\Lambda)/\{(\mathrm{tr}A\Lambda)^{(n-2)/4} I_{(n-2)/2}((\mathrm{tr}A\Lambda)^{\frac{1}{2}})\}$$

where $I_\kappa(z)$ is a Bessel function of purely imaginary argument.

When Λ has rank one and in addition $\Sigma = I_p$, the identity matrix, this becomes $A \sim W(I,n,\lambda^2)$, where

$$(3.1.4)\quad f(A) = (\Gamma_p(\tfrac{1}{2}n).2^{\frac{1}{2}np})^{-1}|A|^{\frac{1}{2}(n-p-1)}\mathrm{etr}(-\tfrac{1}{2}A)$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_0F_1(n/2;\ \tfrac{1}{4}\lambda^2 a_{11})$$

$$= w(A;\ I,n).e^{-\frac{1}{2}\lambda^2}\,{}_0F_1(n/2;\ \tfrac{1}{4}\lambda^2 a_{11})$$

$$= w(A;\ I,n,\lambda^2) \qquad A > 0$$

where a_{11} is the top left hand element of matrix A, and $\lambda^2 = M'M$ is the single noncentrality parameter. We will refer to this from now on as the linear case.

The planar case, where Λ is of rank two, has also been investigated by Anderson and Girshick (1944).

Complex variable

Let $Z = X+iY$ be a $p \times n$ complex variable with columns independently distributed as complex normal with mean vector M and Hermitian covariance matrix Σ, i.e. $\sim CN(M,\Sigma)$. Then $A = Z\bar{Z}'$ has the noncentral complex Wishart distribution with noncentrality parameter $\Lambda = \Sigma^{-1}M\bar{M}'$, i.e. $A \sim CW(\Sigma,n,\Lambda)$ (Goodman (1963)), where

$$(3.1.5)\quad f(A) = (\tilde{\Gamma}_p(n)|\Sigma|^n)^{-1}\mathrm{etr}(-\Sigma^{-1}A)|A|^{n-p}$$

$$\cdot\mathrm{etr}(-\Lambda)\,{}_0F_1(n;\ \Lambda\Sigma^{-1}A)$$

$$= cw(A;\ \Sigma,n,\Lambda) \qquad A = \bar{A}' > 0$$

When M (and Λ) = 0 this reduces to the central complex Wishart distribution $CW(\Sigma,n)$

where

(3.1.6) $$f(A) = (\bar{\Gamma}_p(n)|\Sigma|^n)^{-1}\operatorname{etr}(-\Sigma^{-1}A)|A|^{n-p}$$

$$= cw(A;\ \Sigma,n) \qquad\qquad A = \bar{A}' > 0.$$

3.2 Univariate Beta distributions

Central Case

Let X and Y be independent χ^2 variables with n and m degrees of freedom respectively.

Let $U = X(X+Y)^{-1}$, $\qquad V = XY^{-1}$,

$W = 1-U = Y(X+Y)^{-1}$, and $Z = V^{-1} = YX^{-1}$.

Then U has the Beta type 1 distribution with density function

(3.2.1) $$\beta_1(u;\ m,n) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}u^{\frac{1}{2}n-1}(1-u)^{\frac{1}{2}m-1} \qquad 0 \le u \le 1$$

and hth moment

(3.2.2) $$E(U^h) = \Gamma(\tfrac{1}{2}n+h)\Gamma(\tfrac{1}{2}m+n))/(\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}(m+n)+h))$$

where

(3.2.3) $B(\tfrac{1}{2}m,\tfrac{1}{2}n) = \Gamma(\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}n)/\Gamma(\tfrac{1}{2}(m+n))$ is the Beta function.

Also, V has the Beta type 2 distribution with density function

(3.2.4) $$\beta_2(v;\ m,n) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}v^{\frac{1}{2}n-1}(1+v)^{-\frac{1}{2}(m+n)} \qquad v \ge 0$$

and hth moment

(3.2.5) $$E(V^h) = \Gamma(\tfrac{1}{2}n+h)\Gamma(\tfrac{1}{2}m-h)/(\Gamma(\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}n)).$$

The density functions and moments of W and Z follow by interchanging m and n in equations (3.2.1) - (3.2.5).

Noncentral Case

Now let Y be a noncentral χ^2 variable with non-centrality parameter λ.

Then the distribution of U is defined as the noncentral Beta type 1A distribution, which has density function

$$(3.2.6) \quad \beta_{1A}(u;\, m,n,\lambda^2) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}u^{\frac{1}{2}n-1}(1-u)^{\frac{1}{2}m-1} \cdot e^{-\frac{1}{2}\lambda^2}{}_1F_1(\tfrac{1}{2}(m+n);\, \tfrac{1}{2}m;\, \tfrac{1}{2}\lambda^2(1-u)) \qquad 0 \leq u \leq 1$$

and hth moment

$$(3.2.7) \quad E(U^h) = \Gamma(\tfrac{1}{2}n+h)\Gamma(\tfrac{1}{2}(m+n))/(\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}(m+n)+h)) \cdot e^{-\frac{1}{2}\lambda^2}{}_1F_1(\tfrac{1}{2}(m+n);\, \tfrac{1}{2}(m+n)+h;\, \tfrac{1}{2}\lambda^2).$$

V is defined as having the noncentral Beta type 2A distribution with density function

$$(3.2.8) \quad \beta_{2A}(v;\, m,n,\lambda^2) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}v^{\frac{1}{2}n-1}(1+v)^{-\frac{1}{2}(m+n)} \cdot e^{-\frac{1}{2}\lambda^2}{}_1F_1(\tfrac{1}{2}(m+n);\, \tfrac{1}{2}m;\, \tfrac{1}{2}\lambda^2(1+v)^{-1}) \qquad v \geq 0$$

and hth moment

$$(3.2.9) \quad E(V^h) = \Gamma(\tfrac{1}{2}n+h)\Gamma(\tfrac{1}{2}m-h)/(\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}m)) \cdot e^{-\frac{1}{2}\lambda^2}{}_1F_1(\tfrac{1}{2}m-h;\, \tfrac{1}{2}m;\, \tfrac{1}{2}\lambda^2).$$

It can be seen that a noncentral χ^2 distribution is involved in both the numerator and denominator of $W = Y(X+Y)^{-1}$, which we define as the noncentral Beta type 1B distribution with density function

$$(3.2.10)\quad \beta_{1B}(w;\ m,n,\lambda^2) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}w^{\frac{1}{2}m-1}(1-w)^{\frac{1}{2}n-1}$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2 w) \qquad 0 \leq w \leq 1$$

and hth moment

$$(3.2.11)\quad E(W^h) = \Gamma(\tfrac{1}{2}m+h)\Gamma(\tfrac{1}{2}(m+n))/(\Gamma(\tfrac{1}{2}m)\Gamma(\tfrac{1}{2}(m+n)+h))$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_2F_2(\tfrac{1}{2}(m+n),\tfrac{1}{2}m+h;\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2).$$

$Z = V^{-1} = Y/X$ has a noncentral χ^2 in the numerator and has the noncentral Beta type 2B distribution with density function

$$(3.2.12)\quad \beta_{2B}(z;\ m,n,\lambda^2) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}z^{\frac{1}{2}m-1}(1+z)^{-\frac{1}{2}(m+n)}$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2 z(1+z)^{-1}) \qquad z > 0$$

and hth moment

$$(3.2.13)\quad E(Z^h) = \Gamma(\tfrac{1}{2}m+h)\Gamma(\tfrac{1}{2}n-h)/(\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2}m))$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}m+h;\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2).$$

We note also that $U = V(1+V)^{-1} = (1+Z)^{-1}$
and that $W = (1+V)^{-1} = Z(1+Z)^{-1}$.

3.3 Multivariate Beta type 1 distributions

Real Case

The results of section 3.2 are now extended to the multivariate Beta type 1A and 1B distributions as follows.

Theorem 3.3.1 (Constantine (1963), de Waal (1968))

Let $A \sim W(\Sigma,n)$ and $B \sim W(\Sigma,m,\Lambda)$ independently.
Let $L = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$ and $M = I-L = (A+B)^{-\frac{1}{2}}B(A+B)^{-\frac{1}{2}}$.
Then

(a) L has the noncentral multivariate Beta type 1A distribution with density function

(3.3.1)
$$M\beta_{1A}(L;\ \Sigma,m,n,\Lambda) = (\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}(m+n)})^{-1}$$
$$\cdot\mathrm{etr}(-\tfrac{1}{2}\Lambda)|L|^{\frac{1}{2}(n-p-1)}|I-L|^{\frac{1}{2}(m-p-1)}\int_{T>0}\mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}T)$$
$$\cdot|T|^{\frac{1}{2}(m+n-p-1)}{}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}\Lambda\Sigma^{-1}T^{\frac{1}{2}}(I-L)T^{\frac{1}{2}})dT \qquad 0 < L < I$$

(b) $M = I-L$ has the noncentral multivariate Beta type 1B distribution with density function

(3.3.2)
$$M\beta_{1B}(M;\ \Sigma,m,n,\Lambda) = (\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}(m+n)})^{-1}$$
$$\cdot\mathrm{etr}(-\tfrac{1}{2}\Lambda)|M|^{\frac{1}{2}(m-p-1)}|I-M|^{\frac{1}{2}(n-p-1)}\int_{T>0}\mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}T)$$
$$\cdot|T|^{\frac{1}{2}(m+n-p-1)}{}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}\Lambda\Sigma^{-1}T^{\frac{1}{2}}MT^{\frac{1}{2}})dT \qquad 0 < M < I$$

(c)
(3.3.3)
$$E|L|^h = \Gamma_p(\tfrac{1}{2}n+h)\Gamma_p(\tfrac{1}{2}(m+n))/(\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}(m+n)+h))$$
$$\cdot\mathrm{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\Lambda)$$

and

(d)
(3.3.4)
$$E|I-L|^h = E|M|^h$$
$$= \Gamma_p(\tfrac{1}{2}m+h)\Gamma_p(\tfrac{1}{2}(m+n))/(\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h))$$
$$\cdot\mathrm{etr}(-\tfrac{1}{2}\Lambda)\,{}_2F_2(\tfrac{1}{2}(m+n),\tfrac{1}{2}m+h;\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\Lambda).$$

The above general results have so far proved intractable for computational purposes, and we will therefore examine more closely the "linear case", where $A \sim W(I,n)$ and $B \sim W(I,m,\lambda^2)$.

It should be noted that any matrices $A^* \sim W(\Sigma,n)$ and $B^* \sim W(\Sigma,m,\theta)$ can be expressed in canonical form by transforming by a suitably chosen nonsingular matrix F, so that $F\Sigma F' = I$ and $FMM'F = \Lambda$ (diagonal matrix). Then $A = FA^*F' \sim W(I,n)$ and $B = FB^*F' \sim W(I,m,\Lambda)$. Although the distribution of L will not be invariant under this transformation, that of $|L|$ will, as

$$(3.3.5)\quad \begin{aligned} |L| &= |(A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}| = |A|/|A+B| \\ &= |FA^*F'|/|FA^*F'+FB^*F'| \\ &= |F||A^*||F'|/|F||A^*+B^*||F'| \\ &= |A^*|/|A^*+B^*|. \end{aligned}$$

The various tests of statistical hypotheses based on determinants of Beta distributions will therefore be invariant with respect to such transformations, and the assumption of covariance matrix I does not restrict the generalisation. The assumption that Λ has rank one is obviously more severe, but results for this case may possibly provide bounds for a more general case.

Theorem 3.3.2 (Kshirsagar (1961), Troskie (1966))

Let $A \sim W(I,n)$ and $B \sim W(I,m,\lambda^2)$ independently. Let $L = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$ and $M = I-L = (A+B)^{-\frac{1}{2}}B(A+B)^{-\frac{1}{2}}$. Then

(a) L has the noncentral multivariate Beta type 1A distribution (linear case) with density function

$$(3.3.6)\quad M\beta_{1A}(L;\ I,m,n,\lambda^2) = (B_p(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}|L|^{\frac{1}{2}(n-p-1)} \cdot |I-L|^{\frac{1}{2}(m-p-1)}e^{\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2(1-\ell_{11}))$$

$$0 < L < I$$

where

(3.3.7) $B_p(\frac{1}{2}m,\frac{1}{2}n) = \Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}n)/\Gamma_p(\frac{1}{2}(m+n))$

is the multivariate Beta function and ℓ_{11} is the top left hand element of matrix L.

(b) $M = I-L$ has the noncentral multivariate Beta type 1B distribution (linear case) with density function

(3.3.8) $$M\beta_{1B}(M;I,m,n,\lambda^2) = (B_p(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}|M|^{\frac{1}{2}(m-p-1)}$$

$$\cdot|I-M|^{\frac{1}{2}(n-p-1)}e^{-\frac{1}{2}\lambda^2}{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2 m_{11})$$

$$0 < M < I$$

(c) $E|L|^h = \Gamma_p(\frac{1}{2}n+h)\Gamma_p(\frac{1}{2}(m+n))/(\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}(m+n)+h))$

(3.3.9) $$\cdot e^{-\frac{1}{2}\lambda^2}{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2)$$

and

(d) $E|M|^h = E|I-L|^h$

(3.3.10) $$= (\Gamma_p(\tfrac{1}{2}m+h)\Gamma_p(\tfrac{1}{2}(m+n))/(\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h)))$$

$$\cdot e^{-\frac{1}{2}\lambda^2}{}_2F_2(\tfrac{1}{2}(m+n),\tfrac{1}{2}m+h;\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2).$$

The earlier derived results for the central case follow.

<u>Corollary 3.3.1</u> (Wilks (1932), Anderson (1958))

Let $A \sim W(\Sigma,n)$ and $B \sim W(\Sigma,m)$ independently and let $L = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$. Then

(a) L has the (central) multivariate Beta type 1 distribution with density function

(3.3.11) $$M\beta_1(L;\ \Sigma,m,n) = (B_p(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}|L|^{\frac{1}{2}(n-p-1)}|I-L|^{\frac{1}{2}(m-p-1)}$$

$$0 < L < I$$

(3.3.12)(b) $E|L|^h = \Gamma_p(\frac{1}{2}n+h)\Gamma_p(\frac{1}{2}(m+n))/(\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}(m+n)+h))$

and

(3.3.13)(c) $E|I-L|^h = \Gamma_p(\frac{1}{2}m+h)\Gamma_p(\frac{1}{2}(m+n))/(\Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}(m+n)+h))$,

and (3.3.11), (3.3.12) and (3.3.13) are independent of Σ.

We now define a third noncentral multivariate Beta type 1 distribution, for both the general and linear noncentral cases.

<u>Theorem 3.3.3</u> (de Waal (1968), Pillai, Al-Ani and Jouris (1969))

Let $A \sim W(\Sigma_1,n)$ and $B \sim W(\Sigma_2,m)$ independently and let $L = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$. Then

(a) L has the noncentral multivariate Beta type 1C distribution with density function

(3.3.14) $$M\beta_{1C}(L;\ \Sigma_1,\Sigma_2,m,n) = (\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}n)|2\Sigma_1|^{\frac{1}{2}n}|2\Sigma_2|^{\frac{1}{2}n})^{-1}$$

$$\cdot|L|^{\frac{1}{2}(n-p-1)}|I-L|^{\frac{1}{2}(m-p-1)}\int_{T>0} \operatorname{etr}(-\tfrac{1}{2}\Sigma_2^{-1}T)$$

$$\cdot|T|^{\frac{1}{2}(m+n-p-1)}{}_0F_0(\tfrac{1}{2}T^{\frac{1}{2}}(\Sigma_1^{-1}-\Sigma_2^{-1})T^{\frac{1}{2}}L)dT \qquad 0 < L < I$$

with moments of $|L|$ and $|I-L|$ given by

(3.3.15) (b) $$E|L|^h = \Gamma_p(\tfrac{1}{2}(m+n))\Gamma_p(\tfrac{1}{2}n+h)/(\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}(m+n)+h))$$

$$\cdot|\Sigma_1^{-1}\Sigma_2|^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}(m+n),\tfrac{1}{2}n+h;\ \tfrac{1}{2}(m+n)+h;\ I-\Sigma_1^{-1}\Sigma_2)$$

and

(3.3.16) (c) $$E|I-L|^h = \Gamma_p(\tfrac{1}{2}(m+n))\Gamma_p(\tfrac{1}{2}m+h)/(\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h))$$

$$\cdot|\Sigma_1^{-1}\Sigma_2|^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}(m+n);\ \tfrac{1}{2}(m+n)+h;\ I-\Sigma_1^{-1}\Sigma_2).$$

The noncentral linear case holds when $(I-\Sigma_1^{-1}\Sigma_2)$ has only one nonzero root, $(1-\lambda^2)$, and the moments of $|L|$ and $|I-L|$ then become

(d) $E|L|^h = \Gamma_p(\frac{1}{2}m+n))\Gamma_p(\frac{1}{2}n+h)/(\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}(m+n)+h))$

(3.3.17)

$$\cdot\lambda^n {}_2F_1(\tfrac{1}{2}(m+n),\tfrac{1}{2}n+h;\ \tfrac{1}{2}(m+n)+h;\ (1-\lambda^2))$$

and

(e) $E|I-L|^h = \Gamma_p(\frac{1}{2}(m+n))\Gamma_p(\frac{1}{2}m+h)/(\Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}(m+n)+h))$

(3.3.18)

$$\cdot\lambda^n {}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}(m+n);\ \tfrac{1}{2}(m+n)+h;\ (1-\lambda^2)).$$

(f) The central case holds when $\Sigma_1 = \Sigma_2$.

Note that (3.3.17) may also be written as

(3.3.19) $E|L|^h = \Gamma_p(\frac{1}{2}(m+n))\Gamma_p(\frac{1}{2}n+h)/(\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}(m+n)+h))$

$$\cdot\lambda^{-m} {}_2F_1(\tfrac{1}{2}(m+n),\ \tfrac{1}{2}m;\ \tfrac{1}{2}(m+n)+h;\ 1-\lambda^{-2})$$

by transposing m with n and λ with λ^{-1} in (3.3.18), or by applying the Kummer transformation formula (Erdelyi (1953))

(3.3.20) $${}_2F_1(a,b;\ c;\ z) = (1-z)^{-a} {}_2F_1(a,c-b;\ c;\ z/(z-1))$$

to (3.3.17).

Complex Case

We now consider variables Z which have a complex normal distribution. From section 3.1 we know that $Z\overline{Z}'$ obeys the complex Wishart distribution given by (3.1.5). If A and B are complex Wishart matrices then

$L = (A+B)^{-\frac{1}{2}}\overline{A(A+B)^{-\frac{1}{2}}}'$, which we will write as $(A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$, is Hermitian positive definite, i.e. $L = \overline{L}'$.

Theorem 3.3.4 (Khatri (1965), de Waal (1968))

Let $A \sim CW(\Sigma,n)$ and $B \sim CW(\Sigma,m,\Lambda)$ independently. Let $L = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$. Then

(a) L has the noncentral multivariate complex Beta type 1A distribution with density function

(3.3.21)
$$CM\beta_{1A}(L;\ \Sigma,m,n,\Lambda) = (\overline{\Gamma}_p(m)\overline{\Gamma}_p(n)|\Sigma|^{m+n})^{-1}etr(-\Lambda)$$
$$\cdot|L|^{n-p}|I-L|^{m-p}\int_{T=\overline{T}'>0} etr(-\Sigma^{-1}T)|T|^{m+n-p}$$
$$\cdot{}_0\overline{F}_1(m;\ \Lambda\Sigma^{-1}T^{\frac{1}{2}}(I-L)T^{\frac{1}{2}})dT \qquad 0 < L < I\ ,$$

(b)
(3.3.22)
$$E|L|^h = \overline{\Gamma}_p(n+h)\overline{\Gamma}_p(m+n)/(\overline{\Gamma}_p(n)\overline{\Gamma}_p(m+n+h))etr(-\Lambda)$$
$$\cdot{}_1\overline{F}_1(m+n;\ m+n+h;\ \Lambda)$$

and

(c)
(3.3.23)
$$E|I-L|^h = \overline{\Gamma}_p(m+h)\overline{\Gamma}_p(m+n)/(\overline{\Gamma}_p(m)\overline{\Gamma}_p(m+n+h))$$
$$\cdot etr(-\Lambda){}_2\overline{F}_2(m+n,m+h;\ m+n+h,m;\ \Lambda).$$

The type 1B density $CM\beta_{1B}(M;\ \Sigma,m,n,\Lambda)$ follows by setting $M = I-L$ in (3.3.21).

We shall be mainly concerned with the moments of $|L|$ and $|I-L|$. These follow in the noncentral linear case by setting $\Lambda = \lambda^2$, and in the central case by setting $\Lambda = 0$ in (3.3.22) and (3.3.23). It can be seen that the moments for the complex case follow from those of the real case if m,n and Λ replace $\frac{1}{2}m$, $\frac{1}{2}n$ and $\frac{1}{2}\Lambda$, and the real gamma and hypergeometric functions are replaced by complex ones.

Theorem 3.3.5 (de Waal (1968))

Let $A \sim CW(\Sigma_1,n)$ and $B \sim CW(\Sigma_2,m)$ independently, and let $L = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$. Then

(a) L has the noncentral multivariate complex Beta type 1C distribution with density function

(3.3.24) $$CM\beta_{1C}(L;\ \Sigma_1,\Sigma_2,m,n) = (\bar{\Gamma}_p(m)\bar{\Gamma}_p(n)|\Sigma_1|^n|\Sigma_2|^m)^{-1}$$

$$\cdot|L|^{n-p}|I-L|^{m-p}\int_{T>0} etr(-\Sigma_2^{-1}T)$$

$$\cdot|T|^{m+n-p}\,{}_0\bar{F}_0(-T^{\frac{1}{2}}(\Sigma_1^{-1}-\Sigma_2^{-1})T^{\frac{1}{2}}L)dT, \qquad 0 < L < I,$$

with moments of $|L|$ and $|I-L|$ given by

(b) $$E|L|^h = \bar{\Gamma}_p(m+n)\bar{\Gamma}_p(n+h)/(\bar{\Gamma}_p(n)\bar{\Gamma}_p(m+n+h))|\Sigma_1^{-1}\Sigma_2|^n$$

(3.3.25) $$\cdot{}_2\bar{F}_1(m+n,n+h;\ m+n+h;\ I-\Sigma_1^{-1}\Sigma_2)$$

and

(c) $$E|I-L|^h = \bar{\Gamma}_p(m+n)\bar{\Gamma}_p(m+h)/(\bar{\Gamma}_p(m)\bar{\Gamma}_p(m+n+h))|\Sigma_1^{-1}\Sigma_2|^n$$

(3.3.26) $$\cdot{}_2\bar{F}_1(n,m+n;\ m+n+h;\ I-\Sigma_1^{-1}\Sigma_2).$$

For the noncentral linear case the moments are

(d) $$E|L|^h = \bar{\Gamma}_p(m+n)\bar{\Gamma}_p(n+h)/(\bar{\Gamma}_p(n)\bar{\Gamma}_p(m+n+h))\lambda^{2n}$$

(3.3.27) $$\cdot{}_2\bar{F}_1(m+n,n+h;\ m+n+h;\ 1-\lambda^2)$$

(3.3.28) $$= \bar{\Gamma}_p(m+n)\bar{\Gamma}_p(n+h)/(\bar{\Gamma}_p(n)\bar{\Gamma}_p(m+n+h))\lambda^{-2m}$$

$$\cdot{}_2\bar{F}_1(m+n,m;\ m+n+h;\ 1-\lambda^{-2})$$

and

(e) $$E|I-L|^h = \overline{\Gamma}_p(m+n)\overline{\Gamma}_p(m+h)/(\overline{\Gamma}_p(m)\overline{\Gamma}_p(m+n+h))\lambda^{2n}$$

(3.3.29)

$$\cdot {}_2\overline{F}_1(n,m+n;\ m+n+h;\ 1-\lambda^2).$$

(f) The central case holds when $\Sigma_1 = \Sigma_2$.

3.4 Multivariate Beta type 2 distributions

Real Case

We now extend the univariate results of section 3.2 to two distinct multivariate distributions.

Theorem 3.4.1 (de Waal (1968), (1969))

Let $A \sim W(\Sigma_1,n)$ and $B \sim W(\Sigma_2,m,\Lambda)$ independently. Let $V = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $Z = V^{-1} = B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$. Then

(a) V has the noncentral multivariate Beta type 2A distribution with density function

(3.4.1) $$M\beta_{2A}(V;\ \Sigma_1,\Sigma_2,m,n,\Lambda) = (\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}n)|2\Sigma_1|^{\frac{1}{2}n}|2\Sigma_2|^{\frac{1}{2}m})^{-1}$$

$$\cdot \mathrm{etr}(-\tfrac{1}{2}\Lambda)|V|^{\frac{1}{2}(n-p-1)}\int_{B>0}|B|^{\frac{1}{2}(m+n-p-1)}\mathrm{etr}(-\tfrac{1}{2}\Sigma_2^{-1}B)$$

$$\cdot \mathrm{etr}(-\tfrac{1}{2}B^{\frac{1}{2}}\Sigma_1^{-1}B^{\frac{1}{2}}V)\,{}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}\Lambda\Sigma_2^{-1}B)dB \qquad V > 0,$$

and (b) $|V|$ has hth moment

(3.4.2) $$E|V|^h = \Gamma_p(\tfrac{1}{2}n+h)\Gamma_p(\tfrac{1}{2}m-h)/(\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}n))$$

$$\cdot|\Sigma_1\Sigma_2^{-1}|^h\mathrm{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}m-h;\ \tfrac{1}{2}m;\ \tfrac{1}{2}\Lambda).$$

(c) If $\Sigma_1 = \Sigma_2 = I$

(3.4.3) $$M\beta_{2A}(V;\ I,m,n,\Lambda) = (B_p(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}|V|^{\frac{1}{2}(n-p-1)}$$

$$\cdot|I+V|^{-\frac{1}{2}(m+n)}\mathrm{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\Lambda(I+V)^{-1})$$

$$V > 0$$

(d) Z has the noncentral multivariate Beta type 2B distribution with density function

(3.4.4) $$M\beta_{2B}(Z;\ \Sigma_1,\Sigma_2,m,n,\Lambda) = (\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}m)|2\Sigma_1|^{\frac{1}{2}n}|2\Sigma_2|^{\frac{1}{2}m})^{-1}$$

$$\cdot\mathrm{etr}(-\tfrac{1}{2}\Lambda)|Z|^{-\frac{1}{2}(n+p+1)}\int_{B>0}|B|^{\frac{1}{2}(m+n-p-1)}\mathrm{etr}(-\tfrac{1}{2}\Sigma_2^{-1}B)$$

$$\cdot\mathrm{etr}(-\tfrac{1}{2}B^{\frac{1}{2}}\Sigma_1^{-1}B^{\frac{1}{2}}Z^{-1})\,{}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}\Lambda\Sigma_2^{-1}B)dB \qquad Z > 0$$

and (e) $|Z|$ has hth moment

(3.4.5) $$E|Z|^h = \Gamma_p(\tfrac{1}{2}m+h)\Gamma_p(\tfrac{1}{2}n-h)/(\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}n))$$

$$\cdot|\Sigma_2\Sigma_1^{-1}|^h\mathrm{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}m+h;\ \tfrac{1}{2}m;\ \tfrac{1}{2}\Lambda).$$

(f) If $\Sigma_1 = \Sigma_2 = I$

(3.4.6) $$M\beta_{2B}(Z;\ I,m,n,\Lambda) = (B_p(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}|Z|^{\frac{1}{2}(m-p-1)}$$

$$\cdot|I+Z|^{-\frac{1}{2}(m+n)}\mathrm{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m;\ \tfrac{1}{2}\Lambda Z(I+Z)^{-1})$$

$$Z > 0$$

(g) If $\Sigma_1 = \Sigma_2 = \Sigma$

$$E|(I+Z)^{-1}|^h = E|V(I+V)^{-1}|^h = E|L|^h \quad \text{(equation (3.3.3))}$$

$$E|Z(I+Z)^{-1}|^h = E|(I+V)^{-1}|^h = E|I-L|^h \quad \text{(equation (3.3.4))}$$

<u>Theorem 3.4.2</u> (Troskie (1966))

Let $A \sim W(I,n)$ and $B \sim W(I,m,\lambda^2)$ independently. Let $V = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $Z = B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$. Then the moments of $|V|$ and $|Z|$ are given by

(a) $$E|V|^h = \Gamma_p(\tfrac{1}{2}n+h)\Gamma_p(\tfrac{1}{2}m-h)/(\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}m))$$

(3.4.7) $$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}m-h;\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2)$$

and (b) $E|Z|^h = \Gamma_p(\frac{1}{2}m+h)\Gamma_p(\frac{1}{2}n-h)/(\Gamma_p(\frac{1}{2}m)\Gamma_p(\frac{1}{2}n))$

(3.4.8) $\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\frac{1}{2}m+h;\ \frac{1}{2}m;\ \frac{1}{2}\lambda^2)$

and (c) the densities of V and Z for the noncentral linear case follow similarly from equations (3.4.3) and (3.4.6).

Complex Case

If A and B are complex Wishart matrices the moments of $|V|$ and $|Z|$ may be derived from (3.4.2) and (3.4.5) by changing $\frac{1}{2}m$, $\frac{1}{2}n$ and $\frac{1}{2}\Lambda$ to m,n and Λ respectively and using complex gamma and hypergeometric functions. The densities of V and Z may be found in de Waal (1968).

3,5 Multivariate Dirichlet distributions

Real Case

Theorem 3.5.1 (de Waal (1968), Troskie (1972))

Let $A_j \sim W(\Sigma,n_j)$, $j = 1,\ldots,k$ and $B \sim W(\Sigma,m,\Lambda)$, where A_j, $j = 1,\ldots,k$ and B are independent.

Let L_j be defined by

(3.5.1) $L_j = C^{-1}A_jC'^{-1}$, $j = 1,\ldots,k$

where

(3.5.2) $\sum_{j=1}^{k}A_j+B = CC'$

and C is a lower triangular matrix.

Also let $V_j = B^{-\frac{1}{2}}A_jB^{-\frac{1}{2}}$ $j = 1,\ldots,k$. Then

(a) the joint distribution of $L_1,\ldots,L_k$ is the noncentral multivariate Dirichlet type 1 distribution $MD_1(\Sigma;\ m;\ n_1,\ldots,n_k;\ \Lambda)$ with density function

$$(3.5.3)\qquad (\Gamma_p(\tfrac{1}{2}m)\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j))^{-1}|2\Sigma|^{-\frac{1}{2}(m+n)}\mathrm{etr}(-\tfrac{1}{2}\Lambda)$$

$$\cdot\Pi_{j=1}^{k}|L_j|^{\frac{1}{2}(n_j-p-1)}|I-\textstyle\sum_{j=1}^{k}L_j|^{\frac{1}{2}(m-p-1)}\displaystyle\int_{T>0}\mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}T)$$

$$\cdot|T|^{\frac{1}{2}(m+n-p-1)}{}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}\Lambda\Sigma^{-1}T^{\frac{1}{2}}(I-\textstyle\sum_{j=1}^{k}L_j)T^{\frac{1}{2}})dT\ .$$

$$n = \textstyle\sum_{j=1}^{k}n_j, (I-\sum_{j=1}^{k}L_j) > 0, \qquad 0 < L_j < I.$$

(b) The joint distribution of $V_1,\ldots,V_k$ is the noncentral multivariate Dirichlet type 2 distribution $MD_2(\Sigma;\ m;\ n_1,\ldots,n_k;\ \Lambda)$

with density function

$$(3.5.4)\qquad (\Gamma_p(\tfrac{1}{2}m)\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j))^{-1}|2\Sigma|^{-\frac{1}{2}(m+n)}\mathrm{etr}(-\tfrac{1}{2}\Lambda)$$

$$\cdot\Pi_{j=1}^{k}|V_j|^{\frac{1}{2}(n_j-p-1)}|I+\textstyle\sum_{j=1}^{k}V_j|^{-\frac{1}{2}(m+n)}\displaystyle\int_{T>0}\mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}T)$$

$$\cdot|T|^{\frac{1}{2}(m+n-p-1)}{}_0F_1(\tfrac{1}{2}m;\ \tfrac{1}{4}\Lambda\Sigma^{-1}T^{\frac{1}{2}}(I+\textstyle\sum_{j=1}^{k}V_j)^{-1}T^{\frac{1}{2}})dT,$$

$$n = \textstyle\sum_{j=1}^{k}n_j, \qquad V_j > 0.$$

The above integrals may be simplified when $\Sigma = I$ (de Waal (1968)), when B is a central Wishart matrix (Olkin and Rubin (1964)), or when Λ is of rank one (Troskie (1966), (1967)). When $k = 1$, (3.5.1) and (3.5,2) reduce to the multivariate Beta type 1A and type 2A densities respectively, and when $p = 1$, (3.5.1) becomes the univariate Dirichlet distribution (Wilks (1962)).

Theorem 3.5.2 (Troskie (1966), (1967), (1972))

Let $A_j \sim W(I,n_j)$, $j = 1,\ldots,k$ and $B \sim W(I,m,\lambda^2)$ where A_j, $j = 1,\ldots,k$ and B are independent. Let L_j and V_j be defined as in Theorem 3.5.1. Then

(a) the joint distribution of $L_1,\ldots,L_k$ is the noncentral multivariate Dirichlet type 1 distribution (linear case) $MD_1(I;\, m;\, n_1,\ldots,n_k;\, \lambda^2)$ with density function

$$(\Gamma_p(\tfrac{1}{2}(m+n))/(\Gamma_p(\tfrac{1}{2}m)\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j)))\Pi_{j=1}^{k}|L_j|^{\frac{1}{2}(n_j-p-1)} \tag{3.5.5}$$

$$\cdot|I-\textstyle\sum_{j=1}^{k}L_j|^{\frac{1}{2}(m-p-1)}.e^{-\frac{1}{2}\lambda^2}$$

$$\cdot{}_1F_1(\tfrac{1}{2}(m+n);\, \tfrac{1}{2}m;\, \tfrac{1}{2}\lambda^2(1-\textstyle\sum_{j=1}^{k}\ell_{j11})),$$

$$L_j > 0, \quad I-\textstyle\sum_{j=1}^{k}L_j > 0$$

where $n = \sum_{j=1}^{k}n_j$ and ℓ_{j11} is the top left hand element of matrix L_j.

(b) The joint distribution of $V_1,\ldots,V_k$ is the noncentral multivariate Dirichlet type 2 distribution (linear case) $MD_2(I;\, m;\, n_1,\ldots,n_k;\, \lambda^2)$ with density function

$$(\Gamma_p(\tfrac{1}{2}(m+n))/(\Gamma_p(\tfrac{1}{2}m)\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j)))\Pi_{j=1}^{k}|V_j|^{\frac{1}{2}(n_j-p-1)} \tag{3.5.6}$$

$$\cdot|I+\textstyle\sum_{j=1}^{k}V_j|^{-\frac{1}{2}(m+n)}.e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}(m+n);\, \tfrac{1}{2}m;\, \tfrac{1}{2}\lambda^2\omega^{11}),$$

$$V_j > 0$$

where $\Omega = I+\sum_{j=1}^{k}V_j$ and ω^{11} is the top left hand element of Ω^{-1}.

(c) $E\left\{\Pi_{j=1}^{k}|L_j|^{h_j}\right\}$

$$= \Gamma_p(\tfrac{1}{2}(m+n))\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j+h_j)/(\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j)\Gamma_p(\tfrac{1}{2}(m+n)+h)) \tag{3.5.7}$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}(m+n);\ \tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2),$$

where $n = \sum_{j=1}^{k}n_j$, $h = \sum_{j=1}^{k}h_j$ and $h_j \geq -\frac{1}{2}(n_j-p)$.

(d) $E\left\{\Pi_{j=1}^{k}|V_j|^{h_j}\right\}$

$$= \Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j+h_j)\Gamma_p(\tfrac{1}{2}m-h)/(\Pi_{j=1}^{k}\Gamma_p(\tfrac{1}{2}n_j)\Gamma_p(\tfrac{1}{2}m)) \tag{3.5.8}$$

$$\cdot e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\tfrac{1}{2}m-h;\ \tfrac{1}{2}m;\ \tfrac{1}{2}\lambda^2).$$

(e) $E|I-\sum_{j=1}^{k}L_j|^h = (\Gamma_p(\tfrac{1}{2}(m+n))\Gamma_p(\tfrac{1}{2}m+h)/$

(3.5.9)

$$\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h))e^{-\frac{1}{2}\lambda^2}$$

$$\cdot\ {}_2F_2(\tfrac{1}{2}m+h,\tfrac{1}{2}(m+n);\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2)$$

$$= E|I-L|^h \qquad \text{(equation (3.3.10))}.$$

Note that $V_j = (I-\Sigma L_j)^{-\frac{1}{2}}L_j(I-\Sigma L_j)^{-\frac{1}{2}}$

and that $(I+\Sigma V_j) = (I-\Sigma L_j)^{-1}$.

Complex Case

If A_j, $j = 1,\ldots,k$ and B are complex Wishart matrices the moments of $\Pi_j|L_j|$, $\Pi_j|V_j|$ and $|I-\Sigma_jL_j|$ may be derived from (3.5.7), (3.5.8) and (3.5.9) by changing $\frac{1}{2}m$, $\frac{1}{2}n$, $\frac{1}{2}n_j$ and $\frac{1}{2}\lambda^2$ to m,n, n_j and λ^2 respectively and using complex gamma and hypergeometric functions. The joint densities of the L_j and the V_j may be found in Troskie (1967).

CHAPTER 4

MULTIVARIATE HYPOTHESES AND TEST CRITERIA

4.1 Introduction

Numerous test criteria have been proposed to test a variety of statistical hypotheses. They were at first confined to univariate samples and populations, but were later extended to the multivariate case as knowledge of the appropriate multivariate distributions increased. Important papers in this regard were those of Wishart (1928), Fisher (1928), Hotelling (1931) and Wilks (1932).

One of the most powerful and widely used test criteria is the likelihood ratio criterion, developed by Neyman and Pearson (1928). The multivariate test criteria that we shall examine will all be directly or indirectly based on the Neyman-Pearson principle and will all involve samples taken from a multivariate normal population. We shall examine the distributions of the test criteria under the null and alternative hypotheses and shall be concerned with the associated percentiles and powers.

One of the major problems of statistical hypothesis testing is the evaluation of the percentiles of the distributions, particularly under alternative (non-null) hypotheses. The advent in multivariate analysis of functions such as the zonal polynomial, G-function and hypergeometric function, has meant that many distributions previously only expressed in integral form can be explicitly derived.

Unfortunately this improvement in mathematical elegance has not aided the computational aspects of hypothesis testing, and the numerical evaluation of percentage points and powers still presents enormous problems.

Mathai (1973) lists the main techniques used in deriving exact null and non-null distributions of multivariate test criteria as the methods of direct integration, characteristic functions, convolutions, differential equations, calculus of residues, inverse Mellin transforms, and partial fractions. As regards computation, some of these methods are currently successful only for the null distributions, while others provide results for a restricted class of parameter values.

In this chapter we shall see how the distributions of a number of important test criteria may be found by expressing them in terms of a product of independent Beta distributions. In the chapters that follow we shall discuss how percentiles and powers of each test criterion may be derived, and compare results with those obtained by other methods.

4.2 Testing linear hypotheses about regression coefficients

Consider the linear relationship between N predetermined $(q\times 1)$ variables $X_{(\alpha)}$, $(\alpha = 1,2,\ldots,N)$, and N dependent $(p\times 1)$ variables $Y_{(\alpha)}$, $(\alpha = 1,2,\ldots,N)$, which are normally distributed and depend on the $X_{(\alpha)}$. We may express this relationship as

(4.2.1) $\underset{p\times N}{Y} = \underset{p\times q}{B}\ \underset{q\times N}{X} + \underset{p\times N}{E}$ where $E \sim N(0,\Sigma)$.

If we partition $\underset{p\times q}{B}$ into $(\underset{p\times r}{B_1}\quad \underset{p\times(q-r)}{B_2})$

we may test the null hypothesis

(4.2.2) $H_0 : B_2 = B_2^*$

against the alternative hypothesis

(4.2.3) $H_1 : B_2 \neq B_2^*$.

Let X be similarly partitioned as $\underset{q\times N}{X} = \begin{array}{c} \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix} \end{array} \begin{array}{l} r\times N \\ (q-r)\times N \end{array}$.

Then the special case of (4.2.2) where $\beta_2^* = 0$ is the hypothesis that $X^{(2)}$ does not add significantly to the regression of Y on X.

The likelihood function is given by

(4.2.4) $L(B,\Sigma) = (2\pi)^{-\frac{1}{2}pN}|\Sigma|^{-\frac{1}{2}N}\mathrm{etr}(-\frac{1}{2}\Sigma^{-1}(Y-BX)(Y-BX)')$

and if we let $T = Y-B_2^*X^{(2)}$

then the likelihood ratio criterion of the hypothesis is λ where Wilks' (1932) likelihood criterion

(4.2.5) $U = \lambda^{2/N} = |TAT'|/|TAT'+T(A_1-A)T'|$,

(4.2.6) $A = I-X'(XX')^{-1}X$

and

(4.2.7) $A_1 = I-X^{(1)'}(X^{(1)}X^{(1)'})^{-1}X^{(1)}$.

As A and (A_1-A) are idempotent it follows that

(4.2.7) $TAT' \sim W(\Sigma,N-q)$

and

(4.2.8) $T(A_1-A)T' \sim W(\Sigma,q-r,\Lambda)$

where

(4.2.9) $\Lambda = \Sigma^{-1}(B_2-B_2^*)(S_{22}-S_{21}S_{11}^{-1}S_{12})(B_2-B_2^*)'$

and $S_{ij} = X^{(i)}X^{(j)'}$.

We see that the distribution of the criterion will be central (i.e. $\Lambda = 0$) iff $B_2 = B_2^*$ (as $(S_{22}-S_{21}S_{11}^{-1}S_{12})$ is positive definite). Now, in Theorem 3.3.1, set $m = q-r$ and $n = N-q$. Then Wilks' likelihood criterion $U = |L|$ where $L \sim M\beta_{1A}(\Sigma, q-r, N-q, \Lambda)$. Also,

$$E(U^h) = E|L|^h$$

$$= \Gamma_p(\tfrac{1}{2}(N-q)+h)\Gamma_p(\tfrac{1}{2}(N-r))/(\Gamma_p(\tfrac{1}{2}(N-q))\Gamma_p(\tfrac{1}{2}(N-r)+h)) \tag{4.2.10}$$

$$\cdot \operatorname{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}(N-r);\ \tfrac{1}{2}(N-r)+h;\ \tfrac{1}{2}\Lambda).$$

Hence, by Theorem 2.3.5 we have

<u>Theorem 4.2.1</u> (Greenacre (1972))

The non-null density of Wilks' likelihood criterion U is given by

$$f(u) = \operatorname{etr}(-\tfrac{1}{2}\Lambda)(\Gamma_p(\tfrac{1}{2}(N-r))/\Gamma_p(\tfrac{1}{2}(N-q)))u^{-1} \tag{4.2.11}$$

$$\cdot \Sigma_k\Sigma_\kappa \frac{(\tfrac{1}{2}(N-r))_\kappa}{k!} C_\kappa(\tfrac{1}{2}\Lambda) G_p\left(u \,\middle|\, \begin{matrix} \tfrac{1}{2}(N-r)+k_j-\tfrac{1}{2}(j-1) \\ \tfrac{1}{2}(N-q)-\tfrac{1}{2}(j-1) \end{matrix}\right)$$

$$0 < u < 1$$

Pillai and Nagarsenker (1972) have obtained a general density incorporating an H-function which is applicable to various test criteria, and in this case reduces to (4.2.11). Pillai, Al-Ani and Jouris (1969) obtained the density in the form

$$(4.2.12)\quad f(u) = \operatorname{etr}(-\tfrac{1}{2}\Lambda)(\Gamma_p(\tfrac{1}{2}(N-r))/\Gamma_p(\tfrac{1}{2}(N-q)))u^{\frac{1}{2}(N-p-q-1)}$$

$$\cdot\Sigma_k\Sigma_\kappa(\tfrac{1}{2}(N-r))_\kappa C_\kappa(\tfrac{1}{2}\Lambda)G_p\left(u\,\middle|\,\begin{matrix}\tfrac{1}{2}(q-r)+k_{p-i+1}+\tfrac{1}{2}(i-1)\\ \tfrac{1}{2}(i-1)\end{matrix}\right)$$

There does not currently appear to be a method capable of finding percentiles and powers of this distribution for completely general values of the parameters. We shall now examine the distributions where Λ is of less than full rank.

<u>Theorem 4.2.2</u> (de Waal (1968))

If $L \sim M\beta_{1A}(I,m,n,\Lambda_r)$ where Λ_r denotes that Λ is diagonal $(p\times p)$ and of rank $r < p$, and if $L(p\times p)$ is partitioned as $\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22}\end{pmatrix}$ where L_{11} is an $(r\times r)$ matrix, then

$$(4.2.13)\quad \text{(a)}\quad M\beta_{1A}(L;\ I,m,n,\Lambda_r) = M\beta_{1A}(L_{11};\ I,m,n,\Lambda_r)$$

$$\cdot\Pi_{i=r+1}^{p}\beta_1(u_i;\ m-i+1,n)\Pi_{j=1}^{i-1}\beta_1(w_{ij};\ 1,n-i+j),$$

$$0 < L_{11} < I,\quad 0 < u_i,\quad w_{ij} < 1$$

and

$$(4.2.14)\quad \text{(b)}\quad f(|L|) = f_1(|L_{11}|)\Pi_{i=r+1}^{p}f_i(u_i)$$

where $L_{11} \sim M\beta_{1A}(I,m,n,\Lambda_r)$

and $u_i \sim \beta_1(m-i+1,n)$.

The problem even in this case remains the fact that functions, such as zonal polynomials, of matrix argument are present. This may be overcome when the noncentral linear cases or central cases of the distributions are considered.

Theorem 4.2.3 (Kshirsagar (1961), Troskie (1966))

If $L \sim M\beta_{1A}(I,m,n,\lambda^2)$ (the noncentral linear case), then

(a) $f(|L|) = \Pi_{i=1}^{p} f(\ell_i)$ where the ℓ_i are independent,

$f(\ell_i) = \beta_1(\ell_i;\ m,n-i+1) \qquad i = 2,\ldots,p$ and

$f(\ell_1) = \beta_{1A}(\ell_1;\ m,n,\lambda^2)$

(b) If L is central, $f(\ell_1) = \beta_{1A}(\ell_1;\ m,n)$

(Anderson (1958)).

The density of U follows by setting m and n equal to $(q-r)$ and $(N-q)$ respectively in the above theorem. In Chapter 5 we shall see how percentiles of U, which is the product of a number of univariate β_1 densities, may be found

Canonical Form of the hypothesis

By a suitable transformation of the variables, the null hypothesis $H_0 : \underset{p\times(q-r)}{B_2} = 0$ can be changed to canonical form. If Y,B and X are as in (4.2.1) we may choose a suitable orthogonal matrix V such that

(4.2.15) $Z = (Z_1,\ldots,Z_N) = YV'$ and

(4.2.16) $E(Z) = E(YV') = (\mu_1,\ldots,\mu_r,\ldots,\mu_q,0,\ldots,0)$

Then the joint density of $Z = (Z_1,\ldots,Z_N)$ is

(4.2.17) $f(Z_1,\ldots,Z_N) = (2\pi)^{-\frac{1}{2}pN}|\Sigma|^{-\frac{1}{2}N}$

$$\cdot\exp\left\{-\tfrac{1}{2}\textstyle\sum_{\alpha=1}^{q}(z_\alpha-\mu_\alpha)'\Sigma^{-1}(z_\alpha-\mu_\alpha)-\tfrac{1}{2}\sum_{\alpha=q+1}^{N}z_\alpha'\Sigma^{-1}z_\alpha\right\}$$

and the null hypothesis $H_0 : B_2 = 0$ is equivalent to

(4.2.18) $H_0^* : (\mu_{r+1},\ldots,\mu_q) = (0,\ldots,0)$

Wilks likelihood criterion U then becomes

(4.2.19) $U = |A|/|A+B|$ where

(4.2.20) $A = \sum_{\alpha=q+1}^{N} Z_\alpha Z_\alpha'$ and $B = \sum_{\alpha=r+1}^{q} Z_\alpha Z_\alpha'$

This has the nonnull density (4.2.11) and reduces in the noncentral linear case to

(4.2.21) $f(u) = \beta_{1A}(u_1;\ q-r, N-q, \lambda^2)\Pi_{i=2}^{p}\beta(u_i;\ q-r, N-q1i+1)$.

Note that $N \geq p+q$ and $r \leq q$.

If the nonzero roots of

(4.2.22) $|B-\lambda A| = 0$

are denoted by $0 < \lambda_1 <,\ldots,< \lambda_s < 1$ where $s = \min(p,q-r)$ then U may be written as (Pearson and Wilks (1933))

(4.2.23) $U = |A|/|A+B| = \Pi_{i=1}^{s}(1+\lambda_i)^{-1}$

4.3 Testing the equality of means of several normal distributions with common covariance matrix

Consider q multivariate normal populations with common covariance matrix Σ. Let $y_\alpha^{(i)}$ be an observation from $N(\mu^{(i)},\Sigma)$, $\alpha = 1,\ldots,N_i;\ i = 1,\ldots,q$. The null hypothesis is

(4.3.1) $H_0 : \mu^{(1)} = \mu^{(2)} = \ldots = \mu^{(q)}$.

Anderson (1958) shows how the null hypothesis may be reduced to

(4.3.2) $H_0 : B_1 = 0$

by setting $\underset{p\times q}{B} = (\underset{p\times(q-1)}{B_1} \quad \underset{p\times 1}{B_2})$ where

(4.3.3) $\quad B_1 = (\mu^{(1)}-\mu^{(q)},\ldots,\mu^{(q-1)}-\mu^{(q)})$ and $B_2 = \mu^{(q)}$.

The density of Wilks' likelihood criterion U, for the noncentral linear case, is then given by Theorem 4.2.3 with $m = q-1$ and $n = N-q$. i.e.

(4.3.4) $\quad f(u) = \beta_{1A}(u_1;\ q-1.N-q,\lambda^2)\Pi_{i=2}^{p}\beta_1(u_i;\ q-1,N-q-i+1)$

4.4 Testing the multivariate general linear hypothesis in MANOVA

Consider a p-variate two-way layout with k observations per cell, written in $(p\times 1)$-vector form as

(4.4.1) $\quad y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$

w ere $i = 1,\ldots,I$; $j = 1,\ldots,J$; $k = 1,\ldots,K$; $K > 1$ and $\alpha.(= I^{-1}\Sigma_i\alpha_i) = \beta. = \gamma_i. = \gamma._j = 0$. Assume $e_{ijk} \sim N(0,\Sigma)$ where the e_{ijk} are mutually independent.
Let

(4.4.2) $\quad A = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}(y_{ijk}-y_{ij.})(y_{ijk}-y_{ij.})'$,

(4.4.3) $\quad B_I = JK\sum_{i=1}^{I}(y_{i..}-y_{...})(y_{i..}-y_{...})'$,

(4.4.4) $\quad B_J = IK\sum_{j=1}^{J}(y_{.j.}-y_{...})(y_{.j.}-y_{...})'$ and

(4.4.5) $\quad B_{IJ} = K\sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij.}-y_{i..}-y_{.j.}+y_{...})$

$$\cdot(y_{ij.}-y_{i..}-y_{.j.}+y_{...})'$$

Then A represents the total sample variance within populations (error sum of squares). B_I and B_J represent total sample variances between columns or rows, and B_{IJ} represents sample variance for interaction.
It can easily be shown that

(i) $A \sim W(\Sigma, IJ(K-1))$ for $p < IJ(K-1)$

(ii) Under $H_0^{(I)} : \alpha_i = 0 \quad i = 1,\ldots,I$

$B_I \sim W(\Sigma, I-1)$ for $p < I-1$

(iii) Under $H_0^{(J)} : \beta_j = 0 \quad j = 1,\ldots,J$

$B_J \sim W(\Sigma, J-1)$ for $p \leq J-1$

(iv) Under $H_0^{(IJ)} : \gamma_{ij} = 0 \quad i = 1,\ldots,I; \quad j = 1,\ldots,J$

$B_{IJ} \sim W(\Sigma, (I-1)(J-1))$ for $p \leq (I-1)(J-1)$

Wilks' likelihood criteria for the hypotheses in (ii), (iii) and (iv) above are given by

(4.4.6) $|L_I| = |A|/|A+B_I|$ for $H_0^{(I)}$

(4.4.7) $|L_J| = |A|/|A+B_J|$ for $H_0^{(J)}$

and

(4.4.8) $|L_{IJ}| = |A|/|A+B_{IJ}|$ for $H_0^{(IJ)}$

where L_I, L_J and L_{IJ} are multivariate Beta distributions. The densities of the criteria in (4.4.6), (4.4.7) and (4.4.8) respectively are given by

(4.4.9) $f(|L_I|) = \beta_{1A}(\ell_{I1}; I-1, IJ(K-1), \lambda^2)$

$\cdot \Pi_{i=2}^{p} \beta_1(\ell_{Ii}; I-1, IJ(K-1)-i+1)$

(4.4.10) $f(|L_J|) = \beta_{1A}(\ell_{J1}; J-1, IJ(K-1), \lambda^2)$

$\cdot \Pi_{i=2}^{p} \beta_1(\ell_{Ji}; J-1, IJ(K-1)-i+1)$

and

(4.4.11) $f(|L_{IJ}|) = \beta_{1A}(\ell_{IJ1}; (I-1)(J-1), IJ(K-1), \lambda^2)$

$\cdot \Pi_{i=2}^{p} \beta_1(\ell_{IJi}; (I-1)(J-1), IJ(K-1)-i+1)$, where λ^2 is the appropriate noncentrality parameter in each case and $0 < \ell_{Ii}, \ell_{Ji}, \ell_{IJi} < 1$.

For the two way layout with one observation per cell, the hypotheses $H_0^{(I)} : \alpha_i = 0 \quad i = 1,\ldots,I$ and $H_0^{(J)} : \beta_j = 0 \quad j = 1,\ldots,J$ are appropriate and the densities of Wilks' likelihood criteria for the hypotheses are respectively

$$(4.4.12) \quad f(|L_I|) = \beta_{1A}(\ell_{I1}; I-1,(I-1)(J-1),\lambda^2)$$

$$\cdot\Pi_{i=2}^{p}\beta_1(\ell_{Ii}; I-1,(I-1)(J-1)-i+1)$$

and

$$(4.4.13) \quad f(|L_J|) = \beta_{1A}(\ell_{Ji}; J-1,(I-1)(J-1),\lambda^2)$$

$$\cdot\Pi_{i=2}^{p}\beta_1(\ell_{Ji}; J-1,(I-1)(J-1)-i+1)$$

with appropriate λ^2 and $0 < \ell_{Ii}, \ell_{Ji} < 1$.

4.5 Testing for independence of sets of variables

Consider $\underset{p\times 1}{X} \sim N(\mu,\Sigma)$

where $X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \end{pmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix}$ and $\Sigma = \begin{array}{c} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\begin{matrix} p_1 \\ p_2 \end{matrix} \\ \begin{matrix} p_1 & \quad p_2 \end{matrix} \end{array}$ $\quad p_1 \leq p_2$

We wish to test the hypothesis

$$(4.5.1) \quad H_0 : X^{(1)} \text{ and } X^{(2)} \text{ are independent.}$$

As $X \sim N(\mu,\Sigma)$ this is equivalent to

$$(4.5.2) \quad H_0 : \Sigma_{12} = \Sigma_{21}' = 0$$

With a sample of N vectors $X_{(\alpha)}$, $\alpha = 1,\dots,N$, the likelihood ratio criterion is the $N/2$ power of (Wilks (1932))

$$(4.5.3) \qquad U = |A|/|A_{11}||A_{22}|$$

where $A = \sum_{\alpha=1}^{N}(X_{(\alpha)}-\overline{X})(X_{(\alpha)}-\overline{X})'$ and is partitioned similarly to Σ.

<u>Definition 4.5.1</u> (Khatri (1964), Troskie (1969))

The generalised (sample) multiple correlation matrix is defined as

$$(4.5.4) \qquad R = A_{11}^{-\frac{1}{2}}A_{12}A_{22}^{-1}A_{21}A_{11}^{-\frac{1}{2}}$$

(where we adopt the convention that in an expression such as $A^{\frac{1}{2}}BA^{\frac{1}{2}}$ the postmultiplier is actually $(A^{\frac{1}{2}})'$). From this

$$(4.5.5) \qquad |R| = |A_{12}||A_{21}|/|A_{11}||A_{22}|$$

$$(4.5.6) \qquad |I-R| = |A_{11}||A_{22}|-|A_{12}||A_{21}|/|A_{11}||A_{22}|$$

$$(4.5.7) \qquad = |A|/|A_{11}||A_{22}|$$

$$(4.5.8) \qquad = U = \Pi_{i=1}^{p_1}(1-r_i^2)$$

where r_i^2 are the characteristic roots of R.

We can also write R as $(F+G)^{-\frac{1}{2}}G(F+G)^{-\frac{1}{2}}$ where $F = A_{11\cdot 2} = A_{11}-A_{12}A_{22}^{-1}A_{21}$ and $G = A_{12}A_{22}^{-1}A_{21}$.

The distributions of functions of R depend on whether or not the set $X^{(2)}$ is fixed.

<u>If $X^{(2)}$ is fixed (regression model)</u>

This implies A_{22} fixed and, by Anderson (1958),

$F \sim W(\Sigma_{11\cdot 2},n-p_2)$, $n = N-1$

$G \sim W(\Sigma_{11\cdot 2};p_2,\Lambda)$, F and G independent, and

(4.5.9) $\quad \Lambda = \Sigma_{11\cdot 2}^{-1}\beta A_{22}\beta \quad , \quad \beta = \Sigma_{12}\Sigma_{22}^{-1}$.

G has a central Wishart distribution iff $\beta = 0$ i.e. iff $\Sigma_{12} = 0$. Hence, by Theorem 3.3.1, with m, n, and p replaced by p_2, $n-p_2$ and p_1, we have the conditional distribution

(4.5.10) $\quad R \sim M\beta_{1B}(\Sigma_{11\cdot 2}, p_2, n-p_2, \Lambda)$

and

(4.5.11) $$E|I-R|^h = EU^h = (\Gamma_{p_1}(\tfrac{1}{2}(n-p_2)+h)\Gamma_{p_1}(\tfrac{1}{2}n)/$$

$$\cdot\Gamma_{p_1}(\tfrac{1}{2}(n-p_2))\Gamma_{p_1}(\tfrac{1}{2}n+h))\operatorname{etr}(-\tfrac{1}{2}\Lambda)\,{}_1F_1(\tfrac{1}{2}n;\ \tfrac{1}{2}n+h;\ \tfrac{1}{2}\Lambda)$$

By Theorem 2.3.5 it follows that (Greenacre (1972))

(4.5.12) $$f(u) = (\Gamma_{p_1}(\tfrac{1}{2}n)/\Gamma_{p_1}(\tfrac{1}{2}(n-p_2)))\operatorname{etr}(-\tfrac{1}{2}\Lambda)u^{-1}$$

$$\cdot\Sigma_k\Sigma_\kappa \frac{(\tfrac{1}{2}n)_\kappa C_\kappa(\tfrac{1}{2}\Lambda)}{k!}\, G_{p_1}\left(u \left| \begin{matrix} \tfrac{1}{2}n+k_j-\tfrac{1}{2}(j-1) \\ \tfrac{1}{2}(n-p_2)-\tfrac{1}{2}(j-1) \end{matrix}\right.\right)$$

$$0 < u < 1$$

We now examine the special case of (4.2.11), Wilks' criterion for testing hypotheses about regression coefficients, where (4.2.2) is $H_0 : B_2 = 0$, and $r = 1$ (representing the constant term in B). It can be seen that this case of (4.2.11) is equivalent to (4.5.12) with

$r = 1$ (the constant)

$p = p_1$ (the number of dependent variables)

$q = p_2+1$ (the number of predetermined variables, including the constant).

<u>If $X^{(2)}$ is random (correlation model)</u>

The unconditional distribution of R is found by multiplying the conditional distribution of R by that of $A_{22}(\sim W(\Sigma_{22},n))$ and integrating over A_{22}, but it appears difficult to derive an explicit expression. By using Theorems 2.4.1 and 2.4.2 and appropriate integration we have

<u>Theorem 4.5.1</u> (Troskie (1969))

The hth moments of $|R|$ and $U = |I-R|$ are given by

$$(4.5.13)\quad E|R|^h = (\Gamma_{p_1}(\tfrac{1}{2}n)\Gamma_{p_1}(\tfrac{1}{2}p_2+h)/\Gamma_{p_1}(\tfrac{1}{2}n+h)\Gamma_{p_1}(\tfrac{1}{2}p_2))$$

$$\cdot|I-P|^{\frac{1}{2}n}{}_3F_2(\tfrac{1}{2}n,\tfrac{1}{2}n,\tfrac{1}{2}p_2+h;\ \tfrac{1}{2}n+h,\tfrac{1}{2}p_2;\ P)$$

and

$$(4.5.14)\quad E|I-R|^h = (\Gamma_{p_1}(\tfrac{1}{2}n)\Gamma_{p_1}(\tfrac{1}{2}(n-p_2)+h)/$$

$$\cdot\Gamma_{p_1}(\tfrac{1}{2}(n-p_2))\Gamma_{p_1}(\tfrac{1}{2}n+h))|I-P|^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n;\ \tfrac{1}{2}n+h;\ P)$$

where

$$(4.5.15)\quad P = \Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}}$$

is the population multiple correlation matrix. From (4.5.14) and Theorem 2.3.5 it follows that

<u>Theorem 4.5.2</u> (Greenacre (1972))

The non-null density of U for testing independence is

$$(4.5.16)\quad f(u) = (\Gamma_{p_1}(\tfrac{1}{2}n)/\Gamma_{p_1}(\tfrac{1}{2}(n-p_2)))|I-P|^{\frac{1}{2}n}w^{-1}$$

$$\cdot\Sigma_k\Sigma_\kappa\frac{((\frac{1}{2}n)_\kappa)^2C_\kappa(P)}{k!}\,G_{p_1}\left(u\,\middle|\,\begin{matrix}\frac{1}{2}n+k_j-\frac{1}{2}(j-1)\\ \frac{1}{2}(n-p_2)-\frac{1}{2}(j-1)\end{matrix}\right)\qquad 0 < u < 1$$

This is given in terms of a more general H-function by Pillai and Nagarsenker (1972), whose density contains two errors (a factor of Y^{-1} has been omitted and $\frac{1}{2}(n-2)$ should be $\frac{1}{2}(n-q)$). Pillai, Al-Ani and Jouris (1969) obtained the density in slightly different form.

Because of computational difficulties we again consider the linear case, i.e. when Σ_{12} is of rank one. This implies that P is of rank one and has only one root $\rho_1^2 \neq 0$.

Theorem 4.5.3 (Kshirsagar (1961), Troskie (1969))

In the noncentral linear case, Wilks' likelihood criterion for testing independence of two sets has the density

$$(4.5.17)\quad f(u) = f(|I-R|) = \Pi_{i=1}^{p_1} f(u_i) \quad \text{with } u_i \text{ independent.}$$

$$(4.5.18)\quad f(u_1) = (B(\tfrac{1}{2}p_2,\tfrac{1}{2}(n-p_2)))^{-1} u_1^{\frac{1}{2}(n-p_2)-1}(1-u_1)^{\frac{1}{2}p_2-1}$$

$$\cdot(1-\rho_1^2)^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n;\ \tfrac{1}{2}p_2;\ \rho_1^2(1-u_1))$$

and

$$(4.5.19)\quad f(u_i) = (B(\tfrac{1}{2}p_2,\tfrac{1}{2}(n-p_2-i+1)))^{-1}$$

$$\cdot u_i^{\frac{1}{2}(n-p_2-i+1)-1}(1-u_i)^{\frac{1}{2}p_2-1} \qquad i = 2,\ldots,p_1$$

Similarly

$$(4.5.20)\quad f(|R|) = \Pi_{i=1}^{p_1} f(w_i), \quad \text{where the } w_i \text{ are independent,}$$

$$(4.5.21)\quad f(w_1) = (B(\tfrac{1}{2}(n-p_2),\tfrac{1}{2}p_2))^{-1} w_1^{\frac{1}{2}p_2-1}(1-w_1)^{\frac{1}{2}(n-p_2)-1}$$

$$\cdot(1-\rho_1^2)^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n;\ \tfrac{1}{2}p_2;\ \rho_1^2 w_1)$$

and

$$(4.5.22)\quad f(w_i) = (B(\tfrac{1}{2}(n-p_2),\tfrac{1}{2}(p_2-i+1)))^{-1}$$

$$\cdot w_i^{\frac{1}{2}(p_2-i+1)-1}(1-w_i)^{\frac{1}{2}(n-p_2)-1} \qquad i = 2,\ldots,p_1$$

More than two sets of variables

We give a few results showing how the likelihood ratio criterion U (in the null case) may be expressed as a product of independent univariate Beta distributions (Anderson (1958)).

Let X be partitioned into q subvectors with $p_1, p_2, \ldots, p_q$ components respectively, where $p = \sum_{i=1}^{q} p_i$. Then the central density of U is

(4.5.23) $$f(u) = \Pi_{i=2}^{q} \Pi_{j=1}^{p_i} \beta_1(u_{ij};\ \bar{p}_i, n-\bar{p}_i-j+1)$$

where $\bar{p}_i = p_1+p_2+\ldots+p_{i-1}$. When $q = 2$ this reduces to

(4.5.24) $$f(u) = \Pi_{j=1}^{p_2} \beta_1(u_j;\ p_1, n-p_1+1-j)$$

which Anderson shows is equivalent to

(4.5.25) $$f(u) = \Pi_{i=1}^{p_1} \beta_1(u_i;\ p_2, n-p_2+1-i),$$

the central case of (4.5.17).

If $q = p$, i.e. $p_i = 1$, the density of U is

(4.5.26) $$f(u) = \Pi_{i=1}^{p-1} \beta_1(u_i;\ i, n-i).$$

If $p_1 = p_2 = \ldots = p_q = 2$,

(4.5.27) $$f(u) = \Pi_{i=1}^{q-1} (\beta_1(u_i;\ 4i, 2(n-1-2i)))^2$$

4.6 Testing the equality of two covariance matrices

Assume N_i observations are taken from $N(\underset{p\times 1}{\mu_i}, \underset{p\times p}{\Sigma_i})$ i=1,2

We wish to test the hypothesis

(4.6.1) $$H_0 : \Sigma_1 = \Sigma_2 = \Sigma \quad \text{against} \quad H_1 : \Sigma_1 \neq \Sigma_2.$$

This has likelihood ratio criterion (Anderson (1958))

(4.6.2) $\lambda = (|A_1|^{\frac{1}{2}N_1}|A_2|^{\frac{1}{2}N_2}N^{\frac{1}{2}N}/|A|^{\frac{1}{2}N}N_1^{\frac{1}{2}N_1}N_2^{\frac{1}{2}N_2})$

where A_i/N_i is the maximum likelihood estimate of Σ_i, $i = 1,2$, $A = A_1+A_2$ and $N = N_1+N_2$.

The modified likelihood ratio criterion λ^* (Bartlett (1937)) replaces sample numbers N_i by degrees of freedom of the A_i and omits the numerical constant, i.e.

(4.6.3) $\lambda^* = |A_1|^{\frac{1}{2}n_1}|A_2|^{\frac{1}{2}n_2}/|A|^{\frac{1}{2}n}$

where $n_i = N_i-1$ $(i = 1,2)$ and $n = n_1+n_2$. Its non-null distributional properties are as follows:

<u>Theorem 4.6.1</u> (Khatri and Srivastava (1971))

The modified likelihood ratio criterion for testing equality of two covariance matrices

(4.6.4) $\lambda^* = |L|^{\frac{1}{2}n_1}|I-L|^{\frac{1}{2}n_2}$, where $L \sim M\beta_{1C}(L;\ \Sigma_1,\Sigma_2,n_2,n_1)$,

has hth moment

(4.6.5) $E(\lambda^{*h}) = (\Gamma_p(\frac{1}{2}n)\Gamma_p(\frac{1}{2}n_1(1+h))\Gamma_p(\frac{1}{2}n_2(1+h)/$

$\cdot\Gamma_p(\frac{1}{2}n(1+h))\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2))|\Omega|^{-\frac{1}{2}n_1}$

$\cdot{}_2F_1(\frac{1}{2}n,\frac{1}{2}n_1(1+h);\ \frac{1}{2}n(1+h);\ I-\Omega^{-1})$, $\Omega = \Sigma_2^{-\frac{1}{2}}\Sigma_1\Sigma_2^{-\frac{1}{2}}$

and density function

(4.6.6) $f(\lambda^*) = (\Gamma_p(\frac{1}{2}n)/\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2))\Pi^{\frac{1}{4}p(p-1)}|\Omega|^{-\frac{1}{2}n_1}$

$\cdot(\lambda^*)^{-1}\Sigma_k\Sigma_\kappa(\frac{1}{2}n)_\kappa C_\kappa(I-\Omega^{-1})(k!)^{-1}$

$\cdot H^{2p,0}_{2p,2p}\left(\lambda^*\left|\begin{matrix}(\frac{1}{2}(n-i+1)+k_i,\frac{1}{2}n), & 1 = 1,2,\ldots,p\\ ((\frac{1}{2}(n_1-i+1)+k_i,\frac{1}{2}n_1),(\frac{1}{2}(n_2-i+1),\frac{1}{2}n_2)), & "\end{matrix}\right.\right)$

We now examine another statistic not related to the likelihood ratio criterion of this test, but similar to those of other tests.

<u>Wilks' criterion</u>

Let $\underset{p\times n_1}{X}$ and $\underset{p\times n_2}{Y}$ $p \leq n_i$, $i = 1,2$; $n = n_1+n_2$ be independent matrix variates with columns of X independently $\sim N(0,\Sigma_1)$ and those of Y independently $\sim N(0,\Sigma_2)$. Hence $S_1 = XX'$ and $S_2 = YY'$ are independently $\sim W(\Sigma_i,n_i)$, $i = 1,2$.

Now let $0 < f_1 \leq f_2 \leq \ldots \leq f_p < \infty$ be the characteristic roots of $S_1S_2^{-1}$ and $0 < \omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_p^2 < \infty$ be those of $\Sigma_1\Sigma_2^{-1}$. We consider testing the hypothesis (4.6.1) and will use Wilks' criterion

(4.6.7) $\quad U = \Pi_{i=1}^{p}(1-\theta_i) \qquad \theta_i = f_i/(1+f_i), \quad i = 1,2,\ldots,p$

which has hth moment (Pillai, Al-ani and Jouris (1969))

(4.6.8) $$EU^h = (\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}n_2+h)/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n+h))$$

$$\cdot\ |\Omega|^{-\frac{1}{2}n_1}\,{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n_1;\ \tfrac{1}{2}n+h;\ I-\Omega^{-1}),\ \Omega = \Sigma_1\Sigma_2^{-1}$$

(by applying (3.3.20) with matrix argument and changing n_1 and n_2 to m and n respectively and transposing Σ_1 and Σ_2 this reduces to (3.3.15)), and density function

(4.6.9) $$f(u) = (\Gamma_p(\tfrac{1}{2}n)/\Gamma_p(\tfrac{1}{2}n_2))|\Omega|^{-\frac{1}{2}n_1}u^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot\Sigma_k\Sigma_\kappa(\tfrac{1}{2}n)_\kappa(\tfrac{1}{2}n_1)_\kappa(k!)^{-1}C_\kappa(I-\Omega^{-1})G_p\left(u\left|\begin{matrix}\frac{1}{2}n_1+k_{p-i+1}+\frac{1}{2}(i-1)\\ \frac{1}{2}(i-1)\end{matrix}\right.\right)$$

In the noncentral linear case, i.e. when ω^2 is the only root of Ω that is not equal to one (we shall call this a non-unit root), the moments of U are given by (3.3.17) and alternatively by (3.3.19). We derive the density of U by noting that the hth moment of U (4.6.8) is equivalent to that of U for the correlation model (4.5.14), with the following changes:

$$(4.6.10) \quad p_1 \to p;\ (n-p_2) \to n_2;\ |I-P|^{\frac{1}{2}n} \to |\Omega|^{-\frac{1}{2}n_1};$$

$$\cdot((\tfrac{1}{2}n)_\kappa)^2 \to (\tfrac{1}{2}n)_\kappa(\tfrac{1}{2}n_1)_\kappa;\quad P \to (I-\Omega^{-1}).$$

Noting in the noncentral linear case that ρ_1^2 and ω^2 are the only nonzero and non-unit roots of P and Ω respectively, and that

$$(4.6.11) \quad {}_2F_1(\tfrac{1}{2}n, \tfrac{1}{2}n_1;\ \tfrac{1}{2}n_1;\ z) = {}_1F_0(\tfrac{1}{2}n;\ z),$$

substitution in equations (4.5.17) - (4.5.19) gives:

Theorem 4.6.2

Wilks' criterion for testing equality of covariance matrices in the noncentral linear case has density

$$(4.6.12) \quad f(u) = \Pi_{i=1}^{p} f(u_i), \quad u_i \text{ independent},$$

where

$$(4.6.13) \quad f(u_1) = (B(\tfrac{1}{2}n_1, \tfrac{1}{2}n_2))^{-1} u_1^{\frac{1}{2}n_2-1}(1-u_1)^{\frac{1}{2}n_1-1}$$

$$\cdot\omega^{-n_1}\,{}_1F_0(\tfrac{1}{2}n;\ (1-\omega^{-2})(1-u_1))$$

and

$$(4.6.14) \quad f(u_i) = (B(\tfrac{1}{2}n_1, \tfrac{1}{2}(n_2-i+1)))^{-1} u_i^{\frac{1}{2}(n_2-i+1)-1}(1-u_i)^{\frac{1}{2}n_1-1}$$

$$i = 2, \ldots, p$$

Thus a test criterion for equality of covariance matrices exists which, for the central or noncentral linear case, may be expressed in the computable form of a product of Beta variables.

4.7 Testing whether k multivariate populations are identical

Consider k multivariate normal populations $\sim N(\mu_j,\Sigma_j)$. A modified likelihood ratio criterion λ^* for testing whether the distributions are identical is (Anderson (1958))

$$(4.7.1)\qquad \lambda^* = \Pi_{j=1}^{k}|A_j|^{\frac{1}{2}n_j}/|A+B|^{\frac{1}{2}n} \qquad A = \sum_{j=1}^{k}A_j,\ n = \sum_{j=1}^{k}n_j$$

A_j and B are respectively the "within" and "between" matrices of sums of squares and cross products.

If L_j is defined by

$$(4.7.2)\qquad L_j = (\sum_{j=1}^{k}A_j+B)^{-\frac{1}{2}}A_j(\sum_{j=1}^{k}A_j+B)^{-\frac{1}{2}}, \qquad j = 1,\ldots,k$$

then

$$(4.7.3)\qquad \lambda^* = \Pi_{j=1}^{k}|L_j|^{\frac{1}{2}n_j}$$

Under the null hypothesis $A \sim W(\Sigma,n)$, $A_j \sim W(\Sigma,n_j)$ and $B \sim W(\Sigma,m)$, $m = k-1$, and $(L_1,\ldots,L_k) \sim MD_1(\Sigma;\ m;\ n_1,\ldots,n_k)$.

It can be seen that (4.7.1) is a combination of test criteria (4.2.19) and (4.6.3). By Wilks (1932) it is possible to test the null hypothesis by first testing the hypothesis of equality of covariance matrices and then the hypothesis of equality of mean vectors. If the covariances are identical but means are different, B will have a noncentral Wishart distribution with m degrees of freedom. In the noncentral linear case the moments of λ^* are given by (3.5.7) and the joint distribution of $(L_1,\ldots,L_k)$ is $MD_1(I;\ m;\ n_1,\ldots,n_k;\ \lambda^2)$.

Troskie (1972) and Money (1972) have suggested the use of $|I-\sum_{j=1}^{k}L_j|$ as a test criterion for the hypothesis of identical distributions. The moments of $|I-\sum_{j=1}^{k} L_j|$ are given by (3.5.9) and are shown to be the same as those of $|I-L|$ where $L = \sum_{j=1}^{k} L_j$. Consequently we have

<u>Theorem 4.7.1</u> (de Waal (1968), Troskie (1972))

The test criterion $|I-\sum_{j=1}^{k} L_j|$ in the noncentral linear case has the density

$$(4.7.4) \qquad f(w) = \Pi_{i=1}^{p} f(w_i), \qquad w_i \text{ independent}$$

where

$$(4.7.5) \qquad f(w_1) = (B(\tfrac{1}{2}(k-1), \tfrac{1}{2}\Sigma n_j))^{-1} w_1^{\frac{1}{2}(k-1)-1}(1-w_1)^{\frac{1}{2}\Sigma n_j - 1}$$

$$\cdot e^{-\frac{1}{2}\lambda^2} {}_1F_1(\tfrac{1}{2}(k-1+\Sigma n_j);\ \tfrac{1}{2}(k-1);\ \tfrac{1}{2}\lambda^2 w_1), \quad 0 \leq w_1 \leq 1$$

and

$$(4.7.6) \qquad f(w_i) = (B(\tfrac{1}{2}(k-i), \tfrac{1}{2}\Sigma n_j))^{-1} w_i^{\frac{1}{2}(k-i)-1}(1-w_i)^{\frac{1}{2}\Sigma n_j - 1},$$

$$i = 2,\ldots,p \quad 0 \leq w_i \leq 1$$

4.8 <u>General form for test criteria of sections 4.2-4.7</u>

Test criteria, in the noncentral linear case, for a number of statistical hypotheses have been expressed as a product of univariate Beta distributions, and the results may be generalised as follows:

<u>Theorem 4.8.1</u>

The density of each test criterion Λ considered may be written in the general form

$$(4.8.1) \qquad f(\Lambda) = (\Pi_{i=1}^{p^*} f(u_i)).\Omega^*$$

where

(continued on page 4.22)

TABLE 4.1 PARAMETERS OF BETA DISTRIBUTIONS FOR DIFFERENT TEST CRITERIA Λ IN THE NONCENTRAL LINEAR CASE

TESTS BY SECTION (+ parameter defn.)	p^*	m^*	n^*	EQUATION
4.2 Regression	p	q-r	N-q	(4.2.21)
p dependent variables				
q predetermined vars.				
r 1st subset of q				
4.3 Means	p	q-1	N-q	(4.3.4)
p dimension of vars.				
q no. of populations				
4.4 MANOVA	p	(I-1)	IJ(K-1)	(4.4.9)
I no. of rows	p	(J-1)	IJ(K-1)	(4.4.10)
J no. of columns	p	(I-1)(J-1)	IJ(K-1)	(4.4.11)
K elements per cell	p	(I-1)	(I-1)(J-1)	(4.4.12)
P dimension of vars.	p	(J-1)	(I-1)(J-1)	(4.4.13)
4.5 Independence	p_1	p_2	$n-p_2$	(4.5.17)
p_1 dimn. of 1st subset	p_1	$n-p_2$	p_2	(4.5.20)
p_2 dimn. of 2nd subset				
4.6 Covariances	p	n_1	n_2	(4.6.12)
p dimension of vars.				
4.7 Populations	p	Σn_j	k-1	(4.7.4)
k no. of populations				
p dimension of vars.				

N, and N_i, denote number of observations from the only, and ith, populations respectively; $n = N-1$; $n_i = N_i - 1$.

(See Theorem 4.8.1)

TABLE 4.2 NONCENTRALITY COMPONENT Ω^* OF $f(\Lambda)$ FOR DIFFERENT TEST CRITERIA Λ IN THE NONCENTRAL LINEAR CASE

SECTION	Λ	Ω^*	EQUATION
4.2	$\|L\|$	$e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\frac{1}{2}(m^*+n^*);\ \frac{1}{2}m^*;\ \frac{1}{2}\lambda^2(1-u_1))$	(4.2.21)
4.3	$\|L\|$	" " " "	(4.3.4)
4.4	$\|L\|$	" " " "	(4.4.9) to (4.4.13)
4.5	$\|I-R\|$	$(1-\rho_1^2)^{\frac{1}{2}(m^*+n^*)} \cdot {}_2F_1(\frac{1}{2}(m^*+n^*),\frac{1}{2}(m^*+n^*);\ \frac{1}{2}m^*;\ \rho_1^2(1-u_1))$	(4.5.17)
	$\|R\|$	$(1-\rho_1^2)^{\frac{1}{2}(m^*+n^*)} \cdot {}_2F_1(\frac{1}{2}(m^*+n^*),\frac{1}{2}(m^*+n^*);\ \frac{1}{2}n^*;\ \rho_1^2u_1)$	(4.5.20)
4.6	$\|C\|$	$\omega^{-m^*}\,{}_1F_0(\frac{1}{2}(m^*+n^*);\ (1-\omega^{-2})(1-u_1))$	(4.6.12)
4.7	$\|I-\Sigma L_j\|$	$e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(\frac{1}{2}(m^*+n^*);\ \frac{1}{2}n^*;\ \frac{1}{2}\lambda^2u_1)$	(4.7.4)

(See Theorem 4.8.1)

(4.8.2) $$f(u_i) = (B(\tfrac{1}{2}m^*,\tfrac{1}{2}(n^*-i+1)))^{-1} \cdot u_i^{\frac{1}{2}(n^*-i+1)-1}(1-u_i)^{\frac{1}{2}m^*-1} \qquad i = 1,\ldots,p^*$$

and Ω^*, the noncentrality component of $f(u_1)$ is given in terms of m^*, n^* and p^* in Table 4.2. Table 4.1 gives, for each test, the appropriate value of m^*, n^* and p^*.

This theorem will be useful in establishing a general form for an algorithm used in calculating $f(\Lambda)$. It can be seen that equation (4.8.2) is the density of a $\beta_1(u_i;\ m^*, n^*-i+1)$ variable.

4.9 Tests with complex distributions

Most of the tests mentioned so far may be carried out when the data comes from complex normal populations. Initial work by Goodman (1963), James (1964), Giri (1965) and Khatri (1965) established basic distributional forms and examined several test criteria. Troskie (1967, 1969) and de Waal (1968) derived further results on complex multivariate Beta and Dirichlet distributions, while Pillai and Jouris (1971), Pillai and Nagarsenker (1972), Money (1972) and Troskie and Money (1974) have illustrated the connection between the real and complex distributions of the various test criteria considered, in both the general and the noncentral linear case.

By comparing the moments of the criteria for the real and complex cases, we may express the densities for the complex case in the following general form:

Theorem 4.9.1

The densities of the test criteria Λ considered for the real case in Theorem 4.8.1 may be expressed in general form for the complex case as

(4.9.1) $\quad f(\Lambda) = (\Pi_{i=1}^{p^*} f(u_i')).\Omega^{*\prime}$

where

(4.9.2) $\quad f(u_i') = \beta_1(u_i';\ m^{*\prime}, (n^*-i+1)'), \quad (i = 1,\ldots,p^*),$

and parameters $m^{*\prime}$ and $n^{*\prime}$ in $\Omega^{*\prime}$, which replace parameters m^* and n^* in Ω^*, are such that

(4.9.3) $\quad m^{*\prime} = 2m^*$ and $(n^*-i+1)' = 2(n^*-i+1)$, $(i = 1,2,\ldots,p^*)$

For equations (4.2.21), (4.3.4), (4.4.9) to (4.4.13) and (4.7.4), i.e. those involving λ^2, $\frac{1}{2}\lambda^2$ is replaced by λ^2. As ρ_1^2 and ω^2 are the roots of ratios of covariance matrices they remain unaltered in equations (4.5.17), (4.5.20) and (4.6.12).

In Chapter 9 we shall see how these results lead to an algorithm which enables us to calculate percentage points and powers of the test criteria considered when the variables are complex.

4.10 Testing for reality of a covariance matrix

Let $\underset{p\times n}{Z} = X_1 + {}_iX_2$ have the complex normal distribution with density function

(4.10.1) $\quad \Pi^{-pn}|\Sigma|^{-n}\text{etr}(-\Sigma^{-1}(Z-\mu M)(\overline{Z-\mu M})')$

where $\underset{p\times p}{\Sigma} = \Sigma_1 + i\Sigma_2$ is Hermitian positive definite, $\underset{p\times q}{\mu}$ is a complex matrix and $\underset{q\times n}{M}$ is either a given complex matrix of rank $q \leq n$ or has a distribution which is independent of Σ and μ.

Then the test for reality of Σ has null hypothesis

(4.10.2) $\quad H_0 : \Sigma_2 = 0$

with alternative hypothesis

(4.10.3) $\quad H_1 : \Sigma_2 \neq 0.$

Khatri (1965a) showed that under H_0 the likelihood ratio statistic is

(4.10.4) $\Lambda = |S_1+iS_2|/|S_1|$

where

(4.10.5) $S_1+iS_2 = Z(I-\overline{M}'(M\overline{M}')^{-1}M)\overline{Z}'$

and the likelihood ratio criterion is

(4.10.6) reject H_0 if $\Lambda \leq \lambda_1$, where $\Pr(\Lambda<\lambda_1|H_0) = \alpha$.

By Khatri (1965), $\underset{p\times p}{S} = (S_1+iS_2) \sim CW(\Sigma,n-q)$.

Under $H_0 : \Sigma_2 = 0$, S_1 and S_2 have joint density function

(4.10.7) $(\overline{\Gamma}_p(n-q)|\Sigma_1|^{n-q})^{-1}|S_1+iS_2|^{n-q-p}\mathrm{etr}(-\Sigma_1^{-1}S_1)$.

Setting real skew-symmetric matrix $Q = S_1^{-\frac{1}{2}}S_2S_1^{-\frac{1}{2}}$,

we have

(4.10.8) $\Lambda = |I+iQ| = |I-QQ'|^{\frac{1}{2}}$

and

(4.10.9) $E(\Lambda^h) = \Pi_{i=1}^{p}\big(\Gamma(n-q-\frac{1}{2}(i-1))\Gamma(n-q+h-i+1)/$

$\cdot\Gamma(n-q-i+1)\Gamma(n-q+h-\frac{1}{2}(i-1))\big)$

(4.10.10) $= \Pi_{i=1}^{t'}\big(\Gamma(n-q-i+\frac{1}{2})\Gamma(n-q-t+h-i+1)/$

$\cdot\Gamma(n-q-t-i+1)\Gamma(n-q+h-i+\frac{1}{2})\big)$,

where

(4.10.11) $t' = t \quad = \frac{1}{2}p \quad , \; p$ even

$= t-1 \quad = \frac{1}{2}(p-1), \; p$ odd (>1).

It follows from (4.10.10) that

(4.10.12) $E(\Lambda^h) = \Pi_{i=1}^{t'}E(u_i^h) = E(\Pi_{i=1}^{t'}u_i)^h$

where the u_i are independent Beta variables with density function

(4.10.13) $(B(t-\frac{1}{2}.n-q-t-i+1))^{-1}u_i^{n-q-t-i}(1-u_i)^{t-3/2}$, $0 < u_i < 1$

Consequently the density of the test criterion Λ under

H_0 may be written in the general form

(4.10.14) $f(\Lambda) = (\Pi_{i=1}^{p^*} f(u_i))$

where

(4.10.15) $f(u_i) = \beta_1(u_i; m^*, n^*-i+1)$, $i = 1,2,\ldots,p^*$,

(4.10.16) $p^* = t'$, $m^* = 2t-1$, $n^* = 2(n-q-t)$

and t' and t are defined in (4.10.11).

We shall discuss computation of the percentiles of this criterion in Section 9.2 The noncentral distribution is complicated by the skew-symmetric form of Q, and it is unlikely that it can be reduced to the general form of Section 4.8.

CHAPTER 5

AN ALGORITHM FOR DERIVING NONCENTRAL LINEAR DISTRIBUTIONS BY THE METHOD OF CONVOLUTIONS

5.1 Introduction

In this chapter we shall show how the densities and cumulative distribution functions of the test criteria of Chapter 4 may be found. We shall concentrate our attention on Wilks' likelihood ratio criterion

(5.1.1) $$|L| = |A|/|A+B|$$

where $\underset{p\times p}{A} \sim W(I,n)$ and $\underset{p\times p}{B} \sim W(I,m,\lambda^2)$

and

(5.1.2) $$f(|L|) = \beta_{1A}(x_1;\ m,n,\lambda^2)\Pi_{i=2}^{p}\beta_1(x_i;\ m,n-i+1)$$

and will later show how results for this criterion can be extended to other criteria. We will use $U_{p,m,n}(\lambda^2)$ and $U_{p,m,n}$ to denote $|L|$ in the noncentral linear and central cases respectively.

The exact distribution of $U_{p,m,n}$ under the null hypothesis was found by Wilks (1935) using direct integration for values of p and $m \leq 4$ and general n. Later work by Bartlett (1938), Wald and Brookner (1941), Rao (1948) and Box (1949) developed asymptotic approximations to logarithmic functions of $U_{p,m,n}$. Consul (1966, 1967, 1969 and other papers) used inverse Mellin transforms and operational calculus to derive exact null expressions of

$U_{p,m,n}$ for various values of p,m and n.

Using the method of convolutions, Schatzoff (1964, 1966) derived the central distribution of $U_{p,m,n}$ for either p or m even. He also developed a computer algorithm to tabulate correction factors for converting χ^2 percentiles to exact percentiles of a logarithmic transformation of $U_{p,m,n}$, for m and $p \leq 10$, $mp \leq 70$ and either m or p even. Pillai and Gupta (1969) derived explicit finite series expansions for $U_{p,m,n}$ with either p or m even and $p = 3,4,5$ and 6, also extending Schatzoff's tables, while Lee (1972) simplified the form of the distribution for both p and m odd, further extending the tables to include values for all $p \leq m \leq 20$ and $pm \leq 144$ with omission when p or m is odd and greater than 10.

Mathai and Rathie (1971) used the method of partial fractions to derive explicit series expressions for the null case of $U_{p,m,n}$. Mathai (1971a), using inverse Mellin transforms, calculus of residues and the properties of Psi and Zeta functions, found alternative expressions and published tables of percentage points of $U_{p,m,n}$ for $p \leq 8$, $m \leq 16$, $n \leq 20$, $mp \leq 98$, $mnp \leq 882$ and either m or p even. Mathai (1971a) claimed that the method used was simpler than any other used to date.

Because of the added complication of the noncentrality component, the published work on non-null distributions of test criteria has tended so far to fall into either of three groups. The first involves the derivation of completely general distributions which at the moment offer little hope of computation. The second deals with finding asymptotic representations which are functions of more easily computable terms. In the third group, exact evaluation is made possible by restrictions on the rank and size of noncentrality, or on the size of p,m and n. Papers providing actual numerical values include Roy (1960, 1965), Ito (1962), Posten and Bargmann (1964), Mikhail (1965), Pillai (1965), Pillai and Jayachandran (1967, 1968), de Waal (1968) and Lee (1971, 1971a).

Gupta (1971) used the method of convolutions on equation (5.1.2) to find explicit expressions for the noncentral linear case of $U_{p,m,n}(\lambda^2)$ with $p = 2,3,4$ and 5 and m even, and from these tabulated percentiles for $p = 2$ and 3, $m = 2$, $n \leq 20$ and $\lambda^2 \leq 4,0$.

5.2 Important theorems

The following two theorems are extensions of those given by Anderson (1958) for the central distribution:

Theorem 5.2.1 (Gupta (1971))

In the noncentral linear case $U_{2r,m,n}(\lambda^2)$ is distributed as $X_1(\Pi_{i=1}^{r-1}Y_i^2)X_{2r}$ where X_1 and X_{2r} are independently distributed as $\beta_{1A}(x_1;\, m,n,\lambda^2)$ and $\beta_1(x_{2r};\, m,n-p+1)$ respectively, and the $Y_i(i = 1,2,\ldots,r-1)$ are independently distributed as $\beta_1(y_i;\, 2m,2(n-2i))$. $U_{2s+1,m,n}(\lambda^2)$ is distributed as $X_1\Pi_{i=1}^{s}Y_i^2$ where X_1 is distributed as $\beta_{1A}(x_1;\, m,n,\lambda^2)$ and the $Y_i(i = 1,2,\ldots,s)$ are independently distributed as $\beta_1(y_i;\, 2m,2(n-2i))$. Also, the random variable $Z_i = X_{2i}X_{2i+1}$ has density function

$$(5.2.1)\qquad f(z_i) = (2B(m,n-2i))^{-1} z_i^{\frac{1}{2}(n-2i-2)} (1-z_i^{\frac{1}{2}})^{m-1}$$

<u>Theorem 5.2.2</u> (Das Gupta and Perlman (1973))

In the noncentral linear case

$$(5.2.2)\qquad U_{p,m,n}(\lambda^2) = U_{m,p,n+m-p}(\lambda^2).$$

This follows because

$$(5.2.3)\qquad U_{p,m,n}(\lambda^2) = |\textstyle\sum_{\alpha=1}^{n} X_\alpha X_\alpha'| / |\sum_{\alpha=1}^{n} X_\alpha X_\alpha' + \sum_{\alpha=1}^{m} Y_\alpha Y_\alpha'|$$

$$(5.2.4)\qquad = \Pi_{i=1}^{m} |\textstyle\sum_{\alpha=1}^{n} X_\alpha X_\alpha' + \sum_{\alpha=i+1}^{m} Y_\alpha Y_\alpha'| / |\sum_{\alpha=1}^{n} X_\alpha X_\alpha' + \sum_{\alpha=i}^{m} Y_\alpha Y_\alpha'|$$

$$= U_{m,p,n+m-p}(\lambda^2)$$

We now state a theorem on the convolution of two variables.

<u>Theorem 5.2.3</u> (Schatzoff (1966))

If $V_1 \sim v_1^k e^{sv_1}$, $k > 0$ and integral, and

$V_2 \sim .e^{tv_2}$

then

$$(5.2.5)\qquad V = (V_1+V_2) \sim e^{tv} v^{k+1}/(k+1), \qquad s = t$$

$$(5.2.6)\qquad V \sim e^{sv}\{\textstyle\sum_{r=1}^{k+1} (-1)^{r+1} \dfrac{k!}{(k-r+1)!} \dfrac{v^{k-r+1}}{(s-t)^r}\}$$

$$+ e^{tv}(t-s)^{-(k+1)} k! \qquad , \qquad s \neq t.$$

5.3 <u>The method of convolution</u>

It has been shown in Chapter 4 that the densities of many likelihood ratio criteria reduce to the product of a number of independent Beta densities. The density of

$\log\ U_{p,m,n}(\lambda^2)$ is therefore that of a sum of independent variables, and may be found by using sequential convolution.

We may write $f(x_1) = \beta_{1A}(x_1;\ m,n,\lambda^2)$ as

$$(5.3.1)\qquad (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}x_1^{\frac{1}{2}(n-2)}(1-x_1)^{\frac{1}{2}(m-2)}e^{-\frac{1}{2}\lambda^2}$$

$$\cdot\textstyle\sum_{j=o}^{\infty}(\tfrac{1}{2}(m+n))_j(\tfrac{1}{2}\lambda^2(1-x_1))^j/((\tfrac{1}{2}m)_j j!)\qquad 0 < x < 1$$

$$(5.3.2)\qquad = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}e^{-\frac{1}{2}\lambda^2}\textstyle\sum_{j=o}^{\infty}a_j x_1^{\frac{1}{2}(n-2)}(1-x_1)^{b+j}$$

where

$$(5.3.3)\qquad a_j = ((\tfrac{1}{2}(m+n))_j(\tfrac{1}{2}\lambda^2)^j/(\tfrac{1}{2}m)_j j!)$$

and

$$(5.3.4)\qquad b = \tfrac{1}{2}(m-2)$$

When m is even we may expand powers of $(1-x_1)$ binomially in a finite series to get

$$(5.3.5)\qquad f(x_1) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}e^{-\frac{1}{2}\lambda^2}\textstyle\sum_{j=o}^{\infty}a_j\sum_{k=o}^{b+j}(-1)^k\binom{b+j}{k}x_1^{\frac{1}{2}(n-2+2k)}$$

Putting $y_1 = -\log x_1$ gives

$$(5.3.6)\qquad f(y_1) = (B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}e^{-\frac{1}{2}\lambda^2}\textstyle\sum_{j=o}^{\infty}a_j\sum_{k=o}^{b+j}(-1)^k\binom{b+j}{k}$$

$$\cdot\exp(-\tfrac{1}{2}(n+2k)y_1),\qquad y_1 \geq 0.$$

Similarly, if $f(x_i) = \beta_1(x_i;\ m,n-i+1)$ and $y_i = -\log x_i$,

$$(5.3.7)\qquad f(y_i) = (B(\tfrac{1}{2}m,\tfrac{1}{2}(n-i+1)))^{-1}\textstyle\sum_{\ell=o}^{b}(-1)^{\ell}\binom{b}{\ell}$$

$$\cdot\exp(-\tfrac{1}{2}(n-i+2\ell+1)y_i),\qquad y_i \geq 0$$

If $Z_i = X_{2i}X_{2i+1}$ and if $Y_i' = -\log Z_i$

$$(5.3.8)\qquad f(y_i') = (2B(m,n-2i))^{-1}\textstyle\sum_{\ell=o}^{m-1}(-1)^{\ell}\binom{m-1}{\ell}$$

$$\cdot\exp(-\tfrac{1}{2}(n+\ell-2i)y_i'),\qquad y_i' \geq 0.$$

By using Theorems 5.2.1 and 5.2.3 and equations (5.3.6), (5.3.7) and (5.3.8), Gupta (1971) derived the following results for m even:

Theorem 5.3.1 (Gupta (1971))

Case 1

For $p = 2$ the density of $U_{2,m,n}(\lambda^2)$ is given by

$$(5.3.9)\quad Ke^{-\frac{1}{2}\lambda^2} u^{\frac{1}{2}n-1} \sum_{j=0}^{\infty} a_j \{ \sum_{k=0}^{b+j} \sum_{\ell=0}^{b} ((-1)^{\ell+k}/(2\ell-2k-1)) \cdot \binom{b+j}{k}\binom{b}{\ell}(u^k - u^{\ell-\frac{1}{2}})\}, \qquad 0 \le u \le 1$$

where

$$(5.3.10)\quad K = 2\Pi_{i=1}^{2}(B(\tfrac{1}{2}m, \tfrac{1}{2}(n-i+1)))^{-1}$$

and

$$(5.3.11)\quad a_j = ((\tfrac{1}{2}(m+n))_j (\tfrac{1}{2}\lambda^2)^j / (\tfrac{1}{2}m)_j j!).$$

Case 2

For $p = 3$ the density of $U_{3,m,n}(\lambda^2)$ is given by

$$(5.3.12)\quad Ke^{-\frac{1}{2}\lambda^2} u^{\frac{1}{2}n-1} \sum_{j=0}^{\infty} a_j \{ \sum_{k=0}^{b+j} (-u)^k \binom{m-1}{2k+2}\binom{b+j}{k} \log u + 2\sum_{k=0,\ell\neq 2k+2}^{b+j} \sum_{\ell=0}^{m-1} ((-1)^{\ell+k}/(\ell-2k-2)) \cdot \binom{m-1}{\ell}\binom{b+j}{k}(u^k - u^{\frac{1}{2}\ell-1})\}$$

where

$$(5.3.13)\quad K = (2B(\tfrac{1}{2}m, \tfrac{1}{2}n)B(m, n-2))^{-1}$$

Case 3

For $p = 4$ the density of $U_{4,m,n}(\lambda^2)$ is given by

$$(5.3.14)\quad Ke^{-\frac{1}{2}\lambda^2} u^{\frac{1}{2}n-1} \sum_{j=0}^{\infty} a_j (T_{1j} + 2T_{2j})$$

where

$$(5.3.15)\quad T_{1j} = \Sigma_{k,t}((-1)^{k+t}u^k/(2t-2k-3))f(0,t,k)\{\binom{m-1}{2k+2}$$

$$\cdot((2u^{t-k-3/2}-2)/(2t-k-3)-\log u)-\binom{m-1}{2t-1}u^{t-k-3/2}\log u\},$$

$$(5.3.16)\quad T_{2j} = \Sigma_{k,\ell,\ell\neq 2k+2}((-1)^{\ell+k}/(\ell-2k-2))f(\ell,0,k)$$

$$\cdot\{\Sigma_t((-1)^t/(2t-2k-3))\binom{b}{t}(u^k-u^{t-3/2})$$

$$-\Sigma_{t,t\neq\frac{1}{2}(\ell+1)}((-1)^t/(2t-\ell-1))\binom{b}{t}(u^{\frac{1}{2}\ell-1}-u^{t-3/2})\},$$

$$(5.3.17)\quad K = (B(m,n-2))^{-1}\Pi_{i=1}^{2}(B(\tfrac{1}{2}m,\tfrac{1}{2}(n-3i+3)))^{-1}$$

and

$$(5.3.18)\quad f(\ell,t,k) = \binom{m-1}{\ell}\binom{b}{t}\binom{b+j}{k}$$

<u>Case 4</u>

<u>For $p = 5$</u> the density of $U_{5,m,n}(\lambda^2)$ is given by

$$(5.3.19)\quad Ke^{-\frac{1}{2}\lambda^2}u^{\frac{1}{2}n-1}\Sigma_{j=o}^{\infty}a_j(T_{1j}+4T_{2j}+8T_{3j})$$

where

$$(5.3.20)\quad T_{1j} = \Sigma_k(-u)^k f(2k+2,0,k)\{\binom{m-1}{2k+4}(\log u)^2$$

$$+4\Sigma_{t,t\neq 2k+4}((-1)^t/(t-2k-4))\binom{m-1}{t}((2u^{\frac{1}{2}t-k-2}-2)/$$

$$\cdot(t-2k-4)-\log u)\},$$

$$(5.3.21)\quad T_{2j} = \Sigma_{\ell,k,\ell\neq 2k+2}((-u)^k/(\ell-2k-2))f(\ell,0,k)\log u$$

$$\cdot\{\binom{m-1}{\ell+2}u^{\frac{1}{2}\ell-k-1}-(-1)^\ell\binom{m-1}{2k+4}\},$$

$$(5.3.22)\quad T_{3j} = \Sigma_{\ell,k,\ell\neq 2k+2}((-1)^{\ell+k}/(\ell-2k-2))f(\ell,0,k)$$

$$\cdot\{\Sigma_{t,t\neq 2k+4}(-1)^t\binom{m-1}{t}((u^k-u^{\frac{1}{2}t-2})/(t-2k-4))$$

$$-\Sigma_{t,t\neq \ell+2}(-1)^t\binom{m-1}{t}((u^{\frac{1}{2}\ell-1}-u^{\frac{1}{2}t-2})/(t-\ell-2))\},$$

(5.3.23) $K = (8B(\frac{1}{2}m,\frac{1}{2}n))^{-1}\Pi_{i=1}^{2}(B(m,n-2i))^{-1}$

and

(5.3.24) $f(\ell,t,k) = \binom{m-1}{\ell}\binom{m-1}{t}\binom{b+j}{k}$

The cumulative distribution functions are obtained by straightforward integration of the density functions and are given in Gupta (1970).

We shall now outline the main developments of the convolution approach for central densities, and its associated problems. Mathai (1973) states that the method of successive convolution is usually as difficult as that of direct integration, and points out that a density derived by inverse Mellin transform would similarly be a sum of terms of the type $au^b(-\log u)^c$.

Schatzoff (1964, 1966) obtained his results by using equation (5.1.2), Theorem 5.2.2 and Theorem 5.2.3. Pillai and Gupta (1969) added Theorem 5.2.1 with $U_{2r,m,n} \sim \Pi_{i=1}^{r}Y_i^2$ and $U_{2s+1} \sim \Pi_{i=1}^{s}Y_i^2X_{2s+1}$. They stated "While Schatzoff's method is not suitable for handling the distribution problem for odd values of m, the method of this paper gives the distribution explicitly in all cases. Also, unlike Consul (1966) who gave the distributions for p up to 4 as infinite series, we are giving the distribution here in finite series form except when both p and m are odd in which case alone the series is infinite." The main point, however, is that both Schatzoff's and Pillai and Gupta's methods provide solutions for either p or m even, but Pillai and Gupta's use of Theorem 5.2.1 means fewer convolutions and less compounding of error, and obviates use of Theorem 5.2.2. Also, when m and p are both odd, only the final convolution involves an infinite series. Lee (1972) found percentiles for densities with both p and m odd and less than 10, by expressing them in integral form and using recurrence relations. He noted that the computation could only be carried out for the lower values of p and m, because of the severe loss of significant digits by taking differences of nearly equal numbers. This, together with the infinite series representation, will be seen to be the major computational problem in the non-central linear case.

With successive convolutions the densities and cumulative distribution functions of $U_{p,m,n}$ become increasingly unmanageable, and Schatzoff (1966) gave the following general form for $U_{p,m,n}$.

Theorem 5.3.2 (Schatzoff (1966))

When m is an even integer the density function of $U_{p,m,n}$ is of the form

(5.3.25) $f(u) = \left(\Pi_{i=1}^{p} K_i\right) \sum_{j=1}^{q} c_j u^{\frac{1}{2}(n-\ell_j)} (-\log u)^{k_j}$

where

(5.3.26) $K_i = (\Gamma(\frac{1}{2}(m+n-i+1))/\Gamma(\frac{1}{2}(n-i+1))\Gamma(\frac{1}{2}m))$

and constants c_j and integers q, ℓ_j and k_j are determined from p, m and n.

This is more concisely expressed by

Theorem 5.3.3 (Lee (1972))

If $W = (U_{p,m,n})^{\frac{1}{2}}$, $p \leq m$ and either p or m is even,

(5.3.27) $$f(w) = Kw^{n-p}\sum_{j=o}^{t}(-\log w)^j \sum_{k=2j}^{m+p-2j-3} a_{jk}w^k$$

where K is the normalising constant, and $t = \frac{1}{2}p$ (p even) or $\frac{1}{2}(p-1)$ (p odd). The a_{jk} depend on p and m only, and satisfy the relation

(5.3.28) $$a_{j,m+p-2j-\ell-3} = \sigma(p,m,j)a_{j,2j+\ell}$$

where $$\sigma(p,m,j) = \begin{cases}(-1)^{j+1} & \frac{1}{2}m \text{ even} \\ (-1)^{p+j+1} & \frac{1}{2}m \text{ odd.}\end{cases}$$

For $m \leq p$, m and p may be interchanged.

(Lee (1972) gives the value of t for p odd as $\frac{1}{2}(p+1)$, but examination of the successive densities shows $\frac{1}{2}(p-1)$ to be correct.)

Gupta (1971) proposed the following general form for the noncentral linear case, but used the explicit expressions of Theorem 5.3.1 to calculate percentiles for $p = 2$ and 3.

Theorem 5.3.4 (Gupta (1971))

The probability density function of $U_{p,m,n}(\lambda^2)$ is of the form

(5.3.29) $$(\Pi_{i=1}^{p}K_i)e^{-\frac{1}{2}\lambda^2}\sum_{j=o}^{\infty}a_j\sum_{k=o}^{q_j}C_{jk}u^{\frac{1}{2}(n-b_k)}(-\log u)^{d_k}$$

where

(5.3.30) $$K_i = (B(\tfrac{1}{2}m,\tfrac{1}{2}(n-i+1)))^{-1},$$

a_j is defined as in (5.3.3), and the constants C_{jk} and integers q_j, b_k and d_k are determined from p,m and n.

Gupta stated that Theorem 5.3.4 did not indicate explicitly how to find values for C_{jk}, q_j, b_k and d_k, but provided a basis for a recursive algorithm for deriving the density and distribution function at successive stages of the convolution process.

5.4 An algorithm for deriving density and cumulative distribution functions

For any value of p we shall be concerned with the density of

(5.4.1) $$Y_p = -\log U_{p,m,n}(\lambda^2) = \sum_{i=1}^{p}(-\log X_i).$$

By carefully studying Theorem 5.3.1 or by performing a series of convolutions we may observe that

Theorem 5.4.1

For any p, and even m, the density of $Y_p = -\log U_{p,m,n}(\lambda^2)$ may be expressed in a general form as an infinite series involving integral powers of y_p and $e^{-\frac{1}{2}y_p}$:

(5.4.2) $$f(y_p) = K_p^{-1}e^{-\frac{1}{2}\lambda^2}\sum_{j=0}^{\infty}a_j\sum_{r=1}^{s}\sum_{c=c_c}^{c_f}d_{jrc}e^{-\frac{1}{2}cy_p}y_p^{r-1}$$

where

(5.4.3) $$K_p = B(\tfrac{1}{2}m,\tfrac{1}{2}n)\Pi_{i=1}^{s-1}(2B(m,n-2i)), \qquad p \text{ odd}$$

$$= B(\tfrac{1}{2}m,\tfrac{1}{2}n)(\Pi_{i=1}^{s-2}(2B(m,n-2i)))B(\tfrac{1}{2}m,\tfrac{1}{2}(n-p+1)), \qquad p \text{ even},$$

(5.4.4) $$s = \tfrac{1}{2}(p+1), \ p \text{ odd}; \ = \tfrac{1}{2}(p+2), \ p \text{ even},$$

(5.4.5) $$c_c = n-2s+2r \text{ and } c_f = m+n+2_j-2$$

This differs from Gupta's (1971) general form in that there is a triple, not double summation. But whereas Gupta's form includes four sets of unknowns (powers, constants and summation limits), the density of Theorem 5.4.1 is completely specified once the constants d_{jrc} are known. The density and cumulative distribution functions may then be found by setting $u = e^{-y_p}$ and integrating over the required range of u.

At this stage we point out that in some cases the range of summation of Gupta's explicit densities is implicitly limited. In equation (5.3.12), for example, $\sum_{k=o}^{b+j}(-u)^k\binom{m-1}{2k+2}\binom{b+j}{k}\log u$ is limited by $2k \leq m-1$, i.e. $k \leq \frac{1}{2}m-2$, i.e. $k \leq b-1$. The summation in that case is therefore effectively $\sum_{k=o}^{b-1}$.

The distribution problem reduces to that of finding the matrix of constants $\left(d_{jrc}\right)_p$ for any given values of p,m,n and λ^2. We shall solve it by determining the initial values of $\left(d_{jrc}\right)_p$ for $p = 1$, and then show how the new matrices of constants $\left(d'_{jrc}\right)_{p+1}$ or $\left(d'_{jrc}\right)_{p+2}$ are created at each convolution. We shall use Theorem 5.2.1 by performing a series of "double" convolutions, and finally a "single" convolution only if p is even. Unlike the central case, the general form of $U_{p,m,n}(\lambda^2)$ involves an infinite series expansion in j. Gupta (1971) noted that the total integral of the series determined by taking a few terms only, rapidly approaches the theoretical value one as more terms are taken into account. Our results will confirm this, and show that the rate of convergence depends inversely on λ^2.

Theorem 5.4.2

The matrix of coefficients $\{d_{jrc}\}_p$, for $p = 1$ is given by

$$(5.4.6) \qquad d_{j1c} = (-1)^{\frac{1}{2}(c-n)} \binom{\frac{1}{2}(m-2+2j)}{\frac{1}{2}(c-n)} \qquad (c-n) \text{ even}$$
$$= 0 \qquad (c-n) \text{ odd}$$

By examining carefully the effect of "single" and "double" convolution on the general form of Theorem 5.4.1, we have

Theorem 5.4.3

Let the general form of the jth term of $f(Y_p)$ be

$$(5.4.7) \qquad Y_p(j) = J_p \sum_{r=1}^{s} \sum_{c=c_c}^{c_f} d_{jrc} e^{-\frac{1}{2}cy_p} y_p^{r-1}$$

where

$$(5.4.8) \qquad J_p = K_p^{-1} e^{-\frac{1}{2}\lambda^2} a_j$$

and s, c_c, c_f, K_p and a_j are so defined in (5.4.2).

Then the general form of $Y_{p+1}(j)$ and $Y_{p+2}(j)$, after convolution with (5.3.7) and (5.3.8) respectively, is given by

$$(5.4.9) \qquad Y_{p+i}(j) = J_{p+i} \sum_{r=1}^{s} \sum_{c=c_c}^{c_f} (G_1+G_2+G_3)$$

where

$$(5.4.10) \qquad G_1 = t(-1)^{\ell^*} \binom{b^*}{\ell^*} d_{jrc} e^{-\frac{1}{2}cy_{p+i}} y_{p+i}^r / r \;,$$

$$(5.4.11) \qquad G_2 = \sum_{\substack{\ell=o \\ \ell \neq \ell^*}}^{b^*} (-1)^{\ell+1} \binom{b^*}{\ell} d_{jrc} \sum_{r^*=1}^{r} \left(\frac{2}{c-n^*}\right)^{r^*} \frac{(r-1)!}{(r-r^*)!}$$
$$\cdot e^{-\frac{1}{2}cy_{p+i}} y_{p+i}^{r-r^*} \;,$$

$$(5.4.12) \qquad G_3 = \sum_{\substack{\ell=o \\ \ell \neq \ell^*}}^{b^*} (-1)^{\ell} \binom{b^*}{\ell} d_{jrc} \left(\frac{2}{c-n^*}\right)^{r} (r-1)! e^{-\frac{1}{2}n^* y_{p+i}}$$

and for <u>i = 1</u> or <u>i = 2</u>

$$(5.4.13) \quad J_{p+i} = J_p(B(\tfrac{1}{2}m,\tfrac{1}{2}(n-p)))^{-1} \quad \text{or} \quad J_p(2B(m,n-2s))^{-1}$$

$$\ell^* = \tfrac{1}{2}(c+p-n) \qquad \text{or} \quad c+2s-n$$

$$b^* = \tfrac{1}{2}(m-2) \qquad \text{or} \quad m-1$$

$$n^* = n+2\ell-p \qquad \text{or} \quad n+\ell-2s$$

and

$$t = 1 \quad \text{for} \quad \ell^* \quad \text{integral}$$
$$= 0 \quad \text{for} \quad \ell^* \quad \text{nonintegral}$$

The new matrix of coefficients $\{d'_{jrc}\}_{p+i}$ is the union of sets of contributions from each of the G_k $(k = 1,2,3)$.

From G_1, for $r = 2,\ldots,s$ and $c = c_c,\ldots,c_f$, d'_{jrc} receives

$$(5.4.14) \quad d'(1) = t(-1)^{\ell^*}\binom{b^*}{\ell^*}d_{j(r-1)c}/(r-1).$$

From G_2, for $r = 1,\ldots,s$; $c = c_c,\ldots,c_f$ and $q = 1,\ldots,r$, d'_{jrc} receives

$$(5.4.15) \quad d'(2) = \frac{(r-1)!}{(q-1)!}\, d_{jrc}\sum_{\substack{\ell=o \\ \ell\neq\ell^*}}^{b^*} (-1)^{\ell+1}\binom{b^*}{\ell}\left(\frac{2}{c-n^*}\right)^{r-q+1} .$$

From G_3, for $c = (c_c-i),\ldots,(m+n-2s-1)$, d'_{j1c} receives

$$(5.4.16) \quad d'(3) = t\sum_{r=1}^{s}\sum_{\substack{g=c_c \\ g\neq c}}^{c_f}(-1)^{\ell^*}\binom{b^*}{\ell^*}\left(\frac{2}{g-c}\right)^{r} (r-1)!\,d_{jrg}$$

The convolution procedure therefore determines the coefficients necessary to specify values of the density and cumulative distribution function, which are given in the same general form below.

<u>Theorem 5.4.4</u>

The density and cumulative distribution function of

$U_{p,m,n}(\lambda^2)$ are given by $f(u)$ and $F(u)$ respectively where

$$(5.4.17) \quad f(u) = K_p^{-1} e^{-\frac{1}{2}\lambda^2} \sum_{j=o}^{\infty} a_j \sum_{r=1}^{s} \sum_{c=c_c}^{c_f} d_{jrc} \cdot u^{\frac{1}{2}(c-2)} (-\log u)^{r-1} \qquad 0 \leq u \leq 1$$

and

$$(5.4.18) \quad F(u) = K_p^{-1} e^{-\frac{1}{2}\lambda^2} \sum_{j=o}^{\infty} a_j \sum_{r=1}^{s} \sum_{c=c_c}^{c_f} (r-1)! \cdot u^{\frac{1}{2}c} d_{jrc} \sum_{r^*=1}^{r} \frac{(-\log u)^{r-r^*}}{(r-r^*)!} \left(\frac{2}{c}\right)^{r^*} \qquad 0 \leq u \leq 1$$

and s, c_c, c_f, K_p and a_j are as defined in (5.4.2)

The convolution process obviously involves computation and storage of a large number of coefficients. Careful examination of (5.4.14), (5.4.15) and (5.4.16) enable us to streamline this by noting

<u>Theorem 5.4.5</u>

At any stage of convolution, only elements d_{jrc} with the following subscripts may be non-zero:

<u>$r = 1$</u> and $c = ((m+n-2+2i), \; i = 0,1,\ldots,j)$, all j

or $c = ((n-2s+2),\ldots,(m+n-2))$, all j:

and

<u>$r > 1$</u> and $c = ((n-2s+2r),\ldots,(m+n-2r))$, all j.

It follows that all d^*_{jrc} are zero for $r^* > \frac{1}{2}m$. The results of Lee (1972) given in equations (5.3.27) and (5.3.28) do not hold in the noncentral linear case.

5.5 Computing percentiles and powers : accuracy and convergence

In order to calculate the power of the test criteria for any specified level of significance α, we first need to find the lower (for $|L|$) α-percentile of the central distribution. For a chosen level of noncentrality λ^2, we then have to calculate the value of the noncentral cumulative distribution function at the α-percentage point.

An iteration process is therefore necessary for the first stage, and the Newton-Raphson method (Hildebrand (1956)) was chosen. This guarantees rapid convergence if the following conditions are satisfied:

(i) the sign of the second derivative $F''(u_\alpha^{(o)})$ should be the same as that of $(u_\alpha^{(o)}-u_\alpha^*)$, where $u_\alpha^{(o)}$ is the initial value and u_α^* the actual percentile.

(ii) $F'(u)$ should not change sign in the interval between u_α^* and $u_\alpha^{(o)}$ (with a probability density function this should always be positive). The derivatives are easily determined and we do not list them here.

The initial value $u_\alpha^{(o)} = \alpha^{pm/n}$, suggested by Schatzoff (1964), is often satisfactory, but if not, the $u_\alpha^{(o)}$ can be successively incremented until conditions (i) and (ii) are both satisfied. Alternatively, a starting point may be found for the exact calculation by iterating (more easily) one of the null approximations such as that of Rao (1948):

(5.5.1) $$\Pr(-a \log U_{p,m,n} \leq z) = \Pr(\chi^2_{mp} \leq z)$$
$$+ (\gamma^{2/a^2})\{\Pr(\chi^2_{mp+4} \leq z) - \Pr(\chi^2_{mp} \leq z)\}$$
$$+ a^{-4}\left(\gamma_4\{\Pr(\chi^2_{mp+8} \leq z) - \Pr(\chi^2_{mp} \leq z)\}\right.$$

$$- \gamma_2^2\{(\Pr(\chi^2_{mp+4} \leq z) - \Pr(\chi^2_{mp} \leq z)\}\big) + R_5^v$$

where

(5.5.2) $\quad a = n+\frac{1}{2}(m-p-1),$

(5.5.3) $\quad \gamma_2 = mp(m^2+p^2-5)/48,$ and

(5.5.4) $\quad \gamma_4 = \frac{1}{2}\gamma_2^2+(mp/1920)\{3p^4+3m^4+10m^2p^2-50(m^2+p^2)+159\}$

(The remainder term R_5^v is of order $O(N^{-6})$; if the first term only is used, error is of order $O(N^{-2})$, and if the first two are used, $O(N^{-4})$. Anderson (1958) gives further details.)

The iteration takes place once the necessary set of coefficients d_{jrc} ($j = 0$ here) has been found for the null distribution, by using Theorems 5.4.2 and 5.4.3. Once a percentage point has been found to a predetermined level of accuracy, the iteration stops and the cumulative distribution function is evaluated at that point. This requires a new series of convolutions for the non-null distribution with noncentrality parameter λ^2. At each stage of the convolution the coefficients d_{jrc} are needed, where, by Theorem 5.4.1, $j = 0,1,2,\ldots,\infty$; $r = 1,2,\ldots,s$; $c = n-2s+2r,\ldots,m+n+2j-2$; and $s = \frac{1}{2}(p+1)$ for p odd and $\frac{1}{2}(p+2)$ for p even. The coefficients thus comprise an infinite set (in j) of matrices of size $s \times (m+2j+2s-3)$. Theorem 5.4.5 simplifies matters by listing the only possible non-zero elements, but it can be seen that the first row of each matrix (when $r = 1$) tends to have infinite length as it is directly dependent on $2j$.

The function $F(u)$ increases monotonically with j and converges rapidly for reasonably low values of λ^2. It can therefore be truncated where required, thus limiting the number and maximum dimensions of coefficient matrices. A suitable point for truncation may be found by monitoring the sum of successive terms of the series for the cumulative distribution function when $u = 1$ and $p = 1$: this enables an upper bound on j to be set at the start. Once convergence has started, at roughly $j = (\frac{1}{2}\lambda^2+1)$, the sequence of terms decreases monotonically with j, and if truncated too soon, the power will always be underestimated for $|L|$ (but overestimated for test criteria such as $|I-L|$ where upper percentage points are used).

Examination of the series expansion of $F(u)$ shows that the part most affected by changes in j is a_j, and particularly λ^2. Let

(5.5.5) $$F(u) = K_p^{-1}e^{-\frac{1}{2}\lambda^2}\sum_{j=o}^{\infty}a_jQ_j$$

and let $a_{k+1} \leq qa_k$, $0 < q < 1$.

Then we may show that

(5.5.6) $$k \geq (2q)^{-1}\left(\tfrac{1}{2}\lambda^2-q(\tfrac{1}{2}m+1)+\{(q(\tfrac{1}{2}m+1)-\tfrac{1}{2}\lambda^2)^2-q(2mq-(m+n)\lambda^2)\}^{\frac{1}{2}}\right)$$

As an upper bound for $F(u)$ we have that (assuming $Q_k = Q_{k+1}$)

(5.5.7) $$F(u) = K_p^{-1}e^{-\frac{1}{2}\lambda^2}(\sum_{j=o}^{k}a_jQ_j+(1-q)^{-1}a_{k+1}Q_{k+1})$$

In practise this tends to overestimate $F(u)$ slightly, as both the ratio a_{k+1}/a_k and the series Q_i are monotone decreasing.

The main problem in the use of exact formulae remains the severe loss of significant digits through heavy cancellation as p,m and n increase. (Schatzoff (1964) refers to this as the decimal accuracy problem.) It can be seen that K_p^{-1} increases rapidly as p,m and n become large. As all terms (a_jQ_j) in j are positive it follows that they must become correspondingly small. The individual coefficients d_{jrc}, however, tend to increase both in size and in number as m and p increase. This means that Q_j, a small positive number, is the end result of addition of a large number of sizeable positive and negative terms.

For example, the central cumulative frequency distribution of $U_{5,10,10}$ at $u = 1$, given by

$$(5.5.8) \qquad K_p^{-1}\sum_{r=1}^{s}\sum_{c=c_c}^{c_f}(r-1)!(2/c)^r d_{jrc} = K_p^{-1}\Sigma_r\Sigma_c Q_{rc},$$

has $\max_{r,c} Q_{rc}$ of order 10^5 but

$$\Sigma_r\Sigma_c Q_{rc} = K_p \text{ of order } 10^{-12}.$$

Thus at least 17 significant digits are needed if the answer is to be correct to only one significant digit.

The position is not quite as severe with lower values of u, which aids computation of lower percentiles with small n, but is aggravated by increasing j in the non-central case. Consequently, computation with low values of λ^2 is not only quicker, but also more accurate, if sufficient convergence has taken place before the decimal accuracy problem becomes serious. The problems of convergence and decimal accuracy act in concert so that, although one or two of the parameters p,m,n and λ^2 may be quite large, they may not simultaneously be big.

Without structural changes in the model, this problem appears insoluble. The cancellation was monitored at all stages of convolution and function evaluation, and the truncation error noted at different times. In this way a good indication of the accuracy of each result was obtained. From the above discussion it is obvious that computation requires double precision facilities : multiple precision would be preferable and would increase the applicable parameter range.

CHAPTER 6

RESULTS FOR THE TEST CRITERIA $|L|$, $|I-L|$ AND $|I-\Sigma L_j|$

6.1 Exact noncentral percentiles of $|L|$

Results correct to at least six significant digits have been calculated for various combinations of p up to 13, m up to 50, n up to 120 and λ^2 up to 32. With a lower degree of accuracy they can be extended further, but accurate results with all parameters large cannot be achieved because of the decimal accuracy problems mentioned in section 5.5. All calculations were performed using double precision arithmetic.

The only other published noncentral linear percentiles of $|L|$ known to the author are those of Gupta (1971), calculated from the explicit expressions given in Theorem 5.3.1, which he compares with Posten and Bargmann's (1964) approximations. In Table 6.1 we compare results derived by the method of convolutions with Gupta's comparison. This confirms Gupta's statement that the Posten and Bargmann approximation is not so good for small $t = n+\frac{1}{2}(m-p-1)$ and large λ^2. It also shows that for n (and t) large, the approximation is even better than Gupta's "exact" calculation. This is probably due to Gupta's infinite series being truncated after too few terms - more terms are needed as λ^2 increases, and the greatest discrepancies occur where λ^2 is largest (i.e. 4,0).

TABLE 6.1

Gupta's exact (GE), Posten and Bargmann's approximation (PB) and Hart's exact (HE) percentage points when $\alpha = 0{,}95$

p	m	n	λ^2	GE	PB	HE
2	2	2	0,5	0,571752	0,572598	0,571752
			1,0	0,541292	0,543222	0,541292
			4,0	0,385821	0,398234	0,385821
2	2	8	0,5	0,898552	0,898554	0,898552
			1,0	0,886876	0,886881	0,886876
			4,0	0,802938	0,803031	0,802938
2	2	20	0,5	0,959648	0,959648	0,959648
			1,0	0,954656	0,954656	0,954656
			4,0	0,915845	0,915839	0,915836
3	2	8	0,5	0,775704	0,775729	0,775704
			1,0	0,760957	0,761017	0,760957
			4,0	0,670107	0,676552	0,670098
			8,0			0,559692
			16,0			0,405089
3	2	20	0,5	0,910887	0,910897	0,910897
			1,0	0,904028	0,904040	0,904039
			4,0	0,857452	0,857602	0,857585
			8,0			0,789103
			16,0			0,662312

The main use of the noncentral distribution is in determining the power of various tests and for this reason we shall not discuss it further, but shall list a few tables of percentage points for larger values of p in Appendix 2.

TABLE 6.2

$m = 2 \quad n = 100 \quad \alpha = 0,05$

	p = 2		p = 3		p = 4	
λ^2	ITO	EXACT	ITO	EXACT	ITO	EXACT
1	0,103	0,1030	0,092	0,0912	0,085	0,0844
2	0,165	0,1658	0,141	0,1407	0,126	0,1256
3	0,234	0,2349	0,196	0,1964	0,173	0,1726
4	0,305	0,3073	0,255	0,2564	0,224	0,2240
5	0,377	0,3803	0,317	0,3189	0,278	0,2784
6	0,447	0,4517	0,378	0,3820	0,333	0,3346
7	0,513	0,5197	0,439	0,4442	0,388	0,3912
8	0,576	0,5832	0,498	0,5044	0,443	0,4471

6.2 Comparison of exact and asymptotic powers of $|L|$

In this section we examine the accuracy of some published asymptotic powers of $U = |L|$ in the linear case. J. Roy (1960) obtained an asymptotic expression for the distribution of $-n \log U$ to the order n^{-1}. and tabulated approximate powers for $n = 200$. These were extended by Ito (1962), who compared the approximate powers with those of the Lawley-Hotelling T_o^2 test. Table 6.2 compares Ito's results with those derived by our method and shows that for $\lambda^2 > 2$, Ito's approximation underestimates the exact power by an amount which increases with λ^2 and decreases with p. The exact powers in Table 6.2 are all equal to or greater than Ito's approximate powers for the T_o^2 test.

Mikhail (1965) used a Pearson type III approximation to the moments of $(n(1-U)/mU)$ for $p = 2$, and compared it with an exact power calculation. He used

T A B L E 6.3

Ito's approximate (ITO), Mikhail's exact (ME), Hart's exact (HE) and Patnaik's approximate (PA) powers of $U_{2,2,97}(\lambda^2)$ when α = 0,05

λ^2	ITO	ME	HE	PA
1	0,103	0,105	0,1030	0,105
2	0,165	0,168	0,1656	0,169
4	0,305	0,309	0,3070	0,312

T A B L E 6.4

Mikhail's approximate (MA), Mikhail's adjusted (MB) and Hart's exact (HE) powers of $U_{2,4,n}(\lambda^2)$ when α = 0,05

	n = 45			n = 120		
λ^2	MA	HE	MB	MA	HE	MB
1	0,087	0,0820	0,081	0,087	0,0848	0,085
2	0,134	0,1199	0,118	0,132	0,1266	0,126
4	0,252	0,2096	0,208	0,244	0,2264	0,227

noncentrality parameter $\Lambda = \Sigma\lambda_i^2$ after showing that the first two moments of his statistic did not depend on any other function of the λ_i^2, and concluded that "the approximation is evidently excellent for the larger values of N; for the smaller values of N it becomes poor for large Λ." This conclusion is based on an inaccurate calculation of the exact power. His statement that "Ito's approximation is no better than a simple limit using the Patnaik approximation" is also invalid (see Table 6.3).

Mikhail's approximation consistently overestimates the true power by an amount which increases with Λ and decreases with n. This is shown in Table 6.4, together with an "adjusted" power obtained by subtracting $\Lambda^{3/2}/(4n)$ from Mikhail's approximation.

Posten and Bargmann (1964) have developed approximations of order q^{-1} and q^{-2} to $-q \log U_{p,m,n}(\lambda 2)$ for the linear and planar cases, where $q = n+\frac{1}{2}(m-p-1)$, and from Table 6.1 these appear to give very satisfactory results. Comparisons made in Posten and Bargmann (1964) for either p or $m = 1$ confirm the implied accuracy of their approximations of order q^{-1} and q^{-2}.

Pillai and Jayachandran (1967) obtained an expansion for $U_{2,m,n}(\Lambda)$ and tabulated exact powers for various values

TABLE 6.5

Roy's Gamma approximation (RG), de Waal's χ^2 approximation (deW) and Hart's exact (HE) powers for $U_{p,4,50}(\lambda^2)$ when $\alpha = 0{,}05$.

	p = 3			p = 4		
λ^2	RG	deW	HE	RG	deW	HE
2	0,1053	0,1029	0,1044	0,0960	0,0931	0,0948
6	0,2526	0,2505	0,2558	0,2180	0,2126	0,2194
10	0,4168	0,4262	0,4321	0,3584	0,3610	0,3705

of m, n, λ_1^2 and λ_2^2. Here attention should be drawn to the serious effect that rounding error or cancellation may have on computation and conclusions drawn. Pillai and

Jayachandran (1967) calculated and tabulated percentiles to eight significant digits, yet checks by Lee (1971a) and the author have shown that the associated powers in some cases were correct to only one significant digit. As the powers of different test criteria (such as Hotelling's T_o^2 and Roy's $V^{(p)}$) are often very similar in magnitude, false conclusions regarding superiority of tests may easily be drawn. Pillai and Jayachandran's results appear most seriously affected when m is low and λ^2 high, and in almost every case underestimate the true power.

de Waal (1968) derived a χ^2 approximation of order n^{-1} to $-q \log U_{p,m,n}(\lambda^2)$, where $q = n+\frac{1}{2}(m-p-1)$. Table 6.5 compares this with J. Roy's (1965) Gamma approximations and our exact results. The χ^2 approximation appears better for larger λ^2.

Sugiura and Fujikoshi (1969) found asymptotic expansions of $|L|$ for the general noncentral case to the order of q^{-2}. Using these, Lee (1971a) tabulated approximate powers of $|L|$ for $p = 2,3$ and 4, various values of m and n, and combinations of up to four non-zero values of λ^2. For $p = 2$ and $n = 63$, $m = 3,5$ and 7 and $(\lambda_1^2+\lambda_2^2)$ up to 10, Lee (1971a) showed that the approximation did not differ from the exact value by more than 3 units in the fourth significant digit. The exact checks which can be made on Lee's powers for $p = 3$ and 4 also indicate good approximation. Using the approximation of Sugiura and Fujikoshi (1969), Fujikoshi (1970) tabulated approximate powers of $|L|$ for $p = 3$, $m = 3,5$ and 7, $n = 85$ and 170 and combinations of λ_i^2 up to $\Sigma\lambda_i^2 = 4,5$.

6.3 $|I-L|$ as a test statistic

The statistic $|I-L| = |B|/|A+B|$, where A and B are central and noncentral Wishart matrices with n and m degrees of freedom respectively, was originally proposed by Wilks (1932) in the central case as a generalisation of the correlation ratio. Pearson and Wilks (1933), discussing the bivariate case, stated that $|I-L|$ could not be expressed as a single valued function of $|L|$ and would not be suitable as an alternative test criterion for the hypothesis of equality of means of k populations.

Hsu (1940) examined the moments and distribution of $|I-L|$ for general p and noted that, for $p = 2$, the disadvantage of $|I-L|$ was that it was unlikely to be able to detect the falsehood of the null hypothesis if only one of the two noncentrality parameters λ_i^2 vanished. Ito (1962) recorded its use as a test statistic, and its distributional properties, both exact and asymptotic, have been examined by Troskie (1966, 1972), de Waal (1968, 1968a and 1970) and Money (1972).

Pillai and Nagarsenker (1972) stated that $|I-L|$ could be used as a test criterion for the three tests of (i) equality of two covariance matrices, (ii) canonical correlation and (iii) MANOVA, and gave exact and asymptotic noncentral densities of $|I-L|$ for the three tests. Fujikoshi (1972) noted that $|I-L|$ could be used as a test criterion for the multivariate linear hypothesis or for testing equality of two covariance matrices, and derived asymptotic expansions of $|I-L|$ in the noncentral case.

We shall now examine the suitability of $|I-L|$ as a test criterion in the general p-variate case.

Under the null hypothesis H_0, $|L|$ tends to one and H_0 will be rejected if $|L|$ is significantly close to zero. It seems reasonable, therefore, that under H_0 $|I-L|$ tends to zero and H_o should be rejected when $|I-L|$ is sufficiently close to one. If θ_i, $i = 1,2,\ldots,\ell$ ($\ell = \min(m,p)$) are the non-identically vanishing roots of the equation

$$(6.3.1) \quad |B-\theta(A+B)| = 0$$

we have that

$$(6.3.2) \quad |L| = \Pi_{i=1}^{\ell}(1-\theta_i)$$

and

$$(6.3.3) \quad |I-L| = \Pi_{i=1}^{\ell} \theta_i$$

It can be seen that, for $|L|$ to tend to one, all θ_i must tend to zero, whereas for $|I-L|$ to tend to zero, only one (or more) θ_i need tend to zero.

If we consider the MANOVA case, $|I-L| = 0$ if

$$(6.3.4) \quad |B| = |IK\textstyle\sum_{j=1}^{J}\underset{(p\times 1)}{(y_{.j.}-y_{...})}\underset{(1\times p)}{(y_{.j.}-y_{...})'}| = 0$$

i.e. if the sample variance between rows (or columns) is zero. In the noncentral linear case the j population means for each row (or column) lie on a straight line in p-dimensional space (Anderson (1946)). The j sample means will tend, as the sample size of each population increases, to follow this pattern. If they too lie on a straight line it follows that

$$(6.3.5) \quad \underset{(p\times 1)}{(y_{.j.}-y_{...})} = d_j \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}, \quad \text{for all } j.$$

i.e. there is a constant ratio between coordinate values for each of the j sample means. Then

$$(6.3.6) \quad |B| = \left| IK\textstyle\sum_{j=1}^{J} d_j \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix} (c_1 c_2 \ \dots \ c_p) d_j \right|$$

$$= (IK\textstyle\sum_{j=1}^{J} d_j^2)^p \cdot \left| \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix} (c_1 \ \dots \ c_p) \right|$$

$$= 0, \quad \text{as } CC' \text{ is singular } (C' = (c_1 \ \dots \ c_p)).$$

This implies that we expect $|I-L|$ to tend to zero both under the null hypothesis and in the noncentral linear case. Consequently, use of $|I-L|$ as a test criterion for various hypotheses will result in acceptance of H_0 in certain cases where it should be rejected.

Intuitively, therefore, we would expect that the test criterion $|I-L|$ would not be able to distinguish between the central and noncentral linear distributions, if the sample means approached positions on a straight line. Given the fact that with a noncentral linear alternative distribution the population means lie on a straight line, it follows that the sample means will follow suit once n is reasonably large relative to m. In Section 6.4 we shall show how distributional results can be obtained for $|I-L|$ and in Section 6.5 we shall see how the powers of the statistic support the above argument.

6.4 Distributional results for $|I-L|$

$|I-L|$, which we shall write as $W_{p,m,n}(\lambda^2)$, is the special case of $|I-\Sigma L_j|$ where $j = 1$. de Waal (1968) has shown that $W_{p,m,n}(\lambda^2)$ is distributed as $\Pi_{i=1}^{p} W_i$ where W_1 is independently distributed as $\beta_{1B}(m,n,\lambda^2)$ and the W_i are independently distributed as $\beta_1(m-i+1,n)$, $i = 2,3,\ldots,p$. The central densities are the same as those of $|L|$ with m and n transposed. By comparing equations (3.2.6) and (3.2.10), it can be seen that the only difference in the noncentrality components is that $\frac{1}{2}\lambda^2 w_1$ in the hypergeometric function of $|I-L|$ replaces $\frac{1}{2}\lambda^2(1-u_1)$ in that of $|L|$.

This similarity in form enables the algorithm of Chapter 5 to be used, with certain modifications, to find densities and cumulative distribution functions of $W_{p,m,n}(\lambda^2)$.

When n is even (as opposed to m even for $U_{p,m,n}(\lambda^2)$) the binomial expansion of $(1-u_1)^{\frac{1}{2}n-1}$ is finite, and by setting $Z_i' = W_{2i}W_{2i+1}$, $V_i = -\log W_i$ and $V_i' = -\log Z_i'$ we have

$$(6.4.1)\quad f(v_1) = e^{-\frac{1}{2}\lambda^2}(B(\tfrac{1}{2}m,\tfrac{1}{2}n))^{-1}\sum_{j=0}^{\infty} a_j \sum_{b=0}^{k}(-1)^k\binom{b}{k}$$

$$\cdot\exp(-\tfrac{1}{2}(m+2j+2k)v_1) \qquad v_1 \geq 0$$

where a_j is as in equation (5.3.3) and

$$(6.4.2)\quad b = \tfrac{1}{2}(n-2);$$

$$(6.4.3)\quad f(v_i) = (B(\tfrac{1}{2}(m-i+1),\tfrac{1}{2}n))^{-1}\sum_{\ell=0}^{b}(-1)^{\ell}\binom{b}{\ell}$$

$$\cdot\exp(-\tfrac{1}{2}(m-i+2\ell+1)v_i) \qquad v_i \geq 0,\quad i = 2,\ldots,p$$

and

$$(6.4.4)\quad f(v_i') = (2B(m-2i,n))^{-1}\sum_{\ell=o}^{n-1}(-1)^{\ell}\binom{n-1}{\ell}$$

$$\cdot\exp(-\tfrac{1}{2}(m+\ell-2i)v_i') \qquad v_i' \geq 0.$$

Using equations (6.4.1), (6.4.3) and (6.4.4) with Theorems (5.2.1) and (5.2.3), Money (1972) derived explicit expressions for the densities of $W_{p,m,n}(\lambda^2)$ for general m and λ^2, even n and $p = 2,3,4$ and 5. The cumulative distribution functions are given in Troskie and Money (1974).

The convolution algorithm may be extended to derive the density and cumulative distribution function of $|I-L|$, by making the following changes:

(i) transpose m and n in all formulae, but a_j remains $((\frac{1}{2}(m+n))_j(\frac{1}{2}\lambda^2)^j/(\frac{1}{2}m)_j j!)$

(ii) the matrix of coefficients $\left\{d_{jrc}\right\}_p$, for $p = 1$ is given by

$$(6.4.5)\quad d_{j1c} = (-1)^{\frac{1}{2}(c-m-2j)}\binom{\frac{1}{2}(n-2)}{\frac{1}{2}(c-m-2j)} \quad (c-m-2j) \text{ even}, \geq 0$$

$$= 0 \qquad\qquad \text{otherwise.}$$

Careful observation of the convolution process also shows that, at any stage, only elements d_{jrc} with the following subscripts may be nonzero:

<u>$r = 1$</u> and $c = ((m+2j-2+2i),\ i = 1,2,\ldots,\frac{1}{2}n)$

or $c = ((m-2s+2),\ldots,(m+n-2))$,

<u>$r = 2,\ldots,s-1$</u> and $c = ((m-2s+2r),\ldots,(m+n-2r))$,

and

<u>$r = s$</u> and $c = ((m+2j),\ldots,(m+n-2r))$.

It follows that all d_{jr^*c} are zero for $r^* > \frac{1}{2}n$.

Upper α-percentiles u_α of the central distribution of $|I-L|$ may be calculated by an iteration process and the powers of the test statistic are given by $1-F(u_\alpha)$. Some percentiles of $W_{p,m,n}$ and $W_{p,m,n}(\lambda^2)$ are given in Appendices 1 and 3 respectively.

6.5 Exact powers and power comparisons of $|L|$ and $|I-L|$

The algorithm described in Chapter 5 enables exact powers of $|I-L|$ and of $|L|$ with $p > 2$ to be calculated for the first time. Exact percentiles and powers of $|L|$ can be found for any values of p, even m,n,λ and α, the only restrictions being due to the decimal accuracy problem discussed in Section 5.5. In addition, by Theorem 5.2.2. (Das Gupta and Perlman (1973)), $U_{p,m,n}(\lambda^2) = U_{m,p,m+n-p}(\lambda^2)$ and therefore the algorithm can be used if either p or m is even. For both p and m odd, an extension of Lee's (1972) or Mathai's (1971a) techniques for the central case or interpolation might be used.

We shall now find a relationship between $W_{p,m,n}$ and $W_{n,m+n-p,p}$.

<u>Lemma 6.5.1</u> (Anderson (1958)

$$(6.5.1) \quad \Pi_{i=1}^{p}\big(\Gamma(\tfrac{1}{2}(n+1-i)+h)\Gamma(\tfrac{1}{2}(n+m+1-i))/$$

$$\cdot\Gamma(\tfrac{1}{2}(n+1-i))\Gamma(\tfrac{1}{2}(n+m+1-i)+h)\big)$$

$$= \Pi_{i=1}^{m}\big(\Gamma(\tfrac{1}{2}(n+m+1-i))\Gamma(\tfrac{1}{2}(n+m-p+1-i)+h)/$$

$$\cdot\Gamma(\tfrac{1}{2}(n+m+1-i)+h)\Gamma(\tfrac{1}{2}(n+m-p+1-i))\big)$$

From equation (3.3.10) the hth moment of $W_{p,m,n}(\lambda^2)$ is

(6.5.2) $$E|I-L|^h = \left(\Gamma_p(\tfrac{1}{2}m+h)\Gamma_p(\tfrac{1}{2}(m+n))/\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h)\right)$$

$$\cdot e^{-\frac{1}{2}\lambda^2}{}_2F_2(\tfrac{1}{2}(m+n),\tfrac{1}{2}m+h;\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2)$$

(6.5.3) $$= \Pi_{i=1}^{p}\left(\Gamma(\tfrac{1}{2}(m+n-i+1))\Gamma(\tfrac{1}{2}(m-i+1)+h)/\right.$$

$$\left.\cdot\Gamma(\tfrac{1}{2}(m-i+1))\Gamma(\tfrac{1}{2}(m+n-i+1)+h)\right)$$

$$\cdot e^{-\frac{1}{2}\lambda^2}{}_2F_2(\tfrac{1}{2}(m+n),\tfrac{1}{2}m+h;\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2)$$

and by Lemma 6.5.1 with m and n transposed this is equivalent to

(6.5.4) $$\Pi_{i=1}^{n}\left(\Gamma(\tfrac{1}{2}(m+n-i+1))\Gamma(\tfrac{1}{2}(m+n-p-i+1)+h)/\right.$$

$$\left.\cdot\Gamma(\tfrac{1}{2}(m+n-i+1)+h)\Gamma(\tfrac{1}{2}(m+n-p-i+1))\right)$$

$$\cdot e^{-\frac{1}{2}\lambda^2}{}_2F_2(\tfrac{1}{2}(m+n),\tfrac{1}{2}m+h;\ \tfrac{1}{2}m,\tfrac{1}{2}(m+n)+h;\ \tfrac{1}{2}\lambda^2)$$

Putting $m' = m+n-p$, $n' = p$ and $p' = n$ we have

(6.5.5) $$\Pi_{i=1}^{p'}\left(\Gamma(\tfrac{1}{2}(m'+n'-i+1))\Gamma(\tfrac{1}{2}(m'-i+1)+h)/\right.$$

$$\left.\cdot\Gamma(\tfrac{1}{2}(m'+n'-i+1)+h)\Gamma(\tfrac{1}{2}(m'-i+1))\right)$$

$$\cdot e^{-\frac{1}{2}\lambda^2}{}_2F_2\left(\tfrac{1}{2}(m'+n'),\tfrac{1}{2}(m'+n'-p')+h;\ \tfrac{1}{2}(m'+n'-p'),\right.$$

$$\left.\cdot\tfrac{1}{2}(m'+n')+h;\ \tfrac{1}{2}\lambda^2\right)$$

Comparing equations (6.5.3) and (6.5.5) we see that the sections involving products of gamma functions are equivalent, with p', m' and n' in (6.5.5) replacing p, m and n in (6.5.3). In the noncentrality components, however, $\frac{1}{2}m+h$ and $\frac{1}{2}m$ in (6.5.3) are replaced by $\frac{1}{2}(m'+n'-p)+h$ and $\frac{1}{2}(m'+n'-p')$ in (6.5.5). Hence, unlike $U_{p,m,n}(\lambda^2)$, the noncentral linear distribution of $|I-L|$ cannot directly be expressed in terms of an alternative product of Beta distributions. This provides the following result.

Theorem 6.5.1

In the central case, $W_{p,m,n} = W_{n,m+n-p,p}$. This relationship does not hold for the noncentral linear distribution or for powers of $W_{p,m,n}$.

Powers of $|I-L|$ can therefore only be found for even n, and general p,m,λ^2 and α. We also note that to avoid singularity in $|B|$, $m \geq p$ must hold, and that problems of decimal accuracy arise which are similar in magnitude to those for $|L|$. For $|L|$, approximations up to the order q^{-2}, where $q = n+\frac{1}{2}(m-p-1)$, have been obtained by Posten and Bargmann (1964), Sugiura and Fujikoshi (1969) and Lee (1971). If errors of this order are unacceptable because n is small, exact results are calculable from the algorithm. Using de Waal's (1968a) approximation, Fujikoshi (1972) derived an approximation of $|I-L|$ to the order r^{-3}, where $r = m+\frac{1}{2}(n-p-1)+2\ \mathrm{tr}\Lambda/p$. No numerical results of approximations for $|I-L|$ appear to have been derived.

A selection of exact powers of $U_{p,m,n}$ for various parameters is given in Appendix 4.

Let $\Pi_\alpha(p,m,n,\lambda^2)$ denote the power of $U_{p,m,n}$ at level α, under alternative hypothesis $\lambda_1^2 = \lambda^2$ and

$\lambda_i^2 = 0, i = 2,\ldots,p$. Results obtained all confirm that, for $k > 0$,

$$(6.5.6) \quad \Pi_{\alpha+k}(p,m,n;\ \lambda^2) > \Pi_{\alpha}(p,m,n;\ \lambda^2) \qquad \alpha+k<1$$

$$(6.5.7) \quad \Pi_{\alpha}(p,m,n;\ \lambda^2+k) > \Pi_{\alpha}(p,m,n;\ \lambda^2)$$

$$(6.5.8) \quad \Pi_{\alpha}(p,m+k,n;\ \lambda^2) < \Pi_{\alpha}(p,m,n;\ \lambda^2)$$

$$(6.5.9) \quad \Pi_{\alpha}(p+k,m,n;\ \lambda^2) < \Pi_{\alpha}(p,m,n;\ \lambda^2)$$

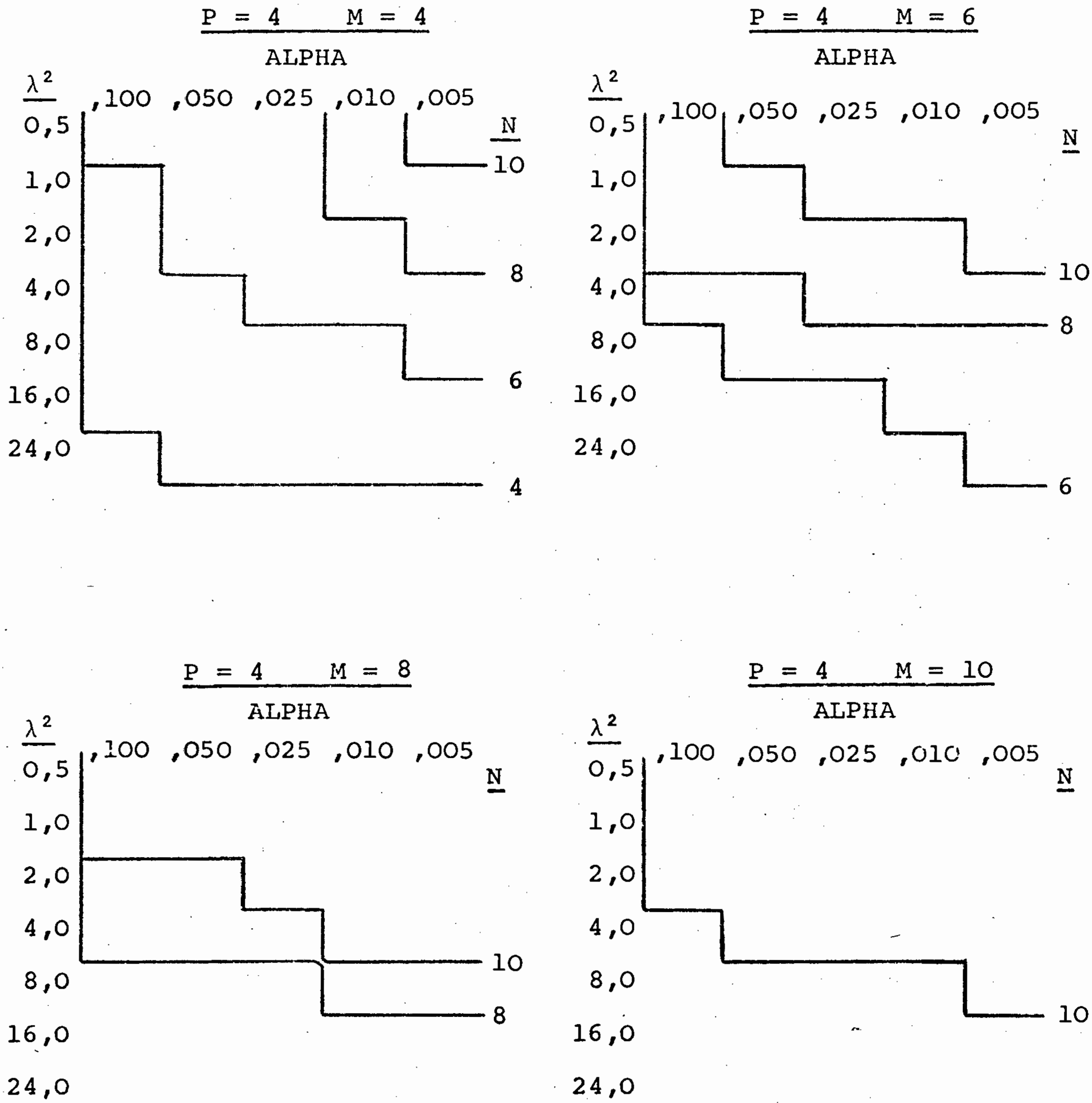
$$(6.5.10) \quad \Pi_{\alpha}(m,p,m+n-p;\ \lambda^2) = \Pi_{\alpha}(p,m,n;\ \lambda^2)$$

$$(6.5.11) \quad \Pi_{\alpha}(p,m,n+k;\ \lambda^2) > \Pi_{\alpha}(p,m,n;\ \lambda^2)$$

(6.5.7) was shown by Das Gupta, Anderson and Mudholkar (1964); (6.5.8), (6.5.9) and (6.5.10) were proved by Das Gupta and Perlman (1973), while (6.5.11) is a conjecture by the latter authors.

Power comparisons of $|L|$ and $|I-L|$ are shown in Appendix 5. These reveal that the power of $|I-L|$ also increases with λ^2 and n and decreases with m and p, but at very different rates from that of $|L|$. To clarify this we shall compare the powers of the two statistics for $p = 4$; $m = 4,6,8$ and 10; $n = 4,6,8$ and 10; $\alpha = 0{,}10$; $0{,}05$; $0{,}025$; $0{,}010$ and $0{,}005$ and $\lambda^2 = 0{,}5$; $1{,}0$; $2{,}0$; $4{,}0$; $8{,}0$; $16{,}0$ and $24{,}0$. Figure 6.1 illustrates the set of values of α and λ^2 for which the power of $|I-L|$ exceeds that of $|L|$, and shows how this set typically changes as m and n vary. Cases with $m > n$ are not shown; for these the set of α and λ^2 will increase in size.

FIGURE 6.1 CONTOURS IN α AND λ^2 ABOVE WHICH POWER $(|I-L|) >$ POWER $(|L|)$



The ratio of the power of $|I-L|$ to that of $|L|$ varies greatly. For $p = m = n = 4$ (i.e. m and n equal and close to p) it is 1,205 for $\lambda^2 = 8,0$ and $\alpha = 0,05$, and reaches 1,708 for $\lambda^2 = 16,0$ and $\alpha = 0,005$. For $p = 4$, $m = n = 10$ the largest ratio is 1,044 at $\lambda^2 = 4,0$ and $\alpha = 0,005$. When $p = m = 4$, $n = 10$, $|L|$ is markedly superior and the ratio drops to under 0,5 for $\lambda^2 = 16,0$ and all α.

A rough generalisation is that $|I-L|$ is more powerful than $|L|$ when m equals or exceeds n or when m is close to n and α and λ^2 are small. (These are the cases where the power of any statistic would be expected to be minimal). In practice, however, it is desirable always to have n large relative to m. If the sample size has to be small it would be unreasonable to choose an alternative hypothesis with low λ^2 because of the low associated power. This supports the statement of Pearson and Wilks (1933) and Hsu (1940) regarding the usefulness of $|I-L|$ as a test statistic, and actually measures the degree to which the similarity of values of the noncentral linear statistic and of the central statistic affects its ability to distinguish between these two cases.

In Table 6.6 we show how the powers of $|L|$ and $|I-L|$ (with $p = 2$, $m = 2$, $\alpha = 0,05$, $\lambda^2 = 0,5$ and $16,0$) change as n increases. This shows that for the low value of $\lambda^2 = 0,5$, there is not much difference between $|L|$ and $|I-L|$ for all n. When, however, λ^2 (= 16,0) is high (and the central and noncentral distributions are distinct) the power of $|I-L|$ is less than half that of $|L|$ from

TABLE 6.6

Power Comparison of $|L|$ and $|I-L|$ with increasing n for $p = 2$, $m = 2$ and $\alpha = 0,05$

	$\lambda^2 = 0,5$			$\lambda^2 = 16,0$		
n	$\|I-L\|$	$\|L\|$	Ratio	$\|I-L\|$	$\|L\|$	Ratio
2	0,0587	0,0540	1,088	0,1742	0,1291	1,349
4	0,0616	0,0598	1,030	0,2164	0,3585	0,604
6	0,0631	0,0634	0,996	0,2482	0,5348	0,464
8	0,0641	0,0657	0,976	0,2728	0,6415	0,425
10	0,0648	0,0673	0,962	0,2921	0,7067	0,413
12	0,0653	0,0685	0,953	0,3076	0,7487	0,411
16	0,0660	0,0702	0,941	0,3310	0,7981	0,414
20	0,0665	0,0712	0,933	0,3478	0,8254	0,421
24	0,0668	0,0720	0,928	0,3604	0,8425	0,428

$n = 6$ onwards. We have also shown broadly that $|I-L|$ is superior to $|L|$ only when $m \geq n$. This implies, in the cases of

(1) Regression : $N \leq 2q-r$;

(2) Means : $N \leq 2q-1$;

(3) MANOVA : $J \leq 2$ or $I \leq 2$ (one element per cell);

(4) MANOVA : $K < 2$ (K elements per cell - contradictory).

These are all statistically unreasonable.

Because of its inability to distinguish between the central and noncentral linear cases we conclude that $|I-L|$ should not be considered as an alternative test statistic to $|L|$.

6.6 Tests using the statistic $|I-\Sigma L_j|$

$|I-\sum_{j=1}^{k} L_j|$ has been proposed as a test for the equality of k multivariate normal populations, and it has been shown that its central moment reduces to that of $|I-L|$. Under the alternative hypothesis of equal covariance matrices and noncentrality (means) matrix of rank one, the density of the statistic (given in Theorem 4.7.1) is equivalent to that of $|I-L|$ with $\Sigma n_j = n$. The properties of and results for $|I-L|$ discussed in Section 6.5 will therefore also hold for $|I-\Sigma L_j|$, and as Σn_j will always tend to be large compared with m, $|I-\Sigma L_j|$ will not be a powerful statistic when the alternative hypothesis is that of unequal means.

de Waal (1970) investigated a similar statistic where the means were all equal, but covariance matrices unequal.

Theorem 6.6.1 (de Waal (1970))

Let A_j, $j = 1,\ldots,k \sim W(\Sigma_1, n_j)$ and $B \sim W(\Sigma_2, m)$ where A_j, $j = 1,\ldots,k$ and B are independent. Let

(6.6.1) $\quad L_j = \left(\sum_{j=1}^{k} A_j + B\right)^{-\frac{1}{2}} A_j \left(\sum_{j=1}^{k} A_j + B\right)^{-\frac{1}{2}}$ where the usual conventions regarding square roots are adopted.

Then

(6.6.2) $$E\left|I-\sum_{j=1}^{k} L_j\right|^h = \Big(\Gamma_p(\tfrac{1}{2}(m+n))\Gamma_p(\tfrac{1}{2}m+h)/$$

$$\cdot\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h)\Big)\,|\Sigma_1^{-1}\Sigma_2|^{\frac{1}{2}n}$$

$$\cdot{}_2F_1\left(\tfrac{1}{2}n, \tfrac{1}{2}(m+n);\ \tfrac{1}{2}(m+n)+h;\ I-\Sigma_1^{-1}\Sigma_2\right)$$

where $\quad n = \sum_{j=1}^{k} n_j$

If we consider the noncentral linear case, i.e. where $(I-\Sigma_1^{-1}\Sigma_2)$ has only one nonzero root $(1-\omega^2)$ we obtain

$$(6.6.3)\qquad E|I-\textstyle\sum_{j=1}^{k}L_j|^h = \big(\Gamma_p(\tfrac{1}{2}(m+n))\Gamma_p(\tfrac{1}{2}m+h)/$$

$$\cdot\Gamma_p(\tfrac{1}{2}m)\Gamma_p(\tfrac{1}{2}(m+n)+h)\big)\omega^n{}_2F_1\big(\tfrac{1}{2}n,\tfrac{1}{2}(m+n);$$

$$\cdot\tfrac{1}{2}(m+n)+h;\ (1-\omega^2)\big)$$

which is the same as the hth moment of $|I-C|$ (given in equation (3.3.18)) where $A \sim W(\Sigma_1,n)$, $B \sim W(\Sigma_2,m)$, A and B independent, and $C = (A+B)^{-\frac{1}{2}}A(A+B)^{-\frac{1}{2}}$.

We shall show in Chapter 8 how $|I-C|$ may be used to test for the equality of two covariance matrices and shall see that under practical conditions the associated powers are sometimes better than for the other proposed test criterion $|C|$. The practical range of parameters m and n tends to be more limited for $|I-\Sigma L_j|$ than for $|I-C|$, as here $n = \Sigma n_j$ will almost always be larger than m. We note also that for computational purposes $n = \Sigma n_j$ must be even, and that the nature of the infinite series will depend on whether ω^2 is < 1 or > 1. If ω^2 is < 1 the series will converge monotonically, but if $\omega^2 > 1$, the series will alternate in sign and will only converge for certain low values of ω^2.

CHAPTER 7

TESTS OF INDEPENDENCE

7.1 Extension of the algorithm to $|I-R|$ and $|R|$

It can be seen that, with appropriate degrees of freedom m and n for hypothesis and error respectively, the central distributions of $|L|$ and $|I-R|$ are equivalent. To extend the results for $|L|$ to the noncentral linear distribution of $|I-R|$, written $U_{p,m,n}(\rho_1^2)$, we need only compare the noncentrality components given in Table 4.2. It follows that to derive $U_{p,m,n}(\rho_1^2)$ from $U_{p,m,n}(\lambda^2)$ we should

(a) replace $e^{-\frac{1}{2}\lambda^2}$ by $(1-\rho_1^2)^{\frac{1}{2}(m+n)}$ and

(b) set $a_j = (((\frac{1}{2}(m+n))_j)^2\rho_1^{2j}/(\frac{1}{2}m)_j j!)$.

Similarly, the central cases of $|I-L|$ and $|R|$, the square of the vector correlation coefficient (Anderson (1958)), are equivalent. To derive the noncentral linear distribution of $|R|$, written $W_{p,m,n}(\rho_1^2)$, from $W_{p,m,n}(\lambda^2)$ we also make changes (a) and (b) above, where m and n are now the degrees of freedom appropriate to these statistics. (See Table 4.1.) Using this method, percentiles of the central and noncentral distributions may be derived, and powers calculated.

Because of the finite series binomial expansion used, we are restricted, for computation, to p_2 even for $|I-R|$ and to $(n-p_2)$ even for $|R|$. We shall now show how these restrictions may be partially overcome.

From Theorem 4.5.1 we have

$$(7.1.1)\qquad E|I-R|^h = \left(\Gamma_{p_1}(\tfrac{1}{2}n)\Gamma_{p_1}(\tfrac{1}{2}(n-p_2)+h)/\Gamma_{p_1}(\tfrac{1}{2}(n-p_2))\right.$$

$$\left.\cdot\Gamma_{p_1}(\tfrac{1}{2}n+h)\right)|I-P|^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n;\ \tfrac{1}{2}n+h;\ P).$$

Using Lemma 6.5.1 with the following changes:

$$p \to p_1;\quad m \to p_2;\quad n \to n-p_2\quad \text{we obtain}$$

$$(7.1.2)\qquad E|I-R|^h = \left(\Gamma_{p_2}(\tfrac{1}{2}n)\Gamma_{p_2}(\tfrac{1}{2}(n-p_1)+h)/\Gamma_{p_2}(\tfrac{1}{2}n+h)\right.$$

$$\left.\cdot\Gamma_{p_2}(\tfrac{1}{2}(n-p_1))\right)|I-P|^{\frac{1}{2}n}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n;\ \tfrac{1}{2}n+h;\ P).$$

This is the distribution of (7.1.1) with p_1 and p_2 transposed. It follows that

<u>Theorem 7.1.1</u>

The likelihood ratio criterion for testing the independence of two sets of variates

$$(7.1.3)\qquad U_{p_1,p_2,n-p_2}(\rho_1^2) = U_{p_2,p_1,n-p_1}(\rho_1^2).$$

The central distributions and powers of $U_{p_1,p_2,n-p_2}$ and $U_{p_2,p_1,n-p_1}$ are equivalent.

From Theorem 4.5.1 we also have

$$(7.1.4)\qquad E|R|^h = \left(\Gamma_{p_1}(\tfrac{1}{2}n)\Gamma_{p_1}(\tfrac{1}{2}p_2+h)/\Gamma_{p_1}(\tfrac{1}{2}n+h)\right.$$

$$\left.\cdot\Gamma_{p_1}(\tfrac{1}{2}p_2)\right)|I-P|^{\frac{1}{2}n}{}_3F_2(\tfrac{1}{2}n,\tfrac{1}{2}n,\tfrac{1}{2}p_2+h;\ \tfrac{1}{2}n+h,\tfrac{1}{2}p_2;\ P),$$

and using Lemma 6.5.1 with

$$p \to p_1;\quad m \to n-p_2;\quad n \to p_2,\quad \text{we obtain}$$

$$(7.1.5)\qquad E|R|^h = \left(\Gamma_{(n-p_2)}(\tfrac{1}{2}n)\Gamma_{(n-p_2)}(\tfrac{1}{2}(n-p_1)+h)/\right.$$

$$\left.\cdot\Gamma_{(n-p_2)}(\tfrac{1}{2}n+h)\Gamma_{(n-p_2)}(\tfrac{1}{2}(n-p_1))\right)$$

$$\cdot|I-P|^{\frac{1}{2}n}\,{}_3F_2(\tfrac{1}{2}n,\tfrac{1}{2}n,\tfrac{1}{2}p_2+h;\ \tfrac{1}{2}n+h,\tfrac{1}{2}p_2;\ P).$$

Putting $p_1' = n-p_2$, $p_2' = n-p_1$, $n' = n$, we obtain

$$(7.1.6)\qquad E|R|^h = \left(\Gamma_{p_1'}(\tfrac{1}{2}n')\Gamma_{p_1'}(\tfrac{1}{2}p_2'+h)/\Gamma_{p_1'}(\tfrac{1}{2}n'+h)\Gamma_{p_1'}(\tfrac{1}{2}p_2')\right)$$

$$\cdot|I-P|^{\frac{1}{2}n'}\,{}_3F_2(\tfrac{1}{2}n',\tfrac{1}{2}n',\tfrac{1}{2}(n'-p_1')+h;\ \tfrac{1}{2}n'+h,\tfrac{1}{2}(n'-p_1');\ P).$$

Comparing (7.1.4) and (7.1.6) we see that the central components are equivalent, with p_1', p_2' and n' in (7.1.6) replacing p_1, p_2 and n in (7.1.4). This correspondence does not extend to the noncentrality component. This gives rise to

<u>Theorem 7.1.2</u>

In the central case $W_{p_1,p_2,n-p_2} = W_{n-p_2,n-p_1,p_1}$. This relationship does not hold for the noncentral linear distribution or for powers of $W_{p_1,p_2,n-p_2}$.

From Theorem 7.1.1 it follows that powers of $|I-R|$ are obtainable in all cases except where both p_1 and p_2 are odd. Theorem 7.1.2 shows that powers and noncentral linear percentiles of $|R|$ can only be determined for $(n-p_2)$ even.

In Appendices 6, 7 and 8 we list some noncentral linear percentiles and powers of $|I-R|$ and some noncentral linear percentiles of $|R|$ respectively. These are found for the first time with $p_1 > 2$.

7.2 Distributional results for the likelihood ratio criterion $|I-R|$

We now outline the distributional results so far obtained for the likelihood ratio test of independence between two sets of variates. As the null distributions of $|L|$ and $|I-R|$ are the same, results derived by authors such as Wilks (1935), Wald and Brookner (1941), Box (1949), Anderson (1958), Consul (1967), Schatzoff (1964), Pillai and Gupta (1969), Mathai (1971a) and Lee (1972) are applicable to both.

Kshirsagar (1961) expressed the noncentral linear case, $U_{p,m,n}(\rho_1^2)$, in terms of a product of Beta functions and Pillai (1965) used the first four moments to calculate powers of the statistic under noncentral linear alternatives with $p = 2$, $m = 7$ and 13, $n = 53$, 83 and 123 and $\rho_1^2 = 0{,}0025$ and $0{,}01$. In Table 7.1 we compare some of Pillai's values with those derived by the algorithm.

TABLE 7.1

Pillai's (1965) approximate (PA) and Hart's exact (HE) powers of $U_{2,7,n}(\rho_1^2)$ when $\alpha = 0{,}01$.

	n = 53			n = 83	
ρ_1^2	PA	HE	ρ_1^2	PA	HE
0,0025	0,0118	0,010854	0,0025	0,0140	0,011375
0,01	0,0190	0,013714	0,01	0,0280	0,016246

It can be seen that this approximation greatly overestimates the true power, which results in false conclusions being drawn as to the superiority of this statistic over others for $n = 83$.

Using Constantine's (1963) joint distribution of the characteristic roots r_i^2 of the generalised multiple correlation matrix R, Pillai and Jayachandran (1967) derived an expression for the noncentral cumulative distribution function of $|I-R|$ with $p = 2$. They also tabulated powers of $U_{2,m,n}(\rho_1^2,\rho_2^2)$ for $m^* = \frac{1}{2}(m-p-1)$ and $n^* = \frac{1}{2}(n-p-1)$; $m^* = 0,1,2$ and 5; $n^* = 5,15,30$ and 40 and $0 \leq \rho_1^2, \rho_2^2 \leq 0{,}15$. It should be noted that Pillai and Jayachandran's (1967) "exact" powers are also based on a truncated series. For small values of ρ_1^2, i.e. $\rho_1^2 = 0{,}0001$ and $0{,}01$, Pillai and Jayachandran's values are all identical to those calculated by the algorithm. For Pillai and Jayachandran's larger parameters, $\rho_1^2 = 0{,}10$ and $0{,}15$, a comparison with the exact values obtained by the algorithm is given in Table 7.2

Asymptotic distributions and powers were derived by Sugiura and Fujikoshi (1969) (neglecting terms of order $q^{-3/2}$, where $q = n+\frac{1}{2}(m-p-1)$) and by Lee (1971), Muirhead (1972) and Sugiura (1973) (all neglecting terms of order q^{-3}). Nagao (1973) has obtained an asymptotic expression for the noncentral distribution (neglecting terms of order $q^{-3/2}$) for the general case of 2 or more sets of variates. Lee (1971) tabulated comparisons of his approximations to powers of $|I-R|$ with the "exact" powers of Pillai and Jayachandran (1967), for $p = 2$, $m = 3,5,7$ and 13, and $n = 63$ and 83. Lee's approximations in the linear case (to 3 significant figures) are in full agreement with

T A B L E 7.2

Pillai and Jayachandran's (1967) exact (PJ) and Hart's exact (HE) powers of $U_{2,m,63}(\rho_1^2)$ when $\alpha = 0{,}05$.

p	m	n	ρ_1^2	PJ	HE
2	3	63	0,10	0,447	0,449
			0,15	0,664	0,669
2	5	63	0,10	0,363	0,365
			0,15	0,574	0,572
2	7	63	0,10	0,314	0,315
			0,15	0,489	0,506

exact powers calculated by the algorithm, and it appears as if they are generally more accurate than Pillai and Jayachandran's powers, for these values of n. Numerical results given by Sugiura (1973) also indicate that his approximations are of the stated level of accuracy for $n = 83$. Using systems of partial differential equations, Muirhead (1972) tabulated approximations of order $q^{-3}(q = n+\frac{1}{2}(m-p-1))$ to $U_{2,m,n}$ with $m = 7$ and 13 and $n =$ 13,33 and 83, and compared his approximations with the powers of Pillai and Jayachandran (1967). For small ρ_1^2, his results appear to be correct to the stated $O(q^{-3})$. Muirhead's values are always less than Pillai and Jayachandran's, but as both become inaccurate with increasing ρ_1^2, no conclusions can be drawn for larger ρ_1^2. Muirhead states that the method used can theoretically give the expansion up to any order of N.

Using the work of Kshirsager (1961), Troskie (1969) and Gupta (1971), Money (1972) showed how explicit series expressions for $U_{p,m,n}(\rho_1^2)$, for general p,n,ρ_1^2 and m even, could be derived. The algorithm used by the author follows these results.

7.3 The statistic $|R|$

Wilks (1932) first defined this statistic in the central case as a generalisation of the correlation ratio. It is, of course, then equivalent to $|I-L|$ of Chapter 6. Anderson (1958) notes that $|R|$ is the sample equivalent of the square of a vector correlation coefficient. Troskie (1969) derived the moments and noncentral linear distribution of $|R|$ and showed that the distribution could be expressed as that of a product of independent Beta variables. He also extended these results to the complex case, where $|R|$ is the sample vector coherence coefficient.

Money (1972) and Troskie and Money (1974) showed how the density of $|R|$ could be derived by using the method of convolutions. Money (1972) gave explicit series expressions for $|R|$ for both the real and complex cases.

The author has seen no reference to $|R|$ as a test statistic other than that of Pillai and Nagarsenker (1972), who refer to it as the Wilks-Lawley U-criterion for canonical correlation and provide a density for the general noncentral case.

For $p = 1$, $|R|$ has the density of the square of the multiple correlation coefficient. Lee (1972a) used termwise integration on Fisher's (1928) density function of the sample multiple correlation coefficient, and by suitable truncation of the infinite series, calculated upper percentiles of the multiple correlation coefficient. These may be easily calculated by using the algorithm to find the noncentral percentiles of $W_{1,m,n}(\rho^2)$, and then taking the square roots of these percentiles. The results obtained agree exactly with those of Lee (1972a).

Using percentage points tabulated by Mijares (1964), Muirhead (1972) calculated asymptotic powers for $W_{1,m,40}$ where $m = 2,3,5,10$ and 15 and $\rho_1^2 = 0{,}03;\ 0{,}05;\ 0{,}07;\ 0{,}10;\ 0{,}20;\ 0{,}30;\ 0{,}40;\ 0{,}50$. The results show that Muirhead's approximation loses accuracy when ρ_1^2 is large, and particularly as m increases, and may in some cases only be accurate to one significant figure. This may be due to the basic decimal accuracy problem encountered in the algorithm. Muirhead also shows how a normal distribution expansion of Sugiura and Fujikoshi (1969) may sometimes be a better approximation.

We shall now show why $|R|$ can not be regarded as a suitable test criterion for the test of independence of two sets of variates. Our argument parallels that for the statistic $|I-L|$.

$|R| = \Pi_{i=1}^{p} r_i^2$ where r_i $(i = 1,\ldots,p)$ are the sample canonical correlation coefficients between the two sets. Consequently, if any r_i are zero, $|R|$ will be zero. If Σ_{12} is of full rank, all of the population canonical correlation coefficients ρ_i will be nonzero. When its rank is not full, one or more of the ρ_i will be zero and the r_i will tend, for moderate sample size, to follow this pattern. An increased sample size should mean increased power, but it will also mean that the r_i will become closer to the (zero) ρ_i.

When the rank of Σ_{12} is less than p, and particularly when it is one, the value of the statistic $|R|$ will approach zero. As $|R|$ should also tend to zero in the central case, it behaves ambiguously as a test criterion for independence and should not be used. Two other test criteria for independence proposed by Pillai (1955), $R^{(q)} = q(\text{tr } R^{-1})^{-1}$ and $T^{(q)} = q(\text{tr}(I-R)R^{-1})^{-1}$ are shown by Troskie (1971) to be unsuitable for similar reasons : Troskie (1971) states that they could only be used for large deviations from the null hypothesis, when the population canonical correlations are all different from zero.

7.4 Powers and power comparisons of $|R|$ and $|I-R|$

Powers of $|I-R|$ and of $|R|$ were calculated for a parameter range similar to that of $|L|$ and $|I-L|$. Powers accurate to 5 decimal places were obtained for values of ρ_1^2 up to 0,7; for larger ρ_1^2 the convergence was too slow to avoid the effect of the decimal accuracy problem. For $U_{2,2,4}(\rho_1^2)$ with $\rho_1^2 = 0,9$ for example, convergence starts only at the 25th term, and by the 55th term an amount of 0,005 is still being contributed to the integral. $|R|$ behaves similarly for large values of ρ_1^2. For large n, however, large values of ρ_1^2 would not be practically necessary. The powers of $|R|$ and $|I-R|$ computed all obeyed equations (6.5.6) to (6.5.11) (where $0 < \rho_1^2 < 1$ replaces $\lambda^2 > 0$).

In Appendices 7 and 9 powers of $|I-R|$ and power comparisons of $|R|$ with $|I-R|$ are given. These confirm the inferiority of $|R|$ as alternative test criterion to $|I-R|$, as stated in Section 7.3. The relative behaviour of the powers of $|I-R|$ and $|R|$ can be seen to closely parallel that of the powers of $|L|$ and $|I-L|$. $|I-R|$ is generally more powerful than $|R|$ except for cases when m is greater than or equal to n. For the test of independence of two sets of variates, $m \geq n$ implies $N \leq 2p^* + 1$, where $p^* = \max(p_1, p_2)$. These are clearly the cases where the lack of data would cause any test criterion to have a very low power. As n becomes large relative to m, the power of $|I-R|$ increases much more rapidly than that of $|R|$. For $p = 1$ the powers of $|I-R|$ and $|R|$, the square of the multiple correlation coefficient, are identical, and $|I-R|$ is therefore an alternative criterion in this case.

CHAPTER 8

TESTING THE EQUALITY OF COVARIANCE MATRICES

8.1 Published results on test criteria

The likelihood ratio test for the hypothesis $\Sigma_1 = \Sigma_2$ against the alternative $\Sigma_1 \neq \Sigma_2$, where Σ_1 and Σ_2 are the covariance matrices of two normal populations, was shown by Das Gupta (1969) to be biased for $N_1 \neq N_2$. (In this section we shall use the notation of Section 4.6.) The modified likelihood ratio test was shown by Sugiura and Nagao (1968) to be unbiased, but exact numerical values have not been derived for this test. Asymptotic non-null distributions under the alternative $\Sigma_1 \neq \Sigma_2$ were derived by Sugiura (1969) and Nagao (1970). These approximations, however, are inaccurate under alternatives close to the null hypothesis, and Pillai and Nagarsenker (1972), Sugiura (1973) and Khatri and Srivastava (1974) have obtained asymptotic expansions which are valid in this case. Approximate numerical results were given by Nagao (1970) and Sugiura (1973), and by Greenstreet and Connor (1974), who used Monte Carlo methods.

Kiefer and Schwartz (1965) suggested a test based on $|S_1+S_2|/|S_2|$ which Giri (1968) noted was invariant and admissible and, by Anderson and Das Gupta (1964), had a monotonically increasing power function in the ω_i^2, where ω_i^2 are the characteristic roots of $\Omega = \Sigma_1\Sigma_2^{-1}$. Giri (1968) pointed out that the test of equality of two covariance matrices could be reduced to that of testing the hypothesis

(8.1.1) $H_0 : \omega_1^2 = \omega_2^2 = \ldots = \omega_p^2 = 1$

against the alternative

(8.1.2) $H_1 : \sum_{i=1}^{p} \omega_i^2 > p$.

Giri (1968) remarked that the dual alternative

(8.1.3) $H_2 : \sum_{i=1}^{p} \omega_i^2 < p$

reduces to (8.1.2) when S_1 and S_2 are interchanged in the statistic $|S_1+S_2|/|S_2|$. (We may rewrite (8.1.3) as

(8.1.4) $H_2^* : \sum_{i=1}^{p}(1/\omega_i)^2 > p$

where $(1/\omega_i)^2$ are the roots of $\Sigma_2\Sigma_1^{-1}$.)

The null hypothesis will be rejected when the value of the statistic is greater than a preassigned constant.

To align this statistic with Wilks' likelihood ratio criteria $|L|$ and $|I-R|$ for other tests, most authors have subsequently used the statistic $|S_2|/|S_1+S_2|$, which has a rejection region close to zero. We shall refer to this statistic as $|C|$ (or $U_{p,m,n(\omega^2)}$ in the noncentral linear case).

Pillai and Jayachandran (1968) proposed the test statistic $|C|$ as a test for hypothesis H_0 against the alternative hypothesis

(8.1.5) $H_A : \omega_i^2 \geq 1$ $(i = 1,\ldots,p)$; $\sum_{i=1}^{p}\omega_i^2 > p$.

They derived the noncentral distribution of the statistic and tabulated power comparisons for $p = 2$, which indicated that $|C|$ was at times more powerful than Roy's largest root, Pillai's $V^{(p)}$ and Hotelling's T_0^2 criteria. Pillai and Gupta (1969) pointed out that $|C|$ was not directly related to the likelihood ratio criterion, and that, unlike the tests for equality of means and independence, large values of both m and n could be expected. Pillai, Al-Ani and Jouris (1969) derived the noncentral distribution of $|S_2|/|\delta S_1+S_2|$ for testing the hypothesis $H_0^* : \delta\Omega = I_p$, δ(known > 0, also finding explicit expressions for the density and cumulative distribution functions when $p = 2$. These results were extended to the complex case by Pillai and Jouris (1971), for $\delta = 1$, i.e. for the test statistic $|C|$.

Nagao (1970) noted that $|S_2(S_1+S_2)^{-1}|^{-1}$ had been proposed as a test criterion for the null hypothesis $\Sigma_1 = \Sigma_2$ against the alternative H_A, and that the null hypothesis would be rejected for large values of the statistic. He further derived an asymptotic expansion of $|S_2(S_1+S_2)^{-1}|$ which neglected terms of order $n^{-3/2}$ $(n = n_1+n_2)$, and observed that it was continuous at the null hypothesis. Nagao (1970) showed that, for $p = 2$, $n_1 = 13$, $n_2 = 63$ and $\omega_1^2 = 1,5$, the asymptotic central percentile and power agreed closely with those of Pillai and Jayachandran (1968), and also obtained asymptotic values for $p = 4$, $n_1 = 50$ and $n_2 = 100$. Pillai and Nagarsenker (1972) considered a general statistic $Y = \Pi_{i=1}^{p}\theta_i^a(1-\theta_i)^b$ in both the real and complex cases, and showed how the noncentral distributions of $|C|$ and $|I-C|$ could be derived by setting $a = 0$, $b = 1$ and $a = 1$, $b = 0$ respectively. They also found asymptotic distributions for $|C|$ and $|I-C|$ which neglected terms of order $n^{-3/2}$.

8.2 The use of $|C|$ and $|I-C|$ as test statistics

In this section we shall examine the behaviour of $|C|$ and $|I-C|$ and assess their suitability as test statistics.

In Theorem 3.3.3 the moments of $|C|$ and $|I-C|$ were derived as $|L|$ and $|I-L|$ using the consistent format : $A \sim W(\Sigma_1,n)$, $B \sim W(\Sigma_2,m)$, $|L| = |A|/|A+B|$. $|C|$ was also expressed in this way in the general form of Section 4.8, and we shall use this format to extend the algorithm to the statistics $|C|$ and $|I-C|$. In this chapter as in Section 4.6 we shall discuss the statistics in the form used by most authors : $|C| = |S_2|/|S_1+S_2|$, where $S_1 \sim W(\Sigma_1,n_1)$, $S_2 \sim W(\Sigma_2,n_2)$. Changes in notation from $|L|$ of Theorem 3.3.3 to that of Chapter 8 are therefore

(8.2.1) $\quad (\Sigma_1,\Sigma_2,n,m) \to (\Sigma_2,\Sigma_1,n_2,n_1)$

We shall refer to the noncentral distributions of $|C|$ and $|I-C|$ with parameters p,m,n and $\omega_i^2(i = 1,\ldots,p)$ as $U_{p,m,n}(\omega_1^2,\ldots,\omega_p^2)$ and $W_{p,m,n}(\omega_1^2,\ldots,\omega_p^2)$ respectively.

The statistic $|C|=|S_1|/|S_1+S_2|$ will tend, as n_1 and n_2 increase, to $|n_2\Sigma_2|/|n_1\Sigma_1+n_2\Sigma_2|$. Under the null hypothesis $H_0 : \Sigma_1 = \Sigma_2$, this will tend to

(8.2.2) $\quad |n_2\Sigma|/|n_1\Sigma+n_2\Sigma| = (n_2/(n_1+n_2))^p.$

Similarly $|I-C| = |S_1|/|S_1+S_2|$ will tend under H_0 to

(8.2.3) $\quad |n_1\Sigma|/|n_1\Sigma+n_2\Sigma| = (n_1/(n_1+n_2))^p.$

Consequently the null hypothesis should be rejected whenever the statistics depart appreciably from these values.

Anderson (1958) notes that the modified likelihood ratio criterion

(8.2.4) $\quad V_1 = |S_1|^{\frac{1}{2}n_1}|S_2|^{\frac{1}{2}n_2}/|S_1+S_2|^{\frac{1}{2}(n_1+n_2)}$

(8.2.5) $\quad = \Pi_{i=1}^{p}\theta_i^{\frac{1}{2}n_1}\Pi_{i=1}^{p}(1+\theta_i)^{-\frac{1}{2}(n_1+n_2)}$

measures how close the θ_i are to n_1/n_2, where the θ_i

are the roots of $|S_1-\theta S_2| = 0$, and points out that any other measure of the closeness of the θ_i to n_1/n_2 will be a test of $H_0 : \Sigma_1 = \Sigma_2$. $|C|$ will satisfy the latter point as the roots of $|S_2|/|S_1+S_2|$ are $(1-\theta_i)^{-1}$, and if under the null hypothesis θ_i is close to n_1/n_2, $(1+\theta_i)^{-1}$ will be close to $n_2/(n_1+n_2)$ as stated. Similarly, the roots of $|I-C|$ are $\theta_i(1+\theta_i)^{-1}$, which will be close to $n_1/(n_1+n_2)$ for θ_i close to n_1/n_2.

If the θ_i are significantly less than or greater than n_1/n_2, the value of the modified likelihood ratio criterion V_1 will be less than a preassigned constant, and a one-tailed test is used. If $|C|$ is used under alternative hypothesis H_A of equation (8.1.5), departure from the null hypothesis will mean (from equation (8.2.2)) that $|C| < (n_2/(n_1+n_2))^p$. If $|C|$ is used under alternative hypothesis

(8.2.6) $\quad H_B : \omega_i^2 \leq 1 \ (i = 1,\ldots,p); \ \sum_{i=1}^{p}\omega_i^2 < p,$

departure from the null hypothesis will lead to $|C| > (n_2/(n_1+n_2))^p$. Thus a two-tailed test may be needed when statistic $|C|$ is used. If $|I-C|$ is used under alternative H_A, departure from H_0 leads (from 8.2.3) to $|I-C| > (n_1/(n_1+n_2))^p$, and if it is used under alternative H_B, departure from H_0 leads to $|I-C| < (n_1/(n_1+n_2))^p$. Thus use of $|I-C|$ may also necessitate a two-taled test, which is the case for other test criteria such as Pillai's $V^{(p)}$, Hotelling's T_o^2 and Roy's largest root.

A further important characteristic of a test statistic is its convergence. The statistics $|L|$, $|I-L|$, $|R|$ and

and $|I-R|$ all converge, at different rates, in the noncentral case. The hth noncentral moment of $|C|$ is given in equation (4.6.8), whose hypergeometric function has the matrix argument $M = (I-\Omega^{-1})$. Under the alternative hypothesis H_A, the absolute value of the largest characteristic root $(1-(\omega_p^2)^{-1})$ of M is < 1 $(0 < \omega_1^2 < \omega_2^2 < \ldots < \omega_p^2)$. Thus, by condition (ii) of equation (2.1.6) the hypergeometric series of $|C|$ will converge for all values of ω_i^2 satisfying equation (8.1.5). Under the dual alternative hypothesis H_B, the absolute value of the largest characteristic root of M is $|1-\omega_1^{-2}|$, and the hypergeometric series in $|C|$ will only converge for $\omega_i^2 > \frac{1}{2}$.

In this case we may use (as Giri (1968) has noted) the statistic $|I-C|$ with transposed n_1 and n_2 and with alternative hypothesis

(8.2.7) $\quad H_A^* : \omega_i^{*2} \geq 1;\ \sum_{i=1}^{p} \omega_i^{*2} > p, \quad \omega_i^* = \omega_i^{-1}.$

It can be seen that

(8.2.8) $\quad U_{p,n_1,n_2}(\omega_1^2,\ldots,\omega_p^2) = W_{p,n_2,n_1}(\omega_p^{-2},\ldots,\omega_1^{-2})$ and

that the upper α-percentiles of either U_{p,n_1,n_2} or W_{p,n_2,n_1} would be used in this case.

In order to examine further the convergence of $|C|$ and $|I-C|$, we must derive the moments and density of $|I-C|$. As $|C| = |S_2|/|S_1+S_2|$ and $|I-C| = |S_1|/|S_1+S_2|$, results for $|I-C|$ will follow from those for $|C|$ by transposing Σ_1 with Σ_2 and n_1 with n_2. From equation (4.6.8) and Theorem 4.6.2 we have

Theorem 8.2.1

The statistic $|I-C| = |S_1|/|S_1+S_2|$ used in testing the equality of two covariance matrices has hth moment

(8.2.9) $$E|I-C|^h = (\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}n_1+h)/\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n+h))$$

$$\cdot|\Omega|^{\frac{1}{2}n_2}\,{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n_2;\ \tfrac{1}{2}n+h;\ I-\Omega), \qquad \Omega = \Sigma_1\Sigma_2^{-1},$$

and density (in the noncentral linear case)

(8.2.10) $$f(w) = \Pi_{i=1}^{p} f(w_i), \qquad w_i \text{ independent},$$

where

(8.2.11) $$f(w_1) = (B(\tfrac{1}{2}n_1,\tfrac{1}{2}n_2))^{-1} w_1^{\frac{1}{2}n_1-1}(1-w_1)^{\frac{1}{2}n_2-1}$$

$$\cdot\omega^{n_2}\,{}_1F_0(\tfrac{1}{2}n;\ (1-\omega^2)(1-w_1))$$

and

(8.2.12) $$f(w_i) = (B(\tfrac{1}{2}n_2,\tfrac{1}{2}(n_1-i+1)))^{-1} w_i^{\frac{1}{2}(n_1-i+1)-1}(1-w_i)^{\frac{1}{2}n_2-1},$$

$$i = 2,\ldots,p.$$

Thus the density of $|I-C|$ may be expressed in the non-central linear case as that of a product of Beta variables.

The hypergeometric function in the hth moment of $|I-C|$ has matrix argument $M = (I-\Omega)$. Under H_A the absolute value of the largest characteristic root of M is $|1-\omega_p^2|$ and the series will therefore only converge for $\omega_i^2 < 2$. Under H_B the series in $|I-C|$ will converge for all $\omega_i^2 \leq 1$.

When testing the hypothesis H_0 of equality of co-variance matrices against each alternative H_A and H_B, we therefore may use either $|C|$ or $|I-C|$. Calculations of power are made difficult for $|I-C|$ when $\omega^2 \geq 2$ and for $|C|$ when $\omega^2 \leq \frac{1}{2}$. Table 8.1 summarises the features so far examined.

TABLE 8.1 PROPERTIES OF TEST STATISTICS UNDER TWO ALTERNATIVE HYPOTHESES

Alternative Hypothesis	Test Statistic	Rejection Region	Convergence Range of ω_i^2
$\omega_i^2 \geq 1$; $\Sigma\omega_i^2 > p$ (H_A)	U_{p,n_1,n_2}	low u	all $\omega_i^2 > 1$
	W_{p,n_1,n_2}	high w	$1 < \omega_i^2 < 2$
$\omega_i^2 \leq 1$; $\Sigma\omega_i^2 < p$ (H_B)	U_{p,n_1,n_2}	high u	$\frac{1}{2} < \omega_i^2 < 1$
	W_{p,n_1,n_2}	low w	all $\omega_i^2 < 1$

From equation (8.2.8) it can be seen that, for alternative hypothesis H_B, the test statistics W_{p,n_2,n_1} and U_{p,n_2,n_1} may be used instead of U_{p,n_1,n_2} and W_{p,n_1,n_2} respectively. The rejection regions will remain high and low respectively and the noncentral distributions will be in terms of $\omega_i^{*2} = \omega_i^{-2}$. Because of this relationship it follows that powers of both test criteria $|C|$ and $|I-C|$ may be calculated against both alternative hypotheses H_A and H_B by using lower percentiles of $|C|$ and upper percentiles of $|I-C|$, and $\omega_i^2 \geq 1$. This reduces the range of parameters needed for power comparisons.

For purposes of computing powers of $|C|$ and $|I-C|$ we shall restrict further comment to the noncentral linear case, i.e. where Ω has only one non-unit root ω^2. As with the other test criteria of this form, results for $|C|$ and $|I-C|$ can only be calculated by the algorithm for even n_1 and n_2 respectively. We showed in Chapters 6 and 7 that for $|L|$ and $|I-R|$

$$(8.2.13) \quad U_{p,m,n(\tau)} = U_{m,p,m+n-p(\tau)}$$

For $|C|$ and $|I-C|$, however, this is not the case.

Applying Lemma 6.5.1 to equation (4.6.8) with the following changes

(8.2.14) $(m;\ n;\ m+n) \rightarrow (n_1;\ n_2;\ n)$,

we may rewrite $E|C|^h$ as:

(8.2.15) $$E|C|^h = (\Gamma_{n_1}(\tfrac{1}{2}n)\Gamma_{n_1}(\tfrac{1}{2}(n-p)+h)/\Gamma_{n_1}(\tfrac{1}{2}(n-p)) \cdot \Gamma_{n_1}(\tfrac{1}{2}n+h))|\Omega|^{-\frac{1}{2}n_1}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n_1;\ \tfrac{1}{2}n+h;\ I-\Omega^{-1}).$$

Because in the noncentral linear case

(8.2.16) $$\omega^{-n_1}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}n_1;\ \tfrac{1}{2}n+h;\ 1-\omega^{-2}) \neq \omega^{-p}{}_2F_1(\tfrac{1}{2}n,\tfrac{1}{2}p;\ \tfrac{1}{2}n+h;\ 1-\omega^{-2}),$$

it follows that

(8.2.17) $$U_{p,n_1,n_2}(\omega^2) \neq U_{n_1,p,n-p}(\omega^2)$$

Consequently we are only able to express $|C|$ as a product of n_1 Beta variables in the central case, and use of the algorithm for the noncentral linear case requires n_1 to be even, irrespective of p. Examination of equation (8.2.9) reveals that $|I-C|$ can only be written as the product of n_2 Beta variables in the central case, and therefore the algorithm cannot be used in the noncentral linear case if p is even and n_2 odd.

de Waal (1970) examined the distribution of $|I-\Sigma L_j|$, where A_j, $j = 1,\dots,k \sim W(\Sigma_1,n_j)$ and $B \sim W(\Sigma_2,m)$, A_j $j = 1,\dots,k$ and B are independent, and $L_j = (\Sigma_j A_j+B)^{-\frac{1}{2}}A_j(\Sigma_j A_j+B)^{-\frac{1}{2}}$.

The hth moment of $|I-\Sigma L_j|$, given in equation (6.6.2), equals that of $|I-C|$ in equation (8.2.9) if the transformation (8.2.1) is made. $|I-\Sigma L_j|$ may be used to test for the equality of covariance matrices of k populations, given that the means are all equal, and that the alternative hypothesis is that one covariance matrix (Σ_2) differs from the others (Σ_1). It was pointed out in Chapter 6 that for powers of the test to be calculated $n = \Sigma n_j$ should be even, and that Σn_j would usually be greater than m. Exact powers of this test in the noncentral linear case, for different values of p,m,n and ω^2 are discussed in the next section.

de Waal (1970) also obtained an asymptotic distribution for $-(m+\frac{1}{2}(n-p-1))\log|I-\Sigma L_j|$ to order m^{-1} for the case where $I-\Sigma_1^{-1}\Sigma_2 = (2/n)P$, where P is fixed as $n \to \infty$.

8.3 Powers and power comparisons of $|C|$ and $|I-C|$

Because $|C|$ and $|I-C|$ can be expressed as products of independent Beta variables, the algorithm of Chapter 5 can be used to derive their central and noncentral linear densities and cumulative distribution functions. Results for $|C|$ can be obtained similarly to those for $|L|$ by

(a) replacing $e^{-\frac{1}{2}\lambda^2}$ by ω^{-m} and

(b) setting $a_j = ((\frac{1}{2}(m+n))_j(1-\omega^{-2})^j/j!)$

Results for $|I-C|$ follow those for $|I-L|$ by

(c) replacing $e^{-\frac{1}{2}\lambda^2}$ by ω^n and

(d) setting $a_j = ((\frac{1}{2}(m+n))_j(1-\omega^2)^j/j!)$

The values of m and n follow in both cases from equation

(8.2.1); n is the number of degrees of freedom of the numerator sum of squares matrix in $|C|$.

Powers of $|C|$ in the noncentral linear case may be derived for a range of parameters p,m and n similar to that of $|L|$ and $|I-R|$. The rate of convergence and accuracy of results decreases with increasing ω^2; accuracy to 5 decimal places was achieved for values of ω^2 up to 4,0. Section 8.2 discusses convergence with respect to ω^2 of the moment hypergeometric series. For power calculations the convergence of the infinite series of the cumulative distribution function is important, and this depends on ω^2, m and n. The above remarks apply also to $|I-C|$, but ω^2 is limited to less than 2 for convergence.

Pillai and Jayachandran (1968) have tabulated powers of $|C|$ for $p = 2$ and m and n both odd, comparing them with powers of other test criteria. As $U_{p,m,n(\omega^2)} \neq U_{m,p,m+n-p(\omega^2)}$ in the noncentral linear case, these results cannot be directly confirmed by the algorithm. Table 8.2 shows Pillai and Jayachandran's (1968) values with those computed by the algorithm for adjacent even values of m.

Power increases monotonically as m,n and ω^2 increase, the only exception being Pillai and Jayachandran's (1968) values for $U_{2,13,13}$ and $U_{2,13,33}$ with $\omega^2 = 2,0$. This is probably caused by rounding error - Pillai and Jayachandran do not tabulate powers of $|C|$ for $(\omega_1^2+\omega_2^2) > 3$ although for Hotelling's T_o^2 $(H^{(2)})$ and Pillai's $V^{(2)}$ criteria results extend as far as $(\omega_1^2+\omega_2^2) = 12$. Pillai and Jayachandran's values in these cases of 0,227 and 0,298 should be nearer 0,215 and 0,322; this does not really

TABLE 8.2 Powers of $U_{2,m,n}(\omega^2)$ obtained by Pillai and Jayachandran (1968) (odd values of m) and by the algorithm (even values of m)

		ω^2				
m	n	1,00001	1,001	1,1	1,5	2,0
2	13	0,0500007	0,050073	0,057541	0,09178	0,1394
3	13	0,0500008	0,050082	0,058538	0,0981	0,154
4	13	0,0500009	0,050089	0,059290	0,10294	0,1664
5	13	0,0500009	0,050094	0,059888	0,106	0,173
6	13	0,0500010	0,050099	0,060380	0,11001	0,1836
7	13	0,0500010	0,050103	0,060793	0,112	0,185
8	13	0,0500011	0,050106	0,061148	0,11508	0,1959
10	13	0,0500011	0,050111	0,061726	0,11893	0,2054
12	13	0,0500012	0,050115	0,062178	0,12198	0,2128
13	13	0,0500012	0,050117	0,062368	0,122	0,227
14	13	0,0500012	0,050119	0,062543	0,12447	0,2189
2	33	0,0500009	0,050087	0,059110	0,10175	0,161
3	33	0,0500010	0,050100	0,060580	0,1119	0,186
4	33	0,0500011	0,050111	0,061755	0,12034	0,208
5	33	0,0500012	0,050119	0,062740	0,127	0,223
6	33	0,0500013	0,050127	0,063595	0,13394	0,242
7	33	0,0500013	0,050133	0,064347	0,139	0,251
8	33	0,0500014	0,050139	0,065021	0,14485	0,270
10	33	0,0500015	0,050149	0,066184	0,15399	0,293
12	33	0,0500016	0,050157	0,067161	0,16183	0,313
13	33	0,0500015	0,050161	0,067594	0,164	0,298
14	33	0,0500016	0,050164	0,068000	0,16867	0,330

affect the conclusions they have drawn.

From their results, Pillai and Jayachandran (1968) observed

(i) For small deviations from the hypothesis, $V^{(2)}$ seems to have more power than $|C|$, in general, and $|C|$ more power than $H^{(2)}$.

(ii) For large deviations from the hypothesis, when values of ω_1^2 and ω_2^2 are far apart (which includes the noncentral linear case with ω^2 large), powers of $H^{(2)}$ seem to exceed those of $|C|$ and $|C|$ those of $V^{(2)}$

(iii) When ω_1^2 and ω_2^2 are close, powers of $V^{(2)}$ in general exceed those of $|C|$ and $|C|$ those of $H^{(2)}$.

Using Pillai and Jayachandran's (1968) results, we shall also examine the relationship between certain linear and planar cases. The above authors tabulated powers for pairs (ω_1^2,ω_2^2) including (1,0; 1,1), (1,05; 1,05), (1,0; 1,5) and (1,25; 1,25). The first and third pairs are examples of the noncentral linear case

(8.3.1) (a) $\omega_1^2 = 1{,}0; \quad \omega_2^2 = \omega^2$

while the second and fourth are of the type

(8.3.2) (b) $\omega_1^2 = \omega_2^2 = \frac{1}{2}(\omega_1^2+\omega_2^2)$.

Except for the case $m = n = 13$, the values of n used (13, 33, 63 and 83) always exceeded m (3, 5, 7 and 13), often by rather unrealistic amounts for the hypothesis being considered. For $(\omega_1^2+\omega_2^2)$ constant, powers of $|C|$ with alternatives of type (a) exceeded those with alternatives of type (b) in most of these cases. For low n or high m

(i.e. m reasonably close to n) this situation was reversed but for all m and n the relative difference in power between cases (a) and (b) was under 5%. From this one might conclude that powers of $|C|$ for case (b) slightly exceed those for case (a) when m is close to n, and perhaps also when m exceeds n. No check has been possible on the accuracy of the powers of type (b), but the accuracy of those of type (a) has been confirmed.

Greenstreet and Connor (1974) used Monte Carlo methods based on 5 000 generated samples to find results for different test statistics. They considered the test for equality of k covariance matrices against the alternative hypothesis $\Sigma_1 = \Sigma_2 = \ldots = \Sigma_{k-1} \neq \Sigma_k$. Greenstreet and Connor found powers for the test criterion $-2\rho_2 \log L$, where

$$(8.3.3) \qquad \rho_2 = 1 - \left\{(\Sigma_{i=1}^{k} n_i^{-1}) - (\Sigma_{i=1}^{k} n_i)^{-1}\right\} \cdot \frac{2p^2+3p-1}{6(k-1)(p+1)} ,$$

L is the modified likelihood ratio criterion, $n_i = N_i - 1$ and N_i is the sample size of the ith population. For simplicity it was assumed that for the p roots ω_i^2 of $\Sigma_1 \Sigma_k^{-1}$,

$$(8.3.4) \qquad \omega_1^2 = \omega_2^2 = \ldots = \omega_p^2 = \omega^*, \ \omega^* > 1,$$

and that $N_1 = N_2 = \ldots = N_k = N$.

We note that, for $p = 2$, the alternative (8.3.4) is of type (b) (equation (8.3.2)), and that here $m = n = N-1$.

The results obtained led Greenstreet and Connor to state that the power increased with N, ω^* and p, and is a concave function of q. In the noncentral linear case the power decreases as p increases (we would expect this as the "total noncentrality" $\sum_{i=1}^{p}(\omega_i^2-1) = (\omega^2-1)$ remains the same as p changes). In Greenstreet and Connor's case the power depends on $\sum_{i=1}^{p}(\omega_i^2-1) = p(\omega^*-1)$; the "total noncentrality" increases with p, and there is no general reason why the power should either increase or decrease with p here. Although Greenstreet and Connor stated that for simplicity they would treat only cases with eigenvalues ω^* of $\Sigma_1\Sigma_k^{-1}$ greater than one, the modified likelihood ratio criterion has a single tailed rejection region for ω^* significantly less than or greater than one. Consequently direct power comparisons are not possible between L and $|C|$ without considering alternatives of the form $\omega^* < 1$.

Greenstreet and Connor (1974) found the power of $-2\rho_2 \log L$, with $N = 10$, $q = 2$, $\omega^* = 1{,}5$, $\alpha = 0{,}05$ and $p = 2$, to be $0{,}074$. The comparable parameters for $|C|$ would be $p = 2$, $m = n = 9$ and $\omega_1^2 = \omega_2^2 = 1{,}5$, but from the earlier discussion on noncentrality types (8.3.1) and (8.3.2), the noncentral linear case with $\omega_1^2 = 1{,}0$ and $\omega_2^2 = 2{,}0$ should provide similar powers. To use the algorithm we would also need to interpolate powers for $m = 8$ and 10, as $m = 9$ is odd.

In Table 8.3 exact powers are given for $U_{2,8,9}$ and $U_{2,10,9}$, with $\alpha = 0{,}05$ and $0{,}025$ and $\omega^2 = 1{,}5$ and $2{,}0$. From this, the powers of $U_{2,9,9(2,0)}$ at the $0{,}025$ and $0{,}05$ α-levels can be seen to be about $0{,}097$ and $0{,}167$ respectively compared with Greenstreet and Connor's (1974) power for $-2\rho_2 \log L$ at the $0{,}05$ level of $0{,}074$.

TABLE 8.3 Powers of $U_{p,m,9}(\omega^2)$

	$p = 2,\ m = 8$		$p = 2,\ m = 10$	
ω^2	$\alpha = 0{,}05$	$\alpha = 0{,}025$	$\alpha = 0{,}05$	$\alpha = 0{,}025$
1,5	0,1020	0,0556	0,1045	0,0570
2,0	0,1635	0,0953	0,1691	0,0986

	$p = 4,\ m = 8$		$p = 4,\ m = 10$	
ω^2	$\alpha = 0{,}05$	$\alpha = 0{,}025$	$\alpha = 0{,}05$	$\alpha = 0{,}025$
1,5	0,079	0,041	0,080	0,042
2,0	0,110	0,060	0,113	0,062
3,0	0,173	0,102	0,180	0,106

Similarly, powers of $U_{4,9,9}(3,0)$ can be seen to be about 0,104 and 0,177 compared with the power for $-2\rho_2 \log L$ at the 0,5 level of 0,071. For both $p = 2$ and 4, the power of $-2\rho_2 \log L$ is less than half of that for the statistic $|C|$ at the same 0,05 level, and less than that for C at the 0,025 level. Sugiura (1973) mentioned a similar case for $p = 2$, $m = 13$, $n = 63$, $\omega_1^2 = \omega_2^2 = 1{,}05$ and $\alpha = 0{,}05$. The asymptotic power for L of 0,05130 was compared with the powers of 0,0670, 0,0701, 0,0703 and 0,0703 obtained by Pillai and Jayachandran (1968) for their four test criteria. Sugiura (1973) stated that this did not mean that the modified likelihood ratio criterion was worse than the others, as the sets of alternatives differed.

Power comparisons of $|C|$ and $|I-C|$ for $\omega^2 > 1$ given in Appendix 12 show that for $n > m$ $|C|$ tends to be more powerful than $|I-C|$, except when m is close to n and ω^2 is low. $|I-C|$ would therefore not seem to be an important alternative in this case. No powers of any test criteria appear to have been published for the case where $\omega^2 > 1$ and $m > n$. Using the notation of Section 8.2, this is the case where $\Sigma_1\Sigma_2^{-1} > 1$ and $n_1 > n_2$. Although in tests for equality of means and independence the case $m > n$ is impractical, it will be as common an occurrence as $m < n$ for the test of equality of covariance matrices. By equation (8.2.8) both cases would occur if a two-tailed test is required. Here the powers of $|I-C|$ tend to exceed those of $|C|$ over the range $1 < \omega^2 < 2$ for which $|I-C|$ is calculable.

For alternatives $\omega^2 < 1$ the reverse holds: $|I-C|$ will generally be more powerful than $|C|$ for $n > m$ and less powerful for $m > n$. Specific comparisons are given in Appendix 12. $|I-C|$ has therefore been shown to be superior to $|C|$ in certain (practical) cases, but lack of published information makes it impossible to assess its performance against other test criteria in these cases. The results for $|I-C|$ also extend to the test statistic $|I-\Sigma L_j|$, for which, as previously stated, we will generally expect to have $n = \Sigma n_j > m$, and may have alternatives $\omega^2 < 1$ or > 1. Selected powers of $|C|$ and $|I-C|$ are given in Appendices 10 and 11.

CHAPTER 9

TESTS WITH COMPLEX VARIABLES

9.1 A general algorithm for test criteria with complex variables

In Section 4.8 the distributions of test criteria for various statistical hypotheses with real variables were expressed in a general form as a product of Beta variables. In Chapter 5 it was shown how percentiles and powers of the general form could be found, and this was carried out for different test criteria in Chapters 6, 7 and 8. Section 4.9 showed how the general form could be extended to the case where the variables were complex.

For the real case the Beta variables have the density

$$(9.1.1) \qquad f(u_i) = (B(\tfrac{1}{2}m^*,\tfrac{1}{2}(n^*-i+1)))^{-1}u_i^{\frac{1}{2}(n^*-i+1)-1}(1-u_i)^{\frac{1}{2}m^*-1}$$

but in the complex case this becomes

$$(9.1.2) \qquad f(u_i') = (B(m^*,n^*-i+1))^{-1}u_i'^{\,n^*-i}(1-u_i')^{m^*-1}.$$

In the complex case the term $2(n^*-i+1)$, and not $(2n^*-i+1)$, replaces (n^*-i+1) of the real case. This means that the results obtained for real variables cannot be extended to complex variables by using distributions with doubled parameter values $2m^*$, $2n^*$ (and $2\lambda^2$ for $|L|$ and $|I-L|$). The process of convolution is different in that the exponent of u_i', i.e. (n^*-i), in the density of successive variables decreases by one and not a half at each stage.

When testing the equality of mean vectors, for example, we have

(9.1.3) $|L| = \Pi_{i=1}^{p} f(u_i')$

where

(9.1.4) $f(u_1') = (B(m^*,n^*))^{-1} u_1'^{\,n^*-1}(1-u_1')^{m^*-1}$

$$\cdot e^{-\lambda^2}\,{}_1F_1(m^*+n^*;\ m^*;\ \lambda^2(1-u_1'))$$

and

(9.1.5) $f(u_i') = (B(m^*,n^*-i+1))^{-1} u_i'^{\,n^*-i}(1-u_i')^{m^*-1}, \quad i = 2,\ldots,p.$

Setting $y_i = -\log u_i'$, and expanding $(1-u_i')^{m^*-1}$ binomially, we have

(9.1.6) $f(y_1) = (B(m^*,n^*))^{-1} e^{-\lambda^2} \sum_{j=o}^{\infty} a_j$

$$\cdot \sum_{k=o}^{b+j} (-1)^k \binom{b+j}{k} \exp(-y_1(n^*+k))$$

and

(9.1.7) $f(y_i) = (B(m^*,n^*-i+1))^{-1} \sum_{\ell=o}^{b} (-1)^{\ell} \binom{b}{\ell}$

$$\cdot \exp(-y_i(n^*+\ell-i+1)), \quad i = 2,\ldots,p,$$

where

(9.1.8) $a_j = ((m^*+n^*)_j \lambda^{2j}/(m^*)_j j!)$

and

(9.1.9) $b = m^*-1$.

From the above it can be seen that

(a) as b is always integral, a finite binomial expansion is always possible (m^* odd or even),

(b) as the exponent of u_i' decreases with each i by a full unit, Theorem 5.2.1 (the method of double convolution) cannot be used and hence the full $(p-1)$ convolutions are necessary, and

(c) the algorithm for complex variables should use equations (9.1.6) and (9.1.7) rather than equations (5.3.6), (5.3.7) and (5.3.8) in the real case.

It follows from (a) that Theorem 5.2.2 is unnecessary; (b) implies that explicit series expressions become unmanageable, and results obtained by a computer algorithm lose accuracy, for comparatively low values of p. From (c) it follows that slight changes will occur in the results summarised in Theorems 5.4.1 - 5.4.5.

Having outlined the main differences of approach in the complex case we shall not proceed further, as the methods used for all test criteria essentially parallel those for the real case. Money (1972) used the method of convolutions to derive explicit series expressions for the complex non-central linear densities of the vector coherence coefficient $|R|$ and $|I-R|$, for $p = 2$ and 3, and showed how they could similarly be obtained for $|L|$, $|I-L|$ and $|I-\Sigma L_j|$.

9.2 Central percentiles of the test criterion for reality of a covariance matrix

In Section 4.10 we showed how the density of the test criterion Λ for reality of a covariance matrix could be expressed as a product of Beta variables

$$(9.2.1) \qquad f(\Lambda) = \Pi_{i=1}^{p^*} f(u_i)$$

where

$$(9.2.2) \qquad f(u_i) = \beta_1(u_i;\ m^*, n^*-i+1) \qquad i = 1,2,\ldots,p^*,$$

$$(9.2.3) \qquad p^* = t',\ m^* = 2t-1 \text{ and } n^* = 2(n-q-t)$$

and

$$(9.2.4) \qquad \begin{aligned} t' = t &= \tfrac{1}{2}p &&,\ p \text{ even} \\ = t-1 &= \tfrac{1}{2}(p-1) &&,\ p \text{ odd } (> 1) \end{aligned}$$

This is the general form of the test criteria for real variables in the central case, where p^*, m^* and n^* have the following values:

		p odd (> 1)	p even
(9.2.5)	p^* =	$\frac{1}{2}(p-1)$	$\frac{1}{2}p$
(9.2.6)	m^* =	p	$p-1$
(9.2.7)	n^* =	$2(n-q)-p-1$	$2(n-q)-p$

From the above we see that m^* is always odd and n^* is always even, and that the algorithm may not be directly applied. Using the fact that under the null hypothesis

(9.2.8) $$U_{p^*,m^*n^*} = U_{m^*,p^*,m^*+n^*-p^*}$$

we may, however, use the algorithm to obtain exact percentiles of the criterion for p even. Percentiles for p odd may be found by interpolation, approximation, or the methods of Lee (1972) and Mathai (1971a). Khatri (1965a) gives the second order approximation

(9.2.9) $$\Pr(-r \log \Lambda \leq \xi) = \Pr(\chi_f^2 \leq \xi) + \gamma_2 r^{-2}\left(\Pr(\chi_{f+4}^2 \leq \xi) - \Pr(\chi_f^2 \leq \xi)\right) + O(r^{-3})$$

where

(9.2.10) $r = 2(n-q)-p-\frac{1}{2}$, $f = \frac{1}{2}p(p-1)$ and

$$\gamma_2 = p(p-1)(p^2+(p-1)^2-8)/48.$$

We notice that, when p is odd, $n^* \geq p^*$ implies

$$2(n-q)-p-1 \geq \tfrac{1}{2}(p-1), \text{ i.e.}$$

(9.2.11) $$n-q \geq \tfrac{3}{4}p + \tfrac{1}{4}.$$

When p is even, $n^* \geq p^*$ implies

$$2(n-q)-p \geq \tfrac{1}{2}p, \text{ i.e.}$$

(9.2.12) $$n-q \geq \tfrac{3}{4}p.$$

For p odd or even, $n-q \geq p$ implies $n^* > p^*$ $(p > 1)$. Thus, when p is even, the central percentiles of Λ may be found by setting

(9.2.13) $p^* = p-1$

(9.2.14) $m^* = \frac{1}{2}p$

and

(9.2.15) $n^* = 2(n-q)-1-\frac{1}{2}p$

and using the central density of U_{p^*,m^*n^*}.

APPENDIX 1 SOME CENTRAL PERCENTILES OF |I-L|

p	m	n	0,90	0,95	0,975	0,99	0,995
4	4	4	0,03965	0,06539	0,09580	0,14144	0,17886
		6	0,01268	0,02209	0,03403	0,05345	0,07062
		8	0,00522	0,00937	0,01485	0,02416	0,03273
		10	0,00252	0,00462	0,00745	0,01241	0,01709
		12	0,00137	0,00253	0,00414	0,00700	0,00975
		14	0,00080	0,00150	0,00248	0,00423	0,00595
4	6	4	0,16480	0,21515	0,26458	0,32765	0,37322
		6	0,07243	0,09905	0,12697	0,16522	0,19474
		8	0,03637	0,05118	0,06733	0,09039	0,10889
		10	0,02015	0,02892	0,03873	0,05312	0,06497
		12	0,01203	0,01752	0,02376	0,03311	0,04095
		14	0,00762	0,01121	0,01535	0,02165	0,02701
4	8	4	0,27745	0,33475	0,38741	0,45075	0,49440
		6	0,14501	0,18195	0,21799	0,26420	0,29795
		8	0,08225	0,10589	0,12980	0,16166	0,18581
		10	0,04987	0,06539	0,08148	0,10349	0,12059
		12	0,03191	0,04241	0,05351	0,06897	0,08120
		14	0,02133	0,02866	0,03650	0,04760	0,05650
4	10	4	0,36803	0,42580	0,47702	0,53670	0,57677
		6	0,21510	0,25739	0,29700	0,34591	0,38059
		8	0,13275	0,16254	0,19141	0,22837	0,25547
		10	0,08594	0,10701	0,12792	0,15536	0,17596
		12	0,05793	0,07307	0,08836	0,10880	0,12441
4	12	6	0,27830	0,32300	0,36376	0,41288	0,44702
		8	0,18277	0,21656	0,24838	0,28805	0,31650
		10	0,12439	0,14969	0,17406	0,20517	0,22798
		12	0,08736	0,10641	0,12507	0,14932	0,16741
4	14	6	0,33401	0,37941	0,42004	0,46818	0,50115
		8	0,23010	0,26633	0,29977	0,34066	0,36952
		10	0,16293	0,19130	0,21806	0,25153	0,27567

APPENDIX 1 SOME CENTRAL PERCENTILES OF |I-L|

p	m	n	0,90	0,95	0,975	0,99	0,995
5	6	6	0,09161	0,03000	0,04211	0,06047	0,07593
		8	0,00771	0,01218	0,01760	0,02618	0,03370
		10	0,00351	0,00566	0,00834	0,01271	0,01663
		12	0,00178	0,00291	0,00435	0,00674	0,00893
5	8	6	0,06361	0,08493	0,10713	0,13749	0,16098
		8	0,02982	0,04099	0,05307	0,07025	0,08405
		10	0,01540	0,02160	0,02848	0,03853	0,04682
		12	0,00857	0,01221	0,01632	0,02245	0,02759
5	10	6	0,11687	0,14605	0,17476	0,21201	0,23961
		8	0,06164	0,07906	0,09682	0,12072	0,13905
		10	0,03481	0,04550	0,05667	0,07210	0,08421
5	12	6	0,17113	0,20537	0,23789	0,27873	0,30817
		8	0,09830	0,12078	0,14284	0,17150	0,19284
		10	0,05942	0,07428	0,08922	0,10911	0,12423
6	6	6	0,00271	0,00501	0,00816	0,01370	0,01897
		6	0,00080	0,00153	0,00256	0,00447	0,00637
		10	0,00029	0,00057	0,00097	0,00174	0,00252
		12	0,00012	0,00024	0,00042	0,00077	0,00113
6	8	6	0,02289	0,03296	0,04423	0,06077	0,07437
		8	0,00866	0,01287	0,01778	0,02530	0,03172
		10	0,00374	0,00569	0,00802	0,01168	0,01487
6	10	6	0,05732	0,07515	0,09360	0,11872	0,13817
		8	0,02543	0,03432	0,04387	0,05738	0,06822
		10	0,01238	0,01703	0,02206	0,02889	0,03356
7	8	8	0,00177	0,00293	0,00441	0,00688	0,00917
		10	0,00063	0,00106	0,00163	0,00259	0,00347
7	10	8	0,00899	0,01287	0,01728	0,02388	0,02943
8	8	8	0,00018	0,00035	0,00060	0,00106	0,00150

APPENDIX 2 SOME NONCENTRAL PERCENTAGES OF |L|

P= 5 M= 6 N=24

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.000	.18397	.15854	.13842	.11727	.10423
.500	.18017	.15506	.13523	.11442	.10161
1.000	.17653	.15176	.13221	.11174	.09915
2.000	.16971	.14559	.12661	.10679	.09462
4.000	.15764	.13477	.11687	.09826	.08688
6.000	.14726	.12557	.10866	.09113	.08046
8.000	.13823	.11763	.10161	.08506	.07501

P= 6 M= 6 N=20

LAMBDA SQUARED	.100	.050	.025	.010	.005
.000	.08522	.06980	.05821	.04666	.03989
.500	.08326	.06811	.05675	.04544	.03881
1.000	.08139	.06651	.05536	.04428	.03780
2.000	.07791	.06354	.05280	.04216	.03594
4.000	.07180	.05838	.04838	.03852	.03277
6.000	.06661	.05403	.04469	.03550	.03016
8.000	.06214	.05031	.04155	.03295	.02797

P= 7 M= 6 N=15

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.000	.01788	.01314	.00993	.00706	.00555
.500	.01739	.01277	.00964	.00685	.00538
1.000	.01693	.01241	.00937	.00665	.00522
2.000	.01608	.01177	.00887	.00629	.00492
4.000	.01461	.01067	.00802	.00567	.00444
6.000	.01340	.00976	.00733	.00517	.00404
8.000	.01237	.00900	.00675	.00475	.00371

P= 8 M= 4 N=25

LAMBDA SQUARED	.100	.050	.025	.010	.005
.000	.14665	.12445	.10715	.08926	.07840
.500	.14356	.12168	.10466	.08708	.07642
1.000	.14061	.11904	.10230	.08503	.07456
2.000	.13507	.11413	.09791	.08123	.07114
4.000	.12529	.10552	.09028	.07467	.06527
6.000	.11689	.09821	.08385	.06919	.06040
8.000	.10959	.09189	.07833	.06453	.05626

APPENDIX 2

SOME NONCENTRAL PERCENTAGES OF |L|

P= 9 M= 4 N=15

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.000	.01497	.01053	.00764	.00515	.00389
.500	.01452	.01020	.00739	.00498	.00376
1.000	.01410	.00990	.00717	.00483	.00364
2.000	.01332	.00934	.00675	.00454	.00342
4.000	.01201	.00839	.00605	.00406	.00306
6.000	.01093	.00762	.00549	.00368	.00277
8.000	.01003	.00699	.00503	.00336	.00253

P=12 M= 4 N=15

LAMBDA SQUARED	.100	.050	.025	.010	.005
.000	.00108	.00063	.00039	.00021	.00014
.500	.00105	.00062	.00038	.00021	.00013
1.000	.00102	.00060	.00037	.00020	.00013
2.000	.00097	.00057	.00035	.00019	.00012
4.000	.00088	.00051	.00031	.00017	.00011
6.000	.00080	.00047	.00028	.00015	.00010
8.000	.00074	.00043	.00026	.00014	.00009

P=12 M= 2 N=24

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.000	.15401	.12785	.10772	.08722	.07499
.500	.15039	.12469	.10494	.08488	.07291
1.000	.14695	.12169	.10232	.08267	.07097
2.000	.14053	.11614	.09750	.07863	.06742
4.000	.12933	.10654	.08920	.07173	.06139
6.000	.11986	.09849	.08230	.06604	.05645
8.000	.11173	.09164	.07646	.06126	.05231

P=13 M= 2 N=15

LAMBDA SQUARED	.100	.050	.025	.010	.005
.000	.00575	.00323	.00188	.00094	.00057
.500	.00557	.00313	.00181	.00091	.00055
1.000	.00540	.00303	.00176	.00088	.00053
2.000	.00508	.00285	.00165	.00083	.00050
4.000	.00455	.00254	.00147	.00074	.00045
6.000	.00411	.00230	.00133	.00067	.00040
8.000	.00376	.00210	.00121	.00061	.00037

APPENDIX 3

SOME NONCENTRAL PERCENTILES OF |I-L|

P= 2 M= 2 N=30

LAMBDA SQUARED	α=.900	.950	.975	.990	.995
.000	.00546	.00903	.01339	.02025	.02621
.500	.00668	.01097	.01614	.02419	.03110
1.000	.00788	.01280	.01866	.02766	.03531
2.000	.01018	.01618	.02317	.03370	.04250
4.000	.01436	.02210	.03084	.04365	.05411
8.000	.02141	.03178	.04310	.05917	.07195
16.000	.03212	.04623	.06110	.08154	.09738

P= 4 M= 4 N= 4

LAMBDA SQUARED	.900	.950	.975	.990	.995
.000	.03965	.06539	.09580	.14144	.17886
.500	.04165	.06841	.09986	.14678	.18506
1.000	.04350	.07117	.10354	.15159	.19062
2.000	.04677	.07604	.10998	.15994	.20019
4.000	.05203	.08376	.12011	.17294	.21503
8.000	.05911	.09413	.13368	.19030	.23479
16.000	.06658	.10519	.14829	.20918	.25643

P= 4 M=10 N= 4

LAMBDA SQUARED	α=.900	.950	.975	.990	.995
.000	.36802	.42580	.47702	.53670	.57677
.500	.37156	.42941	.48061	.54019	.58014
1.000	.37490	.43280	.48399	.54347	.58330
2.000	.38103	.43904	.49019	.54948	.58910
4.000	.39153	.44971	.50079	.55973	.59899
8.000	.40751	.46596	.51694	.57539	.61410
16.000	.42795	.48691	.53788	.59583	.63390
24.000	.44048	.49992	.55103	.60881	.64658

P= 6 M= 6 N= 6

LAMBDA SQUARED	.900	.950	.975	.990	.995
.000	.00271	.00501	.00816	.01370	.01897
.500	.00281	.00519	.00843	.01413	.01954
1.000	.00290	.00535	.00869	.01453	.02008
2.000	.00308	.00565	.00915	.01527	.02104
4.000	.00337	.00616	.00993	.01649	.02265
8.000	.00380	.00691	.01108	.01828	.02501
16.000	.00431	.00780	.01246	.02044	.02785
24.000	.00460	.00831	.01325	.02168	.02950

APPENDIX 4 SOME POWERS OF |L|

P= 5 M= 6 N=16

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10820	.05483	.02774	.01125	.00568
1.000	.11657	.05983	.03061	.01257	.00640
2.000	.13376	.07030	.03671	.01544	.00798
4.000	.16966	.09296	.05033	.02206	.01173
6.000	.20704	.11764	.06572	.02986	.01625
8.000	.24532	.14398	.08272	.03880	.02157

P= 5 M= 6 N=24

.100	.050	.025	.010	.005
.10947	.05564	.02824	.01149	.00582
.11923	.06156	.03167	.01311	.00671
.13956	.07418	.03916	.01670	.00872
.18303	.10240	.05656	.02543	.01377
.22934	.13417	.07708	.03629	.02027
.27749	.16893	.10054	.04931	.02832

P= 6 M= 6 N=20

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10782	.05463	.02763	.01120	.00565
1.000	.11582	.05942	.03039	.01248	.00636
2.000	.13229	.06948	.03628	.01526	.00790
4.000	.16691	.09141	.04951	.02173	.01157
6.000	.20325	.11548	.06459	.02942	.01606
8.000	.24074	.14139	.08139	.03833	.02140

P= 7 M= 2 N=60

.100	.050	.025	.010	.005
.11752	.06078	.03134	.01302	.00669
.13608	.07256	.03848	.01654	.00870
.17593	.09902	.05518	.02517	.01380
.26384	.16239	.09816	.04939	.02899
.35755	.23672	.15300	.08347	.05184
.45144	.31789	.21757	.12727	.08300

APPENDIX 4 SOME POWERS OF |L|

P= 7 M= 6 N=15

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10599	.05350	.02697	.01089	.00548
1.000	.11204	.05706	.02899	.01181	.00598
2.000	.12432	.06440	.03320	.01375	.00704
4.000	.14942	.07980	.04223	.01802	.00941
6.000	.17506	.09606	.05202	.02279	.01210
8.000	.20101	.11301	.06250	.02803	.01512

P= 8 M= 2 N=40

.100	.050	.025	.010	.005
.11498	.05909	.03029	.01248	.00637
.13072	.06890	.03613	.01530	.00796
.16427	.09063	.04951	.02202	.01184
.23788	.14177	.08301	.04007	.02280
.31690	.20149	.12506	.06469	.03860
.39760	.26735	.17460	.09597	.05971

P= 8 M= 4 N=25

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10873	.05518	.02796	.01136	.00574
1.000	.11770	.06059	.03108	.01282	.00654
2.000	.13632	.07205	.03784	.01603	.00834
4.000	.17590	.09746	.05335	.02372	.01274
6.000	.21792	.12584	.07143	.03312	.01831
8.000	.26157	.15676	.09193	.04427	.02510

P= 9 M= 4 N=15

.100	.050	.025	.010	.005
.10580	.05335	.02687	.01084	.00545
.11164	.05677	.02879	.01170	.00591
.12344	.06374	.03275	.01351	.00689
.14742	.07826	.04116	.01742	.00904
.17171	.09343	.05017	.02173	.01144
.19614	.10911	.05969	.02639	.01408

APPENDIX 4 SOME POWERS OF |L|

P=10 M= 2 N=15

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10718	.05409	.02725	.01099	.00552
1.000	.11445	.05827	.02957	.01201	.00606
2.000	.12922	.06690	.03440	.01418	.00722
4.000	.15949	.08507	.04481	.01894	.00980
6.000	.19042	.10429	.05612	.02426	.01272
8.000	.22168	.12437	.06823	.03010	.01599

P=10 M= 4 N=15

LAMBDA SQUARED	.100	.050	.025	.010	.005
.500	.10502	.05289	.02660	.01071	.00538
1.000	.11005	.05581	.02823	.01144	.00577
2.000	.12018	.06174	.03158	.01295	.00658
4.000	.14061	.07396	.03858	.01617	.00833
6.000	.16114	.08657	.04596	.01964	.01025
8.000	.18166	.09948	.05366	.02334	.01231

P=11 M= 2 N=15

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10602	.05340	.02685	.01080	.00542
1.000	.11208	.05684	.02874	.01163	.00586
2.000	.12433	.06389	.03264	.01335	.00677
4.000	.14920	.07853	.04088	.01705	.00874
6.000	.17438	.09377	.04966	.02108	.01092
8.000	.19968	.10951	.05890	.02540	.01329

P=11 M= 4 N=15

LAMBDA SQUARED	.100	.050	.025	.010	.005
.500	.10429	.05245	.02635	.01060	.00532
1.000	.10859	.05492	.02773	.01121	.00564
2.000	.11719	.05992	.03052	.01246	.00631
4.000	.13440	.07008	.03628	.01507	.00772
6.000	.15155	.08044	.04225	.01783	.00922
8.000	.16859	.09094	.04840	.02072	.01082

APPENDIX 4 SOME POWERS OF |L|

P=12 M= 2 N=24

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10866	.05507	.02785	.01129	.00569
1.000	.11755	.06035	.03086	.01267	.00644
2.000	.13599	.07152	.03734	.01569	.00810
4.000	.17509	.09616	.05211	.02283	.01212
6.000	.21650	.12357	.06920	.03146	.01711
8.000	.25946	.15335	.08849	.04161	.02314

P=12 M= 4 N=15

LAMBDA SQUARED	.100	.050	.025	.010	.005
.500	.10360	.05204	.02612	.01049	.00526
1.000	.10719	.05409	.02724	.01098	.00552
2.000	.11435	.05819	.02952	.01199	.00605
4.000	.12855	.06646	.03414	.01406	.00716
6.000	.14259	.07478	.03886	.01620	.00831
8.000	.15643	.08312	.04366	.01840	.00950

P=13 M= 2 N=15

LAMBDA SQUARED	α=.100	.050	.025	.010	.005
.500	.10383	.05210	.02612	.01047	.00524
1.000	.10766	.05421	.02725	.01095	.00549
2.000	.11529	.05844	.02952	.01192	.00599
4.000	.13042	.06694	.03412	.01390	.00702
6.000	.14537	.07548	.03879	.01593	.00808
8.000	.16012	.08403	.04352	.01800	.00917

P=13 M= 4 N=15

LAMBDA SQUARED	.100	.050	.025	.010	.005
.500	.10291	.05163	.02588	.01038	.00520
1.000	.10580	.05325	.02677	.01076	.00540
2.000	.11153	.05650	.02854	.01153	.00580
4.000	.12281	.06294	.03209	.01309	.00662
6.000	.13385	.06934	.03565	.01467	.00746
8.000	.14465	.07568	.03921	.01626	.00831

APPENDIX 5

POWER COMPARISONS OF |L| AND |I-L|

P= 4 M= 4 N= 4

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10600	.05373	.02721	.01105	.00558
.50	MOD L	.10324	.05169	.02586	.01035	.00518
	RATIO	1.0267	1.0395	1.0520	1.0676	1.0786
	MOD I-L	.11151	.05720	.02928	.01205	.00614
1.00	MOD L	.10639	.05333	.02670	.01069	.00535
	RATIO	1.0481	1.0726	1.0956	1.1271	1.1488
	MOD I-L	.12125	.06344	.03307	.01390	.00720
2.00	MOD L	.11241	.05649	.02831	.01134	.00567
	RATIO	1.0786	1.1231	1.1678	1.2257	1.2679
	MOD I-L	.13667	.07360	.03940	.01711	.00906
4.00	MOD L	.12354	.06234	.03132	.01256	.00629
	RATIO	1.1063	1.1807	1.2582	1.3623	1.4408
	MOD I-L	.15687	.08755	.04848	.02196	.01198
8.00	MOD L	.14293	.07264	.03662	.01473	.00738
	RATIO	1.0975	1.2053	1.3238	1.4916	1.6246
	MOD I-L	.17706	.10235	.05868	.02782	.01569
16.00	MOD L	.17423	.08953	.04540	.01832	.00919
	RATIO	1.0162	1.1432	1.2924	1.5182	1.7080
	MOD I-L	.18665	.10973	.06401	.03106	.01785
24.00	MOD L	.19942	.10338	.05267	.02132	.01070
	RATIO	.9360	1.0614	1.2152	1.4572	1.6678

P= 4 M= 4 N= 6

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10688	.05430	.02755	.01122	.00568
.50	MOD L	.10682	.05382	.02707	.01089	.00547
	RATIO	1.0006	1.0088	1.0177	1.0300	1.0395
	MOD I-L	.11325	.05832	.02997	.01239	.00634
1.00	MOD L	.11359	.05765	.02915	.01180	.00594
	RATIO	.9970	1.0117	1.0278	1.0505	1.0681
	MOD I-L	.12462	.06563	.03442	.01460	.00760
2.00	MOD L	.12697	.06528	.03335	.01363	.00690
	RATIO	.9815	1.0053	1.0322	1.0710	1.1018
	MOD I-L	.14294	.07776	.04204	.01849	.00988
4.00	MOD L	.15307	.08049	.04183	.01739	.00889
	RATIO	.9338	.9661	1.0049	1.0633	1.1115
	MOD I-L	.16769	.09495	.05330	.02457	.01358
8.00	MOD L	.20260	.11045	.05900	.02520	.01309
	RATIO	.8277	.8597	.9034	.9748	1.0372
	MOD I-L	.19352	.11403	.06654	.03223	.01848
16.00	MOD L	.29144	.16776	.09341	.04158	.02211
	RATIO	.6640	.6797	.7123	.7753	.8357
	MOD I-L	.20630	.12394	.07374	.03666	.02143
24.00	MOD L	.36814	.22102	.12711	.05843	.03165
	RATIO	.5604	.5608	.5801	.6274	.6770

APPENDIX 5

POWER COMPARISONS OF |L| AND |I-L|

		P= 4 M= 4 N= 8					P= 4 M= 4 N=10				
LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-L	.10750	.05469	.02779	.01134	.00575	.10795	.05499	.02797	.01143	.00580
.50	MOD L	.10884	.05508	.02781	.01124	.00566	.11021	.05595	.02834	.01150	.00580
	RATIO	.9877	.9930	.9993	1.0087	1.0162	.9795	.9828	.9873	.9943	1.0003
	MOD I-L	.11447	.05911	.03046	.01264	.00649	.11538	.05971	.03083	.01283	.00660
1.00	MOD L	.11772	.06024	.03070	.01253	.00635	.12057	.06207	.03181	.01308	.00665
	RATIO	.9724	.9813	.9922	1.0086	1.0219	.9569	.9619	.9691	.9810	.9912
	MOD I-L	.12703	.06721	.03542	.01511	.00790	.12883	.06841	.03613	.01550	.00814
2.00	MOD L	.13560	.07080	.03667	.01524	.00780	.14168	.07480	.03915	.01647	.00851
	RATIO	.9368	.9494	.9658	.9915	1.0130	.9093	.9146	.9240	.9410	.9562
	MOD I-L	.14754	.08087	.04403	.01954	.01051	.15104	.08327	.04558	.02038	.01102
4.00	MOD L	.17154	.09269	.04936	.02112	.01101	.18498	.10193	.05528	.02418	.01281
	RATIO	.8600	.8725	.8919	.9252	.9546	.8165	.8169	.8246	.8426	.8605
	MOD I-L	.17591	.10071	.05712	.02667	.01488	.18235	.10530	.06021	.02840	.01596
8.00	MOD L	.24274	.13851	.07709	.03457	.01855	.27297	.16100	.09239	.04298	.02367
	RATIO	.7247	.7271	.7409	.7716	.8020	.6680	.6540	.6517	.6608	.6744
	MOD I-L	.20655	.12355	.07310	.03603	.02091	.21709	.13142	.07864	.03930	.02303
16.00	MOD L	.37546	.23253	.13830	.06653	.03728	.43939	.28687	.17924	.09152	.05345
	RATIO	.5501	.5313	.5285	.5415	.5607	.4941	.4581	.4387	.4294	.4310
	MOD I-L	.22220	.13582	.08211	.04163	.02468	.23530	.14585	.08934	.04603	.02760
24.00	MOD L	.49000	.32324	.20251	.10296	.05970	.57961	.40892	.27305	.15005	.09182
	RATIO	.4535	.4202	.4055	.4043	.4134	.4060	.3567	.3272	.3068	.3006

APPENDIX 5

POWER COMPARISONS OF |L| AND |I-L|

		P= 4 M= 4 N=12					P= 4 M= 6 N= 4				
LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-L	.10830	.05521	.02811	.01150	.00585	.10554	.05331	.02689	.01087	.00547
.50	MOD L	.11123	.05660	.02873	.01169	.00591	.10252	.05131	.02567	.01027	.00514
	RATIC	.9737	.9755	.9785	.9838	.9886	1.0295	1.0389	1.0475	1.0578	1.0649
	MOD I-L	.11608	.06017	.03111	.01297	.00668	.11079	.05647	.02872	.01171	.00593
1.00	MOD L	.12270	.06346	.03266	.01350	.00690	.10497	.05260	.02633	.01054	.00527
	RATIO	.9461	.9481	.9525	.9609	.9687	1.0554	1.0737	1.0907	1.1114	1.1256
	MOD I-L	.13023	.06935	.03677	.01581	.00832	.12046	.06239	.03218	.01334	.00683
2.00	MOD L	.14628	.07789	.04110	.01747	.00909	.10970	.05508	.02760	.01105	.00553
	RATIC	.8903	.8903	.8947	.9052	.9157	1.0981	1.1328	1.1660	1.2069	1.2355
	MOD I-L	.15380	.08518	.04684	.02106	.01143	.13697	.07278	.03840	.01635	.00853
4.00	MOD L	.19533	.10928	.06012	.02677	.01435	.11855	.05974	.02999	.01203	.00602
	RATIO	.7874	.7795	.7791	.7866	.7966	1.1554	1.2184	1.2806	1.3598	1.4167
	MOD I-L	.18752	.10904	.06276	.02985	.01688	.16145	.08892	.04847	.02146	.01149
8.00	MOD L	.29668	.17949	.10550	.05056	.02842	.13426	.06808	.03429	.01378	.00690
	RATIC	.6321	.6075	.5949	.5905	.5938	1.2025	1.3061	1.4135	1.5573	1.6655
	MOD I-L	.22580	.13803	.08336	.04214	.02491	.19042	.10929	.06195	.02879	.01596
16.00	MOD L	.48891	.33226	.21576	.11557	.06981	.16026	.08208	.04156	.01676	.0084C
	RATIO	.4618	.4154	.3864	.3647	.3568	1.1882	1.3314	1.4904	1.7180	1.9000
	MOD I-L	.24630	.15444	.09563	.04995	.03024	.20628	.12112	.07021	.03359	.01903
24.00	MOD L	.64567	.47882	.33588	.19644	.12564	.18160	.09377	.04769	.01328	.00968
	RATIC	.3815	.3225	.2847	.2543	.2407	1.1359	1.2916	1.4721	1.7422	1.9670

APPENDIX 5   POWER COMPARISONS OF |L| AND |I-L|

P= 4 M= 6 N= 6

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10644	.05387	.02723	.01103	.00556
.50	MOD L	.10520	.05292	.02658	.01068	.00536
	RATIO	1.0118	1.0180	1.0243	1.0326	1.0388
	MOD I-L	.11257	.05760	.02940	.01205	.00613
1.00	MOD L	.11037	.05583	.02817	.01137	.00572
	RATIO	1.0200	1.0316	1.0435	1.0594	1.0713
	MOD I-L	.12395	.06463	.03355	.01403	.00723
2.00	MOD L	.12057	.06165	.03136	.01277	.00645
	RATIO	1.0280	1.0483	1.0697	1.0986	1.1207
	MOD I-L	.14362	.07715	.04113	.01775	.00935
4.00	MOD L	.14052	.07321	.03779	.01561	.00795
	RATIO	1.0221	1.0538	1.0884	1.1371	1.1754
	MOD I-L	.17343	.09703	.05369	.02423	.01317
8.00	MOD L	.17857	.09595	.05073	.02147	.01109
	RATIO	.9712	1.0112	1.0584	1.1289	1.1871
	MOD I-L	.20975	.12290	.07103	.03384	.01911
16.00	MOD L	.24775	.13958	.07655	.03362	.01775
	RATIO	.8466	.8805	.9279	1.0067	1.0767
	MOD I-L	.23018	.13834	.08194	.04029	.02329
24.00	MOD L	.30878	.18049	.10188	.04606	.02474
	RATIO	.7455	.7664	.8043	.8748	.9415

P= 4 M= 8 N= 8

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10614	.05366	.02709	.01096	.00553
.50	MOD L	.10546	.05313	.02673	.01076	.00541
	RATIO	1.0064	1.0100	1.0136	1.0185	1.0221
	MOD I-L	.11208	.05724	.02916	.01193	.00605
1.00	MOD L	.11093	.05628	.02849	.01155	.00582
	RATIO	1.0104	1.0169	1.0237	1.0329	1.0398
	MOD I-L	.12339	.06415	.03321	.01384	.00712
2.00	MOD L	.12189	.06268	.03208	.01316	.00669
	RATIO	1.0123	1.0235	1.0355	1.0518	1.0645
	MOD I-L	.14384	.07702	.04094	.01760	.00924
4.00	MOD L	.14383	.07576	.03954	.01657	.00853
	RATIO	1.0000	1.0167	1.0353	1.0619	1.0830
	MOD I-L	.17730	.09908	.05473	.02463	.01335
8.00	MOD L	.18737	.10274	.05542	.02407	.01267
	RATIO	.9463	.9644	.9875	1.0234	1.0537
	MOD I-L	.22296	.13128	.07610	.03633	.02053
16.00	MOD L	.27090	.15809	.08976	.04120	.02246
	RATIO	.8231	.8304	.8478	.8818	.9140
	MOD I-L	.25151	.15273	.09115	.04514	.02619
24.00	MOD L	.34763	.21303	.12595	.06040	.03385
	RATIO	.7235	.7169	.7237	.7474	.7739

APPENDIX 5 POWER COMPARISONS OF |L| AND |I-L|

		P= 4 M=10 N= 4					P= 4 M=10 N=10				
LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-L	.10403	.05235	.02631	.01059	.00532	.10578	.05343	.02696	.01090	.00549
.50	MOD L	.10174	.05091	.02547	.01019	.00509	.10536	.05310	.02673	.01077	.00541
	RATIO	1.0225	1.0282	1.0333	1.0393	1.0433	1.0040	1.0063	1.0087	1.0118	1.0142
	MOD I-L	.10794	.05464	.02761	.01117	.00563	.11144	.05682	.02891	.01181	.00599
1.00	MOD L	.10345	.05181	.02592	.01037	.00519	.11076	.05624	.02849	.01157	.00584
	RATIC	1.0434	1.0547	1.0649	1.0770	1.0850	1.0061	1.0103	1.0147	1.0207	1.0252
	MOD I-L	.11538	.05906	.03012	.01232	.00625	.12236	.06348	.03279	.01363	.00700
2.00	MOD L	.10678	.05355	.02682	.01074	.00537	.12165	.06267	.03214	.01322	.00673
	RATIO	1.0805	1.1028	1.1233	1.1476	1.1640	1.0058	1.0129	1.0204	1.0309	1.0391
	MOD I-L	.12889	.06726	.03489	.01454	.00747	.14269	.07619	.04039	.01731	.00907
4.00	MOD L	.11310	.05688	.02852	.01143	.00572	.14376	.07601	.03984	.01680	.00869
	RATIC	1.1396	1.1825	1.2229	1.2720	1.3059	.9926	1.0024	1.0138	1.0303	1.0436
	MOD I-L	.15127	.08137	.04334	.01863	.00977	.17766	.09907	.05461	.02451	.01326
8.00	MOD L	.12463	.06298	.03167	.01271	.00636	.18866	.10425	.05671	.02492	.01324
	RATIO	1.2137	1.2919	1.3686	1.4656	1.5349	.9417	.9503	.9629	.9836	1.0016
	MOD I-L	.18261	.10235	.05659	.02543	.01375	.22919	.13519	.07844	.03745	.02117
16.00	MOD L	.14439	.07357	.03715	.01495	.00749	.27780	.16460	.09456	.04453	.02468
	RATIO	1.2647	1.3913	1.5232	1.7007	1.8346	.8250	.8213	.8260	.8412	.8577
	MOD I-L	.20283	.11673	.06617	.03067	.01694	.26394	.16121	.09663	.04805	.02795
24.00	MOD L	.16110	.08265	.04189	.01690	.00848	.36208	.22671	.13707	.05772	.03882
	RATIO	1.2590	1.4124	1.5796	1.8146	1.9991	.7289	.7111	.7050	.7095	.7199

APPENDIX 5 POWER COMPARISONS OF |L| AND |I-L|

LAMBDA SQUARED	ALPHA=	P= 4 M=12 N=12 .100	.050	.025	.010	.005	P= 5 M= 6 N= 6 .100	.050	.025	.010	.005
	MOD I-L	.10546	.05323	.02684	.01084	.00546	.10499	.05304	.02677	.01083	.00546
.50	MOD L	.10517	.05300	.02668	.01076	.00541	.10364	.05200	.02607	.01045	.00523
	RATIO	1.0027	1.0043	1.0060	1.0081	1.0097	1.0130	1.0198	1.0268	1.0359	1.0427
	MOD I-L	.11083	.05645	.02869	.01170	.00593	.10970	.05592	.02846	.01163	.00590
1.00	MOD L	.11039	.05606	.02841	.01154	.00583	.10723	.05399	.02713	.01090	.00546
	RATIO	1.0040	1.0068	1.0099	1.0140	1.0169	1.0230	1.0358	1.0491	1.0667	1.0800
	MOD I-L	.12131	.06281	.03239	.01344	.00689	.11832	.06129	.03164	.01316	.00676
2.00	MOD L	.12098	.06234	.03199	.01318	.00672	.11427	.05790	.02923	.01179	.00592
	RATIO	1.0028	1.0075	1.0127	1.0198	1.0251	1.0355	1.0585	1.0827	1.1155	1.1406
	MOD I-L	.14121	.07520	.03977	.01699	.00889	.13290	.07057	.03727	.01593	.00834
4.00	MOD L	.14265	.07549	.03963	.01675	.00868	.12780	.06550	.03334	.01356	.00684
	RATIO	.9899	.9961	1.0036	1.0145	1.0231	1.0399	1.0774	1.1180	1.1749	1.2196
	MOD I-L	.17664	.09828	.05405	.02419	.01306	.15426	.08469	.04614	.02048	.01101
8.00	MOD L	.18734	.10381	.05667	.02503	.01336	.15295	.07991	.04124	.01699	.00863
	RATIO	.9429	.9467	.9537	.9663	.9774	1.0086	1.0599	1.1186	1.2052	1.2761
	MOD I-L	.23193	.13684	.07938	.03788	.02139	.17930	.10214	.05762	.02672	.01483
16.00	MOD L	.27823	.16597	.09649	.04576	.02559	.19727	.10621	.05603	.02355	.01209
	RATIO	.8336	.8245	.8226	.8278	.8360	.9089	.9617	1.0283	1.1341	1.2261
	MOD I-L	.27144	.16633	.09994	.04980	.02899	.19305	.11216	.06450	.03065	.01733
24.00	MOD L	.36615	.23167	.14170	.07118	.04132	.23564	.12991	.06974	.02960	.01543
	RATIC	.7413	.7180	.7053	.6997	.7016	.8193	.8634	.9249	1.0287	1.1227

APPENDIX 5

POWER COMPARISONS OF |L| AND |I-L|

LAMBDA SQUARED	ALPHA=	P= 5 M= 6 N=12 .100	.050	.025	.010	.005	P= 5 M=10 N=10 .100	.050	.025	.010	.005
	MOD I-L	.10620	.05379	.02722	.01105	.00558	.10489	.05291	.02667	.01077	.00542
.50	MOD L	.10712	.05415	.02733	.01105	.00556	.10445	.05256	.02643	.01064	.00534
	RATIO	.9914	.9933	.9958	.9996	1.0029	1.0042	1.0066	1.0091	1.0124	1.0149
	MOD I-L	.11210	.05744	.02937	.01208	.00616	.10964	.05576	.02831	.01153	.00584
1.00	MOD L	.11432	.05839	.02974	.01214	.00616	.10890	.05515	.02787	.01128	.00569
	RATIO	.9806	.9836	.9877	.9944	1.0000	1.0067	1.0111	1.0157	1.0220	1.0268
	MOD I-L	.12310	.06433	.03350	.01408	.00729	.11873	.06130	.03154	.01305	.00668
2.00	MOD L	.12894	.06714	.03476	.01446	.00742	.11785	.06039	.03083	.01262	.00641
	RATIO	.9547	.9581	.9638	.9737	.9825	1.0075	1.0150	1.0230	1.0342	1.0429
	MOD I-L	.14224	.07666	.04106	.01786	.00947	.13541	.07163	.03772	.01603	.00836
4.00	MOD L	.15885	.08558	.04560	.01960	.01027	.13581	.07112	.03697	.01545	.00794
	RATIO	.8954	.8958	.9005	.9114	.9222	.9970	1.0078	1.0201	1.0379	1.0523
	MOD I-L	.17170	.09647	.05370	.02448	.01342	.16344	.08982	.04889	.02164	.01160
8.00	MOD L	.22009	.12542	.07008	.03176	.01723	.17175	.09334	.05006	.02165	.01138
	RATIO	.7801	.7692	.7662	.7706	.7786	.9516	.9624	.9767	.9997	1.0193
	MOD I-L	.20886	.12297	.07152	.03442	.01962	.20367	.11740	.06676	.03116	.01734
16.00	MOD L	.34099	.21174	.12728	.06259	.03580	.24196	.13947	.07859	.03590	.01956
	RATIO	.6125	.5808	.5619	.5500	.5481	.8417	.8418	.8495	.8680	.8869
	MOD I-L	.23071	.13938	.08308	.04126	.02405	.23032	.13671	.07990	.03858	.02200
24.00	MOD L	.45177	.29982	.19083	.10005	.05967	.30815	.18611	.10909	.05206	.02917
	RATIO	.5107	.4649	.4354	.4123	.4031	.7474	.7346	.7324	.7410	.7541

APPENDIX 5

POWER COMPARISONS OF |L| AND |I-L|

LAMBDA SQUARED	ALPHA=	P= 6 M= 6 N= 6 .100	.050	.025	.010	.005	P= 6 M= 6 N=12 .100	.050	.025	.010	.005
	MOD I-L	.10360	.05224	.02632	.01063	.00535	.10453	.05282	.02667	.01080	.00544
.50	MOD L	.10204	.05107	.02555	.01022	.00511	.10602	.05349	.02696	.01088	.00547
	RATIO	1.0153	1.0228	1.0303	1.0400	1.0470	.9860	.9873	.9893	.9925	.9950
	MOD I-L	.10696	.05434	.02758	.01123	.00569	.10881	.05550	.02827	.01157	.00588
1.00	MOD L	.10404	.05212	.02609	.01044	.00522	.11208	.05705	.02896	.01178	.00596
	RATIO	1.0281	1.0426	1.0571	1.0760	1.0898	.9708	.9729	.9762	.9818	.9865
	MOD I-L	.11305	.05819	.02989	.01236	.00633	.11669	.06049	.03129	.01305	.00672
2.00	MOD L	.10792	.05416	.02713	.01087	.00544	.12432	.06431	.03310	.01368	.00699
	RATIO	1.0475	1.0744	1.1017	1.1376	1.1642	.9386	.9407	.9452	.9538	.9615
	MOD I-L	.12312	.06468	.03336	.01433	.00746	.13010	.06918	.03664	.01573	.00827
4.00	MOD L	.11523	.05802	.02912	.01168	.00584	.14912	.07940	.04187	.01778	.00925
	RATIO	1.0685	1.1147	1.1629	1.2278	1.2772	.8724	.8713	.8749	.8845	.8945
	MOD I-L	.13740	.07415	.03982	.01740	.00927	.15008	.08258	.04515	.02017	.01091
8.00	MOD L	.12839	.06503	.03274	.01315	.00659	.19928	.11134	.06116	.02718	.01455
	RATIO	1.0702	1.1403	1.2164	1.3230	1.4068	.7531	.7417	.7383	.7421	.7496
	MOD I-L	.15350	.08527	.04709	.02132	.01165	.17436	.09963	.05646	.02638	.01474
16.00	MOD L	.15056	.07699	.03896	.01570	.00787	.29765	.17913	.10477	.04993	.02797
	RATIO	1.0196	1.1077	1.2087	1.3574	1.4794	.5858	.5562	.5389	.5283	.5268
	MOD I-L	.16210	.09142	.05123	.02364	.01310	.18829	.10981	.06347	.03041	.01730
24.00	MOD L	.16902	.08711	.04427	.01789	.00898	.38892	.24799	.15238	.07673	.04454
	RATIO	.9591	1.0495	1.1574	1.3214	1.4595	.4841	.4428	.4165	.3963	.3885

APPENDIX 5 POWER COMPARISONS OF |L| AND |I-L|

P= 6 M= 8 N= 8

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10418	.05252	.02646	.01068	.00537
.50	MOD L	.10341	.05193	.02605	.01046	.00524
	RATIO	1.0074	1.0115	1.0156	1.0212	1.0255
	MOD I-L	.10818	.05495	.02787	.01134	.00574
1.00	MOD L	.10680	.05385	.02710	.01091	.00548
	RATIO	1.0129	1.0205	1.0284	1.0391	1.0473
	MOD I-L	.11565	.05953	.03056	.01252	.00645
2.00	MOD L	.11351	.05767	.02920	.01184	.00596
	RATIO	1.0189	1.0323	1.0466	1.0663	1.0816
	MOD I-L	.12874	.06773	.03547	.01500	.00780
4.00	MOD L	.12662	.06524	.03341	.01370	.00695
	RATIO	1.0167	1.0381	1.0616	1.0950	1.1217
	MOD I-L	.14917	.08095	.04361	.01909	.01017
8.00	MOD L	.15174	.08009	.04180	.01748	.00898
	RATIO	.9831	1.0107	1.0433	1.0921	1.1325
	MOD I-L	.17552	.09882	.05510	.02516	.01381
16.00	MOD L	.19798	.10855	.05840	.02520	.01319
	RATIO	.8866	.9104	.9434	.9982	1.0469
	MOD I-L	.19139	.11005	.06259	.02931	.01638
24.00	MOD L	.23960	.13539	.07461	.03302	.01754
	RATIO	.7988	.8128	.8389	.8877	.9338

P= 6 M= 8 N=12

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10472	.05279	.02652	.01051	.00503
.50	MOD L	.10498	.05289	.02662	.01072	.00539
	RATIO	.9975	.9982	.9963	.9799	.9334
	MOD I-L	.10928	.05555	.02809	.01117	.00529
1.00	MOD L	.10999	.05582	.02826	.01147	.00579
	RATIO	.9936	.9952	.9938	.9736	.9139
	MOD I-L	.11791	.06087	.03123	.01267	.00613
2.00	MOD L	.12007	.06177	.03165	.01301	.00663
	RATIO	.9820	.9854	.9868	.9736	.9252
	MOD I-L	.13331	.07061	.03716	.01569	.00800
4.00	MOD L	.14043	.07406	.03875	.01632	.00844
	RATIO	.9493	.9534	.9588	.9616	.9475
	MOD I-L	.15793	.08671	.04718	.02032	.01102
8.00	MOD L	.18146	.09983	.05414	.02373	.01260
	RATIO	.8703	.8686	.8716	.8774	.8750
	MOD I-L	.19094	.10938	.06190	.02864	.01566
16.00	MOD L	.26225	.15415	.08843	.04127	.02283
	RATIO	.7281	.7096	.7000	.6940	.6860
	MOD I-L	.21158	.12424	.07198	.03430	.01916
24.00	MOD L	.33842	.20955	.12569	.06166	.03523
	RATIO	.6252	.5929	.5727	.5562	.5439

APPENDIX 5 POWER COMPARISONS OF |L| AND |I-L|

P= 6 M=10 N=10

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10425	.05264	.02671	.01119	.00613
.50	MOD L	.10372	.05214	.02618	.01053	.00528
	RATIO	1.0051	1.0097	1.0200	1.0636	1.1614
	MOD I-L	.10833	.05517	.02830	.01222	.00703
1.00	MOD L	.10743	.05428	.02738	.01106	.00556
	RATIO	1.0084	1.0164	1.0337	1.1051	1.2635
	MOD I-L	.11607	.05997	.03125	.01394	.00838
2.00	MOD L	.11485	.05860	.02960	.01215	.00615
	RATIO	1.0106	1.0233	1.0436	1.1477	1.3633
	MOD I-L	.13004	.06869	.03653	.01673	.01025
4.00	MOD L	.12962	.06733	.03476	.01441	.00737
	RATIO	1.0032	1.0201	1.0509	1.1611	1.3916
	MOD I-L	.15306	.08348	.04565	.02145	.01325
8.00	MOD L	.15877	.08507	.04507	.01922	.01001
	RATIO	.9641	.9814	1.0128	1.1162	1.3233
	MOD I-L	.18539	.10534	.05975	.02924	.01845
16.00	MOD L	.21483	.12094	.06678	.02980	.01599
	RATIO	.8630	.8710	.8947	.9811	1.1544
	MOD I-L	.20648	.12030	.06981	.03509	.02251
24.00	MOD L	.26733	.15652	.08932	.04134	.02270
	RATIO	.7724	.7686	.7816	.8488	.9916

P= 7 M= 8 N= 8

LAMBDA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-L	.10338	.05206	.02620	.01056	.00531
.50	MOD L	.10252	.05140	.02575	.01032	.00516
	RATIO	1.0084	1.0128	1.0174	1.0236	1.0283
	MOD I-L	.10660	.05402	.02735	.01111	.00561
1.00	MOD L	.10502	.05278	.02649	.01063	.00533
	RATIO	1.0151	1.0235	1.0324	1.0443	1.0534
	MOD I-L	.11256	.05770	.02952	.01214	.00619
2.00	MOD L	.10992	.05551	.02796	.01126	.00565
	RATIO	1.0240	1.0395	1.0557	1.0781	1.0953
	MOD I-L	.12289	.06418	.03340	.01403	.00726
4.00	MOD L	.11941	.06085	.03085	.01251	.00630
	RATIO	1.0291	1.0548	1.0827	1.1219	1.1528
	MOD I-L	.13868	.07438	.03967	.01717	.00907
8.00	MOD L	.13725	.07104	.03644	.01494	.00757
	RATIO	1.0104	1.0469	1.0835	1.1435	1.1993
	MOD I-L	.15857	.08772	.04816	.02161	.01172
16.00	MOD L	.16928	.08986	.04697	.01960	.01003
	RATIO	.9367	.9761	1.0254	1.1026	1.1691
	MOD I-L	.17033	.09589	.05353	.02453	.01351
24.00	MOD L	.19755	.10701	.05679	.02404	.01240
	RATIO	.8622	.8960	.9426	1.0202	1.0896

APPENDIX 6 SOME NONCENTRAL PERCENTILES OF |I-R|

P= 4 M= 4 N=10

RHO SQUARED	α=.100	.050	.025	.010	.005
.000	.07777	.05734	.04319	.03037	.02356
.010	.07676	.05655	.04257	.02991	.02319
.050	.07277	.05345	.04014	.02813	.02177
.100	.06793	.04971	.03722	.02600	.02008
.150	.06323	.04611	.03444	.02399	.01850
.200	.05866	.04265	.03177	.02208	.01699
.250	.05423	.03931	.02922	.02025	.01557

P= 4 M=10 N=10

RHO SQUARED	.100	.050	.025	.010	.005
.000	.01156	.00809	.00584	.00392	.00295
.010	.01142	.00799	.00577	.00387	.00292
.050	.01089	.00761	.00548	.00368	.00277
.100	.01023	.00713	.00514	.00344	.00258
.150	.00959	.00667	.00480	.00320	.00241
.200	.00895	.00622	.00446	.00298	.00223
.250	.00833	.00577	.00414	.00276	.00207

P= 5 M= 6 N=10

RHO SQUARED	α=.100	.050	.025	.010	.005
.000	.01403	.00952	.00667	.00431	.00315
.010	.01386	.00940	.00659	.00425	.00311
.050	.01320	.00894	.00625	.00403	.00295
.100	.01239	.00837	.00584	.00376	.00274
.150	.01159	.00781	.00544	.00350	.00255
.200	.01080	.00727	.00506	.00324	.00236
.250	.01004	.00674	.00468	.00300	.00218

P= 6 M= 6 N= 8

RHO SQUARED	.100	.050	.025	.010	.005
.000	.00070	.00037	.00020	.00010	.00006
.010	.00070	.00036	.00020	.00010	.00006
.050	.00066	.00035	.00019	.00009	.00005
.100	.00063	.00033	.00018	.00008	.00005
.150	.00059	.00031	.00017	.00008	.00005
.200	.00055	.00029	.00016	.00007	.00004
.250	.00051	.00027	.00015	.00007	.00004

APPENDIX 7

SOME POWERS OF |I-R|

P= 6 M= 6 N=20

RHO SQUARED	α=.100	.050	.025	.010	.005
.005	.10203	.05119	.02568	.01031	.00517
.010	.10409	.05241	.02637	.01062	.00534
.050	.12186	.06310	.03254	.01349	.00692
.100	.14759	.07913	.04207	.01809	.00950
.150	.17775	.09869	.05412	.02411	.01297
.200	.21292	.12249	.06931	.03203	.01766

P= 5 M= 6 N=20

.100	.050	.025	.010	.005
.10231	.05136	.02577	.01035	.00519
.10467	.05276	.02657	.01072	.00539
.12507	.06510	.03373	.01407	.00724
.15486	.08385	.04499	.01954	.01034
.18998	.10700	.05944	.02689	.01462
.23106	.13540	.07789	.03671	.02051

P= 8 M= 8 N= 8

RHO SQUARED	α=.100	.050	.025	.010	.005
.005	.10024	.05013	.02507	.01003	.00501
.010	.10048	.05025	.02513	.01005	.00503
.050	.10249	.05131	.02567	.01027	.00514
.100	.10516	.05272	.02640	.01057	.00529
.150	.10805	.05425	.02719	.01089	.00545
.200	.11119	.05591	.02804	.01124	.00562
.250	.11460	.05773	.02898	.01162	.00582

P=10 M= 2 N=15

.100	.050	.025	.010	.005
.10122	.05069	.02538	.01017	.00509
.10246	.05140	.02577	.01034	.00518
.11293	.05740	.02909	.01180	.00595
.12762	.06599	.03390	.01396	.00711
.14435	.07599	.03961	.01656	.00851
.16348	.08768	.04640	.01972	.01023
.18539	.10142	.05455	.02358	.01237

APPENDIX 7 SOME POWERS OF |I-R|

P= 7 M= 6 N=15

RHO SQUARED	α= .100	.050	.025	.010	.005
.005	.10126	.05073	.02541	.01018	.00510
.010	.10253	.05148	.02583	.01037	.00520
.050	.11334	.05784	.02944	.01202	.00609
.100	.12850	.06697	.03470	.01446	.00743
.150	.14578	.07762	.04098	.01745	.00909
.200	.16552	.09011	.04850	.02110	.01116

P= 9 M= 4 N=15

RHO SQUARED	.100	.050	.025	.010	.005
.005	.10110	.05064	.02535	.01016	.00508
.010	.10222	.05128	.02571	.01032	.00517
.050	.11165	.05678	.02880	.01171	.00592
.100	.12479	.06458	.03324	.01374	.00702
.150	.13967	.07359	.03846	.01617	.00835
.200	.15657	.08405	.04464	.01910	.00998

P=10 M= 4 N=15

RHO SQUARED	α=.100	.050	.025	.010	.005
.005	.10096	.05055	.02530	.01013	.00507
.010	.10192	.05110	.02561	.01027	.00514
.050	.11006	.05581	.02824	.01145	.00577
.100	.12133	.06244	.03198	.01314	.00669
.150	.13400	.07002	.03633	.01514	.00777
.200	.14831	.07875	.04141	.01752	.00908

P=11 M= 4 N=15

RHO SQUARED	.100	.050	.025	.010	.005
.005	.10082	.05047	.02526	.01011	.00506
.010	.10165	.05094	.02552	.01023	.00512
.050	.10859	.05493	.02773	.01121	.00564
.100	.11815	.06049	.03085	.01261	.00639
.150	.12883	.06680	.03442	.01423	.00726
.200	.14082	.07399	.03855	.01613	.00830

APPENDIX 7 — SOME POWERS OF |I-R|

P=12 M= 2 N=15

RHO SQUARED	α=.100	.050	.025	.010	.005
.005	.10084	.05047	.02525	.01011	.00506
.010	.10169	.05094	.02551	.01022	.00511
.050	.10881	.05493	.02767	.01115	.00560
.100	.11865	.06050	.03071	.01247	.00629
.150	.12969	.06684	.03420	.01400	.00710
.200	.14212	.07408	.03824	.01580	.00805

P=12 M= 4 N=15

RHO SQUARED	.100	.050	.025	.010	.005
.005	.10069	.05039	.02521	.01009	.00505
.010	.10138	.05078	.02543	.01019	.00510
.050	.10719	.05409	.02725	.01098	.00552
.100	.11513	.05866	.02978	.01211	.00611
.150	.12395	.06379	.03265	.01339	.00680
.200	.13378	.06959	.03593	.01487	.00760

P=11 M= 2 N=15

RHO SQUARED	α=.100	.050	.025	.010	.005
.005	.10103	.05058	.02531	.01014	.00507
.010	.10206	.05116	.02563	.01027	.00514
.050	.11081	.05612	.02835	.01146	.00577
.100	.12299	.06314	.03223	.01318	.00667
.150	.13677	.07121	.03677	.01521	.00776
.200	.15240	.08055	.04209	.01762	.00906
.250	.17023	.09142	.04838	.02053	.01064

P=13 M= 2 N=15

RHO SQUARED	.100	.050	.025	.010	.005
.005	.10066	.05036	.02519	.01008	.00504
.010	.10132	.05072	.02538	.01016	.00508
.050	.10685	.05377	.02701	.01085	.00544
.100	.11443	.05798	.02927	.01181	.00594
.150	.12285	.06269	.03182	.01291	.00651
.200	.13225	.06802	.03472	.01416	.00716
.250	.14280	.07406	.03804	.01561	.00791

APPENDIX 7 SOME POWERS OF |I-R|

P= 7 M= 2 N=60

RHO SQUARED	α=.100	.050	.025	.010	.005
.001	.10212	.05128	.02575	.01035	.00519
.005	.11080	.05660	.02887	.01183	.00602
.010	.12214	.06368	.03309	.01387	.00717
.025	.15931	.08786	.04807	.02145	.01159
.050	.23079	.13809	.08140	.03976	.02288
.100	.39912	.27309	.18224	.10355	.06625

P= 8 M= 2 N=40

RHO SQUARED	.100	.050	.025	.010	.005
.001	.10123	.05074	.02542	.01020	.00511
.005	.10623	.05375	.02717	.01101	.00555
.010	.11267	.05768	.02946	.01209	.00615
.025	.13328	.07055	.03715	.01581	.00825
.050	.17190	.09584	.05288	.02380	.01291
.100	.26431	.16183	.09718	.04839	.02816

P= 8 M= 4 N=25

RHO SQUARED	α=.100	.050	.025	.010	.005
.001	.10050	.05029	.02517	.01008	.00504
.005	.10252	.05149	.02585	.01039	.00521
.010	.10508	.05301	.02671	.01079	.00543
.025	.11308	.05780	.02947	.01206	.00613
.050	.12746	.06658	.03461	.01449	.00748
.100	.16042	.08748	.04724	.02068	.01100
.150	.19956	.11359	.06370	.02915	.01598

P=12 M= 2 N=24

RHO SQUARED	.100	.050	.025	.010	.005
.001	.10044	.05026	.02514	.01007	.00503
.005	.10224	.05130	.02573	.01033	.00518
.010	.10452	.05264	.02648	.01067	.00536
.025	.11161	.05682	.02885	.01174	.00594
.050	.12431	.06442	.03322	.01376	.00704
.100	.15325	.08234	.04380	.01880	.00984
.150	.18747	.10448	.05736	.02551	.01368

APPENDIX 8 SOME NONCENTRAL PERCENTILES OF |R|

P= 4 M= 4 N=10

RHO SQUARED	α=.900	.950	.975	.990	.995
.000	.00252	.00462	.00745	.01241	.01709
.005	.00255	.00467	.00753	.01253	.01725
.010	.00258	.00472	.00761	.01266	.01742
.050	.00282	.00513	.00824	.01365	.01873
.100	.00312	.00565	.00903	.01487	.02033
.150	.00342	.00617	.00982	.01609	.02192
.200	.00373	.00669	.01061	.01730	.02349

P= 4 M=10 N= 4

RHO SQUARED	.900	.950	.975	.990	.995
.000	.36802	.42580	.47702	.53670	.57677
.005	.36854	.42632	.47754	.53721	.57726
.010	.36905	.42684	.47806	.53771	.57775
.050	.37315	.43103	.48223	.54176	.58166
.100	.37832	.43631	.48749	.54687	.58659
.150	.38355	.44164	.49280	.55203	.59159
.200	.38885	.44704	.49818	.55727	.59665

P= 4 M=10 N=10

RHO SQUARED	α=.900	.950	.975	.990	.995
.000	.08594	.10701	.12792	.15536	.17596
.005	.08629	.10743	.12839	.15589	.17653
.010	.08665	.10784	.12886	.15643	.17710
.050	.08951	.11118	.13262	.16067	.18166
.100	.09308	.11533	.13728	.16592	.18730
.150	.09666	.11948	.14193	.17114	.19289
.200	.10024	.12363	.14658	.17634	.19845

P= 6 M= 6 N= 6

RHO SQUARED	.900	.950	.975	.990	.995
.000	.00271	.00501	.00816	.01370	.01897
.005	.00272	.00503	.00819	.01375	.01904
.010	.00274	.00506	.00823	.01381	.01911
.050	.00284	.00523	.00850	.01424	.01968
.100	.00296	.00545	.00884	.01478	.02040
.150	.00309	.00567	.00919	.01533	.02112
.200	.00322	.00590	.00954	.01588	.02165

APPENDIX 9 POWER COMPARISONS OF |R| AND |I-R|

P= 2 M= 2 N= 2

RHO SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD R	.11281	.05764	.02929	.01189	.00599
.100	MOD I-R	.10678	.05349	.02677	.01071	.00536
	RATIO	1.0565	1.0775	1.0941	1.1101	1.1186
	MOD R	.12688	.06631	.03428	.01415	.00719
.200	MOD I-R	.11466	.05756	.02884	.01155	.00578
	RATIO	1.1066	1.1519	1.1888	1.2254	1.2453
	MOD R	.14241	.07623	.04017	.01689	.00867
.300	MOD I-R	.12402	.06241	.03130	.01254	.00628
	RATIO	1.1483	1.2213	1.2832	1.3465	1.3819
	MOD R	.15962	.08769	.04721	.02029	.01054
.400	MOD I-R	.13539	.06834	.03432	.01377	.00689
	RATIO	1.1790	1.2832	1.3753	1.4736	1.5303
	MOD R	.17879	.10107	.05577	.02460	.01298
.500	MOD I-R	.14969	.07583	.03815	.01532	.00767
	RATIO	1.1944	1.3328	1.4617	1.6060	1.6926
	MOD R	.20024	.11690	.06642	.03027	.01629
.600	MOD I-R	.16856	.08579	.04326	.01739	.00871
	RATIO	1.1879	1.3626	1.5352	1.7404	1.8699
	MOD R	.22437	.13589	.08001	.03804	.02103
.700	MOD I-R	.19529	.10007	.05062	.02039	.01022
	RATIO	1.1489	1.3580	1.5805	1.8660	2.0589

P= 2 M= 2 N= 4

RHO SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD R	.12403	.06511	.03396	.01424	.00734
.100	MOD I-R	.12340	.06312	.03207	.01302	.00656
	RATIO	1.0051	1.0315	1.0588	1.0937	1.1182
	MOD R	.14949	.08188	.04434	.01940	.01027
.200	MOD I-R	.15220	.07974	.04121	.01699	.00863
	RATIO	.9822	1.0269	1.0758	1.1418	1.1901
	MOD R	.17637	.10045	.05634	.02569	.01398
.300	MOD I-R	.18825	.10126	.05332	.02236	.01146
	RATIO	.9369	.9920	1.0567	1.1489	1.2197
	MOD R	.20458	.12096	.0702[illegible]	.03338	.01869
.400	MOD I-R	.23425	.12991	.06990	.02989	.01548
	RATIO	.8733	.9311	1.0048	1.1169	1.2074
	MOD R	.23400	.14349	.08627	.04282	.02472
.500	MOD I-R	.29434	.16935	.09356	.04097	.02148
	RATIO	.7950	.8473	.9221	1.0451	1.1505
	MOD R	.26442	.16811	.10475	.05444	.03250
.600	MOD I-R	.37493	.22611	.12927	.05840	.03113
	RATIO	.7052	.7435	.8103	.9321	1.0441
	MOD R	.29548	.19478	.12594	.06880	.04266
.700	MOD I-R	.48621	.31266	.18762	.08866	.04837
	RATIO	.6077	.6230	.6713	.7760	.8818

APPENDIX 9 POWER COMPARISONS OF |R| AND |I-R|

RHO SQUARED		P= 4 M= 4 N= 4					P= 4 M=10 N= 4				
	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
.005	MOD R	.10050	.05031	.02518	.01009	.00505	.10058	.05033	.02519	.01008	.00504
	MOD I-R	.10026	.05014	.02507	.01003	.00501	.10025	.05013	.02507	.01003	.00501
	RATIO	1.0024	1.0034	1.0045	1.0058	1.0067	1.0033	1.0041	1.0048	1.0056	1.0062
.010	MOD R	.10100	.05062	.02537	.01017	.00510	.10115	.05067	.02537	.01017	.00509
	MOD I-R	.10053	.05028	.02514	.01006	.00503	.10050	.05026	.02513	.01005	.00503
	RATIO	1.0047	1.0069	1.0089	1.0115	1.0133	1.0065	1.0082	1.0096	1.0113	1.0124
.050	MOD R	.10504	.05314	.02686	.01088	.00549	.10588	.05343	.02693	.01087	.00546
	MOD I-R	.10273	.05142	.02573	.01029	.00515	.10255	.05133	.02568	.01028	.00514
	RATIO	1.0225	1.0334	1.0439	1.0571	1.0665	1.0324	1.0408	1.0484	1.0573	1.0632
.100	MOD R	.11015	.05636	.02878	.01181	.00601	.11205	.05708	.02900	.01181	.00597
	MOD I-R	.10566	.05295	.02651	.01061	.00531	.10529	.05277	.02642	.01058	.00529
	RATIO	1.0425	1.0643	1.0858	1.1131	1.1326	1.0641	1.0817	1.0977	1.1167	1.1295
.150	MOD R	.11532	.05966	.03078	.01279	.00656	.11852	.06097	.03124	.01284	.00654
	MOD I-R	.10883	.05461	.02736	.01095	.00548	.10826	.05433	.02722	.01090	.00545
	RATIO	1.0597	1.0925	1.1253	1.1675	1.1980	1.0948	1.1222	1.1477	1.1782	1.1990
.200	MOD R	.12057	.06305	.03286	.01382	.00715	.12532	.06511	.03365	.01397	.00716
	MOD I-R	.11226	.05641	.02828	.01133	.00567	.11147	.05602	.02809	.01125	.00563
	RATIO	1.0740	1.1178	1.1621	1.2199	1.2624	1.1242	1.1622	1.1980	1.2415	1.2716
.250	MOD R	.12588	.06653	.03502	.01491	.00778	.13244	.06952	.03625	.01521	.00785
	MOD I-R	.11600	.05838	.02929	.01174	.00587	.11498	.05787	.02904	.01164	.00582
	RATIO	1.0851	1.1397	1.1957	1.2699	1.3252	1.1519	1.2013	1.2484	1.3066	1.3472

APPENDIX 9 POWER COMPARISONS OF |R| AND |I-R|

		P= 4 M= 4 N=10					P= 4 M=10 N=10				
RHO SQUARED	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD R	.10115	.05072	.02543	.01020	.00511	.10117	.05069	.02539	.01018	.00510
.005	MOD I-R	.10143	.05083	.02546	.01021	.00511	.10107	.05062	.02534	.01015	.00508
	RATIO	.9973	.9979	.9986	.9998	1.0008	1.0010	1.0015	1.0020	1.0026	1.0031
	MOD R	.10230	.05144	.02585	.01041	.00523	.10235	.05139	.02579	.01036	.00520
.010	MOD I-R	.10287	.05167	.02593	.01042	.00522	.10216	.05125	.02569	.01031	.00517
	RATIO	.9945	.9956	.9970	.9993	1.0012	1.0019	1.0028	1.0038	1.0052	1.0062
	MOD R	.11146	.05721	.02932	.01209	.00618	.11200	.05717	.02912	.01190	.00604
.050	MOD I-R	.11512	.05885	.02998	.01225	.00621	.11133	.05659	.02869	.01166	.00589
	RATIO	.9682	.9721	.9779	.9873	.9954	1.0059	1.0103	1.0149	1.0211	1.0259
	MOD R	.12277	.06450	.03378	.01430	.00745	.12460	.06488	.03364	.01404	.00723
.100	MOD I-R	.13232	.06917	.03591	.01498	.00769	.12412	.06416	.03300	.01362	.00695
	RATIO	.9279	.9324	.9406	.9550	.9680	1.0039	1.0112	1.0193	1.0306	1.0395
	MOD R	.13393	.07184	.03836	.01664	.00881	.13778	.07314	.03858	.01644	.00858
.150	MOD I-R	.15190	.08123	.04298	.01831	.00954	.13857	.07290	.03806	.01598	.00824
	RATIO	.8817	.8844	.8924	.9085	.9239	.9943	1.0033	1.0137	1.0288	1.0411
	MOD R	.14492	.07922	.04305	.01908	.01026	.15154	.08197	.04397	.01911	.01012
.200	MOD I-R	.17423	.09536	.05146	.02240	.01183	.15498	.08304	.04403	.01881	.00982
	RATIO	.8318	.8308	.8367	.8520	.8679	.9778	.9870	.9986	1.0162	1.0310
	MOD R	.15572	.08663	.04786	.02165	.01181	.16584	.09136	.04982	.02209	.01186
.250	MOD I-R	.19973	.11197	.06166	.02745	.01470	.17368	.09487	.05113	.02224	.01175
	RATIO	.7796	.7736	.7762	.7888	.8036	.9549	.9630	.9745	.9933	1.0096

POWER COMPARISONS OF |R| AND |I-R|

P= 6 M= 6 N= 6

RHO SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD R	.10045	.05028	.02516	.01008	.00504
.005	MOD I-R	.10025	.05013	.02507	.01003	.00501
	RATIO	1.0020	1.0029	1.0039	1.0050	1.0059
	MOD R	.10089	.05055	.02533	.01016	.00509
.010	MOD I-R	.10050	.05026	.02513	.01005	.00503
	RATIO	1.0039	1.0058	1.0077	1.0101	1.0118
	MOD R	.10448	.05279	.02665	.01079	.00544
.050	MOD I-R	.10257	.05135	.02569	.01028	.00514
	RATIO	1.0186	1.0280	1.0374	1.0495	1.0584
	MOD R	.10896	.05561	.02834	.01161	.00590
.100	MOD I-R	.10533	.05280	.02643	.01058	.00529
	RATIO	1.0345	1.0532	1.0721	1.0968	1.1150
	MOD R	.11345	.05846	.03007	.01245	.00638
.150	MOD I-R	.10831	.05437	.02724	.01091	.00546
	RATIO	1.0475	1.0753	1.1038	1.1414	1.1694
	MOD R	.11795	.06135	.03183	.01333	.00688
.200	MOD I-R	.11154	.05607	.02812	.01127	.00564
	RATIO	1.0575	1.0941	1.1321	1.1829	1.2213
	MOD R	.12245	.06427	.03364	.01423	.00741
.250	MOD I-R	.11506	.05794	.02908	.01166	.00583
	RATIO	1.0642	1.1093	1.1567	1.2209	1.2699

P= 6 M= 6 N=10

RHO SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD R	.10071	.05044	.02526	.01012	.00507
.005	MOD I-R	.10084	.05048	.02527	.01012	.00506
	RATIO	.9987	.9991	.9997	1.0005	1.0012
	MOD R	.10142	.05088	.02552	.01025	.00514
.010	MOD I-R	.10169	.05097	.02554	.01024	.00513
	RATIO	.9973	.9982	.9993	1.0009	1.0023
	MOD R	.10705	.05439	.02761	.01125	.00570
.050	MOD I-R	.10881	.05508	.02782	.01126	.00567
	RATIO	.9838	.9875	.9923	.9997	1.0059
	MOD R	.11402	.05880	.03027	.01255	.00644
.100	MOD I-R	.11861	.06081	.03105	.01271	.00645
	RATIO	.9613	.9670	.9749	.9876	.9985
	MOD R	.12090	.06322	.03297	.01389	.00721
.150	MOD I-R	.12954	.06730	.03474	.01440	.00736
	RATIO	.9333	.9394	.9489	.9650	.9793
	MOD R	.12769	.06765	.03571	.01528	.00801
.200	MOD I-R	.14180	.07470	.03901	.01637	.00844
	RATIO	.9005	.9057	.9154	.9332	.9495
	MOD R	.13439	.07208	.03849	.01670	.00885
.250	MOD I-R	.15560	.08317	.04397	.01870	.00972
	RATIO	.8637	.8667	.8754	.8933	.9104

APPENDIX 9 POWER COMPARISONS OF |R| AND |I-R|

RHO SQUARED	ALPHA=	P= 6 M=10 N= 6					P= 8 M= 8 N= 8				
		.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD R	.10057	.05034	.02519	.01009	.00505	.10043	.05028	.02518	.01010	.00508
.005	MOD I-R	.10024	.05013	.02507	.01003	.00501	.10024	.05013	.02507	.01003	.00501
	RATIO	1.0032	1.0042	1.0050	1.0061	1.0069	1.0019	1.0030	1.0044	1.0078	1.0132
	MOD R	.10114	.05067	.02538	.01018	.00510	.10086	.05055	.02535	.01021	.00516
.010	MOD I-R	.10048	.05025	.02513	.01005	.00503	.10048	.05025	.02513	.01005	.00503
	RATIO	1.0065	1.0083	1.0101	1.0122	1.0137	1.0037	1.0059	1.0088	1.0154	1.0260
	MOD R	.10575	.05343	.02696	.01090	.00549	.10427	.05273	.02673	.01101	.00575
.050	MOD I-R	.10249	.05131	.02567	.01027	.00514	.10249	.05131	.02567	.01027	.00514
	RATIO	1.0317	1.0412	1.0502	1.0613	1.0692	1.0174	1.0278	1.0413	1.0717	1.1188
	MOD R	.11165	.05699	.02903	.01187	.00602	.10848	.05544	.02843	.01196	.00642
.100	MOD I-R	.10518	.05273	.02640	.01057	.00529	.10516	.05272	.02640	.01057	.00529
	RATIO	1.0616	1.0810	1.0996	1.1229	1.1396	1.0316	1.0515	1.0768	1.1317	1.2137
	MOD R	.11772	.06071	.03121	.01290	.00660	.11265	.05812	.03010	.01287	.00703
.150	MOD I-R	.10807	.05426	.02719	.01089	.00545	.10805	.05425	.02719	.01089	.00545
	RATIO	1.0893	1.1189	1.1478	1.1845	1.2110	1.0426	1.0713	1.1072	1.1822	1.2909
	MOD R	.12396	.06457	.03350	.01400	.00721	.11679	.06079	.03177	.01377	.00762
.200	MOD I-R	.11121	.05592	.02804	.01124	.00562	.11119	.05591	.02804	.01124	.00562
	RATIO	1.1146	1.1548	1.1945	1.2455	1.2828	1.0504	1.0872	1.1328	1.2250	1.3546
	MOD R	.13035	.06859	.03591	.01517	.00788	.12089	.06346	.03344	.01466	.00819
.250	MOD I-R	.11463	.05774	.02898	.01162	.00582	.11460	.05773	.02898	.01162	.00582
	RATIO	1.1371	1.1880	1.2390	1.3055	1.3547	1.0549	1.0992	1.1537	1.2610	1.4079

APPENDIX 10 SOME POWERS OF |C|

P= 4 M= 4 N=30

OMEGA SQUARED	α=.100	.050	.025	.010	.005
1.001	.10012	.05007	.02504	.01002	.00501
1.100	.11203	.05732	.02927	.01201	.00611
1.200	.12469	.06528	.03404	.01433	.00743
1.300	.13792	.07384	.03932	.01699	.00897
1.400	.15161	.08296	.04509	.01998	.01074
1.500	.16568	.09260	.05135	.02333	.01277
2.000	.23898	.14667	.08893	.04521	.02684

P= 4 M= 8 N=16

.100	.050	.025	.010	.005
.10013	.05007	.02504	.01002	.00501
.11282	.05760	.02934	.01199	.00608
.12613	.06571	.03406	.01421	.00731
.13987	.07427	.03916	.01666	.00869
.15395	.08326	.04461	.01935	.01022
.16832	.09264	.05042	.02227	.01191
.24232	.14402	.08396	.04021	.02273

P= 5 M= 6 N=20

OMEGA SQUARED	α=.100	.050	.025	.010	.005
1.001	.10011	.05006	.02504	.01002	.00501
1.100	.11081	.05644	.02869	.01170	.00593
1.200	.12204	.06330	.03270	.01360	.00698
1.300	.13365	.07055	.03703	.01569	.00816
1.400	.14557	.07817	.04166	.01798	.00947
1.500	.15777	.08613	.04659	.02046	.01091
2.000	.22125	.13008	.07529	.03586	.02023

P= 6 M= 6 N=18

.100	.050	.025	.010	.005
.10009	.05005	.02503	.01001	.00501
.10904	.05535	.02805	.01140	.00576
.11835	.06097	.03130	.01291	.00659
.12789	.06683	.03475	.01455	.00751
.13762	.07293	.03839	.01632	.00850
.14752	.07924	.04223	.01820	.00958
.19867	.11350	.06392	.02942	.01620

APPENDIX 10 SOME POWERS OF |C|

P= 4 M=12 N=12

OMEGA SQUARED	α=.100	.050	.025	.010	.005
1.001	.10012	.05007	.02504	.01002	.00501
1.100	.11252	.05733	.02913	.01187	.00601
1.200	.12534	.06500	.03354	.01391	.00712
1.300	.13841	.07299	.03821	.01611	.00834
1.400	.15166	.08126	.04313	.01847	.00966
1.500	.16505	.08978	.04827	.02098	.01108

P= 7 M= 8 N=12

.100	.050	.025	.010	.005
.10007	.05004	.02502	.01001	.00501
.10683	.05396	.02721	.01099	.00553
.11367	.05798	.02948	.01202	.00609
.12053	.06205	.03181	.01309	.00666
.12738	.06617	.03418	.01418	.00727
.13422	.07033	.03660	.01532	.00789

P= 2 M= 2 N= 4

OMEGA SQUARED	α=.100	.050	.025	.010	.005
1.300	.12101	.06183	.03140	.01274	.00642
1.500	.13478	.06981	.03580	.01466	.00742
2.000	.16810	.08982	.04711	.01969	.01008
2.500	.19964	.10966	.05869	.02500	.01292
3.000	.22933	.12914	.07041	.03051	.01592
3.500	.25720	.14814	.08215	.03617	.01903
4.000	.28334	.16660	.09385	.04194	.02224

P= 2 M= 4 N= 2

.100	.050	.025	.010	.005
.11060	.05548	.02779	.01112	.00556
.11701	.05881	.02948	.01181	.00591
.13137	.06633	.03332	.01337	.00669
.14399	.07298	.03673	.01475	.00738
.15533	.07899	.03983	.01601	.00802
.16569	.08453	.04268	.01717	.00860
.17525	.08967	.04535	.01826	.00915

APPENDIX 11 SOME POWERS OF |I-C|

P= 4 M=14 N= 4

OMEGA SQUARED	α=.900	.950	.975	.990	.995
1.001	.10009	.05005	.02503	.01001	.00501
1.010	.10088	.05050	.02528	.01012	.00507
1.100	.10849	.05492	.02775	.01123	.00566
1.150	.11251	.05729	.02909	.01184	.00598
1.200	.11638	.05959	.03040	.01243	.00631
1.300	.12373	.06401	.03295	.01361	.00695
1.500	.13695	.07215	.03773	.01587	.00820

P= 4 M=16 N= 4

	.900	.950	.975	.990	.995
1.001	.10009	.05005	.02503	.01001	.00501
1.010	.10090	.05051	.02528	.01013	.00507
1.100	.10868	.05502	.02780	.01125	.00567
1.150	.11279	.05744	.02916	.01186	.00600
1.200	.11677	.05979	.03050	.01247	.00633
1.300	.12430	.06431	.03310	.01367	.00698
1.500	.13786	.07265	.03799	.01598	.00826

P= 5 M=12 N= 8

OMEGA SQUARED	α=.900	.950	.975	.990	.995
1.001	.10010	.05006	.02503	.01001	.00501
1.010	.10098	.05058	.02533	.01015	.00508
1.100	.10950	.05564	.02823	.01148	.00581
1.150	.11400	.05836	.02980	.01221	.00621
1.200	.11833	.06100	.03134	.01294	.00661
1.300	.12656	.06607	.03433	.01436	.00740

P= 7 M= 8 N= 8

	.900	.950	.975	.990	.995
1.001	.10006	.05003	.02502	.01001	.00501
1.010	.10055	.05033	.02519	.01009	.00505
1.100	.10528	.05322	.02688	.01088	.00549
1.150	.10773	.05473	.02777	.01131	.00572
1.200	.11007	.05617	.02862	.01172	.00595
1.300	.11442	.05889	.03024	.01250	.00640

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

OMEGA SQUARED		P= 4 M= 4 N= 4					P= 4 M=12 N= 4				
	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-C	.10005	.05003	.02502	.01001	.00500	.10009	.05005	.02503	.01001	.00501
1.001	MOD C	.10003	.05001	.02501	.01000	.00500	.10004	.05002	.02501	.01000	.00500
	RATIO	1.0002	1.0003	1.0004	1.0006	1.0007	1.0005	1.0006	1.0007	1.0008	1.0009
	MOD I-C	.10244	.05152	.02589	.01042	.00523	.10419	.05243	.02636	.01061	.00532
1.050	MOD C	.10131	.05068	.02535	.01014	.00507	.10181	.05094	.02548	.01020	.00510
	RATIO	1.0112	1.0165	1.0216	1.0279	1.0324	1.0234	1.0291	1.0342	1.0402	1.0441
	MOD I-C	.10476	.05297	.02676	.01084	.00546	.10823	.05479	.02769	.01120	.00565
1.100	MOD C	.10259	.05135	.02569	.01028	.00514	.10357	.05187	.02596	.01039	.00520
	RATIO	1.0212	1.0315	1.0416	1.0541	1.0630	1.0450	1.0564	1.0666	1.0786	1.0866
	MOD I-C	.10908	.05569	.02839	.01162	.00591	.11588	.05933	.03027	.01238	.00628
1.200	MOD C	.10508	.05265	.02635	.01055	.00528	.10699	.05366	.02688	.01076	.00538
	RATIO	1.0380	1.0577	1.0772	1.1019	1.1196	1.0831	1.1056	1.1262	1.1507	1.1672
	MOD I-C	.11302	.05821	.02991	.01237	.00633	.12299	.06362	.03274	.01353	.00691
1.300	MOD C	.10750	.05392	.02700	.01081	.00541	.11027	.05539	.02776	.01112	.00556
	RATIO	1.0513	1.0796	1.1078	1.1441	1.1704	1.1153	1.1485	1.1794	1.2166	1.2421
	MOD I-C	.11663	.06054	.03134	.01307	.00673	.12960	.06767	.03512	.01465	.00753
1.400	MOD C	.10984	.05514	.02763	.01107	.00554	.11343	.05706	.02862	.01147	.00574
	RATIO	1.0618	1.0978	1.1341	1.1814	1.2160	1.1425	1.1860	1.2270	1.2769	1.3115
	MOD I-C	.11995	.06270	.03267	.01374	.00712	.13575	.07150	.03738	.01573	.00813
1.500	MOD C	.11211	.05634	.02824	.01131	.00566	.11647	.05867	.02945	.01181	.00591
	RATIO	1.0699	1.1129	1.1568	1.2145	1.2572	1.1655	1.2187	1.2694	1.3321	1.3759

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

OMEGA SQUARED	ALPHA=	P= 4 M= 6 N= 6					P= 4 M= 8 N= 8				
		.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-C	.10008	.05005	.02503	.01001	.00501	.10010	.05006	.02503	.01002	.00501
1.001	MOD C	.10006	.05004	.02502	.01001	.00500	.10009	.05005	.02503	.01001	.00501
	RATIO	1.0002	1.0002	1.0003	1.0004	1.0005	1.0001	1.0002	1.0002	1.0003	1.0004
	MOD I-C	.10388	.05233	.02634	.01062	.00534	.10491	.05293	.02667	.01077	.00542
1.050	MOD C	.10312	.05175	.02595	.01041	.00521	.10437	.05250	.02638	.01061	.00532
	RATIO	1.0074	1.0112	1.0151	1.0202	1.0240	1.0052	1.0081	1.0110	1.0150	1.0180
	MOD I-C	.10762	.05459	.02765	.01123	.00568	.10966	.05579	.02833	.01154	.00584
1.100	MOD C	.10623	.05350	.02690	.01082	.00543	.10874	.05503	.02779	.01124	.00566
	RATIO	1.0131	1.0204	1.0279	1.0379	1.0454	1.0085	1.0139	1.0195	1.0270	1.0328
	MOD I-C	.11466	.05892	.03019	.01243	.00634	.11872	.06132	.03157	.01307	.00669
1.200	MOD C	.11238	.05699	.02881	.01166	.00587	.11751	.06014	.03066	.01253	.00635
	RATIO	1.0203	1.0337	1.0477	1.0667	1.0811	1.0103	1.0196	1.0296	1.0434	1.0542
	MOD I-C	.12120	.06299	.03261	.01360	.00699	.12720	.06660	.03471	.01458	.00754
1.300	MOD C	.11846	.06048	.03073	.01250	.00631	.12628	.06533	.03361	.01387	.00707
	RATIO	1.0231	1.0415	1.0611	1.0879	1.1086	1.0073	1.0194	1.0327	1.0516	1.0665
	MOD I-C	.12727	.06682	.03492	.01472	.00763	.13514	.07163	.03774	.01607	.00839
1.400	MOD C	.12446	.06395	.03266	.01335	.00676	.13504	.07059	.03662	.01525	.00782
	RATIO	1.0225	1.0449	1.0692	1.1030	1.1294	1.0008	1.0147	1.0305	1.0533	1.0716
	MOD I-C	.13292	.07043	.03712	.01581	.00826	.14259	.07641	.04066	.01752	.00922
1.500	MOD C	.13039	.06740	.03459	.01420	.00721	.14378	.07590	.03970	.01668	.00861
	RATIO	1.0194	1.0450	1.0732	1.1130	1.1445	.9918	1.0068	1.0242	1.0503	1.0713

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

OMEGA SQUARED		P= 4 M=10 N=10					P= 5 M=10 N=10				
	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-C	.10012	.05007	.02504	.01002	.00501	.10010	.05006	.02503	.01002	.00501
1.001	MOD C	.10011	.05006	.02503	.01002	.00501	.10009	.05005	.02503	.01001	.00501
	RATIO	1.0001	1.0002	1.0002	1.0003	1.0003	1.0001	1.0002	1.0002	1.0003	1.0003
	MOD I-C	.10577	.05343	.02696	.01090	.00549	.10487	.05290	.02666	.01077	.00542
1.050	MOD C	.10536	.05310	.02673	.01077	.00541	.10445	.05257	.02643	.01064	.00534
	RATIO	1.0039	1.0061	1.0085	1.0117	1.0141	1.0041	1.0064	1.0089	1.0123	1.0148
	MOD I-C	.11139	.05681	.02891	.01181	.00599	.10958	.05574	.02830	.01153	.00584
1.100	MOD C	.11077	.05626	.02851	.01158	.00584	.10891	.05516	.02788	.01129	.00569
	RATIO	1.0056	1.0097	1.0141	1.0200	1.0246	1.0062	1.0105	1.0151	1.0214	1.0262
	MOD I-C	.11685	.06013	.03085	.01272	.00649	.11413	.05851	.02992	.01229	.00626
1.150	MOD C	.11622	.05947	.03033	.01240	.00629	.11338	.05778	.02936	.01196	.00605
	RATIO	1.0055	1.0111	1.0171	1.0255	1.0320	1.0066	1.0126	1.0190	1.0278	1.0346
	MOD I-C	.12216	.06340	.03277	.01364	.00700	.11853	.06121	.03150	.01304	.00668
1.200	MOD C	.12169	.06273	.03219	.01326	.00676	.11786	.06043	.03086	.01264	.00642
	RATIO	1.0038	1.0106	1.0180	1.0284	1.0367	1.0057	1.0129	1.0208	1.0318	1.0401
	MOD I-C	.12732	.06661	.03468	.01455	.00752	.12278	.06384	.03306	.01379	.00709
1.250	MOD C	.12720	.06604	.03410	.01414	.00724	.12235	.06309	.03239	.01334	.00680
	RATIO	1.0009	1.0085	1.0170	1.0292	1.0389	1.0035	1.0118	1.0208	1.0336	1.0431
	MOD I-C	.13233	.06975	.03656	.01547	.00804	.12689	.06640	.03459	.01453	.00750
1.300	MOD C	.13273	.06939	.03604	.01504	.00773	.12683	.06578	.03393	.01405	.00719
	RATIO	.9970	1.0051	1.0145	1.0281	1.0391	1.0004	1.0093	1.0194	1.0336	1.0436

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

P= 4 M= 4 N=12

OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-C	.10007	.05004	.02503	.01001	.00501
1.001	MOD C	.10009	.05005	.02503	.01001	.00501
	RATIO	.9998	.9998	.9999	.9999	1.0000
	MOD I-C	.10336	.05211	.02626	.01060	.00534
1.050	MOD C	.10447	.05262	.02648	.01067	.00536
	RATIO	.9894	.9902	.9916	.9940	.9961
	MOD I-C	.10660	.05415	.02748	.01120	.00568
1.100	MOD C	.10900	.05530	.02800	.01136	.00574
	RATIO	.9780	.9792	.9814	.9856	.9894
	MOD I-C	.11272	.05806	.02986	.01237	.00634
1.200	MOD C	.11822	.06084	.03119	.01284	.00654
	RATIO	.9535	.9543	.9572	.9634	.9694
	MOD I-C	.11842	.06175	.03212	.01350	.00700
1.300	MOD C	.12763	.06660	.03456	.01442	.00742
	RATIO	.9279	.9271	.9295	.9361	.9431
	MOD I-C	.12374	.06523	.03429	.01460	.00764
1.400	MOD C	.13718	.07256	.03810	.01612	.00837
	RATIO	.9020	.8990	.9001	.9060	.9131

P= 5 M= 6 N=12

OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-C	.10008	.05005	.02503	.01001	.00501
1.001	MOD C	.10008	.05005	.02503	.01001	.00501
	RATIO	.9999	.9999	1.0000	1.0000	1.0001
	MOD I-C	.10374	.05228	.02633	.01063	.00535
1.050	MOD C	.10427	.05248	.02639	.01063	.00534
	RATIO	.9949	.9961	.9977	1.0001	1.0021
	MOD I-C	.10733	.05449	.02764	.01125	.00569
1.100	MOD C	.10857	.05501	.02782	.01128	.00569
	RATIO	.9886	.9906	.9933	.9976	1.0013
	MOD I-C	.11413	.05873	.03016	.01246	.00638
1.200	MOD C	.11729	.06020	.03078	.01263	.00643
	RATIO	.9731	.9756	.9797	.9866	.9927
	MOD I-C	.12046	.06273	.03257	.01364	.00705
1.300	MOD C	.12612	.06554	.03388	.01407	.00722
	RATIO	.9551	.9571	.9614	.9694	.9769
	MOD I-C	.12636	.06650	.03488	.01480	.00772
1.400	MOD C	.13504	.07102	.03709	.01559	.00806
	RATIO	.9357	.9364	.9403	.9489	.9579

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

		P= 5 M= 8 N= 8					P= 5 M=12 N= 6				
OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-C	.10008	.05005	.02503	.01001	.00501	.10009	.05005	.02503	.01001	.00501
1.001	MOD C	.10007	.05004	.02502	.01001	.00501	.10005	.05003	.02502	.01001	.00500
	RATIO	1.0001	1.0002	1.0003	1.0003	1.0004	1.0003	1.0004	1.0005	1.0006	1.0007
	MOD I-C	.10405	.05243	.02640	.01065	.00535	.10431	.05254	.02644	.01065	.00535
1.050	MOD C	.10348	.05198	.02609	.01048	.00525	.10269	.05148	.02579	.01034	.00517
	RATIO	1.0055	1.0086	1.0117	1.0159	1.0191	1.0158	1.0205	1.0250	1.0307	1.0349
	MOD I-C	.10795	.05478	.02776	.01128	.00570	.10846	.05500	.02784	.01130	.00570
1.100	MOD C	.10695	.05397	.02719	.01096	.00551	.10534	.05295	.02658	.01067	.00535
	RATIO	1.0094	1.0151	1.0210	1.0291	1.0353	1.0296	1.0387	1.0476	1.0590	1.0672
	MOD I-C	.11529	.05928	.03039	.01253	.00640	.11627	.05971	.03056	.01256	.00640
1.200	MOD C	.11385	.05795	.02941	.01195	.00603	.11054	.05584	.02813	.01133	.00569
	RATIO	1.0127	1.0228	1.0336	1.0485	1.0601	1.0519	1.0692	1.0863	1.1085	1.1249
	MOD I-C	.12210	.06350	.03290	.01374	.00708	.12350	.06413	.03316	.01379	.00708
1.300	MOD C	.12070	.06196	.03165	.01296	.00657	.11560	.05868	.02967	.01199	.00603
	RATIO	1.0116	1.0250	1.0395	1.0601	1.0763	1.0683	1.0929	1.1175	1.1500	1.1743
	MOD I-C	.12842	.06748	.03530	.01490	.00774	.13019	.06829	.03562	.01498	.00774
1.400	MOD C	.12749	.06597	.03392	.01399	.00713	.12054	.06147	.03118	.01264	.00637
	RATIO	1.0073	1.0229	1.0404	1.0655	1.0856	1.0801	1.1110	1.1424	1.1844	1.2162

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

		P= 6 M= 6 N= 6					P= 6 M=12 N= 6				
OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005	.100	.050	.025	.010	.005
	MOD I-C	.10004	.05003	.02502	.01001	.00500	.10008	.05005	.02503	.01001	.00501
1.001	MOD C	.10002	.05001	.02501	.01000	.00500	.10003	.05002	.02501	.01000	.00500
	RATIO	1.0002	1.0003	1.0004	1.0005	1.0006	1.0004	1.0006	1.0007	1.0008	1.0009
	MOD I-C	.10218	.05135	.02580	.01038	.00521	.10375	.05222	.02626	.01058	.00531
1.050	MOD C	.10123	.05064	.02533	.01013	.00507	.10157	.05083	.02543	.01017	.00509
	RATIO	1.0094	1.0140	1.0185	1.0244	1.0286	1.0215	1.0274	1.0329	1.0396	1.0443
	MOD I-C	.10424	.05264	.02656	.01075	.00542	.10734	.05436	.02749	.01114	.00562
1.100	MOD C	.10244	.05128	.02565	.01027	.00513	.10311	.05164	.02584	.01034	.00517
	RATIO	1.0176	1.0265	1.0354	1.0469	1.0554	1.0410	1.0526	1.0636	1.0772	1.0867
	MOD I-C	.10805	.05504	.02800	.01144	.00581	.11405	.05841	.02983	.01223	.00622
1.200	MOD C	.10479	.05252	.02629	.01052	.00526	.10611	.05322	.02665	.01067	.00534
	RATIO	1.0311	1.0480	1.0651	1.0874	1.1039	1.0749	1.0975	1.1191	1.1462	1.1655
	MOD I-C	.11151	.05723	.02933	.01209	.00618	.12021	.06217	.03204	.01328	.00681
1.300	MOD C	.10707	.05372	.02691	.01078	.00539	.10899	.05474	.02744	.01099	.00550
	RATIO	1.0414	1.0654	1.0899	1.1223	1.1465	1.1030	1.1357	1.1676	1.2082	1.2378
	MOD I-C	.11465	.05925	.03056	.01270	.00653	.12587	.06568	.03413	.01430	.00739
1.400	MOD C	.10929	.05489	.02751	.01102	.00551	.11176	.05622	.02820	.01130	.00566
	RATIO	1.0491	1.0794	1.1108	1.1525	1.1839	1.1262	1.1684	1.2101	1.2646	1.3067
	MOD I-C	.11753	.06111	.03170	.01327	.00685	.13109	.06897	.03611	.01531	.00803
1.500	MOD C	.11144	.05603	.02809	.01126	.00563	.11445	.05765	.02894	.01161	.00581
	RATIO	1.0547	1.0907	1.1282	1.1786	1.2168	1.1454	1.1964	1.2479	1.3191	1.3818

POWER COMPARISONS OF |C| AND |I-C|

OMEGA SQUARED	ALPHA=	P= 6 M= 8 N= 8 .100	.050	.025	.010	.005	P= 7 M= 8 N= 8 .100	.050	.025	.010	.005
	MOD I-C	.10007	.05004	.02502	.01001	.00501	.10006	.05003	.02502	.01001	.00501
1.001	MOD C	.10005	.05003	.02502	.01001	.00500	.10004	.05002	.02501	.01001	.00500
	RATIO	1.0001	1.0002	1.0003	1.0004	1.0004	1.0001	1.0002	1.0003	1.0004	1.0005
	MOD I-C	.10335	.05202	.02617	.01054	.00530	.10271	.05164	.02596	.01045	.00525
1.050	MOD C	.10273	.05154	.02584	.01036	.00519	.10202	.05112	.02560	.01025	.00513
	RATIO	1.0060	1.0092	1.0126	1.0171	1.0205	1.0068	1.0103	1.0140	1.0190	1.0227
	MOD I-C	.10655	.05396	.02729	.01107	.00559	.10528	.05322	.02688	.01088	.00549
1.100	MOD C	.10544	.05308	.02668	.01073	.00538	.10401	.05222	.02619	.01051	.00526
	RATIO	1.0105	1.0166	1.0230	1.0318	1.0384	1.0122	1.0190	1.0262	1.0358	1.0431
	MOD I-C	.11253	.05763	.02945	.01210	.00616	.11007	.05617	.02862	.01172	.00595
1.200	MOD C	.11080	.05613	.02836	.01147	.00577	.10794	.05441	.02737	.01101	.00552
	RATIO	1.0156	1.0267	1.0384	1.0547	1.0672	1.0197	1.0323	1.0457	1.0640	1.0781
	MOD I-C	.11803	.06104	.03148	.01307	.00670	.11442	.05889	.03024	.01250	.00640
1.300	MOD C	.11609	.05917	.03004	.01221	.00616	.11178	.05656	.02853	.01151	.00578
	RATIO	1.0167	1.0316	1.0478	1.0703	1.0877	1.0237	1.0411	1.0597	1.0857	1.1062
	MOD I-C	.12309	.06422	.03338	.01398	.00721	.11840	.06137	.03172	.01319	.00676
1.400	MOD C	.12130	.06219	.03173	.01296	.00656	.11553	.05868	.02968	.01201	.00604
	RATIO	1.0148	1.0326	1.0521	1.0790	1.0986	1.0248	1.0459	1.0684	1.0985	1.1190

APPENDIX 12

POWER COMPARISONS OF |C| AND |I-C|

P= 6 M= 8 N=10

OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-C	.10007	.05004	.02503	.01001	.00501
1.001	MOD C	.10007	.05004	.02502	.01001	.00501
	RATIO	1.0000	1.0001	1.0001	1.0002	1.0003
	MOD I-C	.10361	.05218	.02626	.01060	.00533
1.050	MOD C	.10347	.05199	.02611	.01049	.00526
	RATIO	1.0014	1.0036	1.0061	1.0100	1.0140
	MOD I-C	.10707	.05429	.02750	.01120	.00569
1.100	MOD C	.10693	.05400	.02722	.01099	.00553
	RATIO	1.0013	1.0054	1.0103	1.0190	1.0298
	MOD I-C	.11040	.05635	.02875	.01186	.00614
1.150	MOD C	.11040	.05601	.02835	.01150	.00580
	RATIO	1.0000	1.0061	1.0141	1.0316	1.0593
	MOD I-C	.11364	.05841	.03007	.01269	.00685
1.200	MOD C	.11386	.05804	.02949	.01201	.00607
	RATIO	.9980	1.0065	1.0196	1.0568	1.1278
	MOD I-C	.11683	.06056	.03161	.01394	.00817
1.250	MOD C	.11731	.06007	.03064	.01253	.00636
	RATIO	.9958	1.0081	1.0314	1.1127	1.2849

P= 7 M= 8 N=10

OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-C	.10006	.05004	.02502	.01001	.00500
1.001	MOD C	.10006	.05003	.02502	.01001	.00500
	RATIO	1.0000	1.0001	1.0001	1.0001	1.0001
	MOD I-C	.10292	.05177	.02601	.01045	.00521
1.050	MOD C	.10287	.05164	.02590	.01040	.00521
	RATIO	1.0005	1.0025	1.0043	1.0049	1.0008
	MOD I-C	.10572	.05349	.02703	.01093	.00547
1.100	MOD C	.10572	.05327	.02681	.01080	.00542
	RATIO	1.0000	1.0040	1.0083	1.0120	1.0079
	MOD I-C	.10852	.05533	.02828	.01176	.00615
1.150	MOD C	.10857	.05491	.02772	.01120	.00564
	RATIO	.9996	1.0076	1.0200	1.0497	1.0908
	MOD I-C	.11158	.05772	.03035	.01383	.00837
1.200	MOD C	.11140	.05656	.02864	.01161	.00586
	RATIO	1.0017	1.0206	1.0600	1.1905	1.4292
	MOD I-C	.11550	.06158	.03461	.01911	.01463
1.250	MOD C	.11422	.05820	.02956	.01203	.00608
	RATIO	1.0112	1.0581	1.1710	1.5887	2.4069

APPENDIX 12 POWER COMPARISONS OF |C| AND |I-C|

P= 6 M=10 N= 8

OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-C	.10008	.05005	.02503	.01001	.00501
1.001	MOD C	.10006	.05003	.02502	.01001	.00500
	RATIO	1.0002	1.0003	1.0003	1.0004	1.0005
	MOD I-C	.10335	.05230	.02632	.01060	.00532
1.050	MOD C	.10293	.05165	.02590	.01039	.00521
	RATIO	1.0090	1.0124	1.0159	1.0202	1.0227
	MOD I-C	.10754	.05452	.02759	.01119	.00564
1.100	MOD C	.10584	.05330	.02681	.01079	.00541
	RATIO	1.0161	1.0228	1.0294	1.0375	1.0421
	MOD I-C	.11108	.05666	.02884	.01177	.00595
1.150	MOD C	.10872	.05495	.02771	.01118	.00562
	RATIO	1.0217	1.0312	1.0407	1.0522	1.0580
	MOD I-C	.11447	.05873	.03004	.01232	.00624
1.200	MOD C	.11158	.05658	.02861	.01158	.00583
	RATIO	1.0259	1.0379	1.0499	1.0640	1.0696
	MOD I-C	.11772	.06072	.03120	.01284	.00649
1.250	MOD C	.11442	.05821	.02952	.01198	.00604
	RATIO	1.0289	1.0430	1.0571	1.0721	1.0737
	MOD I-C	.12084	.06263	.03229	.01329	.00664
1.300	MOD C	.11724	.05984	.03042	.01238	.00625
	RATIO	1.0307	1.0466	1.0617	1.0737	1.0613

P= 7 M=10 N= 8

OMEGA SQUARED	ALPHA=	.100	.050	.025	.010	.005
	MOD I-C	.10007	.05004	.02502	.01001	.00501
1.001	MOD C	.10004	.05002	.02501	.01001	.00500
	RATIO	1.0002	1.0003	1.0004	1.0005	1.0005
	MOD I-C	.10330	.05197	.02613	.01052	.00528
1.050	MOD C	.10219	.05121	.02565	.01028	.00514
	RATIO	1.0108	1.0148	1.0187	1.0235	1.0267
	MOD I-C	.10644	.05387	.02723	.01104	.00558
1.100	MOD C	.10435	.05241	.02630	.01055	.00529
	RATIO	1.0200	1.0278	1.0355	1.0462	1.0565
	MOD I-C	.10944	.05570	.02831	.01158	.00594
1.150	MOD C	.10648	.05360	.02694	.01083	.00543
	RATIO	1.0278	1.0391	1.0510	1.0700	1.0945
	MOD I-C	.11232	.05747	.02938	.01217	.00638
1.200	MOD C	.10859	.05478	.02757	.01110	.00557
	RATIO	1.0343	1.0491	1.0656	1.0965	1.1457
	MOD I-C	.11507	.05919	.03044	.01278	.00688
1.250	MOD C	.11067	.05595	.02821	.01137	.00571
	RATIO	1.0397	1.0579	1.0791	1.1243	1.2046
	MOD I-C	.11769	.06080	.03139	.01325	.00715
1.300	MOD C	.11273	.05711	.02884	.01164	.00585
	RATIO	1.0439	1.0646	1.0886	1.1378	1.2221

B I B L I O G R A P H Y

1. Anderson, T.W. (1946): The noncentral Wishart distribution and certain problems of multivariate statistics. *Ann. Math. Statist.* 17, 409-431.

2. Anderson, T.W. (1958): *Introduction to Multivariate Statistical Analysis.* John Wiley, New York.

3. Anderson, T.W. and Das Gupta, S. (1964): A monotonicity property of the power functions of some tests of the equality of two covariance matrices. *Ann. Math. Statist.* 35, 1059-1063.

4. Anderson, T.W. and Girshick, M.A. (1944): Some extensions of the Wishart distribution. *Ann. Math. Statist.* 15, 345-357.

5. Bartlett, M.S. (1937): Properties of sufficiency and statistical tests. *Proc. Roy. Soc., A,* 160, 268-282.

6. Bartlett, M.S. (1938): Further aspects of the theory of multiple regression. *Proc. Camb. Phil. Soc.* 34, 33-40.

7. Box, G.E.P. (1949): A general distribution theory for a class of likelihood criteria. *Biometrika* 36, 317-346.

8. Constantine, A.G. (1963): Some noncentral distribution problems in multivariate analysis. *Ann. Math. Statist.* 34, 1270-1285.

9. Constantine, A.G. (1966): The distribution of Hotelling's generalised T_0^2. *Ann. Math. Statist.* 37, 215-225.

10. Consul, P.C. (1966): On the exact distributions of the likelihood ratio criteria for testing linear hypotheses about regression coefficients. *Ann. Math. Statist.* 37, 1319-1330.

11. Consul, P.C. (1967): On the exact distribution of the likelihood ratio criteria for testing independence of sets of variables under the null hypothesis. *Ann. Math. Statist.* 38, 1160-1169.

12. Consul, P.C. (1969): The exact distributions of likelihood criteria for different hypotheses. *Multivariate Analysis II (*Ed.: Krishnaiah, P.R. (1969)), Academic Press, N.Y.

13. Das Gupta, S. (1969): Properties of power functions of some tests concerning dispersion matrices of multivariate normal distributions. *Ann. Math. Statist.* 40, 697-701.

14. Das Gupta, S., Anderson, T.W., and Mudholkar, G.S. (1964): Monotonicity of power functions of some tests of the multivariate linear hypothesis. *Ann. Math. Statist.* 35, 200-205.

15. Das Gupta, S. and Perlman, M.D. (1973): On the power of Wilks' U-Test for MANOVA. *J. Multivariate Anal.* 3, 220-225.

16. de Waal, D.J. (1968): Nie-Sentrale Meerveranderlike Beta-verdelings. Unpublished Ph.D. thesis, University of Cape Town.

17. de Waal, D.J. (1968a): An asymptotic distribution for the determinant of a noncentral B statistic in multivariate analysis. *S. Afr. Statist. J.* 2, 77-84.

18. de Waal, D.J. (1969): The noncentral multivariate Beta type 2 distribution. *S. Afr. Statist. J.* 3, 101-108.

19. de Waal, D.J. (1970): Distributions connected with a multivariate Beta statistic. *Ann. Math. Statist.* 41, 1091-1095.

20. Erdélyi, A. et al. (1953): *Higher Transcendental Functions.* McGraw-Hill, New York.

21. Fisher, R.A. (1928): The general sampling distribution of the multiple correlation coefficient. *Proc. Roy. Soc., A,* 121 654-673.

22. Fujikoshi, Y. (1970): Asymptotic expansions of the distributions of test statistics in multivariate analysis. *J. Sci. Hiroshima Univ. Ser. A-I,* 34, 73-144.

23. Fujikoshi, Y. (1972): Asymptotic formulas for the distributions of the determinant and the trace of a noncentral Beta matrix. *J. Multivariate Anal.* 2, 208-218.

24. Giri, N. (1965): On the complex analogues of T^2 and R^2 tests. *Ann. Math. Statist.* 36, 664-670.

25. Giri, N. (1968): On tests of the equality of two covariance matrices. *Ann. Math. Statist.* 39, 275-277.

26. Goodman, N.R. (1963): Statistical Analysis based on a certain multivariate complex Gaussian distribution (an introduction). *Ann. Math. Statist.* 34, 152-176.

27. Greenacre, M. (1972): Some noncentral distributions and applications in multivariate analysis. Unpublished M.Sc. thesis, University of South Africa, Pretoria.

28. Greenstreet, R.L. and Connor, R.J. (1974): Power of tests for equality of covariance matrices. *Technometrics* 16, 27-30.

29. Gupta, A.K. (1970): Distribution of Wilks' Λ in the non-central linear case. Report no. 128, Department of Mathematics, University of Arizona.

30. Gupta, A.K. (1971): Noncentral distribution of Wilks' statistic in MANOVA. *Ann. Math. Statist.* 42, 1254-1261.

31. Herz, C.S. (1955): Bessel functions of matrix argument. *Ann. Math.* 61, 474-523.

32. Hildebrand, F.B. (1956): *Introduction to numerical analysis.* McGraw-Hill, New York.

33. Hotelling, H. (1931): The generalisation of 'Student's' Ratio. *Ann. Math. Statist.* 2, 359-378.

34. Hsu, P.L. (1940): On generalised analysis of variance (1). *Biometrika* 31, 221-237.

35. Ito, K. (1962): A comparison of the powers of two multivariate analysis of variance tests. *Biometrika* 49, 455-462.

36. James, A.T. (1960): The distribution of the latent roots of the covariance matrix. *Ann. Math. Statist.* 31, 151-158.

37. James, A.T. (1961): Zonal polynomials of the real positive definite symmetric matrices. *Ann. Math.* 74, 456-469.

38. James, A.T. (1964): Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* 35, 475-501.

39. Khatri, C.G. (1964): Distribution of the 'Generalised' multiple correlation Matrix in the Dual Case. *Ann. Math. Statist.* 35, 1801-1806.

40. Khatri, C.G. (1965): Classical Statistical analysis based on a certain multivariate complex Gaussian distribution. *Ann. Math. Statist.* 36, 98-114.

41. Khatri, C.G. (1965a): A test for reality of a covariance matrix in a certain complex Gaussian distribution. *Ann. Math. Statist.* 36, 115-119.

42. Khatri, C.G. and Pillai, K.C.S. (1968): On the noncentral distribution of two test criteria in multivariate analysis of variance. *Ann. Math. Statist.* 39, 215-226.

43. Khatri, C.G. and Srivastava, M.S. (1971): On exact non-null distributions of likelihood ratio criteria for sphericity test and equality of two covariance matrices. *Sankhya, Series A* 33, 201-210.

44. Khatri, C.G. and Srivastava, M.S. (1974): Asymptotic expansions of the non-null distributions of likelihood ratio criteria for covariance matrices. *Ann. Statist.* 2, 109-117.

45. Kiefer, J. and Schwartz, R. (1965): Admissible Bayes character of T^2, R^2, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* 36, 747-770.

46. Kshirsagar, A.M. (1961): The non-central multivariate Beta distribution. *Ann. Math. Statist.* 32, 104-111.

47. Lee, Y-S. (1971): Distribution of the canonical correlations and asymptotic expansions for distributions of certain independence test statistics. *Ann. Math. Statist.* 42, 526-537

48. Lee, Y-S. (1971a): Asymptotic formulae for the distribution of a multivariate test statistic : power comparisons of certain multivariate tests. *Biometrika* 58, 647-651.

49. Lee, Y-S. (1972): Some results on the distribution of Wilks' likelihood-ratio criterion. *Biometrika* 59, 649-664.

50. Lee, Y-S. (1972a): Tables of upper percentage points of the multiple correlation coefficient. *Biometrika* 59, 175-189.

51. Mathai, A.M. (1971): An expansion of Meijer's G-function and the distribution of products of independent Beta variables. *S.Afr. Statist. Jour.* 5, 71-90.

52. Mathai, A.M. (1971a): On the distribution of the likelihood ratio criterion for testing linear hypotheses on regression coefficients. *Ann. Inst. Statist. Math.* 23, 181-197.

53. Mathai, A.M. (1973): A review of the different techniques used for deriving the exact distributions of multivariate test criteria. *Sankhya, Series A,* 35, 39-60.

54. Mathai, A.M. and Rathie, P.N. (1971): The exact distribution of Wilks' criterion. *Ann. Math. Statist.* 42, 1010-1019.

55. Meijer, C.S. (1946): On the G-function I-VIII. *Proc. Kon. Neder. Acad. Van Wet.* 49, 227-237, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072, 1165-1175.

56. Mijares, T.A. (1964): *Percentage points of the sum* $V_1^{(s)}$ *of s roots (s = 1-50).* University of the Philippines, Manila.

57. Mikhail, N.N. (1965): A comparison of tests of the Wilks-Lawley hypothesis in multivariate analysis. *Biometrika* 52, 149-156.

58. Money, A.H. (1972): Noncentral Multivariate Beta distributions and their applications. Unpublished Ph.D. thesis, University of Cape Town.

59. Muirhead, R.J. (1972): On the test of independence between two sets of variates. *Ann. Math. Statist.* 43, 1491-1497.

60. Nagao, H. (1970): Asymptotic expansions of some test criteria for homogeneity of variances and covariance matrices from normal populations. *J.Sci. Hiroshima Univ. Ser. A-I* 34, 153-247.

61. Nagao, H. (1973): Nonnull distributions of two test criteria for independence under local alternatives. *J. Multivariate Anal.* 3, 435-444.

62. Nair, U.S. (1938): The application of the moment function in the study of distribution laws in statistics. *Biometrika* 30, 274-294.

63. Neyman, J. and Pearson, E.S. (1928): On the use and interpretation of certain test criteria for purposes of statistical inference. *Biometrika* 20A, 175-240, 263-294.

64. Olkin, I. and Rubin, H. (1964): Multivariate Beta distributions and independence properties of the Wishart distribution. *Ann. Math. Statist.* 35, 261-269.

65. Pearson, E.S. and Wilks, S.S. (1933): Methods of analysis for k samples of two variables. *Biometrika* 25, 353-378.

66. Pillai, K.C.S. (1955): Some new test criteria in multivariate analysis. *Ann. Math. Statist.* 26, 117-121.

67. Pillai, K.C.S. (1965): Noncentral multivariate Beta distribution and the moments of traces of some matrices. *Multivariate Analysis.* (Ed.: Krishnaiah, P.R. (1966)). Academic Press, New York.

68. Pillai, K.C.S., Al-Ani, S. and Jouris, G.M. (1969): On the distributions of the ratios of the roots of a covariance matrix and Wilks' criterion for tests of three hypotheses. *Ann. Math. Statist.* 40, 2033-2040.

69. Pillai, K.C.S. and Gupta, A.K. (1969): On the exact distribution of Wilks' criterion. *Biometrika* 56, 109-118.

70. Pillai, K.C.S. and Jayachandran, K. (1967): Power Comparisons of tests of two multivariate hypotheses based on four criteria. *Biometrika* 54, 195-210.

71. Pillai, K.C.S. and Jayachandran, K. (1968): Power comparisons of tests of equality of two covariance matrices based on four criteria. *Biometrika* 55, 335-342.

72. Pillai, K.C.S. and Jouris, G.M. (1971): Some distribution problems in the multivariate complex Gaussian case. *Ann. Math. Statist.* 42, 517-525.

73. Pillai, K.C.S. and Nagarsenker, B.N. (1972): On the distribution of a class of statistics in multivariate analysis. *J.Multivariate Anal.* 2, 96-114.

74. Posten, H.O. and Bargmann, R.E. (1964): Power of the likelihood ratio test of the general linear hypothesis in multivariate analysis. *Biometrika* 51, 467-480.

75. Rao, C.R. (1948): Tests of significance in multivariate analysis. *Biometrika* 35, 58-79.

76. Roy, J. (1960): Nonnull distribution of the likelihood ratio in analysis of dispersion. *Sankhya* 22, 289-294.

77. Roy, J. (1965): Power of the likelihood ratio test used in analysis of dispersion. *Multivariate Analysis.* (Ed.: Krishnaiah, P.R. (1966)). Academic Press, N.Y.

78. Schatzoff, M. (1964): Exact distribution of Wilks' likelihood ratio criterion and competitive tests. Unpublished Ph.D. thesis, Harvard University.

79. Schatzoff, M. (1966): Exact distribution of Wilks' likelihood-ratio criterion. *Biometrika* 53, 347-358.

80. Sugiura, N. (1969): Asymptotic expansions of the distributions of the likelihood ratio criteria for covariance matrix. *Ann. Math. Statist.* 40, 2051-2063.

81. Sugiura, N. (1973): Asymptotic non-null distributions of the likelihood ratio criteria for covariance matrix under local alternatives. *Ann. Statist. 1*, 718-728.

82. Sugiura, N. and Fujikoshi, Y. (1969): Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence. *Ann. Math. Statist.* 40, 942-952.

83. Sugiura, N. and Nagao, H. (1968): Unbiasedness of some test criteria for the equality of one of two covariance matrices. *Ann. Math. Statist.* 39, 1686-1692.

84. Troskie, C.G. (1966): Nie-Sentrale Meerveranderlike Betaverdelings. *Tydskrif vir Wetenskappe* 6, 58-71.

85. Troskie, C.G. (1967): Noncentral multivariate Dirichlet distributions. *S.Afr. Statist. J.* 1, 21-32.

86. Troskie, C.G. (1969): The Generalised Multiple Correlation Matrix. *S.Afr. Statist. J.* 3, 109-121.

87. Troskie, C.G. (1971): The distributions of some test criteria in multivariate analysis. *Ann. Math. Statist.* 42, 1752-1757.

88. Troskie, C.G. (1972): The distributions of some test criteria depending on Multivariate Dirichlet distributions. *S.Afr. Statist. J.* 6, 151-163.

89. Troskie, C.G. and Money, A.H. (1974): Noncentral distributions of some test criteria in multivariate analysis. Accepted for publication in S.Afr. Statist. J.

90. Underhill, L.G. (1973): Further distributions of multivariate quadratic forms. Unpublished Ph.D. thesis, University of Cape Town.

91. Wald, A. and Brookner, R.J. (1941): On the distribution of Wilks' statistic for testing the independence of several groups of variates. *Ann. Math. Statist.* 13, 358-372.

92. Wilks, S.S. (1932): Certain generalisations in the analysis of variance. *Biometrika* 24, 471-494.

93. Wilks, S.S. (1962): *Mathematical Statistics.* Wiley, New York.

94. Wishart, J. (1928): The generalised product moment distribution in samples from a normal multivariate population. *Biometrika* 20A, 32-52.