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σ-Spaces and σ-Frames

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Contents

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Synopsis

The study of topological concepts from a lattice theoretical view point was first initiated by Wallman (1938); inspired by the work of Stone (1937). The actual term 'frame' was introduced by Dowker and Papert in the middle sixties to describe a 'local lattice'. A local lattice is defined to be a complete lattice with the appropriate distributivity property (i.e. finite meets distribute over arbitrary joins) and so corresponds to the notion of a generalized topological space. These concepts were pursued by Benabou (1957), Banaschewski (1969), Isbell (1972) and others. Johnstone's book 'Stone Spaces' [17] gives a basic reference for the theory of frames (or locales). The 'Compendium of Continuous Lattices' [13] is a useful reference for many standard lattice theoretical results.

The generalizations of frames to those lattices closed under, at most, countable joins, namely sigma frames, was first considered by Charalambous [8] and has been explored in detail by Reynolds [22] and Banaschewski [2]. Gilmour [11] shows that there exists a dual adjunction between regular sigma frames (a full subcategory of sigma frames) and the zero set spaces (here called the regular sigma spaces, a full subcategory of 'sigma spaces'). Madden and Vermeer [19] obtain a countable version of the Stone Duality for regular sigma frames and regular Lindelöf frames.

σ-spaces and σ-frames are generalized topological spaces and frames, respectively. However, σ-spaces and σ-frames will be developed here independently from topological spaces and frames. On the otherhand, the σ-spaces are the generalized 'zero set spaces' ('Alexandroff spaces') of Gilmour in [11]. The dual adjunction between the two, σ-spaces and σ-frames, is constructed in a manner similar to that between topological spaces and frames and is a generalization of the dual adjunction between zero set spaces and regular σ-frames (as appears in Gilmour [11]). Various notions well understood for topological spaces and frames are imitated, for instance regularity and sobriety for σ-spaces and continuity for σ-frames.

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An outline of the thesis is given below.

Preliminaries The basic definitions of σ-spaces and σ-frames are given. It is observed that σ-spaces are topological over sets and the dual adjunction between σ-spaces and σ-frames is explicitly given.

Chapter 1 The cozero part of a σ-frame is defined and explored, in a similar manner to that for frames in [5]. For a regular σ-frame L, it is observed that the cozero part is the whole of L. The Stone-Čech compactification of regular σ-frames is given, following Gilmour [11]. Pseudocompactness is defined for σ-frames, once again in a fashion similar to that for frames. Countable compactness and pseudocompactness are shown to coincide in σ-frames. The frame envelope of a σ-frame is shown to be pseudocompact if and only if the σ-frame is pseudocompact.

Chapter 2 Separation axioms, including sobriety and regularity, are defined for σ-spaces. The soberification of a σ-space is defined in terms of σ-prime filters of σ-open sets. Sober σ-spaces are observed to be a reflective subcategory of σ-spaces. Soberification is found to distribute over products of σ-spaces - a fact that generalizes the result of Gordon [14], that the analogue of the Hewitt realcompactification of 'zero set spaces' distributes over products.

Chapter 3 Continuous σ-frames are defined in the usual way and some well known results for them are obtained, mainly following Walters [24] and B.Banaschewski [2]. Spectral σ-spaces are defined, essentially as the dual image of continuous σ-frames in the dual adjunction between σ-spaces and σ-frames. The countable version of the Hofmann-Mislove Theorem, due to Gilmour, is given . Spectral σ-spaces are seen to be the equivalent of the σ-spectral topological spaces of B.Banaschewski in [2]. Coherent σ-spaces and stably continuous σ-frames are defined and seen to be dually equivalent.

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Preliminaries

σ-Spaces

A σ-space is a pair $(X, \Sigma(X))$, where X is a set equipped with a collection of subsets $\Sigma(X)$, of X, subject to the following conditions

(1) $\emptyset, X \in \Sigma(X)$.

(2) for $U, V \in \Sigma(X)$, $U \cap V \in \Sigma(X)$.

(3) given $U_n \in \Sigma(X)$, $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} U_n \in \Sigma(X)$.

The members of $\Sigma(X)$ will be referred to as *σ-open* sets. The X in the pair $(X, \Sigma(X))$ will be called the *underlying* set. It should be noted that $\Sigma(X)$ need not be unique to X. For example, given a set X, $\Sigma(X)$ may be the collection of all subsets of X or just the sets $\{\emptyset, X\}$ (this compares to the discrete and indiscrete topology of a set).

When referring to the σ-space $(X, \Sigma(X))$ we shall often refer to it as simply X, where no confusion arises. Also, given $(X, \Sigma(X))$ we will often refer to $\Sigma(X)$ as ΣX instead.

A collection of subsets of a set X is called a *σ-base*, if it is closed under finite intersections and contains the sets $\emptyset$ and X. Of course, the collection of all countable unions of a σ-base will generate a σ-space. Members of a σ-base will be called basic σ-open sets. A collection of subsets of a set will be called a *σ-subbase* if the family of all finite intersections of members is a σ-base. Again, members of a σ-subbase will be called subbasic σ-opens.

A σ-continuous map between two σ-spaces is a set map between the respective underlying sets with the condition that the inverse image *preserves* σ-open sets i.e for a set map $f : X \to Y$, with X and Y σ-spaces, f is σ-continuous if and only if $f^{\leftarrow}(V) \in \Sigma(X)$ for every $V \in \Sigma(Y)$.

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This way we have the category of σ-spaces, which we shall denote $\sigma\mathbf{Sp}$, where the objects are σ-spaces and the maps are σ-continuous maps. We denote the category of sets by **Sets**.

Suppose we have a set X, a collection of σ-spaces $(X_\alpha, \Sigma(X_\alpha))_{\alpha\in I}$ and set maps $f_\alpha : X \to X_\alpha$. We can define a σ-space structure on X that will make each of the maps f_α, σ-continuous. Let $\Sigma(X)$, be the collection of σ-open sets that has $\{f_\alpha^{\leftarrow}(U_\alpha) \mid U_\alpha \in \Sigma_\alpha,\ \alpha \in I\}$ as σ-subbase. Then the σ-open sets of X will be countable unions of finite intersections of this collection. Trivially, this makes each f_α σ-continuous.

This σ-space structure on X is an 'initial' structure, in that it is the smallest collection of subsets of X that will make each of the maps f_α σ-continuous. Furthermore, this defined structure is initial, with respect to the forgetful functor $\mathcal{F} : \sigma\mathbf{Sp} \to \mathbf{Sets}$, in the sense of [1]. That is to say that the category of σ-spaces is topological over the category of sets.

Since **Sets** is complete and co-complete and $\sigma\mathbf{Sp}$ is topological over **Sets**, $\sigma\mathbf{Sp}$ is complete and co-complete as well (see [1]). In particular, products in $\sigma\mathbf{Sp}$ are defined: Given a collection of σ-spaces $(X_\alpha, \Sigma(X_\alpha))$ the σ-space product $(\prod X_\alpha, \Sigma(\prod X_\alpha))$ has as σ-subbase, the collection $\{\pi_\alpha^{\leftarrow}(U_\alpha) \mid U_\alpha \in \Sigma(X_\alpha)\}$, and is the smallest collection of subsets on the set product $\prod X_\alpha$ that make each of the projection maps $\pi_\alpha : \prod X_\alpha \to X_\alpha$, in **Sets**, σ-continuous.

Also, for a σ-space $(Y, \Sigma(Y))$, with subset $X \subseteq Y$, one defines the initial σ-space structure on X for the inclusion map $X \hookrightarrow Y$, making X a *subspace* of Y. Here, the σ-open sets of X are of the form $X \cap V$, $V \in \Sigma(Y)$.

In these respects, σ-spaces are not unlike the topological spaces.

Frames and σ-Frames

A frame L is a bounded lattice where

(1) Given an arbitrary $X \subseteq L$, then $\bigvee X \in L$.

(2) Finite meets distribute over arbitrary joins i.e for any $X \subseteq L$

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}.$$

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A frame homomorphism between two frames is a lattice homomorphism that also preserves arbitrary joins i.e. if $h : L \to M$, for frames L and M with any $X \subseteq L$,

$$h(\bigvee X) = \bigvee \{h(x) \mid x \in X\}.$$

A σ-frame L is a bounded lattice where

(1) Given at most countable $X \subseteq L$, then $\bigvee X \in L$.

(2) Finite meets distribute over countable joins i.e for countable $X \subseteq L$

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}.$$

A σ-frame homomorphism between two σ-frames is a bounded lattice homomorphism that also preserves countable joins i.e. if $h : L \to M$, for σ-frames L and M with any countable $S \subseteq L$,

$$h(\bigvee S) = \bigvee \{h(s) \mid s \in S\}.$$

Again, we have a category(s), here the category of σ-frames (frames), which we shall denote σ**Frm** (**Frm**), with σ-frames (frames) for objects and σ-frame (frame) homomorphisms as the maps.

σ-spaces and σ-frames are generalizations of topological spaces and frames respectively, requiring only closure under at most countable unions and joins, as opposed to closure under arbitrary unions and joins, respectively.

The passage between frames and σ-frames

For a σ-frame L, $J \subseteq L$ is said to be a *σ-ideal* if $0 \in J$, $a \leq b \in J$ implies that $a \in J$ (*closed downwards*) and at most countable $S \subseteq J$ implies $\bigvee S \in J$. Ordering the σ-ideals by inclusion, $\{0\}$ and L are the smallest and largest σ-ideals of L. Meets are given by intersection and for an arbitrary collection $\{J_i\}$ of σ-ideals,

$$\bigvee_i J_i = \left\{ \bigvee S \mid \text{countable } S \subseteq \bigcup_i J_i \right\}.$$

Thus we have a frame that we denote as $\mathcal{H}L$. Given a σ-frame map $h : L \to M$, we have the frame map $\mathcal{H}h : \mathcal{H}L \to \mathcal{H}M$, where if $J \in \mathcal{H}L$, then $\mathcal{H}h(J)$ is the σ-ideal generated (take all countable joins) by $h(J)$ in M.

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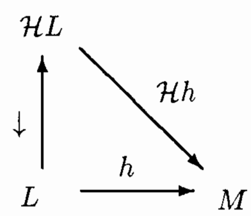
This describes the covariant functor $\mathcal{H} : \sigma\mathbf{Frm} \to \mathbf{Frm}$. Since every frame is a σ-frame, there is a forgetful functor $\mathcal{U} : \mathbf{Frm} \to \sigma\mathbf{Frm}$, taking a frame to its underlying σ-frame, and moreover $\mathcal{H}$ is left adjoint to $\mathcal{U}$ [3].

$\mathcal{H}L$ is also called the *free frame* over L or the *frame envelope* of L. Given $h : L \to M$, to a frame M, there is a unique frame map $\mathcal{H}h : \mathcal{H}L \to M$ that makes the diagram below commute.



Where the map $\downarrow : L \to \mathcal{H}L$ is the σ-frame homomorphism, $a \mapsto \downarrow a = \{x \in L \mid x \leq a\}$ i.e sending elements to their *principal down sets.*

The passage between Topological Spaces and Frames

The contravariant *open* functor $\mathcal{O} : \mathbf{Sp} \to \mathbf{Frm}$ ($\mathbf{Sp}$ denotes the category of topological spaces) associates with a topological space its topology, which is a frame of open sets, and the continuous functions $f : X \to Y$ mapped to the associated inverse image maps $f^{\leftarrow} : \mathcal{O}Y \to \mathcal{O}X$.

The contravariant *spectrum* functor $\mathrm{pt} : \mathbf{Frm} \to \mathbf{Sp}$ may be defined by letting $\mathrm{pt}L$ consist of all completely prime filters of L, (where $F \subseteq L$ is a filter if $a, b \in F$ implies $a \wedge b \in F$, and $a \leq x$ implies $x \in F$, and F is completely prime if for any $X \subseteq L$ with $\bigvee X \in F$ then $X \cap F \neq \emptyset$). The family $\{\mathrm{pt}_a \mid a \in L\}$, where $\mathrm{pt}_a = \{P \in \mathrm{pt}L \mid a \in P\}$, is the *spectral* topology. If $h : L \to M$ is a frame homomorphism then $\mathrm{pt}(h) : \mathrm{pt}M \to \mathrm{pt}L$ is given by $\mathrm{pt}(h)(F) = h^{\leftarrow}(F)$ for each $F \in \mathrm{pt}M$.

Given a topology X, the principal filters of $\mathrm{pt}\mathcal{O}(X)$ are the completely prime filters of open sets of the form $\{O \in \mathcal{O}(X) \mid x \in O\}$. $\mathcal{O}$ and pt are adjoint on the right and restrict to a dual equivalence of the subcategories of sober spaces (those spaces where the completely prime filters of open sets correspond bijectively to the principal filters of open sets) and spatial frames (topologies). (See Johnstone [17] for details of the above results.)

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The passage between σ-spaces and σ-frames

The adjoint situation between σ-spaces and σ-frames mimics that between topological spaces and frames.

Given a σ-frame L, we can define an associated σ-space, which we will call the σ-spectrum of L. A filter $P \subseteq L$ is σ-prime if for each countable $S \subseteq L$ with $\bigvee S \in P$ then $S \cap P \neq \emptyset$. Let ΨL be the collection of all σ-prime filters on L. We define the σ-open sets of ΨL to be of the form $\Psi_a = \{P \in \Psi L \,|\, a \in P\}$ for each $a \in L$. That this forms a σ-space is straightforward:

$\emptyset = \Psi_0$ and $\Psi L = \Psi_e$.
For any $a, b \in L$ and P a σ-prime filter on L, $P \in \Psi_a \cap \Psi_b \Leftrightarrow a \in P$ and $b \in P \Leftrightarrow a \wedge b \in P \Leftrightarrow P \in \Psi_{a\wedge b}$. Hence $\Psi_a \cap \Psi_b = \Psi_{a\wedge b}$. The second equivalence follows from P being closed upwards and closed under finite meets.
Finally, for countable $S \subseteq L$

$$\begin{aligned} P \in \textstyle\bigcup_{a\in S} \Psi_a &\Leftrightarrow P \in \Psi_a, \text{ some } a \in S \\ &\Leftrightarrow a \in P, \text{ some } a \in S \\ &\Leftrightarrow \textstyle\bigvee S \in P, \text{ since } P \text{ is } \sigma\text{-prime and upward closed} \\ &\Leftrightarrow P \in \Psi_{\bigvee S}. \end{aligned}$$

Further, if $h : L \to M$ is a σ-frame morphism then $\Psi h : \Psi M \to \Psi L$ is given by $\Psi h(P) = h^{\leftarrow}(P)$ for each $P \in \Psi M$. Ψh is a σ-continuous map:

Take any $b \in M$. Then

$$\begin{aligned} \Psi(h)^{\leftarrow}(\Psi_b) &= \{P \in \Psi L \,|\, \Psi(h)(P) \in \Psi_b\} \\ &= \{P \in \Psi L \,|\, b \in \Psi(h)(P)\} \\ &= \{P \in \Psi L \,|\, b \in h^{\leftarrow}(P)\} \\ &= \{P \in \Psi L \,|\, h(b) \in P\} \\ &= \Psi_{h(b)}. \end{aligned}$$

Given σ-frame maps h and g that can be composed to form $g \circ h$,
$\Psi(g \circ h) = (g \circ h)^{\leftarrow} = h^{\leftarrow} \circ g^{\leftarrow} = \Psi(h) \circ \Psi(g)$.

In this way we have the contravariant *spectrum* functor $\Psi : \sigma\mathbf{Frm} \to \sigma\mathbf{Sp}$.

On the other hand, given a σ-space X, it is trivial that ΣX is a σ-frame. Of course the countable joins and finite meets in ΣX are the countable unions finite intersections, respectively. Further more, σ-continuity of $f : X \to Y$, says precisely "$f^{\leftarrow} : \Sigma Y \to \Sigma X$ is a σ-frame map". Define $\Sigma(f) = f^{\leftarrow}$.

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Given σ-space maps f and g that can be composed to form $g \circ f$, then $\Sigma(g \circ h) = (g \circ h)^{\leftarrow} = h^{\leftarrow} \circ g^{\leftarrow} = \Sigma(h) \circ \Sigma(g)$.

Thus we have the contravariant σ-*open* functor
$\Sigma : \sigma\mathbf{Sp} \to \sigma\mathbf{Frm}$.

We now show that Ψ and Σ are adjoint on the right with the fixed subcategories consisting of the spatial σ-frames and the sober σ-spaces (later referred to as σ_{sober}-spaces).

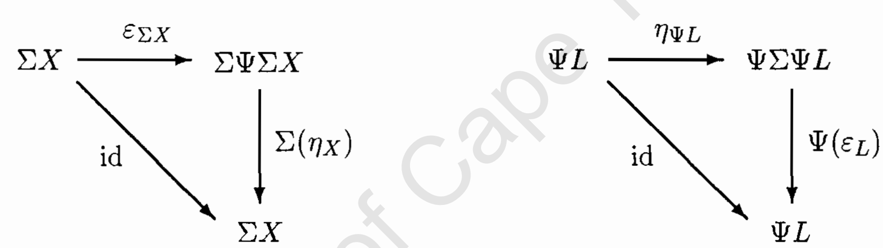
Take a σ-space X and a σ-frame L. Define maps

$$\eta_X : X \to \Psi\Sigma X \text{ by } x \mapsto \Sigma(x) = \{U \in \Sigma X \mid x \in U\}$$
$$\varepsilon_L : L \to \Sigma\Psi L \text{ by } a \mapsto \Psi_a.$$

That ε_L is a σ-frame homomorphism is a reformulation of the result above that the $\{\Psi_a \mid a \in L\}$ form the σ-open sets of the σ-space ΨL. That η_X is σ-continuous, however, is explained below:

Any open set of $\Psi\Sigma X$ is of the form $\Psi_U = \{\mathcal{P} \in \Psi\Sigma X \mid U \in \mathcal{P}\}$, where $U \in \Sigma X$. Now, $x \in \eta_X^{\leftarrow}(\Psi_U) \Leftrightarrow \eta_X(x) \in \Psi_U \Leftrightarrow U \in \eta_X(x) \Leftrightarrow x \in U$. That is, $\eta_X^{\leftarrow}(\Psi_U) = U$.

To show that η_X and ε_L are the unit and co-unit of the dual adjunction, we show that the following two diagrams commute.



For the first diagram, take any $U \in \Sigma X$. Then $\Sigma(\eta_X) \circ \varepsilon_{\Sigma X}(U) = \Sigma(\eta_X)(\Psi_U) = \eta_X^{\leftarrow}(\Psi_U) = U$. For the second, take any $P \in \Psi L$. Then

$$\begin{aligned}
\Psi(\varepsilon_L) \circ \eta_{\Psi L}(P) &= \Psi(\varepsilon_L)(\{\Psi_a \in \Sigma\Psi L \mid P \in \Psi_a\}) \\
&= \Psi(\varepsilon_L)(\{\Psi_a \in \Sigma\Psi L \mid a \in P\}) \\
&= \varepsilon_L^{\leftarrow}(\{\Psi_a \in \Sigma\Psi L \mid a \in P\}) \\
&= \{a \in P \mid a \in P\} \\
&= P.
\end{aligned}$$

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Definition 0.0.1 Given a σ-space X and a σ-frame L, X is called *sober* if η_X is an isomorphism and L is called *spatial* if ε_L is an isomorphism . Further more, $\Psi\Sigma X$ shall be called the *soberification* of X.

Remark 0.0.2 ε_L is always onto. $\eta_X$is one-one when X is a σ_0-space (see Chapter 2 under soberification).

Of course, the functors Ψ and Σ restrict to a dual equivalence between the sober σ-spaces and their σ-continuous maps, and the spatial σ-frames and their σ-frame homomorphisms.

The σ-spectrum of a σ-frame L can equivalently be described in terms of σ-frame homomorphisms to the two element σ-frame $\mathbf{2}$. Given a σ-prime filter $\mathcal{P} \subseteq L$ we can define a σ-frame homomorphism $h : L \to \mathbf{2}$ where $h(a) = 1$ iff $a \in \mathcal{P}$ and the σ-open sets Ψ_a are now defined as $\{h : L \to \mathbf{2} \mid h(a) = 1\}$. Conversely, given any σ-frame homomorphism $h : L \to \mathbf{2}$ we can define a σ-prime filter $\{a \in L \mid h(a) = 1\}$. The functor $\Psi : \sigma\mathbf{Frm} \to \sigma\mathbf{Sp}$ operates on σ-frame maps $g : L \to M$ as $\Psi(g) : \Psi M \to \Psi L$ where $\Psi(g)[h] = h \circ g$ for each $h \in \Psi M$.

This correspondence, between σ-prime filters on L and σ-frame homomorphisms from L to $\mathbf{2}$, is bijective, as can easily be checked. Furthermore, it is equally easy to check that the correspondence is an isomorphism of the two σ-spectrums. The σ-spectrums defined here are freely interchangeable and are equally the σ-spectrum of L.

Finally, it is worth mentioning that the unit of the adjunction, in this alternative description of the σ-spectrum, is now given by
$\eta_X : X \to \Psi\Sigma X = (x \mapsto h_x)$ where $h_x(U) = 1$ iff $x \in U$.

Anyone familiar with the study of regular σ-frames will note that the adjoint situation above is a generalization of that between the categories of regular σ-spaces ($\mathbf{Reg}\sigma\mathbf{Sp}$) and regular σ-frames ($\mathbf{Reg}\sigma\mathbf{Frm}$). See [12].

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Chapter 1

Cozero part of a σ-Frame

1.1 The Cozero part of a σ-Frame

We introduce the rather below relation, $\prec$, and later the completely below relation $\prec\!\prec$. For elements a and b of a bounded lattice L we say that $a \prec b$ if there exists a *separating* element $s \in L$ with $a \wedge s = 0$ and $s \vee b = e$.

The relation $\prec$ has the following properties

(1) If $x \leq a \prec b \leq y$ then $x \prec y$

(2) If $x \prec a$ and $y \prec b$ then $x \wedge y \prec a \wedge b$ and $x \vee y \prec a \vee b$.

Definition 1.1.1 We say a σ-frame L is *regular* if every element $a \in L$, is a join of (at most) countably many elements in L that are rather below a, i.e $a = \bigvee X$, for some countable $X \subseteq L$ with $x \prec a$ for all $x \in X$.

Examples (1) The cozero part of a completely regular frame L is a regular sub σ-frame (in fact the largest regular sub σ-frame). Moreover, a frame is completely regular if and only if it is generated by its cozero part [4].

(2) An Alexandroff space [14] is a σ-space $(X, \Sigma X)$, where X is a set with $\Sigma X \subseteq \mathcal{P}X$, a regular sub σ-frame of $\mathcal{P}X$. The members of ΣX are precisely the cozero sets of X (see [11]).

Definition 1.1.2 A lattice L is said to be normal if for any $a, b \in L$, $a \vee b = e$ implies the existence of $u, v \in L$ with $a \vee u = e = v \vee b$ and $u \wedge v = 0$.

It follows quickly that a lattice is normal if whenever $a, b \in L$ with $a \vee b = e$, there exist $u, v \in L$ with $v \prec a, u \prec b$ and $u \vee v = e$ i.e. the pair cover $\{a, b\}$ is "shrinkable".

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The proof of the following lemma can be found in [11] and was first observed by B.Banaschewski.

Lemma 1.1.3 Every regular σ-frame is normal

Proof: Take any $a, b \in L$. We now proceed to find $u, v \in L$ with $a \vee u = a \vee b = v \vee b$ and $u \wedge v = 0$:

By the regularity of L, we take $a = \bigvee_n a_n$ $(a_n \prec a)$ and $b = \bigvee_n b_n$ $(b_n \prec b)$. By the property of $\prec$ above, we can assume that $a_n \leq a_{n+1}$ and $b_n \leq b_{n+1}$. Choose u_n and v_n such that $a_n \wedge u_n = 0 = b_n \wedge v_n$ and $a \vee u_n = e = v_n \vee b$. Let $u = \bigvee_n u_n \wedge b_n$ and $v = \bigvee_n v_n \wedge a_n$. Then

$$\begin{aligned} a \vee u &= a \vee \bigvee u_n \wedge b_n \\ &= \bigvee (a \vee u_n) \wedge (a \vee b_n) \\ &= \bigvee e \wedge (a \vee b_n) \\ &= a \vee \bigvee b_n \\ &= a \vee b. \end{aligned}$$

Similarly $v \vee b = a \vee b$
If $k \leq n$ then $a_k \wedge u_n \leq a_n \wedge u_n = 0$
If $k \geq n$ then $b_n \wedge v_k \leq b_k \wedge v_k = 0$.
Thus $u \wedge v = 0$.

Now, with the further assumption that $a \vee b = e$, L is normal. □

It is worth noting that the above proof, in choosing the u_n and the v_n, uses the principle of the Axiom of Countable Choice.

We say that a relation R of L interpolates if, for any x and y in L, whenever $x\,R\,y$ there exists $z \in L$ with $x\,R\,z$ and $z\,R\,y$.

Corollary 1.1.4 In any regular σ-frame L, the relation $\prec$ interpolates

Proof: If $a \prec b$ in L then there exists $s \in L$ with $s \wedge a = 0$ and $s \vee b = e$. By normality of L there exist $u, v \in L$ with $s \vee u = e = v \vee b$ and $u \wedge v = 0$. Thus $a \prec u \prec b$. □

For any a and b in L, $a \prec\!\prec b$ (a is *completely below* b) when there exists an *interpolating sequence* $(c_{nk})_{n=0,1,2,\ldots;\,k=0,1,2,\ldots,2^n}$ between a and b, where $c_{00} = a$ and $c_{01} = b$, with $c_{n\,k} = c_{n+1\;2k}$ and $c_{n\,k} \prec c_{n+1\;2k+1} \prec c_{n\;k+1}$.

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Remark 1.1.5 In a regular σ-frame, $\prec = \prec\!\prec$: Given that $\prec$ interpolates in a regular σ-frame, by successive interpolations, and the principle of Countable Dependent Choice (CDC), we can generate an interpolating sequence between a and b.

Now we introduce $\mathcal{L}(\mathbb{R})$, the localic reals, a regular σ-frame. The $\mathcal{L}(\mathbb{R})$ are generated by ordered pairs of rationals (p, q) subject to the relations,

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$
(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,
(R3) $(p,q) = \bigvee\{(r,s) \mid p < r < s < q\}$
(R4) $e = \bigvee\{(p,q) \mid p, q \in \mathbf{Q}\}$.

That is to say, given the ordered pairs, we take the free σ-frame over them and then quotient out by the relations (R1)-(R4). As is the usual case when working with quotients, in constructing a σ-frame homomorphism from the localic reals to a σ-frame, we first construct a set map from $\mathbb{Q} \times \mathbb{Q}$ to the σ-frame (which guarantee a unique σ-frame map from the free σ-frame) and show that this map sends the relations (R1)-(R4) to equalities, which gives a corresponding σ-frame map from the localic reals to the σ-frame, by the universality of quotients. It is because of this ease in constructing σ-frames maps from $\mathcal{L}(\mathbb{R})$, that we opt to work with the $\mathcal{L}(\mathbb{R})$, as opposed to the σ-frame of open sets of $\mathbb{R}$, which is isomorphic to $\mathcal{L}(\mathbb{R})$ [6].

In [3] $\mathcal{L}(\mathbb{R})$ denotes the frame of localic reals. Our construction is of course identical to that of [3] and we will not differentiate between $\mathcal{L}(\mathbb{R})$ as frame or σ-frame.

In $\mathcal{L}(\mathbb{R})$, it is clear that for rationals $p < r < s < q$, $(r,s) \prec (p,q)$ with separating elements $(-,u) \vee (v,-)$ for any u, v where $p < u < r$ and $s < v < q$ with $(-,u) = \bigvee\{(t,u) \mid t < u\}$ and $(v,-) = \bigvee\{(v,w) \mid v < w\}$. In light of this fact, (R3) and (R4), and the fact the all joins need only be countable, it follows that $\mathcal{L}(\mathbb{R})$ is a regular σ-frame.

Definition 1.1.6 A *pre-trail* is a map $t : \mathbb{Q} \to L$, L a σ-frame, together with a second map (the *witness*) $w : \{(r,s) \in \mathbb{Q} \times \mathbb{Q} \mid r < s\} \to L$ where $r < s$ implies that

$$t(r) \wedge w(r,s) = 0 \qquad \text{and} \qquad w(r,s) \vee t(s) = e.$$

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Analogously a *descending pre-trail* is a map $t : \mathbb{Q} \to L$ together with a second map (also called a *witness*) $w : \{(r,s) \in \mathbb{Q} \times \mathbb{Q} \mid r < s\} \to L$ where $r < s$ implies that

$$t(s) \wedge w(r,s) = 0 \qquad \text{and} \qquad w(r,s) \vee t(r) = e.$$

Finally, for any $r, s \in \mathbb{Q}$ define

$$\begin{aligned} t(r)^{\square} &= \bigvee\{w(r,s) \mid r < s,\ s \in \mathbb{Q}\}, && \text{for a pre-trail t and} \\ t(s)^{\square} &= \bigvee\{w(r,s) \mid r < s,\ r \in \mathbb{Q}\}, && \text{for a descending pre-trail t.} \end{aligned}$$

Remark 1.1.7 Given a pre-trail t and $p < q$ in $\mathbb{Q}$ then

(1) $t(p) \prec t(q)$.

(2) $t(q) \wedge t(q)^{\square} = 0$ and $t(q) \vee t(p)^{\square} = e$.

(3) $t(q)^{\square} \prec t(p)^{\square}$.

Proof: The statement (1) is immediate from the definition of a pre-trail. For (2):

$$\begin{aligned} t(q) \wedge t(q)^{\square} &= t(q) \wedge \bigvee\{w(q,r) \mid q < r,\ r \in \mathbb{Q}\} \\ &= \bigvee\{t(q) \wedge w(q,r) \mid q < r,\ r \in \mathbb{Q}\} \quad \sigma\text{-frame distribution law} \\ &= \bigvee\{0\} \\ &= 0, \end{aligned}$$

since $t(q) \wedge w(q,r) = 0$, for each $q < r$.
For any $p < q$, $p, q \in \mathbb{Q}$

$$\begin{aligned} t(q) \vee t(p)^{\square} &= t(q) \vee \bigvee\{w(p,r) \mid p < r,\ r \in \mathbb{Q}\} \\ &= \bigvee\{t(q) \vee w(p,r) \mid p < r,\ r \in \mathbb{Q}\} \\ &= e. \end{aligned}$$

because when $r = q$ then $t(q) \vee w(p,r) = e$.

(3) is immediate because, by (2), $t(q)$ is a separating element of the relation $t(q)^{\square} \prec t(p)^{\square}$.

Remark 1.1.8 From Remark 1.1.7, in the pre-trail t, $t(q)^{\square}$ plays a similar role to that of the 'pseudocomplement' in frames. In general, the pseudocomplement of an element in σ-frames need not exist. Indeed, the need to formally introduce the notion of witness in the definition of a pre-trail is motivated by this fact. In practice the witness will arise naturally in the construction of a trail, no doubt with the application of CDC (see in particular Proposition 1.1.13).

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Definition 1.1.9 A trail (or descending trail) in L is a pre-trail (or descending pre-trail) with the additional condition that

$$\bigvee\{t(r) | r \in \mathbb{Q}\} = e = \bigvee\{t(r)^{\square} | r \in \mathbb{Q}\}$$

Remark 1.1.10 If t is a trail then $t^{\square}$ $(q \mapsto t(q)^{\square})$ is a descending trail, with the same witness.

Lemma 1.1.11 For any trail t in L,

$$\varphi(p,q) = \bigvee\{t(p')^{\square} \wedge t(q') \,|\, p < p' < q' < q\}$$

defines a σ-frame homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$.
For any descending trail t in L,

$$\Psi(p,q) = \bigvee\{t(p') \wedge t(q')^{\square} \,|\, p < p' < q' < q\}$$

defines a σ-frame homomorphism $\Psi : \mathcal{L}(\mathbb{R}) \to L$

Proof: To see that φ indeed determines a σ-frame homomorphism $\varphi : \mathcal{L}(\mathbb{R}) \to L$, it has to be checked that the relations (R1) - (R4), in the definition of $\mathcal{L}(\mathbb{R})$, are transformed into identities in L.

(R1):

$$\varphi(p,q)\wedge\varphi(r,s) = \bigvee\{t(p')^{\square}\wedge t(q')\wedge t(r')^{\square}\wedge t(s') \,|\, p < p' < q' < q, r < r' < s' < s\}$$

and $t(p') \vee t(r') = t(p' \vee r')$, since the $t(r)$ form a trail. Also $t(q')^{\square} \wedge t(s')^{\square} = t(q' \vee s')^{\square}$ since the $t(r)^{\square}$ form a descending trail. But then the right hand side of the above is the desired $\varphi(p \vee r, q \wedge s)$.

(R2): Given $p \leq s < q \leq r$, it is obvious that $\varphi(p,q) \vee \varphi(s,r) \leq \varphi(p,r)$ because

$$\varphi(p,q) \leq \varphi(p,r) \qquad \text{and} \qquad \varphi(s,r) \leq \varphi(p,r)$$

from the definition of φ.
For the reverse inequality, note first that for p' and r' such that $p < p' < r' < r$

$$(*) \qquad t(p')^{\square} \wedge t(r') \leq \varphi(p,q) \vee \varphi(s,r)$$

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whenever $s < p'$ or $r' < q$.
Check :

$$\begin{array}{ll} s < p' & \Rightarrow t(p')^{\square} \wedge t(r') \leq \varphi(s, r) \\ r' < q & \Rightarrow t(p')^{\square} \wedge t(r') \leq \varphi(p, q) \end{array}$$

Hence it remains to consider the case when $p' \leq s$ and $q \leq r'$. For this, pick u and v such that $s < u < v < q$ ($s < q$ by assumption) and compute

$$\begin{aligned} &\varphi(p, q) \vee \varphi(s, r) \geq (t(p')^{\square} \wedge t(v)) \vee (t(u)^{\square} \wedge t(r')) \\ &= (t(p')^{\square} \vee t(u)^{\square}) \wedge (t(v) \vee t(u)^{\square}) \wedge (t(p')^{\square} \vee t(r')) \wedge (t(v) \vee t(r')) \\ &= t(p')^{\square} \wedge e \wedge e \wedge t(r') \\ &= t(p')^{\square} \wedge t(r'). \end{aligned}$$

The second line follows from the third since

$$\begin{array}{ll} p' < u & \Rightarrow t(u)^{\square} \prec t(p')^{\square} \Rightarrow t(p')^{\square} \vee t(u)^{\square} = t(p')^{\square} \\ u < v & \Rightarrow t(u) \prec t(v) \Rightarrow t(v) \vee t(u)^{\square} = e, \\ p' < r' & \Rightarrow t(p') \prec t(r') \Rightarrow t(p')^{\square} \vee t(r') = e, \\ v < r' & \Rightarrow t(v) \prec t(r') \Rightarrow t(v) \vee t(r') = t(r'). \end{array}$$

This proves the statement $(*)$ in general. So finally

$$\varphi(p, q) \vee \varphi(s, r) = \varphi(p, r) \text{ whenever } p \leq s < q \leq r$$

(R3):

$$\begin{aligned} \bigvee\{\varphi(r, s) \mid p < r < s < q\} &= \bigvee\{t(r')^{\square} \wedge t(s') \mid p < r < r' < s' < s < q\} \\ &= \varphi(p, q) \end{aligned}$$

(R4): From the fact that $\varphi(p, q) = 0 = t(p)^{\square} \wedge t(q)$, when $q < p$, and a property of trails

$$\begin{aligned} \bigvee\{\varphi(p, q) \mid p, q \in \mathbb{Q}\} &= \bigvee\{t(p)^{\square} \wedge t(q) \mid p, q \in \mathbb{Q}\} \\ &= \bigvee\{t(p)^{\square} \mid p \in \mathbb{Q}\} \wedge \bigvee\{t(q) \mid q \in \mathbb{Q}\} \\ &= e \wedge e = e. \end{aligned}$$

This proves the fourth identity.

Since $\varphi|_{\mathbb{Q}\times\mathbb{Q}}$ satisfies (i) - (iv), then $\varphi : \mathcal{L}(\mathbb{R}) \to L$ must be a σ-frame homomorphism. That Ψ gives a σ-frame homomorphism as well has a similar proof. $\square$

We define now what we mean by a cozero element of a σ-frame. For σ-frame map $\varphi : \mathcal{L}(\mathbb{R}) \to L$ we say that

$$coz(\varphi) = \varphi(-, 0) \vee \varphi(0, -).$$

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Definition 1.1.12 For L a σ-frame (or frame), we say that $a \in L$ is a cozero element if for some $\varphi : \mathcal{L}(\mathbb{R}) \to L$, $a = coz(\varphi)$.

We put $\mathrm{Coz}L = \{coz(\varphi) \,|\, \varphi : \mathcal{L}(\mathbb{R}) \to L\}$.

Proposition 1.1.13 For any σ-frame L, the following are equivalent for $a \in L$:

(1) $a \in$ CozL.
(2) $a = \bigvee x_n$ where $x_n \prec\!\prec a$, for all $n = 1, 2 \ldots.$
(3) $a = \bigvee a_n$ where $a_n \prec\!\prec a_{n+1}$, for all $n = 1, 2, \ldots.$

Proof: (1) $\Rightarrow$ (2). Any element of $\mathcal{L}(\mathbb{R})$ is a join of countably many elements completely below it, and any $\varphi : \mathcal{L}(\mathbb{R}) \to L$ preserves that fact.
(2) $\Rightarrow$ (3). Given $a = \bigvee x_n$ as stated, define a_n inductively by

$$a_0 = x_0,\; a_n \vee x_{n+1} \prec\!\prec a_{n+1} \prec\!\prec a,$$

using the fact that $\prec\!\prec$ interpolates and is stable under binary joins.
(3) $\Rightarrow$ (1). For each n, let $(c^{(n)}_{kl})_{k=0,1,\ldots;\, l=0,\ldots,2^k}$ be a sequence that allows us to conclude that $a_n \prec\!\prec a_{n+1}$ and define $t : \mathbb{Q} \to L$ by

$$t(r) = \left\{ \begin{array}{ll} 0 & (r > 1) \\ \bigvee\{c^{(n)}_{kl} \mid \frac{l}{2^k} \leq n(n+1)\left(\frac{1}{n} - r\right)\} & \left(\frac{1}{n} \geq r > \frac{1}{n+1}\right) \\ e & (r \leq 0) \end{array} \right\}.$$

We claim that this defines a descending trail (Definition 1.1.9), that is $t(r) \prec t(s)$ whenever $s < r$, with an associated witness.

Trivially, $t(r) \prec t(s)$ when $r > 1$ or $s \leq 0$. If, on the other hand, $r \leq 1$ and $0 < s$, take n such that $\frac{1}{n+1} < r \leq \frac{1}{n}$. Then $0 \leq n(n+1)\left(\frac{1}{n} - r\right) < 1$ and hence $a_n \leq t(r) \prec a_{n+1}$. Now, if $s \leq \frac{1}{n+1}$ then also $a_m \leq t(s)$ for all $m > n$ and consequently $t(r) \prec a_{n+1} \leq t(s)$. Otherwise, if $\frac{1}{n+1} < s$ then both $t(r)$ and $t(s)$ are determined by the sequence $c^{(n)}_{kl}$, as given in the definition of t above, and since there exist k and l such that

$$n(n+1)\left(\frac{1}{n} - r\right) \leq \frac{l}{2^k},\; \frac{l+1}{2^k} \leq n(n+1)\left(\frac{1}{n} - s\right).$$

For such k and l, it follows that

$$t(r) \leq c^{(n)}_{kl} \prec c^{(n)}_{k\,l+1} \leq t(s).$$

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For each $s < r$ is $\mathbb{Q}$, choose $w(s,r) \in L$ to be an element separating $t(r) \prec t(s)$. Thus, we have chosen the associated witness to t, making t a descending pre-trail. However, by its definition, it is trivially a descending trail too.

Now, let $\Psi : \mathcal{L}(\mathbb{R}) \to L$ be the homomorphism provided by Lemma 1.1.11 such that

$$\Psi(p,q) = \bigvee\{t(p') \wedge t(q')^{\square} \mid p < p' < q' < q\};$$

we claim that $coz(\Psi) = a$. Since $t(r)^{\square} = 0$ for $r \leq 0$ we have $\Psi(-,0) = 0$ and hence

$$\begin{aligned} coz(\Psi) = \Psi(0,-) &= \bigvee\{t(p) \wedge t(q)^{\square} \mid 0 < p < q\} \\ &= \bigvee\{t(p) \mid 0 < p \leq 1\} = a, \end{aligned}$$

the final step because all a_n are of the form $t(p)$, for some $0 < p \leq 1$. □.

As pointed out earlier, in a regular σ-frame, $\prec = \prec\!\prec$, via some choice principle, namely that of Countable Dependent Choice. With such choice, the following corollaries become trivial.

Corollary 1.1.14 Coz$L$ is the largest regular sub-σ-frame of L and, in particular, a σ-frame L is regular if and only if Coz$L = L$

Remark 1.1.15 Corollary 1.1.14 is the point-free analogue of the result of Gordon that justifies the use of the word "zero set' in his definition of a zero-set space. (Theorem 3.5 [14])

We denote the full sub-category (of σ**Frm**) of regular σ-frames by **Reg**σ**Frm**, which has as objects regular σ-frames and all σ-frame homomorphisms as the maps.

Proposition 1.1.16 **Reg**σ**Frm** is a full co-reflective sub-category of σ**Frm**, the co-reflective map given by inclusion.

Proof: For any σ-frame L, Coz$L \subseteq L$ is the largest regular sub-σ-frame of L, by Corollary 1.1.14 above.
So let i : Coz$L \to L$, be the usual inclusion map.
Take any regular σ-frame M with $h : M \to L$ a σ-frame map.
The image Im(h) of M is regular, since $a \prec b$ in M implies that $h(a) \prec h(b)$ in Im(h), hence Im(h) $\subseteq$ Coz$L$ and hence h factors through Coz$L$ via $\overline{h}$, where $\overline{h}(a) = h(a)$.

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Uniqueness: Suppose there exists $f : M \to L$ with $h = i \circ f$. Take any $m \in M$, then

$$f(m) = i \circ f(m) = h(m) = \overline{h}(m).$$

Hence $f = \overline{h}$. □

Remark 1.1.17 **RegσFrm** is closed under coproducts

1.2 The Stone-Čech Compactification

We construct the Stone-Čech compactification for σ-frames following [11].

An ideal J is said to be countably generated if there exists an at most countable subset A of J such that for all $x \in J$ there exist finitely many $a_1, \dots a_n$ in A with $x \leq a_1 \vee \dots \vee a_n$.

Given a σ-frame L, we denote by $\mathcal{J}L$ the σ-frame of all countably generated ideals on L, with the bottom given by $\{0\}$, top by L (with countable generating set $\{e\} \subseteq L$), meet by intersection and for joins: given the ideals $J_n \in \mathcal{J}L$, with corresponding generating subsets A_n, $\bigvee_n J_n$ is the ideal generated by $\bigcup_n A_n$.

$\mathcal{J}L$ is compact: If $L = \bigvee_n J_n$ with the ideals J_n generated by the corresponding countable subsets A_n, then $e = a_1 \vee \dots \vee a_k$ for some $a_i \in A_{n_i}$. Then $L = \bigvee_{i=1,\dots,k} J_{n_i}$.

We define, following Gilmour in [11] attributed to B. Banaschewski, the Stone-Čech compactification of a σ-frame L, denoted by KL, as the largest regular sub-σ-frame of $\mathcal{J}L$. In light of Corollary 1.1.13 above, KL = Coz$\mathcal{J}L$.

Lemma 1.2.1 If M is a compact σ-frame and $x \prec \bigvee_{n\in\mathbb{N}} m_n$ in M, then $x \prec m_{n_1} \vee \dots \vee m_{n_l}$, for finitely many $n_1, \dots, n_l$.

Proof: $x \prec m$ implies that there exists $s \in M$ with $x \wedge s = 0$ and $s \vee m = e$. Since M is compact then there exist finitely many $m_{n_1}, \dots, m_{n_l}$ with $s \vee m_{n_1} \vee \dots \vee m_{n_l} = e$. Therefore $x \prec m_{n_1} \vee \dots \vee m_{n_l}$. □

Let $r(x) = \{a \mid a \prec x\}$.

Lemma 1.2.2 If M is a compact regular σ-frame then the correspondence $r : M \to \mathcal{J}M$ is a homomorphism

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Proof: It is clear that $r(e) = L$ and $r(0) = \{0\}$ and by (2), seen at the beginning of section 1.1, $r(x \wedge y) = r(x) \cap r(y)$ allowing us to see that $r(m)$ is an ideal, and further that $r(m) \in \mathcal{J}M$ for each $m \in M$ since M is a regular σ-frame: for any $m \in M$, $m = \bigvee_n m_n$ ($m_n \prec m$), by regularity. Then by the lemma above, for any $x \in r(m)$, $x \leq m_{n_1} \vee \cdots \vee m_{n_l}$, thus $r(m)$ is countably generated.

Thus it only remains to show that r preserves countable joins.

Clearly $\bigvee_n r(x_n) \subseteq r(\bigvee_n x_n)$. For the converse, first we consider the finite case. If $z \in r(a \vee b)$ then $z \prec \bigvee_{n,m} a_n \vee b_n$, where $a = \bigvee_n a_n$ and $b = \bigvee_m b_m$, $a_n \prec a$ and $b_n \prec b$. Hence by Lemma 1.2.1, $z \leq a_{n_1} \vee \cdots \vee a_{n_l} \vee b_{m_1} \vee \cdots \vee b_{m_k}$, and so by the definition of the join of two ideals, $z \in r(a) \vee r(b)$. Thus $r(a \vee b) = r(a) \vee r(b)$.

If $w \in r(\bigvee_n x_n)$, then $w \prec \bigvee_n x_n$ and so by Lemma 1.2.1, $w \prec x_{n_1} \vee \cdots \vee x_{n_k}$, and thus $w \in r(x_{n_1}) \vee \cdots \vee r(x_{n_k})$. Consequently $w \in \bigvee_n r(x_n)$. □

Corollary 1.2.3 If M is a compact regular σ-frame, then $r : M \to \mathrm{K}M$ is an isomorphism.

Proof: Define $k_L : \mathrm{K}L \to L$ as follows: When $J \in \mathrm{K}L$ is countably generated by A, then $k_L(J) = \bigvee A$. Clearly k_L is a homomorphism.

In this case, M is compact and regular and so r is a homomorphism and for each $m \in M$, $\bigvee r(m) = m$. Thus $k_L \circ r = \mathrm{id}$.

It remains to show that $r \circ k_L(J) = J$ for each $J \in \mathrm{K}M$. To this end, assume that A countably generates J.
If $x \in r(k_L(J))$ then $x \prec \bigvee A$ and hence $x \leq a_1 \vee \cdots \vee a_k$ where $a_i \in A \subseteq J$, by Lemma 1.2.1. Thus $x \in J$ and $r(k_L(J)) \subseteq J$.
Now $J = \bigvee K_n$ ($K_n \prec J$), by the observation that $\mathrm{K}M = \mathrm{Coz}\mathcal{J}M$. Take $x \in J$, then $x \leq l_1 \vee \cdots \vee l_m$ where the l_i are generating elements of the K_n. We show that $l_i \prec \bigvee A$ for each i:

If $l_i \in K_p$ say, then there exists $H_p \in \mathcal{J}M$ where

$$K_p \cap H_p = \{0\}, \qquad H_p \vee J = L.$$

Then $e = (a_1 \vee \cdots \vee a_q) \vee h$, where $a_i \in A$ and $h \in H_p$. But $h \wedge l_i = 0$. Thus

$$l_i \prec a_1 \vee \cdots \vee a_q \leq \bigvee A$$

Thus $x \leq l_1 \vee \cdots \vee l_m \prec \bigvee A$ and so finally $J \subseteq r(k_L(J))$. □

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We denote the category of all compact regular σ-frames by **KRegσFrm**. The correspondence σ**Frm** $\to$ **KRegσFrm**, given by K, is functorial: If $h : M \to L$ is a homomorphism in σ**Frm** and $J \in \mathrm{K}M$ is countably generated by A, then $Kh(J)$ is the ideal countably generated by $h(A)$, in L. In the following theorem, the map k_L is as defined in Corollary 1.2.3 above.

Theorem 1.2.4 **KRegσFrm** is a full subcategory of σ**Frm**, with co-reflective map $k_L : \mathrm{K}L \to L$.

Proof: Let $h : M \to L$, with M compact and regular, be a given homomorphism.

$$\begin{array}{ccc} \mathrm{K}M & \overset{k_M}{\underset{\sim}{\longleftrightarrow}} & M \\ \Big\downarrow \mathrm{K}h & & \Big\downarrow h \\ \mathrm{K}L & \overset{k_L}{\longrightarrow} & L \end{array}$$

If $m \in M$ then $m = \bigvee m_n$, $(m_n \prec m)$ and $A = (m_n)_{n \in \mathbb{N}}$ generates $r(m)$. Then $k_L \circ Kh(r(m)) = \bigvee h(A) = h(\bigvee A) = h(m)$.
Uniqueness: Let $f, g : M \to \mathrm{K}L$ with $k_L \circ f = k_L \circ g = h$. Then $f(m) = \bigvee f(m_n)$ with $f(m_n) \prec f(m)$, since f preserves $\prec$.
There exists $H_n \in \mathrm{K}L$ with

$$H_n \cap f(m_n) = \{0\} \text{ and } H_n \vee f(m) = \mathrm{K}L.$$

Then there exists $h \in H_n$ and $a_i \in A$, where A countably generates $f(m)$ such that $(a_1 \vee \cdots \vee a_l) \vee h = e$. Set $a = a_1 \vee \cdots \vee a_l$. Then for each $z \in f(m_n)$, $z \leq a$. Thus

$$\bigvee f(m_n) \leq a \in f(m) \text{ i.e. } k_L f(m_n) \in f(m).$$

But $k_L f(m_n) = k_L g(m_n)$, and $\bigvee g(m_n) \in f(m)$ for each n. It follows that $g(m_n) \subseteq f(m)$, for each n, hence $\bigvee g(m_n) \subseteq f(m)$ i.e $g(m) \subseteq f(m)$.
Similarly $f(m) \subseteq g(m)$. $\square$

Definition 1.2.5 In a lattice, an ideal J is said to be regular if $a \in J \Rightarrow \exists b \in J$ with $a \prec b$.

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Proposition 1.2.6 For regular σ-frame L, $\mathcal{K}L = \{J \in \mathcal{J}L \mid J \text{ regular}\}$

Proof: ($\subseteq$): Take $J \in \mathcal{K}L$. Then $J = \bigvee_n K_n$, $(K_n \prec J)$, $K_n \in \mathcal{J}L$, $\forall n \in \mathbb{N}$, since $\mathcal{K}L$ is regular. Also, assume that A countably generates J.
For any $a \in J$, $a \leq l_1 \vee \cdots \vee l_m$, where the l_i are generating elements of the respective K_{n_i}. From the proof of Corollary 1.2.3, $l_i \prec a_1 \vee \cdots \vee a_p$, some p where $a_i \in A$. Thus $a \prec a_1 \vee \cdots \vee a_p \in J$.

($\supseteq$): Now take a regular ideal $J \in \mathcal{J}L$ and let $\{a_n\}$ countably generate J. By the regularity of J, we can assume that $a_n \prec a_{n+1}$. Because $\prec$ interpolates in L, for each n, we generate a sequence $\{c_k^n\}$ with $a_n \prec c_1^n \prec c_2^n \prec \cdots \prec a_{n+1}$. Put

$$J_n = \{x \in L \mid x \leq c_m^n, \text{ some } m\}.$$

Then J_n is a regular countably generated ideal. To show that $J_n \prec J_{n+1}$ we proceed as follows.
For each n, interpolate $c_1^{n+1} \prec c_2^{n+1}$, by CDC, to get $c_1^{n+1} \prec \cdots \prec b_2 \prec b_1 \prec c_2^{n+1}$. Then choose separating elements $\{d_k\}$, where for each k,

$$d_k \wedge b_{k+1} = 0 \text{ and } d_k \vee b_k = e.$$

Then $d_1 \prec d_2 \prec d_3 \prec \ldots$.
Put $H = \{x \in L \,|\, x \leq d_m, \text{ some } m\}$. It is clear that H is a countably generated regular ideal.

$H \cap \downarrow c_1^{n+1} = \{0\}$: for any $x \in H$, $x \leq d_k$, some k. And so $0 = d_k \wedge b_{k+1} \geq x \wedge c_1^{n+1}$ (since $c_1^{n+1} \leq b_{k+1}$).

$H \vee \downarrow c_2^{n+1} = L$: For any k, $e = d_k \vee b_k \leq d_k \vee c_2^{n+1}$ (since $b_k \leq c_2^{n+1}$).

So $H \cap J_n = \{0\}$ and $H \vee J_{n+1} = L$, because $J_n \subseteq \downarrow c_1^{n+1}$ and $\downarrow c_2^{n+1} \subseteq J_{n+1}$, making it clear that $J_n \prec J_{n+1}$.

$J = \bigvee_n J_n$ since for any $x \in J$, $x \leq a_n \prec c_1^k \Rightarrow x \in J_n$, for some n.
Then by the characterization of Proposition 1.1.13, $J \in \mathrm{Coz}\mathcal{J}L = \mathcal{K}L$. □

The above characterization of $\mathcal{K}L$ is used in the proof of Proposition 3.2.13 and it is this description that is used in [6] to define $\mathcal{K}L$.

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1.3 Pseudocompact σ-Frames

For any σ-frame L, $\varphi : \mathcal{L}(\mathbb{R}) \to L$ is *bounded* if $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$, and L is called *pseudocompact* if all σ-frame maps $\varphi : \mathcal{L}(\mathbb{R}) \to L$ are bounded.

This definition is a direct translation of the definition of pseudocompactness for frames. A result of this, since frames are also σ-frames, is that a frame is pseudocompact as a frame if and only if it is pseudocompact as a σ-frame.

In the case of frames, the one-one map

$$\mathbf{Frm}(\mathcal{L}(\mathbb{R}), \mathcal{O}\mathbf{X}) \to \mathbf{Top}(\mathbf{X}, \mathbb{R})$$

taking each φ to $\widetilde{\varphi}$ such that

$$p < \widetilde{\varphi}(x) < q \text{ iff } x \in \varphi(p,q),$$

expressing $\mathbb{R}$ in terms of Dedekind cuts, shows that φ is bounded (some $\varphi(p,q) = X$) iff $\widetilde{\varphi}$ is bounded (for some p, q, $p < \widetilde{\varphi} < q$ for all x), and hence *a space X is pseudocompact iff the frame $\mathcal{O}X$ is pseudocompact.* In exactly the same fashion a σ-space Y is pseudocompact if and only if its σ-frame of σ-open sets, ΣY, is pseudocompact (for a discussion of σ-spaces, see the following chapter).

We now characterize pseudocompact σ-frames in a variety of ways. The following proof follows closely the lines of the proof of Proposition 2 in [4], which is the corresponding statement for frames. The necessary difference here is that a pseudocomplement is not always available in σ-frames (as it always is in frames), hence the use of descending trails and their corresponding witnesses. The result here however, implies the corresponding result for frames.

Proposition 1.3.1 For any σ-frame L, the following are equivalent:

(1) L is pseudocompact.
(2) Any sequence $a_0 \prec\!\!\prec a_1 \prec\!\!\prec a_2 \prec\!\!\prec \dots$ such that $\bigvee a_n = e$ in L terminates, that is, $a_k = e$ for some k.
(3) The σ-frame Coz$L$ is compact.
(4) The frame $\mathcal{H}$Coz$L$ is compact.

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Proof: (1) $\Rightarrow$ (2) : Given that $a_0 \prec\!\prec a_1 \prec\!\prec a_2 \prec\!\prec \dots$, for each n, let $(c_{kl}^{(n)})_{k=0,1,\dots;\, l=0,\dots,2^k}$ be an interpolating sequence of $a_n \prec\!\prec a_{n+1}$ and define the descending trail $t : \mathbb{Q} \to L$ to be

$$t(r) = \left\{ \begin{array}{ll} 0 & (r > 1) \\ \bigvee\{c_{kl}^{(n)} \mid \frac{l}{2^k} \leq n(n+1)\left(\frac{1}{n} - r\right)\} & \left(\frac{1}{n} \geq r > \frac{1}{n+1}\right) \\ e & (r \leq 0) \end{array} \right\}.$$

as in (3) $\Rightarrow$ (1) of Proposition 1.1.13.
Now let $\varphi : \mathcal{L}(\mathbb{R}) \to L$, be as provided by Lemma 1.1.11 i.e

$$\varphi(p,q) = \bigvee\{t(p') \wedge t(q')^{\square} \mid p < p' < q' < q\}.$$

By pseudocompactness, φ is bounded i.e $\varphi(p,q) = e$ for some $p, q \in \mathbb{Q}$. We can assume that $q > 0$. For any $k - 1 \geq \frac{1}{q}$, $q > \frac{1}{k}$ which implies that $t(q) \prec\ t\left(\frac{1}{k}\right) = a_k$. Then

$$\begin{array}{ll} a_k & \geq t(q) \\ & \geq \bigvee\{t(q') \mid p < q' < q\} \\ & \geq \bigvee\{t(p') \wedge t(q')^{\square} \mid p < p' < q' < q\} = \varphi(p,q), \end{array}$$

and hence $a_k = e$ as desired.

(2) $\Rightarrow$ (3) : If $\bigvee a_n = e$ in Coz$L$ and $a_n = \bigvee a_{nk}$ where $a_{nk} \prec\!\prec a_{nk+1}$ by Proposition 1.1.13, put $c_n = a_{1n} \vee a_{2n} \vee \dots \vee a_{nn}$. Then $c_n \prec\!\prec c_{n+1}$ and $\bigvee c_n = e$, hence $c_k = e$ some k, and consequently also $a_1 \vee \dots \vee a_k = e$, proving compactness.

(3) $\Rightarrow$ (4) : Suppose a σ-frame M is compact, and take $J_s \in \mathcal{H}M$ with $\bigvee_{s \in S} J_n = M$, some index set S. Then there exists countable $A \subseteq \bigcup_s J_s$ with $\bigvee A = e$. By compactness of M there exist $a_i \in A$ with $a_1 \vee \dots \vee a_l = e$, for some $l \in \mathbb{N}$, and $a_i \in J_{s_i}$. But then $\bigvee_i J_{s_i} = M$. Hence if M is a compact σ-frame, $\mathcal{H}M$ is a compact frame. Our result follows from this more general result.

(4) $\Rightarrow$ (1) : First note that any $\varphi : \mathcal{L}(\mathbb{R}) \to L$ lifts through $\mathcal{H}$Coz$L$ (since $\mathcal{L}(\mathbb{R})$ is in fact a frame), that is, there exists $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \to \mathcal{H}CozL$ such that $\bigvee \overline{\varphi} = \varphi$: since $\varphi : \mathcal{L}(\mathbb{R}) \to L$ actually maps into Coz$L$ by the first part of the proof of Proposition 1.1.13, we may define, for any $p, q \in \mathbb{Q}$, by

$$\overline{\varphi}(p,q) = \downarrow \varphi(p,q).$$

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To see that this defines a homomorphism $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \to \mathcal{H}\mathrm{Coz}L$ we note that the defining relations R(1)-R(4) of $\mathcal{L}(\mathbb{R})$ are transformed into identities in $\mathcal{H}\mathrm{Coz}L$ because the composition

$$\mathcal{L}(\mathbb{R}) \xrightarrow{\varphi} \mathrm{Coz}L \xrightarrow{\downarrow} \mathcal{H}\mathrm{Coz}L$$

is a σ-frame homomorphism. Further, $\bigvee \overline{\varphi} = \varphi$ since
$\bigvee \overline{\varphi}(p,q) = \bigvee \downarrow \varphi(p,q) = \varphi(p,q)$ and the (p,q) generate $\mathcal{L}(\mathbb{R})$.
In particular, if $\mathcal{H}\mathrm{Coz}L$ is compact then $\overline{\varphi} : \mathcal{L}(\mathbb{R}) \to \mathcal{H}\mathrm{Coz}L$ is obviously bounded, and this makes $\varphi = \bigvee \overline{\varphi}$ bounded, showing L is pseudocompact. □.

Corollary 1.3.2 A regular σ-frame is compact if and only if it is pseudocompact

Proof: This follows from the previous proposition and Corollary 1.1.14. □

1.4 Regularity and the Frame Envelope

We present some properties of the frame envelope of a regular σ-frame, after the following technical lemmas.

Lemma 1.4.1 In any σ-frame, $a \prec\!\!\prec b$ if and only if there is a trail t with $t(0) = a$ and $t(1) = b$.

Proof: ($\Rightarrow$) Take any σ-frame L with $a \prec\!\!\prec b$ in L. Let $(c_{n\,l})_{n=0,1,\ldots;l=0,1,\ldots,2^n}$ be the interpolating sequence between a and b. Now define, for each $r \in \mathbb{Q}$,

$$t(r) = \left\{ \begin{array}{ll} 0 & (r < 0) \\ \bigvee\{c_{k\,l} \mid \frac{l}{2^k} \leq r\} & (0 \leq r \leq 1) \\ e & (r > 1). \end{array} \right\}.$$

Then, $t(0) = \bigvee\{c_{n\,0} \mid n = 0, 1, \ldots\} = a$ while $t(1) = b$. Further, t is a trail: For any $p < q$, $t(p) \prec t(q)$ trivially whenever $p < 0$ or $1 < q$; on the other hand, if $0 \leq p < q \leq 1$ then $p \leq \frac{l}{2^m} < \frac{l+1}{2^m} \leq q$ for suitable m and l, and consequently

$$t(p) \leq c_{n\,l} \prec c_{n\,l+1} \leq t(q).$$

Given $p < Q$ in $\mathbb{Q}$, choose $w(p,q)$ to be a separating element of $t(p) \prec t(q)$, and we have a witness, making t a pre-trail.

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Also, $\bigvee\{t(r)\,|\,r \in \mathbb{Q}\} = e = \bigvee\{t(r)^{\square}\,|\,r \in \mathbb{Q}\}$ trivially. Thus t is a trail.
($\Leftarrow$) Let $t : \mathbb{Q} \to L$ be a trail such that $t(0) = a$ and $t(1) = b$. Now take $\varphi : \mathcal{L}(\mathbb{R}) \to L$ as given by Lemma 1.1.11. Then

$$\varphi\left(-,\frac{1}{2}\right) = \bigvee\left\{t(p')^{\square} \wedge t(q')\,|\,p < p' < q' < \frac{1}{2}\right\} \geq a$$

since $t(-1)^{\square} \wedge t(0) = a$. Further

$$\varphi\left(-,1\right) = \bigvee\left\{t(p')^{\square} \wedge t(q')\,|\,p < p' < q' < 1\right\} \leq t(1) = b.$$

Finally observe that $\varphi(-,\frac{1}{2}) \prec\!\prec \varphi(-,1)$, since $(-,\frac{1}{2}) \prec\!\prec (-,1)$ and φ preserves that. $\square$.

Lemma 1.4.2 For $J, G \in \mathcal{H}L$ with $J \prec\!\prec G$, $\exists d \in G$ where $J \prec\!\prec \downarrow d$.

Proof: By Lemma 1.4.1 above, we have a trail $t : \mathbb{Q} \to \mathcal{H}L$ with $t(0) = J$ and $t(1) = G$. For $p < q$ in $\mathbb{Q}$, then $t(p) \prec t(q)$ and so we can find $d_{pq} \in t(q)$ with $t(p) \prec \downarrow d_{pq}$:
$\exists H \in \mathcal{H}L$ with $t(p) \cap H = \{0\}$ and $H \vee t(q) = L$. So there exist $a \in H$, $b \in t(q)$ with $\downarrow a \vee \downarrow b = L$ and $t(p) \cap \downarrow a = \{0\}$. Put $d_{pq} = b$ to get $t(p) \prec \downarrow d_{pq}$.
Given any $q \in \mathbb{Q}$ define

$$d_q = \bigvee\{d_{pq}\,|\,p < q,\, p \in \mathbb{Q}\}.$$

Then $d_q \in t(q)$ since $t(q)$ is a σ-ideal. Furthermore, when $p < q$

$$\downarrow d_p \subseteq t(p) \prec \downarrow d_{pq} \subseteq \downarrow d_q.$$

Now define, for $r \in \mathbb{Q}$

$$t'(r) = \left\{\begin{array}{ll} \{0\} & (r < 0) \\ J & (r = 0) \\ \downarrow d_r & (0 < r \leq 1) \\ L & (r > 1) \end{array}\right\}.$$

t' is a trail:
for $r < s$, trivially $t'(r) \prec t'(s)$ whenever $r < 0$ or $s > 1$. Otherwise, $0 \leq r < s \leq 1$. Breaking this into two scenarios, **(1)** $0 = r$ and **(2)** $0 < r$.

(1) take $u \in \mathbb{Q}$ with $r < u < s$ so that $t'(r) = J \prec t(u) \prec \downarrow d_s = t'(s)$.

(2) trivially $t'(r) = \downarrow d_r \prec \downarrow d_s = t'(s)$.

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Either way, $t'(r) \prec t'(s)$.
Given $r < s$ in $\mathbb{Q}$, choose $w(r,s)$ so that it separates $t'(r) \prec t'(s)$. This makes t' a pre-trail.

Trivially $\bigvee\{t'(r) \mid r \in \mathbb{Q}\} = L = \bigvee\{t'(r)^{\square} \mid r \in \mathbb{Q}\}$ and so t' is a trail.
Finally put $d = d_1$. Then $t'(0) = J$ and $t'(1) = \downarrow d$ and so $J \prec\!\prec \downarrow d$ by Lemma 1.4.1. □

Lemma 1.4.3 For $a, b \in L$, $\downarrow a \prec \downarrow b$ in $\mathcal{H}L$ if and only if $a \prec b$ in L.

Proof: Suppose $\downarrow a \prec \downarrow b$. By the definition of $\prec$, $\exists H \in \mathcal{H}L$ with $H \cap \downarrow a = 0$ and $H \vee \downarrow b = L$. Take $h \in H$ with $h \vee b = e$, then $h \wedge a = 0$ giving $a \prec b$. Now suppose that $a \prec b$. Then there exist separating element $s \in L$ with $a \wedge s = 0$ and $s \vee b = e$. But then $\downarrow a \cap \downarrow s = \{0\}$ and $\downarrow s \vee \downarrow b = L$. □

A direct consequence of this is that $\downarrow a \prec\!\prec \downarrow b$ in $\mathcal{H}L$ iff $a \prec\!\prec b$ in L.

Proposition 1.4.4 $\mathcal{H}L$ is completely regular if and only if L is regular.

Proof: Suppose that L is regular and take $J \in \mathcal{H}L$. We know that $J = \bigvee\{\downarrow a \mid a \in J\}$. By regularity, $a = \bigvee a_n$ where $a_n \prec a$ for each $n \in \mathbb{N}$. In fact $a_n \prec\!\prec a$, since in a regular σ-frame, $\prec = \prec\!\prec$. Then $\downarrow a = \bigvee \downarrow a_n$ and $\downarrow a_n \prec\!\prec \downarrow a$, by the preceding Lemma. So

$$J = \bigvee\{\downarrow a \mid \downarrow a \prec\!\prec J\}.$$

Now suppose that $\mathcal{H}L$ is completely regular, hence regular, and take any $a \in L$. Then $\downarrow a \in \mathcal{H}L$ with $\downarrow a = \bigvee J_j$, $J_j \prec \downarrow a$. So $a = \bigvee_n a_n$ where $(a_n)_{n \in \mathbb{N}} \subseteq \bigcup_j J_j$ and $a_n \in J_{j_n}$, by the definition of join in $\mathcal{H}L$. Now for each $n \in \mathbb{N}$,

$$\downarrow a_n \subseteq J_{j_n} \prec \downarrow a$$

$\Rightarrow a_n \prec a$, by the preceding Lemma. □

Proposition 1.4.5 $\mathrm{Coz}\mathcal{H}L \cong \mathrm{Coz}L$.

Proof: We show that $\downarrow\colon \mathrm{Coz}L \to \mathrm{Coz}\mathcal{H}L$ is an isomorphism.
Take any $a \in \mathrm{Coz}L$. From Proposition 1.1.13 $a = \bigvee_n a_n$, $a_n \prec\!\prec a_{n+1}$. Then we have $\downarrow a = \bigvee_n \downarrow a_n$, $\downarrow a_n \prec\!\prec \downarrow a_{n+1}$, by Lemma 1.4.3. Thus $\downarrow a \in \mathrm{Coz}\mathcal{H}L$, by Proposition 1.1.13 again.
So the map $\downarrow$ is well defined and is, trivially, one-one and a homomorphism of σ-frames.

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Now take any $J \in \text{Coz}\mathcal{H}L$. Again by Proposition 1.1.13, $J = \bigvee J_n$, $J_n \prec\prec J_{n+1}$, $J_n \in \mathcal{H}L$. By Lemma 1.4.2, for each $n > 1$ in $\mathbb{N}$, $\exists j_n \in J_n$, where $J_n \prec\prec \downarrow j_{n+1}$. Thus

$$\bigvee_n J_n \subseteq \bigvee_n \downarrow j_{n+1} = \downarrow \bigvee_n j_{n+1}.$$

If we put $j = \bigvee j_{n+1}$, then we see that $j \in J$ and $J = \downarrow j$. Thus $\downarrow$ is onto. $\square$.

Corollary 1.4.6 $\mathcal{H}L$ is pseudocompact if and only if L is pseudocompact.

Proof: By Proposition 1.3.1, a σ-frame L is pseudocompact if and only if Coz$L$ is compact. This together with the preceding proposition proves the result. $\square$.

Maybe it is worth remembering at this point that a frame is pseudocompact as a frame if and only if it is pseudocompact as a σ-frame. Thus, Corollary 1.4.6 is equivalent to saying that $\mathcal{H}L$ is pseudocompact as a frame if and only if L is a pseudocompact σ-frame.

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Chapter 2

σ-Spaces

2.1 Separation Axioms

We aim to develop a theory of σ-spaces in parallel with σ-frames, to mirror the parallels between topological spaces and frames. In this, we cannot avoid comparisons between σ-spaces and classical topological spaces. We make definitions in line with the classical T_2 (Hausdorff), T_1 and T_0 separation axioms.

Given a σ-space X, there is a natural topological space that can be associated with this σ-space. That is to say, if we take the σ-open sets of X as the base for the open sets of a topology on X, then we get what we shall call the *underlying topology* of the σ-space X. We shall denote the underlying topological space of X by $\mathcal{T}X$. Given a σ-continuous map $f : X \to Y$, take $\mathcal{T}f = f$ as the set map between the underlying sets of X and Y. $\mathcal{T}f$ will of course be continuous and we have the functor $\mathcal{T} : \sigma\mathbf{Sp} \to \mathbf{Sp}$.

Recall that for a σ-space X with $x \in X$, we defined

$$\Sigma(x) = \{U \in \Sigma X \mid x \in U\}.$$

This σ-prime filter plays much the same role as that of the neighborhood filter of a point in a topological space.

Definition 2.1.1 A σ-space X is said to be a σ_0-space if whenever $x, y \in X$ with $x \neq y$ then there exists a σ-open set U of X with $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.

Remark 2.1.2 This definition is equivalent to saying that for any $x, y \in X$, $\Sigma(x) = \Sigma(y)$ if and only if $x = y$.

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As we shall see later in the following section, under *Soberification*, that σ_0-spaces are an epi-reflective subcategory of the category of σ-spaces..

Definition 2.1.3 A σ-space X is said to be a σ_1-space if whenever $x, y \in X$ with $x \neq y$ then there exist σ-open sets U and V of X with $x \in U$, $x \notin V$ and $y \in V$, $y \notin U$.

Definition 2.1.4 A σ-space X is said to be a σ_2-space if whenever $x, y \in X$ with $x \neq y$ then there exists σ-open sets U and V of X with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Trivially, $\sigma_2 \Rightarrow \sigma_1 \Rightarrow \sigma_0$, as in the case of the analogous T_2, T_1 and T_0 properties for topological spaces. It is clear that a T_i topological space is also a σ_i-space, $i = 0, 1, 2$. Thus the examples that show that $T_0 \not\Rightarrow T_1 \not\Rightarrow T_2$ will also show that $\sigma_0 \not\Rightarrow \sigma_1 \not\Rightarrow \sigma_2$.

Taking sobriety (σ_{sober}) for a separation axiom (see Definition 0.0.1), it is interesting to investigate how σ_{sober} falls relative to the σ_i. Classically $T_2 \Rightarrow T_{sober} \Rightarrow T_0$ with $T_1 \not\Rightarrow T_{sober} \not\Rightarrow T_1$.

Examples 2.1.5 Let X be an uncountable set, with $\Sigma X = \{\text{all countable subsets}\} \cup \{X\}$. This is clearly σ_2 since all singleton subsets are σ-open. On the other hand, $\mathcal{F} = \{X\}$ is in fact a σ-prime filter of ΣX: because a countable union of countable sets is a countable set. Yet $\mathcal{F} \neq \Sigma(x)$ for any $x \in X$. Thus X is not sober σ-space.

However,

Proposition 2.1.6 A sober σ-space is σ_0.

Proof: Since X is sober, we have the isomorphism $\eta_X : X \to \Psi\Sigma X$. In particular, the map is one-one, so $\Sigma(x) = \eta_X(x) = \eta_X(y) = \Sigma(y)$ if and only if $x = y$. But this says precisely that X is σ_0, by Remark 2.1.2. □

This makes it clear that $\sigma_2 \not\Rightarrow \sigma_{sober} \Rightarrow \sigma_0$.

Proposition 2.1.7 A σ-space X is σ_i if and only if $\mathcal{T}X$ is T_i, for $i = 0, 1, 2$.

Proof: Trivially if X is σ_i then $\mathcal{T}X$ is T_i, since σ-open sets are open in the underlying topology. On the other hand, for any open set O of $\mathcal{T}X$, if $x \in O$ and $y \notin O$, then, by the definition of open set in $\mathcal{T}X$, there exists $U \subseteq O$, a σ-open of X with $x \in U$. From this it is clear that $T_i \Rightarrow \sigma_i$. □

The last proposition leads us to wonder about a similar correspondence between T_{sober} and σ_{sober}. However, in this case we can only say:

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Proposition 2.1.8 If X is σ_{sober} then $\mathcal{T}X$ is T_{sober}.

Proof: We assume that X is σ_{sober} and let $\mathcal{F}$ be a completely prime filter of $\mathcal{T}X$. Further, let $\mathcal{G} = \mathcal{F} \cap \Sigma X$ i.e. all σ-open sets contained in $\mathcal{F}$. Trivially, $\mathcal{G}$ is non-empty, and a filter on ΣX. Furthermore, $\mathcal{G}$ is certainly σ-prime, since $\mathcal{F}$ is completely prime and $\mathcal{G} \subseteq \mathcal{F}$. Hence $\mathcal{G}$ is principal, by the sobriety of X, i.e.

$$\mathcal{G} = \Sigma(x) \qquad \text{for some } x \in X.$$

Finally $\mathcal{F} = \{O \mid x \in O\}$: Take any open set O of $\mathcal{T}X$. Then $O = \bigcup U_\alpha$, some union of σ-open sets U_α.
If we assume that $O \in \mathcal{F}$. Then there exists β with $U_\beta \in \mathcal{F}$ i.e. $U_\beta \in \mathcal{G} = \Sigma(x)$. So $x \in O$.
On the other hand, if $x \in O$ then $x \in U_\beta$, for some β. That is that $U_\beta \in \mathcal{G} \subseteq \mathcal{F}$. But then $O \in \mathcal{F}$ since $U_\beta \subseteq O$ and $\mathcal{F}$ is a principal filter. □

The reverse implication fails: if X is the σ-space described in Example 2.1.5, then X is not sober yet the underlying topology of X is discrete, hence T_2. Thus $\mathcal{T}X$ is sober.

Regular σ-Spaces

Definition 2.1.9 Regular σ-spaces are σ-spaces with their σ-frame of σ-open sets a regular σ-frame. They shall be referred to as σ_{reg}-spaces.

Examples 2.1.10

- Every Boolean σ-algebra is a regular σ-space. A Boolean σ-algebra is a set with a collection of subsets (the σ-open sets) closed under countable intersection and union, as well as complementation. The complements of the σ-open sets are again σ-open and so each σ-open set is rather below itself.

- The cozero sets of a Tychonoff space form a regular σ-frame.

The regular σ-spaces appear in Gilmour [11] as Alexandroff spaces while in Gordon [14] they appear as zero-set spaces. Furthermore, what we call the complements of the σ-open sets of regular σ-spaces have traditionally been called the zero sets. The terminology is justified as the σ-open sets of a regular σ-space turn out to be to be exactly the cozero sets of the σ-maps to the σ-space of real numbers, $\mathbb{R}$, [14]. Note that the open sets of $\mathbb{R}$ are cozero sets in the sense of Gillman & Jerrison [10]

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We now add to our list of separation axioms for the σ-spaces and once again compare them to the topological axioms. Considering Regularity as a separation axiom, σ_{reg}, we see that if X is σ_0 and σ_{reg} then $\mathcal{T}X$ is $T_{3\frac{1}{2}}$ or Tychonoff.

Proposition 2.1.11 The underlying topology of a regular σ-space is completely regular.

Proof: By Corollary 1.1.14, $\Sigma X = \text{Coz}\Sigma X$. These cozero elements are by the above discussion, cozero sets of the underlying topological space. So $\mathcal{T}X$ has cozero sets as basis. This is well known to be equivalent to saying that $\mathcal{T}X$ is completely regular. □

Proposition 2.1.12 If X is σ_{reg} and σ_0 then X is a σ_2-space.

Proof: Suppose that X is a regular σ-space and $x, y \in X$ with $x \neq y$. We can find $U \in \Sigma X$ with $x \in U$ and $y \notin U$, say, since X is σ_0. By regularity of ΣX, there exists $V \in \Sigma X$ with $x \in V \prec U$. Let S be a separating element of $V \prec U$. Then

$$\begin{aligned} U \cup S = X \quad &\Rightarrow U \cap S^c = S^c \\ &\Rightarrow S^c \subseteq U \\ &\Rightarrow y \in S \text{ since } y \notin U. \end{aligned}$$

On the other hand, $S \cap V = \emptyset$ and $x \in V$. But, now we have $x \in V$, $y \in S$ and $V \cap S = \emptyset$. □

It is clear that the underlying topological space of a regular σ-space which is σ_0 is a sober topological space. However, from Gilmour [11], the following example shows that a regular σ-space itself need not be a sober σ-space.

Examples 2.1.13 Let $X = (\mathbb{R}, \mathcal{A})$, where $\mathcal{A}$ is the σ-algebra generated by the countable subsets of $\mathbb{R}$. Then X is a regular σ-space. If P is the collection of all cocountable subsets of $\mathbb{R}$, then we see that P is not of the form $\Sigma(x)$ for any $x \in \mathbb{R}$. Thus X is not sober.

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2.2 Soberification

Taking the σ_0-spaces and the σ_{sober}-spaces with the usual σ-space maps, we have the full subcategories of $\sigma\mathbf{Sp}$, denoted by $\sigma_0\mathbf{Sp}$ and $\mathbf{Sob}\sigma\mathbf{Sp}$ respectively.

Given any σ-space X, if we define X^0 to be the set $\{\Sigma(x) \mid x \in X\}$ with subbasic σ-open sets $U^0 = \{\Sigma(x) \mid x \in U\}$, generating a collection of σ-open sets $\Sigma(X^0)$. We see that $(X^0, \Sigma(X^0))$ is a σ_0 σ-space:
Indeed, if $\Sigma(x) \neq \Sigma(y)$ then $\exists U \in \Sigma(x)$ and $U \notin \Sigma(y)$, say. But then $\Sigma(x) \in U^0$ and $\Sigma(y) \notin U^0$.

If $f : X \to Y$ is a map between σ-spaces, then $f^0 : X^0 \to Y^0$ is the map $\Sigma(x) \mapsto \Sigma(f(x))$. Since $(f^0)^{\leftarrow}(U^0) = (f^{\leftarrow}(U))^0$, f^0 is a σ-space map. X^0 is a quotient of X.

We have a canonical map $\lambda_X : X \to X^0 (= x \mapsto \Sigma(x))$ which is σ-continuous: take any subbasic σ-open $U^0 \in \Sigma X^0$. Then

$$\begin{aligned} \lambda_X^{\leftarrow}(U^0) &= \{x \in X \mid \lambda_X(x) \in U^0\} \\ &= \{x \in X \mid \Sigma(x) \in U^0\} \\ &= U \end{aligned}$$

Of course what we have here is a functor $\_^0 : \sigma\mathbf{Sp} \to \sigma_0\mathbf{Sp}$, between the category of σ-spaces and the category of σ_0-spaces. That the category of σ_0-spaces is a reflective subcategory of the σ-spaces with epi-reflector $\lambda_X : X \to X^0$, follows from the following lemma, with the fact that λ_X is onto, hence epi.

Lemma 2.2.1 For X and Y σ-spaces with Y σ_0, a σ-space map $f : X \to Y$ extends uniquely to $\overline{f} : X^0 \to Y$ where $f = \overline{f} \circ \lambda_X$.

Proof: As Y is σ_0, λ_Y is a bijection and by the formula $(\lambda_Y^{-1})^{\leftarrow}(U) = U^0$, $\lambda_Y^{\leftarrow}$ is σ-continuous. Thus λ_Y is an isomorphism $Y \cong Y^0$. Put $\overline{f} = \lambda_Y^{-1} \circ f^0$. Now

$$\overline{f} \circ \lambda_X(x) = [\lambda_Y^{-1} \circ f^0](\Sigma(x)) = \lambda_Y^{-1}(\Sigma(f(x))) = \lambda_Y^{-1} \circ \lambda_Y(f(x)) = f(x),$$

as claimed. Because λ_X is epi, we see that $\overline{f}$ is unique. □

By categorical arguments, we can conclude that σ_0-spaces are closed under the formation of products. The proof of the following statement is also straightforward.

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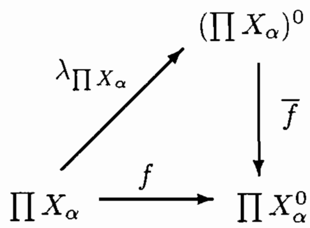
Proposition 2.2.2 $\_^0$ distributes over products.

Proof: Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of σ-spaces. Let the projection maps be $\pi_\alpha : \prod X_\alpha \to X_\alpha$ and $\rho_\alpha : \prod X_\alpha^0 \to X_\alpha^0$. Define

$$\begin{array}{rcl} f : \prod X_\alpha & \longrightarrow & \prod \lambda_{X_\alpha}(X_\alpha) = \prod X_\alpha^0 \\ (x_\alpha) & \longmapsto & (\Sigma(x_\alpha)). \end{array}$$



By the preceding lemma, there exists $\overline{f} : (\prod X_\alpha)^0 \longrightarrow \prod X_\alpha^0$, with $f = \overline{f} \circ \lambda_{\prod X_\alpha}$. Let U be a basic σ-open set of $\prod X_\alpha$. Say $U = \pi_{\alpha_1}^{\leftarrow}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{\leftarrow}(U_{\alpha_k})$. Then

$$\begin{aligned} \overline{f}[U^0] &= \{\overline{f}(\Sigma((x_\alpha)_{\alpha\in I})) \mid (x_\alpha)_{\alpha\in I} \in U\} \\ &= \{\ldots\ldots\ldots\ldots \mid x_{\alpha_i} \in U_{\alpha_i},\ i = 1,\ldots,k\} \\ &= \{\ldots\ldots\ldots\ldots \mid \Sigma(x_{\alpha_i}) \in U_{\alpha_i}^0,\ i = 1,\ldots,k\} \\ &= \{(\Sigma(x_\alpha))_{\alpha\in I} \mid \Sigma(x_{\alpha_i}) \in U_{\alpha_i}^0,\ i = 1,\ldots,k\} \end{aligned}$$

because $f = \overline{f} \circ \lambda_{\prod X_\alpha}$, thus $\overline{f}[U^0] = \rho_{\alpha_1}^{\leftarrow}(U_{\alpha_1}^0) \cap \cdots \cap \rho_{\alpha_k}^{\leftarrow}(U_{\alpha_k}^0)$.
So $\overline{f}$ is a σ-open map i.e. $\overline{f}$ maps σ-open sets to σ-open sets.
$\overline{f}$ is one-one:
Take any $\underline{x} = (x_\alpha)_{\alpha\in I}, \underline{y} = (y_\alpha)_{\alpha\in I} \in \prod X_\alpha$ with $\overline{f}(\Sigma(\underline{x})) = \overline{f}(\Sigma(\underline{y}))$ i.e. $\Sigma(x_\alpha) = \Sigma(y_\alpha)$, $\forall\alpha$. Now take any basic σ-open $U \in \Sigma(\underline{x})$ where $U = \pi_\alpha^{\leftarrow}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_k}^{\leftarrow}(U_{\alpha_k})$, $i = 1, \ldots k$ for respective σ-opens U_{α_i}. Thus $x_{\alpha_i} \in U_{\alpha_i}$

$$\begin{aligned} &\Rightarrow y_{\alpha_i} \in U_{\alpha_i}, \quad \text{because} \quad \Sigma(x_{\alpha_i}) = \Sigma(y_{\alpha_i}), \\ &\Rightarrow \underline{y} \in U \\ &\Rightarrow U \in \Sigma(\underline{y}). \end{aligned}$$

Thus $\Sigma(\underline{x}) \subseteq \Sigma(\underline{y})$ and by symmetry $\Sigma(\underline{x}) = \Sigma(\underline{y})$.

Clearly $\overline{f}$ is onto. □

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We now look at soberification. As already observed, the σ-space of all σ-prime filters on a σ-space X, $\Psi\Sigma X$, is the soberification of X. Moreover, the unit of the dual adjunction between σ-frames and σ-spaces, η_X, is an embedding of X into $\Psi\Sigma X$ if X is σ_0.

From here on, we take all σ-spaces to be σ_0.

For the sake of presentation, we shall denote by sX the soberification of a σ-space, instead of $\Psi\Sigma X$, and take s to be the functor $s : \sigma\mathbf{Sp} \to \mathbf{Sob}\sigma\mathbf{Sp}$. Furthermore, for ease in calculations, we shall take sX to be the σ-space of σ-frame maps $h : \Sigma X \to \mathbf{2}$, which we again call the "points", with the usual σ-open sets $(\Psi_U = \{h \in sX \mid h(U) = 1\} : U \in \Sigma X)$.

For a pair of σ-spaces X and Y, X is said to be *dense* in Y if X is a subspace of Y and if X meets every non-empty σ-open set of Y i.e. $X \cap V \neq \emptyset$ for all $V \in \Sigma Y$.

Given a σ-space X, define on X a new σ-space bX, that has underlying set X and basic σ-open sets $\{U \cap (X \backslash V) \mid U, V \in \Sigma X\}$.

Definition 2.2.3 For σ-spaces $X \subseteq Y$, we say X is *b-dense* in Y if X is dense in bY.

Remark 2.2.4 If X is b-dense in Y, then for each $U, V \in \Sigma Y$, $U \neq V$, $X \cap U \cap (Y \backslash V) \neq \emptyset$. As a result of this, the σ-frame map $h : \Sigma Y \to \Sigma X$ $(V \mapsto V \cap X)$ is one-one, and hence an isomorphism.

This notion of b-dense has been translated from the classical case of topological spaces [15]. With topological spaces, the b-topology on a topological space X is the topology that has as a subbase all the closed and open sets of X. Put more simply, every open set in the b-topology is the union of sets that are finite intersections of open and closed sets. For topological spaces X and Y, with X a subspace, X is said to be b-dense in Y if X is dense in the b-topology of Y.

The definition of b-dense for σ-spaces involves finite intersections of σ-open sets with the complements of σ-open sets, which play the role of the "closed" sets, as in the topological case. Indeed it follows that if, for σ-spaces X and Y, X is b-dense in Y, then the underlying topology of X is b-dense in Y i.e. $\mathcal{T}X$ is b-dense in $\mathcal{T}Y$.

The b-topology for topological spaces, is an essential tool for the investigation of sober spaces and was introduced by L.Skula in [23]. In [15], Hoffmann among other interesting results, shows that the soberification of topological

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spaces preserves products. Here, we seek to emulate Hoffman's approach, following the steps of [16] and [15]. Hoffmann constructs the soberification of a space using join irreducible closed sets, instead of completely prime filters of opens sets. We will work with σ-prime filters of σ-opens.

A σ-space map is said to be a b-dense map if its image is b-dense in the codomain.

Lemma 2.2.5 Any σ-space X is b-dense in its soberification sX.

Proof: By this we mean that $\eta_X(X) \subseteq sX$ is a b-dense subspace. Note that the image $\eta_X(X) = X^0$, the σ_0-ification of X.

Take any non-empty basic σ-open of $b(sX)$, say $\Psi_U \cap (sX \backslash \Psi_V)$ with $U, V \in \Sigma X$. Then $\Psi_U \neq \Psi_V$ and so $U \neq V$, since ΣX is spatial, and in fact $U \cap (X \backslash V) \neq \emptyset$.
From here we simply take an $x \in U \cap (X \backslash V)$. Then clearly

$$\Sigma(x) \in \eta_X(X) \cap \Psi_U \cap (sX \backslash \Psi_V).$$

□

So X^0 is b-dense in sX and $\eta_X : X \to sX$ is a b-dense morphism.

Proposition 2.2.6 Every b-dense map is an epimorphisms in σ_o**Sp**

Proof: Take a b-dense map $r : Z \to X$ and suppose that $f, g : X \to Y$ with $f \circ r = g \circ r$.
Take any $U \in \Sigma Y$. Assume that $f^{\leftarrow}(U) \neq g^{\leftarrow}(U)$. Say $f^{\leftarrow}(U) \cap [g^{\leftarrow}(U)]^c \neq \emptyset$. Then by the b-density of $r(Z)$ in X,

$$f^{\leftarrow}(U) \cap [g^{\leftarrow}(U)]^c \cap r(Z) \neq \emptyset.$$

Take any $z \in Z$ with $r(z) \in f^{\leftarrow}(U) \cap [g^{\leftarrow}(U)]^c \cap r(Z)$. But then $for(z) = gor(z) \in U$ i.e. $r(z) \in g^{\leftarrow}(U)$, a contradiction. So $f^{\leftarrow} = g^{\leftarrow} : \Sigma Y \to \Sigma X$.

$s(f) = \Psi\Sigma(f) = \Psi(f^{\leftarrow}) = \Psi(g^{\leftarrow}) = \Psi\Sigma(g) = s(g)$.
By the naturality,

$$\begin{aligned} \eta_Y \circ f(x) &= s(f) \circ \eta_X(x) \\ &= s(g) \circ \eta_X(x) \\ &= \eta_Y \circ g(x). \end{aligned}$$

Because we are working with σ_0-spaces, η_Y is 1-1 and so $f = g$. □

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Proposition 2.2.7 **SobσSp** is a reflective sub-category of σ**Sp**, $s = \Psi\Sigma$ the reflector, with epi-reflection map $\eta_X : X \to \Psi\Sigma X$, for each σ-space X.

Proof: Take $f : X \to Y$ with Y sober. Because Y is sober, η_Y has an inverse. Define $\overline{f} : sX \to Y$ as $\overline{f} = \eta_Y^{-1} \circ s(f)$. By naturality, $\eta_Y \circ f = s(f) \circ \eta_X$ and so $f = \eta_Y^{-1} \circ s(f) \circ \eta_X = \overline{f} \circ \eta_X$.

That $\overline{f}$ is the unique map with this property follows from the fact that η_X is epi, since it is a b-dense morphism. □

We conclude, by categorical arguments, that products of sober σ-spaces is again a sober σ-space.

Definition 2.2.8 (Gilmour [11]) A regular σ-space X is said to be *realcompact* if every σ-prime filter of σ-open sets of ΣX is of the form $\Sigma(x)$. The Hewitt realcompactification of a regular σ-space X is the σ-space of all σ-prime filters on X, denoted by υX.

Remark 2.2.9 It is clear that a regular σ-space is realcompact if and only of it is sober and the realcompactification is exactly the soberification.

Note that Gordon defines the realcompactification in terms of real ultrafilters of zero sets. The two approaches are shown to be equivalent in [11].

Definition 2.2.10 Given regular σ-spaces X and Y with X a subspace of Y, X is said to be *zero-set dense* in Y if X meets every zero set of Y i.e. For each zero set $Z \subseteq Y$, $X \cap Z \neq \emptyset$.

Remark 2.2.11 Every regular σ-space is zero set dense in its Hewitt realcompactification.

Proof: Given a regular σ-space X, take any σ-open set in υX. It will be of the form Ψ_U, with U some σ-open set in X. We want the complement of Ψ_U in υX to meet $\eta_X(X)$. To this end, take $x \notin U$, which we can do otherwise the exercise is trivial, and see that $U \notin \Sigma(x)$ which entails that $\Sigma(x) \notin \Psi_U$. □

One can say more. Gordon in [14], Theorem 7.13, goes on to characterize realcompactifications with the following statement, which is a consequence of Proposition 2.2.13 and Proposition 2.2.14.

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Theorem 2.2.12 (Gordon [14]) Let X be a subspace of a realcompact zero-set space Y. In order that Y be (isomorphic to) υX it is necessary and sufficient that X be zero-set dense in Y.

Gordon then goes on to use this characterization to show that, in the case of regular σ-frames, the realcompactification of the product is the product of the realcompactifications i.e. that the Hewitt realcompactification distributes over products, $\upsilon \prod X_\alpha \cong \prod \upsilon X_\alpha$. We shall, as a corollary to Proposition 2.2.13, obtains Gordon's result.

Proposition 2.2.13 Let X be embedded as a b-dense subspace of a sober σ-space Y, via the embedding $i : X \hookrightarrow Y$. Then the induced σ-continuous map $s(i) : sX \to sY$ is an isomorphism.

Proof:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & sX \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle s(i)} \\ Y & \xrightarrow{\eta_Y} & sY \end{array}$$

Recall $s = \Psi\Sigma$. Thus for each $h \in sX$, $s(i)(h) = h \circ i^{\leftarrow}$ where $i^{\leftarrow}(U) = X \cap U$. Take any two $h, k \in sX$ and assume that $s(i)(h) = s(i)(k)$. Then $h \circ i^{\leftarrow} = k \circ i^{\leftarrow}$ and consequently $h = k$ since $i^{\leftarrow}$ is an onto σ-frame homomorphism.

For each $V \in \Sigma X$, $V = X \cap U_V$, for some unique $U_V \in \Sigma Y$, suppressing the notation for the obvious embedding i. The U_V is unique because X is b-dense in Y. Thus, given $h \in sY$, we can define $r(h) : \Sigma X \to \mathbf{2}$ by

$$r(h)(V) = 1 \quad \text{if and only if} \quad h(U_V) = 1.$$

Then $r(h) \in sX$ i.e. a σ-frame homomorphism $r : sX \to \mathbf{2}$.
Take any $h \in sY$. Then for any $U \in \Sigma Y$

$$s(i) \circ r(h)(U) = r(h) \circ i^{\leftarrow}(U) = r(h)(U \cap X) = h(U).$$

Now take any $h \in sX$. Then for any $V \in \Sigma X$

$$\begin{aligned} r \circ s(i) \circ (h)(V) &= s(i)(h)(U_V) \\ &= h \circ i^{\leftarrow}(U_V) \\ &= h(U_V \cap X) \\ &= h(V). \end{aligned}$$

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Then $s(i) \circ r = \mathrm{id}_{sY}$ and $r \circ s(i) = \mathrm{id}_{sX}$ and hence $s(i)$ is a bijection. It remains to show that $s(i)$ is a σ-open map.

$$\begin{aligned} h \in \Psi_{U \cap X} &\Leftrightarrow h(U \cap X) = 1 \\ &\Leftrightarrow s(i)(h)(U) = 1 \\ &\Leftrightarrow s(i)(h) \in \Psi_U. \end{aligned}$$

That is to say $s(i)(\Psi_{U \cap X}) = \Psi_U$.
Thus $s(i)$ is an isomorphism. □

This proposition is a generalization of Theorem 2.2.12, as the following proposition shows.

Proposition 2.2.14 If X and Y are regular σ-spaces with X a subspace of Y, X is b-dense in Y if and only if X is zero-set dense in Y.

Proof: By the definition of b-dense and zero-set dense, the forward implication is trivial.
For the reverse implication suppose that X is zero-set dense in Y, take σ-open sets $U, V \in \Sigma Y$ and assume that $U \cap V^c \neq \emptyset$.
By regularity,

$$U = \bigcup U_n, \quad \text{where} \quad \mathrm{U_n \prec U,\ \forall n \in \mathbb{N}.}$$

As $U \cap V^c \neq \emptyset$, there exists $m \in \mathbb{N}$ with $U_m \cap V^c \neq \emptyset$. We can assume the existence of a separating set $S \in \Sigma Y$ with $U_m \cap S = \emptyset$ and $S \cup U = Y$, since $U_m \prec U$. This implies that $U \cap S^c = S^c$. But then

$$U_m \subseteq S^c \subseteq U \Rightarrow \emptyset \neq U_m \cap V^c \subseteq S^c \cap V^c = (S \cup V)^c \neq \emptyset.$$

But $S \cup V \in \Sigma Y$ and because X is assumed to be zero-set dense in Y, $S^c \cap V^c \cap X \neq \emptyset$. Finally $U \cap V^c \cap X \neq \emptyset$ since $S^c \cap V^c \subseteq U \cap V^c$. □

This makes it clear that for regular σ-spaces, zero-set dense and b-dense are the self same condition. That is why the above proposition together with Proposition 2.2.13 implies Gordons Theorem 2.2.12.

Lemma 2.2.15 If X is a σ-space and $\mathcal{U}$ is a σ-base for X, then

$$\mathcal{V} = \{U \cap \bigcap_{s \in S} V_s^c \,|\, U, V_s \in \mathcal{U},\ \mathrm{S\ countable}\}$$

is a σ-base for bX.

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Proof: Take any σ-open $W \in \Sigma(bX)$. Then

$$W = \bigcup_{n\in\mathbb{N}} U_n \cap V_n^c$$

where $U_n = \bigcup_{m\in\mathbb{N}} U_{n_m}$ and $V_n = \bigcup_{l\in\mathbb{N}} V_{n_l}$ with $U_{n_m}, V_{n_l} \in \mathcal{U}$, since $\mathcal{U}$ is a σ-base. For each n,

$$U_n \cap V_n^c = \bigcup_{m\in\mathbb{N}} \left(U_{n_m} \cap \bigcap_{l\in\mathbb{N}} V_{n_l}^c \right).$$

Thus it is clear that W is a countable unions of sets from $\mathcal{V}$. □

Remark 2.2.16 If X is b-dense in Y, then

$$X \cap U \cap \bigcap_{s\in S} V_s^c \neq \emptyset$$

for all $U, V_s \in \Sigma Y$ and at most countable index set S.

Theorem 2.2.17 For σ-spaces X_α, $s(\prod X_\alpha) \cong \prod s(X_\alpha)$ i.e. soberification distributes over products.

Proof: By Proposition 2.2.13, we need only show that $\prod X_\alpha$ is b-dense in $\prod sX_\alpha$. By the definition of products,

$$\mathcal{V} = \{\pi_{\alpha_1}^{\leftarrow}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{\leftarrow}(U_{\alpha_n}) \,|\, U_{\alpha_i} \in \Sigma X_{\alpha_i},\ n \in \mathbb{N}\}$$

is a σ-base for $\prod sX_\alpha$. It will suffice to show that $\prod X_\alpha$ meets every non-empty basic σ-open of $b(\prod sX_\alpha)$. By the preceding lemma, these basic σ-opens will be of the form

$$Z = \bigcap_{s\in S} R_s^c \cap W, \quad R_s, W \in \mathcal{V}.$$

But

$$R_s^c = \bigcup_{k\in H_s} \pi_k^{\leftarrow}(U_k^c) \quad \text{and} \quad W = \pi_{\alpha_1}^{\leftarrow}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{\leftarrow}(U_{\alpha_n})$$

where H_s is a finite collection of indices for each s and $\alpha_1, \ldots, \alpha_n$ are fixed. Because we assume that $Z \neq \emptyset$, there is a sequence k_s, each $k_s \in H_s$, with

$$\bigcap_{s\in S} \pi_{k_s}^{\leftarrow}(U_{k_s}^c) \cap W \neq \emptyset.$$

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The k_s need not all be distinct. Indeed, there may be a countable collection $T \subseteq S$ with $k_{s_1} = k_{s_2}$ for all $s_1, s_2 \in T$ and yet $U_{k_{s_1}} \neq U_{k_{s_2}}$. Accounting for this, we re-write the intersection above as

$$\bigcap_{a \in A} \pi_{k_a}^{\leftarrow} \left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right) \cap W \neq \emptyset$$

where $A \subseteq S$ and is at most countable, and the k_a are all distinct. Each B_a is an at most countable index set and depends on each $a \in A$.

We will exhibit $(x_\alpha)_\alpha \in Z \cap \prod X_\alpha$, to show b-density . To do this, we must specify each x_{k_a}, to ensure $(x_\alpha)_\alpha \in \bigcap_{a \in A} \pi_{k_a}^{\leftarrow} \left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right)$, and specify each x_{α_i}, to ensure $(x_\alpha)_\alpha \in W$. In specifying $(x_\alpha)_\alpha$, either

(1) $k_a = \alpha_i$, some $a \in A$, $i \in 1, \ldots, n$ or

(2) $k_a \neq \alpha_i$ for all a and i.

Because a σ-space is b-dense in its soberification, we know that $\left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right) \cap U_{k_a} \cap X_{k_a} \neq \emptyset$, $\left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right) \cap X_{k_a} \neq \emptyset$ and $U_{\alpha_i} \cap X_{\alpha_i} \neq \emptyset$, making the following choices possible .

In **(1)**, choose $x_{k_a} \in \left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right) \cap U_{k_a} \cap X_{k_a}$, when $k_a = \alpha_i$. When $k_a \neq \alpha_i$, simply choose $x_{k_a} \in \left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right) \cap X_{k_a}$ and $x_{\alpha_i} \in U_{\alpha_i} \cap X_{\alpha_i}$.

In **(2)**, choose $x_{k_a} \in \left(\bigcap_{b \in B_a} U_{k_{a_b}}^c \right) \cap X_{k_a}$ and choose $x_{\alpha_i} \in U_{\alpha_i} \cap X_{\alpha_i}$.

Then $Z \cap \prod X_\alpha \neq \emptyset$. □

Remark 2.2.18 In the above we have restricted ourselves to σ_0-spaces. But in Proposition 2.2.2, we showed that σ_0-ification preserves products. Therefore the above theorem holds for all σ-spaces and not just the σ_0-spaces:

$X^0 = \eta_X(X)$ is b-dense in sX, Lemma 2.2.5, and so $s(X^0) = sX$, by Proposition 2.2.13. Now, for σ-spaces X_α, not necessarily σ_0,

$$s\left(\prod X_\alpha\right) = s\left(\left(\prod X_\alpha\right)^0\right) = s\left(\prod (X_\alpha)^0\right) = \prod s(X_\alpha^0) = \prod s(X_\alpha).$$

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Remark 2.2.19 It is worth noting however, that reflectors do not always distribute over products. For example, the reflector β that is the Stone-Čech compactification, that reflects the category of Tychonoff spaces in the category of compact T_2 topological spaces, is well known to not distribute over products [10].

Remark 2.2.20 The method of proof above follows closely that of Gordon in [14] in proving that $\upsilon \prod X_\alpha \cong \prod \upsilon X_\alpha$, for regular X_α.

Hoffman proves the corresponding result, to the one above, for spaces in [15]. However, our proof differs from his in that he uses the characterization of topological sober spaces in terms of irreducible closed sets.

Finally, we seek to show a property of spatiality for σ-spaces, that is well known of topological spaces [17]. The property is,

$$\mathcal{O}\left(\prod X_\alpha\right) \cong \bigoplus \mathcal{O}(X_\alpha) \quad \text{iff} \quad \bigoplus \mathcal{O}(X_\alpha) \text{ is spatial.}$$

But first,

Proposition 2.2.21 $\Psi \bigoplus L_\alpha \cong \prod \Psi L_\alpha$.

Proof: The contravariant functors Ψ and Σ are adjoint on the right, and this immediately implies the result. □

Corollary 2.2.22 For σ-spaces X_α, $\Sigma\left(\prod X_\alpha\right) \cong \bigoplus \left(\Sigma X_\alpha\right)$ if and only if $\bigoplus \Sigma X_\alpha$ is spatial.

Proof: For the non-trivial implication:

$$\begin{aligned} \Sigma \prod X_\alpha &\cong \Sigma\Psi\Sigma \prod X_\alpha, && \text{by spatiality of } \Sigma \textstyle\prod X_\alpha \\ &\cong \Sigma \prod \Psi\Sigma X_\alpha, && \text{by Theorem 2.2.17} \\ &\cong \Sigma\Psi \bigoplus \Sigma X_\alpha, && \text{by Proposition 2.2.21} \\ &\cong \bigoplus \Sigma X_\alpha, && \text{by assumption.} \end{aligned}$$

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Chapter 3

Continuous σ-Frames

3.1 σ-Scott Topology

In a σ-frame L we say that $a \ll_\sigma b$ (a is σ-well below b) if for each countable $X \subseteq L$ with $b \leq \bigvee X$, there exists finite $E \subseteq X$ with $a \leq \bigvee E$.

Definition 3.1.1 A σ-frame L is continuous if every element, $a \in L$, is a countable join of elements in L that are σ-well below it i.e $a = \bigvee X$, for some countable $X \subseteq L$ with $x \ll_\sigma a$ for all $x \in X$.

It is straightforward to show that if $a \ll_\sigma x$ and $b \ll_\sigma y$ then $a \vee b \ll_\sigma x \vee y$. We say that the $\ll_\sigma$ relation is 'closed under finite joins'.

A useful feature is that in the setting of continuous σ-frames the $\ll_\sigma$ relation interpolates:

if $x \ll_\sigma z$ and $z = \bigvee_n z_n$ ($z_n \ll_\sigma z$) and if, $z_n = \bigvee_k z_{nk}$ ($z_{nk} \ll_\sigma z_n$) then $z = \bigvee_{n,k} z_{nk}$ which implies $x \leq t$ for some $t = \bigvee_i z_{n_i k_i}$, a finite join of the z_{nk}, with $t \ll_\sigma \bigvee_i z_{n_i}$. But $z_{n_i k_i} \ll_\sigma z_{n_i}$. Therefore $x \ll_\sigma \bigvee_i z_{n_i} \ll_\sigma z$.

A subset S of a lattice is said to be an upset if whenever $s \in S$ and $s \leq x$ then $x \in S$.
For a σ-frame L, we say $O \subseteq L$ is *σ-Scott open* if it is an upset and if for every countable $X \subseteq L$ with $\bigvee X \in O$, there exists finite $E \subseteq X$ with $\bigvee E \in O$.

The following propositions and proofs for continuous σ-frames (Propositions 3.1.2, 3.1.3, Lemmas 3.1.4, 3.1.8 and Corollary 3.1.9) follow Walters [24]. The rest follow Banaschewski [2].

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Proposition 3.1.2 In a continuous σ-frame L, an upset O is σ-Scott open if and only if for each $a \in O$, $\exists b \in O$ with $b \ll_\sigma a$.

Proof: ($\Rightarrow$) Take L continuous with $O \subseteq L$ σ-Scott open. For any $a \in O$, $a = \bigvee X$, X a countable set consisting of elements σ-well below a.
$\Rightarrow \exists E \subseteq X$, finite, with $\bigvee E \in O$, since O is σ-Scott open. Put $b = \bigvee E$ and you have that $b \ll_\sigma a$, since trivially, the σ-way below relation is closed under finite joins.
($\Leftarrow$) Now take an upset O, with the property that $a \in O \Rightarrow \exists b \in O$ with $b \ll_\sigma a$, and suppose that $\bigvee X \in O$ for some countable $X \subseteq L$. By assumption, $\exists b \in O$ with $b \ll_\sigma \bigvee X$.
That there exists $E \subseteq X$, finite, with $b \leq \bigvee E$ which shows that $\bigvee E \in O$ since O is an upset. □

Note that a filter is Scott open if it is Scott open as a set.

Proposition 3.1.3 In a continuous σ-frame a filter is prime and σ-Scott open if and only if it is σ-prime.

Proof: ($\Rightarrow$) Take a prime σ-Scott open filter F. Now suppose that $\bigvee X \in F$ for some countable $X \subseteq L$.
$\Rightarrow \exists E \subseteq X$, finite, with $\bigvee E \in F$.
$\Rightarrow \exists a \in E \subseteq X$ such that $a \in F$, since F is prime. So F is σ-prime.
($\Leftarrow$) Take a σ-prime filter F. Trivially it is prime. Now take $a \in F$. $a = \bigvee X$, $x \ll_\sigma a$, $\forall x \in X$, since L is continuous.
It follows that $\exists b \in X$ with $b \in F$ since F is σ-prime. But then F is σ-Scott open by the previous proposition. □.

Lemma 3.1.4 In a continuous σ-frame L:
O is a σ-Scott open set iff O is a union of σ-Scott open filters.

Proof: Take a σ-Scott open set O in a continuous σ-frame L. Take any $a \in O$. $\exists a_1 \in O$ with $a_1 \ll_\sigma a$, since O is σ-Scott open. Continuing in this way we get,

$$a \gg_\sigma a_1 \gg_\sigma a_2 \gg_\sigma \cdots \gg_\sigma a_n \gg_\sigma \cdots$$

all in O and put $F_a = \{x \in L \mid x \geq a_n, \text{ some } n \in \mathbb{N}\}$. Then F_a is a σ-Scott open filter and $a \in F_a \subseteq O$. Then

$$O = \bigcup_{a \in O} F_a.$$

□

The next three statements are due to B.Banaschewski in [2].

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Lemma 3.1.5 For any σ-Scott open filter F disjoint from an ideal J, there exists a σ-Scott open prime filter $G \supseteq F$, disjoint from J.

Proof: Since any updirected union of σ-Scott open filters is a σ-Scott open filter there exists, by Zorns Lemma, a σ-Scott open filter $G \supseteq F$, maximal such that $G \cap J = \emptyset$. We show now that G is prime:
Suppose that G is not prime i.e. take $b, c \notin G$ with $b \vee c \in G$.
Now put $H = \{x \in L \mid b \vee x \in G\}$. $c \in H$, so $H \neq \emptyset$.
H is a σ-Scott open filter : Take countable $X \subseteq L$ with $\bigvee X \in H$,

$$\begin{aligned}
&\Rightarrow b \vee \bigvee X \in G, \\
&\Rightarrow \bigvee\{b \vee x \mid x \in X\} \in G, \\
&\Rightarrow \exists E \subseteq X, \text{ finite, with } \bigvee\{b \vee x \mid x \in E\} \in G, \\
&\Rightarrow \bigvee E \in H.
\end{aligned}$$

Trivially H is a filter with $G \subsetneqq H$, as $c \notin G$. Since G is maximal disjoint from J, $J \cap H \neq \emptyset$. Thus take $a \in J \cap H$ and put $K = \{x \in L \mid x \vee a \in G\}$. By repeating the argument above, it can be shown that K is σ-Scott open with $G \subsetneqq K$ since $b \in K$. Hence there exists $d \in K \cap J$, i.e $d \vee a \in G$ - a contradiction since $d \vee a \in J$. □

Corollary 3.1.6 For any σ-Scott open filter F, the set
$\Psi_F = \{P \mid F \subseteq P \in \Psi L\}$ is compact in the σ-space ΨL, and in fact compact in the underlying topology of ΨL.

Proof: Let $\Psi_F \subseteq \bigcup_{a \in S} \Psi_a$, for any subset $S \subseteq L$. Then also $\Psi_F \subseteq \bigcup_{a \in J} \Psi_a$ for the ideal J generated by S, and this says that, for any σ-prime filter P, $P \supseteq F$ implies that $P \cap J \neq \emptyset$. By the above Lemma 3.1.5, $F \cap J \neq \emptyset$, and therefore $\Psi_F \subseteq \psi_a$, for some $a \in J$. Since each element of J is below the join of finitely many elements of S this shows that Ψ_F is compact in the underlying topology of ΨL, and so compact in ΨL trivially. □

Corollary 3.1.7 Given that L is a continuous σ-frame, the underlying topology of ΨL is locally compact and σ-compact.

Proof: Take any 'point' $P \in \Psi L$ and any $a \in L$ with $P \in \Psi_a$. Since P is σ-Scott open by Proposition 3.1.2, there exists $c \in P$ with $c \ll_\sigma a$. Then the interpolation property of $\ll_\sigma$ leads to the sequence $a = a_1 \gg_\sigma a_2 \gg_\sigma \ldots \gg_\sigma c$ and the filter F generated by $(a_n)_{n \in \mathbb{N}}$ is σ-Scott open. It follows that $P \in \Psi_c \subseteq \Psi_F \subseteq \Psi_a$ where Ψ_F is compact by 3.1.6. Thus the compact neighborhoods of P form a neighborhood base i.e. the underlying topology

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of ΨL is locally compact. Further, if $a \ll_\sigma e$ then one has, as was shown, a compact set Δ such that $\Psi_a \subseteq \Delta \subseteq \Psi_e = \Psi L$, and since e is the join of countably many such a this shows that ΨL is the union of countably many compact sets. □

Lemma 3.1.8 Every σ-Scott open filter is an intersection of σ-Scott open prime filters.

Proof: Given F a σ-Scott open filter, take $a \notin F$, then F and the ideal $\downarrow a$ are disjoint. By the lemma 3.1.5 above, there exists a σ-Scott open prime filter G with $a \notin G$. Thus F is the intersection of such filters. □

Corollary 3.1.9 Every continuous σ-frame is spatial.

Proof: Taking a continuous σ-frame L, all we need to show is that the co-unit map $\varepsilon_L : L \to \Sigma\Psi L$, of the dual adjunction between σ-spaces and σ-frames, is one-one.
To this end, it suffices to show the σ-prime filters of L separate the elements, that is, if $a \not\leq b$ there exists a σ-prime filter G for which $a \in G$ and $b \notin G$. Now, $a \in L \backslash \downarrow b$ and $L \backslash \downarrow b$ is trivially σ-Scott open:
for any countable $S \subseteq L$, if $S \cap (L \backslash \downarrow b) = \emptyset$ then $S \subseteq \downarrow b$ so that $\bigvee S \leq b$ and then $\bigvee S \notin L \backslash \downarrow b$.
Hence by Lemma 3.1.4 there exists a σ-Scott open filter F such that $a \in F \subseteq L \backslash \downarrow b$, and by Lemma 3.1.5 there further exists a prime σ-Scott open filter G for which $F \subseteq G$ and $G \cap \downarrow b = \emptyset$ and hence $a \in G$ and $b \notin G$. Finally, G is a σ-prime filter since L is a continuous, by Proposition 3.1.3, which proves the claim. □

3.2 Spectral σ-Spaces.

A subset of a σ-space is compact if every countable covering by σ-open sets has a finite subcover.

Definition 3.2.1 A σ-space X is said to be *pre-spectral* if ΣX, its σ-frame of σ-open sets is continuous.

In topological spaces, local compactness implies that the frame of open sets is a continuous frame. However, it is not the case that all topological spaces with continuous frames of open sets are locally compact (however, under the condition of sobriety, it is true).

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This contrasts with our situation here, since a σ-space is pre-spectral if and only if its σ-frame of σ-open sets is continuous, by our definition. However, as we shall soon see in the case of sober σ-spaces, there are other similarities between locally compact spaces and pre-spectral σ-spaces.

In the case of a σ-space X with σ-open sets ΣX, we put, for any $A \subseteq X$

(1) $\Sigma(A) = \{B \in \Sigma X \mid A \subseteq B\}$.

(2) A is *saturated* if $A = \bigcap \Sigma(A)$.

We now present, following an unpublished document of C.Gilmour, that was presented at the Aspects of Contemporary Topology Workshop, UA-VUB, Belgium, 2003, the Hofmann-Mislove theorem, as applied to σ-spaces and σ-frames.

Theorem 3.2.2 Given a sober σ-space X, the σ-Scott open filters on ΣX are precisely those of the form $\Sigma(K)$, for some compact saturated subset K of X.

Proof: ($\Leftarrow$) This is straight forward.
($\Rightarrow$) Let $\mathcal{F}$ be σ-Scott open and let $K = \bigcap \mathcal{F}$.
($*$) If $A \in \Sigma X$, $K \subseteq A$ then $A \in \mathcal{F}$:
Suppose not, then by Lemma 3.1.8, there exists σ-prime $\mathcal{G} \supseteq \mathcal{F}$ such that $A \notin \mathcal{G}$. By sobriety, $\mathcal{G} = \{B \in \Sigma X \mid x \in B\}$, some $x \in X$. Since $\mathcal{F} \subseteq \mathcal{G}$, $x \in \bigcap \mathcal{F} = K$. But $x \notin A$, since $A \notin \mathcal{G}$, yet $K \subseteq A$, a contradiction.

K is compact:
Suppose that $K \subseteq \bigcup_{n \in \mathbb{N}} A_n$ with $A_n \in \Sigma X$.

$$\bigcup_n A_n \in \mathcal{F}, \text{ by } (*) \;\Rightarrow\; \exists \text{ finite } I \subset \mathbb{N} \text{ with } \bigcup_{i \in I} A_{n_i} \in \mathcal{F}$$

since $\mathcal{F}$ is σ-Scott open. Finally $K = \bigcap \mathcal{F} \subseteq \bigcup_i A_{n_i}$, and so K is compact.

By ($*$) and the definition of K, $\mathcal{F} = \{A \in \Sigma X \mid K \subseteq A\}$, and so K is saturated. $\square$.

Remark 3.2.3 In the above theorem, the compact K obtained is in fact compact in the underlying topology of X:

It is clear that $\Sigma(K)$ is a σ-Scott open filter in ΣX. By Corollary 3.1.6, the subset of $\Psi\Sigma X$, $\mathcal{K} = \{\mathcal{P} \in \Psi\Sigma X \mid \Sigma(K) \subseteq \mathcal{P}\} = \Psi_{\Sigma(K)}$ is compact

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in the underlying topology of $\Psi\Sigma X$. It remains only to show that K is isomorphic to $\mathcal{K}$, by way of the unit η_X, of the dual equivalence **SobσSp** $\cong$ **SpatialσFrm**. To this end, take any $x \in K$. Then $\Sigma(K) \subseteq \Sigma(x)$ and so $\Sigma(x) \in \mathcal{K}$. Now take any $\mathcal{P} \in \mathcal{K}$. Then $\mathcal{P} = \Sigma(x)$, for some $x \in X$, since X is sober. Since $\Sigma(K) \subseteq \Sigma(x)$, it is clear that

$$x \in \bigcap \Sigma(K) = K.$$

since such a K in the theorem is saturated. Thus we see that $\eta_X(K) = \mathcal{K}$.□

From here on, all the compact subsets mentioned, are in fact compact in the underlying topology, by the above remark.

Definition 3.2.4 A σ-space is *spectral* if it is pre-spectral and sober.

Together with the σ-space maps, the spectral σ-spaces form the category **SpecσSp**. It is clear from the above definition that the dual adjunction between the categories **σSp** and **σFrm** restricts to a dual equivalence **SpecσSp** $\cong$ **ContσFrm**.

Proposition 3.2.5 Given a spectral σ-space X, for any $A, B \in \Sigma X$, $A \ll_\sigma B$ if and only if there exists $K \subseteq X$ compact and saturated with $A \subseteq K \subseteq B$.

Proof: Since $\ll_\sigma$ interpolates in a continuous σ-frame, we have, choosing C_n inductively,

$$A \ll_\sigma C_1 \ll_\sigma C_2 \ll_\sigma \ldots \ll_\sigma C_n \ll_\sigma \ldots \ll_\sigma B.$$

Put $\mathcal{F} = \{D \in \Sigma X \,|\, C_n \subseteq D \text{ some } n \in \mathbb{N}\}$. Then $\mathcal{F}$ is a σ-Scott open filter, and so by the Hofmann-Mislove theorem, $\mathcal{F} = \Sigma(K)$, for some compact saturated $K \subseteq X$. For each $D \in \Sigma(K)$, $A \subseteq D$. This implies that $A \subseteq \bigcap \mathcal{F} = K$ and $B \in \mathcal{F}$ implies that $K \subseteq B$. □

The following corollary suggests that the use of the terminology 'spectral' is justified.

Corollary 3.2.6 In a spectral σ-space, every σ-open set is a countable union of compact sets.

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Proof: Given that X is locally compact, take any $A \in \Sigma X$. Then $A = \bigcup\{C_n \,|\, C_n \ll_\sigma A,\ n \in \mathbb{N}\}$.
For each n there exists compact $K_n \subseteq A$, by Proposition 3.2.5, with $C_n \subseteq K_n \subseteq A$. Hence

$$A = \bigcup_{n \in \mathbb{N}} K_n.$$

□

Corollary 3.2.7 In a spectral σ-space, every σ-open set is a countable union of the interior, in the underlying topology, of compact sets.

Proof: Suppose that, for σ-open sets U, V, $V \subseteq K \subseteq U$. Then $V \subseteq$ int(K) $\subseteq$ U, since the interior of a set is the largest open set contained in a set. Given this and Corollaries 3.2.5 and 3.2.6, the result is now clear. □.

Proposition 3.2.8 The underlying topology of a spectral σ-space is locally compact.

Proof: Take X to be a spectral space. Since X is sober, the underlying topology of X can be identified with the underlying topology of $\Psi\Sigma X$. But by Corollary 3.1.7, $\Psi\Sigma X$ has locally compact underlying topology. □

A Property of Local compactness

It is known that for topological spaces X and Y, if X is locally compact, then $\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \oplus \mathcal{O}(Y)$. We now show that pre-spectral σ-spaces have a similar property. But first we show some preliminary results. We shall follow Pultr's development for frames [21].

For σ-frames L_1 and L_2, let $\iota_i : L_i \to L_1 \oplus L_2$ be the coproduct injections. Furthermore, for $a_1 \in L_1$ and $a_2 \in L_2$, let $a_1 \oplus a_2 = \iota_1(a_1) \wedge \iota_2(a_2)$.

Lemma 3.2.9 Given σ-frames L_1 and L_2, if $a_1 \in L_1$ and $a_2 \in L_2$ then

$$\downarrow a_1 \oplus \downarrow a_2 = \downarrow (a_1 \oplus a_2).$$

More precisely,

$$\iota_i' = (x \mapsto \iota_i(x) \wedge (a_1 \oplus a_2)) : \downarrow a_i \to \downarrow (a_1 \oplus a_2).$$

are coproduct injections to $\downarrow a_1 \oplus \downarrow a_2$.

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Proof: Let $h_i : \downarrow a_i \to M$ be σ-frame homomorphisms. Let $\widehat{a}_i(x) = a_i \wedge x$. By the property of coproducts, there exists $g : L_1 \oplus L_2 \to M$ with $g \circ \iota_i = h_i \circ \widehat{a}_i$.

Now, for any $x_i \in L_i$,

$$\begin{aligned} g(x_1 \oplus x_2) &= g(\iota_1(x_1) \wedge \iota_2(x_2)) \\ &= g\iota_1(x_1) \wedge g\iota_2(x_2) \\ &= h_1\widehat{a}_1(x_1) \wedge h_2\widehat{a}_2(x_2) \\ &= h_1(x_1 \wedge a_1) \wedge h_2(x_2 \wedge a_2). \end{aligned}$$

Now consider $\widehat{a_1 \oplus a_2} : L_1 \oplus L_2 \to \downarrow (a_1 \oplus a_2)$ and suppose that $\widehat{a_1 \oplus a_2}(x_1 \oplus x_2) = \widehat{a_1 \oplus a_2}(y_1 \oplus y_2)$. Then

$$\begin{aligned} g(x_1 \oplus x_2) &= g(x_1 \oplus x_2) \wedge g(e) \\ &= g(x_1 \oplus x_2) \wedge g(\iota_1(e) \wedge \iota_2(e)) \\ &= g(x_1 \oplus x_2) \wedge h_1(e \wedge a_1) \wedge h_2(e \wedge a_2) \\ &= g(x_1 \oplus x_2) \wedge h_1(a_1 \wedge a_1) \wedge h_2(a_2 \wedge a_2) \\ &= g(x_1 \oplus x_2) \wedge g(a_1 \oplus a_2) \\ &= g((y_1 \oplus y_2) \wedge (a_1 \oplus a_2)) \\ &= g(y_1 \oplus y_2), \quad \text{similar as above.} \end{aligned}$$

Thus there exists a unique $h : \downarrow (a_1 \oplus a_2) \to M$ such that $h \circ \widehat{a_1 \oplus a_2} = g$, since $\downarrow (a_1 \oplus a_2)$ is a quotient of $L_1 \oplus L_2$.
Take any $x \in \downarrow a_i$, then

$$\begin{aligned} h \circ \iota_i'(x) &= h(\iota_i(x) \wedge (a_1 \oplus a_2)) \\ &= h \circ \widehat{a_1 \oplus a_2}(\iota_i(x)) \\ &= g(\iota_i(x)) = h_i\widehat{a}_i(x) \\ &= h_i(x \wedge a_i) \\ &= h_i(x) \quad \text{since } x \in \downarrow a_i. \end{aligned}$$

Since h is unique $\iota_i : \downarrow a_i \to \downarrow (a_1 \oplus a_2)$ is a coproduct in σ**Frm**. □

In a σ-frame L, $D \subseteq L$ is said to be countably directed if it is at most countable and for any $x, y \in D$, there exists $z \in D$ with $x \vee y \leq z$.

It is a trivial exercise to show that for any $a, b \in L$

$a \ll_\sigma b$ iff for every countably directed $D \subseteq L$, $b \leq \bigvee D \Rightarrow \exists x \in D$ with $a \leq x$.

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Lemma 3.2.10 **(1)** In any σ-frame, $a \prec b \ll_\sigma e \Rightarrow a \ll_\sigma b$.

(2) If L is regular then $a \ll_\sigma b \Rightarrow a \prec b$.

Proof: (1) suppose $b \leq \bigvee D$, D countably directed. There exists $s \in L$ with $a \wedge s = 0$ and $b \vee s = e$. So $s \vee \bigvee D = e$. Because $b \ll_\sigma e$, there exists $d \in D$ with $b \leq s \vee d$. Finally

$$\begin{aligned} a &= a \wedge b \\ &\leq a \wedge (s \vee d) \\ &= (a \wedge s) \vee (a \wedge d) \\ &= a \wedge d. \quad \text{i.e. } a \leq d. \end{aligned}$$

(2) $b = \bigvee X$, X at most countable with $x \prec b$, $\forall x \in X$. Put $Y = \{\bigvee A \mid \text{finite } A \subseteq X\}$. Then for all $y \in Y$, $y \prec b$ and Y is countably directed. Because $a \ll_\sigma b$, there exists $z \in Y$ with $a \leq z$. So finally $a \leq z \prec b$. □

In a compact regular σ-frame, $e \ll_\sigma e$. So whenever $a \prec b$, we also have that $a \prec b \prec e \ll_\sigma e$, since e is rather above all elements of a σ-frame. But then by (1) of the above lemma, $a \prec b \ll_\sigma e$ and so by applying the lemma again $a \ll_\sigma b$. That is to say that a compact regular σ-frame is continuous.

In fact we can say more, which we do after the following definition,

Definition 3.2.11 Given a σ-frame L, an *open quotient* of L is a σ-frame $\downarrow u$, where $u \in L$, with σ-frame quotient map $\widehat{u} : L \to \downarrow u$, given by $\widehat{u}(-) = - \wedge u$.

Proposition 3.2.12 Any open quotient of a regular compact σ-frame is continuous.

Proof: Let L be a compact regular σ-frame, with $\downarrow u \subseteq L$, an open quotient. $e \ll_\sigma e$, since L is compact, and so by the lemma above, $a \prec b \Rightarrow a \ll_\sigma b$. Furthermore, if $a \ll_\sigma b$ in L and $b \leq u$ then also $a \ll_\sigma b$ in $\downarrow u$. □

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Proposition 3.2.13 A regular σ-frame is continuous if and only if it is an open quotient in its Stone-Čech compactification (that is iff $k_L : \mathrm{K}L \to L$ is an open quotient map).

Proof: ($\Leftarrow$) follows from Proposition 3.2.12.
($\Rightarrow$) We have to show that if L is continuous then there is a regular countably generated ideal A in L such that for any two $J_1, J_2 \in \mathrm{K}L$

$$k_L(J_1) = \bigvee J_1 = \bigvee J_2 = k_L(J_2) \quad \text{if and only if} \quad J_1 \cap A = J_2 \cap A.$$

Let $A = \{x \in L \mid x \ll_\sigma e\}$. A is countably generated since $e = \bigvee_n a_n$, $a_n \ll_\sigma e$, as L is continuous (indeed if $x \in A$, then $x \leq a_{n_1} \vee \cdots \vee a_{n_l}$).
For $x \in A$ then $x \ll_\sigma e$. Since $\ll_\sigma$ interpolates, we can have $y \in L$ with $x \ll_\sigma y \ll_\sigma e$. So $y \in A$ and by Lemma 3.2.10 (2), $x \prec y$. Thus A is regular.
Also, $k_L(A) = e$. Now, assume that $J_1 \cap A = J_2 \cap A$. Then

$$\begin{aligned} k_L(J_1) &= k_L(J_1) \wedge k_L(A) \\ &= k_L(J_1 \cap A) \\ &= k_L(J_2 \cap A) \\ &= k_L(J_2) \wedge k_L(A) \\ &= k_L(J_2). \end{aligned}$$

On the other hand, suppose that $\bigvee J_1 = \bigvee J_2$ and take any $a \in J_1 \cap A$. Because $J_1 \cap A$ is regular, we can find $b \in J_1 \cap A$ with $a \prec b$. Therefore, $a \ll_\sigma b$, since $b \ll_\sigma e$ and by Lemma 3.2.10 (1). Since

$$b \leq \bigvee J_1 = \bigvee J_2,$$

this implies that there exists $x \in J_2$ with $a \leq x$, since ideals are directed. So $a \in J_2$, as ideals are closed downwards. Thus $J_1 \cap A \subseteq J_2 \cap A$ and by symmetry, equality is attained. □

Corollary 3.2.14 The coproduct of two regular continuous σ-frames is continuous.

Proof: Take regular continuous σ-frames L and M. Then by the above theorem, $L \cong \downarrow A \subseteq \mathrm{K}L$ and $M \cong \downarrow B \subseteq \mathrm{K}M$ for countably generated regular ideals A and B, of L and M respectively.
By Lemma 3.2.9,

$$L \oplus M \cong \downarrow A \oplus \downarrow B \cong \downarrow (A \oplus B).$$

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This is an open quotient of $\mathbb{K}L \oplus \mathbb{K}M$, which is compact and completely regular. Thus $L \oplus M$ is continuous by Proposition 3.2.13. □

Finally, the promised result.

Corollary 3.2.15 For X and Y regular and pre-spectral σ-spaces, $\Sigma X \oplus \Sigma Y \cong \Sigma(X \times Y)$.

Proof:

ΣX and ΣY are continuous by the assumption
$\Rightarrow \Sigma X \oplus \Sigma Y$ is continuous by Corollary 3.2.14
$\Rightarrow \Sigma X \oplus \Sigma Y$ spatial
$\Rightarrow \Sigma X \oplus \Sigma Y \cong \Sigma(X \times Y)$ by Corollary 2.2.22

□

σ-Spectral spaces and Spectral σ-Spaces

In [2], B.Banaschewski introduces the σ-spectral topological spaces, which are the sober topological spaces whose topology has as basic sublattice (= base that forms a lattice) the $\mathrm{K}\sigma$-open sets. A set is called $\mathrm{K}\sigma$-open if it is a countable union of the interiors of compact sets contained in it i.e O is $\mathrm{K}\sigma$-open if

$$O = \bigcup_{n \in \mathbb{N}} \mathrm{Int}(K_n) \quad \text{with compact } K_n \subseteq O.$$

Taking these topological spaces, together with all continuous maps with the property that the inverse images $\mathrm{K}\sigma$-opens are $\mathrm{K}\sigma$-open, we have the category we call σ**SpecSp**.

Examples of σ-spectral spaces are the σ-compact locally compact Hausdorff spaces since the intersection of any pair of compact sets is compact in such a space.

Lemma 3.2.16 In a σ-spectral space, the $K\sigma$-open sets form a continuous σ-frame.

Proof: Each $K\sigma$-open set, say V, is a countable union of compact sets (hence Lindelöf) and a countable union of their interiors. The interior of each of these compact sets is a union of $K\sigma$-open sets, because the $K\sigma$-open sets generate the topology. Thus, we can conclude that V is a union of $K\sigma$-opens sets. Let W be one such set. Then W is contained in some compact

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set, say K, contained in V, i.e. $W \subseteq K \subseteq V$. Trivially, $W \ll_\sigma V$, since K is compact. But V is a countable union of such W because V is Lindelöf. □

If, for X a σ-spectral space, we let $\mathcal{K}_\sigma X$ be the σ-space that has the same underlying topology as X with the K_σ-opens as the σ-open sets, then $\mathcal{K}_\sigma X$ is a pre-spectral σ-space. In fact, by the proceeding lemma, it is spectral i.e. sober as well. For Y a spectral σ-space, by Corollary 3.2.7, $\mathcal{T}Y$ is a σ-spectral space.

Lemma 3.2.17 $\mathcal{K}_\sigma X$ is a sober σ-space.

Proof: Take $\mathcal{F} \subseteq \Sigma(\mathcal{K}_\sigma X)$, a σ-prime filter. Clearly $\mathcal{F} \subseteq \Omega X$, that is, it is a collection of open sets in the original topology of X. Let $\mathcal{P} \subseteq \Omega X$ be the filter generated by $\mathcal{F}$. We show that $\mathcal{P}$ is completely prime:

suppose that for open sets O_α,

$$\bigcup O_\alpha \in \mathcal{P}.$$

Then $\exists V \in \mathcal{F}$ with $V \subseteq \bigcup O_\alpha$. Since V is K_σ-open, it is Lindelöf. Thus V is contained in the union of the countable sub-collection $\{O_{\alpha_n} \mid n \in \mathbb{N}\}$. Since $\mathcal{F}$ is σ-prime, there is some $m \in \mathbb{N}$ with $O_{\alpha_m} \in \mathcal{F} \subseteq \mathcal{P}$. And so $\mathcal{P}$ is completely prime.

Thus, by the sobriety of X as a topological space, $\mathcal{P}$ is a principal filter and hence also $\mathcal{F}$ is a principal filter. □

Together with the above lemma, we now see that $\mathcal{K}_\sigma$ and $\mathcal{T}$ are in fact functors $\mathcal{T} : \sigma\mathbf{SpecSp} \to \mathbf{Spec}\sigma\mathbf{Sp}$ and $\mathcal{K}_\sigma : \mathbf{Spec}\sigma\mathbf{Sp} \to \sigma\mathbf{SpecSp}$, with the obvious associated morphism maps :

For a morphism of $f : X \to Y$ of $\sigma\mathbf{SpecSp}$, $\mathcal{T}(f)$ $(= f$ as a set map) is a morphism of $\mathbf{Spec}\sigma\mathbf{Sp}$ by the following argument. Take any K_σ-open set, O, of $\mathcal{T}Y$. Because O is a countable union of σ-open sets, it is Lindelöf. By this fact and the definition of open sets in $\mathcal{T}Y$, O is a countable union of σ-open sets of Y i.e $O \in \Sigma Y$. But then $f^{\leftarrow}(O) \in \Sigma X$ and in fact is K_σ-open in $\mathcal{T}X$ by Corollary 3.2.7. Thus the inverse image maps of $\mathcal{T}(f)$ preserve K_σ-open sets.

That $\mathcal{K}_\sigma(g)$ $(= g$ as a set map) is a morphism of $\sigma\mathbf{SpecSp}$ is straightforward.

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Proposition 3.2.18 σ**SpecSp** $\cong$ **Spec**σ**Sp**.

Proof: We show that the identity maps on the underlying sets,

$$\lambda_X : X \to \mathcal{T}K_\sigma X \qquad \varepsilon_Y : \mathcal{K}_\sigma \mathcal{T}Y \to Y$$

are isomorphisms, for X a σ-spectral space and Y a spectral σ-space. In the case of λ_X it is trivial, by the definition of spectral and underlying topology of a σ-space. As for ε_Y, all we need to show is that $\Sigma Y \cong \Sigma\mathcal{K}_\sigma\mathcal{T}Y$.
To this end, take any $U \in \Sigma\mathcal{K}_\sigma\mathcal{T}Y$. By the definition of $\mathcal{T}Y$, U is a union of σ-open sets in ΣY. But since U is K_σ-open, it is Lindelöf and, as a result, it is a countable union of σ-opens. On the other hand, if $U \in \Sigma Y$, then by Corollary 3.2.7, U is K_σ-open in the underlying topology. □

In [2], it is shown that σ**SpecSp** $\cong$ **Cont**σ**Frm**. Since our definition of σ-spectral spaces immediately imply, by the spatiality of continuous σ-frames, that **Spec**σ**Sp** $\cong$ **Cont**σ**Frm** the equivalence of Proposition 3.2.18 was immediate from the outset. However, we now more explicitly appreciate that the K_σ-opens of a σ-spectral space form a sober σ-space.

Put in another way, the sobriety of the underlying topology of a σ-space does not necessarily imply the sobriety of the original σ-space (for instance look at Example 2.1.5). However, we have here at least one instance where the implication holds.

3.3 Coherence and Stability

We restrict further, the dual equivalence of Proposition 3.2.18.

Definition 3.3.1 A σ-frame L is *stably continuous* if it is continuous and the relation $\ll_\sigma$ forms a sublattice in $L \times L$.

Of particular importance, is the fact that with stability, $a \ll_\sigma b$ and $c \ll_\sigma d$ implies $a \wedge c \ll_\sigma b \wedge d$.

In a lattice L with subset $S \subseteq L$, we denote by $\langle S \rangle$, the subset consisting of all elements greater than or equal to finite meets of elements in S.

Lemma 3.3.2 If F and G are σ-Scott open filters and $\langle F \cup G \rangle \neq L$, then $\langle F \cup G \rangle$ is a σ-Scott open filter.

University of Cape Town

Proof: Take any $x \in \langle F \cup G\rangle$. Then $x \geq a \wedge b$, some $a \in F$ and $b \in G$. Since F and G are σ-Scott open and with Proposition 3.1.2, $\exists c \in F$ and $d \in G$ with $c \ll_\sigma a$ and $d \ll_\sigma b$. By stability, $c \wedge d \ll_\sigma a \wedge b \leq x$. But $c \wedge d \in \langle F \cup G\rangle$ with $c \wedge d \ll_\sigma x$, so $\langle F \cup G\rangle$ is σ-Scott open, again using Proposition 3.1.2. □

Proposition 3.3.3 Compact saturated subsets of ΨL are closed under finite intersection.

Proof: Given compact saturated $\mathcal{K} \subseteq \Psi L$, then $\mathcal{K} = \Psi_G$ where G is the σ-Scott open filter $G = \{a \in L \,|\, \mathcal{K} \subseteq \Psi_a\}$:
Take any $P \in \Psi_G$. Then $P \in \Psi_a$ whenever $\mathcal{K} \subseteq \Psi_a$. Hence

$$P \in \bigcap\{\Psi_a \,|\, \mathcal{K} \subseteq \Psi_a\} = \mathcal{K}$$

since $\mathcal{K}$ is saturated. Otherwise, suppose $P \in \mathcal{K}$. Then take any $a \in G$. But of course

$$P \in \mathcal{K} \subseteq \Psi_a \Rightarrow a \in P,$$

from which we conclude that $G \subseteq P$ i.e. $P \in \Psi_G$. That G is σ-Scott open is trivial.

Now, given compact saturated sets $\mathcal{K}_1, \mathcal{K}_2 \subseteq \Psi L$, $\mathcal{K}_i = \Psi_{G_i}$ with $G_i = \{a \,|\, \mathcal{K}_i \subseteq \Psi_a\}$, $i = 1, 2$. We will show that $\Psi_{G_1} \cap \Psi_{G_2} = \Psi_{\langle G_1 \cup G_2\rangle}$. To this end first take $P \in \Psi_{G_1} \cap \Psi_{G_2}$. Then $G_1 \subseteq P$ and $G_2 \subseteq P \Rightarrow \langle G_1 \cup G_2\rangle \subseteq P$, since P is a filter. On the other hand if $\langle G_1 \cup G_2\rangle \subseteq P$ then trivially $G_1 \subseteq P$ and $G_2 \subseteq P$.

Finally, if $\langle G_1 \cup G_2\rangle = L$ then $\Psi_{\langle G_1 \cup G_2\rangle} = \emptyset$ which is compact and saturated. Otherwise, $\langle G_1 \cup G_2\rangle$ is a σ-Scott open filter, by the previous lemma and by the fact that $0 \notin \langle F \cup G\rangle$. Thus $\Psi_{\langle G_1 \cup G_2\rangle} = \mathcal{K}_1 \cap \mathcal{K}_2$ is compact and saturated by Corollary 3.1.6. □.

Corollary 3.3.4 If L is stably continuous then the compact saturated subsets of ΨL form a generating sublattice for the σ-open subsets of ΨL.

Proof: Take any σ-open set, Ψ_a, in ΨL. By the continuity of L, $\Psi_a = \bigcup_{x \in X} \Psi_x$ where X is countable and $x \ll_\sigma a$ for each $x \in X$. As in Corollary 3.1.7, $\Psi_x \subseteq \mathcal{K} \subseteq \Psi_a$ for some compact saturated $\mathcal{K}$. Thus Ψ_a is a countable union of compact saturated sets. As a finite union of compact saturated sets is trivially compact and saturated, together with Proposition 3.3.3, the proof is complete. □

University of Cape Town

A σ-frame map is said to be proper if it preserves the σ-way below relation. The stably continuous σ-frames together with the proper maps give the category **StContσFrm**.

Definition 3.3.5 A spectral σ-space is said to be *coherent* if its compact saturated sets form a generating sublattice.

Proposition 3.3.6 For a coherent σ-space X, ΣX is a stably continuous σ-frame.

Proof: We already know that ΣX is continuous, since X is locally compact by definition. For $A \ll_\sigma B$ and $C \ll_\sigma D$, by Proposition 3.2.5, there exist compact saturated sets K_1 and K_2 with $A \subseteq K_1 \subseteq B$ and $C \subseteq K_2 \subseteq D$. Since $K_1 \cap K_2$ is compact, then $A \cap C \ll_\sigma B \cap D$. □

The coherent σ-spaces, together with maps with inverse maps that preserve compact saturated sets, form the category **CohσSp**. It is clear from Corollary 3.3.4 that the σ-space spectrum of a stably continuous L is coherent. Hence we have as a restriction of the duality in Proposition 3.2.18,
CohσSp $\cong$ **StContσFrm**.

University of Cape Town

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