

CHARACTERIZATION OF COEXTENSIVE VARIETIES OF UNIVERSAL ALGEBRAS

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ABSTRACT. A coextensive category can be defined as a category \mathcal{C} with finite products such that for each pair X, Y of objects in \mathcal{C} , the canonical functor $\times: X/\mathcal{C} \times Y/\mathcal{C} \longrightarrow (X \times Y)/\mathcal{C}$ is an equivalence. In this thesis we give a syntactic characterization of coextensive varieties of universal algebras. We first show that any such variety must have what we call a diagonalizing term. The existence of such a term is a Mal'tsev condition which is interesting in its own right, and we show that it is sufficient to prove many useful subconditions of coextensivity. We also introduce the notion of a category with upward closed subproducts as a categorical generalization of varieties with diagonalizing terms, which we study in the more general context of Barr-exact categories.

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1. Introduction

A category \mathcal{C} with finite coproducts is said to be extensive if for each pair X, Y of objects in \mathcal{C} , the canonical functor $+ : \mathcal{C}/X \times \mathcal{C}/Y \longrightarrow \mathcal{C}/(X + Y)$ is an equivalence. The definitive account of this conditions was given by A. Carboni, S. Lack, and R.F.C. Walters in their 1992 paper [1]. The term “extensive category” was first used by F. W. Lawvere and S. Schanuel, although “categories with disjoint and universal coproducts” were considered by A. Grothendieck a long time ago, and there are related papers of various authors. Extensive categories can be thought of as categories in which finite coproducts exist and are well behaved. The simplest example of an extensive category is the category **Sets** of sets, in which coproducts are simply disjoint unions.

Coproducts in **Sets** have two useful properties which are enough to make **Sets** extensive. Firstly, they are disjoint in the sense that the pullback of the coproduct diagram $X \xrightarrow{i_X} X + Y \xleftarrow{i_Y} Y$ is the initial object 0 , which is the empty set in **Sets**. When a category has this property we say that it has disjoint coproducts. Secondly, given any map $f : A \longrightarrow X + Y$, the pullbacks of f along the injections i_X and i_Y form a coproduct. In **Sets**, such pullbacks are the inverse images $f^{-1}(X)$ and $f^{-1}(Y)$ of X and Y under f . It is clear that every element of A is mapped by f into either X or Y , and so the set A can be decomposed as $A \simeq f^{-1}(X) + f^{-1}(Y)$ as desired. When a category has this property we say it has universal coproducts. A category is extensive if and only if it has universal and disjoint coproducts (Proposition 2.14 of [1]), and so we can think of extensive categories as those categories with coproducts that interact nicely with pullbacks.

Any topos or Boolean category is extensive [1]. Other notable examples include the categories **Sets** of sets, **Top** of topological spaces, and the opposite category of commutative rings, **CRing**^{op}. That **Top** is extensive is of importance to the study of topological spaces. For example, it is well known that a topological space A is connected if and only if whenever $A \simeq X + Y$ then exactly one of X or Y is trivial, and that for any continuous map f , the image $f(A)$ is also connected. It follows straightforwardly from the definition of extensivity that the same result holds true in any extensive category. The extensivity of **CRing**^{op} is of interest to the study of algebraic geometry for similar reasons, since this category is isomorphic to the category of affine schemes. This example is one motivation for studying coextensive varieties of universal algebras.

The dual condition to extensivity is coextensivity. Specifically, a category \mathcal{C} with finite products is said to be coextensive if for each pair X, Y of objects in \mathcal{C} , the canonical functor $\times : X/\mathcal{C} \times Y/\mathcal{C} \rightarrow (X \times Y)/\mathcal{C}$ is an equivalence. Since coextensivity is the dual condition of extensivity, coextensive categories can be thought of as categories in which finite products exist and are well behaved in the sense that they interact nicely with pushouts. When \mathcal{C} has pushouts, the canonical functor \times has a left adjoint $L : (X \times Y)/\mathcal{C} \rightarrow X/\mathcal{C} \times Y/\mathcal{C}$, which is the functor that sends a morphism $h : X \times Y \rightarrow Z$ to its pushouts across π_X and π_Y . In this case \mathcal{C} is coextensive if and only if the unit η and counit ε of the adjunction are natural isomorphisms. In fact, coextensivity has multiple equivalent definitions. A category with finite products and pushouts along product projections is coextensive if, for any commutative diagram,

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow h & & \downarrow g \\
 A & \xleftarrow{p_1} & Z & \xrightarrow{p_2} & B
 \end{array} \tag{1}$$

the bottom row is a product diagram if and only if both squares are pushouts. Equivalently, a coextensive category can be considered as a category with co-disjoint and co-universal products, where these conditions are dual to the disjoint and universal coproducts of extensive categories. We also introduce the notion of left and right coextensive categories. Here we do not require that the canonical functor \times is an equivalence, but rather that above adjunction has an invertible unit in the case of right coextensive categories and an invertible co-unit in the case of left coextensive categories. It turns out that a category is right coextensive if and only if it has co-universal products. On the other hand, any left coextensive category has co-disjoint products, but the converse is not true. We will first provide a syntactic characterization of left-coextensive varieties of universal algebras, before moving on to characterizing coextensive varieties, which is our main result.

Universal algebra is a field of study introduced as a general context in which to study many well-known algebraic structures such as groups, rings, monoids, lattices and more. In each case, the collection of all such algebras forms what is called a variety of algebras. In categorical algebra it is typical to consider such varieties as categories, with homomor-

phisms of algebras as morphisms. Universal algebra was introduced by Birkhoff in the 1930s in his papers [15] and [16] as a purely algebraic theory. This was later followed by other more categorical approaches to universal algebra, using either Lawvere theories or monads, which are outside the scope of this thesis.

We take the approach of defining a set Ω of operations ω , each of a fixed arity $a\omega$. An Ω -algebra A is then a set $|A|$ equipped with operations $\omega_A : |A|^{a\omega} \rightarrow |A|$ for each $\omega \in \Omega$, and a homomorphism is a map which respects these operations. Such Ω -algebras and homomorphisms form the category $\Omega\text{-alg}$, and the forgetful functor $U_\Omega : \Omega\text{-alg} \rightarrow \mathbf{Sets}$ admits a left adjoint known as the free functor, F_Ω . A term with variables in X is then an element t of the free algebra $F_\Omega(X)$, and an identity is a condition of equality between two such terms $t_1 = t_2$. An algebra A is then said to satisfy an identity $t_1 = t_2$ if $f(t_1) = f(t_2)$ for any morphism $f : F(X) \rightarrow A$. The full subcategory of all Ω -algebras satisfying any fixed set of identities is called a variety, of which the category \mathbf{Grp} of groups is an example. Note that this is equivalent to the other common approaches of defining a term as a ‘derived operation’ or as a well formed string of operations in Ω and variables in X .

It is worth noting that there are several other characterizations of coextensivity for varieties of universal algebras, such as those of [4] and [5], although [5] mentions our characterization since it was obtained in 2020, a few months before [5] was submitted. Our characterization is purely syntactical: we show that a variety is coextensive if and only if it has terms which satisfy certain identities. Conditions of this form are everywhere in universal algebra, and are commonly called Mal’tsev conditions. Mal’tsev conditions take their name from A.I. Mal’tsev in his characterization of varieties with permutable congruence relations. An algebra A is said to have permutable congruence relations if $E \circ R = R \circ E$ for all congruences E, R on A , where \circ denotes the composition of relations. Mal’tsev proved in [9] that every algebra in a variety V has permutable congruence relations if and only if it has some term t that satisfies the identities

$$\begin{aligned} t(x, x, y) &= y, \\ t(x, y, y) &= x, \end{aligned}$$

for all algebras $A \in V$ and elements $x, y \in A$. Our characterization is a condition of the same nature, although it is not strictly speaking a Mal’tsev condition in the sense of [10]. Specifically, it also involves identities to be satisfied by each basic operation of the given

variety. One especially useful Mal'tsev condition which we show every coextensive variety must satisfy is the existence of what we call a diagonalising term. By a diagonalising term, we mean a term t along with some constants $e_1, \dots, e_k, e'_1, \dots, e'_k$ that satisfy the identities

$$\begin{aligned} t(x, y, e_1, \dots, e_k) &= x, \\ t(x, y, e'_1, \dots, e'_k) &= y, \end{aligned}$$

for all algebras $A \in \mathcal{V}$ and elements $x, y \in A$. Furthermore we say a diagonalising term is non-trivial when $k > 0$. Varieties with such terms have also been studied in other contexts under different names. They are called Pierce varieties in [7] and varieties with a (short) decomposition term in [8]. However, these decomposition terms are always considered in the context of a variety with $\vec{0}$ and $\vec{1}$. The approach here is to first fix two sequences of variables $\vec{0} = (0_1, \dots, 0_k)$ and $\vec{1} = (1_1, \dots, 1_k)$ such that any algebra in the variety which satisfies $0_i = 1_i$ for all $i \leq k$ is a singleton. Any diagonalising term can be considered as a (short) decomposition term by fixing $\vec{0} = (e_1, \dots, e_k)$ and $\vec{1} = (e'_1, \dots, e'_k)$.

At this point the reader will be justified in asking why we should speak of diagonalising terms at all rather than speaking of decomposition terms. The main reason is that we prefer not to begin with defining $\vec{0}$ and $\vec{1}$ before defining the diagonalising term. In particular, the distinction between a decomposition term and a short decomposition term only exists if we first fix $\vec{0}$ and $\vec{1}$, and it is unclear how to refer to these terms outside of this context. It is also the case that the literature on decomposition terms was unknown to the author at the time of first publishing results involving diagonalising terms, which makes it awkward to change the terminology.

Varieties with non-trivial diagonalising terms are already quite close to being coextensive. In a variety \mathcal{C} with such a term we can define the map $\delta_X : F(X)^2 \rightarrow F(X) + F(\emptyset)^2$ as $\delta_X(x, y) = t(x, y, (e_1, e'_1), \dots, (e_k, e'_k))$ for any set X . We then show that δ_X is the right inverse of the component η_X at $F(X) + F(\emptyset)^2$ of the unit η of the adjunction $L \dashv (\times)$. It follows that \mathcal{C} is coextensive if and only if δ_X is a homomorphism satisfying the idempotency condition $\delta_X(x, x) = x$ for all $x \in X$. Finally, we show that δ_X satisfies this idempotency condition if and only if it satisfies a specific Mal'tsev condition, and δ_X is a homomorphism if and only if \mathcal{C} satisfies a specific list of identities for each basic operation $\omega \in \Omega$. These conditions together form our characterization of coextensive

varieties.

There are also many useful subconditions of coextensivity which should be mentioned in connection to our characterization. The coextensivity of certain morphisms has been studied in [17] and [18], where connections have been established to classical conditions of universal algebra. Of particular interest, the Fraser-Horn property [6] is shown to be equivalent to the coextensivity of regular epimorphisms, while the strict refinement property is shown to be equivalent to the coextensivity of product projections. We show that any variety with a non-trivial diagonalising term satisfies these and other conditions. Specifically, any such category is left coextensive, has co-disjoint products, has strict terminal objects, satisfies the Fraser-Horn property, and has co-universal surjections. Since diagonalising terms play such an important role in our characterization, it is interesting to consider them from a categorical perspective.

An object X equipped with a monomorphism $m : X \rightarrow A$ is called a subobject of A . Given a pair of subobjects $m : X \rightarrow A$ and $n : Y \rightarrow B$, the product $m \times n : X \times Y \rightarrow A \times B$ is also a monomorphism, and we say $X \times Y$ is a subproduct of $A \times B$. We will be concerned with those categories in which such subproducts of $A \times B$ interact nicely with relations on $A \times B$. Consider a commutative diagram

$$\begin{array}{ccccc}
 Y & \xrightarrow{\quad n \quad} & B & & \\
 \uparrow \pi_Y & & \uparrow \pi_B & & \\
 X \times Y & \xrightarrow{\quad s \quad} & R & \xrightarrow{\quad r \quad} & A \times B \\
 \downarrow \pi_X & & & & \downarrow \pi_A \\
 X & \xrightarrow{\quad m \quad} & A & &
 \end{array} \tag{2}$$

in which all horizontal morphisms are monomorphisms and all vertical morphisms are product projections. Then we say that R is a relation on $A \times B$ containing the subproduct $X \times Y$. We introduce the notion of a category with upward closed subproducts as a category in which any such relation $R \leq A \times B$ which contains a subproduct of $A \times B$ is itself a subproduct of $A \times B$, which is to say that the object R in the above diagram can be seen as a product $R \simeq R_X \times R_Y$, and the morphism r can be seen as a product $r_X \times r_Y : R_X \times R_Y \rightarrow A \times B$. This is a subcondition of coextensivity such that a variety has a diagonalising term if and only if it has upward closed subproducts. We then show that categories with upward closed subproducts satisfy all the same subconditions of coextensivity that varieties with diagonalising terms do, but in the more general categorical

context of Barr exact categories.

Barr exactness, introduced in [11], is important in categorical algebra for a variety of reasons. Exact categories are a suitable categorical setting in which to study various algebraic notions such as congruences and quotients. In particular, it is well known that the familiar definitions of reflexive, symmetric, transitive, and equivalence relations can be generalized categorically as internal relations. Quotient maps are similarly generalized as regular epimorphisms. In Barr exact categories such internal relations and regular epimorphisms are well behaved. Significantly, the kernel pair of any morphism in a Barr exact category is an internal equivalence relation, and any internal equivalence relation is the kernel pair of its coequalizer.

All varieties of universal algebras are exact, as are all abelian and semi-abelian categories. Regularity is a slightly weaker condition than exactness, since regular categories do not require that every internal equivalence relation is a kernel pair, but regularity is still sufficient for most of our results regarding categories with upward closed subproducts.

2. Preliminary results

In this section we will recall some basic definitions and well-known results. We assume basic familiarity with category theory and algebra, but otherwise all necessary definitions are included in the text. In section 2.3 we define coextensivity and give an overview of various well-known results which we will use in our characterisation. Lastly in section 2.4 we define regular and exact categories, and recall various well-known results which we will use in section 4 to generalize many of our results into a purely categorical context. Readers familiar with these topics should feel free to skip the relevant sections, and refer to them only as necessary.

2.1. UNIVERSAL CONSTRUCTIONS. Universal constructions are everywhere in mathematics, with examples including universal arrows, adjunctions, limits and colimits. This section will be devoted to recalling several categorical constructions of which we will make heavy use throughout the later sections of the work. For the definitive treatment of these and other subjects in category theory see [12]. We begin with limits and colimits in the form of various special cases that appear constantly in algebra. The first of these special cases are initial and terminal objects.

2.1.1. **RECALL.** (pg 20, [12]). Let \mathcal{C} be a category and recall that an object $A \in \mathcal{C}$ is called:

- *initial*, if for any object $B \in \mathcal{C}$ there exists a unique morphism $!_B : A \longrightarrow B$,
- *terminal*, if for any object $B \in \mathcal{C}$ there exists a unique morphism $!^B : B \longrightarrow A$.

So an object is initial (terminal) if there exists a unique morphism from (to) that object to (from) every object in the category. As an example, consider the category *Sets* of all small sets. In this category, the empty set is initial, since there exists a unique empty map from it to any other set. On the other hand, any set containing exactly one element is terminal. This illustrates that there can be more than one initial or terminal object in a category, but they will all be the same up to isomorphism if they exist.

2.1.2. **PROPOSITION.** Let $A, B \in \mathcal{C}$ both be initial or both be terminal objects in \mathcal{C} . Then $A \simeq B$.

PROOF. Let A and B be initial, therefore there exist unique morphisms $!_B^A : A \longrightarrow B$ and $!_A^B : B \longrightarrow A$. It immediately follows that the composites $!_A^B \circ !_B^A : A \longrightarrow A$ and $!_B^A \circ !_A^B : B \longrightarrow B$ are the identity morphisms 1_A and 1_B respectively, since there is only one unique morphism from A to A and from B to B respectively. The case in which A and B are both terminal follows by duality. ■

Since the initial and terminal objects are unique up to isomorphism when they exist, it is useful to denote them with the symbols 0 for the initial object and 1 for the terminal object. Note that in the case of *Sets*, 1 could be any singleton, and we do not concern ourselves with picking one in particular, since they are all isomorphic. In fact, all of the universal constructions we address in this section are unique up to isomorphism, since they can all be seen as either limits or colimits, and it is standard to refer to these constructions without specifying exactly which objects and morphisms make up the limit or colimit in question.

2.1.3. **RECALL.** (pg 1, [12]). For any objects $A, B \in \mathcal{C}$, we say that an object $A \times B \in \mathcal{C}$ equipped with morphisms $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$ is a product of A and B if, for any object C equipped with morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$, there exists a unique morphism $h : C \longrightarrow A \times B$ such that $\pi_A h = f$ and $\pi_B h = g$ as in the diagram:

$$\begin{array}{ccccc}
 & & C & & \\
 & f \swarrow & \vdots & \searrow g & \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array} \tag{3}$$

In *Sets*, a product is exactly the usual cartesian product $A \times B$, where π_A and π_B are the usual product projections. In the category *Cat* of small categories, we can define the product of two categories as follows.

2.1.4. RECALL. (pg 36, [12]). Given two categories \mathcal{A} and \mathcal{B} , the product category $\mathcal{A} \times \mathcal{B}$ is simply the category whose objects are all pairs (X, Y) of objects $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, and whose morphisms are all pairs $(f, g) : (X_1, Y_1) \rightarrow (X_2, Y_2)$ of morphisms $f : X_1 \rightarrow X_2$ in \mathcal{A} and $g : Y_1 \rightarrow Y_2$ in \mathcal{B} .

Composition of morphisms in $\mathcal{A} \times \mathcal{B}$ is defined component-wise, and it immediately follows that pairs of identity morphisms are the identity morphisms in $\mathcal{A} \times \mathcal{B}$. The dual construction of the product is the coproduct.

2.1.5. RECALL. (pg 62-63, [12]). For any objects $A, B \in \mathcal{C}$, we say that an object $A + B \in \mathcal{C}$ equipped with morphisms $i_A : A \rightarrow A + B$ and $i_B : B \rightarrow A + B$ is the coproduct of A and B if, for any object C equipped with morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$, there exists a unique morphism $h : A + B \rightarrow C$ such that $hi_A = f$ and $hi_B = g$ as in the diagram:

$$\begin{array}{ccccc}
 & & C & & \\
 & f \swarrow & \vdots & \nwarrow g & \\
 A & \xrightarrow{i_A} & A + B & \xleftarrow{i_B} & B
 \end{array} \tag{4}$$

In *Sets*, the coproduct $A + B$ is exactly the disjoint union of A and B . The morphisms i_A and i_B are the inclusion maps into the disjoint union. We will sometimes refer to these morphisms as coproduct inclusions. Next we will consider pushouts, which are heavily used in many of the conditions we will consider. Recall that given some pair of morphisms $f : A \rightarrow B, g : A \rightarrow C$, their pushout is the colimit of the diagram (pg 65-66, [12]):

$$B \xleftarrow{f} A \xrightarrow{g} C \tag{5}$$

which we denote as a square:

$A \times C \mid f(a) = g(c)\}$. Note that when B is the union $A \cup C$ and f and g are the relevant inclusion maps, then the pullback is simply the intersection $A \cap C$.

One property of both pullbacks and pushouts that makes them especially useful, is that they both satisfy the pasting law:

2.1.8. PROPOSITION. *The pasting law states that given a commutative diagram:*

$$\begin{array}{ccccc}
 A & \longrightarrow & B & \longrightarrow & C \\
 \downarrow & & \downarrow & & \downarrow \\
 & (1) & & (2) & \\
 D & \longrightarrow & E & \longrightarrow & F
 \end{array} \tag{8}$$

- *If the squares (1) and (2) are pullbacks, then so is the rectangle (1) + (2).*
- *If the rectangle (1) + (2) and the square (2) are pullbacks, then so is the square (1).*
- *If (1) and (2) are pushouts, then so is the rectangle (1) + (2).*
- *If the rectangle (1) + (2) and the square (1) are pushouts, then so is the square (2).*

The pullback of a morphism f with itself is called the kernel pair of f (pg 71, [12]). Equivalently we say that the kernel pair of a morphism $f : A \rightarrow B$ is an object $Eq(f)$ equipped with a pair of parallel morphisms $p_1, p_2 : Eq(f) \rightarrow A$ such that $fp_1 = fp_2$, and for any object E equipped with morphisms $e_1, e_2 : E \rightarrow A$ such that $fe_1 = fe_2$, there exists a unique morphism $m : E \rightarrow Eq(f)$ such that $p_1m = e_1$ and $p_2m = e_2$ as in the diagram:

$$\begin{array}{ccccc}
 Eq(f) & \xrightarrow{p_1} & A & \xrightarrow{f} & B \\
 & \xrightarrow{p_2} & \nearrow & & \\
 \uparrow & & e_1 & & \\
 \vdots & & e_2 & & \\
 m & & & & \\
 \vdots & & & & \\
 E & & & &
 \end{array} \tag{9}$$

Note that in *Sets* this works out to be exactly the usual equivalence relation generated by f . $Eq(f) = \{(a_1, a_2) \in A^2 \mid f(a_1) = f(a_2)\}$, with $p_1(a_1, a_2) = a_1$, and $p_2(a_1, a_2) = a_2$. There is also a categorical construction that generalizes the quotient map of an equivalence relation. Specifically, we say that a morphism $f : A \rightarrow B$ is the co-equalizer (pg 64, [12])

of some pair of parallel morphisms $e_1, e_2 : E \longrightarrow A$ if $fe_1 = fe_2$, and for any object C and any morphism $g : A \longrightarrow C$ such that $ge_1 = ge_2$, there exists a unique morphism $p : B \longrightarrow C$ such that $pf = g$ as in the diagram:

$$\begin{array}{ccccc}
 E & \xrightarrow[e_2]{e_1} & A & \xrightarrow{f} & B \\
 & & & \searrow g & \downarrow p \\
 & & & & C
 \end{array} \tag{10}$$

A morphism f is called a regular epimorphism if it is the coequalizer of some parallel pair of morphisms. Kernel pairs and coequalizers are related to each other by the following results

2.1.9. PROPOSITION. *Let $f : A \longrightarrow B$ be a regular epimorphism and $(Eq(f), p_1, p_2)$ its kernel pair. Then f is the coequalizer of (p_1, p_2) .*

PROOF. Let $g : A \longrightarrow C$ be some morphism such that $gp_1 = gp_2$. Since f is regular, we can construct a diagram

$$\begin{array}{ccccc}
 Eq(f) & \xrightarrow[p_2]{p_1} & A & \xrightarrow{f} & B \\
 \uparrow m & & \nearrow e_1 & \searrow g & \downarrow p \\
 E & & & & C \\
 & & \nearrow e_2 & &
 \end{array} \tag{11}$$

for some object E and morphisms $e_1, e_2 : E \longrightarrow A$ such that f is the coequalizer of e_1, e_2 . Note that $fe_1 = fe_2$, and so there exists a unique morphism $m : E \longrightarrow Eq(f)$ such that $p_1m = e_1$ and $p_2m = e_2$. Therefore $ge_1 = gp_1m = gp_2m = ge_2$. Then, since f is the coequalizer of e_1 and e_2 , there exists a unique morphism $p : B \longrightarrow C$ such that $pf = g$. Therefore f is the coequalizer of p_1 and p_2 as desired. ■

2.1.10. PROPOSITION. *Let $p_1, p_2 : Eq(f) \longrightarrow A$ be the kernel pair of some morphism f and let $q : A \longrightarrow A/Eq(f)$ be the coequalizer of p_1 and p_2 . Then p_1, p_2 is the kernel pair of q .*

PROOF. Note that $fp_1 = fp_2$, so there exists a unique morphism $p : A/Eq(f) \longrightarrow B$ such that $pq = f$. Let $e_1, e_2 : E \longrightarrow A$ be some pair of morphisms such that $qe_1 = qe_2$. We can then construct the diagram:

$$\begin{array}{ccccc}
 Eq(f) & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & A & \xrightarrow{f} & B \\
 \uparrow m & & \nearrow e_1 & & \uparrow p \\
 E & & & & A/Eq(f) \\
 & & \nwarrow e_2 & & \downarrow q
 \end{array}
 \tag{12}$$

Note that $fe_1 = pqe_1 = pqe_2 = fe_2$. Then, since p_1 and p_2 are the kernel pair of f , there exists a unique morphism $m : E \rightarrow Eq(f)$ such that $p_1m = e_1$ and $p_2m = e_2$. Therefore p_1, p_2 is the kernel pair of q as desired. ■

In *Sets*, a morphism is a regular epimorphism if and only if it is surjective, and when $E \subseteq A^2$, the co-equalizer of the projections e_1, e_2 is exactly the quotient A/E . A similar story is true in any variety of universal algebras. Next, we introduce a useful well-known lemma relating pushouts and coequalizers.

2.1.11. LEMMA. *Given a diagram*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow g & \curvearrowright s & \downarrow n \\
 C & \xrightarrow{m} & D
 \end{array}
 \tag{13}$$

with $mg = nf$ and $fs = 1_B$, the following conditions are equivalent:

- (a) The square $mg = nf$ is a pushout.
- (b) The diagram

$$A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{gsf} \end{array} C \xrightarrow{m} D
 \tag{14}$$

is a coequalizer diagram.

PROOF. Let $k : C \rightarrow E$ and $h : B \rightarrow E$ be any morphisms such that $kg = hf$. It immediately follows that $kgsf = hfsf = hf = kg$. On the other hand, given any k such that $kgsf = kg$, $h = kgs$ is the unique morphism such that $kg = hf$, since this equality immediately implies $h = hfs = kgs$. It immediately follows that (a) and (b) are equivalent. ■

So far we have defined only those limits and colimits which are required for working with coextensive varieties. In order to define coextensivity, it is also necessary to be familiar with adjunctions, equivalences, and coslice categories.

2.1.12. **RECALL.** (pg 55, [12]). *If $F : A \rightarrow B$ is a functor, then we say that a morphism $u : X \rightarrow F(Y)$ in B is a universal arrow from X to F if, for any object $Z \in A$ and any morphism $g : X \rightarrow F(Z)$, there exists a unique morphism $h : Y \rightarrow Z$ such that $F(h)u = g$. This is to say that morphisms from X to F factor uniquely through u as in the diagram:*

$$\begin{array}{ccc}
 & & F(Y) \\
 & \nearrow u & \vdots \\
 X & & F(h) \\
 & \searrow g & \vdots \\
 & & F(Z)
 \end{array}
 \qquad
 \begin{array}{c}
 Y \\
 \vdots \\
 h \\
 \vdots \\
 Z
 \end{array}
 \qquad (15)$$

2.1.13. **RECALL.** (Theorem 2, pg 83, [12]). *Let A and B be categories. Then recall that an adjunction $F \dashv G$ between A and B is a pair of functors $F : A \rightarrow B$, and $G : B \rightarrow A$ together with a pair of natural transformations $\eta : 1_A \rightarrow GF$, and $\varepsilon : FG \rightarrow 1_B$, such that the components of the transformations $\eta_X : X \rightarrow GF(X)$ and $\varepsilon_Y : FG(Y) \rightarrow Y$ are universal arrows from X to GF and from FG to Y , respectively, for any objects $X \in A$ and $Y \in B$.*

We refer to F as the left adjoint of G , and to G as the right adjoint of F . We also refer to η as the unit and ε as the counit of the adjunction. Adjoint functors have several useful properties, including that the composites

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon^F} F \qquad G \xrightarrow{\eta^G} GFG \xrightarrow{G\varepsilon} G \qquad (16)$$

are identities. In fact, for F and G to form an adjunction, it is sufficient to find any natural transformations $\eta : 1_A \rightarrow GF$, and $\varepsilon : FG \rightarrow 1_B$ satisfying this identity. There are many useful results related to adjoint functors. Famously, right (left) adjoint functors preserve limits (colimits). Furthermore, limit (colimit) construction can be seen as the right (left) adjoint to the constant diagram functor, and consequently limits commute with limits and colimits commute with colimits. We will often make use of the special case that products and pullbacks commute. Next we consider equivalences between categories.

2.1.14. **RECALL.** (pg 18, [12]). Let A and B be categories. We say that a pair of functors $F : A \rightarrow B$, and $G : B \rightarrow A$ form an equivalence if there exist natural isomorphisms $\eta_X : X \rightarrow GF(X)$ and $\varepsilon_Y : FG(Y) \rightarrow Y$.

This is called an adjoint equivalence when $(F, G, \eta, \varepsilon)$ is both an equivalence and an adjunction. Intuitively, two categories are equivalent when their objects and morphisms are ‘the same’ up to isomorphism. In particular, it is well known that a functor F is part of an equivalence if and only if it is fully faithful and essentially surjective. We end this section with slice and coslice categories, which are the final piece of categorical machinery needed to define extensive and coextensive categories.

2.1.15. **RECALL.** Given a category \mathcal{C} and an object $X \in \mathcal{C}$, the coslice category X/\mathcal{C} is the category whose objects are morphisms in \mathcal{C} with domain X . These can also be seen as pairs $(Y, f : X \rightarrow Y)$. Given any two objects (Y, f) and (Z, g) in X/\mathcal{C} , a morphism $h : (Y, f) \rightarrow (Z, g)$ is exactly a morphism $h : Y \rightarrow Z$ such that $hf = g$, or equivalently that the diagram

$$\begin{array}{ccc}
 & X & \\
 f \swarrow & & \searrow g \\
 Y & \xrightarrow{h} & Z
 \end{array} \tag{17}$$

commutes. Composition in the coslice category X/\mathcal{C} is defined to be the same as composition in \mathcal{C} , and it follows that the identity morphisms are the same as well.

The dual of the coslice category X/\mathcal{C} is the slice category \mathcal{C}/X whose objects are morphisms with codomain X instead of domain X .

2.2. **UNIVERSAL ALGEBRA.** The aim of universal algebra is to give a sufficiently general definition of an algebra to encompass most well-known algebraic structures such as groups, rings, and lattices. In this section we will give a brief description of universal algebras as sets equipped with finitary operations. Let Ω be a (finite) set equipped with a map $a : \Omega \rightarrow \mathbb{N}$. We refer to Ω as a signature, its elements are called operations, and for any operation $\omega \in \Omega$, the natural number $a\omega$ is called the arity of ω . An Ω -algebra A is then defined to be a set A which, for each $\omega \in \Omega$ is equipped with an operation $\omega_A : A^{a\omega} \rightarrow A$. As an example, consider groups. A group A is a (small) set A equipped with a binary operation $+$: $A^2 \rightarrow A$, a unary ‘negation’ operation $-$: $A \rightarrow A$ and a constant 0 , which can be seen as a nullary operation $0 : 1 \rightarrow A$. To see groups as Ω algebras we can consider

the signature $\Omega = \{+, -, 0\}$ and $a(+)=2, a(-)=1, a(0)=0$. A homomorphism between two groups A and B is exactly a map which preserves these three operations. Similarly, a homomorphism between Ω -algebras can be defined in the following way.

2.2.1. **RECALL.** *A homomorphism between two Ω -algebras A and B is exactly a map $f : A \rightarrow B$ such that the diagram*

$$\begin{array}{ccc}
 A^{a\omega} & \xrightarrow{f^{a\omega}} & B^{a\omega} \\
 \omega_A \downarrow & & \downarrow \omega_B \\
 A & \xrightarrow{f} & B
 \end{array} \tag{18}$$

commutes for all $\omega \in \Omega$.

Ω -algebras and the homomorphisms between them now form a category $\Omega\text{-Alg}$ for any fixed Ω . Returning to the example of groups, the objects of the category $\Omega\text{-Alg}$ are not necessarily groups, since they are not guaranteed to satisfy the various identities of group theory. Specifically in any group we have that for any group A and any $x, y, z \in A$ the following equations hold.

$$\begin{aligned}
 x + (y + z) &= (x + y) + z, \\
 x + 0 &= x = 0 + x, \\
 x + (-x) &= 0 = (-x) + x.
 \end{aligned}$$

Therefore, the usual category Grp of groups is exactly the full subcategory of $\Omega\text{-Alg}$ containing all objects which satisfy these equalities. This is an example of a variety of universal algebras. More generally, given some set X , we say that a term in an Ω -Algebra is any well formed expression consisting of operations in Ω and variables in $x_1, \dots, x_n \in X$. Then, given two such terms $u(x_1, \dots, x_n)$ and $v(x_1, \dots, x_n)$ we say some algebra A satisfies the identity $u(x_1, \dots, x_n) = v(x_1, \dots, x_n)$ if and only if the equation $u_A(a_1, \dots, a_n) = v_A(a_1, \dots, a_n)$ holds in A for any $a_1, \dots, a_n \in A$. A variety (sometimes called an equational class) can then be defined in one of two ways:

2.2.2. **DEFINITION AND THEOREM.** (Birkhoff, [16]). For some signature Ω , a full subcategory V of $\Omega\text{-Alg}$ is called a variety or an equational class if and only if it satisfies the following propositions, which are equivalent:

- V is closed under subalgebras, homomorphic images, and direct products.
- There is a set of identities I such that for all $A \in \Omega\text{-Alg}$, we have $A \in V$ if and only if A satisfies every identity in I .

Varieties are the main object of study in universal algebra. Note in particular that the above definition is general enough to include many familiar algebraic categories such as the categories of groups, abelian groups, rings, lattices and so on.

A weakness of this classical notion of universal algebra is that to define a variety we need to pick the operations in the signature Ω , and there may be multiple equivalent ways to do so, for example groups can be defined either in terms of a binary ‘multiplication’ operation or in terms of a binary ‘division’ operation. Since it is essentially irrelevant which operations are contained in the signature, it is better to consider not only the operations in the signature, but also all operations that can be ‘derived’ through combining them.

To do so, we will make use of free algebras. A free algebra is essentially an algebra which satisfies only those identities that are common to every algebra in the variety. More formally, in any variety \mathcal{C} , the free algebra $F(X)$ generated on a set X can be defined via the following adjunction.

2.2.3. **RECALL.** *For any variety \mathcal{C} of Ω -Algebras, the following functors exist.*

- *The forgetful functor $U : \mathcal{C} \rightarrow \text{Sets}$, which maps each algebra to its underlying set and each homomorphism to itself considered as a map.*
- *The free functor $F_{\mathcal{C}} : \text{Sets} \rightarrow \mathcal{C}$, which is the left adjoint of U .*

In the case of $\mathcal{C} = \Omega\text{-Alg}$, the elements of the free algebra $F_{\Omega}(X)$ are exactly those terms made up of operations in Ω and ‘variables’ in X . When \mathcal{C} is a subvariety of $\Omega\text{-Alg}$, then the elements of $F_{\mathcal{C}}(X)$ are necessarily equivalence classes of these terms. Specifically, $F_{\mathcal{C}}(X) = F_{\Omega}(X)/E_X$, where E_X is the equivalence relation containing exactly those pairs of terms which are equal in every algebra in the variety. An immediate consequence of this is that a variety V satisfies an identity $u(x_1, \dots, x_k) = v(x_1, \dots, x_k)$ if and only if $u(x_1, \dots, x_k) \simeq_{E_X} v(x_1, \dots, x_k)$.

For example in the case of groups, if $x, y, z \in X$, then the terms $x + (y + z)$ and $(x + y) + z$ are distinct elements of $F_{\Omega}(X)$, while in $F_{\text{Grp}}(X)$ these terms are equal due to associativity.

In the remainder of this thesis, we will often refer to terms and operations interchangeably, and without any consideration for whether such a term is contained in the signature or not. The evaluation of such a term is defined as follows. Let $n = \{1, \dots, n\}$ be the set of natural numbers between 1 and n . By a term of arity n we will mean an element of $F_\Omega(n)$. For any $u \in F_\Omega(n)$ and any algebra $X \in \mathcal{C}$ we will denote by $u(x_1, \dots, x_n)$ the value $e(u)$ where $e : F_\Omega(n) \longrightarrow X$ is the unique homomorphism such that $e(i) = x_i$. Since every term has finite arity, all terms can be seen as elements of $F_\Omega(\mathbb{N})$.

Next we turn our attention to congruences. Congruences are a generalization of equivalence relations from sets to varieties of algebras such that the quotient of an algebra over a congruence is again an algebra in the same variety, and the quotient map is a homomorphism.

2.2.4. RECALL. *A congruence on an algebra $A \in \mathcal{C}$ is an equivalence relation $R \subseteq A^2$ which is also a subalgebra of A^2 .*

Equivalently, an equivalence relation R on some algebra A is a congruence if and only if it is closed under the operations of A in the sense that for any operation ω of A , and any elements $(x_1, y_1), \dots, (x_{a\omega}, y_{a\omega}) \in R$ we have $(\omega(x_1, \dots, x_{a\omega}), \omega(y_1, \dots, y_{a\omega})) \in R$. In general, when a relation R between two algebras A and B is a subalgebra of $A \times B$, then we say R is a homomorphic relation. A congruence can therefore be said to be a homomorphic equivalence relation.

2.2.5. PROPOSITION. *Let X be an algebra in a variety \mathcal{C} , E an equivalence relation on X , and $q : X \longrightarrow X/E$ the quotient map. Then E is a congruence if and only if the quotient set X/E can be made an algebra with operations defined as $\omega_{X/E}(q(x_1), \dots, q(x_{a\omega})) = q(\omega_X(x_1, \dots, x_{a\omega}))$.*

PROOF. Note that if these operations $\omega_{X/E}$ are well defined for all $\omega \in \Omega$, then X/E is immediately an Ω -Algebra in \mathcal{C} , since any identity satisfied by X will be satisfied by X/E . The quotient map q is also immediately a homomorphism. It remains to show that these operations on X/E are well defined if and only if E is a congruence. For any $\omega \in \Omega$, let $(x_1, y_1), \dots, (x_{a\omega}, y_{a\omega}) \in E^{a\omega}$. Then $\omega_{X/E}$ is well defined if and only if $\omega_{X/E}(q(x_1, \dots, x_{a\omega})) = \omega_{X/E}(q(y_1, \dots, y_{a\omega}))$ if and only if $\omega_X(x_1, \dots, x_{a\omega}) \simeq_E \omega_X(y_1, \dots, y_{a\omega})$, which is exactly to say that E is a congruence. ■

It is also often useful to consider homomorphisms in terms of the congruence they generate on their domain. In *Sets*, any map between two algebras $f : A \longrightarrow B$ generates an

equivalence relation on A of the form E_f such that $a \simeq_{E_f} b$ if and only if $f(a) = f(b)$. When f is a homomorphism between two algebras, then E_f is also a congruence, and we will refer to E_f as the congruence generated by f .

Congruences play a central role in universal algebra. Much like equivalence relations, the set of congruences on some algebra form a lattice ordered by inclusion, since they are closed under intersection, and given some relation R on some algebra X , we can define the congruence on X generated by R as the smallest congruence containing R . It is also useful to be able to directly construct the congruence generated by R . To do so, we use the following basic operations of relations.

- Diagonal relation: $\Delta_X = \{(x, x) \in X^2\}$.
- Inverse relation: $R^{-1} = \{(x_1, x_2) \in X^2 \mid (x_2, x_1) \in R\}$.
- Composition: given relations $R \subseteq A \times B$, and $R' \subseteq B \times C$, the composite $R \circ R'$ is defined to be the relation $x_1 \simeq x_2$ if and only if there exists some $x_3 \in B$ such that $x_1 \simeq_R x_3 \simeq_{R'} x_2$.

Now, note that a relation $R \subseteq X^2$ is a congruence if and only if it is reflexive, symmetric, transitive, and homomorphic. For each of these properties we can define a closure operation which gives the smallest relation on X containing R which satisfies the property in the following way.

- Reflexive closure: $Ref(R) = R \cup \{(x, x) \in X^2\}$.
- Symmetric closure: $Sym(R) = R \cup R^{-1}$.
- Transitive closure: $Trans(R) = \bigcup_{i \geq 1}^{\text{inf}} R^i$, where R^i is defined as $R^1 = R$ and $R^{i+1} = R^i \circ R$. Here $\bigcup_{i \geq 1}^{\text{inf}} R^i$ denotes the union of all such sets R^i for naturals $i \geq 1$.
- Homomorphic closure:

$$H(R) = \{(u(x_1, \dots, x_k), u(x'_1, \dots, x'_k)) \mid (x_1, x'_1), \dots, (x_k, x'_k) \in R, u \in F_\Omega(\mathbb{N})\}.$$

To find the congruence generated by R we would like to simply apply all four of these closure operations. Indeed, it is well known that the equivalence relation generated by R is of the form $Eq(R) = Trans(Sym(Ref(R)))$, and we would like to do something similar for congruences. It turns out, however, that the homomorphic closure of a transitive set may not itself be transitive, so $H(Eq(R))$ is not in general a congruence. To resolve this issue, we simply need to adjust the order in which we apply these closure operations.

2.2.6. PROPOSITION. *Let \mathcal{C} be a variety and $X \in \mathcal{C}$. For any relation R on X , the smallest congruence on X containing R is of the form $Eq(R) = Trans(H(Sym(Ref(R))))$.*

PROOF. It is clear that all of the closure operations preserve reflexivity in the sense that if R is a reflexive relation, then $H(R)$, $Trans(R)$, and $Sym(R)$ are all reflexive. Therefore $Eq(R)$ is reflexive.

The transitive closure operation respects symmetry, since if R is a symmetric relation, then for all $(x, y) \in Trans(R)$, there exists some finite sequence $x_1, \dots, x_n \in X$ such that $x_1 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in R$ for all $1 \leq i < n$. Then, by symmetry of R , we have $(x_{i+1}, x_i) \in R$ for all $1 \leq i < n$, and so $(y, x) \in Trans(R)$.

The homomorphic closure operation also respects symmetry, since for all $(x, y) \in H(R)$ there exists some n -ary operation u such that $x = u(x_1, \dots, x_n)$ and $y = u(y_1, \dots, y_n)$ and $(x_1, y_1), \dots, (x_n, y_n) \in R$. If R is symmetric, then $(y_1, x_1), \dots, (y_n, x_n) \in R$, and so $(y, x) \in H(R)$ as desired.

It remains to show that $Trans(H(R))$ is still homomorphic for any symmetric homomorphic relation R . Let $(x_1, y_1), \dots, (x_n, y_n) \in Trans(R)$ be a sequence of length n . Then, for each $1 \leq i \leq n$, there exists some sequence $a_{(i,1)}, \dots, a_{(i,m_i)}$ such that $a_{(i,1)} = x_i$, $a_{(i,m_i)} = y_i$, and $(a_{(i,j)}, a_{(i,j+1)}) \in R$ for all $1 \leq j < m_i$.

A small complication arises here since the various m_i may not be equal. To get around this let m be the maximum of the m_i , and extend each sequence to be of length m . This can be done easily since R is reflexive. Now, note that for any n -ary term u , and all $1 \leq j < m$, we have $(u(a_{(1,j)}, \dots, a_{(n,j)}), u(a_{(1,j+1)}, \dots, a_{(n,j+1)})) \in H(R)$, which implies that $(u(x_1, \dots, x_n), u(y_1, \dots, y_n)) \in Trans(H(R))$, and so $Trans(H(R))$ is homomorphic as desired. ■

The ability to construct congruences in terms of these closure operations will be important for our main result. Next we will give explicit definitions for several useful limits and colimits in a variety. We begin with initial and terminal objects.

2.2.7. PROPOSITION. *The initial object in any variety is the free algebra generated on the empty set $0 \simeq F(\emptyset)$ and the terminal object is any singleton $1 \simeq \{x\}$ with all operations defined as $\omega(x, \dots, x) = x$.*

PROOF. To see that $F(\emptyset)$ is initial it is sufficient to note that \emptyset is initial in *Sets* and F is a left adjoint functor so it preserves initial objects. To see that 1 is terminal consider that for any algebra A there exists a unique map $f : A \rightarrow 1$, which is necessarily the map $f(a) = x$ for all $a \in A$, and this map is immediately a homomorphism. ■

2.2.8. PROPOSITION. *Any finite family of objects $A = (A_1, \dots, A_n)$ in a variety \mathcal{C} has a product $\prod A$ in \mathcal{C} , which up to isomorphism is simply the cartesian product of A_1, \dots, A_n considered as sets, equipped with the usual product projections, and all operations defined componentwise.*

PROOF. Since we already know that varieties have trivial algebras as terminal objects, we need only consider the case of binary products. Given a variety \mathcal{C} and two algebras $X, Y \in \mathcal{C}$, their cartesian product $X \times Y$ can be made an algebra by defining $\omega((x_1, y_1), \dots, (x_{a\omega}, y_{a\omega})) = (\omega(x_1, \dots, x_{a\omega}), \omega(y_1, \dots, y_{a\omega}))$ for any operation ω . It is clear that the usual product projections π_X, π_Y are now homomorphisms, and that $X \times Y$ will satisfy any identity satisfied by both X and Y , so $X \times Y$ must also be in the variety \mathcal{C} . To see that $X \times Y$ satisfies the universal property of a product, note that the diagram

$$\begin{array}{ccccc}
 & & Z & & \\
 & f \swarrow & \vdots & \searrow g & \\
 X & & (f, g) & & Y \\
 & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} &
 \end{array} \tag{19}$$

commutes for any algebra $Z \in \mathcal{C}$ equipped with some homomorphisms $f : Z \rightarrow X, g : Z \rightarrow Y$, where as usual $(f, g)(z) = (f(z), g(z))$ for all $z \in Z$. ■

2.2.9. PROPOSITION. *Any morphism f in a variety \mathcal{C} has a kernel pair, which up to isomorphism is given by the equivalence relation $Eq(f) = \{(x_1, x_2) \in X^2 \mid f(x_1) = f(x_2)\}$ equipped with the projections $p_1, p_2 : Eq(f) \rightarrow X$ such that $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$.*

PROOF. Let \mathcal{C} be a variety and let $f : X \rightarrow Y$ be a homomorphism in \mathcal{C} . We can then construct the diagram

$$\begin{array}{ccccc}
 Eq(f) & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & X & \xrightarrow{f} & Y \\
 \uparrow \text{dotted} & & \nearrow e_1 & & \\
 m & & \nearrow e_2 & & \\
 \text{dotted} & & & & \\
 E & & & &
 \end{array} \tag{20}$$

in which $e_1, e_2 : E \rightarrow X$ are any homomorphisms such that $fe_1 = fe_2$, and $m(a) = (e_1(a), e_2(a))$ is the unique morphism such that the diagram commutes. Since $Eq(f)$ is a subalgebra of X^2 , it is also an object in the variety, and p_1, p_2 are homomorphisms. Therefore $(Eq(f), p_1, p_2)$ is the kernel pair of f as desired. ■

2.2.10. PROPOSITION. *Any pair of parallel homomorphisms $f, g : X \rightarrow Y$ in a variety \mathcal{C} has a co-equalizer, which up to isomorphism is given by the quotient map $p : Y \rightarrow Y/E$ where E is the congruence on Y generated by the relation $f(x) \simeq g(x)$ for all $x \in X$.*

PROOF. Let $f, g : X \rightarrow Y$ be homomorphisms in a variety \mathcal{C} , and E the congruence on Y generated by the relation $f(x) \simeq g(x)$ for all $x \in X$. Then, for any $Z \in \mathcal{C}$ and any homomorphism $h : Y \rightarrow Z$ such that $hf = hg$, there exists a unique homomorphism k such that the diagram commutes

$$\begin{array}{ccccc}
 X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y & \xrightarrow{p} & Y/E \\
 & & \searrow h & & \text{dotted} \\
 & & & & k \\
 & & & & \text{dotted} \\
 & & & & Z
 \end{array} \tag{21}$$

in which $p : Y \rightarrow Y/E$ is the quotient map, and k is necessarily the map $k([y]) = h(y)$. Note that k is well defined since $hf(x) = hg(x)$ for all $x \in X$, which implies $E \leq Eq(h)$, and it follows that $y_1 \simeq_E y_2$ implies $h(y_1) = h(y_2)$. ■

We have seen that kernel pairs are simply congruences and co-equalizers are simply quotient maps. It is also easy to see that any congruence is a kernel pair and any surjective homomorphism is a coequalizer. Specifically, let $p : X \rightarrow Y$ be a surjective homomorphism and let E be the congruence generated by p . Then the diagram

$$E \begin{array}{c} \xrightarrow{e_1} \\ \xrightarrow{e_2} \end{array} X \xrightarrow{p} Y \tag{22}$$

is both a kernel pair diagram and a co-equalizer diagram.

2.2.11. PROPOSITION. *Any finite family of objects $A = (A_1, \dots, A_n)$ in a variety \mathcal{C} has a coproduct $\bigsqcup A$ in \mathcal{C} , which up to isomorphism is of the form $F(\bigsqcup_{i=1}^k U(A_i))/E$. Here F is the free functor, U is the underlying object functor, and E is the congruence generated by the union of the relations $E_i = \{(a, a') \in FU(A_i) \mid \varepsilon_{A_i}(a) = \varepsilon_{A_i}(a')\}$, where $\varepsilon_{A_i} : FU(A_i) \rightarrow A_i$ is the component at A_i of the co-unit of the adjunction $F \dashv U$.*

PROOF. Since we already know that varieties have initial objects isomorphic to $F(\emptyset)$, we need only consider binary coproducts. Given a variety \mathcal{C} and two algebras $X, Y \in \mathcal{C}$, consider the components $\varepsilon_X : F(U(X)) \rightarrow X$ and $\varepsilon_Y : F(U(Y)) \rightarrow Y$ of the co-unit of the free functor adjunction. These homomorphisms are necessarily surjective, and so X and Y are isomorphic to the quotients of $F(U(X))$ and $F(U(Y))$ over the congruences E_X and E_Y generated by ε_X and ε_Y respectively. Next, note that F is a left adjoint, so F preserves colimits, and in particular $F(U(X) + U(Y)) \simeq F(U(X)) + F(U(Y))$. For any algebra $Z \in \mathcal{C}$ and any homomorphisms $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ we can now construct the commutative diagram

$$\begin{array}{ccccc}
 E_X & \xrightarrow{\quad} & E & \xleftarrow{\quad} & E_Y \\
 \downarrow \scriptstyle{x_1, x_2} & & \downarrow \scriptstyle{e_1, e_2} & & \downarrow \scriptstyle{y_1, y_2} \\
 F(U(X)) & \xrightarrow{i'_X} & F(U(X) + U(Y)) & \xleftarrow{i'_Y} & F(U(Y)) \\
 \downarrow \scriptstyle{\varepsilon_X} & & \downarrow \scriptstyle{p} & & \downarrow \scriptstyle{\varepsilon_Y} \\
 X & \xrightarrow{i_X} & F(U(X) + U(Y))/E & \xleftarrow{i_Y} & Y \\
 & \searrow \scriptstyle{f} & \downarrow \scriptstyle{h} & \swarrow \scriptstyle{g} & \\
 & & Z & &
 \end{array} \tag{23}$$

in which E is the congruence on $F(U(X)) + F(U(Y))$ generated by E_X and E_Y considered as relations on $F(U(X)) + F(U(Y))$. Since ε_X and ε_Y are the coequalizers of E_X and E_Y , there exist unique morphisms i_X and i_Y such that $i_X \varepsilon_X = p i'_X$ and $i_Y \varepsilon_Y = p i'_Y$.

To attain the unique morphism h such that the diagram commutes, note that there must

exist a unique morphism $\langle f\varepsilon_X, g\varepsilon_Y \rangle : F(U(X)) + F(U(Y)) \rightarrow Z$ such that $\langle f\varepsilon_X, g\varepsilon_Y \rangle i'_X = f\varepsilon_X$ and $\langle f\varepsilon_X, g\varepsilon_Y \rangle i'_Y = g\varepsilon_Y$ by the universal property of the coproduct. It follows that $\langle f\varepsilon_X, g\varepsilon_Y \rangle e_1 = \langle f\varepsilon_X, g\varepsilon_Y \rangle e_2$, since E is the congruence generated by E_X and E_Y , which must both be subsets of $Eq(\langle f\varepsilon_X, g\varepsilon_Y \rangle)$.

Therefore there exists a unique morphism $h : F(U(X)) + F(U(Y)) \rightarrow Z$ such that $hi_X\varepsilon_X = f\varepsilon_X$ and $hi_Y\varepsilon_Y = g\varepsilon_Y$ since p is the coequalizer of e_1 and e_2 . Lastly, note that ε_X and ε_Y are epimorphisms, so $hi_X = f$ and $hi_Y = g$. Therefore $F(U(X)) + F(U(Y))/E \simeq X + Y$ as desired. ■

2.2.12. PROPOSITION. *Given morphisms $f : A \rightarrow B$ and $g : A \rightarrow C$ in a variety \mathcal{C} , the pushout of f and g exists, and up to isomorphism is of the form $(B + C)/E$, where E is the congruence on $B + C$ generated by the relation $\{(f(a), g(a)) \in B + C \mid a \in A\}$.*

PROOF. Let \mathcal{C} be a variety and consider any homomorphisms $f : A \rightarrow B$ and $g : A \rightarrow C$ in \mathcal{C} . We can then construct the pushout diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & & \\
 \downarrow g & & \downarrow i_B & & \\
 & & B + C & & \\
 \uparrow i_C & & \downarrow p & & \\
 C & \xrightarrow{g} & (B + C)/E & \xrightarrow{h} & D \\
 & \searrow k & \uparrow \phi & & \\
 & & C & &
 \end{array}
 \tag{24}$$

in which p is the coequalizer of $i_B f$ and $i_C g$ and so E is the congruence on $B + C$ generated by the relation $\{(f(a), g(a)) \in B + C \mid a \in A\}$. To see that this is a pushout, let $h : B \rightarrow D$ and $k : C \rightarrow D$ be morphisms such that $hf = kg$ as in the diagram. Then there exists a unique morphism $\langle h, k \rangle : B + C \rightarrow D$ such that $\langle h, k \rangle i_B = h$ and $\langle h, k \rangle i_C = k$. Since p is the coequalizer of $i_B f$ and $i_C g$, there exists a unique morphism $\phi : (B + C)/E \rightarrow D$ such that the diagram commutes, and so the diagram is a pushout as desired. ■

2.2.13. REMARK. Note that when f is surjective, the pushout becomes C/E , where E is the congruence on C generated by the relation $g(a_1) \simeq g(a_2)$ whenever $f(a_1) = f(a_2)$.

Equivalently, let E_f be the congruence on A generated by f with projections $\pi_1, \pi_2 : E_f \rightarrow A$. Then, if f is surjective, the diagram

$$\begin{array}{ccccc}
 E_f & \xrightarrow[\pi_2]{\pi_1} & A & \xrightarrow{f} & B \\
 & & \downarrow g & & \downarrow g' \\
 & & C & \xrightarrow{f'} & B/E
 \end{array} \tag{25}$$

is a pushout if and only if f' is the coequalizer of $g\pi_1$ and $g\pi_2$.

2.3. EXTENSIVITY AND COEXTENSIVITY. In this section we recall several well-known definitions and results regarding extensive and coextensive categories. We will mainly be concerned with coextensivity, but we will begin with a short introduction to extensivity and the related condition of distributivity.

2.3.1. DEFINITION. *A category \mathcal{C} with finite coproducts is extensive if for any objects X, Y in \mathcal{C} , the canonical functor $+: \mathcal{C}/X \times \mathcal{C}/Y \rightarrow \mathcal{C}/(X + Y)$ is an equivalence.*

Here, $+: \mathcal{C}/X \times \mathcal{C}/Y \rightarrow \mathcal{C}/(X + Y)$ is the functor that maps a pair of morphisms (f, g) to their coproduct $f + g$. A very useful equivalent definition of coextensivity is as follows.

2.3.2. RECALL. *(Proposition 2.2, [1]). A category with finite coproducts is extensive if and only if it has pullbacks along coproduct injections and for every commutative diagram*

$$\begin{array}{ccccc}
 A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\
 \downarrow f_1 & & \downarrow f & & \downarrow f_2 \\
 X_1 & \xrightarrow{x_1} & X_1 + X_2 & \xleftarrow{x_2} & X_2
 \end{array} \tag{26}$$

both squares are pullbacks if and only if the top row is a coproduct diagram.

Examples of extensive categories include the category **Cat** of all small categories and the category **Top** of topological spaces. To gain intuition for extensivity, it is useful to consider the case of **Top**. In this category, a coproduct of two spaces is exactly their disjoint union. It can also be seen that if both squares are pullbacks, then $A_1 \simeq f^{-1}(X_1)$ and $A_2 \simeq f^{-1}(X_2)$. Since X_1 and X_2 are clopen in $X_1 + X_2$, it follows that $f^{-1}(X_1)$ and $f^{-1}(X_2)$ are clopen in A , and in particular that $A \simeq f^{-1}(X_1) + f^{-1}(X_2)$. It follows that

Top is extensive. Another well-known and useful property of extensive categories is that connected objects interact nicely with coproducts.

2.3.3. **RECALL.** *An object A in an extensive category \mathcal{C} is called connected if the hom-functor $\text{hom}(A, -) : \mathcal{C} \rightarrow \text{Set}$ preserves finite coproducts.*

Here $\text{hom}(A, B)$ is exactly the set of all morphisms between A and B in \mathcal{C} , so an object A is connected when for every coproduct $X + Y$ every morphism $f : A \rightarrow X + Y$ can be factored uniquely through one of X or Y by way of the coproduct injections. In the case of **Top**, this is equivalent to the usual definition of a connected space, which is a non-empty space that cannot be represented as the union of two disjoint non-empty open subsets. In fact, a similar statement is well known to be true of connected objects in any extensive category.

2.3.4. **PROPOSITION.** *Let \mathcal{C} be an extensive category. An object A in \mathcal{C} is connected if and only if $A \simeq X + Y$ implies exactly one of X and Y is the initial object 0 .*

PROOF. First note that the initial object 0 is never connected, since $\text{hom}(0, X)$ is a singleton for all X . Now let A be connected and let $f : A \rightarrow X + Y$ be an isomorphism. Furthermore, let $i_X : X \rightarrow X + Y$ and $i_Y : Y \rightarrow X + Y$ be coproduct injections. Then, since A is connected, f must factor through either i_X or i_Y . Without loss of generality, assume $f = i_X g$ for some $g : A \rightarrow X$ as in the diagram:

$$\begin{array}{ccccc}
 Y' & \longrightarrow & 0 & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow i_Y \\
 A & \xrightarrow{g} & X & \xrightarrow{i_X} & X + Y
 \end{array} \tag{27}$$

Here both squares are pullbacks. The right square is a pullback since coproducts are disjoint in extensive categories. The left square is a pullback by construction. Note that by the pasting rule, the entire rectangle is then a pullback of an isomorphism, so the top row is also an isomorphism $Y' \simeq Y$. It follows that there is a morphism $Y \rightarrow 0$, which implies $Y \simeq 0$ since extensive categories have strict initial objects. We also know that X is not initial, since then $X + Y \simeq A$ would be initial and thus not connected.

On the other hand, let $A \in \mathcal{C}$ be such that $A \simeq X + Y$ implies one of X and Y is initial, and consider some $f : A \rightarrow X + Y$. Since \mathcal{C} is extensive, we can construct the commutative

diagram

$$\begin{array}{ccccc}
 A_1 & \xrightarrow{a_1} & A & \xleftarrow{a_2} & A_2 \\
 \downarrow f_1 & & \downarrow f & & \downarrow f_2 \\
 X & \xrightarrow{i_X} & X + Y & \xleftarrow{i_Y} & Y
 \end{array} \tag{28}$$

in which the top row is a coproduct diagram. It immediately follows from our assumptions that (without loss of generality) $A_1 \simeq A$ and $A_2 \simeq 0$, and so $f = i_X f_1 a_1^{-1}$ factors uniquely through the coproduct injection i_X as intended. ■

This result makes it convenient to work with connected objects in any extensive category. Later we will see that \mathbf{CRing}^{op} is extensive, and so the above result is also relevant to the study of affine schemes.

2.3.5. RECALL. (*Definition 3.1, [1]*). *A category with finite products and coproducts is distributive if the canonical arrow $\delta : A \times B + A \times C \rightarrow A \times (B + C)$ is an isomorphism.*

It is easy to see why this condition is called distributivity, since it is essentially a categorical version of the familiar distributivity of rings, $ab + ac = a(b + c)$. Extensivity and distributivity are closely related. In particular, any extensive category with products is also distributive, although the converse is not true [1].

2.3.6. RECALL. (*Dual of Definition 2.1, [1]*). *A category \mathcal{C} with products is coextensive if for any objects X, Y in \mathcal{C} , the canonical functor $\times : X/\mathcal{C} \times Y/\mathcal{C} \rightarrow (X \times Y)/\mathcal{C}$ is an equivalence.*

This is exactly the dual condition of extensivity in the sense that a category \mathcal{C} is coextensive if and only if the opposite category \mathcal{C}^{op} is extensive. Here $\times : X/\mathcal{C} \times Y/\mathcal{C} \rightarrow (X \times Y)/\mathcal{C}$ is the functor that maps a pair of morphisms (f, g) to their product $f \times g$. In a variety of algebras, pushouts of arbitrary morphisms across product projections exist, which implies that this functor has a left adjoint.

2.3.7. PROPOSITION. *Let \mathcal{C} be a category in which finite products and pushouts of arbitrary morphisms across product projections exist. Then the left adjoint of $\times : X/\mathcal{C} \times Y/\mathcal{C} \rightarrow (X \times Y)/\mathcal{C}$ is the functor $L : (X \times Y)/\mathcal{C} \rightarrow X/\mathcal{C} \times Y/\mathcal{C}$, which is the functor that sends a morphism $h : X \times Y \rightarrow Z$ to its pushouts across π_X and π_Y .*

PROOF. We will first show that such functors \times and L exist. Since \mathcal{C} has binary products, there must exist a unique morphism $f \times g$ such that the left diagram

$$\begin{array}{ccc}
 X & \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow f \times g \\
 A & \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} & Y \\
 \downarrow L(h)_1 & & \downarrow L(h)_2 \\
 A' & \xleftarrow{p'_1} Z \xrightarrow{p'_2} & B'
 \end{array}
 \quad (29)$$

commutes for any morphisms f, g . Similarly, since \mathcal{C} has pushouts of arbitrary morphisms across product projections, given any $h : X \times Y \rightarrow Z$, we can construct the right diagram in which both squares are pushouts, and it follows that $L(h) \in X/\mathcal{C} \times Y/\mathcal{C}$ as desired. It can be easily checked that identity morphisms are preserved in both cases. To say that L preserves composition is simply the pasting rule for pushouts, while \times preserving composition follows immediately from the universal property of products.

Now that we are satisfied with the existence of these functors, it remains to show that they are adjoint. It is sufficient to find natural transformations $\eta : 1_{(X \times Y)/\mathcal{C}} \rightarrow (\times)(L)$ (the unit) and $\varepsilon : 1_{X/\mathcal{C} \times Y/\mathcal{C}} \rightarrow (L)(\times)$ (the counit), such that such that the composites

$$\times \xrightarrow{\eta \times} (\times)L(\times) \xrightarrow{\times \varepsilon} \times \qquad L \xrightarrow{L\eta} L(\times)L \xrightarrow{\varepsilon L} L \quad (30)$$

are identities. The unit and counit of this adjunction arise quite naturally from the following diagrams.

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X & \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow f \times g \\
 A & \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} & B \\
 \downarrow L(f \times g)_1 & & \downarrow L(f \times g)_2 \\
 C & \xleftarrow{\pi_C} C \times D \xrightarrow{\pi_D} & D
 \end{array}
 &
 \begin{array}{ccc}
 X & \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} & Y \\
 \downarrow L(h)_1 & & \downarrow L(h)_2 \\
 A' & \xleftarrow{p'_1} Z \xrightarrow{p'_2} & B' \\
 \downarrow L(\eta_h)_1 & & \downarrow L(\eta_h)_2 \\
 C' & \xleftarrow{p''_1} A' \times B' \xrightarrow{p''_2} & D'
 \end{array}
 \end{array}
 \quad (31)$$

To construct the left diagram we first take the pushouts, C and D , of the left and right

squares respectively. This gives rise to the unique morphisms ε_f and ε_g such that the diagram commutes. $\eta_{(f \times g)}$ is then the unique morphism such that $\pi_C \eta_{(f \times g)} = p_1$ and $\pi_D \eta_{(f \times g)} = p_2$. It immediately follows that

$$\begin{aligned}\pi_A(\varepsilon_f \times \varepsilon_g)\eta_{(f \times g)} &= \varepsilon_f \pi_C \eta_{(f \times g)} = \varepsilon_f p_1 = \pi_A, \\ \pi_B(\varepsilon_f \times \varepsilon_g)\eta_{(f \times g)} &= \varepsilon_g \pi_D \eta_{(f \times g)} = \varepsilon_g p_2 = \pi_B.\end{aligned}$$

Therefore, by the universal property of $A \times B$, we have that $(\varepsilon_f \times \varepsilon_g)\eta_{(f \times g)} = 1_{(A \times B)}$, and so $(\times \varepsilon)(\eta_\times) = 1$ as desired. On the other hand, to construct the right diagram we construct the product $A' \times B'$, which gives rise to the unique morphism η_h such that the diagram commutes. We then construct the pushouts C' and D' , and note that by the pasting rule, the morphisms $\varepsilon_{L(h)_1}$ and $\varepsilon_{L(h)_2}$ are the unique morphisms such that

$$\begin{aligned}\varepsilon_{L(h)_1} p'_1 &= \pi'_A \text{ and } \varepsilon_{L(h)_1} L(\eta_h)_1 = 1_{A'}, \\ \varepsilon_{L(h)_2} p'_2 &= \pi'_B \text{ and } \varepsilon_{L(h)_2} L(\eta_h)_2 = 1_{B'}.\end{aligned}$$

Which implies that $(\varepsilon_L)(L\eta) = 1$, and so L is the left adjoint of \times as desired. ■

2.3.8. COROLLARY. *Let \mathcal{C} be a category in which finite products and pushouts of arbitrary morphisms across product projections exist. The following conditions are equivalent:*

- (a) \mathcal{C} is coextensive.
- (b) The unit η and counit ε of the adjunction $L \dashv (\times)$ are natural isomorphisms.
- (c) For any objects X and Y , and any commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow f & & \downarrow h & & \downarrow g \\ A' & \xleftarrow{p'_1} & Z & \xrightarrow{p'_2} & B' \end{array} \tag{32}$$

the bottom row of the diagram is a product diagram if and only if, both squares are pushouts.

PROOF. (a) \iff (b) Note that an adjunction whose unit and counit are natural isomorphisms is immediately an equivalence. Such an adjunction is usually called an adjoint equivalence. On the other hand, if \times is an equivalence then it immediately follows that it is part of an adjoint equivalence with some suitable left adjoint. Since adjunctions are unique up to isomorphism, this implies that L forms an adjoint equivalence with \times as desired. (b) \iff (c) follows immediately from the fact that products and pushouts are defined up to isomorphism. \blacksquare

For an alternative (dual) proof which does not rely on the existence of the adjoint functor L , see [1]. When condition (c) holds for some $X, Y \in \mathcal{C}$, we say that \mathcal{C} satisfies the coextensivity condition for any X and Y . This is already a very useful equivalent definition for coextensivity, but it can be simplified even further in the presence of initial objects.

2.3.9. PROPOSITION. *If a category with products has an initial object 0 , then it is coextensive if and only if it satisfies the coextensivity condition for 0 and 0 .*

PROOF. This is simply the dual condition of Proposition 4.1 of [1]. When \mathcal{C} is coextensive then it satisfies the coextensivity condition for any X and Y , including 0 and 0 . On the other hand, consider the diagram

$$\begin{array}{ccccc}
 0 & \xleftarrow{\pi_1} & 0 \times 0 & \xrightarrow{\pi_2} & 0 \\
 \downarrow !_X & & \downarrow !_X \times !_Y & & \downarrow !_Y \\
 & (1) & & (3) & \\
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow h & & \downarrow g \\
 A' & \xleftarrow{p'_1} & S & \xrightarrow{p'_2} & B' \\
 & (2) & & (4) &
 \end{array} \tag{33}$$

where $!_X$ and $!_Y$ are the unique morphisms from 0 to X and Y respectively. If \mathcal{C} satisfies the coextensivity condition for 0 and 0 then (1) and (3) are pushouts for all X, Y in \mathcal{C} . Now, by the pasting law, (2) and (4) are pushouts \iff (1)+(2) and (3)+(4) are pushouts \iff the bottom row is a product diagram. Therefore \mathcal{C} is coextensive. \blacksquare

There are also multiple ways of splitting coextensivity into smaller conditions. We say that \mathcal{C} is:

Left coextensive when the co-unit ε is a natural isomorphism.

Right coextensive when the unit η is a natural isomorphism.

Equivalently \mathcal{C} is left coextensive when, in the previous diagram, both squares are pushouts whenever the bottom row is a product diagram. Similarly \mathcal{C} is right coextensive when, in the previous diagram, the bottom row is a product diagram whenever both squares are pushouts. Clearly \mathcal{C} is coextensive if and only if \mathcal{C} is left and right coextensive. Next, we will recall the dual conditions of several well-known subconditions of extensivity.

2.3.10. **RECALL.** (*Dual of Definition 2.5, [1]*). *In a category with products and pushouts along their projections, a product is said to be co-disjoint if its projections are epimorphisms, and the pushout of its projections is the terminal object as in the diagram:*

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\pi_2} & Y \\
 \downarrow \pi_1 & & \downarrow \\
 X & \xrightarrow{\quad} & 1
 \end{array}
 \quad \square \quad (34)$$

Note that it is not immediately clear why co-disjoint products should require product projections to be epimorphisms, but we include this in the definition in order to be consistent with [1], which requires disjoint coproducts to have monic injections. If every product in a category \mathcal{C} is co-disjoint then we say that \mathcal{C} has co-disjoint products. As an example of a category in which most but not all products are co-disjoint, consider *Sets*. A product $X \times Y$ of two sets is co-disjoint if and only if neither X nor Y is the empty set. In a left coextensive category, all products must be co-disjoint.

2.3.11. **PROPOSITION.** *In a left coextensive category, products are co-disjoint.*

PROOF. Consider the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y \\
 \downarrow 1_X & & \downarrow \pi_1 & & \downarrow \\
 X & \xleftarrow{1_X} & X & \xrightarrow{\quad} & 1
 \end{array}
 \quad \square \quad (35)$$

in which both rows are product diagrams. Then, since \mathcal{C} is coextensive, both squares are pushouts. It is well known that a morphism π_1 in any category is an epimorphism if and only if the left square in the above diagram is a pushout. Therefore π_1 is an epimorphism, and so is π_2 by the same argument. ■

2.3.12. **RECALL.** (*Dual of Definition 2.10, [1]*). *In a category with finite products and pushouts along their projections, a product diagram*

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y \quad (36)$$

is co-universal if given any object $Z \in \mathcal{C}$ equipped with a morphism $h : X \times Y \rightarrow Z$, the pushouts of the projections π_1 and π_2 over h form a product diagram.

When this condition holds for a specific such morphism h and any factorisation of the domain of h as a product two objects X and Y , then we say that \mathcal{C} satisfies the co-universality condition for h . Note that all products are co-universal in a coextensive category. In fact, having co-universal products is exactly ‘half’ of the coextensivity condition in the following sense.

2.3.13. **PROPOSITION.** *A category \mathcal{C} with finite products and pushouts has co-universal products if and only if it is right coextensive.*

PROOF. Consider any product $X \times Y$, and any morphism $h : X \times Y \rightarrow Z$ for some $Z \in \mathcal{C}$. Then we can construct the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow L(h)_1 & & \downarrow h & & \downarrow L(h)_2 \\
 A & \xleftarrow{p_1} & Z & \xrightarrow{p_2} & B \\
 \swarrow \pi_A & & \downarrow \eta_h & & \searrow \pi_B \\
 & & A \times B & &
 \end{array} \quad (37)$$

in which both squares are pushouts, and η_h is the unique morphism such that the diagram commutes. Then \mathcal{C} has co-universal products if and only if the middle row is a product

diagram for any such diagram, which is the case if and only if η is a natural isomorphism as desired. ■

2.3.14. **RECALL.** *(Dual of Definition 2.7, [1]) A terminal object is said to be strict if any morphism from it is an isomorphism.*

2.3.15. **PROPOSITION.** *In a category with co-universal products, terminal objects are strict.*

PROOF. given a morphism $f : 1 \rightarrow A$, let $!^A$ be the unique morphism $!^A : A \rightarrow 1$. It immediately follows that $!^A f$ is the identity morphism of 1, since this is the unique morphism from the terminal object to itself. We can then construct the diagram

$$\begin{array}{ccccc}
 1 & \longleftarrow & 1 & \longrightarrow & 1 \\
 \downarrow f & & \downarrow f & & \downarrow f \\
 A & \xleftarrow{1_A} & A & \xrightarrow{1_A} & A
 \end{array}
 \tag{38}$$

in which both squares are pushouts. Note that the top row is immediately a product diagram, so the bottom row must be a product diagram as well. Then, since the bottom row is a product diagram, we have that $f!^A = 1_A$ by the following lemma. ■

2.3.16. **LEMMA.** *Let $X \times Y$ be a product with invertible product projections π_X, π_Y . Then for any object $Z \in \mathcal{C}$ there exists at most one morphism from Z to $X \times Y$.*

PROOF. Consider any pair of morphisms $f, g : Z \rightarrow X \times Y$. Then, by the universal property of the product, there exists a unique morphism $h : Z \rightarrow X \times Y$

$$\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow & \vdots & \searrow & \\
 & \pi_X f & h & \pi_Y g & \\
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y
 \end{array}
 \tag{39}$$

such that $\pi_X h = \pi_X f$ and $\pi_Y h = \pi_Y g$. But note that π_X and π_Y are invertible, so it immediately follows that $f = h = g$ as desired. ■

Even though \mathcal{C} having co-disjoint products is a weaker condition than \mathcal{C} being left-coextensive, it is still the case that \mathcal{C} is coextensive if and only if \mathcal{C} has co-disjoint and co-universal products. We repeat here the dual of the proof of Proposition 2.14 of [1].

2.3.17. PROPOSITION. *A category \mathcal{C} with finite products and pushouts is coextensive if and only if all products in \mathcal{C} are co-universal and co-disjoint.*

PROOF. We have already shown that products are co-universal and co-disjoint in a coextensive category. It remains to show that if products are co-disjoint and co-universal, then both squares of the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow f \times g & & \downarrow g \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array} \tag{40}$$

are pushouts. We will show that the left square is a pushout. Consider any object $C \in \mathcal{C}$ equipped with morphisms $k : X \rightarrow C$ and $h : A \times B \rightarrow C$ such that $h(f \times g) = k\pi_X$. Then, we can construct the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow f \times g & & \downarrow g \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\
 \downarrow h_1 & & \downarrow h & & \downarrow h_2 \\
 C_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & C_2
 \end{array} \tag{41}$$

in which the bottom two squares are pushouts. \mathcal{C} has co-universal products, so the bottom row is a product diagram. Additionally, $h_2 g \pi_Y = c_2 h(f \times g) = c_2 k \pi_X$, and so there exists a morphism $p : C_1 \rightarrow C_2$ as in the diagram

$$\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_X} & X \\
\downarrow \pi_Y & & \downarrow \\
Y & \xrightarrow{\quad} & 1 \\
& \searrow h_2 g & \swarrow p \\
& & C_2
\end{array}
\quad \begin{array}{l}
\lrcorner \\
c_2 k
\end{array}
\quad (42)$$

since \mathcal{C} has co-disjoint products. But \mathcal{C} also has strict terminal objects, so C_2 is terminal. It immediately follows that $C \simeq C_1 \times 1 \simeq C_1$, and so c_1 is invertible. $c_1^{-1}h_1$ is then the unique morphism required to make the diagram

$$\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_X} & X \\
\downarrow f \times g & & \downarrow f \\
A \times B & \xrightarrow{\pi_A} & A \\
& \searrow h & \swarrow c_1^{-1} h_1 \\
& & C
\end{array}
\quad \begin{array}{l}
\lrcorner \\
k
\end{array}
\quad (43)$$

a pushout as desired. To see that this diagram commutes note that $c_1^{-1}h_1\pi_A = c_1^{-1}c_1h = h$ and that $k\pi_X = h(f \times g) = c_1^{-1}c_1h(f \times g) = c_1^{-1}h_1f\pi_X$. Then, since \mathcal{C} has co-disjoint products, π_X is an epimorphism, and we have $k = c_1^{-1}h_1f$ as desired. Furthermore, $c_1^{-1}h_1$ is the unique morphism such that this diagram commutes, since π_A is an epimorphism. ■

Just as extensivity and distributivity are closely related, so too is coextensivity closely related to codistributivity.

2.3.18. DEFINITION. *A category \mathcal{C} with finite products and coproducts is codistributive if the canonical arrow $A + (B \times C) \longrightarrow (A + B) \times (A + C)$ is an isomorphism for any objects $A, B, C \in \mathcal{C}$.*

In particular, we are interested in codistributivity due to the following result.

2.3.19. PROPOSITION. *Any coextensive category with coproducts is also codistributive.*

PROOF. Note that this is simply the dual of Proposition 4.5 of [1]. Given objects A, B, C of \mathcal{C} , it can be seen that both squares of the diagram

$$\begin{array}{ccccc}
 B & \xleftarrow{\pi_B} & B \times C & \xrightarrow{\pi_C} & C \\
 \downarrow & & \downarrow & & \downarrow \\
 A + B & \xleftarrow{1_A + \pi_B} & A + (B \times C) & \xrightarrow{1_A + \pi_C} & A + C
 \end{array} \tag{44}$$

are pushouts. Since \mathcal{C} is coextensive, it follows that the bottom row is a product diagram, which is exactly the requirement of codistributivity. ■

2.4. REGULAR AND EXACT CATEGORIES. Regular and exact categories play an important role in categorical algebra as the natural context in which to understand various exactness conditions common to all varieties of algebras, including *Sets*. In this section we recall the well-known definitions of strong and regular epimorphisms, internal relations, and regular and exact categories. For an in depth treatment of regularity and internal relations, see [14]. We begin with strong and regular epimorphisms.

2.4.1. RECALL. (*Definition 1.2, [14]*). An epimorphism $f : A \rightarrow B$ is a strong epimorphism if given any monomorphism m in a commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & \searrow \phi & \downarrow h \\
 C & \xrightarrow{m} & D
 \end{array} \tag{45}$$

there exists a unique morphism $\phi : B \rightarrow C$ such that $\phi f = g$ and $m\phi = h$.

2.4.2. PROPOSITION. *The composite of two strong epimorphisms is also a strong epimorphism.*

PROOF. Let $f_1 : A \rightarrow X$ and $f_2 : X \rightarrow C$ be strong epimorphisms. Then for any monomorphism m and any commutative square

$$\begin{array}{ccccc}
 A & \xrightarrow{f_1} & X & \xrightarrow{f_2} & B \\
 \downarrow g & & \swarrow \phi_1 & \searrow \phi_2 & \downarrow h \\
 C & & & & D \\
 & & \swarrow & \searrow & \\
 & & C & \xrightarrow{m} & D
 \end{array}
 \tag{46}$$

there exists a unique morphism $\phi_1 : X \rightarrow C$ such that $\phi_1 f_1 = g$ and $m\phi_1 = hf_2$, since f_1 is strong. But now, since f_2 is strong there exists a unique morphism $\phi_2 : B \rightarrow C$ such that the above diagram commutes. Thus $f_2 f_1$ is a strong epimorphism as desired. ■

2.4.3. PROPOSITION. *A morphism f is both a strong epimorphism and a monomorphism if and only if f is an isomorphism.*

PROOF. It is clear that when f is an isomorphism, then f is a monomorphism and an epimorphism. To see that f is a strong epimorphism, note that if $hf = mg$ and m is a monomorphism, then $g = gf^{-1}f$ and $h = hff^{-1} = mgf^{-1}$, so $\phi = gf^{-1}$ is the unique morphism such that $g = \phi f$ and $h = m\phi$, and f is a strong epimorphism as desired. On the other hand, let f be a strong epimorphism and a monomorphism. Then we can construct a commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow 1_A & & \downarrow 1_B \\
 A & \xrightarrow{f} & B \\
 & \swarrow \phi & \searrow \\
 & A & B
 \end{array}
 \tag{47}$$

for some unique morphism ϕ , which is exactly to say that f is an isomorphism with inverse $f^{-1} = \phi$. ■

2.4.4. RECALL. (*Definition 1.6, [14]*). *A morphism $f : A \rightarrow B$ is a regular epimorphism if it is the coequalizer of any pair of morphisms.*

In a variety of algebras, a regular epimorphism $f : A \rightarrow B$ is again simply a surjective homomorphism, since surjective homomorphisms are isomorphic to quotient maps $f \simeq f/Eq(f)$, and so f is the coequalizer of the projections $\pi_1, \pi_2 : Eq(f) \rightarrow A$.

2.4.5. PROPOSITION. *Every split epimorphism is regular, and every regular epimorphism is strong.*

PROOF. Let $f : A \rightarrow B$ be a split epimorphism. Then there exists some monomorphism $m : B \rightarrow A$ such that $fm = 1_B$. We will show that f is the coequalizer of 1_A and mf . Let g be any morphism such that $gmf = g$ and consider the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow[mf]{1_A} & A & \xrightarrow{f} & B \\
 & & & \searrow g & \downarrow gm \\
 & & & & C
 \end{array} \tag{48}$$

which commutes, since $fmf = 1_A f = f1_A$, and $gmf = g$ by assumption. Furthermore f is an epimorphism, so $gm : B \rightarrow C$ is the only morphism such that the diagram commutes. Therefore the top row is a coequalizer diagram, and f is regular as desired.

Now assume that $f : A \rightarrow B$ is a regular epimorphism, and let $p_1, p_2 : E \rightarrow A$ be any pair of morphisms such that f is the equalizer of p_1 and p_2 . Consider any commutative diagram

$$\begin{array}{ccccc}
 E & \xrightarrow[p_2]{p_1} & A & \xrightarrow{f} & B \\
 & & \downarrow g & \nearrow \phi & \downarrow h \\
 & & C & \xrightarrow{m} & D
 \end{array} \tag{49}$$

in which m is a monomorphism. Note that $mgp_1 = hfp_1 = hfp_2 = mgp_2$, and so $gp_1 = gp_2$ since m is a monomorphism. Therefore, since f is the coequalizer of p_1 and p_2 , there exists a unique morphism $\phi : B \rightarrow C$ such that $\phi f = g$. It follows that $hf = mg = m\phi f$, and so $h = m\phi$ since f is an epimorphism. Therefore the diagram commutes, and f is a strong epimorphism as desired. ■

Note for example that in the category *Sets*, epimorphisms, strong epimorphisms, and regular epimorphisms are all just surjections. Furthermore, the axiom of choice is equivalent to the statement 'all epimorphisms in *Sets* are split'. The result that strong and regular epimorphisms are identical is quite useful, and we will see later that it holds in any regular category. It should come as no surprise that regular epimorphisms are important to the study of regular and exact categories. Another important concept is the notion

of an internal relation, which is a categorical construction that generalizes the familiar relations of sets and algebras.

2.4.6. **RECALL.** (*Definition 1.22, [14]*). *Let \mathcal{C} be a category. An internal relation from an object A to an object B is an object R equipped with two morphisms $d_1 : R \rightarrow A$ and $d_2 : R \rightarrow B$ that are jointly monic.*

By jointly monic, we mean that given any two morphisms $f, g : C \rightarrow R$ such that $d_1 f = d_1 g$ and $d_2 f = d_2 g$, then $f = g$. When \mathcal{C} has binary products, then an internal relation R from A to B is simply a subobject of $A \times B$. In this case we can represent the relation as an object R equipped with a monomorphism $d : R \rightarrow A \times B$ as in the commutative diagram:

$$\begin{array}{ccccc}
 C & \xrightarrow{f} & R & & \\
 & \xrightarrow{g} & \uparrow & & \\
 & & \vdots & & \\
 & & d & & \\
 & & \downarrow & & \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array}
 \tag{50}$$

The morphisms d_1 and d_2 can then be recovered as $d_1 = \pi_A \circ d$ and $d_2 = \pi_B \circ d$. In a variety, a relation R between two algebras A and B is some subset $R \subseteq A \times B$, but not necessarily a subalgebra of $A \times B$. When a relation R is also a subalgebra, then we say that it is a homomorphic relation. It is clear that in a variety all homomorphic relations are internal relations, and all internal relations (R, d_1, d_2) are isomorphic to the homomorphic relation $d_1(r) \simeq d_2(r)$ for all $r \in R$. In a variety relations can be homomorphic, reflexive, symmetric, and transitive. These properties are commonly generalized to internal relations in the following way:

2.4.7. **RECALL.** (*pg 17, [14]*). *An internal relation (R, d_1, d_2) between some object X and itself is: [14]*

- **reflexive** if there exists some morphism $\Delta : X \rightarrow R$ such that $d_1 \Delta = d_2 \Delta = 1_X$.
- **symmetric** if there exists some morphism $\sigma : R \rightarrow R$ such that $d_1 \sigma = d_2$ and $d_2 \sigma = d_1$.

- **transitive** if, where the diagram

$$\begin{array}{ccc}
 R \times_X R & \xrightarrow{p_2} & R \\
 \downarrow p_1 & & \downarrow d_1 \\
 R & \xrightarrow{d_2} & X
 \end{array}$$

is a pullback, there exists some morphism $\tau : R \times_X R \longrightarrow R$ such that $d_1\tau = d_1p_1$ and $d_2\tau = d_2p_2$.

- a **congruence** if it is reflexive, symmetric and transitive.

To see that the above definitions generalize the usual algebraic definitions, let $R \leq X^2$ be a relation on an algebra X in some variety \mathcal{C} . Then there exists some morphism $\Delta : X \longrightarrow R$ such that $d_1\Delta = d_2\Delta = 1_X$ if and only if $\Delta(x) = (x, x) \in R$ for all $x \in X$. Similarly, there exists some morphism $\sigma : R \longrightarrow R$ such that $d_1\sigma = d_2$ and $d_2\sigma = d_1$ if and only if for any $(x, y) \in R$, we have $\sigma(x, y) = (y, x) \in R$. For transitivity, note that $R \times_X R$ is isomorphic to the set $\{((x, y), (y, z)) \in R^2\}$, where p_1 and p_2 can be seen as (restrictions of) the usual product projections $p_1((x, y), (y, z)) = (x, y)$ and $p_2((x, y), (y, z)) = (y, z)$. Then there exists some $\tau : R \times_X R \longrightarrow R$ such that $d_1\tau = d_1p_1$ and $d_2\tau = d_2p_2$ if and only if $\tau((x, y), (y, z)) = (x, z) \in R$. Clearly such a τ exists if and only if R is transitive in the usual sense.

It follows that a relation on an algebra is a congruence in the usual sense if and only if it is a congruence as an internal relation. Recall that congruences play an important role in universal algebra. In particular, for any homomorphism $f : A \longrightarrow B$, there exists a congruence $Eq(f) = \{(a_1, a_2) \in A^2 \mid f(a_1) = f(a_2)\}$, called the congruence generated by f . The projections $p_1, p_2 : Eq(f) \longrightarrow A$ are also the kernel pair of f , which suggests the following property of internal relations.

2.4.8. PROPOSITION. (*Lemma 1.23, [14]*). *In a category with pullbacks the kernel pair of any morphism is an internal congruence.*

PROOF. Let $f : A \longrightarrow B$ be a morphism in some category \mathcal{C} . Then, let $Eq(f)$ be the congruence generated by f . By the universal property of the kernel pair, we can construct the commutative diagram

- regular epimorphisms are pullback stable,
- coequalizers of kernel pairs exist in \mathcal{C} .

Furthermore we say that a regular category \mathcal{C} is exact if every internal equivalence relation is the kernel pair of some morphism.

2.4.10. EXAMPLE. *Sets* is an exact category.

PROOF. It is well known that *Sets* is finitely complete, which is simply to say that it has all small limits. Regular epimorphisms in *Sets* are simply surjective maps. To see that surjective maps are pullback stable, consider the pullback

$$\begin{array}{ccc}
 A \times_B C & \xrightarrow{g'} & A \\
 \downarrow f' & \lrcorner & \downarrow f \\
 C & \xrightarrow{g} & B
 \end{array} \tag{53}$$

where $f : B \rightarrow A$ is surjective. Then for any $c \in C$ there exists some $a \in A$ such that $f(a) = g(c)$, and so $(a, c) \in A \times_B C$, and $f'(a, c) = c$, so f' is surjective as desired. Lastly note that in *Sets* kernel pairs are all equivalence relations, and every equivalence relation E on some set A is the kernel pair of its quotient map $q : A \rightarrow A/E$. ■

Recall that given any algebra A in some variety \mathcal{C} , the congruences on A form a lattice $Con(A)$ in which $E \wedge R = E \cap R$, $E \vee R = \bigcap \{C \in Con(A) \mid C \geq E, R\}$. Naturally the largest element of $Con(A)$ is A^2 and the smallest is Δ_A . In an exact category, $Con(A)$ forms a lattice up to isomorphism in much the same way.

2.4.11. PROPOSITION. Let $A \in \mathcal{C}$ be an object in an exact category with pushouts and let $Con(A)$ be the skeleton of the full subcategory of \mathcal{C}/A^2 containing all internal congruences of A . Then $Con(A)$ is a lattice.

PROOF. First note that by the universal property of a kernel pair, there is at most a single morphism between any two objects in this category. $Con(A)$ is thus a partially ordered set. The congruences (A^2, π_1, π_2) and $(A, 1_A, 1_A)$ are immediately the largest and smallest elements. Given two congruences $E, R \in Con(A)$, with coequalizers p, q , the join $E \vee R$ can be seen as the kernel pair of $p'q = q'p$ as in the left diagram below, which is a pushout, while the meet $E \wedge R$ can be seen as the kernel pair of $\langle p, q \rangle$ as in the right diagram.

$$\begin{array}{ccc}
E \vee R & \begin{array}{c} \xrightarrow{j_1} \\ \xrightarrow{j_2} \end{array} & A & \xrightarrow{p} & B \\
& & \downarrow q & & \downarrow q' \\
& & C & \xrightarrow{p'} & D
\end{array}
\quad \sqcap \quad
\begin{array}{ccc}
X & \xrightarrow{x_1} & A \\
\downarrow x_2 & \lrcorner & \downarrow \langle p, q \rangle \\
E \wedge R & \xrightarrow{m_1} & A \\
\downarrow m_2 & \lrcorner & \downarrow \langle p, q \rangle \\
A & \xrightarrow{\langle p, q \rangle} & B \times C
\end{array}
\tag{54}$$

First note that \mathcal{C} is exact, so (E, e_1, e_2) and (R, r_1, r_2) are the kernel pairs of their coequalizers p and q respectively. Let (X, x_1, x_2) be another congruence on A with coequalizer $y : A \rightarrow Y$. Note that $X \geq E, R$ if and only if $ye_1 = ye_2$ and $yr_1 = yr_2$ if and only if $yj_1 = yj_2$ if and only if $X \geq E \vee R = Eq(q'p)$. On the other hand, $X \leq E, R$ if and only if $x_1p = x_2p$ and $x_1q = x_2q$ if and only if $x_1\langle p, q \rangle = x_2\langle p, q \rangle$ if and only if $X \leq E \wedge R$. It follows that $E \vee X$ and $E \wedge R$ are the join and meet respectively, and $Con(A)$ is a lattice as desired. ■

As mentioned before, any morphism in a regular category can be uniquely factorized into a regular epimorphism composed with a monomorphism. To see how this arises, consider the following well-known propositions.

2.4.12. PROPOSITION. *Any morphism $f : A \rightarrow B$ with a kernel pair $p_1, p_2 : Eq(f) \rightarrow A$ is a monomorphism if and only if $p_1 = p_2$.*

PROOF. Note that $fp_1 = fp_2$, so if f is a monomorphism we immediately have $p_1 = p_2$. On the other hand, if $p_1 = p_2$, then let $e_1, e_2 : E \rightarrow A$ be any pair of morphisms such that $fe_1 = fe_2$. Then, since p_1, p_2 is the kernel pair of f , there exists a unique morphism $m : E \rightarrow Eq(f)$ such that $e_1 = p_1m = p_2m = e_2$, and so f is a monomorphism. ■

2.4.13. PROPOSITION. *Let \mathcal{C} be a regular category. Then any morphism $f : A \rightarrow B$ has a unique up to isomorphism factorization as $f = me$ where m is a monomorphism and e is a regular epimorphism. [14]*

PROOF. Let $e : A \rightarrow A/Eq(f)$ be the coequalizer of the kernel pair $(Eq(f), p_1, p_2)$ generated by f . Then, since $fp_1 = fp_2$, we can construct the commutative diagram

$$\begin{array}{ccccc}
 Eq(f) & \xrightarrow[p_2]{p_1} & A & \xrightarrow{f} & B \\
 & & & \searrow e & \uparrow \text{---} m \\
 & & & & A/Eq(f)
 \end{array} \tag{55}$$

for a unique morphism $m : A/Eq(f) \rightarrow B$. Let $(Eq(m), q_1, q_2)$ be the kernel pair of m . To show that m is a monomorphism it is sufficient to show that $q_1 = q_2$. Consider the diagram

$$\begin{array}{ccccccc}
 Eq(f) & \xrightarrow{b} & Eq(m) \times_{A/Eq(f)} A & \xrightarrow{\pi_2} & A & & \\
 \downarrow a & & \downarrow \pi_1 & & \downarrow e & & \\
 A \times_{A/Eq(f)} Eq(m) & \xrightarrow{\phi_2} & Eq(m) & \xrightarrow{q_2} & A/Eq(f) & & \\
 \downarrow \phi_1 & & \downarrow q_1 & & \downarrow m & & \\
 A & \xrightarrow{e} & A/Eq(f) & \xrightarrow{m} & B & &
 \end{array} \tag{56}$$

in which all four squares are pullbacks by definition. First note that by the pasting rule, the outer square is also a pullback, and in particular $(Eq(f), \phi_1 a, \pi_2 b)$ is the kernel pair of $f = me$. Since kernel pairs are unique up to isomorphism, we can assume $p_1 = \phi_1 a$ and $p_2 = \pi_2 b$. Next note that by the commutativity of the diagram we have

$$q_1 \phi_2 a = e \phi_1 a = e p_1 = e p_2 = e \pi_2 b = q_2 \pi_1 b = q_2 \phi_2 a.$$

Since \mathcal{C} is regular, regular epimorphisms are preserved by pullbacks, so π_1 , a , ϕ_2 , and b are all epimorphisms, and thus $\phi_2 a$ is an epimorphism as well. It follows that $q_1 = q_2$, and so m is a monomorphism as desired. Next, to show that this factorization is unique up to isomorphism, let $f = m'e'$ where e' is a regular epimorphism and m' is a monomorphism. Since m' is a monomorphism $(Eq(f), p_1, p_2)$ is also the kernel pair of e' . Then, since e' is a regular epimorphism, e' is the coequalizer of p_1 and p_2 by Proposition 2.1.9. Therefore, since co-equalizers are unique up to isomorphism, so is the factorization $f = me$.

■

The ability to uniquely factorize any morphism into a regular epimorphism and a monomorphism is in some ways the most useful property of regular categories. One result which immediately follows is that all strong epimorphisms are regular.

2.4.14. PROPOSITION. *Let \mathcal{C} be a regular category. Then a morphism f is a regular epimorphism if and only if it is a strong epimorphism.*

PROOF. We have already shown that any regular epimorphism is strong. On the other hand, let $f : A \rightarrow B$ be a strong epimorphism, and let $f = me$ for some monomorphism m and regular epimorphism e . Then the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow e & \searrow \phi & \downarrow 1_B \\
 C & \xrightarrow{m} & B
 \end{array}
 \tag{57}$$

commutes for some morphism $\phi : B \rightarrow C$. However, note that $m\phi = 1_B$, so m is both a monomorphism and a split epimorphism. It follows that m is an isomorphism, and thus $f \simeq e$, so f is a regular epimorphism as desired. ■

2.4.15. COROLLARY. *The composite of two regular epimorphisms is also a regular epimorphism.*

PROOF. Let \mathcal{C} be a regular category, and $f : A \rightarrow B$ and $g : B \rightarrow C$ regular epimorphisms in \mathcal{C} . Since f and g are regular epimorphisms, they are strong epimorphisms. It follows that their composite gf is a strong epimorphism, and so gf is also a regular epimorphism as desired. ■

Regular epimorphisms are very well behaved in regular categories. We have already seen that they are preserved by pullbacks and composition. Lastly we will show that they are preserved by binary products.

2.4.16. PROPOSITION. *Let \mathcal{C} be a regular category, and let $f : X \rightarrow A$ and $g : Y \rightarrow B$ be regular epimorphisms. Then $f \times g : X \times Y \rightarrow A \times B$ is also a regular epimorphism.*

PROOF. Consider the diagram

$$\begin{array}{ccccc}
 X \times Y & \xrightarrow{f \times 1_Y} & A \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow 1_X \times g & \lrcorner & \downarrow 1_A \times g & \lrcorner & \downarrow g \\
 X \times B & \xrightarrow{f \times 1_B} & A \times B & \xrightarrow{\pi_B} & B \\
 \downarrow \pi_X & \lrcorner & \downarrow \pi_A & \lrcorner & \\
 X & \xrightarrow{f} & A & &
 \end{array} \tag{58}$$

in which all three squares are pullbacks. First note that morphisms $f \times 1_B, f \times 1_Y, 1_A \times g, 1_X \times g$ are all regular epimorphisms, since regular epimorphisms are preserved by pullbacks in regular categories. It immediately follows from the previous corollary that $f \times g = (f \times 1_B) \circ (1_X \times g)$ is a regular epimorphism as desired. ■

3. Coextensivity in varieties of universal algebras

In this section we begin the work of translating coextensivity into the language of universal algebra. We will begin with commutative semirings, which are the motivating example of a coextensive variety of universal algebras. Next, we will characterize some of the subconditions of coextensivity before moving on to the full condition. Of these subconditions, having co-disjoint products turns out to be remarkably simple for varieties, while left coextensivity gives rise to a Mal'tsev condition. We lastly show that right coextensivity is almost the same as coextensivity for varieties, and in fact there are no non-trivial right coextensive varieties that are not also coextensive.

3.1. COMMUTATIVE SEMIRINGS. Recall that a commutative semiring R is an algebra equipped with binary operations $+$ and \cdot , as well as constants 0 and 1 , such that both $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids, and R further satisfies the usual axioms of annihilation $a \cdot 0 = 0 \cdot a = 0$ and distributivity $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$. Consider the variety **CSemiRing** of commutative semirings. Let $h : X \times Y \rightarrow Z$ be a homomorphism in this variety. Then Z contains some elements $e_1 = h(1, 0)$ and $e_2 = h(0, 1)$. Since h is a homomorphism these elements satisfy the identities:

- $e_1 + e_2 = 1$,
- $e_1 \cdot e_2 = 0$,

- $e_1 \cdot e_1 = e_1$,
- $e_2 \cdot e_2 = e_2$.

We can now define the two cosets $e_1Z = \{e_1z \mid z \in Z\}$, $e_2Z = \{e_2z \mid z \in Z\}$. These sets can be made into commutative semirings using the same addition, multiplication and 0 as in Z . The unit is the only thing that changes, specifically for $i \in \{1, 2\}$ and any $e_i z \in e_i Z$ we have $e_i \cdot e_i z = (e_i e_i)z = e_i z$ since both e_1 and e_2 are idempotent. So e_1 and e_2 are the units of $e_1 Z$ and $e_2 Z$ respectively. Note that these operations are defined in just the right way so that the maps $p_i : Z \rightarrow e_i Z$ defined by $p_i(z) = e_i z$ are homomorphisms, and in fact both squares in the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 h_1 \downarrow & & h \downarrow & & h_2 \downarrow \\
 e_1 Z & \xleftarrow{p_1} & Z & \xrightarrow{p_2} & e_2 Z
 \end{array} \tag{59}$$

are pushouts, where $h_1(x) = h((x, 0))$ and $h_2(y) = h((0, y))$. To see that the left square is a pushout, first note that for any $(x, y) \in X \times Y$, we have $p_1 h(x, y) = e_1 h(x, y) = h(1, 0)h(x, y) = h(x, 0) = h_1(x)$, so the left square commutes. π_X is surjective, so by 2.2.13, the left square is a pushout if and only if $e_1 Z \simeq Z/E$ where E is the congruence on Z generated by $(h(x, y_1), h(x, y_2)) \in E$ for all $x \in X$ and $y_1, y_2 \in Y$. This reduces to showing that $E = E'$, where E' is the congruence on Z generated by the relation $e_1 \simeq 1$. Clearly $e_1 = h(1, 0) \simeq_E h(1, 1) = 1$, so $E' \leq E$. On the other hand, given some $x \in X$ and $y_1, y_2 \in Y$, we have $h(x, y_1) \simeq_{E'} h(x, y_1)e_1 = h(x, 0) = h(x, y_2)e_1 \simeq_{E'} h(x, y_2)$, so $E = E'$ as desired. The right square is also a pushout following an identical argument.

It remains to show that the bottom row is a product diagram. To show that $\langle p_1, p_2 \rangle$ is an isomorphism, consider the map $f : e_1 Z \times e_2 Z \rightarrow Z$ defined by $f(e_1 z_1, e_2 z_2) = e_1 z_1 + e_2 z_2$ for all $z_1, z_2 \in Z$. Then, by the above identities, we have:

$$\begin{aligned}
 f\langle p_1, p_2 \rangle(z) &= e_1 z + e_2 z = (e_1 + e_2)z = 1z = z, \\
 \langle p_1, p_2 \rangle f(e_1 z_1, e_2 z_2) &= (e_1 e_1 z_1 + e_1 e_2 z_2, e_2 e_1 z_1 + e_2 e_2 z_2) = (e_1 z_1, e_2 z_2).
 \end{aligned}$$

It follows that f is the inverse of $\langle p_1, p_2 \rangle$, the bottom row of the above diagram is a product diagram, and so **CSemiRing** has co-universal products. The above example

can give us some intuition as to what coextensive varieties look like in general. Clearly the elements $e_1 = h(1, 0)$ and $e_2 = h(0, 1)$ play an important part in turning Z into a product. The images of pairs of constants will prove to be critically important in the general case as well. So far we have only shown that **CSemiRing** has co-universal products. That **CSemiRing** has co-disjoint products follows as a consequence of the following proposition.

3.1.1. PROPOSITION. *A variety \mathcal{C} has co-disjoint products if and only if \mathcal{C} has at least one constant term.*

PROOF. Note that when \mathcal{C} is a variety, the terminal object 1 is any singleton considered as an algebra, where for any $n \in \mathbb{N}$ all operations in \mathcal{C} of arity n are given by the unique map from 1^n to 1 , and all identities are trivially satisfied. Similarly, the empty set 0 can be made an algebra in \mathcal{C} exactly when \mathcal{C} has no constant terms, since there is a unique empty map from 0^n to 0 for all $n > 0$. It follows that the diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\pi_2} & Y \\
 \downarrow \pi_1 & & \downarrow \\
 X & \xrightarrow{\quad} & 1
 \end{array}
 \tag{60}$$

is a pushout as long as neither X nor Y is empty. On the other hand if both X and Y are empty the diagram is never a pushout, since there is no map from 1 to the empty algebra 0 . Therefore \mathcal{C} has co-disjoint products exactly when $\emptyset \notin \mathcal{C}$, which is to say that \mathcal{C} has at least one constant term. ■

3.2. LEFT COEXTENSIVITY. As we have seen, the characterization of varieties with co-disjoint products is remarkably simple. Next we will consider the stronger condition of left coextensivity, which turns out to have a more involved characterization [2]. We first simplify the condition in a familiar way.

3.2.1. PROPOSITION. *A variety \mathcal{C} is left coextensive if and only if the diagram*

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\pi_X} & X \\
 \downarrow f \times g & & \downarrow f \\
 A \times B & \xrightarrow{\pi_A} & A
 \end{array}
 \quad (61)$$

is a pushout for any $f \times g : X \times Y \rightarrow A \times B$.

PROOF. Recall that a variety is left coextensive if the counit ε is a natural isomorphism. Every component $(\varepsilon_f, \varepsilon_g)$ of the counit is an isomorphism if and only if ε_f and ε_g are isomorphisms for any morphisms f, g . Since pushouts are defined up to isomorphism, ε_f is an isomorphism if and only if the above diagram is a pushout. ■

Consider the forgetful functor $U : \mathcal{C} \rightarrow \mathit{Sets}$ taking each algebra to its underlying set, and its left adjoint, the free functor $F : \mathit{Sets} \rightarrow \mathcal{C}$ taking each set X to the free algebra $F(X)$ in \mathcal{C} generated on X . Since \emptyset is the initial object in Sets we have that $F(\emptyset)$ is the initial object in \mathcal{C} . This allows us to consider only the coextensivity condition for $F(\emptyset)$ and $F(\emptyset)$ as follows from Proposition 2.3.9. Given any constant $e \in F(\emptyset)$, we denote by $R_e(A \times B)$ the relation

$$R_e(A \times B) = \{((e_1, e_2), (e_1, e)) \in (A \times B) \times (A \times B) \mid e_1, e_2 \in F(\emptyset)\},$$

and let $C_e(A \times B)$ be the congruence on $A \times B$ generated by R_e . Note also that for any constants e, e' we have $C_e(A \times B) = C_{e'}(A \times B)$, since $R_e \subseteq C_{e'}$ and $R_{e'} \subseteq C_e$ by transitivity and symmetry of congruences. Recall that pushout diagrams of algebras are quotients of coproducts as in Proposition 2.2.12, so calculating the quotient in the above case gives rise to the following proposition.

3.2.2. PROPOSITION. *Let \mathcal{C} be a variety and e any constant term. The following statements are equivalent:*

- (a) \mathcal{C} is left coextensive.
- (b) For all A, B in \mathcal{C} and any $a \in A$ and $b \in B$, $((a, b), (a, e))$ belongs to $C_e(A \times B)$.
- (c) $((x, x), (x, e))$ belongs to $C_e(F(\{x\})^2)$, where $F(\{x\})$ denotes the free algebra on a one element set $\{x\}$.

PROOF. ($a \iff b$) Consider the diagram

$$\begin{array}{ccc}
 F(\emptyset)^2 & \xrightarrow{\pi_1} & F(\emptyset) \\
 \downarrow !_A \times !_B & \swarrow \Delta & \downarrow !_A \\
 A \times B & \xrightarrow{\pi_A} & A
 \end{array} \tag{62}$$

in which Δ is the unique morphism $\Delta : F(\emptyset) \rightarrow F(\emptyset)^2$ such that $\pi_1 \Delta = \pi_2 \Delta = 1_{F(\emptyset)}$. Then by Lemma 2.1.11, the diagram is a pushout if and only if π_A is the coequalizer of $!_A \times !_B$ and $(!_A \times !_B) \Delta \pi_1$. Note that for any $e_1, e_2 \in F(\emptyset)$, we have $(!_A \times !_B)(e_1, e_2) = (e_1, e_2)$, and $(!_A \times !_B) \Delta \pi_1(e_1, e_2) = (e_1, e_1)$, so the coequalizer is isomorphic to the quotient $q : A \times B \rightarrow (A \times B)/C_e(A \times B)$. Therefore π_A is the coequalizer if and only if $((a, b), (a, e))$ belongs to $C_e(A \times B)$ for any $a \in A$ and $b \in B$ as desired.

($b \iff c$) Clearly (b) implies (c). On the other hand, if (c) holds, then since $F(\{x\})$ is a free algebra, for any $(a, b) \in A \times B$, there is a unique map $m_A \times m_B : F(\{x\})^2 \rightarrow (A \times B)$ such that $m_A(x) = a$ and $m_B(x) = b$. Next, let $p : F(\{x\})^2 \rightarrow F(\{x\})^2/C_e(F(\{x\})^2)$ and $q : A \times B \rightarrow (A \times B)/C_e(A \times B)$ be quotient maps. Note that p and q are the coequalizers of R_e considered as a relation on $F(\{x\})$ and $A \times B$ respectively. Therefore we can construct the commutative diagram

$$\begin{array}{ccc}
 F(\{x\})^2 & \xrightarrow{m_A \times m_B} & A \times B \\
 \downarrow p & & \downarrow q \\
 F(\{x\})^2/C_e(F(\{x\})^2) & \xrightarrow{n} & (A \times B)/C_e(A \times B)
 \end{array} \tag{63}$$

where n is the unique such homomorphism such that the diagram commutes. Then $q(a, b) = q(m_A \times m_B)(x, x) = np(x, x) = np(x, e) = q(m_A \times m_B)(x, e) = q(a, e)$, which implies $((a, b), (a, e))$ belongs to $C_e(A \times B)$ as intended. ■

This result allows us to give a complete characterization of left coextensivity in the next theorem. We proceed by unpacking the definitions of the transitive, homomorphic, symmetric, and reflexive closures one by one and write down exactly which terms are in the congruence generated by R . Doing so leaves us with a list of terms and identities equivalent to coextensivity, which is exactly what we mean by a syntactic characterization.

3.2.3. THEOREM. *A variety of universal algebras \mathcal{C} is left coextensive if and only if for some $k, n, m \in \mathbb{N}$, there exist $(m+n)$ -ary terms u_1, \dots, u_k , unary terms t_1, \dots, t_m , $t'_1, \dots, t'_m \in F(\{x\})$, and constants $e, e_1, \dots, e_n, e'_1, \dots, e'_n, e''_1, \dots, e''_n$ such that the following identities hold:*

$$\begin{aligned} u_1(t'_1, \dots, t'_m, e''_1, \dots, e''_n) &= x, \\ u_i(t_1, \dots, t_m, e_1, \dots, e_n) &= x \text{ for all } i \leq k, \\ u_i(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= u_{i+1}(t'_1, \dots, t'_m, e''_1, \dots, e''_n) \text{ for all } i < k, \\ u_k(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= e. \end{aligned}$$

PROOF. By Proposition 3.2.2, \mathcal{C} is left coextensive if and only if $((x, x), (x, e))$ belongs to the congruence $C = C_e(F(\{x\})^2)$. Recall that C is the congruence on $F(\{x\})^2$ generated by the relation

$$R = R_e(F(\{x\})^2) = \{((e_1, e_2), (e_1, e)) \in F(\{x\})^4 \mid e_1, e_2 \in F(\emptyset)\},$$

Recall also that by Proposition 2.2.6, we have $C = \text{Trans}(Q)$, where $Q = H(\text{Sym}(\text{Ref}(R)))$. It follows that $((x, x), (x, e)) \in C$ if and only if there exist $a_0, \dots, a_k \in F(\{x\})$ such that $a_0 = x$, $a_k = e$ and $((x, a_{i-1}), (x, a_i)) \in Q$ for $i = 1, \dots, k$. Observe that Q is the subalgebra of $F(\{x\})^4$ generated by all elements of the form $((t, t'), (t, t'))$ and $((e, e'), (e, e''))$ for all $t, t' \in F(\{x\})$ and $e, e', e'' \in F(\emptyset)$. Therefore $((x, a_{i-1}), (x, a_i)) \in Q$ if and only if there exist $t_1, \dots, t_m, t'_1, \dots, t'_m \in F(\{x\})$, constants $e_1, \dots, e_n, e'_1, \dots, e'_n, e''_1, \dots, e''_n \in F(\emptyset)$ and an $(m+n)$ -ary term u_i such that

$$\begin{aligned} (x, a_{i-1}) &= u_i((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e''_1), \dots, (e_n, e''_n)), \\ (x, a_i) &= u_i((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e'_1), \dots, (e_n, e'_n)). \end{aligned}$$

Note that not only u_i , but also the other terms and constants, as well as the number of such terms and constants in the above identities should depend on i . However, we can ignore this by adding irrelevant terms and constants to each of the u_i such that they are of the same arity, and satisfy the above equations for the same terms and constants. We are now dealing with some sequence

$$(x, x) = (x, a_0)Q(x, a_1)Q \dots Q(x, a_{k-1})Q(x, a_k) = (x, e),$$

which can be expressed as:

$$\begin{array}{r}
 (x, x) = (x, a_0) = u_1((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e''_1), \dots, (e_n, e''_n)) \\
 \qquad \qquad \qquad u_1((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e'_1), \dots, (e_n, e'_n)) \\
 (x, a_1) \begin{array}{l} \nearrow \\ \searrow \end{array} \\
 \qquad \qquad \qquad u_2((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e''_1), \dots, (e_n, e''_n)) \\
 \qquad \qquad \qquad u_2((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e'_1), \dots, (e_n, e'_n)) \\
 (x, a_2) \begin{array}{l} \nearrow \\ \searrow \end{array} \\
 \qquad \qquad \qquad u_3((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e''_1), \dots, (e_n, e''_n)) \\
 \qquad \qquad \qquad \dots \dots \dots \\
 \qquad \qquad \qquad u_{k-1}((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e'_1), \dots, (e_n, e'_n)) \\
 (x, a_{k-1}) \begin{array}{l} \nearrow \\ \searrow \end{array} \\
 \qquad \qquad \qquad u_k((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e''_1), \dots, (e_n, e''_n)) \\
 (x, e) = (x, a_k) = u_k((t_1, t'_1), \dots, (t_m, t'_m), (e_1, e'_1), \dots, (e_n, e'_n))
 \end{array} \tag{64}$$

It follows from the above equalities that $((x, x), (x, e)) \in C$ if and only if there exist terms $t_1, \dots, t_m, t'_1, \dots, t'_m \in F(\{x\})$, and constants $e, e_1, \dots, e_n, e'_1, \dots, e'_n, e''_1, \dots, e''_n$ such that the desired identities hold:

$$\begin{aligned}
 u_1(t'_1, \dots, t'_m, e''_1, \dots, e''_n) &= x, \\
 u_i(t_1, \dots, t_m, e_1, \dots, e_n) &= x \text{ for all } i \leq k, \\
 u_i(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= u_{i+1}(t'_1, \dots, t'_m, e''_1, \dots, e''_n) \text{ for all } i < k, \\
 u_k(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= e.
 \end{aligned}$$

■

3.2.4. EXAMPLE. For the equalities above, let us consider the case $k = 1$. This is an especially important case, since if \mathcal{C} is a Mal'tsev variety, then every homomorphic,

reflexive relation is a congruence, and so we immediately have $k = 1$. Writing $u_1 = u$, we have:

$$\begin{aligned} u(t_1, \dots, t_m, e_1, \dots, e_n) &= x, \\ u(t'_1, \dots, t'_m, e''_1, \dots, e''_n) &= x, \\ u(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= e. \end{aligned}$$

Next, note that it is sufficient for \mathcal{C} to satisfy only the first and third equations above, since the second equation will be satisfied by the choice of $e''_i = e_i$ for all $i \leq n$, which reduces our system of identities to

$$\begin{aligned} u(t_1, \dots, t_m, e_1, \dots, e_n) &= x, \\ u(t'_1, \dots, t'_m, e'_1, \dots, e'_n) &= e. \end{aligned}$$

These, in addition to the usual requirement of a term t such that $t(x, x, y) = y$ and $t(x, y, y) = x$, give a simple characterization of left coextensive Mal'tsev varieties. We have a further special case, where $m = n = 1$, $t_1 = t'_1 = x$, and $e'_1 = e$. We write $e = 0$ and $e_1 = 1$. Our system of identities then becomes

$$\begin{aligned} u(x, 1) &= x, \\ u(x, 0) &= 0, \end{aligned}$$

which is familiar from the example of rings. Note that such a variety \mathcal{C} which satisfies these equalities is not necessarily coextensive, i.e., $\times : X/\mathcal{C} \times Y/\mathcal{C} \rightarrow (X \times Y)/\mathcal{C}$ is fully faithful but not necessarily an equivalence. One example is the variety **CRing** of commutative rings, which is coextensive following the same argument as **CSemiRing**. **CRing** is also Mal'tsev, since the term $t(x, y, z) = x - y + z$ satisfies the required identities $t(x, x, y) = y$ and $t(x, y, y) = x$. Another example is the variety **Heyt** of Heyting algebras, which is Mal'tsev, since the term $t(x, y, z) = ((z \implies y) \implies x) \wedge ((x \implies y) \implies z)$ satisfies the required identities. **Heyt** is also left coextensive, since we have the identities $x \wedge 1 = x$ and $x \wedge 0 = 0$.

Next we will characterize the full condition of coextensivity. One might imagine that the full characterization will be the same as that of left coextensivity with some added conditions, but in fact much of the above complexity will be simplified by the existence

of what we call a diagonalising term.

3.3. DIAGONALISING TERMS.

3.3.1. DEFINITION. *We say that a diagonalising term is any term t of arity $k + 2$ for some $k \in \mathbb{N}$, equipped with two sequences of k constants $e_1, \dots, e_k, e'_1, \dots, e'_k$ such that the identities hold:*

$$\begin{aligned} t(x, y, e_1, \dots, e_k) &= x, \\ t(x, y, e'_1, \dots, e'_k) &= y. \end{aligned}$$

Note that if $k = 0$, then the variety satisfies the identity $x = t(x, y) = y$. In this case we say that the diagonalising term t is trivial.

3.3.2. PROPOSITION. *Let \mathcal{C} be a variety with a non-trivial diagonalising term $(t, e_1, \dots, e_k, e'_1, \dots, e'_k)$. Then \mathcal{C} is left coextensive.*

PROOF. We start by fixing any constant as e . This is always possible, since \mathcal{C} has a non-trivial diagonalising term, and so \mathcal{C} has at least one constant e_1 . Note that we have the equalities:

$$\begin{aligned} t(x, e, e_1, \dots, e_k) &= x, \\ t(x, e, e'_1, \dots, e'_k) &= e. \end{aligned}$$

It follows from Theorem 3.2.3 that \mathcal{C} is left coextensive, since the above equalities are exactly those required by the theorem when we choose $u_1 = t$ as the only operation in the sequence, as well as letting $e'_i = e_i$ for all $i \leq n$. ■

3.3.3. PROPOSITION. *Any coextensive variety contains a non-trivial diagonalising term.*

PROOF. Let $F(\{x, y\})$ be the free algebra generated on the set $\{x, y\}$. Since \mathcal{C} is coextensive, it is also codistributive by Proposition 2.3.19, and so there exists a canonical isomorphism

$$\eta : F(\{x, y\}) + F(\emptyset)^2 \longrightarrow (F(\{x, y\}) + F(\emptyset))^2 \longrightarrow F(\{x, y\})^2,$$

where the first isomorphism is given directly by codistributivity, and the second follows from the fact that $F(\emptyset)$ is initial. Note that $F(\{x, y\}) + F(\emptyset)^2$ is generated by the set $\{x, y\} \cup U(F(\emptyset)^2)$, where U is the underlying set functor. It follows that all elements of

$F(\{x, y\}) + F(\emptyset)^2$ are of the form $t(x, y, (e_1, e'_1), \dots, (e_k, e'_k))$ for some term t and constants $e_1, \dots, e_k, e'_1, \dots, e'_k$.

Furthermore, we have $\eta(t(x, y, (e_1, e'_1), \dots, (e_k, e'_k))) = (t(x, y, e_1, \dots, e_k), t(x, y, e'_1, \dots, e'_k))$. Since η is an isomorphism, there must be some element in $F(\{x, y\}) + F(\emptyset)^2$ whose image in $F(\{x, y\})^2$ is (x, y) . This immediately gives the desired equalities

$$t(x, y, e_1, \dots, e_k) = x \text{ and } t(x, y, e'_1, \dots, e'_k) = y,$$

Note that since \mathcal{C} is coextensive, \mathcal{C} has co-disjoint products and so \mathcal{C} has at least one constant term. We can therefore assume that t is non-trivial as desired. ■

3.3.4. EXAMPLE. The variety of semirings has a diagonalising term $t(a, b, c, d) = ac + bd$, with constants $(e_1, e_2) = (1, 0)$ and $(e'_1, e'_2) = (0, 1)$. Then it follows that

$$t(x, y, e_1, e_2) = x \cdot 1 + y \cdot 0 = x,$$

$$t(x, y, e'_1, e'_2) = x \cdot 0 + y \cdot 1 = y,$$

as desired. Note that while the variety of commutative semirings is coextensive, the variety of semirings is not, since in the daigram

$$\begin{array}{ccccc} \mathbb{N} & \xleftarrow{\pi_1} & \mathbb{N}^2 & \xrightarrow{\pi_2} & \mathbb{N} \\ \downarrow & & \downarrow & & \downarrow \\ F(\{x\}) & \xleftarrow{\quad} & F(\{x\}) + \mathbb{N}^2 & \xrightarrow{\quad} & F(\{x\}) \end{array} \quad (65)$$

the bottom row is not a product. Note that the natural numbers \mathbb{N} are initial in the variety of semirings, so both squares in the above diagram are pushouts. To see that the bottom row is not a product, note that $x \cdot (1, 0)$ and $(1, 0) \cdot x$ are distinct elements of $F(\{x\}) + \mathbb{N}^2$ which are both mapped to the same elements x and 0 on either side of the diagram. The variety of semirings is therefore an example of a non-coextensive variety having a diagonalising term.

The existence of a diagonalising term turns out to be a fairly powerful subcondition of coextensivity, and we explore it in greater detail in section 4. The most immediately useful consequence of the diagonalising term is the effect it has on congruences in the

variety.

3.3.5. PROPOSITION. *Let \mathcal{C} be a variety with a diagonalising term t . then \mathcal{C} satisfies the following conditions:*

- (a) *Any reflexive homomorphic relation R on some product $A \times B \in \mathcal{C}$ is of the form $R_A \times R_B$ for some reflexive homomorphic relations R_A on A and R_B on B .*
- (b) *\mathcal{C} satisfies the Fraser-Horn property [6], which states that any congruence E on some product $A \times B \in \mathcal{C}$ is of the form $E_A \times E_B$ for some congruences E_A on A and E_B on B .*
- (c) *\mathcal{C} satisfies the co-universality condition for any surjective homomorphism q .*

PROOF. (a) Let R be a reflexive homomorphic relation on $A \times B$. First note that if either A or B is the empty set, then so is $A \times B$, and immediately $R = (A \times B)^2 \simeq A^2 \times B^2$, which satisfies (a). We can therefore assume that neither A nor B is empty. Let $\pi_A : (A \times B)^2 \rightarrow A^2$ and $\pi_B : (A \times B)^2 \rightarrow B^2$ be product projections. Then the images of R under π_A and π_B , $R_A = \pi_A(R)$, and $R_B = \pi_B(R)$, are of the form:

$$R_A = \{(a, c) \in A^2 \mid ((a, b'), (c, d')) \in R \text{ for some } b', d' \in B\},$$

$$R_B = \{(b, d) \in B^2 \mid ((a', b), (c', d)) \in R \text{ for some } a', c' \in A\}.$$

It immediately follows that R_A and R_B are reflexive, since neither A nor B is empty. Next, consider the map $\phi : R \rightarrow R_A \times R_B$ such that $\phi((a, b), (c, d)) = ((a, c), (b, d))$. This map is clearly well defined and injective. To see that ϕ is surjective, note that whenever $((a, c), (b, d)) \in R_A \times R_B$, there exist some tuples $((a, b'), (c, d')) \in R$ and $((a', b), (c', d)) \in R$. Then, since R is a reflexive homomorphic relation, we have that

$$t(((a, b'), (c, d')), ((a', b), (c', d)), ((e_1, e'_1), (e_1, e'_1)), \dots, ((e_k, e'_k), (e_k, e'_k))) = ((a, b), (c, d)),$$

using the equalities

$$\begin{aligned} t(a, a', e_1, \dots, e_k) &= a, & t(b', b, e'_1, \dots, e'_k) &= b, \\ t(c, c', e_1, \dots, e_k) &= c, & t(d', d, e'_1, \dots, e'_k) &= d. \end{aligned}$$

Therefore ϕ is an isomorphism, and $R \simeq R_A \times R_B$ as desired. Condition (b) then easily follows from (a).

(c) Consider some surjective homomorphism $q : A \times B \rightarrow C$ for some algebras $A, B, C \in \mathcal{C}$. Note that $C \simeq (A \times B)/E$ where E is the congruence on $A \times B$ generated by q . We can now construct the diagram

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \\
 \downarrow & & \downarrow q & & \downarrow \\
 A/E_A & \xleftarrow{\lrcorner} & (A \times B)/E & \xrightarrow{\lrcorner} & B/E_B
 \end{array} \tag{66}$$

in which both squares are pushouts by Remark 2.2.13. Note that $E \simeq E_A \times E_B$ by the previous result. Furthermore, it is true in general that the map $\phi : (A/E_A) \times (B/E_B) \rightarrow (A \times B)/(E_A \times E_B)$ such that $\phi([a]_{E_A}, [b]_{E_B}) = [a, b]_{E_A \times E_B}$ is a well defined isomorphism. It follows that the bottom row of the above diagram is a product diagram, and so \mathcal{C} satisfies the co-universality condition for q as desired. ■

3.4. COEXTENSIVE VARIETIES. The previous result suggests that a variety with a diagonalising term is already fairly close to being coextensive. We will now consider what additional properties such a variety needs in order to become coextensive.

3.4.1. PROPOSITION. *A variety \mathcal{C} with a non-trivial diagonalising term t is coextensive if and only if $\eta_X : F(X) + F(\emptyset)^2 \rightarrow F(X)^2$ is an isomorphism for all sets X .*

PROOF. If \mathcal{C} is coextensive, then η_X is immediately an isomorphism for all sets X . On the other hand, note that if \mathcal{C} has a diagonalising term, then \mathcal{C} is immediately left coextensive, and it remains to show that \mathcal{C} is right coextensive. Given a morphism $i : F(\emptyset)^2 \rightarrow A$ for some algebra A , let $U(A)$ be the underlying set of A and $\varepsilon_A : FU(A) \rightarrow A$ the canonical morphism from the free algebra on $U(A)$ to A . Then i can be factorized as the composite of the morphisms:

$$F(\emptyset)^2 \xrightarrow{\subset} FU(A) + F(\emptyset)^2 \xrightarrow{[\varepsilon_A, i]} A \tag{67}$$

Furthermore, since ε_A is surjective, so is $[\varepsilon_A, i]$. We can now construct the diagram

$$\begin{array}{ccccc}
 F(\emptyset) & \xleftarrow{\pi_1} & F(\emptyset) \times F(\emptyset) & \xrightarrow{\pi_2} & F(\emptyset) \\
 \downarrow & & \downarrow & & \downarrow \\
 FU(A) & \xleftarrow{[id, \pi_1]} & FU(A) + F(\emptyset)^2 & \xrightarrow{[id, \pi_2]} & FU(A) \\
 \downarrow & & \downarrow [\varepsilon_A, i] & & \downarrow \\
 A_1 & \xleftarrow{\quad} & A & \xrightarrow{\quad} & A_2
 \end{array} \tag{68}$$

in which all four squares are pushouts. Note that if η_X is an isomorphism for all sets X , then $\eta_{U(A)}$ is an isomorphism and the middle row is a product diagram. Then since $[\varepsilon_A, i]$ is surjective, the bottom row is also a product diagram as by the Proposition 3.3.5. It follows that \mathcal{C} is coextensive as desired. ■

This result establishes the diagram

$$\begin{array}{ccccc}
 F(\emptyset) & \xleftarrow{\pi_1} & F(\emptyset) \times F(\emptyset) & \xrightarrow{\pi_2} & F(\emptyset) \\
 \downarrow & & \downarrow i & & \downarrow \\
 F(X) & \xleftarrow{[id, \pi_1]} & F(X) + F(\emptyset)^2 & \xrightarrow{[id, \pi_2]} & F(X) \\
 \swarrow p_1 & & \downarrow \eta_X & & \searrow p_2 \\
 & & F(X)^2 & &
 \end{array} \tag{69}$$

as the only diagram we need to consider. Recall that η_X is a component of the unit of the adjunction $L \dashv (\times)$, and is defined as

$$\begin{aligned}
 \eta_X(s(x_1, \dots, x_n, (w_1, e'_1), \dots, (w_m, w'_m))) &= \\
 (s(x_1, \dots, x_n, w_1, \dots, w_m), s(x_1, \dots, x_n, w'_1, \dots, w'_m)), &
 \end{aligned}$$

for any $(k+m)$ -ary term s , elements $x_1, \dots, x_n \in X$ and constants $w_1, \dots, w_m, w'_1, \dots, w'_m$. Furthermore, recall that whenever \mathcal{C} has a diagonalising term t equipped with constants $e_1, \dots, e_k, e'_1, \dots, e'_k$, we can define the map $\delta_X : F(X)^2 \longrightarrow F(X) + F(\emptyset)^2$ as

$\delta_X(x, y) = t(x, y, (e_1, e'_1), \dots, (e_k, e'_k))$, and so $\eta_X \delta_X((x, y)) = (x, y)$. This map is not necessarily a homomorphism in \mathcal{C} , but it turns out that \mathcal{C} is coextensive exactly when δ_X is a homomorphism satisfying the condition $\delta_X(x, x) = x$ for all $x \in X$.

3.4.2. PROPOSITION. *A variety \mathcal{C} is coextensive if and only if for any set X :*

- (a) \mathcal{C} has a non-trivial diagonalising term t ,
- (b) $\delta_X(x, x) = x$ for all $x \in X$,
- (c) $\delta_X : F(X)^2 \longrightarrow F(X) + F(\emptyset)^2$ is a homomorphism.

PROOF. It follows from Propositions 3.3.3 and 3.4.1 that a variety \mathcal{C} is coextensive if and only if \mathcal{C} has a non-trivial diagonalising term t and η_X is an isomorphism for any set X . Since δ_X is a right inverse of η_X considered as a map, η_X is an isomorphism if and only if δ_X is an isomorphism. In this case δ_X is immediately a homomorphism, and $\delta_X(x, x) = \delta_X \eta_X(x) = x$ as desired.

On the other hand, assume that δ_X is a homomorphism and that $\delta_X(x, x) = x$. We already have $\eta_X \delta_X = 1$, so it remains to show that $\delta_X \eta_X = 1$. Since δ_X is a homomorphism with $\delta_X \eta_X(x) = \delta_X(x, x) = x$ for all $x \in X$, it suffices to show that $\delta_X \eta_X(w, w') = (w, w')$ for any $(w, w') \in F(\emptyset)^2$. But note that

$$\begin{aligned} \delta_X \eta_X(w, w') &= t((w, w), (w', w'), (e_1, e'_1), \dots, (e_k, e'_k)) \\ &= (t(w, w', e_1, \dots, e_k), t(w, w', e'_1, \dots, e'_k)) \\ &= (w, w') \end{aligned}$$

and so $\delta_X \eta_X = 1$, and η_X is an isomorphism for any set X as desired. ■

When converting a categorical condition into a syntactic characterization it is often useful to first convert the condition into a statement about free algebras, since they obey only the identities that are satisfied by the entire variety. In our case, we have converted coextensivity into a statement about the algebra $F(X) + F(\emptyset)^2$. This is not quite a free algebra, so instead we consider it as a quotient of a free algebra.

3.4.3. PROPOSITION. *Let \mathcal{C} be a variety with a diagonalising term $(t, e_1, \dots, e_k, e'_1, \dots, e'_k)$. Let X be a set, and let $C = \{c_1, \dots, c_k\}$ be a set with k elements disjoint from X . Then $F(X) + F(\emptyset)^2 \simeq F(X \cup C)/E$, where E is the congruence generated by the relation $R = \{v(c_1, \dots, c_k) \simeq t(v(e_1, \dots, e_k), v(e'_1, \dots, e'_k), c_1, \dots, c_k) \text{ for all } k\text{-ary operations } v\}$.*

PROOF. We begin with the special case in which X is empty, and in this case denote E as E_C . We will show that $F(C)/E_C \simeq F(\emptyset)^2$. Let p be the unique morphism $p : F(C) \rightarrow F(\emptyset)^2$ such that $p(c_i) = (e_i, e'_i)$ for all $i \leq k$. For any $\omega, \omega' \in F(\emptyset)$ we have:

$$\begin{aligned} p(t(\omega, \omega', c_1, \dots, c_k)) &= t((\omega, \omega'), (\omega', \omega'), (e_1, e'_1), \dots, (e_k, e'_k)) \\ &= (\omega, \omega') \end{aligned}$$

Therefore p is surjective and so $F(\emptyset)^2 \simeq F(C)/E_p$, where E_p is the congruence generated by p . We will show that $E_p = E_C$. Note that

$$\begin{aligned} p(v(c_1, \dots, c_k)) &= (v(e_1, \dots, e_k), v(e'_1, \dots, e'_k)) \\ &= p(t(v(e_1, \dots, e_k), v(e'_1, \dots, e'_k), c_1, \dots, c_k)) \end{aligned}$$

so $R \subseteq E_p$. On the other hand, whenever $p(v(c_1, \dots, c_k)) = p(u(c_1, \dots, c_k))$ we have $v(e_1, \dots, e_k) = u(e_1, \dots, e_k)$ and $v(e'_1, \dots, e'_k) = u(e'_1, \dots, e'_k)$. Since these identities hold in $F(\emptyset)$, they must hold in $F(C)$. Therefore:

$$\begin{aligned} v(c_1, \dots, c_k) &\simeq_{E_C} t(v(e_1, \dots, e_k), v(e'_1, \dots, e'_k), c_1, \dots, c_k) \\ &= t(u(e_1, \dots, e_k), u(e'_1, \dots, e'_k), c_1, \dots, c_k) \\ &\simeq_{E_C} u(c_1, \dots, c_k) \end{aligned}$$

and so $E_C = E_p$ as desired. Next let us consider the general case for any set X . First note that from our above result $F(\emptyset)^2 \simeq F(C)/E_C$, and $F(X \cup C) \simeq F(X) + F(C)$ since left adjoints preserve coproducts. Therefore it remains to show that $F(X) + F(C)/E_C \simeq (F(X) + F(C))/E$. Note, however, that E is simply the congruence generated by E_C considered as a relation on $F(X) + F(C)$. The result then follows immediately, since coproducts commute with co-equalizers. Specifically, one can construct the diagram

$$\begin{array}{ccccc}
E_C & \rightrightarrows & F(C) & \longrightarrow & F(C)/E_C \\
\downarrow & & \downarrow & & \downarrow \\
E_C & \rightrightarrows & F(X) + F(C) & \longrightarrow & (F(X) + F(C))/E \\
\uparrow & & \uparrow & & \uparrow \\
F(\emptyset) & \rightrightarrows & F(X) & \longrightarrow & F(X)
\end{array} \tag{70}$$

in which all three rows are coequalizer diagrams, and all three columns are coproduct diagrams. ■

We can now give our characterization of coextensive varieties of universal algebras.

3.4.4. THEOREM. *Let \mathcal{C} be a variety with a diagonalising term $(t, e_1, \dots, e_k, e'_1, \dots, e'_k)$, and let $c = (c_1, \dots, c_k)$ be a sequence of k variables. Given two terms α and β , we write $\alpha \simeq_c \beta$ if for some $l, m, n \in \mathbb{N}$, \mathcal{C} admits $(l + 2m)$ -ary terms u_1, \dots, u_n and k -ary terms $v_1, \dots, v_m, w_1, \dots, w_m$ with*

$$\begin{aligned}
& u_1(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)) = \alpha, \\
& \forall_{i < n} u_i(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) = \\
& u_{i+1}(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)), \\
& u_n(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) = \beta,
\end{aligned}$$

where $w_j(c) = t(v_j(e_1, \dots, e_k), v_j(e'_1, \dots, e'_k), c_1, \dots, c_k)$ for all $j \leq m$.

Such a non-trivial variety \mathcal{C} of Ω -algebras is coextensive if and only if it satisfies the following conditions:

- (a) $t(x, x, c_1, \dots, c_k) \simeq_c x$ for any variable x and sequence $c = (c_1, \dots, c_k)$ of k variables,
- (b) $s(t(x_1, y_1, c_1, \dots, c_k), \dots, t(x_p, y_p, c_1, \dots, c_k)) \simeq_c t(s(x_1, \dots, x_p), s(y_1, \dots, y_p), c_1, \dots, c_k)$ for any $p \in \mathbb{N}$, any p -ary operation $s \in \Omega$ and variables $x_1, \dots, x_p, y_1, \dots, y_p, c_1, \dots, c_k$.

PROOF. It follows from Proposition 3.4.2 that \mathcal{C} is coextensive if and only if \mathcal{C} has a diagonalising term $(t, e_1, \dots, e_k, e'_1, \dots, e'_k)$, and the map $\delta_X : F(X)^2 \rightarrow F(X) + F(\emptyset)^2$ such that $\delta_X(x, y) = t(x, y, (e_1, e'_1), \dots, (e_k, e'_k))$ is a homomorphism satisfying the condition

$\delta_X(x, x) = x$ for all $x \in X$ for any set X . Let $C = \{c_1, \dots, c_k\}$, and let E be the congruence generated on $F(X \cup C)$ by the relation

$$R = \{v(c_1, \dots, c_k) \simeq t(v(e_1, \dots, e_k), v(e'_1, \dots, e'_k), c_1, \dots, c_k) \text{ for all } k\text{-ary operations } v\}.$$

It follows by Proposition 3.4.3, that $\delta_X(x, x) = x$ if and only if $t(x, x, c_1, \dots, c_k) \simeq_E x$, and similarly δ_X is a homomorphism if and only if:

$$\begin{aligned} \delta_X(s(x_1, \dots, x_p), s(y_1, \dots, y_p)) = s(\delta_X((x_1, y_1), \dots, \delta_X(x_p, y_p))) &\iff \\ t(s(x_1, \dots, x_p), s(y_1, \dots, y_p), c_1, \dots, c_k) \simeq_E s(t(x_1, y_1, c_1, \dots, c_k), \dots, t(x_p, y_p, c_1, \dots, c_k)). \end{aligned}$$

for all $p \in \mathbb{N}$, p -ary operations s and all $x_1, \dots, x_p, y_1, \dots, y_p \in X$. It remains to show that given $\alpha, \beta \in F(X \cup C)$, we have $\alpha \simeq_E \beta$ if and only if $\alpha \simeq_c \beta$. E is the congruence generated by R . Therefore $(\alpha, \beta) \in E$ if and only if $(\alpha, \beta) \in Q^n$ for some $n \in \mathbb{N}$, where Q is the homomorphic, symmetric, reflexive closure of R . But $(\alpha, \beta) \in Q^n$ if and only if there exists a sequence $a_0, \dots, a_n \in F(X \cup C)$ such that $a_0 = \alpha$, $a_n = \beta$, and $(a_{i-1}, a_i) \in Q$ for all $i = 1, \dots, n$.

Next, let us consider exactly which pairs belong to the relation Q . We can write $Q = H(\text{Sym}(\text{Ref}(R))) = H(R \cup R^{-1} \cup \{(x, x) \mid x \in X \cup C\})$. It follows that $(a_{i-1}, a_i) \in Q$ if and only if for some $l, m \in \mathbb{N}$ there exist $x_1, \dots, x_l \in X \cup C$, k -ary terms v_1, \dots, v_m , and a $(l + 2m)$ -ary operation u_i such that

$$\begin{aligned} a_{i-1} &= u_i(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)), \\ a_i &= u_i(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)), \end{aligned}$$

where $w_j(c) = t(v_j(e_1, \dots, e_k), v_j(e'_1, \dots, e'_k), c_1, \dots, c_k)$ for each $j \leq m$. Note that not only u_i , but also the other terms and the number of such terms in the above identities should depend on i . However, we can ignore this by adding irrelevant terms and constants to each of the u_i such that they are of the same arity, and satisfy the above equations for the same terms and variables. We are now dealing with some sequence

$$\alpha = a_0 \simeq_Q a_1 \simeq_Q \cdots \simeq_Q a_{n-1} \simeq_Q a_n = \beta,$$

which can be expressed as:

$$\begin{array}{r}
\alpha = a_0 = u_1(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)) \\
\quad u_1(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) \\
a_1 \begin{array}{l} \nearrow \\ \searrow \end{array} \\
\quad u_2(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)) \\
\quad u_2(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) \\
a_2 \begin{array}{l} \nearrow \\ \searrow \end{array} \\
\quad u_3(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)) \\
\quad \dots \dots \dots \\
\quad u_{n-1}(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) \\
a_{n-1} \begin{array}{l} \nearrow \\ \searrow \end{array} \\
\quad u_n(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)) \\
\beta = a_n = u_n(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c))
\end{array} \tag{71}$$

It follows from the above equalities that $\alpha \simeq_E \beta$ if and only if \mathcal{C} satisfies the identities

$$\begin{aligned}
& u_1(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)) = \alpha, \\
& \forall_{i < n} u_i(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) = \\
& \quad u_{i+1}(x_1, \dots, x_l, w_1(c), \dots, w_m(c), v_1(c), \dots, v_m(c)), \\
& \quad u_n(x_1, \dots, x_l, v_1(c), \dots, v_m(c), w_1(c), \dots, w_m(c)) = \beta,
\end{aligned}$$

which is exactly to say that $\alpha \simeq_c \beta$, as desired. ■

This concludes the characterization of coextensive varieties of universal algebras, which is our main result. Since we also already know the characterizations of left-coextensive varieties, as well as varieties with co-disjoint products, it is natural to wonder about right coextensive varieties that aren't coextensive. In fact there is only one such variety up to

equivalence.

3.4.5. PROPOSITION. *A variety is right coextensive without being coextensive if and only if it has no constant terms and satisfies the identity $x = y$. Equivalently, the variety contains only the empty algebra and singleton algebras.*

PROOF. Let \mathcal{C} be right coextensive without being coextensive, this is to say that \mathcal{C} has co-universal products but not co-disjoint products. Therefore \mathcal{C} has no constant terms and so the initial object 0 is the empty algebra. Then for any $X \in \mathcal{C}$ we construct the diagram:

$$\begin{array}{ccccc}
 0 & \xleftarrow{id_0} & 0 & \xrightarrow{id_0} & 0 \\
 \downarrow !_x & & \downarrow !_x & & \downarrow !_x \\
 X & \xleftarrow{id_X} & X & \xrightarrow{id_X} & X
 \end{array} \tag{72}$$

In which the top row is a product diagram and both squares are pushouts, so the bottom row is also a product diagram. Then, for any $x, y \in X$, there must exist some $z \in X$ such that $x = id_X(z) = y$, and so $x = y$ for all $x, y \in X$ and X is either empty or a singleton as desired. Note that up to isomorphism there are only two objects $0, 1 \in \mathcal{C}$ and so it is easy to check in each case that right coextensivity holds. ■

3.5. COEXTENSIVITY VIA CENTRAL ELEMENTS. Recently there has been renewed interest in coextensive varieties, and there are several other characterizations of coextensive varieties, although none are syntactic in nature. In the recent paper, coextensive varieties via central elements [5], the author takes an approach to coextensivity which places emphasis on what are called central elements. This approach has several things in common with our syntactic approach, and so it is useful to give a comparison between these two characterizations.

A variety with $\vec{0}$ and $\vec{1}$ is defined to be a variety \mathcal{C} with n -tuples $\vec{0} = (0_1, \dots, 0_n)$ and $\vec{1} = (1_1, \dots, 1_n)$ of constants such that if an algebra $A \in \mathcal{C}$ satisfies $\vec{0} = \vec{1}$, then A is terminal, or equivalently $x = y$ for all $x, y \in A$ [5]. This is a generalization of the 0 and 1 elements of semirings and (bounded) lattices, which are examples of varieties with $\vec{0}$ and $\vec{1}$.

A central element of an algebra A is then defined to be any element $\vec{e} = (e_1, \dots, e_n) \in A^n$ such that there exists some isomorphism $\tau : A \rightarrow A_1 \times A_2$ such that $\tau(e_i) = (0_i, 1_i)$ for all $i \leq n$ [5]. Furthermore, a pair of central elements \vec{e}, \vec{f} are called complementary if $\tau(e_i) = (0_i, 1_i)$ and $\tau(f_i) = (1_i, 0_i)$ for all $i \leq n$ [5]. In this case we write $\vec{e} \diamond_A \vec{f}$. The set of central elements in A is denoted as $Z(A)$.

The relevance of such central elements to coextensivity is clear from the standard proof that **CSemiRing** is coextensive. In this variety, for any morphism $h : X \times Y \rightarrow Z$, we have that $e_1 = h(0, 1)$ and $e_2 = h(1, 0)$ are complementary central elements. Furthermore, Z can be uniquely decomposed into a product in terms of these elements as $Z \simeq e_1 Z \times e_2 Z$. It is also the case that any coextensive variety has $\vec{0}$ and $\vec{1}$ as follows from the following proposition.

3.5.1. PROPOSITION. *A variety \mathcal{C} with at least one constant has $\vec{0}$ and $\vec{1}$ if and only if \mathcal{C} has strict terminal objects.*

PROOF. Assume \mathcal{C} is a variety with $\vec{0}$ and $\vec{1}$, and that there exists a morphism $f : 1 \rightarrow A$. Then for all $1 \leq i \leq n$, we have $0_i = f(0_i) = f(1_i) = 1_i$ in A , and so by assumption A is terminal.

On the other hand, assume \mathcal{C} has strict terminal objects, and let E be the congruence on $F(\{x, y\})$ generated by $F(\emptyset)^2$ considered as a relation. Since $F(\emptyset)$ is non-empty, $1 \simeq F(\emptyset)/(F(\emptyset)^2)$, and there exists a unique morphism $f : 1 \rightarrow F(\{x, y\})/E$. It follows that $F(\{x, y\})/E \simeq 1$ and therefore $E = F(\{x, y\})^2$.

In particular we have $(x, y) \in E$, which implies the existence of some finite set of pairs of constants $S = \{(0_1, 1_1), \dots, (0_n, 1_n)\} \subseteq R$ such that $x \simeq y$ in the congruence generated by S . It immediately follows that \mathcal{C} is a variety with $\vec{0}$ and $\vec{1}$. ■

Furthermore, any variety with a diagonalising term $(t, e_1, \dots, e_k, e'_1, \dots, e'_k)$ is clearly a variety with $\vec{0}$ and $\vec{1}$, since we can set $\vec{0} = e_1, \dots, e_k$ and $\vec{1} = e'_1, \dots, e'_k$ such that if $0_i = 1_i$ for all $i \leq k$, then

$$x = t(x, y, e_1, \dots, e_k) = t(x, y, e'_1, \dots, e'_k) = y.$$

In fact, what we call diagonalising terms appear in [5] under another name. Recall that a variety \mathcal{V} with $\vec{0}$ and $\vec{1}$ is a Pierce variety if there exists a term $U(x, y, z_1, \dots, z_k, w_1, \dots, w_k)$

such that the following identities hold in \mathcal{V} .

$$\begin{aligned} t(x, y, 0_1, \dots, 0_k, 1_1, \dots, 1_k) &= x, \\ t(x, y, 1_1, \dots, 1_k, 0_1, \dots, 0_k) &= y. \end{aligned}$$

Such a term is called a decomposition term. A distinction is made between decomposition terms and short decomposition terms, which are of the form $u(x, y, z_1, \dots, z_k)$ and satisfy the identities

$$\begin{aligned} t(x, y, 0_1, \dots, 0_k) &= x, \\ t(x, y, 1_1, \dots, 1_k) &= y. \end{aligned}$$

Varieties with such a term are called Church varieties. Notice that such a short decomposition term is exactly what we call a diagonalising term. Pierce and Church varieties are also in some sense identical to each other. It is clear that Church varieties are always Pierce varieties, while on the other hand a Pierce variety can always be seen as a Church variety by considering $\vec{0}' = (0_1, \dots, 0_n, 1_1, \dots, 1_n)$, and $\vec{1}' = (1_1, \dots, 1_n, 0_1, \dots, 0_n)$. The distinction between these varieties only matters when we first require \mathcal{C} to be a variety with $\vec{0}$ and $\vec{1}$ and go on to hold $\vec{0}$ and $\vec{1}$ constant when requiring \mathcal{V} to be a Pierce variety.

Since Pierce varieties can be seen as varieties with a diagonalising term, it immediately follows from Proposition 3.3.3 (or Lemma 4.3 of [5]) that they satisfy the Fraser-Horn property, which states that any congruence E on some product $A \times B \in \mathcal{C}$ is of the form $E_A \times E_B$ for some congruences E_A on A and E_B on B . This property plays an important role in both approaches to coextensivity. In particular it is already strong enough that any variety which satisfies the Fraser-Horn property satisfies the co-universality condition for surjective homomorphisms.

Up until this point, the approach to coextensivity using central elements has much in common with the syntactic approach. The first point of divergence is the use of Boolean factor congruences and their relationship with central elements. By factor congruences we mean those congruences which are the kernel pair of some product projection. The set of factor congruences on an algebra A is denoted as $FC(A)$, and it is said that a variety \mathcal{C} has Boolean factor congruences if the $FC(A)$ forms a distributive sublattice of the usual lattice of congruences on A for all $A \in \mathcal{C}$.

It was shown in [8] that any Pierce variety has Boolean factor congruences, while the connection between Boolean factor congruences and central elements comes from Theorem 1.1 of [13], which establishes a bijection between $FC(A)$ and $Z(A)$ in varieties with Boolean factor congruences. We will show in 4.2.9 that the result that any Pierce variety has Boolean factor congruences can be generalized to exact categories with upward closed subproducts.

If \mathcal{C} is a variety with Boolean factor congruences, then it is said that a homomorphism $f : A \rightarrow B$ in \mathcal{C} preserves central elements if $\vec{e} \in Z(A)$ implies $f(\vec{e}) = (f(e_1), \dots, f(e_n)) \in Z(B)$. Furthermore, f is said to preserve complementary central elements if, whenever $\vec{e}_1, \vec{e}_2 \in Z(A)$ are complementary, then so are $f(\vec{e}_1)$ and $f(\vec{e}_2)$. A variety \mathcal{C} is stable by complements if every homomorphism preserves complementary central elements [5].

The main result of [5] is then to show that a variety is coextensive if and only if it is a Pierce variety satisfying one of several concrete conditions as follows:

3.5.2. THEOREM. *Let \mathcal{V} be a variety. Then the following are equivalent:*

- (a) \mathcal{V} is coextensive.
- (b) \mathcal{V} is a Pierce variety in which the relation $\vec{e} \diamond_A \vec{f}$ is equationally definable.
- (c) \mathcal{V} is a Pierce variety in which the class of directly indecomposable members of \mathcal{V} is a universal class.
- (d) \mathcal{V} is a Pierce variety stable by complements.
- (e) \mathcal{V} is a variety with the Fraser-Horn property such that the stalks of the Pierce sheaf of every element of \mathcal{V} are directly indecomposable.

For a full description of each of these conditions, see the original paper. For our purposes it is sufficient to note that each of these conditions is distinct from the characterization of Theorem 3.4.4, which is syntactic in the sense that it describes coextensive varieties in terms of identities that must be satisfied by the algebraic theory of \mathcal{C} . This difference is perhaps due to the fact that the central elements are not directly part of the algebraic theory of \mathcal{C} , which consists of the operations of Ω and the identities satisfied by \mathcal{C} . The central elements of an algebra A are instead determined by the product projections of A , so they naturally lead to characterizations in terms of concrete conditions satisfied by the variety rather than syntactic conditions satisfied by its algebraic theory.

4. Diagonalising terms and upward closed subproducts

Varieties with diagonalising terms play an important role in the story of coextensive varieties, and are sufficient to imply various useful subconditions of coextensivity. Seeing this, it is natural to ask whether there is a suitable categorical condition, equivalent in a variety to the existence of such a term, which is sufficient to imply the same subconditions in a purely categorical context. In this section, we will find such a condition in the context of exact categories. We begin by taking a closer look at varieties with diagonalising terms. Recall that we say a variety \mathcal{C} has a diagonalising term t equipped with some constants $e_1, \dots, e_k, e'_1, \dots, e'_k \in F(\emptyset)$ when the two identities hold:

$$\begin{aligned} t(x, y, e_1, \dots, e_k) &= x, \\ t(x, y, e'_1, \dots, e'_k) &= y. \end{aligned}$$

Recall that the motivating example of a coextensive variety is the variety **CSemiRing** of commutative semirings. In a similar fashion, the best examples of varieties with diagonalising terms are the variety **SemiRing** of semirings and the variety **Lat** of (bounded) lattices.

4.0.1. **RECALL.** *A (bounded) lattice L can be defined as a set L equipped with two constants 0 and 1 , and two commutative, associative binary operations \vee, \wedge , satisfying the four identities:*

$$\begin{aligned} a \vee (a \wedge b) &= a, \\ a \wedge (a \vee b) &= a, \\ a \vee 0 &= a, \\ a \wedge 1 &= a. \end{aligned}$$

From these identities it follows that \vee and \wedge are idempotent and that they satisfy the identities:

$$\begin{aligned} a \vee 1 &= 1, \\ a \wedge 0 &= 0. \end{aligned}$$

Then it is simple to show the following proposition.

4.0.2. **PROPOSITION.** *The variety **Lat** of (bounded) lattices has a diagonalising term, $t(a, b, c, d) = (a \wedge c) \vee (b \wedge d)$, with constants $e_1, e_2 = (1, 0)$, and $e'_1, e'_2 = (0, 1)$.*

PROOF. For any (bounded) lattice $L \in \mathbf{Lat}$ and any $x, y \in L$ we have

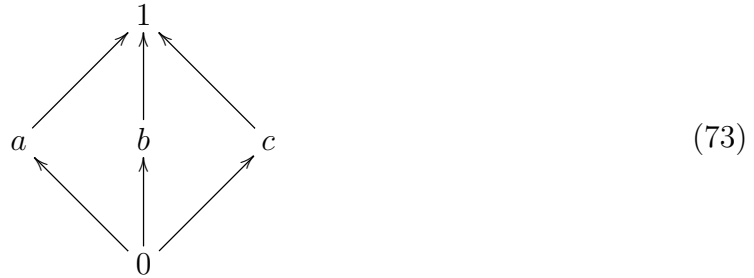
$$\begin{aligned} t(x, y, e_1, e_2) &= (x \wedge 1) \vee (y \wedge 0) = x \vee 0 = x, \\ t(x, y, e'_1, e'_2) &= (x \wedge 0) \vee (y \wedge 1) = 0 \vee y = y. \end{aligned}$$

■

While **Lat** has a non-trivial diagonalising term, it is not coextensive. Note however, that the category **DLat** of distributive (bounded) lattices is a subvariety of **CSemiRing** closed under products, so **DLat** is coextensive. Our other example, **SemiRing**, also has a diagonalising term without being coextensive, but it is closer to being coextensive in the sense that it satisfies condition (b) of Proposition 3.4.2, that $\delta_X(x, x) = x$. Indeed, we have

$$\delta_X(x, x) = t(x, x, (1, 0), (0, 1)) = x(1, 0) + x(0, 1) = x((1, 0) + (0, 1)) = x(1, 1) = x,$$

where the last equality follows from the fact that $(1, 1)$ plays the role of 1 in $F(\emptyset)^2$ and thus also in $F(X) + F(\emptyset)^2$. For (bounded) lattices we can consider the example of $L =$



and for any non-empty X consider the unique homomorphism $f : F(X) + F(\emptyset)^2 \rightarrow L$ such that $f(x) = a$ for all $x \in X$, $f(1, 0) = b$, and $f(0, 1) = c$. Then, for any $x \in X$ we have

$$f\delta_X(x, x) = f((x \wedge (1, 0)) \vee (x \wedge (0, 1))) = (a \wedge b) \vee (a \wedge c) = 0 \neq f(x),$$

and so $\delta_X(x, x) \neq x$. This demonstrates that while **CSemiRing** is coextensive,

(a) if we remove either distributivity or commutativity of multiplication, then we still

have a diagonalising term but not coextensivity;

- (b) if we remove only commutativity of multiplication, we still have the condition $\delta_X(x, x) = x$, while if we remove only distributivity then we don't, even after requiring all the (bounded) lattice axioms.

Still, any variety with a diagonalising term is in some sense fairly close to being coextensive. In fact, having a non-trivial diagonalising term is sufficient to imply the following properties:

4.0.3. PROPOSITION. *Let \mathcal{C} be a variety with a non-trivial diagonalising term. Then the following conditions hold:*

- (a) \mathcal{C} has co-disjoint products.
- (b) \mathcal{C} is left coextensive.
- (c) Any homomorphic relation $R \leq A \times B$ which contains a subproduct of $A \times B$ is itself a subproduct of $A \times B$ for any $A, B \in \mathcal{C}$.
- (d) For any $A \in \mathcal{C}$, if R is a reflexive homomorphic relation on A which contains a subproduct of A^2 , then $R \simeq A^2$.
- (e) For any $A \in \mathcal{C}$, if R is a congruence on A which contains a subproduct of A^2 , then $R \simeq A^2$.
- (f) \mathcal{C} has strict terminal objects.
- (g) Any reflexive homomorphic relation R on some product $A \times B \in \mathcal{C}$ is of the form $R_A \times R_B$ for some reflexive homomorphic relations R_A on A and R_B on B .
- (h) Any congruence E on some product $A \times B \in \mathcal{C}$ is of the form $E_A \times E_B$ for some congruences E_A on A and E_B on B .
- (i) \mathcal{C} satisfies the co-universality condition for any surjective homomorphism q .

PROOF. Condition (a) holds since t is non-trivial and so \mathcal{C} has at least one constant term, which implies $F(\emptyset)$ is not empty. Condition (b) is simply Proposition 3.3.2. To prove condition (c), let $R \leq A \times B$ contain some subproduct of $A \times B$. Then for any $(a_1, b_1), (a_2, b_2) \in R$ we have $(a_1, b_2) = t((a_1, b_1), (a_2, b_2), (e_1, e'_1), \dots, (e_k, e'_k)) \in R$, which is sufficient to make R a subproduct of $A \times B$.

Conditions (d) and (e) follow from (c), since A^2 is the only reflexive subproduct of A^2 for any $A \in \mathcal{C}$, and all congruences are reflexive homomorphic relations. For condition (f), note that the terminal object in any variety is an algebra with a single element. If a morphism $f : 1 \rightarrow A$ exists, then all constants are equal in A , and so for any $x, y \in A$ we have $x = t(x, y, e_1, \dots, e_k) = t(x, y, e'_1, \dots, e'_k) = y$. Therefore terminal objects are strict. Lastly, conditions (g), (h), and (i) are simply Proposition 3.3.5. ■

Note that many of the conditions in the above proposition refer to relations and congruences, which are algebraic notions. In order to find a categorical generalization of this proposition it is therefore necessary to define relations and congruences categorically. For this reason we choose to work in the context of exact categories, in which such relations are well behaved.

4.1. UPWARD CLOSED SUBPRODUCTS. We now turn our attention to defining a suitable categorical condition equivalent to the existence of a diagonalising term. Consider the following definitions.

4.1.1. DEFINITION. *Let (R, d_1, d_2) be a relation between some objects A and B in a category \mathcal{C} . We say that R contains a subproduct of $A \times B$ if the diagram*

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow m & & \downarrow g \\
 A & \xleftarrow{d_1} & R & \xrightarrow{d_2} & B
 \end{array} \tag{74}$$

*commutes for some objects X, Y and monomorphisms f, g, m in \mathcal{C} . Furthermore, if m is an isomorphism then we say R is itself a subproduct of $A \times B$. We say that a category has **upward closed subproducts** if, whenever any relation (R, d_1, d_2) on some objects $A, B \in \mathcal{C}$ contains a subproduct of $A \times B$, then R is a subproduct of $A \times B$.*

4.1.2. PROPOSITION. *A variety \mathcal{C} has a diagonalising term if and only if \mathcal{C} has upward closed subproducts.*

PROOF. Since \mathcal{C} is a variety, an internal relation $R \leq A \times B$ is a subalgebra of $A \times B$. The smallest possible subproduct of $A \times B$ is the set C of all pairs of constants in $A \times B$, so R contains a subproduct of if and only if it contains C . Note that R is a product of the form $R_A \times R_B$ if and only if for any $(a_1, b_1), (a_2, b_2) \in R$ we have $(a_1, b_2) \in R$. This

is true for all such R if and only if it is true when $A = F(\{a_1, a_2\})$, $B = F(\{b_1, b_2\})$, and R is the subalgebra generated by C . In this case it is clear that the elements of R are all of the form $t((a_1, b_1), (a_2, b_2), (e_1, e'_1), \dots, (e_k, e'_k))$ for some term t and sequences of constants $e_1, \dots, e_k, e'_1, \dots, e'_k$. Therefore $(a_1, b_2) \in R$ if and only if \mathcal{C} has a diagonalising term. ■

Furthermore, any coextensive category \mathcal{C} satisfies this property, even when \mathcal{C} is not a variety.

4.1.3. PROPOSITION. *Let \mathcal{C} be a coextensive category. Then \mathcal{C} has upward closed subproducts.*

PROOF. Let $d : R \longrightarrow A \times B$ be a relation containing some subproduct $(X \times Y, f \times g)$. Then there exists $m : X \times Y \longrightarrow R$ such that $dm = f \times g$, and since \mathcal{C} is coextensive, $R \simeq R_A \times R_B$ where R_A and R_B are the pushouts of R as in the commutative diagram:

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow m_A & & \downarrow m & & \downarrow m_B \\
 R_A & \xleftarrow{\quad} & R & \xrightarrow{\quad} & R_B \\
 \downarrow d_A & & \downarrow d & & \downarrow d_B \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array} \tag{75}$$

where d_A, d_B are the unique morphisms such that $d_A m_A = f$, $d_B m_B = g$ and the diagram commutes. Since the middle row is also a product diagram, we have $m = m_A \times m_B$ and $d = d_A \times d_B$ as desired. d_A, d_B are themselves monomorphisms by the following proposition. ■

4.1.4. PROPOSITION. *Let \mathcal{C} be a category in which all product projections are epimorphisms. Then for any morphisms $f, g \in \mathcal{C}$, $f \times g$ is a monomorphism if and only if both f and g are monomorphisms.*

PROOF. It is clear that in any category if $f : X \longrightarrow A$ and $g : Y \longrightarrow B$ are monomorphisms, then so is $f \times g : X \times Y \longrightarrow A \times B$. On the other hand, if $f \times g$ is a monomorphism, then

for any $h, h' : Z \rightarrow X$ such that $fh = fh'$ we can construct the diagram:

$$\begin{array}{ccccc}
 Z & \xleftarrow{\pi_Z} & Z \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow h & & \downarrow h \times 1_Y & & \downarrow 1_Y \\
 \downarrow h' & & \downarrow h' \times 1_Y & & \downarrow 1_Y \\
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow f & & \downarrow f \times g & & \downarrow g \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array} \tag{76}$$

In which $(f \times g) \circ (h \times 1_Y) = fh \times g = fh' \times g = (f \times g) \circ (h' \times 1_Y)$, so the two parallel morphisms $h \times 1_Y$ and $h' \times 1_Y$ are equal, since $f \times g$ is a monomorphism. Therefore $h\pi_Z = \pi_X \circ (h \times 1_Y) = \pi_X \circ (h' \times 1_Y) = h'\pi_Z$ and since π_Z is an epimorphism we have $h = h'$. Therefore f is a monomorphism and likewise so is g by repeating the same argument. ■

We have shown that the existence of upward closed subproducts is a subcondition of coextensivity, equivalent to the existence of a diagonalising term when \mathcal{C} is a variety. Our next goal is to show that the conditions of Proposition 4.0.3 hold not only when \mathcal{C} is a variety, but in the more general context of exact categories. We start with the following lemmas.

4.1.5. LEMMA. *Let $\pi_Y : X \times Y \rightarrow Y$ be a product projection. Then the diagram*

$$X^2 \times Y \begin{array}{c} \xrightarrow{\pi_1 \times 1_Y} \\ \rightrightarrows \\ \xrightarrow{\pi_2 \times 1_Y} \end{array} X \times Y \xrightarrow{\pi_Y} Y \tag{77}$$

is a kernel pair diagram.

PROOF. for any $f, g : A \rightarrow X \times Y$ such that $\pi_Y f = \pi_Y g$, there exists a unique morphism $h = (\pi_X f, \pi_X g, \pi_Y f) : A \rightarrow X^2 \times Y$ from the universal property of the product such that $(\pi_1 \times 1_Y) \circ h = \langle \pi_X f, \pi_Y f \rangle = f$ and $(\pi_2 \times 1_Y) \circ h = \langle \pi_X g, \pi_Y g \rangle = g$. ■

4.1.6. LEMMA. Consider the diagram

$$\begin{array}{ccccc}
 Eq(f) & \xrightarrow[p_2]{p_1} & A & \xrightarrow{f} & B \\
 \downarrow h & & \downarrow g & & \downarrow g' \\
 Eq(f') & \xrightarrow[q_2]{q_1} & C & \xrightarrow{f'} & D
 \end{array} \tag{78}$$

in which both rows are kernel pair diagrams. The following conditions hold:

- (a) If f is a regular epimorphism, then the right square is a pushout if and only if f' is the coequalizer of q_1h and q_2h .
- (b) If g is a monomorphism, then so is h .

PROOF. (a) let f be a regular epimorphism. It follows that f is the coequalizer of p_1 and p_2 . Now let f' be the coequalizer of q_1h and q_2h , and consider the diagram

$$\begin{array}{ccccc}
 Eq(f) & \xrightarrow[p_2]{p_1} & A & \xrightarrow{f} & B \\
 \downarrow h & & \downarrow g & & \downarrow g' \\
 Eq(f') & \xrightarrow[q_2]{q_1} & C & \xrightarrow{f'} & D \\
 & & \searrow k & & \downarrow l \\
 & & & & X
 \end{array} \tag{79}$$

ϕ (dotted arrow from D to X)

for any morphisms $k : C \rightarrow X$ and $l : B \rightarrow X$ such that $kg = lf$. It follows that $kq_1h = kgp_1 = lfp_1 = lfp_2 = kgp_2 = kq_2h$, so there exists a unique morphism $\phi : D \rightarrow X$ such that $\phi f' = k$. Note also that $\phi g' f = \phi f' g = kg = lf$, and f is an epimorphism, so $\phi g' = l$, and the right square is a pushout as desired.

On the other hand, let the right square be a pushout, and let $k : C \rightarrow X$ be some morphism such that $kq_1h = kq_2h$. It follows that $kgp_1 = kq_1h = kq_2h = kgp_2$, and since f is the coequalizer of p_1 and p_2 there exists a unique morphism $l : B \rightarrow X$ such that $kg = lf$. Then, since the right square is a pushout, there exists a unique morphism $\phi : D \rightarrow X$ such that $\phi f' = k$ and $\phi g' = l$. Lastly note that for any other morphism $\psi : D \rightarrow X$ such that $\psi f' = k$, we again have $\psi g' = l$ and thus $\psi = \phi$ since f' is an

epimorphism.

(b) Let g be a monomorphism, and let $a, b : E \rightarrow Eq(f)$ be any pair of morphisms such that $ha = hb$. It can be seen in the diagram

$$\begin{array}{ccccc}
 E & \xrightarrow[a]{b} & Eq(f) & \xrightarrow[p_2]{p_1} & A \\
 & & \downarrow h & & \downarrow g \\
 & & Eq(f') & \xrightarrow[q_2]{q_1} & C
 \end{array} \tag{80}$$

that $gp_1a = q_1ha = q_1hb = gp_1b$ and $gp_2a = q_2ha = q_2hb = gp_2b$. Since g is a monomorphism we have $p_1a = p_1b$ and $p_2a = p_2b$, which implies $a = b$ since p_1 and p_2 are a kernel pair and are therefore jointly monomorphic. It follows that h is a monomorphism as desired. ■

4.1.7. LEMMA. *Let \mathcal{C} be a regular category. Then \mathcal{C} has co-disjoint products if and only if product projections in \mathcal{C} are regular epimorphisms.*

PROOF. First let us assume that \mathcal{C} has co-disjoint products, then the diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow \pi_X & & \downarrow \\
 X & \xrightarrow{\quad} & 1
 \end{array} \tag{81}$$

is a pushout for any $X, Y \in \mathcal{C}$. In the case when $Y = X$ this immediately implies that $!_X$ is the coequalizer of the product projections in the above diagram and so $!_X$ is a regular epimorphism for any $X \in \mathcal{C}$. On the other hand, the above diagram is also a pullback for any $X, Y \in \mathcal{C}$. Since \mathcal{C} is regular we have that regular epimorphisms are pullback stable, so the product projections π_X, π_Y are regular epimorphisms as desired.

On the other hand, assume product projections in \mathcal{C} are regular epimorphisms. Note that for any $X \in \mathcal{C}$, we have $X \simeq X \times 1$, and the unique morphism $!_X : X \rightarrow 1$ is a product projection, so it is a regular epimorphism. Any regular epimorphism is the co-equalizer

of its kernel pair, so in the diagram

$$\begin{array}{ccccc}
 X^2 \times Y & \begin{array}{c} \xrightarrow{\pi_1 \times 1_Y} \\ \xrightarrow{\pi_2 \times 1_Y} \end{array} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow \pi_{X^2} & & \downarrow \pi_X & & \downarrow \\
 X^2 & \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} & X & \longrightarrow & 1
 \end{array} \tag{82}$$

we have that both rows are coequalizer diagrams. Here π_{X^2} is also a product projection, and thus a regular epimorphism. Therefore $!_X$ is also the coequalizer of $\pi_1 \pi_{X^2}$, and $\pi_2 \pi_{X^2}$, and it immediately follows from Lemma 4.1.6 (a) that the right square is a pushout as desired. ■

4.1.8. LEMMA. *Let \mathcal{C} be a regular category with co-disjoint products and let $d : R \rightarrow A \times B$ be a monomorphism. Then the factorization of (R, d) as some product $(R_A \times R_B, d_A \times d_B)$ for monomorphisms $d_A : R_A \rightarrow A$ and $d_B : R_B \rightarrow B$ is unique up to isomorphism if it exists.*

PROOF. Let $d : R \rightarrow A \times B$ be a monomorphism, and consider any commutative diagram

$$\begin{array}{ccccc}
 R_A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & R_B \\
 \downarrow d_A & & \downarrow d & & \downarrow d_B \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array} \tag{83}$$

in which the top row is a product diagram. By the previous two lemmas d_A and d_B are monomorphisms while π_1, π_2 are regular epimorphisms. Therefore (R_A, π_1, d_A) and (R_B, π_2, d_B) are the factorizations of $\pi_A d$ and $\pi_B d$ as regular epimorphisms followed by a monomorphism, and since \mathcal{C} is a regular category these factorizations are unique up to isomorphism. ■

4.2. GENERALIZING SUBCONDITIONS OF COEXTENSIVITY. Using the categorical machinery we have developed so far, we can now generalize several useful results from varieties of algebras to exact categories. We begin with the familiar equivalence of two conditions related to coextensivity.

4.2.1. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products. Then the following conditions are equivalent.*

- (a) *For any $A \in \mathcal{C}$, if R is a congruence on A which contains a subproduct of A^2 , then $R \simeq A^2$.*
- (b) *\mathcal{C} has strict terminal objects.*

PROOF. ($a \implies b$) Let $f : 1 \longrightarrow X$ be some morphism, and recall that \mathcal{C} has strict terminal objects if and only if any such morphism f is an isomorphism. Any morphism from the terminal object is immediately a monomorphism, so 1 is a subobject of X . Consider the diagram

$$1 \xrightarrow{f} X \begin{array}{c} \xleftarrow{\pi_1} \\ \xleftarrow{\Delta_X} \\ \xrightarrow{\pi_2} \end{array} X^2 \quad (84)$$

in which $1 \simeq 1^2$ since 1 is terminal, and so $\Delta_X f \simeq f \times f$. Then, by (a), Δ_X must be an isomorphism, since the diagonal relation $(X, 1_X, 1_X)$ is a congruence containing the subproduct 1^2 . Therefore $X \xleftarrow{1_X} X \xrightarrow{1_X} X$ is a product diagram. This implies that for any $Y \in \mathcal{C}$ there is at most one morphism $g : Y \longrightarrow X$. On the other hand there is always a unique morphism $!_Y : Y \longrightarrow 1$, and so $f!_Y$ is the unique morphism in $\text{hom}(Y, X)$, which implies X is terminal as desired.

($b \implies a$) Let (R, d_1, d_2) be any equivalence relation on A containing some subproduct $X \times Y$ and let q be the coequalizer of d_1, d_2 in the diagram:

$$\begin{array}{ccccc} & X & & & \\ & \uparrow \pi_1 & & m_1 & \\ X \times Y & \xrightarrow{m} & R & \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} & A \xrightarrow{q} B \\ & \downarrow \pi_2 & & m_2 & \\ & Y & & & \end{array} \quad (85)$$

Then $qm_1\pi_1 = qd_1m = qd_2m = qm_2\pi_2$. Since \mathcal{C} has co-disjoint products there exists some

morphism $f : 1 \rightarrow B$ as in the diagram:

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\pi_2} & Y \\
 \downarrow \pi_1 & & \downarrow \\
 X & \xrightarrow{\quad} & 1 \\
 & \searrow qm_1 & \downarrow qm_2 \\
 & & B
 \end{array}
 \tag{86}$$

Therefore $B \simeq 1$ by (b). Since \mathcal{C} is exact, (R, d_1, d_2) is an effective equivalence relation, and so it must be the kernel pair of its coequalizer q . Since B is terminal, the kernel pair of q is (A^2, π_1, π_2) , and so $R \simeq A^2$ as desired. ■

Note that the above result is essentially the categorical equivalent of the result that a variety with at least one constant has strict terminal objects if and only if it has $\vec{0}$ and $\vec{1}$. Note that in a variety, having co-disjoint products is equivalent to the fairly trivial condition of having at least one constant term, so a variety with a non-trivial diagonalising term is equivalent to a variety with co-disjoint products and upward closed subproducts. We can now start converting the results of Proposition 4.0.3 from results about varieties with a non-trivial diagonalising term into results about exact categories with co-disjoint products and upward closed subproducts.

4.2.2. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then the following properties hold:*

- (a) *For any $A \in \mathcal{C}$, if R is a reflexive relation on A which contains a subproduct of A^2 , then $R \simeq A^2$.*
- (b) *For any $A \in \mathcal{C}$, if R is a congruence on A which contains a subproduct of A^2 , then $R \simeq A^2$.*
- (c) *\mathcal{C} has strict terminal objects.*

PROOF. (a) Let R be a reflexive relation on A which contains a subproduct of A^2 . Then, since \mathcal{C} has upward closed subproducts and R contains a subproduct of A^2 , we can con-

struct the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & R & \xrightarrow{\pi_Y} & Y \\
 \downarrow i_X & & \uparrow r & & \downarrow i_Y \\
 A & \xleftarrow{\pi_1} & A^2 & \xrightarrow{\pi_2} & A \\
 & & \downarrow d & & \\
 & & & &
 \end{array}
 \tag{87}$$

for some monomorphisms $i_X : X \rightarrow A$, $i_Y : Y \rightarrow A$ such that $d = i_X \times i_Y$. Since R is reflexive there exists some $r : A \rightarrow R$ such that $i_X \pi_X r = \pi_1 d r = 1_A$ and $i_Y \pi_Y r = \pi_2 d r = 1_A$. Hence i_X and i_Y are both monomorphisms and split epimorphisms, and so $R \simeq A^2$. (b) follows from (a) since any congruence is a reflexive relation, and (c) is equivalent to (b) by the previous proposition. \blacksquare

It is worth comparing the above proof with the algebraic proof of the same proposition in the context of varieties with non-trivial diagonalising terms. In this context, a reflexive relation R containing a subproduct of A^2 must contain all pairs of constants $(e_1, e'_1), \dots, (e_k, e'_k)$, as well as all pairs $(a, a) \in A^2$. Let t be a diagonalising term equipped with constants $e_1, \dots, e_k, e'_1, \dots, e'_k$. It follows immediately that for any $(a, b) \in A^2$ we have $(a, b) = t((a, a), (b, b), (e_1, e'_1), \dots, (e_k, e'_k)) \in R$ and so $R = A^2$.

This is very similar to the above proof, in which the existence of r plays the role of requiring $(a, a) \in R$ for all $a \in A$, and we use upward closed subproducts to show that R itself is a subproduct of A^2 , which is the same role played by the diagonalising term in the algebraic case. Next, let us consider how reflexive relations interact with products in the categorical case.

4.2.3. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then any reflexive relation R on some product $A \times B \in \mathcal{C}$ is of the form $R_A \times R_B$ for some reflexive relations R_A on A and R_B on B .*

PROOF. Let (R, d_1, d_2) be a reflexive relation on $A \times B$. Note that such a relation can also be seen as a relation between A^2 and B^2 . We will first show that R contains a subproduct of $A^2 \times B^2$. Since R is reflexive, there exists $\Delta : A \times B \rightarrow R$ such that $d_1 \Delta = d_2 \Delta = 1_{A \times B}$. Since R is a relation on $A \times B$, it can be seen as a subobject of $(A \times B)^2$, with $d : R \rightarrow (A \times B)^2$ the unique monomorphism such that $\pi_1 d = d_1$ and $\pi_2 d = d_2$. It follows that $d \Delta = \Delta_A \times \Delta_B : A \times B \rightarrow A^2 \times B^2$ where Δ_A, Δ_B are the diagonal morphisms for A and B respectively. Now, since \mathcal{C} has upward closed subproducts, we

can construct the diagram

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\
 \downarrow r_A & & \downarrow \Delta & & \downarrow r_B \\
 \Delta_A & \xleftarrow{\quad} & R & \xrightarrow{\quad} & R_B & \Delta_B \\
 \downarrow d_A & & \downarrow d & & \downarrow d_B \\
 A^2 & \xleftarrow{\pi'_A} & A^2 \times B^2 & \xrightarrow{\pi'_B} & B^2
 \end{array} \tag{88}$$

in which all rows are product diagrams and d_A, d_B are monomorphisms. To attain the morphisms r_A and r_B , note that the product projections π_A and π_B are strong epimorphisms, since product projections are regular epimorphisms in regular categories with co-disjoint products. It follows that there exist unique morphisms r_A and r_B such that the diagram commutes with $d_A r_A = \Delta_A$, and $d_B r_B = \Delta_B$. It follows that R_A and R_B are reflexive relations on A and B as desired. ■

It is again interesting to compare the above proof with the proof of the equivalent algebraic condition 3.3.5 (a). In the algebraic proof, we first define R_A and R_B as the images of R under the projections π'_A and π'_B , see that they are immediately reflexive, and then use the diagonalising term to show that $R \simeq R_A \times R_B$.

In the above categorical proof, note that R_A and R_B are again the images of R under π'_A and π'_B in the sense that they are given by the unique factorisation of $\pi'_A d$ and $\pi'_B d$ as the composite of an epimorphism and a monomorphism. These proofs may appear very different at first glance, but many details are in direct correspondance. For example, in the algebraic case, the reflexivity of R_A relies on B being non-empty. This is equivalent to π_A being a regular epimorphism, which we use in the categorical proof to show the existence of r_A , and thus the reflexivity of R_A .

Next, recall that any variety with a non-trivial diagonalising term is left coextensive. We will ultimately show that the same is true of exact categories with co-disjoint products and upward closed subproducts, however it is useful to begin with a subcondition that only applies to monomorphisms.

4.2.4. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then given any monomorphisms $f : X \rightarrow A$ and $g : Y \rightarrow B$ in \mathcal{C} , the diagram*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow f \times g & & \downarrow g \\ A \times B & \xrightarrow{\pi_B} & B \end{array} \quad (89)$$

is a pushout diagram.

PROOF. Given any such monomorphisms f, g we can construct the diagram

$$\begin{array}{ccccc} X^2 \times Y & \begin{array}{c} \xrightarrow{\pi_1 \times 1_Y} \\ \xrightarrow{\pi_2 \times 1_Y} \end{array} & X \times Y & \xrightarrow{\pi_Y} & Y \\ \downarrow \psi & & \downarrow f \times g & (1) & \downarrow m \\ R & \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} & A \times B & \xrightarrow{h} & C \\ \downarrow \phi & & \downarrow 1_A \times 1_B & (2) & \downarrow k \\ A^2 \times B & \begin{array}{c} \xrightarrow{\pi_1 \times 1_B} \\ \xrightarrow{\pi_2 \times 1_B} \end{array} & A \times B & \xrightarrow{\pi_B} & B \end{array} \quad (90)$$

in which the square (1) is a pushout, (R, d_1, d_2) is the kernel pair of h and $k : C \rightarrow B$ is the unique morphism such that $km = g$ and $kh = \pi_B$. We will show that the right rectangle (1) + (2) is a pushout, which is true if and only if k is an isomorphism.

First note that by Lemma 4.1.5, the top and bottom rows are kernel pair diagrams, while the middle row is a kernel pair diagram by definition. Therefore there exist unique morphisms ϕ and ψ such that the left squares in the diagram commute, and we have that $\phi\psi = f \times f \times g$.

Furthermore, by Lemma 4.1.6, ϕ and ψ are monomorphisms, and h is the coequalizer of $d_1\psi$ and $d_2\psi$. Since both h and π_B are regular epimorphisms, they are the coequalizers of their kernel pairs, and thus k is an isomorphism if and only if the kernel pairs of h and π_B are isomorphic. It remains to show that ϕ is an isomorphism. Our next step is to uniquely factor R , ϕ , and ψ into products. First note that (R, ϕ) is a relation between

A^2 and B , which contains the subproduct $(X^2 \times Y, f^2 \times g)$. Since \mathcal{C} has upward closed subproducts, R is also a subproduct of $A^2 \times B$, and so we construct the diagram

$$\begin{array}{ccccc}
 X^2 & \xleftarrow{\pi_{X^2}} & X^2 \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow \psi_{X^2} & & \downarrow \psi & & \downarrow \psi_Y \\
 R_{A^2} & \xleftarrow{p_{A^2}} & R & \xrightarrow{p_B} & R_B \\
 \downarrow \phi_{A^2} & & \downarrow \phi & & \downarrow \phi_B \\
 A^2 & \xleftarrow{\pi_{A^2}} & A^2 \times B & \xrightarrow{\pi_B} & B
 \end{array}
 \quad (91)$$

in which all rows are product diagrams and all vertical arrows are monomorphisms. Here the monomorphisms ϕ_{A^2} and ϕ_B exist due to R being a subproduct of $A^2 \times B$, while ψ_{X^2} and ψ_Y exist because the product projections π_{x^2} and π_Y are strong epimorphisms. Next we will show that ϕ_{A^2} and ϕ_B are isomorphisms.

Recall that (R, d_1, d_2) is a reflexive relation on $A \times B$, and so by Proposition 4.2.3 $R \simeq R'_A \times R'_B$ for some reflexive relations R'_A on A and R'_B on B . However, we have already factorised this same relation in the above diagram as the product of R_{A^2} and R_B , where (R_{A^2}, ϕ_{A^2}) is already a relation on A , and $(R_B, \Delta_B \phi_B)$ is a relation on B . Furthermore, such a factorisation must be unique up to isomorphism by Lemma 4.1.8, so we have that R_{A^2} and R_B are reflexive.

It follows that there exists some morphism $\Delta_{R_B} : B \rightarrow R_B$ such that $\Delta_B \phi_B \Delta_{R_B} = \Delta_B$, and thus $\phi_B \Delta_{R_B} = 1_B$. Therefore ϕ_B is both a split epimorphism and a monomorphism, so it is an isomorphism. On the other hand, we have $\phi_{A^2} \psi_{X^2} = f \times f$, so R_{A^2} is a reflexive relation on A containing a subproduct of A^2 , and so ϕ_{A^2} is an isomorphism by Proposition 4.2.2 condition (a). Therefore ϕ is an isomorphism as desired. \blacksquare

Note that in the algebraic case, the existence of a diagonalising term immediately implies left coextensivity using the characterisation of left coextensivity as in 3.3.2. However, it is also simple to prove directly. We first note that π_B is the quotient map of $A \times B$ over the congruence $\{((a_1, b), (a_2, b)) \in (A \times B)^2\}$, while the pushout h of π_Y over $f \times g$ is the quotient map of $A \times B$ over the congruence generated by the relation

$\{((f(x_1), g(y)), (f(x_2), g(y))) \in (A \times B)^2 \mid x_1, x_2 \in X, y \in Y\}$. It remains to show that these congruences are equal, but that follows simply from the existence of the diagonalising term.

In the above categorical proof, we proceed in a similar manner, by showing that π_B is the coequalizer of $\pi_1 \times 1_B$ and $\pi_2 \times 1_B$, while the pushout h can be seen as the coequalizer of $f\pi_1 \times g = d_1\psi$ and $f\pi_2 \times g = d_2\psi$. It remains to show that the kernel pairs (R, d_1, d_2) and $(A^2 \times B, \pi_1 \times 1_B, \pi_2 \times 1_B)$ are equal, but here we run into some difficulties. To show this using upward closed subproducts turns out to be significantly more complicated than using a diagonalising term. We need to assume f and g are monomorphisms so that ψ will be a monomorphism, which allows us to decompose R into a subproduct of $A^2 \times B$ and finally show that they are isomorphic as desired. We will later cover the case in which f and g are not monomorphisms, but to do so we must first prove the categorical equivalents of conditions (b) and (c) of Proposition 3.3.5.

4.2.5. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then any congruence E on some product $A \times B \in \mathcal{C}$ is of the form $E_A \times E_B$ for some congruences E_A on A and E_B on B .*

PROOF. Given some congruence (E, d_1, d_2) on some product $A \times B \in \mathcal{C}$, we have by Proposition 4.2.3 that $(E, d_1, d_2) \simeq (E_A \times E_B, d_{A,1} \times d_{B,1}, d_{A,2} \times d_{B,2})$ for some reflexive relations $(E_A, d_{A,1}, d_{A,2})$ on A and $(E_B, d_{B,1}, d_{B,2})$ on B . It remains to show that E_A and E_B are symmetric and transitive. To do so, we will consider those morphisms σ and τ which make E symmetric and transitive, and argue using Proposition 4.2.4 that these morphisms can be decomposed into products, after which it will be simple to see that each component satisfies the equalities of symmetry and transitivity. We begin with symmetry as follows.

Since E is symmetric there exists a morphism $\sigma : E \rightarrow E$ such that $d_1\sigma = d_2$ and $d_2\sigma = d_1$. It immediately follows that $d_1\sigma\Delta = d_2\Delta = d_1\Delta = d_2\sigma\Delta$ and since d_1, d_2 are

jointly monic, we have $\sigma\Delta = \Delta$. We can now construct the diagram

$$\begin{array}{ccccc}
 A \times B & \xrightarrow{\Delta} & E & \xrightarrow{\sigma} & E \\
 \downarrow \pi_A & & \downarrow \pi_{E_A} & & \downarrow \pi_{E_A} \\
 A & \xrightarrow{r_A} & E_A & \xrightarrow{\sigma_A} & E_A
 \end{array} \tag{92}$$

in which $\pi_{E_A} : E \rightarrow E_A$ is the product projection from E to E_A , and the left square is a pushout by Proposition 4.2.4, since Δ is a monomorphism. Here σ_A is defined to be the unique morphism such that $\pi_{E_A}\sigma = \sigma_A\pi_{E_A}$ and $\sigma_A r_A = r_A$. It follows that $\sigma = \sigma_A \times \sigma_B$ where σ_B is defined similarly.

Note that $d_{A,2} \times d_{B,2} = d_2 = d_1\sigma = d_{A,1}\sigma_A \times d_{B,1}\sigma_B$, so $d_{A,2} = d_{A,1}\sigma_A$ and $d_{B,2} = d_{B,1}\sigma_B$ as desired since product projections are epimorphisms. Similarly we have $d_{A,1} = d_{A,2}\sigma_A$ and $d_{B,1} = d_{B,2}\sigma_B$, so E_A and E_B are symmetric. It remains to show that E_A and E_B are transitive. Recall that a relation (R, d_1, d_2) is transitive if, where the diagram

$$\begin{array}{ccc}
 R \times_X R & \xrightarrow{p_2} & R \\
 \downarrow p_1 & \lrcorner & \downarrow d_1 \\
 R & \xrightarrow{d_2} & X
 \end{array}$$

is a pullback, there exists some morphism $\tau : R \times_X R \rightarrow R$ such that $d_1\tau = d_1p_1$ and $d_2\tau = d_2p_2$. Since E is a congruence, it is transitive, and there exists such a morphism $\tau : E \times_{A \times B} E \rightarrow E$. We will show that $\tau = \tau_A \times \tau_B$ for some morphisms $\tau_A : E_A \times_A E_A \rightarrow E_A$ and $\tau_B : E_B \times_B E_B \rightarrow E_B$ that satisfy the above equations. To do so, let us first note that all of the relations E , E_A and E_B are reflexive relations. In general, for any reflexive relation (R, d_1, d_2) on some object X we can construct the diagram

$$\begin{array}{ccccc}
 R & \xrightarrow{m_X} & R \times_X R & \xrightarrow{p_2} & R \\
 \downarrow d_1 & \lrcorner & \downarrow p_1 & \lrcorner & \downarrow d_1 \\
 X & \xrightarrow{\Delta} & R & \xrightarrow{d_2} & X
 \end{array} \tag{93}$$

in which both squares are pullbacks and $p_2m_X = 1_R$. To construct the above diagram, note that pullback of the bottom row $d_2\Delta = 1_X$ along d_1 must also be an isomorphism, so we can assume the top row composes to 1_R . Then since the right square is a pullback, there exists a unique morphism m_X such that the diagram commutes, and the left square is also a pullback by the pasting rule. Furthermore, note that m_X is a monomorphism, since Δ is a monomorphism, and monomorphisms are preserved by pullback.

Next, we apply the above result to each of E , E_A , and E_B , and note in particular that $m_{(A \times B)} = m_A \times m_B$ since pullbacks and products commute. Note also that $\tau m_{(A \times B)} = 1_E$, since d_1 and d_2 are jointly monic, and we have $d_2\tau m_{(A \times B)} = d_2p_2m_{(A \times B)} = d_2$ and $d_1\tau m_{(A \times B)} = d_1p_1m_{(A \times B)} = d_1\Delta d_1 = d_1$. We can now construct the diagram

$$\begin{array}{ccccc}
 E_A & \xleftarrow{\pi_{E_A}} & E & \xrightarrow{\pi_{E_B}} & E_B \\
 \downarrow m_A & & \downarrow m_{(A \times B)} & & \downarrow m_B \\
 E_A \times_A E_A & \xleftarrow{\pi'_A} & E \times_{A \times B} E & \xrightarrow{\pi'_B} & E_B \times_B E_B \\
 \downarrow \tau_A & & \downarrow \tau & & \downarrow \tau_B \\
 E_A & \xleftarrow{\pi_{E_A}} & E & \xrightarrow{\pi_{E_B}} & E_B
 \end{array} \tag{94}$$

in which all rows are products, and so the upper two squares are pushouts by Proposition 4.2.4. We can then define τ_A and τ_B to be the unique morphisms from these pushouts such that the diagram commutes and $\tau_A m_A = 1_{E_A}$ and $\tau_B m_B = 1_{E_B}$. It follows that $\tau = \tau_A \times \tau_B$. Lastly, we need to show that τ_A and τ_B satisfy the relevant equations to make E_A and E_B transitive. This follows simply from E being transitive, since $\tau = \tau_A \times \tau_B$ satisfies these same equations $d_1\tau = d_1p_1$ and $d_2\tau = d_2p_2$. Note that every morphism in these equations is the product of its counterparts for E_A and E_B . Then, since product projections are epimorphisms, the same equations must hold true for E_A and E_B , which is enough to make them transitive as desired. ■

Note that the algebraic version of the above condition, that if $E_A \times E_B$ is a congruence then so are E_A and E_B , is immediately clear. On the other hand, the above proof is significantly more complex. The reason for this difference in complexity is that in the

algebraic case we already know that the operations of symmetry and transitivity are defined component-wise. This is in some sense equivalent to knowing that the above morphisms σ and τ can be seen as products, which is what we spend the majority of our time on in the categorical proof. Next, we complete our categorical translation of Proposition 3.3.5 as follows.

4.2.6. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then \mathcal{C} satisfies the co-universality condition for any regular epimorphism q .*

PROOF. Let $q : A \times B \rightarrow C$ be a regular epimorphism and $d_1, d_2 : E \rightarrow A \times B$ the kernel pair of q . (E, d_1, d_2) is a congruence on $A \times B$, so it is immediately the product of two congruences $(E_A, d_{A,1}, d_{A,2})$ on A and $(E_B, d_{B,1}, d_{B,2})$ on B by our Proposition 4.2.5. Since \mathcal{C} is an exact category, all congruences are effective, and so we can construct the diagram

$$\begin{array}{ccccc}
 E_A & \xleftarrow{\pi_{E_A}} & E & \xrightarrow{\pi_{E_B}} & E_B \\
 \downarrow d_{A,1} & & \downarrow d_1 & & \downarrow d_{B,1} \\
 & & & & \downarrow d_{B,2} \\
 & & & & \downarrow d_{A,2} \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\
 \downarrow q_A & & \downarrow q & & \downarrow q_B \\
 X & \xleftarrow{p_X} & C & \xrightarrow{p_Y} & Y \\
 & \swarrow \pi_X & \downarrow \eta_q & \searrow \pi_Y & \\
 & & X \times Y & &
 \end{array} \tag{95}$$

in which $(E_A, d_{A,1}, d_{A,2})$ and $(E_B, d_{B,1}, d_{B,2})$ are both the kernel pairs of their coequalizers q_A and q_B . The top and middle rows are products by construction, while p_X , p_Y , and η_q are the unique morphisms such that the bottom squares and triangles commute. Note that since product projections are epimorphisms, q_A is the coequalizer of $d_{A,1}\pi_{E_A} = \pi_A d_1$ and $d_{A,2}\pi_{E_A} = \pi_A d_2$, while q_B is the coequalizer of $d_{B,1}\pi_{E_B} = \pi_B d_1$ and $d_{B,2}\pi_{E_B} = \pi_B d_2$.

Then, since q is a regular epimorphism, it follows from Lemma 4.1.6 (a) that both bottom squares are pushouts. \mathcal{C} is a regular category, so $\eta_q q = q_A \times q_B$ is regular by Proposition 2.4.16 since q_A and q_B are both regular. The kernel pair of $q_A \times q_B$ is necessarily $(E_A \times$

$(E_B, d_{A,1} \times d_{B,1}, d_{A,2} \times d_{B,2}) \simeq (E, d_1, d_2)$. Therefore q and $q_A \times q_B$ are regular epimorphisms with the same kernel pair, and it follows that η_q is an isomorphism as desired. ■

Interestingly, the above proof is an almost direct categorical translation of the proof of Proposition 3.3.5 condition (c). Armed with this result, we can return to considering left coextensivity without restricting ourselves to monomorphisms as in Proposition 4.2.4.

4.2.7. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then \mathcal{C} is left coextensive.*

PROOF. For any morphisms $f : X \rightarrow A$ and $g : Y \rightarrow B$ let $me = f \times g$ be the factorization of $f \times g$ as a regular epimorphism followed by a monomorphism. We can now construct the diagram

$$\begin{array}{ccccc}
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\
 \downarrow e_A & & \downarrow e & & \downarrow e_B \\
 Z_A & \xleftarrow{\quad} & Z & \xrightarrow{\quad} & Z_B \\
 \downarrow m_A & & \downarrow m & & \downarrow m_B \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array} \tag{96}$$

in which both upper squares are defined to be pushouts and the morphisms m_A and m_B are the unique morphisms from these pushouts such that $m_A e_A = f$, $m_B e_B = g$, and both bottom squares commute. Note that by Proposition 4.2.6 the middle row is a product diagram, so $m_A \times m_B = m$. m is a monomorphism and hence so are m_A and m_B . Then by Proposition 4.2.4 both bottom squares are pushouts, and so by the pasting rule the left and right rectangles are pushouts as desired. ■

Putting the above propositions together into a single theorem, we obtain a categorical version of Proposition 4.0.3.

4.2.8. THEOREM. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then the following properties hold:*

- (a) *For any $A \in \mathcal{C}$, if R is a reflexive relation on A which contains a subproduct of A^2 , then $R \simeq A^2$.*
- (b) *For any $A \in \mathcal{C}$, if R is a congruence on A which contains a subproduct of A^2 , then $R \simeq A^2$.*
- (c) *\mathcal{C} has strict terminal objects.*
- (d) *Any reflexive relation R on some product $A \times B \in \mathcal{C}$ is of the form $R_A \times R_B$ for some reflexive relations R_A on A and R_B on B .*
- (e) *Any congruence E on some product $A \times B \in \mathcal{C}$ is of the form $E_A \times E_B$ for some congruences E_A on A and E_B on B .*
- (f) *\mathcal{C} is left coextensive.*
- (g) *\mathcal{C} satisfies the co-universality condition for any regular epimorphism q .*

We have shown that much of the relationship between diagonalising terms and coextensivity can be generalized to exact categories. Recall that a variety has Boolean factor congruences if for any object A the subset $Fac(A)$ of $Con(A)$ containing only the kernel pairs of product projections is a distributive sublattice of $Con(A)$. Recall also that any variety with a non-trivial diagonalising term has Boolean factor congruences. This result can be decomposed into two parts.

Firstly, the result that every variety with a non-trivial diagonalising term satisfies the Fraser-Horn property is generalized by condition (e) of the above proposition. Secondly, the result that every variety satisfying the Fraser-Horn property has Boolean factor congruences is generalized by the results of [17], as pointed out by an examiner of this thesis. Here it is shown that the Fraser-Horn property is equivalent to coextensivity of regular epimorphisms, and having Boolean factor congruences is equivalent to coextensivity of product projections. The result follows immediately, since product projections in regular categories with co-disjoint products are regular epimorphisms by Lemma 4.1.7.

It is very interesting that multiple useful subconditions of coextensivity are equivalent to restrictions of the coextensivity condition to specific families of morphisms, and such

connections should be studied further. To see how the distributive lattice $Fac(A)$ can be constructed categorically, we can also prove the same result directly in the following way.

4.2.9. PROPOSITION. *Let \mathcal{C} be an exact category with co-disjoint products and upward closed subproducts. Then \mathcal{C} has Boolean factor congruences.*

PROOF. let $A \in \mathcal{C}$ and let $Con(A)$ be the congruence lattice of A . We immediately have that the largest and smallest objects (A^2, π_1, π_2) and $(A, 1_A, 1_A)$ are the kernel pairs of the product projection $A \simeq A \times 1$, so they are in $Fac(A)$. Next we will show that the join and meet of factor congruences are themselves factor congruences. Let $X_1 \leftarrow^{x_1} A \rightarrow^{x_2} X_2$ and $Y_1 \leftarrow^{y_1} A \rightarrow^{y_2} Y_2$ be product diagrams. We can now construct the diagram

$$\begin{array}{ccccc}
 A_{1,1} & \longleftarrow & X_1 & \longrightarrow & A_{1,2} \\
 \uparrow & \lrcorner & \uparrow & & \lrcorner \uparrow \\
 Y_1 & \longleftarrow^{y_1} & A & \xrightarrow{y_2} & Y_2 \\
 \downarrow & & \downarrow^{x_2} & & \downarrow \\
 A_{2,1} & \longleftarrow & X_2 & \longrightarrow & A_{2,2} \\
 & & & & \lrcorner \downarrow
 \end{array} \tag{97}$$

in which all four squares are pushouts. Now, since \mathcal{C} is exact and has co-disjoint products, all morphisms in the diagram are regular epimorphisms. Then, since \mathcal{C} has upward closed subproducts, we have that all rows and columns in the diagram are product diagrams by condition (g) of the previous result. It follows that $A \simeq \prod_{i,j \in \{0,1\}} A_{i,j}$. Let $a_{i,j} : A \rightarrow A_{i,j}$ be the product projections, and let $E = Eq(x_1)$ and $R = Eq(y_1)$ be factor congruences. It immediately follows that $E \vee R = Eq(a_{1,1})$, and $E \wedge R = Eq(\langle x_1, y_1 \rangle) = Eq(\langle a_{1,2}, a_{1,1}, a_{2,1} \rangle)$ by Proposition 2.4.11, so $Fac(A)$ is a sublattice of $Con(A)$.

It remains to show that $Fac(A)$ is distributive. Let $Z_1 \leftarrow^{z_1} A \rightarrow^{z_2} Z_2$ be a third product diagram. Taking all pushouts of the various product projections x_i, y_i, z_i we can again decompose A into a product $A \simeq \prod_{i,j,k \in \{0,1\}} A_{i,j,k}$ with projections $a_{i,j,k}$. It is clear that the kernel pairs of these projections form a sublattice of $Fac(A)$ which is isomorphic to the Boolean lattice 2^8 . Then, writing $Q = Eq(z_1)$, it follows that $Q \vee (E \wedge R) = (Q \vee E) \wedge (Q \vee R)$, and so $Fac(\mathcal{C})$ is distributive as desired. \blacksquare

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