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ON THE THEORY OF KRULL RINGS

AND

INJECTIVE MODULES

by

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CONTENTS

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INTRODUCTION

In the first chapter we give an outline of classical KRULL rings as in SAMUEL [1964], BOURBAKI [1965] and FOSSUM [1973]. In the second chapter we introduce two notions important to our treatment of KRULL theory. The first is injective modules and the second torsion theories. We then look at injective modules over Noetherian rings as in MATLIS [1958] and then over KRULL rings as in BECK [1971]. We show that for a KRULL ring there is a torsion theory $(\mathcal{N},\mathcal{M})$ where $\mathcal{N}$ is the pseudo-zero modules and $\mathcal{M}$ the set of $\mathcal{N}$-torsion-free (BECK calls these co-divisorial) modules. From LAMBEK [1971] there is a full abelian sub category $\mathcal{C}$, namely the category of $\mathcal{N}$-torsion-free, $\mathcal{N}$-divisible modules, with exact reflector. We show in $\mathcal{C}$: (I) every direct sum of injective modules is injective

and (II) $\mathcal{C}$ has global dimension at most one.

It is these two properties that we exploit in the third chapter to give another characterization of KRULL rings. Then we generalize this to rings with zero-divisors and find that (i) R has to be reduced (ii) the ring is KRULL if and only if it is a finite product of fields and KRULL domains (iii) the injective envelope of the ring is semi-simple artinian. We then generalize the ideas to rings of higher dimension. Here R_p regular for p ht n generalizes the classical R_p a discrete valuation ring for p ht 1 .

Finally we propose an analogue of the Divisor Class group for KRULL rings.

We use the following conventions :

All rings are commutative with identity and domains are not equal to their fields of fractions, unless otherwise stated.

If R is a ring R^* denotes the multiplicative group of units of R. The symbols $\subset$ or $\subseteq$ mean inclusion, proper or otherwise while $\subsetneqq$ mean proper inclusion.

We will use A to indicate a KRULL ring, unless otherwise stated.

CHAPTER ONE

KRULL RINGS

1.1 INTRODUCTION

In this chapter we give an account of the classical theory of commutative KRULL domains. PIERRE SAMUEL [1964], using divisor theory, defined a KRULL domain as a domain which has a set of divisors that are freely generated by prime elements. BOURBAKI [1965] on the other hand, made use of valuation theory and defined a KRULL domain as a domain having a family of discrete valuations on its field of fractions so that (i) the intersection of their associated valuation rings is the ring itself and (ii) for non-zero elements the value is zero for almost all valuations. Similarly ROBERT FOSSUM [1973] defined KRULL domains via two properties (I) the INTERSECTION property, that is, the domain being the intersection of principal valuation rings and (II) the FINITE CHARACTER property, that is, a non-zero element in the ring is a unit in almost all the principal valuation rings. Since a set of valuations on the quotient field of a ring under certain conditions induces a theory of divisors on the ring, these approaches are readily compatible. From these definitions various other characterizations of KRULL domains are derived. But basically there are two notable characterizations of KRULL domains: Firstly the ring being completely integrally closed and having the maximum condition on divisorial ideals; Secondly the localizations of the ring at its height one primes being discrete valuation rings, the ring being equal to the intersection of these valuation rings and non-zero elements in the ring being contained in only finitely many height one primes.

1.2 DIVISORIAL IDEALS AND DIVISORS

The theory of divisors is very useful in branches of mathematics such as Algebraic Geometry and Algebraic Number Theory. In algebraic geometry the notion of divisor forms an important tool for studying the intrinsic geometry on a variety or a scheme. In number theory it is used to study the question of decomposition of algebraic numbers into prime fractors. (The problems of factorization are very closely connected historically with FERMAT's last theorem. For example see the chapter on The Theory of Divisibility in Number Theory by Z.I. BOREVICH and I.R. SHAFAREVICH.) On the other hand the divisor class group, which is the group of divisors modulo the principal divisors, is an important arithmetic invariant of an algebraic number field and an important special case of the more general notion of PICARD group. If, for example, the number of divisor classes equals one then this means that every divisor is principal, which is equivalent to the maximal order of the field having unique factorization. In this section we develop the notion of divisorial ideal and show that there is a one-to-one correspondence between the set of divisorial ideals and the set of divisors. These notions are required for the definition of KRULL domain as in SAMUEL [1964] and BOURBAKI [1965].

Definition 1 : Let R be an integral domain and Q its field of fractions. Every sub-R-module $\mathfrak{a}$ of Q such that there exists a $0 \neq d \in R$ for which $d\,\mathfrak{a} \subset R$ is termed a *fractional* or *fractionary* ideal of R

So elements of a fractionary ideal can be thought of as having a "common denominator".

A fractionary ideal is *principal* if it is generated by a single element.

A fractionary ideal is *integral* or an R-*ideal* if it is contained in the domain R .

Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractionary ideals of R . Then their sum

$$\mathfrak{a} + \mathfrak{b} = \{a + b : a \in \mathfrak{a} , b \in \mathfrak{b}\}$$

and their product

$$\mathfrak{a}\mathfrak{b} = \{\text{finite sums } \Sigma a_i b_i : a_i \in \mathfrak{a} , b_i \in \mathfrak{b}\}$$

and their intersection

$$\mathfrak{a} \cap \mathfrak{b} = \{a : a \in \mathfrak{a} \text{ and } a \in \mathfrak{b}\}$$

as well as the residual quotient of $\mathfrak{b}$ by $\mathfrak{a}$

$$\mathfrak{b} : \mathfrak{a} = \{x \in Q : x\,\mathfrak{a} \subset \mathfrak{b}\}$$

are all fractionary ideals.

Notation :

We will use $\mathfrak{b} :_R \mathfrak{a}$ to represent $(\mathfrak{b} : \mathfrak{a}) \cap R = \{x \in R : x\,\mathfrak{a} \subset \mathfrak{b}\}$.

Proposition 1 : Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractionary ideals of the integral domain R then $\mathfrak{b} : \mathfrak{a}$ is isomorphic to $\mathrm{Hom}_R(\mathfrak{a},\mathfrak{b})$

Proof : There is a canonical homomorphism from $\mathfrak{b} : \mathfrak{a}$ to $\mathrm{Hom}_R(\mathfrak{a},\mathfrak{b})$: with every $b \in (\mathfrak{b}:\mathfrak{a})$ associate the homomorphism $h_b : \mathfrak{a} \to \mathfrak{b}$ given by $x \longmapsto bx$. If $b \in (\mathfrak{b}:\mathfrak{a})$ such that $bx = 0$ for all $x \in \mathfrak{a}$, then $ba = 0$ for $a \in R \cap \mathfrak{a}$ hence $b = 0$ since a is not a zero divisor in R . Let $f \in \mathrm{Hom}_R(\mathfrak{a},\mathfrak{b})$ and set $b = f(a)/a$; for all $x \in \mathfrak{a}$ there exists $d \in R$ such that $dx \in R$. Then $f(x) = a^{-1}d^{-1}f(adx) = a^{-1}d^{-1}dx\, f(a) = bx$, so that $b \in (\mathfrak{b} : \mathfrak{a})$. □

Notation :

The set of non-zero fractionary ideals of the integral domain R will be denoted I(R).

A fractional ideal $\mathfrak{a}$ of R is *invertible* if there exists a fractional ideal $\mathfrak{b}$ of R such that $\mathfrak{a}\mathfrak{b} = R$.

Proposition 2 : Let R be an integral domain with field of fractions Q.

(1) If $\mathfrak{a} \in I(R)$ is invertible, then $\mathfrak{a}$ is a finitely generated R-module.

(2) If $\mathfrak{a},\mathfrak{b} \in I(R)$ and $\mathfrak{a} \subset \mathfrak{b}$ and $\mathfrak{b}$ is invertible, then there is an R-ideal $\mathfrak{c}$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$.

(3) If $\mathfrak{a} \in I(R)$, then $\mathfrak{a}$ is invertible if and only if there is a fractionary ideal $\mathfrak{b}$ of R such that $\mathfrak{a}\mathfrak{b}$ is principal.

Proof: (1) Let $\mathfrak{b} \in I(R)$ such that $\mathfrak{a}\mathfrak{b} = R$. Then there exist $a_1,\ldots,a_n \in \mathfrak{a}$ and $b_1,\ldots,b_n \in \mathfrak{b}$ such that $1 = \sum_{i=1}^{n} a_i b_i$. For each $x \in \mathfrak{a}$, $xb_i \in R$, for $i = 1,\ldots,n$, and $x = \sum_{i=1}^{n} a_i (xb_i)$. Thus, $a_1,\ldots,a_n$ generate $\mathfrak{a}$ as an R-module.

(2) Let $\mathfrak{b}' \in I(R)$ such that $\mathfrak{b}\mathfrak{b}' = R$, and $\mathfrak{c} = \mathfrak{a}\mathfrak{b}'$, then $\mathfrak{c} \subseteq R$ and $\mathfrak{b}\mathfrak{c} = \mathfrak{b}\mathfrak{b}'\mathfrak{a} = \mathfrak{a}$.

(3) Clearly if $\mathfrak{a}$ is invertible then $\mathfrak{a}\mathfrak{b}$ is principal where $\mathfrak{b}$ is an inverse for $\mathfrak{a}$.

If $\mathfrak{b} \in I(R)$ such that $\mathfrak{a}\mathfrak{b} = (x)$, $x \in Q$, then $\mathfrak{a}(\mathfrak{b}x^{-1}) = R$. □

Lemma 1 : If $\mathfrak{a}_i$ for $i \in I$, $\mathfrak{b}_j$ for $j \in J$, $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{b}'$ are frationary ideals of R then the following hold :

(1) $(\bigcap_{i\in I} \mathfrak{a}_i) : (\sum_{j\in J} \mathfrak{b}_j) = \bigcap_{i\in I, j\in J} (\mathfrak{a}_i : \mathfrak{b}_j)$

(2) $\mathfrak{a} : \mathfrak{b}\mathfrak{b}' = (\mathfrak{a} : \mathfrak{b}) : \mathfrak{b}'$

(3) If $0 \neq x \in Q$, then $\mathfrak{b} : Rx = x^{-1}\mathfrak{b}$. □

Definition 2: A non-zero fractionary ideal is called a *divisorial ideal* of R if it is the intersection of principal fractionary ideals.

Lemma 2 : If $\mathfrak{b}$ is a divisorial ideal and $(0) \neq \mathfrak{a} \in I(R)$. Then $\mathfrak{b} : \mathfrak{a}$ is divisorial.

Proof : Let $\mathfrak{b} = \bigcap_i Rx_i$, $x_i \in Q$. Then

$$\mathfrak{b} : \mathfrak{a} = (\bigcap_i Rx_i) : \mathfrak{a} = \bigcap_i (Rx_i : \mathfrak{a}) = \bigcap_i (\bigcap_{0\neq a\in\mathfrak{a}} Rx_i a^{-1}) \qquad □$$

Proposition 3 : If $\mathfrak{a} \in I(R)$ is invertible and if $\mathfrak{b} \in I(R)$ such that $\mathfrak{a}\mathfrak{b} = R$, then $\mathfrak{b} = (R : \mathfrak{a})$.

Proof : Since $\mathfrak{a}\mathfrak{b} = R$, $\mathfrak{b} \subseteq (R : \mathfrak{a})$. Also, $\mathfrak{a}(R : \mathfrak{a}) \subseteq R$ so that $R : \mathfrak{a} = \mathfrak{a}\mathfrak{b}(R : \mathfrak{a}) \subseteq \mathfrak{b}R = \mathfrak{b}$. □

The fractional ideal $R : \mathfrak{a}$ is called the *quasi-inverse* of $\mathfrak{a}$.

Two non-zero fractionary ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *quasi-equal* if their quasi-inverses are equal.

Let $\mathfrak{a} \in I(R)$, then the smallest divisorial ideal containing $\mathfrak{a}$, denoted $\bar{\mathfrak{a}}$, is the intersection of all principal fractional ideals containing $\mathfrak{a}$.

Proposition 4 : (1) If $\mathfrak{a} \in I(R)$, then $\bar{\mathfrak{a}} = R : (R : \mathfrak{a})$

(2) If $\mathfrak{a}, \mathfrak{b} \in I(R)$, then $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$ if and only if $R : \mathfrak{a} = R : \mathfrak{b}$.

Proof : (1) Since R is divisorial , $R : (R : \mathfrak{a})$ is divisorial by the lemma above. Clearly $\mathfrak{a} \subset [R : (R : \mathfrak{a})]$. Suppose that $\mathfrak{a} \subset Rx$, $0 \neq x \in Q$.

Then $(R : \mathfrak{a}) \supset (R : Rx) = Rx^{-1}$, thus

$$[R : (R : \mathfrak{a})] \subset (R : Rx^{-1}) = Rx .$$

(2)
$$\begin{aligned} \bar{\mathfrak{a}} = \bar{\mathfrak{b}} &\iff R : (R : \mathfrak{a}) = R : (R : \mathfrak{b}) \\ &\iff R : [R : (R : \mathfrak{a})] = R : [R : (R : \mathfrak{b})] \\ &\iff R : \mathfrak{a} = R : \mathfrak{b} . \end{aligned}$$ □

On $I(R)$ we introduce the ARTIN equivalence relation , $\sim$, as follows : $\mathfrak{a} \sim \mathfrak{b}$ if and only if $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$.

From the proposition above $\mathfrak{a} \sim \mathfrak{b}$ if and only if $R : \mathfrak{a} = R : \mathfrak{b}$.

$\bar{\mathfrak{a}}$ is the maximal fractional ideal in the same class as $\mathfrak{a}$.

The quotient set of $I(R)$ by the equivalence relation $\sim$, $I(R)/\sim$, is called the set of divisors of R and is denoted by $\mathrm{Div}(R)$

There is thus a one-to-one correspondence between the set $\mathrm{Div}(R)$ of divisors and the set of divisorial ideals of R .

Let div (for the divisor) denote the canonical mapping $\mathrm{div} : I(R) \to I(R)/\sim$. Since $I(R)$ is partially ordered by inclusion and since $a \subset b \Rightarrow \bar{a} \subset \bar{b}$, the partial order goes down to the quotient set $I(R)/\sim$ by div . If $a \subset b$, we write $\mathrm{div}(b) \leqslant \mathrm{div}(a)$.

$I(R)$ has the structure of a partially ordered commutative monoid with law of composition given by $(a,b) \to a\,b$, with R acting as the unit element.

If $a \sim a'$ then $ab \sim a'b$ since $R : ab = (R : a) : b = (R : a') : b$

$$= R : a'b \ .$$

Since the order in $\mathrm{Div}(R)$ is compatible with the composition law in $\mathrm{Div}(R)$, $\mathrm{Div}(R)$ acquires the structure of a commutative partially ordered monoid with unit R , with the composition law $(\bar{a},\bar{b}) \to \overline{a\,b}$. The composition law in $\mathrm{Div}(R)$ is written additively so that $\mathrm{div}(ab) = \mathrm{div}(a) + \mathrm{div}(b)$ for $a, b \in I(R)$, and $\mathrm{div}(R) = 0$.

Note that $\mathrm{div}(a \cap b) \geqslant \sup\{\mathrm{div}(a), \mathrm{div}(b)\}$

and $\mathrm{div}(a + b) = \inf\{\mathrm{div}(a), \mathrm{div}(b)\}$.

For $0 \neq x \in Q$ we write $\mathrm{div}(x)$ instead of $\mathrm{div}(Rx)$ and $\mathrm{div}(x)$ is called a *principal divisor*.

We now need to develop the theory of almost integral closure. This generalizes the classical theory of integrality which we now recall.

An R-module M is *faithful* if whenever $a \in R$ is such that $aM = 0$, then $a = 0$.

Proposition 5 : Let R be a subring of a ring R' and let $\alpha \in R'$. Then the following statements are equivalent :

(1) α is a root of a polynomial $X^n + a_{n-1}X^{n-1} + \ldots + a_0$ with coefficients $a_i \in R$, and degree $n \geqslant 1$.

(2) The subring $R[\alpha]$ is a finitely generated R-module.

(3) There exists a faithful module over $R[\alpha]$ which is a finitely generated R-module.

(4) There exists a finitely generated R-module M so that $\alpha M \subset M$.

Proof: Assume (1). Let $g(X)$ be a polynomial in $R[X]$ of degree $\geqslant 1$ with leading coefficient 1 such that $g(\alpha) = 0$. If $f(X) \in R[X]$ then $f(X) = q(X)g(X) + r(X)$ with $q, r \in R[X]$ and $\deg r < \deg g$. Hence $f(\alpha) = r(\alpha)$, and so if $\deg g = n$, then $1, \alpha, \ldots, \alpha^{n-1}$ are generators of $R[\alpha]$ as a module over R ; hence (2) holds. Assume (2). Then $R[\alpha]$ is a faithful module over itself. Assume (3), and let M be the faithful module over $R[\alpha]$ which is finitely generated over R, say $M = \sum_{i=1}^{n} Ru_i$. Since $\alpha M \subset M$ there exist $q_{ij} \in R$ such that $\alpha u_i = \sum_{j=1}^{n} q_{ij}u_j$ for $i = 1, \ldots, n$. Therefore if d is the determinant of the matrix $(q_{ij} - \delta_{ij}\alpha)$ with elements in $R[\alpha]$ (δ_{ij} denoting the KRONECKER index $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$), then $du_i = 0$ for all i and hence $dM = 0$. Since M is faithful $d = 0$. Hence α is a root of the polynomial $\det(q_{ij} - \delta_{ij}\alpha)$ which gives an integral equation for α over R.

(3) and (4) are equivalent. □

If the equivalent statements of the above proposition hold for α , then α is said to be *integral* over R . If every element of R' is integral over R , then R' is said to be *integral* over R . If the elements of R are the only elements of R' which are integral over R , then R is said to be *integrally closed in* R' .

If R is integrally closed in its total field of fractions Q then R is just said to be *integrally closed.*

Definition 3 : An element $x \in Q$ is said to be *almost integral over* R if the ideal $\sum_{i=0}^{\infty} Rx^i$ is a fractionary ideal.

This means that the powers of x have a common denominator.

Lemma 3 : An element x is almost integral over R if and only if there is a fractionary ideal a such that $xa \subseteq a$.

Proof: If x is almost integral over R , then $a = \sum_{i=0}^{\infty} Rx^i$ is a fractionary ideal with the desired property. Conversely if $xa \subseteq a$, then $x^i \in (a : a)$ which is a fractionary ideal. Thus the powers of x generate a fractionary ideal and x is almost integral over R . □

Corollary 1 : The set $\tilde{R}$ of elements almost integral over R forms a ring.

Proof: Suppose x, y are almost integral over R, say $x \in (\mathfrak{a} : \mathfrak{a})$ and $y \in (\mathfrak{b} : \mathfrak{b})$ for fractionary ideals $\mathfrak{a}$ and $\mathfrak{b}$. Then $(x + y) \in [(\mathfrak{a} \cap \mathfrak{b}) : (\mathfrak{a} \cap \mathfrak{b})]$ and $xy \in (\mathfrak{a}\mathfrak{b} : \mathfrak{a}\mathfrak{b})$. Since $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}$ are fractionary ideals, $x + y$ and xy are almost integral over R. So $\tilde{R}$ is a ring. □

The ring $\tilde{R}$ is called the *complete integral closure* of R and R is said to be *completely integrally closed* if $R = \tilde{R}$.

Corollary 2 : The integral domain R is completely integrally closed if and only if $R = (\mathfrak{a} : \mathfrak{a})$ for all fractionary ideals $\mathfrak{a}$. □

Proposition 6 : The monoid Div(R) is a group if and only if R is completely integrally closed.

Proof (After FOSSUM [1973]) :

Suppose Div(R) is a group. Let $\mathfrak{a}$ be a fractionary ideal. Suppose $\mathfrak{c} = \mathfrak{a} : \mathfrak{a}$. Let $\mathfrak{b} = R : (R : \mathfrak{a})$. If $x \in \mathfrak{c}$ and $y\mathfrak{a} \subseteq R$, then $xy\mathfrak{a} \subseteq R$. Hence $x \in \mathfrak{c}$ implies $x \in [(R : \mathfrak{a}) : (R : \mathfrak{a})]$. So $\mathfrak{c} \subseteq (\mathfrak{b} : \mathfrak{b})$. Let $B = \mathfrak{b} : \mathfrak{b}$ be in Div(R). Then the product, in Div(R), of B with itself is $R : (R : B^2)$. But B is a ring, so $B^2 = B$. Hence $R : (R : B^2) = R : (R : B) = B$. Hence B is an idempotent in the group Div(R) so B is the identity element R. Hence $R \subseteq (\mathfrak{a} : \mathfrak{a}) \subseteq B = R$ so R is completely integrally closed. Conversely, if R is completely integrally closed and $\mathfrak{a}$ is a divisorial fractionary ideal, then $[R : \mathfrak{a}(R : \mathfrak{a})] = [(R : \mathfrak{a}) : (R : \mathfrak{a})] = R$. Hence $R : [R : \mathfrak{a}(R : \mathfrak{a})] = R$ which shows that $R : \mathfrak{a}$ is an inverse to $\mathfrak{a}$ in Div(R). So Div(R) is a group. □

Note : $\mathrm{div}(\mathfrak{a}) + \mathrm{div}(R : \mathfrak{a}) = 0$ if we use additive notation for divisors (as we may since all rings considered here are commutative).

Definition 4 : If $\mathrm{Div}(R)$ is a group and if we denote by $\mathrm{Prin}(R)$ the subgroup of $\mathrm{Div}(R)$ generated by the principal divisors, then the quotient group $\mathrm{Div}(R)/\mathrm{Prin}(R)$ is the *divisor class group* of R and is denoted by $\mathrm{Cl}(R)$.

If R is a Principal Ideal Domain (abbreviated PID) then $\mathrm{Cl}(R) = 0$ so that roughly speaking $\mathrm{Cl}(R)$ measures how far a domain is from being principal and having a simple factorization theory.

1.3 KRULL DOMAINS

On this approach a KRULL domain is simply a domain in which the set of divisors Div(R) forms not merely a monoid but a free abelian group.

Let $\mathbb{Z}$ denote the ring of integers and I an index set. Then the free abelian group on the set I , $\mathbb{Z}^{(I)} = \bigoplus_{i \in I} \mathbb{Z}_i$ where $\mathbb{Z}_i = \mathbb{Z}$, is partially ordered by means of the relation : $(\alpha_i) \geqslant (\beta_i)$ if $\alpha_i \geqslant \beta_i$ for all $i \in I$ where (α_i) , $(\beta_i) \in \mathbb{Z}^{(I)}$. For (α_i) , (β_i) , $(\gamma_i) \in \mathbb{Z}^{(I)}$ with $(\alpha_i) \geqslant (\beta_i)$ implies $(\alpha_i) + (\gamma_i) \geqslant (\beta_i) + (\gamma_i)$.

The ordered group $\mathbb{Z}^{(I)}$ has the following properties :

(1) Any two elements of $\mathbb{Z}^{(I)}$ have a least upper bound and a greatest lower bound. This means $\mathbb{Z}^{(I)}$ is an ordered lattice.

(2) The positive elements of $\mathbb{Z}^{(I)}$ satisfy the minimum condition, that is, given a non-empty subset of positive elements of $\mathbb{Z}^{(I)}$, there exists a minimal element in that set.

Conversely any ordered abelian group satisfying conditions (1) and (2) is of the form $\mathbb{Z}^{(I)}$ for some indexing set I . See BOURBAKI [1952] Algebra, Chapter VI, Section 1, Number 13, Theorem 2.

Definition 5 : Let R be an integral domain. R is a KRULL domain if $\mathrm{Div}(R) \cong \mathbb{Z}^{(I)}$ for some indexing set I , the isomorphism being order-preserving.

Theorem 1 : Let R be an integral domain. Then R is a KRULL domain if and only if the following two conditions are satisfied.

(1) R is completely integrally closed.

(2) The divisorial ideals satisfy the maximum condition.

Proof : This theorem is an immediate consequence of Proposition 6 and the characterization of the ordered group $\mathbb{Z}^{(I)}$. □

Remark : If R is completely integrally closed, then it is integrally closed.

Corollary 1 : For a NOETHERIAN ring to be a KRULL domain, it is necessary and sufficient that it be an integrally closed domain.

Proof : By the remark above a KRULL domain is integrally closed. Since R Noetherian $\Rightarrow \sum_{n=1}^{\infty} Rx^n$ is finitely generated $\Rightarrow x$ integral $\Rightarrow x \in R$ (if R is integrally closed) so R is completely integrally closed. □

We now introduce some valuation theory, in particular the notion of discrete valuation. The divisor theory of a KRULL domain has associated with it a family of discrete valuations called the essential valuations. For more detail on valuation theory BOURBAKI [1964] is a good reference.

For Γ a totally ordered abelian group let Γ_∞ be the set obtained by adjoining an element ∞ to Γ and then give Γ_∞ :

(1) a total ordering for which ∞ is the greatest element, that is, $\alpha < \infty$ for all $\alpha \in \Gamma$.

(2) a commutative monoid structure which induces on Γ the given group law defined by the equations $(\infty) + (\infty) = \infty$, $\alpha + (\infty) = \infty$ for all $\alpha \in \Gamma$.

It is evident that Γ_∞ is totally ordered.

Let C be a (not necessarily commutative) ring and Γ a totally ordered abelian group written additively. A *valuation* on C with values in Γ is any mapping $v : C \to \Gamma_\infty$ which satisfies the following conditions :

(1) $v(xy) = v(x) + v(y)$ for $x,y \in C$.

(2) $v(x + y) \geqslant \inf\{v(x), v(y)\}$ for $x,y \in C$.

(3) $v(1) = 0$ and $v(0) = \infty$.

Let F be a (not necessarily commutative) field, v a valuation on F and Γ the order group of v . v is called *discrete* if there exists a (necessarily unique) isomorphism of the ordered group Γ onto $\mathbb{Z}$. Let γ be the element of Γ corresponding to 1 under this isomorphism; every element $u \in F$ such that $v(u) = \gamma$ is called a *uniformizer* (or *parameter*) of v . A discrete valuation is called *normed* if its order group is $\mathbb{Z}$.

Let $e_i = (\delta_{ij}) \in \mathbb{Z}^{(I)}$ where $i,j \in I$ and δ_{ij} is the Kronecker delta symbol. The e_i are minimal among the strictly positive elements.

Let R be a KRULL domain and let φ be the order-preserving isomorphism $\varphi: \mathrm{Div}(R) \rightarrow \mathbb{Z}^{(I)}$. Let $P_i = \varphi^{-1}(e_i)$, the *prime divisors* and let $P(R)$ be the set of prime divisors.

Any $\mathfrak{a} \in \mathrm{Div}(R)$ can be uniquely written as

$$\mathfrak{a} = \Sigma_{P \in P(R)} \; n_P \, . \, P ,$$

where $n_P \in \mathbb{Z}$ and $n_P = 0$ for almost all P .

Let $0 \neq x \in Q$ and consider the representation

$$\mathrm{div}(x) = \Sigma_{P \in P(R)} \; v_P(x) \, . \, P \quad ,$$

where $v_P(x) \in \mathbb{Z}$ and $v_P(x) = 0$ for almost all $P \in P(R)$.

Since $\mathrm{div}(xy) = \mathrm{div}(x) + \mathrm{div}(y)$ we have $v_P(xy) = v_P(x) + v_P(y)$ for all $P \in P(R)$.

Further $\mathrm{div}(x + y) \geqslant \mathrm{div}(Rx + Ry) = \inf \{\mathrm{div}(x) , \mathrm{div}(y)\}$ so that $v_P(x + y) \geqslant \inf \{v_P(x) , v_P(y)\}$.

Set $v_P(0) = \infty$.

Thus the v_P are all discrete valuations on Q and are called the *essential valuations* of R .

Let P be a prime divisor and $\bar{P}$ the divisorial ideal corresponding to it. Since P is positive $\bar{P}$ is an integral ideal. $\bar{P}$ is a prime ideal since if $x,y \in R$ such that $xy \in \bar{P}$ then $\mathrm{div}(xy) \geqslant P$, that is $\mathrm{div}(x) + \mathrm{div}(y) \geqslant P$, so that $v_P(x) + v_P(y) \geqslant 1$. Since $v_P(x)$ and $v_P(y)$ are both greater than or equal to zero, $v_P(x)$ or $v_P(y)$ are greater than or equal to 1 and $x \in \bar{P}$ or $y \in \bar{P}$.

The divisorial ideal corresponding to nP, $n \geqslant 0$ is $\{x \in R : v_P(x) \geqslant n\}$. The prime ideal $\bar{P}$ is the centre of the valuation v_P on R i.e. $\{x \in R : v_P(x) > 0\}$.

Since the prime divisors are minimal among the set of positive divisors, the corresponding divisorial ideals, the *prime divisorial ideals*, are maximal among the integral divisorial ideals.

Lemma 4 : Let q be a non-zero prime ideal. Then q contains some non-zero prime divisorial ideal.

Proof : Let $0 \neq x \in q$. Then $\mathrm{div}(x) = \sum_i n_i P_i$ (finite sum) $n_i \geqslant 0$ and $P_i \in P(R)$. Let $\bar{P}_i$ be the prime divisorial ideal corresponding to P_i. Let $0 \neq y \in \prod_i \bar{P}_i^{n_i}$ then $v_{P_i}(y) \geqslant n_i$. Hence $\mathrm{div}(y) \geqslant \mathrm{div}(x)$, that is, $Ry \subset Rx$. Thus $\prod_i \bar{P}_i^{n_i} \subset Rx \subset q$, and since q is prime, $\bar{P}_i \subset q$ for some i.

□

Recall that a prime ideal is of *height one* (abbreviated ht 1) if it is minimal among the non-zero prime ideals of R. (Similarly a prime ideal is of height two if it is minimal over a height one prime, etc.)

Notation: The set of prime ideals of ht 1 will be denoted by β .

Corollary 1 : A prime ideal is prime divisorial if and only if it is of ht 1 .

Proof : Let p be a prime divisorial ideal. If p is not of ht 1 , then $p \supsetneqq q$, where q is a non-zero prime ideal. By the above lemma q contains a prime divisorial ideal q' . Thus $p \supsetneqq q'$ contradicting the maximality of q' among integral divisorial ideals.

Conversely let p be a prime ideal of ht 1 . Then, by the above lemma, p contains a non-zero prime divisorial ideal p'. Hence $p = p'$. □

Lemma 5 : Let $\bar{P}$ be a divisorial ideal corresponding to a prime divisor P . Then the ring of quotients $R_{\bar{P}}$ is the ring of v_P.

Proof : Let $\frac{a}{s} \in R_{\bar{P}}$, $a \in R$ and $s \in R - \bar{P}$. Then $v_P(s) = 0$ and $v_P(a) \geqslant 0$, so that $v_P(\frac{a}{s}) \geqslant 0$. Conversely let $0 \neq x \in Q$ with $v_P(x) \geqslant 0$. Let $\mathrm{div}(x) = \Sigma_{S \in P(R)} n(S).S$, and let $\bar{S}$ be the prime divisorial ideal corresponding to S . Let $b = \prod_{n(S)<0} \bar{S}^{-n(S)}$. As the prime divisors S with $n(S) < 0$ are different from P , we have $b \not\subset \bar{P}$. Take $t \in b$, $t \notin \bar{P}$, then $v_S(tx) \geqslant 0$ for all S, that is, $\mathrm{div}(tx) \geqslant 0$ and $tx \in R$. Hence $x \in R_{\bar{P}}$. □

Corollary 1 : $R = \bigcap_{\bar{P} \in \beta} R_{\bar{P}}$. □

We can now give a characterization of KRULL domains making use of valuation theory. This characterization is essentially the way in which BOURBAKI [1965] and FOSSUM [1973] define KRULL domains.

Theorem 2 (Valuation Criterion) : Let R be a domain. Then R is a KRULL domain if and only if there exists a family $\{v_i : i \in I\}$ of discrete valuations on Q such that

(1) $R = \bigcap_{i \in I} R_{v_i}$, where R_{v_i} is the ring of v_i

(2) For every $x \in R$, $v_i(x) = 0$, for almost all $i \in I$.

Proof : Suppose R is a KRULL domain. Then $R = \bigcap_{P \in P(R)} R_{v_P}$ and condition (2) is obvious from the way in which v_P was defined. Conversely, since a discrete valuation ring is completely integrally closed and the intersection of completely integrally closed domains is completely integrally closed, R is completely integrally closed. Let $0 \neq x \in Q$. Then $Rx = \{y \in Q : v_i(y) \geqslant v_i(x) , \text{ for } i \in I\}$. Because of condition (2) any divisorial ideal is of the form $\{x \in Q : v_i(x) \geqslant n_i , i \in I , (n_i) \in \mathbb{Z}^{(I)}\}$, and conversely. Hence R is a KRULL domain. □

It is now clear that the preceding theorem gives another characterization of KRULL domains if one takes the valuations to be the p-adic valuations and R_p being the associated valuation rings for p the ht 1 primes.

Theorem 3 : Let R be an integral domain and β the set of its prime ideals of height 1. For R to be a KRULL domain, it is necessary and sufficient that the following properties are satisfied :

(i) For all $p \in \beta$, R_p is a discrete valuation ring.

(ii) $R = \bigcap_{p\in\beta} R_p$

(iii) For all $0 \neq x \in R$, there exists only a finite number of ideals $p \in \beta$ such that $x \in p$.

Moreover, the valuations corresponding to the R_p for $p \in \beta$ are the essential valuations of R . □

CHAPTER TWO

INJECTIVE MODULES

2.1 INTRODUCTION

In this chapter we will give an account of injective modules in general. The classical results in this area are mainly due to REINHOLD BAER [1940], B. ECKMANN and A. SCHOPF [1953]. These results are to be found in various texts such as SAUNDERS MACLANE [1963], HENRI CARTAN and SAMUEL EILENBERG [1956] and D.W. SHARPE and P. VAMOS [1972]. We then look at injective modules over NOETHERIAN rings, first studied in EBEN MATLIS [1958]. Finally we look at injective modules over KRULL domains as in ISTVAN BECK [1971]. BECK showed that, even if the KRULL domain is not NOETHERIAN, there is an analogue of the NOETHERIAN result that the direct sums of injectives are injective. For a KRULL domain the direct sum of co-divisorial (or N-torsion free) injectives is injective. We also show that the set of almost-null modules (when finitely generated BOURBAKI calls these modules pseudo-zero) and the co-divisorial modules form a torsion theory.

2.2 GENERALITIES ABOUT INJECTIVE MODULES

Definition 1 : A module E is injective if, for every module M and every submodule N of M , every f : N → E can be extended to a map g : M → E so that the following diagram is commutative

$$\begin{tikzcd}
N \arrow[r, tail] \arrow[dr, "f"'] & M \arrow[d, dashed, "g"] \\
 & E
\end{tikzcd}$$

Proposition 1 : A module E is injective if and only if the functor $\mathrm{Hom}(-,E)$ is exact.

Proof : For every module M and every submodule N of M apply the functor $\mathrm{Hom}(-,E)$ to the short exact sequence $N \rightarrowtail M \twoheadrightarrow M/N$ to get $\mathrm{Hom}(M/N,E) \to \mathrm{Hom}(M,E) \to \mathrm{Hom}(N,E)$. Since the induced maps are defined by composition, left exactness is clear for arbitrary E . That the second induced map is onto is just the lifting condition for E . □

A short exact sequence $N \overset{i}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} L$ is *split* if there is a map $j : L \to M$ with $pj = 1_L$ or, equivalently, if there is a map $q : M \to N$ with $qi = 1_N$.

It is not difficult, using the biproduct diagram (see MACLANE [1963] pg 15), to show that if the above short exact sequence is split then $M \cong N \oplus L$.

Given a pair of maps

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

the *pushout* is a module L and maps α, β such that the following diagram is commutative

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\
C \arrow[r, "\alpha"'] & L
\end{tikzcd}$$

and universal: given any other pair of maps α' , β' and module D making the diagram commute, there exists a unique φ making the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"'] \arrow[ddr, "\beta'"] & \\
C \arrow[r, "\alpha"] \arrow[drr, "\alpha'"'] & L \arrow[dr, dashed, "\varphi"] & \\
 & & D
\end{tikzcd}$$

Proposition 2 : A module E is injective if and only if every short exact sequence $E \rightarrowtail B \twoheadrightarrow C$ splits. In particular E is a summand of B .

Proof : If E is injective then from the diagram

$$\begin{tikzcd}
E \arrow[rr, tail, "i"] \arrow[dr, "1_E"'] & & B \arrow[dl, dashed, "g"] \\
 & E &
\end{tikzcd}$$

there exists a map $g : B \to E$ with $gi = 1_E$, so that the sequence splits.

Conversely, if we have the diagram

$$\begin{tikzcd}
N \arrow[r, tail, "\alpha"] \arrow[d, "f"'] & M \\
E &
\end{tikzcd}$$

we can construct the pushout

$$\begin{tikzcd}
N \arrow[r, tail, "\alpha"] \arrow[d, "f"'] & M \arrow[d, "f'"] \\
E \arrow[r, "\alpha'"'] & P
\end{tikzcd} .$$

Since α is monic, α' is also monic and so by assumption there is a map $\beta : P \to E$ with $\beta\alpha' = 1_E$. Then $\beta f' : M \to E$ and $(\beta f')\alpha = \beta(f'\alpha) = \beta\alpha' f = f$ so that E is injective. □

Proposition 3 (Baer Criterion) : An R-module E is injective if and only if every map $f : I \to E$, where I is a left R-ideal, can be extended to R .

Proof : The condition is clearly necessary.

Conversely, consider the diagram where i is the inclusion map

$$\begin{tikzcd}
N \arrow[r, tail, "i"] \arrow[dr, "f"'] & M \\
& E
\end{tikzcd}$$

Let F be the set of all pairs (N',g') where $N \subset N' \subset M$ and $g' : N' \to E$ extends f . Since $(N,f) \in F$, $F \neq \emptyset$. Partially order F : $(N',g') \leqslant (N'',g'')$ if $N' \subset N''$ and g'' extends g' . By ZORN's lemma there is a maximal pair (N_0,g_0) in F . If $N_0 = M$ we are done. So suppose $N_0 \neq M$ and $x \in M \setminus N_0$. Let $I = \{r \in R : rx \in N_0\}$. Then I is a left ideal of R . Define $h : I \to E$ by $h(r) = g_0(rx)$. By hypothesis there is a map $h' : R \to E$ extending h . Define $N_1 = N_0 + Rx$ and $g_1 : N_1 \to E$ by $n_0 + rx \longmapsto g_0 n_0 + rh'(1)$, $r \in R$. g_1 is well-defined and extends g_0 so that $(N_1,g_1) \in F$ and $(N_0,g_0) \leqslant (N_1,g_1)$ contradicting (N_0,g_0) being maximal. Therefore $N_0 = M$ and E is injective. □

Proposition 4 : The direct product of a family of injective modules is injective.

Proof : Since $\mathrm{Hom}(-,\Pi E_i) \cong \Pi\,\mathrm{Hom}(-,E_i)$ □

Proposition 5 : Every summand of an injective module is itself injective.

Proof : Every lifting to a containing module can be pushed down to a direct summand. □

Lemma 1 : The diagram with exact row.

$$\begin{tikzcd}
L \arrow[r, tail, "\alpha"] \arrow[d, "\gamma"'] & N \arrow[r, two heads, "\beta"] & M \\
E & &
\end{tikzcd}$$

can be completed to a commutative diagram with exact rows.

$$\begin{tikzcd}
L \arrow[r, tail, "\alpha"] \arrow[d, "\gamma"'] & N \arrow[r, two heads, "\beta"] \arrow[d, "\gamma'"'] & M \arrow[d, equal] \\
E \arrow[r, tail, "\alpha'"'] & P \arrow[r, two heads, "\beta'"'] & M
\end{tikzcd}$$

in which the first square is a pushout.

Proof : Form the pushout $P = (E \oplus N)/W$, where $W = \{(\gamma a, -\alpha a) : a \in L\}$, $\gamma' : b \longmapsto (0,b) + W$ and $\alpha' : e \longmapsto (e,0) + W$. Define $\beta' : P \to M$ by $(e,b) + W \longmapsto \beta b$. β' is well defined, the diagram commutes and the bottom row is exact. □

Proposition 6 : A module E is injective if and only if every short exact sequence $E \rightarrowtail N \twoheadrightarrow C$, with C cyclic, splits.

Proof: If E is injective the sequence splits for any module C. Conversely, consider the following diagram with I a left R-ideal and $f : I \to E$ an R-map.

$$\begin{tikzcd}
I \arrow[r, tail, "i"] \arrow[d, "f"'] & R \arrow[r, two heads] \arrow[d, "f'"'] & R/I \arrow[d, equal] \\
E \arrow[r, tail, "\alpha'"'] & P \arrow[r, two heads] & R/I
\end{tikzcd}$$

Since R/I is cyclic the bottom row splits, that is, there is a map $\beta : P \to E$ with $\beta\alpha' = 1_E$. Since $\beta f' : R \to E$, such that $\beta f' i = f$, E is injective. □

We now recall the concept of divisible module.

An element $r \in R$ is called a *zero-divisor* if there is a $0 \neq s \in R$ such that $rs = 0$. An element $r \in R$ which is not a zero-divisor is called a *non-zero divisor* or a *regular* element. An ideal of R is called *regular* if it contains a regular element.

Definition 2 : Let M be an R-module, $m \in M$ and $r \in R$. Then the element m is *divisible* by r if $m = rm'$ for some $m' \in M$. The module M is termed divisible if every $m \in M$ is divisible by every non-zero divisor $r \in R$.

Example : The quotient field Q of an integral domain R is divisible.

An R-module D is divisible if it has the lifting property for

$$\begin{tikzcd} Rr \arrow[r, tail] \arrow[dr, "f"'] & R \\ & D \end{tikzcd}$$

where r is a non-zero divisor of R .

Alternatively, D is divisible if multiplication by a non-zero divisor, $D \xrightarrow{.r} D$, is onto.

In the case of an integral domain we can simplify the definition of divisible by replacing the phrase non-zero divisor by non-zero element.

Proposition 7 : Every injective module E is divisible.

Proof : Let $e \in E$ and $r \in R$ a non-zero divisor. Consider the diagram

$$\begin{tikzcd} Rr \arrow[r, tail] \arrow[dr, "f"'] & R \\ & E \end{tikzcd}$$

where $f : Rr \to E$ is defined by $f(sr) = se$, $s \in R$. f is well defined since r is not a zero-divisor. Since E is injective there exists $g : R \to E$ extending f .

Thus $e = f(r) = g(r) = rg(1)$ and so E is divisible. □

Lemma 2 : Every quotient of a divisible module is divisible.

Proof : Let N be a submodule of a divisible R-module D and $r \in R$ a non-zero divisor. From the diagram

$$\begin{tikzcd} D \arrow[r, two heads, ".r"] \arrow[d, two heads] & D \arrow[d, two heads] \\ D/N \arrow[r, ".r"'] & D/N \end{tikzcd}$$

the bottom map is necessarily onto. □

Lemma 3 : Every summand of a divisible module is divisible.

Proof : Suppose D is a divisible R-module and S a summand of D. If $r \in R$ is a non-zero divisor and $s \in S$ then there exists $s' \in D$ such that $s = rs'$, since D is divisible. Since S is a summand of D, $s' \in S$. □

Lemma 4 : Let $\{E_i : i \in I\}$ be a family of divisible R-modules. Then $\prod_{i \in I} E_i$ and $\bigoplus_{i \in I} E_i$ are divisible R-modules.

Proof : Let $\{e_i\} \in \prod_{i \in I} E_i$, where $e_i \in E_i$, and let $r \in R$ be a non-zero divisor. Then for each $i \in I$ there exists $e_i' \in E_i$ such that $e_i = re_i'$. Then $\{e_i\} = r\{e_i'\}$, so that $\prod_{i \in I} E_i$ is divisible. If $\{e_i\} \in \bigoplus_{i \in I} E_i$, then we can arrange for $\{e_i'\}$ to belong to $\bigoplus_{i \in I} E_i$ by insisting $e_i' = 0$ if $e_i = 0$. □

Proposition 8 : Let R be a commutative domain and E a torsion-free divisible R-module. Then E is injective.

Proof : Consider the following diagram with I an R-ideal and $f : I \to E$ an R-homomorphism.

$$\begin{tikzcd}
I \arrow[r, tail] \arrow[dr, "f"'] & R \arrow[d, dashed, "g"] \\
 & E
\end{tikzcd}$$

If $I = (0)$ the diagram can be completed. So assume $I \neq (0)$. Consider a $0 \neq s \in I$. Because E is divisible there exists $e \in E$ such that $f(s) = se$. Define the R-homomorphism $g : R \to E$ by $g(r) = re$ $(r \in R)$. Then if $t \in R$ $g(ts) = tse = tf(s) = f(ts)$ so that g agrees with f on I. □

Example : The quotient field Q of an integral domain R is injective since it is torsion-free and divisible.

An R-module E is said to be an essential extension of the R-module M if

(i) M is a submodule of E and

(ii) for every non-zero submodule S of E, $S \cap M \neq 0$.

This is equivalent to the condition that, for every $0 \neq e \in E$ there exists an $r \in R$ such that $0 \neq re \in M$.

Proposition 9 : A module is injective if and only if it has no proper essential extensions.

Proof : Suppose M is injective and E is a proper essential extension of M. Then M is a direct summand of E, that is, there is a submodule F of E so that $E = M + F$ and $M \cap F = 0$, contradicting the essentiality of E.

Conversely, suppose M has no proper essential extensions, and E an injective module containing M. The existence of E is guaranteed since every module is a submodule of an injective module - MACLANE [1963] page 93, Theorem 7.4. By ZORN's lemma, there is a submodule N of E which is maximal and with $M \cap N = 0$. The composite map $M \hookrightarrow E \rightarrow E/N$ is monic since $M \cap N = 0$. E/N is essential over M: if S/N is a non-zero submodule of E/N, then $S \supsetneqq N$ and maximality of N forces $S \cap M \neq 0$. By hypothesis $M \cong E/N$, hence $E = M + N$ and since $M \cap N = 0$, M is a summand of E and so M is injective. □

Proposition 10 : The following conditions on a module E containing a module M are equivalent

(1) E is a maximal essential extension of M.

(2) E is an essential extension of M and E is injective.

(3) E is injective and there is no injective E' with $M \subset E' \subsetneqq E$.

Moreover for any R-module M such a module E exists and is unique up to isomorphism.

Proof (ECKMANN and SCHÖPF) : (1) ⇒ (2) : Assume (1). Then E has no essential extension and so is injective by Proposition 9.

(2) ⇒ (3) : Assume (2) and suppose $M \subset E' \subset E$, E' injective. Then E' is a summand of E. But E is an essential extension of E', hence $E' = E$.

(3) ⇒ (1) : Assume (3) and let E' be a maximal essential extension of M in E. Then E' is injective and so equal to E. □

The module E in the above theorem is called the *injective envelope* or the *injective hull* of M and is usually denoted by E(M) or I(M) .

Definition 3 : An R-module M is said to be *indecomposable,* if its only direct summands are 0 and M .

Proposition 11 : Let M be an R-module and E = E(M). Then the following statements are equivalent :

(1) E(M) is an injective envelope of every one of its non-zero submodules.

(2) M contains no non-zero submodules S and T such that $S \cap T = 0$.

(3) E(M) is indecomposable.

Proof : Assume (1) and suppose that S and T are submodules of M such that $S \cap T = 0$. Suppose that $S \neq 0$. Then E = E(S) and T = 0 .

Assume (2). Suppose that there exist non-zero submodules S, T of E such that $E = S \oplus T$. Then $S \cap T = 0$, contradicting the assumption.

Assume (3). Let N be a non-zero submodule of E . Then E has a submodule E' which is an injective envelope of N , and so E' is a direct summand of E . Since $E' \neq 0$ it follows that E' = E . □

Let I be a left R-ideal such that $I = J_1 \cap J_2 \cap \ldots \cap J_n$, where the J_i are left R-ideals. We call this a *decomposition* of I and say the decomposition is *irredundant*, if no J_i contains the intersection of the others.

Definition 4 : Let I be a left R-ideal. Then I is said to be irreducible if for left R-ideals K and L $I = K \cap L$ implies $I = K$ or $I = L$.

Notation :

Let S be a subset of an R-module M. Then the annihilator or order ideal of S, denoted by $O(S)$, is given by $\{r \in R : rS = 0\}$. $O(S)$ is clearly a left R-ideal.

Theorem 1 (MATLIS) : An R-module E is an indecomposable injective module if and only if $E \cong E(R/J)$, where J is an irreducible left R-ideal. In this case, for every $0 \neq x \in E$, $O(x)$ is an irreducible left ideal and $E \cong E(R/O(x))$.

Proof : Let J be an irreducible left R-ideal, and K, L left R-ideals such that $K/J \cap L/J = 0$ where K/J and L/J are considered as submodules of $E(R/J)$. Then $K \cap L = J$, and since J is irreducible either $K = J$ or $L = J$. Hence $E(R/J)$ is indecomposable. Conversely, let E be an indecomposable injective module, $0 \neq x \in E$ and $J = O(x)$, so that the cyclic module Rx is $R/O(x) = R/J$. Now $E \cong E(R/J)$. Suppose that $J = K \cap L$ is an irredundant decomposition of J by left ideals K, L. Imbed R/J in $E(R/K) \oplus E(R/L)$ and let D be an injective envelope of R/J in the direct sum. Due to the irredundancy of the decomposition of J, $R/J \cap R/K \neq 0$. Therefore D projects monomorphically into $E(R/K)$. The image of D is an injective module containing R/K, and hence is equal to $E(R/K)$. Thus $E(R/K)$ is indecomposable; similarly, $E(R/L)$ is indecomposable. Thus $E(R/J) \cong E(R/K) \oplus E(R/L)$. This contradicts the indecomposability of $E(R/J)$, and thus J is irreducible. □

Proposition 12 : Let M be an R-module and $0 \neq a \in M$. Then there is a simple module S and a homomorphism $\varphi : M \to E(S)$ such that $\varphi(a) \neq 0$.

Proof : Consider the proper left ideal $O(a)$. The $O(a)$ is contained in a maximal left ideal M of R. Define a mapping $\varphi' : Ra \to R/M$ by $\varphi'(ra) = r + M$. φ' is well-defined, because if $ra = 0$ then $r \in O(a) \subseteq M$ and $r + M = 0_{R/M}$ where $0_{R/M}$ is the zero of R/M. Further φ' is an R-homomorphism and $\varphi'(a) = 1 + M \neq 0_{R/M}$. Put $S = R/M$, so that R/M is a simple R-module, and consider the diagram

$$\begin{tikzcd}
Ra \arrow[r, tail, "\text{inc.}"] \arrow[d, "\varphi'"'] & M \arrow[ddl, dashed, "\varphi"] \\
S \arrow[d, "\text{inc}"'] & \\
E(S) &
\end{tikzcd}$$

Since $E(S)$ is injective, there exists a homomorphism $\varphi : M \to E(S)$ such that $\varphi(a) = \varphi'(a) \neq 0$. □

Definition 5 : An R-module E is said to be an *injective cogenerator* of R if

(i) E is injective, and

(ii) for every R-module M and every $0 \neq a \in M$ there is an R-homomorphism $\varphi : M \to E$ such that $\varphi(a) \neq 0$.

(So homomorphisms to E *separate points.*)

Remark : This definition is justified by the next proposition which is the dual of the representation of an arbitrary module M as a quotient of a sum of copies of the generator R :

Proposition 13 : Let E be an injective cogenerator of R. Then every R-module can be embedded in a direct product of copies of E.

Proof : It is sufficient to consider a non-zero R-module M. Let $0 \neq a \in M$ then there is an R-homomorphism $\varphi_a : M \to E$ such that $\varphi_a(a) \neq 0$. Define $\varphi : M \to \prod_{0 \neq a \in M} E$ by $\varphi(x) = \{\varphi_a(x)\}$ for $x \in M$. Then φ is a homomorphism. $\square$

Corollary 1 : Every R-module can be embedded in a direct product of injective envelopes of simple modules.

Proof : An injective co-generator for Mod-R can be taken to be $\prod E(S)$, S simple by Proposition 12 above. $\square$

Theorem 2 : The following statements are equivalent :

(a) R is a Noetherian ring.

(b) every direct sum of injective R-modules is injective

(c) every direct sum of a countably infinite family of injective envelopes of simple R-modules is injective.

Proof : (a) ⇒ (b) Assume R is a Noetherian ring. Let $\{E_i : i \in I\}$ be a family of injective R-modules, and put $E = \bigoplus_{i \in I} E_i$. Consider the following diagram where $\mathfrak{b}$ is a left ideal of R .

$$\begin{tikzcd}
\mathfrak{b} \arrow[r, tail] \arrow[d, "f"'] & R \arrow[dl, dashed, "\varphi'"'] \arrow[ddl, dashed, "\varphi"] \\
E' \arrow[d, "inc"'] & \\
E &
\end{tikzcd}$$

Since R is Noetherian $\mathfrak{b}$ is finitely generated and so there exists a finite subset J of I such that $f(\mathfrak{b}) \subseteq E' = \bigoplus_{j \in J} E_j$. Now E' is injective so there exists an R-homomorphism $\varphi' : R \to E'$ such that the resulting diagram is commutative. We complete the diagram by $\varphi : R \to E$ which is φ' followed by the inclusion mapping. Hence E is injective.

(b) ⇒ (c) Is obvious.

(c) ⇒ (a) Assume (c) and the existence of an infinite strictly increasing sequence $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_n \subset \ldots$ of left ideals of R . Put $\mathfrak{a} = \bigcup_{n=1}^{\infty} \mathfrak{a}_n$. Then $\mathfrak{a}$ is a left ideal of R and, for each k , $\mathfrak{a}/\mathfrak{a}_k \neq 0$. Proposition 12 shows that there exists a simple module S_k and a non-zero homomorphism $\alpha_k : \mathfrak{a}/\mathfrak{a}_k \to E(S_k)$. Denote by $\varphi_k : \mathfrak{a} \to E(S_k)$ the mapping obtained by composing the natural mapping $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_k$ with α_k . Then φ_k is non-zero.

Now we define the mapping $\varphi : \mathfrak{a} \to \bigoplus_{k=1}^{\infty} E(S_k)$ by $\varphi(r) = \{\varphi_k(r)\}_{k=1}^{\infty}$, $r \in \mathfrak{a}$. Note, for any $r \in \mathfrak{a}$, there exists k_0 such that $r \in \mathfrak{a}_k$ for all $k \geqslant k_0$, so that $\varphi_k(r) = 0$ for all $k \geqslant k_0$ and φ is a well defined R-homomorphism. By hypothesis, $\bigoplus_{k=1}^{\infty} E(S_k)$ is injective, so φ can be extended to a homomorphism $\psi : R \to \bigoplus_{k=1}^{\infty} E(S_k)$. But R is singly generated as an R-module, so there exists an integer n such that $\psi(R) \subseteq \bigoplus_{k=1}^{n} E(S_k)$. This means that φ_k is the zero mapping for all $k > n$, which is a contradiction. □

Theorem 3 : Let R be a left Noetherian ring. Then every injective R-module has a decomposition as a direct sum of indecomposable, injective submodules.

Proof : Let M be an injective R-module. By ZORN's lemma we can find a submodule C of M which is maximal with respect to the property of being a direct sum of indecomposable, injective submodules. Suppose $C \neq M$. Then C is injective and there is a non-zero submodule D of M such that $M = C \oplus D$. Let $0 \neq x \in D$. Then, by Lemma 5, $O(x)$ is an intersection of a finite number of irreducible left ideals. Therefore $E(R/O(x))$ is a direct sum of a finite number of indecomposable injective R-modules. Since $Rx \cong R/O(x)$ we can consider $E(R/O(x))$ embedded in D. But then $C \oplus E(R/O(x))$ contradicts the maximality of C, and thus $C = M$. □

An ideal $\mathfrak{a}$ is called a *primary ideal* if $ab \in \mathfrak{a}$, $a \notin \mathfrak{a} \Rightarrow \exists\ n$ such that $b^n \in \mathfrak{a}$.

EMMY NOETHER pointed out that with every primary ideal $\mathfrak{a}$ there is associated a unique prime ideal $P = \sqrt{\mathfrak{a}} = \{x : x^n \in \mathfrak{a}$ for some positive $n\}$. We then say that $\mathfrak{a}$ is P-primary.

From the definitions it follows that if $\mathfrak{a}$ is P-primary then

(a) $ab \in \mathfrak{a}$, and $a \notin P \Rightarrow b \in \mathfrak{a}$.

(b) $IJ \subseteq \mathfrak{a}$ and $I \not\subseteq P \Rightarrow J \subseteq \mathfrak{a}$.

(c) $I \not\subseteq P \Rightarrow \mathfrak{a} :_R I = \mathfrak{a}$.

EMMY NOETHER showed that there are P-primary ideals that are not a power of P, e.g. for the polynomial ring $F[x,y]$ over the field F it is easy to see that (x,y^2) is an (x,y)-primary ideal which is not a power of (x,y). NORTHCOTT [1953] gives an example to show that there are prime powers that are not primary. He used the ring $F[x,y,z]$ where F is a field to show that for the prime ideal $P = \{f(x,y,z) : f(t^3,t^4,t^5) = 0$ as a polynomial in $t\}$ P^2 is not P-primary.

Lemma 6 (EMMY NOETHER) : If R is a Noetherian ring, then every irreducible ideal is primary.

Proof : We will prove the contra-positive. Suppose that $\mathfrak{a}$ is a non-primary ideal in R. Then there exist elements b, c such that $bc \in \mathfrak{a}$, $c \notin \mathfrak{a}$, and no power of b is in $\mathfrak{a}$. From $bc \in \mathfrak{a}$ and $c \notin \mathfrak{a}$, it follows that $\mathfrak{a} \subsetneqq \mathfrak{a} :_R (b)$. Now for every $r \geq 0$, $\mathfrak{a} :_R (b^r) \subseteq [\mathfrak{a} :_R (b^r)] :_R (b) = \mathfrak{a} :_R (b^{r+1})$ hence $\mathfrak{a} \subset \mathfrak{a} :_R (b) \subseteq \mathfrak{a} :_R (b^2) \subseteq \mathfrak{a} :_R (b^3) \subseteq \ldots$. By the chain condition there exists an integer m such that $\mathfrak{a} :_R (b^n) = \mathfrak{a} :_R (b^m)$, for all $n > m$. We claim that $\mathfrak{a}$ is reducible and that $\mathfrak{a} = [\mathfrak{a} :_R (b^m)] \cap [\mathfrak{a} + (b^m)]$.

By construction, both $\mathfrak{a} :_R (b^m)$ and $\mathfrak{a} + (b^m)$ strictly contain $\mathfrak{a}$.

Let $x \in [\mathfrak{a} :_R (b^m)] \cap [\mathfrak{a} + (b^m)]$. Then $x \in \mathfrak{a} + (b^m)$ and so $x = a + rb^m$, where $a \in \mathfrak{a}$ and $r \in R$. We also have $x \in \mathfrak{a} :_R (b^m)$; accordingly $xb^m = ab^m + rb^{2m}$ belongs to $\mathfrak{a}$, which shows that $rb^{2m} \in \mathfrak{a}$, and therefore that $r \in \mathfrak{a} :_R (b^{2m})$. By the choice of m, $\mathfrak{a} :_R (b^{2m}) = \mathfrak{a} :_R (b^m)$, hence $r \in \mathfrak{a} :_R (b^m)$ and $rb^m \in \mathfrak{a}$. Thus $x = a + rb^m \in \mathfrak{a}$. □

Proposition 15 (MATLIS) : Let R be a Noetherian ring. Then there is a one-to-one correspondence between the prime ideals of R and the indecomposable R-modules given by $P \to E(R/P)$, where P is a prime ideal of R. If $\mathfrak{a}$ is an irreducible ideal, then $E(R/\mathfrak{a}) \cong E(R/P)$ (where $\mathfrak{a}$ is P-primary).

Proof : Since a prime ideal is irreducible $E(R/P)$ is an indecomposable injective module by Theorem 1. Let P_1, P_2 be two prime ideals of R such that $E(R/P_1) \cong E(R/P_2)$. If we consider R/P_1 and R/P_2 imbedded in $E(R/P_1)$ then $R/P_1 \cap R/P_2 \neq 0$. However, every non-zero element of R/P_1 (resp. R/P_2) has order ideal P_1 (resp. P_2). Thus $P_1 = P_2$ and the mapping $P \to E(R/P)$ is 1-1. Let E be an indecomposable, injective R-module. Then there is an irreducible R-ideal $\mathfrak{a}$ such that $E \cong E(R/\mathfrak{a})$. Now $\mathfrak{a}$ is a P-primary ideal for a unique associated prime ideal P. If $\mathfrak{a} = P$ we are done; hence assume $\mathfrak{a} \neq P$. Then there is a smallest integer $n > 1$ such that $P^n \subset \mathfrak{a}$. Take $b \in P^{n-1}$ such that $b \notin \mathfrak{a}$ and denote the image of b in $R/\mathfrak{a}$ by $\bar{b}$. Clearly $O(\bar{b}) \supset P$; on the other hand if $a \in O(\bar{b})$, then $ab \in \mathfrak{a}$, and so $a \in P$, showing that $O(\bar{b}) = P$. Therefore, there is an element of $E(R/\mathfrak{a})$ with order ideal P , and thus $E(R/\mathfrak{a}) \cong E(R/P)$. □

Lemma 7 : Let P be a prime ideal in a Noetherian ring R and $E = E(R/P)$. Then :

(1) $\mathfrak{a}$ is an irreducible, P-primary ideal if and only if there is an $0 \neq x \in E$ such that $O(x) = \mathfrak{a}$.

(2) If $r \in R \smallsetminus P$, then $O(rx) = O(x)$ for all $x \in E$, and the homomorphism : $E \to E$ defined by $x \mapsto rx$ is an automorphism of E .

Proof : (1) This is an immediate consequence of Theorem 1 and Proposition 15.

(2) Let $r \in R \smallsetminus P$; the map : $E \to E$ defined by $x \mapsto rx$, for $x \in E$ has kernel 0 by (1); therefore, it is an automorphism. It follows that $O(x) = O(rx)$ for every $x \in E$. □

The i-th symbolic prime power, $P^{(i)}$, of the prime ideal P is given by $P^{(i)} = \{x \in R : \exists\, c \in R \smallsetminus P \text{ with } cx \in P^i\}$. Alternatively it is the pull-back of the localization of P^i : $P^{(i)} = R \cap (P^i R_P)$.

The following Proposition due to E. MATLIS shows that for R Noetherian, indecomposable injectives are essentially generalizations of the PRÜFER groups from p-torsion abelian group theory.

Theorem 4 : Let P be a prime ideal in a Noetherian ring R, $E = E(R/P)$, and $A_i = 0 :_E P^i$. Then :

(1) A_i is a submodule of E, $A_i \subset A_{i+1}$, and $E = \bigcup A_i$.

(2) $O(A_i) = \bigcap_{x \in A_i} O(x) = P^{(i)}$.

(3) The non-zero elements of A_{i+1}/A_i form the set of elements of E/A_i having order ideal P.

(4) Let K be the quotient field of R/P. Then A_{i+1}/A_i is a vector space over K, and $A_1 \cong K$.

Proof : (1) Clearly A_i is a submodule of E, and $A_i \subset A_{i+1}$. Let $0 \neq x \in E$; then by Lemma 7(1) $O(x)$ is a P-primary ideal. Thus there exists a positive integer i such that $P^i \subset O(x)$, and so $x \in A_i$. Therefore $E = \bigcup A_i$.

(2) By the Lemma 7(1) $\bigcap_{x \in A_i} O(x)$ is the intersection of all irreducible, P-primary ideals containing P^i. The intersection of P-primary ideals is again P-primary - NORTHCOTT [1953], page 12, Proposition 4. $\bigcap_{x \in A_i} O(x)$ is the smallest P-primary ideal containing P^i, hence equal to $P^{(i)}$.

(3) Since $PA_{i+1} \subset A_i$, it follows from Lemma 7(2) that every non-zero element of A_{i+1}/A_i has order ideal P. Conversely, if $x \in E$ is an element such that $Px \subset A_i$, then $x \in A_{i+1}$.

(4) If $r \in R$, denote its image in R/P by $\bar{r}$. Similarly, if $x \in A_{i+1}$, denote its image in A_{i+1}/A_i by $\bar{x}$. If $s \in R \setminus P$, then by Lemma 7(2) there exists a unique $y \in A_{i+1}$ such that $x = sy$. Define an operation of K on A_{i+1}/A_i by $(\bar{r}/\bar{s}) . \bar{x} = \overline{ry}$. With this definition A_{i+1}/A_i becomes a vector space over K.

Take $0 \neq x \in A_1$. Since $A_0 = 0$, A_1 is a vector space over K ; and so we can define a K-monomorphism $g : K \to A_1$ by $g(\bar{r}/\bar{s}) = (\bar{r}/\bar{s}) . x$, for $\bar{r}/\bar{s} \in K$. Let $0 \neq z \in A_1$. Since E is an essential extension of A_1 , there exist $t,w \in R \setminus P$ such that $tx = wz$. Thus $g(t/w) = z$, and g is an isomorphism. □

Corollary 1 : Let M be a maximal ideal in a Noetherian ring R , $E = E(R/M)$ and $A_i = 0 :_E P^i$. Then

$$0(A_i) = M^i \quad \text{and} \quad 0(E) = \bigcap_{i=1}^{\infty} M^i = 0 .$$

Proof : Since M is a maximal ideal $M^i = M^{(i)}$ and so $0(A_i) = M^i$. Now $\bigcap_{i=1}^{\infty} M^i = 0[E(R/M)]$ and this is zero by Corollary 1 to Proposition 14. □

2.4 TORSION AND TORSION-FREE MODULES OVER NOETHERIAN KRULL DOMAINS

BOURBAKI [1965] defined the notion of pseudo-zero modules for finitely generated modules over Noetherian KRULL domains. We give his approach, introduce the notion of torsion theory and show that the study of finitely generated modules over Noetherian KRULL domains reduces to the study of torsion and torsion-free modules.

Proposition 16 : Let A be a Noetherian KRULL domain. Let M be a finitely generated A-module. The following conditions are equivalent :

(a) $M_p = 0$ for every prime ideal p of height ≤ 1 .

(b) The annihilator $\mathfrak{a}$ of M is an ideal $\neq (0)$ and $A : \mathfrak{a} = A$.

Proof : $M_p = 0 \iff \mathfrak{a} \not\subset p \iff \mathfrak{a}A_p = A_p$; on the other hand, for every integral ideal $\mathfrak{b} \neq 0$ of A , $\mathfrak{b}A_p = A_p$ for all $p \in \beta$ is equivalent to $\operatorname{div} \mathfrak{b} = \operatorname{div} A = 0$ in Div(A) , or also to $\operatorname{div}(A : \mathfrak{b}) = 0$ and, as $A : \mathfrak{b}$ is divisorial, this relation is also equivalent to $A : \mathfrak{b} = A$. Noting that $\mathfrak{a} \not\subset p$ for $p = (0)$ we see that $\mathfrak{a} \neq (0)$. □

The equivalent conditions of this Proposition also implies that Ass (M) contains no prime ideal of height $\leqslant 1$.

Definition 6 : An A-module M is called *pseudo-zero* if it is finitely generated and it satisfies the equivalent conditions of Proposition 16.

From the definition and Proposition 16 we can see that a pseudo-zero A-module is a torsion A-module.

Definition 7 : Let M and N be two A-modules and $f : M \to N$ a homomorphism. f is called *pseudo-injective* (resp. *pseudo-surjective, pseudo-zero*) if Ker (f) (resp. coker (f) , Im (f)) is pseudo-zero; f is called *pseudo-bijective* if it is both pseudo-injective and pseudo-surjective.

A pseudo-bijective homomorphism is also called a *pseudo-isomorphism* Suppose M and N are finitely generated; then, for $f : M \to N$ to be pseudo-injective (resp. pseudo-surjective, pseudo-zero), it is necessary and sufficient that, for all $p \in \beta \cup \{(0)\}$, $f_p : M_p \to N_p$ be injective (resp. surjective, zero) ; this follows from the flatness of the A-module A_p .

Lemma 8 : Let A be Noetherian KRULL domain.

Let $\{p_i\}_{1 \leqslant i \leqslant k}$ be a non-emptyfinite family of prime ideals of ht 1 and let $S = \bigcap_{i=1}^{k} (A \setminus p_i)$; then the ring $S^{-1}A$ is a principal ideal domain.

Proof : $S^{-1}A$ is a semi-local ring whose maximal ideals are the $m_i = p_i S^{-1}A$ for $1 \leqslant i \leqslant k$, the local ring $(S^{-1}A)_{m_i}$ being isomorphic to A_{p_i} and hence a discrete valuation ring. The ring $S^{-1}A$ is therefore a DEDEKIND domain and, as it is semi-local it is a principal ideal domain. □

Lemma 9 : Let A be a Noetherian KRULL domain and let E be a finitely generated A-module, T the torsion submodule of E and $M = E/T$. There exists a homomorphism $g : E \to T$ whose restriction to T is both a homothety and a pseudo-isomorphism.

Proof : $O(T) \neq 0$ since T is a finitely generated torsion A-module. Let $p_i (1 \leqslant i \leqslant k)$ be the prime ideals of ht 1 containing $O(T)$; if this number is zero, T is pseudo-zero and we may take $g = 0$. Otherwise, let $S = \bigcap_{i=1}^{k} (A \setminus p_i)$; then $S^{-1}A$ is a principal ideal domain and hence $S^{-1}M$, which is a torsion-free finitely generated $S^{-1}A$-module, is free and, as $S^{-1}M = (S^{-1}E)/(S^{-1}T)$, $S^{-1}T$ is a direct factor of $S^{-1}E$. Now $\mathrm{Hom}_{S^{-1}A}(S^{-1}E, S^{-1}T) = S^{-1}\mathrm{Hom}_A(E,T)$; hence there exist $s_0 \in S$ and $g_0 \in \mathrm{Hom}_A(E,T)$ such that $s_0^{-1} g_0$ projects $S^{-1}E$ onto $S^{-1}T$. If $h_0 \in \mathrm{Hom}_A(T,T)$ denotes the restriction of g_0 to T , there therefore exists $s_1 \in S$ such that $s_1 h_0(x) = s_1 s_0 x$ for all $x \in T$; writing $s = s_1 s_0$, $g = s_1 g_0$; $h = s_1 h_0$, h is therefore the homothety of ratio s on T and is the restriction of g to T .

h is a pseudo-isomorphism : If $p = 0$ or if $p \in \beta$ is distinct from the p_i $(1 \leqslant i \leqslant k)$, $T_p = 0$ and $h_p : T_p \to T_p$ is an isomorphism; if on the other hand p is equal to one of the p_i $(1 \leqslant i \leqslant k)$, s is invertible in A_{p_i} and h_{p_i} , the homothety of ratio s on T_{p_i} , is also an isomorphism. □

Theorem 5 : Let A be a Noetherian KRULL domain. Let E be a finitely generated A-module, T the torsion submodule of E and $M = E/T$. There exists a pseudo-isomorphism

$$f : E \to T \times M .$$

Proof : Let $g : E \to T$ be a homomorphism satisfying the properties of Lemma 9. Let h be the restriction of g to T and let π be the canonical projection of E onto M . The homomorphism $f = (g, \pi) : E \to T \times M$ solves the problem: We have the following commutative diagram with exact rows.

$$\begin{tikzcd}
T \arrow[r, tail] \arrow[d, "h"'] & E \arrow[r, two heads, "\pi"] \arrow[d, "f"'] & M \arrow[d, "1_M"'] \\
T \arrow[r, tail] & T \times M \arrow[r, two heads] & M
\end{tikzcd}$$

The serpent lemma gives the exact sequence :

$$\mathrm{Ker}(h) \rightarrowtail \mathrm{Ker}(f) \to 0 \to \mathrm{Coker}(h) \twoheadrightarrow \mathrm{Coker}(f)$$

Hence $\mathrm{Ker}(h) \cong \mathrm{Ker}(f)$ and $\mathrm{Coker}(h) \cong \mathrm{Coker}(f)$ and h is a pseudo-isomorphism and so f is also a pseudo-isomorphism. □

So "to within a pseudo-isomorphism" the study of finitely generated modules over a Noetherian KRULL domain reduces to the study of torsion modules on the one hand and torsion-free on the other.

2.5 TORSION THEORIES

We will now give an exposition of torsion theories following JOACHIM LAMBEK [1971] and then show that in the case of KRULL rings there is a torsion theory similar to BOURBAKI's but working over an arbitrary KRULL domain and without the finiteness condition on modules.

Let R be an associative ring with unity and Mod-R the category of right R-modules.

If $\mathcal{B}$ and $\mathcal{C}$ are classes of R-modules, we let

$$\mathcal{B}^r = \{C \in \text{Mod-R} : \forall_{B \in \mathcal{B}} \text{Hom}_R[B,C] = 0\} \quad \text{and}$$
$$\mathcal{C}^\ell = \{B \in \text{Mod-R} : \forall_{C \in \mathcal{C}} \text{Hom}_R[B,C] = 0\} .$$

Since $\forall_{C \in \mathcal{C}} \forall_{B \in \mathcal{B}} \text{Hom}_R[B,C] = 0 \Leftrightarrow \forall_{B \in \mathcal{B}} \forall_{C \in \mathcal{C}} \text{Hom}_R[B,C] = 0$, $\mathcal{C} \subseteq \mathcal{B}^r \Leftrightarrow \mathcal{B} \subseteq \mathcal{C}^\ell$ and such a situation is called a *polarity*.

One can easily see that :

$$\mathcal{B} \subseteq \mathcal{B}^{r\ell} , \quad \mathcal{B}_1 \subseteq \mathcal{B}_2 \Rightarrow \mathcal{B}_2^r \subseteq \mathcal{B}_1^r , \quad \mathcal{B}^{r\ell r} \subseteq \mathcal{B}^r ,$$

as well as the dual statements with r and ℓ interchanged.

An object function T : Mod-R → Mod-R is called a *radical* if for all R-modules M and N

(1) $T(M) \subseteq M$

(2) for all homomorphisms $f : M \to N$, $f(T(M)) \subseteq T(N)$ and

(3) $T(M/T(M)) = 0$.

A class $\mathcal{B}$ is said to be *closed under group extensions* if whenever N is a submodule of M and N and M/N are in $\mathcal{B}$, then so is M .

Proposition 17 : If $\mathcal{B}$ and $\mathcal{C}$ are classes of R-modules, the following statements are equivalent :

(1) $\mathcal{B} = \mathcal{C}^{\ell}$ and $\mathcal{C} = \mathcal{B}^{r}$

(2) $\mathcal{B}$ is closed under isomorphic images, factor modules, group extensions, and direct sums; moreover, $\mathcal{C} = \mathcal{B}^{r}$.

(3) $\mathcal{C}$ is closed under isomorphic images, submodules, group extensions, and direct products; moreover $\mathcal{B} = \mathcal{C}^{\ell}$.

(4) There is a radical T on Mod-R such that $T(T(M)) = T(M)$ for all modules M ; moreover $\mathcal{B} = \{M : T(M) = M\}$, $\mathcal{C} = \{M : T(M) = 0\}$.

Proof : (4) ⇒ (1) ⇒ (2) and (3) follows from basic properties of the Hom functor.

To show (2) ⇒ (4) let $T(M)$ be the sum of all submodules of M which are in $\mathcal{B}$.

To show (3) ⇒ (4) let $T(M)$ be the intersection of all submodules N of M for which M/N is in $\mathcal{C}$. □

A pair $(\mathcal{B},\mathcal{C})$ satisfying the equivalent conditions in Proposition 17 is called a *pre-torsion theory.*

The correspondence between pre-torsion theories and *idempotent* radicals, that is, radicals T satisfying $T(T(M)) = T(M)$, is a one-to-one correspondence.

Note that : If $(\mathcal{B},\mathcal{C})$ is a pre-torsion theory, then $\mathcal{C}$ is a reflective and $\mathcal{B}$ a coreflective subcategory of Mod-R .

Given any pre-torsion theory, $(\mathcal{B},\mathcal{C})$, we construct a closure operation C on the lattice of submodules of any R-module M , as follows : If N is any sub-module of M , C(N) is that sub-module of M containing N for which $C(N)/N = T(M/N)$.

Relative to the given pre-torsion theory, we shall call N *closed* in M if $C(N) = N$, that is, if $M/N \in \mathcal{C}$, and we shall call N *dense* in M if $C(N) = M$, that is, if $M/N \in \mathcal{B}$.

Proposition 18 : If $\mathcal{B}$ and $\mathcal{C}$ are R-modules, the following statements are equivalent :

(1) $\mathrm{Hom}_R[B,E(C)] = 0$

(2) $\mathrm{Hom}_R[B,C'] = 0$ for any essential extension C' of C

(3) $\mathrm{Hom}_R[B_1,C] = 0$ for any sub-module B_1 of B .

(4) $\forall_{b \in B} \forall_{0 \neq c \in C} \exists_{r \in R}\ br = 0$ and $cr \neq 0$

Proof : (1) ⇒ (2) ⇒ (3) ⇒ (4) can easily be checked.

(4) ⇒ (1) : Suppose $0 \neq f : B \to E(C)$. Then $\mathrm{Im} f \cap C \neq 0$ hence there exists $b \in B$ such that $0 \neq f(b) = c \in C$; but then, for any $r \in R$, $br = 0 \Rightarrow cr = f(b)r = f(br) = 0$, contradicting (4). □

If $\mathcal{B}$ and $\mathcal{C}$ are classes of R-modules, we let

$$\mathcal{B}^R = \{C \in \text{Mod-}R : \forall_{B \in \mathcal{B}}\ \mathrm{Hom}_R[B,E(C)] = 0\} \quad \text{and}$$

$$\mathcal{C}^L = \{B \in \text{Mod-}R : \forall_{C \in \mathcal{C}}\ \mathrm{Hom}_R[B,E(C)] = 0\} .$$

Clearly $\mathcal{C} \subseteq \mathcal{B}^R \Leftrightarrow \mathcal{B} \subseteq \mathcal{C}^L$ and so we have another polarity. The two polarities are related : $C \in \mathcal{B}^R \Leftrightarrow E(C) \in \mathcal{B}^r$

and $B \in \mathcal{C}^L \Leftrightarrow B_1 \in \mathcal{C}^\ell$ for all $B_1 \subseteq B$.

Proposition 19 : If $\mathcal{B}$ and $\mathcal{C}$ are classes of R-modules, the following statements are equivalent :

(1) $\mathcal{B} = \mathcal{C}^L$ and $\mathcal{C} = \mathcal{B}^R$

(2) $\mathcal{B}$ is closed under isomorphic images, factor modules, group extensions, direct sums, and sub-modules; moreover $\mathcal{C} = \mathcal{B}^r$.

(3) $\mathcal{C}$ is closed under isomorphic images, sub-modules, direct products and injective hulls; moreover $\mathcal{B} = \mathcal{C}^\ell$

(4) There is a radical T on Mod-R such that $N \subseteq M \Rightarrow N \cap T(M) = T(N)$; moreover $\mathcal{B} = \{M \mid T(M) = M\}$, $\mathcal{C} = \{M \mid T(M) = 0\}$.

Proof : As for pre-torsion with the exception of two points:

(3) implies that $\mathcal{C}$ is closed under group extensions: let C be a sub-module of M such that C and M/C are in $\mathcal{C}$. Pick K maximal among the submodules of M such that $C \cap K = 0$. Then $K \cong (C + K)/C \subseteq M/C$ is in $\mathcal{C}$; hence so is $C + K \cong C \times K$. But M is an essential extension of $C + K$, and so, $M \subseteq E(C+K)$, hence also $M \in \mathcal{C}$.

(4) Let N be a sub-module of M and take $m \in N \cap T(M)$. Then $mR \subseteq T(M)$, hence $mR \in \mathcal{B}$, hence $mR \subseteq T(N)$. □

A pair $(\mathcal{B},\mathcal{C})$ satisfying these equivalent conditions will be called a *torsion theory*. The elements of $\mathcal{B}$ are called *torsion modules* and the modules in $\mathcal{C}$ are called *torsion free*. And $T(M)$ is called the *torsion submodule* of M .

A radical T satisfying the new condition (4). is called a *torsion radical*. As in the pre-torsion theory case, there is a one-to-one correspondence between torsion theories and torsion radicals.

We say the torsion theory $T = (\mathcal{B},\mathcal{C})$ is *smaller* than the torsion theory $T' = (\mathcal{B}',\mathcal{C}')$ if $\mathcal{B} \subseteq \mathcal{B}'$, or equivalently, $\mathcal{C}' \subseteq \mathcal{C}$.

Notation:

Let A be a KRULL domain and Q its quotient field and let Mod-A denote the category of all unitary A-modules. Let $\mathcal{N}$ (almost null) denote the subcategory of Mod-A consisting of all A-modules N such that $N_p = 0$ for all prime ideals p of ht 1. Let $\mathcal{M}$ be the full subcategory of all $M \in$ Mod-A such that M has no subobject other than 0 belonging to $\mathcal{N}$. Hence, $M \in \mathcal{M}$ if and only if $\mathrm{Hom}_A(N,M) = 0$ for all modules $N \in \mathcal{N}$. Thus $\mathcal{M} = \mathcal{N}^r$.

We will assume that the set of prime ideals of ht 1, $\beta \neq \emptyset$. This is the same as assuming $A \neq Q$. We will also consider the zero ideal in A as a divisorial ideal, and then a is divisorial if and only if

$$a = \bigcap_{p\in\beta} a_p$$

A *Serre subcategory* F is a non empty subcategory of Mod-R such that whenever $M' \rightarrowtail M \twoheadrightarrow M''$ is exact and two of the modules are in F, then the third one is also in F.

Trivially :

Every Serre subcategory F contains the zero module. If $M \in F$ then $M \overset{\cong}{\rightarrowtail} M \twoheadrightarrow 0$ is exact and $0 \in F$.

Every Serre subcategory F is closed under isomorphic images. If $M \in F$ and $M \cong M'$ then $M \overset{\cong}{\rightarrowtail} M' \twoheadrightarrow 0$ is exact and so $M' \in F$.

N is a Serre Subcategory :

$N \neq \emptyset$ since $0 \in N$. Since tensoring with A_p is exact, if we have $M' \rightarrowtail M \twoheadrightarrow M''$ exact with two of the modules in N, then the third one will also be in N.

We can now show that the pair (N,M) determines a torsion theory in which N is the set of torsion modules and M the set of torsion-free modules.

Theorem 6 : Let A be a KRULL domain and N and M the subcategories of Mod-A defined earlier. The pair (N,M) is a torsion theory where N is the set of torsion modules and M the set of torsion-free modules for this torsion theory.

Proof : Since $- \otimes_A A_p$ is exact for each p, N is closed under submodules, factor modules, isomorphic images and group extensions. N is also closed under direct sums since $- \otimes_A A_p$ commutes with direct sums. Moreover $M = N^r = \{M \in \text{Mod-A} : \forall_{N \in N} \text{Hom}_A[N,M] = 0\}$.

□

We will also refer to M as being N-torsion free.

Every torsion module is the union of principal submodules, which are also torsion. Hence every torsion module is isomorphic to a factor of a direct sum of cyclic torsion modules : the set of cyclic torsion modules generates the class of all torsion modules. Since this class is closed under isomorphic images, direct sums and factors, it follows that any torsion theory has the form (A^{RL}, A^{R}) , where A is the set of all cyclic torsion modules. In fact, it follows that we may take $A = \{M\}$, where M is the direct sum of all R/D , D being any dense right ideal.

A dual result was first pointed out in JANS [1965]. Let M be any torsion-free injective. With any $0 \neq m \in M$ associate a submodule K_m of M which is maximal such that $mR \cap K_m = 0$. Clearly K_m is injective and $M = K_m \oplus E(mR)$. Since $m \notin K_m$, $\bigcap_{0 \neq m \in M} K_m = 0$ and the canonical mapping

$$M \to \prod_{0 \neq m \in M} M/K_m \cong \prod_{0 \neq m \in M} E(mR)$$

is an isomorphism. Therefore the class of torsion-free injectives is cogenerated by the set of torsion-free injectives of the form E(R/F) , F being a closed right ideal. Since the class of torsion-free modules is closed under direct products, submodules and injective envelopes, it follows that any torsion theory has the form (A^{L}, A^{LR}) , where A is the set of torsion-free injectives of the form E(R/F) . In fact, we may take $A = \{M\}$, where M is the direct product of all E(R/F) , F being any closed right ideal.

We will now define the concept of "divisible" relative to a torsion theory.

Proposition 20 : Given a torsion theory on Mod-R , and a module M , the following statements are equivalent :

(1) For any dense right ideal D of R and any homomorphism $f : D \to M$, $\exists_{m \in M} \forall_{d \in D}\ f(d) = md$.

(2) If K/L is torsion, any homomorphism $L \to M$ can be extended to a homomorphism $K \to M$.

(3) $E(M)/M$ is torsion-free.

Proof : (1) $\Rightarrow$ (2). By ZORN's lemma, we can extend $f : L \to M$ to $g : N \to M$ so that it cannot be extended any further, $L \subseteq N \subseteq K$. We claim that $N = K$. Otherwise there exists $b \in K \setminus N$, and we consider the right ideal $D = (N :_R b) = (0 :_R \bar{b})$, where $\bar{b} \in K/N$. Now K/N is isomorphic to a factor module of K/L , hence it is torsion, and therefore D is dense. Consider the homomorphism $D \to M$ given by $d \mapsto g(bd)$. By (1) there exists $m \in M$ such that $g(bd) = md$ for all $d \in D$. Thus $g(br) = mr$ for all $r \in R$ such that $br \in N$. We may therefore extend g to $h : N + bR \to M$ by $h(c + br) = g(c) + mr$, a contradiction.

(2) $\Rightarrow$ (3). Let $D(M)$ be the closure of M in $E(M)$. Then $D(M)/M$ is torsion, and we have the diagram where $f : D(M) \to M$ extends 1_M .

$$\begin{tikzcd}
M \arrow[r, tail, "\text{inc.}"] \arrow[dr, "1_M"'] & D(M) \arrow[d, dashed, "f"] \\
& M
\end{tikzcd}$$

Since $D(M)$ is an essential extension of M , f is monic. It follows that the inclusion $M \to D(M)$ is a surjection, hence M is closed in $E(M)$.

(3) $\Rightarrow$ (1). Let $f : D \to M$, where D is a dense right ideal. Then there exists $i \in E(M)$ such that $f(d) = id$ for all $d \in D$. Since $M :_R i$ contains D , $(iR + M)/M \cong R/(M :_R i)$ is torsion. But $E(M)/M$ is torsion-free, hence $i \in M$. $\square$

When M satisfies the equivalent conditions of Proposition 20 we call it *divisible*.

Proposition 21 : For a given torsion theory on Mod-R , let $D(M)$ be the closure of M in $E(M)$. Then the extension $M \to D(M)$ is characterized uniquely up to isomorphism by the following properties :

(1) $M \to D(M)$ is an essential extension

(2) $D(M)/M$ is torsion

(3) $D(M)$ is divisible.

Proof : (1) Suppose S is a submodule of $D(M) \subseteq E(M)$ such that $S \cap M = 0$. Then $S = 0$ since $S \subseteq E(M)$.

(2) $T(D(M)/M) = D(M)/M \cap T(E(M)/M)$. But $D(M)/M = T(E(M)/M)$ so that $D(M)/M$ is torsion.

(3) $D(M)$ is closed so that $E(D(M))/D(M)$ is torsion-free and so $D(M)$ is divisible.

To show the uniqueness of D(M) we show that for any X such that $M \rightarrowtail X$ has X/M torsion any $X \to E(M)$ factors uniquely through D(M) . Consider the following diagram :

$$\begin{tikzcd}
M \arrow[rr, tail] \arrow[dr, tail] \arrow[ddr, tail] & & X \arrow[rrr, two heads] \arrow[dl, dashed] \arrow[ddl] & & & X/M \arrow[dddlll, dashed] \arrow[ddddll, "0\text{-map}"] \\
& D(M) \arrow[d, tail] & & & & \\
& E(M) \arrow[dr, two heads] & & & & \\
& & \frac{E(M)}{M} \arrow[dr, two heads] & & & \\
& & & F(E(M)/M) & &
\end{tikzcd}$$

where F(E(M)/M) is the maximal torsion-free quotient of E(M)/M. The map $X/M \to F(E(M)/M)$ is zero since X/M is torsion and F(E(M)/M) is torsion-free. If $D(M) = \ker(E(M) \to F(E(M)/M))$ then the map from X to D(M) is unique from unique factorization through the kernel. D(M) now is the universal solution to a mapping problem so unique up to isomorphism. □

D(M) is called the *divisible hull* of M .

The restriction of D to the subcategory $\mathcal{C}$ of torsion-free modules is a functor, in fact the left adjoint of the inclusion functor $\mathcal{A} \to \mathcal{C}$ where $\mathcal{A}$ is the subcategory of torsion-free divisible modules.

Indeed, let $C \in \mathcal{C}$. Since C is a dense submodule of D(C) , every homomorphism $f : C \to A$ with A in $\mathcal{A}$ has an extension $f' : D(C) \to A$. Moreover, this extension is uniquely determined by f , since

$\mathrm{Hom}_R[D(C)/C, A] = 0$. Furthermore, $D(C)$ is not only divisible but also torsion-free, being an essential extension of C . Thus $C \to D(C)$ is a best approximation of C in $\mathcal{A}$.

Composing the left adjoints of the inclusion $\mathcal{A} \to \mathcal{C} \to$ Mod-R , we have that $\mathcal{A} \to$ Mod-R has a left adjoint Q , whose object function is given by $Q(M) = D(M/T(M))$.

$Q(M)$ is called the *module of quotients* of M , and Q the *quotient functor* relative to the given torsion theory.

Proposition 22 : Let there be a given torsion theory on Mod-R . Then the category $\mathcal{A}$ of torsion-free divisible modules is abelian and the inclusion functor $\mathcal{A} \to$ Mod-R has a left adjoint Q which is exact.

Proof : Since $\mathcal{A}$ is additive and it contains the zero module, it is closed under finite direct sums, it has kernels and every monomorphism is a kernel, and it has cokernels and every epimorphism is a cokernel. Indeed, the cokernel of $f : A \to A'$ in $\mathcal{A}$ is $A' \to C \to Q(C)$, where $A' \to C$ is the cokernel of f in Mod-R. Moreover the map $f : A \to A'$ is an epimorphism if and only if its cokernel in Mod-R is torsion, that is, $\mathrm{Im}\, f$ is dense in A' . Then f is the cokernel in $\mathcal{A}$ of ker f .

□

Since $\mathcal{A}$ is abelian, the following properties concerning the left adjoint Q of the inclusion functor $U : \mathcal{A} \to$ Mod-R are equivalent :

Q preserves monomorphisms.

Q is left exact.

Q is exact.

UQ preserves monomorphisms

UQ is left exact.

It is easily seen that the functors $M \mapsto M/T(M)$ and $C \mapsto D(C)$ for torsion-free C preserves monomorphisms seperately.

Proposition 23 : There is a one-to-one correspondence between torsion theories and full abelian subcategories of Mod-R with exact reflectors : to every torsion theory corresponds the category of torsion-free divisible modules.

Proof : Let $\mathcal{A}$ be a full abelian subcategory of Mod-R and assume the inclusion functor U has an exact left adjoint Q . Let $\eta : I \to UQ$ be the adjunction. Putting $T(M) = \operatorname{Ker} \eta(M)$, then T is a torsion radical and in the corresponding torsion theory for any torsion-free module C the divisible hull is Q(C) . □

2.6 INJECTIVE MODULES OVER A KRULL DOMAIN

We now discuss some of the properties of the category $\mathcal{M}$, where $\mathcal{M}$ is the category of all A-modules M such that M has no subobject other than 0 belonging to $\mathcal{N}$. So that $M \in \mathcal{M}$ if and only if $\operatorname{Hom}_A(N,M) = 0$ for all modules $N \in \mathcal{N}$. ($\mathcal{M}$ is the category of $\mathcal{N}$-torsion-free objects for the torsion theory $\mathcal{N}$ - see previous section.)

Proposition 24 : An A-module $M \in \mathcal{M}$ if and only if $O(m)$ is a divisorial ideal for all $m \in M$.

Proof : Suppose $M \in \mathcal{M}$, and let $m \in M$. Let $\mathfrak{a} = O(m)$. Let $r \in \cap_{\beta} \mathfrak{a}_p$. This implies that for each $p \in \beta$ there is an element $z \in A \smallsetminus p$ such that $zr \in \mathfrak{a}$. Hence $O(rm) \not\subset p$ when $p \in \beta$. This implies that $Arm \in \mathcal{N}$, hence $rm = 0$. Thus $r \in \mathfrak{a}$ and $\mathfrak{a} = \cap_{\beta} \mathfrak{a}_p$, so $\mathfrak{a}$ is a divisorial ideal. The other direction is clear. □

Corollary 1 : Let $M \in \mathcal{M}$. Then the annihilator of M, $O(M)$, is a divisorial ideal. □

Corollary 2 : Let $\mathfrak{a}$ be an ideal in A. Then $A/\mathfrak{a} \in \mathcal{M}$ if and only if $\mathfrak{a}$ is a divisorial ideal.

Proof : From Corollary 1 if a cyclic module $A/\mathfrak{a}$ belongs to $\mathcal{M}$ for some ideal $\mathfrak{a} \subset A$, then $\mathfrak{a}$ is a divisorial ideal. Conversely, let $\mathfrak{a}$ be a divisorial ideal in A and let $x \in A$. Then $\mathfrak{a} :_A x = \{r \in A : rx \in \mathfrak{a}\}$ is a divisorial ideal.

BECK says that an A-module M is *co-divisorial* iff $M \in \mathcal{M}$. Apart from torsion-theoretic considerations we shall follow BECK closely.

Proposition 25 : Let $M \in \mathcal{M}$. Then the injective envelope of M, $E(M) \in \mathcal{M}$.

Proof : Since the pair $(\mathcal{N},\mathcal{M})$ is a *torsion theory*, with $\mathcal{M}$ the $\mathcal{N}$-torsion-free modules, $\mathcal{M}$ is closed under injective envelopes. □

Proposition 26 : Let $M' \rightarrowtail M \twoheadrightarrow M''$ be an exact sequence of A-modules. If M' and M" are co-divisorial, then M is co-divisorial.

Proof : Since the $\mathrm{Hom}_A(N,-)$ functor is left exact and $\mathrm{Hom}\,(N,M') = \mathrm{Hom}_A(N,M'') = 0$ for all modules $N \in \mathcal{N}$ we get $\mathrm{Hom}_A(N,M) = 0$ for all modules $N \in \mathcal{N}$ and so M is co-divisorial. □

Corollary 1 : Let $f : \mathfrak{a} \to M$ be an A-homomorphism from a divisorial ideal $\mathfrak{a}$ to a co-divisorial module M . Then ker f is a divisorial ideal.

Proof : Let $\mathfrak{b} = \ker f$. We have an injection $\mathfrak{a}/\mathfrak{b} \to M$, so $\mathfrak{a}/\mathfrak{b}$ is co-divisorial. We also have the exact sequence

$\mathfrak{a}/\mathfrak{b} \rightarrowtail A/\mathfrak{b} \twoheadrightarrow A/\mathfrak{a}$.

Since $\mathfrak{a}$ is divisorial, $A/\mathfrak{a}$ is co-divisorial. Since $\mathfrak{a}/\mathfrak{b}$ is also co-divisorial it follows that $A/\mathfrak{b}$ is co-divisorial, so $\mathfrak{b}$ is divisorial. □

Proposition 27 : Let $M \overset{f}{\rightarrowtail} M' \overset{g}{\twoheadrightarrow} N$ be an exact sequence. Assume that M' is co-divisorial, and that $N \in \mathcal{N}$. Then f is an essential extension of M .

Proof : We identify M with a submodule of M'. Let $0 \neq x \in M'$ and let $\mathfrak{a} = \{r \in A : rx = 0\}$ and let $\mathfrak{b} = \{r \in A : rg(x) = 0\}$. Indeed $\mathfrak{b} = \{r \in A : rx \in M\}$. Since $x \in M' \in \mathcal{M}$ $\mathfrak{a} = O(x)$ is a divisorial ideal and $O(x) \neq A$ since $x \neq 0$. Now $\mathfrak{b} = O(g(x))$, and $g(x) \in N \in \mathcal{N}$. Hence $\mathfrak{b} \not\subset \mathfrak{p}$ when $\mathfrak{p} \in \beta$ Thus $\mathfrak{b} \neq \mathfrak{a}$. Let $r \in \mathfrak{b} \setminus \mathfrak{a}$. Then $0 \neq rx \in M$ and M' is an essential extension of M. □

If M is an A-module let $d_M : M \to \prod_\beta M_\mathfrak{p}$ be the canonical mapping. The kernel of the map d_M is the maximal subobject of M belonging to $\mathcal{N}$, so that d_M is an injection if and only if M is co-divisorial.

If M is a torsion module, the map d_M is actually a mapping into $\coprod_\beta M_\mathfrak{p}$.

Proposition 28 : Let M be a co-divisorial torsion module, and let $d_M : M \to \coprod_\beta M_\mathfrak{p}$ be the canonical mapping. Then d_M defines an essential extension of M, and $\operatorname{Coker} d_M \in \mathcal{N}$.

Proof : It is sufficient to prove $L = \operatorname{Coker} d_M \in \mathcal{N}$. We have an exact sequence $M \rightarrowtail \coprod_\beta M_\mathfrak{p} \twoheadrightarrow L$. Applying the exact functor $- \otimes_A A_\mathfrak{p}$ $(\mathfrak{p} \in \beta)$ to this sequence we get $M_\mathfrak{p} \rightarrowtail (\coprod_\beta M_{\mathfrak{p}'}) \otimes_A A_\mathfrak{p} \twoheadrightarrow L_\mathfrak{p}$. Since tensor product commutes with direct limits, the middle term is $\coprod_{\mathfrak{p}' \in \beta} (M_{\mathfrak{p}'} \otimes_A A_\mathfrak{p})$. Furthermore, $M_{\mathfrak{p}'} \otimes_A A_\mathfrak{p} \cong M \otimes_A (A_{\mathfrak{p}'} \otimes_A A_\mathfrak{p})$ and $A_{\mathfrak{p}'} \otimes A_\mathfrak{p} = Q$ unless $\mathfrak{p}' = \mathfrak{p}$. Since M is a torsion module, $M \otimes_A Q = 0$, and so $M_{\mathfrak{p}'} \otimes_A A_\mathfrak{p} = 0$ if $\mathfrak{p}' \neq \mathfrak{p}$. Hence $(\coprod_\beta M_{\mathfrak{p}'}) \otimes_A A_\mathfrak{p} \cong M_\mathfrak{p}$, and $L_\mathfrak{p} = 0$. Since this is true for all $\mathfrak{p} \in \beta$, it follows that $L \in \mathcal{N}$. □

We now look at co-divisorial injective A-modules.

Recall BOURBAKI [1961].

A prime ideal P of R is an associated prime of an R-module M, if one of the following equivalent properties holds :

(1) there exists an element $x \in M$ with $O(x) = P$

(2) M contains a submodule isomorphic to R/P.

Notation : The set of associated primes of an R-module M is denoted by Ass (M).

Proposition 29 : Let p be a prime ideal of height one, and let $f : \mathfrak{a} \to E(A/p)$ be a homomorphism. Suppose that $f(x) = 0$ for some $x \in \mathfrak{a} \setminus p$. Then $f \equiv 0$.

Proof : Since $E(A/p)$ is injective we can extend the map f to a map $\bar{f} : A \to E(A/p)$. Let $b = \ker \bar{f}$. If $b = A$, there is nothing to prove. Otherwise $E(A/p) \cong E(A/b)$ since $E(A/p)$ is indecomposable. Hence $p \in \operatorname{Ass} A/b$, so $b \subset p$ and $f \equiv 0$. □

Corollary 1 : Let p be a prime ideal of height one, and let $f : \mathfrak{a} \to E(A/p)$ be a homomorphism. Suppose that $f(x) = 0$ for some $x \in \mathfrak{a}$, such that $v_p(x) \leqslant v_p(a)$ for all $a \in \mathfrak{a}$. Then $f \equiv 0$.

Proof : Let $b \in \mathfrak{a}$. Define a map $g_b : A \to E(A/p)$ by letting $g_b(t) = f(bt)$. Then $g_b(t) = 0$ for some $t \notin p$, hence $g_b \equiv 0$ according to Proposition 29. In particular, $g_b(1) = f(b) = 0$. Hence $f \equiv 0$. □

Let $\mathfrak{a}$ be a non-zero fractionary ideal in Q . Let v_p be the valuation associated to the principal valuation ring A_p . Define $v_p(\mathfrak{a}) = \inf \{v_p(a) : a \in \mathfrak{a}\}$. Since $\mathfrak{a}_p = (pA_p)^r$ for some r in $\mathbb{Z}$, it follows that $v_p(\mathfrak{a}) > -\infty$. In fact $v_p(\mathfrak{a}) = r$. Also $v_p(\mathfrak{a}) = 0$ for almost all $p \in \beta$. Recall from commutative algebra that for $p \in \beta$ the *symbolic n-th power* of p is the divisorial ideal $p^{(n)} = \{a \in Q : v_p(a) \geqslant n\}$.

Lemma 10 : Let $\mathfrak{a}$ be a fractionary ideal in the quotient field of A . Let $n_p = v_p(\mathfrak{a})$. Then $A : (A : \mathfrak{a}) = \bigcap_{p \in \beta} p^{(n_p)}$.

Proof : We have $A : (A : \mathfrak{a}) = \bigcap_{\beta} \mathfrak{a}_p = \bigcap_{\beta} (pA_p)^{n_p} = \bigcap_{\beta} p^{(n_p)}$. □

So for a divisorial ideal $\mathfrak{a}$ we have $\mathfrak{a} = \bigcap_{\beta} p^{(n_p)}$.

Proposition 30 : Let $E \neq (0)$ be a co-divisorial indecomposable injective A-module. Then $E \cong Q$ or $E \cong E(A/p)$ for some prime ideal $p \in \beta$.

Proof : If E is not a torsion module, $E \cong Q$. Hence assume that E is a torsion module. Let $0 \neq x \in E$ and let $\mathfrak{a} = O(x)$. We have an injection $A/\mathfrak{a} \to E$ and since E is indecomposable, and $A/\mathfrak{a} \neq 0$, it follows that $E \cong E(A/\mathfrak{a})$. Since $A/\mathfrak{a}$ is a submodule of E , $A/\mathfrak{a}$ is co-divisorial, so $\mathfrak{a}$ is a divisorial ideal. Furthermore, $\mathfrak{a}$ is an irreducible ideal in A since $E(A/\mathfrak{a})$ is indecomposable. Since $\mathfrak{a} \neq 0$, $\mathfrak{a} = p^{(n)}$ (symbolic power) of some prime ideal $p \in \beta$. We can find an element

$x \in A$ such that $p^{(n)} :_A x = p$ by letting $v_p(x) = n - 1$. Hence we have an injection $A/p \xrightarrow{x.} A/p^{(n)} \to E$. Since E is indecomposable this gives $E \cong E(A/p)$. □

Proposition 31 : Let p be a prime ideal of height one. Then any direct sum $\coprod E(A/p)$ is injective.

Proof : Let $f : \mathfrak{a} \to \coprod E(A/p)$. Since v_p is a discrete valuation, we can find an element $a \in \mathfrak{a}$ such that $v_p(a) \leqslant v_p(b)$ for all $b \in \mathfrak{a}$. Since $f(a)$'s components in the direct sum are zero almost everywhere, the same has to hold for $f[\mathfrak{a}]$. Hence f maps into a finite direct sum of injectives, which certainly is injective. Hence f can be extended to A. □

Proposition 32 : Let E be a co-divisorial module which is a direct sum of indecomposable injective modules. Then E is injective.

Proof : Let $f : \mathfrak{a} \to E$. Let $0 \neq x \in \mathfrak{a}$. Then x is contained in just a finite number of prime ideals of height one. Furthermore, $f(x)$'s components in the direct sum are zero almost everywhere. If $x \notin p$ $(p \in \beta)$, and $g : \mathfrak{a} \to E(A/p)$ (or Q), and $g(x) = 0$, then $g[\mathfrak{a}] = 0$. The proof then follows from Propositions 30 and 31 and the observation that any direct sum $\coprod Q$ is injective. □

The following proposition is an analogue of MATLIS's result for left-Noetherian rings namely that every injective R-module has a decomposition as a direct sum of indecomposable injective submodules.

Proposition 33 : Let E be a co-divisorial injective A-module. Then E is a direct sum of indecomposable injective modules.

Proof : Let E be a co-divisorial injective module. Let C be a maximal submodule of E with respect to the property of being a direct sum of indecomposable injective submodules. Then C is injective by Proposition 32, hence $E \cong C \oplus D$ for some D . Let $0 \neq x \in D$. Then $O(x) = p_1^{(n_1)} \cap \ldots \cap p_k^{(n_k)}$ (or $O(x) = 0$) for some prime ideals $p_i \in \beta$, since E is co-divisorial. Hence $E(A/O(x)) \cong E(A/p_1) \oplus \ldots \oplus E(A/p_k)$. This contradicts the maximality of C and hence $C = E$. □

Proposition 34 : Let $\{E_i : i \in I\}$ be a family of co-divisorial injective A-modules. Then $\coprod_{i \in I} E_i$ is injective.

Proof : This follows from Propositions 32 and 33 . □

The characterization of Dedekind domains as those commutative domains for which every divisible R-module is injective is due to H. CARTAN and S. EILENBERG [1956]. The original proof used Hereditary rings and projective modules. We will follow the proof of D.W. SHARPE and P. VÁMOS [1972].

Theorem 7 : Let R be a commutative domain. Then the following statements are equivalent :

(a) R is a DEDEKIND domain

(b) every divisible R-module is injective.

Proof : (a) ⇒ (b). Let R be a DEDEKIND domain and let E be a divisible R-module. Further let I be an integral ideal of R and consider the diagram

$$\begin{tikzcd}
I \arrow[r, tail] \arrow[dr, "f"'] & R \arrow[d, dashed, "g"] \\
& E
\end{tikzcd}$$

Clearly it is sufficient to consider the case $I \neq 0$. Then I has a fractional inverse F where $IF = R$. Thus there exists non-zero elements $s_1, \dots, s_n \in I$ and $k_1, \dots, k_n \in F$ such that $\sum_{i=1}^{n} s_i k_i = 1$. Since E is divisible, for each i $(1 \leqslant i \leqslant n)$ there exists $e_i \in E$ such that $f(s_i) = s_i e_i$.

Consider $s \in I$ and since $sk_i \in IF = R$,

$$f(s) = f\,(\sum_{i=1}^{n} sk_i s_i) = \sum_{i=1}^{n} (sk_i)\, f(s_i) = s \sum_{i=1}^{n} k_i s_i e_i .$$ Now $k_i s_i \in FI = R$, so that $\sum_{i=1}^{n} k_i s_i e_i \in E$ and we can define an R-homomorphism $g : R \to E$ by $g(r) = r \sum_{i=1}^{n} k_i s_i e_i$. Then g agrees with f on I and E is injective by the BAER Criterion.

(b) ⇒ (a). Since every direct sum of divisible modules is divisible, we have every direct sum of injective modules is injective and so R is a Noetherian ring.

We will show that every non-zero integral ideal of R is invertible. If R is a field this is true, so we suppose R is not a field. Consider first a maximal ideal M of R. As an R-module the field of fractions K of R is divisible with R and M as submodules. Then K/M is divisible and so injective. Thus $E = E(R/M)$ is a submodule of K/M.

Put $A_n = 0 :_E M^n$ $(n = 1, 2, \dots)$. Then $MA_2 \subseteq A_1$, and

$A_1 = R/M$ by Theorem 4. Since R/M is a simple R-module either $MA_2 = 0$ or $MA_2 = A_1$.

Consider the case $MA_2 = 0$. Then $A_2 \subseteq 0 :_E M = A_1$, so $A_2 = A_1$. But Corollary 1 to Theorem 4 gives $M = 0 :_R A_1$ and $M^2 = 0 :_R A_2$, so that $M = M^2 = M^3 = \dots$ and so on. But from the Corollary 1 to Theorem 4 $\bigcap_{n=1}^{\infty} M^n = 0$, so that $M = 0$. Since we are assuming R is not a field we must have $MA_2 = A_1$. Also, $K/M \supseteq E \supseteq A_2 \supseteq A_1 = R/M$, so there is an R-submodule T of K such that $T \supseteq R$ and $A_2 = T/M$. But $M(T/M) = R/M$, so that $(MT)/M = R/M$. Thus $MT = R$ and M is invertible since T is a fractional ideal of R.

Consider the collection of all integral ideals of R which do not have inverses. This collection is not empty since the zero ideal is a member. Since R is Noetherian the collection possesses a maximal member say I. Now $I \subset M$ for some maximal ideal M and M has an inverse M^{-1}.

Now $I = IM^{-1}M \subseteq IM^{-1} \subseteq MM^{-1} = R$, so that IM^{-1} is an integral ideal containing I. If $I \subsetneq IM^{-1}$, then IM^{-1} has an inverse. But then $IM^{-1}(IM^{-1})^{-1} = R$, and I has an inverse $M^{-1}(IM^{-1})^{-1}$. It follows that $I = IM^{-1}$, whence $IM = I$. Thus $I = IM = IM^2 = \dots$, whence

$$I \subseteq \bigcap_{n=1}^{\infty} M^n = 0 .$$

Thus every non-zero integral ideal of R has an inverse. □

Proposition 35 : Let E be any A-module, and let $p \in \beta$. Then the following statements are equivalent :

(1) E_p is an injective A_p-module.

(2) E_p is an injective A-module.

Furthermore (1) and (2) will hold if E is an injective A-module ((1) and (2) will also hold if E is a divisible A-module).

Proof : Let E be an injective A-module. Then E is a divisible A-module. This implies E_p is a divisible A-module for all prime ideals p . If the prime ideal p is of height one, E_p is an injective A_p-module since A_p is a DEDEKIND domain.

Conversely, let E_p be an injective A_p-module and consider the diagram where a is an A-ideal.

$$\begin{tikzcd}
a \arrow[rr, tail] \arrow[dd] \arrow[dr] & & A \arrow[dd] \arrow[dl, dashed] \\
& E_p & \\
a_p \arrow[rr, tail] \arrow[ur] & & A_p \arrow[ul, dashed]
\end{tikzcd}$$

So E_p is an injective A-module. □

For A = R a DEDEKIND domain Q/R is an injective module since it is a divisible module.

Proposition 36 : For any KRULL ring A the injective envelope of Q/A is isomorphic to the direct sum $\coprod_{p \in \beta} Q/A_p$.

Proof : Since Q/A is co-divisorial and $Q/A_p \cong Q/A \otimes_A A_p$ by flatness of A_p $\coprod_{p \in \beta} Q/A_p$ is an essential extension of Q/A by Proposition above. Furthermore for all $p \in \beta$ Q/A_p is an injective A-module, since Q/A_p is an injective A_p-module. Q/A_p is also co-divisorial so that $\coprod_{p \in \beta} Q/A_p$ is an injective A-module. Hence $E(Q/A) \cong \coprod_{p \in \beta} Q/A_p$. □

Corollary 1 : The injective envelope of Q/A is isomorphic to the direct sum $\coprod_{p \in \beta} E(A/p)$.

Proof : Since $Q/A_p \cong E(A/p)$, the result is clear from Proposition 33.

□

Proposition 37 : Let M be any A-module. Then

$$\text{inj. dim}_A \left(\coprod_{\text{ht}\, p=1} M_p \right) \leqslant 1 .$$

Proof : We can consider M_p as an A_p-module, and we can find an exact sequence $M_p \rightarrowtail (E_0)_p \twoheadrightarrow (E_1)_p$ where $(E_0)_p$ and $(E_1)_p$ are injective A_p-modules. Hence $\coprod_\beta M_p \rightarrowtail \coprod_\beta (E_0)_p \twoheadrightarrow \coprod_\beta (E_1)_p$ is exact, and the two right direct sums are injective. □

For $A = R$ a DEDEKIND domain with quotient field Q , Q/R is an injective co-generator in the category of all R-modules. So $M = 0 \Longleftrightarrow \text{Hom}_R(M,Q/R) = 0$.

We now give a corresponding proposition for any KRULL domain.

Proposition 38 : Let A be a KRULL domain, $A \neq Q$, and let N be any A-module. Then the following two statements are equivalent:

(1) $N \in \mathcal{N}$

(2) $\text{Hom}_A(N,E(Q/A)) = 0$.

Proof : (1) ⇒ (2). $E(Q/A) \in \mathcal{M}$ since $Q/A \in \mathcal{M}$ and $\mathcal{M}$ is closed under injective envelopes.

(2) ⇒ (1). Suppose that $N \notin \mathcal{N}$ and that $\text{Hom}_A(N,E(Q/A)) = 0$. Consider the canonical map $d : N \to \prod_\beta N_p$, and let $D(N) = \text{Im}(d)$. Then $D(N) \subset \prod_\beta N_p$ is co-divisorial, and $D(N) \neq 0$ since $N \notin \mathcal{N}$. Furthermore $d : N \to D(N)$ is a surjection. Then $E(D(N))$ is also co-divisorial and we therefore have an injection

$E(D(N)) \to \coprod_J Q \oplus \coprod_I E(Q/A)$ for some indexing sets I and J. Since $\mathrm{Hom}_A(N,E(Q/A)) = 0$ we have an injection $D(N) \to \coprod_J Q$. Hence $\mathrm{Hom}_A(N,Q) \neq 0$. Let $f : N \to Q$, $f(N) \neq 0$. Since $\mathrm{Hom}_A(N,Q/A) = 0$ we can conclude that $f(N) \subset A$. This implies that $f(N) \subset \cap_\beta\ p$. If not, f continued by $A \longrightarrow A/p \to E(Q/A)$ would be a non-zero map from N to $E(Q/A)$. If the family β is infinite we have deduced the contradiction that $f(N) = 0$, since 0 is the only element in A which is contained in an infinite number of prime ideals of height one. If β is finite, A is a DEDEKIND domain and the equivalence (2) ⇒ (1) is valid as remarked above. □

Given a SERRE subcategory N of the abelian category Mod-A we can form a new abelian category Mod-A/N, the *quotient category* of Mod-A by taking the category whose objects are the objects of Mod-A and with

$$\mathrm{Hom}_{\text{Mod-}A/N}(M,N) = \varinjlim \mathrm{Hom}_{\text{Mod-}A}(M',N/N')$$

where the limit is taken over the pair (M',N') such that M' (resp. N') is a subobject of M (resp. N) with M/M' and N' in N. The functor $T : \text{Mod-}A \to \text{Mod-}A/N$ defined by $T(M) = M$ on objects and $T(f)$ = class of f in $\mathrm{Hom}_{\text{Mod-}A/N}(M,N)$ for f in $\mathrm{Hom}_{\text{Mod-}A}(M,N)$ is an exact functor from Mod-A to Mod-A/N and gives the universal exact mapping of Mod-A to an abelian category with kernel N.

The Proposition 38 then shows that $E(Q/A)$ is a co-generator in the quotient category Mod-A/N.

Let M be any A-module. We have a canonical mapping $\lambda_M : M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M,E(Q/A)), E(Q/A)))$. Let $f \in \mathrm{Hom}_A(M,E(Q/A))$. Then define $\lambda_M(m)$ by the equation $\lambda_M(m)(f) = f(m)$ for all $m \in M$.

Proposition 39 : Let M be any A-module. The following two properties are equivalent.

(1) M is co-divisorial.

(2) $\lambda_M : M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M,E(Q/A)), E(Q/A))$ is an injection.

Proof : (2) $\Rightarrow$ (1) is trivial since $E(Q/A)$ is co-divisorial and $\mathrm{Hom}(L,N)$ is co-divisorial whenever N is co-divisorial.

(1) $\Rightarrow$ (2). For $0 \neq x \in M$ Ax is not in $\mathcal{N}$ and so we can find a homomorphism $f : Ax \to E(Q/A)$ such that $f(x) \neq 0$. Since $E(Q/A)$ is injective we can lift f to a mapping $\bar{f} : M \to E(Q/A)$. This shows that λ_M is an injection since $\lambda_M(x)(\bar{f}) = \bar{f}(x) = f(x) \neq 0$. Hence $\lambda_M(x) \neq 0$. □

We now show that the Serre subcategory $\mathcal{N}$ is closed under injective envelopes. The quotient category Mod-A/$\mathcal{N}$ is shown to be isomorphic to the category $\mathcal{C}$ of $\mathcal{N}$-torsion-free, $\mathcal{N}$-divisible modules. We then show that $\mathcal{C}$ has homological dimension at most one.

Definition 8 : Let M be any A-module. Define

$$D(M) = \mathrm{im}\,(M \xrightarrow{d_M} \Pi_{\beta} M_p) = \mathrm{im}\, d_M ,$$

where d_M is the canonical map.

The map $D : M \to D(M)$ is a functor from Mod-A to $\mathcal{M}$, and D is the reflector of the inclusion functor $i : \mathcal{M} \to$ Mod-A. That is : Given any map $f : M \to N \in \mathcal{M}$, there exists a unique map $g : D(M) \to N$ such that the following diagram commutes

$$\begin{tikzcd}
M \arrow[r, "d_M"] \arrow[dr, "f"'] & D(M) \arrow[d, dashed, "g"] \\
 & N
\end{tikzcd}$$

If $f : M \to N$, define Df to be the unique morphism making the following diagram commute

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "d_M"'] & N \arrow[d, "d_N"] \\
D(M) \arrow[r, "Df"] & D(N)
\end{tikzcd}$$

Note that Df is an injection (surjection) if f is an injection (surjection).

BECK's use of the quotient category $\text{Mod-}A/\mathcal{N}$ is rather abstract, but fortunately LUTHER CLABORN and ROBERT FOSSUM [1968] showed that the functor $T : \text{Mod-}A \to \text{Mod-}A/\mathcal{N}$ has a right adjoint, right inverse $S : \text{Mod-}A/\mathcal{N} \to \text{Mod-}A$. CLABORN and FOSSUM constructed the adjoint S by letting $S(M) = \{y \in \Pi_\beta M_p : \forall p \in \beta, \exists s \notin p$ such that $sy \in M\}$.

The quotient category $\text{Mod-}A/\mathcal{N}$ can therefore be identified, up to isomorphism, with a subcategory of Mod-A, namely the subcategory of A-modules which are torsion-free divisible in terms of the torsion theory $\mathcal{N}$.

We get this subcategory $\mathcal{C}$ of Mod-A by setting

$\mathcal{C} = \{X \in \text{Mod-}A : \text{Hom}_A(N,X) = 0$ ("$\mathcal{N}$-torsion-free") and $\text{Ext}^1(N,X) = 0$ ("$\mathcal{N}$-divisible") for all $N \in \mathcal{N}\}$.

A more constructive way of defining it is via an injective cogenerator $E = \prod_{\beta} E(A/p)$.

In terms of E the $\mathcal{N}$-torsion-free objects are $\mathcal{M} = \{X \in \text{Mod-}A : \exists\, X \rightarrowtail \prod E\}$. That the subcategory of $\mathcal{N}$-torsion-free and the subcategory of co-divisorial modules are the same can best be seen from describing the subcategory $\mathcal{N}$ as $N \in \mathcal{N}$ iff $\mathrm{Hom}_A(N, E(A/p)) = 0$ for all p of ht 1.

We could now equally have defined the functor $D : \text{Mod-}A \to \mathcal{M}$ as $D(M) = \mathrm{im}\,(M \to \prod^{\mathrm{hom}(M,E)} E)$ where $E = \prod_{\beta} E(A/p)$ since this $D(M)$ will also have the universal property for maps from M into $\mathcal{M}$. Note that $M \twoheadrightarrow D(M)$.

We can now reflect into the category of $\mathcal{N}$-torsion-free $\mathcal{N}$-divisible modules $\mathcal{C} = \{X \in \text{Mod-}A : \mathrm{Hom}_A(N,X) = 0$ and $\mathrm{Ext}^1(N,X) = 0$ for all $N \in \mathcal{N}\}$ by filling in the following diagram :

$$\begin{tikzcd}
 & C(M) \arrow[dr] & & & \\
D(M) \arrow[ur, dashed, tail] \arrow[rr, tail, "\theta"] & & \prod E \arrow[rr, two heads] \arrow[drr, "\varphi"'] & & \mathrm{Coker}\ \theta \arrow[d, two heads] \\
 & & & & D(\mathrm{Coker}\ \theta)
\end{tikzcd}$$

where $C(M) = \ker \varphi$ and where we get $M \xrightarrow{\psi} C(M)$ (and from which it is evident that $C(M) \rightarrowtail \prod^{u} E \to \prod^{v} E$ for some cardinals u, v so $\mathcal{C}$ can be defined as the category "co-presented" by E .)

From our earlier results on torsion-free divisible modules it is evident that the functor $C : \text{Mod-}A \to \mathcal{C}$ is exact and kills $\mathcal{N}$, so there is a functor $\text{Mod-}A/\mathcal{N} \to \mathcal{C}$ which may now be checked to be a category equivalence.

Proposition 40 : Let $a \neq (0)$ be an A-ideal and $p \in \beta$ a prime ideal containing a. Then $p \in \text{Ass}\,(A/a)$.

Proof : Since it can be shown that $a :_A p \not\subset a_p$, if we take $z \in a :_A p$ such that $z \notin a_p$ then $a :_A z = p$ and so $p \in \text{Ass}\, A/a$. □

Proposition 41 : Let N be any A-module and $p \in \beta$. The following statements are equivalent :

(1) $N_p = 0$

(2) N is a torsion module and $p \notin \text{Ass}\, N$.

Proof : (1) ⇒ (2). Suppose $N_p = 0$ and $n \in N$ then $(An)_p = 0$ and this implies that $O(n) \not\subset p$. So $O(n) \neq 0$ and $p \notin \text{Ass}\, N$.

(2) ⇒ (1). Suppose N is a torsion module such that $N_p \neq (0)$. Then there exists an $n \in N$ such that $0 \neq a = O(n) \subset p$. We then have an injection $A/p \to An \subset N$, so $p \in \text{Ass}\, N$. □

Corollary 1 : Let $p \in \beta$, and let N be an A-module. If $N_p = 0$ then $[E(N)]_p = 0$.

Proof : If $N_p = 0$ then N is a torsion module and $p \notin \mathrm{Ass}\, N$. However if N is a torsion module then E(N) is a torsion module and since Ass N = Ass E(N) we have E(N) is a torsion module and $p \notin \mathrm{Ass}\, E(N)$. So $[E(N)]_p = 0$. □

Corollary 2 : The SERRE subcategory $\mathcal{N}$ is closed under injective modules. □

Proposition 42 : Let E be an injective A-module. Let $d_E : E \to D(E)$ be the canonical map. Then $E \cong \ker d_E \oplus D(E)$.

Proof : $\ker d_E$ is the maximal subobject of E belonging to $\mathcal{N}$. Since E is injective and by the Corollary above $\ker d_E$ is injective, the sequence $\ker d_E \rightarrowtail E \twoheadrightarrow D(E)$ splits. □

Corollary 1 : D(E) is injective if E is injective. □

Proposition 43 : Let $M \xrightarrow{f} M'$ be an essential extension of M. Then $M_p \xrightarrow{f_p} M'_p$ is an essential extension of M_p for all $p \in \beta$.

Proof : Follows from $M_p = 0 \Rightarrow [E(M)]_p = 0$. □

Proposition 44 : Let $M \rightarrowtail E_0 \to E_1 \to E_2 \to \dots$ be a minimal injective resolution of an A-module M. Then $D(E_i) = 0$ for $i \geqslant 2$.

Proof : Let $p \in \beta$, and tensor the resolution with the flat A-module A_p to get $M_p \rightarrowtail [E_0]_p \to [E_1]_p \to [E_2]_p \to \dots$. Since each E_i is an injective A-module, $[E_i]_p$ is an injective A_p-module. Since essential extensions are preserved by $- \otimes_A A_p$ it follows that the derived injective resolution of A_p-module M_p is a minimal injective resolution. Since the global dimension of A_p is one, it follows that $[E_i]_p = 0$ for $i \geqslant 2$. Hence $D(E_i) = 0$ for $i \geqslant 2$. □

Proposition 45 : The injective dimension of the category $\mathcal{C}$ is at most one.

Proof : Let C be the functor $C : \text{Mod-R} \to \mathcal{C}$. Then C is exact and kills N . For E an injective A-module $C(E)$ is injective in $\mathcal{C}$. Take any A-module M , and let $M \rightarrowtail E_0 \to E_1 \to E_2 \to \dots$ be a minimal injective resolution of M . Then $C(M) \rightarrowtail C(E_0) \twoheadrightarrow C(E_1)$ is exact since $E_2 \in N$, and this shows that $\text{inj. dim}_{\mathcal{C}}\, C(M) \leqslant 1$. □

We now have an analogue of the well known Theorem 7 : A domain is a DEDEKIND domain iff every divisible module is injective.

Proposition 46 : Let M be an A-module, and suppose that $D(M/aM) = 0$ for all $0 \neq a \in A$. Then $D(E(M)/M) = 0$.

Proof : Let $0 \neq a \in A$ and consider $aM \rightarrowtail M \twoheadrightarrow M/aM$. Applying the exact functor $- \otimes_A A_p$ gives : $[aM]_p \rightarrowtail M_p \twoheadrightarrow [M/aM]_p$. Now M_p is a divisible A-module so that it is an injective A_p-module for all $p \in \beta$. Consider $M \overset{f}{\rightarrowtail} E(M) \twoheadrightarrow E(M)/M$. Applying the exact functor $- \otimes_A A_p$ gives : $M_p \overset{f_p}{\rightarrowtail} [E(M)]_p \twoheadrightarrow [E(M)/M]_p$. Since M_p is an injective A_p-module and f_p is an essential extension $[E(M)/M]_p = 0$ for all $p \in \beta$ and so $D[E(M)/M] = 0$. □

Proposition 47 : Let C be the functor $C : \text{Mod } A \to \mathcal{C}$. Let M be an A-module such that $C(M/aM) = 0$ for all $0 \neq a \in A$. Then $C(M)$ is injective in the category $\mathcal{C}$.

Proposition 48 : Let $M \overset{f}{\rightarrowtail} M'$ be an essential extension of the A-module M . Then $D(M) \overset{Df}{\rightarrowtail} D(M')$ is an essential extension of $D(M)$.

Proof : It is sufficient to consider the case where M' is the injective envelope of M . Consider the following diagram :

$$\begin{tikzcd}
M \arrow[r, tail, "f"] \arrow[d, "d_M"'] & E(M) \arrow[d, "d_{E(M)}"] & \\
D(M) \arrow[r, tail, "Df"] \arrow[d, "f'"'] & D(E(M)) \arrow[d, equal] & \\
E(D(M)) \arrow[r, tail, "F"] & D(E(M)) \arrow[r, two heads] & L
\end{tikzcd}$$

The maps f and f' are the essential extensions of the modules M and D(M) into their injective envelopes and d_M and $d_{E(M)}$ are the canonical maps. Since $D(E(M))$ is injective there exists a map F making the lower square commutative and $L = \text{Coker } F$. Let $p \in \beta$, and apply the functor $- \otimes_A A_p$ to the diagram. The maps $(d_M)_p$ and $(d_{E(M)})_p$ are isomorphisms since ker d_M and ker $d_{E(M)}$ are in $\mathcal{N}$. By Proposition 43 f_p is an essential extension so that $(Df)_p$ is also an essential extension. This implies F_p is an essential extension, and therefore an isomorphism since $[E(D(M))]_p$ is an injective A_p-module. Hence $L_p = 0$ for all $p \in \beta$, i.e. $D(L) = 0$. On the other hand, the sequence $E(D(M)) \rightarrowtail D(EM)) \twoheadrightarrow L$ splits since $E(D(M))$ is injective. Hence L is a sub-object of $D(E(M)) \in \mathcal{M}$, so $L = 0$. Since F is an isomorphism it follows that $D(M) \overset{Df}{\rightarrowtail} D(E(M))$ is an essential extension. □

We have also proved the following Corollary :

Corollary 1 : Let M be an A-module. Then $E(D(M)) \cong D(E(M))$, □

CHAPTER THREE

PUTTING INJECTIVES TO WORK

3.1 INTRODUCTION

In this chapter we give a characterization of KRULL domains using the abelian subcategory $\mathcal{C}$ of Mod-R, that is, the category of $\mathcal{N}$-torsion-free, $\mathcal{N}$-divisible modules. We then use this characterization to generalize KRULL domains in two ways. Firstly we drop the condition that the ring must be an integral domain. This leads to the results that (i) a ring will be KRULL if and only if it is a finite product of fields and KRULL domains and (ii) the injective envelope of the ring R will be semi-simple artinian. The second generalization is to rings of higher dimension than one which we call n-KRULL rings. Here we find that the classical R_p, (for p a prime ideal of ht 1) is a discrete valuation ring generalizes to R_p (for p now a prime ideal of ht n) is a regular ring. Finally we propose an analogue of the Divisor Class group.

3.2 A CHARACTERIZATION OF KRULL DOMAINS

In Chapter Two we have shown that : If R is a KRULL domain and $\mathcal{N}$ is the Serre subcategory of almost-null modules then there is a torsion theory $(\mathcal{N},\mathcal{M})$ where $\mathcal{M}$ is the $\mathcal{N}$-torsion-free modules. With this torsion theory we then associate as usual a full abelian subcategory $\mathcal{C}$ of Mod-R with exact reflector, namely, the category of $\mathcal{N}$-torsion-free, $\mathcal{N}$-divisible modules.

We showed that

(I) In $\mathcal{C}$ every direct sum of injective modules is injective

(II) $\mathcal{C}$ has global dimension at most one. (in fact, global dimension exactly one for a proper KRULL domain).

We now give a characterization of KRULL domains via these two properties.

Proposition 1 : Let R be an integral domain and $\mathcal{M}$ the category of co-divisorial modules as above. If every direct sum of injective R-modules in $\mathcal{M}$ is injective then R has A.C.C. on divisorial ideals.

Proof : Suppose there exists an infinite strictly increasing sequence of divisorial ideals of R . $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots \subset \mathfrak{a}_n \subset \ldots .$ Let $\mathfrak{a} = \bigcup_{k=1}^{\infty} \mathfrak{a}_k$. Then $\mathfrak{a}$ is an ideal of R and for each k , $\mathfrak{a}/\mathfrak{a}_k$ is a non-zero R-module in $\mathcal{M}$ since it is a submodule of $R/\mathfrak{a}_k \in \mathcal{M}$. Now $E(\mathfrak{a}/\mathfrak{a}_k) \in \mathcal{M}$ since $\mathcal{M}$ is closed under injective envelopes. Denote by φ_k the mapping from $\mathfrak{a} \to E(\mathfrak{a}/\mathfrak{a}_k)$ by composing the natural mapping $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_k$ with the embedding of $\mathfrak{a}/\mathfrak{a}_k$ into its injective envelope. By hypothesis $\bigoplus_{k=1}^{\infty} E(\mathfrak{a}/\mathfrak{a}_k)$ is injective. We now define the mapping $\varphi : \mathfrak{a} \to \bigoplus_{k=1}^{\infty} E(\mathfrak{a}/\mathfrak{a}_k)$ by $\varphi(r) = \{\varphi_k(r)\}_{k=1}^{\infty}$, $r \in \mathfrak{a}$. Note that, for any $r \in \mathfrak{a}$, there exists k_0 such that $r \in \mathfrak{a}_k$ for all $k \geqslant k_0$, so that $\varphi_k(r) = 0$ for all $k \geqslant k_0$ and φ is a well-defined R-homomorphism. Since $\bigoplus_{k=1}^{\infty} E(\mathfrak{a}/\mathfrak{a}_k)$ is injective φ can be extended to a homomorphism $\psi : R \to \bigoplus_{k=1}^{\infty} E(\mathfrak{a}/\mathfrak{a}_k)$. Since R is singly generated as an R-module from the structure of direct sum in $\mathcal{M}$ there exists an integer n such that

$\psi(R) \subseteq \bigoplus_{k=1}^{n} E(a/a_k)$. This means that φ_k is the zero mapping for all $k > n$, which is a contradiction if $a/a_k \neq 0$. Thus we have the A.C.C. on divisorial ideals. □

Remark : Since injectives have the lifting property for all submodules they are N-divisible, so one could equally state this for C in place of M . The vital point is the nature of direct sums in each of these categories.

Lemma 1 : Let R be an integral domain and let p be a prime ideal of R . Then $R_p = \{x \in Q^* : Rx^{-1} \cap R \not\subset p\} \cup \{0\}$.

Proof (due to BECK, see FOSSUM [1973]) : If $0 \neq x \in R_p$, then there exist an $s \in R \setminus p$ such that $sx \in R$. Hence $s \in Rx^{-1} \cap R$ and the set described contains R_p . The other inclusion holds since if $Rx^{-1} \cap R \not\subset p$ then there is an $a \in R$ and an $s \in R \setminus p$ so that $ax^{-1} = s$ or $x = a/s \in R_p$. □

Proposition 2 : Let R be a proper integral domain such that C the category of N-torsion-free, N-divisible modules has global dimension at most one and the A.C.C. on divisorial ideals. Then R_p , for p a prime ideal of ht 1 , is a discrete valuation ring and $R = \bigcap_{p\,\mathrm{ht}\,1} R_p$. So R is completely integrally closed.

Proof : Let $M \rightarrowtail E_0 \to E_1 \to E_2 \to E_3 \to \ldots$ be an injective resolution of R_p-modules. This resolution can be considered as an injective resolution of R-modules. Applying the exact functor $C : \text{Mod-}R \to \mathcal{C}$ to this resolution gives $C(M) \rightarrowtail C(E_0) \twoheadrightarrow C(E_1)$, $C(E_2) = 0$ since $\mathcal{C}$ has homological dimension at most one. Therefore $(E_2)_p = 0 \quad \forall\, p \in \beta$. Since $M \otimes_{R_p} R_p = M$ for any R_p-module M , $E_2 = 0$ and R_p also has homological dimension at most one, and so R_p is a DEDEKIND domain if it is not a field. (In fact R_p is a discrete valuation ring since it is a local ring by (CARTAN-EILENBERG [1956])). Suppose $0 \neq x \in Q$, $x \in \bigcap_{p\,ht\,1} R_p$. Then $Rx^{-1} \cap R \not\subset p \quad \forall\, p$ of ht 1 from lemma 1. Hence $Rx^{-1} \cap R$ is a divisorial ideal not contained in a maximal divisorial ideal. Thus $Rx^{-1} \cap R = R$. In particular $\exists\, s \in R$ such that $s/x = 1$ so $x \in R$. Hence $R = \bigcap_{p\,ht\,1} R_p$. But the R_p for $p\,ht\,1$ are discrete valuation rings and the intersection of discrete valuation rings is completely integrally closed. So R is completely integrally closed. □

We can now use Propositions 1 and 2 to give a novel characterization of KRULL domains.

Theorem 1 : An integral domain R is a KRULL domain if and only if it satisfies the following conditions :

(I) Every direct sum of injective R-modules in $\mathcal{C}$ is injective.

(II) The category $\mathcal{C}$ has global homological dimension one.

Proof : From Propositions 1 and 2 we have that the conditions (I) and (II) imply that R is completely integrally closed and has A.C.C. on divisorial ideals.

Conversely Proposition 34 and 45 of chapter 2 show that if R is a KRULL domain then (I) and (II) hold. □

3.3 KRULL RINGS WITH ZERO DIVISORS

In the classical case KRULL rings are assumed to be integral domains. We now drop this condition by allowing the ring to have zero-divisors.

If a ring R is GORENSTEIN, that is, Noetherian with finite cohomological dimension then it has an injective resolution of the form :

$$R \rightarrowtail \bigoplus_{p\,\mathrm{ht}\,0} E(R/p) \to \bigoplus_{p\,\mathrm{ht}\,1} E(R/p) \to \bigoplus_{p\,\mathrm{ht}\,2} E(R/p) \to \dots$$

In the case of R a domain the resolution now starts

$$R \rightarrowtail Q \to \bigoplus_{p\,\mathrm{ht}\,1} E(R/p)$$

and in the more specific case of R a Dedekind domain it is just

$R \rightarrowtail Q \twoheadrightarrow \bigoplus_{p\,\mathrm{ht}\,1} R(p^\infty)$ where $R(p^\infty) = Q/R_p$.

Writing the kernel of $Q \twoheadrightarrow \bigoplus_{p\,\mathrm{ht}\,1} R(p^\infty)$ in terms of the intersection of the kernels of $Q \twoheadrightarrow Q/R_p$ we get $R = \bigcap_{p\,\mathrm{ht}\,1} R_p$ inside Q .

The role of Q here is to act as the injective envelope of R , $E(R)$. To generalize it would seem that we should want to have Q replaced by $E(R) \cong \bigoplus_{p\,\mathrm{ht}\,0} E(R/p)$. We now look at the conditions this will impose on the ring R .

Recall that a ring R is called *reduced* if the set of nilpotent elements of R $\mathrm{nil}(R)$ also denoted by $\sqrt{0}$ is equal to (0). ($\mathrm{nil}(R) = \sqrt{0} = \{a \in R : a^n = 0$ for some $n > 0\}$).

Proposition 3 : $E(R) \cong \bigoplus_{p\,\mathrm{ht}\,0} E(R/p)$ implies that R is reduced.

Proof : Since for p a prime ideal of $\mathrm{ht}\,0$ $R \twoheadrightarrow R/p \rightarrowtail E(R/p)$ is the map on the p-th component of $E(R)$ we have

$0 = \mathrm{Ker}(R \rightarrowtail E(R)) = \mathrm{Ker}(R \rightarrowtail \prod_{p\,\mathrm{ht}\,0} R/p)$ or $0 = \bigcap_{p \text{ minimal}} p$.

However $\bigcap_{p \text{ minimal}} p = \bigcap_{\text{all primes}} p = \sqrt{0}$.

Hence R is reduced. □

E. MATLIS [1983] gives the following proposition for a reduced ring (This seems to be FOLKLORE and this is one of the few places where this result is explicitly set down.)

Proposition 4 : Let R be a reduced ring and $\{P_\alpha : \alpha \in I\}$ the set of minimal prime ideals of R.

(1) R_{P_α} is the quotient field of R/P_α, hence an injective R-module.

(2) $E(R)$ is a direct summand of $\prod_{\alpha \in I} R_{P_\alpha}$

(3) $\bigcup_{\alpha \in I} P_\alpha$ is the set of all zero divisors of R.

Proof : (1) Let $0_\alpha = \{r \in R : ur = 0$ for some $u \in R \setminus P_\alpha\}$.

Clearly 0_α is an ideal in R and $0_\alpha \subset P_\alpha$ since P_α is prime. Now since P_α is minimal $P_\alpha R_{P_\alpha}$ is the only prime ideal of R_{P_α}

and so must be its own radical, that is, nilpotent. Thus if $p \in P_\alpha$, there exists $u \in R \setminus P_\alpha$ so that $up^n = 0$ for some integer $n > 0$. Hence $(up)^n = 0$, and since R is reduced $up = 0$. Thus for all $p \in P_\alpha$ we have $p \in 0_\alpha$, that is, $P_\alpha = 0_\alpha$ and hence $P_\alpha R_{P_\alpha} = 0$. But in that case R_{P_α} is the quotient field of R/P_α and an injective R/P_α-module as $(R/P_\alpha)_0$, so by pullback to R an injective R-module : we have $\hom_R[-,(R/P_\alpha)_0] \cong \hom_{R_{P_\alpha}}[- \otimes R_P, (R/P_\alpha)_0]$ exact by flatness of R_{P_α}

(2) From (1) we clearly get $\prod_{\alpha \in I} R_{P_\alpha}$ an injective R-module and R embedded in it, so $E(R)$ embeds and is a retract since it is injective.

(3) From (1) every element of $\bigcup_{\alpha \in I} P_\alpha$ is a zero-divisor in R. Conversely if $0 \neq x \in R$ is a zero-divisor then $\exists\, 0 \neq y \in R$ such that $xy = 0$. But since $\bigcap_{\alpha \in I} P_\alpha = 0$ $\exists\, \beta \in I$ such that $y \notin P_\beta$; and hence $x \in P_\beta$. □

This is a partial converse to Proposition 3. Hence to get $E(R) \cong \bigoplus_{p\,\mathrm{ht}\,0} E(R/p)$ we need R-reduced and presumably a Noetherian condition on (unmixed) height 0 ideals.

In fact $E(R) \cong \bigoplus_{p\,\mathrm{ht}\,0} E(R/p)$ will actually force $E(R)$ to be semi-simple artinian, that is, a finite product of fields.

To see this we consider the image of R in $E(R)$. We must have $R \cong \bigoplus_{p\,\mathrm{ht}\,0} R/p$ (this is another way of seeing that R must be reduced).

The nature of direct sums in Mod-R implies only finitely many images of 1 Mod p are non-zero, that is, only finitely many p count. Thus R is a finite product of domains (some of which - those corresponding to minimal ideals which are also maximal - may be fields).

For a domain R/p we have $E_{R/p}(R/p) = (R/p)_0 = Q(R/p)$ where $E_{R/p}(R/p)$ is the injective envelope of R/p as a module over the domain R/p and Q(R/p) is the total quotient field of R/p. A change of rings argument (localization is exact) gives $E_R(R/p) \cong R_p = Q(R/p)$.
(We could also have seen this by using a partition of 1 into orthogonal idempotents $1 = \bigoplus_{\alpha \in I} e_\alpha$ where $e_\alpha = \mathrm{Im}\ (1 \text{ in } R/P_\alpha)$ or $P_\alpha = (1 - e_\alpha)R$ and $\{P_\alpha : \alpha \in I\}$ is the set of minimal prime ideals of R .) Thus $E(R) = \bigoplus_{p\,\mathrm{ht}\,0} E(R/p) = \bigoplus_{p\,\mathrm{ht}\,0} Q(R/p)$ is semi-simple artinian since it is a finite product of fields. This gives the following :

Proposition 5 : $E(R) \cong \bigoplus_{p\,\mathrm{ht}\,0} E(R/p)$ implies that E(R) is semi-simple artinian. □

We can also write the decomposition $R \cong \bigoplus_{P_\alpha\,\mathrm{ht}\,0} R/P_\alpha$ in terms of localization since $R[e^{-1}] \cong R/(1-e)R$ for any idempotent e .

This is vital since any localization of a module category satisfying the KRULL properties :

(I) $\mathcal{C} = \text{Mod-}R/\mathcal{N}$ has global dimension $\leqslant 1$

(II) $\mathcal{C}$ has all injectives Σ-injectives. (closed under Σ)

must be again satisfying the KRULL properties. Hence we can conclude the following :

Theorem 2 : A ring R is a KRULL ring if and only if it is a finite product of fields and KRULL domains. □

This allows one to reduce the most general KRULL ring to the domain case. R. KENNEDY [1980] has shown that it is possible to develop the theory of KRULL rings from the start via *regular divisorial ideals* and similar notions but his version of KRULL ring is more general in that he allows components which are not fields but maybe von-Neumann regular rings it appears.

3.4 n-KRULL RINGS

K. HUGHES has suggested how to generalize the theory of KRULL domains to effective dimension greater than one. We now give an outline of what we call n-KRULL rings.

We first introduce the subcategory $\mathcal{C}_n$ of Mod-R. Similarly to the situation in section 3.2 we let

$\mathcal{C}_n = \{X \in \text{Mod-R} : \exists\ X \rightarrowtail \prod^N E \to \prod^M E\}$ N,M cardinals but where E is now equal to $\prod_{\text{p ht n}} E(R/p)$ in place of $\prod_{\text{p ht 1}} E(R/p)$.

We now define n-KRULL rings by generalizing the characterization for KRULL domains given in Theorem 1.

Definition 1 : A ring R will be called n-*KRULL* if the following two conditions hold :

(I) $\mathcal{C}_n$ has global homological dimension n

(II) In $\mathcal{C}_n$ every sum of injective R-modules is injective.

We will now call an R-ideal n-*divisorial* if $R/\mathfrak{a} \in \mathcal{C}_n$.

Remark : One *could* use a weaker property merely requiring that unmixed injectives of height n be closed under taking direct sums, but it seems that this would not give a nice theory.

In the classical theory of KRULL rings R_p , for p a prime ideal of ht 1 , is a Discrete valuation ring. We now show that for n-KRULL rings this generalizes to R_p , for p now a prime ideal of ht n , being regular.

Proposition 6 : Let R be an n-KRULL ring. Then R_p is regular for p a prime ideal of ht n .

Proof : For each prime ideal p of height n $R_p \in \mathcal{C}_n$ (in fact R_p is the reflection of R into the category cogenerated by E(R/p) alone) and all R_p-modules can be similarly embedded in $\mathcal{C}_n$. Hence R_p has global dimension n and is Noetherian. Consequently R_p has finite global homological dimension and so it is regular by the AUSLANDER-BUCHSBAUM-SERRE theorem (SERRE [1975]). □

3.5 HIGHER ANALOGUES OF DIVISOR CLASS GROUP

The real motivation for developing KRULL theory is to be able to say something about factorization theory and the Class group. Hence having a good functorial notion of the Class group is important.

The theory of the invariant class attached to a finitely generated module originates in the classic result, due to STEINITZ [1912] (in a slightly different form. See MILNOR [1971], pg.11.), that for R a Dedekind domain $K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R)$ where $K_0(R)$ denotes the GROTHENDIECK group of R and Pic(R) the PICARD group of R .

BOURBAKI [1965] refines this to the case of a Noetherian KRULL domain by K_0 (f.g.-Mod-R/N) $\cong \mathbb{Z} \oplus \mathrm{Cl}(R)$ where we use f.g.-Mod-R to denote the subcategory of Mod-R of finitely generated modules and N the pseudo-zero modules.

CLABORN and FOSSUM [1967], [1968(a)] generalize this in two ways.

Let $N_i = \{N \in \text{f.g Mod-R} : N_p = 0$ for prime ideals of ht i-1$\}$ so $N = N_2$. It is clear that we have a filtration $\ldots N_n \subset \ldots \subset N_2 \subset N_1 \subset N_0$ of f.g. Mod-R. By the category analogue of the Noether isomorphism one gets a sequence of categories $0 \to N_1/N_2 \to N_0/N_2 \to N_0/N_1 \to 0$ whence an exact sequence of GROTHENDIECK groups

$$K_0(N_1/N_2) \to K_0(N_0/N_2) \to K_0(N_0/N_1) \to 0 .$$

If R is an integral domain then $K_0(N_0/N_1) \cong \mathbb{Z}$ with the isomorphism being given by $M \to \dim_Q(Q \otimes_R M)$. Hence $K_0(N_0/N_2) \cong \mathbb{Z} \oplus \mathrm{Im}\ (K_0(N_1/N_2) \to K_0(N_0/N_2))$ and in the classic case of R Noetherian BOURBAKI gives the image isomorphic to Cl(R) .

For their first generalization CLABORN and FOSSUM introduces, what they call the i-th homological class group given by $W_i(R) = \mathrm{Im}\ (K_0(N_i/N_{i+1}) \to K_0(N_{i-1}/N_{i+1}))$.

The other construction of CLABORN and FOSSUM is a more refined invariant in that there is an epimorphism $C_i(R) \twoheadrightarrow W_i(R)$ for each i . However $C_i(R)$ is only defined for locally MACAULAY rings.

The basic idea is to write $Cl(R) = Div(R)/Prin(R)$ where $Div(R)$ is the free group on the ht 1 prime ideals p and $Prin(R)$ is given by $\sum_{p\,ht\,1} v_p(x) . p$ for $x \in R^*$ where we interpret $v_p(x)$ as $\ell_{R_p}(R_p/xR_p)$ and ℓ is the length function.

CLABORN and FOSSUM replaces non-zero divisors x by R-sequences $(x_1, \ldots, x_n)$, that is, $x_i \in R$ such that $[\sum_{j=1}^{k} x_j R : x_{j+1}R] = \sum_{j=1}^{k} x_j R$ for $k = 0, \ldots, n-1$ and then introduces, what they call the n-th class group of R ,

$$C_n(R) = \left\{ \frac{\text{free group on } [p],\ [p] \text{ prime ideals of ht} n}{\sum_{p\,ht\,n} \ell_p\,(R_p/(x_1R_p + \ldots + x_nR_p)\,.\,[p]} \right\}$$

From FOSSUM [1973] it appears that all that is necessary to extend to non-Noetherian KRULL rings is : The replacing of BOURBAKI's finitely generated modules by submodules of finitely generated modules (we will abbreviate this to sub.f.g. modules) and interpreting $\mathbb{Z} \oplus Cl(R)$ as $K_0(\text{f.g. Mod-}Q) \oplus Cl(R)$ we get $Cl(R) = Ker(K_0(\text{sub.f.g.Mod-}R/N_1) \to K_0(\text{sub.f.g.Mod-}R/N_0))$ and then to use the short exact sequence of K-theory to get equally BOURBAKI's result for a Noetherian KRULL ring R that $Cl(R) = K_0(N_0/N_1)$.

There is a slightly different way to approach this generalized Class group which may be more appropriate for KRULL rings and stays closer to BOURBAKI's treatment, namely we introduce the class of a sub.f.g. module as follows :

Consider the situation $K_0(\text{sub.f.g.Mod-}R) \to K_0(\text{f.g.Mod-}Q)$ obtained by localization, $[M] \to [M \otimes Q]$ in the Dedekind case. In general if we use $\mathcal{C}_1$ for reflection into the $\mathcal{N}_1$-torsion-free divisible category we have $[M] \to [\mathcal{C}_1(M)]$.

Now $\mathcal{C}_1(M)$ will contain a lattice L namely a sub.f.g. torsion-free module L (in the classical case we can actually choose L to be free and isomorphic to $\overset{n}{\Sigma}\, R$, where $n = \dim_Q(M \otimes Q)$) with $\mathcal{C}_1(L) = \mathcal{C}_1(M)$.

Now if we consider the square

$$\begin{tikzcd}
L \cap M \arrow[r, tail] \arrow[d, tail] & M \arrow[d] \\
L \arrow[r] & L \cup M
\end{tikzcd}$$

This gives a short exact sequence $L \cap M \rightarrowtail M \oplus L \twoheadrightarrow L \cup M$. Applying $\mathcal{C}_1$ we get

$$\begin{tikzcd}
\mathcal{C}_1(L \cap M) \arrow[r, tail] & \mathcal{C}_1(M) \oplus \mathcal{C}_1(L) \arrow[r, two heads] \arrow[d, equal] & \mathcal{C}_1(L \cup M) \arrow[d, equal] \\
 & \mathcal{C}_1(L) \oplus \mathcal{C}_1(L) & \mathcal{C}_1(L)
\end{tikzcd}$$

from exactness. So

$$\begin{tikzcd}
\mathcal{C}_1(L \cap M) \arrow[r, tail] \arrow[d] & \mathcal{C}_1(L) \arrow[d] \\
\mathcal{C}_1(L) \arrow[r] & \mathcal{C}_1(L)
\end{tikzcd}$$

is a pullback and $\mathcal{C}_1(L \cap M) \cong \mathcal{C}_1(L)$ also.

Now we send $[M] \to [M/(L \cap M)]$ which has $\mathcal{C}_1(M/(L \cap M)) = \mathcal{C}_1(M)/\mathcal{C}_1(L \cap M) = 0$ by exactness.

So that we get an element in $K_0(N_0)$ modulo the uncertainty coming from different choice of lattices like $L \cap M$ inside M. Let us restrict the uncertainty further by always taking free lattices.

In the GROTHENDIECK group these will generate a subgroup $\{[L_1/L_2]\}$ and $[M/L_2] = [M/L_1] + [L_1/L_2]$ for two different free lattices $L_2 \subset L_1 \subset M$. so that the class group should be defined by

$$K_0[N_1] \ / \ \{\text{subgroup coming from pairs of free lattices}\}$$

This generalizes to higher dimensions by replacing the C_1 reflection by C_n and defining higher "free" lattices to be submodules which are finite sums of generators $L = \Sigma\, R/P$, P prime ideals unmixed of height $n-1$ and with $C_n(L) \cong C_n(M)$ for M a sub.f.g. C_n-torsion-free module.

In the classical case $sR \rightarrowtail R$, for s a non-zero divisor, gives such a pair of lattices so giving principal ideals vanishing in the class group (or principal cyclics in K_0(torsion)). We get a similar conclusion now for z a non-zero divisor mod P.

K. HUGHES conjectures that our notion of Divisor Class group and the Class groups $C_i(R)$ and $W_i(R)$ of CLABORN and FOSSUM will all be the same for n-KRULL rings.

3.6 A NOTE ON NON-COMMUTATIVE KRULL RINGS

There are various authors who have defined non-commutative analogues of classical KRULL rings. Some of these are M. CHAMARIE [1981], R. FOSSUM [1968], E. JESPERS, L. LE BRUYN and P. WAUTERS [1982], H. MARUBAYASHI [1980] and H.P. REHM [1977]. The theory that we have developed in this chapter seems to fit best with that of CHAMARIE who works with KRULL rings which embed in a simple ARTIN ring and are defined through the use of a torsion theory in a similar way to our use of N-torsion. Nobody however seems to have explored higher dimensional as well as non-commutative analogues of KRULL rings.

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