FORCED OSCILLATIONS IN SIMPLE AND BINARY GAS ATMOSPHERIC MODELS

by

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to my parents

P R E F A C E

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C O N T E N T S

C H A P T E R 1

INTRODUCTION

The presence of periodic oscillations in the earth's atmosphere has been confirmed in recent years by analysis of satellite drag data. The amplitudes of these oscillations vary with height and time in a complex manner with the underlying physical mechanism of this behaviour not fully understood.

Classical studies have been limited to the lower levels of the atmosphere, but these have neglected to include the damping effects of heat conduction and viscosity. These also ignored the second order terms in the equations of motion which, in effect, treats an otherwise singular perturbation problem as regular.

Upper atmosphere studies of the diurnal density oscillations were discussed by Nicolet, on the basis of the mutual diffusion of the components of a binary gas system, where he compared different equilibrium configurations. This statical treatment again ignores the damping effects on mass flow. D.G. Parkyn reduced the problem in idealised form to that of investigating the effect of a travelling temperature wave at the base of a viscous, heat conducting, diffusing gas atmosphere. This model excludes molecular dissociation and ionization in the upper regions and absorption of solar energy. Incorporation of all these properties would render the problem impossibly difficult.

As a first step to the development of the analysis for the

complex spherical atmosphere, Parkyn simplified the model to that of a cylindrical homogeneous atmosphere, and he considered the effect of forced oscillations about an isothermal equilibrium state. Parkyn showed that this idealised problem is capable of explicit solution and, contrary to the result of Wilkes, he found that the amplitudes of the forced oscillations decrease with height in the lower atmospheric regions. This implies the importance of heat conduction and viscosity as damping effects in these regions.

It is proposed to extend the analysis of Parkyn by simplifying the geometry, treating one space dimension, so giving more flexibility to the assumed physical properties.

Continuum equations of motion will be taken to hold throughout the range of investigation.

CHAPTER 2

ATMOSPHERIC MODELS AND THE IDEALIZED PROBLEM

As a general investigation is intended, the geometry of the atmospheric models will be limited to that of a flat earth with an associated gravitational field. This simplification is not severe as comparative behaviour between the models is sought and, it further allows the retention of more physical properties in the model.

Ignoring the Coriolis force and absorption of solar energy, we will investigate atmospheric models comprising one and two diatomic gases in a single dimensional frame of reference, perpendiclar to a flat earth. The model base level will be chosen at a convenient altitude where relatively steady atmospheric conditions can be found.

Perhaps the most severe restriction is the specification that the gases are diatomic. It is well known that, in the higher regions of the atmosphere, gas molecules dissociate into atoms and ionization occurs to a significant degree.

The continuum equations of motion hold throughout the dense lower regions to the base of the exosphere where the mean free path and the scale height are of the same order of magnitude. Analysis will therefore be valid only in this lower region. However, useful predictions can be made for the higher regions.

CHAPTER 3

ISOTHERMAL SIMPLE GAS MODEL

As a basis for comparison the first step is to consider a simple homogeneous gaseous atmosphere undergoing forced oscillations about an isothermal equilibrium state as a result of periodic boundary conditions.

3.1 *EQUATIONS OF MOTION*

We apply the Navier-Stokes equations to the one-dimensional model and proceed to derive the equations of motion in the (x) co-ordinate system. All dependent variables will be functions of x and t only. The equations of motion and energy (Chapman and Cowling) reduce to

Continuity:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nU) = 0 \qquad 3.1.1$$

Equation of State:

$$P = nkT \qquad 3.1.2$$

Momentum:

$$\frac{\partial P}{\partial x} - \frac{4}{3}\frac{\partial}{\partial x}(\mu\frac{\partial U}{\partial x}) + \rho g + \rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = 0 \qquad 3.1.3$$

Energy:

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial x} + \frac{2}{Nkn}P\frac{\partial U}{\partial x} - \frac{8}{3N}\frac{\mu}{kn}(\frac{\partial U}{\partial x})^2 - \frac{2}{Nkn}\frac{\partial}{\partial x}(\lambda\frac{\partial T}{\partial x}) = 0 \qquad 3.1.4$$

T is the temperature, P the pressure, n the particle number density, U the mass velocity, ρ the density and k is Boltzman's constant. The gravitational field has its only component as -g. N is a polyatomicity parameter which has a value 5 for a diatomic gas; t is the time variable and the particle mass m is defined by

$$\rho = nm \qquad 3.1.5$$

and assumed to be constant. The molecular coefficients of heat conduction and viscosity λ, μ are dependent only on temperature and vary as its square root. Thus we can express

$$\lambda = \lambda^* T^{\frac{1}{2}}$$
$$\mu = \mu^* T^{\frac{1}{2}} \qquad 3.1.6$$

but these can be taken as constant in this isothermal atmosphere. The preceding equations are accurate to a second approximation with respect to the pressure tensor and thermal vector.

3.2 *STEADY STATE*

We shall consider the effect of small oscillations about a steady state solution of the equations of motion in which

$$\frac{\partial}{\partial t} \equiv 0 \text{ and } U = \frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial x^2} \equiv 0 \qquad 3.2.1$$

We note that the energy equation reduces to

$$\frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x}) = 0 \qquad 3.2.2$$

which can permit a non-isothermal equilibrium state. This will be investigated in detail in a later chapter. We are now concerned with the isothermal solution for which $T = T_o$ a constant. Equations 3.1.2 and 3.1.3 become

$$P_o = n_o k T_o$$

$$\text{and } \frac{dP_o}{dx} + \rho_o g = 0 \qquad 3.2.3$$

where the suffix 'o' denotes the equilibrium base state. From 3.2.3 we obtain the familiar exponential function for the particle number density

$$n_o(x) = n_o(o)e^{-x/H} \qquad 3.2.4$$

where $H = \frac{kT_o}{mg}$, the atmospheric scale height. This is constant for this particular model.

3.3 *SMALL OSCILLATIONS*

We discuss the case of small oscillations about this isothermal steady state and we define τ, π and η by the relations

$$T = T_o(1 + \tau)$$

$$P = P_o(1 + \pi) \qquad 3.3.1$$

$$n = n_o(1 + \eta)$$

and treat τ, π, η, U and their derivatives as small quantities. For diatomic gases N = 5. On substitution and retention of first

order terms, the equations 3.1.1 through 3.1.4 reduce to

$$\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} - \frac{U}{H} = 0$$

$$\pi = \tau + \eta$$

$$n_o \frac{\partial U}{\partial x} - \frac{n_o}{H}\tau - \frac{4\mu}{3kT_o}\frac{\partial^2 \tau}{\partial x^2} + \frac{n_o}{Hg}U = 0$$

$$\frac{\partial \tau}{\partial t} + \frac{2}{5}\frac{\partial U}{\partial x} - \frac{2\lambda}{5kn_o}\frac{\partial^2 \tau}{\partial x^2} = 0 \qquad \text{3.3.2 through 3.3.5}$$

A time dependence of the form $\exp(i\sigma t)$ is assumed for each of the dependent variables where σ is the angular frequency of the resulting oscillation. Thus, in general, each of τ, π, η and U will take the form $f(x,t) = f^*(x)e^{i\sigma t}$ where the $f^*(x)$ may be complex. After substitution the asterisk will be dropped and the complex amplitude factors implied throughout the ensuing analysis. On substitution, differentiation with respect to t, and dropping $e^{i\sigma t}$, we can write

$$i\eta - \delta V - V = 0$$

$$\pi = \tau + \eta$$

$$\delta\pi + \tau + \frac{\varepsilon}{y}\delta^2 V - i\alpha V = 0$$

$$i\frac{5}{2}\tau - \delta V - \frac{\omega}{y}\delta^2 \tau = 0 \qquad \text{3.3.6 through 3.3.9}$$

where we have defined $V = \frac{U}{\sigma H}$

$y = e^{-x/H}$ such that

$$\frac{d}{dx} = -\frac{1}{H} y \frac{d}{dy} \equiv -\frac{1}{H}\delta$$

$$\varepsilon = 4\sigma\mu/(3kn_o(o)T_o)$$

$$\alpha = H\sigma^2/g$$

$$\omega = \lambda/(kn_o(o)H^2\sigma) \qquad 3.3.10$$

The equations carry only three real dimensionless parameters ε, α, ω at this stage. We eliminate η between 3.3.6 and 3.3.7 and substitution of π into 3.3.8 leaves two equations in τ, V and their derivatives. By repeated differentiation and elimination of V and its derivatives, the single equation in τ remains:

$$\left[\delta^2 + \delta + \frac{5}{7}\alpha - \frac{2\varepsilon\omega}{7y^2}\delta^2(\delta - 2)(\delta - 1)\right]\tau$$

$$= -\frac{i}{y}\left[\frac{5\varepsilon}{7}\delta(\delta - 1) + \frac{2\omega}{7}\delta^2(\delta^2 - \delta + \alpha)\right]\tau \qquad 3.3.11$$

We shall now proceed to investigate solutions of equation 3.3.11.

Defining $\frac{1}{z} = \frac{2\omega}{7iy}$ and $\gamma = \frac{7\varepsilon}{2\omega}$ 3.2.12

we note that $\gamma = \frac{14\sigma^2H^2\mu}{3T_o\lambda}$ which is zero only if $\mu = 0$ (which ignores viscosity as a damping effect).

For typical gas parameter values and assuming a diurnal oscillation $|z| >> 1$ at the lower boundary, which suggests that an asymptotic solution of 3.3.11 will suffice to determine the general behaviour of the amplitude of the temperature oscillations in the lower regions.

Equation 3.3.11 can be rewritten as

$$\left[(\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)\right]\tau$$

$$= \frac{\gamma}{z}\left[\frac{7}{5}\delta(\delta - 1) - \frac{d}{dz}\delta(\delta - 1)(\delta - 2)\right]\tau \qquad 3.3.13$$

or $A(\delta)\tau = \frac{\gamma}{z}B(\delta)\tau$

where $A(\delta) \equiv (\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)$

and $B(\delta) \equiv \frac{7}{5}\delta(\delta - 1) - \frac{d}{dz}\delta(\delta - 1)(\delta - 2)$ 3.3.14

3.4 *NON-VISCOUS HEAT CONDUCTING MODEL*

We shall discuss the case where heat conduction acts alone as a damping effect. Viscosity is absent and $\mu = 0$. This means that $\gamma = 0$ and we are left with the degenerate equation:

$$\left[(\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)\right]\tau = 0 \qquad 3.4.1$$

This is a generalized hypergeometric equation and is capable of explicit solution.

To summarize the results of Barnes, the generalized hypergeometric equation

$$\left[(\vartheta + \alpha_1)\ldots(\vartheta + \alpha_p) - \frac{d}{dy}(\vartheta + \rho_1 - 1)\ldots(\vartheta + \rho_q - 1)\right]\tau = 0, \quad 3.4.2$$

where $\vartheta \equiv y(d/dy)$ and $q \geq p$, has $(q + 1)$ linearly independent solutions

$$ {}_pF_q\left[\alpha_1 \ldots \alpha_p;\ \rho_1 \ldots \rho_q;\ y\right]$$

together with

$$y^{1-\rho_m}\, {}_pF_q\left[\alpha_1 - \rho_m + 1,\ \ldots \alpha_p - \rho_m + 1;\right.$$

$$\left. 2 - \rho_m,\ \rho_1 - \rho_m + 1,\ \ldots \rho_q - \rho_m + 1;\ y\right]$$

where $m = 1, 2, \ldots q$ and among the quantities $\rho_r - \rho_m + 1$, that corresponding to $m = r$ is omitted. The function

$$ {}_pF_q\left[\alpha_1 \ldots \alpha_p;\ \rho_1 \ldots \rho_q;\ y\right]$$

is defined by

$${}_pF_q = \frac{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + n)\ldots\Gamma(\alpha_p + n)}{\Gamma(\rho_1 + n)\ldots\Gamma(\rho_q + n)}\, y^n.$$

Thus all solutions of 3.4.2 will be bounded as $y \to 0$, if

$$\rho_m \leq 1 \quad (m = 1, \ldots q). \qquad 3.4.3$$

Further, for large values of $|y|$ the asymptotic behaviour of ${}_pF_q$ is given by

$$\frac{\prod_{r=1}^{p}\Gamma(\alpha_r)}{\prod_{r=1}^{q}(\rho_r)}\,{}_pF_q\left[\alpha_1\ldots\alpha_p;\ \rho_1\ldots\rho_q;\ y\right]$$

$$\sim \mu^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}(1-\mu)}\exp\left[\mu y^{1/\mu}\right]y^{\left[\Sigma\alpha-\Sigma\rho+\frac{1}{2}(\mu-1)\right]/\mu} \qquad 3.4.4$$

where $\mu = q + 1 - p$ and $|\arg y| < \pi$.

Equation 3.4.1 is in the correct form with $p = 2$, $q = 3$, and $\mu = 2$. We have shown that $|z|$ is very large in the lower regions and it decreases with increasing height. Thus, whatever the boundary conditions

$$\tau \sim \exp\,(2z^{\frac{1}{2}})\,z^{-\frac{1}{4}} \qquad 3.4.5$$

This is the exact form that Parkyn obtained from his cylindrical model, justifying to some extent the use of a flat model in this context. Thus $|\tau|$ decreases with height in the range of validity of the asymptotic expansion. In the higher regions one will need to resort to the series solutions of the equation or perform numerical integration of equation 3.4.1.

We shall consider next the second degenerate case of the viscous, non heat conducting model.

3.5 *VISCOUS NON HEAT CONDUCTING MODEL*

We refer to equation 3.3.11 and allow $\lambda = \omega = 0$ while $\mu \neq 0$. Thus $\epsilon \neq 0$, and we have

$$\left[\delta^2 + \delta + \frac{5}{7}\alpha\right]\tau = -\frac{i}{y}\frac{5\epsilon}{7}\delta(\delta - 1)\tau \qquad 3.5.1$$

Define $z = \frac{7i\gamma}{5\varepsilon}$, and let $\delta \equiv z\frac{d}{dz}$. Thus 3.5.1 reduces to

$$(\delta^2 + \delta + \tfrac{5}{7}\alpha)\tau = \frac{d}{dz}(\delta - 1)\tau \qquad 3.5.2$$

We are unable to adopt the same approach as before in finding the asymptotic expansion as the requirement that $q \geq p$ is not satisfied. However, with reference to 3.4.3, since $q = 1$ and $\rho_1 = 0$, all solutions of 3.5.2 are finite as $|z| \to 0$ suggesting that the heat conduction effect will be dominant. Parkyn's cylindrical model gave, for this case:

$$\tau \sim \exp(2z^{\frac{1}{2}})z^{-\frac{3}{4}}$$

Investigation of the equation which contains both heat conduction and viscosity will show the heat conduction effect to be dominant.

3.6 *VISCOUS HEAT CONDUCTING ATMOSPHERE*

We shall now examine the full equation 3.3.13 and discuss the combined effect of viscosity and heat conduction. We have that

$$\left[(\delta^2 + \delta + \tfrac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)\right]\tau = \frac{\gamma}{z}\left[\frac{7}{5}\delta(\delta - 1) - \frac{d}{dz}\delta(\delta - 1)(\delta - 2)\right]\tau$$

Both sides of this equation have the same order so that a perturbation procedure would not lead to omission of high order derivatives and thus does not result in a singular perturbation problem.

Let $\tau = \tau_o + \gamma\tau_1 + \gamma^2\tau_2 + \ldots\ldots$ 3.6.1

Substitution yields

$$\left[(\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)\right]\tau_o = 0 \qquad 3.6.2$$

and τ_o is in fact the solution of the non-viscous heat conducting case already considered. The other members τ_n, are obtained by solution of

$$\left[(\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)\right]\tau_n$$

$$= \frac{1}{z}\left[\frac{7}{5}\delta(\delta - 1) - \frac{d}{dz}\delta(\delta - 1)(\delta - 2)\right]\tau_{n-1}$$

or, in general, the particular integral of

$$\left[(\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + \alpha\delta)\right]\tau_n = F_{n-1}(z) \qquad 3.6.3$$

where $F_{n-1}(z) = \frac{1}{z}\left[\frac{7}{5}\delta(\delta - 1) - \frac{d}{dz}\delta(\delta - 1)(\delta - 2)\right]\tau_{n-1}$ with $n \geqslant 1$.

The above procedure would lead to an explicit solution of our generalised problem covering the asymptotic range of the atmosphere. We can avoid this heavy analysis by re-arranging equation 3.3.13 as follows:

$$\left[(\delta^2 + \delta + \frac{5}{7}\alpha) - \frac{d}{dz}(\delta^3 - \delta^2 + (\alpha + \frac{2}{5}\gamma)\delta - \frac{2}{5}\gamma)\right]\tau$$

$$= \frac{\gamma}{z}\left[\delta(\delta - 1) - \frac{d}{dz}\delta(\delta - 2)(\delta - 1)\right]\tau \qquad 3.6.4$$

OR we can write

$$A^*(\delta)\tau = \frac{\gamma}{z}B^*(\delta)\tau \qquad 3.6.5$$

Now $A^*(\delta)T_1 = 0$ and $B^*(\delta)T_2 = 0$ are both generalized hypergeometric equations. It is easily shown that their asymptotic solutions are the same as that found for τ_o. This means that in the lower regions where $|z|$ is large

$$\tau = \tau_o$$

satisfies 3.6.4 exactly as an asymptotic solution, and hence, due to the uniqueness of asymptotic expansions, it is the asymptotic solution of 3.6.4.

This result has the implication that heat conduction dominates the viscosity effect in the lower regions and permits a detailed discussion of this case alone.

3.7 *DISCUSSION*

We have considered a dynamic atmospheric model and shown that forced temperature oscillations about an isothermal steady state have decreasing amplitudes in the lower regions. We have also shown that heat conduction has the major damping effect in this region, while viscosity has a lesser importance. These results are valid only in the range of validity of the asymptotic solution, which implies a critical level where $|z| \sim O(1)$. We can thus estimate this critical height as being

$$X_c \sim H \log\left(\frac{7}{2} k n_o H^2 \sigma/\lambda\right) \qquad 3.7.1$$

There is still uncertainty about amplitude variation in the higher regions of the atmosphere and this will be discussed in the next chapter.

CHAPTER 4

NUMERICAL INTEGRATION OF THE SIMPLE GAS EQUATIONS

We have shown that the analytic treatment of the previous chapter could be applied successfully to the small oscillation equations 3.3.6 through 3.3.9 only in the lower regions of our atmospheric model.

Throughout this lower region the transformed independent variable z (definition 3.3.12) is large enabling us to adopt the asymptotic solution for τ.

In the regions above the critical height at which $|z| \sim O(1)$, we must resort to numerical integration of the equations or examine their series solutions.

This chapter will treat the techniques of, and the results obtained by numerical methods.

4.1 *OUTLINE OF THE METHOD EMPLOYED*

The system of small oscillation equations of Chapter 3 can be reduced to a single fourth-order ordinary differential equation in the complex variable τ. (See 3.3.11.) Solution of this system in real arithmetic implies the solution of two simultaneous fourth-order equations in the real and imaginary parts of τ.

Further, this is essentially an initial value problem as we shall assume lower boundary values at or below the critical level, to obtain continuity with the asymptotic solution which we have

shown to hold from the base of our model to this critical height.

The Runge-Kutta fourth-order integration procedure will be adopted and it is incorporated into a completely generalised computer program to perform the integration for various values of the gas parameters, initial values, height ranges and accuracy control parameters. This stable, self-starting method has decided advantages over other methods and considerable background work was done in order to develope techniques of accuracy control, which is perhaps its greatest difficulty. Furthermore, accuracy is given priority over speed of execution.

These techniques were developed specifically for the equations under consideration, but they should otherwise have good general application.

One restriction of some importance is that the standard Runge-Kutta procedure had to be adopted whereas the modification of Gill (IBM 1130 Scientific Subroutine Package Manual) is preferred. This version compensates for accumulated rounding error. The equations in their transformed form required the superior floating point facilities of the ICT 1301, which, due to shortage of core storage, rendered Gill's modification uneconomical. However, we will show that excellent accuracy is still obtainable in regions of stability.

In our efforts to maintain a high degree of accuracy, separate consideration must be given to the minimisation of the accumulated round-off and truncation errors.

The former can be achieved to the extent that, with careful programming one can minimise the number of round-off errors per step by ensuring minimal arithmetic operations. This further implies maximum simplification of the initial set of equations by transformation, non-dimensionalisation of parameters and similar techniques.

Considerable calculation could then be required to transform the physically meaningful initial values to those now acceptable to the simplified equations. This applies equally to output, but this is considered worthwhile as round-off errors, so introduced, are not accumulated and propagated through the remaining range of integration. This becomes increasingly important when many integration steps are required.

Control of accumulated truncation error is handled as follows: given a step size h, and the vector of dependent variables $\underset{\sim}{y}(x_o)$, an approximation for $\underset{\sim}{y}(x_o+h)$ called $\underset{\sim}{y}^{(1)}(x_o+h)$ is computed. The step size is then halved, and two integration steps are performed from x_o, giving a second approximation $\underset{\sim}{y}^{(2)}(x_o+h)$. From these two approximations a test value δ is calculated as follows:

$$\delta = \frac{1}{m} \sum_{i=1}^{m} \left| y_i^{(1)} / y_i^{(2)} - 1 \right|$$

where only the m non-zero y_i are included in the summation to determine δ, the average relative error. If none is zero, m = n, the number of first order equations. A given tolerance ε_o is specified as a parameter for the procedure and, if

$$\delta < \varepsilon_o$$

then satisfactory convergence has been achieved, and $\underset{\sim}{y}^{(1)}(x_o+h)$ is available both for output, and as the initial vector for the next stage.

On failure of the test, the step-size is halved, and the procedure starts again at the point x_o. A limit L should be set to the number of permitted binary subdivisions of the initial step-size h, to error-flag an unstable region.

We will show that, although the preceding techniques ensure control of local truncation error, over many integration steps this is not sufficient to maintain good accuracy. This is due to the opposing effect of accumulated rounding error which becomes dominant if ε_o is chosen too small, or if L is allowed to be too large. Ideally the total number of integration steps should be minimized, and local truncation error should be of the same order as local round-off error. Optimum values in the choice of ε_o and L can be found by inspection of the average relative error, at each output value of x, obtained by forward integration to x_1, the end of the range, and then integrating back to the point x_o. This is discussed later in further detail.

If L is chosen to be relatively large a significant accumulated round-off error is introduced by repeated subdivision or doubling of h or both. This is avoided by keeping h constant and using the value h/k as required. Here

$$k = 2^m \text{ and } m \leqslant L.$$

This introduces only one round-off error. This can be important if full precision is required to represent h, often the case where a transformation has been performed on the independent variable.

Further, the calculation of the function derivatives of the dependent variables was arranged so that the non-constant coefficients were computed separately, and stored for use by the second approximation stage. This led to the saving of considerable time, and can better be appreciated when applied to the binary gas equations in the next chapter.

Much sophistication can be employed in this context if the set of equations under consideration is sufficiently complex to warrant its implementation.

4.2 *PREPARATION OF THE EQUATIONS*

We refer to Chapter 3 where the equations of motion are: (cf. equations 3.3.6 through 3.3.9.)

$$i\eta - \delta V - V = 0$$

$$\pi = \tau + \eta$$

$$\delta\pi = \tau + \frac{\varepsilon}{y}\,\delta^2 V - i\alpha V = 0$$

$$i\frac{5}{2}\tau - \delta V - \frac{\omega}{y}\,\delta^2\tau = 0$$

These are easily shown to reduce to the following:

$$(\frac{\varepsilon}{y} - i)\,\delta^2 V = -\delta\tau - \tau + i\delta V + i\alpha V$$

$$\frac{\omega}{y}\,\delta^2\tau = i\frac{5}{2}\tau - \delta V$$

or, further we can write

$$\frac{d}{dy}(\delta\tau) = \frac{1}{\omega}\left[i\frac{5}{2}\tau - \delta V\right] \qquad 4.2.1$$

$$\frac{d}{dy}(\delta V) = \frac{\varepsilon + iy}{\varepsilon^2 + y^2}\left[-\delta\tau - \tau + i\delta V + i\alpha V\right] \qquad 4.2.2$$

Now, if we define

$$\tau = y_1 + iy_2$$

$$V = y_3 + iy_4$$

$$\delta\tau = y_5 + iy_6$$

$$\delta V = y_7 + iy_8 \qquad 4.2.3$$

we have four first order differential equations

$$\frac{dy_1}{dy} = \frac{1}{y}\,y_5$$

$$\frac{dy_2}{dy} = \frac{1}{y}\,y_6$$

$$\frac{dy_3}{dy} = \frac{1}{y}\,y_7$$

$$\frac{dy_4}{dy} = \frac{1}{y}\,y_8 \qquad 4.2.4 \text{ through } 4.2.7$$

by definition. If for convenience, we write

$$A = \frac{\varepsilon}{\varepsilon^2 + y^2}, \text{ and } B = \frac{y}{\varepsilon^2 + y^2},$$

we have the remaining four d.es from 4.2.1 and 4.2.2.

$$\frac{dy_5}{dy} = - A\{y_7 + y_1 + y_6 + \alpha y_4\} - B\{y_5 + \alpha y_3 - y_8 - y_2\}$$

$$\frac{dy_6}{dy} = A\{y_5 + \alpha y_3 - y_8 - y_2\} - B\{y_7 + y_1 + y_6 + \alpha y_4\}$$

$$\frac{dy_7}{dy} = - \frac{1}{\omega} \frac{5}{2} y_2 - \frac{1}{\omega} y_5$$

$$\frac{dy_8}{dy} = \frac{1}{\omega} \frac{5}{2} y_1 - \frac{1}{\omega} y_6 \qquad \text{4.2.8 through 4.2.11}$$

These eight equations are in a form suitable to the Runge-Kutta procedure. Probably the most meaningful initial values for this system of equations are those of τ, π, U and $\frac{d\tau}{dx}$. These are easily transformed to satisfy equations 4.2.4 through 4.2.11 through the following relations

$$V = \frac{U}{\sigma H}$$

$$\delta\tau = - H \frac{d\tau}{dx}$$

$$\delta V = \lambda(\pi - \tau) - \frac{U}{\sigma H} \qquad \text{4.2.12}$$

However, as it is also desirable to study the behaviour of equation

3.3.11 by specifying initial values of τ, $\delta\tau$, $\delta^2\tau$ and $\delta^3\tau$, we can easily calculate the equivalent initial values of π, U and $\frac{d\tau}{dx}$ using the relations

$$V = \frac{1}{\alpha}\left[\frac{\omega}{y}\delta^3\tau + \frac{5}{2}\frac{\varepsilon}{y}\delta\tau\right] + \frac{i}{\alpha}\left[\frac{\varepsilon\omega}{y^2}\delta^3\tau - \frac{\varepsilon\omega}{y^2}\delta^2\tau - \frac{7}{2}\delta\tau - \frac{7}{2}\tau\right]$$

$$\text{and } \pi = \frac{7}{2}\tau + i\frac{\omega}{y}\delta^2\tau - iV$$

$$\text{with } U = \sigma HV \text{ and } \frac{d\tau}{dx} = -\frac{1}{H}\delta\tau \qquad 4.2.13$$

On output, examination of τ, π, U and η will suffice. These satisfy the following relations:

$$U = \sigma HV$$

$$\eta = -i(\delta V + V)$$

$$\pi = \eta + \tau \qquad 4.2.14$$

4.3 *INITIAL FINDINGS*

We will perform numerous integrations of the equation set, varying the parameters to obtain some idea of their respective effects. Reference is made to the U.S. Standard Atmosphere, (1962, USSA) for the structure and composition parameter values, and we will take the altitude 160 km as our model base level ($x = 0$). Values for the coefficients of molecular viscosity and thermal conduction are given by Chapman and Cowling for air at 0^0C, and we adjust these for the increased temperature at 160 km. The effect of a diurnal oscillation will be of prime interest.

Assuming reasonable values for the tolerance ε_o, and a limit L to the subdivision of a step-size h equivalent to a transformed altitude increment of 10 km at the base level, we find that the integration procedure is unstable.

It is easily shown that, for this particular parameter set, the region of attempted integration is far below the critical height, discussed in Section 3.7, in which region the rapidly decreasing asymptotic solution exists.

An extremely small step-size is therefore required for convergence which renders the method totally uneconomical.

All attempts to remove this partial instability by transformation of the equations, and other methods proved fruitless. However, we will show that accurate numerical solutions can be found which overlap the known solutions of the lower regions, giving a clear overall picture of the entire range.

One feature, however, is clear: there is a definite 'start height' of integration near the critical height 3.7.1, and it is dependent on the accuracy control parameters ε_o and L. This height can also be expected to vary with the various gas parameters in a manner similar to that of the critical height.

Optimisation of ε_o and L, and the associated error analysis is now discussed.

4.4 *ERROR ANALYSIS*

The model critical height was estimated as being

$$X_c \sim H \log\left(\frac{7}{2} k n_o H^2 \sigma / \lambda\right)$$

in Section 3.7. Now, expecting the start height to have a similar logarithmic dependence on $n_o H^2 \sigma/\lambda$, we perform a set of backward integrations for various values of σ to determine the point before which the limit L is exceeded. This defines the start height for the particular values of $n_o H^2 \sigma/\lambda$, ε_o and L, but we still have no assurance that accuracy is under control.

Table 4.1 presents the start heights found for $\varepsilon_o = \cdot 01$, and L = 4, allowing thus a maximum of 16 integration steps per 10 km altitude increment. The critical heights are shown for reference. Both heights are relative to the model base.

TABLE 4.1

ANGULAR FREQUENCY σ rad./sec	START HEIGHT Km	CRITICAL HEIGHT Km
$\cdot 727 \times 10^{-2}$	550	785
$\cdot 727 \times 10^{-3}$	500	707
$\cdot 727 \times 10^{-4}$	400	628
$\cdot 727 \times 10^{-5}$	320	549
$\cdot 727 \times 10^{-6}$	290	471
$\cdot 727 \times 10^{-7}$	210	392
$\cdot 727 \times 10^{-8}$	110	313
$\cdot 727 \times 10^{-9}$	20	235

We notice that the start heights are approximately 200 km below the critical heights which will ensure overlap of the analytic and

numerical solutions.

We investigate the overall accuracy of the procedure for $\sigma = \cdot 727 \times 10^{-8}$ rad./sec. This is a representative frequency since, having a low start height, relatively many steps are required to reach the arbitary ceiling height of 840 km. (1000 km physical altitude).

Assuming typical gas parameters, $\varepsilon_o = \cdot 01$, a set of simple initial values at 110 km and the above value for σ, we integrate forward to 840 km and then back to 110 km with the initial values for the second run being the final values of the forward run. With the superscripts '1' and '2' denoting the forward and backward dependent variable values respectively at some altitude x_k, we generate the average relative error δ at x_k defined by

$$\delta = \frac{1}{m} \sum_{i=1}^{m} \left| \frac{y_i^{\,1}(x_k)}{y_i^{\,2}(x_k)} - 1 \right|, \quad m \leqslant n$$

with only the m non-zero y_i permitted in the summation. There are n first-order equations.

Log δ and the number of integrations required per 10 km increment, are represented graphically in Figure 4.1.

The error function δ, besides incorporating the overall round-off error comprises also the sum of the accumulated truncation error of both phases. This is true to the extent that it can be shown to include, per step, the sum of the next orders of h in the Taylor series expansion approximated by the Runge-Kutta

LOG δ X10⁻¹
-2
-1
0
1
2
3
4
100
200
300
400
500
600
700
800
MODEL ALTITUDE KM.
2
4
6
8
10
12
14
NO. OF INTEGRATIONS PER 10 KM INCREMENT
ACCUMULATED AVE RELATIVE ERROR

FIGURE 4·1

procedure. We thus have effectively an average fifth order post-mortem error term for the fourth order procedure.

Examination of the results obtained shows that accumulated error completely swamps the solution in the lower region. This is obviously due to the increased number of integrations per 10 km increment in this region. If the limit L is reduced to unity, which increases the start height to 230 km for this case, the accumulated error will be kept within satisfactory limits. This is easily verified by reference to Figure 4.1, where the dotted line indicates the increased start height. We notice that this is still well below the associated critical height for this parameter set.

Having thus decided on a value for L we investigate the role of ε_o in the overall maintainance of good accuracy. It will suffice to discuss our findings.

Retaining the parameters of the previous example, two further error runs were performed; one with $\varepsilon_o = \cdot 01$ and the other with $\varepsilon_o = \cdot 001$, where the increased start height was taken. No great difference was found in the higher regions where local truncation error is less than $\cdot 001$. However, in the lower regions, the stricter tolerance in the second run forced the number of integration steps performed to exceed unity (limit L was increased for this investigation) introducing more round-off error to actually decrease the accuracy fractionally compared with the results obtained for $\varepsilon_o = \cdot 01$. We conclude that a one percent tolerance

is adequate.

Finally, if all start heights are increased sufficiently so that no binary subdivision of the initial 10 km equivalent step-size is required, and if these start heights are re-calculated for any change in its functionally dependent parameters, no further consideration need be given to accuracy.

4.5 *PRODUCTION AND RESULTS*

Resulting from the requirements of the previous section the re-calculated start heights for the various angular frequencies are presented in Table 4.2. Also calculated is the critical height, the constant $\frac{\varepsilon}{2\omega}$, the value of the independent variable $\frac{7y}{2\omega}$ at the start height, and finally the ratio

$$\frac{d|\tau|}{dx}/|\tau| = -\frac{1}{H}\left[\left(\frac{7y}{2\omega}\right)^{\frac{1}{2}} - \frac{1}{4}\right]$$

at the start height. This is obtainable from the asymptotic solution and is valid wherever the solution holds.

TABLE 4.2

σ rad./sec	START HEIGHT Km	CRITICAL HEIGHT Km	$\frac{\varepsilon}{2\omega}$	$\lvert z \rvert$	$\frac{d\lvert\tau\rvert}{dx}/\lvert\tau\rvert$
·727,-2	680	785	·121, 7	21·8	-·13,-5
·727,-3	600	707	·121, 5	22·7	-·13,-5
·727,-4	540	628	·121, 3	13·1	-·99,-6
·727,-5	470	549	·121, 1	10·2	-·86,-6
·727,-6	380	471	·121,-1	14·2	-·10,-5
·727,-7	300	392	·121,-3	14·7	-·11,-5
·727,-8	230	313	·121,-5	11·4	-·92,-6
·727,-9	150	235	·121,-7	11·9	-·94,-6

(The notation a, b means a x 10^{-b}). It is noticeable that a significant range of overlap (the altitude difference between the critical and start heights) still exists and further, that $|z|$ is sufficiently large at the start height to justify the use of the asymptotic solution, and hence of the ratio $\frac{d|\tau|}{dx}/|\tau|$ at this point.

Finally, the relatively large variation of $\frac{\varepsilon}{2\omega}$ compared with $|z|$, having no noticeable effect on the stability of the equations, confirms to some extent the negligible influence of viscosity.

A. *Initial Value Variation*

It is essential to study the behaviour of the equation set under various sets of initial values at the start height. As we

are essentially interested in the amplitude of the resulting temperature oscillation, input of values for τ, $\frac{d\tau}{d}$, $\frac{d^2\tau}{dx^2}$ and $\frac{d^3\tau}{dx^3}$ is preferred.

The diurnal oscillation frequency $\sigma = \cdot 727 \times 10^{-4}$ is chosen.

Since the equations are linear and homogeneous, it is sufficient to limit consideration to those sets of initial values which are linearly independent.

A consistent overall behaviour pattern can be established with the results of the following initial value sets. (The second and third τ derivatives are kept zero for all cases).

1. $\tau \neq 0$, $\frac{d\tau}{dx} = 0$.

This is found to have the trivial solution τ = constant.

2. $\tau = 0$, $\frac{d\tau}{dx} \neq 0$.

Examination of Figure 4.2 shows the solution to be strongly increasing with altitude. A slight indentation occurs in the region of the critical height.

3. Here $\frac{d\tau}{dx}/\tau = 10^{-m} e^{in\pi/4}$, with n = 0, 1, ... 7; and m = 6, 7 and 8. We thus cover the full relative phase range for three different relative amplitude values. Reference to Table 4.2 shows that an appropriate value for m is 6. Figures 4.3 through 4.5 present the sets of varying relative phase curves for the various m values.

LOG τ VERSUS ALTITUDE

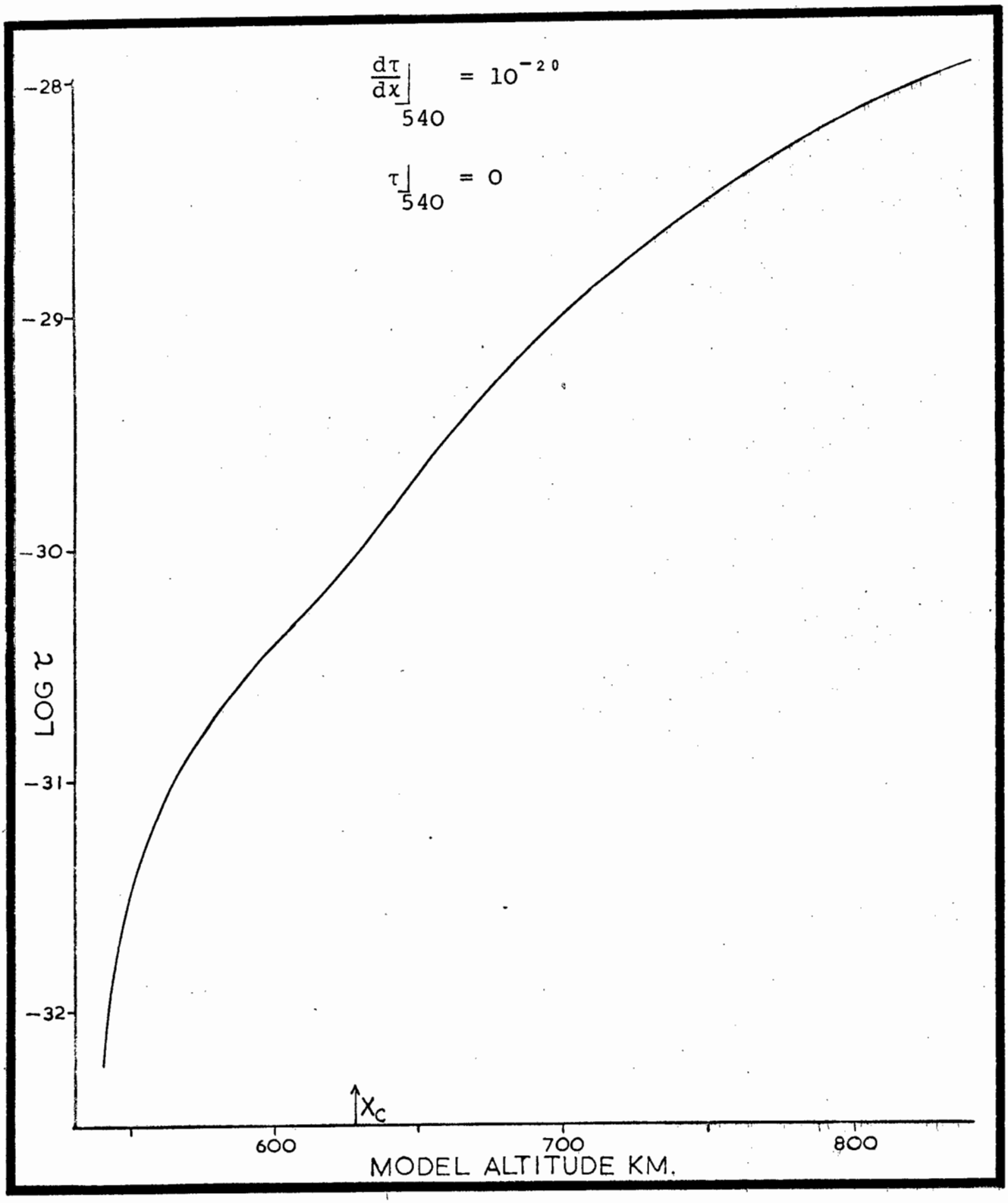


FIGURE 4·2

LOG τ VERSUS ALTITUDE

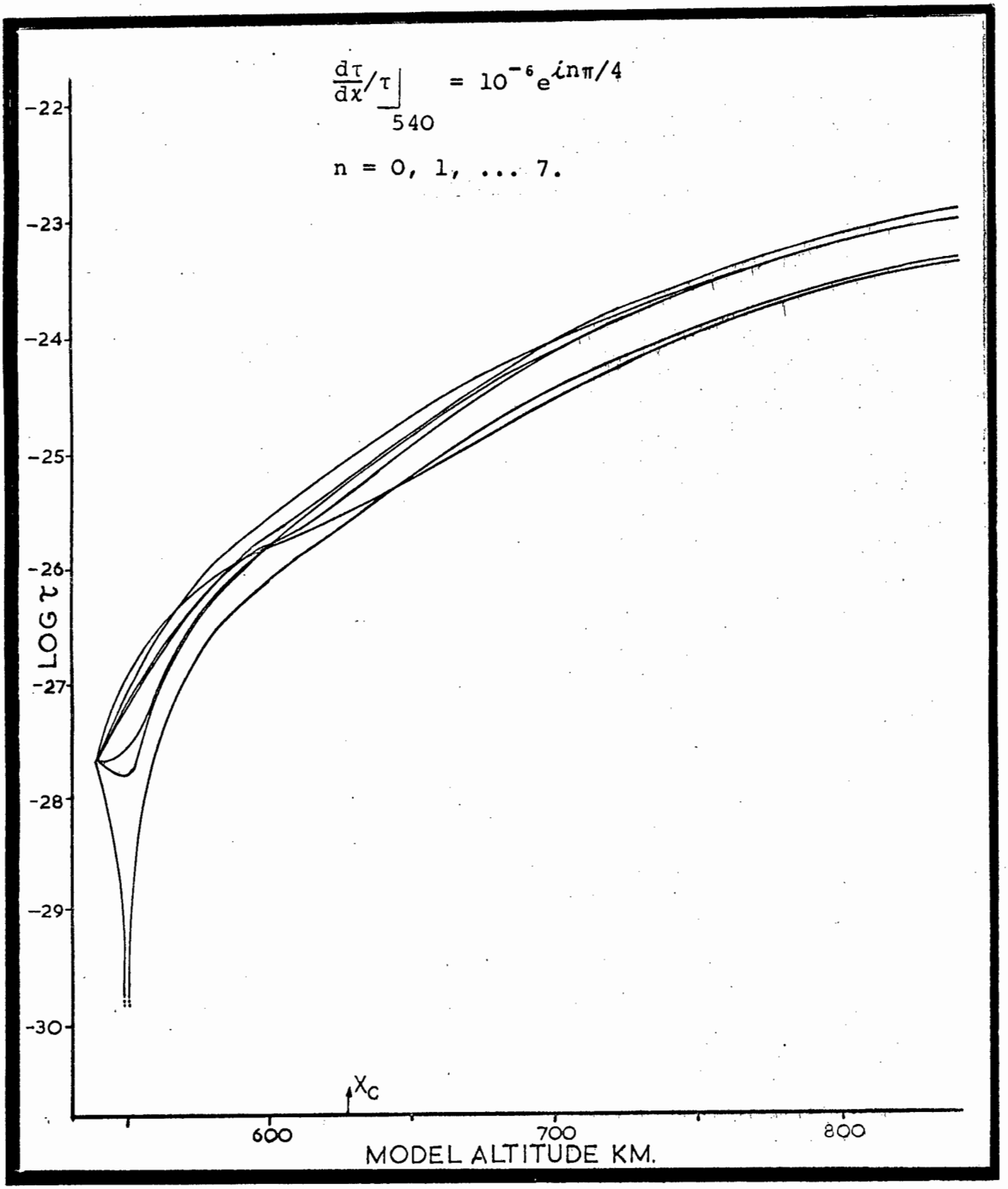


FIGURE 4·3

LOG τ VERSUS ALTITUDE

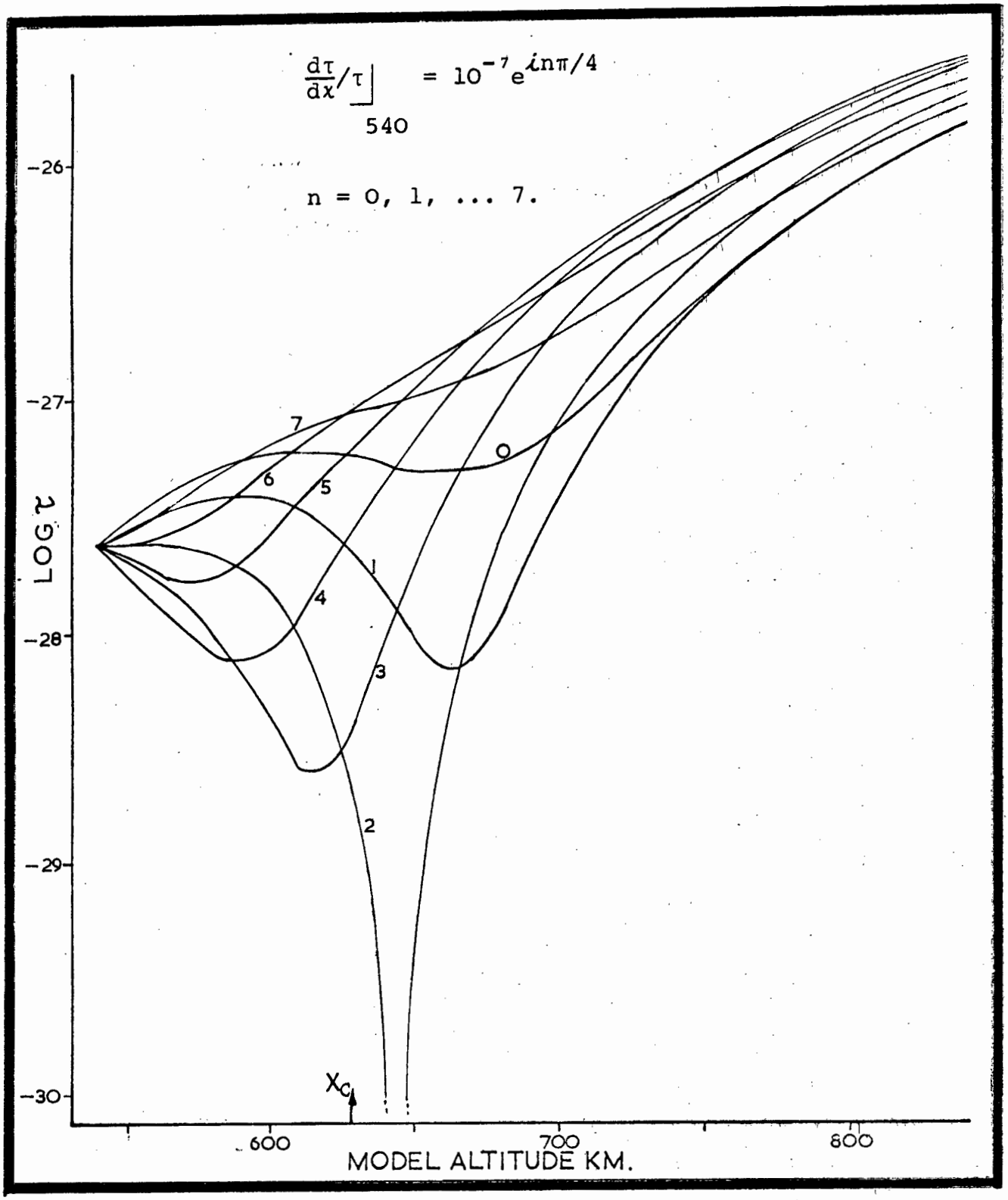


FIGURE 4·4

LOG τ VERSUS ALTITUDE

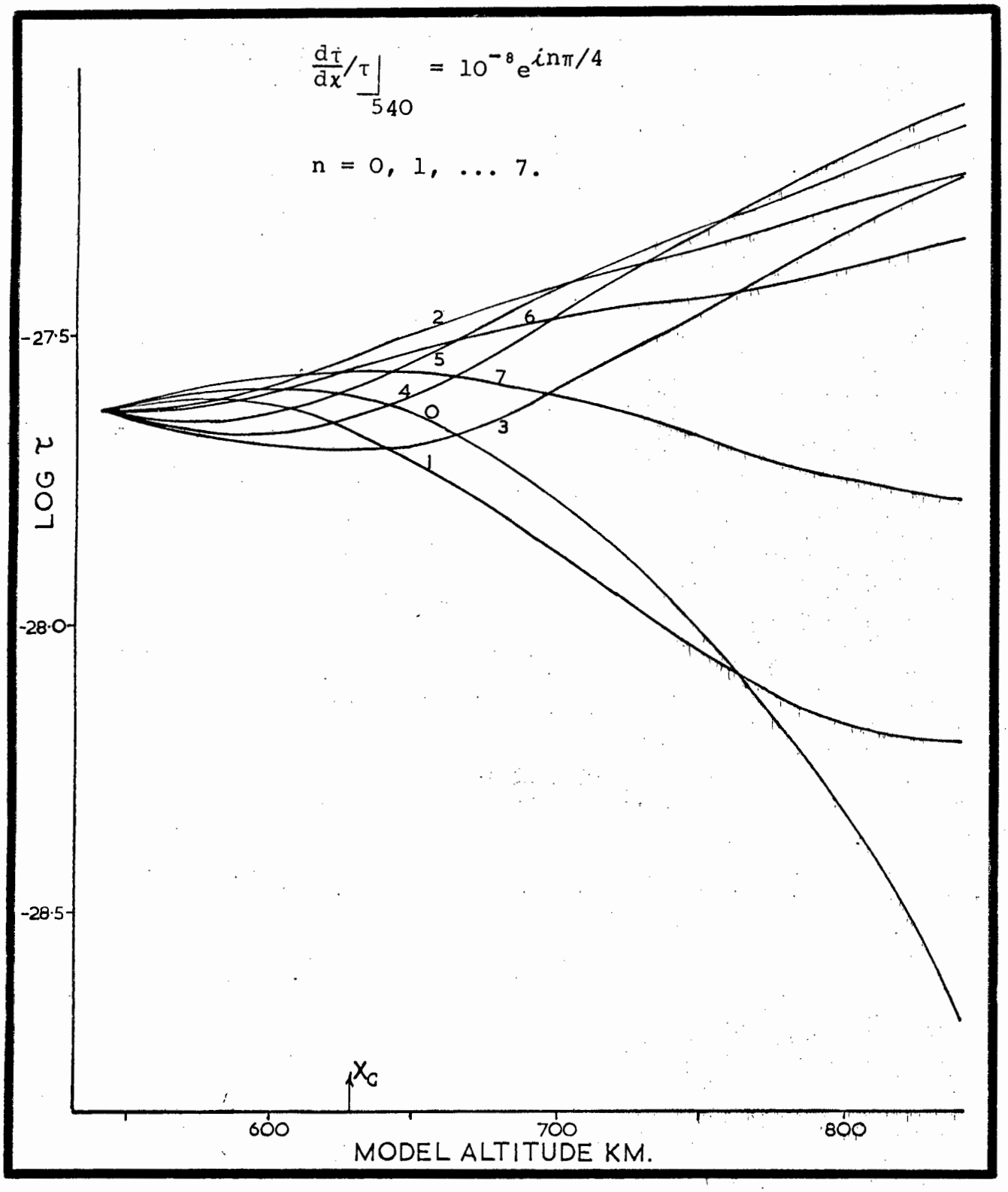


FIGURE 4·5

Two features are prominent: the first being that, irrespective of the phase difference between τ and $\frac{d\tau}{dx}$ an increasing solution exists in the higher regions. Secondly, the smaller m (and hence the greater the amplitude ratio) the stronger the variation. This is to be expected.

Figure 4.4 illustrates best the effect of varying the initial phase difference and further reflects the presence of the minimum which exists in the broad region where the solutions of the two outlined approaches must be joined.

B. *Angular Frequency Variation*

We have seen the important role that σ plays in the location of the critical and start heights (see Table 4.2).

Figure 4.6 represents graphically the family of log τ curves of differing angular frequency. The same set of initial values has been used for each curve and integration was performed from each respective start height.

The solution in general is increasing although an indentation is again present and this is exaggerated for $\sigma = \cdot 727 \times 10^{-5}$. It appears that the angular frequency has a varying damping effect with varying magnitude. This is better illustrated by Figure 4.7 where, again, a family of different curves is plotted.

However, these curves are drawn from a common base altitude and they extend through a common altitude range, thus enabling us better to study this relative effect. Unfortunately this investigation is restricted to the range having as its base the

LOG τ VERSUS ALTITUDE

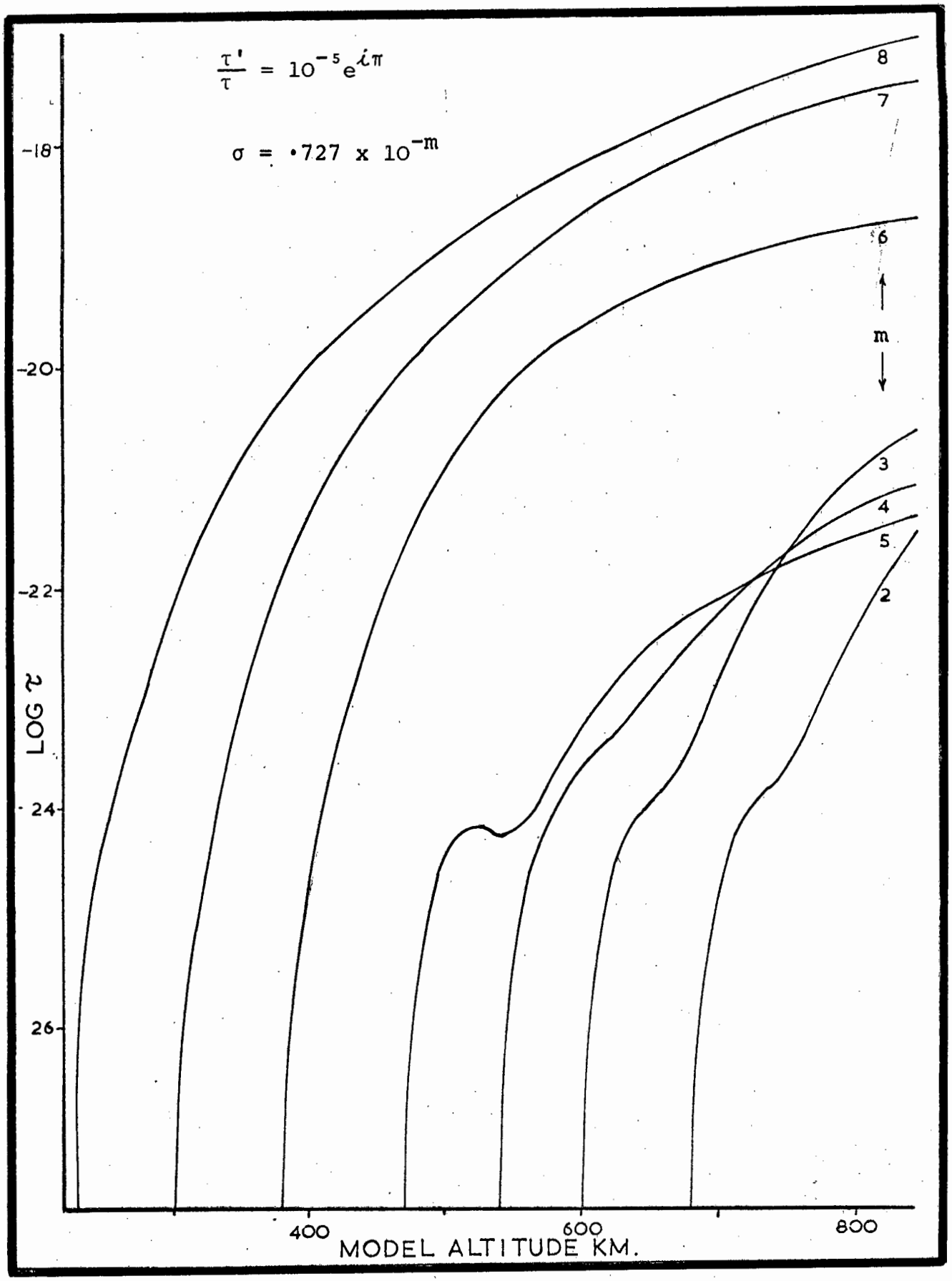


FIGURE 4·6

LOG τ VERSUS ALTITUDE

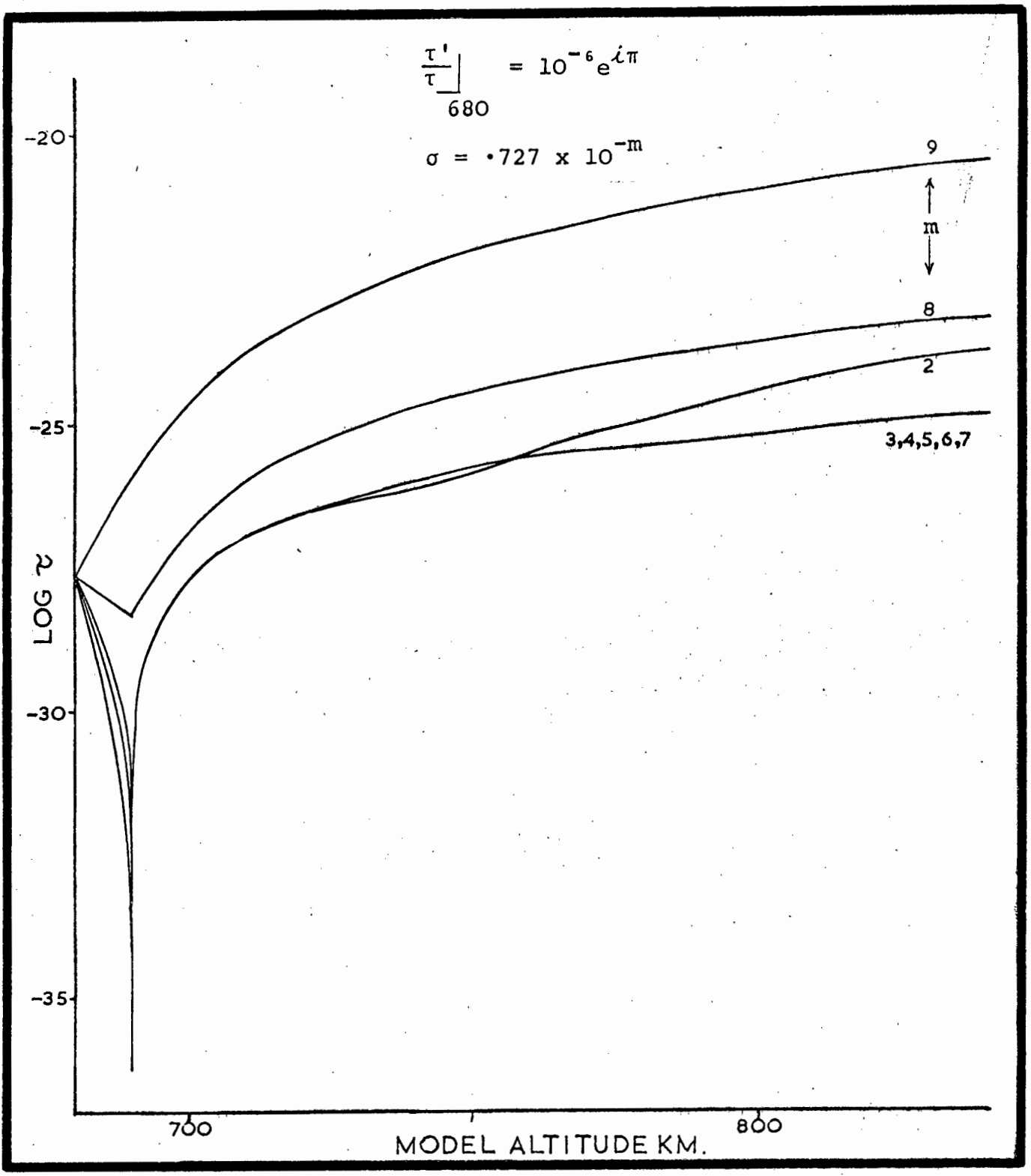


FIGURE 4·7

largest start height (680 km for $\sigma = \cdot 727 \times 10^{-2}$). Further, there are insufficient points in the region immediately above this base height for any general trend to be discernible in this small region. However, in the regions above, we notice that maximum damping is achieved by values of σ in the range $\cdot 727 \times 10^{-7}$ to $\cdot 727 \times 10^{-3}$, and further that in this region their respective effects are relatively indistinguishable.

Smaller and larger values of angular frequency lead to larger τ amplitudes.

C. *Thermal Conduction Coefficient Variation*

A graphical representation of the effects of varying λ, the thermal conduction coefficient, is given in Figure 4.8. Four different values of λ are considered.

The critical height has been shown to vary with λ, and we can write

$$X_2 - X_1 = H \log (\lambda_1/\lambda_2)$$

which enables us to determine the critical height shift if λ is altered from λ_1 to some value λ_2. (X_1, X_2 are the associated critical heights respectively). For the scale height H = 34·17 km applied, we have a critical height shift of approximately 79 km between adjacent curves in Figure 4.8 where the ratio λ_1/λ_2 is 10. The short vertical lines drawn at specific altitudes through each curve indicate their respective shifted critical heights. (For m = 1, $X_c \simeq 471$ km and it can therefore not be accomodated on this

LOG τ VERSUS ALTITUDE

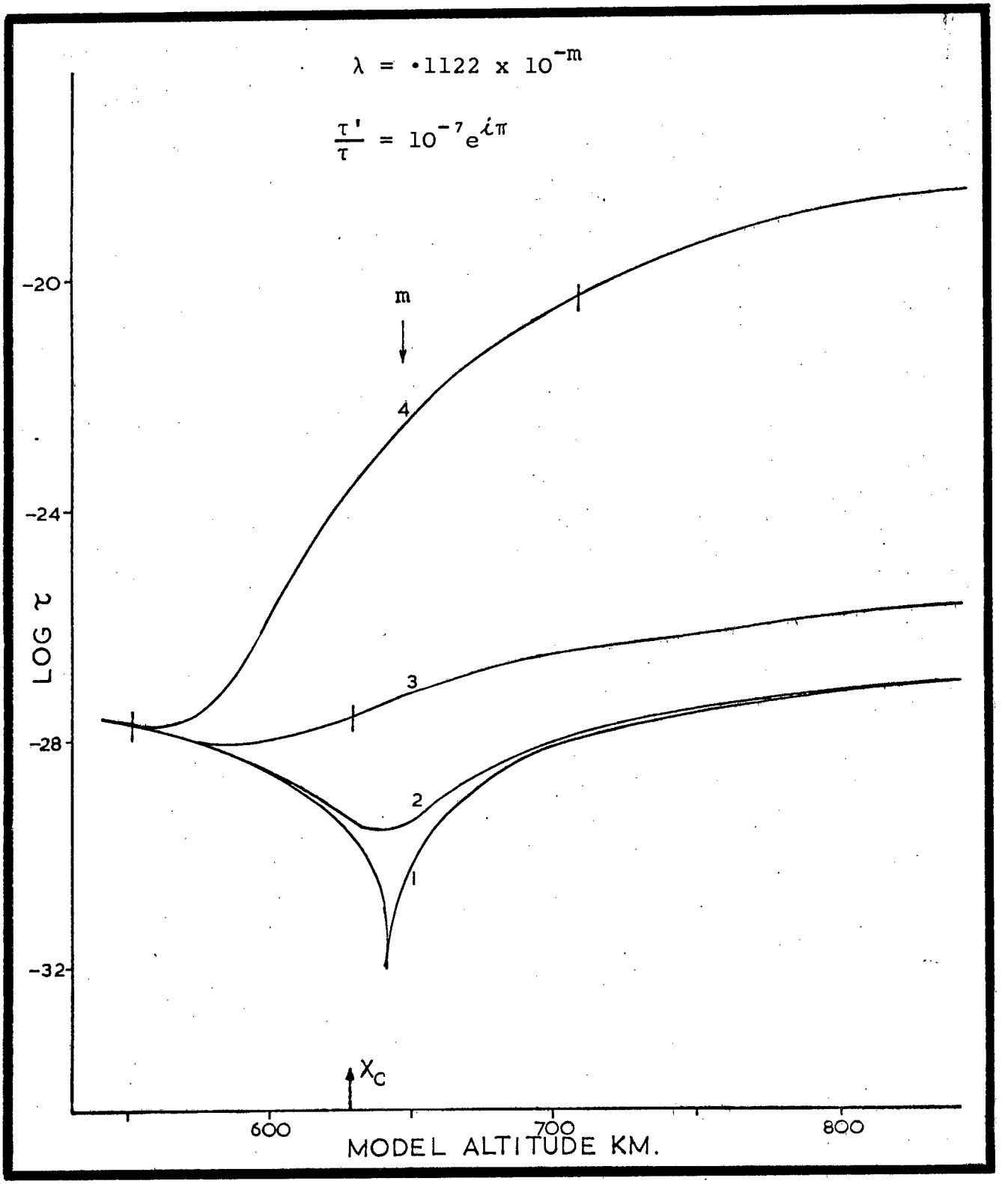


FIGURE 4·8

scale).

Noticeable is the effect of greater damping with increasing magnitude of the thermal conduction coefficient. Further, this effect carries the minimum into higher regions, and in a direction opposite to the related motion of the critical height.

Finally, the effect of increasing the magnitude of the coefficient beyond a certain value appears to accentuate the damping in a decreasing region around the minimum only.

These results confirm the importance of damping by heat conduction in these lower regions, and further that, although the range of damping increases with increasing magnitude of the coefficient, an upper limit to the location of the minimum is indicated, above which only increasing solutions are found.

D. Viscosity Coefficient Variation

The effect of increasing the magnitude of the coefficient of viscosity is similar to that of the thermal conduction coefficient already discussed, in that both are predominantly damping factors.

However, with reference to Figure 4.9, it is evident that the location of the minimum is virtually fixed, and that damping increases with the increasing magnitude of the coefficient. Further, we notice that with the coefficient zero ($m = \infty$) a well behaved variation exists which is consistent with the pattern of decreasing magnitudes.

LOG τ VERSUS ALTITUDE

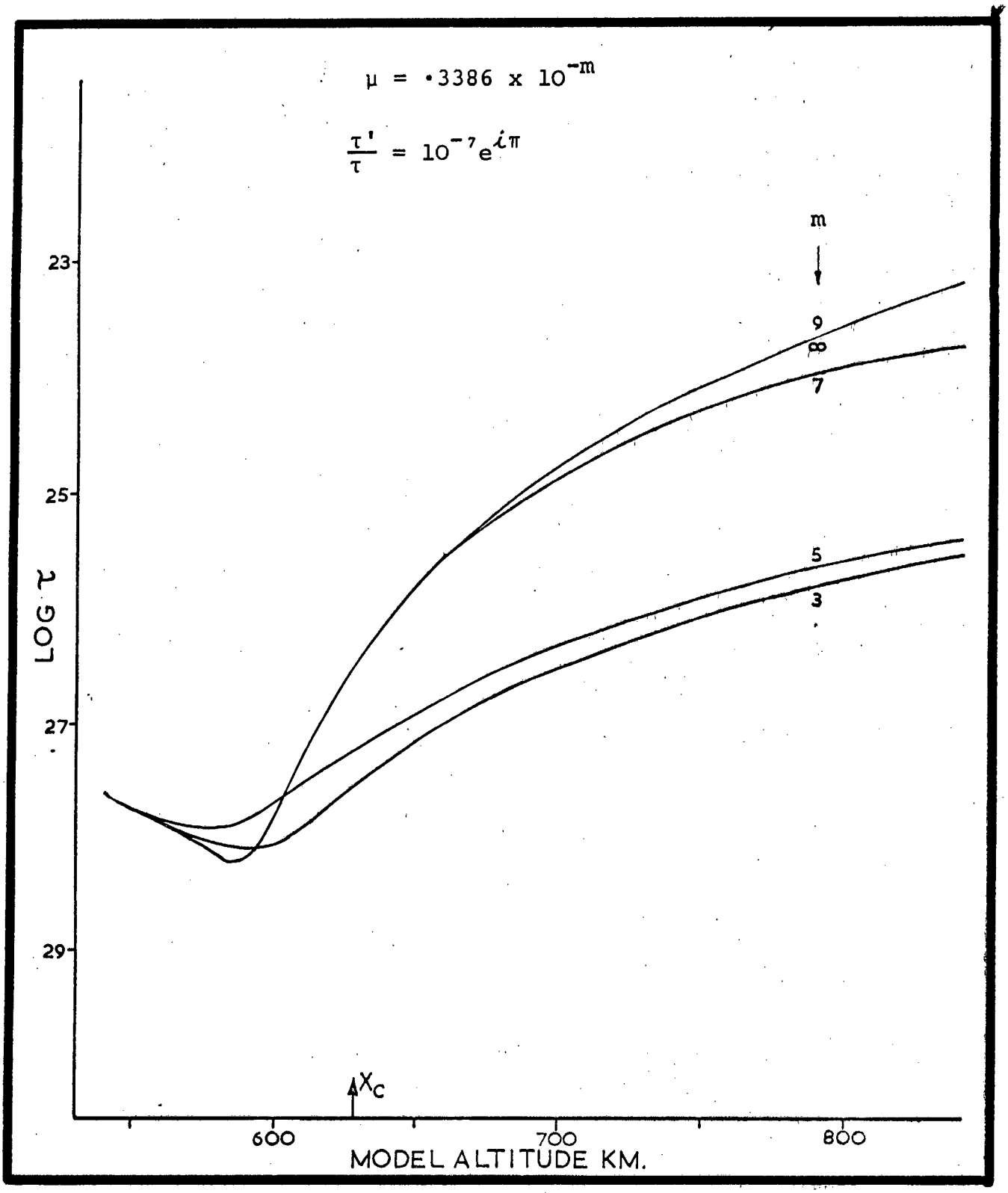


FIGURE 4·9

E. *Scale Height Variation*

The critical height variation with scale height can be represented by

$$X_c \sim H \log (AH^2)$$

for some constant A. Or, we can write

$$X_2 = (\frac{H_2}{H_1})X_1 + 2H_2 \log (\frac{H_2}{H_1})$$

enabling us to calculate a critical height X_2 associated with the scale height H_2, given X_1 and H_1. The respective critical heights are again drawn with each of the different H curves. (Figure 4.10).

The critical height decreases with scale height, and in this respect its variation is similar to that of angular frequency.

Increasing the magnitude of the scale height stimulates the amplitude of the temperature oscillation to larger magnitudes. Again, the minimum moves in a direction opposite to that of the increasing critical height.

Since $H \equiv \frac{kT}{mg}$, any increase in the temperature T, or decrease in the molecular mass m and the gravitational potential g, will counteract the damping influences of viscosity and heat conduction.

In the earth's atmosphere each of these components varies in such a way as to increase the scale height with increasing altitude in the higher regions.

LOG τ VERSUS ALTITUDE

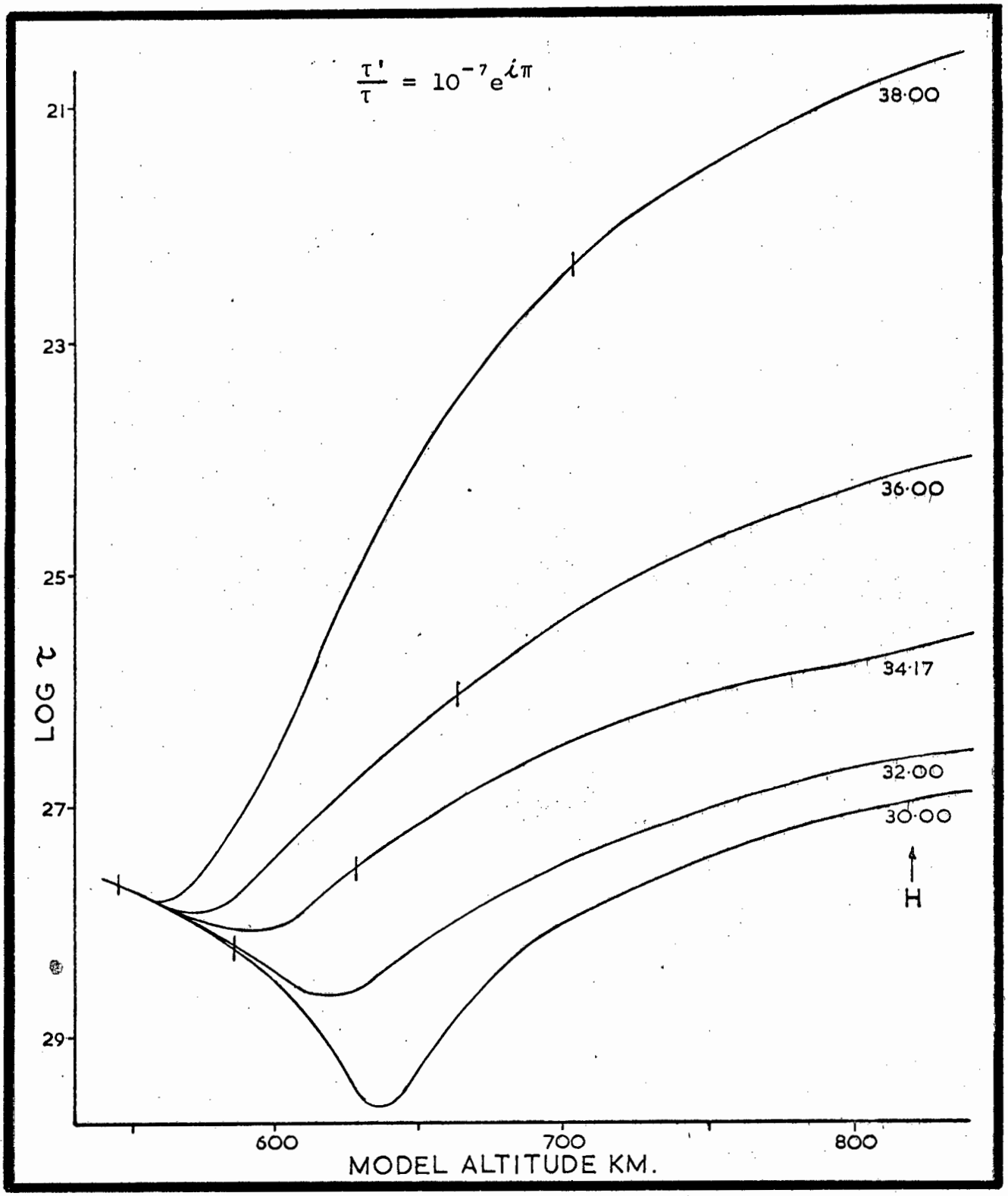


FIGURE 4·10

4.6 *DISCUSSION*

We have shown that in the upper regions of our simple gas atmospheric model, solutions of the forced temperature oscillation about an isothermal steady state differ markedly from those arising in the lower asymptotic regions discussed in Chapter 3.

These higher region solutions of the τ amplitude are strongly increasing compared with the strongly decreasing asymptotic solutions of the denser lower regions. We have shown conclusively the existence of a minimum in the overall solutions, and that its location is in the region of the critical height. This is where the two outlined methods of solution overlap.

Further, it has been established that both heat conduction and viscosity act in such a way as to damp these oscillations, even in the higher regions with the former effect again dominant.

The location of the minimum is largely governed by the frequency of the associated oscillation through its control of the critical height, with minor displacements arising through related parameter choice.

Finally, the increasing higher region solutions can only be accentuated by the effect of local atmospheric scale height increase with height as found in the earth's atmosphere.

CHAPTER 5

NON-ISOTHERMAL SIMPLE GAS SYSTEM

One of the more severe restrictions imposed on the simple gas model discussed hitherto is the assumption of constancy of steady state temperature with height.

We now investigate the effects on the solutions of a simple gas atmospheric model introduced by extending the model to the case of a temperature gradient in the steady state.

It will suffice to compare asymptotic solutions of the two systems under the damping influences of thermal conduction only.

5.1 *STEADY STATE*

Since $\frac{dT_o}{dx} \neq 0$, we must allow for the molecular coefficient of heat conduction λ, to vary thus

$$\lambda = \lambda^* T_o^{\frac{1}{2}} \qquad 5.1.1$$

where λ^* is a constant.

The energy equation 3.2.2 yields

$$\frac{d}{dx}(T_o^{\frac{1}{2}} \frac{dT_o}{dx}) = 0 \qquad 5.1.2$$

and its solution can be expressed as

$$T_o = \theta(1 - \alpha x)^{2/3} \qquad 5.1.3$$

where θ is the lower boundary temperature, and α is a constant. Now, if $|\alpha x| << 1$, we can write

$$T_o \simeq \theta(1 - \beta x) \qquad 5.1.4$$

where $\beta = -\frac{1}{\theta}\frac{dT_o}{dx}$, 5.1.5

the negative sign being chosen in keeping with standard meteorological notation. Further,

$$|\beta x| << 1 \qquad 5.1.6$$

The local scale height H_o, at the lower boundary is defined by

$$H_o = \frac{k\theta}{mg} \qquad 5.1.7$$

The momentum equation now has the steady state solution

$$n_o(x) = n_o(0)(1 - \beta x)^{\frac{1}{\beta H_o} - 1} \qquad 5.1.8$$

This is not readily suitable for algebraic manipulation so we approximate with an exponential fit of the form

$$n_o(x) = n_o(0)e^{-x/H(x)} \qquad 5.1.9$$

which defines some non-constant function $H(x)$. Now, since $|\beta x| << 1$, we find that

$$H(x) \simeq H_o\left[1 + \beta(H_o - \frac{x}{2})\right] \qquad 5.1.10$$

We define the two dimensionless quantities

$$y = x/H_o \text{ and } p = \frac{1}{\beta H_o} \qquad 5.1.11$$

and note that 5.1.6 is equivalent to the assumption that

$$|p| >> y \qquad 5.1.12$$

p is the number of scale heights H_o, at which the lower bound temperature θ is reduced to zero. For typical atmospheric values $|p| >> 1$ and, for the isothermal case, $|p|$ is infinite. On substitution, 5.1.9 can be written as

$$n_o(y) = n_o^* \, e^{\frac{y}{p}(1 - \frac{y}{2})} \qquad 5.1.13$$

where $n_o^* = n(0)e^{-y}$, the isothermal model's distribution function 3.2.4.

5.2 *SMALL OSCILLATIONS*

The reduction of the small oscillation equations is similar to that of Chapter 3. However, we remember that

$$T_o = \theta(1 - y/p)$$

$$\frac{\partial T_o}{\partial x} \neq 0 \quad \text{and that}$$

$$\lambda \simeq \lambda^* \theta^{\frac{1}{2}}(1 - \frac{y}{2p}), \text{ to first order in } \frac{1}{p}.$$

The equations of motion and energy are:

$$\frac{\partial}{\partial y}\{n_o(1 - \frac{y}{p})\pi\} + n_o\eta + \frac{n_o}{H_o g}\frac{\partial V}{\partial t} = 0$$

$$\frac{\partial \eta}{\partial t} + \frac{1}{n_o}\frac{\partial}{\partial y}(n_o V) = 0$$

$$\pi = \tau + \eta$$

$$(1 - \frac{y}{p})\frac{\partial \tau}{\partial t} - \frac{V}{p} + \frac{2}{5}(1 - \frac{y}{p})\frac{\partial V}{\partial y} - \omega^*\frac{\partial}{\partial y}\{(1 - \frac{y}{2p})\frac{\partial}{\partial y}(1 - \frac{y}{p})\tau\} = 0$$

where $V = \frac{U}{H_o}$ and $\omega^* = \frac{2\lambda^*\theta^{\frac{1}{2}}}{5kn_oH_o^2}$ 5.2.1 through 5.2.4

Define $W = i\frac{V}{\sigma}$

$$\alpha = \frac{\sigma^2}{gH_o}$$

$$\tau^* = (1 - \frac{y}{p})\tau$$

and $\omega = \frac{i\omega^*}{\sigma}$ 5.2.5

σ is introduced as the angular frequency in the elimination of the time derivatives as in Chapter 3. We substitute the quantities 5.2.5 into equations 5.2.1 through 5.2.4, and elimination of π leaves the three equations

$$\eta = \frac{1}{n_o}\frac{d}{dy}(n_oW)$$

$$\frac{d}{dy}(n_o\tau) + \frac{d}{dy}\left[n_o(1 - \frac{y}{p})\eta\right] + n_o\eta + \alpha n_oW = 0$$

$$\tau + \frac{W}{p} - \frac{2}{5}(1 - \frac{y}{p})\frac{dW}{dy} + \frac{\omega}{n_o}\frac{d}{dy}\left[(1 - \frac{y}{2p})\frac{d\tau}{dy}\right] = 0$$

5.2.6 through 5.2.8

where the asterisk has been dropped from τ^* and it is implied hereafter.

Define $\theta = e^{-(1 - \frac{1}{p})y - \frac{y^2}{2p}}$, and

$$\delta = \theta\frac{d}{d\theta} \qquad 5.2.9$$

(θ is not to be confused with the lower bound temperature which no longer appears explicitly.)

Elimination of W, η and their derivatives, to first order in $\frac{1}{p}$, leaves the equation

$$P(\tau) + \omega Q(\tau) = 0$$

with $$P(\tau) = \left[1 + \frac{2}{p}(y - 1)\right]A(\tau)$$

$$Q(\tau) = \left[1 + \frac{1}{2p}(7y - 8)\right]\frac{5}{7n_o}B(\tau)$$

where $$A(\tau) = \left[\delta^2 + (1 - \frac{4}{7p})\delta - \frac{4}{7p}\right]\tau$$

$$B(\tau) = \left[\delta^2 + q_1\delta + q_2\right]\delta^2\tau$$

and $$q_1 = -\left[1 + \frac{1}{14p}(7y + 71)\right]$$

$$q_2 = \frac{1}{2p}(y + 5) \qquad 5.2.10$$

Thus $A(\tau) = -\frac{5\omega}{7n_o}\left[1 + \frac{1}{2p}(3y - 4)\right]B(\tau)$ for which we require the asymptotic solution.

Now, to first order in $\frac{1}{p}$,

$$\left[1 + \frac{1}{2p}(4 - 3y)\right] n_o(y) \simeq n_o(0)\left(1 + \frac{2}{p}\right)\phi$$

where $\phi = \theta^{1 + \frac{3}{2p}}$ 5.2.11

Also $\delta_\theta = \left(1 + \frac{3}{2p}\right)\delta_\phi$, with the sub-scripts θ and ϕ denoting the respective independent variables.

Defining $\psi = -\frac{7n_o(0)}{5\omega}\left(1 - \frac{1}{p}\right)\phi$ 5.2.12

we can write

$$\left[\delta^2 + \beta_1\delta + \beta_2\right]\tau = \frac{1}{\psi}\left[\delta^2 + \alpha_1\delta + \alpha_2\right]\delta^2\tau$$

where $\delta = \delta_\psi \equiv \psi\frac{d}{d\psi}$, and

$$\beta_1 = 1 - \frac{29}{14p}, \quad \beta_2 = -\frac{4}{7p},$$

$$\alpha_1 = -\left[1 + \frac{1}{14p}(7y + 50)\right] \text{ and } \alpha_2 = \frac{1}{2p}(y + 5) \qquad 5.2.13$$

Now, since 5.1.12 requires that $|p| >> y$, we have an upper bound imposed on the model and, if $\bar{y}$ is defined as an average value of y in this range, we can safely substitute $\bar{y}$ for y in α_1 and α_2 without significantly affecting the form of equation 5.2.13. This is due to the relatively slow linear variation of y compared with the exponential form of ψ.

With α_1 and α_2 now constant, we can show 5.2.13 has the

asymptotic solution

$$\tau^* \sim \exp(2\psi^{\frac{1}{2}})\psi^{-\frac{1}{4}\left[1 - \frac{1}{p}(\bar{y} + 3)\right]} \qquad 5.2.14$$

with the asterisk reinstated.

But, to the first order in $\frac{1}{p}$,

$$\tau \simeq (1 + \frac{y}{p})\tau^*, \text{ or}$$

$$\tau \simeq \psi^{-\frac{1}{p}}.\tau^*$$

Substitution for τ^* in 5.2.14 gives

$$\tau \sim \exp(2\psi^{\frac{1}{2}})\psi^{-\frac{1}{4}\left[1 - \frac{1}{p}(\bar{y} - 1)\right]} \qquad 5.2.15$$

Reconstruction of ψ yields

$$\psi = z.\exp(-\frac{1}{2p}Y)$$

where $z = \frac{7n_o(0)}{5\omega}e^{-y}$ is the independent variable in the isothermal asymptotic solution, and

$$Y = 2 + y + y^2 \qquad 5.2.16$$

We can now write 5.2.15 in terms of z for comparison with the isothermal result. We find

$$\tau \sim \exp(2z^{\frac{1}{2}}\varepsilon_1).z^{-\frac{1}{4}\varepsilon_2}$$

where $\varepsilon_1 = \exp(-\frac{1}{4p}Y)$, and

$$\varepsilon_2 = z^{\frac{\bar{y} - 1}{4p}} . \exp(\frac{1}{8p}Y) \qquad 5.2.17$$

As expected, if $p \to \infty$, then $\varepsilon_1, \varepsilon_2 \to 1$. This agrees with the isothermal case. We define a ratio function

$$R(p,y) = \frac{\tau_{NI}}{\tau_I}$$

where τ_{NI} is the non-isothermal solution and τ_I is the corresponding isothermal asymptotic solution of the τ amplitude factor. Thus

$$R(p,y) = C.\varepsilon_2 \exp\left[2z^{\frac{1}{2}}(\varepsilon_1 - 1)\right] \qquad 5.2.18$$

C is chosen so that $R(p,o) = 1$. We will compute values of this function R to study its behaviour for different steady state temperature gradients.

5.3 *RANGES OF VALIDITY*

The asymptotic solution 5.2.15 is valid only for $|\psi| > 1$, and the range of y over which this holds is dependent on p. Further, 5.1.12 requires that $y << |p|$. Let us assume that the latter requirement is satisfied provided that $y \leqslant \varepsilon|p|$, where ε is a small tolerance such that $\varepsilon < 1$. Finally, we define y_{max} to be the largest value of y to satisfy the stricter requirement.

We note that $\psi = \frac{7n_o(0)}{5\omega} e^{-\left[y + \frac{1}{2p}(2 + y + y^2)\right]}$ and consider separately the cases of $p > 0$ and $p < 0$.

The case of p positive:

For $p > 0$, $\frac{d\psi}{dy} < 0$ always and y_{max}, where $|\psi| = 1$, is given as the positive root of

$$y^2 + (1 + 2p)y + 2(1 - pM) = 0 \qquad 5.3.1$$

where $M = \log(\frac{7n_o(0)}{5|\omega|})$. However, the restriction that $y \leqslant \varepsilon|p|$ might be the stricter, giving $y_{max} = \varepsilon p$.

We can show that there exists a critical value p^* such that, for

$0 < p \leqslant p^*$, $y_{max} = \varepsilon p$, whereas for $p > p^*$,

y_{max} is the positive root of 5.3.1 5.3.2

For typical values, $M >> \varepsilon$ and

$$p^* \simeq M(\frac{2 - \varepsilon}{2\varepsilon}) \qquad 5.3.3$$

The case of p negative:

For $p < 0$, or $\frac{dT_o}{dx} > 0$, ψ decreases with y to its minimum at $|p| - \frac{1}{2}$. But since $y << |p|$ and $|p| >> 1$, ψ remains a decreasing function in the range of validity of the asymptotic solution. Only for large $|p|$ does ψ ever approach unity, and only then, provided that $p \leqslant p_1$, can we derive a value y_{max} from 5.3.1.

P_1 is the negative root of

$$(1 + p)^2 = 8(1 - pM) \qquad 5.3.4$$

Therefore, for $p \leqslant p^* \leqslant p_1 < 0$, y_{max} is the smaller positive root of 5.3.1, whereas for $p > p^*$,

$$y_{max} = \varepsilon|p| \qquad 5.3.5$$

Similarly $p^* \simeq - M(\frac{2 + \varepsilon}{2\varepsilon})$ 5.3.6

These ranges and conditions apply to 5.2.15. For the range of validity of the ratio function R(p,y), we must include the restriction on the isothermal solution, i.e.

$$y \leqslant M \qquad 5.3.7$$

which can restrict further only the case $p < 0$.

5.4 *COMPUTATION OF R(p,y) AND RESULTS*

A short computer program was written to tabulate log R for various values of p; y was limited to the range $0 \leqslant y \leqslant y_{max}$.

Reference is made to the COSPAR International Reference Atmosphere, 1961 (CIRA) for structure parameter values. A diurnal oscillation is considered and the altitude 160 km is chosen as the model base. For the parameter set chosen M = 17·7 and we take ε = ·2.

Figure 5.1 presents a family of log R curves for a number of p values. Two features are clear: the first is that the sign of

LOG R VERSUS Y

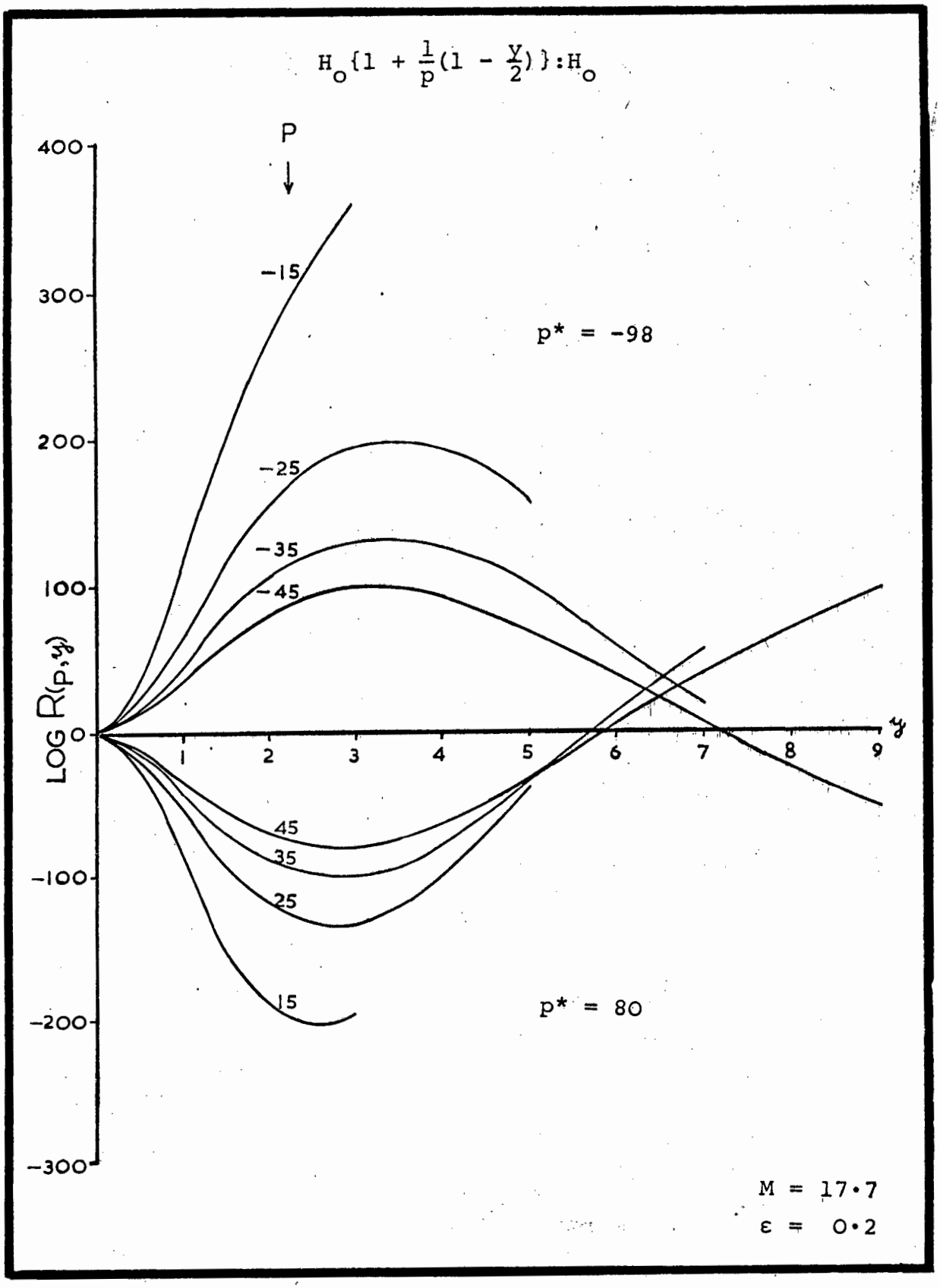


FIGURE 5·1

p separates almost inverse effects and secondly, an inversion is present in each p curve. This latter effect is interesting and can be explained on examination of the function H(y). From 5.1.10, rewritten, we have

$$H(y) = H_o\left[1 + \frac{1}{p}(1 - \frac{y}{2})\right] \qquad 5.4.1$$

where $H_o = \frac{k\theta}{mg}$, the local scale height at the model base. We distinguish between the local scale height $H_o(y)$ at any point and the function H(y). $H_o(y)$ is, by definition, such that

$$H_o(y) = \frac{kT}{mg} = \frac{k\theta}{mg}(1 - \frac{y}{p}) = H_o(1 - \frac{y}{p}) \qquad 5.4.2$$

whereas the latter, H(y), is that value defined by 5.1.9.

At the lower boundary

$$H(0) = H_o(1 + \frac{1}{p}) \qquad 5.4.3$$

and $H_o(0) = H_o$, so that for

$\frac{dT}{dx} < 0$, $H(0) > H_o(0)$, and for

$\frac{dT}{dx} > 0$, $H(0) < H_o(0)$

Further, we note that the gradient of H(y) is one half that of $H_o(y)$. From 5.4.1 it is evident that $H(y) = H_o$ at $y = 2$, and in the higher regions, the above inequalities are reversed.

Figure 5.1 presents curves derived by comparing a non-isothermal simple gas model with an isothermal model, each having the same lower bound local scale height H_o. However, the lower bound values H(O) differ by a term $\frac{1}{p}H_o$, which gives rise to the rapid increase of $|\log R|$. This rise is halted and a maximum is reached soon after H(y) and H_o are equal. In the higher regions the effect is inverted.

As the inversion can be accounted for by the initial difference $\frac{1}{p}H_o$ between H(y) and H_o, it is interesting to compare the non-isothermal model with an isothermal system, the latter having a constant scale height $H_o(1 + \frac{1}{p})$. The lower bound difference is thus zero.

The results are illustrated in Figure 5.2 where the inversion is noticeably reduced. This suggests that in the immediate lower regions of the model, an isothermal model with an adjusted scale height of $H_o(1 + \frac{1}{p})$ gives a far better representation than does one with scale height H_o.

Finally, in these dense lower regions the presence of a positive gradient in the steady state temperature is a damping effect reducing in magnitude with increasing p, and vice versa. The opposite is true for a negative gradient. Typical values of p are discussed in the next section.

5.5 *GENERALISATION OF SCALE HEIGHT VARIATION*

Since the whole effect is virtually explained by examination of H(x), we can make reasonable preditions by generalising the

LOG R VERSUS Y

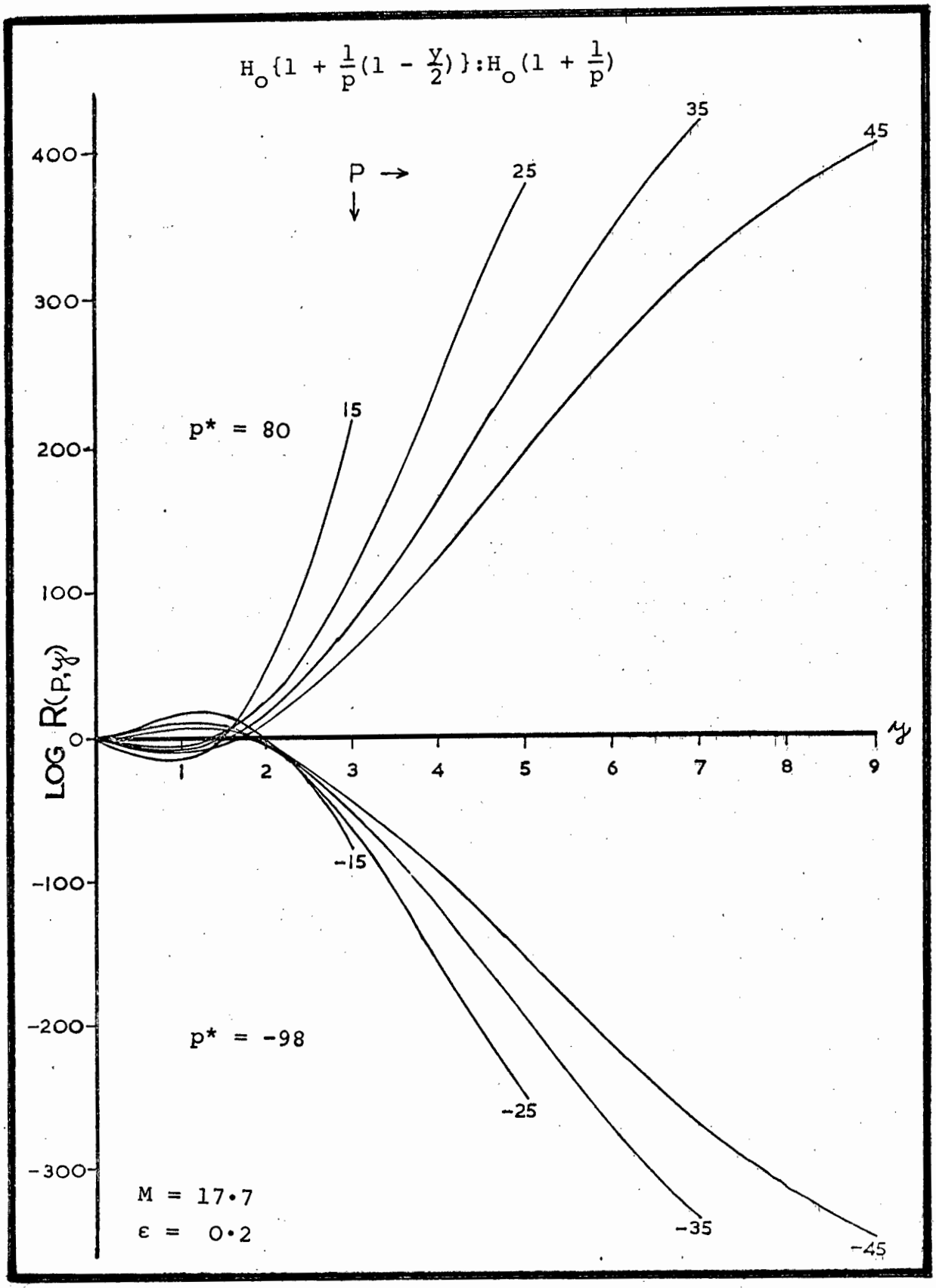


FIGURE 5·2

model somewhat.

Increase of scale height with altitude in the thermosphere is a result of some combination of temperature increase, molecular weight decrease and gravitation decrease.

If we allow these to vary linearly with height, such that

$$T = \theta(1 + \beta x) \qquad \text{(note the change in sign)}$$

$$m = m_o(1 - \alpha x)$$

$$g = g_o(1 - \nu x) \qquad 5.5.1$$

where α, β and ν are small quantities, we can show that, to first order of smallness, the steady state number density

$$n(x) = n(0)e^{-x/H(x)}$$

$$\text{where } H(x) = H_o\left[(1 - \beta H_o) + \frac{\alpha + \nu + \beta}{2}x\right] \qquad 5.5.2$$

and, $H_o = \frac{k\theta}{mg}$ as before.

The only generalisation to $H(x)$ is in the gradient which is now one half the sum of the contributing gradients.

We should therefore expect the set of log R curves arising from this model to have much the same form as those of Figure 5.1, with the turning points nearer the base level, and having increased magnitudes overall. To obtain an estimate of the magnitude of this effect, we will fit a set of typical atmospheric parameters to the model.

The COSPAR International Reference Atmosphere 1961 (CIRA) is used for reference. (Craig p. 268). Figure 5.3 shows the variation of temperature and mean molecular weight. The variation of g is well known, and it can be represented by a linear function to a high degree of accuracy.

Specific regions of the model will be examined by applying the preceding theory assuming that, within these regions, the variation of T, m and g is linear.

We define the theoretical density $\rho^*(x)$ such that

$$\rho^*(x) = \rho(0).(1 - \alpha x)e^{-\frac{x}{H(x)}} \qquad 5.5.3$$

allowing for the variation in m, since $\rho = nm$.

In the CIRA, temperature rises rapidly from 120 km to 160 km; in this region $p \sim 1{\cdot}5$ which is unsuitable. Further $|\beta| >> \alpha$ and the temperature increase dominates the mass decrease, to increase $H(x)$.

However, from 160 km to 250 km, excellent linear representation can be made. Using the upper and lower bound parameter values, we find that

$$p = -19{\cdot}27$$
$$|\beta| = {\cdot}001406$$
$$\alpha = {\cdot}000936$$
$$\nu = {\cdot}000301$$

which satisfies all restrictions of the analysis. Table 5.1 presents values of $\rho^*(x)$ and $H(x)$ for comparison with the observed

COSPAR INTERNATIONAL REFERENCE ATMOSPHERE

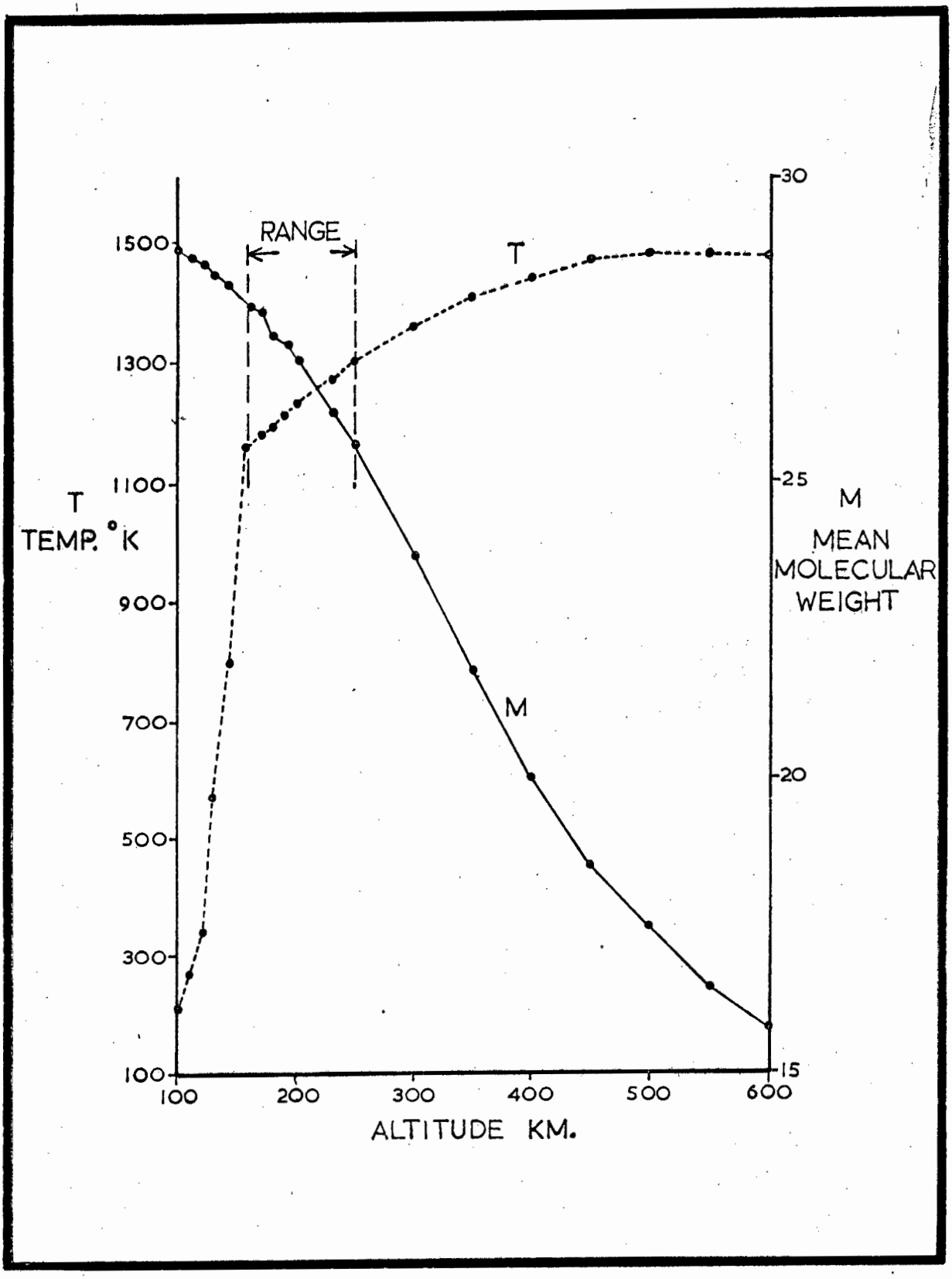


FIGURE 5·3

density values $\rho(x)$.

TABLE 5.1

ALTITUDE km	OBSERVED VALUES	GENERALISED MODEL		NON-ISOTHERMAL MODEL	
	$\rho(x)$	$\rho^*(x)$	$H(x)$	$\rho^*(x)$	$H(x)$
160	·111,-13	·111,-13	34·98	·111,-13	34·98
170	·826,-14	·829,-14	35·47	·836,-14	35·24
180	·659,-14	·625,-14	35·96	·632 -14	35·50
190	·473,-14	·474,-14	36·44	·480,-14	35·76
200	·361,-14	·362,-14	36·93	·366,-14	36·02
230	·167,-14	·168,-14	38·39	·166,-14	36·80
250	·103,-14	·103,-14	39·37	·995,-15	37·32

(The notation a, -b denotes a x 10^{-b}).

We notice that both models represent the observed density values exceptionally well, confirming the validity of the analysis for this case. A better fit is obtained with the generalised model however, due to the inclusion of the variation of m, and to a lesser extent, that of g.

Finally, in the higher regions, the temperature gradient is small and, although reasonable results are obtainable if all effects are considered, the mass variation dominates rendering a purely non-isothermal model inadequate.

CHAPTER 6

BINARY GAS SYSTEM

The previous three chapters have treated atmospheric models consisting of only one diatomic gas, while, as it is well known, our earth's atmosphere comprises a mixture of gases.

In the regions immediately above the earth's surface, the proportion of water vapour is significant and this fluctuates wildly with the prevailing local weather conditions. In the higher regions however, a binary gas model, consisting of the two major atmospheric components, can be adopted with a reasonably good representation.

In this chapter we will consider a model comprising two diatomic gases in a single dimensional frame of reference, with the model base level ($x = 0$) chosen at a convenient altitude where relatively stable atmospheric conditions are found. The gravitational potential $-g$, is constant, and we employ a method of analysis similar to that of Chapter 3.

6.1 *DEFINITIONS AND EQUATIONS OF MOTION*

We define the following:

U, the mass-velocity of the system; V_1, V_2, the mean velocities of the constituent gases; m_1, m_2, the particle masses; n_1, n_2, the particle number densities; T, P, respectively the temperature and the pressure; ρ_1, ρ_2, the partial densities

respectively; k, Boltzman's constant; λ, μ, respectively the coefficients of molecular thermal conduction and viscosity; D_{12}, the coefficient of mutual diffusion; K_T, the thermal diffusion ratio.

The total number density

$$n = n_1 + n_2,$$

and, the total density

$$\rho = \rho_1 + \rho_2.$$

The mass difference Δm, is defined by

$$\Delta m = m_2 - m_1.$$

Further, we shall use the diffusion factor α, defined by

$$\alpha = \frac{n^2}{n_1 n_2} K_T, \qquad 6.1.1$$

throughout as it varies less with density than does K_T.

On reduction of the continuum equations of motion and energy for a binary gas mixture in a non-uniform state (Chapman and Cowling) for our one-dimensional model, we have

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x}\left[n_1(u + V_1)\right] = 0, \qquad 6.1.2$$

and

$$\frac{\partial n_2}{\partial t} + \frac{\partial}{\partial x}\left[n_2(u + V_2)\right] = 0 \qquad 6.1.3$$

from continuity considerations.

6.2 STEADY STATE

We consider the steady state for which

$$\frac{\partial}{\partial t} \equiv 0,$$

and $U = V_1 = V_2 \equiv 0.$

The equations of motion reduce to

$$\frac{dP}{dx} + \rho g = 0, \qquad 6.2.1$$

$$P = nkT, \qquad 6.2.2$$

$$\frac{d}{dx}\left(\frac{n_1}{n}\right) + \frac{n_1 n_2}{n\rho} \Delta m \frac{1}{p} \frac{dP}{dx} + \frac{n_1 n_2}{n^2} \alpha \frac{1}{T} \frac{dT}{dx} = 0, \qquad 6.2.3$$

and $$\lambda \frac{d^2T}{dx^2} = 0, \qquad 6.2.4$$

where λ is assumed to be constant. This is true if we consider the isothermal solution of 6.2.4.

It can be shown that for $T = T_o$, a constant, equations 6.2.1 to 6.2.3 are satisfied with

$$n_1(x) = n_1(0)e^{-x/H_1},$$

and $$n_2(x) = n_2(0)e^{-x/H_2} \qquad 6.2.5$$

as the respective particle number densities for the two constituent gases. The scale heights H_1, H_2 are defined by

$$H_1 = \frac{kT_o}{m_1 g}, \text{ and } H_2 = \frac{kT_o}{m_2 g}. \qquad 6.2.6$$

6.3 SMALL OSCILLATION EQUATIONS

By defining a set of small, dimensionless quantities η_1, η_2, π and τ, we can linearize the original set of equations. (The 'o' in the following relations indicates the steady state solutions).

$$n_1 = n_1^{o}(1 + \eta_1)$$

$$n_2 = n_2^{o}(1 + \eta_2)$$

$$P = P_o(1 + \pi)$$

$$T = T_o(1 + \tau) \qquad 6.3.1$$

Each of η_1, η_2, π and τ is dependent on both x and t, and we also have that $U(x,t)$, $V_1(x,t)$ and $V_2(x,t)$ must be small quantities.

We substitute these dependent variables into the original set of equations and retain terms to the first order of small quantities. This gives, on substitution and deletion of the suffix 'o',

$$\frac{\partial \eta_1}{\partial t} + \frac{1}{n_1}\frac{\partial}{\partial x}\{n_1(U + V_1)\} = 0, \qquad 6.3.2$$

$$\frac{\partial \eta_2}{\partial t} + \frac{1}{n_2}\frac{\partial}{\partial x}\{n_2(U + V_2)\} = 0, \qquad 6.3.3$$

$$\pi = \tau + \frac{n_1}{n}\eta_1 + \frac{n_2}{n}\eta_2, \qquad 6.3.4$$

$$\frac{\partial}{\partial x}(n\pi) - \omega\frac{\partial^2 U}{\partial x^2} - \varepsilon(\rho_1\eta_1 + \rho_2\eta_2) - \varepsilon/g\ \rho\frac{\partial U}{\partial t} = 0, \qquad 6.3.5$$

$$\rho_1 V_1 + \rho_2 V_2 = 0, \qquad 6.3.6$$

$$V_1 - V_2 = - D_{12}\left[\left(\frac{\partial \eta_1}{\partial x} - \frac{\partial \eta_2}{\partial x}\right) + \varepsilon \Delta m^2 \frac{n_1 n_2}{n\rho}(\eta_1 - \eta_2)\right.$$

$$\left. + \Delta m \frac{n}{\rho}\frac{\partial \pi}{\partial x} + \alpha \frac{\partial \tau}{\partial x}\right], \qquad 6.3.7$$

$$\text{and,}\quad n\frac{\partial \tau}{\partial t} = \beta \frac{\partial^2 \tau}{\partial x^2} - \gamma \frac{\partial}{\partial x}\left\{\frac{n_1 n_2}{n}(V_1 - V_2)\right\} - \frac{2}{5} n \frac{\partial U}{\partial x}, \qquad 6.3.8$$

where we have defined

$$\omega = 4\mu/3kT_o$$

$$\beta = 2\lambda/5k$$

$$\gamma = 2\alpha/5$$

$$\varepsilon = - g/kT_o = - \frac{1}{m_1 H_1} = \frac{1}{m_2 H_2}$$

and N has taken the value 5. 6.3.9

A procedure similar to that adopted for the simple gas is applied to remove the time dependence.

A dependence of the form

$$f(x,t) = f^*(x) e^{i\sigma t}$$

is assumed for all the dependent variables with the angular frequency σ being constant. The amplitude factor $f^*(x)$ is now allowed to be complex.

Labelling the gases in such a way that m_1 is the particle mass of the lighter constituent, we define

$$y = e^{-x/H_1}, \qquad 6.3.10$$

and proceed to transform the independent variable. Therefore, we can write

$$\frac{d}{dx} = -\frac{1}{H_1} y \frac{d}{dy} \equiv -\frac{1}{H_1} \delta, \qquad 6.3.11$$

which defines the differential operater δ. The following definitions are required to simplify the parameter problem:

$$s = m_2/m_1,$$

the mass ratio which is now greater than unity, and

$$R = n_2(0)/n_1(0),$$

the lower bound particle number density ratio. Further, we non-dimensionalise the following by defining

$$U^* = -\frac{1}{H_1\sigma}U,$$

$$V_1^* = -\frac{1}{H_1\sigma}V_1,$$

$$V_2^* = -\frac{1}{H_1\sigma}V_2,$$

$$\omega^* = \sigma\omega/n_1(0),$$

$$\xi = -\sigma^2 H_1/g,$$

$$D_{12}^* = D_{12}/\sigma H_1^2,$$

and $\beta^* = \beta/\sigma n_1(0) H_1^2$. $\qquad 6.3.12$

We note that all variables and parameters are now dimensionless. The asterisks will be dropped and implied hereafter.

Further, we note that

$$n_1(0) = \frac{n(0)}{1 + R},$$

and $$n_2(0) = \frac{Rn(0)}{1 + R}.$$

The equations can now be written in a greatly simplified form

$$\eta_1 = i\frac{d}{dy}(yU + yV_1), \tag{6.3.13}$$

$$\eta_2 = iy^{1 - s}(y^s U + y^s V_2), \tag{6.3.14}$$

$$\pi = \tau + \frac{y}{y + Ry^s}\eta_1 + \frac{Ry^s}{y + Ry^s}\eta_2, \tag{6.3.15}$$

$$y\frac{d}{dy}\left[(y + Ry^s)\pi\right] - \omega y\frac{d}{dy}\left[y\frac{\partial U}{dy}\right] - (y\eta_1 + sRy^s\eta_2) - i\xi(y + sRy^s)U = 0, \tag{6.3.16}$$

$$V_1 + Rsy^{s - 1}V_2 = 0, \tag{6.3.17}$$

$$V_1 - V_2 = D_{12}\left[y\frac{d}{dy}(\eta_1 - \eta_2) + \frac{(s - 1)^2 Ry^{s + 1}}{(y + Ry^s)(y + Rsy^s)}(\eta_1 - \eta_2)\right.$$

$$\left. + (s - 1)\frac{y + Ry^s}{y + sRy^s}\, y\, \frac{d\pi}{dy} + \alpha y\frac{d\tau}{dy}\right], \tag{6.3.18}$$

$$\text{and} \quad i(y + Ry^s)\tau = \beta y\frac{d}{dy}(y\frac{d\tau}{dy}) - \gamma y\frac{d}{dy}\left[\frac{Ry^{s+1}}{y + Ry^s}(V_1 - V_2)\right]$$

$$- \frac{2}{5}(y + Ry^s)y\frac{dU}{dy}. \qquad 6.3.19$$

By defining

$$V = V_1 - V_2, \text{ and}$$

$$X = V_1 + V_2,$$

we use equation 6.3.17 to obtain X in terms of V and thus find

$$V_1 = \frac{Rsy^s}{y + Rsy^s}V$$

$$V_2 = \frac{-y}{y + Rsy^s}V \qquad 6.3.20$$

It is convenient to introduce

$$W = \frac{Ry^{s+1}}{y + sRy^s}V \qquad 6.3.21$$

and using equations 6.3.13, 6.3.14, 6.3.15, 6.3.20 and 6.3.21, we can construct, for later elimination, the following relationships:

$$\eta_1 - \eta_2 = -i(s - 1)U + i\frac{y + Rsy^s}{Ry^{s+1}}y\frac{dW}{dy}, \qquad 6.3.22$$

$$\pi = \tau + \frac{i}{y + Ry^s}y\frac{d}{dy}\left[(y + Ry^s)U + (s - 1)W\right], \qquad 6.3.23$$

$$y\eta_1 + sRy^s\eta_2 = iy\frac{d}{dy}\left[(y + sRy^s)U\right]. \qquad 6.3.24$$

By multiplying equation 6.3.23 by $(y + Ry^s)$ and then operating with $y\frac{d}{dy}$, we obtain

$$y\frac{d}{dy}\left[(y + Ry^s)\pi\right] = y\frac{d}{dy}\left[(y + Ry^s)\tau\right] + iy\frac{d}{dy}\left[y\frac{d}{dy}\{(y + Ry^s)U + (s-1)W\}\right]$$

which, on substitution together with 6.3.24 into 6.3.16 to eliminate π, η_1 and η_2, leaves an equation of the form

$$F_1(\tau) + G_1(U) + H_1(W) = 0, \qquad 6.3.25$$

with $F_1(\tau) = y\frac{d}{dy}\left[(y + Ry^s)\tau\right]$,

$$G_1(U) = y\frac{d}{dy}\left[\{i(y + Ry^s) - \omega\}y\frac{dU}{dy}\right] - i\xi(y + sRy^s)U,$$

and $H_1(W) = i(s-1)y\frac{d}{dy}\{y\frac{dW}{dy}\}$.

Equation 6.3.19 reduces to

$$F_2(\tau) + G_2(U) + H_2(W) = 0, \qquad 6.3.26$$

with $F_2(\tau) = i(y + Ry^s)\tau - \beta y\frac{d}{dy}(y\frac{d\tau}{dy})$,

$G_2(U) = \frac{2}{5}(y + Ry^s)y\frac{dU}{dy}$, and

$$H_2(W) = \gamma y\frac{d}{dy}\left(\frac{y + Rsy^s}{y + Ry^s}W\right).$$

And 6.3.18, once η_1, η_2, π and $(V_1 - V_2)$ are eliminated, can be written:

$$F_3(\tau) + G_3(U) + H_3(W) = 0, \qquad 6.3.27$$

with $F_3(\tau) = \left[(s-1)\frac{y + Ry^s}{y + sRy^s} + \alpha\right] y\frac{d\tau}{dy}$,

$$G_3(U) = i(s-1)\frac{y + Ry^s}{y + sRy^s}\, y\,\frac{d}{dy}\left(y\frac{dU}{dy}\right), \text{ and}$$

$$H_3(W) = \frac{y + sRy^s}{D_{12}Ry^{s+1}}W + i\left[\frac{y + sRy^s}{Ry^{s+1}} + \frac{(s-1)^2}{y + sRy^s}\right] y\frac{d}{dy}\left(y\frac{dW}{dy}\right)$$

$$- is\frac{(y + Ry^s)}{Ry^{s+1}}\, y\,\frac{dW}{dy}.$$

Examining the form of the $\{G_i(U)\}$ it is easily verified that the following are two consistent and independent relationships:

$$G_3(U) = i(s-1)\frac{y + Ry^s}{y + sRy^s}\, y\,\frac{d}{dy}\left[\frac{5}{2(y + Ry^s)}\, G_2(U)\right] \qquad 6.3.28$$

and $$G_2(U) = \frac{2}{5}(y + Ry^s)\, y\frac{d}{dy}\left[\frac{i}{\xi(y + sRy^s)}\left[G_1(U)\right.\right.$$

$$\left.\left. - \frac{5}{2}\, y\,\frac{d}{dy}\left\{\frac{1}{y + Ry^s}\left(i(y + Ry^s) - \omega\right) G_2(U)\right\}\right]\right]. \qquad 6.3.29$$

These produce two final equations of elimination in τ and W, since

$$G_i(U) = -\left[F_i(\tau) + H_i(W)\right], \quad i = 1, 2 \text{ and } 3.$$

The substitution of the $\{G_i(U)\}$ in these equations followed by the subsequent simplification is an extremely laborious task. Clearing the denominators equation 6.3.28 can be written in the form:

$$F_4(\tau) + H_4(W) = 0, \qquad 6.3.30$$

where

$$F_4(\tau) = f_{43}\delta^3\tau + f_{42}\delta^2\tau + f_{41}\delta\tau, \text{ and}$$

$$H_4(W) = h_{42}\delta^2 W + h_{41}\delta W + h_{40}W.$$

$\delta \equiv y\frac{d}{dy}$ was defined earlier. The sets $\{f_4\}$, $\{h_4\}$ are given in Appendix A.

Similarly, we obtain from equation 6.3.29 the equation:

$$F_5(\tau) + H_5(W) = 0, \qquad 6.3.31$$

where

$$F_5(\tau) = f_{54}\delta^4\tau + f_{53}\delta^3\tau + f_{52}\delta^2\tau + f_{51}\delta\tau + f_{50}\tau, \text{ and}$$

$$H_5(W) = h_{53}\delta^3 W + h_{52}\delta^2 W + h_{51}\delta W + h_{50}W.$$

Again the sets $\{f_5\}$, $\{h_5\}$ are presented in Appendix A.

6.4 *NON-HEAT CONDUCTING VISCOUS ATMOSPHERE*

We shall consider two degenerate cases only:

(i) A non-heat conducting, viscous model; and

(ii) A non-viscous, heat conducting model.

The analysis becomes very heavy so case (i) will be discussed first and the ground work laid for the more involved case (ii).

We discuss the binary gas atmosphere for which

$$\beta = \gamma = 0, \ \omega \neq 0 \text{ and } 1/D_{12} \neq 0.$$

This is a non-heat conducting, viscous diffusing binary atmosphere with no thermal diffusion. The reason for ignoring the latter is to reduce problem complexity. We assume that thermal diffusion is a relatively small effect, and we shall endeavour to justify this later.

For further simplifying the equations of elimination, we define

$$\xi^* = 5\xi,$$

$$W^* = \frac{iW}{y},$$

$$\alpha_o = -i/D_{12},$$

$$\theta = Ry^{s-1},$$

and $$y^* = -iy/5\omega. \qquad 6.4.1$$

This involves no change to δ, and further

$$\delta(\theta^n) = n(s-1)\theta^n \qquad 6.4.2$$

Apply 6.4.2 to 6.3.30 and write

$$a_1\delta\tau + a_2\delta^2 W + a_3\delta W + a_4 W = 0, \qquad 6.4.3$$

where the asterisks have been dropped, and they will be implied hereafter. The non-constant coefficients a_i are represented by

$$a_1 = E_1^3$$

$$a_2 = E_2^3$$

$$a_3 = E_3^3$$

$$a_4 = \alpha_o E_4^3 + E_5^3, \qquad 6.4.4$$

where the set $\{E_q^p\}$ is a subset of a general set of functions $\{\phi_n^m\}$ of the form

$$\phi_n^m(s,\theta) = {}_nP_1(s) + {}_nP_2(s)\theta + \ldots + {}_nP_m\theta^{m-1}, \qquad 6.4.5$$

with $\{{}_nP_i(s)\}$, $i = 1, \ldots m$; a set of ordinary polynomials in s having real, integral, constant coefficients. The set $\{E_i^3\}$, $i = 1, 2 \ldots 5$; is given in Appendix A.

Similarly, 6.3.31 can be written in the form

$$b_1\delta^2\tau + b_2\delta\tau + b_3\tau + b_4\delta^3 W + b_5\delta^2 W + b_6\delta W + b_7 W = 0, \qquad 6.4.6$$

where $b_1 = \frac{1}{y}\Sigma_1^2 + \Sigma_2^3$

$$b_2 = \frac{1}{y}\Sigma_3^2 + \Sigma_4^3$$

$$b_3 = \xi\Sigma_5^3$$

$$b_4 = \Sigma_6^2$$

$$b_5 = \Sigma_7^2$$

$$b_6 = \Sigma_8^2$$

$$b_7 = \Sigma_9^2 \qquad 6.4.7$$

with the set $\{\Sigma_q^p\}$ a similar subset of $\{\Phi_n^m\}$. The $\{\Sigma_q^p\}$, q = 1, 2 ... 9; are also to be found in Appendix A.

The elimination of W and its derivatives from 6.4.3 and 6.4.6 is greatly simplified with the aid of a computer program to perform the algebraic and differential operations on and between members of the set $\{\Phi_n^m\}$. Factorisation for factors of the form $s^n(s-1)^m.2^q.7^p$ is included to remove those factors common throughout any one elimination equation. This has the further advantage of increasing the throughput, and decreasing the storage requirements.

By repeated differentiation and elimination of W variables we can verify that equations 6.4.3 and 6.4.6 reduce to:

$$c_1\delta^4\tau + c_2\delta^3\tau + c_3\delta^2\tau + c_4\delta\tau + c_5\tau = 0, \qquad 6.4.8$$

where the coefficients c_i are written in the form

$$c_1 = E_1^{31} + \frac{1}{y}E_2^{30} + \frac{\alpha_o}{y}E_3^{30} + \alpha_o E_4^{31},$$

$$c_2 = E_5^{31} + \frac{1}{y}E_6^{30} + \frac{\alpha_o}{y}E_7^{30} + \alpha_o E_8^{31},$$

$$c_3 = E_9^{31} + \frac{1}{y}E_{10}^{30} + \frac{\alpha_o}{y}E_{11}^{30} + \frac{\alpha_o^2}{y}E_{12}^{30} + \xi E_{13}^{31} + \alpha_o \xi E_{14}^{31} + \alpha_o E_{15}^{31} + \alpha_o^2 E_{16}^{31},$$

$$c_4 = E_{17}^{31} + \frac{1}{y}E_{18}^{30} + \frac{\alpha_o}{y}E_{19}^{30} + \frac{\alpha_o^2}{y}E_{20}^{30} + \xi E_{21}^{31} + \alpha_o \xi E_{22}^{31} + \alpha_o E_{23}^{31} + \alpha_o^2 E_{24}^{31},$$

$$\text{and } c_5 = \xi E_{25}^{31} + \alpha_o \xi E_{26}^{31} + \alpha_o^2 \xi E_{27}^{31} \qquad 6.4.9$$

and the set $\{E_k^m\}$, $k = 1, 2, \ldots 27$; is determined.

To obtain any analytic solution of 6.4.8 we must simplify 6.4.9 by retaining only the dominant terms in each E_n^m. Now,

$$\theta = Re^{\frac{\chi}{H_1}(s - 1)}$$

and, $s = m_2/m_1 > 1$,

by specifically labelling the gases to ensure $m_2 > m_1$.

Therefore θ decreases with increasing χ. This implies the existance of a critical height X_o, above which all of the E_n^m can adequately be represented by their respective dominant terms.

We examine

$$E_n^m(s,\theta) = {}_nP_1(s) + {}_nP_2(s)\theta + \ldots + {}_nP_m(s)\theta^{m-1}$$

$$= {}_nP_k(s)\left[1 + {}_nA_1(s)\theta + \ldots {}_nA_{m-k}(s)\theta^{m-k}\right]\theta^{k-1}$$

where ${}_nP_k(s)$ is the first non-zero ${}_nP_j(s)$, $j = 1, \ldots m$;

$$\text{and } {}_nA_j = \frac{{}_nP_{k+j}(s)}{{}_nP_k(s)} \qquad 6.4.10$$

Consider now the set $\{{}_jA_1(s_o)\}$, $j = 1, \ldots 27$; at some value of $s = s_o$. Let ${}_rA_1(s_o)$ be the maximum, and ${}_tA_1(s_o)$ be the minimum element.

Further, let the condition for adequate representation of $\{E_n^m\}$ by $\{{}_nP_k\theta^{k-1}\}$ be that

$$\forall m,n; \quad \left| \sum_{k=1}^{m-k} {}_nA_k\theta^k \right| \leqslant \varepsilon \qquad 6.4.11$$

where ε is a small tolerance. We can show that on multiplication of each of the E_n^m by

$$1 + \beta_1\theta + \beta_2\theta^2,$$

$$\text{where } \beta_1 = -\tfrac{1}{2}\left[{}_rA_1(s_o) + {}_tA_1(s_o)\right]$$

$$\beta_2 = \beta_1{}^2 - \tfrac{1}{2}\left[{}_rA_2(s_o) + {}_tA_2(s_o)\right] \qquad 6.4.12$$

the maximum and minimum elements of the adjusted set $\{{}_jA_1{}^*(s_o)\}$, $j = 1, 2, \ldots 27$; will have the same modulus. The full set $\{{}_jA_i{}^*(s_o)\}$ can be derived from the original set $\{{}_jA_i(s_o)\}$ using the recurrence relation:

$${}_jA_i{}^* = {}_jA_i + \beta_1 \cdot {}_jA_{i-1} + \beta_2 \cdot {}_jA_{i-2},$$

with ${}_jA_{-1} = 0$, and ${}_jA_0 = 1$. 6.4.13

Further, we can expect the maximum element of

$$\left\{ \left| \sum_{k=1}^{m-k} {}_n A_k^* \theta^k \right| \right\}$$

to be at approximately its minimum value for all choices of β_1 and β_2, since $\theta < 1$, and the major contributions to the sum thus come from the adjusted lower end of the series.

Applying this technique to the homogeneous equation 6.4.8 we can expect 6.4.11 to be satisfied at approximately the smallest value of the critical height X_o. Thus, retaining only the dominant terms in each element of E_n^m, the coefficients simplify to

$$c_1 = 7 + \frac{1}{y}$$

$$c_2 = 21 - \frac{1}{y}$$

$$c_3 = 14 + \frac{\alpha_o}{y} + 7(1 + \alpha_o) - \xi$$

$$c_4 = 7 - \frac{\alpha_o}{y} + 7\alpha_o - 2\xi$$

$$c_5 = -(1 + \alpha_o)\xi$$

where the terms of the form $\frac{1}{y}O(\theta)$ were dropped since

$$|\alpha_o^2|\frac{1}{y} >> O(\theta)\frac{1}{y}$$

for typical values of α_o. We also removed a factor $(1 + \alpha_o)$ common throughout the coefficients. If we now write $\xi^* = -\xi/7$,

$y^* = -7y$, and drop the asterisks on substitution, we have equation 6.4.8 in the form:

$$\left[(\delta + 1)^2 + \alpha_o\right]\left[(\delta^2 + \delta + \xi)\right]\tau = \frac{d}{dy}(\delta^2 + \alpha_o)(\delta - 1)\tau$$

or, on rearrangement

$$\left[(\delta + 1)^2 + \alpha_o\right]\left[(\delta^2 + \delta + \xi) - \frac{d}{dy}(\delta - 1)\right]\tau = 0 \qquad 6.4.14$$

or, $A(\delta)\left[B(\delta)\tau\right] = 0.$

Now, $B(\delta)\tau = 0$ is identical in form to 3.5.2 found on treating the simple gas model for the same degenerate case. These two equations are brought exactly in line if we choose $H = H_1$, and the adjusted base number density $n_o(0) = n_1^0(0)$ for the simple gas model.

The implication thus is that, for the region above the critical height X_o, a comparative simple gas model is one with a constant scale height H_1 and an adjusted base number density $n_1^0(0)$. Further, the only additional effects on τ are those introduced by diffusion in the binary gas model. This is clarified if we investigate 6.4.14, i.e.

$$A(\delta)\left[B(\delta)\tau\right] = 0$$

Now, if $A(\delta)V = 0$, then

$B(\delta)\tau = V$, which, since

$A(\delta) \equiv (\delta + 1)^2 + \alpha_o$ and $\alpha_o \propto \frac{1}{D_{12}}$,

adds a particular integral dependent on D_{12} to the simple gas solution of $B(\delta)\tau = 0$.

We are unable to determine the asymptotic solution of 6.4.14 for the same reasons as given in Section 3.5.

6.5 *DETERMINATION OF* X_o/H_1

X_o was introduced in Section 6.4 as the critical height above which the coefficients of equation 6.4.8 could be replaced by their dominant terms, within a maximum relative error of ε.

We proceeded to show that multiplication of the entire equation by a quadratic function in θ is used to minimise X_o for any particular choice of s, ε and R.

We shall now determine X_o/H_1 numerically for various values of R, s, and ε. A computer program was written to perform this task. The coefficients $\{E_n^m\}$ calculated in the reduction process, were adjusted by multiplication with $(1 + \beta_1\theta + \beta_2\theta^2)$ for various mass ratios, s. Hereafter, X_o/H_1 was determined for a set of R values to three different tolerances.

From 6.4.11 we can write

$$\left|{}_nA_1\right|\theta + \left|\sum_{k=2}^{m-k} \theta^k {}_nA_k\right| = \varepsilon$$

in the worst case. But since $\theta < 1$, the summation will become less important near the critical level. We can therefore write

$$e^{(s-1)X_o/H_1} = \rho_o C(s, X_o, \varepsilon)$$

where $\rho = R/\varepsilon$, and C is a slowly varying function of X_o. We illustrate graphically log $(s - 1)$ versus X_o/H_1 for a set of ρ values. (See Figure 6.1).

6.6 *NON-VISCOUS HEAT CONDUCTING ATMOSPHERE*

The second case for discussion is that for which $\omega = \gamma = 0$, $\beta \neq 0$ and $1/D_{12} \neq 0$. This specifies a heat-conducting, non-viscous diffusing atmosphere with no thermal diffusion. Define

$$W' = \frac{iW}{y},$$

$$\alpha_o = - i/D_{12},$$

$$\theta = Ry^{s-1},$$

and $$\frac{1}{y'} = i5\beta/y. \qquad 6.6.1$$

With the above considerations, equation 6.3.30 can be written as

$$a_1\delta^3\tau + a_2\delta^2\tau + a_3\delta\tau + a_4\delta^2 W + a_5\delta W + a_6 W = 0 \qquad 6.6.2$$

where we drop the dashes and imply them hereafter. The set $\{a_i\}$ is expressed in terms of a set $\{E_i^j\}$ which, again, is a sub-set of $\{\phi_n^m\}$. (Refer to 6.4.5). Thus

$$a_1 = \frac{1}{y} E_1^3$$

$$a_2 = \frac{1}{y} E_2^3$$

LOWER BOUND OF VALIDITY

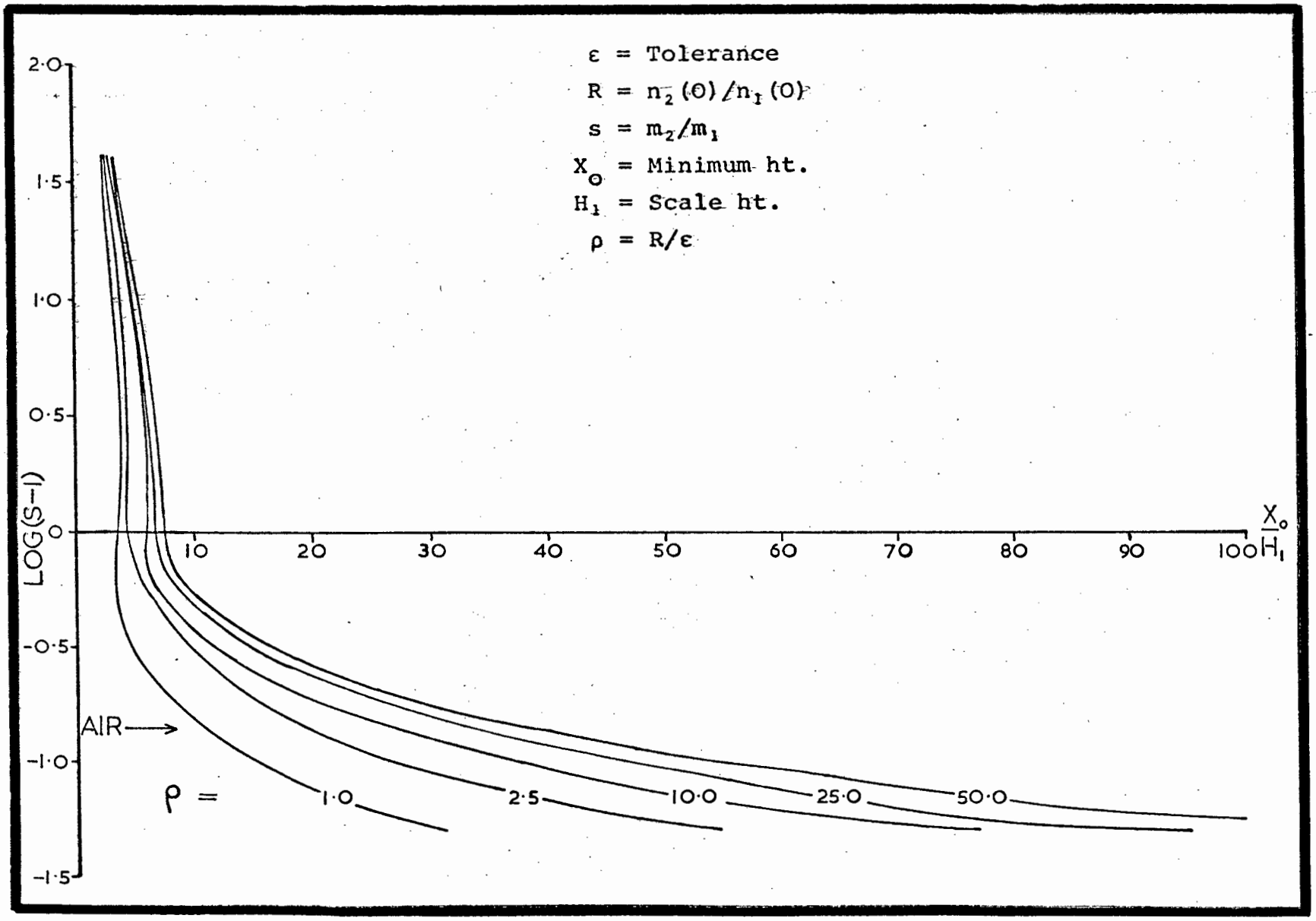


FIGURE 6·1

$$a_3 = E_3^4$$

$$a_4 = E_4^4$$

$$a_5 = E_5^4$$

$$a_6 = E_6^4 + \alpha_o E_7^4, \qquad 6.6.3$$

where the set $\{E_i^j\}$ is given in Appendix A. Similarly equation 6.3.31 reduces to

$$b_1\delta^4\tau + b_2\delta^3\tau + b_3\delta^2\tau + b_4\delta\tau + b_5\tau + b_6\delta^3 W + b_7\delta^2 W + b_8\delta W + b_9 W = 0, \qquad 6.6.4$$

with the set $\{b_i\}$ in terms of a set $\{\Sigma_i^j\}$, a subset of $\{\Phi_n^m\}$. Here

$$b_1 = \frac{1}{y}\Sigma_1^3$$

$$b_2 = \frac{1}{y}\Sigma_2^3$$

$$b_3 = \xi/y\Sigma_3^3 + \Sigma_4^4$$

$$b_4 = \Sigma_5^4$$

$$b_5 = \xi\Sigma_6^4$$

$$b_6 = \Sigma_7^3$$

$$b_7 = \Sigma_8^3$$

$$b_8 = \Sigma_9^3$$

$$b_9 = \Sigma^3_{10}, \qquad 6.6.5$$

with the set $\{\Sigma^j_i\}$ again presented in Appendix A.

Elimination of W and its derivatives between these two equations, using the methods of section 6.5, leads to the final equation:

$$p_1\delta^6\tau + p_2\delta^5\tau + p_3\delta^4\tau + p_4\delta^3\tau + p_5\delta^2\tau + p_6\delta\tau + p_7\tau = 0 \qquad 6.6.6$$

with the coefficients $\{p_i\}$ such that

$$p_1 = \frac{\alpha_o}{y}E^{41}_1 + \frac{\alpha_o^2}{y}E^{41}_2$$

$$p_2 = \frac{1}{y}E^{41}_3 + \frac{\alpha_o}{y}E^{41}_4 + \frac{\alpha_o^2}{y}E^{41}_5$$

$$p_3 = \frac{1}{y}E^{41}_6 + \frac{\alpha_o}{y}E^{41}_7 + \frac{\alpha_o^2}{y}E^{41}_8 + \frac{\alpha_o\xi}{y}E^{41}_9 + \frac{\alpha_o^2\xi}{y}E^{41}_{10} + \alpha_o E^{42}_{11} + \alpha_o^2 E^{42}_{12}$$

$$p_4 = \frac{1}{y}E^{41}_{13} + \frac{\alpha_o}{y}E^{41}_{14} + \frac{\alpha_o^2}{y}E^{41}_{15} + \frac{\xi}{y}E^{41}_{16} + \frac{\alpha_o\xi}{y}E^{41}_{17} + \frac{\alpha_o^2\xi}{y}E^{41}_{18}$$

$$+ E^{42}_{19} + \alpha_o E^{42}_{20} + \alpha_o^2 E^{42}_{21}$$

$$p_5 = \frac{1}{y}E^{40}_{22} + \frac{\alpha_o}{y}E^{40}_{23} + \frac{\alpha_o^2}{y}E^{40}_{24} + \frac{\xi}{y}E^{41}_{25} + \frac{\alpha_o\xi}{y}E^{41}_{26} + \frac{\alpha_o^2\xi}{y}E^{41}_{27} + E^{42}_{28}$$

$$+ \alpha_o E^{42}_{29} + \alpha_o^2 E^{42}_{30} + \alpha_o\xi E^{42}_{31} + \alpha_o^2\xi E^{42}_{32}$$

$$p_6 = E^{42}_{33} + \alpha_o E^{42}_{34} + \alpha_o^2 E^{42}_{35} + \xi E^{42}_{36} + \alpha_o\xi E^{42}_{37} + \alpha_o^2\xi E^{42}_{38}$$

$$p_7 = \xi E^{42}_{39} + \alpha_o\xi E^{42}_{40} + \alpha_o^2\xi E^{42}_{41} \qquad 6.6.7$$

and the set $\{E_i^j\}$ is determined.

Retaining only the dominant terms in each of the E_i^j and neglecting those terms of $O(\theta)$ as we did previously, we can remove a factor $(1 + \alpha_o)$ common throughout the set $\{p_i\}$ which then reduces to

$$p_1 = -\frac{\alpha_o}{y}$$

$$p_2 = \frac{1}{y}(3\alpha_o - 2)$$

$$p_3 = \frac{1}{y}(3 - 4\alpha_o) + \frac{\alpha_o \xi}{y} - 7\alpha_o$$

$$p_4 = \frac{1}{y}(2\alpha_o - 1) + \frac{2\xi}{y}(1 - \alpha_o) - 7(\alpha_o + 2)$$

$$p_5 = \frac{\xi}{y}(2\alpha_o - 1) - 7(\alpha_o + 3) + 5\alpha_o\xi$$

$$p_6 = -7(1 + \alpha_o) + 10\xi$$

$$p_7 = 5\xi(1 + \alpha_o) \qquad 6.6.8$$

Let $\beta_o = 1/\alpha_o$, $\xi' = -\xi$ and $y' = -7y$, so that

$$p_1 = \frac{1}{y}$$

$$p_2 = -\frac{1}{y}(3 - 2\beta_o)$$

$$p_3 = \frac{1}{y}(4 - 3\beta_o) + \frac{\xi}{y} - 1$$

$$p_4 = -\frac{1}{y}(2 - \beta_o) + \frac{2\xi}{y}(\beta_o - 1) - (1 + 2\beta_o)$$

$$p_5 = \frac{\xi}{y}(2 - \beta_o) - (1 + 3\beta_o) - \frac{5\xi}{7}$$

$$p_6 = -(1 + \beta_o) - 10\xi\beta_o/7$$

$$p_7 = -\frac{5\xi}{7}(1 + \beta_o) \qquad 6.6.9$$

where the dashes are once again dropped but implied. Equation 6.6.6 can be written

$$\left[\delta^2 + 2\beta_o\delta + (1 + \beta_o)\right]\left[\delta^2 + \delta + \frac{5\xi}{7}\right]\tau = \frac{d}{dy}\delta\left[\delta^2 + 2(\beta_o - 1)\delta + (2 - \beta_o)\right]\left[\delta^2 - \delta + \xi\right]\tau$$

or, on rearrangement

$$\left[\delta^2 + 2\beta_o\delta + (1 + \beta_o)\right]\left[(\delta^2 + \delta + \frac{5\xi}{7}) - \frac{d}{dy}\delta(\delta^2 - \delta + \xi)\right]\tau = 0 \qquad 6.6.10$$

This is again equivalent to the solution of

$$A(\delta)\left[B(\delta)\tau\right] = 0$$

or, of $B(\delta)\tau = V$, where $A(\delta)V = 0$, which introduces a particular integral over and above the general solution of

$$B(\delta)\tau = 0 \qquad 6.6.11$$

where X_o has been discussed in the previous section as the lower critical height. X_c is determined where the asymptotic solution breaks down, or where

$$|y| \sim O(1)$$

However, equation 6.6.10 is valid for all $x \geq X_o$, and a series solution is required to examine the amplitude of the temperature oscillation in the regions where $x > X_c$.

6.7 *DISCUSSION*

We are now able to draw certain conclusions regarding the comparative behaviour of a binary and a single gas system.

The general problem of a binary system where both heat-conduction and viscosity act together to damp the oscillations was not attempted. This would prove exceptionally difficult.

However, for both degenerate cases, it appears that in the lower regions, below the critical height X_o, the effect of the heavier constituent gas cannot be ignored. Analysis in this region is extremely difficult and no explicit solution was found.

In the regions above X_o, where the lighter gas dominates, we have that the temperature oscillation is further affected only by mutual diffusion. The simple gas model otherwise adequately represents the system. Further, we see that under heat-conduction alone, which proved to be the dominant damping effect in this lower region of the simple gas model, the effect of diffusion is negligible. This suggests that the general problem should

similarly be unaffected by diffusion in this range of the atmosphere.

Finally, Fig. 6.1 indicates that the nearer the mass ratio s is to unity, the higher the critical height X_o which leaves unexplained a larger lower region. But the nearer s is to unity, the better a simple gas representation should be which, we have shown, is still a solution of the binary gas equations provided that a representative scale height is taken.

Therefore, we can safely propose that, in the lower atmospheric regions irrespective of the particle masses of the constituent gases, the variation with height in the amplitude of a temperature perturbation behaves similarly to that of a representative single gas.

CHAPTER 7

NUMERICAL INTEGRATION OF THE BINARY GAS EQUATIONS

7.1 *SIMILARITY WITH THE SIMPLE GAS SYSTEM*

We can show that the set of perturbation equations describing a binary gas system in which (see definitions 6.1.1)

$$m_1 = m_2$$

$$\alpha = 0$$

and $D_{12} = \infty$

reduces identically to that of a simple gas system. Further, we have shown that the asymptotic solutions for the τ amplitude terms of the simple and binary gas systems are identical in the lower regions. Intuitively therefore, we can expect the behaviour of the binary gas system to vary only slightly from that of the simple gas system in the upper regions where, as for the simple gas system, numerical integration of the perturbation equations is required for accurate solution.

Finally, if we consider the techniques, methods and results of Chapter 4 to have been directed towards the solution of a binary gas system in the limiting case of a unit mass ratio m_1/m_2, and the absence of diffusive effects, then it should suffice to devote this chapter to the study of the effects of perturbations of the mass ratio and the diffusion parameters α and D_{12}. We

should examine also the effect of varying the number density ratio at the base level.

7.2 *PREPARATION OF THE EQUATIONS*

The binary gas small oscillation equations derived in Chapter 6 can be written as:

$$i\sigma\eta_1 + \frac{d}{dx}(U + V_1) + \varepsilon m_1 (U + V_1) = 0$$

$$i\sigma\eta_2 + \frac{d}{dx}(U + V_2) + \varepsilon m_2 (U + V_2) = 0$$

$$\pi = \tau + \frac{n_1}{n_o}\eta_1 + \frac{n_2}{n_o}\eta_2$$

$$\frac{d}{dx}(n_o\pi) - \omega\frac{d^2U}{dx^2} - \varepsilon(\rho_1\eta_1 + \rho_2\eta_2) - i\frac{\varepsilon}{g}\rho_o\sigma U = 0$$

$$\rho_1 V_1 + \rho_2 V_2 = 0$$

$$V_1 - V_2 = -D_{12}\left[\left(\frac{d\eta_1}{dx} - \frac{d\eta_2}{dx}\right) + \varepsilon\Delta m^2\frac{n_1 n_2}{n_o\rho_o}(\eta_1 - \eta_2) + \Delta m\frac{n_o}{\rho_o}\frac{d\pi}{dx} + \alpha\frac{d\tau}{dx}\right]$$

$$i\sigma n_o\tau = \beta\frac{d^2\tau}{dx^2} - \gamma\frac{d}{dx}\left[\frac{n_1 n_2}{n_o}(V_1 - V_2)\right] - \frac{2}{5}n_o\frac{dU}{dx}$$

7.2.1 through 7.2.7

Using equations 7.2.1 and 7.2.2 we form the following relationships in order to eliminate η_1, η_2 and their derivatives from the remaining equations 7.2.3 through 7.2.7:

$$\rho_1\eta_1 + \rho_2\eta_2 = \frac{i}{\sigma}\frac{d}{dx}(\rho_o U)$$

$$n_1\eta_1 + n_2\eta_2 = \frac{i}{\sigma}\frac{d}{dx}(n_o U) + \frac{i}{\sigma}\frac{d}{dx}(n_1V_1 + n_2V_2)$$

$$\frac{d\eta_1}{dx} - \frac{d\eta_2}{dx} = \frac{i}{\sigma}\left(\frac{d^2V_1}{dx^2} - \frac{d^2V_2}{dx^2}\right) + \frac{i\varepsilon}{\sigma}\frac{d}{dx}(m_1V_1 - m_2V_2 - \Delta mU)$$

$$\eta_1 - \eta_2 = \frac{i}{\sigma}\left(\frac{dV_1}{dx} - \frac{dV_2}{dx}\right) + \frac{i\varepsilon}{\sigma}(m_1V_1 - m_2V_2 - \Delta mU)$$

7.2.8 through 7.2.11

It is convenient to define

$$V = (V_1 - V_2)/2$$

and $W = (V_1 + V_2)/2$, which now allows equation 7.1.5 to be written as

$$W = \frac{\Delta\rho}{\rho_o}V, \quad \text{with } \Delta\rho = \rho_2 - \rho_1 \qquad 7.2.12$$

We proceed with the elimination of η_1, η_2, $\frac{d\eta_1}{dx}$ and $\frac{d\eta_2}{dx}$. Further, W is eliminated using equation 7.2.12 leaving the four equations in τ, π, U and V. With the dashes representing differentiation with respect to x, we can write:

$$\pi = \tau + \frac{i}{\sigma}U' + \frac{i}{\sigma}\frac{\varepsilon\rho_o}{n_o}U + \frac{2i\Delta m n_1 n_2}{\sigma n_o \rho_o}V' + \frac{2i\Delta m\varepsilon\rho_1\rho_2}{\sigma\rho_o^2}V$$

$$\frac{d}{dx}(n_o\pi) - \omega U'' - \frac{i\varepsilon}{\sigma}\frac{d}{dx}(\rho_o U) - \frac{i\varepsilon\sigma}{g}\rho_o U = 0$$

$$V_{,} = - D_{12}\left[\frac{i}{\sigma}V'' + \frac{i\varepsilon}{2\sigma}\frac{d}{dx}\left(2m_1m_2\frac{n_o}{\rho_o}V - \Delta mU\right) + \frac{i\varepsilon\Delta m^2}{\sigma}\frac{n_1n_2}{n_o\rho_o}V'\right.$$

$$\left. + \frac{i\varepsilon^2\Delta m^2 n_1n_2}{2\sigma n_o\rho_o}\left(2m_1m_2\frac{n_o}{\rho_o}V - \Delta mU\right) + \frac{\Delta m}{2}\frac{n_o}{\rho_o}\pi' + \frac{\alpha}{2}\tau'\right]$$

and $$i\sigma n_o\tau = \beta\tau'' - 2\gamma\frac{d}{dx}\left(\frac{n_1n_2}{n_o}V\right) - \frac{2}{5}n_oU'$$

7.2.13 through 7.2.16

Finally, elimination of π and π' from the above equations, and algebraic manipulation on the remaining 3 equations enables us to formulate the following:

$$\tau'' = ia_1\tau + a_2U' + a_3V' + a_4V \qquad 7.2.17$$

$$U'' = (b_1 + ib_2)\tau' + (b_3 + ib_4)\tau + (b_5 + ib_6)U' + (b_7 + ib_8)U + (b_9 + ib_{10})V' + (b_{11} + ib_{12})V \qquad 7.2.18$$

$$V'' = (c_1 + ic_2)\tau' + (c_3 + ic_4)\tau + (c_5 + ic_6)U' + (c_7 + ic_8)U + (c_9 + ic_{10})V' + (c_{11} + ic_{12})V \qquad 7.2.19$$

The coefficients $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ are presented in Appendix B.

By applying the transformation

$$y = e^{\varepsilon m_1 x}$$

to the equations 7.2.17 through 7.2.19, we can write:

$$\tau'' = d_1\tau' + id_2\tau + d_3U' + d_4V' + d_5V \qquad 7.2.20$$

$$U'' = (e_1 + ie_2)\tau' + (e_3 + ie_4)\tau + (e_5 + ie_6)U' + (e_7 + ie_8)U + (e_9 + ie_{10})V' + (e_{11} + ie_{12})V \qquad 7.2.21$$

$$V'' = (f_1 + if_2)\tau' + (f_3 + if_4)\tau + (f_5 + if_6)U' + (f_7 + if_8)U + (f_9 + if_{10})V' + (f_{11} + if_{12})V \qquad 7.2.22$$

where differentiation is now with respect to y. Again, the coefficients $\{d_i\}$, $\{e_i\}$, and $\{f_i\}$ are presented in Appendix B. These final three equations are easily prepared for the Runge-Kutta integration procedure emerging finally as a set of twelve first-order real ordinary differential equations. This preparation, similar in method to that of section 4.2, will not be discussed.

7.3 *SECONDARY EQUATIONS*

We require the initial values of τ, $\frac{d\tau}{dy}$, U, $\frac{dU}{dy}$, V and $\frac{dV}{dy}$ at the lower boundary of the model. Given the physically meaningful values of τ, π, U, $(V_1 - V_2)$, $\frac{d\tau}{dx}$ and $\frac{d}{dx}(V_1 - V_2)$ we can determine the initial set of values using the relations which follow:

$$\frac{d\tau}{dy} = \frac{1}{\epsilon m_1 y}\frac{d\tau}{dx}$$

$$V = \frac{V_1 - V_2}{2}$$

$$\frac{dV}{dy} = \frac{1}{2\epsilon m_1 y}\frac{d}{dx}(V_1 - V_2)$$

$$\frac{dU}{dy} = \frac{i\sigma}{\varepsilon m_1 y}(\tau - \pi) - \frac{\rho_o}{m_1 n_o y}U - \Delta m \frac{n_1 n_2}{\varepsilon m_1 y n_o \rho_o}\frac{d}{dx}(V_1 - V_2)$$

$$- \Delta m\, m_2 \frac{n_1 n_2}{y\rho_o^2}(V_1 - V_2) \qquad 7.3.1 \text{ through } 7.3.4$$

Similarly, output of the variables τ, U, $V_1 - V_2$, η_1, η_2 and π can be achieved by calculation of the right hand sides of the following relations:

$$V_1 - V_2 = 2V$$

$$\pi = \tau + \frac{i\varepsilon m_1 y}{\sigma}\frac{dU}{dy} + i\frac{\varepsilon\rho_o}{\sigma n_o}U + i\frac{2\varepsilon\Delta m m_1}{\sigma n_o \rho_o}n_1 n_2 y \frac{dV}{dy} + i\frac{2\Delta m \varepsilon \rho_1 \rho_2}{\sigma \rho_o^2}V$$

$$\eta_1 = \frac{i}{\sigma}\varepsilon m_1 y \frac{dU}{dy} + i\frac{\varepsilon m_1}{\sigma}U + i\frac{2\varepsilon m_1 \rho_2 y}{\sigma\rho_o}\frac{dV}{dy} + \frac{2i\varepsilon m_1 m_2 \rho_2 n_o}{\sigma\rho_o^2}V$$

$$\eta_2 = \frac{i}{\sigma}\varepsilon m_1 y \frac{dU}{dy} + i\frac{\varepsilon m_2}{\sigma}U - i\frac{2\varepsilon m_1 \rho_1 y}{\sigma\rho_o}\frac{dV}{dy} - \frac{2i\varepsilon m_1 m_2 \rho_1 n_o}{\sigma\rho_o^2}V$$

7.3.5 through 7.3.8

7.4 GAS PARAMETERS

From definition 6.3.12, we have that

$R = n_2(0)/n_1(0)$, the base number density ratio, and

$s = m_2/m_1$, the molecular mass ratio. It is convenient to choose m_1 as the mass of the lighter constituent so that $s \geq 1$.

Thus given lower boundary values for the local scale height H, and the total number density $n_o(0)$, we find that

$$m_1 = \frac{1 + R}{1 + sR} \frac{kT_o}{Hg} \text{ and } m_2 = sm_1 \qquad 7.4.1$$

$$\text{Further, } n_1(0) = \frac{n_o(0)}{1 + R} \text{ and } n_2(0) = Rn_1(0) \qquad 7.4.2$$

where T_o and g are respectively the isothermal temperature, and the gravitational constant.

Using equations 7.4.1 and 7.4.2 to determine these four structure constants, we are free to vary s and R, and investigate the effects of a binary mixture on the amplitude of the τ oscillation.

Further, the coefficient of mutual diffusion D_{12}, varies according to

$$D_{12} \propto \frac{T^s}{n},$$

with index s = ·792. (Chapman and Cowling, p. 248). This must be applied to adjust for the base value of D_{12}.

Finally, for rigid elastic spherical molecules K_T, (and hence α) has its sign dependent on the mass ratio s, or on the molecular diameter ratio when s is near unity (Chapman and Cowling, p. 254). Therefore investigation of both the positive and negative values of α, the thermal diffusion factor, is required.

7.5 *INITIAL PREPARATION*

The simple gas special case discussed in Section 7.1 allows a most convenient check to be performed between the two entirely

different theory reductions, and between the computer programs written to integrate numerically the perturbation equations of each system. This check proved successful although rounding error increased somewhat with the added complexity of the binary gas equations.

Background work with regard to error analysis and start height determination discussed in Section 4.4, applies fully to the numerical solution of the binary gas equations.

Further, as the solution for τ can be expressed as a linear sum of small quantities arising from the component variations, those variations attributable to the binary gas mixture and diffusive effects can safely be taken to be small independent linear terms.

The effects of varying the following four parameters will be studied independently:

1. mass ratio s;
2. base number density ratio R;
3. thermal diffusion factor α;
4. mutual diffusion coefficient D_{12}.

7.6 *PRODUCTION AND RESULTS*

Structure and parameter values for the binary gas model are those applied to the simple gas model discussed in Section 4.3. The model base level ($x = 0$) is again chosen at 160 km physical altitude and the remaining parameter values are given by Chapman

and Cowling and these are adjusted for the temperature and density differences at this lower boundary.

Values of R and s, the base level number density ratio and the molecular mass ratio respectively are not accurately known, and we will assume 80% molecular nitrogen and a 20% molecular oxygen composition to be representative. Atomic oxygen is well known to be present in a significant percentage but this is ignored for the purpose of this investigation.

Again, a diurnal oscillation will be considered.

A. *Molecular Mass Ratio Variation*

It is convenient to choose the same initial value set as that for a representative simple gas model with, in addition, V and $\frac{dV}{dy}$ zero at the lower bound. This choice allows a direct correlation between the solutions of the binary gas model and that of the simple gas model, besides representing the relative effects of the mass ratio variation.

Figure 7.1 presents graphically the various solutions for log τ plotted against the model altitude. With the exception of the simple gas curve all other curves are solutions of the binary gas equations and these curves differ only in the molecular mass ratio of the constituent gases. Further, typical diffusion constants are present and there is a four to one dominance of lighter gas molecules at the base level of the model.

The results are not surprising: increasing values of s cause the upper regions to comprise relatively lighter molecules which,

LOG τ VERSUS ALTITUDE

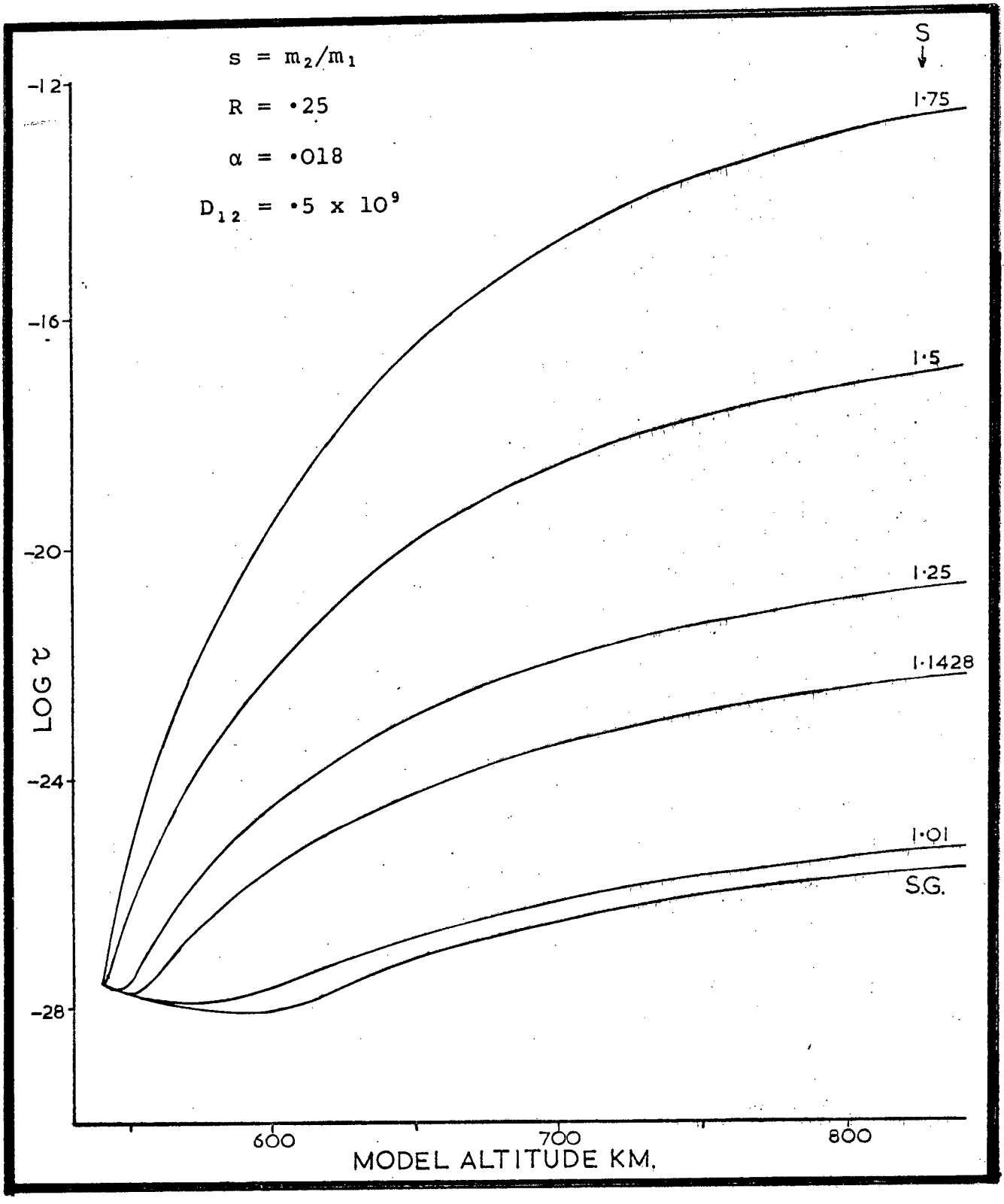


FIGURE 7·1

in turn, gives rise to an increased local scale height H, the effects of which have been discussed . (See Figure 4.10).

It is however, interesting to note that the deviation away from the simple gas curve is attributable very largely to the non-unit mass ratio. This is indicated by the comparative closeness of the s = 1·01 curve to that of the simple gas. The graph s = 1·1428 is a solution of a O_2 - N_2 model. However, a O - N_2 model comprising 80% atomic oxygen at the base level could (if we ignore the fact that one constituent is no longer diatomic) be represented by the curve s = 1·75.

However, we need to investigate the role of R variation before we draw any final conclusions regarding the influence of a binary mixture on the amplitude of the τ perturbation.

B. Number Density Ratio Variation

Figure 7.2 presents a family of log τ curves differing only in the value of R, the lower bound number density ratio. Also plotted is the representative simple gas solution of the same parameter set.

We notice that the slightest presence of a heavier constituent acts in such a way as to increase the magnitude of τ. This continues with an increasing percentage presence of this heavier constituent. However the increase slows down when the heavier gas becomes the major constituent and we can expect that, for large R, the τ amplitude will again approximate to that of the simple gas solution for τ. This is easily seen to be true if we

LOG τ VERSUS ALTITUDE

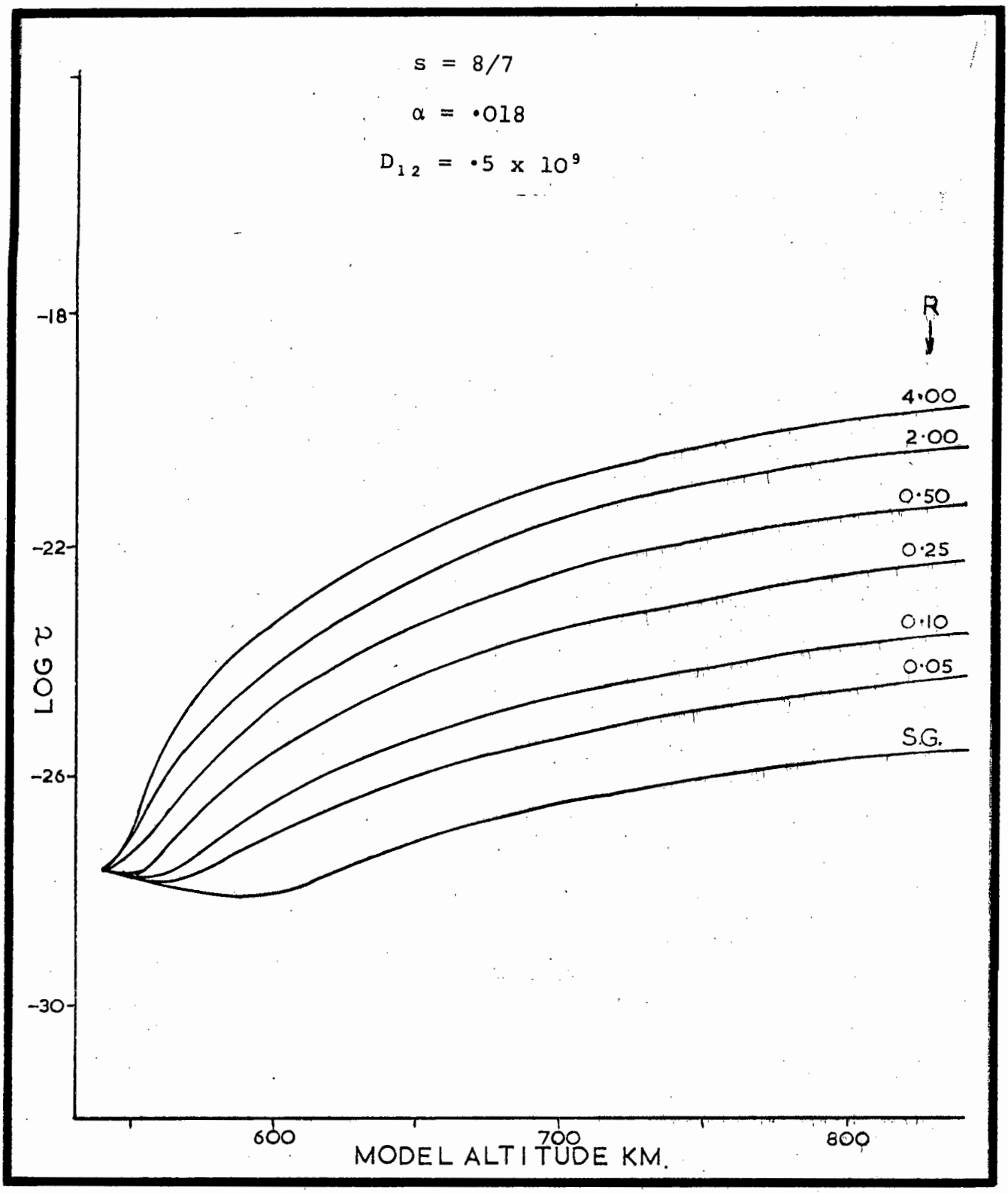


FIGURE 7·2

consider that a simple gas model is the extreme case of the binary gas model with R = 0 or ∞.

We can conclude that variations in both s and R induce behaviour consistent with that of local scale height variation.

If the local scale height at the base level is H_o, we can write

$$H(x) = H_o \left(\frac{1 + sR}{1 + R}\right) \left(\frac{1 + Ry^{s-1}}{1 + sRy^{s-1}}\right) \qquad 7.5.1$$

from the definitions of the local scale height $H = \frac{kT}{mg}$ and $m = \rho/n$, the mean molecular mass.

We can show that H(x) is never a decreasing function of x, the rate of increase being dependent on both s and R.

This means that the presence of a second constituent in an atmospheric model of the upper regions will always lead to solutions of the perturbation which are larger in amplitude than those of the original constituent.

C. *Thermal Diffusion Factor Variation*

We find that, for the previous initial value set where both V and $\frac{dV}{dy}$ are zero, no significant change in the τ amplitude is discernible with variations in the magnitude of α, the thermal diffusion factor.

Allowing V to be non-zero however, while keeping the remaining initial values unchanged, uncovers the nature of the induced effect.

Figure 7.3 presents the family of curves which arises from varying α.

LOG τ VERSUS ALTITUDE

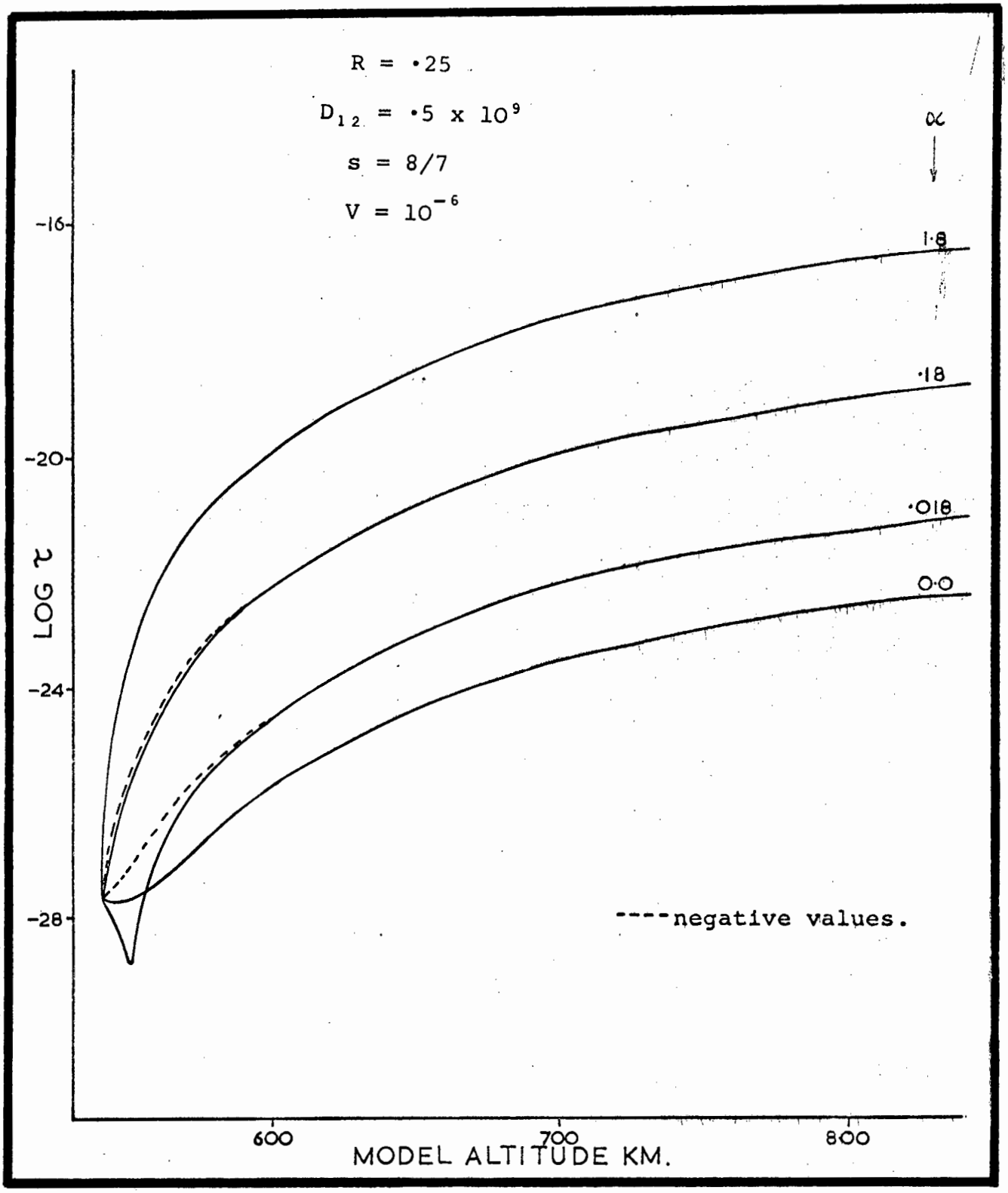


FIGURE 7·3

A typical value of $|\alpha|$ is $\cdot 018$ while the single constituent model has α effectively zero.

Again we find that thermal diffusion tends to increase the magnitude of solutions.

Further, it appears that the sign of α is not too influential in affecting the shape of the log τ curve except in the lower regions and for the small values of α.

D. Mutual Diffusion Coefficient Variation

Varying the coefficient D_{12} through a great range of values from $0 \cdot 5 \times 10^{9}$, its typical magnitude, to 10^{80} gave rise to no noticeable effect. As the simple gas model can be approximated with D_{12} approaching infinity, it appears that in this altitude range diffusion is relatively unimportant. A slight increasing trend was discernible when D_{12} was reduced in value to $\cdot 5 \times 10^{3}$. This result is obtained with both initial value sets previously applied to the model.

We can safely conclude that mutual diffusion of the constituent gases has no effect on the magnitude of the τ amplitude factor in the upper regions of a typical binary gas atmospheric model.

7.7 *DISCUSSION*

We have studied in some detail the independent role of each of s, R, α and D_{12} in affecting the amplitude of the τ perturbation.

The extension from a single to a binary gas atmospheric model, incorporating all the above parameters gives rise to solutions for

τ in the region of discussion which are larger in magnitude than those of the simple gas system. Further, the dominant effect is due to the increase in the local scale height H in the regions immediately below the start height.

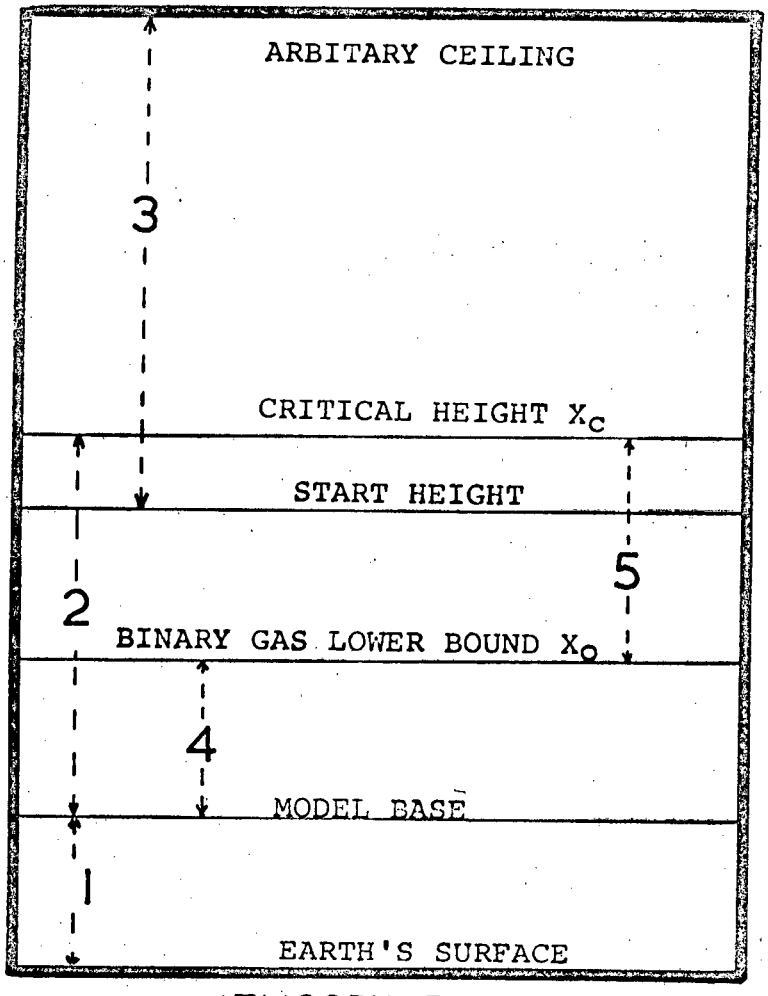
S U M M A R Y

In an attempt to tie together loose ends we will consider whether, to what extent and in which atmospheric regions the various models discussed are representative.

We have shown, contrary to the results of Parkyn, that viscosity plays only a minor role in atmospheric response to forced oscillations. His analysis for the degenerate case of a non-heat conducting viscous cylindrical atmosphere gives rise to fourth order derivatives of τ and the subsequent asymptotic solution of the perturbation equation shows the effect of viscous damping to be dominant. In our treatment of flat earth models these higher derivatives associated with viscosity are absent. Greater validity of solution is surely credited to the cylindrical model, it being geometrically closer to the idealised spherical case.

However, the single dimension limitation of the models we have discussed does not detract from the validity of conclusions drawn from their comparative behaviour.

A diagram is included (overleaf) to identify the various atmospheric regions referenced hereafter. This diagram is not to scale.



ATMOSPHERIC REGIONS

Region 1 is a layer of the atmosphere which is in a state of turbulence and of wildly fluctuating composition. As a result no investigation was attempted in this region, and our model base level was chosen at 160 km physical altitude where steadier conditions are found.

The simple gas atmospheric model under forced periodic oscillations at its base level required two distinct methods of solution; an analytic treatment leading to an asymptotic solution for the temperature perturbation in region 2, and a numerical

approach in region 3. Unfortunately neither technique could treat the full model atmospheric region but a significant area of overlap is found. Thus a complete picture of atmospheric response to forced oscillations in a simple gas model is at hand. Strongly decreasing amplitudes of the temperature perturbation are found in region 2, substantiating the results of Parkyn, and in the higher regions, strongly increasing solutions exist. These increasing amplitudes are not entirely from viscous effects as suggested, but they result from various mechanisms.

The binary gas model similarly led to both the analytic and the numerical approaches. In region 4, the depth of which varies considerably with the ratio of the particle masses, no solution was found possible, and in this region alone is a binary gas model required. However, we did find that in region 5 a representative simple gas model is adequate, and further that, although the depth of region 5 decreases with increasing Xo as the mass ratio approaches unity, this is compensated for largely by the fact that the system as a whole is rapidly approaching that of a simple gas.

Solutions of the binary gas system in region 3 were shown, in general, to have amplitudes more strongly increasing than those of the comparative simple gas model. This effect is very largely due to increase of local scale height with altitude. Diffusive effects were shown to be relatively negligible, except for very specialised boundary conditions, throughout the entire model atmosphere.

Perhaps the greatest limitation of both the simple and the binary gas models summarised this far is the assumption of constancy of the steady state temperature with height. This defect is high-lighted in Chapter 5 where a non-isothermal atmospheric model is considered first before the generalisation to include both the molecular mass and the gravitational potential decrease with altitude. Unfortunately this model has a restricted atmospheric domain but, as a representative model, it is extremely good.

A P P E N D I X A

A.1 The sets $\{f_4\}$, $\{h_4\}$ of equation 6.3.30 are:

$$f_{43} = i5D_{12}\beta(s-1)(y+Ry^s)^3Ry^{s+1},$$

$$f_{42} = -i5D_{12}\beta(s-1)(y+Ry^s)^2(y+sRy^s)Ry^{s+1},$$

$$f_{41} = D_{12}(y+Ry^s)^3Ry^{s+1}\left[7(s-1)(y+Ry^s) + 2\alpha(y+sRy^s)\right],$$

$$h_{42} = iD_{12}(y+Ry^s)^2\left[2(y+Ry^s)(y+sRy^s)^2 + 2(s-1)^2(y+Ry^s)Ry^{s+1} - 5\gamma(s-1)(y+sRy^s)Ry^{s+1}\right],$$

$$h_{41} = iD_{12}5\gamma(s-1)(y+Ry^s)\left[(y+sRy^s)^2Ry^{s+1} - 2(s-1)^2R^2y^{2s+2} - s(y+Ry^s)^3(y+sRy^s)\right],$$

$$h_{40} = 2(y+Ry^s)^3(y+sRy^s)^2 - 5i\gamma D_{12}(s-1)^3R^2y^{2s+2}\left[(s-2)y + (1-2s)Ry^s\right],$$

provided that $D_{12} \neq 0$.

A.2 Similarly the sets $\{f_5\}$, $\{h_5\}$ of equation 6.3.31 are:

$$f_{54} = 5\beta\left[(y+Ry^s) + i\omega\right](y+Ry^s)^4(y+sRy^s),$$

$$f_{53} = -5\beta(y+Ry^s)^5(y+s^2Ry^s) - i5\omega\beta(y+Ry^s)^3\left[2(y+sRy^s)^2 + (y+Ry^s)(y+s^2Ry^s)\right],$$

$$f_{52} = 5(y+Ry^s)^4(y+sRy^s)\left[\omega(y+Ry^s) - \xi\beta(y+sRy^s)\right] + i(y+Ry^s)^2(y+sRy^s)\left[10\omega\beta(y+sRy^s)^2 - 7(y+Ry^s)^4\right],$$

$$f_{51} = -5\omega(y + Ry^s)^5(y + s^2Ry^s)$$
$$+ 7i(y + Ry^s)^5\left[(y + Ry^s)(y + s^2Ry^s) - 2(y + sRy^s)^2\right],$$

$$f_{50} = i5\xi(y + Ry^s)^5(y + sRy^s)^2,$$

$$h_{53} = (y + Ry^s)^3(y + sRy^s)\left[2(s - 1)(y + Ry^s)^2\right.$$
$$\left. - 5\gamma(y + Ry^s)(y + sRy^s) - i5\gamma\omega(y + sRy^s)\right],$$

$$h_{52} = (y + Ry^s)^3\left[5\gamma(y + sRy^s)^3 - 10\gamma(s-1)^2(y + sRy^s)Ry^{s+1}\right.$$
$$\left. - 2(s - 1)(y + s^2Ry^s)(y + Ry^s)^2\right]$$
$$+ i5\gamma\omega(y + Ry^s)^2(y + sRy^s)\left[3(y + sRy^s)^2 - 2R(s - 1)^2y^{s+1}\right],$$

$$h_{51} = 5\gamma\xi(y + Ry^s)^3(y + sRy^s)^3 + 5\gamma R(s - 1)^2y^{s+1}(y + Ry^s)^2$$
$$\times \left[(5 - 3s)y^2 - (s^2 - 6s + 1)Ry^{s+1} + s(5s - 3)R^2y^{2s}\right]$$
$$- i5\gamma\omega\left[2(y + Ry^s)(y + sRy^s)^4 + (s - 1)^2(y + Ry^s)Ry^{s+1}\right.$$
$$\left.\times \{3(s - 3)y^2 + (s^2 - 14s + 1)Ry^{s+1} + 3s(1 - 3s)y^{2s}\}\right],$$

$$h_{50} = 5\xi\gamma(s - 1)^2Ry^{s+1}(y + Ry^s)^2(y + sRy^s)^2$$
$$- 5\gamma(s - 1)^3Ry^{s+1}(y + Ry^s)\left[(s - 2)y^3 + (4 - 5s)Ry^{s+2}\right.$$
$$\left. + s(5 - 4s)R^2y^{2s+1} - s(1 - 2s)R^3y^{3s}\right]$$
$$- i5\gamma\omega(s - 1)^2Ry^{s+1}\left[(s - 2)(s - 3)y^3\right.$$
$$- 3R(3s - 1)(s - 2)y^{s+2} - 3s(2s - 1)(s - 3)R^2y^{2s+1}$$
$$\left. + s(2s - 1)(3s - 1)R^3y^{3s}\right].$$

A.3 The set $\{E_i\}$ of equation 6.4.3 are:

$$E_1^3 = (0) + (-7,7)\theta + (-7,7)\theta^2$$

$$E_2^3 = (2) + (2,0,2)\theta + (0,0,2)\theta^2$$

$$E_3^3 = (4) + (4,0,4)\theta + (0,0,4)\theta^2$$

$$E_4^3 = (2) + (0,4)\theta + (0,0,2)\theta^2$$

$$E_5^3 = (2) + (2,0,2)\theta + (0,0,2)\theta^2$$

where the commas separate coefficients of increasing powers of s.

The set $\{\Sigma_i\}$ is similarly:

$$\Sigma_1^2 = (1) + (0,1)\theta$$

$$\Sigma_2^3 = (7) + (7,7)\theta + (0,7)\theta^2$$

$$\Sigma_3^2 = (-1) + (0,0,-1)\theta$$

$$\Sigma_4^3 = (7) + (-7,28,-7)\theta + (0,0,7)\theta^2$$

$$\Sigma_5^3 = (-1) + (0,-2)\theta + (0,0,-1)\theta^2$$

$$\Sigma_6^2 = (-2,2) + (0,-2,2)\theta$$

$$\Sigma_7^2 = (-4,4) + (0,-6,8,-2)\theta$$

$$\Sigma_8^2 = (-2,2) + (0,-6,10,-4)\theta$$

$$\Sigma_9^2 = (0) + (0,-2,4,-2)\theta.$$

A.4 The set $\{E_i\}$ of equation 6.6.2 are:

$$E_1^3 = (0) + (-1,1)\theta + (-1,1)\theta^2$$

$$E_2^3 = (0) + (1,-1)\theta + (0,1,-1)\theta^2$$

$$E_3^4 = (0) + (-7,7)\theta + (-14,14)\theta^2 + (-7,7)\theta^3$$

$$E_4^4 = (2) + (4,0,2)\theta + (2,0,4)\theta^2 + (0,0,2)\theta^3$$

$$E_5^4 = (4) + (8,0,4)\theta + (4,0,8)\theta^2 + (0,0,4)\theta^3$$

$$E_6^4 = (2) + (4,0,2)\theta + (2,0,4)\theta^2 + (0,0,2)\theta^3$$

$$E_7^4 = (2) + (2,4)\theta + (0,4,2)\theta^2 + (0,0,2)\theta^3.$$

and the set $\{\Sigma_i\}$ are:

$$\Sigma_1^3 = (1) + (1,1)\theta + (0,1)\theta^2$$

$$\Sigma_2^3 = (-1) + (-1,0,-1)\theta + (0,0,-1)\theta^2$$

$$\Sigma_3^3 = (-1) + (0,-2)\theta + (0,0,-1)\theta^2$$

$$\Sigma_4^4 = (7) + (14,7)\theta + (7,14)\theta^2 + (0,7)\theta^3$$

$$\Sigma_5^4 = (7) + (0,28,-7)\theta + (-7,28)\theta^2 + (0,0,7)\theta^3$$

$$\Sigma_6^4 = (-5) + (-5,-10)\theta + (0,-10,-5)\theta^2 + (0,0,-5)\theta^3$$

$$\Sigma_7^3 = (-2,2) + (-2,0,2)\theta + (0,-2,2)\theta^2$$

$$\Sigma_8^3 = (-4,4) + (-4,-2,8,-2)\theta + (0,-6,8,-2)\theta^2$$

$$\Sigma_9^3 = (-2,2) + (-2,-4,10,-4)\theta + (0,-6,10,-4)\theta^2$$

$$\Sigma_{10}^3 = (0) + (0,-2,4,-2)\theta + (0,-2,4,-2)\theta^2 .$$

A P P E N D I X B

B.1 The coefficients $\{a_i\}$ of equation 7.2.17 are:

$$a_1 = \frac{\sigma}{\beta} n_o$$

$$a_2 = \frac{2}{5\beta} n_o$$

$$a_3 = \frac{2\gamma}{\beta} \frac{n_1 n_2}{n_o}$$

$$a_4 = \frac{2\gamma}{\beta} \frac{\epsilon n_1 n_2}{n_o^2} (\rho_o - \Delta m \Delta n).$$

B.2 The coefficients $\{b_i\}$ of equation 7.2.18 can be simplified by defining

$$N(x) \equiv 1 + \Delta m^2 \frac{n_1 n_2}{\rho_o^2},$$

$$Q(x) \equiv 1 - \alpha \Delta m \frac{n_1 n_2}{n_o \rho_o}, \text{ and,}$$

$$D(x) \equiv \frac{1}{\sigma^2 \omega^2 N^2 + n_o^2}.$$

We can then write:

$$b_1 = \omega \sigma^2 N^2 Q n_o D$$

$$b_2 = \sigma n_o^2 Q D$$

$$b_3 = \epsilon \omega \sigma^2 N^2 \rho_o D$$

$$b_4 = \varepsilon\sigma N n_o \rho_o D$$

$$b_5 = - b_4/\sigma$$

$$b_6 = \frac{b_3}{\sigma}$$

$$b_7 = \frac{\sigma}{g} b_4$$

$$b_8 = -\frac{\sigma}{g} b_3$$

$$b_9 = - 2\varepsilon\Delta m \frac{\rho_1\rho_2}{\rho_o^2} n_o^2 D$$

$$b_{10} = 2\varepsilon\Delta m\omega\sigma N \frac{\rho_1\rho_2}{\rho_o^2} n_o D$$

$$b_{11} = - 2\Delta m \frac{n_o}{\rho_o}(\varepsilon^2 m_1 m_2 \frac{\rho_1\rho_2}{\rho_o^2} n_o^2 + \omega\sigma^2 N \frac{n_1 n_2}{D_{12}})D, \text{ and}$$

$$b_{12} = 2\Delta m \frac{n_o^2}{\rho_o}(\varepsilon^2\omega\sigma m_1 m_2 N \frac{\rho_1\rho_2}{\rho_o^2} - \frac{\sigma}{D_{12}} n_1 n_2)D.$$

B.3 We define the further two relations to simplify the coefficients of equation 7.2.19:

$$P(x) \equiv \alpha + \Delta m \frac{n_o}{\rho_o}$$

$$M(x) \equiv 2N(x) - 1.$$

The coefficient set $\{c_i\}$ can now be written as:

$$c_1 = - \omega\sigma^2 \frac{\Delta m}{2} Q \frac{n_o^2}{\rho_o} D$$

$$c_2 = \frac{\sigma}{2}(\omega^2\sigma^2 NP + \alpha n_o^2)D$$

$$c_3 = -\frac{\varepsilon\omega}{2}\Delta m\sigma^2 N n_o D$$

$$c_4 = -\varepsilon\frac{\Delta m}{2}\sigma n_o^2 D$$

$$c_5 = -\frac{c_4}{\sigma}$$

$$c_6 = \frac{c_3}{\sigma}$$

$$c_7 = \frac{\sigma}{g}c_4$$

$$c_8 = -\frac{\sigma}{g}c_3$$

$$c_9 = -\varepsilon m_1 m_2 \frac{n_o}{\rho_o}(\omega^2\sigma^2 NM + n_o^2)D$$

$$c_{10} = -\varepsilon\sigma\omega\Delta m^2 \frac{\rho_1\rho_2}{\rho_o^3} n_o^2 D$$

$$c_{11} = \left[\varepsilon^2\Delta m^2 \frac{\rho_1\rho_2}{\rho_o^2}(\omega^2\sigma^2\Delta m\Delta\rho \frac{n_o}{\rho_o^2}N + n_o^2) + \frac{\omega\sigma^2}{D_{12}}\Delta m^2\frac{n_o^2}{\rho_o^2} n_1 n_2\right]D$$

$$c_{12} = -\left[\varepsilon^2\omega\sigma m_1 m_2\Delta m^2 \frac{\rho_1\rho_2 n_o^3}{\rho_o^4} - \frac{\sigma n_o}{D_{12}}(\sigma^2\omega^2 N + n_o^2)\right]D.$$

B.4 The coefficient set $\{d_i\}$ of equation 7.2.20 are obtainable from the set $\{a_i\}$, if we define

$$X \equiv \frac{1}{\varepsilon m_1 y},$$

and write

$$d_1 = -\frac{1}{y}$$

$$d_2 = X^2 a_1$$

$$d_3 = X a_2$$

$$d_4 = X a_3$$

$$d_5 = X^2 a_4 .$$

B.5 Similarly, the coefficient set $\{e_i\}$ of equation 7.2.21 can be expressed as:

$$e_1 = X b_1$$

$$e_2 = X b_2$$

$$e_3 = X^2 b_3$$

$$e_4 = X^2 b_4$$

$$e_5 = X b_5 - \frac{1}{y}$$

$$e_6 = X b_6$$

$$e_7 = X^2 b_7$$

$$e_8 = X^2 b_8$$

$$e_9 = Xb_9$$

$$e_{10} = Xb_{10}$$

$$e_{11} = X^2b_{11}$$

$$e_{12} = X^2b_{12}$$

B.6 Finally, the set $\{f_i\}$ can be written as:

$$f_1 = Xc_1$$

$$f_2 = Xc_2$$

$$f_3 = X^2c_3$$

$$f_4 = X^2c_4$$

$$f_5 = Xc_5$$

$$f_6 = Xc_6$$

$$f_7 = X^2c_7$$

$$f_8 = X^2c_8$$

$$f_9 = Xc_9 - \frac{1}{y}$$

$$f_{10} = Xc_{10}$$

$$f_{11} = X^2c_{11}$$

$$f_{12} = X^2c_{12}.$$

R E F E R E N C E S

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