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UNBOUNDED LINEAR OPERATORS IN OPERATOR RANGES

by

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SYNOPSIS

Many results in operator theory for example some perturbation results, are at present known only in the Banach space case. The aim of this work is to provide a natural generalisation of such results by considering operator ranges (the image of a bounded operator defined everywhere on a Banach space) as well as investigating and characterizing some of the properties of operator ranges. For the sake of generality we will for the most part be considering unbounded or closed linear operators instead of continuous everywhere defined linear operators. We will not be attempting to give exhaustive coverage of unbounded linear operators but will try to give some insight into the use of operator range techniques in the theory of unbounded linear operators. The first chapter will be aimed mainly at defining and introducing concepts used in later chapters. In the second chapter we turn our attention to the conjugate of a linear operator whilst also briefly looking at projections in an operator range. Chapter three is concerned mainly with investigating and characterizing the closed range property of linear operators whereas in the first part of chapter four we will be proving some fairly well known results on compact, precompact and strictly singular operators to be used in chapter five. In the second half of chapter four we will investigate the relationship between weakly compact operators and pre-reflexive spaces. Chapter five will be dealing with perturbation of semi-Fredholm operators by first of all continuous and then by strictly singular operators. We close with a discussion of the instability of non-semi-Fredholm operators under compact and α-compact perturbations.

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CHAPTER I

INTRODUCTION

Before defining and exploring the concept of operator ranges we first develop the concept of normed spaces, Banach spaces and that of linear operators. From these the definition of operator ranges will follow naturally; after which we will briefly explore the properties of operator ranges, before turning our attention to the closed graph and open mapping theorems. These we will generalise (partially) to operator ranges. The remainder of the chapter contains an account of dual spaces and of reflexive and prereflexive spaces.

I.1 Normed Spaces

I.1.1 Definition (Vector space).

A *vector space* X over a field K is a set admitting two algebraic operations, vector addition (with any two elements x,y of X there corresponds an element $x + y$ of X which is called the sum of x and y) and scalar multiplication (with any element x of X and λ of K we can associate a vector $\lambda x \in X$ called the product of λ and x), such that the following properties hold :

(1) $x + y = y + x$

(2) $x + (y + z) = (x + y) + z$

(3) there is a zero element in X , denoted by 0 , such that for all $x \in X$, $0 + x = x$

(4) for each $x \in X$ there is an element $-x \in X$ such that

$$x + (-x) = 0$$

(we write $x - y$ for $x + (-y)$)

(5) $(\lambda\mu)x = \lambda(\mu x)$

(6) $(\lambda + \mu)x = \lambda x + \mu x$

(7) $\lambda(x + y) = \lambda x + \lambda y$

(8) $1.x = x$

where $x,y,z \in X$, $\lambda, \mu, 1 \in K$.

I.1.2 Remark.

(i) From the definition it follows that the zero element of X is unique and for each $x \in X$, the element $-x \in X$ is unique. It is easy to show that

$$0.x = 0 \quad \text{and} \quad (-1)x = -x \quad \text{for all} \quad x \in X$$

$$\text{and} \quad \lambda.0 = 0 \quad \text{for all} \quad \lambda \in K .$$

(ii) K is called the scalar field of X and X is called a real or complex vector space, depending on whether the scalar field K of X is the field of real or complex numbers.

I.1.3 Examples.

(i) Space $\mathbb{R}^n$, i.e. the set of all n-tuples of real numbers, written $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, etc. is a real vector space with the two algebraic operations defined in the usual way,

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \qquad \alpha \in \mathbb{R} .$$

(ii) The space $\mathbb{C}^n$ consisting of all n-tuples of complex numbers, written $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, etc. is a complex vector space with algebraic operations defined as in the previous example where now $\alpha \in \mathbb{C}$.

I.1.4 Definition (Normed Space).

Let X be a vector space over the field of real or complex numbers. A norm on X, denoted by $\| \ \|$, is a real-valued function on X such that the following properties hold for all $x \in X$ and scalars α:

(1) $\|x\| \geqslant 0$ for all $x \in X$

(2) $x \neq 0 \Rightarrow \|x\| \neq 0$

(3) $\|\alpha x\| = |\alpha| \, \|x\|$

(4) $\|x + y\| \leqslant \|x\| + \|y\|$ (triangle inequality).

The vector space X, together with a norm on X, is called a *normed space* or *a normed linear space*. We will sometimes denote a normed space together with its norm as $(X, \| \ \|)$.

Throughout this work X, Y, Z, ... will denote infinite dimensional normed spaces unless stated otherwise.

I.1.5 Definition ($B_X(x,r)$, $U_X(x,r)$, $S_X(r)$).

Let X be a metric space and r a positive number.

We define the set $B_X(x,r) = \{y \in X \mid d(x,y) \leqslant r\}$ to be the *closed r-ball about* $x \in X$.

$U_X(x,r) = \{y \in X \mid d(x,y) < r\}$ will be called the *open r-ball about* $x \in X$.

For $x = 0$ we will sometimes use the notation $B_X(r)$ and $U_X(r)$ and refer to the *closed-* or *open r-ball*. When $x = 0$ and $r = 1$ we will sometimes just refer to the *closed* (B_X) or *open ball* (U_X).

$S_X(r)$ will be used to denote the set $S_X(r) = \{y \in X \mid d(0,y) = r\}$ and will be referred to as the *r-sphere* of X.

I.1.6 Definition (Subspace, span).

Let X be an arbitrary vector space. $Y \subset X$ is defined to be a *subspace* of X if for all $y_1, y_2 \in Y$ and $\alpha, \beta \in K$, $\alpha y_1 + \beta y_2 \in Y$. For M a subset of X, the *span of M* written sp(M), is the subspace consisting of all linear combinations of elements of M.

I.2 Banach Spaces

I.2.1 Definition (completeness, Cauchy and convergent sequences).

Let $\{x_n\}$ be a sequence contained in the normed linear space X. We define $\{x_n\}$ to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists a positive integer N such that

$$\|x_m - x_n\| < \varepsilon \quad \text{whenever} \quad n,m \geqslant N .$$

We say that $\{x_n\}$ is convergent if there is an element x in X such that for every $\varepsilon > 0$ there exists N for which

$$\|x_n - x\| < \varepsilon \quad \text{when} \quad n \geqslant N .$$

We will use the notation $x_n \to x$ to indicate that $\{x_n\}$ converges to $x \in X$. The space X is said to be *complete* if every Cauchy sequence is convergent. A complete normed linear space is called a *Banach space*.

I.2.2 Examples.

(i) $\mathbb{R}^n$ and $\mathbb{C}^n$

(ii) $\ell_p(\mathbb{N})$ $1 \leqslant p < +\infty$

By definition each element in the space ℓ_p is a sequence $x = (\lambda_i) = (\lambda_1, \lambda_2, \ldots)$ such that

$$\|x\| = \left(\sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p} < \infty$$

(iii) $\ell_\infty(\mathbb{N})$. The elements of ℓ_∞ are all bounded sequences, i.e. sequences $x = (\lambda_1, \lambda_2, \dots)$ such that $\sup_{i \in \mathbb{N}} |\lambda_i| < \infty$. Let $\|x\| = \sup_{i \in \mathbb{N}} |\lambda_i|$.

For proofs of the completeness of the above spaces and of some of their basic properties see for example Kreyszig [16].

I.2.3 Proposition [16].

Let M *be a nonempty subset of a metric space* (X,d). *Then* $x \in \bar{M}$ *iff there exists a sequence* $\{x_n\}$ *in* M *such that* $\{x_n\}$ *converges to* x.

Proof: Let $x \in \bar{M}$. If $x \in M$ the result is obvious. If $x \in \bar{M} \setminus M$, x is a point of accumulation of M. Hence for each $n = 1, 2, \dots$ the ball $U_X(x, \frac{1}{n})$ contains an $x_n \in M$ and $\{x_n\}$ converges to x since $\frac{1}{n}$ converges to 0 as n increases.

Conversely if $\{x_n\}$ is in M and $\{x_n\}$ converges to x then either $x \in M$ or x is obviously a point of accumulation of M and thus $x \in \bar{M}$. □

I.2.4 Corollary. *A subspace* M *of a complete metric space* (X,d) *(and thus also of a Banach space) is complete iff* M *is closed in* X.

Proof: Follows from I.2.3 and the fact that a convergent sequence will have a unique limit. □

I.2.5 Definition (X/M).

Let M be a closed subspace of a normed space X. Define an equivalence relation R on X by $x R y$ if $x - y \in M$. Let X/M denote the corresponding set of equivalence classes and let $[x]$ denote the set of elements equivalent to x. Thus

$$[x] = \{x + m \mid m \in M\} = x + M .$$

Vector addition and scalar multiplication on X/M are defined by

$$[x] + [y] = [x + y]$$

$$\alpha[x] = [\alpha x] .$$

Define a norm on X/M by

$$\|[x]\| = \inf_{m \in M} \|x - m\| = d(x,M)$$

where $d(x,M)$ is the distance from x to M. It is easy to verify that X/M is a normed space. M is required to be closed so that $\|[x]\| = 0$ implies $[x] = [0]$.

I.2.6 Theorem [11]. *If* X *is a Banach space and* M *is a closed subspace of* X*, then* X/M *is a Banach space.*

Proof: Let $\{[x_n]\}$ be a Cauchy sequence in X/M. There exists a subsequence $\{[y_n]\}$ of $\{[x_n]\}$ such that

$$\|[y_{n+1}] - [y_n]\| < 2^{-n} \qquad 1 \leqslant n .$$

Hence, since $\|[y_{n+1}] - [y_n]\| = \|[y_{n+1} - y_n]\| = \inf_{m \in M} \|y_{n+1} - y_n - m\|$, we may choose $v_n \in [y_n]$ such that

$$\|v_{n+1} - v_n\| < 2^{-n} \qquad 1 \leqslant n .$$

The sequence $\{v_n\}$ is Cauchy sequence, since

$$\|v_{n+i} - v_n\| \leqslant \|v_{n+1} - v_n\| + \|v_{n+2} - v_{n+1}\| + \ldots + \|v_{n+i} - v_{n+i-1}\|$$

$$\leqslant \sum_{k=0}^{\infty} 2^{-n-k} = 2^{-n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus since X is a Banach space, there exists a $v \in X$ such that $v_n \to v$. But

$$\|[v_n] - [v]\| = \|[v_n - v]\| = \inf_{m \in M} \|v_n - v - m\| \leqslant \|v_n - v\| .$$

Thus $[v_n]$ converges to $[v]$ in X/M. Since $\{[v_n]\} = \{[y_n]\}$ is a subsequence of the Cauchy sequence $\{[x_n]\}$, $\{[x_n]\}$ also converges to $[v]$. Hence X/M is complete.

I.2.7 Definition. (Infinite series, convergence, absolute convergence).

We say that an *infinite series* $\sum_{i=1}^{\infty} x_i$ of elements x_i in a normed space X *converges* in X if there exists an $x \in X$ such that $s_n = \sum_{i=1}^{n} x_i$ converges to x. We write $x = \sum_{i=1}^{\infty} x_i$.

The series *converges absolutely* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

I.2.8 Theorem [11]. *A normed linear space* X *is complete if and only if every series in* X *which converges absolutely also converges in* X.

Proof: Suppose X is complete and $\sum_{i=1}^{\infty} \|x_i\| < \infty$, $x_i \in X$. For $s_n = \sum_{i=1}^{n} x_i$,

$$\|s_{n+k} - s_n\| = \|\sum_{i=n+1}^{n+k} x_i\| \leqslant \sum_{i=n+1}^{n+k} \|x_i\| \to 0 \quad \text{as} \quad n \to \infty .$$

Thus $\{s_n\}$ is a Cauchy sequence and therefore converges in the space X. Conversely, assume absolute convergence implies convergence. Let $\{y_n\}$ be a Cauchy sequence in X. There exists a subsequence $\{x_n\}$ of $\{y_n\}$ such that

$$\|x_{k+1} - x_k\| \leqslant 2^{-k} \qquad 1 \leqslant k .$$

Now

$$x_k = x_1 + (x_2 - x_1) + \dots + (x_k - x_{k-1}) \qquad 2 \leqslant k$$

and

$$\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| \leqslant \sum_{i=1}^{\infty} 2^{-1} = 1 .$$

Thus, by hypothesis, the sequence of partial sums $x_k - x_1$ converges in X, or equivalently the sequence $\{x_k\}$ converges to some $x \in X$. Since $\{x_k\}$ is a subsequence of the Cauchy sequence $\{y_k\}$, it is easy to see that $\{y_k\}$ also converges to x. Hence X is complete. □

I.3 Linear Operators

I.3.1 Definition (linear operators).

Let X and Y be vector spaces over the same field of scalars. An operator T with domain X and range in Y is called *linear* if for all x and z in X and all scalars α and β,

$$T(\alpha x + \beta z) = \alpha Tx + \beta Tz .$$

We will sometimes refer to T as a map instead of an operator.

We will use the following notations :

$D(T)$ denotes the domain of T

$R(T)$ denotes the range of T

$N(T)$ denotes the subspace $\{x \in D(T) \mid Tx = 0\}$

$N(T)$ is called the *null space* or *kernel* of T.

We will sometimes refer to $R(T)$ as the image or map of $D(T)$.

We will denote the set of all linear operators with domain contained in the normed space X and range contained in the normed space Y by $L(X,Y)$. We write $L[X,Y]$ if $D(T) = X$ for all

$T \in L(X,Y)$. T refers to a non-zero element of $L(X,Y)$ unless stated otherwise.

I.3.2 Definition (injective, surjective, bounded).

A linear operator T is called 1-1 or *injective* if distinct elements in $D(T)$ are mapped onto distinct elements in $R(T)$. Since a linear operator has the property $T0 = 0$, T is 1-1 iff $N(T) = \{0\}$.

T is called onto or *surjective* if $R(T) = Y$. If X and Y are normed spaces, $T \in L(X,Y)$ is defined to be *bounded on its domain* if there exists a positive number M such that

$$\|Tx\| \leqslant M\|x\| \qquad \text{for all} \quad x \in D(T) .$$

We will denote the set of all such operators by $B(X,Y)$.
We shall use $B[X,Y]$ to denote the linear space of all bounded everywhere defined linear operators from the normed space X into the normed space Y, with norm defined as follows:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

For $X = Y$ we will use $B[X]$ instead of $B[X,X]$. (In general we will let $\|T\| = \sup \{\|Tx\|: x \in D(T), \|x\| = 1\}$ for $T \in B(X,Y)$.)

I.3.3 Theorem [11].

Let $T \in L(X,Y)$. *The following statements are equivalent:*

(i) T *is continuous at a point in* $D(T)$.

(ii) T *is uniformly continuous on* $D(T)$.

(iii) T *is bounded on its domain; i.e., there exists a number* M *such that for all* $x \in D(T)$

$$\|Tx\| \leqslant M\|x\| .$$

Proof:

(i) implies (ii). Suppose T is continuous at $x_0 \in D(T)$. Given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ such that

(1) $$\|Tx - Tx_0\| < \varepsilon \text{ if } \|x - x_0\| < \delta \quad .$$

Let u be any point in $D(T)$. Then for $\|u - v\| < \delta$ it follows from (1) and the additivity of T that

$$\|Tv - Tu\| = \|Tx_0 - T(x_0 + u - v)\| < \varepsilon \qquad v \in D(T) .$$

Hence (ii) is proved.

(ii) implies (iii). The continuity of T at 0 implies the existence of a $\delta > 0$ such that

$$\|Tz\| = \|Tz - T0\| \leqslant 1 \qquad \|z\| \leqslant \delta$$

Thus, for $x \neq 0$ in $D(T)$, $\delta = \|\delta x/\|x\|\|$, and therefore

$$1 \geqslant \left\|T\left(\frac{\delta x}{\|x\|}\right)\right\| = \frac{\delta\|Tx\|}{\|x\|} \quad .$$

Hence $\|Tx\| \leqslant \delta^{-1}\|x\|$ for all $x \in D(T)$.

(iii) implies (i). The inequality $\|Tx - Tz\| \leqslant M\|x - z\|$ for $x,z \in D(T)$ implies (i). □

We adopt the following convention: We refer to the elements of $B(X,Y)$ as continuous operators and the elements of $B[X,Y]$ as bounded operators.

I.3.4 **Theorem.** *If Y is a Banach space then $B[X,Y]$ is a Banach space.*

Proof: Suppose Y is complete and let $\{T_n\}$ be a Cauchy sequence in $B[X,Y]$. Given $\varepsilon > 0$, there exists a positive integer N such that

$$\|T_m - T_n\| < \varepsilon \qquad m,n \geqslant N .$$

For all $x \in X$ and $m,n \geqslant N$ we thus obtain

$$(1) \qquad \|T_m x - T_n x\| \leqslant \|T_m - T_n\| \, \|x\| < \varepsilon \|x\| .$$

It is easy to see that this will imply that $\{T_n x\}$ is a Cauchy sequence in Y. By the completeness of Y, $T_n x \to y \in Y$. Thus for each $x \in X$, there exists $y \in Y$ such that $T_n x \to y$. Hence we can define a linear operator by letting $Tx = \lim_{n\to\infty} T_n x$ for each $x \in X$. It remains to be verified that $T \in B[X,Y]$ and that $T_n \to T$. Since $\{T_n\}$ converges pointwise to T, it follows from (1), after fixing $n \geqslant N$ and letting $m \to \infty$, that for all $x \in X$

$$(2) \qquad \|Tx - T_n x\| = \|(T - T_n)x\| \leqslant \varepsilon \|x\| .$$

Hence $T - T_n$ is bounded and thus so is

$$T = (T - T_n) + T_n \text{ , i.e. } T \in B[X,Y] .$$

It follows from (2) that

$$\|T - T_n\| = \sup_{\|x\| = 1} \|Tx - T_n x\| \leqslant \varepsilon \quad \text{if} \quad n \geqslant N .$$

Thus $T_n \to T$ in $B[X,Y]$. □

I.3.5 Definition (inverse).

Let T be an injective linear operator with $D(T) \subset X$ and $R(T) \subset Y$, X and Y normed linear spaces. The *inverse* of T, written T^{-1}, is the map from subspace $R(T)$ into X given by $T^{-1}(Tx) = x$. It is clear that T^{-1} is linear.

For a function f which is not necessarily 1-1, $f^{-1}(B)$ will denote the set $\{x \mid f(x) \in B\}$.

T^{-1} will not be assumed continuous unless specifically stated.

I.3.6 Theorem [11]. *Let* $T \in L(X,Y)$. *Then* T^{-1} *exists and is continuous iff there is an* $m > 0$ *such that*

$$\|Tx\| \geq m\|x\| \quad (x \in D(T)).$$

Proof: Suppose $\|Tx\| \geq m\|x\|$ for all $x \in D(T)$. Then $x \neq 0$ implies $Tx \neq 0$. Hence T is 1-1 . Since

$$\|T^{-1}Tx\| = \|x\| \leq m^{-1}\|Tx\|$$

T^{-1} is bounded on $R(T)$ and therefore continuous. On the other hand, if T^{-1} is continuous, then

$$\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\|\|Tx\| \qquad x \in X$$

The theorem follows upon taking $m = 1/\|T^{-1}\|$. □

A bounded linear operator with a continuous inverse has some very special properties. We make the following definition.

I.3.7 Definition (isomorphism).

A linear operator mapping a normed space into a normed space is called an *isomorphism* if it is continuous and has a continuous inverse.

The normed spaces X and Y are said to be *isomorphic* if there exists an isomorphism mapping X onto Y . We write $X \cong Y$.

We immediately note the following result.

I.3.8 Proposition [11]. *If the normed space* Y *is isomorphic to a Banach space, then* Y *is also a Banach space.*

Proof: Let T be an isomorphism from the Banach space X onto Y. Suppose $\{y_n\}$ is a Cauchy sequence in Y. Since T^{-1} is bounded,

$$\|T^{-1}y_n - T^{-1}y_m\| \leqslant \|T^{-1}\|\|y_n - y_m\| .$$

Hence $\{T^{-1}y_n\}$ is also a Cauchy sequence and therefore converges to some $x \in X$. By the continuity of T, $y_n = TT^{-1}y_n \to Tx$. Therefore Y is complete. □

I.3.9 Definition (X × Y).

Let X and Y be normed spaces. $X \times Y$ is defined as being the normed space of all ordered pairs (x,y), $x \in X$, $y \in Y$, with vector addition and scalar multiplication defined as follows :

$$\alpha(x_1,y_1) + \beta(x_2,y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

for $x_1, x_2 \in X$, and $y_1, y_2 \in Y$ and all scalars α, β. We define a norm on $X \times Y$ as follows :

$$\|(x,y)\| = \|x\| + \|y\| .$$

I.3.10 Remark.

The norm $|||(x,y)||| = \max\{\|x\|, \|y\|\}$ defines the same topology on $X \times Y$. (See Kreyszig [16], p.75.)

I.3.11 Definition (Graph, closed operators).

The *graph* $G(T)$ of $T \in L(X,Y)$ is the set $\{(x,Tx) \mid x \in D(T)\}$. Since T is linear, $G(T)$ is a subspace of $X \times Y$. If the graph of T is closed in $X \times Y$, then T is called *closed*. We will denote the set of all closed operators with domain in X and range in Y by $C(X,Y)$; or by $C[X,Y]$ in case $D(T) = X$.

I.3.12 Remark [11].

(i) T is closed if and only if $\{x_n\}$ in $D(T)$, $x_n \to x$, $Tx_n \to y$, imply $x \in D(T)$ and $Tx = y$.

(ii) If T is 1-1 and closed, then T^{-1} is closed.

(iii) The null space of a closed operator is closed.

(iv) If $D(T)$ is closed and T is continuous, then T is closed.

(v) The continuity of T does not necessarily imply that T is closed. Conversely, T closed does not necessarily imply that T is continuous. This statement can be verified by the following examples.

I.3.13 Example.

(i) Let $X = Y$ be an arbitrary infinite dimensional normed space. Let M be an arbitrary proper dense subspace of X . We define an operator $T \in L(X,Y)$ by $D(T) = M$ and $Tm = m$ $(\forall m \in M)$. T is obviously continuous, but not closed.

(ii) [11]. Let $X = Y = C([0,1])$ be the space of all functions which continuously map the closed interval $I = [0,1]$ into the set of scalars. Let $C^1([0,1])$ be the subspace of X consisting of the functions with continuous first derivatives. Define the linear differential operator T mapping $C^1([0,1])$ into Y by $(Tx)(t) = x'(t), \; t \in [0,1] \; , \; x \in C^1([0,1])$.
T is closed : Let $x_n \to x$, $Tx_n \to y$ $(\{x_n\} \subset C^1([0,1]))$. Then $\{x_n\}$ converges uniformly to x and $\{x'_n\}$ converges uniformly to y on $[0,1]$. It follows from taking antiderivatives of x'_n and y that $x \in C^1([0,1])$ and that $Tx = x' = y$ on $[0,1]$. Thus T is closed. However T is unbounded, since the

sequence $\{y_n(t)\} = \{t^n\}$ $(\{y_n\} \subset C^1([0,1]))$ has the properties $\|Ty_n\| = n$ and $\|y_n\| = 1$.

We saw in I.3.12 that T continuous ⇒ T closed if D(T) is closed. The question now arises: When does T closed ⇒ T continuous? We give the following three well known results which we will state without proof. Two of these will however later be generalised to operator ranges.

I.3.14 Definition (Open map).

An operator T is called *open* if T maps open sets onto open sets. (If T 1-1, T open ⇒ T^{-1} continuous by definition of continuity.)

I.3.15 Definition (interior point, rare, meagre, non-meagre).

We define an element x of a subset M of X to be an *interior point* of M if for some $r > 0$, $U_X(x,r) \subset M$.

We say that M is *rare* in X if its closure $\bar{M}$ has no interior points.

The set M is said to be *meagre* in X if $M = \bigcup_{k=1}^{\infty} A_k$ where A_k is rare in X for each $k \geqslant 1$.

If M is not meagre in X we refer to M as being *non-meagre*.

We note that some authors use the terminology nowhere dense, of the first category and of the second category instead of rare, meagre and non-meagre respectively.

I.3.16 Theorem [11] (Open mapping theorem).

Let $T \in C[X,Y]$. *If* X *is complete,* Y *non-meagre and* T *surjective, then* T *is open.*

We note the following:

I.3.17 Theorem [16](Baire's category theorem).

If a metric space is complete, it is non-meagre.

I.3.18 Theorem [11] (Closed graph theorem).

Let $T \in C(X,Y)$. *If* X *and* Y *are complete and* $D(T)$ *is closed in* X, *then* T *is continuous.*

The next three theorems will be needed in subsequent chapters.

I.3.19 Theorem [11] (Uniform boundedness principle or the Banach-Steinhaus theorem).

Let X *be complete and* F *a subset of* $B[X,Y]$ *such that for each* $x \in X$, $\sup_{T \in F} \|Tx\| < \infty$. *Then* $\sup_{T \in F} \|T\| < \infty$.

Note that if $N(T)$ is closed for $T \in L(X,Y)$, $X/N(T)$ will be a normed space and T thus induces an injective operator $\hat{T} \in L(X/N(T),Y)$ where $\hat{T}[x] = Tx$ for each $x \in D(T)$.

I.3.20 Theorem [11]. *Let* $T \in L(X,Y)$ *with* $N(T)$ *closed. Then*

(i) T *is closed iff* $\hat{T}$ *is closed*

(ii) T *is continuous iff* $\hat{T}$ *is continuous; in which case* $\|T\| = \|\hat{T}\|$.

Proof:

(i) Let T be closed. Suppose $[x_n] \to [x]$, $[x_n] \in D(T)/N(T)$, and $\hat{T}[x_n] \to y$. Then there exists a sequence $v_n \in N(T)$ such that $x_n - v_n \to x$. Since $T(x_n - v_n) = \hat{T}[x_n] \to y$ and T is closed, x is in $D(T)$ and $Tx = y$. Thus $[x] \in D(\hat{T})$, $\hat{T}[x] = y$ and $\hat{T}$ is closed. Conversely, let $\hat{T}$ be closed. Suppose $x_n \to x$ and $Tx_n \to y$. Then $[x_n] \to [x]$ and $\hat{T}[x_n] = Tx_n \to y$. Hence $[x] \in D(\hat{T})$ and $\hat{T}[x] = y$, i.e. $x \in D(T)$ and $Tx = y$. Therefore T is closed.

(ii) Suppose T is continuous. Then for each $x \in D(T)$

$$\|\hat{T}[x]\| = \|Tz\| \leqslant \|T\|\|z\| \qquad z \in [x] .$$

Hence

$$\|\hat{T}[x]\| \leqslant \|T\| \inf_{z \in [x]} \|z\| = \|T\|\|[x]\| .$$

Thus $\|\hat{T}\| \leqslant \|T\|$. Conversely, if $\hat{T}$ is continuous, then

$$\|Tx\| = \|\hat{T}[x]\| \leqslant \|\hat{T}\|\|[x]\| \leqslant \|\hat{T}\|\|x\| .$$

Therefore $\|T\| \leqslant \|\hat{T}\|$. Thus we obtain $\|T\| = \|\hat{T}\|$. □

We now define a very special kind of operator.

I.3.21 Definition (projection).

Let M be a subspace of X . An operator $P \in L[X,Y]$ is called a *projection* of X onto M if P is a linear map of X onto M such that $P^2 (=PP) = P$. (Our projections need not be bounded, unlike e.g. [11] .)

I.3.22 Remark. It follows easily that P closed implies M closed.

As an application of the closed graph theorem we give the following result.

I.3.23 Theorem [11]. *Let* M *be a closed subspace of a Banach space* X . *There exists a bounded projection from* X *onto* M *if and only if there exists a closed subspace* N *of* X *such that* $X = M \oplus N$ $(M \cap N = \{0\}$ *and* $X = M + N)$. *In this case, there exists a* $c > 0$ *such that*

$$\|m + n\| \geqslant c\|m\| \qquad m \in M , n \in N .$$

Proof: Suppose P is a projection from X onto M . Let $N = N(P)$. Since P is bounded, N is closed. Furthermore, $N \cap M = \{0\}$, since $v \in N \cap M$ implies $v = Pv = 0$. Since any $x \in X$ may be written $x = Px + (x - Px)$ and $x - Px$ is in N, $X = M + N$.

Conversely, suppose N satisfies the hypotheses of the theorem. Then each $x \in X$ can be uniquely written as follows

$$x = m + n \qquad m \in N , n \in N .$$

Define the operator P from X onto M by $P(m + n) = m$. Clearly, P is linear and $P^2 = P$. P is closed: Suppose

$$x_k = m_k + n_k \to x \qquad m_k \in M , n_k \in N$$

$$Px_k = m_k \to y .$$

Since M is closed, $y \in M$. Therefore $y = Py$ and $n_k \to x - y$. Since N is also closed, $x - y \in N$. Hence $0 = P(x - y) = Px - y$. Thus P is closed. By the closed-graph theorem, P is bounded and

$$\|P\|\|m + n\| \geqslant \|P(m + n)\| = \|m\| \qquad m \in M , n \in N . \qquad \square$$

I.4 Operator Ranges

I.4.1 Definition (Operator Ranges).

A normed space R is called an *operator range* if it is the image of a bounded injective operator defined everywhere on a Banach space R_1. We will call R_1 the *pre-image space* of R. In general if X is an operator range we will denote its pre-image space by X_1. We will denote the operator mapping R_1 onto R by α_R and the inverse of α_R by β_R. In general if $\|.\|$ is the norm on R, then $\|.\|_1$ will be used to denote the norm on R_1.

I.4.2 Remark.

It follows easily that α_R and β_R are injective, surjective, everywhere defined, (i.e. $D(\alpha_R) = R_1$ and $D(\beta_R) = R$) and closed with α_R bounded and β_R open.

Note that the injectivity condition in Definition I.4.1 is not too restrictive since if Y was the image of a bounded linear operator T, with $D(T)$ a Banach space, say X, we could consider the injective operator $\hat{T}$ which is bounded by I.3.20 with $D(\hat{T}) = X/N(T)$, a Banach space by I.2.6. Therefore Y is an operator range with $Y_1 = X/N(T)$.

We will refer to injective surjective operators as being *bijective*.

I.4.3 Remark.

Operator ranges need not be complete. Consider the natural injection $I \in B[\ell_1, \ell_2]$ of ℓ_1 into ℓ_2. I is bounded: Let

$x = (\gamma_1, \gamma_2, \ldots) \in \ell_1$

$$\|Ix\|_2^2 = \sum_{i=1}^{\infty} |\gamma_i|^2 \leq \sup_{j \in \mathbb{N}} |\gamma_j| \left(\sum_{i=1}^{\infty} |\gamma_i| \right)$$

$$\leq \left(\sum_{i=1}^{\infty} |\gamma_i| \right)^2 = \|x\|_1^2 .$$

$R(I)$ is dense but not complete in ℓ_2 since, by the open mapping theorem, that would imply $\ell_1 \cong \ell_2$, an obvious contradiction.

I.4.4 Proposition. *The following are equivalent :*

(i) X *is an operator range.*

(ii) There exist a stronger norm $|||.|||$ *on* X *such that* $(X, |||.|||)$ *is a Banach space and* $|||x||| \geq \|x\|$ *for every* $x \in X$ *.*

Proof: Assume X to be an operator range.

Renorm X with $|||x||| = \|x\| + \|\beta_X x\|_1$ for $x \in X$. Suppose $\{x_n\}$ is a Cauchy sequence in $(X, |||.|||)$. Since $|||x||| \geq \|\beta_X x\|_1$ for every $x \in X$, $\{\beta_X x_n\}$ is Cauchy and thus convergent in the pre-image space X_1 . Thus there exists $\beta_X x_0$ in X_1 s.t. $\beta_X x_n \to \beta_X x_0$. Since α_X is bounded , $x_n \to x_0$ in $(X, \|.\|)$. Thus $x_n \to x_0$ in $(X, |||.|||)$.

Now (ii) obviously implies (i) since the identity map from $(X, |||.|||)$ onto $(X, \|.\|)$ is bounded.

□

We note the following two results due to R W Cross [04].

I.4.5 Proposition. *Let* R *be a subspace of a Banach space* Y *. The following are equivalent :*

(i) R *is an operator range;*

(ii) R *is the domain of an operator* $T \in C(Y,X)$ *where* X *is complete;*

(iii) there is a norm $|||.|||$ *on* R *such that* $(R,|||.|||)$ *is a Banach space and* $|||r||| \geq \|r\|$ *for* $r \in R$.

Proof: Suppose R is a subspace of a Banach space Y . The equivalence of (i) and (iii) is immediate from Proposition I.4.4. Suppose (i) holds. By definition β_R satisfies (ii). Next assume that (ii) holds and let $T \in C(Y,X)$ with $D(T) = R$. Renorm R as follows ;

$$|||y||| = \|y\| + \|Ty\| \quad \text{for} \quad y \in R .$$

Let $\{y_n\}$ be a Cauchy sequence in $(R,|||.|||)$. It follows that $\{y_n\}$ and $\{Ty_n\}$ are Cauchy, and thus convergent sequences in Banach spaces Y and X respectively. Suppose

$$y_n \to y \quad \text{in} \quad Y \quad \text{and} \quad Ty_n \to x \quad \text{in} \quad X .$$

From the fact that T is closed it follows that $y \in R$ and $Ty = x$. Thus by definition of $|||.|||$, $y_n \to y$ in $(R, |||.|||)$, establishing the completeness of $(R, |||.|||)$. □

I.4.6 Proposition. *Let* E *and* F *be operator ranges in a Banach space* X . *Then* $E + F$ *and* $E \cap F$ *are operator ranges.*

Proof: Suppose E and F are operator ranges contained in a Banach space X . It is an easily verified fact that for Banach spaces E_1 and F_1 , the space $E_1 \times F_1$ is also complete. Let $T \in B[E_1 \times F_1, X]$ be defined by

$$T(x,y) = \alpha_E x + \alpha_F y \quad \text{where} \quad x \in E_1,\ y \in F_1 .$$

It now follows that $E + F$ is an operator range with pre-image space $E_1 \times F_1/N(T)$ and $\alpha_{E+F} = \hat{T}$. By Proposition I.4.5 there exist stronger norms $\|.\|_E$ and $\|.\|_F$ such that $(E,\|.\|_E)$ amd $(F,\|.\|_F)$ are complete. We now define the stronger norm $|||.|||$, on $E \cap F$ as follows :

$$|||z||| = \|z\|_E + \|z\|_F \quad \text{for} \quad z \in E \cap F .$$

It is easily verified that $(E \cap F, |||.|||)$ is complete, thus establishing the fact that $E \cap F$ is an operator range. □

I.4.7 Proposition. *Let* $T \in C(X,Y)$.

(i) *If* X *is an operator range,* $T\alpha_X \in C(X_1,Y)$.

(ii) *If* Y *is an operator range,* $\beta_Y T \in C(X,Y_1)$.

Proof:

(i) This is a consequence of the easily verified fact that for any bounded operator $A \in B[Z,X]$, the operator TA is closed.

(ii) Assume $T \in C(X,Y)$ and Y an operator range. By I.3.20 and I.3.12, $\hat{T}$ and thus $\hat{T}^{-1}$, is closed. By (i) $\hat{T}^{-1}\alpha_Y$ is closed and thus by I.3.20 and I.3.12 $\beta_Y\hat{T} = (\beta_Y T)\hat{}$ and thus $\beta_Y T$ is closed. □

I.4.8 Proposition. *Let* X *be an operator range. The pre-image space* X_1 *is unique up to an isomorphism.*

Proof: Suppose X has another pre-image space, say X_0, besides X_1. Let T be the injective bounded operator mapping X_1 onto X i.e. $T = \alpha_X$, and let U be the injective closed operator mapping X onto X_0. By I.4.7 $S = UT$ is closed and injective. From the closed graph and open mapping theorems it now follows that S is an isomorphism between X_1 and X_0 . □

We now give a generalisation of the closed graph and open mapping theorems.

I.4.9 Proposition (generalised closed graph theorem).

Let $T \in C(X,Y)$ *with* $D(T)$ *closed in the Banach space* X *and* Y *an operator range. Then* $T \in B(X,Y)$.

Proof: Let $T \in C(X,Y)$ with $D(T)$ closed, X a Banach space and Y an operator range. Then by I.4.7, $\beta_Y T \in C(X,Y_1)$ with $D(T) = D(\beta_Y T)$ and Y_1 obviously complete. Then by I.3.18, $\beta_Y T$ is continuous. Since α_Y is bounded, $T = \alpha_Y(\beta_Y T)$ is continuous. □

I.4.10 Proposition (generalised open mapping theorem).

Let $T \in C[X,Y]$. *If* X *is an operator range,* Y *non-meagre and* T *surjective, then* T *is open.*

Proof: Let $T \in C[X,Y]$ with X an operator range, Y non-meagre and T surjective. By I.4.7 $T\alpha_X \in C[X_1,Y]$ with $R(T) = R(T\alpha_X) = Y$. Therefore by I.3.16, $T\alpha_X$ is open and thus, since β_X is open by definition, $T = (T\alpha_X)\beta_X$ is open. □

I.4.11 Corollary. *An operator range* X *is non-meagre iff* X *is complete.*

Proof: Suppose X is a non-meagre operator range. Consider the bounded surjective map $\alpha_X \in C[X_1,X]$. By I.4.10 α_X is open and $\beta_X = \alpha_X^{-1}$ is thus bounded. Thus $X \cong X_1$ implying, by I.3.8 that X is complete. The converse follows from I.3.17. □

I.4.12 Definition (restriction, quotient map).

If T is an operator from X into Y and M is a subspace of X, we will call the operator $T|_M$ defined by $T|_M(m) = T(m)$ for all $m \in M$ the *restriction* of T to M, denoted by $T|_M$.

Suppose M is a closed subspace of X. We define the *quotient map* π_M to be the operator mapping $x \in X$ onto $[x] \in X/M$. Since $\|x\| \geq \|[x]\|$ for all $x \in X$, π_M is bounded.

I.4.13 Proposition. *Let* M *be a closed subspace of an operator range* X. *Then*

(i) M *is an operator range*

(ii) X/M *is an operator range.*

Proof: Consider $\beta_X \in C[X,X_1]$. By I.4.2, β_X is open and thus maps the open set $X \setminus M$ onto the open set $X_1 \setminus M_1$ where $M_1 = \beta_X M$. Thus M_1 is closed in X_1 and therefore complete. Consequently M is an operator range with $\alpha_M = \alpha_X|_{M_1}$ and pre-image space M_1, proving (i). Since M_1 is closed, X_1/M_1 is a Banach space by I.2.6. Notice that $\pi_M\alpha_X$ is a bounded operator mapping X_1 onto X/M with $N(\pi_M\alpha_X) = M_1$. Thus $\alpha_{X/M} = (\pi_M\alpha_X)^\wedge$ is a bounded injective operator from X_1/M_1 onto X/M, proving (ii). □

I.5 Dual Spaces

These are also referred to in the literature as conjugate or adjoint spaces.

I.5.1 Definition (bounded linear functional, dual or conjugate space).

A *functional* on a vector space V is an operator mapping V into the space of scalars. The *dual* of a normed space X, denoted by X', is the normed space of all bounded linear functionals on X, i.e. the space $B[X,K]$ where K is the space of scalars. Since both $\mathbb{R}$ and $\mathbb{C}$ are complete (see I.2.2) it follows trivially from I.3.4 that X' is a Banach space.

I.5.2 Lemma [11]. *Let* M *be a dense subspace of* X. *If* $T \in B(X,Y)$ *with* $D(T) = M$ *and* Y *a Banach space, then there exists a unique operator* $\bar{T} \in B[X,Y]$, *which we will call the extension of* T, *such that* $\bar{T}|_M = T$ *and* $\|T\| = \|\bar{T}\|$.

Proof: Suppose $T \in B(X,Y)$ is defined as above. Then it follows by I.2.3 that for any $x \in X$, there exists a sequence $\{x_n\}$ in M such that $x_n \to x$. Since T is continuous, $\{Tx_n\}$ is a Cauchy sequence in Y. Since Y is complete, $Tx_n \to y$ for some $y \in Y$. Let $\bar{T}x = y$. To show that $\bar{T}$ is well defined we suppose that there exists another sequence $\{z_n\}$ in M, such that $z_n \to x$ and as before $Tz_n \to w$ for some $w \in Y$, i.e. $\bar{T}x = w$. Consider

$$\|Tx_n - Tz_m\| \leqslant \|T\|\|x_n - z_m\| .$$

As $m,n \to \infty$ we obtain

$$\|y - w\| \leqslant \|T\|\|x - x\| = 0 .$$

Therefore $y = w$.

Now

$$\|\bar{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leqslant \|T\| \lim_{n\to\infty} \|x_n\| = \|T\| \|x\| .$$

Hence $\|\bar{T}\| \leqslant \|T\|$. However since $M \subset X$, $\|\bar{T}\| = \sup \{\|\bar{T}x\| : x \in X , \|x\| = 1\}$ $\geqslant \sup \{\|Tx\| : x \in M , \|x\| = 1\} = \|T\|$. Thus we obtain $\|\bar{T}\| = \|T\|$. □

I.5.3 Definition (isometry).

If I is an isomorphism between X and Y , we say that I is an *isometry* if $\|x\| = \|Ix\|$ for each $x \in X$. X and Y are called *isometric.*

I.5.4 Proposition. *If M is a dense subspace of X , M' is isometric to X' .*

Proof: Suppose M is a dense subspace of X . Let $m' \in M' = B[M,K]$, where K is the Banach space of scalars. By Lemma I.5.2 it follows that there exists a unique bounded extension $\bar{m}'$ of m' , to all of X such that $\bar{m}' \in X'$ and $\|m'\| = \|\bar{m}'\|$. Let I be the map which takes each $m' \in M'$ onto its unique bounded extension in X' . Note that I is everywhere defined on M' . From the uniqueness of the extension and the equality of norms of say $m' \in M'$ and $\bar{m}' \in X'$, it follows that I is well defined and injective. We show that I is surjective. Let $x' \in X'$. Then $x'_M = x'|_M \in M'$. Since by Lemma I.5.2 x'_M has a unique bounded extension, it follows that $Ix'_M = x'$. Thus since $\|Im'\| = \|\bar{m}'\|$ for each $m' \in M'$, it now follows that I is an isometry between M' and X' . □

We state the following important theorem without proof. For a proof the reader is referred to any of the standard references, e.g. Simmons [20]. A proof also appears in Goldberg [11], p.17.

I.5.5 Theorem (The Hahn-Banach theorem).

Let M *be a subspace of* X *, a real or complex vector space. Let* p *be a real-valued function defined on* X *with the following properties:*

(i) $p(x + y) \leqslant p(x) + p(y)$ *for* $x, y \in X$.

(ii) $p(\lambda x) = |\lambda| p(x)$ *for* $x \in X$, λ *a scalar.*

If f *is a linear functional on* M *such that*

$$|f(m)| \leqslant p(m) \quad \textit{for} \quad m \in M ,$$

then there exist a linear extension $\tilde{f}$ *of* f *to all of* X *such that*

$$|\tilde{f}(x)| \leqslant p(x) \quad \textit{for} \quad x \in X .$$

I.5.6 Corollary. *Let* f *be a bounded linear functional on* M *a subspace of the normed space* X . *Then there exists* $\tilde{f} \in X'$ *which is an extension of* f *and for which*

$$\|\tilde{f}\| = \|f\| .$$

Proof: Let p be the function which is defined as follows ; $p(x) = \|f\|\|x\|$ for all $x \in X$. Note that f and p satisfy the conditions in I.5.5, thus proving the existence of $\tilde{f}$. Also $\|\tilde{f}\| \leqslant \|f\|$ by choice of p . However since $\tilde{f}$ is an extension of f , $\|\tilde{f}\| \geqslant \|f\|$. The corollary follows.

I.5.7 Corollary [11]. *Let* M *be a subspace of* X . *Given* $x \in X$ *with* $d = d(x,M) > 0$, *there exists an* $x' \in X'$ *such that*

$$\|x'\| = 1 \ , \quad x'm = 0 \ \text{for} \quad m \in M \ , \ \text{and} \quad x'x = d(x,M) \ .$$

Proof: Let M_1 be the subspace spanned by x and the elements of M . Define linear functional v' on M_1 by

$$v'(\alpha x + m) = \alpha d \qquad m \in M \ .$$

Then $v'M = 0$ and $v'x = d$. We assert that v' is in M_1' with $\|v'\| = 1$. For λ a non-zero scalar and $m \in M$,

$$\|\lambda x + m\| = |\lambda| \|x + \tfrac{m}{\lambda}\| \geqslant |\lambda| \ d \ .$$

Thus for all scalars λ ,

$$|v'(\lambda x + m)| = |\lambda| d \leqslant \|\lambda x + m\| \quad \text{for} \quad m \in M \ .$$

Hence $v' \in M_1'$ and $\|v'\| \leqslant 1$. There exists a sequence $\{m_k\}$ in M such that $\|x - m_k\| \to d$. Since

$$d = v'(x - m_k) \leqslant \|v'\| \|x - m_k\| \to \|v'\| d$$

it follows that $\|v'\| \geqslant 1$. Thus $\|v'\| = 1$. The corollary follows upon taking $x' \in X'$ to be an extension of v' so that $\|x'\| = \|v'\| = 1$.

□

I.5.8 Corollary [11]. *Given* $x \in X$, *there exists an* $x' \in X'$ *such that* $\|x'\| = 1$ *and* $x'x = \|x\|$. *In particular, if* $x \neq y$, *there exists an* $x' \in X'$ *such that* $0 \neq \|x - y\| = x'x - x'y$.

Proof: Take $M = \{0\}$ in Corollary I.5.7. □

I.5.9 Corollary [11]. *For any* $x \in X$,

$$\|x\| = \sup\{|x'x| : x' \in X' \ , \|x'\| = 1\} \ .$$

Proof: For $x' \in S_{X'}(1)$,

(1) $$|x'x| \leq \|x'\|\|x\| \leq \|x\| \ .$$

By Corollary I.5.8 there exists a z' in the 1-sphere of X' such that

(2) $$z'x = \|x\| \ .$$

The corollary follows from (1) and (2). □

As an application of the Hahn-Banach theorem we give the converse to I.3.4.

I.5.10 Corollary [11]. *If* $B[X,Y]$ *is complete, then* Y *is complete.*

Proof: Let $\{y_n\}$ be a Cauchy sequence in Y . Choose $x_0 \in X$ such that $\|x_0\| = 1$. There exists an $x' \in X'$ such that $x'x_0 = \|x_0\| = 1$. Define $T_n \in B[X,Y]$ by

$$T_n x = x'(x)y_n \quad \text{for} \quad x \in X \ .$$

Now

$$\|(T_n - T_m)x\| = |x'x|\|y_n - y_m\| \leq \|x'\|\|y_n - y_m\|\|x\| \quad \text{for} \quad x \in X \ .$$

Hence $\|T_n - T_m\| \leq \|x'\|\|y_n - y_m\|$ which implies that $\{T_n\}$ is a Cauchy sequence in $B[X,Y]$. Thus $\{T_n\}$ converges in $B[X,Y]$ to some $T \in B[X,Y]$. Since

$$\|y_n - Tx_0\| = \|T_n x_0 - Tx_0\| \leq \|T_n - T\|\|x_0\| \ ,$$

$\{y_n\}$ converges to Tx_0 and therefore Y is complete.

□

I.5.11 Definition (natural map, reflexive space).

The linear operator J_X from X into X'', defined by

$$(J_X x)x' = x'x \quad \text{for all} \quad x \in X \ , \ x' \in X'$$

is called the *natural map* of X into its second dual X''. (J_X is obviously bounded since $\|(J_X x)x'\| = \|x'x\| \leqslant \|x'\|\|x\|$.)

If $R(J_X) = X''$, the space X is said to be *reflexive*.

I.5.12 Remark.

(i) J_X is an isometry between X and $R(J_X)$. From Corollary I.5.9 we have that

$$\|J_X x\| = \sup_{\|x'\|=1} |(Jx)x'| = \sup_{\|x'\|=1} |x'x| = \|x\| \quad .$$

(ii) Reflexive spaces are complete. This follows from I.3.8.

From the uniform boundedness principle we now obtain the following proposition.

I.5.13 Proposition [11]. *Suppose* K *is a subset of* X *such that*

$$\sup_{k \in K} |x'k| < \infty \quad \textit{for all} \quad x' \in X' \ .$$

Then

$$\sup_{k \in K} \|k\| < \infty.$$

Proof: By considering J_X, the natural map of X into X'' the proof follows immediately from I.3.19 since

$$\sup_{k \in K} |(J_X k)x'| = \sup_{k \in K} |x'k| < \infty \quad \text{for} \quad x' \in X'$$

and since

$$\|J_X k\| = \|k\| \quad \text{for all} \quad k \in K \ .$$

□

I.5.14 Definition (orthogonal, orthogonal complement).

A subset K of X is said to be *orthogonal* to a subset F of X' if $x'k = 0$ for all $k \in K$ and $x' \in F$.
We define the set $K^{\perp} = \{x' \in X' : x'k = 0$ for all $k \in K\}$ to be the *orthogonal complement* of K.

I.5.15 Remark.

(i) Irrespective of $K \subset X$, $K^{\perp}$ is a closed subspace of X.

(ii) $K^{\perp} = (\bar{K})^{\perp} = \overline{(K^{\perp})}$ by continuity of the elements of X' and $J_X X$.

The following theorem will be used in subsequent chapters.

I.5.16 Theorem [11]. *Let* M *be a subspace of* X. *Then*

(i) $X'/M^{\perp}$ *is isometric to* M' *under the map* U *defined by* $U[x'] = x'_M$ *where* $[x']$ *is in* $X'/M^{\perp}$ *and* x'_M *is the restriction of* x' *to* M.

(ii) If M *is closed (so that* X/M *is a normed space), then* $(X/M)'$ *is isometric to* $M^{\perp}$ *under the map* V *defined by*

$$(Vz')x = z'[x] \quad \text{for } z' \in \left(\frac{X}{M}\right)' .$$

Proof:

(i) Note that U is well defined, since $[y'] = [x']$ implies $0 = y'm - x'm$, $m \in M$. Clearly U is linear with range in M'. Given $m' \in M'$, there exists by Corollary I.5.6, an $x' \in X'$ which is an extension of m'. Hence $U[x'] = x'_M = m'$ which shows that $R(U) = M'$. For any $y' \in [x']$,

$$\|U[x']\| = \|y'_M\| \leqslant \|y'\| .$$

Thus

$$(1) \qquad \|U[x']\| \leqslant \inf_{y' \in [x']} \|y'\| = \|[x']\| .$$

On the other hand, there exists a $v' \in X'$ which is an extension of x'_M such that $\|v'\| = \|x'_M\|$. Therefore v' is in $[x']$ and

$$(2) \qquad \|U[x']\| = \|x'_M\| = \|v'\| \geqslant \|[x']\| .$$

By (1) and (2), $\|U[x']\| = \|[x']\|$.

(ii) For $z' \in (X/M)'$,

$$|(Vz')x| = |z'[x]| \leqslant \|z'\| \|[x]\| \leqslant \|z'\| \|x\| \quad \text{for} \quad x \in X$$

and

$$(Vz')m = z'[m] = z'[0] = 0 \quad \text{for} \quad m \in M .$$

Thus Vz' is in $M^{\perp}$ with

$$(3) \qquad \|Vz'\| \leqslant \|z'\| .$$

Since

$$|z'[x]| = |(Vz')y| \leqslant \|Vz'\| \|y\| \quad \text{for} \quad y \in [x] ,$$

it follows that

$$|z'[x]| \leqslant \|Vz'\| \; \|[x]\| .$$

Thus

$$\|z'\| \leqslant \|Vz'\|$$

which, together with (3), proves that V is an isometry. Given $x' \in M^{\perp}$, let z' be the linear functional on X/M defined by $z'[x] = x'x$. Since

$$|z'[x]| = |x'y| \leqslant \|x'\| \|y\| \quad \text{for} \quad y \in [x]$$

it follows that $|z'[x]| \leqslant \|x'\| \|[x]\|$. Hence z' is in $(X/M)'$. Furthermore, $Vz' = x'$, proving that $R(V) = M^{\perp}$. □

I.5.17 Proposition. *Every normed space* X *is a dense subspace of a Banach space.*

Proof: [11]. Given X , consider $\overline{R(J_X)} \subset X''$. $\overline{R(J_X)}$ is a closed subspace of X" and thus complete. Let $\tilde{X} = X \cup (\overline{R(J_X)} \setminus R(J_X))$ with the norm and vector addition and scalar multiplication defined as in $\overline{R(J_X)}$. □

I.5.18 Definition (completion).

We define $\tilde{X}$ in the proof of I.5.17 to be the *completion* of X .

I.6 Finite Dimensional Normed Spaces

I.6.1 Theorem [05]. *If two real (complex) normed spaces are of the same finite dimension* n *, they are isomorphic.*

Proof: Let X be a real (complex) normed space of dimension n . Consider ℓ_1^n , the space of all real (complex) n-tuples normed by

$$\|x\|_1 = |\lambda_1| + |\lambda_2| + \dots + |\lambda_n| \text{ for}$$
$$x = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \ell_1^n .$$

Let $e_1, e_2, \dots, e_n$ be a basis for X . We now define $I \in [\ell_1^n, X]$ as follows;

$$I(\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n \quad \text{for}$$
$$(\lambda_1, \lambda_2, \dots, \lambda_n) \in \ell_1^n .$$

I is obviously, injective, surjective and linear. Since

$$\|\lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_n e_n\| \leqslant \max_{1 \leqslant i \leqslant n} \|e_i\| \, (|\lambda_1| + |\lambda_2| + \dots + |\lambda_n|) ,$$

I is bounded. It remains to show that I^{-1} is bounded. Define a real valued function f on $S_{\ell_1^n}(1)$ by

$$f((\lambda_1, \lambda_2, \ldots, \lambda_n)) = \|\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n\| \ .$$

By continuity of the norm and of I, f is continuous on the closed and bounded and therefore compact subset $S_{\ell_1^n}(1)$ of $\mathbb{R}^n(\mathbb{C}^n)$. Thus f attains a minimum, say m , at some point y of $S_{\ell_1^n}(1)$. Furthermore $m \neq 0$ since $m = 0$ would imply that

$$f(y) = \|\gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_n e_n\| = 0 \quad \text{for } y = (\gamma_1, \gamma_2, \ldots, \gamma_n).$$

Thus $\gamma_1 e_1 + \gamma_2 e_2 + \ldots + \gamma_n e_n = 0$ implying, by the linear independance of $e_1, e_2, \ldots, e_n$, that $\gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$. An obvious contradiction. Now for any $x \in \ell_1^n$ such that $\|x\| = \mu > 0$ we have

$$\|I(x)\| = \mu \, \|I(\tfrac{x}{\mu})\| \geqslant \mu \, m = m\|x\|$$

since $(\frac{x}{\mu}) \in S_{\ell_1^n}(1)$.

Thus, by I.3.6, X is isomorphic to ℓ_1^n and the result follows. □

I.6.2 Corollary. *All finite dimensional normed spaces are complete.*

Proof: Follows from I.6.1, I.3.8 and the completeness of spaces like ℓ_1^n, $\mathbb{R}^n$ and $\mathbb{C}^n$. □

I.6.3 Corollary. *A finite dimensional subspace* N *of* X *is complete and therefore closed in* X .

Proof: Follows immediately from I.6.2 and I.2.4. □

We state the following lemma, often called Riesz's lemma, without proof. For proofs refer to Diestel [05], p.2 or Kreyszig [16], p.78.

I.6.4 Lemma. *Let* Y *be a closed subspace of* X *such that* $Y \neq X$. *Then for any* $0 < \theta < 1$, *there is an* $x_\theta \in S_X(1)$ *such that* $\|x_\theta - y\| > \theta$ *for every* $y \in Y$.

I.6.5 Proposition. *The normed space* X *is finite dimensional iff each closed, bounded subset of* X *is compact.*

Proof: If X is finite dimensional, it is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$. Since closed bounded subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$ are compact and since an isomorphism maps closed bounded sets onto closed bounded sets, compactness of closed bounded subsets of X follows.

Suppose now X is infinite dimensional. Let $x_1 \in S_X(1)$. By I.6.4 there exists an $x_2 \in S_X(1)$ such that

$$\|x_2 - y\| \geqslant \frac{3}{4} \quad \text{for} \quad y \in sp(x_1) ,$$

since $sp(x_1)$ is closed by I.6.3.

Proceeding likewise there exists $x_3 \in S_X(1)$ such that

$$\|x_3 - y\| \geqslant \frac{3}{4} \quad \text{for} \quad y \in sp(x_1, x_2) .$$

Thus by induction there exists an infinite sequence $\{x_n\}$ in $S_X(1)$ such that $\|x_m - x_n\| > \frac{1}{2}$ for $m \neq n$. Thus from the definition of compactness, $S_X(1)$ is closed and bounded but not compact. □

I.6.6. Remark.

Proposition I.6.5 implies that if the closed unit ball of a normed space X is compact then X is finite dimensional.

I.6.7 Theorem [11]. *The sum of two closed subspaces of a normed space is closed whenever one of the subspaces is finite-dimensional.*

Proof: Let M and N be a closed subspace and a finite-dimensional subspace, respectively, of the normed space X. We define the linear map A from X onto X/M by $Ax = [x]$. Since $\| Ax \| = \| [x] \| \leqslant \| x \|$, A is continuous. Moreover, the linearity of A and the finite-dimensionality of N imply the finite-dimensionality of AN. Hence, by Corollary I.6.3, AN is closed and therefore $A^{-1}AN = M + N$ is closed. (Note that A^{-1} is used in the set theoretic sense.) □

I.6.8 Definition (Kronecker delta).

We define δ_{ij}, the *Kronecker delta* as follows;

$$\delta_{ij} = 1 \text{ if } i = j , \quad \delta_{ij} = 0 \text{ if } i \neq j .$$

I.6.9 Theorem [11]. *There is always a bounded projection from a normed linear space onto any one of its finite-dimensional subspaces.*

Proof: Suppose M is a finite-dimensional subspace of X with basis $x_1, x_2, \ldots, x_n$. Let $x_1', x_2', \ldots, x_n'$ be elements in X' such that $x_i'x_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. The x_i' may be constructed as follows. Let $M_i = \text{sp}\,\{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. Since M_i is finite-dimensional and therefore closed, there exists a $v_i' \in X'$ such that $v_i'x_i = \lambda \neq 0$ and $v_i'M_i = 0$. Choose $x_i' = \frac{1}{\lambda} v_i'$. It is easy to verify that the operator P defined by $Px = \sum_{i=1}^{n} x_i'(x)x_i$ is a bounded projection from X onto M.

□

I.6.10 Proposition. *If* X *is of dimension* $n < \infty$ *,* X' *is also of dimension* n .

Proof: Let X be of dimension $n < \infty$ and let $e_1, e_2, \ldots, e_n$ be a basis for n . It is easy to see that the functionals

x_i' for $1 \leqslant i \leqslant n$ defined by

$$x_i' \ (\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n) = \lambda_i$$

for $(\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n) = x \in X$,

are linear, bounded, and form a basis for X' . Thus the dimension of X' is n . □

I.6.11 Corollary. *If* X *is finite dimensional it is reflexive.*

Proof: By I.6.10 if X is of dimension n , X' and thus X'' is of dimension n . Thus the natural map , J_X , is surjective, proving the result.

I.7 Reflexive and Pre-Reflexive Spaces

I.7.1 Definition (pre-reflexive).

We define an operator range X to be *pre-reflexive* if its pre-image space X_1 is reflexive. (This definition is the author's own).

I.7.2 Example. Consider $T \in B[\ell_2, \ell_1]$ defined by

$$T(\{x_n\}) = \{x_n/n\} \qquad \text{for } \{x_n\} \in \ell_2 .$$

The operator T is continuous.

$$\|\{x_n/n\}\|_1 = \sum_{n=1}^{\infty} |x_n/n| \leqslant \left(\sum_{n=1}^{\infty} |\tfrac{1}{n}|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$$

$$\leqslant K \|\{x_n\}\|_2 \quad \text{where} \quad K = \left(\sum_{n=1}^{\infty} |\tfrac{1}{n}|^2\right)^{\frac{1}{2}} < \infty \,.$$

The range of T is dense in ℓ_1. Consider elements of ℓ_2, say $y = \{y_n\}$, of the form

$$y_n = 0 \quad \text{whenever} \quad n > K \quad \text{for some} \quad K \in \mathbb{N} \,.$$

These, when mapped onto ℓ_1, form a dense subset of ℓ_1. We can now conclude that $R(T)$ is a non-complete pre-reflexive space since if $R(T) = \ell_1$ we obtain the contradictory statement $\ell_2 \cong \ell_1$ by appealing to the open-mapping theorem.

We will only make a few introductory remarks on reflexive and pre-reflexive spaces in this section. More results on pre-reflexive spaces will be proved in subsequent chapters once the necessary theory has been developed.

I.7.3 Theorem [11]. *A closed subspace of a reflexive space is reflexive.*

Proof: Let M be a closed subspace of reflexive space X. Given $m'' \in M''$, define $x'' \in X''$ by

$$x''x' = m''x'_M$$

where x'_M is the restriction of $x' \in X'$ to M. Let $m = J_X^{-1}x''$. Suppose $m \notin M$. Then there exists an $x' \in X'$ such that $x'm \neq 0$ while $x'M = 0$. Thus $x'_M = 0$ and

$$0 \neq x'm = x'J_X^{-1}x'' = x''x' = m''x'_M = m''0 = 0 \,,$$

an obvious contradiction. Hence $m \in M$. For each $m' \in M'$, let m'_e be an element in X', which is an extension of m'. Then

$$m''m' = x''m'_e = m'_e J_X^{-1} x'' = m'_e m = m'm \ .$$

Thus $J_M m = m''$, proving that J_M is surjective and M therefore reflexive. □

I.7.4 Corollary. *A closed subspace of a pre-reflexive space is pre-reflexive.*

Proof: Let M be a closed subspace of R , a pre-reflexive space. Since $R \setminus M$ is open and since β_R is an open operator, $\beta_R(R \setminus M) = R_1 \setminus \beta_R M$ is open and $\beta_R M$ thus closed in R_1 . By I.7.3 $\beta_R M$, which is the pre-image space of M , is reflexive, thus proving the corollary. □

I.7.5 Corollary [07]. *A Banach space* X *is reflexive iff its dual* X' *is reflexive.*

Proof: Suppose X is reflexive. For any $x_0''' \in X'''$, the functional x'_0 , defined by $x'_0 = x_0''' J_X$ is an element of X' (since J_X is an isometry from X onto X''). Thus $x''x'_0 = (J_X x)x'_0 = x'_0 x = (x_0''' J_X)x = x_0''' x''$ for any $x'' \in X''$. Since x_0''' was chosen arbitrarily it follows that $J_{X'}(X') = X'''$, i.e. X' is reflexive. Conversely, if X' is reflexive, X'' is reflexive by what has just been proven. Thus the closed subspace $J_X(X)$ of X'' , and therefore X , is reflexive. □

We state the following theorem, which will be needed later, without proof. For a proof the reader is referred to Diestel [05], p.18 or Robertson and Robertson [17] , p.73.

I.7.6 Definition (weak convergence).

A sequence $\{x_n\} \subset X$ is said to *converge weakly* to $x \in X$ if

$$x'x_n \to x'x \quad \text{for any} \quad x' \in X' .$$

I.7.7 Theorem. *The space* X *is reflexive iff every bounded sequence has a weakly convergent subsequence.*

CHAPTER II

THE CONJUGATE OF AN UNBOUNDED LINEAR OPERATOR

We will not attempt to cover all aspects of conjugate operators but will only investigate the relationship between an operator $T \in L(X,Y)$, α_X , β_Y and their respective conjugates. For further information on conjugate operators the reader is referred to Goldberg [11] , pp. 49-75.

II.1 The Operators T', α'_X and β'_Y

We appeal to I.5.2 to make our definition of the conjugate.

II.1.1 Definition. (Conjugate of $T \in L(X,Y)$).

Let $T \in L(X,Y)$ have dense domain in X . The *conjugate* T' of T , is defined to be the operator with $D(T') = \{y' \in Y' : y'T \text{ continuous on } D(T)\}$. For $y' \in D(T')$, let T' be the operator which maps y' onto $\overline{y'T}$, the unique continuous linear extension of $y'T$ to all of X . The operator T' is obviously linear and $D(T')$ is thus a subspace of Y' . We obtain $T' \in L(Y',X')$

II.1.2 Remark. Let $T \in L(X,Y)$.

The condition that $D(T)$ be dense in X for T' to exist is not too restrictive since if $\overline{D(T)} \subsetneqq X$, we could redefine T to be an element of $L(X_0,Y)$ where $X_0 = \overline{D(T)}$. Throughout the remainder of the chapter we will assume $D(T)$ to be dense in X , unless otherwise stated. We present the following example, from Goldberg [11], p.51, of a conjugate operator.

II.1.3 Example. Let $X = Y = \ell_p$, $1 \leqslant p < \infty$, and let

$$u_1 = (1, 0, 0, \ldots), \; u_2 = (0, 1, 0, \ldots), \text{ etc.}$$

be the unit vectors in ℓ_p . Define T by

$$D(T) = \operatorname{sp} \{u_k : k \in \mathbb{N}\}$$

$$T(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \left(\sum_{j=1}^{n} jx_j, x_2, x_3, \ldots, x_n, 0, 0, \ldots \right) .$$

Suppose $y' = (a_1, a_2, \ldots) \in D(T')$. Then for $k \geqslant 2$,

$$|y'Tu_k| = |a_1 k + a_k| \geqslant |a_1|k - |a_k| \geqslant |a_1|k - \|y'\| .$$

Since $\|u_k\| = 1$ and $y'T$ is bounded on $D(T)$, $a_1 = 0$. Also, any element $(0, b_1, b_2, \ldots) \in \ell_{p'} = \ell'_p$ is in $D(T')$. Hence the domain of T' consists of all the elements in $\ell_{p'}$ which have zero as their first term. Suppose $T'y' = (c_1, c_2, \ldots)$, where $y' = (0, a_2, a_3, \ldots)$ $\in D(T')$. Then

$$c_k = T'y'u_k = y'Tu_k = a_k , \text{ for } k \geqslant 2$$

and $c_1 = 0$. Thus $T'y' = y'$.

II.1.4 Remark [11].

It follows easily that $T' \in L(Y',X')$ is always a closed linear operator. Suppose $\{y'_n\}$ is a sequence in $D(T')$ such that $y'_n \to y' \in Y'$ and $T'y'_n \to x' \in X'$. Then $y'_n Tx \to y'Tx$ and $\overline{y'_nT}x = y'_nTx = T'y'_n x \to x'x$ for each $x \in X$. Thus it follows from the density of $D(T)$ in X and the boundedness of x' that since $y'T = x'|_{D(T)}$, $y' \in D(T')$ and $T'y' = x'$.

II.1.5 Theorem [11]. *Let* $T \in L(X,Y)$.
Then $D(T') = Y'$ *iff* $T \in B(X,Y)$. *Also* $T \in B(X,Y)$ *implies* $T' \in B[Y',X']$ *and* $\|T\| = \|T'\|$.

Proof: Clearly, if T is continuous, then $y'T$ is continuous for each $y' \in Y'$. Thus $D(T') = Y'$. Suppose $D(T') = Y'$. Let $S = S_{D(T)}(1)$. For each $y' \in Y'$, $\sup_{x \in S} |y'Tx| \leq \|T'y'\|$. Hence, by Proposition I.5.13, $\|T\| = \sup_{x \in S} \|Tx\| < \infty$. Now, for each $x \in S$, $|T'y'x| \leq \|y'\|\|T\|$. Thus $\|T'y'\| \leq \|T\|\|y'\|$, and therefore $\|T'\| \leq \|T\|$. By Corollary I.5.9

$$\|Tx\| = \sup_{\|y'\|=1} |y'Tx| = \sup_{\|y'\|=1} |T'y'x| \leq \|T'\|\|x\| \quad \text{for} \quad x \in D(T) .$$

Hence $\|T\| \leq \|T'\|$ and the theorem follows. □

II.1.6 Corollary. *Let* X *be an operator range. The following are equivalent :*

(i) β_X *bounded*

(ii) β'_X *bounded*

(iii) X *is complete.*

Proof: The equivalence of (i) and (iii) follows trivially from the closed graph theorem. We now see that (i) $\Longleftrightarrow$ (ii) by Theorem II.1.5 since $D(\beta_X) = X$ by definition. □

II.1.7 Corollary. *Let* X *be an operator range. Then* α'_X *is bounded and injective.*

Proof: Recall that $\alpha_X \in B[X_1, X]$ and is bijective.

$$\begin{aligned} \text{Thus} \quad \alpha'_X x' = 0 &\Rightarrow x'\alpha_X x_1 = 0 \quad (\text{for every } x_1 \in X_1) \\ &\Rightarrow x'x = 0 \quad (\text{for every } x \in X) \\ &\Rightarrow x' = 0 \text{ , i.e., } \alpha'_X \text{ is injective.} \end{aligned}$$

The boundedness of α'_X follows from II.1.5. □

Note that in the corollary the surjectivity of α_X implied the injectivity of α'_X . A stronger result holds, that is Lemma II.1.10.

II.1.8 Definition. (orthogonal complement).

Let W be a subset of X' . The *orthogonal complement* of W , written ${}^{\perp}W$, is defined to be the set ${}^{\perp}W = \{x: x \in X \ , \ w'x = 0 \ \text{ for all } \ w' \in W\}$.

Compare with Definition I.5.14. Note that ${}^{\perp}W$ is a closed subspace of X and that ${}^{\perp}W = {}^{\perp}(\bar{W})$.

We prove the following two results, stated without proof in Goldberg [11], p.59.

II.1.9 Lemma. *If* M *is a subspace of* X , *then* ${}^{\perp}(M^{\perp}) = \bar{M}$.

Proof: Since $\bar{M}^{\perp} = M^{\perp}$, we only consider

$$\begin{aligned} {}^{\perp}(\bar{M}^{\perp}) &= \{x: x \in X \ , \ x'x = 0 \ \text{ for all } \ x' \in \bar{M}^{\perp}\} \\ &= \{x: x \in X \ , \ x'x = 0 \ \text{ for all } \ x' \in X' \ \text{ such that } \ x'\bar{M} = 0\} \ . \quad (1) \end{aligned}$$

(Note that by $x'\bar{M} = 0$ we mean $x'm = 0$ for all $m \in \bar{M}$)

It is clear from (1) that $x \in \bar{M} \Rightarrow x \in {}^{\perp}(\bar{M}^{\perp})$.

Conversely if $x \notin \bar{M}$, there exists $x'_0 \in X'$ such that $x'_0\bar{M} = 0$ and $x'_0x = 1$.

Clearly $x'_0 \in \bar{M}^{\perp}$ and thus $x \notin {}^{\perp}(\bar{M}^{\perp})$, proving the lemma. □

II.1.10 Lemma. *Let* $T \in L(X,Y)$. *Then*

(i) $\overline{R(T)}^{\perp} = R(T)^{\perp} = N(T')$;

(ii) $\overline{R(T)} = {}^{\perp}N(T')$.

Therefore T' *is injective iff* T *has dense range.*

Proof:

(i) Note that

$\overline{R(T)}^{\perp} = R(T)^{\perp} = \{y' : y' \in Y' , y'Tx = 0$ for each $x \in D(T)\}$.

Thus $y' \in R(T)^{\perp}$ iff $y'Tx = 0$ for each $x \in D(T)$

iff $T'y'x = \overline{y'T}x = 0$ for each $x \in X$

by Lemma I.5.2. Thus $\overline{R(T)}^{\perp} = R(T)^{\perp} = N(T')$.

(ii) From (i) we have

$$R(T)^{\perp} = N(T') .$$

Thus

$$^{\perp}(R(T)^{\perp}) = {}^{\perp}N(T') .$$

Applying Lemma II.1.9 the result follows. □

We give the following result which appears in Goldberg [11].

II.1.11 Theorem. *If* T *and* T' *each have an inverse, then* $(T^{-1})' = (T')^{-1}$.

Proof: By Lemma II.1.10, $D(T^{-1}) = R(T)$ is dense in Y. Hence $(T^{-1})'$ is defined. Suppose $x' \in D((T')^{-1}) = R(T')$. Then there exists a $y' \in D(T')$ such that $T'y' = x'$. To show $x' \in D((T^{-1})')$, it suffices to prove that $x'T^{-1}$ is continuous on $R(T)$. This is certainly the case since

(1) $x'T^{-1}(Tx) = x'x = T'y'x = y'Tx$ for $x \in D(T)$

(and thus $D((T')^{-1}) \subset D((T^{-1})'))$. From (1) it now follows that $(T^{-1})'x' = y'$ on $R(T)$ whence $y' = (T^{-1})'x'$ since $R(T)$ is dense in Y . It follows that $(T^{-1})' = (T')^{-1}$ on $D((T')^{-1})$ since initially we assumed $T'y' = x'$, i.e. $y' = (T')^{-1}x'$ for $x' \in D((T')^{-1})$. It remains to prove $D((T^{-1})') \subset D((T')^{-1})$.

Suppose $z' \in D((T^{-1})')$. Define $v' \in Y'$ to be the continuous linear extension of $z'T^{-1}$ to all of Y . Thus $T'v'x = v'Tx = z'T^{-1}(Tx) = z'x$ for all $x \in D(T)$. Since $D(T)$ is dense in X , it follows that $T'v' = z'$ indicating that $z' \in D((T')^{-1})$, thus proving the result. □

II.1.12 Corollary. *Let* X *be an operator range. Then* $\alpha'_X = (\beta'_X)^{-1}$ *and* $(\alpha'_X)^{-1} = \beta'_X$.

Proof: Follows trivially from II.1.10 and the definition of α_X and β_X .

We give a generalisation of a result which appears as a corollary in Goldberg [11] , p.55.

II.1.13 Definition (total set of linear functionals).

A set V^* of linear functionals on a vector space V is called *total* if for any $v \in V$ such that $v \neq 0$, there exists $v^* \in V^*$ such that $v^*(v) \neq 0$.

II.1.14 Theorem. *Let* $T \in L[X,Y]$ *with* X *complete and* Y *an operator range. Then* T *is continuous iff* $D(T')$ *is total.*

Proof: Suppose $D(T')$ is total.

Let $\{x_n\}$ be a sequence in $D(T) = X$ such that $x_n \to x$ and $Tx_n \to y$. Then for every $y' \in D(T')$ we have that $y'Tx_n = T'y'x_n \to T'y'x = y'Tx$ and $y'Tx_n \to y'y$. Thus $y'(Tx - y) = 0$ for every $y' \in D(T')$. Since $D(T')$ is total however, it follows that T is closed. Therefore by Proposition I.4.9, T is continuous. Conversely if $T \in B[X,Y]$,

$D(T') = Y'$ by Theorem II.1.5, which is clearly total. □

II.1.15 Remark. The completeness condition on X in Theorem II.1.14 cannot be weakened to operator ranges. For any operator range X, $R(\alpha_X') = D((\alpha_X')^{-1}) = D(\beta_X') \subset X_1$ is total. Let $x_1 \in X_1$ such that $x_1 \neq 0$. Thus $0 \neq \alpha_X x_1 \in X$ and hence $x'\alpha_X x_1 = \alpha_X' x' x_1 \neq 0$ for some $x' \in D(\alpha_X') = X'$. Suppose now that X is a non-complete operator range. Then β_X is an operator from an operator range X onto a Banach space X_1 with $D(\beta_X')$ total. Note however that since X is not complete, β_X is obviously not bounded.

II.1.16 Definition (weak* topology).

We define the *weak* topology* on X' to be the weakest topology such that every $x'' \in J_X X$ is continuous on X', i.e. the topology with sets of the form

$$S(x,x_0',\varepsilon) = \{x' : x' \in X' \text{ and } |x'x - x_0'x| < \varepsilon\}$$

$$\text{for all } x \in X ,\ x' \in X' \text{ and } \varepsilon > 0 ,$$

forming a subbase. (See Simmons [20], p.233). Note that this topology is weaker than the norm topology.

We note the following result appearing in Dunford and Schwartz [07], p.439.

II.1.17 Proposition. *A linear subspace of* X' *is total iff it is weak*-dense in* X'.

II.1.18 Corollary. *Let* X *be an operator range. Then* $R(\alpha_X') = D(\beta_X')$ *is weak*-dense in* X_1' *(i.e. dense in the weak*-topology).*

Proof: Follows from Proposition II.1.17 and Remark II.1.15.

II.2 The Existence of β_X'' and $\beta_X^{(n)}$

We note from Theorem II.1.5 that for any $n \in \mathbb{N}$, we will always have the existence of $\alpha_X^{(n)}$ where $\alpha_X^{(n)}$ denotes the n^{th} conjugate of α_X. In trying to establish the existence of β_X'' we find that $D(\beta_X')$ is only weak*-dense in X_1' and not necessarily dense in the metric topology, as is required. In this section we set out to establish conditions for the existence of β_X'' and $\beta_X^{(n)}$ for any $n \in \mathbb{N}$ respectively.

II.2.1 Theorem. *Let* X *be reflexive. A subspace* M' *of* X' *is total iff* M' *is dense in* X'.

Proof: Suppose $M \subset X'$ is total but not dense in X'. Then there exists an $x'' \in X''$ such that $x'' \neq 0$ and $x''m' = 0$ for all $m' \in M$. An obvious contradiction by the reflexivity of X and the totality of M'. Conversely if M' is dense in X', then $x''m' = 0$ for all $m' \in M'$ implies that $x'' = 0$ where $x'' \in X''$. Thus by reflexivity of X, the totality of M' is established. □

II.2.2 Corollary. *Let* X *be a pre-reflexive space. Then* $D(\beta_X')$ *is dense in* X_1'.

Proof: Follows from Theorem II.2.1 and Remark II.1.15. □

Thus we see that β''_X exists if X is a pre-reflexive space.

II.2.3 Remark. Let R be a pre-reflexive space. Then

(i) $\beta''_R = (\alpha''_R)^{-1}$ (obvious from Lemma II.1.10 and Theorem II.1.11).

(ii) $\beta''_R = J_{R_1}\beta_R J_R^{-1}$ (follows since $D(\beta''_R) = \alpha''_R R''_1 = (J_R \alpha_R J_{R_1}^{-1})(J_{R_1}\beta_R R) = J_R R$).

II.2.4 Theorem. *Let* R *be a pre-reflexive space. Then* $\beta_R^{(n)}$ *exists for every* $n \in \mathbb{N}$ *iff* $\tilde{R}$ *is reflexive.*

Proof: Suppose R is a pre-reflexive operator range such that $\tilde{R}$ is reflexive. By Remark II.2.3 and Corollary II.1.12 we have that $(\alpha'_R)^{-1} = \beta'_R$ and $(\alpha''_R)^{-1} = \beta''_R$. Since by Corollary I.7.5, R'_1 and $\tilde{R}'$ are also reflexive, $\beta_R^{(n)}$ exists for every $n \in \mathbb{N}$. Conversely, suppose $\beta_R^{(n)}$ exists for every $n \in \mathbb{N}$. Since $\beta_R^{(3)}$ exists, we must have $D(\beta''_R)$ dense in $R'' = (\tilde{R})''$. But

$$D(\beta''_R) = \alpha''_R(R''_1) = \alpha''_R(J_{R_1}R_1) = \alpha''_R(J_{R_1}\beta_R R) = (J_R\alpha_R J_{R_1}^{-1})(J_{R_1}\beta_R R) = J_R R .$$

Thus $J_{\tilde{R}}\tilde{R} = \overline{D(\beta''_R)} = (\tilde{R})''$, proving the theorem. □

II.2.5 Corollary. *Let* R *be a pre-reflexive space. Then* $\alpha_R^{(n)}$ *is injective for every* $n \in \mathbb{N}$ *iff* $\tilde{R}$ *is reflexive.*

Proof: Follows from Theorem II.1.11 and II.2.4. □

II.3 States of Linear Operators and Characterization of States

We use the following classification of states, the state diagrams of which were originally obtained by Goldberg [10] . For $T \in L(X,Y)$ we have the following states :

I : $R(T) = Y$;

II : $R(T) \neq Y$, $\overline{R(T)} = Y$;

III : $\overline{R(T)} \neq Y$;

1 : T^{-1} exists and is continuous;

2 : T^{-1} exists but is not continuous;

3 : T^{-1} does not exist.

Thus $T \in I_1$, iff $R(T) = Y$ and T^{-1} exists and is continuous. Also if $T \in II_2$ and $T' \in III_2$ we write $(T,T') \in (II_2, III_2)$.

II.3.1 Remark. Let X be a non-complete operator range. From our previous results we have

(i) $\alpha_X \in I_2$, $\beta_X \in I_1$,

(ii) $\beta'_X \in I_1$, $\alpha'_X \in (II_2 \cup III_2)$, and $\alpha'_X \in II_2$ if X is pre-reflexive.

II.3.2 Proposition. *Let* $T \in L(X,Y)$ *where* X *and* Y *are operator ranges. We then obtain the following :*

(i) $T \in 1$ *if* $T\alpha_X \in 1$

(ii) $T \in 2$ *implies* $T\alpha_X \in 2$

(iii) $T \in 1$ *implies* $\beta_Y T \in 1$

(iv) $T \in 2$ *if* $\beta_Y T \in 2$

(v) $T \in 3$ *iff* $T\alpha_X \in 3$ *iff* $\beta_Y T \in 3$.

Proof:

(i) Suppose $T\alpha_X \in 1$. Then $(T\alpha_X)^{-1} = (\alpha_X)^{-1}T = \beta_X T^{-1}$ is continuous by definition. Thus $T^{-1} = \alpha_X(\beta_X T^{-1})$, the product of two continuous operators, is continuous and hence $T \in 1$ by definition.

(ii) Suppose $T \in 2$. Then T is injective and T^{-1} is not continuous. From the injectivity of T and α_X it follows that $T\alpha_X$ is injective, and hence $(T\alpha_X)^{-1}$ exists implying that either $T\alpha_X \in 1$ or $T\alpha_X \in 2$. Suppose $T\alpha_X \in 1$. Then by (i) $T \in 1$, an obvious contradiction. We therefore obtain $T\alpha_X \in 2$.

(iii) Suppose $T \in 1$. Then T^{-1} exists and is continuous. Now since $\alpha_Y = \beta_Y^{-1}$ is continuous and injective, it follows that $T^{-1}\alpha_Y = T^{-1}\beta_Y^{-1} = (\beta_Y T)^{-1}$ is continuous, and hence $\beta_Y T \in 1$.

(iv) Suppose $\beta_Y T \in 2$. Then $\beta_Y T$ is injective and $(\beta_Y T)^{-1}$ is not continuous. It follows that T is injective and hence either $T \in 1$ or $T \in 2$. Since by (iii), $T \in 1$ would imply $\beta_Y T \in 1$, we must have $T \in 2$.

(v) Suppose $T \in 3$, that is T not injective. Note that since both α_X and β_Y are bijective, T is injective iff $\beta_Y T$ is injective iff $T\alpha_X$ is injective, and hence equivalently $T \in 3$ iff $\beta_Y T \in 3$ iff $T\alpha_X \in 3$. □

II.3.3 Proposition. *Let* $T \in L(X,Y)$ *where* X *and* Y *are operator ranges. Then*

(i) $T' \in 1$ *if* $\alpha_X' T' \in 1$

(ii) $T' \in 2$ *implies* $\alpha_X' T' \in 2$

(iii) $T' \in 1$ *implies* $T'\beta_Y' \in 1$

(iv) $T' \in 2$ *if* $T'\beta_Y' \in 2$

(v) $T' \in 3$ *iff* $\alpha_X' T' \in 3$ *iff* $T'\beta_Y' \in 3$.

Proof: Recall that for X a non-complete operator range, α'_X and β'_X are injective with $\alpha'_X = (\beta'_X)^{-1}$ and $(\alpha'_X)^{-1} = \beta'_X$ (Corollary II.1.12), α'_X continuous and everywhere defined on X' and β'_Y surjective and unbounded (Corollary II.1.6, II.1.7 and II.1.12).

(i) Suppose $\alpha'_X T' \in 1$. Then $(\alpha'_X T')^{-1}$ exists and is continuous. However since α'_X is bounded and injective, $(\alpha'_X T')^{-1} = (T')^{-1}(\alpha'_X)^{-1}$ and hence $(\alpha'_X T')^{-1}\alpha'_X = (T')^{-1}(\alpha'_X)^{-1}\alpha'_X = (T')^{-1}\beta'_X\alpha'_X = (T')^{-1}I_{X'} = (T')^{-1}$ is injective and continuous, and hence $T' \in 1$.

(ii) Suppose $T' \in 2$, that is T' is injective but $(T')^{-1}$ is not continuous. Since α'_x is injective, it follows that $\alpha'_X T'$ is injective, that is either $\alpha'_X T' \in 1$ or $\alpha'_X T' \in 2$. But since by (i) $\alpha'_X T' \in 1$ would imply $T' \in 1$ we obtain $\alpha'_X T' \in 2$.

(iii) Let $T' \in 1$. Then $(T')^{-1}$ exists and is continuous. We know however that $(\beta'_Y)^{-1} = \alpha'_Y$ is bounded and obviously injective. Hence $\alpha'_Y(T')^{-1} = (\beta'_Y)^{-1}(T')^{-1} = (T'\beta'_Y)^{-1}$ is continuous, implying that $T'\beta'_Y \in 1$.

(iv) Assume $T'\beta'_Y \in 2$, that is $(T'\beta'_Y)^{-1}$ exists but is not continuous. Now since β'_Y is bijective, $(T')^{-1}$ exists, and hence either $T' \in 1$ or $T' \in 2$. However from (iii) we note that $T' \in 1$ implies $T'\beta'_Y \in 1$, a contradiction. Therefore $T' \in 2$.

(v) Let $T \in L(X,Y)$ with X and Y operator ranges. Note that since α'_X is injective and everywhere defined with β'_Y bijective it therefore follows that T' is injective iff $T'\beta'_Y$ is injective iff $\alpha'_X T'$ is injective; or equivalently, $T' \in 3$ iff $T'\beta'_Y \in 3$ iff $\alpha'_X T' \in 3$, proving (v). □

II.3.4 Proposition. *Let* $T \in L(X,Y)$ *and* Y *an operator range. Then*

(i) $T \in I$ *iff* $\beta_Y T \in I$

(ii) $T \in II$ *if* $\beta_Y T \in II$

(iii) $T \in III$ *implies* $\beta_Y T \in III$.

Proof: Follows easily from the properties of α_Y and β_Y . □

II.3.5 Proposition. *Let* $T \in L(X,Y)$ *and* X *a non-complete operator range. Then for* $T' \in C(Y',X')$ *we have*

(i) $\alpha_X' T' \notin I$

(ii) $T' \in III$ *if* $\alpha_X' T' \in III$ *and* X *is pre-reflexive.*

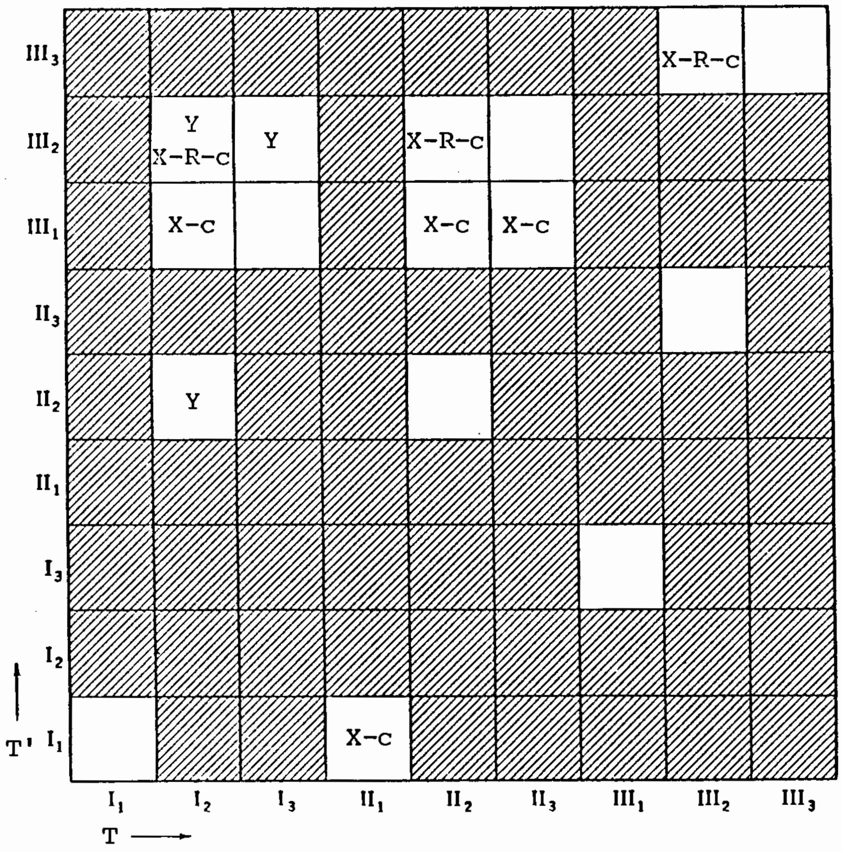
Proof:

(i) Suppose $\alpha_X' T' \in I$. It is easily seen that this can only be the case when $\alpha_X' \in I$. In which case we have that $\beta_X' = (\alpha_X')^{-1}$ is bounded by the closed graph theorem. But by Corollary II.1.6 this implies that X is complete, a contradiction.

(ii) Suppose X is pre-reflexive. From II.2.2 it follows that $R(\alpha_X') = D(\beta_X')$ is dense in X_1' . Thus if $T' \in (I \cup II)$, i.e. if $R(T')$ is dense in X' , by the continuity of α_X' it follows that $\alpha_X'(R(T')) = R(\alpha_X' T')$ is dense in $R(\alpha_X')$, a dense subspace of X_1' . Thus it follows that $R(\alpha_X' T')$ is dense in X_1' , and hence $\alpha_X' T' \in II$, proving the result. □

For the sake of completeness we give the following state diagram taken from Goldberg [11], first established by the same in [10]. We will give a generalisation of a characterisation of states appearing in Goldberg [11] , pp.72-74.

II.3.6 State Diagram for Closed Linear Operators.



T' ↑ T →	I_1	I_2	I_3	II_1	II_2	II_3	III_1	III_2	III_3
III_3	▨	▨	▨	▨	▨	▨	▨	X-R-c	
III_2	▨	Y X-R-c	Y	▨	X-R-c		▨	▨	▨
III_1	▨	X-c		▨	X-c	X-c	▨	▨	▨
II_3	▨	▨	▨	▨	▨	▨	▨		▨
II_2	▨	Y	▨	▨		▨	▨	▨	▨
II_1	▨	▨	▨	▨	▨	▨	▨	▨	▨
I_3	▨	▨	▨	▨	▨	▨		▨	▨
I_2	▨	▨	▨	▨	▨	▨	▨	▨	▨
I_1		▨	▨	X-c	▨	▨	▨	▨	▨

Y : Cannot occur if Y is complete.

X - c : Cannot occur if X is complete and T is closed.

X - R - c : Cannot occur if X is reflexive and T is closed.

From the state diagram we see for example that $T \in III_1$ iff $T' \in I_3$. For examples of states the reader is referred to Goldberg [11], pp.66-70.

II.3.7 Definition $(R_\ell, R_r, D_\ell, D_r, Z_\ell, Z_r)$.

(i) T is called *left (right) regular* provided there exists an operator $A \in B[Y,X]$ such that $AT = I$, $(TA = I)$, where I is the identity map on $D(T)$ (Y). The set of all left (right) regular elements will be denoted by $R_\ell(R_r)$.

(ii) T is called a *left (right) divisor of zero* if there exists an $A \neq 0$ in $B[Y,X]$ such that $TA = 0$ $(AT = 0)$ on Y $(D(T))$. The set of all left (right) divisors of zero is denoted by $D_\ell(D_r)$.

(iii) T is called a *left (right) topological divisor of zero* if there exists a sequence $\{A_n\} \subset B[Y,X]$ such that $\|A_n\| = 1$ and $TA_n \to 0$ $(A_nT \to 0)$ in $B[Y]$ $(B[D(T)X])$. The set of all left (right) topological divisors of zero is denoted by $Z_\ell(Z_r)$.

We will use the notation $3 = D_\ell$ to mean that $T \in 3$ iff $T \in D_\ell$. We give the following theorem from Goldberg [11], p.72.

II.3.8 Theorem. *Let* $T \in L(X,Y)$. *Then*

(i) $III = D_r$

(ii) $1 = CZ_\ell$

(iii) $3 = D_\ell$

(iv) $2 = Z_\ell \cap CD_\ell$.

Proof:

(i) If $T \in III$, there exists some $y' \in Y'$ such that $y' \neq 0$ and $y'R(T) = 0$. Define $A \in B[Y,X]$ by $Ay = y'(y)x_0$, where $x_0 \neq 0$ is fixed in $D(T)$. Then $A \neq 0$ and $ATx = 0$ for all $x \in D(T)$. Thus $T \in D_r$. Conversely, suppose $T \in D_r$. Let $A \neq 0$ be in $B[Y,X]$ and let $AT = 0$. Then $R(T) \subset N(A) \neq Y$, whence $T \in III$.

(ii) Suppose $T \in 1$ and suppose there exists a sequence $\{A_n\} \subset B[Y,X]$ such that $A_n \neq 0$ and $TA_n \to 0$ in $B[Y]$. Now there exists an $m > 0$ such that for each y in the 1-sphere of Y ,

$$\|TA_n\| \geq \|TA_ny\| \geq m\|A_ny\| .$$

Hence $A_n \to 0$ in $B[Y,X]$ and therefore $T \notin Z_\ell$. On the other hand, suppose $T \notin 1$. Then by Theorem I.3.6, there is a sequence $\{x_n\}$ in $D(T)$ such that $\|x_n\| = 1$ and $Tx_n \to 0$. Fix $y' \in Y'$ with $\|y'\| = 1$. For each $n \in \mathbb{N}$, define $A_n \in B[Y,X]$ by $A_n y = y'(y)x_n$. Then $\|A_n\| = 1$ and $\|TA_n\| \leqslant \|Tx_n\| \to 0$. Hence $T \in Z_\ell$.

(iii) Suppose there exists an $x \neq 0$ such that $Tx = 0$. Define $A \in B[Y,X]$ by $Ay = y'(y)x$, where y' is some nonzero element in Y' . Then $A \neq 0$ but $TA = 0$. Hence $T \in D_\ell$. If $T \in D_\ell$, there exists an $A \neq 0$ in $B[Y,X]$ such that $TA = 0$. Consequently, $\{0\} \neq R(A) \subset N(T)$.

(iv) is a trivial consequence of (ii) and (iii). □

II.3.9 Corollary. *Let* $T \in L(X,Y)$. *Then*

(i) $III_1 = D_r \cap CZ_\ell$

(ii) $III_2 = D_r \cap Z_\ell \cap CD_\ell$

(iii) $III_3 = D_r \cap D_\ell$.

Proof: Obvious from Theorem II.3.8. □

II.3.10 Theorem. *Let* $T \in C(X,Y)$, X *an operator range and* Y *a normed space. Then*

(i) Y *complete implies* $CZ_r \supset I$

(ii) X *complete implies* $I \supset CZ_r$.

Proof:

(i) Suppose Y is complete, $T \in I$ and of class Z_r . Then there exist sequences $\{A_n\}$ in $B[Y,X]$ and $\{y_n\}$ in Y such that

$$\|A_n\| = 1 \ , \ \|y_n\| = 1 \ , \ \|A_n y_n\| \geq 1 - \frac{1}{n}$$

and $\|A_n T\| \to 0$ in $B[D(T),X]$,

by definition of Z_r . Let $\hat{T}$ be the closed injective operator induced by T . For each $x \in D(T)$ and $n \in \mathbb{N}$ we have that

$$\|(A_n\hat{T})[x]\| = \|A_n Tz\| \leq \|A_n T\| \|z\| \quad \text{for} \quad z \in [x] \ .$$

It follows that

$$\|(A_n\hat{T})[x]\| \leq \|A_n T\| \inf_{z \in [x]} \|z\| = \|A_n T\| \|[x]\| \ ,$$

i.e. that

$$\|A_n\hat{T}\| \leq \|A_n T\| \quad .$$

Conversely however,

$$\|A_n Tx\| = \|(A_n\hat{T})[x]\| \leq \|A_n\hat{T}\| \ \|[x]\| \leq \|A_n\hat{T}\| \|x\| \ .$$

Thus $\|A_n T\| \leq \|A_n\hat{T}\|$ implying that $\|A_n T\| = \|A_n\hat{T}\|$. Since $T \in I$, there exists a sequence $\{x_n\}$ in $D(T)$ such that $Tx_n = y_n$. It follows that $\{[x_n]\}$ is unbounded in $D(T)/N(T)$ since $\|A_n T\| \to 0$ and

$$1 - \frac{1}{n} \leq \|A_n y_n\| = \|A_n Tx_n\| = \|A_n\hat{T} \ [x_n]\| \leq \|A_n\hat{T}\| \ \|[x_n]\| \ .$$

Thus for $z_n = [x_n]/\|[x_n]\|$

$$\|\hat{T}z_n\| = \frac{\|\hat{T}[x_n]\|}{\|[x_n]\|} = \frac{\|Tx_n\|}{\|[x_n]\|} = \frac{\|y_n\|}{\|[x_n]\|} = \frac{1}{\|[x_n]\|} \to 0 \ ,$$

implying by Theorem I.3.6 that $\hat{T}$ doesn't have a bounded inverse. This however contradicts Proposition I.4.9.. It follows that $T \in I$ implies $T \in CZ_r$.

(ii) Suppose X complete and $T \notin Z_r$. To prove that $T \in I$ it suffices to show, by II.3.6 , that $T' \in 1$. Assume T' does not have a continuous inverse. Then there exists a sequence $\{y_n'\} \in D(T')$ such that $\|y_n'\| = 1$ and $T'y_n' \to 0$. Let x_0 be an element of norm 1 in $D(T)$. Define $A_n \in B[Y,X]$ by

$$A_n y = y_n'(y) x_0 \ .$$

Then $\|A_n\| = 1$, and for all x in the 1-sphere of $D(T)$,

$\|A_nTx\| = \|y_n'(Tx)x_0\| = \|T'y_n'(x)x_0\| = |T'y_n'x|\,\|x_0\| = |T'y_n'x| \leqslant \|T'y_n'\| \to 0.$

Hence $\|A_nT\| \to 0$, which contradicts the supposition $T \notin Z_r$, thus establishing the result. □

II.3.11 Theorem. *Let* $T \in C(X,Y)$ *where* X *is an operator range. Then*

(i) X *complete implies* $I_1 \supset C(Z_r \cup Z_\ell)$

(ii) Y *complete implies* $I_1 = R_\ell \cap R_r \subset C(Z_r \cup Z_\ell)$.

Proof: Suppose that X is an operator range, Y complete and T closed. Let $T \in I_1$. Then $T^{-1} \in B[Y,X]$, $TT^{-1} = I_Y$, the identity on Y , and $T^{-1}T = I_{D(T)}$, the identity on $D(T)$. Thus by definition $T \in R_\ell \cap R_r$. Suppose now $T \in R_\ell \cap R_r$. In that case T must be injective and surjective. Thus since T^{-1} is closed with $D(T^{-1}) = Y$ it follows from the generalised closed graph theorem I.4.9, that T^{-1} is bounded, i.e. $T \in I_1$. The rest of the theorem now follows from Theorems II.3.8 and II.3.10. □

II.3.12 Proposition. *Suppose* $T \in C(X,Y)$ *with* X *an operator range and* Y *complete. The states of* T *are then characterized as follows:*

(i) $I_1 = R_\ell \cap R_r \subset C(Z_r \cup Z_\ell)$

(ii) I_2 *cannot exist*

(iii) $I_3 \subset CZ_r \cap D_\ell$

(iv) $II_1 \supset CZ_\ell \cap Z_r \cap CD_r$

(v) $II_2 \supset Z_\ell \cap Z_r \cap C(D_r \cup D_\ell)$

(vi) $II_3 \supset Z_r \cap D_\ell \cap CD_r$.

Proof:

(i) Follows from Theorem II.3.11.

(iii) - (vi) Follows from Theorems II.3.8 and II.3.10.

(ii) Suppose $T \in I_2$. But then since $R(T) = Y$ and T is closed and injective, it follows from the generalized closed graph theorem I.4.9, that $T \in 1$, a contradiction. □

II.3.13 Proposition. *Let* $T \in C(X,Y)$ *with* X *complete. The characterization of the states of* T *is then as follows :*

(i) $I_1 \supset \mathcal{C}(Z_r \cup Z_\ell)$

(ii) $I_2 \supset \mathcal{C}Z_r \cap Z_\ell \cap \mathcal{C}D_\ell$

(iii) $I_3 \supset \mathcal{C}Z_r \cap D_\ell$

(iv) II_1 *cannot exist*

(v) $II_2 \subset Z_\ell \cap Z_r \cap \mathcal{C}(D_r \cup D_\ell)$

(vi) $II_3 \subset Z_r \cap D_\ell \cap \mathcal{C}D_r$.

Proof:

(ii), (iii), (v) and (vi) follow from Theorems II.3.8 and II.3.10.

(i) Follows from Theorem II.3.11.

(iv) It follows from II.3.6 that II_1 cannot exist. □

II.4 Projection Spaces

We now take a break from the conjugate operator to take a closer look at the implications of Definitions I.5.14 and II.1.8. It will be shown that this is closely related to the existence of bounded projections in operator ranges, viz. the generalization of Theorem I.3.23 to operator ranges.

II.4.1 Definition (projection spaces).

Let X be an operator range and M, N any two closed subspaces of X such that $X = M \oplus N$, i.e. $M \cap N = 0$ and $M + N = X$. We say that X is a *projection space*, written $X \in P$, if

$$D(\beta_X') \cap [(\beta_X M)^\perp \oplus (\beta_X N)^\perp]$$
$$= [D(\beta_X') \cap (\beta_X M)^\perp] \oplus [D(\beta_X') \cap (\beta_X N)^\perp] \quad .$$

II.4.2 Theorem. *Let X be an operator range. A necessary and sufficient condition for X to be a projection space is that the following statement hold :*

If M and N are closed subspaces of X such that $X = M \oplus N$, then $X' = M^\perp \oplus N^\perp$.

Proof: Suppose $X \in P$. Let M and N be any two closed subspaces of X such that $X = M \oplus N$. It follows easily that $X_1 = (\beta_X M) \oplus (\beta_X N)$ where $\beta_X M$ and $\beta_X N$ are closed in X_1. (M closed in X implies $X \setminus M$ open in X. Since β_X is open however it follows that $\beta_X(X \setminus M) = X_1 \setminus \beta_X M$ is open in X_1, i.e. $\beta_X M$ closed in X_1). Clearly $(\beta_X M)^\perp \cap (\beta_X N)^\perp = \{0\}$. By Theorem I.3.23 there exists a bounded projection P, from X_1 onto $\beta_X M$ with $N(P) = \beta_X N$. Given $x_1' \in X_1'$, let $u' = x_1'P$ and $v' = x_1'(I_{X_1} - P)$ where I_{X_1} is the identity operator mapping X_1 onto X_1. Then $x_1' = u' + v'$ with $u' \in (\beta_X N)^\perp$ and $v' \in (\beta_X M)^\perp$. Therefore $X_1' = (\beta_X M)^\perp \oplus (\beta_X N)^\perp$. Now since X is a projection space, it follows that

$$D(\beta_X') = [D(\beta_X') \cap (\beta_X M)^\perp] \oplus [D(\beta_X') \cap (\beta_X N)^\perp] .$$

Since β_X' is linear, injective and surjective by Corollaries II.1.7 and II.1.12, we obtain $X' = \beta_X' [D(\beta_X')] = \beta_X'(\beta_X M)^\perp \oplus \beta_X'(\beta_X N)^\perp$ where

$\beta'_X(\beta_X M)^\perp$ is used to denote $\beta'_X[D(\beta'_X) \cap (\beta_X M)^\perp]$. The sufficiency of $X \in \mathcal{P}$ follows upon showing that

$$\beta'_X(\beta_X M)^\perp = M^\perp .$$

Recall that

$$M^\perp = \{x' : x' \in X' \, , \, x'm = 0 \quad \text{for all} \quad m \in M\}$$

and $\beta'_X(\beta_X M)^\perp = \beta'_X \{x'_1 : x'_1 \in D(\beta'_X) \, , \, x'_1(\beta_X m) = 0 \text{ for all } m \in M\}$.

Suppose now that $\beta'_X x'_1 \in \beta'_X(\beta_X M)^\perp$. Then for every $m \in M$ we have that

$$\beta'_X x'_1 m = x'_1 \beta_X(m) = x'_1(\beta_X m) = 0$$

by definition of $(\beta_X M)^\perp$. Thus $\beta'_X x'_1 \in M^\perp$, i.e. $\beta'_X(\beta_X M)^\perp \subset M^\perp$. Conversely suppose $x' \in M^\perp$. Then, by Corollaries II.1.7 and II.1.12 $(\beta'_X)^{-1}x' = \alpha'_X x' \in D(\beta'_X)$ and hence for every $\beta_X m \in \beta_X M$ it follows that $\alpha'_X x'(\beta_X m) = x'\alpha_X(\beta_X m) = x'm = 0$.

Thus by definition $\alpha'_X x' \in (\beta_X M)^\perp$, i.e. $\beta'_X \alpha'_X x' = x' \in \beta'_X(\beta_X M)^\perp$, implying $M^\perp \subset \beta'_X(\beta_X M)^\perp$. Therefore $M^\perp = \beta'_X(\beta_X M)^\perp$, and the sufficiency of $X \in \mathcal{P}$ follows. Conversely suppose that $X = M \oplus N$ implies $X' = M^\perp \oplus N^\perp$ for M and N closed subspaces of X .

Suppose now that $X \notin \mathcal{P}$. By definition there exist suitable closed subspaces E and F of X such that $X = E \oplus F$ and $D(\beta'_X) = D(\beta'_X) \cap [(\beta_X E)^\perp \oplus (\beta_X F)^\perp] \supsetneq [D(\beta'_X) \cap (\beta_X E)^\perp] \oplus [D(\beta'_X) \cap (\beta_X F)^\perp]$. Hence $E^\perp \oplus F^\perp = X' = \beta'_X(D(\beta'_X)) \supsetneq \beta'_X(\beta_X E)^\perp \oplus \beta'_X(\beta_X F)^\perp = E^\perp \oplus F^\perp$, an obvious contradiction, thus establishing the necessity of $X \in \mathcal{P}$. □

II.4.3 Remark.

All Banach spaces are projection spaces.

This follows trivially from the fact that if X is complete, β_X would be continuous , and thus $D(\beta'_X) = X'_1$. Obviously $X'_1 \cap [A \oplus B] = [X'_1 \cap A] \oplus [X'_1 \cap B]$ for any two subspaces of X'_1 such that $A \cap B = \{0\}$.

II.4.4 Corollary. *Let* X *be an operator range. The following are equivalent:*

(i) *If* M, N *are closed subspaces of* X *such that* $X = M \oplus N$ *, then* $X' = M^{\perp} \oplus N^{\perp}$.

(ii) *If* M *is a closed subspace of* X *, then there exists a bounded projection from* X *onto* M *iff there exists a closed subspace* N *of* X *such that* $X = M \oplus N$.

Proof:

(i) ⇒ (ii) Suppose (i) holds. Let $X = M \oplus N$ for M and N closed subspaces of X . Then by (i) , $X' = M^{\perp} \oplus N^{\perp}$. But then since $X' \in \mathcal{P}$ by Remark II.4.3, it follows from Theorem II.4.2 that $X'' = (M^{\perp})^{\perp} \oplus (N^{\perp})^{\perp}$. Thus by Theorem I.3.23 there exists a bounded projection, say $P^{\perp\perp}$ of X'' onto $(M^{\perp})^{\perp}$. Let $P = J_X^{-1} P^{\perp\perp}|_{J_X X}$. We note that $J_X M \subset (M^{\perp})^{\perp}$ and consequently $P \in B[X,M]$. Conversely if there existsa bounded projection P of X onto M , let $N = N(P)$. It easily follows that $X = M \oplus N$ and that $N = N(P)$ is closed, thus establishing (ii).

(ii) ⇒ (i) Suppose (ii) holds and let the closed subspace M be given. Let N be closed and satisfy $M \oplus N = X$. Then $M^{\perp} \cap N^{\perp} = \{0\}$. By (ii) however, there exists a bounded projection P from X onto M . Given $x' \in X'$, let $u' = x'P$ and $v' = x'(I_X - P)$. Then $x' = u' + v'$ with $u' \in N^{\perp}$ and $v' \in M^{\perp}$. Thus $X' = M^{\perp} \oplus N^{\perp}$, proving the corollary. □

II.4.5 Corollary. *Let* X *be an operator range. If* M *is a subspace of* X *and* N_1 *a subspace of* X_1 *, then*

(i) $M^{\perp} = \beta_X'(\beta_X M)^{\perp}$

(ii) $N_1^{\perp} \supset N_1^{\perp} \cap R(\alpha_X') = \alpha_X'(\alpha_X N_1)^{\perp}$.

Proof:

(i) Suppose M is a subspace of X. It then follows from the proof of Theorem II.4.2 that $M^{\perp} = \beta'_X(\beta_X M)^{\perp}$.

(ii) Suppose N_1 is a subspace of X_1. Then from (i)

$$\beta'_X(N_1^{\perp}) = \beta'_X(\beta_X \alpha_X N_1)^{\perp} = (\alpha_X N_1)^{\perp} .$$

From Corollaries II.1.7 and II.1.12 we therefore obtain

$$\alpha'_X(\alpha_X N_1)^{\perp} = \alpha'_X \beta'_X(N_1^{\perp}) = N_1^{\perp} \cap R(\alpha'_X) \subset N_1^{\perp} \quad . \qquad \square$$

Definition II.4.1, Theorem II.4.2 and Corollaries II.4.4 and II.4.5 are by the author.

II.4.6 Remark. We note that for A, B and C subspaces of X with A and B closed, $C \cap [A \oplus B]$ need not equal $[C \cap A] \oplus [C \cap B]$. Let $X = \mathbb{R}^2$, the real plane, and $A = \{(x,y) : (x,y) \in \mathbb{R}^2 , y = 0\}$, $B = \{(x,y) : (x,y) \in \mathbb{R}^2 , x = 0\}$. Then $X = A \oplus B$ with both A and B closed subspaces of X. Consider the closed subspace $C = \{(x,y) : (x,y) \in \mathbb{R}^2 , x = y\}$. Then $C \cap [A \oplus B] = C \neq [C \cap A] \oplus [C \cap B] = \{0\}$. In fact for $D = \{(x,y) : (x,y) \in \mathbb{R}^2 , x \neq 0 , y \neq 0\}$ a dense subset of $X = \mathbb{R}^2$ we have that

$$D \cap [A \oplus B] = D \neq [D \cap A] \oplus [D \cap B] = \emptyset .$$

CHAPTER III

OPERATORS WITH CLOSED RANGE

Without attempting to give complete coverage, we will investigate the closed range property of operators before defining classes of operators with respect to this property and examining their characteristics. Most of the work will be centered around generalising results previously only known for Banach spaces (cf. [11]) to operator ranges. Note that in general we will not assume $D(T)$ dense in X unless stated.

III.1 The Minimum Modulus of an Operator

III.1.1 Theorem. *Let* $T \in C(X,Y)$ *with* X *an operator range. Then*

(i) X *complete and* T^{-1} *continuous implies* $R(T)$ *closed*

(ii) Y *complete,* T *injective and* $R(T)$ *closed implies* T^{-1} *continuous.*

Proof:

(i) Suppose X is complete and T^{-1} is continuous. By Theorem I.3.6 there exists an $m > 0$ such that

(1) $\quad \|Tx\| \geq m\|x\|$ for all $x \in D(T)$.

Now suppose $y \in \overline{R(T)}$. Then there exists a sequence $\{x_n\}$ in $D(T)$ such that $Tx_n \to y$ in Y .

From (1) we have

$$\|Tx_n - Tx_m\| \geq m\| x_n - x_m\| .$$

Hence $\{x_n\}$ is Cauchy and thus a convergent sequence in the Banach space X . Suppose $x_n \to x$, $x \in X$. Since T is closed it follows that $x \in D(T)$ and $Tx = y \in R(T)$. Thus $R(T)$ is closed.

(ii) Next suppose $T \in C(X,Y)$ is an injective operator with $R(T)$ closed and Y complete. It follows from remark I.3.12 that T^{-1} is closed. Since $D(T^{-1}) = R(T)$ is closed in Y , a Banach space, it follows from Proposition I.4.9 (the generalised closed graph theorem) that T^{-1} is continuous. □

Motivated by the above theorem, we look for a similar way to characterise the closed range property for $T \in L(X,Y)$ when T is not necessarily injective. We make the following definition:

III.1.2 Definition (minimum modulus).

Let $T \in L(X,Y)$ be a non-zero operator with $N(T)$ closed. Then the *minimum modulus* of T , written $\gamma(T)$ is defined by

$$\gamma(T) = \inf_{x \in D(T) \setminus N(T)} \frac{\|Tx\|}{d(x,N(T))} .$$

III.1.3 Theorem. *Suppose* $T \in L(X,Y)$ *with* $N(T)$ *closed . Then* $\gamma(T) > 0$ *iff* $\hat{T}^{-1}$ *is continuous.*

Proof: We have

$$\begin{aligned}\gamma(T) &= \inf \{\frac{\|Tx\|}{d(x,N(T))} : x \in D(T) , x \notin N(T)\} \\ &= \inf \{\frac{\|\hat{T}[x]\|}{\|[x]\|} : [x] \in \frac{D(T)}{N(T)} , x \notin N(T)\} \\ &= \frac{1}{\|(\hat{T})^{-1}\|} ,\end{aligned}$$

thus establishing the result. □

III.1.4 Theorem. *Let* $T \in C(X,Y)$ *with* X *an operator range. Then*

(i) if X *is complete*

$$\gamma(T) > 0 \textit{ implies } R(T) \textit{ closed ;}$$

(ii) if Y *is complete*

$$R(T) \textit{ closed implies } \gamma(T) > 0 .$$

Proof:

(i) Suppose $T \in C(X,Y)$ with X complete and $\gamma(T) > 0$. Then by Theorem I.3.20, $\hat{T}$ is closed and by Theorem III.1.3, $\hat{T}^{-1}$ is continuous. Thus since X and therefore $X/N(T)$ is complete by Theorem I.2.6, it follows from Theorem III.1.1 that $R(\hat{T}) = R(T)$ is closed

(ii) Conversely, suppose $T \in C(X,Y)$ with Y complete and $R(T)$ closed. By Theorem I.3.20, $\hat{T}$ is closed, and trivially $R(\hat{T}) = R(T)$ is closed. It therefore follows from Theorem III.1.1 that $\hat{T}^{-1}$ is continuous, i.e. $\gamma(T) > 0$ by Theorem III.1.3. □

III.1.5 Theorem [11]. *Let* $T \in L(X,Y)$ *with* $N(T)$ *closed. Then* $\gamma(T) > 0$ *and* $R(T)$ *closed imply* T *closed.*

Proof: Suppose $T \in L(X,Y)$ with $N(T)$ closed, $\gamma(T) > 0$ and $R(T)$ closed. Then, by Theorem III.1.3 $(\hat{T})^{-1}$ is continuous with $D((\hat{T})^{-1}) = R(T)$ closed. Thus by Remark I.3.12, $(\hat{T})^{-1}$ and thus $\hat{T}$ is closed. It now follows from Theorem I.3.20 that T is closed. □

III.1.6 Remark. It is a trivial consequence of Theorems III.1.4 and III.1.5 that for $T \in L(X,Y)$ with $N(T)$ closed and X, Y Banach spaces, that any two of the following three conditions will imply the third.

(a) T closed, (b) $R(T)$ closed and (c) $\gamma(T) > 0$.

III.1.7 Theorem. *Let* $T \in L(X,Y)$ *with* $N(T)$ *closed. Then*

(i) if X *is an operator range* $\gamma(T\alpha_X) \leqslant \|\alpha_X\| \, \gamma(T)$;

(ii) if Y *is an operator range* $\gamma(T) \leqslant \|\alpha_Y\| \, \gamma(\beta_Y T)$.

Proof:

(i) Suppose $T \in L(X,Y)$, $N(T)$ closed and X an operator range. Then since β_X is an open bijection from X to X_1 , it follows that $\beta_X N(T) = N(T\alpha_X)$ is closed, and hence $\gamma(T\alpha_X)$ exists.

By definition we have

$$\|\alpha_X x\| \leqslant \|\alpha_X\| \, \|x\|_1 \quad \text{for} \quad x \in X_1$$

and hence

$$\|\alpha_X x - \alpha_X n\| \leqslant \|\alpha_X\| \|x - n\|_1 \quad \text{for} \quad x \in X_1 \, , \; n \in N(T\alpha_X) \, ,$$

i.e.
$$\begin{aligned} d(\alpha_X x, N(T)) &= \inf \{ \|\alpha_X x - \alpha_X n\| : x \in X_1, \; n \in N(T\alpha_X)\} \\ &\leqslant \|\alpha_X\| \; (\inf\{ \|x - n\|_1 : x \in X_1 \, , \; n \in N(T\alpha_X)\}) \\ (1) \qquad &= \|\alpha_X\| d(x, N(T\alpha_X)) \; . \end{aligned}$$

Hence from (1) we obtain

$$\gamma(T\alpha_X) = \inf_{x \in D(T\alpha_X) \setminus N(T\alpha_X)} \frac{\|T\alpha_X x\|}{d(x, N(T\alpha_X))}$$

$$\leqslant \|\alpha_X\| \left(\inf_{\alpha_X x \in D(T) \setminus N(T)} \frac{\|T\alpha_X x\|}{d(\alpha_X x, N(T))} \right) = \|\alpha_X\| \; \gamma(T) \; .$$

(ii) Suppose now $T \in L(X,Y)$, $N(T)$ closed with Y an operator range.

As in (i), we have

$$(2) \qquad \|y\| \leqslant \|\alpha_Y\| \; \|\beta_Y y\|_1 \, , \quad \text{for} \quad y \in Y \, .$$

Thus by (2)

$$\gamma(T) = \inf_{x \in D(T) \setminus N(T)} \frac{\|Tx\|}{d(x, N(T))} \leqslant \|\alpha_Y\| \inf_{x \in D(T) \setminus N(T)} \frac{\|\beta_Y Tx\|_1}{d(x, N(T))}$$

$$= \|\alpha_Y\| \; \gamma(\beta_Y T) \; .$$

□

III.1.8 Definition (closable operator).

We define $T \in L(X,Y)$ to be *closable* if there exists a linear extension of T which is a closed operator.

III.1.9 Lemma [11]. *For* $T \in L(X,Y)$ *the following are equivalent :*

(i) T *is closable.*

(ii) T *has a minimal closed linear extension; i.e. there exists a closed linear extension* $\bar{T}$ *of* T *such that any closed linear extension of* T *is a closed linear extension of* $\bar{T}$.

(iii) For any $y \neq 0$ *in* Y *,* $(0,y) \notin \overline{G(T)}$.

Proof: (i) implies (iii). Let $\bar{T}$ be a closed linear extension of T. If $y \in Y$ and $y \neq 0$, then $(0,y) \notin G(\bar{T}) \supset G(T)$. Hence $(0,y) \notin \overline{G(T)}$, since $G(\bar{T})$ is closed in $X \times Y$.

(iii) implies (ii). Suppose $(0,y) \notin \overline{G(T)}$ for any $y \neq 0$ in Y. Define $\bar{T}$ as the operator whose graph is $\overline{G(T)}$; that is,

$$D(T) = \{x : (x,z) \in \overline{G(T)} \text{ for some } z \in Y\} \text{ and } \bar{T}x = z .$$

Then $\bar{T}$ is well defined and is a closed linear extension of T . Furthermore, $\bar{T}$ is the minimal closed linear extension of T ; for if T_1 is a closed linear extension of T , then $G(T_1) \supset \overline{G(T)} = G(\bar{T})$.

(ii) implies (i) trivially.

For T a closable operator we will denote the minimal closed linear extension of T by $\bar{T}$. Note from proof of Lemma III.1.9 that $\overline{G(T)} = G(\bar{T})$.

III.1.10 Theorem. *Let* $T \in L(X,Y)$ *be closable with* $N(T)$ *closed. Then* $\gamma(T) \leqslant \gamma(\bar{T})$ *with equality holding if* $N(T) = N(\bar{T})$.

Proof: Assume $T \in L(X,Y)$ with T closable and $N(T)$ closed. We will show that

$$(1) \qquad \gamma(T) = \inf_{x \in D(\bar{T}) \setminus N(T)} \frac{\|\bar{T}(x)\|}{d(x,N(T))}$$

from which the result will follow. (Note that in (1) we use $d(x,N(T))$, not $d(x,N(\bar{T}))$.) Observe that since $D(T) \setminus N(T) \subset D(\bar{T}) \setminus N(T)$ and $\frac{\|Tx\|}{d(x,N(T))} = \frac{\|\bar{T}x\|}{d(x,N(T))}$ for $x \in D(T) \setminus N(T)$

$$(2) \qquad \inf_{x \in D(\bar{T}) \setminus N(T)} \frac{\|\bar{T}x\|}{d(x,N(T))} \leqslant \inf_{x \in D(T) \setminus N(T)} \frac{\|\bar{T}x\|}{d(x,N(T))}$$

$$= \inf_{x \in D(T) \setminus N(T)} \frac{\|Tx\|}{d(x,N(T))} = \gamma(T) .$$

Let $\delta = \inf_{D(\bar{T}) \setminus N(T)} \frac{\|\bar{T}x\|}{d(x,N(T))}$. Now choose a sequence $\{\tilde{x}_n\}$ in $D(\bar{T}) \setminus N(T)$ such that $\frac{\|\bar{T}\tilde{x}_n\|}{d(\tilde{x}_n,N(T))} \to \delta$. (Note that for δ we take the infinum over $D(\bar{T}) \setminus N(T)$, not $D(\bar{T}) \setminus N(\bar{T})$.) Now since $G(T)$ is dense in $G(\bar{T})$ and $N(T)$ is closed, it follows that the set $G(T) \setminus [N(T) \times Y]$ is relatively dense in $G(\bar{T}) \setminus [N(T) \times Y]$ and hence there exists a sequence $\{x_n\}$ in $D(T) \setminus N(T)$ such that

$$\|(\tilde{x}_n,\bar{T}\tilde{x}_n) - (x_n,Tx_n)\| \leqslant \frac{1}{n} \quad \text{for} \quad n \in \mathbb{N} .$$

Hence

$$\|[\tilde{x}_n] - [x_n]\| \leqslant \|\tilde{x}_n - x_n\| \leqslant \frac{1}{n} \quad \text{and} \quad \|\bar{T}\tilde{x}_n - Tx_n\| \leqslant \frac{1}{n} \ ; \quad \text{hence}$$

$$\big|\|[\tilde{x}_n]\| - \|[x_n]\|\big| \leqslant \frac{1}{n} \quad \text{and} \quad \big|\|\bar{T}\tilde{x}_n\| - \|Tx_n\|\big| < \frac{1}{n} \ ; \quad \text{and therefore}$$

$$(3) \qquad \left|d(\tilde{x}_n,N(T)) - \frac{1}{n}\right| = \left|\|[\tilde{x}_n]\| - \frac{1}{n}\right| \leqslant \|[x_n]\| = d(x_n,N(T))$$

$$\text{and} \quad \|Tx_n\| \leqslant \|\bar{T}x_n\| \leqslant \|\bar{T}\tilde{x}_n\| + \frac{1}{n} \quad \text{for} \quad n \in \mathbb{N} ,$$

where $\|[\tilde{x}_n]\|$ and $\|[x_n]\|$ are taken over the normed space $X/N(T)$.

Note that from (2) it follows that $\delta \leqslant \gamma(T) \leqslant \frac{\|Tx\|}{d(x,N(T))}$ for $x \in D(T) \setminus N(T)$ ·

and hence from (3) we obtain

$$(4) \qquad \delta \leqslant \frac{\|Tx_n\|}{d(x_n,N(T))} \leqslant \frac{\|\bar{T}\tilde{x}_n\| + \frac{1}{n}}{\left|d(\tilde{x}_n,N(T)) - \frac{1}{n}\right|} \quad \text{for} \quad n \in \mathbb{N} .$$

But since by choice of $\{\tilde{x}_n\}$, $\dfrac{\|\bar{T}\tilde{x}_n\|}{d(\tilde{x}_n,N(T))}$ and thus $\dfrac{\|\bar{T}\tilde{x}_n\| + \frac{1}{n}}{\left|d(\tilde{x}_n,N(T)) - \frac{1}{n}\right|}$

converges to δ , it follows from (4) that $\gamma(T) = \inf\limits_{x \in D(T) \setminus N(T)} \dfrac{\|Tx\|}{d(x,N(T))} \leqslant \delta$

thus implying by (2) that $\gamma(T) = \delta$. Now obviously by choice of δ , if $N(T) = N(\bar{T})$, then

$$\gamma(T) = \delta = \inf_{x \in D(\bar{T}) \setminus N(T)} \frac{\bar{T}x}{d(x,N(T))} = \inf_{x \in D(\bar{T}) \setminus N(\bar{T})} \frac{\bar{T}x}{d(x,N(\bar{T})} = \gamma(\bar{T}) .$$

If however $N(T) \neq N(\bar{T})$, then by virtue of the fact that $\bar{T}$ is an extension of T we must have $N(T) \subsetneq N(\bar{T})$, that is

$$d(x,N(\bar{T})) \leqslant d(x,N(T)) \quad \text{for} \quad x \in X .$$

Hence

$$\frac{\|\bar{T}x\|}{d(x,N(T))} \leqslant \frac{\|\bar{T}x\|}{d(x,N(\bar{T}))} \quad \text{for} \quad x \in D(\bar{T}) \setminus N(\bar{T}) ,$$

and since $D(\bar{T}) \setminus N(\bar{T}) \subset D(\bar{T}) \setminus N(T)$, we obtain

$$\begin{aligned} \gamma(T) = \delta = \inf_{x \in D(\bar{T}) \setminus N(T)} \frac{\|\bar{T}x\|}{d(x,N(T))} &\leqslant \inf_{x \in D(\bar{T}) \setminus N(\bar{T})} \frac{\|\bar{T}x\|}{d(x,N(T))} \\ &\leqslant \inf_{x \in D(\bar{T}) \setminus N(\bar{T})} \frac{\|\bar{T}x\|}{d(x,N(\bar{T}))} = \gamma(\bar{T}) \end{aligned}$$

which proves the theorem. □

III.1.11 Remark. Suppose we define convergence on $L(X,Y)$ as follows. For $\{T_n\} \subset L(X,Y)$

$$T_n \to T \in L(X,Y) \quad \text{if} \quad D(T_n) \supset D(T) \quad \text{for each} \quad n \in \mathbb{N}$$

and $\qquad \|T_n - T\| \to 0 \quad \text{in} \quad B(X,Y)$.

It will be shown in chapter V, Lemma V.3.5, that if $R(T)$ is closed with the dimension of both $N(T)$ and $Y/R(T)$ infinite, then we may construct a sequence $\{T_n\} \subset C(X,Y)$ with $T_n \to T$ as defined above but with $\gamma(T_n) = 0 \not\to \gamma(T) \neq 0$ provided X and Y are Banach spaces.

The author believes Theorem III.1.10 to be his own.

III.2 Closed Operators with Closed Range

We start off by generalising a theorem appearing in Goldberg [11].

III.2.1 Theorem. *Let* $T \in C(X,Y)$ *with* X *an operator range.*

(i) If X *is complete, and if* T *maps bounded closed sets onto closed sets, then* $R(T)$ *is closed.*

(ii) If $R(T)$ *is closed, and if* Y *is complete and* $N(T)$ *is finite dimensional, then* T *maps bounded closed sets onto closed sets.*

Proof:

(i) Suppose X is complete and $T \in C(X,Y)$ maps bounded closed sets onto closed sets. If $R(T)$ were not closed, then by III.1.4 $\gamma(T) = 0$, which implies the existence by I.3.6 of a sequence $\{[x_n]\}$ in $D(T) \setminus N(T)$ such that

(1) $$\|[x_n]\| = 1 \text{ and } \hat{T}[x_n] = Tx_n \to 0$$

since $\hat{T}^{-1}$ is unbounded by III.1.3. We will show that the existence of such a sequence leads to a contradiction hence proving (i). Let $\{z_n\}$ be a sequence such that $z_n \in [x_n]$ and $\|z_n\| \leqslant 2$. Note that $Tz_n = \hat{T}[x_n] = Tx_n \to 0$. If $\{z_n\}$ has no convergent subsequence, it is a closed bounded set and therefore by hypothesis, $\{Tz_n\}$ is a closed set, and hence $Tz_N = 0$ for some N. But then $[x_N] = [z_N] = 0$ contradicting (1). If $z_{n'} \to z$ for some subsequence $\{z_{n'}\}$ of $\{z_n\}$, then $Tz = 0$ since $Tz_{n'} \to 0$ and T is closed. Hence $[x_{n'}] = [z_{n'}] \to [z] = 0$ contradicting (1) and establishing (i).

(ii) Suppose Y is complete, $N(T)$ finite dimensional and $R(T)$ closed. Let B be a closed bounded set in X and let $Tx_n \to y$, where $\{x_n\} \subset B \cap D(T)$. As $R(T)$ is closed $Tx = y$ for some $x \in D(T)$. From Theorem III.1.4 and III.1.3 it now follows that $\hat{T}$ has a bounded inverse, and hence $[x_n] = \hat{T}^{-1}(Tx_n) \to \hat{T}^{-1}(Tx) = [x]$. Therefore there exists a sequence $\{z_n\}$ in $N(T)$ such that $x_n + z_n \to x$. Note that $\{x_n + z_n\}$ is therefore bounded and since $\{x_n\} \subset B$, $\{z_n\}$ is then also bounded. We note that since $N(T)$ is finite dimensional, there exists by Proposition I.6.5 a subsequence $\{z_{n'}\}$ of $\{z_n\}$ such that $z_{n'} \to z \in N(T)$. Hence $x_{n'} \to x - z$. Since B is closed and T is closed, it follows that $x - z \in D(T) \cap B$ and $y = Tx = T(x - z) \in TB$. Hence TB is closed and (ii) is proved. □

III.2.2 Theorem. *Let* $T \in C(X,Y)$ *with* X *an operator range and* Y *complete and let* N *be a closed subspace of* Y *such that* $R(T) \oplus N$ *is closed. Then* $R(T)$ *is closed and if* T *is densely defined* $T'Y' = T'N^{\perp}$.

Proof: Assume $T \in C(X,Y)$, with X an operator range and Y complete. Suppose now that $R(T) \oplus N$ is closed for some closed subspace N of Y. Let T_1 be the operator $T\alpha_X$ so that $R(T_1) = R(T)$. Note that T_1 is closed by Proposition I.4.7. Now define $T_0 \in L(X_1 \times Y, Y)$ as follows:

$$D(T_0) = D(T_1) \times N \subset X_1 \times Y$$

and $T_0(x,n) = T_1x + n$ for each $(x,n) \in D(T_0)$. Suppose that $\{(x_k,n_k)\}$ is a sequence in $D(T_0)$ such that

$$(x_k,n_k) \to (x,y) \in X_1 \times Y$$

$$\text{and} \qquad T_0(x_k,n_k) \to z \in Y .$$

It follows by definition of the norm on $X_1 \times Y$ that $x_k \to x$ and $n_k \to y$ with $T_1x_k + n_k \to z$. Since N is a closed subspace of the Banach space Y , and T_1 a closed operator, it follows that $y \in N$ and that $T_1x_k \to z - y$ implies $T_1x = z - y$. Hence $(x,y) \in D(T_1) \times N = D(T_0)$ and $T_0(x,y) = T_1x + y = z$ implying that T_0 is closed. But since $R(T_0) = R(T) + N$ is closed by hypothesis, we have by Theorem III.1.4 that $\gamma(T_0) > 0$. Since $R(T) \cap N = \{0\}$ by hypothesis, $N(T_0) = N(T_1) \times \{0\}$. Hence $\|T_1x\| = \|T_0(x,0)\| \geq \gamma(T_0)\, d((x,0),N(T_0)) = \gamma(T_0)\, d(x,N(T_1))$ which obviously implies that $\dfrac{\|T_1x\|}{d(x,N(T_1))} \geq \gamma(T_0)$ for any $x \in D(T_1)\setminus N(T_1)$. Therefore $\gamma(T_1) \geq \gamma(T_0)$. Now since $\gamma(T_1) \geq \gamma(T_0) > 0$ and T_1 is closed, we conclude from Theorem III.1.4 that $R(T_1) = R(T)$ is closed thus establishing the first part of the theorem.

Now suppose that $D(T)$ is dense in X and let $y' \in D(T')$. Consider Z to be $Z = R(T) \oplus N$. Notice that Z is closed in the space Y by hypothesis and is therefore also a Banach space. Now by Theorem II.4.2 and Remark II.4.3 we see that $Z' = R(T)^{\perp Z} \oplus N^{\perp Z}$ where $R(T)^{\perp Z}$ and $N^{\perp Z}$ are the orthogonal complements of $R(T)$ and N w.r.t. Z. Hence if z' is $y' \in Y'$ restricted to Z , there exists $z_1' \in R(T)^{\perp Z}$ and $z_2' \in N^{\perp Z}$ such that $z' = z_1' + z_2'$. Let v' be a continuous linear extension of z_1' to all of Y . Then $v' \in R(T)^{\perp Y}$

and therefore $v' \in N(T')$ by Theorem II.1.10. Hence $T'y' = T'(y' - v')$. Moreover $y' - v' \in N^{\perp Y}$ since $y' - v'$ restricted to N is just z'_2. Therefore $T'Y' \subset T'N^{\perp}$. □

III.2.3 Definition (finite deficiency).

A subspace W of a vector space V is said to have *finite deficiency* if the dimension of V/W is finite. We write $\dim \frac{V}{W} < \infty$.

III.2.4 Corollary. *Let* $T \in C(X,Y)$ *with* X *an operator range and* Y *complete. If the range of* T *has finite deficiency then it is closed.*

Proof: Suppose $R(T)$ has finite deficiency in Y. Then $R(T) \oplus N = Y$ for some finite dimensional and therefore closed subspace of Y. Hence by Theorem III.2.2 $R(T)$ is closed. □

Note that Corollary III.2.4 need not hold if T is not closed. We give the following counter example from Goldberg [11], p.101.

III.2.5 Example. The above corollary need not hold if T is not closed. To show this, we first make the following observations:

(i) Every linear operator defined on a finite-dimensional normed linear space V is continuous. This can be shown by first taking V to be the unitary n-space U^n and then using the fact that U^n is isomorphic to V.

(ii) Every linear functional f on X with closed kernel is continuous. This follows from the fact that the 1-1 operator $\hat{f}$ induced by f maps the one-dimensional space $X/N(f)$ (assuming $f \neq 0$) onto the scalars. Hence $\hat{f}$ is continuous and therefore f is continuous by Theorem I.3.20.

(iii) There exists an unbounded linear functional on X, provided X is infinite-dimensional. Take $\{v_1, v_2, \ldots\}$ to be an infinite subset of a Hamel basis $\{x_\alpha\}$ of X, where $\|x_\alpha\| = 1$. Let f be the linear functional on X defined by $f(x_\alpha) = 0$ when $x_\alpha \neq v_k$, $1 \leqslant k$, and $f(v_k) = k$. Then f is clearly unbounded.

By the above results, $N(f)$ is not closed and $X = N(f) + \text{sp}\,\{x\}$, $x \notin N(f)$. The linear operator T defined on X by $T(u + \alpha x) = u$, $u \in N(f)$, has range $N(f)$.

III.2.6 Corollary. *Let* $T \in L(X,Y)$ *with* Y *a Banach space,* $\overline{D(T)} = X$ *and* $R(T)$ *closed. Then for any closed subspace* M *of* Y *such that* $R(T) \oplus M$ *is closed,* $T'Y' = T'M^\perp$.

Proof: Let $T \in L(X,Y)$ with Y complete, $\overline{D(T)} = X$ and $R(T)$ closed. The corollary now follows from the second part of the proof of III.2.2 since for that part of the proof we don't need X to be an operator range or T to be closed. □

III.2.7 Proposition. *Let* $T \in L(X,Y)$ *with* Y *a projection space,* $\overline{D(T)} = X$ *and* $R(T)$ *closed. If there exists a closed subspace* M *of* Y *such that* $Y = R(T) \oplus M$, *then* $T'Y' = T'M^\perp$.

Proof: Suppose $T \in L(X,Y)$, $Y \in \mathcal{P}$, $\overline{D(T)} = X$ and $R(T)$ closed. Let M be a closed subspace of Y such that $Y = R(T) \oplus M$. Then since $Y \in \mathcal{P}$, it follows from Theorem II.4.2 that $Y' = R(T)^\perp \oplus M^\perp$. Hence for $y' \in D(T')$ we can write $y' = u' + v'$ where $u' \in R(T)^\perp$ and $v' \in M^\perp$. From Theorem II.1.10 it therefore follows that $u' \in N(T')$, that is

$T'y' = T'(u' + v') = T'v' \in T'M^{\perp}$, proving that

$T'Y' \subset T'M^{\perp}$, i.e. $T'Y' = T'M^{\perp}$.

III.2.8 Lemma [11]. *Let* $T \in L(X,Y)$ *with* $D(T)$ *dense in* X *and* $N(T)$ *closed. Then* $D(T') = D(\hat{T}')$ *and* $\|\hat{T}'y'\| = \|T'y'\|$.

Proof: Suppose $y' \in D(T')$. Then by definition $y'T$ is continuous on $D(T)$. Hence

$$|y'\hat{T}[x]| = |y'Tz| \leqslant \|y'T\| \,.\, \|z\| \quad \text{for} \quad z \in [x] \quad \text{and} \quad [x] \in D(T)/N(T)$$

from which we obtain

$$|y'\hat{T}[x]| \leqslant \|y'T\| \inf_{z \in [x]} \|z\| = \|y'T\| \, \|[x]\| ,$$

that is $\|y'\hat{T}\| \leqslant \|y'T\|$ which implies that $y' \in D(\hat{T}')$. Suppose now that $y' \in D(\hat{T}')$, i.e. $y'\hat{T}$ is continuous. Then

$$|y'Tx| = |y'\hat{T}[x]| \leqslant \|y'\hat{T}\| \|[x]\| \leqslant \|y'\hat{T}\| \, \|x\| .$$

Hence $\|y'T\| \leqslant \|y'\hat{T}\|$ and $y' \in D(T')$ implying that $\|y'T\| = \|y'\hat{T}\|$ and $D(T') = D(\hat{T}')$. From I.5.2 we note that $\|T'y'\| = \|y'T\| = \|y'\hat{T}\| = \|\hat{T}'y'\|$ which proves the lemma. □

III.2.9 Theorem. *Let* $T \in C(X,Y)$ *with* $D(T)$ *dense in* X . *Consider the following statements :*

(i) $R(T)$ *is closed*

(ii) $R(T) = {}^{\perp}N(T')$

(iii) $R(T') = N(T)^{\perp}$

(iv) $R(T')$ *is closed.*

Then :

(a) *if* X *and* Y *are normed spaces, (i) and (ii) are equivalent and (iii) implies (iv)* *(b) if* X *is an operator range and* Y *complete, then we also have (i) implying (iii)*

(c) if X *is complete and* Y *a normed space, (iv) implies (ii).*

Proof:

(a) Assume $T \in C(X,Y)$ with $\overline{D(T)} = X$. It is obvious from Theorem II.1.10 that (i) and (ii) are equivalent and since $N(T)^{\perp}$ is always a closed subspace, (iii) implies (iv) trivially.

(b) Assume also that X is an operator range and Y complete. Then $\hat{T}$ has a continuous inverse by Theorems III.1.3 and III.1.4, that is $\hat{T} \in 1$. We note by II.3.6 that this can only be the case if $\hat{T}' \in I$, that is $R(\hat{T}') = (X/N(T))'$. Therefore from Theorem I.5.16 it follows that $(X/N(T))'$ is isometric to $N(T)^{\perp}$ under the map V defined by $(Vz')x = z'[x]$ for $z' \in (X/N(T))'$ and $x \in X$, that is $V(R(\hat{T}')) = N(T)^{\perp}$. Now by Lemma III.2.8 $D(T') = D(\hat{T}')$ and since $\hat{T}'y'[x] = T'y'x$ for each $y' \in D(T')$ it follows from the definition of V that

$$R(T') = V(R(\hat{T}')) = N(T)^{\perp}.$$

(c) Suppose now that $T \in C(X,Y)$, $D(T)$ is dense in Banach space X and $R(T')$ is closed. Let $Y_1 = \overline{R(T)}$ and let T_1 be T considered as a map into Y_1 . T_1 is obviously closed and it therefore follows from II.3.6 that in order to show $R(T_1) = R(T) = Y_1$, i.e. $T_1 \in I$, we need only show that $T_1' \in 1$, that is $(T_1')^{-1}$ is continuous. Now since $\overline{R(T_1)} = Y_1$, it follows from Theorem II.1.10 that T_1' is injective. We prove that $R(T_1') = R(T')$. Let $T_1'v' \in R(T_1')$. Then $v'T_1$ and thus trivially $v'T$ is continuous on $D(T) = D(T_1)$. However from Corollary I.5.6 it follows that there exist an extension y' of v' to all of Y.

Trivially $y'T$ is still continuous on $D(T)$ since $y'|_{R(T)} = v'$. It follows from Lemma I.5.2 that $T_1'v' = \overline{y'T} = T'y' \in R(T')$. Now let $T'y'$ be an arbitrary element of $R(T')$. Then $y'T$ and therefore $y'T_1$ is continuous on $D(T_1) = D(T)$. It follows that $y'|_{\overline{R(T_1)}} = u'$ is an element of Y_1' with $y'T_1 = u'T_1$, and again by Lemma I.5.2 we note that $T_1'u' = T'y' \in R(T_1')$. Hence $R(T_1') = R(T')$ is closed by hypothesis and we see from Theorem III.1.1 that $(T_1')^{-1}$ is continuous, that is $T_1' \in 1$. Now since X is complete, we see by II.3.6 that $T_1 \in I$, that is $R(T) = R(T_1) = Y_1 = \overline{R(T)}$ implying that $R(T)$ is closed and hence proving the theorem. □

We note that Theorems III.2.2 and III.2.9 as well as Corollary III.2.4 are generalisations of results appearing in Goldberg

III.3 Products of Operators with Closed Range

We first make the following definitions.

III.3.1 Definition (kernel index, deficiency index, index).

Let $T \in L(X,Y)$. We define the dimension of $N(T)$ to be the *kernel index* of T, written $a(T)$. The dimension of the deficiency of $R(T)$ is defined to be the *deficiency index* of T, written $b(T)$. Hence $a(T)$ and $b(T)$ will be either non-negative integers or ∞. If $a(T)$ and $b(T)$ are not both infinite, we say that T has an *index* written $K(T)$ and defined by $K(T) = a(T) - b(T)$ where $r - \infty$ and $\infty - r$ are considered to be $-\infty$ and $+\infty$ respectively for $r \in \mathbb{R}$.

III.3.2 Definition (normally solvable, Fredholm- and semi-Fredholm operators).

Let $T \in C(X,Y)$. If $R(T)$ is closed, we say that T is *normally solvable*. We denote the class of all such operators by $NS(X,Y)$; $NS[X,Y]$ if $D(T) = X$.

If T is a normally solvable operator with an index, we say that T is *semi-Fredholm*, denoted by $T \in SF(X,Y)$ and $T \in SF[X,Y]$ if $D(T) = X$.

If T is a normally solvable operator with a finite index, we say that T is *Fredholm*, written $T \in F(X,Y)$ and $T \in F[X,Y]$ if $D(T) = X$.

III.3.3 Proposition. *Let* $T \in NS(X,Y)$. *Then*

(i) if X *is an operator range* $T\alpha_X \in NS(X_1,Y)$

(ii) if Y *is an operator range* $\beta_Y T \in NS(X,Y_1)$.

Proof:

(i) Suppose $T \in NS(X,Y)$ with X an operator range. Since $R(T) = R(T\alpha_X)$, it follows trivially from Proposition I.4.7 that $T\alpha_X \in NS(X,Y)$.

(ii) Assume $T \in NS(X,Y)$ with Y an operator range. By Proposition I.4.7 it follows that $\beta_Y T$ is closed, and since by definition β_Y is open, everywhere defined and surjective, it follows that $\beta_Y R(T) = R(\beta_Y T)$ is closed, hence proving the proposition. □

III.3.4 Remark. Let $T \in L(X,Y)$. We note the following:

(i) If X is an operator range, $a(T) = a(T\alpha_X)$ and $b(T) = b(T\alpha_X)$ since α_X is bijective.

(ii) If Y is an operator range, $a(T) = a(\beta_Y T)$ and $b(T) = b(\beta_Y T)$ since β_Y is bijective and everywhere defined.

(iii) An obvious conseqence of (i) and (ii) combined with Proposition III.3.3 is the following:

(a) If X is an operator range, then

$$T \in F(X,Y) \text{ implies } T\alpha_X \in F(X_1,Y)$$

$$\text{and } T \in SF(X,Y) \text{ implies } T\alpha_X \in SF(X_1,Y)$$

with $K(T) = K(T\alpha_X)$ in both cases

(b) If Y is an operator range, then

$$T \in F(X,Y) \text{ implies } \beta_Y T \in F(X,Y_1)$$

$$\text{and } T \in SF(X,Y) \text{ implies } \beta_Y T \in SF(X,Y_1)$$

with $K(T) = K(\beta_Y T)$ in both cases.

Before coming to the main theorem of this section, we need the following lemmas. Note that Lemma III.3.6 is a generalisation of a lemma appearing in Goldberg [11], p.104. The lemma in its current form is due to J. Jaftha.

III.3.5 Lemma [11]. *Let* M *be a closed subspace having finite deficiency in* X . *Then*

(i) for any subspace V *of* X *there exists a finite-dimensional subspace* N *contained in* V *such that*

$$\bar{V} = \bar{V} \cap M \oplus N$$

(ii) if V *is dense in* X , *then* $V \cap M$ *is dense in* M .

Proof:

(i) The dimension of $\bar{V}/\bar{V} \cap M$ does not exceed the dimension of the finite-dimensional space X/M , since the linear map η from

$\bar{V}/\bar{V} \cap M$ to X/M defined by $\eta(x + \bar{V} \cap M) = x + M$ is 1-1 . Hence there exists a finite-dimensional subspace W of $\bar{V}$ such that

$$(1) \qquad \bar{V} = \bar{V} \cap M \oplus W$$

By Theorem I.6.9, there exists a bounded projection P from $\bar{V}$ onto W with $P(\bar{V} \cap M) = 0$. Since PV is finite-dimensional and P is continuous, it follows that

$$(2) \qquad W = P\bar{V} \subset \overline{PV} = PV$$

Let $w_1, w_2, \ldots, w_n$ be a basis for W . Then, by (2), there exist elements $v_1, v_2, \ldots, v_n$ in V such that $Pv_i = w_i$, $1 \leqslant i \leqslant n$. Since the set $\{w_i\}$ is linearly independent, the space $N = \mathrm{sp}\,\{v_1, v_2, \ldots, v_n\}$ is an n-dimensional subspace of V and $\{0\} = N \cap (M \cap \bar{V})$. By (1),

$$\dim \frac{\bar{V}}{M \cap \bar{V}} = \dim W = \dim N .$$

Hence it follows that $\bar{V} = M \cap \bar{V} \oplus N$.

(ii) Suppose $\bar{V} = X$. Then from (i), $X = M \oplus N$, where N is a finite-dimensional subspace of V . Therefore $V = M \cap V \oplus N$. It follows from Theorem I.6.9 that there exists a bounded projection P from X onto N and hence $I_X - P$ will be a bounded projection from X onto M . Now by I.3.20 $(I_X - P)^\wedge$ will be bounded. Note that $(I_X - P)^\wedge$ maps X/N onto M by mapping each $m + N \in X/N$ onto $m \in M$. Now let $\eta^{-1} = (I_X - P)^\wedge$. It is clear that since $\|[m]\| \leqslant \|m\|$ for each $m \in M$, η is bounded. It therefore follows that $X/N \cong M$. For clarity we write

$$(3) \qquad M \xrightarrow{\eta} \frac{X}{N} \supset \frac{V}{N} = \frac{(M \cap V) \oplus N}{N} \xrightarrow{\eta^{-1}} M \cap V .$$

Since V is dense in X, V/N is dense in X/N. Noting that

$$\eta(M \cap V) = \frac{V}{N}$$

it follows from (3) and the continuity of η and η^{-1} that

$$M \cap V = \eta^{-1}\eta(M \cap V)$$

is dense in M. □

III.3.6 Lemma. *Let* $T \in NS(X,Y)$ *with* $\gamma(T) > 0$. *If* M *is a subspace of* X *such that* $M + N(T)$ *is closed then* TM *is closed.*

Proof: Assume $\gamma(T) > 0$ for $T \in NS(X,Y)$. Let M be a subspace of X such that $M + N(T)$ is closed and let $\{m_n\}$ be a sequence in $M \cap D(T)$ such that $Tm_n \to y$. Since $R(T)$ is closed, $y \in R(T)$ and hence $y = Tx$ for some $x \in D(T)$. We obtain $\hat{T}[m_n] = Tm_n \to \hat{T}[x] = Tx$. From Theorem III.1.3 we notice that $\hat{T}^{-1}$ is continuous, hence implying that $[m_n] \to [x]$ in $D(\hat{T}) = D(T)/N(T)$. We can therefore choose a sequence $\{k_n\}$ in $N(T)$ such that $m_n + k_n \to x$ and hence since $M + N(T)$ is closed and $\{m_n + k_n\} \subset M + N(T)$, it follows that $x \in M + N(T)$. Consequently $y = Tx \in TM$ proving that TM is closed. □

III.3.7 Corollary. *Let* $T \in C(X,Y)$ *with* X *an operator range. If* M *is a subspace of* X *such that* $M + N(T)$ *is closed then* TM *is closed if either of the following two conditions are satisfied*

(i) X *is a Banach space and* $\gamma(T) > 0$

(ii) Y *is complete and* $R(T)$ *closed.*

Proof: Assume $T \in C(X,Y)$ with X an operator range and M a subspace of X such that $M + N(T)$ is closed. Now if X is complete and $\gamma(T) > 0$, $R(T)$ is closed by Theorem III.1.4 and therefore by Lemma III.3.6, TM is closed. Alternatively if Y is complete and $R(T)$ closed, $\gamma(T) > 0$ by Theorem III.1.4 and hence TM is closed by Lemma III.3.6. □

III.3.8 Lemma. *Suppose* $T \in L(X,Y)$, $B \in L(Z,X)$, $\overline{D(T)} = X$ *and* $R(B)$ *closed with* $K(T)$ *and* $K(B)$ *both finite. Then* $K(TB) = K(T) + K(B)$.

Proof: Assume $T \in L(X,Y)$ and $B \in L(Z,X)$ with $R(B)$ closed and $K(T)$ and $K(B)$ both finite. It follows that $a(B) < \infty$ implying that $N(B)$ is finite dimensional and hence closed. Consequently $N(TB)/N(B)$ is a normed space. Now define $\eta \in L(N(TB)/N(B)\ ,\ R(B) \cap N(T))$ by $\eta([x]) = Bx$. It is clear that η is a linear bijection from $N(TB)/N(B)$ onto the finite dimensional space $N_1 = R(B) \cap N(T)$ and hence it becomes clear that

(1) $$a(TB) = a(B) + n_1 \quad \text{with} \quad n_1 = \dim N_1 \ .$$

Now let N_2 be a subspace of $N(T)$ such that $N(T) = N_1 \oplus N_2$. Then

(2) $$a(T) = n_1 + n_2 \quad \text{with} \quad n_2 = \dim N_2 \ .$$

Notice that $R(B)$ and N_2 are linearly independent since if $Bx \in R(B) \cap N_2 \subset N(T)$, then obviously

$$Bx \in R(B) \cap N(T) \cap N_2 = N_1 \cap N_2 = \{0\} \ .$$

Since $R(B)$ is closed by hypothesis it follows from the finite dimensionality of N_2 and from Theorem I.6.7 that $R(B) \oplus N_2$ is closed. By hypothesis $D(T)$ is dense in X and hence it follows from Lemma III.3.5 that

(3) $$R(B) \oplus N_2 \oplus N_3 = X$$

for some finite dimensional subspace N_3 of $D(T)$. Consequently

(4) $$b(B) = n_2 + n_3 \quad \text{with} \quad n_3 = \dim N_3 .$$

Now $N(T) = N_1 \oplus N_2 \subset R(B) \oplus N_2$. This together with (3) implies that T is injective on all of N_3 and hence

(5) $$TX = TR(B) \oplus TN_3 .$$

The injectivity of T on N_3 together with (5) imply that

(6) $$b(TB) = b(T) + n_3 .$$

From (1), (2), (4) and (6) it follows that

$$\begin{aligned} K(TB) &= a(B) + n_1 - b(T) - n_3 \\ &= a(B) + a(T) - n_2 - b(T) - n_3 \\ &= a(B) + a(T) - b(B) - b(T) \\ &= K(T) + K(B) \end{aligned}$$

hence proving the lemma. □

The following theorem was first proved by Gohberg and Krein [08] for T and B Fredholm operators. A generalisation of this to normally solvable operators for the Banach space case appears in Goldberg [11], p.103. We give a generalisation of the version appearing in Goldberg by removing the completeness restriction from Y . Notice that although we make the extra demand that $\gamma(T) > 0$ this is still a true generalisation since the fact that $\gamma(T) > 0$ is implicit in the Banach space case by Theorem III.1.4.

III.3.9 Theorem. *Let* $T \in NS(X,Y)$, $B \in C(Z,X)$ *with* X *complete,* $a(T) < \infty$ *and* $\gamma(T) > 0$. *Then*

(i) TB *is closed*

(ii) if B *is normally solvable,* TB *is normally solvable*

(iii) if $D(T)$ *is dense in* X *and* T *and* B *are Fredholm operators, then* TB *is a Fredholm operator with*

$$K(TB) = K(T) + K(B) .$$

Proof:

(i) Assume $T \in NS(X,Y)$, $B \in C(Z,X)$, X complete, $a(T) < \infty$ and $\Upsilon(T) > 0$. Now suppose $\{z_n\}$ is a sequence in $D(TB)$ such that $z_n \to z$ and $TBz_n \to y$. Since $\Upsilon(T) > 0$, we know by Theorem III.1.3 that $\hat{T}$ has a continuous inverse and hence it follows that the sequence $\{[Bz_n]\}$ is Cauchy in $D(T)/N(T)$ and therefore convergent in $X/N(T)$ by the completeness of X . Suppose $[Bz_n] \to [x] \in X/N(T)$. Then there exists a sequence $\{x_n\} \subset N(T)$ such that $Bz_n + x_n \to x$. Assume $\{x_n\}$ to be unbounded and choose a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$(1) \qquad \|x_{n'}\| \to \infty \quad \text{and} \quad \frac{Bz_{n'} + x_{n'}}{\|x_{n'}\|} \to 0$$

Since $\{x_{n'}/\|x_{n'}\|\}$ is a bounded sequence in the finite dimensional space $N(T)$, it follows by Proposition I.6.5 that there exists a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ and a $v \in N(T)$ such that

$$x_{n''}/\|x_{n''}\| \to v ,$$

and hence from (1) it is obvious that $Bz_{n''}/\|x_{n''}\| \to -v$ and $z_{n''}/\|x_{n''}\| \to 0$. Therefore since B is closed, $v = B(0) = 0$ contradicting the fact that $\|v\| = 1$ since $\{x_{n''}/\|x_{n''}\|\}$ is a sequence of norm one elements. We conclude that that $\{x_n\}$ is bounded and consequently, again by Proposition I.6.5, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that

$$x_{n'} \to w \in N(T) .$$

implying that $Bz_{n'} \to x - w$. Since $z_{n'} \to z$ it follows from the closedness of B that $z \in D(B)$ and $Bz = x - w = \lim_{n'\to\infty} Bz_{n'}$.

As T is also closed, we notice that since $TBz_{n'} \to y$ and $Bz_{n'} \to Bz = x - w$, therefore $Bz \in D(T)$, i.e. $z \in D(TB)$ and $TBz = y$, proving TB to be closed.

(ii) If in addition we assume B to be normally solvable, that is BZ closed, then since $a(T) < \infty$, $BZ + N(T)$ would be closed by Theorem I.6.7. Hence from (i) and from Lemma III.3.6 it follows that TB is closed with $TBZ = R(TB)$ closed; that is, TB is normally solvable.

(iii) If $T \in F(X,Y)$, $B \in F(Z,X)$ with X complete, $\overline{D(T)} = X$ and $\Upsilon(T) > 0$, it follows from (ii) and Lemma III.3.8 that TB is a Fredholm operator with

$$K(TB) = K(T) + K(B) . \qquad \square$$

III.3.10 Remark. Notice that in (ii) and (iii) of Theorem III.3.9 we don't need X to be complete or B to be closed for $R(TB)$ to be closed If $\Upsilon(T) > 0$, $a(T) < \infty$, $T \in NS(X,Y)$ and $B \in L(Z,X)$ with $R(B)$ closed, then $R(B) + N(T)$ would be closed, and the closedness of $R(TB)$ would now follow from Lemma III.3.6.

The next proposition appears as part of Theorem IV.2.7 in Goldberg [11]. In this case we have managed to remove the completeness restriction from the spaces X, Y, and Z at the same time requiring only that T and B be linear with $b(B) < \infty$ instead of needing T and B to be semi-Fredholm and closed respectively with $a(T) < \infty$ and $b(B) < \infty$. As with III.3.9, this is a true generalisation, for although we make the added assumptions that $\Upsilon(B) > 0$ and $R(B)$ shall be closed, these facts are implicit in the Banach space case by Corollary III.2.4 and Theorem III.1.4.

III.3.11 Proposition. *Let* $T \in L(X,Y)$, $B \in L(Z,X)$ *with* $N(B)$ *closed,* $\overline{D(T)} = X$, $\overline{D(B)} = Z$, $b(B) < \infty$, $\gamma(B) > 0$ *and* $R(B)$ *closed in* X . *Then* $D(TB)$ *is dense in* Z .

Proof: Assume $T \in L(X,Y)$, $B \in L(Z,X)$, $b(B) < \infty$ and $\gamma(B) > 0$ with $D(T)$ and $D(B)$ dense in X and Z respectively. Then $\hat{B}$ has a continuous inverse on the closed subspace $R(B)$ of X by Theorem III.1.3. Since $D(T)$ is dense in X , $D(T) \cap R(B)$ will be dense in $R(B)$ by Lemma III.3.5. By the continuity of $\hat{B}^{-1}$ we now obtain

$$(1) \qquad D(B)/N(B) = \hat{B}^{-1}R(B) = \hat{B}^{-1}\,\overline{D(T) \cap R(B)}$$
$$\subset \overline{\hat{B}^{-1}\,D(T) \cap R(B)} = \overline{D(TB)}/N(B) \; .$$

Consequently since $D(B)$ is dense in Z , we conclude from (1) that $D(TB)$ is dense in Z . □

We now ask ourselves whether or not we can possibly remove the completeness restriction from X in III.3.9. The answer is yes but unfortunately we have to sacrifice the generality of requiring Y to be only a normed space. We obtain the following proposition:

III.3.12 Proposition. *Let* $T \in NS(X,Y)$, $B \in C(Z,X)$ *with* X *an operator range and* Y *complete. If* $a(T) < \infty$ *then*

(i) TB *is closed*

(ii) TB *is normally solvable if* B *is normally solvable*

(iii) if T *and* B *are Fredholm with* $D(T)$ *dense in* X , *then* TB *is Fredholm and* $K(TB) = K(T) + K(B)$.

Proof:

(i) Assume $T \in NS(X,Y)$, $B \in C(Z,X)$, X an operator range, Y complete and $a(T) < \infty$. It follows from Propositions I.4.7 and III.3.3 that $T\alpha_X$ is normally solvable and $\beta_X B$ is closed. Also by Remark III.3.4 and Theorem III.1.4 $a(T\alpha_X) = a(T) < \infty$ and $\gamma(T\alpha_X) > 0$ since X_1 and Y are complete. Hence by Theorem III.3.9 it follows that $(T\alpha_X)(\beta_X B) = TB$ is closed.

(ii) If in addition we assume $R(B)$ to be closed then $R(B) + N(T)$ will be closed by Theorem I.6.7. Hence by Corollary III.3.7 $T(R(B) + N(T)) = R(TB)$ will be closed and since by (i) TB is closed the result follows.

(iii) Suppose $T \in F(X,Y)$, $B \in F(Z,X)$ with Y complete, X an operator range and $D(T)$ dense in X . From (ii) and Lemma III.3.8 it follows that TB is normally solvable and that $K(TB) = K(T) + K(B)$ is finite since both $K(T)$ and $K(B)$ are finite by definition implying that TB is Fredholm. □

III.3.13 Definition (polynomials of operators).

Let $X = Y$. Given a polynomial $p(\lambda) = \sum_{k=0}^{n} a_k\lambda^k$ define $p(T) = \sum_{k=0}^{n} a_k T^k$ where T^0 is considered to be the identity operator on all of X . The domain of $p(T)$ is the domain of T^n .

III.3.14 Corollary. *Let* $X = Y$ *be an operator range. If there exists a scalar* λ_0 *such that* $\lambda_0 I - T$ *is normally solvable,* $a(\lambda_0 I - T) < \infty$ *and* $\gamma(\lambda_0\alpha_X - T\alpha_X) > 0$ *, then* $p(T)$ *is closed for any polynomial* $p(\lambda)$.

Proof: Assume there exists a scalar λ_0 such that $\lambda_0 I - T$ is normally solvable, $a(\lambda_0 I - T) < \infty$ and $\gamma(\lambda_0\alpha_X - T\alpha_X) > 0$. Let $p(\lambda)$ be a polynomial of degree n . The corollary is trivial if $n = 0$. Suppose the corollary holds for polynomials of degree k . Let $p(\lambda)$ be of degree $n = k + 1$. Consequently we may write

$$p(\lambda) = (\lambda_0 - \lambda)q(\lambda) + c$$

where $q(\lambda)$ is a polynomial of degree k and c a constant. Hence we have

$$p(T) = (\lambda_0 I - T)q(T) + cI \tag{1}$$

with $q(T)$ a closed operator by the induction hypothesis. Since $\lambda_0 I - T$ is normally solvable it follows from Propositions I.4.7 and III.3.3 that $\lambda_0\alpha_X - T\alpha_X$ and $\beta_X q(T)$ are normally solvable and closed respectively. We conclude that since $\gamma(\lambda_0\alpha_X - T\alpha_X) > 0$ and $a(\lambda_0 I - T) = a(\lambda_0\alpha_X - T\alpha_X) < \infty$, $(\lambda_0\alpha_X - T\alpha_X)\beta_X q(T) = (\lambda_0 I - T)q(T)$ is closed by Theorem III.3.9. Now let $\{x_n\} \subset D(p(T))$ such that

$$x_n \to x \quad \text{and} \quad p(T)x_n \to y . \tag{2}$$

Since $(\lambda_0 I - T)q(T)$ is closed and $D(p(T)) = D((\lambda_0 I - T)q(T))$ by definition, it follows from (1) and (2) that $x_n \to x$ and $(\lambda_0 I - T)q(T)x_n \to y - cx$, therefore implying that $x \in D((\lambda_0 I - T)q(T)) = D(p(T))$ and $(\lambda_0 I - T)q(T)x = y - cx$, that is $p(T)x = y$. Hence $p(T)$ is closed. □

III.3.15 Definition (Fredholm resolvent, Fredholm spectrum).

Let $X = Y$. The set of scalars λ for which $\lambda I - T$ is a Fredholm operator is called the *Fredholm resolvent* of T , denoted by $F\rho(T)$. The set of scalars λ for which $\lambda \notin F\rho(T)$ is called the *Fredholm spectrum* of T , denoted by $F\sigma(T)$.

The following theorem appears in Goldberg [11] in the Banach space case. It was first proved for differential operators by Rota [18] and later generalised to Fredholm operators by Balslev and Gamelin [01]. We give a partial generalisation to operator ranges.

III.3.16 Theorem. *Let* $X = Y$ *be a complex operator range and let* $T \in C(X)$. *Then for any polynomial* p

(i) $p(F\sigma(T)) \subset F\sigma(p(T))$ *if* $\gamma(\lambda\alpha_X - T\alpha_X) > 0$ *for every* $\lambda \in F\sigma(T)$

(ii) $F\sigma(p(T)) \subset p(F\sigma(T))$ *if* $D(T\alpha_X)$ *is dense in* X_1 *and* $\gamma(\lambda\alpha_X - T\alpha_X) > 0$ *for every* $\lambda \in F\rho(T)$

(iii) suppose $D(T\alpha_X)$ *is dense in* X_1 *and* $\gamma(\lambda\alpha_X - T\alpha_X) > 0$ *for any* λ . *If there exists a* $\mu \in p(F\sigma(T))$, *then* $p(T)$ *is closed and* $K(\mu I - p(T)) = \sum_{i=1}^{n} K(\lambda_i I - T)$ *where* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the zero's of the polynomial counted according to their multiplicity.*

Proof:

(i) Assume $\gamma(\lambda\alpha_X - T\alpha_X) > 0$ for every $\lambda \in F\sigma(T)$. Given ν , let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be scalars such that

(1) $$\nu - p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

and hence

(2) $$\nu I - p(T) = (\lambda_1 I - T)(\lambda_2 I - T) \ldots (\lambda_n I - T) .$$

Suppose $\nu \in p(F\sigma(T))$. Then by (1) one of the λ_i's is in $F\sigma(T)$, say λ_k . Since $\lambda_k I - T$ commutes with $\lambda_i I - T$, $1 \leqslant i \leqslant n$, we conclude from (2) that

(3) $$N(\lambda_k I - T) \subset N(\nu I - p(T))$$

and $R(\nu I - p(T)) \subset R(\lambda_k I - T)$.

Now $\lambda_k I - T$ and hence $\lambda_k\alpha_X - T\alpha_X$ is closed by Proposition I.4.7.

Consequently since $\gamma(\lambda_k\alpha_X - T\alpha_X) > 0$ by hypothesis it follows from Theorem III.1.4 that $R(\lambda_k I - T) = R(\lambda_k\alpha_X - T\alpha_X)$ is closed. Therefore since $\lambda_k I - T$ is not a Fredholm operator, $a(\lambda_k I - T) = \infty$ and/or $b(\lambda_k I - T) = \infty$. We conclude from (3) that $\nu I - p(T)$ is not Fredholm and hence $\nu \in F\sigma(p(T))$.

(ii) Suppose that $\gamma(\lambda\alpha_X - T\alpha_X) > 0$ for every $\lambda \in F\rho(T)$ and $D(T\alpha_X)$ is dense in X_1. As in (i) consider

$$\nu - p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

$$\text{and} \qquad \nu I - p(T) = (\lambda_1 I - T)(\lambda_2 I - T) \ldots (\lambda_n I - T) .$$

Since $D(T\alpha_X)$ is dense in X_1 we conclude that $D(\lambda_i\alpha_X - T\alpha_X)$ is dense in X_1. Assuming all operators $\lambda_i I - T$ to be Fredholm it follows from Remark III.3.4 that $\lambda_i\alpha_X - T\alpha_X$ and $\beta_X\lambda_i - \beta_X T$ are Fredholm for $1 \leqslant i \leqslant n$. Now since $\gamma(\lambda_i\alpha_X - T\alpha_X) > 0$ for $1 \leqslant i \leqslant n$ by hypothesis, we see from Theorem III.3.9 that

$$(\lambda_{n-1}\alpha_X - T\alpha_X)(\beta_X\lambda_n - \beta_X T) = (\lambda_{n-1}I - T)(\lambda_n I - T) \quad \text{is Fredholm.}$$

Moreover $\beta_X[(\lambda_{n-1}I - T)(\lambda_n I - T)]$ is Fredholm by Remark III.3.4. As before we notice that

$$\begin{aligned}(\lambda_{n-2}\alpha_X - T\alpha_X)\beta_X[(\lambda_{n-1}I - T)(\lambda_n I - T)]\\ = (\lambda_{n-2}I - T)(\lambda_{n-1}I - T)(\lambda_n I - T)\end{aligned}$$

is therefore Fredholm. Proceeding likewise we conclude that $\nu I - p(T)$ is Fredholm. Hence if $\nu \in F\sigma(p(T))$ then for some λ_k, $\lambda_k I - T$ is not a Fredholm operator. Therefore $F\sigma(p(T)) \subset p(F\sigma(T))$.

(ii) Suppose there exists $\mu \in p(F\sigma(T))$ and that $\gamma(\lambda\alpha_X - T\alpha_X) > 0$ for every λ with $D(T\alpha_X)$ dense in X_1. Then if we have the representations given in (1) and (2) for μ instead of ν we conclude from the proofs of (i) and (ii) that $\mu I - p(T)$ is

Fredholm iff each $\lambda_k I - T$ is Fredholm. Hence from Remark III.3.4 it follows that $\lambda_1\alpha_X - T\alpha_X$ is Fredholm with $D(T\alpha_X) = D(\lambda_1\alpha_X - T\alpha_X)$ dense in X_1 and $\gamma(\lambda_1\alpha_X - T\alpha_X) > 0$ by hypothesis. From the proof of (ii) we see that

$$\beta_X[(\lambda_2 I - T)(\lambda_3 I - T) \ldots (\lambda_n I - T)] \text{ is Fredholm}$$

and consequently it follows from Theorem III.3.9 that

$$\begin{aligned} K(p(T)) &= K(\lambda_1\alpha_X - T\alpha_X) + K(\beta_X[(\lambda_2 I - T)(\lambda_3 I - T) \ldots (\lambda_n I - T)] \\ &= K(\lambda_1 I - T) + K[(\lambda_2 I - T)(\lambda_3 I - T) \ldots (\lambda_n I - T)] . \end{aligned}$$

Proceeding likewise we obtain

$$K(p(t)) = \sum_{i=1}^{n} K(\lambda_i I - T) .$$

The fact that $p(T)$ is closed follows immediately from Corollary III.3.14 since $\lambda_i I - T$ is Fredholm and $\gamma(\lambda_i\alpha_X - T\alpha_X) > 0$ for $1 \leq i \leq n$. □

III.3.17 Definition (K-resolvent, K-spectrum).

Let $X = Y$. The set of scalars λ for which $\lambda I - T$ has $K(\lambda I - T) < \infty$ is called the K-*resolvent* of T , denoted by $K\rho(T)$. The set of λ for which $\lambda \notin K\rho(T)$ is called the K-*spectrum* of T , denoted by $K\sigma(T)$. (This definition is the authors own.)

III.3.18 Proposition. *Let* $X = Y$ *be a complex normed space with* $T \in L(X)$. *Then for any polynomial* p , $p(K\sigma(T)) \subset K\sigma(p(T))$.

Proof: Assume $X = Y$ to be a complex normed space and $T \in L(X)$. Then for ν an arbitrary scalar, we have $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

(1) $$\nu - p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

and hence

(2) $$\nu I - p(T) = (\lambda_1 I - T)(\lambda_2 I - T) \ldots (\lambda_n I - T) .$$

Suppose now $\nu \in p(K\sigma(T))$. Then by (1) one of the λ_i's is in $K\sigma(T)$, say λ_k . Consequently since $\lambda_k I - T$ commutes with $\lambda_i I - T$, $1 \leqslant i \leqslant n$, it follows that $N(\lambda_k I - T) \subset N(\nu I - p(T))$ and $R(\nu I - p(T)) \subset R(\lambda_k I - T)$ and hence we conclude that $\nu \in K\sigma(p(T))$.

III.3.19 Remark. Suppose we define the sets $L^+(X,Y)$ and $L^-(X,Y)$ to be the sets of linear operators with $a(T) < \infty$ and $b(T) < \infty$ respectively. Now if $X = Y$ let $K\overset{+}{\rho}(T)$ $(K\overset{-}{\rho}(T))$ be the set of all scalars λ such that $a(\lambda I - T) < \infty$ $(b(\lambda I - T) < \infty)$ and $K\overset{+}{\sigma}(T)(K\overset{-}{\sigma}(T))$ the set of scalars $\lambda \notin K\overset{+}{\rho}(T)(\lambda \notin K\overset{-}{\rho}(T))$. We refer to $K\overset{+}{\rho}(T)(K\overset{-}{\rho}(T))$ as the K^+*-resolvent* *(K^--resolvent)* and to $K\overset{+}{\sigma}(T)(K\overset{-}{\sigma}(T))$ as the K^+*-spectrum* *(K^--spectrum)*. Proceeding as in Proposition III.3.18 it is obvious that $p(K\overset{+}{\sigma}(T)) \subset K\overset{+}{\sigma}(p(T))$ and $p(K\overset{-}{\sigma}(T)) \subset K\overset{-}{\sigma}(p(T))$.

The idea for the generalisation of Theorems III.3.9 and III.3.11 is mainly due to J. Jaftha. Corollaries III.3.7 and III.3.14, Lemma III.3.8, Propositions III.3.12 and III.3.18, Theorem III.3.16 and Remark III.3.19 are generalisations by the author.

CHAPTER IV

WEAKLY COMPACT, PRECOMPACT AND COMPACT OPERATORS AND PREREFLEXIVE SPACES

In the first section we will prove some results which will be needed in the next chapter. In the second section we will take a closer look at the close relationship between weakly compact operators and pre-reflexivity.

IV.1 Compact and Precompact Operators

IV.1.1 Definition (totally bounded, precompact operators, compact operators).

We define a subset S of a metric space to be *totally bounded* if for every $\varepsilon > 0$ there exists a finite number of balls of radius ε such that S is contained in the union of these balls.

For T a linear operator with domain in X and range in Y, we say that T is precompact if TB_X is totally bounded. T is said to be compact if $\overline{TB_X}$ is compact in Y. We will denote the set of precompact (compact) operators by $PK(X,Y)$ $(K(X,Y))$; $PK[X,Y]$ $(K[X,Y])$ if $D(T) = X$.

IV.1.2. Definition (strictly singular operators).

Let $T \in B(X,Y)$. We say that T is *strictly singular* if it does not have a continuous inverse on any infinite dimensional subspace of its domain. We denote the set of strictly singular operators in $B(X,Y)$ by $SS(X,Y)$; $SS[X,Y]$ if $D(T) = X$.

The concept of strictly singular operators was introduced by Kato [13], a concept which has proved to be very useful in perturbation theory, as will be seen in the next chapter.

IV.1.3 Proposition [11]. *Every precompact operator is strictly singular.*

Proof: Let $T \in PK(X,Y)$. T is continuous since a totally bounded set is bounded. Suppose T has a continuous inverse on a subspace $M \subset D(T)$. Then $TB_M \subset TB_X$ is totally bounded and hence since T has a continuous inverse on M , it follows that B_M is totally bounded in M . It is obvious from the proof of I.6.5 that this can only be the case if M is finite dimensional and hence we conclude that T is strictly singular. □

IV.1.4 Remarks [11].

(i) Every bounded linear operator with finite-dimensional range is compact. For if $K: X \to Y$ is such an operator, then letting B be the 1-ball of X, KB is a bounded set in the finite-dimensional subspace $R(K)$ of Y . The compactness of $\overline{KB}$ is therefore a consequence of Theorem I.6.5.

(ii) An operator is compact if and only if it takes every bounded sequence into a sequence which has a convergent subsequence.
An operator is precompact if and only if it takes every bounded sequence into a sequence which has a Cauchy subsequence.

IV.1.5 Proposition [11]. *The set* PK[X,Y] *is a closed subset of* B[X,Y] .

Proof: Let T_n be a sequence in PK[X,Y] such that $T_n \to T \in B[X,Y]$ and let $\varepsilon > 0$ be arbitrary. It is obvious that

$$\| T_N x - Tx \| < \varepsilon/3 \quad \text{for some} \quad N \in \mathbb{N} \quad \text{and} \quad x \in B_X .$$

Since T_N is precompact, there exist $x_1, x_2, \dots, x_m$ in B_X such that

$$\| T_N x - T_N x_i \| < \varepsilon/3 \quad \text{for} \quad x \in B_X \quad \text{and some} \quad x_i .$$

Hence it follows that

$$\| Tx - Tx_i \| \leqslant \| Tx - T_N x \| + \| T_N x - T_N x_i \| + \| T_N x_i - Tx_i \| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

showing that T is precompact. □

IV.1.6 Theorem [11]. *A continuous linear operator with dense domain is precompact iff its conjugate is compact.*

Proof: Let K be a precompact linear operator which maps X into Y. We shall prove that K' is precompact and therefore compact since X' is complete. Given $\varepsilon > 0$, there exist $x_1, x_2, \dots, x_n$ in B_X such that for each $x \in B_X$ there is an x_i for which

(1) $$\| Kx - Kx_i \| < \frac{\varepsilon}{3} .$$

Let A be the map from Y' to the unitary n-space given by

$$Ay' = (y'Kx_1, \dots, y'Kx_n) .$$

Since A is clearly bounded and linear, we know by Remark IV.1.4 that A is compact. Hence there exist $y'_1, \dots, y'_m$ in $B_{Y'}$ such that for each $y' \in B_{Y'}$ there is a y'_j for which

$$\| Ay' - Ay'_j \| < \frac{\varepsilon}{3} .$$

In particular,

$$(2) \qquad |y'Kx_i - y_j'Kx_i| < \frac{\varepsilon}{3} \qquad 1 \leqslant i \leqslant n \ .$$

From (1) and (2),

$$\begin{aligned} |K'y'x - K'y_j'x| &\leqslant |y'Kx - y'Kx_i| + |y'Kx_i - y_j'Kx_i| + |y_j'Kx_i - y_j'Kx| \\ &\leqslant \|Kx - Kx_i\| + \frac{\varepsilon}{3} + \|Kx_i - Kx\| < \varepsilon \ . \end{aligned}$$

Thus $\|K'y' - K'y_j'\| \leqslant \varepsilon$ whence K' is precompact.

Conversely, suppose K' is compact. Then by what has just been shown, K'' is compact. Now, $K''J_X = J_Y K$, where J_X and J_Y are the natural maps from X into X'' and Y into Y'' , respectively. Since J_X is bounded, it is clear that $K''J_X$ is compact and, in particular, $J_Y K$ is precompact. Since J_Y has a continuous inverse, it is easy to see that K is precompact.

The theorem now follows since for any $T \in PK(X,Y)$ with $D(T)$ dense in X , we can consider T to be in $PK[D(T),Y]$. Hence $T \in PK(X,Y)$ iff $T \in PK[D(T),Y]$ iff $T' \in K[Y',D(T)'] = K[Y',X']$. □

IV.1.7 Corollary [11]. *Let* $T \in PK(X,Y)$. *If* $R(T)$ *is complete, it is finite dimensional.*

Proof: Let $T \in PK(X,Y)$ with $R(T)$ complete. Consider T_1 to be T from $D(T)$ onto $Y_1 = R(T)$, that is $T_1 \in PK[D(T),Y_1]$. Hence $T_1 \in I$ and from II.3.6 it follows that the compact operator T_1' has a continuous inverse. From Proposition IV.1.3 and the definition of strict singularity it follows that Y_1' and hence $Y_1 = R(T)$ is finite dimensional. □

IV.1.8 Theorem [11]. *Let* Z *be a normed space. If* T *is in* $K[X,Y]$ $PK[X,Y]$ *, or* $SS[X,Y]$ *, then for* $A \in B[Z,X]$ *,* TA *is in* $K[Z,Y]$ *,* $PK[Z,Y]$ *, or* $SS[Z,Y]$ *, respectively. Similarly, if* A *is in* $B[Y,Z]$ *then* AT *is in* $K[X,Z]$ *,* $PK[X,Z]$*, or* $SS[X,Z]$ *, respectively.*

Proof: We shall only prove the theorem for $T \in SS[X,Y]$. The proofs for T in $K[X,Y]$ or $PK[X,Y]$ are also easy. Suppose $TA \in B[Z,Y]$ has a continuous inverse on a subspace M of Z . Then there exists a $c > 0$ such that for all $x \in M$

$$\|TAx\| \geq c\|x\| \geq \frac{c}{\|A\|}\|Ax\| \quad .$$

Thus T has a continuous inverse on AM , and therefore AM is finite-dimensional. But since TA and hence A is injective on M , it follows that M is finite dimensional. Conversely if $A \in B[Y,Z]$, suppose AT has a continuous inverse on a subspace N of X . Then there exists a $c > 0$ such that for all $x \in N$,

$$\|A\| \ \|Tx\| \geq \|ATx\| \geq c\|x\| \ .$$

Hence T has a continuous inverse on N , whence N is finite-dimensional. Therefore AT is strictly singular. □

The following theorem, which appears in Goldberg [11] is due to Kato [13] except for the assertion that the operator is precompact. The theorem and its corollary will be needed in chapter V.

IV.1.9 Theorem. *Let* $T \in L(X,Y)$ *. Suppose that* T *does not have a continuous inverse when restricted to any closed subspace having finite deficiency in* X *. Then given* $\varepsilon > 0$ *, there exists an infinite-dimensional subspace* $M(\varepsilon)$ *contained in* $D(T)$ *such that* T *restricted to* $M(\varepsilon)$ *is precompact and has norm not exceeding* ε *.*

Proof: The hypothesis implies the existence of an $x_1 \in X$ such that $\|x_1\| = 1$ and $\|Tx_1\| < 3^{-1}\varepsilon$. There is an $x_1' \in X'$ such that $\|x_1'\| = 1$ and $x_1'x_1 = \|x_1\| = 1$. Since $N(x_1')$ has deficiency 1 in X, there exists an $x_2 \in N(x_1')$ such that $\|x_2\| = 1$ and $\|Tx_2\| < 3^{-2}\varepsilon$. There exists an $x_2' \in X'$ such that $\|x_2'\| = 1$ and $x_2'x_2 = \|x_2\| = 1$. Since $N(x_1') \cap N(x_2')$ has finite deficiency in X , there exists an $x_3 \in N(x_1') \cap N(x_2')$ such that $\|x_3\| = 1$ and $\|Tx_3\| < 3^{-3}\varepsilon$. Inductively, sequences $\{x_k\}$ and $\{x_k'\}$ are constructed in X and X' , respectively, with the following properties.

(1) $$\|x_k\| = \|x_k'\| = x_k'x_k = 1 \qquad \|Tx_k\| < 3^{-k}\varepsilon , \quad 1 \leq k < \infty$$

(2) $$x_k \in \bigcap_{i=1}^{k-1} N(x_i') \quad \text{or, equivalently,} \quad x_i'x_k = 0 \qquad 1 \leq i < k \quad .$$

It is easy to verify that the set of x_k is linearly independent, whence $M = \operatorname{sp}\{x_1, x_2, \ldots\}$ is an infinite-dimensional subspace of $D(T)$. It will now be shown that the restriction T_M of T to M has norm not exceeding ε. Suppose $x = \sum_{i=1}^{m} \alpha_i x_i$. Then from (1) and (2),

$$|\alpha_1| = |x_1'x| \leq \|x_1'\|\|x\| = \|x\| \quad .$$

In fact,

(3) $$|\alpha_k| \leq 2^{k-1}\|x\| \qquad 1 \leq k < m$$

For suppose (3) is true for $k \leq j < m$. Then from (1) and (2),

(4) $$x_{j+1}'x = \sum_{i=1}^{j} \alpha_i x_{j+1}'(x_i) + \alpha_{j+1} \quad .$$

Hence, by (4) and the induction hypothesis,

$$|\alpha_{j+1}| \leq |x_{j+1}'x| + \sum_{i=1}^{j} |\alpha_i|\,|x_{j+1}'x_i| \leq \|x\| + \sum_{i=1}^{j} 2^{i-1}\|x\| = 2^j\|x\| \quad .$$

Thus (3) follows by induction and

$$\|Tx\| \leqslant \sum_{i=1}^{m} |\alpha_i| \|Tx_i\| \leqslant \sum_{i=1}^{m} 2^{i-1}3^{-i}\varepsilon \|x\| \leqslant \varepsilon\|x\| .$$

Hence $\|T_M\| \leqslant \varepsilon$. To prove that T_M is precompact, it suffices to show that T_M is the limit in $B[M,Y]$ of a sequence of precompact operators. For each positive integer n , define $T_n^M : M \to Y$ to be T on $sp\{x_1, x_2, \ldots, x_n\}$ and 0 on $sp\{x_{n+1}, x_{n+2}, \ldots\}$. Clearly T_n^M is linear and has finite-dimensional range. Moreover, T_n^M is bounded on M ; for if $x = \sum_{i=1}^{n+k} \alpha_i x_i$, then by (1) and (3) ,

$$\|T_n^M x\| \leqslant \sum_{i=1}^{n} |\alpha_i| \|Tx_i\| \leqslant \sum_{i=1}^{n} 2^{i-1}3^{-i}\varepsilon \|x\| .$$

Thus T_n^M is precompact by IV.1.4. Since

$$\|T_M x - T_n^M x\| \leqslant \sum_{i=n+1}^{\infty} |\alpha_i| \|Tx_i\|$$

$$\leqslant \varepsilon\|x\| \sum_{i=n+1}^{\infty} 2^{i-1}3^{-i} \to 0 \quad \text{as} \quad n \to \infty$$

it follows that T_n^M converges to T_M in $B[M,Y]$, completing the proof of the theorem. □

IV.1.10 Corollary [11]. *Let* X *and* Y *be complete. If* $T \in C(X,Y)$ *but* $R(T)$ *is not closed, then for each* $\varepsilon > 0$ *there exists an infinite-dimensional closed subspace* $M(\varepsilon)$ *contained in* $D(T)$ *such that* T *restricted to* $M(\varepsilon)$ *is compact with norm not exceeding* ε *.*

Proof: Let W be a closed subspace having finite deficiency in X . Assume T has a continuous inverse on W . Then TW is closed; for if $Tx_n \to y$, $x_n \in W$, then $\{x_n\}$ is a Cauchy sequence and so converges to some x in the Banach space W. Since T is closed, x is in $D(T)$ and $Tx = y$. By hypothesis, there exists a finite-dimensional subspace N of X such that $X = W + N$. Hence $TX = TW + TN$. Since TW is closed and TN is finite-dimensional, TX is also closed by Theorem I.6.7. But this contradicts the hypothesis that $R(T)$ is not closed. Therefore T does not have a continuous inverse on W. Hence there exists an $M = M(\varepsilon)$ with the properties described in Theorem IV.1.9. Since Y is complete and T is closed, and continuous on M, it follows that $\bar{M}$ is contained in $D(T)$. Moreover, $\|T_{\bar{M}}\| \leqslant \varepsilon$ and $\overline{TB_{\bar{M}}} = \overline{TB_M}$ where $T_{\bar{M}}$ is the restriction of T to $\bar{M}$. The precompactness of T and the completeness of Y imply that $\overline{TB_M}$ is compact. Hence $T_{\bar{M}}$ is compact. □

IV.1.11 Corollary. *Let* $T \in C(X,Y)$ *with* X *complete and* $R(T)$ *not closed. Then for each* $\varepsilon > 0$ *there exists an infinite-dimensional subspace in* $D(T)$, *say* $M(\varepsilon)$, *such that* T *restricted to* $M(\varepsilon)$ *is precompact with norm not exceeding* ε.

Proof: Assume $T \in C(X,Y)$ with X complete and $R(T)$ not closed. Suppose now that W is a closed subspace having finite deficiency in X. Proceeding as in the first part of the proof of IV.1.10, the corollary now follows from Theorem IV.1.9. □

IV.2 Pre-Reflexivity and Weakly Compact Operators

There are quite a few results linking weak compactness of operators to reflexivity. In this section we give more general results linking weakly compact operators to pre-reflexivity.

IV.2.1 Definition (weakly compact operators).

A linear operator which takes bounded sequences onto sequences with weakly convergent subsequences is said to be *weakly compact*. We will denote the set of weakly compact operators in $L(X,Y)$ by $WK(X,Y)$; $WK[X,Y]$ if $D(T) = X$.

IV.2.2 Proposition. *If* $T \in WK(X,Y)$ *then* $T \in B(X,Y)$.

Proof: Suppose $T \in WK(X,Y)$ with $T \notin B(X,Y)$. Then there exists a bounded sequence $\{x_n\} \subset D(T)$ such that $\{Tx_n\}$ converges weakly and $\|Tx_n\| \to \infty$. But this is an obvious contradiction to the uniform boundedness principle (Proposition I.5.13). □

IV.2.3 Proposition. *If either* X *or* Y *is reflexive then* $B(X,Y) = WK(X,Y)$.

Proof: Suppose either X or Y is reflexive. It follows easily from Theorem I.7.7 and Proposition IV.2.2 that $B(X,Y) = WK(X,Y)$. □

IV.2.4 Proposition. *If* X *is complete and* Y *pre-reflexive then* $B[X,Y] = WK[X,Y]$.

Proof: Suppose $T \in B[X,Y]$ with X complete and Y pre-reflexive. Since $D(T) = X$ we conclude that T and therefore $\beta_Y T$ is closed by Proposition I.4.7. By the closed graph theorem $\beta_Y T$ is bounded and hence since α_Y is weakly compact by Proposition IV.2.3, it follows that $T = \alpha_Y(\beta_Y T)$ is weakly compact. □

We give an example to show that the completeness on X cannot be weakened.

IV.2.5 Example. Consider the map T from $\ell_2(\mathbb{N})$ into $\ell_1(\mathbb{N})$ defined by $T(\{x_n\}) = \{x_n/n\}$. It was shown in Example I.7.2 that T is bounded with $R(T) \subset \ell_1$ being a non-complete pre-reflexive space. Banach [02] showed that on ℓ_1 convergence in norm is the same as weak-convergence. Hence $B_{R(T)}$ is obviously not weakly compact since that would imply norm compactness which by Proposition I.6.5 would imply the finite dimensionality of R(T). This is an obvious contradiction since in Example I.7.2, R(T) was shown to be dense in ℓ_1. We conclude that the identity map on R(T) is not weakly compact.

IV.2.6 Lemma [11]. *Suppose* X *is reflexive. If there exists a bounded linear operator which maps* X *onto a Banach space* Y *, then* Y *is reflexive.*

If M *is a closed subspace of* X *, then* X/M *is reflexive.*

Proof: Assume that T is an isomorphism from X onto Y. It follows from Theorems II.1.5 and II.1.11 that T" is an isomorphism. Since $T''J_X = J_Y T$, and both J_X and T" are surjective, it follows that

$$Y'' = R(T''J_X) = R(J_Y T) \subset R(J_Y) .$$

Thus Y is reflexive. Let us only assume that R(T) = Y. Then by II.3.6 T' has a continuous inverse and from II.1.5 we conclude that Y' is isomorphic to T'Y'. Since Y' is complete, T'Y' is also complete and therefore a closed subspace of the reflexive space X'. Hence T'Y' is reflexive by Theorem I.7.3. Thus, by what was just proved, Y' is reflexive and therefore so is Y. The last statement of the theorem now follows, since the map from X onto X/M defined by $x \to [x]$ is bounded and linear. □

IV.2.7 Corollary. *Let the operator range* R *be pre-reflexive. If there exists a bounded linear operator mapping* R *onto a Banach space* Y*, then* Y *is reflexive.*

Proof: Follows trivially from IV.2.6 since if $T \in B[R,Y]$ with $R(T) = Y$, $T\alpha_R$ is a bounded operator from reflexive R_1 onto Y. Hence Y is reflexive. □

IV.2.8 Corollary. *If* M *is a closed subspace of a pre-reflexive space* R*, then* R/M *is pre-reflexive.*

Proof: Assume M is a closed subspace of pre-reflexive space R. Then $\beta_R M$ is a closed subspace of the reflexive space R_1. By Lemma IV.2.6 $R_1/\beta_R M$ is reflexive. It is easily seen that $R_1/\beta_R M$ is the pre-image of R/M hence establishing the corollary. □

The following lemma which we state without proof appears as an exercise in Diestel [05],p.237.

IV.2.9 Lemma. *Let* $T \in WK[X,Y]$ *with* X *and* Y *complete. Then there exists a reflexive space* Z *and operators* $S \in B[X,Z]$, $R \in B[Z,Y]$ *such that* $T = RS$.

We obtain the following "converse" to Proposition IV.2.4.

IV.2.10 Theorem. *Let* R *be an operator range. Then* R *is the image of a weakly compact operator defined on a Banach space iff there is a pre-reflexive space containing* R *and contained in* $\tilde{R}$.

Proof: Suppose $T \in WK[X,\tilde{R}]$ with X complete and $R(T) = R$. From Lemma IV.2.9 it follows that there exist $S \in B[X,Z]$, $U \in B[Z,\tilde{R}]$ such that $T = US$ with Z reflexive. Hence by Lemma IV.2.6 it follows that $Z/N(U)$ is reflexive and since $\hat{U} \in B[Z/N(U),\tilde{R}\}$ is bounded by Theorem I.3.20, we conclude that $R(\hat{U}) = R(U)$ is a pre-reflexive space with pre-image $Z/N(U)$ with $R \subset R(U) \subset \tilde{R}$. Conversely suppose $R \subset W \subset \tilde{R}$ with W pre-reflexive. The map α_W is weakly compact by IV.2.3. Consider the subspace $S = \beta_W(R)$ of W_1. We renorm S with the norm $|||\,.\,||| = \|\,.\,\|_1^W + \|\beta_R\alpha_W\,.\,\|_1^R$ where $\|\,.\,\|_1^W$ and $\|\,.\,\|_1^R$ are the norms on R_1 and W_1 respectively. The injective map, say T , from S into W_1 is easily seen to be bounded, viz.

$$|||y||| = \|y\|_1^W + \|\beta_R\alpha_W y\|_1^R \geqslant \|Ty\|_1^W \quad \text{for} \quad y \in S$$

where Ty is just y considered as an element of W_1 . Since α_W is weakly compact, it follows that $\alpha_W T \in WK[S,R]$ with $R(\alpha_W T) = R$. Consequently if we can prove S to be complete under the norm $|||\,.\,|||$, we are done. Let $\{x_n\}$ be a Cauchy sequence in $(S, |||.|||)$. From the definition of $|||\,.\,|||$ it follows easily that $\{x_n\}$ and $\{\beta_R\alpha_W x_n\}$ are Cauchy sequences in both W_1 and R_1 respectively. It follows that

$$\beta_R\alpha_W x_n \to x \in R_1 \quad \text{and} \quad x_n \to w \in W_1$$

and hence

$$\alpha_W x_n \to \alpha_R x \in R \subset W$$

$$\text{and} \quad \alpha_W x_n \to \alpha_W w \in W .$$

Obviously $\alpha_R x = \alpha_W w \in R \subset W$ and hence $x = \beta_R \alpha_W w$, from which it easily follows that $x_n \to w$ in S with w considered as an element of S. We conclude that S must be complete. □

IV.2.11 Remark. Theorem IV.2.10 immediately implies the factorisability of a weakly compact operator as stated in Lemma IV.2.9. Let $T \in WK[X,Y]$ with X and Y complete. By Theorem IV.2.10 it now follows that there exists a pre-reflexive subspace W of Y such that

$$R(T) \subset W \subset \overline{R(T)} \subset Y .$$

Let U be α_W considered as a map into Y and let S be $\beta_W T$. U is obviously an element of $B[W_1,Y]$ and since T is closed $(D(T) = X)$, $\beta_W T$ is therefore closed by I.4.7. Hence by the closed graph theorem $S \in B[X,W_1]$ with $T = US$ and W_1 reflexive.

We state the following theorem which appears in Goldberg [11] without proof. For a proof the reader is referred to Dunford and Schwartz [07], Theorem VI.8.12, the remark preceding the theorem on p.508, the remark following Theorem VI.8.14 on pp.510-511 and the representation of conjugate spaces in tables on pp.374-379.

IV.2.12 Theorem. *Every weakly compact map from* X *or* X' *into a Banach space takes weakly convergent sequences onto norm convergent sequences whenever* X *is any one of the following Banach spaces:*

(i) $L_1(S,\Sigma,\mu)$ *and* $L_\infty(S,\Sigma,\mu)$, *where* (S,Σ,μ) *is a positive measure space.*

(ii) $C(S)$, *where* S *is a compact Hausdorff space.*

(iii) $B(S)$, *where* S *is an arbitrary set.*

Thus every weakly compact operator from X *or* X' *into a Banach space is strictly singular.*

IV.2.13 Corollary. *Let* X *be any one of the spaces in Theorem IV.2.12 Then every bounded linear map from* X *or* X' *into a pre-reflexive space is strictly singular.*

Proof: Follows immediately from Proposition IV.2.4 and Theorem IV.2.12. □

IV.2.14 Lemma. *Let* X *and* Y *be reflexive. Then* $X \times Y$ *is reflexive.*

Proof: Assume X and Y to be reflexive and let $\{(x_n, y_n)\}$ be a bounded sequence. From the definition of the norm on $X \times Y$ (I.3.9) it follows easily that $\{x_n\}$ and $\{y_n\}$ are bounded in X and Y respectively. From Theorem I.7.7 it is clear that there exists a subsequence $\{(u_n, v_n)\}$ of $\{(x_n, y_n)\}$ such that $x' u_n \to x'x$ for each $x' \in X'$ and some $x \in X$ and hence a subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(u_n, v_n)\}$ such that $x'u_{n_k} \to x'x$ and $y'v_{n_k} \to y'y$ for each $x' \in X'$ and $y' \in Y'$ and x and y elements of X and Y respectively. Now let z' be an arbitrary element of $(X \times Y)'$ and let x' and y' be elements of X' and Y' defined by

$$z'(p,0) = x'p \quad \text{for} \quad p \in X$$

$$\text{and} \qquad z'(0,q) = y'q \quad \text{for} \quad q \in Y \ . \ .$$

It is now obvious that for the subsequence $\{(u_{n_k}, v_{n_k})\}$ of $\{(x_n, y_n)\}$ we have

$$z'(u_{n_k}, v_{n_k}) \to x'x + y'y = z'(x,y) \quad \text{where}$$

$(x,y) \in X \times Y$. Consequently since $z' \in (X \times Y)'$ was arbitrary we conclude from Theorem I.7.7 that $X \times Y$ is reflexive. □

IV.2.15 Proposition. *Let* X *and* Y *be pre-reflexive subspaces of a Banach space* Z . *Then* $X + Y$ *and* $X \cap Y$ *are pre-reflexive.*

Proof: Let X and Y be pre-reflexive subspaces of a Banach space Z . By definition the pre-images X_1 and Y_1 of X and Y are reflexive and hence from the lemma it follows that $X_1 \times Y_1$ is reflexive. Now define the map T from $X_1 \times Y_1$ onto $X + Y$ as follows

$$T(x_1,y_1) = \alpha_X x_1 + \alpha_Y y_1 \quad \text{with } x_1 \in X_1 \text{ and } y_1 \in Y_1 .$$

$(X_1 \times Y_1)/N(T)$ is reflexive by Lemma IV.2.6 and since $\hat{T}$ is bounded by Theorem I.3.20 it follows easily that $X + Y$ is pre-reflexive with pre-image $(X_1 \times Y_1)/N(T)$. Now consider the subspace

$$D = \{(x_1,y_1): x_1 \in X_1,\ y_1 \in Y_1,\ \alpha_X x_1 = \alpha_Y y_1 \in X \cap Y\} .$$

It is easily verified that D is a complete and therefore closed subspace of $X_1 \times Y_1$ and hence from Theorem I.7.3 we conclude that D is reflexive. Since $T|_D$ is a bounded surjection from the space D onto $X \cap Y$ it follows as before that $X \cap Y$ is pre-reflexive with reflexive pre-image $D/N(T|_D)$. □

Corollaries IV.2.7, IV.2.8 and IV.2.13 as well as Proposition IV.2.4 are by the author. Theorem IV.2.10, Remark IV.2.11 and Proposition IV.2.15 were established with the assistance of Dr. R. W. Cross.

CHAPTER V

PERTURBATION THEORY

As in chapter III most of the results in this chapter are generalisations of Banach space results appearing in Goldberg [11]. We start off by looking at perturbation by continuous operators, and then by strictly singular operators, after which we take a closer look at the instability of the set $C(X,Y) \setminus SF(X,Y)$ under compact and α-compact perturbations.

V.1 Perturbation by Continuous Operators

V.1.1 Lemma. *Let* $T \in L(X,Y)$ *with* $D(T)$ *dense in* X . *Then*

(i) $a(T') = \dim Y/\overline{R(T)}$ (= $b(T)$ *if* $R(T)$ *is closed)*

(ii) if X *is an operator range,* Y *is complete and* $T \in NS(X,Y)$, *then* $a(T) = b(T')$ *and hence if* T *has an index,* $K(T) = -K(T')$ *where we let* $\infty = -(-\infty)$.

Proof:

(i) Assume $T \in L(X,Y)$ with $D(T)$ dense in X . Then

(1) $\dim Y/\overline{R(T)} = \dim (Y/\overline{R(T)})' = \dim R(T)^{\perp} = \dim N(T') = a(T')$

by Theorem I.5.16 and II.1.10.

(ii) Assume also that X is an operator range, Y complete and $T \in NS(X,Y)$. Then from Theorems I.5.16 and III.2.9 we conclude that

(2) $b(T') = \dim X'/R(T') = \dim X'/N(T)^{\perp} = \dim N(T)' = \dim N(T) = a(T)$.

Hence if $K(T)$ exists we see from (1) and (2) that

$K(T) = a(T) - b(T) = b(T') - a(T') = -(a(T') - b(T')) = -K(T')$. □

V.1.2 Lemma [11]. *Let* $N(T)$ *be closed and let* $D(T)$ *be dense in* X. *If* $\gamma(T) > 0$, *then* $\gamma(T) = \gamma(T')$ *and* T' *has a closed range.*

Proof: Let $Y_1 = R(T)$ and let T_1 be the operator T considered as a map onto Y_1. Then by Theorem III.1.3 the bijection $\hat{T}_1$ has a bounded inverse, since $\gamma(T_1) = \gamma(T) > 0$. Hence, by Theorems II.1.10 and II.1.11, $(\hat{T}_1')^{-1}$ exists and is equal to $(\hat{T}_1^{-1})'$, and by Theorem II.1.5 it now follows that $(\hat{T}_1')^{-1}$ is bounded. Before evaluating $\gamma(T')$ we need the following observations:

(i) By Theorems I.5.16 and II.1.10, $Y'/N(T') = Y'/R(T)^{\perp}$ and $Y'/R(T)^{\perp}$ is isometric to Y_1' under the map $[y'] \to y_R'$, where y_R' is the restriction of y' to Y_1. In particular $\|[y']\| = \|y_R'\|$.

(ii) It is clear that $\|(\hat{T})'y'\| = \|(\hat{T}_1)'y_R'\|$. Thus, for $y' \in D(T')$, $\|(\hat{T}_1)'y_R'\| = \|T'y'\|$ by Lemma III.2.8.

(iii) By Theorems II.1.5 and II.1.11, $\|((\hat{T}_1)')^{-1}\| = \|(\hat{T}_1^{-1})'\| = \|\hat{T}_1^{-1}\|$.

From (i), (ii), and (iii) it follows that

$$\gamma(T') = \inf_{[y'] \in D(\hat{T}')} \frac{\|T'y'\|}{\|[y']\|} = \inf_{y_R' \in Y_1'} \frac{\|(\hat{T}_1)'y_R'\|}{\|y_R'\|} = \frac{1}{\|((\hat{T}_1)')^{-1}\|}$$

$$= \frac{1}{\|\hat{T}_1^{-1}\|} = \gamma(T)$$

Thus by III.1.4, T' has a closed range since $\gamma(T') > 0$ and T' is closed.

We will state the following lemma, due to Krein, Krasnoselskii and Milman ([15]; cf. [11], p.110), without proof.

V.1.3 Lemma. *Let* M *and* N *be subspaces of* X *with* $\dim M > \dim N$ *(hence* $\dim N < \infty$ *). Then there exists an* $m \neq 0$ *in* M *such that*

$$\|m\| = d(m,N) .$$

V.1.4 Lemma [11]. *Suppose* $\gamma(T) > 0$. *Let* B *be continuous with* $D(B) \supset D(T)$. *If* $\|B\| < \gamma(T)$ *, then*

(i) $a(T + B) \leqslant a(T)$

(ii) $\dim Y/\overline{R(T + B)} \leqslant \dim Y/\overline{R(T)}$.

Proof:

(i) For $x \neq 0$ in $N(T + B)$ and $\|B\| < \gamma = \gamma(T)$,

$$\gamma \|[x]\| \leqslant \|Tx\| = \|Bx\| \leqslant \|B\| \|x\| < \gamma \|x\|$$

where $[x] \in X/N(T)$. Thus $\|x\| > \|[x]\| = d(x,N(T))$, and therefore, by Lemma V.1.3, $a(T + B) \leqslant a(T)$.

(ii) Let $X_0 = \overline{D(T)}$ and let B_1 be B restricted to $D(T)$. Considering T and B_1 as operators with domains dense in X_0 the conjugates T' and B_1' exist with domains in Y' and ranges in X_0' . By Theorem II.1.5 and Lemma V.1.2

$$\gamma(T') = \gamma(T) > \|B\| \geqslant \|B_1\| = \|B_1'\| .$$

Hence, it follows from (i) applied to T' and B_1' and from Lemma V.1.1 that

$$\dim \frac{Y}{\overline{R(T+B)}} = \dim \frac{Y}{\overline{R(T+B_1)}} = a(T' + B_1') \leqslant a(T') = \dim \frac{Y}{\overline{R(T)}} .$$

□

V.1.5 Lemma [11]. *Suppose* T *has a continuous inverse. If* B *is continuous with* $\|B\| < \gamma = 1/\|T^{-1}\|$ *, then* $T + B$ *has a continuous inverse. If, in addition,* $D(B) \supset D(T)$ *, then*

$$\dim \frac{Y}{\overline{R(T)}} = \dim \frac{Y}{\overline{R(T+B)}} .$$

Proof: For $x \in D(T + B)$ choose a positive integer N so that

$$\frac{\|B\|}{N} < \gamma - \|B\| .$$

Now for $0 \leqslant k \leqslant N$,

$$\|(T + B - \frac{k}{N} B)x\| \geqslant \|Tx\| - \|(1 - \frac{k}{N}) Bx\| \geqslant (\gamma - \|B\|) \|x\| .$$

Thus $T + B$ has a continuous inverse and

$$\gamma(T + B - \frac{k}{N} B) \geqslant \gamma - \|B\| \quad \text{for} \quad 0 \leqslant k \leqslant N . \tag{1}$$

Since $\|(1/N)B\| < \gamma - \|B\|$, Lemma V.1.4, together with (1), implies that for $0 \leqslant k \leqslant N$ and $\bar{b}(T) = \dim Y/\overline{R(T)}$,

$$\begin{aligned} \bar{b}(T + B - \frac{k+1}{N} B) &= \bar{b}(T + B - \frac{k}{N} B - \frac{1}{N} B) \\ &\leqslant \bar{b}(T + B - \frac{k}{N} B) . \end{aligned} \tag{2}$$

It follows from (2) and Lemma V.1.4 that

$$\bar{b}(T) \leqslant \bar{b}(T + B) \leqslant \bar{b}(T)$$

and hence proving the lemma. □

V.1.6 Lemma [11]. *Suppose that* T_1 *is a linear extension of* T *such that* $\infty > n = \dim D(T_1)/D(T)$. *Then*

(i) if T *is closed,* T_1 *is closed*

(ii) if T *has a closed range,* T_1 *has a closed range*

(iii) if T *has an index,* $K(T_1) = K(T) + n$.

Proof:

(i) By hypothesis, $D(T_1) = D(T) \oplus N$, where N is a finite-dimensional subspace. Hence, $G(T_1) = G(T) + Z$, where $G(T)$ and $G(T_1)$ are the graphs of T and T_1 respectively, and $Z = \{(n, T_1 n) \mid n \in N\}$. Thus, if $G(T)$ is closed, then $G(T_1)$ is closed, since Z is finite-dimensional.

(ii) If $R(T)$ is closed, then $R(T_1)$ is closed, since

$$R(T_1) = R(T) + T_1N$$

and T_1N is finite-dimensional.

(iii) It is easy to see that it suffices to prove (iii) for the case when $n = 1$. Suppose that $D(T_1) = D(T) \oplus \mathrm{sp}\,\{x\}$, for some $x \in D(T_1)$. Then $T_1X = TX \oplus V$, where $V = \mathrm{sp}\,\{T_1x\}$ when $T_1x \notin R(T)$ or $V = \{0\}$ when $T_1x \in R(T)$.

If $T_1x \notin R(T)$, then it is easy to verify that $b(T) = b(T_1) + 1$ and that $N(T) = N(T_1)$. Therefore $K(T_1) = K(T) + 1$.

If $T_1x \in R(T)$, then $R(T_1) = R(T)$ and there exists a $z \in D(T)$ such that $Tz = T_1x$. Hence $N(T_1) = N(T) \oplus \mathrm{sp}\,\{x - z\}$. Thus $a(T_1) = a(T) + 1$ and $K(T_1) = K(T) + 1$. □

We now give a partial generalisation to operator ranges of a theorem due to Kato (c f. [11] , p.112).

V.1.7 Theorem. *Let* $T \in SF(X,Y)$ *and* $S \in B(X,Y)$ *with* $D(T) \subset D(S)$. *If* X *is an operator range and if* $\|S\| < \frac{1}{\|\alpha_X\|}\,\gamma(T\alpha_X)$ *then*

(i) $T + S$ *is closed if either* Y *is complete or if* $\overline{D(T)} \subset D(S)$

(ii) $T + S$ *closed implies* $T + S$ *normally solvable*

(iii) $T + S$ *closed implies* $a(T + S) \leqslant a(T)$ *and* $b(T + S) \leqslant b(T)$

(iv) $K(T + \lambda S) = K(T)$ *for any* $\lambda \in [0,1]$ *whenever* $\|S\| < \frac{1}{\|\alpha_X\|}\,\gamma(T\alpha_X)$ *and* Y *is complete.*

Proof:

(i) Assume either Y to be complete or $\overline{D(T)} \subset D(S)$ where $T \in C(X,Y)$ and $S \in B(X,Y)$ such that $D(T) \subset D(S)$. It is then easy to verify that $T + S$ is closed.

(ii) Assume $T + S$ to be closed and X an operator range. Now let $a(T) = a(T\alpha_X) < \infty$. If $R(T + S) = R(T\alpha_X + S\alpha_X)$ is not closed it follows from Corollary IV.1.11 that there exists a subspace M of $D(T\alpha_X + S\alpha_X)$ such that $\dim M = \infty$ and

$$\|(T + S)\alpha_X x\| < (\gamma(T\alpha_X) - \|S\|\|\alpha_X\|)\|x\| \quad \text{for all} \quad 0 \neq x \in M .$$

Hence for each $x \neq 0$ in M

$$\gamma(T\alpha_X)\ d(x,N(T\alpha_X)) \leq \|T\alpha_X x\| \leq \|S\alpha_X x\| + \|(T + S)\alpha_X x\|$$
$$< (\|S\|\|\alpha_X\| + \gamma(T\alpha_X) - \|S\|\|\alpha_X\|)\ \|x\| .$$

Consequently $d(x,N(T\alpha_X)) < \|x\|$ for each $0 \neq x \in M$ and hence $\dim M \leq a(T\alpha_X) < \infty$ by Lemma V.1.3, contradicting the fact that $\dim M = \infty$. Hence $T + S$ is normally solvable.

Suppose now $b(T) = b(T\alpha_X) < \infty$. Let $X_0 = \overline{D(T\alpha_X)}$ and let S_1 be $S\alpha_X$ restricted to $D(T\alpha_X)$ and let T_1 be $T\alpha_X$ considered as an operator from X_0 into Y. It now follows from II.1.5 and V.1.2 that

$$\gamma(T_1') = \gamma(T_1) = \gamma(T\alpha_X) > \|S\|\|\alpha_X\| \geq \|S\alpha_X\| \geq \|S_1\| = \|S_1'\| .$$

Moreover since $a(T_1') = b(T_1) = b(T\alpha_X)$ by V.1.1, it follows from the first part of the proof that $T_1' + S_1'$ is normally solvable since T_1' is normally solvable by Lemma V.1.2. It is obvious that $D(T_1' + S_1') \subset D((T_1 + S_1)')$. Now let $y' \in D((T_1 + S_1)')$. Then $y'(T_1 + S_1)$ is continuous on $D(T_1) = D(T_1 + S_1)$ by definition. But since S_1 is continuous, it follows that

$$|y'T_1x| \leq |y'(T_1 + S_1)x| + |y'S_1x| \leq (\|y'(T_1 + S_1)\| + \|y'S_1\|)\ \|x\|$$

$$\text{for} \quad x \in D(T_1)$$

and hence $y'T_1$ is continuous, i.e. $y' \in D(T_1' + S_1')$. We conclude thet $T_1' + S_1' = (T_1 + S_1)'$ and consequently, since $X_0 = \overline{D(T_1)} \subset X_1$ is complete, $R(T_1 + S_1) = R(T + S)$ is closed by Theorem III.2.9. Hence (ii) is proved.

(iii) Suppose X is an operator range. If $T + S$ is closed, (iii) is a trivial consequence of Lemma V.1.4 and the fact that $R(T)$ and $R(T + S)$ are closed by (ii).

(iv) Assume Y complete. We start off by showing that $K(T + S) = K(T)$ for $\|S\|$ sufficiently small.

Suppose $a(T) = a(T\alpha_X) < \infty$. It then follows from I.6.9 that there exists a bounded projection $P \in B[X, N(T)]$ and a closed subspace $M = N(P)$ such that $P(X) = N(T)$ and hence $X = M \oplus N(T)$. Let T_M be T restricted to $M \cap D(T)$. T_M is obviously closed and injective with $R(T_M) = R(T)$ closed and by the generalised closed graph theorem T_M^{-1} is continuous. Now since $X = M \oplus N(T)$, it follows easily that $X_1 = \beta_X M \oplus N(T\alpha_X)$ with $\beta_X M$ and $N(T\alpha_X)$ both closed. $\beta_X M$ is obviously the pre-image of M. It now follows easily from (i), (ii), (iii) and Lemma V.1.5 that

(1) $\quad a(T_M + S) = a(T_M) = 0 \qquad b(T_M + S) = b(T)$

provided $\|S\| < \frac{1}{\|\alpha_M\|} \gamma(T_M\alpha_M)$. ($\gamma(T_M\alpha_M)$ is obviously greater than zero by III.1.4 since X_1 and Y are complete, $R(T_M) = R(T)$ closed and $T_M\alpha_M$ closed.) Now

$$D(T + S) = D(T) = D(T) \cap M \oplus N(T)$$

$$D(T_M + S) = D(T_M) = D(T) \cap M .$$

Hence from Lemma V.1.6 and (1) we have that

$$K(T) = K(T_M) + a(T) = K(T_M + S) + a(T) = K(T + S) .$$

Suppose $b(T) = b(T\alpha_X) < \infty$.

Define T_1 and S_1 as in the proof of (ii). As in (ii) we have $a(T_1') = b(T_1) = b(T) < \infty$ and hence from the first part of the proof applied to T_1' and S_1' we have

$$K(T_1') = K(T_1' + S_1') \quad \text{for } \|S\| \text{ sufficiently small.}$$

Consequently from Lemma V.1.1 we conclude that

$$K(T) = K(T\alpha_X) = -K(T_1') = -K(T_1' + S_1') = K(T\alpha_X + S\alpha_X)$$
$$= K(T + S)$$

for $\|S\|$ sufficiently small.

Now let $I = [0,1]$ and let Z denote the set of integers together with ∞ and $-\infty$. Define $\phi: I \to Z$ by $\phi(\lambda) = K(T + \lambda S)$ and let I have the usual topology and Z the discrete topology. From what was shown above we see that

$$\phi(\lambda) = K(T + \lambda_0 S + (\lambda - \lambda_0)S) = K(T + \lambda_0 S) = \phi(\lambda_0)$$

for λ sufficiently close to λ_0. Hence ϕ is continuous from which it follows that $\phi(I)$ is a connected set consisting of only one point, that is

$$K(T) = K(T + \lambda S) \quad \text{for } \lambda \in [0,1] \text{ if } \|S\| < \frac{1}{\|\alpha_X\|} \gamma(T\alpha_X) .$$

□

V.1.8 Corollary. *Let* $T \in L(X,Y)$ *with* $D(T)$ *dense,* X *an operator range,* Y *complete and* $N(T)$ *closed. If* $\gamma(T) > 0$ *and* $S \in B(X,Y)$ *satisfies* $D(S) \supset D(T)$ *and* $\|S\| < \gamma(T)$, *then* $T' + S'$ *is normally solvable with* $a(T') \geqslant a(T' + S')$, $b(T') \geqslant b(T' + S')$, *and* $K(T') = K(T' + \lambda S')$ *for* $\lambda \in [0,1]$ *whenever* $K(T')$ *exists.*

Proof: It follows from Theorem II.1.5 and Lemma V.1.2 that $\gamma(T') = \gamma(T) > \|S\| = \|S'\|$ with T' normally solvable. The corollary now follows on applying the theorem. □

We can now state the following corollary, a special case of which appears in Goldberg [11], p.114. We reproduce the proof given in [11] with minor modifications.

V.1.9 Corollary. *Let* $T \in SF(X,Y)$ *with* X *an operator range and* Y *complete. If* $S \in B(X,Y)$ *with* $D(S) \supset D(T)$, *then there exists a number* $\rho > 0$ *such that* $a(T + \lambda S)$ *and* $b(T + \lambda S)$ *are constant in the annulus* $0 < |\lambda| < \rho$.

Proof: The proof of the corollary is a modification of Lemma 8.1 in Gokhberg and Krein [08], where S was taken to be the identity operator. We first assume $a(T) < \infty$. For $x \in N(T + \lambda S)$ and $\lambda \neq 0$,

$$Tx = -\lambda Sx$$

whence

$$Sx \in R(T) = R_1 \quad \text{and} \quad x \in S^{-1}R_1 = D_1 .$$

Thus

$$-\lambda Sx = Tx \in TD_1 = R_2 \quad \text{and} \quad x \in S^{-1}R_2 = D_2 .$$

It follows that

$$(1) \qquad N(T + \lambda S) \subset \bigcap_{k=1}^{\infty} D_k$$

where

$$D_k = S^{-1}R_k \qquad R_1 = R(T)$$

and

$$R_{k+1} = TD_k .$$

It is easy to see that

$$R_1 \supset R_2 \supset \dots \quad \text{and} \quad D_1 \supset D_2 \supset \dots .$$

We show by induction that R_n and D_n are closed subspaces of Y and $D(S)$ respectively. R_1 is closed by hypothesis, and D_1 is closed since S is continuous. Suppose R_k and D_k are closed. By Corollary III.3.7,

$R_{k+1} = TD_k$ is closed and therefore $D_{k+1} = S^{-1}R_{k+1}$ is closed since S is continuous. Hence D_n and R_n are closed by induction. Define

$$X_0 = \bigcap_{k=1}^{\infty} D_k \qquad \text{and} \qquad Y_0 = \bigcap_{k=1}^{\infty} R_k \ .$$

It follows from the definition of D_k and R_k that $TX_0 \subset Y_0$ and $SX_0 \subset Y_0$. Let T_1 and S_1 be the operators T and S respectively, restricted to $D(T) \cap X_0$ with ranges in Y_0. Since T is a closed operator and X_0 is closed in $D(T)$, T_1 is also closed. We show that $R(T_1) = Y_0$.

Let y be an element in $Y_0 = \bigcap_{n=1}^{\infty} TD_n$. Then for each $n \geqslant 1$ there exists an $x_n \in D_n$ such that $Tx_n = y$. Since $N(T)$ is finite-dimensional and $D_n \supset D_{n+1}$, there exists an integer k_0 such that

$$N(T) \cap D_{k_0} = N(T) \cap D_k \quad \text{for} \quad k \geqslant k_0 \ .$$

From the way the sequence $\{x_k\}$ was chosen, together with the fact that $D_k \subset D_{k_0}, k \geqslant k_0$, it follows that

$$x_k - x_{k_0} \in N(T) \cap D_{k_0} = N(T) \cap D_k \subset D_k \qquad k \geqslant k_0 \ .$$

Hence $x_{k_0} \in \bigcap_{k \geqslant k_0} D_k = X_0$ and $Tx_{k_0} = y$, which shows that T_1 is surjective. Thus there exists by Theorem V.1.7, a number $\rho > 0$ such that for $|\lambda| < \rho$,

(2) $$K(T + \lambda S) = K(T)$$

(3) $$b(T_1 + \lambda S_1) = b(T_1) = 0 \quad \text{and} \quad a(T_1 + \lambda S_1) = K(T_1) = a(T_1) \ .$$

Now, (1) implies that $N(T + \lambda S) = N(T_1 + \lambda S_1)$ for $\lambda \neq 0$. In particular,

(4) $$a(T + \lambda S) = a(T_1 + \lambda S_1) \qquad \lambda \neq 0 \ .$$

It is clear from (2), (3) and (4) that $a(T + \lambda S)$ and $b(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \rho$ under the assumption $a(T) < \infty$. If $a(T) = \infty$, then $b(T) < \infty$ by hypothesis. As in the proof of Theorem V.1.7 we apply the result just proved to the appropriate conjugate operators in order to prove the corollary when $a(T) = \infty$.

V.1.10 Corollary. *Let* $T \in L^{-}(X,Y)$*,* $S \in B(X,Y)$*, with* $N(T)$ *closed,* $\gamma(T) > 0$ *and* $D(S) \supset D(T)$. *If there exists a number* $\mu > 0$ *such that* $R(T + \lambda S)$ *is closed for* $|\lambda| < \mu$ *then there exists a number* $\rho > 0$ *such that* $b(T + \lambda S)$ *is constant in the annulus* $0 < |\lambda| < \rho$.

Proof: Suppose there exists a $\mu > 0$ such that $R(T + \lambda S)$ is closed for $|\lambda| < \mu$. Let $X_0 = \overline{D(T)}$ and let S_1 be S restricted to $D(T)$ and consider T to be from X_0 to Y. By Lemmas V.1.2 and V.1.1 it now follows that $b(T) = a(T') < \infty$, $\gamma(T') = \gamma(T) > 0$ and T' is normally solvable. From Theorem II.1.5 we see that $\|S_1'\| = \|S_1\|$, so S_1' is bounded and hence from Corollary V.1.9 applied to T' and S' we conclude that there exists a $\nu > 0$ such that $a(T' + \lambda S_1')$ is constant in the annulus $0 < |\lambda| < \nu$. Let $\rho = \text{minimum } \{\nu, \mu\}$. It then follows from the hypothesis and Lemma V.1.1 that $b(T + \lambda S) = a(T' + \lambda S_1')$ is constant in the annulus $0 < |\lambda| < \rho$ thereby proving the corollary. □

The following proposition is a generalisation of ([11], V.1.8, p.116):

V.1.11 Proposition. *Let* $S \in B(X,Y)$*,* $T \in L(X,Y)$ *with* $D(S) \supset D(T)$, Y *complete and* X *an operator range. If* U *is the set of* λ *for which* $T + \lambda S \in SF(X,Y)$*, then* *(i)* U *is an open set*

(ii) if C *is a component of* U *(largest connected subset of* U *), then* $a(T + \lambda S)$ *and* $b(T + \lambda S)$ *will have constant values on* C *, say* n_1 *and* n_2 *, except for isolated points. At the isolated points*

$$a(T + \lambda S) > n_1 \quad \textit{and} \quad b(T + \lambda S) > n_2$$

Proof:

(i) Assume $\lambda \in U$, i.e. $T + \lambda S$ normally solvable with an index. (i) is now a trivial consequence of Theorem V.1.7 applied to $T + \lambda S$ since $\gamma(T\alpha_X + \lambda S\alpha_X) > 0$ by III.1.4 and hence we can find an $\varepsilon > 0$ such that $T + (\lambda + \varepsilon)S \in SF(X,Y)$.

(ii) Assume C to be a component of U . The component C is open, since any component of an open set in the space of scalars is open. Let $a(\lambda_0) = n_1$ be the smallest integer which is attained by $a(\lambda) = a(T + \lambda S)$ on C . Suppose $a(\lambda') \neq n_1$. Owing to the connectivity of C , there exists an arc Γ lying in C with endpoints λ_0 and λ' . It follows from Corollary V.1.9 and the fact that C is open, that about each $\mu \in \Gamma$ there exists an open ball $U_C(\mu,r)$ contained in C such that $a(\lambda)$ is constant on the set $U_C(\mu,r)$ with the point μ deleted.
Since Γ is compact and connected, there exist points

$$\lambda_1, \lambda_2, \ldots, \lambda_n = \lambda'$$

on Γ such that

(1) $U_C(\lambda_0,r_0), U_C(\lambda_1,r_1) \ldots, U_C(\lambda_n,r_n)$ cover Γ and
$U_C(\lambda_i,r_i) \cap U_C(\lambda_{i+1},r_{i+1}) \neq \emptyset \qquad 0 \leqslant i \leqslant n-1$.

We assert that $a(\lambda) = a(\lambda_0)$ on all of $U_C(\lambda_0,r_0)$. It follows from Theorem V.1.7 that $a(\lambda) \leqslant a(\lambda_0)$ for λ sufficiently close to λ_0. Therefore, since $a(\lambda_0)$ is the minimum of $a(\lambda)$ on C, $a(\lambda) = a(\lambda_0)$ for λ sufficiently close to λ_0. Since $a(\lambda)$ is constant for all $\lambda \neq \lambda_0$ in $U_C(\lambda_0, r_0)$, this constant must be $a(\lambda_0)$. Now $a(\lambda)$ is constant on the set $U_C(\lambda_i, r_i)$ with the point λ_i deleted, $1 \leqslant i \leqslant n$. Hence, it follows from (1) and the observation $a(\lambda) = a(\lambda_0)$ for all $\lambda \in U_C(\lambda_0, r_0)$, that $a(\lambda) = a(\lambda_0)$ for all $\lambda \neq \lambda'$ in $U_C(\lambda', r_n)$ and $a(\lambda') > n_1$. The result just obtained can be applied to $T_1' + \lambda S_1'$, as in Theorem V.1.7, in order to prove the analogous results for

$$b(T + \lambda S) = a(T_1' + \lambda S_1') \quad . \qquad \square$$

V.1.12 Corollary. *Let* $T \in L(X,Y)$, $S \in B(X,Y)$ *with* $D(S) \supset \overline{D(T)}$, X *an operator range and* $\gamma(T\alpha_X + \lambda S\alpha_X) > 0$ *for all scalars* λ.

If U *is the set of all scalars* λ *such that* $T + \lambda S \in SF(X,Y)$, *then*

(i) U *is open*

(ii) if C *is a component of* U, *then* $b(T + \lambda S)$ *will have a constant value on* C, *say* n, *except for isolated points. At the isolated points*

$$b(T + \lambda S) > n \quad \textit{provided} \quad n < \infty \, .$$

Proof:

(i) Assume U to be the set of all λ such that $T + \lambda S \in SF(X,Y)$). Since $\gamma(T\alpha_X + \lambda S\alpha_X) > 0$ we can apply Theorem V.1.7 as $\frac{1}{\|\alpha_X\|}\, \gamma(T\alpha_X + \lambda S\alpha_X)$ will still be greater than zero. Hence since $T + \lambda S$ is closed by hypothesis the result now follows from Theorem V.1.7.

(ii) Let C be any component of U. Observe that $N(T\alpha_X + \lambda S\alpha_X)$ must be closed for each $\lambda \in U$ since $T\alpha_X + \lambda S\alpha_X \in NS(X_1,Y)$ by III.3.3. From Lemma V.1.1 we notice that $b(T + \lambda S) = b(T\alpha_X + \lambda S\alpha_X) = a(T_1' + \lambda S_1')$ where T_1 and S_1 are defined as in the proof of V.1.7. By an argument analogous to that in Proposition V.1.11 being applied to T_1' and S_1' the result now follows with use being made of Corollary V.1.10 instead of V.1.9. □

Corollaries V.1.12, V.1.10 and V.1.8, the generalisation of Theorem V.1.7, Lemma V.1.1, Corollary V.1.9 and Proposition V.1.11 are by the author.

V.2 Perturbation by Strictly Singular Operators

The following theorem, due to Kato, is known for Banach spaces (cf. [11]). We give a partial generalisation to operator ranges.

V.2.1 Theorem. *Let* $T \in NS(X,Y)$ *with* X *an operator range,* Y *complete and* $a(T) < \infty$. *If* $B \in SS(X,Y)$ *with* $D(B) \supset D(T)$ *then*

(i) $T + B \in NS(X,Y)$

(ii) $K(T + B) = K(T)$

(iii) $a(T + \lambda B)$ *and* $b(T + \lambda B)$ *have constant values* n_1 *and* n_2 *respectively, except perhaps for isolated points. At the isolated points*

$$\infty > a(T + \lambda B) > n_1 \text{ and } b(T + \lambda B) > n_2 .$$

Proof:

(i) Assume $B \in SS(X,Y)$ with $D(T) \subset D(B)$. $T + B$ is obviously closed since Y is complete. Since $a(T) = a(T\alpha_X) < \infty$, there exists a closed subspace M_1 of X_1 such that $X_1 = M_1 \oplus N(T\alpha_X)$. Let T_{M_1} be $T\alpha_X$ restricted to M_1. T_{M_1} is obviously closed since $T\alpha_X$ is closed and M_1 is closed and hence since

$$R(T_{M_1}) = R(T\alpha_X) = R(T) ,$$

T_{M_1} is normally solvable with $\gamma(T_{M_1}) > 0$ by III.1.4. Suppose that $R(T + B) = R(T\alpha_X + B\alpha_X)$ is not closed. It follows from Lemma V.1.6 that $R(T_{M_1} + B\alpha_X)$ is then also not closed. Therefore since $T + B$ and thus $T\alpha_X + B\alpha_X$ and $T_{M_1} + B\alpha_X$ are closed, we conclude from Corollary IV.1.11 that there exists an infinite-dimensional subspace M_0 of $D(T_{M_1}) = D(T_{M_1} + B\alpha_X)$ such that

$$\|(T_{M_1} + B\alpha_X)x\| < (\gamma(T_{M_1})/2)\|x\| \quad \text{for} \quad x \in M_0 .$$

Moreover since T_{M_1} is injective, it follows that

$$\|B\alpha_X x\| \geq \|T_{M_1} x\| - \|(T_{M_1} + B\alpha_X)x\| \geq (\gamma(T_{M_1})/2)\|x\| \text{ for any } x \in M_0 ,$$

and hence $B\alpha_X$ is not strictly singular since it has a continuous inverse on M_0. Since $B \in SS(X,Y)$ iff $B \in SS[D(B),Y]$ and $\alpha_X|_{D(B\alpha_X)} \in B[D(B\alpha_X),D(B)]$ we conclude from Theorem IV.1.8 that $B\alpha_X$ is strictly singular, a contradiction.. Hence $R(T + B)$ is closed.

(ii) We now prove $a(T + B) < \infty$. Since $N(T + B) \cap N(T)$ is finite-dimensional, we conclude from I.6.9 that there exists a closed subspace N of X such that

$$N(T + B) = N(T + B) \cap N(T) \oplus N .$$

Let T_N be T restricted to N. Since $\gamma(T) > 0$ by Theorem III.1.4, we conclude from III.3.6 that TN is a closed and hence complete subspace of the Banach space Y . Therefore since T_N is injective we conclude from the generalised closed graph theorem that T_N^{-1} is continuous. Since $T_N = -B$ on N it is clear from the strict singularity of B that $\dim N < \infty$ and hence $a(T + B) < \infty$. Since B is strictly singular iff λB is strictly singular we conclude from (i) and the above that $T + \lambda B$ is normally solvable with $a(T + \lambda B) < \infty$ for all λ . Consequently applying Theorem V.1.7 to $T + \lambda B$, it follows that $\phi(\lambda) = K(T + \lambda B)$ is continuous from $[0,1]$ into Z where $[0,1]$ has the usual topology and Z is the set of integers and $-\infty$ with the discrete topology. Hence ϕ is a constant function. In particular,

$$K(T) = \phi(0) = \phi(1) = K(T + B) .$$

(iii) Since $T + \lambda B \in SF(X,Y)$ with $a(T + \lambda B) < \infty$ for all λ , (iii) is a trivial consequence of Proposition V.1.11. □

V.2.2 Corollary. *Let* $T \in NS(X,Y)$ *with* $a(T) < \infty$, X *an operator range and* $\Upsilon(T\alpha_X) > 0$. *If* $B \in SS(X,Y)$ *and* $D(B) \supset \overline{D(T)}$, *then* $T + B$ *is normally solvable.*

Proof: Assume $B \in SS(X,Y)$ and $D(B) \supset \overline{D(T)}$. Since $a(T) = a(T\alpha_X) < \infty$, it follows from I.6.9 that there exists a bounded projection P from X_1 onto $N(T\alpha_X)$. Let M_1 be the closed subspace $N(P)$ of $X_1 = M_1 \oplus N(T\alpha_X)$. Obviously $I_{X_1} - P$ is a bounded projection from X_1 onto M_1 with $N(I_{X_1} - P) = N(T\alpha_X)$ and hence from Theorem I.3.20 we conclude that $(I_{X_1} - P)^\wedge$ is a bounded bijection from the Banach space $X_1/N(T\alpha_X)$ onto the Banach space M_1 . It now follows from the open mapping theorem that $X_1/N(T\alpha_X) \cong M_1$. Since $\Upsilon(T\alpha_X) > 0$ we conclude from III.1.3 that $(T\alpha_X)^\wedge$ has a continuous inverse and as

$X_1/N(T\alpha_X)$ is isomorphic to M_1 it therefore follows that $T\alpha_X|_{M_1}$ has a continuous inverse as it is obviously injective. Hence from Theorem III.1.3 we conclude that $\gamma(T\alpha_X|_{M_1}) > 0$. Proceeding as in the proof of (i) of Theorem V.2.1, the corollary follows. □

The following corollary as well as Corollary V.2.5 appear in Goldberg [11] for the Banach space case.

V.2.3 Corollary. *Let* $T \in SF(X,Y)$ *with* Y *complete,* X *an operator range and* $D(T\alpha_X)$ *dense in* X_1 . *Suppose* $B \in SS(X,Y)$, $D(B) \supset D(T)$ *and* $(B\alpha_X)'$ *strictly singular. Then*

(i) $T + B$ *is normally solvable*

(ii) $K(T + B) = K(T)$

(iii) $a(T + \lambda B)$ *and* $b(T + \lambda B)$ *have constant values* n_1 *and* n_2 *respectively, except perhaps for isolated points. At the isolated points*

$$a(T + \lambda B) > n_1 \quad \text{and} \quad b(T + \lambda B) > n_2 .$$

Proof: Assume $B \in SS(X,Y)$, $D(B) \supset D(T)$ and $(B\alpha_X)'$ strictly singular. The requirement that $(B\alpha_X)'$ must be strictly singular is not too restrictive since we see from Theorems IV.1.6 and IV.1.8 that we could consider for example compact operators. Suppose $b(T) < \infty$ (the result follows trivially from V.2.1 if $a(T) < \infty$). From Lemmas V.1.1 and V.1.2 we see that

$$a(T) = a(T\alpha_X) = b((T\alpha_X)')$$

$$\text{and} \quad b(T) = b(T\alpha_X) = a((T\alpha_X)') < \infty$$

and that $(T\alpha_X)'$ is normally solvable since $\gamma(T\alpha_X) > 0$ by III.1.4. The theorem now follows by applying V.2.1 to $(T\alpha_X)'$ and $(B\alpha_X)'$ and by considering Theorem III.2.9. □

By discarding the completeness restriction on Y we get a similar result, namely the following:

V.2.4 Corollary. *Let* $T \in SF(X,Y)$ *with* $b(T) < \infty$, X *complete,* $\gamma(T) > 0$ *and* $D(T)$ *dense in* X. *Suppose* $B \in SS(X,Y)$, $B' \in SS[Y',X']$ *and* $D(B) \supset \overline{D(T)}$. *Then*

(i) $T + B$ *is normally solvable*

(ii) $b(T + \lambda B)$ *has a constant value* n *except perhaps at isolated points; at the isolated points we have*

$$\infty > b(T + \lambda B) > n_2 .$$

Proof: Assume $B \in SS(X,Y)$ with $B' \in SS[Y',X']$ and $D(B) \supset \overline{D(T)}$. From Lemmas V.1.1 and V.1.2 we see that $b(T) = a(T')$ and that T' is normally solvable. Since by III.2.9 $R(T' + B') = R((T + B)')$ closed implies $R(T + B)$ closed, the corollary now follows by applying Theorem V.2.1 to T' and B'. □

V.2.5 Corollary. *Let* $X = Y$ *be an operator range. Suppose there exists a* λ_0 *such that* $\beta_X(\lambda_0 I - T)^{-1}$ *is strictly singular on all of* Y. *Then for every* λ, $K(\lambda I_X - T) = 0$ *and* $\beta_X(\lambda I_X - T)$ *is a Fredholm operator.*

Proof: Assume we have a λ_0 such that $\beta_X(\lambda_0 I - T)^{-1}$ is strictly singular on all of Y. Then for any λ

(1) $$(\lambda I_X - T) = (I_X + (\lambda - \lambda_0)(\lambda_0 I_X - T)^{-1})(\lambda_0 I_X - T) .$$

Since $\beta_X(\lambda_0 I_X - T)^{-1}$ is strictly singular and everywhere defined, it follows from IV.1.8 that $\alpha_X \beta_X(\lambda_0 I_X - T)^{-1} = (\lambda_0 I_X - T)^{-1}$ is strictly singular and everywhere defined. Hence $(\lambda_0 I_X - T)^{-1}$ and thus $(\lambda_0 I_X - T)$ is closed with $N(\lambda_0 I_X - T) = \{0\}$ and $R(\lambda_0 I_X - T) = Y$. It follows trivially that $K(\lambda_0 I_X - T) = 0$ and $\lambda_0 I_X - T$ is Fredholm. From Theorem V.2.1 we now conclude that

$$\begin{aligned} &K(I_X + (\lambda - \lambda_0)(\lambda_0 I_X - T)^{-1}) \\ &\quad = K(\beta_X I_X + (\lambda - \lambda_0)\beta_X (\lambda_0 I_X - T)^{-1}) \\ &\quad = K(\beta_X I_X) = K(I_X) = 0 \end{aligned}$$

with $\beta_X I_X + (\lambda - \lambda_0)\beta_X (\lambda_0 I_X - T)^{-1}$ a Fredholm operator. Consequently it follows from Proposition III.3.12 that

$$\begin{aligned} &(\beta_X I_X + (\lambda - \lambda_0)\beta_X (\lambda_0 I_X - T)^{-1})(\lambda_0 I_X - T) \\ &\qquad = \beta_X(\lambda I_X - T) \quad \text{is Fredholm} \end{aligned}$$

and that

$$\begin{aligned} K(\lambda I_X - T) &= K(\beta_X(\lambda I_X - T)) \\ &= K(\beta_X I_X + (\lambda - \lambda_0)\beta_X (\lambda_0 I_X - T)^{-1}) + K(\lambda_0 I_X - T) \\ &= 0 . \end{aligned}$$

□

Corollaries V.2.4 and V.2.2 as well as the generalisations of Theorem V.2.1 and Corollaries V.2.3 and V.2.5 are by the author.

V.3 The Instability of Non-Semi-Fredholm Closed Operators under Compact and α-Compact Perturbations.

V.3.1 Definition (α - compact operators).

Let X be an operator range. We define an operator $T \in L(X,Y)$ to be α-*compact* if $T\alpha_X \in K(X_1,Y)$. We will denote the set of all α-compact operators in $L(X,Y)$ by $\alpha K(X,Y)$; $\alpha K[X,Y]$ if $D(T) = X$. (This definition is the author's own.)

It follows trivially from Theorem IV.1.8 that if X is an operator range, $T \in K(X,Y)$ implies $T \in \alpha K(X,Y)$.

V.3.2 Definition (separable).

We define a metric space X to be *separable* if it has a countable dense subset.

V.3.3 Lemma [11]. *The dual space of a separable normed space has a countable total subset.*

Proof: Assume $\{x_k\}$ to be a countable set dense in the normed space X . For each k there exists an $x_k' \in X'$ such that $\|x_k'\| = 1$ and $x_k'x_k = \|x_k\|$. We show that $\{x_k'\}$ is total. Assume there exists an $x \neq 0$ in X such that $x_k'x = 0$ for each k . Since $\{x_k\}$ is dense in X , there is an x_N such that $\|x - x_N\| < \|x\|/2$. Hence

$$\|x_N\| \geqslant \|x\| - \|x - x_N\| > \frac{\|x\|}{2} .$$

But

$$\|x_N\| = x_N'x_N - x_N'x \leqslant \|x_N - x\| < \frac{\|x\|}{2}$$

which is a contradiction. Thus $\{x_k'\}$ is total. □

V.3.4 Lemma [11]. *If* N *is a finite-dimensional subspace of* X' , *then* $({}^{\perp}N)^{\perp} = N$.

Proof: Assume N to be a finite-dimensional subspace of X' , the dual of X . It is easily seen that $N \subset ({}^{\perp}N)^{\perp}$ and hence from Theorem I.5.16 we have

(1) $\dim X/{}^{\perp}N = \dim (X/{}^{\perp}N)' = \dim ({}^{\perp}N)^{\perp} \geqslant \dim N$.

Now let $x_1', x_2', \dots, x_n'$ be a basis for N. The map A from $X/{}^{\perp}N$ into $\mathbb{R}^n$ or $\mathbb{C}^n$ defined by

$$A[x] = (x_1'x, x_2'x, \dots, x_n'x) \quad \text{for} \quad [x] \in X/{}^{\perp}N$$

is easily seen to be linear and injective and hence

$$\dim X/{}^{\perp}N \leqslant n = \dim N . \tag{2}$$

From (1) and (2) we conclude that $N = ({}^{\perp}N)^{\perp}$. □

We observe that this is a special case of a theorem by Dieudonné ([06]; cf. [11] , p.59) who showed that $N = ({}^{\perp}N)^{\perp}$ if N is a reflexive subspace of X' .

V.3.5 Theorem. *Let* $T \in (NS(X,Y) \setminus SF(X,Y))$ *with* X *and* Y *operator ranges. Then there exists a compact operator* $B \in K[X,Y]$ *such that for any* $\lambda \neq 0$, $T + \lambda B$ *does not have a closed range. If* $N(T)$ *is separable then* B *can be chosen so that* $T + \lambda B$ *is injective.*

Proof: Assume $T \in NS(X,Y) \setminus SF(X,Y)$ with X and Y operator ranges. Suppose first that Y is complete. Since $\dim R(T)^{\perp} = \dim (Y/R(T))' = b(T) = \infty$ by Theorem I.5.16, there is an infinite linearly independent set $y_1', y_2', \dots \subset R(T)^{\perp}$. Now choose $y_1 \in Y$ such that $y_1'y_1 \neq 0$. For $k \in \mathbb{N}$ let y_{k+1} be an element in $\bigcap_{i=1}^{k} N(y_i') = {}^{\perp}\mathrm{sp}\{y_1', y_2', \dots, y_n'\}$ such that $y_{k+1} \notin N(y_{k+1}')$. The existence of y_{k+1} is assured by Lemma V.3.4 and the fact that $y_1', y_2', \dots$ are linearly independent. Hence

$$\begin{aligned} y_j' \, R(T) &= 0 \quad \text{for} \quad j \geqslant 1 \\ y_j'y_i &= 0 \quad \text{for} \quad i > j \geqslant 1 \\ y_j'y_j &\neq 0 . \end{aligned} \tag{1}$$

Now let $\{x_1, x_2, \ldots\}$ be an infinite linear independent set in $N(T)$. It follows that $X_0 = \overline{sp}\,\{x_1, x_2, \ldots\}$ is separable and by Lemma V.3.3 there is a countable total set $v_1, v_2, \ldots \subset X_0'$ where we assume that $v_i \neq 0$ for $i \in \mathbb{N}$. Let $x_i' \in X'$ be an extension of v_i to all of X. We define $B \in L[X,Y]$ as follows

$$(2) \qquad Bx = \frac{\sum_{i=1}^{\infty} x_i'(x)y_i}{2^i\|x_i'\|\|y_i\|} .$$

The existence of Bx is assured by Theorem I.2.8 since the series is absolutely convergent and Y is complete. It is also easily seen that the absolute convergence of the series implies that B is bounded and that B is the limit in $B[X,Y]$ of the operators B_n defined by

$$B_n x = \frac{\sum_{i=1}^{n} x_i'(x)y_i}{2^i\|x_i'\|\|y_i\|} .$$

B_n is easily seen to be compact by Remark IV.1.4 and the completeness of Y, and since $B_n \to B$ in $B[X,Y]$ it follows from Proposition IV.1.5 that B is precompact and thus compact since Y is complete. Suppose $Bx \in R(T) \cap R(B)$. Then by (1) and (2) it follows that

$$0 = y_1'Bx = x_1'(x)y_1'y_1 \quad \text{whence} \quad x_1'x = 0 .$$

Similarly $$0 = y_2'Bx = x_2'(x)y_2'y_2 \quad \text{whence} \quad x_2'x = 0 .$$

Continuing inductively we conclude that $x_i'x = 0$ for $i \in \mathbb{N}$ and consequently $Bx = 0$. Hence $R(T) \cap R(B) = \{0\}$. In particular we notice that B is injective on $X_0 \subset N(T)$ since $Bx = 0 \in R(T)$ implies $x_i'x = 0$ for $i \in \mathbb{N}$ and therefore $x = 0$ since $\{x_1', x_2', \ldots\}$ is total on X_0.

Suppose now that $R(T + \lambda B) = R((T + \lambda B)\alpha_X)$ is closed for some $\lambda \neq 0$. Since it can easily be shown that $T + \lambda B$ and thus $(T + \lambda B)\alpha_X$ is closed, we conclude from Theorem III.1.4 that $\gamma((T + \lambda B)\alpha_X) > 0$.

Hence

$$(3) \qquad \|(T+\lambda B)\alpha_X\beta_X x\| \geqslant \gamma . d(\beta_X x, N((T+\lambda B)\alpha_X))$$

where $\gamma = \gamma((T+\lambda B)\alpha_X) > 0$ and $x \in D(T+\lambda B)$,

i.e. $\beta_X x \in D((T+\lambda B)\alpha_X)$.

Now $N(T+\lambda B) = N(B) \cap N(T)$ since $(T+\lambda B)x = 0$ implies $Bx \in R(T) \cap R(B) = \{0\}$ and thus $Tx = 0$. Hence

$$\begin{aligned} N((T+\lambda B)\alpha_X) &= \beta_X N(T+\lambda B) = \beta_X N(B) \cap \beta_X N(T) \\ &= N(B\alpha_X) \cap N(T\alpha_X) . \end{aligned}$$

Defining B_1 to be the restriction of B to $N(T)$ it follows that

$$N(B_1) = N(T) \cap N(B) = N(T+\lambda B)$$

and hence

$$N(B_1\alpha_X) = N((T+\lambda B)\alpha_X) .$$

Therefore for $\beta_X x \in N(T\alpha_X)\setminus N(B_1\alpha_X)$ we see from (3) that

$$\begin{aligned} \|B_1\alpha_X\beta_X x\| &= \frac{1}{|\lambda|} \|(T+\lambda B_1)\alpha_X\beta_X x\| \\ &\geqslant \frac{\gamma}{|\lambda|} d(\beta_X x, N(B_1\alpha_X)) > 0 \end{aligned}$$

which implies that $\gamma(B_1\alpha_X) > 0$ since

$$\frac{\|B_1\alpha_X\beta_X x\|}{d(\beta_X x, N(B_1\alpha_X))} \geqslant \frac{\gamma}{|\lambda|} > 0 \quad \text{for} \quad \beta_X x \in D(B_1\alpha_X)\setminus N(B_1\alpha_X).$$

Consequently $R(B_1\alpha_X) = R(B_1)$ is closed by Theorem III.1.4 since $B_1\alpha_X$, a continuous operator with closed domain, is obviously closed. Since $R(B_1)$ is closed in the Banach space Y it must be complete and hence from Corollary IV.1.7 it follows that $R(B_1)$ is finite dimensional. But this is impossible since $X_0 \subset N(T)$ while $\dim X_0 = \infty$ and B is injective on X_0 . Hence $R(T+\lambda B)$ is not closed.

Suppose now that Y is a non-complete operator range. From Proposition III.3.3 and Remark III.3.4 we see that $T \in NS(X,Y)\setminus SF(X,Y)$ implies $\beta_Y T \in NS(X,Y_1)\setminus SF(X,Y_1)$ and hence from what has just been proven we we conclude that there exists $C \in K[X,Y_1]$ such that for any $\lambda \neq 0$ $R(\beta_Y T + \lambda C)$ is not closed. It is easily seen that this implies $\alpha_Y R(\beta_Y T + \lambda C) = R(T + \lambda\alpha_Y C)$ not closed since for any $y \in \overline{R(\beta_Y T + \lambda C)} \setminus R(\beta_Y T + \lambda C)$ there exists $\{y_n\} \subset R(\beta_Y T + \lambda C)$ such that $y_n \to y$ and hence $\{\alpha_Y y_n\} \subset R(T + \lambda\alpha_Y C)$ with $\alpha_Y y_n \to \alpha_Y y \notin R(T + \lambda\alpha_Y C)$. $\alpha_Y C$ is clearly compact by Theorem IV.1.8 and now by letting $B = \alpha_Y C$ the first part of the theorem is proven.

If $N(T)$ is separable, then $\{x_1, x_2, \ldots\}$ may be chosen so that $X_0 = N(T)$, in which case $T + \lambda B$ is injective since by construction B is injective on X_0 and hence $N(T + \lambda B) = N(B) \cap N(T) = N(B) \cap X_0 = \{0\}$.

□

The theorem just proven is a generalisation by the author of a Banach space result appearing in Goldberg [11] . Essentially the same construction is used, suitably modified. The next two theorems and the corollary were proven by Bouldin ([03], Theorem 2.1) for bounded operators in Hilbert spaces and later generalised by Gonzalez and Onieva [12] to the Banach space case. We give a generalisation to operator ranges for closed operators.

We notice from the above that if X and Y were Banach spaces and $T \in NS(X,Y) \setminus SF(X,Y)$, then $\gamma(T) > 0$ and $\gamma(T + \lambda B) = 0$ by III.1.4, where $T + \lambda B \in C(X,Y) \setminus NS(X,Y)$ for any $\lambda \neq 0$, and hence Remark III.1.11 is applicable. Recalling the definition of convergence we gave in Remark III.1.11, we obtain the following :

V.3.6 Theorem. *Let* $T \in C(X,Y) \setminus SF(X,Y)$ *where* X *and* Y *are operator ranges. Then*

(i) T *is the limit in* $C(X,Y)$ *of a sequence of operators* $\{T_n\}$ *in* $C(X,Y) \setminus NS(X,Y)$

(ii) there exists $B \in K[X,Y]$ *and* $A \in C(X,Y) \setminus NS(X,Y)$ *such that* $T = A + B$.

Proof:

(i) Suppose $T \in C(X,Y) \setminus SF(X,Y)$. If $T \in C(X,Y) \setminus NS(X,Y)$ let $T = T_n$ for every $n \in \mathbb{N}$. If $T \in NS(X,Y) \setminus SF(X,Y)$ there exists by Theorem V.3.5 a compact operator $B \in K[X,Y]$ such that $T + \lambda B \notin NS(X,Y)$ for every $\lambda \neq 0$. It is easy to verify that $T + \lambda B \in C(X,Y)$ since B is everywhere defined and hence choosing a sequence $\lambda_n \to 0$ such that $\lambda_n \neq 0$ for $n \in \mathbb{N}$ it follows that $T + \lambda_n B \to T$ where $\{T + \lambda_n B\} \subset C(X,Y) \setminus NS(X,Y)$.

(ii) Assume $T \in C(X,Y) \setminus SF(X,Y)$. Obviously if $T \notin NS(X,Y)$, $T = T + 0$. If $T \in NS(X,Y) \setminus SF(X,Y)$ then by Theorem V.3.5 there exists a compact operator $B \in K[X,Y]$ such that $T - B = A \in C(X,Y) \setminus NS(X,Y)$ and $T = T - B + B = A + B$.

V.3.7 Corollary. *Let* X *be an operator range and let* Y *be complete. Then* $T \in C(X,Y) \setminus SF(X,Y)$ *iff there exists* $B \in K[X,Y]$ *and* $A \in C(X,Y) \setminus NS(X,Y)$ *such that* $T = A + B$.

Proof: Suppose there exists $B \in K[X,Y]$ and $A \in C(X,Y) \setminus NS(X,Y)$ such that $T = A + B$. It is easy to verify that T is closed. Suppose $T \in SF(X,Y)$. Then by Theorem IV.1.6 and IV.1.8 $(-B\alpha_X)'$ is compact and since the hypothesis implies that $D(T) = D(A)$ (because $T = A + B$ and

B is everywhere defined) we see from Corollary V.2.3 that $T - B = A \in SF(X,Y) \subset NS(X,Y)$ since if $D(T\alpha_X)$ was not dense in X_1, we could replace X_1 and X by $\overline{D(T\alpha_X)}$ and $\alpha_X \overline{D(T\alpha_X)}$ respectively and consider T, A and B to be from $\alpha_X \overline{D(T\alpha_X)}$ into Y. □

V.3.8 Definition ($\bar{b}(T)$).

Given $T \in L(X,Y)$, $\bar{b}(T)$ is defined to be the deficiency of $\overline{R(T)}$ in Y, that is

$$\bar{b}(T) = \dim Y/\overline{R(T)} .$$

Obviously $\bar{b}(T) \leq b(T)$.

We now come to the major result of this section :

V.3.9 Theorem. *Let* X *be an operator range and let* Y *be complete. Then* $T \in C(X,Y) \setminus SF(X,Y)$ *iff there exist* $B \in \alpha K[X,Y]$ *and* $A \in L(X,Y)$ *such that* $A\alpha_X \in C(X_1,Y)$, $a(A) = \bar{b}(A) = \infty$ *and* $T = A + B \in C(X,Y)$.

Proof: Assume there exist $B \in \alpha K[X,Y]$ and $A \in L(X,Y)$ such that $A\alpha_X \in C(X_1,Y)$ with $a(A) = \bar{b}(A) = \infty$ and $T = A + B \in C(X,Y)$. Then $a(T - B) = \bar{b}(T - B) = \infty$. Suppose $T \in SF(X,Y)$. Then by Remark III.3.4 $T\alpha_X \in SF(X,Y)$ and hence considering $T\alpha_X$ as an operator from $\overline{D(T\alpha_X)} = X_0 \subset X_1$ into Y and taking the restriction of $-B\alpha_X$, a compact operator, to X_0, it follows from Corollary V.2.3 that $T\alpha_X - B\alpha_X \in SF(X_1,Y)$; that is either $a(T - B) = a(T\alpha_X - B\alpha_X) < \infty$ or $\bar{b}(T - B) \leq b(T - B) = b(T\alpha_X - B\alpha_X) < \infty$, a contradiction.

Conversely, assume $T \in C(X,Y)$. If $T \in NS(X,Y) \setminus SF(X,Y)$ set $B = 0$. If $T \notin NS(X,Y)$ we shall show that there exist two α-compact operators B_1 and B_2 such that $a(T - B_1 - B_2) = \bar{b}(T - B_1 - B_2) = \infty$. Hence

$$T = (T - B_1 - B_2) + (B_1 + B_2) .$$

Since $T \notin NS(X,Y)$, $R(T)$ is not closed. Suppose $\bar{b}(T) < \infty$. Let a_n be the sequence of integers defined inductively by

$$(1) \qquad a_1 = 2 \qquad a_n = 2\,(1 + \sum_{k=1}^{n-1} a_k) \quad \text{for} \quad n = 2, 3, 4, \dots .$$

We claim that there are two sequences $\{y_k\} \subset Y$ and $\{y'_k\} \subset D((T\alpha_X)') \subset Y'$ such that

$$(2) \qquad \|y_k\| \leqslant a_k \qquad \|y'_k\| = 1 \qquad \|(T\alpha_X)'y'_k\| < \frac{1}{2^k a_k} \qquad y'_j(y_k) = \delta_{jk}$$

$$\text{for} \quad j,k = 1, 2, 3, \dots .$$

Assuming this for the moment we define

$$V_n x = \sum_{k=1}^{n} (T\alpha_X)'y'_k\,(\beta_X x)\; y_k \quad \text{for} \quad n = 1, 2, \dots \text{ and for } x \in X .$$

It is easy to check that $V_n\alpha_X$ will be a continuous everywhere defined operator with finite dimensional range and hence compact by Remark IV.1.4.

For $n > m$ we have

$$\begin{aligned}\|V_n\alpha_X\beta_X x - V_m\alpha_X\beta_X x\| &= \|V_n x - V_m x\| \\ &\leqslant \sum_{k=m+1}^{n} \|(T\alpha_X)'y'_k\|\,\|\beta_X x\|\,\|y_k\| \\ &\leqslant \left(\sum_{k=m+1}^{n} 2^{-k}\right) \|\beta_X x\| \\ &\leqslant \|\beta_X x\|/2^m \to 0 \qquad \text{as} \quad m \to \infty .\end{aligned}$$

Hence $\{V_n\alpha_X\}$ is a Cauchy and thus convergent sequence in the Banach space $B[X_1,Y]$. In fact if $B_1 x = \sum_{n=1}^{\infty} (T\alpha_X)'y'_k\,(\beta_X x) y_k$ for $x \in X$ it follows from the absolute convergence of the series for $B_1\alpha_X$ that $B_1\alpha_X(\beta_X x)$ and hence $B_1 x$ exists for each $x \in X$. It is easily seen that $V_n\alpha_X \to B_1\alpha_X$ in $B[X_1,Y]$ and hence $B_1\alpha_X$ is compact by Theorem

IV.1.5 and the completeness of Y . Thus by definition B_1 is α-compact. Now for each $x \in D(T)$ and each k we have by (2) that

$$\begin{aligned} y'_k(B_1x) &= (T\alpha_X)'y'_k\,(\beta_X x) \\ &= y'_k\,(Tx) \end{aligned}$$

and hence $y'_k\,(\overline{R(T - B_1)}) = 0$ for each k . Since $y_1, y_2, \dots$ are linearly independent and by (2) obviously not in $\overline{R(T - B_1)}$, it follows that $\bar{b}(T - B_1) = \infty$.

It remains to find sequences satisfying (2). The proof is by induction. Since $R(T) = R(T\alpha_X)$ is not closed, $R((T\alpha_X)')$ is not closed (Theorem III.2.9). We verify that there exists $y'_1 \in D((T\alpha_X)')$ such that $\|y'_1\| = 1$ and $\|(T\alpha_X)'y'_1\| < \frac{1}{4}$. For suppose on the contrary that $\|(T\alpha_X)'y'\| \geq \frac{1}{4}$ for all $y' \in D((T\alpha_X)')$ such that $\|y'\| = 1$. Then $(T\alpha_X)'$ has a continuous inverse by Theorem I.3.6 and since $(T\alpha_X)'$ is a closed operator Theorem III.1.1 implies that $R((T\alpha_X)')$ is closed, a contradiction. Hence y'_1 exists. We can also verify that $y_1 \in Y$ exists with $\|y_1\| < 2$ and $y'_1(y_1) = 1$. Suppose not. Then for every $y \in Y$ such that $\|y\| < 2$ we have $y'_1(y) \neq 1$. If $|y'_1(y)| > 1$ let $y_1 = y/y'_1y$. Obviously $\|y_1\| = \frac{1}{|y'_1y|}\,\|y\| < 2$ and $y'_1y_1 = y'_1y/y'_1y = 1$. Suppose now that $|y'y| < 1$ for every $y \in Y$ such that $\|y\| < 2$. But then

$$\begin{aligned} 1 = \|y'_1\| &= \sup_{\|y\| = 1} |y'y| = \sup_{\|y\| = 3/2} \left|y'\left(\frac{2y}{3}\right)\right| \\ &= \sup_{\|y\| = 3/2} \frac{2}{3}\,|y'_1(y)| = \frac{2}{3}\left(\sup_{\|y\| = 3/2} |y'(y)|\right) \leq \frac{2}{3} \end{aligned}$$

by our supposition. This contradiction establishes the existence of y_1.

Suppose $y_1, y_2, \dots, y_{n-1}, y'_1, y'_2, \dots, y'_{n-1}$ have been found satisfying (2). We claim that there exists $y'_n \in D((T\alpha_X)') \cap \{y_1, y_2, \dots, y_{n-1}\}^{\perp}$ such that $\|y'_n\| = 1$ and $\|(T\alpha_X)'y'_n\| < \frac{1}{2^n a_n}$. Suppose not. Let

$M = \operatorname{sp}\{y_1, y_2, \ldots, y_{n-1}\}$. M is thus finite dimensional and from Theorems I.6.9 and I.3.23 we conclude that $Y = M \oplus N$ with N a closed subspace of Y . Hence by Remark II.4.3 and Theorem II.4.2, $Y' = M^{\perp} \oplus N^{\perp}$. But $\dim M = \dim Y/N = \dim (Y/N)' = \dim N^{\perp}$ by I.5.16. Hence $N^{\perp}$ is finite dimensional. Since $R((T\alpha_X)')$ is not closed, $D((T\alpha_X)')$ is not finite dimensional and therefore $D((T\alpha_X)') \cap M^{\perp}$ is infinite dimensional. If we were now to suppose that $(T\alpha_X)'M^{\perp}$ is closed then $R((T\alpha_X)') = (T\alpha_X)'M^{\perp} + (T\alpha_X)'N^{\perp}$ would be closed by the finite dimensionality of $(T\alpha_X)'N^{\perp}$. Therefore $(T\alpha_X)'M^{\perp}$ is not closed. Since $(T\alpha_X)'|_{M^{\perp}}$ is closed by virtue of the fact that $M^{\perp}$ is a closed and thus complete subspace of Y' , we can now follow an argument similar to that used in establishing the existence of y_1' , with use being made of $M^{\perp} \cap D((T\alpha_X)')$, $(T\alpha_X)'|_{M^{\perp}}$ and $(T\alpha_X)'M^{\perp}$ instead of $D((T\alpha_X)')$, $(T\alpha_X)'$ and $R((T\alpha_X)')$ respectively. Hence y_n' exists. As before we can find a $y \in Y$ such that $y_n'(y) = 1$ and $\|y\| < 2$. Let

$$y_n = y - \sum_{k=1}^{n-1} y_k'(y)y_k \quad .$$

Then

$$\|y_n\| \leqslant \|y\| \left(1 + \sum_{k=1}^{n-1} \|y_k\|\right)$$

$$\leqslant 2\left(1 + \sum_{k=1}^{n-1} a_k\right) = a_n$$

by (1) and the induction hypothesis. Moreover $y_n'(y_n) = 1$, $y_n'(y_k) = 0$ for $1 \leqslant k \leqslant n-1$ by virtue of the fact that $y_n' \in M^{\perp}$ and by construction of y_n . We also have

$$y_k'(y_n) = y_k'(y) - y_k'(y) = 0 \quad \text{for} \quad 1 \leqslant k < n$$

and hence (2) holds by induction.

Suppose now that $a(T - B_1) < \infty$ since the assertion is trivial otherwise. We claim that there exist sequences

Footnote: Note that in line 6 $\dim Y'/M^{\perp} < \infty$ and hence since $\dim D((T\alpha_X)') = \infty$, $\dim D((T\alpha_X)') \cap M^{\perp} = \infty$.

$\{x_k\} \subset D(T) = D(T - B_1)$ and $\{x'_{1k}\} \subset X'_1$ such that

$$(3) \qquad \| \beta_X x_k \| = 1 , \quad \| x'_{1k} \| \leqslant 2^{k-1} , \quad \| (T - B_1) x_k \| \leqslant 2^{1-2k}$$

$$x'_{1k}(\beta_X x_j) = \delta_{kj} \quad \text{for} \quad k,j = 1, 2, \ldots .$$

Assuming this for the moment we define

$$U_n x = \sum_{k=1}^{n} x'_{1k}(\beta_X x)(T - B_1)x_k \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and } x \in X.$$

It is easily seen that each $U_n\alpha_X$ is continuous, everywhere defined and has finite dimensional range. Hence since Y is complete, it follows from Remark IV.1.4 that $\{U_n\alpha_X\}$ is a sequence of compact operators in $B[X,Y]$. Consider

$$B_2 x = \sum_{k=1}^{\infty} x'_{1k}(\beta_X x) \ (T - B_1)x_k .$$

It now follows from Theorem I.2.8, the completeness of X_1 and the absolute convergence of the series for $B_2\alpha_X$, that $B_2\alpha_X(\beta_X x)$ and thus $B_2 x$ exists for each $x \in X$. It is easily seen that $\{U_n\alpha_X\}$ converges to $B_2\alpha_X$ in $B[X_1,Y]$ and hence $B_2\alpha_X$ is compact by Proposition IV.1.5 and the completeness of Y. Hence B_2 is α-compact.

We observe that B_2 coincides with $T - B_1$ on sp $\{x_1, x_2, \ldots\}$ and hence $a(T - B_1 - B_2) = \infty$ since $\{x_1, x_2, \ldots\}$ is a linearly independent set. Moreover $R(B_2) \subset \overline{R(T - B_1)}$ and since $\bar{b}(T - B_1) = \infty$ we therefore have $\bar{b}(T - B_1 - B_2) = \infty$. It remains to find sequences satisfying (3). We start off by showing that $R(T - B_1)$ is not closed. Suppose $R(T - B_1) = R((T - B_1)\alpha_X)$ is closed. Since T is closed, $T\alpha_X$ and thus $T\alpha_X - B_1\alpha_X$ will be closed since α_X and $B_1\alpha_X$ are bounded. Let $X_0 = \overline{D(T\alpha_X - B_1\alpha_X)}$. Then $T\alpha_X - B_1\alpha_X \in NS(X_0,Y)$ with $a(T - B_1) < \infty$. Hence by considering $B_1\alpha_X|_{X_0}$ it follows from Corollary V.2.3 that

$T\alpha_X = T\alpha_X - B_1\alpha_X + B_1\alpha_X \in NS(X_0,Y)$, i.e. $R(T\alpha_X) = R(T)$ is closed, a contradiction since initially we assumed $T \in C(X,Y) \setminus NS(X,Y)$. Since $T\alpha_X - B_1\alpha_X \in C(X,Y) \setminus NS(X,Y)$ there exists $x_1 \in D(T) = D(T - B_1)$ such that $\|\beta_X x_1\| = 1$ and $\|(T - B_1)x\| < \frac{1}{2}$. To see this, suppose the contrary. Then for every $\beta_X x \in D(T\alpha_X - B_1\alpha_X)$ for which $\|\beta_X x\| = 1$ we have

$$\|(T - B_1)x\| = \|(T\alpha_X - B_1\alpha_X)\beta_X x\| \geq \frac{1}{2} \text{ and}$$

$T\alpha_X - B_1\alpha_X$ therefore has a continuous inverse by Theorem I.3.6 whence $R(T\alpha_X - B_1\alpha_X)$ is closed by Theorem III.1.1, an obvious contradiction.

Now by Corollary I.5.8 there exists $x'_{11} \in X'_1$ such that $\|x'_{11}\| = 1$ and $x'_{11}(\beta_X x_1) = 1 = \|\beta_X x\|$. Suppose we have constructed $x_1, x_2, \ldots, x_{n-1}$ and $x'_{11}, x'_{12}, \ldots, x'_{1n-1}$ in X and X'_1 respectively such that (3) is satisfied. Let $M = sp\,\{x'_{11}, x'_{12}, \ldots, x'_{1n-1}\}$. M is obviously finite dimensional and thus closed. From Theorem I.5.16 and Lemma V.3.4 we see that $\dim (X_1/{}^{\perp}M) = \dim (X_1/{}^{\perp}M)' = \dim ({}^{\perp}M)^{\perp} = \dim M$. Hence ${}^{\perp}M$ is a closed subspace of X_1 with finite deficiency. Consequently $(T\alpha_X - B_1\alpha_X)|_{{}^{\perp}M} \in C(X,Y) \setminus NS(X,Y)$ by Lemma V.1.6. Arguing as before with $(T\alpha_X - B_1\alpha_X)|_{{}^{\perp}M}$ instead of $T\alpha_X - B_1\alpha_X$ we obtain $\beta_X x_n \in D(T\alpha_X - B_1\alpha_X) \cap {}^{\perp}M$ such that $\|\beta_X x_n\| = 1$ and $\|(T - B_1)x_n\| < 2^{1-2n}$. Now let x'_1 be an arbitrary element of X'_1 for which $x'_1(\beta_X x_n) = 1$ and $\|x'_1\| = 1$.

Then the functional

$$x'_{1n} = x'_1 - \sum_{k=1}^{n-1} x'_1(\beta_X x_k)x'_{1k}$$

has the properties that

$$x'_{1n}(\beta_X x_k) = \delta_{nk} \quad \text{for} \quad k = 1, 2, \ldots, n$$

and $\|x'_{1n}\| \leq 2^{n-1}$.

Footnote (line 1): Recall that $D(T\alpha_X) = D(T\alpha_X - B_1\alpha_X) \subset X_0$ and hence $T\alpha_X(X_0) = R(T\alpha_X) = R(T)$.

Hence by induction we can construct the sequences in (3) thus proving the theorem. □

It is easily seen that Theorem V.3.6 and V.3.9 and Corollary V.3.7 contain the result obtained by Gonzalez and Onieva [12] , Theorem 2.1. In fact we obtain the following partial generalisation to operator ranges of their result.

V.3.10 Corollary. *Let* X *be an operator range and let* Y *be complete. Then*

(i) the closure of the set $B[X,Y]\setminus NS[X,Y]$ *is the set* $B[X,Y]\setminus SF[X,Y]$

(ii) $T \in B[X,Y]\setminus SF[X,Y]$ *iff there exists* $B \in K[X,Y]$ *and* $A \in B[X,Y]\setminus NS[X,Y]$ *such that* $T = A + B$

(iii) if X *is complete,* $T \in B[X,Y]\setminus SF[X,Y]$ *iff there exists* $B \in K[X,Y]$ *and* $A \in B[X,Y]$ *such that* $a(A) = \bar{b}(A) = \infty$ *and* $T = A + B$.

Proof:

(i) Assume X to be an operator range and Y to be complete. Obviously since $T \in B[X,Y]\setminus SF[X,Y]$ implies $T \in C[X,Y]\setminus SF[X,Y]$ it follows from Theorem V.3.6 that there exists a sequence $\{T_n\}$ in $C[X,Y]\setminus NS[X,Y]$ such that $\|T_n - T\| \to 0$. Moreover since T and, from the definition of convergence, $T_n - T$ are bounded, we conclude that $\{T_n\} \subset B[X,Y]\setminus NS[X,Y]$. From Theorem V.1.7 we see that $B[X,Y] \cap SF[X,Y]$ is an open subset of $B[X,Y]$ since $\gamma(T\alpha_X) > 0$ for $T \in SF[X,Y]$ by III.1.4 and consequently $B[X,Y]\setminus SF[X,Y]$ is a closed subset of $B[X,Y]$. From what was shown above it now easily follows that $\overline{B[X,Y]\setminus NS[X,Y]} = B[X,Y]\setminus SF[X,Y]$.

(ii) This is an immediate consequence of Corollary V.3.7 and the fact that $T \in B[X,Y]$ implies $T \in C[X,Y]$.

(iii) Assume that X is complete. It is obvious that $B \in \alpha K[X,Y]$ iff $B \in K[X,Y]$ if X is complete. Since $T \in B[X,Y]$ iff $T \in C[X,Y]$ by the closed graph theorem, the result now follows from Theorem V.3.9. □

Theorems V.3.6 and V.3.9 as well as Corollary V.3.7 are by the author.

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