Gleason solutions and canonical models for row contractions

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Abstract

This thesis extends the deBranges-Rovnyak model for completely non-coisometric (CNC) contractions to the setting of row contractions from several copies of a Hilbert space into itself. It is shown that a large class of of row contractions (including all CNC row contractions with commuting components) can be represented as extremal Gleason solutions in the de Branges-Rovnyak space associated to a contractive multiplier between vector-valued Drury-Arveson spaces. Here, a Gleason solution is the appropriate several-variable analogue of the adjoint of the restricted backward shift. Given such a row contraction T, the corresponding multiplier b_T, that is, the characteristic function of T, is shown to be unitary invariant. We further characterise a natural sub-class of row contractions for which it is a complete unitary invariant.

I know the meaning of plagiarism and declare that all of the work in the dissertation, save for that which is properly acknowledged, is my own.

Signature:

Signed by candidate

Andriamanankasina Ramanantoanina

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List of symbols

$\mathcal{A}_d$	Non-commutative disk algebra
$\mathcal{B}(\mathcal{H})$	Bounded linear operators on $\mathcal{H}$
$\bigvee \mathcal{H}$	Closed linear span of $\mathcal{H}$
$\mathscr{H}(b)$	The de Branges-Rovnyak space of the Schur multiplier b
$\mathbb{F}_d^+$	Unital free semigroup in d generators
$\mathcal{H}(k)$	RHKS with positive kernel k
$\mathcal{H}^d$	Direct sum of d copies of the Hilbert space $\mathcal{H}$
$\mathscr{L}(b)$	Herglotz space of b
$\mathcal{L}_d$	Non-commutative Toeplitz algebra
$\mathcal{O}(\mathbb{B}_d)$	Analytic function in the unit ball $\mathbb{B}_d$
$\mathbb{N}^d$	Unital additive semigroup of d-tuples of non-negative integers
$\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$	Schur class multipliers
F_d^2	Fock space
G_b	Herglotz function of the square Schur multiplier b
$H_d^\infty(\mathcal{U}, \mathcal{Y})$	Multipliers between the Drury-Arveson spaces
k^b	The de Branges-Rovnyak kernel of the Schur function b
L	The left free shift
V^b	The canonical row partial isometry on the Herglotz space of b

Introduction

A powerful and fruitful approach to studying linear operators on Hilbert space has been to construct unitarily equivalent model operators which have a canonical form, or act on a space with extra structure, such as a reproducing kernel Hilbert space of analytic functions. For example, there are two equivalent models for contractions on Hilbert space that have been used extensively: the Nagy-Foias model [36] and the de Branges-Rovnyak model [17, 8]. We mention the work of Garcia, Martin and Ross [21] in which the authors develop a related model for partial isometries. This model essentially reduces to the de Branges-Rovnyak model in the case of a completely non-coisometric (CNC) partial isometry. One of the main goals of this thesis is to extend this model for partial isometries on Hilbert space to a large class of CNC row contractions from several copies of a Hilbert space into itself.

A contraction, T, on a Hilbert space $\mathcal{H}$ is CNC if there is no subspace of $\mathcal{H}$ which is co-invariant for T on which T^* is isometric. It can be shown that a contraction on a Hilbert space $\mathcal{H}$ is CNC if and only if

$$\bigvee_{z \in \mathbb{D}} (I - \bar{z}T)^{-1} \mathrm{Ran}\,(D_{T^*}) = \mathcal{H}; \qquad D_{T^*} := (I - TT^*)^{1/2}.$$

The de Branges-Rovnyak model originated essentially in [17], and provided a representation theory for CNC Hilbert space contractions. The model spaces are de Branges-Rovnyak spaces in the Hardy-Hilbert space and the model operator is the adjoint of the restriction of the backward shift.

Let $\mathcal{U}, \mathcal{Y}$ be separable Hilbert spaces. Recall that the $\mathcal{Y}$-valued Hardy-Hilbert space, $H^2(\mathcal{Y})$, is the reproducing kernel Hilbert space (RKHS) of $\mathcal{Y}$-valued analytic functions on the open complex unit disk, $\mathbb{D}$, associated to the Szegö kernel

$$k(z, w) := \frac{1}{1 - z\overline{w}} I_{\mathcal{Y}}; \qquad z, w \in \mathbb{D}.$$

We denote by $S = M_z$ the linear isometry of multiplication by the independent variable on $H^2(\mathcal{Y})$. This operator is called the *shift* and it plays a central role in the operator theory in the Hardy space [36, 26]. Let $H^\infty(\mathcal{U}, \mathcal{Y})$ denote the set of bounded multipliers from $H^2(\mathcal{U})$ into $H^2(\mathcal{Y})$. The elements of $H^\infty(\mathcal{U}, \mathcal{Y})$ are the $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued analytic functions bounded on the unit disk $\mathbb{D}$. Let $\mathcal{S}(\mathcal{U}, \mathcal{Y})$, the *Schur class*, be the closed unit ball of $H^\infty(\mathcal{U}, \mathcal{Y})$. This is the convex set of all the contraction-valued analytic functions. The de Branges-Rovnyak space, $\mathscr{H}(b)$, associated to the Schur class function $b \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ is the (unique) RKHS associated to the positive kernel function

$$k^b(z,w) := \frac{I - b(z)b(w)^*}{1 - z\overline{w}} \in \mathcal{B}(\mathcal{Y}); \qquad z, w \in \mathbb{D}.$$

The space $\mathscr{H}(b)$ is always contractively contained in $H^2(\mathcal{Y})$, and it is invariant with respect to S^*, the *backward shift*, which acts on $f \in H^2(\mathcal{Y})$ as the difference quotient

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}; \qquad z \in \mathbb{D}.$$

Denote by $X_b := (S^*|_{\mathscr{H}(b)})^*$ the adjoint of the restriction of the backward shift to $\mathscr{H}(b)$. It provides a model for CNC contractions. Namely, any CNC contraction, T, on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the adjoint of the restriction of S^* to a certain de Branges-Rovnyak space uniquely determined by T.

Theorem 0.1 ([17], Appendix, Theorem 1)**.** *If a contraction, T, on a Hilbert space, $\mathcal{H}$, is CNC then there exists a Schur multiplier b_T, uniquely determined by T, such that T is unitarily equivalent to X_{b_T} on $\mathscr{H}(b_T)$.*

One of the main goal of this thesis is to develop a multivariable analogue of Theorem 0.1 above. Assume that $d \geq 1$ and consider the tuple T, $T = [T_1, \ldots, T_d]$, of operators acting on a common Hilbert space $\mathcal{H}$. The tuple T can be thought of as an operator from $\mathcal{H}^d$, the direct sum of d copies of $\mathcal{H}$, into $\mathcal{H}$:

$$[T_1, \ldots, T_d] \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} = T_1 h_1 + \ldots + T_d h_d \in \mathcal{H}$$

Such tuple T is called a *row contraction* when it is a contractive element of $\mathcal{B}(\mathcal{H}^d, \mathcal{H})$. Denote $D_{T^*} = \sqrt{I - TT^*}$ and $D_T = \sqrt{I - T^*T}$ the defect operators of T. A row contraction T is said to be CNC when there is no non-trivial subspace of $\mathcal{H}$ which is invariant for each T_k^* on which T^* is an isometry.

In the multivariable setting, we consider a natural several complex variable analogue of the Hardy space: the Drury-Arveson space H^2_d. It is the unique RKHS corresponding to the several complex variable Szegö kernel

$$k(z,w) := \frac{1}{1-\langle z,w\rangle_{\mathbb{C}^d}}; \qquad z,w \in \mathbb{B}_d,$$

where $\mathbb{B}_d$ is the unit ball of the d dimensional complex euclidian space $\mathbb{C}^d$. The role of the shift is played by the Arveson d-Shift $S = (S_1, \ldots, S_d)$ where $S_k = M_{z_k}$ is the multiplication by the coordinate function z_k. The de Branges-Rovnyak spaces are defined as before using Schur multipliers, which are contractive multipliers between vector-valued Drury-Arveson spaces.

A significant complication that occurs is that the several-complex variable de Branges-Rovnyak spaces are not generally co-invariant for the component operators of the Arveson d-shift. Moreover, in case where the de Branges-Rovnyak space is co-invariant for S, the component operators of S are commuting contractions so that the row contractions that can be represented are limited to commutative row contractions only. We overcome this obstacle by considering a natural multivariable analogue of the adjoint of the restricted backward shift. In our model, the role of X_b is played by *contractive Gleason solutions*. Given a Schur multiplier b, a Gleason solution in $\mathscr{H}(b)$ is a row contraction X which satisfies the multivariable difference quotient

$$(zX^*h)(z) = h(z) - h(0); \qquad z \in \mathbb{B}_d, h \in \mathscr{H}(b).$$

Note that for any row contraction T and any element $z = (z_1, \ldots, z_d) \in \mathbb{B}_d$, $\mathbb{B}_d$ the unit ball of the d-dimensional complex euclidean space $\mathbb{C}^d$, we make sense of zT^* and Tz^* as composition of operators where we identify $z \in \mathbb{B}_d$ to the row contraction $[z_1 I_H, \ldots, z_d I_{\mathcal{H}}]$ on $\mathcal{H}$.

Definition 0.2 (Commutatively completely non-coisometric row contractions)**.** We say that a row contraction, T, on a Hilbert space, $\mathcal{H}$, is *commutatively completely non-coisometric* (CCNC) when

$$\bigvee_{z\in\mathbb{B}_d} (I - Tz^*)^{-1}\mathrm{Ran}\,(D_{T^*}) = \mathcal{H}.$$

Any CCNC row contraction is necessarily CNC. If T is a d-contraction, a row contractions whose components are mutually commuting operators, then T is CCNC if and only if it is CNC.

We say two row contractions T and R are unitarily equivalent if $UT_iU^* = R_i$, or equivalently $UTU^* \otimes I_d = R$, for a unitary operator U. Our analogue of Theorem 0.1 is the following.

Theorem 0.3 (Theorem 4.20)**.** *A row contraction, T, on a Hilbert space $\mathcal{H}$ is a CCNC row contraction if and only if there exists a Schur multiplier b_T, uniquely determined by T up to a natural notion of equivalence, such that T is unitarily equivalent to an extremal Gleason solution for $\mathscr{H}(b_T)$.*

The function b_T in the theorem is the *characteristic function* of T, it is unique up to a natural notion of equivalence and it is a unitary invariant. A unitary invariant is an object associated to the row contraction T which behaves naturally under unitary equivalence. However, the characteristic function b_T is not a complete unitary invariant for general CCNC row contractions, mainly because contractive Gleason solutions are in general not unique. The de Branges-Rovnyak spaces which admit a unique contractive Gleason solution are the ones which correspond to *quasi-extreme* (QE) Schur multipliers. In the scalar case, $d = 1$, the QE multipliers are the extreme points of the unit ball of H^∞. We could identify a large class of row contractions T whose characteristic functions are QE: we call these row contractions *QE row contractions*.

Theorem 0.4 (Theorem 4.26)**.** *A CCNC row contraction T is QE, i.e., its characteristic function b_T is a QE Schur multiplier, if and only if*

$$\mathrm{Ker}(T)^\perp \subset \bigvee_{z\in\mathbb{B}_d} z^*(I - Tz^*)^{-1}\mathrm{Ran}\,(D_{T^*})\,.$$

The characteristic function b_T in Theorem 0.3 is then a complete unitary invariant for QE row contractions, i.e., two QE row contractions T and R are unitarily equivalent if and only if their characteristic functions b_T and b_R are equivalent.

The development of model theory for general (not necessarily commuting) row contractions was first considered by Popescu [31, 29]. His results extend the Nagy-Foias model to CNC row contractions and, in the process, defines a characteristic function which is a complete unitary invariant. The characteristic function is a free Schur class multiplier between vector-valued Fock spaces [10].

The seminal work of Arveson [4] initiated an extensive investigation of commuting row contractions, i.e, row contractions formed by commuting operators. Bhattacharyya, Eschmeier and Sarkar [11, 12] defined a multivariable generalisation of the Nagy-Foias characteristic function and extended the Nagy-Foias model to commuting row contractions. The characteristic function is an operator-valued holomorphic function on the unit ball,

$\mathbb{B}_d$, of the euclidian space $\mathbb{C}^d$. It is a contractive multiplier between vector-valued Drury-Arveson spaces and it is proved to be a complete unitary invariant for CNC commuting row contractions [12, Theorem 3.6]. This was also proved latter by Ball-Bolotnikov [7, Theorem 5.2]. The multivariable deBranges-Rovnyak colligation theory of Ball-Bolotnikov-Fang [7] shows that a Gleason solution is a natural multivariable extension of the concept of restriction of the backwards shift to a deBranges-Rovnyak space. It was also proven in [7, Theorem 5.7] that any CNC commutative row contraction is unitarily equivalent to a contractive Gleason solution on the de Branges-Rovnyak space corresponding to its characteristic function. Such commuting contractive Gleason solutions have been shown to be unitarily equivalent to the restriction of the adjoint of the Arveson shift to a de Branges-Rovnyak space [2, Theorem 4.1].

In comparison, this thesis extends the de Branges-Rovnyak model for single contraction to CCNC row contractions in the multivariable setting. We develop a functional model for this class of row contractions, this includes all the commutative CNC row contractions. Our approach to model theory is based on the relation between row partial isometries and their extensions. The thesis is organised as follows:

– In the first Chapter, we recall some facts about multivariable operator theory. We briefly review the basic results on dilation theory for row contractions and some properties of the Drury-Arveson space. In Section 4, we study row partial isometries which is one of the main foci of this thesis.

– We dedicate the second Chapter to explore different tools and objects associated with Schur multipliers. We study two RKHSs associated to Schur multipliers: the de Branges-Rovnyak space and the Herglotz space. We show that contractive Gleason solutions are CCNC (Proposition2.5). We also review the concept of QE Schur multipliers and its relation with the solution to the Gleason problems which is central to the model theory we develop.

– The third Chapter serves two purposes, defining the characteristic function for CCNC row partial isometries and realising such row partial isometries as solution of the Gleason problem (which is again a row partial isometry) that act as multiplication by the coordinate functions on its initial space. In order to do that, we extend the model defined for single partial isometry in [21] to the setting of row partial isometry. We construct a characteristic function and prove an analogue of Theorem 0.1 above for Gleason row partial isometries (Theorem 3.15). The characteristic function is proved to be a complete unitary invariant for the class of quasi-extreme (QE) row partial isometry (Definition 3.19).

– In the fourth Chapter, in order to extend the model from Chapter 3 to CCNC row contractions, we start, in Secion 4.1, by computing the characteristic function of the isometric part of an extremal Gleason solution in $\mathscr{H}(b)$ (Theorem 4.4). We see that as a several-complex variable analogue of the Frostman shift [20]. We study the Frostman transformation for Schur multipliers in Section 4.2 and by mean of this transformation we define a characteristic function for CCNC row contractions in Section 4.3. This characteristic function is proven to be unitary invariant for CCNC row contractions (Theorem 4.17) and a complete unitary invariant for QE row contractions. We give an abstract characterisation of all QE row contractions (Theorem 4.26).

Chapter 1
Partial isometric tuples

In multivariable operator theory, one considers a linear map, $T : \mathcal{H}^d \longrightarrow \mathcal{H}$, from several copies of a Hilbert space $\mathcal{H}$ into itself. Here $d > 0$ is an integer and $\mathcal{H}^d := \mathcal{H} \otimes \mathbb{C}^d$ is the direct sum of d copies of $\mathcal{H}$. We write such a map as a row of operators $T = (T_1, \dots, T_d)$ and the T_k are bounded operators on $\mathcal{H}$. The theory of bounded linear maps from several copies of a Hilbert space into itself bifurcates into the cases of mutually commuting component operators and non-commuting component operators. We will be interested in both cases.

The first section below consists of a brief account of the results on dilation of row contractive in the general non-commutative setting that will be useful in the sequel. Next we present the Drury-Arveson space which lies at the heart of commutative multivariable operator theory. In the last section we introduce one of the central objects of study in this thesis, a row partial isometry, i.e., a partial isometry from several copies of a Hilbert space into itself.

1.1 Dilation of row contractions

Let $\mathcal{H}$ be a Hilbert space. A tuple $T := (T_1, T_2, \dots, T_d)$ is a *row contraction on* $\mathcal{H}$ when $T_k \in \mathcal{B}(\mathcal{H})$ and

$$I_{\mathcal{H}} - TT^* = I_{\mathcal{H}} - T_1T_1^* - T_2T_2^* - \dots - T_dT_d^* \geq 0, \tag{1.1}$$

or equivalently when T is a contractive element of $\mathcal{B}(\mathcal{H}^d, \mathcal{H})$. We say that a row contraction T is *purely contractive* if

$$||T\mathbf{h}||_{\mathcal{H}} < ||\mathbf{h}||_{\mathcal{H}^d}\,; \qquad \mathbf{h} \in \mathcal{H}^d,$$

and we say that T is *stricly contractive* when

$$||T||_{\mathcal{B}(\mathcal{H}^d,\mathcal{H})} < 1.$$

We use the term *d-contraction* to refer to row contractions whose components are mutually commuting operators. If T is a row contraction then necessarily

$$I_{\mathcal{H}^d} - T^*T = \left[\delta_{ij} I_{\mathcal{H}} - T_i^* T_j\right]_{i,j=1}^{d} \geq 0, \tag{1.2}$$

and T is a *row isometry* when equality holds. It is straightforward to verify that T is a row isometry if and only if the components of T are isometries with orthogonal ranges. The component operators cannot commute in this case. A row isometry is called a *Cuntz unitary* if it is onto, i.e, equality holds in the equations 1.1 and 1.2.

Let $\mathbb{F}_d^+$ be the set of all finite words $\alpha := \alpha_1\alpha_2 \ldots \alpha_n, n \in \mathbb{N}$, where each α_k is chosen from the set $\{1, 2, \ldots, d\}$. We can endow $\mathbb{F}_d^+$ with the semigroup structure by defining $\alpha \cdot \beta$ by concatenation. If we assume that $\emptyset$, the *empty word* containing no letters, belongs to $\mathbb{F}_d^+$, then $\emptyset$ is a unit for $\mathbb{F}_d^+$ and $\mathbb{F}_d^+$ is a unital semigroup or monoid called the *free semigroup* on d letters. If $\alpha \in \mathbb{F}_d^+$, the length of α is denoted $|\alpha|$:

$$|\alpha| = \begin{cases} n & \text{if } \alpha = \alpha_1 \ldots \alpha_n \\ 0 & \text{if } \alpha = \emptyset \end{cases} .$$

We use the notation $\mathbb{N}^d$ to refer to the unital semigroup of d-tuples of non-negative integers. The group operation is the entry-wise addition. We denote $(\alpha) = (n_1, n_2, \ldots, n_d) =: \mathbf{n} \in \mathbb{N}^d$ where n_k is the number of occurrence of the letter k in the word α. This letter counting map defines a surjective unital semigroup homomorphism of $\mathbb{F}_d^+$ onto $\mathbb{N}^d$.

Let T be a row contraction on a Hilbert space $\mathcal{H}$. If $\alpha = \alpha_1 \ldots \alpha_n \in \mathbb{F}_d^+$ with $n > 0$ then

$$T^\alpha := T_{\alpha_1} \ldots T_{\alpha_n} \in \mathcal{B}(\mathcal{H}),$$

and by convention $T^\emptyset = I_{\mathcal{H}}$. For $\mathbf{n} \in \mathbb{N}^d$,

$$T^{\mathbf{n}} := \sum_{(\alpha)=\mathbf{n}} T^\alpha \in \mathcal{B}(\mathcal{H}),$$

and by convention $T^{(0,\ldots,0)} = I_{\mathcal{H}}$.

Definition 1.1. A tuple $R = (R_1, \ldots, R_d)$ of operators acting on a common Hilbert space $\mathcal{K}$ is a dilation of $T := (T_1, \ldots, T_d)$ acting on $\mathcal{H}$ when

- $\mathcal{H}$ is a subspace of $\mathcal{K}$, and,

- compression to $\mathcal{H}$ is a unital epimorphism of the unital operator algebra generated by R onto that generated by T. Equivalently, $T^\alpha = P_\mathcal{H} R^\alpha|_\mathcal{H}$ for all $\alpha \in \mathbb{F}_d^+$, where P_H is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

The basic philosophy of dilation theory is to study row contractions (and more general linear maps) by viewing them as compressions of dilations which act on larger Hilbert spaces, and which enjoy several nice additional properties. For example, the dilation R can be a row isometry or a Cuntz unitary in which case the dilation is said to be isometric or Cuntz dilation, respectively. The dilation is said to be minimal when the space $\mathcal{K}$ is the smallest subspace containing $\mathcal{H}$ which is invariant for each R_k, $1 \leq k \leq d$. Every row contraction has a minimal isometric dilation which is unique up to isomorphism:

Theorem 1.2 ([30], Theorem 2.1). *Let $T = (T_1, \ldots, T_d)$ be a row contraction acting on a Hilbert space $\mathcal{H}$. There is a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a row isometry $V = (V_1, \ldots, V_d)$ acting on $\mathcal{K}$ such that $\mathcal{H}$ is invariant for each V_k^* and*

$$T_k^* = V_k^*|_\mathcal{H}; \qquad (k = 1, \ldots, d).$$

This row isometry V is necessarily a dilation of T. Moreover $\mathcal{K}$ can be chosen to be minimal in the sense that $\mathcal{K}$ is the minimal V-invariant subspace containing $\mathcal{H}$ i.e.

$$\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_d^+} V^\alpha \mathcal{H}.$$

The minimal isometric dilation is unique up to isomorphism.

Remark 1.3. The existence of minimal isometric dilations for finite row contractions is proved in [13]. This result also holds for row contractions from infinite copies of a Hilbert space into itself [30, Theorem 2.1], and the dilation can be chosen to be minimal. It is also proved in [30, Proposition 2.6] that any row contraction has a Cuntz unitary dilation.

Let V be a row isometry acting on a Hilbert space $\mathcal{K}$. A Wold decomposition for $\mathcal{K}$ is proved in [33]: $\mathcal{K}$ admits a decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ where $\mathcal{K}_0$ and $\mathcal{K}_1$ are reducing for the component operators of V, $\mathcal{K}_1$ is the biggest subspace of $\mathcal{H}$ where

$$V^* = \begin{pmatrix} V_1^* \\ V_2^* \\ \vdots \\ V_d^* \end{pmatrix}$$

is isometric. V is said to be *pure* when $\mathcal{K}_1 = \{0\}$.

Definition 1.4. A row contraction T on a Hilbert space $\mathcal{H}$ is said to be *completely non co-isometric* (CNC) if there is no non trivial subspace of $\mathcal{H}$ which is invariant for each T_k^* on which T^* acts isometrically.

Remark 1.5. For a row contraction, T, acting on a Hilbert space, $\mathcal{H}$, with minimal isometric dilation V acting on $\mathcal{K} \supset H$, the Wold decomposition of $\mathcal{K}$ induces a decomposition of $\mathcal{H}$. Namely, $\mathcal{H}$ decomposes into $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are reducing for the component operators of T, $\mathcal{H}_1$ is the largest co-invariant subspace of $\mathcal{H}$ on which T^* acts isometrically and $T|_{\mathcal{H}_0}$ is CNC [30, Remark 2.7]. That is, T is CNC if and only if $\mathcal{H}_1 = \{0\}$.

There exists a row contraction, L, which is universal in the following sense. Given any row contraction T, the map $L^\alpha \mapsto T^\alpha$ defines a completely contractive unital operator algebra epimorphism. This row contraction L acts on the *Fock space*. The row contraction L is a non-commutative and multi-variable generalisation of the shift operator acting on the Hilbert-Hardy space. Let $d \in \mathbb{N}$ be a positive integer and consider the Fock space

$$F_d^2 := \bigoplus_{k=0}^{\infty} (\mathbb{C}^d)^{\otimes k} = \mathbb{C} \oplus (\mathbb{C}^d) \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots,$$

where

$$(\mathbb{C}^d)^{\otimes k} = \underbrace{\mathbb{C}^d \otimes \dots \otimes \mathbb{C}^d}_{k \text{ times}}, \qquad \text{and,} \qquad (\mathbb{C}^d)^{\otimes 0} = \mathbb{C},$$

is the k-fold tensor product of $\mathbb{C}^d$. Let $(\xi_1, \dots, \xi_d)$ be a fixed orthonormal basis of $\mathbb{C}^d$. Then,

$$\left\{\xi_\alpha := \xi_{\alpha_1} \otimes \dots \otimes \xi_{\alpha_n};\ \alpha = \alpha_1 \dots \alpha_n \in \mathbb{F}_d^+\right\}$$

is an orthonormal basis of F_d^2, where $\xi_\emptyset = 1 \in \mathbb{C}^{\otimes 0}$. The row contraction, $L := (L_1, \dots, L_d)$, is naturally defined on F_d^2 by

$$L_k \xi_\alpha := \xi_k \otimes \xi_\alpha; \qquad k = 1, \dots, d.$$

The operators L_k are called the *left creation operators* and L is the *left free shift*. Each component operator, L_k, is isometric and their ranges are pairwise orthogonal so that L is a row isometry. The weakly closed subalgebra of $\mathcal{B}(F_d^2)$ generated by L, the *non-commutative analytic Toeplitz algebra*, is denoted $\mathcal{L}_d$. The norm closed algebra generated by L, the *non-commutative disk algebra*, is denoted by $\mathcal{A}_d$ and the operator system $\overline{\mathcal{A}_d + \mathcal{A}_d^*}$ is universal for row contractions in the following sense.

Theorem 1.6 ([33], Theorem 2.1). *If T is a row contraction on a Hilbert space $\mathcal{H}$ then there exists a (necessarily unique) completely positive unital map*

$$\Psi : \overline{\mathcal{A}_d + \mathcal{A}_d^*} \longrightarrow \mathcal{B}(\mathcal{H})$$

that sends L_k to T_k: $\Psi(L_k) = T_k$. All row contractions arise in this way.

The map Ψ induces a completely contractive unital homomorphism from the unital operator algebra generated by $L_1, \ldots, L_d$ onto the operator algebra generated by $T_1, \ldots, T_d$. This yields a von Neumann-type inequality for row contractions:

$$||p(T_1, \ldots, T_d)|| \leq ||p(L_1, \ldots, L_d)|| \,,$$

for a polynomial, p, in d non-commuting variables.

The above correspondence between completely positive maps and row contractions was proved in [33, Theorem 2.1] and the generalisation of the von Neumann inequality by the same author in [32].

1.2 Reproducing kernel Hilbert spaces

Many of the important spaces we encounter in this thesis are reproducing kernel Hilbert spaces (RKHS). We recall some facts about these spaces here and set up some notation we will be using throughout the thesis. References for the theory of RKHS include the original paper of Aronszajn [3] (for the scalar valued case) and the book of Agler and McCarthy [1]. Paulsen's textbook [28] also contains a comprehensive account of the theory.

Let $\mathcal{Y}$ be a Hilbert space. A positive $\mathcal{B}(\mathcal{Y})$-valued kernel function on a set $X \subset \mathbb{C}^d$ is a function $k : X \times X \longrightarrow \mathcal{B}(\mathcal{Y})$ such that for any finite subset $\{x_1, ..., x_m\} \subset X$, the matrix

$$\Big[k(x_i, x_j)\Big]_{i,j=0}^{m} \in \mathcal{B}(\mathcal{Y}) \otimes \mathbb{C}^{m \times m}$$

is non-negative. Equivalently,

$$\sum_{i,j=0}^{m} \langle k(x_i, x_j) y_j, y_i \rangle \geq 0,$$

for any finite subsets $\{x_1, \ldots, x_m\} \subset X$ and $\{y_1, \ldots, y_m\} \subset \mathcal{Y}$.

Theorem 1.7. *Let X be a set and $\mathcal{Y}$ be a Hilbert space. A function $k : X \times X \longrightarrow \mathcal{B}(\mathcal{Y})$ is a positive kernel if and only if it has a Kolmogorov factorisation $k(z,w) = h(z)h(w)^*$ for some function $h : X \to \mathcal{B}(\mathcal{Y}, \mathcal{J})$ where $\mathcal{J}$ is some auxiliary Hilbert space.*

It is clear that for any $\mathcal{B}(\mathcal{Y})$-valued function ϕ, $\phi(z)k(z,w)\phi(w)^*$ is still a positive kernel.

A $\mathcal{Y}$-valued RKHS $\mathcal{H}$ is a Hilbert space of functions on a set $X \subset \mathbb{C}^d$ such that the evaluation at any point $z \in X$, $f \in \mathcal{H} \longmapsto f(z) \in \mathcal{Y}$, is a linear bounded operator from $\mathcal{H}$ to $\mathcal{Y}$. Denote $k_z \in \mathcal{B}(\mathcal{Y}, \mathcal{H})$ the adjoint of the evaluation operator at a point $z \in X$:

$$k_z^* f = f(z) \in \mathcal{Y}, \qquad f \in \mathcal{H}.$$

The function

$$k(z,w) = k_z^* k_w \in \mathcal{B}(\mathcal{Y}); \qquad z, w \in X,$$

is then a $\mathcal{B}(\mathcal{Y})$-valued positive kernel on $X \times X$. Conversely, a construction illustrated in [1, Chapter 2.] associates a unique RKHS to any given positive kernel.

Theorem 1.8. *Let $\mathcal{Y}$ be a Hilbert space. Given a set X, there is a natural bijection between $\mathcal{B}(\mathcal{Y})$-valued positive kernel functions on $X \times X$ and $\mathcal{Y}$-valued RKHS on the set X.*

We will write $\mathcal{H}(k)$ for the RKHS associated uniquely to the positive kernel function k. We have the following characterisation of the functions that belong to a RKHS.

Theorem 1.9. *Let $\mathcal{Y}$ be a Hilbert space and k be a $\mathcal{B}(\mathcal{Y})$-valued positive kernel on a set X. A $\mathcal{Y}$-valued function f on the set X belongs to $\mathcal{H}(k)$ if and only if there is a $\lambda > 0$ such that*

$$\langle \cdot, f(w) \rangle f(z) - \lambda^2 k(z,w); \qquad z, w \in X,$$

defines a positive kernel on X. The least of all such λ is $||f||_{\mathcal{H}(k)}$.

Any $\mathcal{Y}$-valued RKHS, $\mathcal{H}(k)$, on a set X, is equipped with a *multiplier algebra*, $\mathcal{M}(\mathcal{H}(k))$. It consists of all the functions that multiply $\mathcal{H}(k)$ into itself:

$$\mathcal{M}(\mathcal{H}(k)) := \{\phi : X \longrightarrow \mathcal{B}(\mathcal{Y}) | \quad \phi h \in \mathcal{H}(k), \ \forall h \in \mathcal{H}(k)\}.$$

Here, ϕh denotes the point-wise multiplication. Multiplication by an element of $\mathcal{M}(\mathcal{H})$ defines a bounded multiplication operator on $\mathcal{H}(k)$. The subalgebra $\mathcal{M}(\mathcal{H}(k))$ is a Weak

Operator Topology (WOT)-closed subspace of $\mathcal{B}(\mathcal{H}(k))$. More generally, given a pair of RKHS, a $\mathcal{U}$-valued RKHS $\mathcal{H}(k)$ and a $\mathcal{Y}$-valued RKHS $\mathcal{H}(k')$ on the same set X, the space of multipliers, $\mathcal{M}(\mathcal{H}(k), \mathcal{H}(k'))$, between $\mathcal{H}(k)$ and $\mathcal{H}(k')$ consists of $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued functions ϕ such that point wise multiplication by ϕ maps $\mathcal{H}(k)$ into $\mathcal{H}(k')$. It is a consequence of the closed graph theorem that a multiplier ϕ in $\mathcal{M}(\mathcal{H}(k), \mathcal{H}(k'))$ defines a bounded multiplication operator, M_ϕ, in $\mathcal{B}(\mathcal{H}(k), \mathcal{H}(k'))$. We often identifies the multiplier ϕ and the corresponding multiplication operator.

Simple computation shows that $M_\phi^* k'_w y = k_w \phi(w)^* y$, $w \in \mathbb{B}_d$ and $y \in \mathcal{Y}$. Indeed, for all $h \in \mathcal{H}(k)$,

$$\begin{aligned} \left\langle M_\phi^* k'_w y, h \right\rangle &= \langle k'_w y, M_\phi h \rangle = \langle y, \phi(w) h(w) \rangle \\ &= \langle \phi(w)^* y, h(w) \rangle = \langle k_w \phi(w)^* y, h \rangle . \end{aligned}$$

In fact, a bounded linear operator $T \in \mathcal{B}(\mathcal{H}(k), \mathcal{H}(k'))$ is a multiplication operator if and only if there is a $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued function ϕ such that $T^* k'_w y = k_w \phi(w)^* y$, $w \in \mathbb{B}_d$ and $y \in \mathcal{Y}$.

Theorem 1.10. *Let k and k' be two positive kernels on a set X and let ϕ be a $\mathcal{B}(\mathcal{U}, \mathcal{Y})$-valued function on the same set. A function $\phi \in \mathcal{M}(\mathcal{H}(k), \mathcal{H}(k'))$ if and only if there is positive real number λ such that*

$$\lambda^2 k'(z, w) - \phi(z) k(z, w) \phi(w)^*$$

defines a positive kernel. The norm of M_ϕ is the least of all such λ.

1.3 The Drury-Arveson space and d-contractions

The Drury-Arveson space arises naturally in the commutative, multivariable generalisation of operator theory on the classical Hilbert-Hardy space H^2 on the unit disk $\mathbb{D}$ ([4] or the survey paper of Shalit [35]). The Drury-Arveson space was first introduced in the work of Drury [19] seeking a generalisation of the von Neumann inequality for commuting contractions [27].

Following [4], we will use the term *d-contraction* to refer to row contraction on a Hilbert space $\mathcal{H}$ whose components mutually commute. That is $T = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H}^d, \mathcal{H})$:

$$I_{\mathcal{H}} - TT^* = I_{\mathcal{H}} - T_1 T_1^* - \ldots - T_d T_d^* \geq 0, \qquad \text{and,} \qquad T_i T_j = T_j T_j; \qquad i, j, = 1, \ldots, d.$$

There are two objects naturally associated to a d-contraction, a completely positive map,

$$\phi_T : X \in \mathcal{B}(\mathcal{H}) \longmapsto T(X \otimes I_d)T^* = T_1 X T_1^* + \ldots + T_d X T_d^* \in \mathcal{B}(\mathcal{H}),$$

and a defect operator,

$$D_{T^*} = (I_{\mathcal{H}} - TT^*)^{1/2} = (I_{\mathcal{H}} - \phi(I_{\mathcal{H}}))^{1/2}.$$

The tuple T being contractive guarantees that

$$\phi_T^k(I_{\mathcal{H}}) \geq \phi_T^{k+1}(I_{\mathcal{H}}) \geq 0.$$

The sequence $\phi_T^n(I_{\mathcal{H}})$ then has a strong limit in $\mathcal{B}(\mathcal{H})$. T is said to be pure when the limit is 0. For example a strict d-contraction, i.e., a d-contraction with norm $||T||_{\mathcal{B}(\mathcal{H}^d,\mathcal{H})} < 1$, is pure.

Let $z_1, z_2, \ldots, z_d$ be the coordinate functions on $\mathbb{C}^d$. For a $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we denote $\mathbf{n}! = n_1! \ldots n_d!$, and $|\mathbf{n}| = n_1 + \ldots + n_d$. We use z to denote the tuple $(z_1, \ldots, z_d)$ and $z^{\mathbf{n}} = z_1^{n_1} \ldots z_d^{n_d}$.

Definition 1.11 (Drury-Arveson space). Let $\mathcal{O}(\mathbb{B}_d)$ be the space of analytic function in the unit ball, $\mathbb{B}_d$, of $\mathbb{C}^d$. The Drury-Arveson space was first introduced in [19] as

$$H_d^2 := \left\{ f = \sum_{\mathbf{n}\in\mathbb{N}^d} c_{\mathbf{n}} z^{\mathbf{n}} \in \mathcal{O}(\mathbb{B}_d); \text{ where } (c_{\mathbf{n}}) \subset \mathbb{C} \text{ and } \sum_{\mathbf{n}\in\mathbb{N}^d} |c_{\mathbf{n}}|^2 \frac{\mathbf{n}!}{|\mathbf{n}|!} < \infty \right\}; \qquad d \in \mathbb{N}.$$

The space is named after S.W. Drury and W. Arveson. Drury [19] introduced this space as a tool in his several-variable generalisation of the celebrated von Neumann inequality for commuting contractions on Hilbert space. Arveson [4] placed H_d^2 in the center of modern multivariable operator theory in the commutative setting. The notation H_d^2 is suggested by the fact that the Drury-Arveson space for the univariate case ($d = 1$) is the classical Hardy-Hilbert space H^2 on the unit disk $\mathbb{D}$ of $\mathbb{C}$.

Arveson [4] observed that H_d^2 is a RKHS. Indeed, the inner product on H_d^2 is such that two different monomials are orthogonal and

$$\langle z^{\mathbf{n}}, z^{\mathbf{n}} \rangle_{H_d^2} = \frac{|\mathbf{n}|!}{\mathbf{n}!}; \quad (\mathbf{n} \in \mathbb{N}^d).$$

From the definition of H^2_d one sees that

$$\left\{ \left(\frac{|\mathbf{n}|!}{\mathbf{n}!} \right)^{1/2} z^{\mathbf{n}}, \ \mathbf{n} \in \mathbb{N}^d \right\}$$

is an orthonormal basis of H^2_d. Let $w \in \mathbb{B}_d$ and $f = \sum_{\mathbf{n} \in \mathbb{N}^d} c_{\mathbf{n}} z^{\mathbf{n}} \in H^2_d$.

$$\begin{aligned} |f(w)|^2 &\leq \sum_{\mathbf{n} \in \mathbb{N}^d} |c_{\mathbf{n}} w^{\mathbf{n}}|^2 = \sum_{\mathbf{n} \in \mathbb{N}^d} |c_{\mathbf{n}}|^2 \frac{|\mathbf{n}|!}{\mathbf{n}!} |w^{\mathbf{n}}|^2 \frac{\mathbf{n}!}{|\mathbf{n}|!} \\ &\leq \sum_{\mathbf{n} \in \mathbb{N}^d} |c_{\mathbf{n}}|^2 \frac{|\mathbf{n}|!}{\mathbf{n}!} \sum_{\mathbf{n} \in \mathbb{N}^d} |w^{\mathbf{n}}|^2 \frac{\mathbf{n}!}{|\mathbf{n}|!} = ||f||^2_{H^2_d} \frac{1}{1 - ||w||^2}. \end{aligned}$$

The functional evaluation at a point of $\mathbb{B}_d$ is then a bounded operator which makes H^2_d a RHKS. Given the above orthonormal basis, the kernel function of H^2_d is

$$\begin{aligned} k(z,w) &= \sum_{\mathbf{n} \in \mathbb{N}^d} \left\langle \left(\frac{|\mathbf{n}|!}{\mathbf{n}!} \right)^{1/2} z^{\mathbf{n}}, \left(\frac{|\mathbf{n}|!}{\mathbf{n}!} \right)^{1/2} w^{\mathbf{n}} \right\rangle \\ &= \sum_{\mathbf{n} \in \mathbb{N}^d} \left(\frac{|\mathbf{n}|!}{\mathbf{n}!} \right)^{1/2} z^{\mathbf{n}} \left(\frac{|\mathbf{n}|!}{\mathbf{n}!} \right)^{1/2} \overline{w^{\mathbf{n}}} \\ &= \sum_{\mathbf{n} \in \mathbb{N}^d} \left(\frac{|\mathbf{n}|!}{\mathbf{n}!} \right) z^{\mathbf{n}} \overline{w^{\mathbf{n}}} \\ &= \sum_{k \in \mathbb{N}} \sum_{\mathbf{n} \in \mathbb{N}^d, |\mathbf{n}| = k} \langle z, w \rangle^k \\ &= \frac{1}{1 - \langle z, w \rangle}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{C}^d$. This is clearly a several-variable generalisation of the Szegö kernel for $H^2(\mathbb{D}) = H^2_1$.

Proposition 1.12 ([4], Proposition 1.8 and Proposition 1.12)**.** *H^2_d is the reproducing kernel Hilbert space of analytic functions on $\mathbb{B}_d$ corresponding to the positive sesqui-analytic kernel function*

$$k(z,w) := \frac{1}{1 - \langle z, w \rangle}; \qquad z, w \in \mathbb{B}_d.$$

Denote $k_w : z \in \mathbb{B}_d \longmapsto k(z,w)$ the evaluation kernel at $w \in \mathbb{B}_d$, that is for all $f \in H^2_d$

$$\langle f, k_w \rangle = f(w) \quad (w \in \mathbb{B}_d).$$

The set $\{k_w| \quad w \in \mathbb{B}_d\}$ is total in H^2_d. The algebra of analytic polynomial, $\mathbb{C}[z]$, is also a dense subspace in H^2_d.

We will need to work with vector-valued Drury-Arveson spaces. If $\mathcal{Y}$ is a separable Hilbert space, the dimension of $\mathcal{Y}$ can be finite or infinite, then

$$k(z,w) := \frac{1}{1-\langle z,w\rangle} I_{\mathcal{Y}}; \qquad (z,w \in \mathbb{B}_d),$$

defines a $\mathcal{B}(\mathcal{Y})$-valued positive kernel and the corresponding RKHS is the $\mathcal{Y}$-valued Drury-Arveson space:

$$H^2_d(\mathcal{Y}) := \left\{ f = \sum_{\nu \in \mathbb{N}^d} e_\nu z^\nu \in \mathcal{O}(\mathbb{B}_d);\ \text{where } (e_\nu) \subset \mathcal{Y} \text{ and } \sum_{\nu \in \mathbb{N}^d} ||e_\nu||^2 \frac{\nu!}{|\nu|!} < \infty \right\}.$$

The set $\{k_w y, w \in \mathbb{B}_d, y \in \mathcal{Y}\}$ is total in $H^2_d(\mathcal{Y})$. The space $H^2_d(\mathcal{Y})$ can be identified with $H^2_d \otimes \mathcal{Y}$. The correspondence

$$k_w \otimes y \in H^2_d \otimes \mathcal{Y} \longmapsto k_w y \in H^2_d(\mathcal{Y})$$

defines a unitary map between $H^2_d \otimes \mathcal{Y}$ and $H^2_d(\mathcal{Y})$. Recall that, as discussed in the previous section, the RKHS $H^2_d(\mathcal{Y})$ comes along with a multiplier algebra which is

$$H^\infty_d(\mathcal{Y}) := \left\{ \phi : \mathbb{B}_d \longrightarrow \mathcal{B}(\mathcal{Y}); \phi h \in H^2_d(\mathcal{Y}) \text{ for all } h \in H^2_d(\mathcal{Y}) \right\},$$

where ϕh denotes the point-wise multiplication. More generally, given a pair of Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, the space of multipliers from $H^2_d(\mathcal{U})$ into $H^2_d(\mathcal{Y})$ will be denote $H^\infty_d(\mathcal{U}, \mathcal{Y})$. When $\mathcal{Y} = \mathbb{C}$, we simply denote H^∞_d the algebra $H^\infty_d(\mathbb{C})$.

For example:

- The coordinate functions, $z_k : x := (x_1, \ldots, x_d) \in \mathbb{B}_d \longmapsto x_k$ for $k = 1, \ldots, d$, are multipliers in H^∞_d. The multiplication M_{z_k} by the coordinate function z_k is actually a contraction on H^2_d. The tuple $S := (S_1, \ldots, S_d) = (M_{z_1}, \ldots, M_{z_d})$ is commonly called the d-shift and plays the role of the unilateral shift on the classical Hardy-Hilbert in the commutative multivariable operator theory. It is easily checked that $I - SS^* = P_0$, where $P_0 = k_0 k_0^*$ is the orthogonal projection onto $\bigvee k_0$. S is then a row partial isometry on H^2_d.
- For a polynomial $p \in \mathbb{C}[z]$, $p(z)h(z) = (p(S)h)(z)$ so that $M_p = p(S)$ and there-

fore multiplications by polynomials define bounded operators on H^2_d and therefore, polynomials are multipliers. In fact, H^∞_d is the closure of the polynomials in the weak operator topology of $\mathcal{B}(H^2_d)$.

- If $\mathcal{Y} = \mathbb{C}$ then $1 \in H^2_d$. H^∞_d is then contained in H^2_d so that H^∞ functions are holomorphic functions on $\mathbb{B}_d$. For any $f \in H^\infty_d$, $M_f = f(S)$, the Cesáro sums of the series of f converge in the strong operator topology (SOT) in $\mathcal{B}(H^2_d)$ so that one can make sense of the operator $f(S)$, and $M_f = f(S)$ [16]. For example, if f is holomorphic in a neighbourhood of $\overline{\mathbb{B}}_d$, which is the Taylor spectrum, $\sigma_T(S)$, of S [14, 38], then $f(S)$ is well defined according to Taylor's functional calculus [37] so that $f(S)$ commutes with every element of H^∞_d and therefore $f \in H^\infty_d$.

We will be interested in the analogue of Schur class multipliers in the Drury-Arveson space. We say that a multiplier $b \in H^\infty_d(\mathcal{U}, \mathcal{Y})$ is a *Shur class multiplier* or simply a *Schur multiplier* when the corresponding multiplication operator, M_b, is a contraction. We denote $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ the set of all Schur multiplier of $H^2_d(\mathcal{U})$ into $H^2_d(\mathcal{Y})$.

The canonical analogue of the shift, S, on the Hardy space, $H^2 (= H^2_1)$, is the Arveson d-shift, $S = (S_1, \ldots, S_d)$, is the d-partial isometry whose components act as multiplication by the independent variables. The multipliers are the bounded operators in $\mathcal{B}(H^2_d(\mathcal{U}), H^2_d(\mathcal{Y}))$ which intertwine the d-shifts $S \otimes I_{\mathcal{U}}$ and $S \otimes I_{\mathcal{Y}}$ [9, Theorem 5.1]. The d-shift also enjoys the following universal property.

Theorem 1.13 ([4], Theorem 6.2). *Let $\mathcal{T}_d$, the* several-variable Toeplitz C^*-algebra, *be the C^*-algebra generated by $(I, S_1, \ldots, S_d)$. $\mathbb{C}[z]$ is contained in $\mathcal{T}_d$ as a subalgebra. Given a d-contraction $T = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{H}$, the map $S_k \longmapsto T_k$, $(k = 1, \ldots, d)$ defines a completely positive unital map*

$$\psi : \mathcal{T}_d \longrightarrow \mathcal{B}(\mathcal{H}).$$

Conversely, any such completely positive map defines a d-contractions $T = (\psi(S_1), \ldots, \psi(S_d))$ on $\mathcal{H}$. The map ψ is a completely contractive homomorphism on $\mathbb{C}[z]$:

$$||p(T)|| \leq ||p(S)|| \,; \quad \textit{for all} \quad p \in \mathbb{C}[z].$$

This is the Drury-von Neumann inequality for d-contractions.

For example, if T is a pure d-contraction on a Hilbert space $\mathcal{H}$ then the completely positive map ψ can be described explicitly [4, 35]. Indeed, if T is pure then the equation

$$V^*(z^{\mathbf{n}} \otimes g) = T^{\mathbf{n}} D_{T^*} g; \qquad (\mathbf{n} \in \mathbb{N}^d, \quad g \in \mathrm{Ran}\,(D_{T^*})),$$

defines a co-isometry from $\mathcal{H}$ onto $H_d^2 \otimes \overline{\mathrm{Ran}\,(D_{T^*})}$. In this case $V^*\mathcal{H}$ is invariant for each $S_k^* \otimes I_{D_{T^*}}$ and T is unitarily equivalent to the compression of the d-shift of $H_d^2 \otimes \mathrm{Ran}\,(D_{T^*})$ to the subspace $V^*\mathcal{H}$. The map ψ in the above theorem is then given by

$$\psi(S_k) = V^*\left(S_k \otimes I_{D_{T^*}}\right) V; \qquad 1 \le k \le d.$$

1.4 Row partial isometries

This final section describes basic properties of row partial isometries, which constitutes one of the main objects of this thesis. A row partial isometry on a Hilbert space $\mathcal{H}$, is simply a partial isometry from several copies of $\mathcal{H}$ into itself, i.e. it is a row contraction on $\mathcal{H}$ which is also a partial isometry. We have already seen two important examples of row partial isometries. The d-shift, S, on the Drury-Arveson space is a row partial isometry, and the left creation tuple L, is a row isometry on the Fock space. In this section we collect some basic facts about row partial isometries.

We start with the following characterisation of CNC row partial isometries.

Theorem 1.14 ([25], Theorem 2.2)**.** *A row partial isometry X on a Hilbert space $\mathcal{H}$ is CNC if and only if*

$$\mathcal{H} = \bigvee_{\alpha \in \mathbb{F}_d^+} X^\alpha \mathrm{Ran}(X)^\perp.$$

A d-partial isometry X on a Hilbert space $\mathcal{H}$ is CNC if and only if

$$\mathcal{H} = \bigvee_{\mathbf{n} \in \mathbb{N}_d} X^{\mathbf{n}} \mathrm{Ran}(X)^\perp = \bigvee_{z \in \mathbb{B}_d} (I - Xz^*)^{-1} \mathrm{Ran}(X)^\perp.$$

These statements can be found in [6] for general row contractions and in [12] for d-contractions. Full proof of these facts can be found in [25, Section 2]. These can be extended to CNC row contractions.

Definition 1.15. Let X be a row partial isometry on $\mathcal{H}$. We say that an operator tuple T on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ is an extension of X when T coincides with X on the initial space of X: $T|_{\mathrm{Ker}(X)^\perp} = X$, equivalently $TX^*X = X$. We say that T is a co-extension when T^* agrees with X^* on the final space of X, i.e., $T^*XX^* = X^*$.

For example, any dilation of X, and in particular, its isometric dilation is an extension of X.

Lemma 1.16 ([25], Lemma 2.9). *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be a contraction and $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be a partial isometry. The following are equivalent:*

(i) T is an extension of X.

(ii) T has a decomposition $T = X + Y$ where Y maps $\mathrm{Ker}(X)$ into $\mathrm{Ran}\,(X)^\perp$.

(iii) T is a co-extension of X.

Proof. $(i) \Leftrightarrow (ii)$: If $T \supset X$ then there is a $Y : \mathrm{Ker}(X) \longrightarrow \mathcal{H}$ such that $T = X + Y$. If $f \in \mathrm{Ran}\,(X)$ then $||X^*f||^2 = ||f||^2 \geq ||T^*f||^2 = ||X^*f + Y^*f||^2$. As $\mathrm{Ran}\,(X^*) = \mathrm{Ker}(X)^\perp \perp \mathrm{Ker}(X) \supseteq \mathrm{Ran}\,(Y^*)$,

$$||X^*f + Y^*f||^2 = ||X^*f||^2 + ||Y^*f||^2 \,,$$

so that

$$||X^*f||^2 \geq ||X^*f||^2 + ||Y^*f||^2 \,.$$

It follows that $||Y^*f|| = 0$ and therefore $f \in \mathrm{Ker}(Y^*) = \mathrm{Ran}(Y)^\perp$. We proved that $Y\mathrm{Ker}(X) \subset \mathrm{Ran}\,(X)^\perp$.

Conversely $X + Y$ extends X for any $Y : \mathrm{Ker}(X) \longrightarrow \mathrm{Ran}\,(X)^\perp$.

$(ii) \Leftrightarrow (iii)$: If $T = X + Y$ then $T^* = X^* + Y^*$ extends X^*. And a similar argument as above proves that any co-extension of X can be written this way. ■

In particular, a row partial isometry on a Hilbert space $\mathcal{H}$ is a partial isometry from $\mathcal{H}^d$ to $\mathcal{H}$, and it satisfies the properties of the above lemma.

Let T be a row contraction. We denote by X_T the row partial isometry that acts like T on its initial space,

$$\mathrm{Ker}(X_T)^\perp := \left\{ \mathbf{h} \in \mathcal{H}^d | \quad ||T\mathbf{h}|| = ||\mathbf{h}|| \right\} .$$

Lemma 1.17. *The defect spaces of T and X_T satisfy*

$$\overline{\mathrm{Ran}\,(D_T)} = \mathrm{Ran}\,(D_{X_T}) = \mathrm{Ker}(X_T),$$

$$\overline{\mathrm{Ran}\,(D_{T^*})} = \mathrm{Ran}\left(D_{X_T^*}\right) = \mathrm{Ran}(X_T)^\perp.$$

Any row contraction, T, is a contractive extension of its isometric part X_T, and since $TD_T = D_{T^*}T$, it follows that T maps $\mathrm{Ran}\,(D_T)$ into $\mathrm{Ran}\,(D_{T^*})$. By definition of X_T, $\mathrm{Ker}(X_T)^\perp = \mathrm{Ran}(D_T)^\perp$, $X_T = T|_{\mathrm{Ran}(D_T)^\perp}$, and hence, $C_T := T|_{\overline{\mathrm{Ran}(D_T)}}$ is a pure contraction, i.e.,

$$||T\mathbf{h}|| < ||\mathbf{h}||\,, \qquad \text{for all } \mathbf{h} \in \mathrm{Ker}(X_T).$$

Remark 1.18. A row contraction T can then be decomposed into its isometric part, X_T, and its purely pure part, C_T,

$$T = X_T + C_T, \quad X_T = T|_{\mathrm{Ran}(D_T)^\perp}, \quad C_T = T|_{\overline{\mathrm{Ran}(D_T)}}$$

We omit the subscript T when the context is clear. We shall refer to this decomposition as the *isometric-pure decomposition of* T.

Let T be a row contraction on a Hilbert space $\mathcal{H}$, $T = X + C$ its isometric-pure decomposition. By definition of X, given a subspace $\mathcal{E}$ of $\mathcal{H}$, T^* is isometric on $\mathcal{E}$ if and only if X^* is isometric on $\mathcal{E}$. Hence, $\mathcal{E}$ is contained in $\mathrm{Ran}\,(X)$. Since T extends X, X^* acts like T^* on $\mathcal{E}$ and therefore the subspace $\mathcal{E}$ is co-invariant for T if and only if it is for X. Hence, the largest co-invariant subspace on which T^* isometrically is the largest co-invariant subspace on which X^* acts isometrically.

Proposition 1.19. *Let T be a row contraction on a Hilbert space $\mathcal{H}$, $T = X + C$ its isometric-pure decomposition. Then, T is CNC if and only if its isometric part X is CNC.*

Proposition 1.20 ([25], Corollary 2.4). *If T is a row contraction on a Hilbert space $\mathcal{H}$ and $T = X + C$ its isometric-pure decomposition, then*

$$\bigvee_{\alpha\in\mathbb{F}_d^+} T^\alpha \mathrm{Ran}\,(D_{T^*}) = \bigvee_{\alpha\in\mathbb{F}_d^+} X^\alpha \mathrm{Ran}(X)^\perp.$$

Therefore T is CNC if and only if

$$\mathcal{H} = \bigvee_{\alpha\in\mathbb{F}_d^+} T^\alpha \mathrm{Ran}\,(D_{T^*})\,.$$

A d-partial isometry X on a Hilbert space $\mathcal{H}$ is CNC if and only if

$$\mathcal{H} = \bigvee_{\mathbf{n}\in\mathbb{N}_d} T^{\mathbf{n}} \mathrm{Ran}\,(D_{T^*}) = \bigvee_{z\in\mathbb{B}_d} (I - Tz^*)^{-1} \mathrm{Ran}\,(D_T{}^*)\,.$$

If a row contraction T, not necessarily commutative, satisfies

$$\mathcal{H} = \bigvee_{z\in\mathbb{B}_d} (I - Tz^*)^{-1}\mathrm{Ran}\,(D_{T^*})\,,$$

then it is CNC [25, Corollary 2.4]. This motivates our definition of CCNC row contractions (Definition 0.2). Recall that a row contraction T is said to be CCNC if

$$\bigvee_{z\in\mathbb{B}_d} (I - Tz^*)^{-1}\mathrm{Ran}\,(D_{T^*}) = \mathcal{H}.$$

If X is the isometric part of T then the subspaces $(I - Tz^*)^{-1}\mathrm{Ran}\,(D_{T^*})$ can be expressed in terms of the *restricted range spaces* of $X - z$, for $z \in \mathbb{B}_d$, defined below.

Definition 1.21. Let X be a row partial isometry on $\mathcal{H}$. For $z \in \mathbb{B}_d$, we define the restricted range

$$\mathcal{R}(X - z) := (X - z)\mathrm{Ker}(X)^{\perp}.$$

It is clear that $\mathcal{R}(X - z) = \mathrm{Ran}\,((X - z)X^*X)$. XX^* is the orthogonal projections on the range of X, therefore $XX^*\mathrm{Ran}\,(X) = \mathrm{Ran}\,(X)$ and consequently,

$$\mathcal{R}(X - z) = (X - z)X^*\mathrm{Ran}\,(X) = (I - zX^*)\mathrm{Ran}\,(X)\,.$$

Lemma 1.22 ([25], Lemma 2.10)**.** *Let X be a row partial isometry on $\mathcal{H}$ and T be a contractive extension of X to a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. For $w \in \mathbb{B}_d$,*

$$(I - Tw^*)^{-1} : \mathcal{K} \ominus \mathrm{Ran}\,(X) \longrightarrow \mathcal{K} \ominus \mathcal{R}(X - w),$$

is an isomorphism, and hence for $z, w \in \mathbb{B}_d$,

$$(I - Tw^*)^{-1}(I - Tz^*) : \mathcal{K} \ominus \mathcal{R}(X - z) \longrightarrow \mathcal{K} \ominus \mathcal{R}(X - w),$$

is also an isomorphism.

Proof. $(I - Tw^*)^{-1} \in \mathcal{B}(\mathcal{K})$ is well defined since Tw^* is a strict contraction. For all $f \in \mathcal{K} \ominus \mathcal{R}(X - w)$ and $g \in \mathrm{Ran}\,(X)$ where $g = XG$ for some $G \in \mathcal{H}^d \ominus \mathrm{Ker}(X)$, the following computation shows that $(I - Tw^*)[\mathcal{K} \ominus \mathcal{R}(X - w)] \subset \mathcal{K} \ominus \mathrm{Ran}\,(X)$:

$$\begin{aligned}
\langle (I - Tw^*)f, g \rangle &= \langle f, (I - wT^*)XG \rangle \\
&= \langle f, (X - wX^*X)G \rangle \qquad \text{by Lemma 1.16,} \\
&= \langle f, (X - w)X^*XG \rangle \\
&= 0.
\end{aligned}$$

Since $(I - Tw^*)$ is invertible $\mathcal{K} \ominus \mathcal{R}(X - w) \subset (I - Tw^*)^{-1} [\mathcal{K} \ominus \mathrm{Ran}\,(X)]$.

Conversely, given any $f \in \mathcal{K} \ominus \mathrm{Ran}\,(X)$ and any $g \in \mathcal{R}(X - w)$ where $g = (X - w)G$ for some $G \in \mathcal{H}^d \ominus \mathrm{Ker}(X)$, we have:

$$\begin{aligned}
\left\langle (I - Tw^*)^{-1} f, g \right\rangle &= \left\langle (I - Tw^*)^{-1} f, (X - w)G \right\rangle \\
&= \left\langle (I - Tw^*)^{-1} f, (I - wX^*)XG \right\rangle \\
&= \left\langle f, (I - wT^*)^{-1}(I - wT^*)XG \right\rangle \qquad \text{by Lemma 1.16,} \\
&= \langle f, XG \rangle \\
&= 0,
\end{aligned}$$

This proves the reverse containment.

Using the fact that

$$(I - Tw^*)^{-1}[\mathcal{K} \ominus \mathrm{Ran}\,(X)] = \mathcal{K} \ominus \mathcal{R}(X - w)$$

and

$$(I - Tz^*)[\mathcal{K} \ominus \mathcal{R}(X - z)] = \mathcal{K} \ominus \mathrm{Ran}\,(X)\,,$$

it is then clear that

$$(I - Tw^*)^{-1}(I - Tz^*)\;[\mathcal{K} \ominus \mathcal{R}(X - z)] = \mathcal{K} \ominus \mathcal{R}(X - w).$$

■

The above lemma proves that $(I - Xw^*)^{-1}\mathrm{Ran}(X)^\perp = \mathcal{R}(X - z)^\perp$ so that the partial isometry X is CCNC if

$$\bigvee_{z \in \mathbb{B}_z} \mathcal{R}(X - z)^\perp = \mathcal{H}. \tag{1.3}$$

Proposition 1.23. *Let T be a row contraction on a Hilbert space $\mathcal{H}$, $T = X + C$ its isometric-pure* *decomposition. Then, T is CCNC if and only its isometric part X is CCNC.*

Proof. Recall from Lemma 1.22 that since $T \supset X$ then

$$\mathcal{R}(X - w)^{\perp} = (I - Tw^{*})^{-1}\mathrm{Ran}(X)^{\perp}.$$

Therefore

$$\begin{aligned}\bigvee_{w \in \mathbb{B}_d} \mathcal{R}(X - w)^{\perp} &= \bigvee_{w \in \mathbb{B}_d} (I - Tw^{*})^{-1}\mathrm{Ran}(X)^{\perp} \\ &= \bigvee_{w \in \mathbb{B}_d} (I - Tw^{*})^{-1}\overline{\mathrm{Ran}\,(D_{T^{*}})} \\ &= \bigvee_{w \in \mathbb{B}_d} (I - Tw^{*})^{-1}\mathrm{Ran}\,(D_{T^{*}})\,.\end{aligned}$$

■

Chapter 2
Schur class multiplier of the Drury-Arveson space

The goal of this chapter is to develop several equivalent characterisations of Schur multipliers for vector-valued Drury-Arveson space. Section 2.1 and 2.2 consist of a brief account of some useful properties of the de Branges-Rovnyak and the Herglotz space associated to a Schur multiplier. Section 2.3, develops the theory of quasi-extreme Schur multipliers, a subclass of Schur multipliers related to extreme points of the Schur class. We relate the theory of quasi-extreme functions to the representation of Schur multipliers as transfer function of a colligation operator.

As discussed in Section 1.3 in the previous chapter, a Schur multiplier is a contractive element of $H_d^\infty(\mathcal{U},\mathcal{Y})$, the space of multipliers from $H_d^2(\mathcal{U})$ into $H_d^2(\mathcal{Y})$. That is a function $b \in H_d^\infty(\mathcal{U},\mathcal{Y})$ such that the corresponding multiplication operator has norm $||M_b|| \leq 1$. We denote the set of all such functions $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$. An *inner* multiplier is a multiplier b whose corresponding multiplication operator M_b is a partial isometry.

2.1 de Branges-Rovnyak spaces

In this section, $\mathcal{U}$ and $\mathcal{Y}$ are fixed separable or finite dimensional Hilbert spaces.

Consider a Schur multiplier $b \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$. There is a $\mathcal{Y}$-valued Hilbert function space associated to b. It is a consequence of the following characterisation of the function b which satisfies $||M_b|| \leq 1$ or equivalently $I - M_bM_b^* \geq 0$.

Theorem 2.1 (Theorem 2.1. [9]). *A multiplier, $b \in H_d^\infty(\mathcal{U},\mathcal{Y})$, is a Schur multiplier if and only if*

$$(z,w) \longmapsto \frac{I_{\mathcal{Y}} - b(z)b(w)^*}{1-\langle z,w\rangle}; \qquad z,w \in \mathbb{B}_d, \tag{2.1}$$

defines a positive kernel on $\mathbb{B}_d \times \mathbb{B}_d$.

To any Schur class multiplier $b \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$, we can then associate a positive $\mathcal{B}(\mathcal{Y})$-valued kernel function on $\mathbb{B}_d \times \mathbb{B}_d$, the *de Branges-Rovnyak kernel*:

$$k^b(z,w) := \frac{I_{\mathcal{Y}} - b(z)b(w)^*}{1 - \langle z, w \rangle} \in \mathcal{B}(\mathcal{Y}); \qquad z, w \in \mathbb{B}_d.$$

Definition 2.2. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. The $\mathcal{Y}$-valued RKHS associated to the positive kernel k^b is denoted $\mathscr{H}(b) := \mathcal{H}(k^b)$. The space $\mathscr{H}(b)$ is called the *de Branges-Rovnyak space* associated to the Schur multiplier b.

As in Section 1.2 on RKHS, we will denote

$$k^b_w : z \in \mathbb{B}_d \longmapsto k^b(z,w),$$

so that the evaluation map at a point w of the unit ball $\mathbb{B}_d$ is given by $(k^b_w)^*$:

$$(k^b_w)^* h = h(w); \qquad w \in \mathbb{B}_d, h \in \mathscr{H}(b).$$

The space $\mathscr{H}(b)$ is contractively contained in $H^2_d(\mathcal{Y})$ [34], i.e., the inclusion map $i : \mathscr{H}(b) \longrightarrow H^2_d(\mathcal{Y})$ is a contraction. It follows from the fact that $k - k^b$ is a positive kernel, where k is the Drury-Arveson kernel. The space $\mathscr{H}(b)$ is a linear subspace of $H^2_d(\mathcal{Y})$ but the norm on $\mathscr{H}(b)$ is bigger than the norm on $H^2_d(\mathcal{Y})$.

Remark 2.3. Given $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, the de Branges-Rovnyak space $\mathscr{H}(b)$ can be described as a complementary range space in the sense of Sarason [34]. Let $\mathfrak{M}(b) = \operatorname{Ran}\left((I - M_b M_b^*)^{1/2}\right)$ endowed with the inner product, $\langle \cdot, \cdot \rangle_b$:

$$\left\langle (I - M_b M_b^*)^{1/2} f, (I - M_b M_b^*)^{1/2} g \right\rangle_b = \langle Pf, g \rangle ; \quad f, g \in H^2_d(\mathcal{Y}), \tag{2.2}$$

where P is the orthogonal projection of $H^2_d(\mathcal{Y})$ onto $\operatorname{Ker}((I - M_b M_b^*)^{1/2})^{\perp}$. $\mathfrak{M}(b)$ is a RKHS. To see this, let us compute $h(z)$ for an element $h = (I - M_b M_b^*)^{1/2} f \in \operatorname{Ran}\left((I - M_b M_b^*)^{1/2}\right)$. For $y \in \mathcal{Y}$:

$$\begin{aligned}
\langle h(z), y \rangle_{\mathcal{Y}} &= \left\langle (I - M_b M_b^*)^{1/2} f, k_z y \right\rangle_{H^2_d(\mathcal{Y})} \\
&= \left\langle f, (I - M_b M_b^*)^{1/2} k_z y \right\rangle_{H^2_d(\mathcal{Y})} \\
&= \left\langle (I - M_b M_b^*)^{1/2} f, (I - M_b M_b^*) k_z y \right\rangle_b \\
&= \langle h, (I - M_b M_b^*) k_z y \rangle_b .
\end{aligned}$$

This shows that the inner product defined in equation 2.2 endows $\mathfrak{M}(b)$ with a RKHS structure, where the evaluation map is given by

$$h \in \operatorname{Ran}\left((I - M_b M_b^*)^{1/2}\right) \longrightarrow h(z) = k_z^*(I - M_b M_b^*)h.$$

The following computation shows that the corresponding kernel is the de Branges-Rovnyak kernel:

$$\begin{aligned}\langle (I - M_b M_b^*)k_w y, (I - M_b M_b^*)k_z x\rangle_b &= \left\langle (I - M_b M_b^*)^{1/2} k_w y, (I - M_b M_b^*)^{1/2} k_z x\right\rangle_{H_d^2(\mathcal{Y})} \\ &= \langle k_z^*(I - M_b M_b^*)k_w y, x\rangle_{\mathcal{Y}} \\ &= \left\langle \frac{I - b(z)b(w)^*}{1 - \langle z, w\rangle} y, x \right\rangle_{\mathcal{Y}}.\end{aligned}$$

According to the bijective correspondence between positive kernel and RKHS, we infer that $\mathscr{H}(b) = \mathfrak{M}(b)$ with the inner product in equation 2.2. In this, $(I - M_b M_b^*)^{1/2}$ is a co-isometry onto $\mathfrak{M}(b)$.

When b is inner then $(I - M_b M_b^*)^{1/2}$ is already a partial isometry. The de Branges-Rovnyak space $\mathscr{H}(b) = \operatorname{Ran}\left((I - M_b M_b^*)^{1/2}\right) = \operatorname{Ran}\left((I - M_b M_b^*)\right)$ is then the orthogonal complement of the final space of M_b. Hence, $\mathscr{H}(b)$ is isometrically contained in $H_d^2(\mathcal{Y})$.

In the univariate case, de Branges-Rovnyak spaces are subspaces of the Hardy space H^2 and are invariant for the adjoint of the shift operator. When $d > 1$, this may not be the case. In the Drury-Arveson space, the Arveson d-shift S is the analogue of the shift operator, and, a de Branges-Rovnyak space is in general not invariant for the adjoint of the component operators of S. We consider a natural analogue of the adjoint of the restriction of the adjoint of the shift operator, which will play the role of the adjoint of the restricted backward shift in our model.

Definition 2.4. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. A row-operator $X \in \mathcal{B}(\mathscr{H}(b)^d, \mathscr{H}(b))$ solves the Gleason problem in $\mathscr{H}(b)$ if, for all $f \in \mathscr{H}(b)$, the adjoint operator satisfies

$$z(X^* f)(z) = f(z) - f(0), \qquad z \in \mathbb{B}_d. \tag{2.3}$$

We say that the Gleason solution X in $\mathscr{H}(b)$ is contractive (resp. extremal) if

$$XX^* \leq I - k_0^b (k_0^b)^* \qquad \text{(resp. } XX^* = I - k_0^b (k_0^b)^*\text{)}. \tag{2.4}$$

A column operator $\mathbf{b} : \mathcal{U} \longrightarrow \mathscr{H}(b)^d$ solves the Gleason problem for b if

$$z\mathbf{b}(z) = b(z) - b(0). \tag{2.5}$$

We also say that the Gleason solution for b is contractive (resp. extremal) when

$$\mathbf{b}^*\mathbf{b} \leq I - b(0)^*b(0) \qquad (\text{resp. } \mathbf{b}^*\mathbf{b} = I - b(0)^*b(0)). \tag{2.6}$$

For the scalar case $d = 1$, the de Branges-Rovnyak spaces are co-invariant for the shift operator S, and S^* is the unique solution of the Gleason problem:

$$z(S^*f) = f(z) - f(0); \quad (z \in \mathbb{D}).$$

For $d > 1$, contractive Gleason solutions always exist, but in general they are not unique [5, Theorem 2.7]. In the next section on Herglotz spaces, we will prove their existence and give an explicit construction of all Gleason solutions for $\mathscr{H}(b)$ and b.

Contractive Gleason solutions consist of a very important class of CCNC row contractions. In fact, the model theory that we will develop in Chapter 3 and Chapter 4 represents a CCNC row contraction as extremal Gleason solution in some de Branges-Rovnyak spaces. Using our RKHS notation from Section 1.2, equation 2.3 can be written as

$$(k_z^b)^* zX^*f = (k_z^b)^* - (k_0^b)^*f, \qquad f \in \mathscr{H}(b), z \in \mathbb{B}_d$$

In other words, X solves the Gleason problem for $\mathscr{H}(b)$ if and only if, for all $z \in \mathbb{B}_d$,

$$Xz^*k_z^b = k_z^b - k_0^b \qquad \text{or equivalently} \qquad k_z^b = (I - Xz^*)^{-1}k_0^b. \tag{2.7}$$

Proposition 2.5. *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and let X be a contractive Gleason solution in $\mathscr{H}(b)$: X is a CCNC row contraction.*

Proof. It is clear that X is a row contraction. Indeed, $I - XX^* \geq k_0^b(k_0^b)^* \geq 0$. Moreover, by the Douglas Factorization Lemma [18], this identity means that $\operatorname{Ran}(D_{X^*}) \supset k_0^b\mathcal{Y}$. By the above discussion, $Xw^*k_w^b = k_w^b - k_0^b$, for all $w \in \mathbb{B}_d$ or equivalently

$$k_z^b = (I - Xw^*)^{-1}k_0^b.$$

Therefore $(I - Xw^*)^{-1}\mathrm{Ran}\,(D_{X^*}) \supset (I - Xw^*)^{-1}k_0^b\mathcal{Y} = k_w^b\mathcal{Y}$. We then have

$$\mathscr{H}(b) \supset \bigvee_{w\in\mathbb{B}_d} (I - Xw^*)^{-1}\mathrm{Ran}\,(D_{X^*}) \supset \bigvee_{w\in\mathbb{B}_d} k_w^b\mathcal{Y} = \mathscr{H}(b).$$

It follows that

$$\mathscr{H}(b) = \bigvee_{w\in\mathbb{B}_d} (I - Xw^*)^{-1}\mathrm{Ran}\,(D_{X^*})\,,$$

which shows that X is CCNC. ■

There is a close relationship between contracive Gleasons solution for b and contractive Gleason solutions for $\mathscr{H}(b)$. In fact, under an additional assumption on b, there is a bijective correspondence between those two objects. This assumption needs the following definition.

Definition 2.6. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. We define the subspace $null(b) \subset \mathcal{U}$ as the set of all vectors $u \in \mathcal{U}$ which is annihilated by all the $b(z)$, $z \in \mathbb{B}_d$. That is

$$null(b) := \bigcap_{z\in\mathbb{B}_d} \mathrm{Ker}(b(z)).$$

Its orthogonal complement, the *support of* b, is denote $supp(b) := null(b)^{\perp}$:

$$supp(b) := \bigvee_{z\in\mathbb{B}_d} \mathrm{Ran}\,(b(z)^*)\,.$$

The following theorem is proved in [23] for square multipliers b (i.e. $b \in \mathcal{S}_d(\mathcal{Y})$). However, we note that the proof in [23] works for general rectangular Schur multipliers.

Theorem 2.7 ([23], Theorem 4.4)**.** *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Given a contractive solution* $\mathbf{b}$ *to the Gleason problem for b:*

$$X(\mathbf{b})^*k_w^b := w^*k_w^b - \mathbf{b}b(w)^*$$

defines a contractive Gleason solution, $X(\mathbf{b})$, for $\mathscr{H}(b)$, and any contractive Gleason solution for $\mathscr{H}(b)$ arises this way. The correspondence $\mathbf{b} \longmapsto X(\mathbf{b})$ is then a surjective map from the set of contractive Gleason solutions for b onto the set of contractive Gleason solutions for $\mathscr{H}(b)$. It preserves extremal Gleason solutions. Moreover, if we assume that $null(b) = \{0\}$ then the map is bijective.

For $d > 1$, contractive solutions of the Gleason problems are in general not unique. We examine uniqueness in the last section in this chapter. As a consequence of the above theorem, if the solution to the Gleason problem for b is unique then so is the solution to the Gleason problem for $\mathscr{H}(b)$.

2.2 Herglotz spaces

Recall that $\alpha \in \mathcal{B}(\mathcal{U}, \mathcal{Y})$ is a *strict contraction* if it is a contraction of norm strictly less than 1, $||\alpha|| < 1$, and α is a *pure contraction* if $||\alpha u|| < ||u||$ for all $u \in \mathcal{U}$, $u \neq 0$.

Definition 2.8. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ be a Schur multiplier. We say that:

1. b is *strictly contractive* if $b(z)$ is a strict contraction for all $z \in \mathbb{B}_d$.
2. b is *purely contractive* if $b(z)$ is a pure contraction for all $z \in \mathbb{B}_d$.
3. b is *non unital* if $\mathcal{U} = \mathcal{Y}$ and $I - b(z)$ is invertible for all $z \in \mathbb{B}_d$.

We easily see that strictly contractive square multipliers are non unital. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ be a Schur multiplier. One can show that b is strictly (resp. purely) contractive if and only if $b(0)$ is a strict (resp. pure) contraction [23, Remark 1.10]. In particular, if $b(0) = 0$ then b is strictly contractive. This easily follows from the relation: $b(z) = z\mathbf{b}(z)$ for all $z \in \mathbb{B}_d$ and for a contractive Gleason solution for b. Since $\mathbf{b}$ is a contraction, we have

$$||b(z)|| \leq ||z|| \, ||\mathbf{b}(z)|| \leq ||z|| < 1.$$

Proposition 2.9 ([23]). *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. If $b(0) = 0$ then b is strictly contractive.*

Lemma 2.10. *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Then $b|_{\mathrm{Ker}(D_{b(0)})} \in \mathcal{B}(\mathrm{Ker}(D_{b(0)}), \mathrm{Ker}(D_{b(0)^*}))$ is a constant unitary, and, $b|_{\mathrm{Ker}(D_{b(0)})^\perp} \in \mathcal{S}_d(\mathrm{Ker}(D_{b(0)})^\perp, \mathrm{Ker}(D_{b(0)^*})^\perp)$ is purely contractive. Any Schur multiplier $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ can be decomposed as the sum of a constant unitary b_u and a purely contractive Schur multiplier b_p.*

Proof. Let $\mathbf{b}$ be a contractive Gleason solution for b. Since $\mathbf{b}^*\mathbf{b} \leq I - b(0)^*b(0) = D_{b(0)}^2$ then we have $\mathrm{Ker}(\mathbf{b}) \supset \mathrm{Ker}(D_{b(0)})$. Hence, for all $u \in \mathrm{Ker}(D_{b(0)})$ and for all $z \in \mathbb{B}_d$

$$(b(z) - b(0))u = (k_z^b)^* z\mathbf{b}u = 0, \qquad \text{i.e.,} \qquad b(z)u = b(0)u.$$

Moreover, $b(0) : \mathrm{Ker}(D_{b(0)}) \longrightarrow \mathrm{Ker}(D_{b(0)^*})$ is unitary and $b(0) : \mathrm{Ker}(D_{b(0)})^\perp \longrightarrow \mathrm{Ker}(D_{b(0)^*})^\perp$ is a pure contraction. According to the discussion in the above paragraph, this is equivalent to $b(z)|_{\mathrm{Ker}(D_{b(0)})^\perp}$ being a pure contraction for each $z \in \mathbb{B}_d$. ■

Recall that one can define the de Branges-Rovnyak kernel, k^b, for a $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. When b is a non unital square multiplier, $b \in \mathcal{S}_d(\mathcal{Y})$, one can associate another positive kernel to b, besides the de Branges-Rovnyak kernel.

Definition 2.11. Let $b \in \mathcal{S}_d(\mathcal{Y})$ be non-unital, i.e., $I - b(z)$ is invertible for all $z \in \mathbb{B}_d$. We define the *Herglotz kernel* associated to b

$$K^b(z, w) := (I - b(z))^{-1} k^b(z, w)(I - b(w)^*)^{-1}; \qquad z, w \in \mathbb{B}_d. \tag{2.8}$$

The RKHS, $\mathcal{H}(K^b)$, associated to this positive kernel is called the *Herglotz space of* b, and we denote it $\mathscr{L}(b)$.

Lemma 2.12. *Multiplication by $(I - b)^{-1}$ defines a unitary operator from $\mathscr{H}(b)$ onto $\mathscr{L}(b)$, we denote*

$$U_b := M_{(I-b)^{-1}} : \mathscr{H}(b) \longrightarrow \mathscr{L}(b).$$

It acts on kernel functions as

$$U_b k^b_w = K^b_w (I - b(w)^*); \qquad w \in \mathbb{B}_d.$$

It is straightforward to see that K^b is a positive kernel on $\mathbb{B}_d$. A little bit more algebraic manipulations shows that

$$\begin{aligned} K^b(z, w) &= (I - b(z))^{-1} \frac{I - b(z)b(w)^*}{1 - zw^*} (I - b(w)^*)^{-1} \\ &= (I - b(z))^{-1} \frac{1}{2} \frac{(I + b(z))(I - b(w)^*) + (I - b(z))(I + b(w)^*)}{1 - zw^*} (I - b(w)^*)^{-1} \\ &= \frac{1}{2} \frac{(I - b(z))^{-1}(I + b(z)) + (I + b(w)^*)(I - b(w)^*)^{-1}}{1 - zw^*}. \end{aligned} \tag{2.9}$$

The function, $G_b(z)$, defined by

$$G_b(z) := (I - b(z))^{-1}(I + b(z)); \qquad z \in \mathbb{B}_d, \tag{2.10}$$

is called the *Herglotz function* of associated to b. We see that

$$K^b(z, w) = \frac{1}{2} \frac{G_b(z) + G_b(w)^*}{1 - zw^*}; \qquad z, w \in \mathbb{B}_d.$$

Definition 2.13. Generally, we say that a $\mathcal{B}(\mathcal{Y})$-valued analytic function, $G : \mathbb{B}_d \longrightarrow \mathcal{B}(\mathcal{Y})$, is a Herglotz-Schur class function if

$$K^G(z,w) := \frac{1}{2}\frac{G(z)+G(w)^*}{1-\langle z,w\rangle}; \qquad z,w \in \mathbb{B}_d,$$

defines a positive kernel on $\mathbb{B}_d$. We denote $\mathcal{S}^+(\mathcal{Y})$ the set of all $\mathcal{B}(\mathcal{Y})$-valued Herglotz-Schur functions on $\mathbb{B}_d$. We immediately see that the map

$$b \in \mathcal{S}_d(\mathcal{Y}) \longrightarrow G_b \in \mathcal{S}_d^+(\mathcal{Y}),$$

maps non-unitals Schu,r multipliers to Herglotz-Schur functions. This map is actually a bijection.

A densely defined operator $A \in \mathcal{B}(\mathcal{H})$ (bounded or not), on a Hilbert space $\mathcal{H}$, is said to be *accretive* if $\mathrm{Re}(\langle Ah, h\rangle) \geq 0$ for all $h \in \mathcal{H}$, and for such an operator, $A+I$ is invertible and $(A-I)(A+I)^{-1}$ is a contraction on $\mathcal{H}$ (See [36, Chapter IV, Section 4]).

Given a Herglotz-Schur function, $G \in \mathcal{B}(\mathcal{Y})$, since K^G is a positive kernel then $G(z)$ is a bounded accretive operator for all $z \in \mathbb{B}_d$. The function

$$b_G(z) := (G(z)-I)(G(w)+I)^{-1}; \qquad z \in \mathbb{B}_d,$$

is a well defined contractive $\mathcal{B}(\mathcal{Y})$-valued function on $\mathbb{B}_d$ and $I - b_G(z) = 2(G(z)+I)^{-1}$ which is invertible. Reversing the steps in the equations 2.9, we see that

$$\frac{I - b_G(z)b_G(w)^*}{1-\langle z,w\rangle} = \frac{1}{2}\frac{G(z)+G(w)^*}{1-\langle z,w\rangle}; \qquad z,w \in \mathbb{B}_d.$$

It follows that $b_G \in \mathcal{S}_d(\mathcal{Y})$.

Now we go back to describing the Gleason solutions for de Branges-Rovnyak spaces. Solutions to the Gleason problem for $\mathscr{H}(b)$ has been defined in the previous section as a row contraction X on $\mathscr{H}(b)$ such that

$$z(X^*h)(z) = h(z) - h(0); \qquad z \in \mathbb{B}_d, h \in \mathscr{H}(b),$$

or, equivalently,

$$Xz^*k_z^b = k_z^b - k_0^b; \qquad z \in \mathbb{B}_d.$$

Solutions to the Gleason problem in $\mathscr{L}(b)$ can be defined similarly: A Gleason solution for $\mathscr{L}(b)$ is a tuple D satisfying

$$Dz^* K_z^b = K_z^b - K_0^b; \qquad z \in \mathbb{B}_d.$$

We say that D is a contractive Gleason solution when it is a row contraction, and extremal when it is a co-isometry.

Theorem 2.14. *Let $b \in \mathcal{S}_d(\mathcal{Y})$ be non unital, and denote*

$$\mathfrak{D} := \bigvee_{z \in \mathcal{B}, h \in \mathcal{Y}} z^* K_z^b h \subset \mathscr{L}(b)^d.$$

The row contraction $V^b : \mathscr{L}(b)^d \longrightarrow \mathscr{L}(b)$ defined by

$$V^b z^* K_z^b y = (K_z^b - K_0^b) y; \qquad z \in \mathbb{B}_d, y \in \mathcal{Y}, \qquad \textit{and} \qquad V^b \mathbf{h} = 0; \qquad \mathbf{h} \in \mathfrak{D}^\perp, \quad (2.11)$$

is a row partial isometry. A tuple D is a contractive Gleason solution for $\mathscr{L}(b)$ if and only if it is a contractive extension of V^b.

Proof. We have

$$\begin{aligned} 2(1 - zw^*)(K_z^b)^* K_w^b &= G_b(z) + G_b(w)^* \\ 2(K_0^b)^* K_w^b &= G_b(0) + G_b(w)^* \\ 2(K_z^b)^* K_0^b &= G_b(z) + G_b(0)^* \\ 2(K_0^b)^* K_0^b &= G_b(0) + G_b(0)^*. \end{aligned}$$

Adding the first and the last equations and substracting the middle two yield

$$\begin{aligned} 0 &= 2((K_z^b)^* K_w^b - zw^*(K_z^b)^* K_w^b - (K_0^b)^* K_w^b - (K_z^b)^* K_0^b + (K_0^b)^* K_0^b) \\ &= 2((K_z^b)^*(K_w^b - K_0^b) - (K_0^b)^*(K_w^b - K_0^b) - zw^*(K_z^b)^* K_w^b) \\ &= 2([(K_z^b)^* - (K_0^b)^*][K_w^b - K_0^b] - zw^*(K_z^b)^* K_w^b). \end{aligned}$$

Therefore for all $z, w \in \mathbb{B}_d$

$$\begin{aligned} ((z^* K_z^b)^*(w^* K_w^b) &= zw^*(K_z^b)^* K_w^b \\ &= [(K_z^b)^* - (K_0^b)^*][K_w^b - K_0^b], \end{aligned}$$

so that for all $y \in \mathcal{Y}$

$$\left\langle w^* K_w^b y, z^* K_z^b y \right\rangle_{\mathscr{L}(b)^d} = \left\langle (K_w^b - K_0^b)y, (K_z^b - K_0^b)y \right\rangle_{\mathscr{L}(b)}. \qquad (2.12)$$

This shows that V^b is a row partial isometry with initial space $\mathfrak{D}$ and final space

$$\bigvee_{z \in \mathcal{B}, h \in \mathcal{Y}} (K_z^b - K_0^b)h.$$

■

We immediately see that a Gleason solution for $\mathscr{L}(b)$ always exists. The equation 2.11 can be written as

$$K_z^b = (I - V^b z^*)^{-1} K_0^b; \qquad (z \in \mathbb{B}_d), \qquad (2.13)$$

and if W is an extension of V^b then W satisfies the equation 2.11 and hence

$$K_z^b = (I - W z^*)^{-1} K_0^b; \qquad (z \in \mathbb{B}_d).$$

It follows that any contractive Gleason solutions for $\mathscr{L}(b)$, or equivalently, any contractive extensions of V^b, is a CCNC row contraction. We also notice that the solutions of the Gleason problem in $\mathscr{L}(b)$ are in general not unique. When V^b is not onto, then it admits a non-trivial co-extension, D, provided that $\mathrm{Ker}(V^b)$ is not trivial. The row partial isometry and its extension, D, then define two different contractive Gleason solutions in $\mathscr{L}(b)$.

Remark 2.15. In general, $\mathrm{Ker}(V^b)$ is not trivial. This has been proven in [23] for several interesting cases and conjectured that this is always the case: for example this is the case when $b \in \mathcal{S}_d(\mathcal{Y})$ where $\mathcal{Y}$ is finite dimensional.

We have the following relation between Gleason solutions in $\mathscr{L}(b)$ and Gleason solutions in $\mathscr{H}(b)$. This proves, in particular, that Gleason solutions for $\mathscr{H}(b)$ always exist, when b is non unital.

Theorem 2.16 ([23], Theorem 4.6)**.** *Let $b \in \mathcal{S}_d(\mathcal{Y})$ be a non unital square multiplier. The map*

$$D \longmapsto \mathbf{b}_D := U_b^* D^* K_0^b (I - b(0))$$

is a bijection between contractive Gleason solutions for $\mathscr{L}(b)$ and contractive Gleason solutions for b. It preserves extremal Gleason solutions.

Taking account of Theorem 2.7, we see that, if $b \in \mathcal{S}_d(\mathcal{Y})$ is such that $supp(b) = \mathcal{Y}$, then there is a bijective correspondence between

- contractive Gleason solutions for $\mathscr{H}(b)$,
- contractive Gleason solutions for b,
- contractive extensions of V^b in $\mathscr{L}(b)$ (these are the contractive Gleason solutions for $\mathscr{L}(b)$).

When $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, $\mathcal{U} \neq \mathcal{Y}$, then the Herglotz kernel for b is no longer defined. Instead, one can define a square multiplier uniquely determined by b.

Definition 2.17. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Without loss of generality, we can assume that either $\mathcal{U} \supset \mathcal{Y}$ or $\mathcal{Y} \supset \mathcal{U}$. The *square extension of* b is the function, $[b]$, defined by

$$[b] = \begin{cases} \begin{bmatrix} b & 0_{\mathcal{Y}\ominus\mathcal{U}} \end{bmatrix}, & \text{if } \mathcal{U} \subset \mathcal{Y}, \\ \begin{bmatrix} b \\ 0_{\mathcal{U}\ominus\mathcal{Y}} \end{bmatrix}, & \text{if } \mathcal{Y} \subset \mathcal{U}. \end{cases} \tag{2.14}$$

2.2.1 The case $\mathcal{Y} \supset \mathcal{U}$. For simplicity, let us denote $a := [b]$. We have

$$\begin{aligned} \frac{I - a(z)a(w)^*}{1 - zw^*} &= \frac{1}{1 - zw^*}\left(I - \begin{bmatrix} b(z) & 0_{\mathcal{Y}\ominus\mathcal{U}} \end{bmatrix} \begin{bmatrix} b(w)^* \\ 0_{\mathcal{Y}\ominus\mathcal{U}} \end{bmatrix}\right) \\ &= \frac{I - b(z)b(w)^*}{1 - zw^*}; \qquad z, w \in \mathbb{B}_d. \end{aligned}$$

It follows that $a \in \mathcal{S}_d(\mathcal{Y})$ and $\mathscr{H}(b) = \mathscr{H}(a)$, $\mathscr{H}(b)$ and $\mathscr{H}(a)$ have the same contractive Gleason solutions. However, it is not hard to see that, for any contractive Gleason solution, $\mathbf{a}$, for a, $\mathbf{a}|_{\mathcal{U}}$ is a contractive Gleason solution for b. Since $b = a|_{\mathcal{U}}$ then each contractive Gleason solution for b is defined in this way. Indeed, for a contractive Gleason solution, $\mathbf{b}$, for b, chose a bounded operator $\mathbf{a}' : \mathcal{U}^{\perp} \longrightarrow \mathscr{H}(b)^d$ such that $z\mathbf{a}'(z) = 0$ for all $z \in \mathbb{B}_d$ (Such $\mathbf{a}'$ always exists, take for example $\mathbf{a}' = \mathbf{0}$). Therefore, if $y = u + u_{\perp} \in \mathcal{Y}$, $u \in \mathcal{U}$ and $u_{\perp} \in \mathcal{U}^{\perp}$, then

$$z\mathbf{a}(z)y = (b(z) - b(0))u + z\mathbf{a}'(z)u_{\perp} = (a(z) - a(0))y.$$

We see that $\mathbf{a} = [\mathbf{b} \quad \mathbf{a}']$ solves the Gleason problem for a and $\mathbf{b} = \mathbf{a}|_{\mathcal{U}}$. There are "more" contractive Gleason solutions for a than contractive Gleason solutions for b in the sense that the map $\mathbf{a} \longrightarrow \mathbf{a}|_{\mathcal{U}}$ takes the set of contractive Gleason solutions for a onto the set of contractive Gleason solutions for b.

Proposition 2.18 ([25], Remark 5.2). *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $\mathcal{U} \subset \mathcal{Y}$. The square extension, $[b]$, of b coincides weakly with b:*

- $\mathscr{H}(b) = \mathscr{H}([b])$
- *$\mathscr{H}([b])$ and $\mathscr{H}(b)$ have the same contractive Gleason solutions*
- *$\mathbf{b}$ is a contractive Gleason solution for b if and only if there exists a contractive Gleason solution $[\mathbf{b}]$ for $[b]$ such that $\mathbf{b} = [\mathbf{b}]|_{\mathcal{U}}$.*

2.2.2 The case $\mathcal{U} \supset \mathcal{Y}$. Let $a := [b]$ be the square extension of b. We have

$$\begin{aligned}\frac{I - a(z)a(w)^*}{1 - zw^*} &= \frac{1}{1 - zw^*}\left(I - \begin{bmatrix} b(z) \\ 0_{\mathcal{U}\ominus\mathcal{Y}} \end{bmatrix} \begin{bmatrix} b(w)^* & 0_{\mathcal{U}\ominus\mathcal{Y}} \end{bmatrix}\right) \\ &= \begin{bmatrix} \frac{I - b(z)b(w)^*}{1 - zw^*} & 0 \\ 0 & \frac{1}{1 - zw^*} I_{\mathcal{U}\ominus\mathcal{Y}} \end{bmatrix} \\ &= \begin{bmatrix} k^b(z,w) & 0 \\ 0 & k(z,w) I_{\mathcal{U}\ominus\mathcal{Y}} \end{bmatrix},\end{aligned}$$

where k is the Drury-Arveson kernel. It follows that $a \in \mathcal{S}_d(\mathcal{U})$ and $\mathscr{H}(a) = \mathscr{H}(b) \oplus H^2_d(\mathcal{U} \ominus \mathcal{Y})$. We then have $\mathscr{H}(b) \subset \mathscr{H}(a)$ and for all $h \in \mathscr{H}(b)$,

$$(k^a_z)^* h = h(z) = (k^b_z)^* h.$$

Therefore, if X^a is a contractive Gleason solution in $\mathscr{H}(a)$, then, for all $h \in \mathscr{H}(b)$,

$$(k^a_z)^* z (X^a)^* h = (k^a_z - k^a_0)^* h = h(z) - h(0) = (k^b_z - k^b_0)^* h.$$

This means that $X^b := X^a|_{\mathscr{H}(b) \otimes \mathbb{C}^d}$ is a Gleason solution in $\mathscr{H}(b)$. Since S is a contractive Gleason solution in $H^2_d(\mathcal{U} \ominus \mathcal{Y})$, we see that $X^a = X^b \oplus S$, where S is the d-shift in $H^2_d(\mathcal{U} \ominus \mathcal{Y})$.

Conversely, if X^b is a contractive Gleason solution for $\mathscr{H}(b)$ then it is clear that $X^b \oplus S$ is a contractive Gleason solution in $\mathscr{H}(a)$. If $u = y + y_\perp \in \mathcal{U}; y \in \mathcal{Y}, y_\perp \in \mathcal{Y}^\perp$, then $k^a_z u = k^b_z y + k_z y_\perp$. As $k^b_z y \in \mathscr{H}(b)$ and $k_z y_\perp \in H^2_d(\mathcal{U} \ominus \mathcal{Y})$, we can infer that $k^b_z y = P_{\mathscr{H}(b)} k^a_z u$

and $k_z y_\perp = P_{H^2_d(\mathcal{U}\ominus\mathcal{Y})} k^a_z u$. Clearly, $X^b \oplus S$ is a contarctive Gleason solution:

$$\begin{aligned}
I_{\mathscr{H}(a)} - (X^b \oplus S)(X^b \oplus S)^* &= I_{\mathscr{H}(a)} - (X^b(X^b)^* \oplus SS^*) \\
&= (I_{\mathscr{H}(b)} - X^b(X^b)^*) \oplus (I - SS^*) \\
&\leq (I_{\mathscr{H}(b)} - k^b_0(k^b_0)^*) \oplus (I - k_0 k^*_0) \\
&= I_{\mathscr{H}(a)} - (k^b_0(k^b_0)^* \oplus k_0 k^*_0) \\
&= I_{\mathscr{H}(a)} - (P_{H^2_d(\mathcal{U}\ominus\mathcal{Y})} k^a_0 (k^a_0)^* P_{H^2_d(\mathcal{U}\ominus\mathcal{Y})} + P_{\mathscr{H}(b)} k^a_0 (k^a_0)^* P_{\mathscr{H}(b)}) \\
&= I_{\mathscr{H}(a)} - k^a_0 (k^a_0)^*.
\end{aligned}$$

Consider $u = y + y_\perp \in \mathcal{U}; y \in \mathcal{Y}, y_\perp \in \mathcal{Y}^\perp$,

$$\begin{aligned}
(X^b \oplus S) z^* k^a_z u &= (X^b \oplus S)(z^* k^b_z y + z^* k_z y_\perp) \\
&= X^b z^* k^b_z y + S z^* k_z y_\perp \\
&= (k^b_z y - k^b_0 y) + (k_z y_\perp - k_0 y_\perp) \\
&= (k^b_z y + k_z y_\perp) - (k^b_0 y + k_0 y_\perp) \\
&= k^a_z u - k^a_0 u.
\end{aligned}$$

It follows that $X^b \oplus S$ is a contractive Gleason solution in $\mathscr{H}(a)$.

Since S is the unique Gleason solution for $H^2_d(\mathcal{Y}^\perp)$, we see that $X^b \longmapsto X^b \oplus S$ is a bijection between contractive Gleason solutions for $\mathscr{H}(b)$ and $\mathscr{H}(a)$.

Proposition 2.19 ([25], Remark 5.2)**.** *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $\mathcal{Y} \subset \mathcal{U}$. We have:*

- $\mathscr{H}([b]) = \mathscr{H}(b) \oplus H^2_d(\mathcal{Y}^\perp)$
- *The map $X^{[b]} \longmapsto X^{[b]}|_{\mathscr{H}(b)\otimes\mathbb{C}^d}$ is a bijection between contractive Gleason solutions for $\mathscr{H}([b])$ and contractive Gleason solutions for $\mathscr{H}([b])$, it preserves extremal solutions.*

Assuming that the square extension of b is non unital, the Herglotz kernel $K^{[b]}$ is well defined and we can have a complete list of the contractive Gleason solutions for $\mathscr{H}(b)$ and for b from Theorem 2.7, Theorem 2.16 and the above Propositions 2.18 and 2.19.

2.3 Quasi-extreme Schur multipliers

For $d = 1$, the study of Schur multipliers splits into two cases: the extreme and the non-extreme points [34]. Extreme points of $[H^\infty]_1$ are described in [22], Chapter 9. Namely, b is an extreme point of the convex set, $[H^\infty]_1$, of Schur multipliers if and only if

$$\int_{\mathbb{T}} \ln(1 - |b(z)|)dm(z) = -\infty.$$

Let μ be an AC measure of b. The Radon-Nykodim derivative of μ with respect to the Lebesgue measure is

$$h(z) = \frac{1 - |b(z)|^2}{|1 - b(z)|^2}.$$

It follows that b is extreme if and only if

$$\int_{\mathbb{T}} \ln(h(z))dm(z) = -\infty.$$

Denote $\mathcal{P}(\mu)$ the closure of the polynomials in $L^2(\mu)$ and $\mathcal{P}_0(\mu)$ the closed linear span of the non constant analytic monomials. Szegö's theorem ([22, Chapter 4]) asserts that

$$\inf_{p \in \mathcal{P}_0(\mu)} \int_{\mathbb{T}} |1 - p|^2 d\mu = \exp\left(\int_{\mathbb{T}} \ln(h(z))dm(z) \right).$$

It is an immediate consequence of Szegö's theorem that b is an extreme point if and only if

$$\inf_{p \in \mathcal{P}_0(\mu)} \int_{\mathbb{T}} |1 - p|^2 d\mu = 0,$$

i.e, $\mathcal{P}_0(\mu)$ is dense in $\mathcal{P}(\mu)$.

Theorem 2.20. *The following assertions are equivalent:*

(i) b *is an extreme point of* $[H^\infty]_1$.

(ii) $\mathrm{dist}(1, \mathcal{P}_0(\mu)) = \inf_{p \in \mathcal{P}_0(\mu)} \int_{\mathbb{T}} |1 - p|^1 d\mu = 0.$

(iii) b *is not an element of* $\mathscr{H}(b)$.

Jury and Martin [23] studied a natural analogue of the Szegö theorem and obtained a necessary and sufficient condition for a non unital square Schur multiplier, $b \in \mathcal{S}_d(\mathcal{Y})$, to have a unique contractive Gleason solution. Namely, the several-variable analogue of the

Szegö infimum is zero if and only if there is no non-zero $y \in \mathcal{Y}$ such that $by \in \mathscr{H}(b)$ [23, Theorem 3.22]. We take this as our definition of quasi-extreme Schur multipliers.

Definition 2.21. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. We say that b is *quasi-extreme* (QE) if there is no non-zero $u \in \mathcal{U}$ such that $bu \in \mathscr{H}(b)$.

In the classical case, see item (iii) in the above theorem, this is equivalent to the extremality of b. Also, in the several-variable, scalar-valued case, it has been shown that QE always implies extreme [24]. It turns out that b being QE is the necessary and sufficient condition to guarantee the uniqueness of contractive Gleason solution in $\mathscr{H}(b)$.

Theorem 2.22 ([23], Theorem 4.4)**.** *Let $b \in \mathcal{S}_d(\mathcal{Y})$ be non unital. The following assertions are equivalent:*

(i) b is QE.

(ii) There is a unique contractive Gleason solution for $\mathscr{H}(b)$ (this solution is extremal) and $supp(b) = \mathcal{Y}$.

(iii) There is a unique contractive Gleason solution for b (this solution is extremal).

(iv) The canonical partial isometry, V^b, on $\mathscr{L}(b)$ is onto (and therefore co-isometric).

(v) There is no non zero constant function in $\mathscr{L}(b)$.

These can be extended to rectangular (not necessarily square) Schur multipliers. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $\mathcal{U} \supset \mathcal{Y}$ and $[b]$ is non unital. Therefore $[b] \in \mathcal{S}_d(\mathcal{U})$ and $[b]$ is QE if and only if it satisfies one (and therefore all) of the equivalent assertions in the previous theorem. By definition, $supp([b]) = supp(b)$, and according to Proposition 2.19, there is a bijective correspondence between contractive Gleason solutions for $\mathscr{H}([b])$ and $\mathscr{H}(b)$. To prove the above theorem for rectagular b such that $\mathcal{Y} \subset \mathcal{U}$, it is then enough to prove that b is QE if and only if $[b]$ is QE. This is the case as it is shown in the following lemma.

Lemma 2.23 ([25])**.** *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $\mathcal{U} \supset \mathcal{Y}$ and $[b]$ is non unital. Then, b is QE if and only if $[b]$ is QE.*

Proof. For simplicity, let us denote $a := [b]$. For all $u \in \mathcal{U}$, $au = bu$. If b is not QE, i.e., there is a non zero $u \in \mathcal{U}$ such that $bu \in \mathscr{H}(b)$, then $au = bu \in \mathscr{H}(b) \subset \mathscr{H}(a)$ (See Proposition 2.19) so that a is not QE.

Conversely, if there is no non zero u such that $au \in \mathscr{H}(a)$, then it is enough to prove that $bu = au \in \mathscr{H}(b)$. Let $\lambda = ||bu||$. Since $au \in \mathscr{H}(a)$, Theorem 1.9 implies that

$$\langle \cdot, (au)(w)\rangle\,(au)(z) - \lambda^2 k^a(z,w) = \begin{bmatrix} (bu)(z)(bu)(w)^* & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda^2 k^b(z,w) & 0 \\ 0 & \lambda^2 k(z,w) \end{bmatrix}$$

is a positive kernel on $\mathbb{B}_d$. It follows that

$$\langle \cdot, (bu)(w)\rangle\,(bu)(z) - \lambda^2 k^b(z,w)$$

is a positive kernel on $\mathbb{B}_d$. According to Theorem 1.9, $bu \in \mathscr{H}(b)$ and therefore b is not QE. ■

If $\mathcal{Y} \supset \mathcal{U}$, then $[b] \in \mathcal{S}_d(\mathcal{Y})$ but $\mathcal{Y}$ is strictly larger than $supp([b]) \subset \mathcal{U}$. We see that $[b]y = \in \mathscr{H}(b)$ for a $y \in \mathcal{U}^\perp$. Hence, $[b]$ is not QE by definition. However, one can still prove that there is unique contractive Gleason solution for $\mathscr{H}(b)$.

Theorem 2.24 ([25], Theorem 5.3)**.** *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that its square extension, $a := [b]$, is non unital. The following assertions are equivalent:*

- *(i) There is no non zero $u \in \mathcal{U}$ such that $bu \in \mathscr{H}(b)$, i.e., b is QE.*
- *(ii) $K_0^a(I - a(0))\mathcal{U} \subset \mathrm{Ran}\,(V^a)$.*
- *(iii) There is no non zero $\mathcal{U}$-valued constant function in $\mathscr{L}(a)$.*
- *(iv) There is a unique contractive Gleason solution for b (and this solution is extremal).*
- *(v) There is a unique contractivice Gleason solution for $\mathscr{H}(b)$ (which is also extremal) and $supp(b) = \mathcal{U}$.*

Proof. Assume first that $\mathcal{Y} \subset \mathcal{U}$. The previous Lemma says that b is QE if and only if $a := [b]$ is QE. By Theorem 2.22, a being QE is equivalent to (ii) which simply means that V^a is a co-isometry, i.e., there is a unique contractive and extremal Gleason solution for $\mathscr{H}(a)$. This in turns is equivalent to (iii) which, in the case $\mathcal{Y} \subset \mathcal{U}$, means that there is no non zero constant function in $\mathscr{L}(a)$. The bijection correspondence in Proposition 2.19 accomplishes the uniqueness and extremality of Gleason solution for $\mathscr{H}(b)$ and b.

Now assume that $\mathcal{Y} \supset \mathcal{U}$ and let us denote $a := [b] = [b \quad 0]$. Recall that in this case $\mathscr{H}(b) = \mathscr{H}(a)$.

(i) ⇒ (ii) **:** We prove the contrapositive. Assume that there is a non zero $u \in \mathcal{U}$ such that $K_0^a(I - a(0))u \notin \operatorname{Ran}(V^a)$. Chose a $\mathcal{U}$-valued constant function $f \equiv v \in \operatorname{Ran}(V^a)^\perp$ such that $\langle f, K_0^a(I - a(0))u\rangle \neq 0$. We have

$$0 \neq \langle f, K_0^a(I - a(0))u\rangle = \langle v, (I - a(0))u\rangle = \langle (I - a(0)^*)v, u\rangle ,$$

which shows that $(I - a(0)^*)v$ can not be zero. Since

$$k_0^a v = (I - aa(0)^*)v = (I - a)v + a(I - a(0)^*)v,$$

we find that $b(I - a(0)^*)v = a(I - a(0)^*)v = (I - a)v - k_0^a v = U_a^* f - k_0^a v \in \mathscr{H}(a) = \mathscr{H}(b)$.

(ii) ⇒ (iii) **:** Assume that $K_0^a(I-a(0))\mathcal{U} \subset \operatorname{Ran}(V^a)$ and let $h \equiv u$ be a $\mathcal{U}$-valued constant function in $\mathscr{L}(a)$. Since $\operatorname{Ran}(V^a)^\perp$ consists of the constant functions, $h \perp K_0^a(I - a(0))\mathcal{U}$. Therefore

$$\begin{aligned} 0 &= \langle K_0^a(I - a(0))u, h\rangle \\ &= \langle (I - a(0))u, u\rangle \\ &= \left\langle (I - a(0))u, (I - a(0))^{-1}u\right\rangle \\ &= ||u||^2 , \end{aligned}$$

proving that $u = 0$ and $h \equiv 0$.

(iii) ⇒ (i) **:** Assume that there is no non zero $\mathcal{U}$-valued constant function in $\mathscr{L}(a)$. Let $u \in \mathcal{U}$ such that $bu \in \mathscr{H}(b)$. Let $v = (I - a(0)^*)^{-1}u \in \mathcal{U}$. We have that $(I - a)v = k_0^a v - a(I - a(0)^*)v = k_0^a v - au = k_0^a v - bu \in \mathscr{H}(b)$. Therefore, the constant function $f = U_a(I - a)v \equiv v \in \mathscr{L}(a)$. It follows that $v = 0$ and hence $u = 0$.

(iii) ⇒ (iv) **:** From Theorem 2.16, we know that a contractive extension, W, of V^a defines a contractive Gleason solution for a: $\mathbf{a}_W := U_a^* W^* K_0^a(I - a(0)^*)$, where U_a is the multiplication by $(I - a)^{-1}$ which is unitary from $\mathscr{H}(a)$ onto $\mathscr{L}(a)$ (Lemma 2.12). Any Gleason solution for b is the restriction of a Gleason solution for a to $\mathcal{U}$ (Proposition 2.18). To prove that there is a unique contractive Gleason solution for b (and hence there is a unique contractive Gleason solution for $\mathscr{H}(b)$), it is enough to prove that

$$W^* K_0^a(I - a(0))u = (V^a)^* K_0^a(I - a(0))u,$$

for all $u \in \mathcal{U}$ and a contractive extension, W, of V^a. Since we assume that $K_0^a(I-a(0))\mathcal{U} \subset$

$\operatorname{Ran}(V^a)$ and W is a contractive extension of the partial isometry V^a, we have $W^* K_0^a(I - a(0))u = (V^a)^* K_0^a(I - a(0))u$, for all $u \in \mathcal{U}$. It follows that $\mathbf{b} := \mathbf{a}_{V^a} P_{\mathcal{U}}$ is the unique contractive Gleason solution for b.

Now we prove that $\mathbf{b}$ is extremal:

$$\begin{aligned}
\mathbf{b}^*\mathbf{b} &= P_{\mathcal{U}}(I - a(0)^*)(K_0^a)^* V^a U_a U_a^* (V^a)^* K_0^a (I - a(0)) P_{\mathcal{U}} \\
&= (I - b(0)^*)(K_0^a)^* P_{\operatorname{Ran}(V^a)} K_0^a (I - b(0)) \\
&= (I - b(0)^*)(K_0^a)^* K_0^a (I - b(0)) \qquad (\text{since } K_0^a(I - b(0))\mathcal{U} \subset \operatorname{Ran}(V^a)) \\
&= (I - b(0)^*) K^a(0,0)(I - b(0)) \\
&= I_{\mathcal{U}} - b(0)^* b(0).
\end{aligned}$$

$(\boldsymbol{iv}) \Leftrightarrow (\boldsymbol{v})$ is already proven in Theorem 2.7.

$(\boldsymbol{iv}) \Rightarrow (\boldsymbol{ii})$ Assume that there is a unique contractive Gleason solution for b and it is extremal. We know that this solution is $\mathbf{b} := \mathbf{b}_{V^a} P_{\mathcal{U}} = (V^a)^* K_0^a (I - a(0)) P_{\mathcal{U}}$. Therefore, we have

$$I - b(0)^* b(0) = \mathbf{b}^*\mathbf{b} = P_{\mathcal{U}}(I - a(0)^*)(K_0^a)^* V^a (V^a)^* K_0^a (I - a(0)) P_{\mathcal{U}},$$

and also

$$I - b(0)^* b(0) = (I - b(0)^*) K^a(0,0)(I - b(0)) = P_{\mathcal{U}}(I - a(0)^*)(K_0^a)^* K_0^a (I - a(0)) P_{\mathcal{U}}.$$

Combining these two identities, we get

$$P_{\mathcal{U}}(I - a(0)^*)(K_0^a)^* P_{\operatorname{Ran}(V^a)^\perp} K_0^a (I - a(0)) P_{\mathcal{U}} = 0,$$

and it follows that $K_0^a(I - a(0))\mathcal{U} \subset \operatorname{Ran}(V^a)$. ■

2.4 Transfer function representation

The following representation theorem for Schur multipliers is proved in [1, Section 8.2] and in [9, Theorem 2.1].

Theorem 2.25. *The function b is a Schur multiplier ($\in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$) if and only if there is a Hilbert* *space $\mathcal{E}$, and a unitary operator $U : \mathcal{E} \oplus \mathcal{Y} \longrightarrow \mathcal{E}^d \oplus \mathcal{U}$ such that, decomposing U as*

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{E} \oplus \mathcal{U} \longrightarrow \mathcal{E}^d \oplus \mathcal{Y},$$

then

$$b(z) = D + C(I - zA)^{-1} zB; \qquad z \in \mathbb{B}_d.$$

The operator matrix U is called a *colligation operator* and the function

$$z \longmapsto D + C(I - zA)^{-1} zB; \qquad z \in \mathbb{B}_d$$

is called the *transfer function* of the colligation U. The theorem then says that any Schur multiplier can be realised as the transfer function of some unitary colligation.

Example 2.26. Given a row contraction T on a Hilbert space $\mathcal{H}$, recall that $D_{T^*} = (I - TT^*)^{1/2}$, $D_T = (I - T^*T)^{1/2}$ and the defect spaces $\mathcal{D}_{T^*} = \overline{\operatorname{Ran}(D_{T^*})}$, $\mathcal{D}_T = \overline{\operatorname{Ran}(D_T)}$, the Nagy-Foias characteristic function of T is the $\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$-valued function defined by

$$\theta_T(z) = -T + D_{T^*}(I - zT^*)^{-1} zD_T; \qquad z \in \mathbb{B}_d.$$

θ_T is the transfer function of

$$\begin{pmatrix} -T^* & D_T \\ D_{T^*} & T \end{pmatrix} : \mathcal{H} \oplus \mathcal{D}_T \longrightarrow \mathcal{H}^d \oplus \mathcal{D}_{T^*},$$

which is unitary.

Remark 2.27. It is sufficient that a colligation U be a contraction for its transfer function to be a Schur multiplier [1, Remark 8.28][5, Remark 2.2]. The theorem is then still true if we replace the unitary colligation with a contractive one. Namely, a function b belongs to $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ if and only if there is a Hilbert space $\mathcal{E}$ and a contractive colligation W such that, decomposing W as

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{E} \oplus \mathcal{U} \longrightarrow \mathcal{E}^d \oplus \mathcal{Y},$$

then $b(z) = D + C(I - zA)^{-1} zB; \qquad z \in \mathbb{B}_d$.

Definition 2.28. Given a $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, we say that a colligation W, decomposed as

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{E} \oplus \mathcal{Y} \longrightarrow \mathcal{E}^d \oplus \mathcal{Y},$$

is a *canonical functional model (c.f.m)* colligation for b if

- The space $\mathcal{E} = \mathscr{H}(b)$.
- $A^* : \mathscr{H}(b)^d \longrightarrow \mathscr{H}(b)$ is a contractive Gleason solution for $\mathscr{H}(b)$,
- $B : \mathcal{U} \longrightarrow \mathscr{H}(b)^d$ is a contractive Gleason solution for b, and,
- The operators $C : \mathscr{H}(b) \longrightarrow \mathcal{Y}$ and $D : \mathcal{U} \longrightarrow \mathcal{Y}$ are given by

$$C^* = k_0^b, \qquad \text{and} \qquad D^* = b(0)^*.$$

It is clear that a c.f.m. is always a contractive colligation and that the transfer function of a c.f.m. for a given Schur multiplier b is b itself.

Denote

$$\mathbf{S}_\infty(b) := \bigvee_{w \in \mathbb{B}_d} w^* k_w^b \mathcal{Y}.$$

Proposition 2.29 ([5], Proposition 2.4, Remark 2.8). *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Define the operators $A_{dBR} : \mathscr{H}(b) \longrightarrow \mathscr{H}(b)^d$ and $B_{dBR} : \mathcal{U} \longrightarrow \mathscr{H}(b)^d$ by*

$$A_{dBR}^* z^* k_z^b y = (k_z^b - k_0^b)y; \quad y \in \mathcal{Y} \qquad \text{and} \qquad A_{dBR}^*|_{\mathbf{S}_\infty(b)^\perp} = 0$$
$$B_{dBR}^* z^* k_z^b y = (b(z)^* - b(0)^*)y; \quad y \in \mathcal{Y} \qquad \text{and} \qquad B_{dBR}^*|_{\mathbf{S}_\infty(b)^\perp} = 0.$$

A_{dBR} and B_{dBR} are minimal solutions in the sense that the operator $A : \mathscr{H}(b)^d \longrightarrow \mathscr{H}(b)$ is such that A^* solves the Gleason problem for $\mathscr{H}(b)$ if and only if $A^*|_{\mathbf{S}_\infty(b)} = A_{dBR}^*$ and $B : \mathcal{U} \longrightarrow \mathscr{H}(b)$ is a Gleason solution for b if and only if $B^*|_{\mathbf{S}_\infty(b)} = B_{dBR}^*$.

Theorem 2.30. *Every Schur multiplier, $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, has a c.f.m. colligation. In fact, there is a c.f.m. colligation, W_{dBR}, minimal in the sense that every c.f.m. colligation, W, for b is a contractive extension of W_{dBR}.*

This was proved in [5, Theorem 2.7]. We outline the proof here.

Proof of Theorem 2.30. We have the Kolmogorov factorisation (Theorem 1.7) for the de Bragnes-Rovnyak kernel, k^b:

$$k^b(z,w) = (k^b_z)^* k^b_w; \qquad z, w \in \mathbb{B}_d.$$

This factorisation can be written as

$$I_{\mathcal{Y}} - b(z)b(w)^* = (k^b_z)^* k^b_w - (k^b_z)^* z w^* k^b_w; \qquad z, w \in \mathbb{B}_d,$$

This can be interpreted in inner product form as

$$\left\langle \begin{bmatrix} w^* k^b_w x \\ x \end{bmatrix}, \begin{bmatrix} z^* k^b_z y \\ y \end{bmatrix} \right\rangle_{\mathscr{H}(b)^d \oplus \mathcal{Y}} = \left\langle \begin{bmatrix} k^b_w x \\ b^*(w)x \end{bmatrix}, \begin{bmatrix} k^b_z y \\ b^*(z)y \end{bmatrix} \right\rangle_{\mathscr{H}(b) \oplus \mathcal{U}}. \tag{2.15}$$

Let $C_{dBR} = (k^b_0)^*$ and $D_{dBR} = b(0)$ and let A_{dBR} and B_{dBR} be as defined Proposition 2.29. The operator colligation W_{dBR} defined by

$$W_{dBR} = \begin{pmatrix} A_{dBR} & B_{dBR} \\ C_{dBR} & D_{dBR} \end{pmatrix} : \mathscr{H}(b) \oplus \mathcal{U} \longrightarrow \mathscr{H}(b)^d \oplus \mathcal{Y},$$

acts as

$$W^*_{dBR} \begin{bmatrix} z^* k^b_z y \\ y \end{bmatrix} \in \mathscr{H}(b)^d \oplus \mathcal{Y} \longmapsto \begin{bmatrix} k^b_z y \\ b^*(z)y \end{bmatrix} \in \mathscr{H}(b) \oplus \mathcal{U}.$$

According to equation 2.15, W_{dBR} is a partial isometry from $\mathscr{H}(b) \oplus \mathcal{U}$ into $\mathscr{H}(b)^d \oplus \mathcal{Y}$, with final space

$$\mathrm{Ran}\,(W_b) = \mathbf{S}_\infty(b) \bigoplus \mathcal{Y}.$$

We have

$$\begin{aligned} A^*_{dBR} w^* k^b_w + C^*_{dBR} y &= k^b_w y, \\ B^*_{dBR} w^* k^b_w y + D^*_{dBR} y &= b(w)^* y. \end{aligned}$$

Solving these equations for $b(w)^*$, we have

$$b(w)^* y = B^*_{dBR} w^* (I - A^*_{dBR} w^*)^{-1} C^*_{dBR} y + D^*_{dBR} y,$$

which holds for any arbitrary $y \in \mathcal{Y}$, therefore

$$b(w) = D_{dBR} + C_{dBR}(I - wA_{dBR})^{-1} w B_{dBR}.$$

We see that if W is another c.f.m. colligation for b, W decomposes as

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{E} \oplus \mathcal{Y} \longrightarrow \mathcal{E}^d \oplus \mathcal{Y},$$

then necessarily we have $C = (k_0^b)^* = C_{dBR}$, $D = b(0) = D_{dBR}$, and A^* and B^* are extensions of A_{dBR}^* and B_{dBR}^* respectively (Proposition 2.29). It follows that W^* extends W_{dBR}^*, and since W_{dBR} is a partial isometry, W extends W_{dBR} (Lemma 1.16). ■

The fact that W_{dBR} is contractive (partial isometry) implies that the Gleason solutions A_{dBR} and B_{dBR} are contractive. We obtain a necessary and sufficient condition for the uniqueness of contractive Gleason solution for $\mathscr{H}(b)$.

Theorem 2.31. *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ with minimal c.f.m. colligation W_{dBR}. Then b coincides weakly with a QE Schur multiplier, i.e., $b|_{supp(b)}$ is QE, if and only if W_{dBR} is an isometry.*

Chapter 3
Model for row partial isometries

A characteristic function for contractions was defined by Sz. Nagy and Foias [36]. It is a complete unitary invariant for CNU contractions. A multivariable generalisation of characteristic function for d-contractions was also defined by Bhattacharyya, Eschmeier and Sakar [11, 12]. This multivariable characteristic function is a complete unitary invariant for pure d-contractions [11] and for CNC d-contractions [12]. In this chapter we define a characteristic function for any CCNC row partial isometry and prove that it is a unitary invariant. The row partial isometries considered here are not necessarily commutative, but when they are commutative then the newly defined characteristic function is equivalent to the multivariable Nagy-Foias characteristic function for d-contractions in [12].

In Section 3.1, we define an abstract model for a CCNC row partial isometry X on a Hilbert space $\mathcal{H}$ and we define a characteristic function b_X using a multivariable generalisation of the formulation in [21] of Livsic characteristic functions for partial isometries. In Section 3.2, we establish that any such row partial isometry, X, is unitarily equivalent to some extremal Gleason solution for $\mathscr{H}(b_X)$ which is then a partial isometry and acts as multiplication by coordinate functions on its initial space (Theorem 3.15). We use this model to show that the characteristic function is not a complete unitary invariant for general CCNC row partial isometries and to find, in Section 3.3, a large class of CCNC row partial isometries for which the characteristic function is a complete unitary invariant.

3.1 Livsic characteristic function

In this section we will construct a characteristic function for any CCNC row partial isometry by developing a multivariable generalisation of the approach in [21].

Definition 3.1. Let X be a row partial isometry on a Hilbert space $\mathcal{H}$. A *model triple* for X is a triple $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ where $\mathcal{J}_0, \mathcal{J}_\infty$ are separable Hilbert spaces which are respectively isomorphic to $\mathrm{Ran}(X)^\perp$ and $\mathrm{Ker}(X)$, and $j := ((j_z)_{z\in\mathbb{B}_d}, j_\infty)$ is a family of maps satisfying

1. $j_0 : \mathcal{J}_0 \longrightarrow \mathcal{H}$ is isometric and $\mathrm{Ran}\,(j_0) = \mathrm{Ran}(X)^\perp$;
2. $j_\infty : \mathcal{J}_\infty \longrightarrow \mathcal{H}^d$ is isometric and $\mathrm{Ran}\,(j_\infty) = \mathrm{Ker}(X)$;

3. $j_z : \mathcal{J}_0 \longrightarrow \mathcal{R}(X-z)^\perp$ is an isomorphism for all $z \in \mathbb{B}_d$.

We say that the model triple $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ is *analytic* when $z \in \mathbb{B}_d \longmapsto j_z$ is an anti-analytic function on $\mathbb{B}_d$.

Example 3.2. For a row partial isometry X on a Hilbert space $\mathcal{H}$, recall that a contractive extension T of X on $\mathcal{H}$ is a row contraction on $\mathcal{H}$ such that $TX^*X = X$. For such an extension, we can consider the model triple $(j^T, \mathcal{J}_0, \mathcal{J}_\infty)$ where $j_0^T : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(X)^\perp$ and $j_\infty^T : \mathcal{J}_\infty \longrightarrow \mathrm{Ker}(X)$ are onto isometries and we take

$$j_z^T := (I - Tz^*)^{-1} j_0^T; \qquad (z \in \mathbb{B}_d).$$

Since j_0^T is an onto isometry, it is invertible and, by the property of the family of maps $(I - Tz^*)^{-1}$ in Lemma 1.22, so is j_z for each $z \in \mathbb{B}_d$.

One can chooses the extension T to be X itself. We refer to this model as the *canonical model triple* for X. For any extension T of X on $\mathcal{H}$, the model triple $(j^T, \mathcal{J}_0, \mathcal{J}_\infty)$, and in particular the canonical model triple, is analytic.

Now we can define the Livsic-type characteristic function using any fixed model triple. Let X be a row partial isometry on a Hilbert space $\mathcal{H}$ and $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ a model triple for X. Define

$$D(z) := j_z^* j_0 \quad \text{and} \quad N(z) = (j_z^* \otimes I_d) j_\infty; \qquad (z \in \mathbb{B}_d).$$

Lemma 3.3. *Let X be a row partial isometry on a Hilbert space $\mathcal{H}$. Given any model triple $(j, \mathcal{J}_0, \mathcal{J}_\infty)$, $D(z) := j_z^* j_0 \in \mathcal{B}(\mathcal{J}_0)$ is invertible for any $z \in \mathbb{B}_d$.*

Proof. By definition of a model triple, $j_0 : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(X)^\perp$ is unitary. Also, the map $(I - Xz^*)^{-1} : \mathrm{Ran}(X)^\perp \longrightarrow \mathcal{R}(X-z)^\perp$ is an isomorphism (Lemma 1.22). Hence, for any $x, y \in \mathcal{J}_0$, since $j_z y \in \mathcal{R}(X-z)^\perp$, there exists a $y' \in \mathcal{J}_0$ such that $y = (I - Xz^*)^{-1} j_0 y'$, we have

$$\langle D(z)x, y \rangle = \langle j_0 x, j_z y \rangle = \left\langle j_0 x, (I - Xz^*)^{-1} j_0 y' \right\rangle.$$

Since $\mathrm{Ran}\,(j_0) = \mathrm{Ran}(X)^{\perp}$, we have

$$\begin{aligned}\langle D(z)x, y\rangle &= \left\langle j_0 x, \sum_{k=0}^{\infty}(Xz^*)^k j_0 y'\right\rangle \\ &= \sum_{k=0}^{\infty}\left\langle j_0 x, (Xz^*)^k j_0 y'\right\rangle \\ &= \langle j_0 x, j_0 y'\rangle \\ &= \langle x, y'\rangle .\end{aligned}$$

If we chose $x = y'$ then $\langle y', D^*(z)y\rangle = \langle D(z)y', y\rangle = ||y'||^2$, and hence $D(z)y = 0$ implies that $y' = 0$, which in turns implies that $y = 0$. It shows that $D(z)$ has trivial kernel, i.e., $\mathrm{Ran}\,(D(z))$ is dense in $\mathcal{J}_0$. Next we prove that $D(z)$ is bounded below.

Assume there is a sequence of unit vectors $(y_n)_n \subset \mathcal{J}_0$ such that $||D(z)y_n|| \longrightarrow 0$. We know that

$$(I - Xz^*)^{-1} j_0 : \mathcal{J}_0 \longrightarrow \mathcal{R}(X - z)^{\perp}$$

is an isomorphism so that the sequence $H_n = (I - Xz^*)^{-1} j_0 y_n$ is bounded away from 0: say $0 < c \leq ||H_n|| \leq C$, for some positive constant c, C. Similarly, the sequence $h_n = j_z^{-1} H_n$ is bounded away from zero: $0 < b \leq ||h_n|| \leq B$, for some positive constant b, B. On one hand, we have

$$\begin{aligned}|\langle j_z h_n, j_0 y_n\rangle| &= |\langle H_n, (I - Xz^*)H_n\rangle| \\ &= |\langle H_n, (I - zX^*Xz^*)H_n\rangle + \langle H_n, (X - zX^*X)z^*H_n\rangle| \\ &= |\langle H_n, (I - zX^*Xz^*)H_n\rangle + \langle H_n, (X - zX^*X)z^*H_n\rangle| \\ &= |\langle H_n, (I - zX^*Xz^*)H_n\rangle + \langle H_n, (I - zX^*)Xz^*H_n\rangle|.\end{aligned}$$

Since $H_n \perp \mathcal{R}(X - z) = (I - zX^*)\mathrm{Ran}\,(X)$, we have $\langle H_n, (I - zX^*)Xz^*H_n\rangle = 0$. Also

$$\begin{aligned}|\langle H_n, (I - zX^*Xz^*)H_n\rangle| &= \left|\left\langle H_n, (I - z(P_{\mathrm{Ker}(X)^{\perp}} \otimes I_d)z^*)H_n\right\rangle\right| \\ &= \left|\left\langle H_n, (H_n - ||z||^2\, P_{\mathrm{Ker}(X)^{\perp}} H_n)\right\rangle\right| \\ &= \left|\, ||H_n||^2 - ||z||^2 \left|\left|P_{\mathrm{Ker}(X)^{\perp}} H_n\right|\right|^2 \right| \\ &\geq (1 - ||z||^2)\, ||H_n|| \\ &\geq b^2(1 - ||z||^2).\end{aligned}$$

On the other hand, we have

$$|\langle j_z h_n, j_0 y_n\rangle| = |\langle h_n, D(z)y_n\rangle| \leq ||h_n|| \, ||D(z)y_n|| \leq B \, ||D(z)y_n|| \, .$$

The last quantity goes to 0 as n goes to infinity, which contradict $|\langle H_n, (I - zX^*Xz^*)H_n\rangle| \geq b^2(1 - ||z||^2)$. Therefore $D(z)$ is bounded below. ■

If $\mathrm{Ker}(X) = \{0\}$ then $\mathcal{J}_\infty = \{0\}$ and $N(z) = 0$. To avoid this case, we assume that X has non-trivial kernel.

Definition 3.4. We define the characteristic function of X as the $\mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$-valued function

$$b_X(z) := D(z)^{-1}zN(z); \quad (z \in \mathbb{B}_d).$$

If one chooses a different model $(l, \mathcal{L}_0, \mathcal{L}_\infty)$ for X and defines

$$d(z) := l_z^* l_0 \quad \text{and} \quad n(z) = (l_z^* \otimes I_d) l_\infty; \qquad (z \in \mathbb{B}_d),$$

then this model is said to be equivalent to $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ if there is a family of isomorphisms $\{C_z \in \mathcal{B}(\mathcal{J}_0, \mathcal{L}_0), (z \in \mathbb{B}_d); C_\infty\}$ such that C_0 and C_∞ are unitary and satisfy

$$l_z = j_z C_z \quad (z \in \mathbb{B}_d), \qquad \text{and} \qquad l_\infty = j_\infty C_\infty.$$

Therefore ,

$$\begin{aligned} d(z)^{-1} z n(z) &= (C_z^* j_z^* j_0 C_0)^{-1} z (C_z j_z \otimes I_d) j_\infty C_\infty \\ &= C_0^{-1} (j_z j_0)^{-1} C_z^{-1} C_z z (j_z \otimes I_d) j_\infty C_\infty \\ &= C_0^* D(z)^{-1} z N(z) C_\infty. \end{aligned}$$

That is, any two models for the same row partial isometry produce two characteristic functions that are conjugate to each other by fixed unitary operators. It is natural to identify two such characteristic functions and view them as the same object.

Note that if we chose $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ to be analytic then b_X is analytic on $\mathbb{B}_d$. Since any model triple (analytic or not) produces the same characteristic function up to conjugation by constant unitaries. it follows that any choice of model triple will yield an analytic characteristic function b_X. In fact, we will shortly prove that b_X is a Schur multiplier.

Definition 3.5. Two operator-valued functions a, b are said to coincide when there are unitary operators U, V satisfying

$$a(z) = Ub(z)V; \quad (z \in \mathbb{B}_d).$$

They are said to coincide weakly if

$$a(z)a(w)^* = Ub(z)b(w)^*U^*; \quad (z, w \in \mathbb{B}_d),$$

for some constant (independent of z) unitary operator U. Coincidence and weak coincidence define two equivalence relations for Schur multipliers. The characteristic function we define is actually an equivalence class for the weak coincidence relation.

Two Schur class functions a, b coincide weakly when there is a constant unitary operator U satisfying $k^a(z, w) = Uk^b(z, w)U^*$ or equivalently $\mathscr{H}(a) = \mathscr{H}(UbV)$ for some fixed unitaries U, V. Recall that the support of b is

$$supp(b) := \bigvee_{z \in \mathbb{B}_d} \mathrm{Ran}\,(b(w)^*) = \left(\bigcap \mathrm{Ker}(b(w))\right)^{\perp},$$

then a and b coincide weakly if and only if $a|_{supp(a)}$ coincides with $b|_{supp(b)}$ [12, Lemma 2.5].

Example 3.6. Let X be a CNC row partial isometry on a Hilbert space $\mathcal{H}$, and let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ be the canonical model triple for X, i.e., we chose $j_z = (I - Xz^*)^{-1} j_0$. Denote

$$\mathbf{S}_\infty(X) := \bigvee_{z \in \mathbb{B}_d} z^*(I - Xz^*)^{-1}\mathrm{Ran}(X)^{\perp} = \bigvee_{z \in \mathbb{B}_d} z^* \mathcal{R}(X - z)^{\perp}.$$

The support of b_X is $supp(b_X) = j_\infty^*(\mathbf{S}_\infty(X) \ominus \mathrm{Ker}(X)^{\perp})$.

Proof. Let $D(z) = j_z^* j_0$ and $N(z) = (j_z^* \otimes I_d) j_\infty$, $b_X(z) = D^{-1}(z) z N(z)$. Since $D(z)$ is invertible,

$$\begin{aligned} supp(b_X) &= \bigvee_{z\in\mathbb{B}_d} j_\infty^*(j_z\otimes I_d)^* z^* \mathcal{J}_0 \\ &= j_\infty^* \bigvee_{z\in\mathbb{B}_d} z^* j_z \mathcal{J}_0 \\ &= j_\infty^* \bigvee_{z\in\mathbb{B}_d} z^*(I - Xz^*)^{-1} j_0 \mathcal{J}_0 \\ &= j_\infty^* \bigvee_{z\in\mathbb{B}_d} z^*(I - Xz^*)^{-1} \mathrm{Ran}(X)^\perp \\ &= j_\infty^* \mathbf{S}_\infty(X) \\ &= j_\infty^*(\mathbf{S}_\infty(X) \ominus \mathrm{Ker}(X)^\perp). \end{aligned}$$

■

The characteristic function we defined is a Schur class function and it behaves naturally under unitary equivalence, i.e., it is a unitary invariant. The following tool will be useful to prove these facts.

Lemma 3.7. *Let X be a row partial isometry on a Hilbert space $\mathcal{H}$ and $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ any model triple for X. The operator*

$$\Xi(j) := \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \mathcal{H}\oplus\mathcal{J}_\infty \longrightarrow \mathcal{H}^d \oplus \mathcal{J}_0$$

is an onto isometry.

Proof. It is a straightforward computation:

$$\begin{aligned} \Xi^*\Xi &= \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix}^* \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} = \begin{pmatrix} X & j_0 \\ j_\infty^* & 0 \end{pmatrix} \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} XX^* + j_0 j_0^* & X j_\infty \\ j_\infty^* X^* & j_\infty^* j_\infty \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{J}_\infty} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \Xi\Xi^* &= \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix}^* = \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \begin{pmatrix} X & j_0 \\ j_\infty^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} X^*X + j_\infty^* j_\infty & X^* j_0 \\ j_0^* X & j_0^* j_0 \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}^d} & 0 \\ 0 & I_{\mathcal{J}_0} \end{pmatrix}. \end{aligned}$$

In the above, we used the fact that $\mathrm{Ran}\,(j_0) = \mathrm{Ran}(X)^\perp = \mathrm{Ker}(X^*)$ so that $X^* j_0 = 0$ and similarly $X j_\infty = 0$. ∎

Theorem 3.8. *The characteristic function, b_X, of a row partial isometry, X, is a Schur class multiplier.*

Proof. Let X be a row partial isometry on a Hilbert space $\mathcal{H}$ and $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ the canonical model for X (Example 3.2), b_X denotes the characteristic function. For each $z \in \mathbb{B}_d$, $D(z) = j_z^* j_0 \in \mathcal{B}(\mathcal{J}_0)$ and $zN(z) = z(j_z^* \otimes I_d) j_\infty \in \mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$. The matrix operator

$$\Xi := \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \mathcal{H} \oplus \mathcal{J}_\infty \longrightarrow \mathcal{H}^d \oplus \mathcal{J}_0$$

satisfies

$$\Xi^* \left(\begin{bmatrix} w^* j_w y \\ D(w)^* y \end{bmatrix} \right) = \left(\begin{bmatrix} j_w y \\ (wN(w))^* y \end{bmatrix} \right); \quad (w \in \mathbb{B}_d, y \in \mathcal{J}_0).$$

Indeed, for each $w \in \mathbb{B}_d$ and $y \in \mathcal{J}_0$, one has

$$\begin{aligned} & X w^* j_w^* y + j_0 D(w)^* y \\ & \quad = X w^* (I - X w^*)^{-1} j_0 y + j_0 j_0^* (I - X w^*)^{-1} j_0 y \\ & \quad = X w^* (I - X w^*)^{-1} j_0 y + (I - X w^*)^{-1} j_0 y - X X^* (I - X w^*)^{-1} j_0 y \\ & \quad = X w^* (I - X w^*)^{-1} j_0 y + (I - X w^*)^{-1} j_0 y - X X^* j_0 y - X X^* X w^* (I - X w^*)^{-1} j_0 y \\ & \quad = (I - X w^*)^{-1} j_0 y - X X^* j_0 y \\ & \quad = j_w y, \end{aligned}$$

since $X^* j_0 = 0$, and

$$\begin{aligned} j_\infty^* w^* j_w y &= j_\infty^* (j_w \otimes I_d) w^* y \\ &= (N(w)^* w^*) y = (wN(w))^* y. \end{aligned}$$

Let us now consider those two equations

$$X w^* j_w^* y + j_0 D(w)^* y = j_w y,$$

and

$$j_\infty^* w^* j_w y = (wN(w))^* y.$$

One can solve those equations for $(wN(w))^* y$. First extract an expression of $j_w y$ in the first equation to obtain

$$j_w y = (I - Xw^*)^{-1} j_0 D(w)^* y$$

and then substitute in the second equation

$$(wN(w))^* y = j_\infty^* w^* (I - Xw^*)^{-1} j_0 D(w)^* y.$$

The above equality holds for all $y \in \mathcal{J}_0$ so that one has

$$(wN(w))^* = j_\infty^* w^* (I - Xw^*)^{-1} j_0 D(w)^*.$$

Consequently,

$$\begin{aligned} wN(w) &= D(w) j_0^* (I - wX^*)^{-1} w j_\infty, \\ D(w)^{-1} wN(w) &= j_0^* (I - wX^*)^{-1} w j_\infty, \\ b_X(w) &= j_0^* (I - wX^*)^{-1} w j_\infty. \end{aligned}$$

Since Ξ is unitary, its transfer function $t(w) = j_0^*(I - wX^*)^{-1} w j_\infty$ is a Schur class multiplier (Theorem 2.25). ■

Recall that the Nagy-Foias characteristic function defined for CNC d-contractions is defined in [11] by

$$\theta_T(z) := -T|_{\overline{\text{Ran}(D_T)}} + D_{T^*}(I - zT^*)^{-1} z D_T.$$

If we denote $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ the defect spaces $\overline{\text{Ran}\,(D_T)}$ and $\overline{\text{Ran}\,(D_{T^*})}$ respectively, where $D_T = \sqrt{I - T^*T}$ and $D_{T^*} = \sqrt{I - TT^*}$ are the defect operators, then θ_T is the transfer function of the colligation

$$\begin{pmatrix} T^* & D_T \\ D_{T^*} & -T \end{pmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{D}_{T^*} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H}^d \\ \mathcal{D}_T \end{bmatrix},$$

considered as multiplier in $H_d^\infty(\mathcal{D}_T, \mathcal{D}_{T^*})$ (Example 2.26). If X is a CNC commutative row partial isometry then $D_T = P_{\text{Ran}(X)^\perp}$ is the orthogonal projection of $\mathcal{H}$ onto the orthogonal complement of the range of X, $D_{T^*} = P_{\text{Ker}(X)}$ is the orthogonal projection of $\mathcal{H}^d$ onto the kernel of X, therefore $-T|_{\text{Ran}(D_{T^*})} = 0$, so that the Nagy-Foias characteristic function of X is simply

$$\theta_X(z) = P_{\text{Ran}(X)^\perp}(I - zX^*)^{-1} z P_{\text{Ker}(X)}.$$

Proposition 3.9. *Given any commutative CNC row partial isometry X, which is then CCNC,* *the Schur multiplier b_X coincides weakly with the Nagy-Foias characteristic function θ_X.*

Proof. Let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ a model for X. We have seen in the proof of the previous theorem that b_X is the characteristic function of the matrix operator

$$\begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \mathcal{H} \oplus \mathcal{J}_\infty \longrightarrow \mathcal{H}^d \oplus \mathcal{J}_0$$

$$b_X(z) = D(z)^{-1} z N(z) = j_0^*(I - wX^*) w j_\infty.$$

By definition of the model, $j_0 : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(X)^\perp$ is an onto isometry so that $j_0 j_0^* = P_{\mathrm{Ran}(X)^\perp}$ and $j_\infty : \mathcal{J}_\infty \longrightarrow \mathrm{Ker}(X)$ is such that $j_\infty j_\infty^* = P_{\mathrm{Ker}(X)}$. Therefore one has

$$\theta_X(z) = P_{\mathrm{Ran}(X)^\perp}(I - zX^*)^{-1} z P_{\mathrm{Ker}(X)} = j_0 j_0^* I - zX^*)^{-1} z j_\infty j_\infty^* = j_0 b_X(z) j_\infty^*.$$

■

Now let us consider two partial row isometries X on $\mathcal{H}$ and Y on $\mathcal{K}$ which are unitarily equivalent. There is a unitary operator

$$U : \mathcal{H} \longrightarrow \mathcal{K}$$

such that $UX_iU^* = Y_i$ for $1 \leq i \leq d$ or equivalently

$$UY(U^* \otimes I_d) = X.$$

Let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ be a model for X, since $U^*\mathrm{Ran}(X)^\perp = \mathrm{Ran}(Y)^\perp$ and $U^*\mathrm{Ker}(X) = \mathrm{Ker}(Y)$, $l_0 := U^* j_0$ maps $\mathcal{J}_0$ isometrically onto $\mathrm{Ran}(Y)^\perp$ and $l_\infty := U^* j_\infty$ maps $\mathcal{J}_\infty$ isometrically onto $\mathrm{Ker}(Y)$. $l_z := (I - Yz^*)^{-1} l_0$. $(l, \mathcal{J}_0, \mathcal{J}_\infty)$ is a model for Y and

$$\begin{aligned} D_Y(z) &= l_z^* l_0 = l_0^*(I - zY^*)^{-1} l_0 \\ &= j_0^* U (I - zY^*)^{-1} U^* j_0 \\ &= j_0^*((I - zUY^*U^*)^{-1} j_0 = D_X(z), \end{aligned}$$

$$\begin{aligned} N_Y(z) &= (l_z^* \otimes I_d) U^* j_\infty \\ &= (j_0^* U (I - zY^*)^{-1} U^* \otimes I_d) j_\infty \\ &= (j_z \otimes I_d) j_\infty = N_X(z) \end{aligned}$$

Hence the characteristic functions of X and Y are the same.

Theorem 3.10. *The characteristic function b_X is invariant under unitary equivalence. That is, given two CCNC row partial isometries Y and X, if Y and X are unitarily equivalent then b_Y coincides with b_X.*

The characteristic function b_X is a unitary invariant for CCNC row partial isometry but, as we will show in the following section, it is not a complete unitary invariant meaning that there are non unitary equivalent partial row isometries that have the same characteristic function.

3.2 Model for CCNC row partial isometries

In [21], a model as defined in 3.2 was used to construct an *abstract model space* for partial isometries. In this section we construct a multivariable generalisation of this concept. Recall that X is CCNC row partial isometry on a Hilbert space $\mathcal{H}$ if

$$\mathcal{H} = \bigvee_{z \in \mathbb{B}_d} \mathcal{R}(X - z)^{\perp}.$$

Lemma 3.11. *Let X be a CCNC row partial isometry acting on a Hilbert space $\mathcal{H}$. Let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ be an analytic model triple for X. Denote $\mathcal{H}_j := \left\{ \hat{h} | \quad h \in \mathcal{H} \right\}$ where $\hat{h}$ is the $\mathcal{J}_0$-valued holomorphic function defined by*

$$\hat{h}(z) := j_z^* h; \qquad z \in \mathbb{B}_d.$$

Let $\left\langle \hat{f}, \hat{g} \right\rangle := \langle f, g \rangle$; for $f, g \in \mathcal{H}$. This defines a inner product on $\mathcal{H}_j$ which satisfies:

1. *$\mathcal{H}_j$ is a RKHS, the evaluation map at a point z of $\mathbb{B}_d$ is given by $(K_z^j)^* = j_z^* U_j^*$, where U_j is the unitary transformation*
$$U_j : h \in \mathcal{H} \longmapsto \hat{h} \in \mathcal{H}_j,$$
2. *the reproducing kernel of $\mathcal{H}_j$ is given by $K(z, w) = j_z^* j_w$.*

Proof. We first prove that $\left\langle \hat{f}, \hat{g} \right\rangle := \langle f, g \rangle$ defines a inner product on $\mathcal{H}_j$. The only non-trivial thing to check is that if a function $\hat{h} \in \mathcal{H}_j$ is zero everywhere then $\hat{h} = 0$. If for all

$z \in \mathbb{B}_d$, we have $0 = \hat{h}(z) = j_z^* h$, then

$$h \in \bigcap_{z\in\mathbb{B}_d} \operatorname{Ker}(j_z^*) = \bigcap_{z\in\mathbb{B}_d} \operatorname{Ran}(j_z)^\perp = \bigcap_{z\in\mathbb{B}_d} \mathcal{R}(X-z)^\perp = \{0\}\,.$$

The last equality is a consequence of the fact that X is CCNC. Therefore $\left|\left|\hat{h}\right|\right| = ||h|| = 0$, i.e., $\hat{h} = 0$.

It is clear that the operators $(K_z^j)^*\hat{h} = j_z^* h = j_z^* U_j^* U_j h = j_z^* U_j^* \hat{h}$ are bounded and they satisfy

$$\left\langle \hat{h}, \hat{K}_z^j y\right\rangle = \left\langle \hat{h}, U_j j_z y\right\rangle = \left\langle j_z^* U_j^* \hat{h}, y\right\rangle = \langle j_z^* h, y\rangle = \left\langle \hat{h}(z), y\right\rangle; \qquad y \in \mathcal{J}_0.$$

The point evaluation map is the map $K_z^* = j_z^* \hat{U}^*$ and the reproducing kernel is $K^j(z,w) = (K_z^j)^* K_w^j = j_z^* U_j^* U_j j_w = j_z^* j_w$. ∎

Recall that the canonical model triple for X (Example 3.2) is given by the model triple $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ with $j_z = (I - Xz^*)^{-1} j_0$.

Definition 3.12. Let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ be the canonical model for X. We will denote $\hat{K}$ the positive kernel function

$$\hat{K}(z,w) := j_z^* j_w = j_0^* (I - zX^*)^{-1} (I - Xw^*)^{-1} j_0; \quad (z, w \ \in \mathbb{B}_d).$$

We let $\hat{\mathcal{H}} := \mathcal{H}(\hat{K})$ be the $\mathcal{J}_0$-valued RKHS corresponding to $\hat{K}$ and we write $\hat{U}$ for the natural unitary operator mapping $\mathcal{H}$ onto $\hat{\mathcal{H}}$. We denote $\hat{X}$ the image of X under the unitary transformation $\hat{U}$: $\hat{X} := \hat{U} X \hat{U}^*$.

A natural question to ask is: how does $\hat{X}$ behave on the space $\hat{\mathcal{H}}$?

Lemma 3.13. *The row partial isometry $\hat{X}$ acts as multiplication by coordinate functions on its initial space, i.e., for $\hat{\mathbf{h}} = (\hat{h}_1, \ldots, \hat{h}_d) \in \operatorname{Ker}(\hat{X})^\perp \subset \mathcal{H}(\hat{K})^d$,*

$$(\hat{X}\hat{\mathbf{h}})(z) = z\hat{\mathbf{h}}(z) = z_1\hat{h}_1(z) + \ldots + z_d\hat{h}_d(z); \qquad z \in \mathbb{B}_d.$$

The initial space of $\hat{X}$ is a subset of $\left\{\hat{\mathbf{h}} \in \hat{\mathcal{H}}^d,\ z\hat{\mathbf{h}} \in \hat{\mathcal{H}} \text{ and } \left|\left|z\hat{\mathbf{h}}\right|\right| = \left|\left|\hat{\mathbf{h}}\right|\right|\right\}$, and

$$\operatorname{Ran}\left(\hat{X}\right) = \left\{\hat{h} \in \hat{\mathcal{H}},\ \hat{h}(0) = 0\right\} = \operatorname{Ker}(j_0^*) = \operatorname{Ran}(j_0)^\perp = (\hat{K}_0 \mathcal{J}_0)^\perp,$$

where $z\hat{\mathbf{h}}$ is the function defined on $\mathbb{B}_d$ by: $z \longmapsto z\hat{\mathbf{h}}(z)$.

Proof. For a $\hat{\mathbf{h}} \in \mathrm{Ker}(\hat{X})^{\perp}$

$$\begin{aligned}(\hat{X}\hat{\mathbf{h}})(z) &= \hat{K}_z^*(\hat{U}X\hat{U}\hat{U}^*\mathbf{h}) \\ &= j_z^*\hat{U}^*(\hat{U}X\hat{U}^*\hat{U}\mathbf{h}) \\ &= j_z^*((X-z)\mathbf{h} + z\mathbf{h}).\end{aligned}$$

Since $\mathrm{Ker}(j_z^*) = \mathrm{Ran}(j_z)^{\perp} = \mathcal{R}(X-z) = (X-z)|_{\mathrm{Ker}(X)^{\perp}}$ and $(X-z)h \in (X-z)|_{\mathrm{Ker}(X)^{\perp}}$, then we have $j_z^*(X-z)h = 0$. It follows that

$$(\hat{X}\hat{\mathbf{h}})(z) = j_z^*(z\mathbf{h}) = z(\hat{h})(z).$$

We then have the inclusion. This also implies that $\mathrm{Ran}\left(\hat{X}\right) \subset \left\{h \in \mathcal{H}(\hat{K}),\ h(0) = 0\right\}$. The equality is a consequence of the fact that if $0 = h(0) = j_0^* h$ then

$$\begin{aligned}(\hat{X}\hat{X}^*h)(z) &= j_z^*(\hat{X}\hat{X}^*h) = j_z^*(z\hat{X}^*h) \\ &= (j_z^* - j_0^*)h = h(z) - h(0) \\ &= h(z),\end{aligned}$$

i.e., $\hat{X}\hat{X}^*h = h$ so that $h \in \mathrm{Ran}\left(\hat{X}\right)$ and we have the reverse inclusion. ∎

Recall that, for any model triple $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ for X, the matrix Ξ,

$$\Xi := \begin{pmatrix} X^* & j_\infty \\ j_0^* & 0 \end{pmatrix} \hat{\mathcal{H}} \oplus \mathcal{J}_\infty \longrightarrow \hat{\mathcal{H}}^d \oplus \mathcal{J}_0,$$

is unitary. Moreover, if $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ is the canonical model triple as in the proof of Theorem 3.8 then Ξ satisfies

$$\Xi^*\left(\begin{bmatrix} w^* j_w y \\ D(w)^* y \end{bmatrix}\right) = \left(\begin{bmatrix} j_w y \\ (wN(w))^* y \end{bmatrix}\right); \quad (w \in \mathbb{B}_d, y \in \mathcal{J}_0),$$

where $D(z) = j_z^* j_0$ and $N(z) = (j_z^* \otimes I_d) j_\infty$. For any $z, w \in \mathbb{B}_d$ and $x, y \in \mathcal{J}_0$

$$\left\langle \begin{bmatrix} w^* j_w y \\ D(w)^* y \end{bmatrix}, \begin{bmatrix} z^* j_z x \\ D(z)^* x \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} j_w y \\ (wN(w))^* y \end{bmatrix}, \begin{bmatrix} j_z x \\ (zN(z))^* x \end{bmatrix} \right\rangle,$$

or equivalently,

$$\langle w^* j_w y, z^* j_z x\rangle + \langle D(w)^* y, D(z)^* x\rangle = \langle j_w y, j_z x\rangle + \langle N(w)^* w^* y, N(z) z^* x\rangle .$$

Solving for $\langle j_z^* j_w y, x\rangle$ yields

$$\langle j_w y, j_z x\rangle \langle w^* j_w y, z^* j_z x\rangle = \langle D(w)^* y, D(z)^* x\rangle - \langle N(w)^* w^* y, N(z) z^* x\rangle ,$$

$$\langle j_z^* j_w - j_z^* z w^* j_w y, x\rangle = \langle D(z)D(w)^* y - zN(z)N(w)w^* y, x\rangle ,$$

$$(1 - zw^*) \langle j_z^* j_w y, x\rangle = \langle D(z)D(w)^* y - zN(z)N(w)w^* y, x\rangle .$$

Theorem 3.14. *The reproducing kernel for $\hat{\mathcal{H}}$ is*

$$\hat{K}(z,w) = j_z^* j_w = \frac{D(z)D(w)^* - zN(z)N(w)w^*}{1 - zw^*} = D(z)\frac{I - b_X(z)b_X(w)^*}{1 - zw^*}D(w)^*.$$

Multiplication by $D(z)^{-1}$ defines a unitary mapping of $\hat{\mathcal{H}}$ onto $\mathscr{H}(b_X)$.

Recall that an extremal Gleason solution, Z, in the de Branges-Rovnyak space $\mathscr{H}(b)$ is a Gleason solution which satisfies $k_0^b(k_0^b)^* = I - ZZ^*$. The following theorem shows that any CCNC row partial isometry is unitarily equivalent to an extremal Gleason solution for a multivariable de Branges-Rovnyak space.

Theorem 3.15. *Denote $U_D : \mathscr{H}(b_X) \longrightarrow \hat{\mathcal{H}}$ the multiplication by $D(z)$. The row partial isometry $Z := U_D^* \hat{X} U_D$ acts as multiplication by coordinate functions on its initial space:*

$$\mathrm{Ker}(Z)^\perp \subset \{\mathbf{h} \in \mathscr{H}(b_X),\ z\mathbf{h} \in \mathscr{H}(b_X) \textit{ and } ||z\mathbf{h}|| = ||\mathbf{h}||\} ,$$

$$\mathrm{Ran}\,(Z) = \{h \in \mathscr{H}(b_X),\ h(0) = 0\} = (k_0^{b_X} \mathcal{J}_0)^\perp.$$

Moreover Z is an extremal solution to the Gleason problem for $\mathscr{H}(b_X)$.

Proof. Since the multiplication by D^{-1} is unitary from $\hat{\mathcal{H}}$ onto $\mathscr{H}(b_X)$, it is clear from the previous lemma that Z has the initial and final space as in the theorem, and Z acts as multiplication by z on its initial space.

Let us prove that Z is an extremal Gleason solution for $\mathscr{H}(b_X)$. Given that $\mathrm{Ran}\,(Z) =$

$(k_0^{b_X}\mathcal{J}_0)^\perp$, the projection onto the range of Z is

$$ZZ^* = I - k_0^{b_X}(k_0^{b_X})^*.$$

As Z acts by multiplication by the coordinates functions on its initial space which is $\mathrm{Ran}\,(Z^*)$, we have

$$\begin{aligned} z(Z^*h)(z) = (ZZ^*h)(z) &= [(I - k_0^{b_X}(k_0^{b_X})^*)h](z) \\ &= h(z) - k^{b_X}(z,0)h(0) = h(z) - h(0), \end{aligned}$$

since $k^{b_X}(z,0) = I - b_X(z)b_X(0)^* = I$. This completes the proof. ∎

The above theorem shows that the characteristic function of any row partial isometry, X, is a Schur class multiplier such that $b_X(0) = 0$. This prompts the natural question: Can any Schur function b such that $b(0) = 0$ arise as the characteristic function of some CCNC row partial isometry? The following results prove that this is indeed the case. The map $X \longrightarrow b_X$ is a surjection of the set of all CCNC row partial isometries onto the (weak coincidence classes) of Shur class functions that vanish at the origin.

Lemma 3.16. *Let $b \in \mathscr{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ such that $b(0) = 0$. Any extremal Gleason solution, X, for $\mathscr{H}(b)$ is a CCNC row partial isometry.*

Proof. We proved in Proposition 2.5 that any contractive Gleason solution in $\mathscr{H}(b)$ is CCNC, in particular any extremal Gleason solution is CCNC. It remains to prove that an extremal Gleason solution is a row partial isometry when $b(0) = 0$.

Let $X = Z - C$ be the isometric-pure decomposition of X. The Gleason solution X is extremal so that $\mathrm{Ran}(Z)^\perp = k_0^b\mathcal{J}_0$. Since $b(0) = 0$, $(k_0^b)^*k_0^b = k^b(0,0) = I_{\mathcal{Y}} - b(0)b(0)^* = I_{\mathcal{Y}}$, $k_0^b : \mathcal{Y} \longrightarrow \mathrm{Ran}(Z)^\perp$ is an isometry and therefore $P_{\mathrm{Ran}(Z)^\perp} = k_0^b(k_0^b)^*$. We then have

$$X^* = X^*P_{\mathrm{Ran}(Z)} + X^*P_{\mathrm{Ran}(Z)^\perp} = X^*P_{\mathrm{Ran}(Z)}.$$

We used the fact that there exists an extremal Gleason solution, $\mathbf{b}$, for b satisfying $X^*k_w^b = w^*k_w^b - \mathbf{b}b(0)^*$ so that $X^*P_{\mathrm{Ran}(Z)^\perp} = X^*k_0^b(k_0^b)^* = \mathbf{b}b(0)^*(k_0^b)^* = 0$. X is an extension of Z, it is therefore a co-extension, hence $X^*P_{\mathrm{Ran}(Z)} = Z^*$. X is then equal to its isometric part Z, this proves that X is a row partial isometry. ∎

We establish in the next Proposition that if $b(0) = 0$ then the characteristic function of an extremal Gleason solution for $\mathscr{H}(b)$ is the multiplier b itself.

Proposition 3.17. *Let $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ vanish at the origin and let X be an extremal Gleason solution for $\mathscr{H}(b)$. The characteristic function of X coincides weakly with b.*

Proof. Let X be an extremal Gleason solution for $\mathscr{H}(b)$. According to Theorem 2.7, there is an extremal Gleason solution, $\mathbf{b}$, for b such that

$$X^* k_w^b = w^* k_w^b - \mathbf{b} b(w)^*; \qquad (w \in \mathbb{B}_d).$$

Writing this relation for $w = 0$, we have $X^* k_0^b = 0$.

Since X an extremal Gleason solution in $\mathscr{H}(b)$ we have $\mathrm{Ran}(X)^\perp = k_0^b \mathcal{J}_0$, and

$$k_w^b = (I - Xw^*)^{-1} k_0^b, \text{ and } \mathcal{R}(X - w) = k_w^b \mathcal{J}_0; \quad (w \in \mathbb{B}_d).$$

Since $b(0) = 0$ then $(k_0^b)^* k_0^b = I_{\mathcal{J}_0}$ so that $j_0 := k_0^b : \mathcal{Y} \longrightarrow k_0^b \mathcal{J}_0 = \mathrm{Ran}(X)^\perp$ is an onto isometry. Let j_∞ be an onto isometry of $\mathcal{J}_\infty$ onto $\mathrm{Ker}(X)$ and $j_z := k_z$. The model $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ is then the canonical model for X and the associated model space is the RKHS associated to the kernel $\hat{K}(z, w) := j_z^* j_w = (k_z^b)^* k_w^b = k^b(z, w)$ which is the de Branges-Rovnyak space $\mathscr{H}(b)$ itself. Multiplication by $D(z) = j_z^* j_0 = (k_z^b)^* k_0^b = k^b(z, 0)$ defines a unitary operator from $\mathscr{H}(b)$ onto $\mathscr{H}(b_X)$ satisfying

$$k^b(z, w) = D(z) k^{b_X}(z, w) D(w)^*; \quad z, w \in \mathbb{B}_d,$$

i.e.

$$b(z)b(w)^* = D(z) b_X(z) b_X(w)^* D(w)^*; \quad (z, w \in \mathbb{B}_d).$$

The fact that b coincides weakly with b_X is a consequence of the fact that $D(z)$ does not depend on z, namely

$$D(z) = k^b(z, 0) = I_{\mathcal{J}_0}; \quad (z \in \mathbb{B}_d),$$

since $b(0) = 0$. ■

Remark 3.18.
- The above two results prove that the map $X \mapsto b_X$ is a bijection from CCNC row partial isometries onto weak coincidence classes of Schur functions which vanish at the origin.
- If X is any CCNC row partial isometry, Proposition 3.8 proves that our Livsic characteristic function, b_X, for X coincides weakly with the Nagy-Foias characteristic

function for X, as defined in [12]. It was proven in [12, Theorem 3.6] that the Nagy-Foias characteristic function of any CNC d-contraction (a commutative row contraction) is a complete unitary invariant for CNC d-contractions. The above theorem shows that if b is a Schur multiplier with two extremal Gleason solutions X^b and Y^b for $\mathscr{H}(b)$ which are not unitarily equivalent then X^b and Y^b still have the same characteristic function (which is b). The existence of such Schur multiplier is proven in [25, Claim 3.20]. It follows that, in contrast with the commutative setting, the Livsic type characteristic function defined here is not a complete unitary invariant for CCNC row contractions.

3.3 Quasi-extreme row partial isometries

Definition 3.19. A row partial isometry is said to be quasi-extreme (QE) when it is CCNC and its characteristic function is QE.

Recall that if $b \in \mathscr{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$, $b(0) = 0$, is QE then there is a unique contractive solution to the Gleason problem in $\mathscr{H}(b)$ and this solution is extremal (Theorem 2.24). Any row partial isometry such that its characteristic function is (or coincides weakly with) b is unitarily equivalent to this Gleason solution. It follows that the characteristic function we define here is a complete unitary invariant for row partial isometries with quasi-extreme characteristic function.

Theorem 3.20. *Two QE row partial isometries are unitarily equivalent if and only if their characteristic functions coincide weakly.*

Consider a Schur multiplier $b \in \mathscr{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ which vanishes at the origin. Any extremal Gleason solution for $\mathscr{H}(b)$ is a CCNC row partial isometry that acts as multiplication by the coordinate functions on its initial space. If b is QE, then such a extremal Gleason solution, Z^b, is guaranteed to exist and its initial space satisfies

$$\mathrm{Ker}(Z^b)^\perp \subset \mathbf{S}_\infty(b) := \bigvee_{w \in \mathbb{B}_d} w^* k_w^b \mathcal{J}_0. \tag{3.1}$$

This is proved in the next chapter, Theorem 4.23, for a QE multiplier b which does not necessarily vanish at the origin.

Conversely, the existence of an extremal Gleason solution for $\mathscr{H}(b)$ satisfying equation 3.1 implies that b coincides weakly with a QE Schur multiplier, i.e., $b|_{supp(b)}$ is QE.

Lemma 3.21. *Let $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ such that $b(0) = 0$ and $supp(b) = \mathcal{J}_\infty$. Then, b is QE if and only if there exists an extremal Gleason solution, Z^b, for $\mathscr{H}(b)$ satisfying*

$$\mathrm{Ker}(Z^b)^\perp \subset \bigvee_{w \in \mathbb{B}_d} w^* k_w^b \mathcal{J}_0.$$

Proof. Assume that there exists an extremal Gleason solution, Z^b, which satisfies $\mathrm{Ker}(Z^b)^\perp \subset \mathbf{S}_\infty(b)$. Let X be any other contractive Gleason solution for $\mathscr{H}(b)$ and $X = Z - C$ its isometric-pure decomposition. Our goal is to prove that X is extremal so that it is a row partial isometry which extends Z^b and has the same range as Z^b. This will prove that X and Z^b are the same and therefore there is a unique contractive Gleason solution for $\mathscr{H}(b)$ and b is QE.

Now let us prove that X is extremal. On one hand, since X is a contractive solution, then $I - XX^* \geq k_0^b(k_0^b)^* = I - Z^b(Z^b)^*$, i.e., $\mathrm{Ran}\,(D_{X^*}) \supset k_0^b \mathcal{J}_0 = \mathrm{Ran}(Z^b)^\perp$. On the other hand, X and Z^b are both Gleason solution so their restriction to any subspace of $\mathbf{S}_\infty(b)$ are the same. Therefore, $Z^b|_{\mathrm{Ker}}(Z^b)^\perp = X|_{\mathrm{Ker}}(Z^b)^\perp$ which shows that X acts isometrically on $\mathrm{Ker}(Z^b)^\perp$. It follows that the row partial isometry Z is an extension of Z^b, or equivalently, Z^* is an extension of $(Z^b)^*$. Consequently, $\mathrm{Ran}\left(Z^b\right) \subset \mathrm{Ran}\,(Z)$, i.e., $\mathrm{Ran}\,(D_{X^*}) \subset \mathrm{Ran}(Z)^\perp \subset \mathrm{Ran}(Z^b)^\perp = k_0^b \mathcal{J}_0$. It follows that $\mathrm{Ran}\,(D_{X^*}) = \mathrm{Ran}\left({}_0^b(k_0^b)^*\right)$, i.e., $I - XX^* = k_0^b(k_0^b)^*$. ■

This allows us to identify the largest class of row partial isometry for which the characteristic function b_X is a complete unitary invariant.

Theorem 3.22. *Let X be a row partial isometry on a Hilbert space $\mathcal{H}$. Then, X is QE if and only if X is CCNC and*

$$\mathrm{Ker}(X)^\perp \subset \bigvee_{w \in \mathbb{B}_d} w^* \mathcal{R}(X - w)^\perp.$$

Proof. Let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ be the canonical model triple (Example 3.2) for X and $b := b_X \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ be the characteristic function of X. There is a unitary operator $U : \mathcal{H} \longrightarrow \mathscr{H}(b)$ such that the row partial isometry X is unitarily equivalent to an extremal Gleason solution, Z^b, on $\mathscr{H}(b)$: $Z^b = UXU^*$. On one hand, we have

$$k_w^b \mathcal{J}_0 = (I - Z^b w^*)^{-1} k_0^b \mathcal{J}_0 = (I - Z^b w^*)^{-1} \mathrm{Ran}(Z^b)^\perp = \mathcal{R}(Z^b - w)^\perp = U^* \mathcal{R}(X - w)^\perp,$$

i.e.,

$$\mathbf{S}_\infty(b) = U^* \otimes I_d \left(\bigvee_{w \in \mathbb{B}_d} w^* \mathcal{R}(X - w)^\perp \right).$$

On the other hand, $\mathrm{Ker}(Z^b)^\perp = (U^* \otimes I_d)\mathrm{Ker}(X)^\perp$. It follows that $\mathrm{Ker}(Z^b)^\perp \subset \mathbf{S}_\infty(b)$ if and only if

$$\mathrm{Ker}(X)^\perp \subset \bigvee_{w \in \mathbb{B}_d} w^* \mathcal{R}(X - w)^\perp.$$

■

3.4 Examples

We compute the characteristic function of some examples of row partial isometries. The first example is a pure partial 2-isometry and we see that it is unitarily equivalent to the restriction of the 2-shift to $\mathscr{H}(b_X)$ which acts as multiplication by $z = (z_1, z_2)$ on its initial space. In the second example, we consider a square Schur multiplier b that vanishes at the origin and compute the characteristic function of the partial isometry V^b in Theorem 2.14.

Example 3.23. Let us consider the space $\mathcal{H} := \mathbb{C}^3$ and the tuple

$$X := \left[X_1 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right].$$

Denote by e_0, e_1, e_2 the canonical basis of $\mathcal{H}$. The defect operator of X,

$$D_{X^*} = I - XX^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \langle \cdot, e_0 \rangle \, e_0,$$

is a projection so that X is a row partial isometry and $X_1 X_2 = 0 = X_2 X_1$. $\mathrm{Ran}(X)^\perp$ is then one dimensional and is generated by e_0 so that

$$j_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \mathbb{C} \longrightarrow \mathcal{H}$$

is an isometry of $\mathbb{C}$ onto $\mathrm{Ran}(X)^{\perp}$. $\mathrm{Ker}(X)$ is a 4 dimensional subspace of H_2^2 generated by

$$\left\{ \begin{bmatrix} e_2 \\ 0 \end{bmatrix}, \begin{bmatrix} e_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_2 \end{bmatrix}, \begin{bmatrix} 0 \\ e_3 \end{bmatrix} \right\},$$

so

$$j_\infty = \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix} : \mathbb{C}^4 \longrightarrow H^2$$

is isometric onto $\mathrm{Ker}(X)$.

The corresponding canonical model for X is given by

$$j_z = (I - \bar{z}_1 X_1 - \bar{z}_2 X_2)^{-1} j_0 = \begin{pmatrix} 1 & 0 & 0 \\ -\bar{z}_1 & 1 & 0 \\ -\bar{z}_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic function is then

$$\begin{aligned} b_X(z) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & -z_1 & -z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \\ & \left[z_1 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -z_1 & -z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right. \\ & \left. + z_2 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -z_1 & -z_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \\ & = \begin{pmatrix} z_1^2 & z_1 z_2 & z_1 z_2 & z_2^2 \end{pmatrix} \in H_d^\infty(\mathbb{C}^4, \mathbb{C}). \end{aligned}$$

b_X is inner and

$$\mathscr{H}(b_X) = \mathrm{Ran}(M_{b_X})^{\perp} = \left\{ a + bz_1 + cz_2, (a,b,c) \in \mathbb{C}^3 \right\} \subset H_2^2.$$

We have that X is unitarily equivalent to the unique solution for the Gleason problem in $\mathscr{H}(b_X)$ which is the compressed 2-shift: it has the same matrix representation in the orthonormal basis $\{1, z_1, z_2\}$ of $\mathscr{H}(b_X)$.

Example 3.24. Consider $b \in \mathcal{S}_d(\mathcal{Y})$ such that $b(0) = 0$ which is not QE and such that V^b has a co-isometric extension that we will denote W. Recall that V^b is CCNC and $\mathrm{Ran}(V^b)^\perp = K_0^b\mathcal{Y}$, $\mathrm{Ker}(V^b) = \mathrm{Ran}((V^b)^*)^\perp = (V^b)^*|_{\mathrm{Ran}(V^b)^\perp} = W^*K_0^b\mathcal{Y}$ so if we let $K_\infty^b = W^*K_0^b$ then $(K^b, \mathcal{Y}, \mathcal{Y})$ is a model for V^b,

$$D(z) = (K_z^b)^*K_0^b = K^b(z,0) = (I - b(z))^{-1}$$

since $b(0) = 0$ and

$$\begin{aligned} zN(z) &= (K_z^b)^*zW^*K_0^b = \big((K_0^b)^*Wz^*K_z^b\big)^* \\ &= \big((K_0^b)^*(K_z^b - K_0^b)\big)^* = K^b(z,0) - K^b(0,0) \\ &= (I - b(z))^{-1} - I. \end{aligned}$$

The characteristic function of V^b is then

$$b_{V^b}(z) = D(z)^{-1}zN(z) = (I - b(z))((I - b(z))^{-1} - I) = b(z).$$

Chapter 4
Model for row contractions

In this Chapter we extend the de Branges-Rovnyak model for CCNC contractions to the several-variable setting. Namely we prove that any CCNC row contraction is unitarily equivalent to a contractive (and extremal) Gleason solution on a several-variable de Branges-Rovnyak space $\mathscr{H}(b_T)$ (Theorem 4.19), and we prove that the Schur multiplier b_T is a unitary invariant for CCNC row contractions (Theorem 4.17). This Schur function b_T will be called the characteristic function b_T is an operator-analogue of a Frostman shift of b_X.

Recall by Remark 3.18, the characteristic function b_X is generally not a complete unitary invariant for CCNC row partial isometries. For it to be a complete unitary invariant, one needs to restrict to quasi-extreme (QE) row contractions: we will say that a CCNC row contraction is QE if its characteristic function b_T is QE, and as for the case of row partial isometries.

4.1 Isometric part of extremal Gleason solutions

The goal of this section is to study the relation between an extremal Gleason solution for $\mathscr{H}(b)$ and its isometric part in order to extend the model in the previous chapter for CCNC row partial isometries to CCNC row contractions.

Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Let Z be the isometric part of an extremal Gleason solution, X, for $\mathscr{H}(b)$. There exists an extremal Gleason solution, $\mathbf{b}$, for b which satisfies $X^* k_w = w^* k_w - \mathbf{b} b(w)^*$, $w \in \mathbb{B}_d$ (Theorem 2.7). Write $X = Z - C$ the isometric-pure decomposition of X (Remark 1.18). By Proposition 2.5, the row contraction X is CCNC and so is its isometric part Z. It provides a large class of CCNC row partial isometries.

Let us assume first that $||b(0)|| < 1$ so that $D_{b(0)^*} = \sqrt{I - b(0)b(0)^*} = \sqrt{k^b(0,0)}$ is invertible. The operator $k_0^b \sqrt{k^b(0,0)}^{-1}$ is isometric and the orthogonal projection onto its range, $k_0^b \mathcal{Y}$, is

$$P_0 := k_0^b k^b(0,0)^{-1} (k_0^b)^* = k_0^b D_{b(0)^*}^{-2} (k_0^b)^*.$$

Denote $P_0^\perp = I - P_0 = I - k_0^b D_{b(0)^*}^{-2} (k_0^b)^*$. Since X is extremal then $I - XX^* = k_0^b (k_0^b)^*$ or equivalently $\mathrm{Ran}\,(D_{X^*}) = k_0^b \mathcal{Y}$. Hence, $\mathrm{Ran}\,(Z) = \mathrm{Ran}(D_{X^*})^\perp = (k_0^b\mathcal{Y})^\perp$ and $Z^* = X^*P_0^\perp$, it follows that

$$\begin{aligned} X^* &= X^*P_0^\perp + X^*P_0 \\ &= Z^* + X^*k_0^b D_{b(0)^*}^{-2}(k_0^b)^* \\ &= Z^* - \mathbf{b}b(0)^* D_{b(0)^*}^{-2}(k_0^b)^* \\ &= Z^* - \mathbf{b}D_{b(0)}^{-1}b(0)^* D_{b(0)^*}^{-1}(k_0^b)^*, \end{aligned}$$

i.e $C^* = \mathbf{b}D_{b(0)}^{-1}b(0)^*D_{b(0)^*}^{-1}(k_0^b)^*$.

Proposition 4.1. *Let $b \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$, $||b(0)|| < 1$, and let X be an extremal Gleason solution in $\mathscr{H}(b)$, with isometric-pure decomposition $X = Z - C$. There exists an extremal Gleason solution,* $\mathbf{b}$*, for b satisfying $X^*k_w = w^*k_w - \mathbf{b}b(w)^*$, $w \in \mathbb{B}_d$. The pure part of X is given by*

$$C = k_0^b D_{b(0)^*}^{-1} b(0) D_{b(0)}^{-1}\mathbf{b}^*.$$

Remark 4.2. Since $\mathbf{b}D_{b(0)}^{-1}$ and $k_0^bD_{b(0)^*}^{-1}$ are isometries, assuming that $b(0)$ is a strict contraction, we can infer that the pure part of an extremal Gleason solution in $\mathscr{H}(b)$ is a strict row contraction.

Lemma 4.3. *Let $b \in \mathcal{S}_d(\mathcal{U},\mathcal{Y})$, $||b(0)|| < 1$, and let X be an extremal Gleason solution in $\mathscr{H}(b)$, with isometric-pure decomposition $X = Z - C$. There is a Gleason solution,* $\mathbf{b}$*, for b which satisfies the following inclusions:*

$$\mathrm{Ker}(C)^\perp \subset \mathrm{Ran}\,(\mathbf{b}) \subset \mathrm{Ker}(Z), \qquad \textit{and} \qquad X^*k_w^b = w^*k_w^b - \mathbf{b}b(w)^*.$$

Notice that since $||b(0)|| < 1$, we have $\mathbf{b}D_{b(0)}^{-1}$ is an isometry, as in Remark 4.2. Therefore $\mathrm{Ran}\,(\mathbf{b}) = \mathrm{Ran}\,\big(\mathbf{b}D_{b(0)}\big)$ is closed so that $\mathrm{Ran}\,(\mathbf{b}) = \overline{\mathrm{Ran}\,(\mathbf{b})}$.

Proof. Chose an extremal Gleason solution for b, $\mathbf{b}'$, which satisfies $X^*k_w^b = w^*k_w^b - \mathbf{b}'b(w)^*$. Let P be the orthogonal projection onto $supp(b)$. Then $\mathbf{b} := \mathbf{b}'P$ is still a Gleason solution for b:

$$b(z) - b(0) = (b(z) - b(0))P = z\mathbf{b}'(z)P = z\mathbf{b}(z); \qquad z \in \mathbb{B}_d.$$

Moreover, $\mathbf{b}$ is contractive. In fact,

$$I - b(0)^*b(0) \geq P(I - b(0)^*b(0))P = P(\mathbf{b}'^*\mathbf{b}')P = \mathbf{b}^*\mathbf{b}.$$

The Gleason solution $\mathbf{b}$ still satisfies $X^* k_w^b = w^* k_w^b - \mathbf{b} b(w)^*$. By Proposition 4.1, $C^* = \mathbf{b} b(0)^* D_{b(0)^*}^{-2} (k_0^b)^*$, and it is clear that we have $\mathrm{Ker}(C)^\perp = \overline{\mathrm{Ran}\,(C^*)} \subset \overline{\mathrm{Ran}\,(\mathbf{b})}$. It remains to prove the second containment. On one hand we have

$$\begin{aligned}\mathrm{Ran}\,(\mathbf{b}) &= \mathbf{b}\mathcal{U} = \mathbf{b}' P \mathcal{U} = \mathbf{b}' supp(b) \\ &= \mathbf{b}' \bigvee_{w \in \mathbb{B}_d} b(w^*)\mathcal{Y} = \bigvee_{w \in \mathbb{B}_d} \mathbf{b}' b(w)^* \mathcal{Y} \\ &= \bigvee_{w \in \mathbb{B}_d} (w^* - X^*) k_w^b \mathcal{Y} \\ &= \bigvee_{w \in \mathbb{B}_d} (w^* - X^*)(I - X w^*)^{-1} k_0^b \mathcal{Y} \\ &= \bigvee_{w \in \mathbb{B}_d} (w^* - X^*)(I - X w^*)^{-1} D_{X^*} \mathscr{H}(b).\end{aligned}$$

On the other hand, if we denote θ_X, $\theta_X(z) := -X + D_{X^*}(I - wX)^{-1} w D_X$, the Nagy-Foias characteristic function of X, then,

$$D_X \theta_X(w)^* = (w^* - X^*)(I - X w^*)^{-1} D_{X^*},$$

hence $\mathrm{Ran}\,(\mathbf{b}) \subset \mathrm{Ran}\,(D_X) \subset \mathrm{Ker}(Z)$. The above identity is a consequence of the following computation:

$$\begin{aligned}\theta_X(w) D_X &= -X D_X + D_{X^*}(I - wX^*)^{-1} w D_X^2 \\ &= -D_{X^*} X + D_{X^*}(I - wX^*)^{-1} w (I \otimes I_d - X^* X) \\ &= D_{X^*}(I - wX^*)^{-1}(-(I - wX^*) + ww(I \otimes I_d - X^* X) \\ &= D_{X^*}(I - wX^*)^{-1}(-I + wX^* X + w - wX^* X) \\ &= D_{X^*}(I - wX^*)^{-1}(w - X).\end{aligned}$$

■

Now we construct the model that we are going to use to compute the characteristic function of the isometric part, Z, of an extremal Gleason solution, X, in $\mathscr{H}(b)$. Recall that assuming $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ and $b(0)$ pure is enough, we assume that $||b(0)|| < 1$ to avoid complications. Denote $supp(b) = \mathcal{J} \subset \mathcal{J}_\infty$.

Chose a contractive Gleason solution for b, $\mathbf{b}$, such that $X^* k_w^b = w^* k_w^b - \mathbf{b} b(w)^*$ and $\mathrm{Ran}\,(\mathbf{b}) \subset \mathrm{Ker}(X)$ (Lemma 4.3). According to Remark 4.2, $k_0^b D_{b(0)^*}^{-1}$ and $\mathbf{b} D_{b(0)}^{-1}|_{\mathcal{J}}$ are (uniquely extend to some) isometries that we will denote $\jmath_0$ and $\jmath_\infty$ respectively. Therefore we can choose a Hilbert space $\mathcal{J}'_\infty$ and $j'_\infty : \mathcal{J}'_\infty \longrightarrow \mathrm{Ker}(Z)$ such that $\mathcal{J}_\infty = \mathcal{J} \oplus \mathcal{J}'_\infty$ and

$$j_\infty := \jmath_\infty \oplus j'_\infty : \mathcal{J} \oplus \mathcal{J}'_\infty \longrightarrow \mathrm{Ker}(Z),$$

is an onto isometry. Also, $\mathrm{Ran}(Z)^\perp = \bigvee k_0 \mathcal{J}_0$ so we define an onto isometry j_0 by

$$j_0 := k_0^b D_{b(0)^*}^{-1} : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(Z)^\perp,$$

Since $\mathcal{R}(Z-z)^\perp = (I - Xz^*)^{-1}\mathrm{Ran}\,(Z) = \bigvee k_z^b \mathcal{J}_0$, we define

$$j_z := (I - Xz^*)^{-1} j_0 = k_z^b D_{b(0)^*}^{-1},$$

and we see that $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ is the canonical model triple for Z.

Theorem 4.4. *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $||b(0)|| < 1$. Let $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ be the model defined above, if b_Z is the resulting characteristic function of Z then*

$$b_Z(z) = D_{b(0)^*}(I - b(z)b(0)^*)^{-1}(b(z) - b(0))D_{b(0)}^{-1}.$$

Proof. Using the model $(j, \mathcal{J}_0, \mathcal{J}_\infty)$, we have

$$D(z) = j_z^* j_0 = D_{b(0)^*}^{-1}(k_z^b)^* k_0^b D_{b(0)^*}^{-1} = D_{b(0)^*}^{-1}(I - b(z)b(0)^*)D_{b(0)^*}^{-1},$$

and

$$zN(z) = j_z^* z j_\infty = D_{b(0)^*}^{-1}(k_z^b)^* z j_\infty.$$

Recall that

$$\begin{aligned} supp(b_Z) &= j_\infty^* \bigvee_{z \in \mathbb{B}_d} z^* \mathcal{R}(Z - z)^\perp \\ &= j_\infty^* \mathbf{S}_\infty(b) \\ &= j_\infty^* (\mathbf{S}_\infty(b) \ominus \mathrm{Ker}(Z)^\perp) \\ &= j_\infty^* \mathrm{Ran}\,(\mathbf{b}) \\ &= \mathcal{J}, \end{aligned}$$

hence $zN(z)|_{supp(b_Z)} = (k_z^b)^* z\mathbf{b}D_{b(0)}^{-1} = (b(z) - b(0))D_{b(0)}^{-1}$. Therefore

$$\begin{aligned} b_Z(z)|_{supp(b_Z)} &= D(z)^{-1} zN(z)|_{supp(b_Z)} \\ &= D_{b(0)^*}(I - b(z)b(0)^*)^{-1}(b(z) - b(0))D_{b(0)}^{-1}. \end{aligned}$$

■

This formula is an operator version of the Frostman shift of a Schur class function on the disk first introduced in [20]. In Section 2 below, we will analyse the Frostman shift transformation of Schur class multipliers in detail.

4.2 Frostman shift

We let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces and let $\alpha, \zeta \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be contractions. We make the additional assumption that $\alpha \neq 0$ and the defect operators of α are invertible. This happens for example when $||\alpha|| < 1$, i.e., when α is a strict contraction. With this additional assumption we can define two transformations of ζ:

$$\mathfrak{M}_\alpha(\zeta) := D_{\alpha^*}(I_{\mathcal{H}_2} - \zeta\alpha^*)^{-1}(\zeta - \alpha)D_\alpha^{-1},$$

and,

$$\Phi_\alpha(\zeta) := (\mathfrak{M}_{-\alpha^*}(\zeta^*))^* = D_{\alpha^*}^{-1}(\zeta + \alpha)(I_{\mathcal{H}_1} + \alpha^*\zeta)^{-1}D_\alpha.$$

Recall that $D_\alpha = \sqrt{I - \alpha\alpha^*}$ denote the defect operator of α. Both $\mathfrak{M}_\alpha(\zeta)$ and $\Phi_\alpha(\zeta)$ belong $(\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2))_1$, This will follow from the following lemmas.

Lemma 4.5. *The transformation $\mathfrak{M}_\alpha$ satisfies the following identities:*

(i) $I_{\mathcal{H}_2} - \mathfrak{M}_\alpha(\zeta)\mathfrak{M}_\alpha(\lambda)^* = D_{\alpha^*}(I - \zeta\alpha^*)^{-1}(I - \zeta\lambda^*)(I - \alpha\lambda^*)^{-1}D_{\alpha^*}.$

(ii) $I_{\mathcal{H}_1} - \mathfrak{M}_\alpha(\lambda)^*\mathfrak{M}_\alpha(\zeta) = D_\alpha(I - \lambda^*\alpha)^{-1}(I - \lambda^*\zeta)(I - \alpha^*\zeta)^{-1}D_\alpha.$

Proof. Proof of (i).

$$\begin{aligned}
&I_{\mathcal{H}_2} - \mathfrak{M}_\alpha(\zeta)\mathfrak{M}_\alpha(\lambda)^* \\
&= I_{\mathcal{H}_2} - D_{\alpha^*}(I-\zeta\alpha^*)^{-1}(\zeta-\alpha)D_\alpha^{-1}D_\alpha^{-1}(\lambda^*-\alpha^*)(I-\alpha\lambda^*)^{-1}D_{\alpha^*} \\
&= D_{\alpha^*}(I-\zeta\alpha^*)^{-1}\left[(I-\zeta\alpha^*)D_{\alpha^*}^{-2}(I-\alpha\lambda^*) - (\zeta-\alpha)D_\alpha^{-2}(\lambda^*-\alpha^*)\right](I-\alpha\lambda^*)^{-1}D_{\alpha^*} \\
&= D_{\alpha^*}(I-\zeta\alpha^*)^{-1}\left[D_{\alpha^*}^{-2} - \zeta\alpha^* D_{\alpha^*}^{-2} - D_{\alpha^*}^{-2}\alpha\lambda^* + \zeta\alpha^* D_{\alpha^*}^{-2}\alpha\lambda^*\right. \\
&\left. -\zeta D_\alpha^{-2}\lambda^* + \zeta D_\alpha^{-2}\alpha^* + \alpha D_\alpha^{-2}\lambda^* - \alpha D_\alpha^{-2}\alpha^*\right](I-\alpha\lambda^*)^{-1}D_{\alpha^*}
\end{aligned}$$

Using the relations $D_{\alpha^*}^{-2}\alpha = \alpha D_\alpha^{-2}$ and $\alpha^* D_\alpha^{-2} = D_{\alpha^*}^{-2}\alpha^*$, some of the terms cancel out and we are left with

$$\begin{aligned}
&= D_{\alpha^*}(I-\zeta\alpha^*)^{-1}\left[D_{\alpha^*}^{-2} - \alpha D_{\alpha^*}^{-2}\alpha^* + \zeta\alpha^* D_{\alpha^*}^{-2}\alpha\lambda^* - \zeta D_\alpha^{-2}\lambda^*\right](I-\alpha\lambda^*)^{-1}D_{\alpha^*} \\
&= D_{\alpha^*}(I-\zeta\alpha^*)^{-1}\left[D_{\alpha^*}^{-2}(I_{\mathcal{H}_2} - \alpha\alpha^*) + \zeta D_{\alpha^*}^{-2}(\alpha^*\alpha - I_{\mathcal{H}_1})\lambda^*\right](I-\alpha\lambda^*)^{-1}D_{\alpha^*} \\
&= D_{\alpha^*}(I-\zeta\alpha^*)^{-1}(I-\zeta\lambda^*)(I-\alpha\lambda^*)^{-1}D_{\alpha^*}
\end{aligned}$$

Proof of (ii).

Now we prove the second identity. Start by pulling D_α^{-1} to the left and right.

$$I - \mathfrak{M}_\alpha(\lambda)^*\mathfrak{M}_\alpha(\zeta) = D_\alpha^{-1}[(I-\alpha^*\alpha) - (\lambda^*-\alpha^*)(I-\alpha\lambda^*)^{-1}D_{\alpha^*}^2(I-\zeta\alpha^*)^{-1}(\zeta-\alpha)]D_\alpha^{-1}.$$

One can easily check that

$$(I-\zeta\alpha^*)^{-1}(\zeta-\alpha) = \zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 - \alpha$$

by multiplying with $(I-\zeta\alpha^*)$ from the left on both sides of the identity. We then have

$$\begin{aligned}
&[(\lambda^*-\alpha^*)(I-\alpha\lambda^*)^{-1}D_{\alpha^*}^2(I-\zeta\alpha^*)^{-1}(\zeta-\alpha)] \\
&\qquad = [-\alpha^* + D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^*]D_{\alpha^*}^2[\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 - \alpha] \\
&\qquad = \alpha^* D_{\alpha^*}^2\alpha - \alpha^* D_{\alpha^*}^2\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 \\
&\qquad + D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^* D_{\alpha^*}^2\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 - D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^* D_{\alpha^*}^2\alpha \\
&\qquad = D_\alpha^2\alpha^*\alpha - D_\alpha^2\alpha^*\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 \\
&\qquad + D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^* D_{\alpha^*}^2\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 - D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^*\alpha D_\alpha^2,
\end{aligned}$$

using the relations $D_{\alpha^*}^{-2}\alpha = \alpha D_\alpha^{-2}$ and $\alpha^* D_\alpha^{-2} = D_\alpha^{-2}\alpha^*$. Therefore

$$\begin{aligned} I - \mathfrak{M}_\alpha(\lambda)^*\mathfrak{M}_\alpha(\zeta) = D_\alpha^{-1}[&(I-\alpha^*\alpha) - D_\alpha^2\alpha^*\alpha \\ &+ D_\alpha^2\alpha^*\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2 + D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^*\alpha D_\alpha^2 \\ &- D_\alpha^2(I-\lambda^*\alpha)^{-1}\lambda^* D_{\alpha^*}^2\zeta(I-\alpha^*\zeta)^{-1}D_\alpha^2]D_\alpha^{-1}, \end{aligned}$$

and factorising $D_\alpha^2(I-\lambda^*\alpha)^{-1}$ and $(I-\alpha^*\zeta)^{-1}D_\alpha^2$ to the left and right respectively, we finally get

$$\begin{aligned} I - \mathfrak{M}_\alpha(\lambda)^*\mathfrak{M}_\alpha(\zeta) = &D_\alpha^{-1}D_\alpha^2(I-\lambda^*\alpha)^{-1} \\ &\big[(I-\lambda^*\alpha)(I-\alpha^*\zeta) - \lambda^* D_{\alpha^*}^2\zeta + \lambda^*\alpha(I-\alpha^*\zeta) + (I-\lambda^*\alpha)\alpha^*\zeta\big] \\ &(I-\alpha^*\zeta)^{-1}D_\alpha^2 D_\alpha^{-1} \\ &= D_\alpha(I-\lambda^*\alpha)^{-1}(I-\lambda^*\zeta)(I-\alpha^*\zeta)^{-1}D_\alpha. \end{aligned}$$

■

Lemma 4.6. *The transformation Φ_α satisfies:*

(i) $I - \Phi_\alpha(\zeta)^*\Phi_\alpha(\lambda) = D_\alpha(I+\zeta^*\alpha)^{-1}(I-\zeta^*\lambda)(I+\alpha^*\lambda)^{-1}D_\alpha.$

(ii) $I - \Phi_\alpha(\zeta)\Phi_\alpha(\lambda)^* = D_{\alpha^*}(I+\zeta\alpha^*)^{-1}(I-\zeta\lambda^*)(I+\alpha\lambda^*)^{-1}D_{\alpha^*}.$

(iii) $\mathfrak{M}_\alpha(\Phi_\alpha(\zeta)) = \zeta$, *i.e., $\mathfrak{M}_\alpha$ and Φ_α are compositional inverse.*

Proof. (i) and (ii) are immediate consequences of (i) and (ii) in the previous lemma.

(iii) The second one is a straightforward computation:

$$\mathfrak{M}_\alpha(\Phi_\alpha(\zeta)) = D_{\alpha^*}(I - \Phi_\alpha(\zeta)\alpha^*)^{-1}(\Phi_\alpha(\zeta) - \alpha)D_\alpha^{-1}.$$

On one hand, we have

$$\begin{aligned} \Phi_\alpha(\zeta) - \alpha &= D_{\alpha^*}^{-1}(\zeta+\alpha)(I+\alpha^*\zeta)^{-1}D_\alpha - \alpha \\ &= D_{\alpha^*}^{-1}(\zeta+\alpha-\alpha(I+\alpha^*\zeta))(I+\alpha^*\zeta)^{-1}D_\alpha \\ &= D_{\alpha^*}^{-1}(\zeta-\alpha\alpha^*\zeta)(I+\alpha^*\zeta)^{-1}D_\alpha \\ &= D_{\alpha^*}^{-1}(I-\alpha\alpha^*)\zeta(I+\alpha^*\zeta)^{-1}D_\alpha \\ &= D_{\alpha^*}\zeta(I+\alpha^*\zeta)^{-1}D_\alpha. \end{aligned}$$

On the other hand

$$\begin{aligned}
(I-\Phi_\alpha(\zeta)\alpha^*)^{-1} &= \left(I - D_{\alpha^*}^{-1}(\zeta+\alpha)(I+\alpha^*\zeta)^{-1}D_\alpha\alpha^*\right)^{-1} \\
&= \left(I - D_{\alpha^*}^{-1}(\zeta+\alpha)(I+\alpha^*\zeta)^{-1}\alpha^* D_{\alpha^*}\right)^{-1} \\
&= \left(D_{\alpha^*}^{-1}[I-(\zeta+\alpha)(I+\alpha^*\zeta)^{-1}\alpha^*]D_{\alpha^*}\right)^{-1} \\
&= \left(D_{\alpha^*}^{-1}[I-(\zeta+\alpha)\alpha^*(I+\zeta\alpha^*)^{-1}]D_{\alpha^*}\right)^{-1} \\
&= \left(D_{\alpha^*}^{-1}[(I+\zeta\alpha^*)-(\zeta+\alpha)\alpha^*](I+\zeta\alpha^*)^{-1}D_{\alpha^*}\right)^{-1} \\
&= \left(D_{\alpha^*}^{-1}[I-\alpha\alpha^*](I+\zeta\alpha^*)^{-1}D_{\alpha^*}\right)^{-1} \\
&= \left(D_{\alpha^*}(I+\zeta\alpha^*)^{-1}D_{\alpha^*}\right)^{-1} \\
&= D_{\alpha^*}^{-1}(I+\zeta\alpha^*)D_{\alpha^*}^{-1}.
\end{aligned}$$

Put them together to get

$$\begin{aligned}
\mathfrak{M}_\alpha(\Phi_\alpha(\zeta)) &= D_{\alpha^*}D_{\alpha^*}^{-1}(I+\zeta\alpha^*)D_{\alpha^*}^{-1}D_{\alpha^*}\zeta(I+\alpha^*\zeta)^{-1}D_\alpha D_\alpha^{-1} \\
&= (I+\zeta\alpha^*)\zeta(I+\alpha^*\zeta)^{-1} \\
&= \zeta.
\end{aligned}$$

■

We will call Φ_α the *α-Frostman transformation*, and $\Phi_\alpha(\zeta)$ the *α-Frostman shift* of ζ.

Remark 4.7. By the above Lemma, for any strict contraction $\alpha \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the transformation Φ_α maps strict contractions $\zeta \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ onto strict contractions, Φ_α is a bijection of the open unit ball of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ onto itself. We also notice that Φ_α sends isometries and unitaries to isometries and unitaries respectively.

The α-Frostman transformation of a strict contraction ζ is well defined for a pure contraction α. If α is pure then D_{α^*} and D_α have dense ranges and the above Lemmas show that $\mathfrak{M}_\alpha(\zeta)$ is contractive on the dense subset $\operatorname{Ran}(D_\alpha)$ so that it extends uniquely to a contraction on $\mathcal{H}_1$. Hence $\Phi_\alpha(\zeta)$ is a well defined bounded (contractive) operator for a pure contraction α and a strict contraction ζ.

Now we can apply these transformations to define and study the Frostman shifts for Schur multipliers.

Definition 4.8. Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $||b(0)|| < 1$, i.e., b is strictly contractive. We define the 0-Frostman shift of b by :

$$b_0(z) := \mathfrak{M}_{b(0)}(b(z)) = D_{b(0)^*}(I - b(z)b(0)^*)^{-1}(b(z) - b(0))D_{b(0)}^{-1}; \quad (z \in \mathbb{B}_d).$$

Given a non zero pure contraction $\alpha \in \mathcal{B}(\mathcal{U}, \mathcal{Y})$, we define the α-Frostman shift of b by:

$$b_\alpha(z) := \Phi_\alpha(b_0(z)) = D_{\alpha^*}^{-1}(b_0(z) + \alpha)(I + \alpha^* b_0(z))^{-1} D_\alpha; \quad (z \in \mathbb{B}_d).$$

The 0-Frostman shift, b_0, is well defined since $||b(0)|| < 1$, b_0 is strictly contractive according to Proposition 2.9 and therefore b_α is well defined as well for a pure contraction α (See Remark 4.7 above). For $d = 1$, when $\mathcal{U} = \mathcal{Y} = \mathbb{C}$ then b_0 and b_α are the Frostman shifts of Schur class function on the complex unit disk introduced in [20].

In the previous section, we computed the characteristic function of the isometric part of an extremal Gleason solution for in $\mathscr{H}(b)$ and we have seen that it coincides with the 0-Frostman shift, b_0, of b.

Lemma 4.9. *Let $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ such that $||b(0)|| < 1$ and let $\alpha \in \mathcal{B}(\mathcal{U}, \mathcal{Y})$ be a pure contraction, the Frostman shifts b_0 and b_α are also Schur multipliers: $b_0, b_\alpha \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Multiplication by*

$$m_0(z) := k^b(z, 0)k^b(0, 0)^{-1/2} = (I - b(z)b(0)^*)D_{b(0)^*}^{-1}$$

and

$$m_\alpha(z) := k^{b_\alpha}(z, 0)k^{b_\alpha}(0, 0)^{-1/2} = (I - b_\alpha(z)\alpha^*)D_{\alpha^*}^{-1}$$

map $\mathscr{H}(b_0)$ unitarily onto $\mathscr{H}(b)$ and $\mathscr{H}(b_\alpha)$, respectively.

In the classical case, these multipliers reduce to the Crofoot transformations [15].

Proof. From Lemma 4.5, for all $z, w \in \mathbb{B}_d$, we have

$$\begin{aligned} I - b_0(z)b_0(w)^* &= I - \mathfrak{M}_{b(0)}(b(z)\mathfrak{M}_{b(0)}(b(w))^* \\ &= D_{b(0)^*}(I - b(z)b(0)^*)^{-1}(I - b(z)b(w)^*)(I - b(0)b(w)^*)^{-1}D_{b(0)^*}. \end{aligned}$$

Since $b \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ then so does b_0. The fact that $||b(0)|| < 1$ implies that $m_0(z)$ is invertible for all $z \in \mathbb{B}_d$. It follows from the identity above that m_0 defines a unitary multiplication operator from $\mathscr{H}(b_0)$ onto $\mathscr{H}(b)$.

Since $b_0 \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, the following calculations shows that b_α also is a Schur class multiplier:

$$\begin{aligned} I - b_\alpha(z)b_\alpha(w)^* &= \Phi_\alpha(b_0(z))\Phi_a(b_0(w))^* \\ &= D_{\alpha^*}(I + b_0(z)\alpha^*)^{-1}(I - b_0(z)b_0(w)^*)(I + \alpha b_0(w)^*)^{-1}D_{\alpha^*}. \end{aligned}$$

Notice that b_α is defined so that $(b_\alpha)_0 = b_0$. It is a consequence of the identity $\mathfrak{M}_\alpha(\Phi_\alpha(b_0(z))) = b_0(z)$ and the fact that $b_\alpha(0) = \alpha$. Indeed,

$$\begin{aligned} (b_\alpha)_0(z) &= \mathfrak{M}_{b_\alpha(0)}(\Phi_\alpha(b_0(z))) \\ &= \mathfrak{M}_\alpha(\Phi_\alpha(b_0(z))) \\ &= b_0(z), \end{aligned}$$

according to Lemma 4.6. Substitute $b := b_\alpha$ in the expression of m_0 above and we get a unitary multiplication operator m_α that maps $\mathscr{H}(b_0)$ onto $\mathscr{H}(b_\alpha)$. ∎

We made an important observation in the above proof, that $(b_\alpha)_0 = b_0$ and $b_\alpha = (b_0)_\alpha$. It is natural to expect that intrinsic properties of a given Schur class funtion b, such as quasi-extremity, will be preserved under the Frostman transform. This is indeed the case, as the following results show.

Theorem 4.10. *There is a bijection between contractive Gleason solutions for b and contractive Gleason solutions for b_0. The correspondence is given by*

$$\mathbf{b} \longrightarrow \mathbf{b}_0 = (m_0 \otimes I)^{-1}\mathbf{b}D_{b(0)}^{-1},$$

where $\mathbf{b}$ is a Gleason solution for b and $\mathbf{b}_0$ is a Gleason solution for b_0, it preserves extremal Gleason solutions.

Proof. If $\mathbf{b}$ is a contractive Gleason solution for b then

$$\begin{aligned} z\mathbf{b}_0(z) &= z(m_0(z)^{-1} \otimes I)\mathbf{b}(z)D_{b(0)}^{-1} \\ &= m_0(z)^{-1}z\mathbf{b}(z)D_{b(0)}^{-1} \\ &= D_{b(0)^*}(I - b(z)b(0)^*)^{-1}(b(z) - b(0))D_{b(0)}^{-1} \\ &= b_0(z), \end{aligned}$$

and if $\mathbf{b}_0$ is a contractive Gleason solution for b_0 then

$$\begin{aligned} z\mathbf{b}(z) &= z(m_0(z) \otimes I)\mathbf{b}_0(z)D_{b(0)} \\ &= m_0(z)z\mathbf{b}_0(z)D_{b(0)} \\ &= (I - b(z)b(0)^*)D_{b(0)^*}^{-1}b_0(z)D_{b(0)} \\ &= (I - b(z)b(0)^*)D_{b(0)^*}^{-1}\left[D_{b(0)^*}(I - b(z)b(0)^*)^{-1}(b(z) - b(0))D_{b(0)}^{-1}\right]D_{b(0)} \\ &= b(z) - b(0). \end{aligned}$$

Moreover $\mathbf{b}_0^*\mathbf{b}_0 = D_{b(0)}^{-1}\mathbf{b}^*\mathbf{b}D_{b(0)}^{-1}$ so that $\mathbf{b}_0^*\mathbf{b}_0 \leq I$ if and only if $\mathbf{b}^*\mathbf{b} \leq I - b(0)^*b(0) = D_{b(0)}^2$ and $\mathbf{b}_0^*\mathbf{b}_0 = I$ if and only if $\mathbf{b}^*\mathbf{b} = I - b(0)^*b(0) = D_{b(0)}^2$. It follows that $\mathbf{b}_0$ is a contractive (respectively extremal) Gleason solution for b_0 if and only if $\mathbf{b}$ is a contractive (respectively extremal) Gleason solution for b. ■

Now we can apply the remark preceding the previous theorem to prove the following corollary.

Corollary 4.11. *A Schur multiplier $b \in \mathscr{S}_d(\mathcal{U}, \mathcal{Y})$ is QE if and only if any and therefore all of its Frostman shifts b_α are QE.*

Proof. Replace m_0 by m_α and substitute $\mathbf{b}$ by a Gleason solution $\mathbf{b}_\alpha$ for b_α in the above theorem to get a bijective correspondence between contractive Gleasons solution for b_0 and b_α. Hence there is a unique contractive Gleason solution for b_0 if and only if there is a unique contractive Gleason solution for b_α. According to theorem 2.24, the Schur function b_α is therefore QE if and only if b_0 is. In particular, since $b = (b_0)_{b(0)}$, it follows that b is QE if and only if b_0 is QE (which happens if and only if b_α is QE for some strictly contractive α). ■

4.3 Model for CCNC row contractions

In this section, we extend the Gleason solution model that we developed for CCNC row partial isometries to CCNC row contractions. We make the additional assumption that the purely pure part of the row contractions we consider here are strict row contractions. That is we consider row contractions, T, on a Hilbert space $\mathcal{H}$ with isometric-pure decomposition, $T = Z - C$, where $||C||_{\mathcal{B}(\mathcal{H}^d,\mathcal{H})} < 1$, and we prove that such row contraction is unitarily equivalent to an extremal Gleason solution for a multivariable de Branges-Rovnyak space.

Let $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$, $||b(0)|| < 1$, and X be an extremal Gleason solution in $\mathscr{H}(b)$. We know that X is a CCNC row contraction (Proposition 2.5) and its isometric part, Z, is a CCNC partial row isometry (Proposition 1.23). We proved, in Section 1 of the present chapter, that the characteristic function of Z coincides with b_0. The CCNC row partial isometry Z is then unitarily equivalent to an extremal Gleason solution, Z_0, in $\mathscr{H}(b_0)$ (Theorem 3.15). This unitary equivalence is implemented by the Crofoot transformation, m_0, of Lemma 4.9.

Lemma 4.12. *Let $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ such that $||b(0)|| < 1$. Let X be an extremal Gleason solution in $\mathscr{H}(b)$, and let $\mathbf{b}$ be an extremal Gleason solution for b satisfying $X^* k_w^b = w^* k_w^b - \mathbf{b} b(w)^*$; $w \in \mathbb{B}_d$. Let $\mathbf{b_0} = (m_0^{-1} \otimes I_d)\mathbf{b} D_{b(0)}^{-1}$ and Z_0 the extremal Gleason solution in $\mathscr{H}(b_0)$ defined by*

$$Z_0^* k_w^{b_0} = w^* k_w^{b_0} - \mathbf{b_0} b_0(w)^*.$$

Then, the isometric part, Z, of X is given by:

$$Z = m_0 Z_0 (m_0 \otimes I_d)^*.$$

Proof. From the computations in the paragraph preceding Proposition 4.1, we have $Z^* = X^*(I - k_0^b D_{b(0)^*}^{-2} (k_0^b)^*)$, and we need to verify that this is the same as $m_0 Z_0 (m_0 \otimes I_d)^*$. To prove this, we compare $Z^* k_w^b$ and $(m_0 \otimes I_d) Z_0^* m_0^* k_w^b$. On one hand, we have

$$\begin{aligned} Z^* k_w^b &= w^* k_w^b - \mathbf{b} b(w)^* + \mathbf{b} b(0)^* D_{b(0)^*}^{-2} (I - b(0) b(w)^*) \\ &= w^* k_w^b - \mathbf{b} D_{b(0)}^{-2} \left((I - b(0)^* b(0)) b(w)^* - b(0)^* (I - b(0) b(w)^*)\right) \\ &= w^* k_w^b - \mathbf{b} D_{b(0)}^{-2} (b(w)^* - b(0)^*). \end{aligned}$$

On the other hand, using the property that m_0 is a multiplier from $\mathscr{H}(b_0)$ onto $\mathscr{H}(b)$, we have $m_0^* k_w^b = k_w^{b_0} m_0(w)^*$. Therefore

$$\begin{aligned}
(m_0 \otimes I_d) Z_0^* m_0^* k_w^b &= (m_0 \otimes I_d) Z_0^* k_w^{b_0} m_0(w)^* \\
&= (m_0 \otimes I_d) \left(w^* k_w^{b_0} m_0(w)^* - \mathbf{b}_0 b_0(w)^* m_0(w)^* \right) \\
&= \left(w^* m_0 k_w^{b_0} m_0(w)^* - (m_0 \otimes I_d) \mathbf{b}_0 b_0(w)^* m_0(w)^* \right) \\
&= \left(w^* m_0 m_0^* k_w^b - (m_0 \otimes I_d) \mathbf{b}_0 b_0(w)^* m_0(w)^* \right) \\
&= w^* k_w^b - \mathbf{b} D_{b(0)}^{-1} b_0(w)^* m_0(w)^* \\
&= w^* k_w^b - \mathbf{b} D_{b(0)}^{-2} \left((b(w)^* - b(0)^*)(I - b(0)b(w)^*)^{-1} D_{b(0)^*} D_{b(0)^*}^{-1} (I - b(0)b(w)^*) \right) \\
&= w^* k_w^b - \mathbf{b} D_{b(0)}^{-2} (b(w)^* - b(0)^*).
\end{aligned}$$

This is the same as the above and completes the proof. ∎

Starting with an extremal Gleason solution, Z_0, in $\mathscr{H}(b_0)$, there exists an extremal Gleason solution, $\mathbf{b}_0$, for b_0 such that $Z_0^* k_w^{b_0} = w^* k_w^{b_0} - \mathbf{b}_0 b(w)^*$. The bijective correspondence in Theorem 4.10 guarantees that $\mathbf{b} = (m_0 \otimes I_d)^{-1} \mathbf{b}_0 D_{b(0)^*}^{-1}$ defines an extremal Gleason solution for b. Setting $X k_w^b := w^* k_w^b - \mathbf{b} b(w)^*$ for $w \in \mathbb{B}_d$ yields an extremal Gleason solution X for $\mathscr{H}(b)$. The above Lemma shows that $m_0 Z_0 (m_0^* \otimes I_d)$ is the isometric part of some Gleason solution, X, in $\mathscr{H}(b)$. Therefore, any extremal Gleason solutions for $\mathscr{H}(b_0)$ is untarily equivalent to the isometric part of an extremal Gleason solution for $\mathscr{H}(b)$.

If $b_Z \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ is the characteristic function of a CCNC row partial isometry, Z, and $\alpha \in \mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$ is a pure contraction then Z is unitarily equivalent to an extremal Gleason solution, Z_0, for $\mathscr{H}(b_Z)$. According to the above discussion, since $b_Z = ((b_Z)_\alpha)_0$, then Z_0 is unitarily equivalent to the isometric part of an extremal Gleason solution for $\mathscr{H}((b_Z)_\alpha)$. We have proven the following theorem.

Theorem 4.13. *Any CCNC row partial isometry, Z, with characteristic function $b_Z \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$, is unitarily equivalent to the isometric part, Z_α, of an extremal Gleason solution, X_α, for $\mathscr{H}((b_Z)_\alpha)$, $\alpha \in \mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$:*

$$Z_\alpha = m_\alpha Z (m_\alpha^* \otimes I_d).$$

This gives us a hint of how to construct a characteristic function for CCNC row contractions. Naturally we want the characteristic function of any extremal Gleason solution, X^b, in $\mathscr{H}(b)$ to be b, but we already know that the characteristic function of its isometric part, Z^b, is $b_{Z^b} = b_0$. So, to define a characteristic function that reduces to the one defined in the previous chapter for CCNC row partial isometry, we need a transformation that reverses the Frostman shift. Fortunately, we have studied such transformation in the previous section so we will just put that in use.

Let T be a CCNC row contraction with isometric-pure decomposition $T = Z - C$. We already know how to define the characteristic function of the CCNC row partial isometry Z. We just need to define a pure contraction α that is uniquely determined by T and we will define the characteristic function of T as the α-Frostman shift of b_Z.

Lemma 4.14. *Let Z be a CCNC row partial isometry and fix a model $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ for Z.*

$$\delta \in (\mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0))_1 \longmapsto Z_\delta := Z - j_0 \delta j_\infty^* \in (\mathcal{B}(\mathcal{H}^d, \mathcal{H}))_1$$

is a bijection between pure contractions in $\mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$ and row contractions with isometric part Z. A row contraction, T, with isometric part Z is uniquely determined by its defect contraction $\delta_T = -j_0 T j_\infty^*$.

Proof. Notice first that the maps $j_0 : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(Z)^\perp$, and, $j_\infty : \mathcal{J}_\infty \longrightarrow \mathrm{Ker}(Z)$ are onto isometries. Therefore $\delta \longmapsto j_0 \delta j_\infty^*$ defines a bijection between pure contractions in $\mathcal{B}(\mathrm{Ker}(Z), \mathrm{Ran}(Z)^\perp)$ and $\mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$. The lemma is a consequence of the fact that any row contraction with isometric part Z can be written $Z - C$ where C is a pure contraction in $\mathcal{B}(\mathrm{Ker}(Z), \mathrm{Ran}(Z)^\perp)$ (Remark 1.18).

If $T = Z - C$ is the isometric contractive decomposition of a row contraction T then $C = j_0^* \delta_T j_\infty$ for a unique pure contraction $\delta_T \in \mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$. We necessarily have $\delta_T = j_0^* C j_\infty = j_0^*(Z - T) j_\infty = -j_0^* T j_\infty$ since $j_0^* Z j_\infty = 0$ as $\mathrm{Ran}\,(j_\infty) = \mathrm{Ker}(Z)$. ∎

We now have all the ingredients to define the characteristic function of a row contraction.

Definition 4.15. Let T be a CCNC row contraction on a Hilbert space $\mathcal{H}$ with isometric part Z. If b_Z is the characteristic function of T computed with the model $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ then define the characteristic function of T as

$$b_T := (b_Z)_{\delta_T},$$

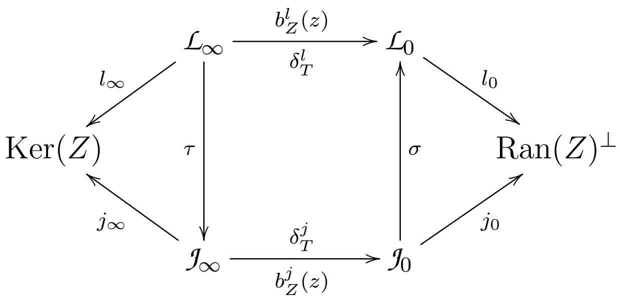
the δ_T-Frostman shift of b_Z (Definition 4.8), where δ_T is computed using the model $(j, \mathcal{J}_0, \mathcal{J}_\infty)$.

Recall that b_Z vanishes at the origin and any such function is strictly contractive, i.e., $b_Z(z)$ is a strict contraction for all $z \in \mathbb{B}_d$. Hence the characteristic function b_T is well defined for any CCNC row contraction.

If T is a row partial isometry, then $T = Z$ so that $b_T = b_Z$ is the characteristic function we defined in the previous chapter for row partial isometries.

Remark 4.16. When we write $b_T = (b_Z)_{\delta_T^j}$, we assumed that we have chosen two onto isometries $j_0 : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(Z)^\perp$ and $j_\infty : \mathcal{J}_\infty \longrightarrow \mathrm{Ker}(Z)$. A different choice of model triple $(l, \mathcal{L}_0, \mathcal{L}_\infty)$ gives two unitary operators σ, τ satisfying $l_0 = j_0\sigma^*$ and $l_\infty = j_\infty\tau$. Therefore

$$\delta_T^l = -l_0^* T l_\infty = \sigma(-j_0^* T j_\infty)\tau = \sigma\delta_T^j\tau.$$



It follows that $(b_Z^j)_{\delta_T^j}$ coincides with $(b_Z^l)_{\delta_T^l}$, b_Z^j is the characteristic function computed using the model $(j, \mathcal{J}_0, \mathcal{J}_\infty)$ and b_Z^l the one using $(l, \mathcal{L}_0, \mathcal{L}_\infty)$. Indeed,

$$l_z^* l_0 = \sigma j_z^* j_0 \sigma^*,$$

$$z(l_z^* \otimes I_d) l_\infty = \sigma z(j_z^* \otimes I_d) j_\infty \tau.$$

Hence we have

$$b_Z^l(z) = \sigma(j_z^* j_0)^{-1}\sigma^*\sigma z(j_z^* \otimes I_d) j_\infty \tau = \sigma b_Z^j(z)\tau.$$

Therefore

$$(b_Z^l)_{\delta_T^l}(z) = (\sigma b_Z^j \tau)_{\sigma\delta_T^j\tau}(z).$$

It is a straightforward computation to show that this coincides with $(b_Z^j)_{\delta_T^j}$:

$$\begin{aligned}
(\sigma b_Z^j \tau)_{\sigma\delta_T^j\tau}(z) &= D^{-1}_{\tau^*(\delta_T^j)^*\sigma^*}(\sigma b_Z^j(z)\tau + \sigma\delta_T^j\tau)(I + \tau^*(\delta_T^j)^*\sigma^*\sigma b_Z^j(z)\tau)^{-1} D_{\sigma\delta_T^j\tau} \\
&= \sigma D^{-1}_{\delta_T^j}\sigma^*\sigma(b_Z^j(z) + \delta_T^j)\tau\tau^*(I + (\delta_T^j)^* b_Z^j(z))^{-1}\tau\tau^* D_{\delta_T^j}\tau \\
&= \sigma(b_Z^j)_{\delta_T^j}(z)\tau.
\end{aligned}$$

For the characteristic function, b_T, to be invariant under a change of model, it is then important that δ_T and b_Z are computed using the same model.

Theorem 4.17. *The characteristic function, b_T, for CCNC row contractions is invariant under unitary equivalence.*

Proof. Let T, T' be two CCNC row contractions unitaliry equivalent, i.e., there is a unitary operator U such that $T' = UT(U^* \otimes I_d)$. Let $T = Z - C$ and $T' = Z' - C'$ be the isometric-pure decomposition of T and T', respectively. Therefore $Z' = UZ(U^* \otimes I_d)$. Let $(j, \mathcal{J}_\infty, \mathcal{J}_0)$ be a model for Z'. It follows that $l_0 := U^* j_0 : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(Z)^\perp$ and $l_\infty := (U^* \otimes I_d) j_\infty : \mathcal{J}_\infty \longrightarrow \mathrm{Ker}(Z)^\perp$ are onto isometries. Let $l_z = U^* j_z$, $z \in \mathbb{B}_d$, $(l, \mathcal{J}_\infty, \mathcal{J}_0)$ is then a model for Z. If we compute $b_{Z'}$ using $(j, \mathcal{J}_\infty, \mathcal{J}_0)$ and b_Z using $(l, \mathcal{J}_\infty, \mathcal{J}_0)$ then we have

$$l_z^* l_0 = j_z^* U U^* j_0 = j_z j_0,$$

and

$$z(l_z^* \otimes I_d) l_\infty = z(j_z^* U \otimes I_d)(U^* \otimes I_d) j_\infty = z(j_z^* \otimes I_d) j_\infty.$$

Therefore $b_{Z'}^j = b_Z^l$. If we compute $\delta_{T'}$ using $(j, \mathcal{J}_\infty, \mathcal{J}_0)$ and δ_T using $(l, \mathcal{J}_\infty, \mathcal{J}_0)$ then we have

$$\delta_{T'}^j = -j_0^* T' j_\infty = -j_0^* U T U^* j_\infty = -l_0^* T l_\infty = \delta_T^l.$$

Therefore $b_{T'} = (b_{Z'}^j)_{\delta_{T'}^j} = (b_Z^l)_{\delta_T^l} = b_T$. ■

In the following Lemma, we construct the model triple we are going to use to compute the characteristic function of a CCNC row contraction T. Recall that $m_0 : \mathscr{H}(a) \longrightarrow \mathscr{H}(b)$ is the Crofoot transformation in Lemma 4.9, $M_{D^{-1}} : \hat{\mathcal{H}} \longrightarrow \mathscr{H}(a)$ is the unitary multiplication operator in Theorem 3.14 ($D(z) = j_z^* j_0$), and $\hat{U} : \mathcal{H} \longrightarrow \hat{\mathcal{H}}$ the identification of $\mathcal{H}$ with the abstract model space defined by the model triple $J.\mathcal{J}_\infty, \mathcal{J}_0)$ (Definition 3.12).

Lemma 4.18. *Let T be a row contraction with isometric-pure decomposition: $T = Z - C$. Let $(j, \mathcal{J}_\infty, \mathcal{J}_0)$ be the canonical model triple for Z and denote $a := b_Z^j$ and $b := b_T^j$. Denote*

$$A := m_0 M_{D^{-1}} \hat{U} : \mathcal{H} \longrightarrow \mathscr{H}(b),$$

Then $k_0^b D_{b(0)^*}^{-1} = A j_0$ and $\mathbf{b} := A j_\infty D_{b(0)}$ is an extremal solution to the Gleason problem for b.

Proof. The following computation shows that $k_0^b D_{b(0)^*}^{-1} = Aj_0$:

$$\begin{aligned} j_0^* A^* k_z^b &= j_0^* \hat{U}^* M_{D^{-1}}^* m_0^* k_z^b \\ &= j_0^* \hat{U}^* M_{D^{-1}}^* k_z^a m_0(z) \\ &= j_0^* \hat{U}^* \hat{K}_z (D(z)^*)^{-1} m_0(z)^* \\ &= j_0^* \hat{U}^* \hat{U} j_z (j_0^* j_z)^{-1} m_0(z)^* \\ &= m_0(z)^* = D_{b(0)^*}^{-1} k^b(0,z) \\ &= D_{b(0)^*}^{-1} (k_0^b)^* k_z^b. \end{aligned}$$

Since $a = b_0$, to prove that $\mathbf{b} := Aj_\infty D_{b(0)}$ is an extremal solution to the Gleason problem for b, we need to show that $\mathbf{a} := M_{D^{-1}} \hat{U} j_\infty$ is an extremal Gleason solution for $\mathscr{H}(a)$ and apply Theorem 4.10. We have that $\mathbf{a} := M_{D^{-1}} \hat{U} j_\infty$ is an extremal Gleason solution for $\mathscr{H}(a)$ since $j_\infty, M_{D^{-1}}$ and $\hat{U}$ are unitaries, hence $\mathbf{a}'^* \mathbf{a}' = I_{\mathcal{J}_\infty}$, and,

$$\begin{aligned} z\mathbf{a}(z) &= (z^* k_z^b)^* \mathbf{a} \\ &= \left(z^* \hat{U}^* (M_{D^{-1}}^*) k_z^b \right)^* j_\infty \\ &= \left(z^* \hat{U}^* \hat{K}_z (D(z)^{-1})^* \right)^* j_\infty \\ &= \left(z^* j_z (D(z)^{-1})^* \right)^* j_\infty \\ &= D(z)^{-1} z (j_z^* \otimes I_d) j_\infty \\ &= D(z)^{-1} z N(z) = a(z). \end{aligned}$$

■

Theorem 4.19. *Let T be a CCNC row contraction on a Hilbert space $\mathcal{H}$. Then, T is unitarily equivalent to an extremal Gleason solution, X^{b_T}, on $\mathscr{H}(b_T)$.*

Proof. Let $T = Z - C$ be the isometric-pure decomposition of T. Let $(j, \mathcal{J}_\infty, \mathcal{J}_0)$ be the canonical model triple for Z and denote respectively $a := b_Z^j$ and δ_T^j the characteristic function of Z and the defect contraction of T computed with this model triple. The characteristic function of T computed with this model triple is $b := b_T^j = a_{\delta_T^j}$.

By Theorem 3.15, $Z^a := M_{D^{-1}} \hat{U} Z \hat{U}^* M_{D^{-1}}^*$ is an extremal Gleason solution for $\mathscr{H}(a)$, where $D(z) = j_z^* j_0$. Let $\mathbf{a} := M_{D^{-1}} \hat{U} j_\infty$, it is an extremal Gleason solution for a. Moreover it satisfies $(Z^a)^* k_w^a = w^* k_w^a - \mathbf{a}a(w)^*$ for all $w \in \mathbb{B}_d$. To see this, compute

$$\begin{aligned} ((Z^a)^* - w^*)k_w^a &= \left((M_{D^{-1}}\hat{U} \otimes I_d)(Z^* - w^*)\hat{U}^* M_{D^{-1}}^*\right) k_w^a \\ &= (M_D^{-1}\hat{U} \otimes I_d)(Z^* - w^*)j_w(D(w)^{-1})^*. \end{aligned}$$

Compare this to

$$\begin{aligned} \mathbf{a}a(w)^* &= (M_{D^{-1}}\hat{U} \otimes I)j_\infty N(w)^* w^*(D(w)^{-1})^* \\ &= (M_{D^{-1}}\hat{U} \otimes I_d)j_\infty j_\infty^*(j_w \otimes I_d)w^*(D(w)^{-1})^* \\ &= (M_{D^{-1}}\hat{U} \otimes I_d)P_{\mathrm{Ker}(Z)} w^* j_w(D(w)^{-1})^* \\ &= (M_{D^{-1}}\hat{U} \otimes I_d)(I - Z^*Z)w^* j_w(D(w)^{-1})^* \\ &= (M_{D^{-1}}\hat{U} \otimes I_d)\left[w^* j_w - Z^*Zw^* j_w\right](D(w)^{-1})^* \\ &= (M_{D^{-1}}\hat{U} \otimes I_d)\left[w^* j_w - Z^*(j_w - j_0)\right](D(w)^{-1})^* \\ &= (M_{D^{-1}}\hat{U} \otimes I_d)\left[(w^* - Z^*)j_w\right](D(w)^{-1})^*. \end{aligned}$$

We used the fact that $j_w = (I - Zw^*)^{-1}j_0$ and that $Z^*j_0 = 0$. Since by construction $b_0 = a$, then $\mathbf{b} := m_0\mathbf{a}D_{b(0)}^{-1}$ is an extremal Gleason solution for b. According to Theorem 4.13, the isometric part of the corresponding extremal Gleason solution, $X^b = Z^b - C^b$, for $\mathscr{H}(b)$ is

$$Z^b = m_0 Z^a m_0^* = AZA^*,$$

where

$$A := m_0 M_{D^{-1}}\hat{U} : \mathcal{H} \longrightarrow \mathscr{H}(b).$$

On the other hand, the isometric part of $T' := ATA^*$ is $Z' = AZA^* = Z^b$. We will prove that $T' = X^b$ so that T is unitarily equivalent to the extremal Gleason solution X^b for $\mathscr{H}(b)$, since $A : \mathcal{H} \longrightarrow \mathscr{H}(b)$ is unitary. Since T' and X^b are pure extensions of Z^b, according to Lemma 4.14, it is enough to prove that $\delta_{T'}^l = \delta_{X^b}^l$ for a model triple $(l, \mathcal{J}_\infty, \mathcal{J}_0)$ for Z^b.

We know from the proof of Theorem 4.4 that the family of maps defined by

$$l_\infty := \mathbf{b}D_{b(0)}^{-1} \oplus l_\infty' : \mathcal{J}_\infty \longrightarrow \mathrm{Ker}(Z^b),$$

$$l_0 := k_0^b D_{b(0)^*}^{-1} : \mathcal{J}_0 \longrightarrow \mathrm{Ran}(Z^b)^\perp,$$

$$l_z := (I - Z^b z^*)^{-1} l_0 = k_z^b D_{b(0)^*}^{-1} : \mathcal{J}_0 \longrightarrow k_z^b \mathcal{J}_0,$$

defines an analytic model triple, $(l, \mathcal{J}_\infty, \mathcal{J}_0)$, for Z^b. The previous Lemma shows that the model triple $(l, \mathcal{J}_\infty, \mathcal{J}_0)$ and $(Aj, \mathcal{J}_\infty, \mathcal{J}_0)$ are the same. Indeed, $l_0 = k_0^b D_{b(0)^*}^{-1} = Aj_0$, $l_\infty = \mathbf{b} D_{b(0)}^{-1} = Aj_\infty$ and

$$l_z = (I - Z^b z^*)^{-1} l_0 = (I - AZ(A^* \otimes I_d)z^*)^{-1} Aj_0 = A(I - Zz^*)^{-1} A^* Aj_0 = Aj_z.$$

Recall that $(C^b)^* = \mathbf{b} D_{b(0)}^{-1} b(0)^* l_0^* = l_\infty b(0)^* l_0^*$. Therefore the defect operator of X^b and T', computed with the model $(l, \mathcal{J}_0, \mathcal{J}_\infty)$, is

$$\delta^l_{X^b} = -l_0^* X^b l_\infty = 0 + l_0^* l_0 b(0) l_\infty^* l_\infty = b(0) = b_T(0) = \delta^j_T = \delta^{Aj}_{T'} = \delta^l_{T'}.$$

■

We have proven that any CCNC row contractions are represented by extremal Gleason solutions on de Branges-Rovnyak spaces.

If $b(0)$ is pure then the characteristic function of an extremal Gleason solution in $\mathscr{H}(b)$ is b. As discussed in Remark 3.18, in general, Gleason solutions are not unique so that this characteristic function fails to be a complete unitary invariant. The following theorem summarizes what we know so far.

Theorem 4.20. *A row contraction, T, on a Hilbert space $\mathcal{H}$ is CCNC, i.e.,*

$$\mathcal{H} = \bigvee_{z \in \mathbb{B}_d} (I - Tz^*)^{-1} \mathrm{Ran}\,(D_{T^*})\,,$$

if and only if there is a purely contractive Schur multiplier, $b_T \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ ($\mathcal{J}_\infty \simeq \mathrm{Ran}\,(D_T)$ and $\mathcal{J}_0 \simeq \mathrm{Ran}\,(D_{T^*})$), which is uniquely determined by T up to conjugation by constant unitaries, such that T is unitarily equivalent to an extremal Gleason solution for $\mathscr{H}(b_T)$.

If $T = Z - C$ is the isometric-pure decomposition of T then $b_T = (b_Z)_{\delta_T}$, the δ_T-Frostman shift of the characteristic function of the row partial isometry Z, where δ_T is a pure contraction in $\mathcal{B}(\mathcal{J}_\infty, \mathcal{J}_0)$.

The characteristic function, b_T, is a unitary invariant for CCNC row contractions: if T and R are unitarily equivalent CCNC row contractions then b_T and b_R coincide.

4.4 QE row contractions

In this section, we identify a class of row contractions for which the characteristic function is a complete unitary invariant. For this characteristic function to be a complete unitary invariant, we will restrict to row contractions such that $\mathscr{H}(b_T)$ has a unique contractive Gleason solution. These are the row contractions whose characteristic function coincide weakly with a QE Schur multiplier.

Definition 4.21. Let T be a CCNC row contraction on a Hilbert space $\mathcal{H}$. We say that T is QE when its characteristic function b_T is QE.

We know that such row contraction is unitarily equivalent to the unique contractive Gleason solution, X^{b_T}, on $\mathscr{H}(b_T)$. If T and R are two unitarily equivalent QE row contractions then their characteristic function coincide weakly, i.e., $b_T|_{supp(b_T)}$ coincides with $b_R|_{supp(b_R)}$. Therefore $\mathscr{H}(b_T|_{supp(b_T)}) = \mathscr{H}(b_R|_{supp(b_R)})$, and, T and R are unitarily equivalent to the unique contractive Gleason solution on $\mathscr{H}(b_T|_{supp(b_T)}) = \mathscr{H}(b_R|_{supp(b_R)})$, and hence they are unitarily equivalent.

Theorem 4.22. *Two QE row contractions are unitarily equivalent if and only if their characteristic functions coincide weakly. The (QE) characteristic function b_T is a complete unitary invariant for QE row contractions.*

This result is not entirely satisfactory since the only way to recognise a QE row contraction is to compute its characteristic function. It is natural to ask which row contractions are QE. By definition, QE row partial isometries are QE row contractions. We already know of a concrete description of the class of QE row partial isometries. Recall that

$$\mathbf{S}_\infty(Z) := \bigvee_{w\in\mathbb{B}_d} w^*\mathcal{R}(Z-w)^\perp = \bigvee_{w\in\mathbb{B}_d} (I-Zw^*)^{-1}\mathrm{Ran}(Z)^\perp.$$

Then the row partial isometry Z is QE if and only if it is CCNC and $\mathrm{Ker}(Z)^\perp \subset \mathbf{S}_\infty(Z)$. This is a consequence of Lemma 3.21. To give a satisfactory description of QE row contractions, we prove the following characterisation of QE Schur multiplier.

Lemma 4.23. *Let $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ with $||b(0)|| < 1$. If b is QE then the unique contractive Gleason solution, X^b, which is extremal, for $\mathscr{H}(b)$ satisfies*

$$\mathrm{Ker}(X^b)^\perp \subset \bigvee_{w\in\mathbb{B}_d} w^* k_w^b \mathcal{J}_0 =: \mathbf{S}_\infty(b). \tag{4.1}$$

Moreover, the pure part, C^b, of X^b is a strict row contraction.

Proof. We assume without loss of generality that either $\mathcal{J}_\infty \subset \mathcal{J}_0$ or $\mathcal{J}_0 \subset \mathcal{J}_\infty$, and take $\mathcal{J}$ to be the largest of $\mathcal{J}_0, \mathcal{J}_\infty$. Let $a := [b]$ be the square extension of b, the function a belongs to $\mathcal{S}_d(\mathcal{J})$, and $\mathscr{H}(b) \subseteq \mathscr{H}(a)$.

There is a bijective correspondence between contractive Gleason solutions in $\mathscr{H}(b)$ and contractive Gleason solutions in $\mathscr{H}(a)$. If b is QE then $\mathscr{H}(b)$ has a unique contractive Gleason solution, X^b, which is extremal, and so does a. If X^a is the unique extermal Gleason solution in $\mathscr{H}(a)$ then either $X^a = X^b$ or $X^a = X^b \oplus S$ (Propositions 2.18 and 2.19). Therefore $\mathrm{Ker}(X^b)^\perp = P_{\mathscr{H}(b)^d}\mathrm{Ker}(X^a)^\perp$.

Recall that

$$\mathbf{S}_\infty(a) = \bigvee_{z \in \mathbb{B}_d} z^* k_z^a \mathcal{J}.$$

If $\mathcal{J}_\infty \subset \mathcal{J}_0$ then $\mathcal{J} = \mathcal{J}_0$. It follows that $k^a(z,w) = k^b(z,w)$ for all $z, w \in \mathbb{B}_d$ and $\mathscr{H}(b) = \mathscr{H}(a)$. Hence,

$$\begin{aligned}
\mathbf{S}_\infty(a)^\perp &= \left\{\mathbf{h} \in \mathscr{H}(a) \otimes \mathbb{C}^d \middle| \quad 0 = (k_z^a)^* z\mathbf{h} = z\mathbf{h}(z); \forall z \in \mathbb{B}_d\right\} \\
&= \left\{\mathbf{h} \in \mathscr{H}(b) \otimes \mathbb{C}^d \middle| \quad 0 = (k_z^b)^* z\mathbf{h} = z\mathbf{h}(z); \forall z \in \mathbb{B}_d\right\} \\
&= \mathbf{S}_\infty(b)^\perp,
\end{aligned}$$

i.e., $\mathbf{S}_\infty(a) = \mathbf{S}_\infty(b)$. If $\mathcal{J}_0 \subset \mathcal{J}_\infty$ then $\mathcal{J}_0 \subset \mathcal{J}$ and $\mathscr{H}(a) = \mathscr{H}(b) \oplus H^2_d(\mathcal{J} \ominus \mathcal{J}_0)$. For $x \in \mathcal{J}$, $x = y + y_\perp$ for some $y \in \mathcal{J}_0$ and $y_\perp \in \mathcal{J}_0^\perp$, and, $k_z^a x = k_z^b y + k_z y_\perp$ for all $z \in \mathbb{B}_d$. It follows that

$$\begin{aligned}
\mathbf{S}_\infty(a) &= \bigvee_{z \in \mathbb{B}_d} z^* k_z^a \mathcal{J} \\
&= \left(\bigvee_{z \in \mathbb{B}_d} z^* k_z^b \mathcal{J}_0\right) \bigvee \left(\bigvee_{z \in \mathbb{B}_d} z^* k_z (\mathcal{J} \ominus \mathcal{J}_0)\right) \\
&= \mathbf{S}_\infty(b) \bigvee \left(\bigvee_{z \in \mathbb{B}_d} z^* k_z (\mathcal{J} \ominus \mathcal{J}_0)\right) \\
&= \mathbf{S}_\infty(b) \oplus \left(\bigvee_{z \in \mathbb{B}_d} z^* k_z (\mathcal{J} \ominus \mathcal{J}_0)\right),
\end{aligned}$$

since $\left(\bigvee_{z \in \mathbb{B}_d} z^* k_z (\mathcal{J} \ominus \mathcal{J}_0)\right) \subset H^2_d(\mathcal{J} \ominus \mathcal{J}_0)$. Therefore $\mathbf{S}_\infty(b) = P_{\mathscr{H}(b)^d}\mathbf{S}_\infty(a)$. If we can prove that $\mathrm{Ker}(X^a)^\perp \subset \mathbf{S}_\infty(a)$, then the claim will follow since

$$\mathrm{Ker}(X^b)^\perp = P_{\mathscr{H}(b)^d}\mathrm{Ker}(X^a)^\perp \subset P_{\mathscr{H}(b)^d}\mathbf{S}_\infty(a) = \mathbf{S}_\infty(b).$$

To prove this, let $\mathbf{a} := U_a^*(V^a)^*U_a k_0^a(I - a(0)^*)^{-1}(I - a(0))$, a contractive Gleason solution for a, and

$$X^* k_w^a = w^* k_w^a - \mathbf{a}a(w)^*$$

defines a contractive Gleason solution for $\mathscr{H}(a)$. Since X^a is the unique Gleason solution for $\mathscr{H}(a)$ then it satisfies

$$(X^a)^* k_w^a = w^* k_w^a - \mathbf{a}a(w)^*,$$

As $\mathrm{Ran}\,(\mathbf{a}) \subset U_a^*\mathrm{Ran}\,((V^a)^*) \subset \mathbf{S}_\infty(a)$, where U_a is the canonical unitary multiplier from $\mathscr{H}(a)$ onto $\mathscr{L}(a)$ which acts on kernel vectors as $U_a^* K_z^a = k_z^a(I - b(z)^*)^{-1}$ (Theorem 2.12), we have $\mathrm{Ran}\,(\mathbf{a}) \subset \mathbf{S}_\infty(a)$. This proves that $\mathrm{Ker}(X^a)^\perp = \overline{\mathrm{Ran}\,((X^a)^*)} \subset \mathbf{S}_\infty(a)$.

■

This can be also directly proved.

Alternative proof. We know that there is a minimal contractive Gleason solution, X_{dBR}, for $\mathscr{H}(b)$ (Proposition 2.29) and it satisfies $\mathrm{Ker}(X_{dBR})^\perp \subset \mathbf{S}_\infty(b)$. If b is QE then X_{dBR} is the unique contractive Gleason solution for $\mathscr{H}(b)$. ■

The above Lemma allows us to identify the class of row contractions whose characteristic functions are QE Schur multipliers. If $b \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$, note that $\mathbf{S}_\infty(b) = \mathbf{S}_\infty(X^b)$. Assuming b is QE and $b(0)$ is a strict contraction, then the unique contractive Gleason solution, X^b, is CCNC (Proposition 2.5), it satisfies $\mathrm{Ker}(X^b)^\perp \subset \mathbf{S}_\infty(b) = \mathbf{S}_\infty(X^b)$. When a CCNC row contraction T satisfies $\mathrm{Ker}(T)^\perp \subset \mathbf{S}_\infty(T)$ then it is unitarily equivalent to an extremal Gleason solution on $\mathscr{H}(b_T)$ which satisfies equation 4.1. Therefore b_T coincide with a QE Schur multiplier and any row contraction whose characteristic function coincides with b_T is unitarily equivalent to the unique extremal Gleason solution in $\mathscr{H}(b_T)$.

Theorem 4.24. *Let T be row contraction on a Hilbert space $\mathcal{H}$ which is a strict extension of its isometric part. We denote*

$$\mathbf{S}_\infty(T) := \bigvee_{w \in \mathbb{B}_d} w^*(I - Tw^*)^{-1}\mathrm{Ran}\,(D_{T^*})\,.$$

Hence T QE if and only if T is CNC and $\mathrm{Ker}(T)^\perp \subset \mathbf{S}_\infty(T)$.

Proof. The proof is very similar to the proof of Theorem 3.22. Let $b_T \in \mathcal{S}_d(\mathcal{J}_\infty, \mathcal{J}_0)$ be the characteristic function of T. There is an extremal Gleason solution in $\mathscr{H}(b_T)$, denote it X^{b_T}, and a unitary operator $U : \mathcal{H} \longrightarrow \mathscr{H}(b_T)$ such that $X^{b_T} = UTU^*$. Denote $k := k^{b_T}$ the de Branges-Rovnyak kernel corresponding to b_T. On one hand, we have

$$k_w \mathcal{J}_0 = (I - X^{b_T} w^*)^{-1} k_0 \mathcal{J}_0 = (I - X^{b_T} w^*)^{-1} \overline{\mathrm{Ran}\left(D_{(X^{b_T})^*}\right)},$$

and consequently,

$$\mathbf{S}_\infty(b) = U^* \otimes I_d \left(\bigvee_{w \in \mathbb{B}_d} w^*(I - X^{b_T} w^*)^{-1} \overline{\mathrm{Ran}\left(D_{(X^{b_T})^*}\right)} \right).$$

On the other hand, $\mathrm{Ker}(X^{b_T})^\perp = U^* \otimes I_d \mathrm{Ker}(T)^\perp$. Therefore $\mathrm{Ker}(X^{b_T})^\perp \subset \mathbf{S}_\infty(b)$ if and only if $\mathrm{Ker}(T)^\perp \subset \mathbf{S}_\infty(T)$. It then follows that b_T is QE if and only if $\mathrm{Ker}(T)^\perp \subset \mathbf{S}_\infty(T)$. ∎

Remark 4.25. Note that it T is a row contraction and $T = Z - C$ its isometric-pure decomposition, then

$$\begin{aligned}
\mathbf{S}_\infty(Z) &= \bigvee_{w \in \mathbb{B}_d} w^*(I - Zw^*)^{-1} \mathrm{Ran}\left(D_{Z^*}\right) \\
&= \bigvee_{w \in \mathbb{B}_d} w^*(I - Zw^*)^{-1} \mathrm{Ran}(Z)^\perp \\
&= \bigvee_{w \in \mathbb{B}_d} w^*(I - Tw^*)^{-1} \mathrm{Ran}(Z)^\perp.
\end{aligned}$$

We have $\overline{\mathrm{Ran}\left(D_{T^*}\right)} = \mathrm{Ran}(Z)^\perp$. Consequently

$$\begin{aligned}
\bigvee_{w \in \mathbb{B}_d} w^*(I - Tw^*)^{-1} \mathrm{Ran}(Z)^\perp &= \bigvee_{w \in \mathbb{B}_d} w^*(I - Tw^*)^{-1} \mathrm{Ran}\left(D_{T^*}\right) \\
&= \mathbf{S}_\infty(T).
\end{aligned}$$

Therefore, of T is QE then $\mathrm{Ker}(Z)^\perp \subset \mathrm{Ker}(T)^\perp \subset \mathbf{S}_\infty(T) = \mathbf{S}_\infty(Z)$ so that Z is also QE. The converse is not true. One can chose the pure part of T so that $\mathrm{Ker}(T) \not\subseteq \mathbf{S}_\infty(T)$ and get a non QE row contraction T.

We gather what we have proven about QE row contractions in the following theorem.

Theorem 4.26. *The characteristic function, b_T, of a CCNC row contraction T is QE if and only* *if*

$$\mathrm{Ker}(T) \subset \mathbf{S}_\infty(T) = \bigvee_{w \in \mathbb{B}_d} w^*(I - Tw^*)^{-1}\mathrm{Ran}\,(D_{T^*})\,.$$

If T and R are such row contractions then T is unitarily equivalent to R if and only if b_T coincides weakly with b_R.

Conclusion

In this thesis, we identify the class of row contractions that can be represented by contractive Gleason solution in a de Branges-Rovnyak space: the CCNC row contractions. This class includes all the CNC d-contractions, which are row contractions with mutually commuting components.

The model theory we develop here produces a characteristic function, which is a unitary invariant, for CCNC row partial isometry, and more generaly for CCNC row contractions whose purely pure part is a strict row contraction. The characteristic function is an operator valued Schur multiplier of the Drury-Arveson space. The model spaces are de Branges-Rovnyak spaces, which are contractively contained in the Drury-Arveson space, and we represent any CCNC row contractions as an extremal Gleason solution in the de Branges-Rovnyak space corresponding to its characteristic function. The uniqueness of extremal Gleason solution for quasi-extreme characteristic function allows us to identify a class of row contractions for which the characteristic function is a complete unitary invariant.

There are several further investigations that can be pursued after this thesis.

If X is an extremal Gleason solution in $\mathscr{H}(b)$ then $b_X = b$. From Proposition 2.5, we know that contractive (not necessarily extremal) Gleason solutions are CCNC row contractions. So given a Schur multiplier b, The characteristic function of an extremal Gleason solution in $\mathscr{H}(b)$ is b itself. Given a contractive, non extremal, Gleason solution X in $\mathscr{H}(b)$, it is natural to ask how are b_X and b related.

We expect that a fully general non-commutative or "free" de Branges-Rovnyak model can be constructed for any CNC row contraction using the free deBranges-Rovnyak spaces, $\mathscr{H}(B)$, corresponding to a free Schur multiplier B. Here, recall, that the full Fock space, F^2_d, can be naturally viewed as the free several-variable Hardy space, the free formal RKHS corresponding to the "non-commutative" or "free Szego kernel" [10]. The free Schur class is then defined as the closed unit ball of the formal multiplier algebra between vector-valued Fock spaces, and the corresponding de Branges-Rovnyak spaces $\mathscr{H}(B)$, for free Schur multiplier B, are formal free RKHS contractively contained in F^2_d. In this setting the characteristic function B_T, of a CNC row contrction T, will belong to the free Schur class.

One can continue on a slightly different subject besides model theory for row contractions. In the article [15], the Crofoot transformation was used to study multipliers between model spaces in the classical case. We have proved in Lemma 4.9 that there is an isometric multiplier from $\mathscr{H}(b)$ onto $\mathscr{H}(b_0)$. Using the multivariate generalisation of Crofoot transformation defined in this thesis, one could study the multipliers from one de Branges-Rovnyak spaces onto another.

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