EXACT NON-EQUILIBRIUM SOLUTIONS

OF THE

EINSTEIN-BOLTZMANN EQUATIONS

by

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Thesis Presented for the Degree of
DOCTOR OF PHILOSOPHY
in the Department of Applied Mathematics
UNIVERSITY OF CAPE TOWN

Brussels, January 1994

ABSTRACT

Exact Non-Equilibrium Solutions of the Einstein-Boltzmann Equations

In this thesis we use the exact solution of the Boltzmann equation, with a relaxation-time model of collisions, to find solutions of the Einstein-Boltzmann system of equations. A covariant harmonic decomposition of the distribution function is used to obtain exact results. The conditions imposed by the conservation of particle number and energy-momentum, and by the H-theorem are determined. The properties of exact truncated Boltzmann solutions with first and second order anisotropies are investigated. Exact entropy results are obtained for the solution with first order anisotropy, and the solution with second order anisotropy is shown to obey exact thermodynamics laws. The Einstein-Boltzmann equations with relaxation-time model of collisions are solved in FRW and Bianchi I spacetime. In FRW spacetime, a general anisotropic solution and an isotropic solution are obtained. The non-equilibrium anisotropic solution with arbitrary isotropic relaxation function has vanishing particle flux and an equilibrium energy-momentum tensor. Specific forms of the relaxation function permit tilted solutions and solutions with non-zero bulk viscosity. Exact entropy results are derived for the isotropic solution showing that the H-theorem is satisfied. The non-equilibrium isotropic solution has vanishing non-equilibrium pressures and fluxes. The FRW and Bianchi I solutions are used to demonstrate the generation of anisotropy in FRW cosmologies. A relaxation length model of collisions is introduced. This model is used to obtain solutions of the Einstein-Boltzmann equations in static spherically symmetric spacetime. In this static model, anisotropic pressure comes from the bulk viscosity.

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January 1994

ACKNOWLEDGEMENT

I express my deep gratitude to Dr. R. Maartens for his constant support, assistance, and breadth of vision that has helped make the Ph. D. experience stimulating and enriching. His critical appraisal of the original manuscript has substantially improved the final version. I wish to thank Prof. G. F. R. Ellis for his continuous support and encouragement and Prof. D. R. Matravers for suggesting the subject.

To my family.

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INTRODUCTION

In many cosmological and astrophysical situations, an idealised fluid model of the matter is inappropriate, and the self-consistent microscopic model of relativistic kinetic theory [Ehlers (1971), Israel and Stewart (1980), De Groot, Van Leeuwen, and Van Weert (1980), and Israel (1989)] gives a more detailed physical description. An obvious example is the case of collision-free particles, such as cosmological neutrinos (when the temperature is $\leq 10^{10}\ K$) and photons (after recombination) [Thorne (1967), Ehlers, Geren, and Sachs (1968), Misner (1968), Matzner (1969) and (1972), and Stewart (1972)], or stellar clusters in equilibrium [Zeldovich and Podurets (1966), Fackerell (1968), and Ipser and Thorne (1968)]. Other examples arise in transitional situations, such as the transition from the early-universe collision-dominated era to a collision-free phase for photons [Peebles and Yu (1970), and Peebles (1980)] or neutrinos [Stewart (1968)]. There are also other non-equilibrium evolutions, such as stellar clusters with collisions [Podurets (1970)], or the evolution of an FRW early universe into an anisotropic Bianchi universe or an inhomogeneous universe via a disturbance of the equilibrium collisional balance [Matravers and Ellis (1989), Ellis and Matravers (1992)]. Other non-equilibrium situations suited to a kinetic approach are interaction problems, such as the relativistic transport of photons [Lindquist (1966), Dautcort (1969), De Groot et al. (1980), Kichenassamy and Krikorian (1985), Schweizer (1988)] and cosmic rays [Webb (1985)], or mixtures of cosmic elementary particles [De Groot et al. (1980), Bernstein (1988)]. Further details of and references to these and other applications may be found in Israel and Stewart (1979, 1980), De Groot et al. (1980), Israel (1989), and Schweizer (1988).

A kinetic approach is in general more fundamental and consistent than the phenomenological fluid picture and its associated thermodynamics. For example, the standard (Eckart) thermodynamics of fluids violates causality and is unstable [Hiscock and Lindblom (1983)]. A causal and stable generalisation emerges clearly from kinetic theory, as developed by Israel and Stewart (1976). In some cases, a fluid model leads to the loss of information and certain effects. For example, a Landau-type damping of gravitational perturbations by a kinetic gas is not present in the fluid models [Stewart (1972)]. Kinetic theory incorporates the particle structure of matter, dispenses with the additional phenomenological equations required in the fluid model, and allows a unified treatment of massive and massless particles (radiation).

Much of the work referred to thus far, and indeed most of the major results in relativistic kinetic theory, rely on near-equilibrium approximations, since the full equations are enormously complex. The foundations of relativistic kinetic approximation theory were laid by Israel (1963), Marle (1966), Anderson (1971), Stewart (1971), and others (see De Groot et al. (1980)). Very few exact solutions are known and nearly all of them equilibrium solutions. Equilibrium arises either from detailed balancing of collisions, or the absence of collisions. In the former case, it is a standard result [Ehlers (1971)] that the distribution must be relativistic Maxwell-Boltzmann (or its quantum counter part), and that such solutions can only occur if the spacetime admits a timelike Killing (non-zero mass particles) or conformal Killing (zero mass) vector.

The exact collision-free equilibrium solutions of the Einstein-Boltzmann equations known to us are: isotropic [Ehlers, Geren, and Sachs (1968), Hakim (1968), Bel (1969), and Treciokas and Ellis (1971)] and anisotropic [Ellis, Matravers, and Treciokas (1983a), and Maharaj and Maartens (1987)] solutions in Robertson-Walker spacetime; isotropic [Zeldovich and Podurets (1966) and Fackerell (1968)] and anisotropic [Ray (1982) and Maharaj and Maartens (1986)] solutions in static spherically symmetric spacetime; anisotropic solutions in spatially homogeneous spacetimes with local rotational symmetry [Stewart (1973), Berezdivin and Sachs (1973), Ray and Zimmerman (1977), Maartens and Maharaj (1985)]. The general solutions of the Liouville or collisionless Boltzmann equation has been found in Robertson-Walker [Maartens and Maharaj (1987) and Maharaj and Maartens (1987)], static spherically symmetric [Maharaj and Maartens (1986)], and Bianchi I (non-axisymmetric) [Maartens and Maharaj (1990)] spacetimes.

We know of only two exact non-equilibrium Einstein-Boltzmann solutions. Stewart (1968) presented a solution in Bianchi I axisymmetric spacetime. The harmonic expansion methods of Ellis [Ellis, Matravers, and Treciokas (1983a,b) and Ellis, Treciokas, and Matravers (1983c)] were used to outline a (formal) solution in Bianchi I (non-axisymmetric) spacetime [Matravers and Ellis (1989)]. In this thesis we give some new exact non-equilibrium solutions of the Einstein-Boltzmann equations.

In order to have any hope of finding an exact non-equilibrium solution, simplifying assumptions must be imposed on the spacetime geometry and on the collision term. In most physical applications, approximation methods are needed. The applicability of exact solutions is limited. However, exact non-equilibrium solutions and exact properties of non-equilibrium solutions are important for other reasons also. Unlike the approximate solutions, they are not limited to be close to equilibrium. Thus one can hope to gain a better insight into the nature of fully non-equilibrium situations, or transitional situations that contain a highly non-equilibrium phase. Furthermore, there are a number of fundamental questions, not resolved by approximation methods, about the general effect of collisions on the distribution. In particular, there is the question of whether and under what conditions collisions tend to isotropise the distribution [Ellis et al. (1983c)]. Another issue is the relation between approximations and exact results, involving questions about the consistency of the approximations and the assumptions involved in them [Ellis et al. (1983c)].

The basic laws of non-equilibrium kinetic theory, such as the H-theorem (positivity of entropy production) for the Boltzmann collision term, were established by Ehlers (1961), Tauber and Weinberg (1961), Israel (1963), Chernikov (1963), and others (see De Groot et al. (1980)). There has been much progress in approximation methods, but very few further exact properties of non-equilibrium solutions have been derived. Most of those known to us are due to Ellis and co-workers [Treciokas and Ellis (1971), Ellis et al. (1983a,b)]. Their work is a large part of the foundation for this thesis. However the aim of this thesis is more modest than to advance on the major established results. Our aim is to derive some exact properties of Einstein-Boltzmann solutions with a relaxation time (relativistic BGK) model of the collision term [Marle (1969), and Anderson and Witting (1974)].

In Chapter 1 we start by presenting those general aspects of relativistic kinetic theory on which the subsequent chapters depend. We present the concept of a particle distribution function, the Liouville operator, the relativistic Boltzmann transport equation, and the associated concepts such as the phase space and the use of a tetrad basis. The use of the first two moments of the distribution function to determine the kinematics and the dynamics of the gas is presented along with a discussion of the Boltzmann and other collision functionals. An improved and clearer approach is used to introduce the concepts of equilibrium and non-equilibrium solutions. The possible approaches to solving the relativistic transport equation, such as truncation methods and the covariant harmonic expansion method, are presented. Chapter 1 also contains new relations relating the dynamical and kinematical quantities as measured by different observers, as well as the harmonic forms of the H-theorem, entropy density, and entropy flux. This chapter is concluded with a discussion of the known exact solutions of the Boltzmann equation.

In Chapter 2 we discuss the general integro-differential system of equations that must be solved in order to obtain equilibrium Einstein-Boltzmann solutions. In order to obtain exact Einstein solutions it is necessary to apply simplifying assumptions to the spacetime. We present the metrics describing the Robertson-Walker, Bianchi I, and static spherically symmetric spacetimes for which exact Einstein-Boltzmann solutions are presented in the remaining chapters. We also present the field equations in the above-mentioned spacetimes in harmonic form. As the exact solutions are based on exact isotropic and anisotropic equilibrium solutions, we also include a summary of those solutions used in this thesis.

Chapter 3 contains the first comprehensive exact analysis of the BGK model. We introduce the BGK collision functional and derive a formal solution of the Boltzmann equation with BGK collision functional. We also introduce the Anderson-Witting (AW) form of the BGK relaxation function. The restrictions placed on the solution by the conditions for the conservation of particles, energy and momentum are obtained in harmonic form. These restrictions give matching conditions for linear relaxation functions and allow the definition of a new local rest frame. We derive harmonic forms for the entropy quantities and show that, to the lowest order (and using the AW BGK model), anisotropy in the distribution tends to increase the entropy production rate. We find closed form entropy expressions for a first order truncated AW BGK solution and show that the H-theorem is identically satisfied. Furthermore, we use a second order truncated AW BGK solution to derive the harmonics of the Boltzmann equation and show that these equations take the form of thermodynamics laws. The consistency conditions resulting from the truncation of the distribution are determined. This solution is further investigated on a FRW background. The chapter is concluded with a discussion of the Einstein-Boltzmann problem with a BGK collision functional.

In Chapter 4 we find an anisotropic, non-equilibrium Einstein-BGK solution in FRW spacetime. We start by deriving the results using an arbitrary, isotropic relaxation function. We show that the conditions for the conservation of energy and particles lead to vanishing bulk viscosity and an equilibrium energy-momentum tensor. The conditions for conservation of momentum and zero particle drift lead to vanishing first order anisotropy in the distribution function. The field equations require zero energy flux and anisotropic stress which, in turn, also require vanishing first order anisotropy as well as vanishing second order anisotropy. We restrict the relaxation mechanism and show that solutions exist allowing non-zero bulk viscosity for a massless gas. Using the

AW relaxation model we also show that it is possible to obtain tilted solutions and non-equilibrium isotropic solutions for massless gases.

In Chapter 5 we use the isotropic, non-equilibrium AW BGK solution and derive exact properties. This solution is a special case of the anisotropic solution of Chapter 4. We find that the solution has zero bulk viscosity, number flux, energy flux, and anisotropic stress with equilibrium energy density, but that it is still non-equilibrium. An explicit expression for the entropy production rate is obtained. We show that the entropy production rate satisfies the H-theorem and that it could be large (even in the case of a massless gas).

In Chapter 6 we obtain an anisotropic Einstein-BGK solution in Bianchi I spacetime. Using an AW collision model we show that the solution has equilibrium particle number density, energy density and energy flux. We present a particular solution that would satisfy all restrictions due to the field and conservation equations. We also derive the first two harmonic equations determining the inter-dependence existing for the harmonics in Bianchi I geometry. We furthermore use the AW FRW solution from Chapter 4 and the particular Bianchi I solution to provide a non-equilibrium model for the evolution of anisotropy in FRW geometry. This mechanism can force an isotropic universe to evolve to an anisotropic universe. We also suggest a physical mechanism that could trigger this evolution.

In Chapter 7 we obtain a non-equilibrium, anisotropic Einstein-BGK solution in static spherically symmetric spacetime. In this case the relaxation occurs with changing radius and we introduce the concept of relaxation length. The condition for the conservation of particles requires that the solution is non-tilted, and along with the condition for conservation of energy, leads to vanishing first order anisotropy. The field equations require vanishing heat flow and we found that anisotropic stress comes from bulk viscosity.

For all of the solutions we also discuss possible physical applications. The thesis is concluded in Chapter 8 with a summary of the main results.

The content of Chapters 1 and 2 is a review of existing material except for the relations for the dynamic quantities, the entropy quantities in harmonic form, and the improved discussion on equilibrium (taken from Maartens and Wolvaardt (1994a)). The new work developed during the course of the project is given in Chapters 3 through 7. Chapter 3 is taken largely from Maartens and Wolvaardt (1994a), while Chapter 5 is based partly on Maartens and Wolvaardt (1994a). Joint publications are also in preparation based on Chapters 4, 6 and 7 [Maartens and Wolvaardt (1994b,c), Wolvaardt and Maartens (1994)].

With regard to conventions and notation, we follow Ellis et al. (1983a,b). In particular, the metric signature is taken as $(-,+,+,+)$, the curvature tensor is given by $R^{i}{}_{jkl} = -\Gamma^{i}{}_{jk,l} + \ldots$, and the Ricci tensor by $R_{ij} = R^{k}{}_{ikj}$. Late Latin indices $(i,j,\ldots = 0,1,2,3)$ denote coordinate components with late Greek for spatial components $(\rho,\nu,\ldots = 1,2,3)$. Early Latin indices $(a,b,\ldots = 0,1,2,3)$ denote components of a general or tetrad basis with early Greek for spatial components $(\alpha,\beta,\ldots = 1,2,3)$. We use units in which Einstein's gravitational constant, the speed of light in vacuum, and Boltzmann's constant are all unity. Also note that throughout this thesis the terms Robertson-Walker, RW, or FRW are used when we refer to Friedman-Lemaitre-Robertson-Walker spacetime geometry.

1.0 RELATIVISTIC KINETIC THEORY

1.1 Introduction

In this chapter we present an outline of relativistic kinetic theory. For a full treatment see Ehlers (1971) and De Groot et al. (1980). Only those issues relevant to this thesis will be discussed more fully. Some new approaches and results will be given. We will only consider a simple gas of uncharged identical classical particles with mass m ($m \geq 0$). The extensions to the charged case, to quantum statistics, and to mixtures with a range of masses are fairly straightforward [Ehlers (1971) and De Groot et al. (1980)].

Any attempt to go beyond the highly idealised fluid picture of matter to a self-consistent microscopic picture faces the enormously complex problem of modelling the collective interactions of particles with the gravitational field. The actual 'gravitational force' on each particle is a very complicated functional of the world lines of all the other particles. Although progress can be made in attempts to deal with particle correlations and field fluctuations [Israel and Kandrup (1984), Kandrup (1984, 1989)] one needs more drastic simplifying assumptions in order to set up and solve physical models. This leads to the no-correlation, mean-field description of general relativistic kinetic theory. Essentially we neglect particle correlations and use a one-particle distribution function f [Ehlers (1971) and Stewart (1971)].

The particles are assumed to move in the mean field generated by themselves collectively (or in a background field in the case of a test gas), and to interact directly only at very short range. The effect of the short-range forces is approximated by instantaneous point binary collisions (and annihilation and creation processes). In between collisions, the particles' motion is approximated by test particle motion (geodesic) in the mean field. This mean field is the approximation of the effect of the very long range force of gravity. In order for these assumptions to be reasonable (although it can be argued that some of them are inherently unreasonable - see Kandrup (1984)), the system cannot be too far from equilibrium. Furthermore, the mean free path of particles must be much greater than the range of interparticle forces, and the gravitational field should not vary significantly across this range. Also, the gas cannot be too dense and cold, so that particles which are about to collide have uncorrelated momenta.

We can describe the range of possible behaviours of the gas qualitatively as follows [Stewart (1969), De Groot et al. (1980)]. If ϵ is the ratio of the mean free path to a characteristic macroscopic length (or the mean collision time to a characteristic macroscopic time), then:

(a) $\epsilon \gg 1$ is the free flow (or Knudsen) regime, where a fluid description is completely inapplicable. In the limit $\epsilon \to \infty$ we get collision-free equilibrium.

(b) $\epsilon \ll 1$ is the hydrodynamic regime, where collisions dominate. This is where the Chapman-Enskog approximation is applied, giving a kinetic foundation to first-order Eckart thermodynamics [Stewart (1971), De Groot et al. (1980), Israel (1989)]. The limit $\epsilon \to 0$ describes collision-dominated equilibrium ('detailed balancing').

(c) $\epsilon \sim 1$ is the transition regime. The Grad method of approximation applies here, giving a kinetic theory foundation to the second-order Israel thermodynamics [Stewart (1971), De Groot et al. (1980), Israel (1989)].

Note that a fluid description is only valid for $\epsilon < 1$.

1.2 Relativistic Boltzmann Equation

Given a point x in a spacetime manifold X, one can define a vector space P_x as the space spanned by the tangent vectors to all particle worldlines through x. The tangent vectors can be identified with the particle 4-momenta, making P_x the momentum space at x. From the relation

$$g_{ij}(x)p^i p^j = -m^2, \tag{1.1}$$

where g_{ij} is the metric tensor, m the particle rest mass, and p^i the components of particle 4-momentum, we can define the mass shell $P_x(m)$ as a hypersurface in P_x. (For a simple gas, we are assuming that the particle restmass remains unchanged.)

The next step is to introduce at x an arbitrary observer moving with a 4-velocity u^a where $u_a u^a = -1$. We can now define an invariant 3-dimensional volume element dV

in the spacelike hypersurface orthogonal to u^a (ie. in the rest space of u^a) as

$$dV = \sqrt{|g|}\epsilon_{abcd}u^a d_1x^b d_2x^c d_3x^d \tag{1.2}$$

where ϵ_{abcd} is the Levi-Civita tensor density, $|g| = \det(g_{ab})$, and d_1x^b, d_2x^c, and d_3x^d are arbitrary displacement vectors at x which span an element of the hypersurface orthogonal to u^a. We also define an invariant 3-dimensional volume element on the mass shell as

$$d\mathcal{P} = \frac{\sqrt{|g|}}{|-p_0|}\epsilon_{\alpha\beta\gamma}d_1p^\alpha d_2p^\beta d_3p^\gamma. \tag{1.3}$$

where d_1p^α, d_2p^β, and d_3p^γ are arbitrary and independent momentum displacement vectors on $P_x(m)$ and $p^0 > 0$.

We are now in a position to introduce the one-particle distribution function $f(x,p)$, which determines the number of particles near event x with 4-momenta near p. More specifically,

$$dN = f(x^i,p^j)(-u_kp^k)d\mathcal{P}dV \tag{1.4}$$

is the number of particle worldlines crossing a volume element dV in the rest space of the observer u^i at event x with 4-momenta in the range $d\mathcal{P}$ about p^j.

If the phase space density of collisions is given by the collision term $C[f]$, then the change in dN due to collisions is

$$C[f](-u_kp^k)d\mathcal{P}dV. \tag{1.5}$$

Furthermore, Liouville's theorem states that for an arbitrary observer u^a, $(-u_kp^k)d\mathcal{P}dV$ is conserved along the worldlines. As the particles follow geodesics (free fall motion in the mean field or background field) in between collisions, the change in dN along a worldline is, by (1.4)

$$\frac{d}{dv}(dN) = \mathbf{\mathcal{L}}(f)(-u_kp^k)d\mathcal{P}dV, \tag{1.6}$$

where the Liouville operator $\mathbf{\mathcal{L}}$ is the derivative along the geodesics followed by the particles in between collisions and is given by

$$\mathcal{L} = \frac{d}{dv} = \frac{dx^i}{dv}\frac{\partial}{\partial x^i} + \frac{dp^i}{dv}\frac{\partial}{\partial p^i}$$

$$= p^i\frac{\partial}{\partial x^i} - \Gamma^i{}_{jk}p^j p^k \frac{\partial}{\partial p^i}\,. \tag{1.7}$$

The affine parameter is given by $v = (\text{proper time})/m$ for $m > 0$, and is any suitable affine parameter for $m = 0$. Note that $\mathcal{L}$ is a coordinate-independent operator as it is the directed derivative along the world lines. Also, since (1.1) is a first integral of geodesic motion, $\mathcal{L}(m) = 0$. Thus at each event x, the $\partial/\partial p^i$ in (1.7) may be restricted to the mass shell (1.1). Usually, as in (1.3), the spatial components p^μ are taken as coordinates on the mass shell, and the time component p^0 is then fixed by (1.1).

If we now equate the change in dN along the particle trajectories (1.6) to the change in dN due to collisions (1.5), we get

$$\mathcal{L}(f)(-u_k p^k)d\mathcal{P}dV = C[f](-u_k p^k)d\mathcal{P}dV,$$

and the Boltzmann transport equation governing the evolution of f can be written as

$$\mathcal{L}(f) = C[f]. \tag{1.8}$$

1.3 $3+1$ Decomposition

One or more 4-velocity vector fields are usually picked out by the geometry or the dynamics. Given any 4-velocity u^a, where $u^a u_a = -1$, we can project orthogonal to u^a into the instantaneous rest space of a co-moving observer with the tensor

$$h_{ab} = g_{ab} + u_a u_b \tag{1.9}$$

and thus define a $3+1$ splitting of space-time [Ellis et al. (1983b)]. In particular, any particle 4-momentum p^a may be decomposed as

$$p^a = Eu^a + \lambda e^a, \;\; e^a e_a = 1, \;\; e^a u_a = 0, \;\; \lambda \geq 0, \tag{1.10}$$

where $\lambda e^a = h^a{}_b p^b$ is the relativistic 3-momentum and E the energy, both as measured by a comoving observer:

$$E = -u_a p^a, \ \ \lambda = (E^2 - m^2)^{1/2}. \tag{1.11}$$

As a result, the distribution function $f(x^i, p^a)$ can be written as a function $f(x^i, m, E, e^a)$.

Any vector has a decomposition of the form (1.10). Any symmetric tensor T_{ab} can be decomposed as

$$T_{ab} = \mu u_a u_b + p h_{ab} + \pi_{ab} + q_a u_b + q_b u_a \tag{1.12}$$

where $\pi_{[ab]} = 0 = \pi_{ab} u^b = \pi_a{}^a = q_a u^a$. The covariant derivative of u^a itself splits into kinematic tensors describing the evolution of u^a world-lines:

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - \dot{u}_a u_a \tag{1.13}$$

where ω_{ab} is the rate of vorticity tensor $(\omega_{(ab)} = 0 = \omega_{ab} u^b)$, σ_{ab} is the rate of shear tensor $(\sigma_{[ab]} = 0 = \sigma_{ab} u^b = \sigma_a{}^a)$, θ is the rate of expansion $(\theta = u^a{}_{;a})$, and $\dot{u}_a = u_{a;b} u^b$ is the 4-acceleration $(\dot{u}_a u^a = 0)$. The tensors π_{ab} and σ_{ab} are examples of the projected symmetric trace-free (PSTF) parts of rank two tensors. In general, if A_{ab} is any such tensor, its PSTF part is

$$A_{<ab>} = h_a{}^c h_b{}^d A_{(cd)} - \tfrac{1}{3} A_{cd} h^{cd} h_{ab} \tag{1.14a}$$

so that $\pi_{ab} = T_{<ab>}$, $\sigma_{ab} = u_{<a;b>}$. The definition (1.14) may be extended [De Groot et al. (1980), Schweizer (1988)] to higher rank tensors $A_{ab\ldots c}$, whose PSTF parts satisfy

$$A_{<ab\ldots c>} = A_{<(ab\ldots c)>}, \quad A_{<ab\ldots c>} u^c = 0 = A_{<ab\ldots c>} h^{ab}. \tag{1.14b}$$

1.4 Orthonormal Tetrad Approach

A tetrad basis is used in most of this thesis. In most cases it is specialised to an orthonormal basis. By attaching an orthonormal spatial triad $\{\boldsymbol{E}_\alpha\}$ to u^a we get an orthonormal tetrad $\{\boldsymbol{E}_a\} = \{\boldsymbol{u}, \ \boldsymbol{E}_\alpha\}$. (Notice that indices a, b, c,... are used for the tetrad basis and indices $i, j, k, \ldots$ are used for the coordinate basis.) We obtain the following tetrad components:

$$\boldsymbol{E}_a \cdot \boldsymbol{E}_b \equiv g_{ab} = \text{diag}\,(-1, 1, 1, 1), \quad u^a = \delta^a{}_0\,, \quad h_{ab} = \text{diag}\,(0, 1, 1, 1). \tag{1.15}$$

Tetrad and coordinate components of tensors are related by the quantities $E_a{}^i$, $E^a{}_i$, where

$$\boldsymbol{E}_a = E_a{}^i \frac{\partial}{\partial x^i}, \quad E^a{}_i E_a{}^j = \delta_i{}^j, \quad E^a{}_i E_b{}^i = \delta^a{}_b. \tag{1.16}$$

The connection components and the tetrad commutation functions, defined by

$$\nabla_a \boldsymbol{E}_b = \Gamma^c{}_{ab} \boldsymbol{E}_c, \quad [\boldsymbol{E}_a, \boldsymbol{E}_b] = \gamma^c{}_{ab} \boldsymbol{E}_c$$

are related by

$$\Gamma_{abc} = \tfrac{1}{2}(\gamma_{abc} + \gamma_{cab} - \gamma_{bca})$$

and have the symmetries

$$\Gamma_{cba} = -\Gamma_{abc}\,, \quad \gamma^a{}_{cb} = -\gamma^a{}_{bc}.$$

These tetrad equations together with (1.11) and (1.13) allow us to express the Liouville operator (1.7) as [Ellis et al. (1983b)]:

$$\begin{aligned} \boldsymbol{\mathcal{L}} = {} & E\partial_0 + \lambda e^\alpha \partial_\alpha - \lambda\Big\{\tfrac{1}{3}\lambda\theta + E\dot{u}_\alpha e^\alpha + \lambda\sigma_{\alpha\beta}e^\alpha e^\beta\Big\}\frac{\partial}{\partial E} + \Big\{-\lambda^{-1}(E^2\dot{u}^\alpha + \lambda^2 a^\alpha) - \\ & E(\sigma^\alpha{}_\beta + \omega^\alpha{}_\beta + \epsilon^\alpha{}_{\beta\gamma}\Omega^\gamma)e^\beta + \lambda^{-1}(E^2\dot{u}_\beta + \lambda^2 a_\beta)e^\alpha e^\beta - \lambda\epsilon^\alpha{}_{\beta\delta}n^\delta{}_\gamma e^\beta e^\gamma + \\ & E\sigma_{\beta\gamma}e^\alpha e^\beta e^\gamma\Big\}\frac{\partial}{\partial e^\alpha}\,. \end{aligned} \tag{1.17a}$$

The additional quantities in (1.17a) are the tetrad derivatives

$$\partial_a f = E_a{}^i \frac{\partial f}{\partial x^i}\,,$$

the rate of rotation of the triad

$$\Omega^{\alpha} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}\boldsymbol{E}_{\beta} \cdot \dot{\boldsymbol{E}}_{\gamma} \tag{1.17b}$$

and the spatial parts of the commutation functions

$$a_{\alpha} = \gamma^{\beta}{}_{\alpha\beta} \ , \quad n^{\alpha\beta} = \gamma^{(\alpha}{}_{\gamma\delta}\epsilon^{\beta)\gamma\delta} \ . \tag{1.17c}$$

We can also use (1.11) to express the mass shell measure (1.3) in tetrad form:

$$d\mathcal{P} = \lambda dE d\Omega \tag{1.18}$$

where $d\Omega$ is the solid angle spanned by two independent de^{a}, which are tangent to the unit 2-sphere in the rest space. A useful integral identity over this sphere is (in any basis)

$$\begin{aligned} \int d\Omega e^{a_1} \ldots e^{a_r} &= \frac{4\pi}{(r+1)} h^{(a_1 a_2} h^{a_3 a_4} \ldots h^{a_{r-1} a_r)}, \quad r \text{ even} \\ &= 0 \qquad\qquad\qquad\qquad\qquad r \text{ odd.} \end{aligned} \tag{1.19}$$

1.5 Kinematics and Dynamics of the Gas

The kinematics and dynamics of the gas are determined by the moments of the distribution function. For the distribution f to be physical, it must be nonnegative and vanishing for $E \to \infty$ so that its moments on the mass shell are bounded. The fundamental macroscopic tensors defined by the microscopic distribution are the particle 4-current vector

$$n^{a} = \int p^{a} f d\mathcal{P} \ , \tag{1.20}$$

the energy-momentum tensor

$$T^{ab} = \int p^{a} p^{b} f d\mathcal{P}, \tag{1.21}$$

and the entropy 4-current vector

$$S^{a} = -\int p^{a} f(\log f - 1) d\mathcal{P}. \tag{1.22}$$

For a given 4-velocity u^a, T^{ab} has therefore the form (1.12), with μ the energy density, p the isotropic pressure, π_{ab} the anisotropic stress tensor and q_a the energy flux, all measured by a comoving observer. The vectors (1.20), (1.22) decompose as

$$n^a = Nu^a + j^a, \quad j^a u_a = 0 \tag{1.23}$$

$$S^a = su^a + \Sigma^a, \quad \Sigma^a u_a = 0 \tag{1.24}$$

where N is the number density, s the entropy density, j^a the number flux, and Σ^a the entropy flux, all as measured by a u^a observer.

If $\tilde{u}^a$ is any other 4-velocity, then

$$u_a \tilde{u}^a = -\cosh\psi$$

where ψ is the hyperbolic angle between u^a and $\tilde{u}^a$ ($\cosh\psi = (1-v^2)^{-1/2}$ where v is the instantaneous relative speed of $\tilde{u}^a$ in the u^a rest space). The tensors (1.20) - (1.22) may also be decomposed with $\tilde{u}^a$ and $\tilde{h}_{ab} = g_{ab} + \tilde{u}_a \tilde{u}_b$, leading to different kinematic and dynamic quantities measured by a $\tilde{u}_a$ observer. It is possible to derive the relationships between the two sets of kinematic and dynamic quantities by following the procedure outlined below.

Let c_a and $\tilde{c}_a$ be spacelike unit vectors orthogonal to u_a and $\tilde{u}_a$ respectively (i.e. $u^a c_a = 0$, $c_a c^a = 1$, $\tilde{u}^a \tilde{c}_a = 0$, and $\tilde{c}_a \tilde{c}^a = 1$). Now, using $u_a \tilde{u}^a = -\cosh\psi$ we find

$$u_a = \cosh\psi\, \tilde{u}_a - \sinh\psi\, \tilde{c}_a$$

$$\tilde{u}_a = \cosh\psi\, u_a + \sinh\psi\, c_a$$

$$c_a = -\sinh\psi\, \tilde{u}_a + \cosh\psi\, \tilde{c}_a$$

$$\tilde{c}_a = \sinh\psi\, u_a + \cosh\psi\, c_a\,.$$

We now equate the decompositions of the tensors T^{ab}, n^a, and S^a:

$$T_{ab} = \mu u_a u_b + p h_{ab} + \pi_{ab} + q_a u_b + q_b u_a$$

$$= \tilde{\mu}\, \tilde{u}_a \tilde{u}_b + \tilde{p}\tilde{h}_{ab} + \tilde{\pi}_{ab} + \tilde{q}_a \tilde{u}_b + \tilde{q}_b \tilde{u}_a$$

$$n^a = N u^a + j^a = \tilde{N}\tilde{u}^a + \tilde{j}^a$$

$$S^a = s u^a + \Sigma^a = \tilde{s}\tilde{u}^a + \tilde{\Sigma}^a.$$

Using the relations between u_a and $\tilde{u}_a$, all u^a in the above three equations can be expressed in terms of $\tilde{u}^a$. Next, by applying the projections g^{ab}, $\tilde{u}^b$ and $\tilde{u}^a\tilde{u}^b$ respectively to the above equations it is possible, through some algebraic manipulation, to determine the relationships between the dynamic quantities [Maartens and Wolvaardt (1994a)]:

$$\tilde{N} = N \cosh\psi - j_a \tilde{u}^a \tag{1.25a}$$

$$\tilde{j}_a = j_a + (j_b \tilde{u}^b - N \cosh\psi)\tilde{u}_a + N u_a \tag{1.25b}$$

$$\tilde{s} = s \cosh\psi - \Sigma_a \tilde{u}^a \tag{1.25c}$$

$$\tilde{\Sigma}_a = \Sigma_a + (\Sigma_b - s \cosh\psi)\tilde{u}_b + s u_a \tag{1.25d}$$

$$\tilde{\mu} = \mu + (\mu + p)\sinh^2\psi + \pi_{ab}\tilde{u}^a\tilde{u}^b - 2q_a\tilde{u}^a\cosh\psi \tag{1.25e}$$

$$3\tilde{p} = 3p + (\mu + p)\sinh^2\psi + \pi_{ab}\tilde{u}^a\tilde{u}^b - 2q_a\tilde{u}^a\cosh\psi \tag{1.25f}$$

$$\tilde{q}_a = q_a\cosh\psi - \pi_{ab}\tilde{u}^b + [2q_b\tilde{u}^b\cosh\psi - (\mu + p)\cosh^2\psi - \pi_{bc}\tilde{u}^b\tilde{u}^c]\tilde{u}_a + [(\mu + p)\cosh\psi - q_b\tilde{u}^b]u_a \tag{1.25g}$$

$$\tilde{\pi}_{ab} = \pi_{ab} + 2\tilde{u}_{(a}\pi_{b)c}\tilde{u}^c + \tfrac{1}{3}[2q_c\tilde{u}^c\cosh\psi - (\mu + p)\sinh^2\psi - \pi_{cd}\tilde{u}^c\tilde{u}^d]\tilde{h}_{ab} + [(\mu + p)\cosh^2\psi + \pi_{cd}\tilde{u}^c\tilde{u}^d - 2q_c\tilde{u}^c\cosh\psi]\tilde{u}_a\tilde{u}_b + 2[q_c\tilde{u}^c - (\mu + p)\cosh\psi]\,\tilde{u}_{(a}u_{b)} + (\mu + p)u_a u_b + 2q_{(a}u_{b)} - 2q_{(a}\tilde{u}_{b)}\cosh\psi \tag{1.25h}$$

The exact relations (1.25) generalise the previously given special cases of $q_a = 0 = \pi_{ab}$

('tilted' perfect fluid spacetimes [King and Ellis (1973), Maharaj and Maartens (1987)]) and of near-equilibrium first-order relations between the Eckart and Landau-Lifshitz frames [Israel (1989)].

The distribution defines at least two 4-velocities. A kinematic average 4-velocity, defined by the particle flow, is the Eckart 4-velocity u^a_E :

$$n^a = N_E u^a_E \quad (j^a_E = 0). \tag{1.26a}$$

Alternatively, a dynamic average 4-velocity, defined as the unit timelike eigenvector of T_{ab}, is the Landau-Lifshitz 4-velocity $u^a{}_L$:

$$T^a{}_b u^b_L = -\mu_L u^a_L \quad (q^a_L = 0) \tag{1.26b}$$

The relations between the two sets of kinematic and dynamic quantities may be deduced from (1.25), using $j^a_E = 0 = q^a_L$. In equilibrium, $u^a_E = u^a_L$, and this remains true to first order near equilibrium [Israel (1989)]. Also near equilibrium, the Eckart energy flux is related to the Landau-Lifshitz number flux by [Israel and Stewart (1979)]

$$q^a_E \approx -N_L^{-1}(\mu_L + p_L) j^a_L \, . \tag{1.27a}$$

Using (1.25), we can give the exact relation of which (1.27a) is a limiting case. Writing $\tilde{q}^a = q^a_E$ and $u^a = u^a_L$, and using $\cosh\psi = N/\tilde{N}$, we get

$$-\tilde{N}\tilde{q}_a = N[\tilde{N}^{-2}(\mu N^2 + p j_b j^b + \pi_{bc} j^b j^c) - \mu]u_a +$$

$$[\tilde{N}^{-2}(\mu N^2 + p j_b j^b + \pi_{bc} j^b j^c) + p] j_a + \pi_{ab} j^b \tag{1.27b}$$

which gives (1.27a) on neglecting terms of second order in j_a, π_{ab}, $N/\tilde{N}$.

1.6 Equilibrium State and Isotropic Distribution Functions

The first two moments (1.20), (1.21) of f are sufficient to determine an equilibrium distribution, but for non-equilibrium all the higher moments in general are needed for a complete specification of f. For small deviations from equilibrium, such complete knowledge is unnecessary, and one uses truncated expansions. In any case, the following identity is crucial [Ehlers (1971), Stewart (1971)]:

$$\left(\int p^a \ldots p^b p^c \Psi(f) d\mathcal{P}\right)_{;c} = \int p^a \ldots p^b \Psi'(f) \mathbf{L}(f) d\mathcal{P} \tag{1.28}$$

where Ψ is any scalar function of f. Using the Boltzmann equation (1.8), (1.28) gives the following divergence relations for (1.20)-(1.22):

$$n^a{}_{;a} = \int C[f] d\mathcal{P} \tag{1.29}$$

$$T^{ab}{}_{;b} = \int p^a C[f] d\mathcal{P} \tag{1.30}$$

$$S^a{}_{;a} = -\int \log f \; C[f] d\mathcal{P}. \tag{1.31}$$

By the definition of $C[f]$ (leading to (1.8)), it follows that $\int \Phi C[f] d\mathcal{P}$ is the rate of production per unit volume of the property Φ by collisions. If this rate is zero, then Φ is known as a collisional invariant. Thus, $n^a{}_{;a}$ is the particle number production rate, $T^{ab}{}_{;b}$ is the 4-momentum production rate. If particle number and p^a are collisional invariants, then by (1.29), (1.30) we get conservation of particles and of energy-momentum in the form

$$n^a{}_{;a} = 0 \tag{1.32}$$

$$T^{ab}{}_{;b} = 0 \tag{1.33}$$

An equilibrium distribution generates no entropy and thus satisfies

$$S^a{}_{;a} = 0. \tag{1.34}$$

Following Maartens and Wolvaardt (1994a), we will call this *kinematic equilibrium* (it is sometimes called 'local' equilibrium [Israel (1989)]). By (1.29)-(1.33), we see that the Maxwell-Boltzmann distribution is a kinematic equilibrium for any collision term $C[f]$, provided there is conservation of particle number and energy-momentum:

$$\bar{f} = \exp[\alpha(x) + \beta_a(x) p^a] \;\Rightarrow\; S^a{}_{;a} = -\alpha n^a{}_{;a} - \beta_a T^{ab}{}_{;b}\,. \tag{1.35}$$

For the Boltzmann collision term, Chernikov's theorem [De Groot et al. (1980)] shows that (1.35) is the only possible kinematic equilibrium. This is not true for other collision terms. The Maxwell-Boltzmann distribution (1.35) is a particular case of an *isotropic distribution* $f = F(x, u_a p^a)$, which has, by (1.20)-(1.22), isotropic moments (with $u^a = u^a_E = u^a_L$):

$$n^a = Nu^a, \quad T^{ab} = \mu u^a u^b + ph^{ab}, \quad S^a = su^a \; . \tag{1.36}$$

A distribution is unchanged by collisions if

$$C[f] = 0 \Leftrightarrow \mathbf{L}(f) = 0 \; . \tag{1.37}$$

Following Maartens and Wolvaardt (1994a), we will call this *dynamic equilibrium* since it obeys the Boltzmann equation (sometimes it is called global equilibrium [Israel (1989)]). By (1.31), dynamic equilibrium implies kinematic equilibrium, but the converse is not true. There are two types of dynamic equilibrium: (a) Liouville equilibrium due to the absence of collisions; (b) collision-dominated equilibrium due to the detailed balancing of collisions. In case (a), many distributions other than (1.35) may arise. In case (b), for elastic binary point collisions (Boltzmann model) only (1.35) is possible. In this case $C[f]$ vanishes identically and $\mathbf{L}(f) = 0$ implies

$$\alpha_{;a} = 0, \quad \beta_{(a;b)} = \delta_{mo}\phi g_{ab} \; . \tag{1.38}$$

which lead to the restrictions ($\beta^a = T^{-1}u^a$, $T =$ temperature):

$$\sigma_{ab} = 0, \;\; \theta = -3\frac{\dot{T}}{T} \, , \;\; \dot{T} = -\delta_{mo}\phi T^2 \; . \tag{1.39}$$

In particular, an expanding gas cannot be in collision-dominated equilibrium unless $m = 0$. These restrictive results have led some authors to question their foundations. For example, the possibility of further collisional invariants is raised by Schucking and Spiegel (1970) and removed in Abellan, Navarro, and Alvarez (1977). More fundamentally, Alvarez (1976a) postulates a new scalar entropy for which the equilibrium is the Bel distribution [Bel (1969)], allowing for $\theta \neq 0$ when $m > 0$. The definition (1.22) of entropy is certainly open to question [Schucking and Spiegel (1970)], but we will follow the standard approach of accepting it. The circumstances under which the equilibrium distribution can be used approximately in cosmological calculations, and how departures from this distribution can be calculated, are investigated in Bernstein (1988).

We now turn our attention to isotropic distribution functions. A distribution function is said to be isotropic if a 4-velocity vector field u^a exists such that

$$f = F(x, E) \; , \quad E = -u_a p^a \; .$$

The distribution function is isotropic about u_a in momentum space as all components of p^a not parallel to u_a cancel in the averaging process. The isotropic distribution function has special properties, some of which are presented by Treciokas and Ellis (1971) and Ehlers, Geren, and Sachs (1968). A particularly interesting result which we shall just state here, is that a solution of the Boltzmann transport equation (1.8) with an isotropic distribution function implies that the 4-velocity u^a is shear free: $\sigma_{ab} = 0$; furthermore, if the solution also satisfies the Einstein field equations, then $\theta\omega_{ab} = 0$ [Treciokas and Ellis (1971)]. The motion is either non-expanding ($\theta = 0$) or non-rotating ($\omega_{ab} = 0$).

Isotropic distribution functions may be applicable to the early stages of cosmological models, where high collision rates are assumed to isotropise the distribution, or in later stages (like the current epoch) where the non-interacting background radiation has a high degree of isotropy. The isotropisation of the microwave background radiation is a motivation for the study of non-equilibrium relativistic kinetic theory. These issues are further discussed in Chapters 4 and 5 of this thesis. Some exact equilibrium solutions in different spacetime geometries are presented in Chapter 2.

1.7 Boltzmann Collision Functional

The Boltzmann model of elastic binary point collisions with incoming momenta uncorrelated leads to the Boltzmann collision term

$$C[f] = \int (f_* f'_* - f f') W \, d\mathcal{P}' d\mathcal{P}_* d\mathcal{P}'_* \tag{1.40a}$$

where p, p' are the incoming and p_*, p'_* are the outgoing 4-momenta, $f_* = f(x, p_*)$ etc., and $W(pp', p_*p'_*)$ is the transition probability (containing the cross-section) which is taken to obey microscopic reversibility

$$W(pp', \, p_*p'_*) = W(p_*p'_*, \, pp') \tag{1.40b}$$

(The consequences of relaxing (1.40b) are investigated by Alvarez (1976b).) If $h(x,p)$ is any phase space function then (1.40) implies

$$\int hC[f]d\mathfrak{P} = \tfrac{1}{4}\int (h + h' - h_* - h'_*)W(f_*f'_* - ff')d\mathfrak{P}d\mathfrak{P}'d\mathfrak{P}_*d\mathfrak{P}'_*$$

This identity together with (1.29) and (1.30) show that the conservation equations (1.32) and (1.33) hold identically for the Boltzmann collision term (1.40). Furthermore, (1.31) gives

$$S^a{}_{;a} = \tfrac{1}{4}\int W(\log f_*f'_* - \log ff')(f_*f'_* - ff')d\mathfrak{P}d\mathfrak{P}'d\mathfrak{P}_*d\mathfrak{P}'_*$$

so that

$$S^a{}_{;a} \geq 0 \tag{1.41}$$

since $(x-1)\log\, x \geq 0$ for all positive x. Thus the H-theorem (1.41) follows identically from (1.40). Note that (1.41) does not tell us whether the entropy increase reaches a maximum, i.e. whether there will be an approach to (kinematic) equilibrium. It is usually assumed that collisions (i.e. in the Boltzmann model (1.40)) tend to bring about (kinematic) equilibrium, but we know of a proof only in the very special case of a test gas in a static background [Alves (1985)].

For small deviations from equilibrium

$$f = \bar{f}(1+\epsilon) \tag{1.42}$$

where $\epsilon(x,p)$ is small and $\bar{f}$ is the kinematic equilibrium (1.35) (with α, β_a not subject to the global conditions (1.38)). Then (1.42) in (1.22) gives

$$S^a - \bar{S}^a = -\alpha(n^a - \bar{n}^a) + \beta_b(T^{ba} - \bar{T}^{ba}) - Q^a \tag{1.43a}$$

using (1.20), (1.21), where $\bar{n}^a$, $\bar{T}^{ab}$ are given by (1.36). The second order deviations from equilibrium are contained in the last term of (1.43a), which is to lowest order

$$Q^a \approx -\tfrac{1}{2}\int \bar{f}\epsilon^2 p^a d\mathfrak{P}. \tag{1.43b}$$

This is independent of the form of $C[f]$.

For the Boltzmann collision model (1.40), the approximation (1.42) leads to the linearised Boltzmann equation

$$-\bar{f}^{-1}\mathfrak{L}(f) = \mathfrak{C}[\epsilon] \equiv \int W\bar{f}'(\epsilon + \epsilon' - \epsilon_* - \epsilon'_*)d\mathcal{P}d\mathcal{P}'d\mathcal{P}_*d\mathcal{P}'_* \qquad (1.43c)$$

on which the Chapman-Enskog and Grad methods are based.

In the Chapman-Enskog kinetic theory approximation, or in quasi-stationary thermodynamics, Q^a is assumed to be zero, leading to acausal equations for viscosity and heat flow of Eckart and Landau-Lifshitz. In transient thermodynamics or in the Grad 14-moment approximation in kinetic theory, the causal Israel-Stewart equations arise when Q^a is not set to zero [Israel and Stewart (1979)]:

$$\Pi = -\xi[\theta + \beta_1\dot{\Pi} - \alpha_1 q^a{}_{;a} - \gamma_2 q^a \dot{u}_a] \qquad (1.43d)$$

$$q_a = -\chi h_a{}^b[T_{;b} + T\dot{u}_b + \beta_2\dot{q}_b - \alpha_1\Pi_{;b} - \alpha_2\pi_b{}^c{}_{;c}]$$

$$+\gamma_1\Pi\dot{u}_a + \gamma_3\pi_{ab}\dot{u}^b + \beta_2\omega_{ab}q^b \qquad (1.43e)$$

$$\pi_{ab} = -\eta[\sigma_{ab} + \beta_3 h_a{}^c h_b{}^d\dot{\pi}_{cd} - \gamma_3 q_{<a;b>} - \gamma_4 q_{<a}\dot{u}_{b>} - 2\beta_3\pi^c{}_{<a}\omega_{b>c}] \qquad (1.43f)$$

Equations $(1.43d-f)$ hold for small deviations from equilibrium. They are given in the Landau-Lifshitz frame (so that $(1.27a)$ holds), with $\Pi = p - \bar{p}$, ξ the bulk viscosity, χ the thermal conductivity, η the shear viscosity, and α_i, β_i, γ_i, the transient coefficients. Note that the assumptions involved in the Israel-Stewart (and non-causal) theories imply that a non-equilibrium state is determined by Π, π_{ab}, q_a (or j_a). In particular if $\Pi = \pi_{ab} = q_a = j_a = 0$, then $S^a{}_{;a} = 0$. It is important to realise that this is an approximation which is not an exact result in kinetic theory (and this opens up questions about phenomenological thermodynamics that assumes it is an exact result). In general a non-equilibrium state cannot fully be specified by Π, π_{ab}, q_a (or j_a). Consequently (and this is often obscured by extrapolating near-equilibrium results beyond their validity), *it is possible for a non-equilibrium distribution to have n^a, T^{ab} of equilibrium form* (1.36) *(with $\Pi = 0$ i.e. $p = \bar{p}$)*. In particular, *a gas may behave on the average like a perfect fluid (with $p = \bar{p}$) and not be in equilibrium.* In Section 5.5 we will present an exact Einstein-Boltzmann solution with just this property (see also Section

4.6 and Maartens and Wolvaardt (1994a)).

An alternative approach is to use an exact linear collision term, as in the relativistic BGK model:

$$C[f] = \gamma(\bar{f} - f) \tag{1.44}$$

where γ determines the average collision rate, and f is relaxing towards the given fixed distribution $\bar{f}$, which is independent of f. We shall discuss this model in detail in Chapter 3.

Apart from the Boltzmann and BGK collision terms, there is also the transport collision term, modelling the passage of particles through an emitting and absorbing medium. The general form of this term is [Ellis et al. (1983b)]

$$C[f] = n(x)[Q(x,p) - \kappa(x,p)f(x,p) + \int \Sigma(p,p')f(x,p')d\mathcal{P}'] \tag{1.45}$$

where n is the medium's number density, Q describes spontaneous particle emission, κ describes absorption, and Σ describes scattering.

1.8 Covariant Harmonic Decomposition

A useful technique is Ellis' harmonic decomposition method [Ellis et al. (1983b,c)]. Given any 4-velocity field u^a, with its associated $3+1$ splitting (1.10), the distribution is expanded as

$$f(x,p) = F(x,E) + F_a(x,E)e^a + F_{ab}(x,E)e^a e^b + \dots \tag{1.46a}$$

where the harmonic coefficients are isotropic PSTF tensors ($F_{a\dots b} = F_{<a\dots b>}$) with suitable asymptotic behaviour to ensure convergence of the series. Note that $F \geq 0$, but the higher harmonics need not be positive. If collisions tend to isotropise the distribution (which is usually assumed, but not proven [Ellis et al. (1983c)]), then the harmonics F_a, F_{ab}, ... above the zero order harmonic F will die out as the gas evolves. (Also, the shear of the preferred 4-velocity will have to die out [Treciokas and Ellis (1971)].)

Using a harmonic expansion of the collision term,

$$C[f] = b(x,E) + b_a(x,E)e^a + \ldots \tag{1.46b}$$

and the tetrad form (1.17) of $\mathbf{\mathcal{L}}$, the Boltzmann equation splits into a sequence of harmonic equations [Ellis et al. (1983c)]. By (1.18), (1.19), and (1.46), the moments (1.20) and (1.21) give the following harmonic forms [Ellis et al. (1983b)]

$$N = 4\pi \int_m^\infty E\lambda F dE \tag{1.47a}$$

$$j_a = \tfrac{4\pi}{3} \int_m^\infty \lambda^2 F_a dE \tag{1.47b}$$

$$\mu = 4\pi \int_m^\infty E^2 \lambda F dE \tag{1.47c}$$

$$p = \tfrac{4\pi}{3} \int_m^\infty \lambda^3 F dE \tag{1.47d}$$

$$q_a = \tfrac{4\pi}{3} \int_m^\infty E\lambda^2 F_a dE \tag{1.47e}$$

$$\pi_{ab} = \tfrac{8\pi}{15} \int_m^\infty \lambda^3 F_{ab} dE \tag{1.47f}$$

$$\Pi = \tfrac{4\pi}{3} \int_m^\infty \lambda^3 (F - \bar{F}) dE \tag{1.47g}$$

It is striking that the kinematics and dynamics are determined directly only by the first three harmonics (scalars are determined by F, vectors (fluxes) by F_a and second rank tensors (anisotropic stresses) by F_{ab}). The higher harmonics play an indirect role via any interaction with F, F_a, F_{ab} in the Boltzmann equation. Because it is nonlinear in f, the entropy 4-current involves in general all the harmonics. By using the covariant harmonic decomposition (1.46), we can derive useful expressions for the entropy quantities s, Σ_a, and $S^a{}_{;a}$ in (1.24) and (1.31) [Maartens and Wolvaardt (1994a)]. Starting with the definition of the entropy 4-current vector (1.22) and using (1.46a) we find

$$\begin{aligned} \log f &= \log F + \log(1 + \tfrac{F_a}{F}\, e^a + \ldots) \\ &= \log F + \tfrac{F_a}{F}\, e^a + \tfrac{F_{ab}}{F}\, e^a e^b - \tfrac{1}{2}\tfrac{F_a}{F}\tfrac{F_b}{F}\, e^a e^b + \ldots \end{aligned}$$

where the series expansion $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$ is utilised (note that this series expansion converges for $-1 < x \leq 1$). After some algebraic manipulation we find

$$f - f \log f = (F - F \log F) - (\log F)F_a e^a - [(\log F)F_{ab} + \tfrac{1}{2F}F_a F_b]e^a e^b + \ldots$$

Now, inserting this into (1.22) and using (1.18), (1.19), and (1.10) for p^a, we find

$$s = 4\pi \int_m^\infty E\lambda \left[(F - F \log F) - (1/6F)F_a F^a + \ldots\right] dE \tag{1.48a}$$

$$\Sigma_a = -\tfrac{4\pi}{3} \int_m^\infty \lambda^2 \left[(\log F)F_a + \ldots\right] dE. \tag{1.48b}$$

Taking a similar approach we can write $S^a{}_{;a}$ in harmonic form. The integrand of (1.31) becomes

$$\log f\ C[f] = b \log F + \left(b_a \log F + b\,\frac{F_a}{F}\right)e^a + \left[b\left(\frac{F_{ab}}{F} - \frac{1}{2}\frac{F_a}{F}\frac{F_b}{F}\right) + b_a\frac{F_b}{F} + b_{ab}\log F\right]e^a e^b + \ldots$$

Inserting this into (1.31) and using (1.18) and (1.19) then gives

$$S^a{}_{;a} = -4\pi \int_m^\infty \lambda\left[b \log F + (1/3F^2)(Fb_a - \tfrac{1}{2}bF_a)F^a + \ldots\right] dE\ . \tag{1.48c}$$

Notice that the deviations in entropy density (1.48a) and entropy production rate (1.48c) from the isotropic values are of second order. Furthermore, (1.48a) shows that to the lowest order

$$s < s_{iso} \tag{1.48d}$$

i.e. the entropy density for the isotropic distribution (not necessarily equilibrium) is a maximum. *Anisotropy in a distribution therefore tends to decrease the entropy density.* Conversely, the isotropisation of the distribution is accompanied by a growth in entropy density.

Note that by (1.47e,f), $F_a = 0 = F_{ab}$ implies that the energy-momentum tensor has perfect fluid form. The converse is not true, since non-zero F_a, F_{ab} can sometimes lead to the integrals in (1.47e,f) vanishing (see [Ellis et al. (1983a), Maharaj and Maartens (1986)]). Note also that (1.47c,d) imply the useful identity

$$p = \tfrac{1}{3}(\mu - m^2 M), \qquad M = 4\pi \int_m^\infty \lambda F dE \tag{1.47h}$$

where $M = \int f d\mathcal{P} \geq 0$ is the zeroth moment of f, and it follows that $0 < p \leq \frac{1}{3}\mu$.

1.9 Main Exact Results

We conclude this review by citing the main exact results of relevance to our work. (Thus, we leave aside the general theorems on existence and uniqueness of solutions to the full Einstein-Boltzmann equations [Bancel and Choquet-Bruhat (1973)], and on the properties of approximations [De Groot et al. (1980), Majorana (1988)]). Treciokas and Ellis (1971) showed that for isotropic solutions of the Boltzmann equation, with any (isotropic) collision term, the shear of the preferred 4-velocity vanishes (see also Section 1.6):

$$\mathbf{L}(f) = C[f] \text{ and } f = F(x, -u_a p^a) \Rightarrow \sigma_{ab} = 0. \tag{1.49}$$

If $m = 0$, then u^a is parallel to a conformal Killing vector. If f satisfies the Einstein equations

$$G_{ab} = T_{ab} = \int p_a p_b f d\mathcal{P} \tag{1.50}$$

then $\theta\omega_{ab} = 0$. (These results generalise the collision-free results of Ehlers, Geren and Sachs (1968)).

Ellis et al. (1983c) showed that for solutions of the Boltzmann equation with BGK collision term (1.44)

$$\left.\begin{array}{l} f \text{ has a finite number of harmonics} \\ \text{or: } F_a = F_{ab} = F_{abc} = 0 \\ \text{or: } f \text{ has 4 consecutive harmonics zero} \end{array}\right\} \Rightarrow \sigma_{ab} = 0 \tag{1.51}$$

We turn now to a detailed study of the BGK model (Chapter 3).

1.10 Summary

In this chapter we presented the main principles of Relativistic Kinetic Theory which are required for the rest of the thesis. The Boltzmann transport equation and the related issues such as the Liouville operator and the collision functional (in full, linearised or approximate form) were presented. Useful techniques such as the $3+1$ decomposition, the orthonormal tetrad approach, and the covariant harmonic decomposition were outlined. The moment approach to the determination of dynamic quantities was presented along with a discussion of equilibrium and isotropic distribution functions.

Original contributions of this chapter include the equations (1.25), relating the dynamic quantities as measured by different observers u_a and $\tilde{u}_a$, as well as the harmonic forms of the entropy quantities. This allowed us to show that anisotropy in a distribution tends to decrease the entropy density. We also presented an improved and clearer discussion of equilibrium.

2.0 EXACT EQUILIBRIUM EINSTEIN-BOLTZMANN SOLUTIONS

2.1 Introduction

In this chapter we present the relevant known exact equilibrium (collision-free) solutions to the Einstein-Boltzmann equations (i.e. the Einstein-Liouville equations) that will form the basis for the exact non-equilibrium solutions presented in the remainder of this thesis. We start by presenting the coupled set of integro-differential equations that must be solved. In order to obtain solutions, it is necessary to apply simplifying assumptions to the spacetime. The Robertson-Walker (k=0), the Bianchi I, and the Static Spherically Symmetric (SSS) spacetimes are considered. For each case the spacetime metric, the Einstein field equations in harmonic form, the moments of the distribution function, and the solutions of the Einstein-Liouville equations are presented.

2.2 Einstein-Liouville Model

The discussion of Chapter 1 concerned the solution of the Boltzmann equation in a given spacetime geometry with metric g_{ij}. For the self-gravitating gas, the spacetime geometry is determined by the mean field generated collectively by the gas particles themselves. The interaction between the gas (represented by the one-particle distribution function), and the spacetime geometry (represented by the metric g_{ij}), is described mathematically by the coupled Einstein-Boltzmann system of integro-differential equations and the contracted Bianchi integrability conditions. We restrict ourselves to collision-free distribution functions satisfying the Liouville equation as discussed in Section 1.6. The discussion of Section 1.6 showed that the conservation equations (1.32, 1.33), and thus the contracted Bianchi identities, are identically satisfied as $C[f] = 0$ (for the Liouville case) in equations (1.29, 1.30).

The Einstein-Liouville set of equations is given by

$$\mathcal{L}(f) = 0 \tag{2.1a}$$

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = T_{ab} \tag{2.1b}$$

$$T^{ab} = \int p^a p^b f d\mathcal{P} \tag{2.1c}$$

where R_{ab} is the Ricci tensor ($R = R^a{}_a$). Equation (2.1a) is known as the Liouville equation (see also (1.35)). Also note that the entropy production rate is zero for this equilibrium model as $C[f] = 0$ in equation (1.31).

For the discussion in this chapter it will be necessary to decompose the Einstein field equations (2.1b) with respect to some preferred 4-velocity. We usually consider those 4-velocities picked out by the geometry of the spacetimes under consideration. Using (2.1b) and the dynamic quantities of (1.12), the field equations can be decomposed as:

$$R_{ab}u^a u^b = \tfrac{1}{2}(\mu + 3p) \tag{2.2a}$$

$$R_{ab}u^a h^b{}_c = -q_c \tag{2.2b}$$

$$R_{ab}h^a{}_c h^b{}_d = \tfrac{1}{2}(\mu - p)h_{cd} + \pi_{cd} \ . \tag{2.2c}$$

2.3 Exact Equilibrium Solutions in Robertson-Walker ($k = 0$) Spacetime

The discussion of this section is largely based on that of Ellis et al. (1983a) and Maartens and Maharaj (1987). We consider the Robertson-Walker metric with flat spatial sections and standard coordinates $x^i = (t, x^\nu)$:

$$ds^2 = -dt^2 + R^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \ . \tag{2.3}$$

The symmetry of Robertson-Walker spacetime defines a preferred 4-velocity

$$u^i = \delta^i{}_0 \ . \tag{2.4}$$

The orthonormal tetrad of (1.16) can now be chosen as

$$\mathbf{E}_a = \{\mathbf{u}, R^{-1}(t)\partial/\partial x^\nu\} \ \Leftrightarrow \ E_0{}^a = \delta^a{}_0 \ , \quad E_\nu{}^a = R^{-1}\delta^a{}_\nu \ . \tag{2.5}$$

This allows one to write the particle direction of motion as

$$e^a = (0,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\phi),$$

where θ, ϕ represent the direction of $\boldsymbol{e}$ relative to the axes $\boldsymbol{E}_\nu$.

The Ricci tensor for (2.3) is given by

$$R_{ab}u^a u^b = -3\ddot{R}(t)/R(t) \tag{2.6a}$$

$$R_{ab}u^a h^b{}_c = 0 \tag{2.6b}$$

$$R_{ab}h^a{}_c h^b{}_d = [R(t)\ddot{R}(t) + 2\dot{R}^2(t)]R^{-2}(t)h_{cd}\ . \tag{2.6c}$$

For the discussion here, we furthermore decompose f using the covariant harmonic expansion of (1.46) and we make use of the harmonic forms (1.47) of the dynamic quantities. The general approach for finding the Einstein-Liouville solutions is to find restrictions on the dynamic quantities using Ricci tensor symmetries in the decomposed Einstein field equations (2.2). The conditions on the dynamic quantities result in conditions on the distribution function f which are imposed by the Einstein equations. It is during this process that the important feature of the covariant harmonic decomposition, i.e. the non-mixing of harmonics in the integrals determining the quantities (1.47) (as discussed in Section 1.7), allows us to obtain exact solutions. However, it is first necessary to find f satisfying the Liouville equation (2.1a).

To construct solutions of the Liouville equation we note that $f(x^i, p^a)$ solves this equation if and only if it is constant on all geodesics. The particle mass $m \equiv (-p^a p_a)^{1/2}$ is constant on all geodesics $x^i(v)$ with tangent vectors $p^a = dx^a/dv$. Furthermore, if $\boldsymbol{\xi}$ is a Killing vector ($\xi_{(a;b)} = 0$), then $\xi_a(x^i)p^a$ is constant along geodesics since

$$\mathcal{L}(\xi_a(x)p^a) = \xi_{a;b}p^a p^b = \xi_{(a;b)}p^a p^b = 0. \tag{2.7}$$

Thus, in a spacetime admitting r Killing vectors $\xi_\Lambda(\Lambda = 1 \text{ to } r)$, any positive function h of r variables defines a solution $f(x^i, p^a) = h(m, \xi_\Lambda \cdot \boldsymbol{p})$ of the Liouville equation [Ehlers (1971)].

The Robertson-Walker metric admits six Killing vectors, three of which are $\xi_\nu = \partial/\partial x^\nu$ ($\Leftrightarrow \xi_\nu{}^i = \delta^i{}_\nu$). Thus, the components $R^2 p^\nu$ are constants of the motion; equivalently we can take the direction (relative to $\boldsymbol{E}_\nu$) and spatial magnitude as the constants: $\boldsymbol{\mathcal{L}}(\theta) = \boldsymbol{\mathcal{L}}(\phi) = \boldsymbol{\mathcal{L}}(P) = 0$. P represents the total linear 3-momentum of the gas particles ($P = \lambda R(t)$). As a result any function $f = h(m, P, \theta, \phi)$ is a solution of the Liouville equation. In this case we are interested in the covariant harmonic decomposition of f. Using (1.46a) it is clear that $f(x^i, p^a)$ is a solution of the Liouville equation if [Ellis et al. (1983a)]

$$F_{a\ldots b}(x^i, E) = F_{a\ldots b}(P), \quad P \equiv \lambda R(t) = (E^2 - m^2)^{1/2} R(t) \,. \tag{2.8}$$

P and therefore $F_{a\ldots b}(P)$ are spatially homogeneous ($\partial P/\partial x^\nu = 0$) and isotropic ($\partial P/\partial e^a = 0$). Thus, f is a spatially homogeneous distribution function. However, f is dynamically anisotropic ($\partial f/\partial e^a \neq 0$).

We are now in a position to specialise the equations (1.47) to the Robertson-Walker case. Using the solution (2.8) we obtain

$$N = \frac{4\pi}{R^3(t)} \int_0^\infty P^2 F(P) dP \tag{2.9a}$$

$$\mu = \frac{4\pi}{R^4(t)} \int_0^\infty P^2 [P^2 + m^2 R^2(t)]^{1/2} F(P) dP \tag{2.9b}$$

$$p = \frac{4\pi}{3R^4(t)} \int_0^\infty P^4 [P^2 + m^2 R^2(t)]^{-1/2} F(P) dP \tag{2.9c}$$

$$j_0 = 0, \quad j_\nu = \frac{4\pi}{3R^3(t)} \int_0^\infty P^3 [P^2 + m^2 R^2(t)]^{-1/2} F_\nu(P) dP \tag{2.9d}$$

$$q_0 = 0, \quad q_\nu = \frac{4\pi}{3R^4(t)} \int_0^\infty P^3 F_\nu(P) dP \tag{2.9e}$$

$$\pi_{0a} = 0, \; \pi_{\mu\nu} = \frac{8\pi}{15R^4(t)} \int_0^\infty P^4 [P^2 + m^2 R^2(t)]^{-1/2} F_{\mu\nu}(P) dP \tag{2.9f}$$

Equations (2.8) and (2.9) represent the Liouville solution. Notice that for the Liouville case the kinematics and the dynamics are determined directly only by the first three harmonics. The interaction between the harmonics does not occur as in the general Boltzmann equation. The next step is to find the Einstein-Liouville solution.

In order to ensure that the structure of the kinetic energy-momentum tensor (1.12), (1.21) matches the geometrically determined structure of the Einstein tensor we restrict

f by restricting the dynamic quantities (2.9). This is done by comparing the Ricci tensor (2.6) with the Einstein field equations (2.2), obtaining

$$q_\nu = 0 \tag{2.10a}$$

$$\pi_{\mu\nu} = 0 \tag{2.10b}$$

$$\mu = 3\dot{R}^2/R^2 \tag{2.10c}$$

$$\mu + 3p = -6\ddot{R}/R \quad . \tag{2.10d}$$

The conservation equations (1.33) then become

$$\dot{\mu} + (\mu + p)3\dot{R}/R = 0 \tag{2.11a}$$

$$(\mu + p)\dot{u}^a + h^{ab}p_{,b} = 0 \tag{2.11b}$$

and are identically satisfied by virtue of Liouville's equation. Equation (2.11*b*) is satisfied trivially by symmetry, while (2.11*a*) can replace (2.10*d*). (Equation (2.11*a*) implies that (2.10*c*) is the first integral of (2.10*d*)).

By (2.9*b*) and (2.10*c*) $R(t)$ is implicitly determined by

$$t = (3^{1/2}/2) \int_0^{R^2} \left[4\pi \int_0^\infty P^2(m^2u + P^2)^{1/2} F(P) dP\right]^{-1/2} du \, . \tag{2.12}$$

Equation (2.12) implies that for any choice of $F(P)$, the two field equations (2.10*c*) and (2.10*d*) are satisfied. This is somewhat analogous to choosing an equation of state $p = p(\mu)$ in a perfect fluid solution. Furthermore, it is clear from equations (2.9*e*) and (2.9*f*) that the remaining two field equations (2.10*a*), (2.10*b*) are satisfied if

$$\int_0^\infty P^3 F_\nu(P) dP = 0 \tag{2.13a}$$

$$\int_0^\infty P^4[P^2 + m^2R^2(t)]^{-1/2} F_{\mu\nu}(P) dP = 0. \tag{2.13b}$$

It is remarkable that the full Einstein-Liouville solution can be obtained by appropriately restricting the first and second order harmonics only. Also note that as the restrictions (2.13) apply to the integrals of F_ν and $F_{\nu\mu}$, it is possible to find non-zero solutions for F_ν and $F_{\nu\mu}$. This and the fact that no restrictions are placed on the higher order harmonics, allow the existence of a dynamically anisotropic particle distribution function in a spatially homogeneous and isotropic Robertson-Walker spacetime. However, although the distribution of particles may be anisotropic, the average energy, momentum, and stress distributions must be homogeneous and isotropic to satisfy the field equations for a Robertson-Walker universe.

We now turn our attention to finding F_ν and $F_{\nu\mu}$ that satisfy (2.13). For the case $m = 0$, the equations (2.13) reduce to

$$\int_0^\infty P^3 F_\nu(P) dP = 0 \tag{2.14a}$$

$$\int_0^\infty P^3 F_{\mu\nu}(P) dP = 0. \tag{2.14b}$$

A large number of non-zero F_ν and $F_{\nu\mu}$ can be found to satisfy (2.14). For $m \neq 0$, non-zero $F_{\mu\nu}$ can be found for a specific t to satisfy (2.13*b*). However, when t changes this is no longer true. The general solution therefore requires that $F_{\mu\nu} = 0$ when $m > 0$.

We have reviewed exact anisotropic Einstein-Liouville solutions in Robertson-Walker spacetimes; the properties of these solutions are discussed further in Ellis et al. (1983a).

2.4 Exact Equilibrium Solutions in Bianchi I Spacetime

The discussion of this section is based on that of Matravers and Ellis (1989). We consider the Bianchi I metric with standard coordinates $x^i = (t, x, y, z)$:

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2 \ . \tag{2.15}$$

The Bianchi I symmetry defines the preferred 4-velocity

$$u^i = \delta^i{}_0 \ . \tag{2.16}$$

This vector field is geodesic and irrotational, and is orthogonal to the surfaces $t =$ constant. In general the shear is non-zero and its evolution is given by

$$\dot{\sigma}_{ij} - \sigma_{ik}\sigma^k{}_j + \tfrac{2}{3}h_{ij}\sigma^2 - \tfrac{1}{2}\pi_{ij} + E_{ij} = 0 \tag{2.17}$$

where $E_{ik} = C_{ijkl}u^j u^l$ is the electrical part of the Weyl tensor C_{ijkl} and the dot denotes the covariant derivative of the tensor along the 4-velocity field:

$$\dot{\sigma}_{ij} = (\sigma_{ij})_{;k}u^k \ . \tag{2.18}$$

The Einstein field equations for this metric are

$$-\left(\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z}\right) = \tfrac{1}{2}(\mu + 3p) \tag{2.19a}$$

$$q_j = 0 \tag{2.19b}$$

$$X^2\left(\frac{\ddot{X}}{X} + \frac{\dot{Y}\dot{X}}{YX} + \frac{\dot{Z}\dot{X}}{ZX}\right) = \tfrac{1}{2}(\mu - p)X^2 + \pi_{11} \tag{2.19c}$$

$$Y^2\left(\frac{\ddot{Y}}{Y} + \frac{\dot{Y}\dot{X}}{YX} + \frac{\dot{Z}\dot{Y}}{ZY}\right) = \tfrac{1}{2}(\mu - p)Y^2 + \pi_{22} \tag{2.19d}$$

$$Z^2\left(\frac{\ddot{Z}}{Z} + \frac{\dot{Z}\dot{X}}{ZX} + \frac{\dot{Z}\dot{Y}}{ZY}\right) = \tfrac{1}{2}(\mu - p)Z^2 + \pi_{33} \tag{2.19e}$$

with a first integral

$$\frac{\dot{Y}\dot{X}}{YX} + \frac{\dot{Z}\dot{X}}{ZX} + \frac{\dot{Z}\dot{Y}}{ZY} = \mu \ . \tag{2.19f}$$

Unlike the Robertson-Walker case, the trace-free pressure tensor $\pi_{\mu\nu}$ is not zero for all μ, ν ; instead it is given by

$$\pi_{ij} = \mathrm{diag}(0,\ \pi_{11},\ \pi_{22},\ \pi_{33}) \quad \Rightarrow \pi_{ij} = 0 \quad \text{for} \quad i \neq j \ . \tag{2.19g}$$

As π_{ij} is trace-free

$$X^{-2}\pi_{11} + Y^{-2}\pi_{22} + Z^{-2}\pi_{33} = 0 \ . \tag{2.19h}$$

Furthermore, the conservation equations give

$$\dot{\mu} + (\mu + p)\theta + \pi_{ij}\sigma^{ij} = 0 \ . \tag{2.19i}$$

The dynamic quantities are given by (1.47). Also, (2.19i) is identically satisfied if

(2.19a-f) are satisfied.

The approach to finding solutions for the Einstein-Liouville equations in Bianchi I spacetime is analogous to the Robertson-Walker case. The 3 Killing vectors $\xi_\nu = \partial/\partial x^\nu$ of Bianchi I spacetime generate three constants of the motion $p_\nu = \xi_\nu \cdot \boldsymbol{p}$. Again, any function $f = h(p_1, p_2, p_3)$ then represents a spatially homogeneous solution to the Liouville equation for Bianchi I spacetimes.

For our discussion on non-equilibrium Bianchi I solutions, it is also useful to define the solution $f = K(P)$, where P is the constant of the motion given, as in Robertson-Walker spacetime, by

$$P^2 = (p_1)^2 + (p_2)^2 + (p_3)^2$$

$$= X^4(t)(p^1)^2 + Y^4(t)(p^2)^2 + Z^4(t)(p^3)^2 \tag{2.20}$$

(It is interesting to note that $K(P)$ is also a spatially homogeneous solution in Bianchi IX spacetimes [Maartens and Maharaj (1990)]). However, unlike the Robertson-Walker case where P is dynamically isotropic $(R = X = Y = Z)$, P is clearly anisotropic (f is thus also anisotropic).

In order to investigate the restrictions placed on the dynamic quantities by the Einstein field equations (2.19), we use the general expressions (1.47) for the dynamic quantities and obtain (from (2.19b), (1.47e))

$$\int_m^\infty E\lambda^2 F_a dE = 0 \tag{2.22a}$$

and from (2.19g), (1.47e)

$$\int_m^\infty \lambda^3 F_{ab} dE = 0 \quad \text{for} \quad a \neq b \ . \tag{2.22b}$$

X, Y, and Z are in principle determined, once F and F_{aa} are specified, by the remaining field equations. Of course we are unable to give explicit solutions analogous to (2.12) in the Robertson-Walker case.

2.5 Exact Equilibrium Solutions in Static Spherically Symmetric Spacetime

The discussion of this section is based on Maartens and Maharaj (1985), Maharaj and Maartens (1986), and Misner et al. (1973, p679). The metric for static spacetimes with spherical symmetry is given in standard coordinates $x^i = (t, r, \theta, \phi) = (t, x^\alpha)$ by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\psi(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.23}$$

The static spherically symmetric (SSS) geometry defines a preferred 4-velocity

$$u = e^{-\nu/2}\partial_t \, . \tag{2.24}$$

Then an invariant orthonormal tetrad is

$$\omega^a = \{e^{\nu/2}dt, e^{\psi/2}dr, rd\theta, r\sin\theta d\phi\} \, , \tag{2.25}$$

giving the Einstein tensor components

$$G^{00} = r^{-2} - e^{-\psi}(r^{-2} - r^{-1}\psi') \tag{2.26a}$$

$$G^{11} = -r^{-2} + e^{-\psi}(r^{-2} + r^{-1}\nu') \tag{2.26b}$$

$$G^{22} = G^{33} = e^{-\psi}[2\nu'' + (\nu' - \psi')(\nu' + 2r^{-1})]/4 \tag{2.26c}$$

$$G^{ab} = 0 \quad \text{for} \quad a \neq b \, . \tag{2.26d}$$

The Killing vectors of the metric (2.23) are given by [Maharaj and Maartens (1986)]

$$\begin{aligned}
\xi_1 &= \cos\phi \, \partial_\theta - \sin\phi \, \cot\theta \, \partial_\phi \\
\xi_2 &= \sin\phi \, \partial_\theta + \cos\phi \, \cot\theta \, \partial_\phi \\
\xi_3 &= \partial_\phi \\
\xi_4 &= \partial_t \, .
\end{aligned} \tag{2.27}$$

The static spherically symmetric solution of Liouville's equation in SSS spacetime is then given as $f(x^i, p^a) = K(Y_4, Y_1^2 + Y_2^2 + Y_3^2)$, with $Y_a = \xi_a \cdot \boldsymbol{p}$, and can be written

$$f = K[e^{\nu} p^t, r^4((p^\theta)^2 + \sin^2\theta (p^\phi)^2)]$$

$$= K[e^{\nu/2} p^0, r^2((p^2)^2 + (p^3)^2)] \tag{2.28}$$

using (2.23), (2.25), and (2.27) [Maharaj and Maartens (1986), Misner et al. (1973, p680)].

By equations (1.20) and (1.21), with $d\mathcal{P} = dp^{123}/p^0$, we obtain

$$n^a = \int p^a \frac{1}{p^0} K[e^{\nu/2} p^0, r^2((p^2)^2 + (p^3)^2)] dp^{123} \tag{2.29}$$

$$T^{ab} = \int p^a p^b \frac{1}{p^0} K[e^{\nu/2} p^0, r^2((p^2)^2 + (p^3)^2)] dp^{123} \ . \tag{2.30}$$

Considering the integrand of (2.29) for $a = 1, 2, 3$ it is clear that it is odd in p^1, p^2, and p^3; which implies that $n^1 = n^2 = n^3 = 0$. As a result the kinematic average 4-velocity is $u = e^{-\nu/2}\partial_t$, the geometrically defined 4-velocity. It is the collision-free property that forces the 4-velocity to be orthogonal to the surfaces $t =$ constant.

Similarly, since K in (2.30) is even in p^1, p^2, and p^3 and symmetric in p^2 and p^3, the following holds

$$T^{ab} = T^{ab}(r) = \mathrm{diag}(T^{00}, T^{11}, T^{22}, T^{33}) \ , \quad \text{with} \quad T^{22} = T^{33} \ . \tag{2.31}$$

The form of (2.31) is consistent with the Einstein field equations (2.26) without additional restrictions on the functional form of K. In fact, any function $K[e^{\nu/2} p^0, r^2((p^2)^2 + (p^3)^2)]$ gives a solution of the Einstein-Liouville system of equations, i.e., if K is specified then T^{ab} and G^{ab} are consistently determined by (2.26) and (2.30) in terms of ν, ψ, and so in principle we get a solution ν, ψ for any choice of K. (This is analogous to specifying an equation of state in a fluid model).

It is possible to obtain expressions for the dynamic quantities as defined by (1.12), (1.21). The symmetry of (2.31) implies that the heat flow $q^i = 0$, without restricting f.

By defining new variables

$$\widetilde{E} = e^{\nu/2}p^0 , \qquad J^2 = r^2((p^2)^2 + (p^3)^2) , \qquad e^{i\Phi} = rJ^{-1}(p^2 + ip^3) , \tag{2.32}$$

new coordinates can be introduced on the mass shell

$$p^1 = [e^{-\nu}\widetilde{E}^2 - r^{-2}J^2 - m^2]^{1/2}, \quad p^2 = r^{-1}J \cos \Phi, \quad p^3 = r^{-1}J \sin \Phi,$$

with $\widetilde{E}$ representing the energy at infinity and J the angular momentum (see also [Misner et al. (1973, p681)]). It is now possible to write the dynamic quantities as

$$\mu(r) = 2\pi r^{-2}e^{-3\nu/2}\int\int \widetilde{E}J(e^{-\nu}\widetilde{E}^2 - r^{-2}J^2 - m^2)^{-1/2}K(\widetilde{E},J)d\widetilde{E}\,dJ \ , \tag{2.33}$$

$$p(r) = \tfrac{2}{3}\pi r^{-2}e^{-\nu/2}\int\int J(e^{-\nu}\widetilde{E}^2 - m^2)(e^{-\nu}\widetilde{E}^2 - r^{-2}J^2 - m^2)^{-1/2}K(\widetilde{E},J)d\widetilde{E}\,dJ, \tag{2.34}$$

$$\pi^{ij} = (c^ic^j - \tfrac{1}{3}h^{ij})\pi r^{-2}e^{-\nu/2}\int\int \frac{(2e^{-\nu}\widetilde{E}^2 - 3r^{-2}J^2 - 2m^2)}{(e^{-\nu}\widetilde{E}^2 - r^{-2}J^2 - m^2)^{1/2}}\,JK(\widetilde{E},J)d\widetilde{E}\,dJ \ , \tag{2.35}$$

with $c = e^{-\psi/2}\partial_r$. Note that this model has non-zero (but static and spherically symmetric) anisotropic stress. Perfect fluid results (i.e. zero anisotropic stress) can be obtained by taking f as dynamically isotropic: $f = F(\widetilde{E})$ [Maharaj and Maartens (1986)]. Note that we could rewrite the above expressions in terms of the harmonics of f:

$$f = K[e^{\nu/2}E, J] = F(r,E) + F_a(r,E)e^a + \ldots$$

In principle one can determine the form of the harmonic coefficients by harmonically expanding the angular momentum J. One would then find that

$$j_a = 0 = q_a \ , \qquad \pi_{ab} = P(r)(c_ac_b - \tfrac{1}{3}h_{ab})$$

follow without restrictions on F_a, F_{ab}. Essentially this happens because K is invariant under the full group of motions (unlike the Robertson-Walker distribution discussed in Section 2.3)

2.6 Summary

In this chapter we presented the equilibrium solutions that will be used in the remainder of the thesis as part of the relaxation collision term (1.44). The relaxation collision term consists of a distribution function f, typically taken to be dynamically anisotropic, relaxing towards an equilibrium distribution function $\bar{f}$, with characteristic relaxation time τ. The relaxation collision functional is investigated in general in Chapter 3, while its application to the Robertson-Walker, Bianchi I, and SSS spacetimes is presented in the subsequent chapters.

We have concentrated here on specific dynamically anisotropic solutions. Further details on the solutions for Robertson-Walker spacetimes can be found in Maartens and Maharaj (1987a, 1987b,), and Maharaj and Maartens (1987b). An exact inhomogeneous Robertson-Walker solution is presented in Maharaj and Maartens (1987a), while a detailed discussion of Liouville solutions in Bianchi spacetimes is given in Maartens and Maharaj (1990).

3.0 THE RELATIVISTIC BGK COLLISION TERM

3.1 Introduction

Since $C[f]$ models the self-interaction of the gas, it should in general be nonlinear in f. The linear BGK model (1.44) is certainly a drastic simplification. It is a macroscopic relaxation-time approximation to the detailed, microscopic collision mechanism. However, the BGK approximation, though rough, gives sufficiently accurate order of magnitude results to deal with many physical problems [Marle (1969), Anderson and Witting (1974), Anderson and Payne (1976), Dominguez-Tenreiro and Hakim (1977), Majorana (1990), Nightingale (1973)]. Approximations based on the BGK model give results close to those based on the Boltzmann model (1.40) [Anderson and Witting (1974), Anderson and Payne (1976), Majorana (1990), Nightingale (1973)]. It is therefore worthwhile to investigate exact solutions and their properties in the BGK model - an aim which is impossible for the microscopic Boltzmann model. These exact results could then provide indications of the behaviour of the Boltzmann model, and hopefully throw light on some of the unresolved questions. Furthermore, the BGK model arises from the transport collision model (1.45) when scattering is elastic and isotropic - for example, isotropic Thomson scattering [Peebles and Yu (1970)] and free-free Bremstrahlung [Schweizer (1988)].

In (1.44) the distribution $\bar{f}$ is often taken to be dynamic equilibrium: $\mathcal{L}(\bar{f}) = 0$. If this is a collision-dominated equilibrium the spacetime will have to conform to the tight restrictions of (1.37), (1.39). If it is more generally an isotropic equilibrium, then by (1.49) the shear must vanish. Alternatively, Stewart (1968) has $\bar{f}$ isotropic but non-equilibrium (necessary since $\sigma_{ab} \neq 0$ in Bianchi I spacetime). Consequently, Stewart modifies the Boltzmann equation so as to preserve a simple exact solution:

$$\mathcal{L}(f - \bar{f}) = \gamma(f - \bar{f}) \ .$$

This amounts to taking $\bar{f} = 2f$ in (1.44) (let $f \to f - \bar{f}$) so it is in fact a particular BGK model. Anderson and Witting (1974) take $\bar{f}$ to be a kinematic equilibrium (1.36) which is not dynamic ($\mathcal{L}(\bar{f}) \neq 0$) so that (1.37), and (1.39) do not apply. They use (1.44) to find approximate solutions f of (1.8) close to local equilibrium $\bar{f}$. We will not impose these restrictions on $\bar{f}$.

The results of this chapter are based on Maartens and Wolvaardt (1994a).

3.2 Formal Solution of BGK Equation

The Boltzmann equation (1.8), with BGK collision term (1.44), is a linear first order differential equation given by

$$\frac{df}{dv} + \gamma f = \gamma \bar{f} \ .$$

The exact formal solution of this equation is (writing $f(v)$ for $f(x(v), p(v))$):

$$\begin{aligned} f(v) &= he^{-\Gamma(v)} + e^{-\Gamma(v)} \int^{v} \bar{f}(u) \frac{de^{\Gamma(u)}}{du} du \ , \\ &= he^{-\Gamma(v)} + e^{-\Gamma(v)} \left[\bar{f}(v) e^{\Gamma(v)} - \int^{v} e^{\Gamma(u)} \frac{d\bar{f}(u)}{du} du \right] \\ &= \bar{f}(v) + he^{-\Gamma(v)} + k(v), \end{aligned} \tag{3.1}$$

where

$$\mathbf{\pounds}(h) = 0, \quad \Gamma(v) = \int^{v} \gamma(u) du, \quad k(v) = -e^{-\Gamma(v)} \int^{v} e^{\Gamma(u)} \mathbf{\pounds}(\bar{f}(u)) du.$$

If $\bar{f}$ is dynamic equilibrium, then we get the simple solution

$$f(v) = \bar{f} + he^{-\Gamma(v)} \tag{3.2a}$$

In (3.1) and (3.2a), h and the arbitrary constants implicit in the integrals are determined by the initial conditions. For example with (3.2a), if f is initially $f_{in}(=f(0))$ and relaxes towards $\bar{f}$, then

$$h = f_{in} - \bar{f}, \quad \Gamma(v) = \int_{0}^{v} \gamma(w) dw \tag{3.2b}$$

and $\Gamma(v)$ represents essentially *the number of collisions that have occurred.* If f relaxes *away* from $\bar{f}$ towards f_{out} $(= f(\infty))$ then

$$h = f_{out} - \bar{f}, \quad \Gamma(v) = \int_v^\infty \gamma(w)dw \tag{3.2c}$$

and $\Gamma(v)$ represents essentially *the number of collisions still to occur.*

3.3 BGK Relaxation Function

The relaxation function γ contains all the information about the collisions. Marle (1969) initiated the relativistic version of the BGK model, and took $\gamma = \gamma(x)$. As pointed out by Anderson and Witting (1974), this leads to an energy-dependent relaxation time - and furthermore, gives functionally different transport coefficients from the Grad results in the relativistic limit. They propose

$$\gamma(x,p) = (-u_a p^a)/\tau(x) \equiv E/\tau(x) \tag{3.3}$$

where u^a is a mean 4-velocity and τ is the mean collision time or relaxation time. This can be seen clearly in the flat space case with τ constant. Since $E = p^t = dt/dv$, (3.2a) and (3.3) give $\Gamma = t/\tau$ so that τ is the relaxation time. Anderson and Witting take u^a to be defined by $\bar{f}$ which is a kinematic equilibrium. We will not make these restrictions on u^a or $\bar{f}$. The AW form (3.3) of the BGK model (1.44) gives transport coefficients that functionally agree with the Grad results in the relativistic regime [Anderson and Witting (1974)]. We will usually use the AW form when it is necessary to specify γ.

The harmonic decomposition (1.46b) of the BGK collision term (1.44) with isotropic $\gamma(x,E)$ (including (3.3)) gives

$$b_{a_1\ldots a_r} = \gamma(\bar{F}_{a_1\ldots a_r} - F_{a_1\ldots a_r}) \;. \tag{3.4}$$

If $\bar{f}$ is isotropic then $\bar{F}_{a_1\ldots a_r} = 0, \; r \geq 1$. Otherwise we can rewrite (3.4) as

$$b_{a_1\ldots a_r} = -\gamma_r F_{a_1\ldots a_r}$$

with $\gamma_r = \gamma(1 - \bar{F}_{a_1\ldots a_r}/F_{a_1\ldots a_r})$. This may be interpreted as the different harmonics of f having different relaxation times. Ellis et al. (1983b) refer to this as the generalised BGK model, but it can always be put into a BGK form with anisotropic $\bar{f}$.

In any case, (3.4) reflects the simplicity of the BGK model through the absence of 'harmonic mixing' - the collisional harmonics depend only on the distribution harmonics of the same order at each level.

3.4 Conservation of Particles and Energy-Momentum

The microscopic Boltzmann collision model (1.40) guarantees that the conservation equations (1.32), (1.33) and H-theorem (1.41) are identically satisfied. This is *not* the case with the macroscopic BGK model. Additional conditions on the relaxation function γ and distribution f arise from imposing (1.32), (1.33) and (1.41). In certain cases, these conditions serve to fix arbitrary parameters. For example, Anderson and Witting (1974) take $\bar{f}$ to be of the form (1.35) and the conservation equations fix the arbitrary quantities α, β_a. In the general case, the conservation equations (1.32), (1.33) for the BGK model (1.44) give

$$\int \gamma(f-\bar{f})d\mathcal{P} = 0 = \int p^a\gamma(f-\bar{f})d\mathcal{P}\ . \tag{3.5a}$$

The restrictions imposed by (3.5a) depend heavily on the relaxation mechanism γ. Roughly, (3.5a) limits the arbitrariness in the local rest frame and temperature and density of the non-equilibrium distribution. It is reasonable to assume that γ is isotropic. For a general isotropic relaxation function $\gamma(x,E)$, (1.46a) in (3.5a), using (1.18) and (1.19), gives the harmonic forms of particle number and energy-momentum conservation:

$$\int_m^\infty \gamma(F-\bar{F})\lambda dE = 0 \tag{3.5b}$$

$$\int_m^\infty \gamma(F-\bar{F})\lambda E dE = 0 \tag{3.5c}$$

$$\int_m^\infty \gamma(F_a-\bar{F}_a)\lambda^2 dE = 0 \tag{3.5d}$$

(where we have used $u_aT^{ab}{}_{;b}=0$ and $h_{ab}T^{bc}{}_{;c}=0$ to obtain the conditions for energy and momentum conservation respectively).

If we further take γ to be linear, we can encompass the Marle and AW models:

$$\gamma(x,E) = E/\tau(x) + \nu(x)\ . \tag{3.6a}$$

Using (1.18)-(1.21), (1.46), ($1.47a-g$), ($1.47h$) we find that ($3.6a$) and (3.5) give *matching conditions* that link f and $\bar{f}$:

$$\tau^{-1}(N-\bar{N}) + \nu(M-\bar{M}) = 0 \tag{3.6b}$$

$$\tau^{-1}(\mu-\bar{\mu}) + \nu(N-\bar{N}) = 0 \tag{3.6c}$$

$$\tau^{-1}(q_a-\bar{q}_a) + \nu(j_a-\bar{j}_a) = 0. \tag{3.6d}$$

For the Marle model ($\tau^{-1}=0$), taking $\bar{j}_a=0$, we see that momentum conservation determines the local rest frame to be the Eckart frame ($1.26a$). On the other hand, in the AW model ($\nu=0$) with $\bar{q}_a=0$ (when, for example, $\bar{f}$ is isotropic), the Landau-Lifshitz frame ($1.26b$) is picked out. In both cases the number densities of f and $\bar{f}$ are matched: $N=\bar{N}$. Also note that for the AW case, ($1.47h$) shows that $\mu=\bar{\mu}$ implies $p=\bar{p}$ when $m=0$ (i.e. zero bulk viscosity). The general linear model ($3.6a$) allows the interesting case (with $\bar{j}_a=0=\bar{q}_a$)

$$q_a = -\tau\nu j_a$$

which defines a new local rest frame, neither Eckart nor Landau-Lifshitz, in which the energy and particle flux directions coincide. In summary:

The conservation of particles gives ($3.5b$) *for a general isotropic relaxation function and* $N=\bar{N}$ *for the AW relaxation function. The conservation of energy gives* ($3.5c$) *for a general isotropic relaxation function and* $\mu=\bar{\mu}$ *for the AW relaxation function. The conservation of momentum gives* ($3.5d$) *for a general isotropic relaxation function and* $q_a=\bar{q}_a$ *for the AW relaxation function.*

3.5 Entropy

By (1.31) and (1.41), the H-theorem for the BGK model implies

$$S^a{}_{;a} = \int \gamma(f-\bar{f})\log f\ d\mathcal{P}\ \geq\ 0 \tag{3.7a}$$

which in general is a further restriction on γ and f. In the AW case (3.3), (3.7a) implies

$$
\begin{aligned}
S^a{}_{;a} &= -\tau^{-1}u_a\Big[\int p^a f(\log f-1)d\mathcal{P} - \int p^a \bar f(\log f-1)d\mathcal{P}\Big] \\
&= \tau^{-1}u_a\Big[S^a + \int p^a\,\bar f\big[\log[\,\bar f(1+\epsilon)]-1\big]d\mathcal{P}\Big] \\
&= \tau^{-1}u_a\Big[S^a - \bar S^a + \int p^a\,\bar f\log(1+\epsilon)d\mathcal{P}\Big] \\
&= \tau^{-1}u_a(S^a-\bar S^a+\eta^a)\geq 0 \qquad (3.7b)
\end{aligned}
$$

where

$$\eta^a = \int p^a\,\bar f\,\log(1+\epsilon)d\mathcal{P},\quad \epsilon = \frac{(he^{-\Gamma}+k)}{\bar f}$$

and we used (1.11), (1.20), (1.22), (1.24), (3.1) and (3.5b). (Note that ϵ need not be small, unlike in (1.42). If ϵ is small, then (3.7b) is consistent with (1.43a,b).) The H-theorem therefore imposes restrictions on $\bar f$, f in general. Unlike the restrictions (3.5b), (3.6) from the conservation equations, it is not possible to say in general what these restrictions are. What we have achieved in (3.7b) is to separate the term containing only f (i.e. S^a) from the term containing only $\bar f$ and τ (i.e. $\bar S^a+\eta^a$).

The harmonic form (1.48c) for the AW BGK case gives

$$S^a{}_{;a} = 4\pi\tau^{-1}\int_m^\infty E\lambda\Big\{(F-\bar F)\log F + (1/3F^2)\big[\tfrac{1}{2}(F+\bar F)F^a - F\bar F^a\big]F_a + \ldots\Big\}dE \geq 0 \quad (3.7c)$$

using (3.3), (3.4). From (3.7c) we can deduce the conditions for the H-theorem to lowest anisotropic order. If $\bar f$ is isotropic ($\bar F_a = 0$), then (3.7c) shows that *to lowest order, anisotropy in f tends to increase the entropy production rate. Conversely, in the AW BGK model, isotropisation of f is accompanied by a decrease in the entropy production rate,* as might be expected in general.

We can give exact entropy results for a particular AW BGK model [Maartens and Wolvaardt (1994a)]. If f has only first order ('dipole') anisotropy and is relaxing towards the isotropic $\bar f$, then

$$f = F + F_a e^a, \quad \bar{f} = \bar{F} = F \,. \tag{3.8}$$

The conditions for (3.8) to satisfy the Boltzmann equation are dealt with below. Now with (3.8) we can evaluate the series in the entropy density (1.48*a*) and entropy production rate (3.7*c*). The entropy density becomes:

$$\begin{aligned} \mathrm{s} &= 4\pi \int_m^\infty F(1 - \log F)\lambda E dE - 4\pi \int_m^\infty F\left(\frac{1}{6}\frac{F_a F^a}{F^2} + \frac{1}{60}\frac{(F_a F^a)^2}{F^4} + \ldots\right)\lambda E dE \\ &= \bar{\mathrm{s}} - 4\pi \int_m^\infty \lambda E F \xi(F_a F^a / F^2) dE \end{aligned} \tag{3.7d}$$

while the entropy production rate becomes:

$$\begin{aligned} S^a{}_{;a} &= 4\pi\tau^{-1} \int_m^\infty \Big((F - \bar{F})\log F + \frac{1}{6}\frac{F_a F^a}{F^2}(F + \bar{F}) + \\ &\qquad \frac{1}{60}\frac{(F_a F^a)^2}{F^4}(F + 3\bar{F}) + \frac{1}{210}\frac{(F_a F^a)^3}{F^6}(F + 5\bar{F}) + \ldots\Big)\lambda E dE \\ &= \tau^{-1}(\bar{\mathrm{s}} - \mathrm{s}) + 4\pi\tau^{-1} \int_m^\infty \lambda E F \zeta(F_a F^a / F^2) dE \end{aligned} \tag{3.7e}$$

where the functions ξ, ζ are defined by the power series

$$\xi(x) = \sum_1^\infty \frac{x^n}{(2n-1)2n(2n+1)} \;, \quad \zeta(x) = \sum_1^\infty \frac{x^n}{2n(2n+1)} = 2x\xi'(x) - \xi(x) \,. \tag{3.7f}$$

Then from $(3.7d - f)$ it follows that for a first order ('dipole') anisotropic distribution (3.8) in the AW BGK model, the relation (1.48*d*) for entropy density holds exactly and *the entropy production rate satisfies the H-theorem identically.* Furthermore, *the isotropisation of the distribution strictly increases the entropy density and decreases the the entropy production rate.* These results hold for any F, F_a (noting that $F \geq 0$, $F_a F^a \geq 0$ always). The forms of F, F_a will be determined by the Boltzmann equation.

It is striking that the H-theorem is satisfied (and that isotropisation increases entropy density) without restrictions. However the distribution (3.8) is very special: by $(1.47f, d)$ it has no anisotropic stress ($\pi_{ab} = 0$) or bulk viscous pressure ($\Pi = p - \bar{p} = 0$). Without these conditions we are unable to find the closed forms $(3.7d, e)$ from the series in (1.48*a*), (3.7*c*). We cannot extrapolate from these special conditions to find explicitly the conditions for a general distribution to obey the H-theorem.

3.6 Exact Truncated BGK Solutions

In the approximate solutions of the linearised Boltzmann equation (1.43c) (with the Boltzmann term (1.40)), the deviation from the local equilibrium distribution is taken as quadratic in the 4-momenta [De Groot et al. (1980)]. Using the harmonic decomposition (1.46), this corresponds to the truncated distribution

$$f = F + F_a e^a + F_{ab} e^a e^b \tag{3.9}$$

(of which (3.8) is a special case). In fact, (3.9) is *more general* than the usual approximations, in which the energy dependence of the coefficients is restricted. By investigating exact truncated Boltzmann solutions in the BGK case, we can hope to get an idea of some of the (currently unknown) consistency conditions that operate for truncated solutions of the full Boltzmann collision model [Ellis et al. (1983c)].

The first condition is given by (1.51), which shows that the shear of the preferred 4-velocity vanishes: $\sigma_{ab} = 0$. In other words, *truncated Boltzmann solutions in the BGK model are only possible in shear-free flows* (and this holds true even if a finite number of higher order terms are added in (3.9), by (1.51)). This is a severe consistency constraint, and although it does not carry over exactly to the full Boltzmann collision model, it indicates that stringent consistency conditions are also hidden in the standard approximate solutions (see [Ellis et al.(1983c)]).

Suppose that (3.9) is relaxing towards the isotropic distribution $\bar{f} = \bar{F}$, with AW BGK collision term (3.3), (3.4). Now it follows from (1.17a) that the Liouville operator raises the order of anisotropy by 2. Thus $\mathbf{\pounds}(f)$ has 4 harmonics and is, by (1.17a), (3.9)

$$\begin{aligned}
\mathbf{\pounds}(f) = &\left[\triangle(F) - 2k^a F_a\right] + \left[\triangle_a(F) + \triangle(F_a) - l^b{}_a F_b - 2k^b F_{ab}\right] e^a + \\
&\left[-\lambda^2 \sigma_{ab} \partial F/\partial E + \triangle_b(F_b) + \triangle(F_{ab}) - 2l^c{}_a F_{cb}\right] e^a e^b + \\
&\left[-\lambda^3 \sigma_{bc} \partial(\lambda^{-1} F_a)/\,\partial E + \triangle_c(F_{ab}) + 2k_c F_{ab} - 2\lambda \epsilon^d{}_{ce} n^e{}_a F_{bd}\right] e^a e^b e^c + \\
&\left[-\lambda^4 \sigma_{cd} \partial(\lambda^{-2} F_{ab})/\partial E\right] e^a e^b e^c e^d
\end{aligned} \tag{3.10}$$

where

$$k_a = E^2\dot{u}_a + \lambda^2 a_a\ , \qquad\qquad l_{ab} = E(\sigma_{ab} + \omega_{ab} + \epsilon_{abc}\Omega^c),$$

$$\triangle\ = E_{\perp}\partial_0 - \tfrac{1}{3}\lambda^2\theta\partial/\partial E,\ \ \triangle_a = \lambda(_{\perp}\partial_a - E\dot{u}_a\partial/\partial E),$$

and $\perp(\ldots)$ denotes projection into the rest space after differentiation. Now the coefficients in (3.10) are by construction symmetric and spatially projected. However, the last three are not trace free. The trace of the e^ae^b term feeds into the zero-order term, that of the 3rd order feeds into the first-order term, and that of the 4th order feeds into the second- and zero-order terms. The result is the harmonic decomposition of the Boltzmann equation. The complete set of equations representing the harmonic decomposition of the Boltzmann equation is given by Ellis et al. (1983c, equation (21)). Using this result or by implementing (3.10) directly, the 4th harmonic gives:

$$\lambda^{-2}\sigma_{<ab}F_{cd>} = \alpha_{abcd}(x)$$

Now $\alpha_{abcd} = 0$, otherwise F_{ab} will not die away when $E \to \infty$. Thus $\sigma_{<ab}F_{cd>} = 0$, and since $\sigma_{ab} = \sigma_{<ab>}$, $F_{ab} = F_{<ab>} \neq 0$, this implies $\sigma_{ab} = 0$ (if $F_{ab} = 0$, then the 3rd harmonic gives $\sigma_{ab} = 0$). This is a particular case of the general result of [Ellis et al. (1983c)]. Now with $\sigma_{ab} = 0$, the harmonics of the Boltzmann equation (1.8) with AW BGK collision term (1.44), (3.3) give, by (3.9), (3.10), (1.46), (3.4) and (1.17b,c):

$$D(F) - \tfrac{1}{3}E\lambda^{-1}\dot{u}^a(\lambda^2F_a)' + \tfrac{1}{3}\lambda h^{ab}F_{a;b} = \tau^{-1}E(\bar{F} - F) \tag{3.11a}$$

$$D(F_a) - 2E\omega^b{}_aF_b - E\lambda\dot{u}_aF' + \lambda h^b{}_aF_{;b} - \tfrac{2}{5}E\lambda^{-2}\dot{u}^b(\lambda^3F_{ba})' + \tfrac{2}{5}\lambda h_a{}^bh^{cd}F_{bc;d} = -\tau^{-1}EF_a \tag{3.11b}$$

$$D(F_{ab}) - 2E\omega^c{}_{(a}F_{b)c} + \lambda F_{<a;b>} - E\lambda^2(\lambda^{-1}\dot{u}_{<a}F_{b>})' = -\tau^{-1}EF_{ab} \tag{3.11c}$$

$$F_{<ab;c>} - E\lambda^2(\lambda^{-2}\dot{u}_{<a}F_{bc>})' = 0 \tag{3.11d}$$

where a prime denotes $\partial/\partial E$ and

$$D = E_{\perp}(u^a\nabla_a) - \tfrac{1}{3}\lambda^2\theta\partial/\partial E$$

The equations (3.11) are in agreement with the general results of Ellis et al. (1983c, p492-4, p501-2). (Note the third term on the right of their equation (24a) (p494) is identically zero.)

The exact truncated solution for the AW BGK collision term therefore satisfies $\sigma_{ab} = 0$ and equations (3.11) - in addition to the matching conditions imposed by the conservation equations. These are given by (3.6b) with $\nu = 0$ and $\bar{q}_a = 0$:

$$N = \bar{N}, \quad \mu = \bar{\mu}, \quad q_a = 0 \quad (\Rightarrow u^a = u^a_L) \ . \tag{3.12a}$$

The Eckart frame energy flux $\tilde{q}_a$ is given in terms of the number flux j_a by (1.27b). By (1.47), the conditions (3.12a) may be expressed as integral constraints on F, F_a:

$$\int_m^\infty E(F - \bar{F})\lambda dE = 0 = \int_m^\infty E^2(F - \bar{F})\lambda dE \tag{3.12b}$$

$$\int_m^\infty EF_a\lambda^2 dE = 0 \ . \tag{3.12c}$$

The energy conservation result $\mu = \bar{\mu}$ shows by (3.12b), (1.47d) that for $m = 0$, we have $p = \bar{p}$. Thus in the AW BGK model we have the (exact) result:

$$m = 0 \ \Rightarrow \ \Pi = 0 \ . \tag{3.13}$$

Before tackling equations (3.11) in general, we look at some particular solutions. If $F_a = 0$, then (3.11a) integrates (assuming $\mathbf{\pounds}(\bar{F}) = 0$ which requires $\sigma_{ab} = 0$ and u_a parallel to a conformal Killing vector [Ehlers et al. (1968)]):

$$F(t, x^\rho, E) = \bar{F} + H(x^\rho, R\lambda)e^{-\Gamma}, \quad \Gamma = \int dt/\tau \ , \quad \mathbf{\pounds}(H) = 0$$

where H is arbitrary, t is proper time along the mean flow $(u^i = \delta^i{}_t)$ and $R(t)$ is the average expansion scale factor: $\theta = 3\dot{R}/R$ [Ellis (1971)]. (Note that this also holds if $\dot{u}_a = 0 = h^{ab}F_{a;b}$.)

In FRW spacetime (with orthogonal u_a) $\dot{u}_a = \omega_{ab} = \sigma_{ab} = 0$. Taking the spatial sections to be flat $(k = 0)$, we have the preferred triad $\boldsymbol{E}_\alpha = R(t)\partial/\partial x^\alpha$, so that $\gamma^\alpha{}_{\beta\gamma} = 0$ $(\Rightarrow a_\alpha = 0 = n_{\alpha\beta})$ and $\Omega_\alpha = 0$ (see (1.17b,c)). Thus

$$h^{ab}F_{a;b} = h^{ab}\partial_b F_a, \qquad h_a{}^b h^{cd}F_{bc;d} = h_a{}^b h^{cd}\partial_d F_{bc},$$

$$F_{<a;b>} = \partial_{<a}F_{b>}, \qquad F_{<ab;c>} = \partial_{<a}F_{bc>},$$

with the metric given by (2.3). If the harmonics are spatially homogeneous, i.e. $f = f(t, E, e^a)$, then all these quantities vanish, together with $h_a{}^b F_{;b}$, so that (3.11d) is identically satisfied, and (3.11$a-c$) decouple, each being of the form (assuming $\bar{F}$ is dynamic equilibrium, $\mathcal{L}(\bar{F}) = 0$):

$$\mathcal{L}(K) = E\partial K/\partial t - \lambda^2(\dot{R}/R)\partial K/\partial E = -\tau^{-1}EK .$$

These equations integrate to give

$$F = \bar{F}(R\lambda) + H(R\lambda)e^{-\Gamma}, \;\; F_a = H_a(R\lambda)e^{-\Gamma}, \;\; F_{ab} = H_{ab}(R\lambda)e^{-\Gamma} \qquad (3.14a)$$

where $\Gamma(t) = \int_0^t dt'/\tau(t')$, and H, H_a, H_{ab} are arbitrary solutions of $\mathcal{L}(K) = 0$.

Thus there are *no integrability conditions for the exact truncated solution in $k = 0$ FRW spacetime.* (In fact, (3.14a) holds also for $k = 1$, but not for $k = -1$, since in the latter case $a_b \neq 0$. (Compare [Ellis et al. (1983a, c)]). If $H_{ab} = 0$ and $F = \bar{F}$ then the entropy density and production rate are given by (3.7$d-f$).

By (1.47), the dynamics of the exact truncated FRW solution (3.14a) are given by

$$\Pi = \tfrac{4\pi}{3}R^{-4}e^{-\Gamma}\int_0^\infty (P^2 + m^2R^2)^{-1/2}P^4H(P)dP \qquad (3.14b)$$

$$j_a = \tfrac{4\pi}{3}R^{-3}e^{-\Gamma}\int_0^\infty (P^2 + m^2R^2)^{-1/2}P^3H_a(P)dP \qquad (3.14c)$$

$$\pi_{ab} = \tfrac{8\pi}{15}R^{-4}e^{-\Gamma}\int_0^\infty (P^2 + m^2R^2)^{-1/2}P^4H_{ab}(P)dP . \qquad (3.14d)$$

The Boltzmann solution (3.14) is further subject to the conservation conditions (3.12). Energy conservation ($\mu = \bar{\mu}$) implies by (3.12b), (3.14a) that

$$\int_0^\infty (P^2 + m^2R^2)^{1/2}P^2H(P)dP = 0 \qquad (3.14e)$$

which for $m > 0$ (and $\dot{R} \neq 0$) forces $H = 0$ ($\Leftrightarrow F = \bar{F}$) and for $m = 0$ implies $\Pi = 0$ by (3.14b) (thus confirming the general result (3.13)). Hence $\Pi = 0$ for all m. Number conservation ($N = \bar{N}$) follows from $H = 0$ for $m > 0$ while for $m = 0$ it gives, by (3.12b)

and (3.14a)

$$\int_0^\infty P^2 H(P)dP = 0 \quad (m=0). \tag{3.14f}$$

Momentum conservation ($q_a = 0$) implies by (3.12c), (3.14a):

$$\int_0^\infty P^3 H_a(P)dP = 0\ . \tag{3.14g}$$

Note that H_{ab} is unrestricted. Einstein's field equations will place conditions on H_{ab} and further conditions on H (for $m = 0$) and H_a (see Section 3.7 for a general discussion). In summary:

For a test gas relaxing towards isotropic equilibrium on an FRW ($k = 0$) background, the exact truncated solution of the Boltzmann equation with AW BGK collision term, subject to conservation of particle number and energy-momentum, is given by (3.14a, f, g) (*with* $H = 0$ *for* $m > 0$). *The bulk viscosity vanishes* ($\Pi = 0$) *for all* $m \geq 0$. *The anisotropic pressure* π_{ab} *is non-zero even though* $\sigma_{ab} = 0$. *Both* π_{ab} *and the heat flow diverge at the big bang and die away with the expansion of the universe.*

The result that $\Pi = 0$, although emerging from a relaxation-time model, has interesting implications for FRW solutions with bulk viscosity, in both the non-causal and causal thermodynamics [Pavon, Bavaluy and Jou (1991)].

We turn now to (3.11) in the general case (without assuming $\mathcal{L}(\bar{F}) = 0$). Equations (3.11) are equations for the harmonics of the truncated distribution. By integrating out the energy over the mass shell at each spacetime event, we can derive equations in the moments of the distribution, some of which will take the form of thermodynamics laws. Multiplying (3.11) by appropriate factors before integrating leads to the following relevant set of equations, on using (1.47) and (3.12) [Maartens and Wolvaardt (1994a)]:

$$\Pi + \xi\theta + \tfrac{1}{3}m^2\tau\dot{M} = m^2 B \tag{3.15a}$$

$$h_{ab}\dot{j}^b + (\tau^{-1} + \theta)j_a + 2\omega_{ab}j^b + \tfrac{1}{3}h_a{}^b n_{;b} + n\dot{u}_a + \pi^{(2)}_{ab;c}h^{bc} = m^2 C_a \tag{3.15b}$$

$$h_a{}^c h_b{}^d \dot{\pi}_{cd} + (\tau^{-1} + \tfrac{4}{3}\theta)\pi_{ab} + 2\omega_{(a}{}^c\pi_{b)c} + \tfrac{1}{3}m^2\theta\pi^{(1)}_{ab} = m^2 D_{ab} \tag{3.15c}$$

$$\pi_{<ab;c>} + 6\dot{u}_{<a}\pi_{bc>} = 5m^2\dot{u}_{<a}\dot{\pi}^{(1)}_{bc>} \tag{3.15d}$$

where

$$\xi = \tfrac{1}{9}m^2\tau[2M + 15m^2A^{(2,1)}] \tag{3.15e}$$

$$B = 5h^{ab}A^{(2,2)}_{a;b} \tag{3.15f}$$

$$C_a = 5h_a{}^bA^{(1,1)}_{;b} - \tfrac{5}{3}\theta A^{(2,2)}_a + 2h_a{}^bh^{cd}A^{(1,1)}_{bc;d} \tag{3.15g}$$

$$D_{ab} = A^{(1,2)}_{<a;b>} + \tfrac{1}{3}m^2\theta A^{(2,1)}_{ab} \tag{3.15h}$$

and we have defined the moments $(r, s = 1, 2)$

$$A^{(r,s)}_{a\ldots b} = \tfrac{4\pi}{15}\int_m^\infty E^{-r}\lambda^s F_{a\ldots b}dE \tag{3.15i}$$

and the π_{ab}-like non-equilibrium quantities

$$\pi^{(r)}_{ab} = \tfrac{8\pi}{15}\int_m^\infty E^{r-1}\lambda F_{ab}dE\ . \tag{3.15j}$$

Comparing with the Israel-Stewart thermodynamic laws (1.43), we see that $(3.15a-c)$ have a similar structure. In particular, they also involve *a coupling of non-equilibrium quantities to vorticity and acceleration, and transient time-derivative terms*, both of which are absent in the non-causal theories. The equation for Eckart-frame energy flux follows from (3.15*b*) on using the (exact) relation (1.27*b*). For *near-equilibrium* conditions, we can use (1.27*a*) to get the approximate equation

$$h_{ab}\dot{q}^b + [\tau^{-1} + \tfrac{4}{3}\theta + \tfrac{1}{3}m^2(\mu+p)^{-1}\dot{M}]q_a + 2\omega_{ab}q^b - \tfrac{1}{3}hh_a{}^bn_{;b} - nh\dot{u}_a - \pi^{(2)}_{ab;c}h^{bc} = -m^2hC_a \tag{3.15b'}$$

where $h = (\mu+p)/n$. Our exact coefficient of bulk viscosity (3.15*e*) vanishes when $m = 0$, like the approximate coefficient [Israel and Stewart (1979)]. Close to equilibrium, (3.15*e*) could be evaluated using $F \approx \exp(\alpha - \beta E)$.

Comparing (3.15 b') with (1.43*e*) and (3.15*c*) with (1.43*f*), we get equations for our coefficients of conductivity and 'shear ' viscosity (noting that $\sigma_{ab} = 0$):

$$\chi = \beta_2^{-1}[\tau^{-1} + \tfrac{4}{3}\theta + \tfrac{1}{3}m^2(\mu+p)^{-1}\dot{M}]^{-1}$$

$$\eta = \beta_3^{-1}(\tau^{-1} + \tfrac{4}{3}\theta)^{-1}$$

where β_2, β_3 are the Israel-Stewart transient coefficients.

The non-equilibrium pressure excess Π does not vanish for $\theta = 0$ (unlike the non-causal $\Pi = -\xi\theta$ law). But it does vanish for $m = 0$ (see (3.13)), an exact result that is in line with the approximate results. Furthermore, $m = 0$ reduces the right sides of $(3.15b - d)$ to zero, so that the thermodynamic laws for a massless gas are considerably simplified:

$$\Pi = 0 \tag{3.16a}$$

$$h_a{}^b \dot{q}_b + \nu q_a + 2\omega_{ab} q^b - \tfrac{1}{3} h h_a{}^b n_{;b} - n h \dot{u}_a - h \pi^{(2)}_{ab;c} h^{bc} = 0 \tag{3.16b}$$

$$h_a{}^c h_b{}^d \dot{\pi}_{cd} + \nu \pi_{ab} + 2\omega_{(a}{}^c \pi_{b)c} = 0 \tag{3.16c}$$

$$\pi_{<ab;c>} + 6\dot{u}_{<a}\pi_{bc>} = 0 \tag{3.16d}$$

where $\nu = \tau^{-1} + \frac{4}{3}\theta$. Equation $(3.16b)$ only holds near equilibrium, the exact relation following from $(3.15b)$ with $(1.27b)$ and $m = 0$.

The Israel-Stewart approximate laws (1.43) carry implicit but unknown consistency conditions following from truncation of the distribution. In our exact model, the consistency conditions are explicit: $\sigma_{ab} = 0$ and equation $(3.15d)$ governing π_{ab}. The existence of anisotropic stress π_{ab} in the absence of shear, although impossible in the non-causal theories, is also admitted in the Israel-Stewart theory. In summary:

The exact truncated solution of the Boltzmann equation with Anderson-Witting BGK collision term, satisfying conservation of particle number and energy-momentum, obeys the thermodynamic laws $(3.15a - c)$, *and is subject to the consistency conditions* $\sigma_{ab} = 0$ *and equation* $(3.15d)$ *for* π_{ab}.

3.7 Einstein-BGK Model

The considerations of the previous section apply to a gas in a given spacetime with metric g_{ij}. When g_{ij} is the mean field generated collectively by the gas itself (as discussed in Chapter 1), we have the self-gravitating case. Further dynamical restrictions, in addition to the kinematic restrictions discussed in Section 3.4 and 3.5, arise from the Einstein equations. In the Einstein-Boltzmann system (1.8), (1.50) with BGK model (1.44), the integrability conditions (1.33) do not follow identically from the field equations (1.50). As pointed out in Section 3.4, this is a consequence of the macroscopic relaxation approximation imposed upon the microscopic collision mechanism (which, in the Boltzmann model (1.40), does lead identically to (1.33)). Energy-momentum conservation imposes the condition (3.5a) as an additional, unavoidable component of the model. Therefore the Einstein-BGK model is described by the system

$$\mathcal{L}(f) = \gamma(\bar{f} - f) \tag{3.17a}$$

$$G_{ab} = \int p_a p_b f d\mathcal{P} \tag{3.17b}$$

$$\int p^a \gamma(f - \bar{f}) d\mathcal{P} = 0 \tag{3.17c}$$

Note that (3.17c) may be replaced by the equivalent equations following from the decomposition (1.12) of T_{ab} [Ellis (1971)]:

$$\dot{\mu} + (\mu + p)\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a \dot{u}_a = 0 \tag{3.17c'}$$

$$(\mu + p)\dot{u}_a + h_a{}^b(p_{;b} + \pi_b{}^c{}_{;c} + \dot{q}_b) + (\omega_{ab} + \sigma_{ab} + \tfrac{4}{3}\theta h_{ab})q^b = 0 \tag{3.17c''}$$

The point is that these do not follow identically from (3.17a, b). If (3.17c) is satisfied, then (3.17c', c'') do follow identically as a consequence of (3.17b).

Now from a physical point of view, the system (3.17) should be supplemented by the further, independent kinematical restrictions of particle number conservation (3.5a) and the H-theorem (3.7a):

$$\int \gamma(f - \bar{f}) d\mathcal{P} = 0 \tag{3.17d}$$

$$S^a{}_{;a} = \int \gamma(f - \bar{f}) \log f \; d\mathcal{P} \geq 0 \tag{3.17e}$$

(Recall that both of these hold identically in the microscopic Boltzmann collision model.)

The full Einstein-BGK system is completed by a prescription of the relaxation model $(\gamma, \bar{f})$. Then the method of solution is clear in principle. We start with an appropriate symmetric geometry so that the form of g_{ij} is known in terms of as yet undetermined metric functions. Using this form, we solve the Boltzmann equation (3.17a) for f. The solution involves g_{ij} implicitly, and is taken into Einstein's equations (3.17b) to solve for g_{ij}, with restrictions on f. These restrictions arise in general from the need to match the structure of the kinetic energy-momentum tensor to the geometrically determined structure of the Einstein tensor. The degree of arbitrariness involved in f reflects the fact that it is not f directly, but only an integrated average of f, that enters the field equations. The Einstein-Boltzmann solution f is then subject to the further restrictions imposed by the energy-momentum conservation (3.17c) and the kinematical conditions (3.17d, e). These restrictions may make the Einstein-Boltzmann solution impossible. It would be possible to argue for suspending the kinematical conditions (3.17d, e), but not the integrability condition (3.17c).

The only exact Einstein-BGK solution known to us, namely Stewart's (1968), was not subjected to the integrability conditions (3.17c), or the kinematical restrictions (3.17d, e). This solution needs to be completed by applying these conditions. The only other exact non-equilibrium Einstein-Boltzmann solution known to us, that of Matravers and Ellis (1989), does include the conservation equations (but not the H-theorem). However this solution remains in a sense formal since the collision term is not specified and consequently the restrictions from energy-momentum and number conservation are not explicit. In the exact solutions which we present in subsequent chapters, we aim to give explicit solutions to the full Einstein-BGK system (3.17$a-e$).

In Section 3.2 we have discussed the solutions of (3.17a) (see (3.1), (3.2)). In tackling Einstein's equations (3.17b), it is often useful to decompose them using a preferred 4-velocity (for example, the one occurring in the AW relaxation model (3.3)). By (3.17b) and (1.12) we get the decomposed Einstein field equations (2.2).

The right hand sides of (2.2) are given by phase space integrals of f, using (1.10), (1.12) and (1.21):

$$\mu = \int E^2 f d\mathfrak{P} \tag{3.18a}$$

$$p = \tfrac{1}{3}\int \lambda^2 f d\mathfrak{P} \tag{3.18b}$$

$$q_a = \int E\lambda f e_a d\mathcal{P} \tag{3.18c}$$

$$\pi_{ab} = \int \lambda^2 f(e_a e_b - \tfrac{1}{3}h_{ab}) d\mathcal{P} \tag{3.18d}$$

The harmonic forms $(1.47c - f)$ follow from (3.18), and these are usually the only practicable way of approaching the solution of the field equations.

Finally, the kinematical conditions $(3.17d, e)$ have already been dealt with in Section 3.4 (see (3.5), (3.7)). This completes our general discussion of the Einstein-BGK model. In the subsequent chapters of this thesis, we will derive specific exact solutions and investigate their properties, using the formalism and insights from the general discussion of this chapter. In Chapter 5, we look at isotropic non-equilibrium solutions in FRW spacetime, which could provide a kinetic theory model for the viscous fluid early universe models (see Pavon, Bafaluy, and Jou (1991) and references quoted therein). In Chapters 4 and 6 we find anisotropic non-equilibrium solutions in FRW and Bianchi I spacetimes, and use these to generalise the anisotropy-production model of Matravers and Ellis (1989). (A special case of the anisotropic Einstein-BGK solution of Chapter 4 is given in Maartens and Wolvaardt (1994a) using the truncated solution (3.9)). In Chapter 7 we consider non-equilibrium solutions in inhomogeneous static spherically symmetric spacetime, which apply in the study of non-equilibrium stellar clusters.

3.8 Summary

Closely following the results of Maartens and Wolvaardt (1994a), the foundations for this thesis were presented in Chapter 3. It gives a comprehensive exact analysis of the BGK relaxation-time collision term (complementing the work of Anderson and Witting (1974) which is in approximations). We found the matching conditions imposed by the conservation equations for linear relaxation, including an interesting case which defines a new local rest frame, neither Eckart nor Landau-Lifshitz. The entropy was investigated and conditions were given for the H-theorem to be satisfied. These show that to lowest order (if $\bar{f}$ is isotropic), anisotropy in f tends to increase the entropy production rate and that in the AW BGK model, isotropisation of f is accompanied by a decrease in the entropy production rate, as might be expected in general. In addition, closed-form expressions for a distribution with only 'dipole' anisotropy were derived. These show that the entropy production rate satisfies the H-theorem identically and that the isotropisation of the distribution strictly increases the entropy density and decreases the the entropy production rate. Finally, the main results of Chapter 3 were the conditions on an exact truncated distribution that obeys the Boltzmann conservation equations in the Anderson-Witting BGK model. The conditions were integrated for $k = 0$ FRW spacetime, and the solution was shown to have zero bulk viscosity for $m > 0$ and $m = 0$ (and no integrability conditions are imposed). In the general case, the conditions were used to derive a new set of exact thermodynamic laws (equations (3.15)). These laws have similarities with the Israel-Stewart extended thermodynamic laws. In particular, they contain time derivatives of the non-equilibrium quantities, unlike the Eckart/Landau-Lifshitz laws (which are consequently acausal). Unlike the Israel-Stewart laws, ours do not require near-equilibrium conditions. Furthermore, the consistency conditions following from truncation of the distribution are given explicitly. They are $\sigma = 0$ (shear-free flow) and a condition on π_{ab}. Finally, the further conditions imposed when Einstein's field equations are added to the Boltzmann and conservation equations, were derived.

4.0 EXACT NON-EQUILIBRIUM EINSTEIN-BGK SOLUTIONS IN ROBERTSON-WALKER SPACETIME

4.1 Introduction

In this chapter we consider Einstein-BGK solutions in Robertson-Walker ($k = 0$) spacetime. We seek to extend the work of Ellis et al. (1983a) and Maartens and Maharaj (1987) on equilibrium solutions in Robertson-Walker spacetime to the BGK case. The solution we obtain is spatially homogeneous and dynamically anisotropic, and relaxes towards isotropic equilibrium. The relaxation mechanism is taken to be an arbitrary, spatially homogeneous, isotropic function. We investigate the restrictions placed on the solution by the conditions for the conservation of particles, of energy-momentum, and zero particle drift. In order to obtain an Einstein solution, we also determine the conditions placed on the distribution function by the field equations. We find that a solution that satisfies these conditions must have vanishing dipole and quadropole anisotropy, vanishing bulk viscosity and an equilibrium energy-momentum tensor. We also show that an isotropic, non-equilibrium solution can exist only for a massless gas with a restricted relaxation function. Restricting the relaxation mechanism, also allows an example of a massless gas which produces non-zero bulk viscosity. We show that for the FRW metric, the AW form of the BGK collision functional allows the existence of tilted solutions. The chapter is concluded with a discussion of possible applications of the solution.

4.2 BGK solution in Robertson-Walker Spacetime

In Robertson-Walker spacetime with standard coordinates $x^i = (t, x^1, x^2, x^3)$, the metric is (2.3)

$$ds^2 = -dt^2 + R^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \ ,$$

with a preferred 4-velocity (2.4): $u^i = \delta^i{}_0$, and an orthonormal tetrad basis given by (2.5).

We are seeking a solution representative of the physical situation where a spatially homogeneous, dynamically anisotropic distribution function f is relaxing toward an equilibrium solution $\bar{f}$ with increasing time. The equilibrium solution may be taken as dynamically isotropic. While we present the BGK solution for a general anisotropic $\bar{f}$, the properties of the solution are investigated using an isotropic $\bar{f}$ for the sake of clarity. The possible applications of this solution are numerous; one example being the relaxation, due to the decrease of collisions via expansion, of the distribution function describing the distribution of particles in the early universe. The formal solution of the Boltzmann equation with BGK collision functional is given by (3.2a) for the case where $\bar{f}$ is a dynamic equilibrium solution. For the Robertson-Walker case we use the Killing vector first integrals $p_\nu = \xi_\nu{}^a p_a$ to define dynamic equilibrium functions

$$h \equiv h(p_\nu) \quad \text{and} \quad \bar{f} \equiv \bar{f}(p_\nu). \tag{4.1}$$

From (2.7) we know that h and $\bar{f}$ defined in this way are dynamically anisotropic spatially homogeneous solutions of the Liouville equation (i.e. satisfying the conditions of (3.2a) and (3.2b)). If the particular application of the solution requires the relaxation toward an isotropic (dynamic) equilibrium distribution, $\bar{f}$ may be defined as

$$\bar{f} \equiv \bar{f}(P)\ , \tag{4.2}$$

where P is isotropic and is defined by (2.8).

We next investigate the term $\Gamma(v)$ in the solution (3.2a). We assume that the collision rate γ is isotropic and spatially homogeneous, but not restricted in any other way. Thus, we are not imposing the AW model (3.3) or the linear model (3.6a) of γ. For this model Γ is taken to represent the number of collisions that have occurred in the relaxation toward the equilibrium $\bar{f}$ (3.2b). Furthermore, because P is a constant of the motion ($\mathcal{L}(P) = 0$), we can write Γ as follows

$$\begin{aligned}\Gamma(t,P) &= \int_0^v \gamma(t,P)dv' \\ &= \int_0^t \frac{\gamma(t',P)}{p^0(t',P)}\,dt' \\ &= \int_0^t \gamma(t',P)[m^2 + R^{-2}(t')P^2]^{-1/2}dt',\end{aligned} \tag{4.3a}$$

where we used (1.1), (2.3), and (2.8). Notice that Γ is isotropic and spatially homogeneous. We are finally in a position to write the solution of the BGK equation in Robertson-Walker spacetimes. By (4.1), (4.3) and (3.2a):

$$f(t,p_\nu) = \bar{f}(p_\nu) + h(p_\nu)\exp\Big[-\int_0^t \gamma(t',P)[m^2 + R^{-2}(t')P^2]^{-1/2}dt'\Big]. \tag{4.4}$$

Clearly f is spatially homogeneous but dynamically anisotropic. Note that f must depend explicitly on the cosmic time if it is non-equilibrium.

Using (4.3a), (4.4) the following useful relation can be obtained:

$$\frac{\partial e^{-\Gamma}}{\partial t} = -e^{-\Gamma(t,P)}\gamma(t,P)[m^2 + R^{-2}(t)P^2]^{-1/2}$$

giving

$$-\gamma^{-1}R^{-1}\frac{\partial e^{-\Gamma}}{\partial t} = [P^2 + m^2R^2(t)]^{-1/2}e^{-\Gamma(t,P)}\,. \tag{4.3b}$$

The approach to finding the solution presented here, is to specialise the general BGK solution of (3.1); an alternative approach would be to first specialise the Boltzmann equation (1.8) with BGK collision term (1.44) for Robertson-Walker spacetimes, and then to solve it. The discussion around (3.14) illustrates a similar approach for the exact truncated case. In that case the harmonic decomposition of the BGK equation is obtained for the truncated case, specialised to the FRW case, and then the solution is obtained.

For the remainder of this chapter it is assumed that $\bar{f}$ is dynamically isotropic ($\bar{f} = \bar{F}(P)$), i.e. we are investigating the case where an anisotropic distribution function is relaxing towards an isotropic distribution function.

In order to find exact solutions to the Einstein-BGK problem and to investigate the kinematics and the dynamics of the solutions it is necessary to decompose $f(t,p_\nu)$ using the covariant harmonic decomposition. For clarity both sides of equation (4.4) are decomposed and then the terms of equal order are equated. The decomposition of $f(t,p_\nu)$ yields

$$f(t,p_\nu) = F(t,P) + F_a(t,P)e^a + F_{ab}(t,P)e^ae^b + \ldots\,. \tag{4.5}$$

To decompose the right hand side of (4.4) it is necessary to decompose $\bar{f}$ and $h(p_\nu)$.

This decomposition yields

$$\bar{f}(P) + h(p_\nu)e^{-\Gamma(t,P)} = \bar{F}(P) + e^{-\Gamma(t,P)}[H(P) + H_a(P)e^a + H_{ab}(P)e^a e^b + \dots] \tag{4.6}$$

Equating (4.5) and (4.6) we obtain

$$F(t,P) = \bar{F}(P) + H(P)e^{-\Gamma(t,P)} \ ,$$

$$F_{a\dots b}(t,P) = H_{a\dots b}(P)e^{-\Gamma(t,P)}. \tag{4.7}$$

It is exactly this decomposition of the solution that allows us in the following section to investigate the kinematics and the dynamics of the particle distribution.

4.3 Kinematics, Dynamics and Entropy

The kinematics of the distribution function are determined by the particle 4-current vector as given by the first moment of the distribution function (1.20). With respect to a given 4-velocity this vector may be decomposed as given by equation (1.23)

$$n^a = Nu^a + j^a \ , \quad j^a u_a = 0 \ ,$$

where N and j^a are determined by the general integrals (1.47a) and (1.47b). These integrals may be rewritten by changing the variable of integration and by using the BGK solution in Robertson-Walker spacetimes (4.7). Thus we obtain

$$N = \frac{4\pi}{R^3(t)} \int_0^\infty P^2[\bar{F}(P) + H(P)e^{-\Gamma(t,P)}]dP \tag{4.8a}$$

$$j_\nu = \frac{4\pi}{3R^3(t)} \int_0^\infty P^3[P^2 + m^2R^2(t)]^{-1/2} H_\nu(P)e^{-\Gamma(t,P)}dP \tag{4.8b}$$

where $j_0 = 0$ as $u^a = \delta^a{}_0$ for the Robertson-Walker case. A non-zero number flux gives a particle drift that is out of keeping with FRW symmetry, and although it is possible to satisfy the field equations for $j_a \neq 0$ (see the fluid solutions of Calvao and Salim (1992)), we regard this as unnatural. From equation (4.8b) it is clear that in order to get a zero number flux which gives a non-tilted kinematic average 4-velocity either the integral in (4.8b) must vanish or H_ν must vanish. For any particular t, non-zero harmonic coefficients H_ν may be found that give a zero integral in (4.8b); however as t changes the condition will no longer be satisfied. In order to have a vanishing number flux for all t, (4.8b) requires that H_ν vanishes:

$$j_a = 0, \quad \text{no restrictions on } \gamma(t, P) \Rightarrow H_a(P) = 0. \tag{4.9}$$

Thus, a zero H_ν will give a non-tilted 4-velocity. This condition is investigated further in Sections 4.5 where it is shown that the Einstein field equations require a zero energy flux vector which in turn requires that $H_\nu = 0$.

We next investigate the dynamics of the solution as determined by the quantities μ, p, q_a, and π_{ab} as given in the decomposition of the energy-momentum tensor (1.12), (1.21). These dynamic quantities are determined in general by the integrals of (1.47$c-g$) (including the bulk viscosity). These integrals can be specialised to the Robertson-Walker case by changing the variable of integration and by using the solution (4.7) as follows

$$\mu = \frac{4\pi}{R^4(t)} \int_0^\infty P^2[P^2 + m^2R^2(t)]^{1/2}[\bar{F}(P) + H(P)e^{-\Gamma(t,P)}]dP, \tag{4.10a}$$

$$p = \frac{4\pi}{3R^4(t)} \int_0^\infty P^4[P^2 + m^2R^2(t)]^{-1/2}[\bar{F}(P) + H(P)e^{-\Gamma(t,P)}]dP, \tag{4.10b}$$

$$q_\nu = \frac{4\pi}{3R^4(t)} \int_0^\infty P^3 H_\nu(P)e^{-\Gamma(t,P)}dP, \qquad q_0 = 0 \tag{4.10c}$$

$$\pi_{\mu\nu} = \frac{8\pi}{15R^4(t)} \int_0^\infty P^4[P^2 + m^2R^2(t)]^{-1/2}H_{\mu\nu}(P)e^{-\Gamma(t,P)}dP, \qquad \pi_{0a} = 0, \tag{4.10d}$$

$$\Pi = \frac{4\pi}{3R^4(t)} \int_0^\infty P^4[P^2 + m^2R^2(t)]^{-1/2}H(P)e^{-\Gamma(t,P)}dP. \tag{4.10e}$$

It is the Einstein field equations that determine these dynamic quantities via interaction with the distribution function. The discussion of the properties of the dynamics is therefore postponed to Section 4.5 where we present the Einstein solution. The discussion of the conservation of energy and momentum is also delayed to Section 4.4.

Next, we investigate the entropy quantities: the entropy density s, the entropy flux Σ^a, and the entropy production rate $S^a{}_{;a}$ (as given by the general expressions (1.22) and (1.24)). This investigation is necessary as we are dealing with a non-equilibrium solution in which the entropy production is expected to be positive.

The entropy density s is given by (1.48a). By changing the variable of integration and introducing the Robertson-Walker solution (4.7), we can write it as

$$s=\frac{4\pi}{R^3(t)}\int_0^\infty P^2\Big\{[\bar{F}(P)+H(P)e^{-\Gamma(t,P)}][1-\log(\bar{F}(P)+H(P)e^{-\Gamma(t,P)})]-$$

$$\tfrac{1}{6}[\bar{F}(P)+H(P)e^{-\Gamma(t,P)}]^{-1}e^{-2\Gamma(t,P)}H_aH^a+\ldots\Big\}dP, \tag{4.11a}$$

The entropy flux Σ_a is given by (1.48b). Again, by changing the variable of integration and using the solution (4.7), we can write

$$\Sigma_a=-\frac{4\pi}{3R^3(t)}\int_0^\infty P^3\left\{H_ae^{-\Gamma(t,P)}\log[\bar{F}(P)+H(P)e^{-\Gamma(t,P)}]+\ldots\right\}[P^2+m^2R^2]^{-1/2}dP \tag{4.11b}$$

Similarly, the entropy production rate (1.48c) can be written as

$$S^a{}_{;a}=\frac{4\pi}{R^2}\int_0^\infty\gamma(t,P)P^2[P^2+m^2R^2]^{-1/2}\Big\{H(P)e^{-\Gamma(t,P)}\log[\bar{F}(P)+H(P)e^{-\Gamma(t,P)}]+$$

$$\tfrac{1}{6}H_aH^ae^{-2\Gamma}[\bar{F}(P)+H(P)e^{-\Gamma(t,P)}]^{-2}[2\bar{F}(P)+H(P)e^{-\Gamma(t,P)}]+\ldots\Big\}dP \tag{4.11c}$$

It is not clear in general from (4.11c) whether the H-theorem (1.41) is satisfied for the BGK solution (4.7). If $H_a=0$ (see equation (4.9)), then to first anisotropic order (4.11c) gives

$$S^a{}_{;a}\approx\frac{4\pi}{R^2}\int_0^\infty\gamma(t,P)P^2[P^2+m^2R^2]^{-1/2}H(P)e^{-\Gamma(t,P)}\log[\bar{F}(P)+H(P)e^{-\Gamma(t,P)}]dP. \tag{4.11d}$$

In Section 5.6 we show that with the AW relaxation function, $\bar{F}$ Maxwell-Boltzmann and $m=0$, (4.11d) satisfies the H-theorem identically.

4.4 Conservation of Particles, Energy and Momentum

In establishing the conditions placed on the distribution function (4.7) by the conservation equations several different approaches are possible. One can start with equations (2.11a) and (2.11b) and insert the appropriate expressions for the dynamic quantities using equations (4.10), or one can start from the conservation equations (1.32), (1.33). We follow the alternative of directly implementing the results derived in Chapter 3 $(3.5b-d)$ for a general isotropic relaxation function. In Appendix A, we derive the energy-momentum conservation equations for FRW spacetime using (1.32), (1.33), as an alternative check of the results of Chapter 3.

The harmonic form for particle number conservation is given by $(3.5b)$ for a general isotropic relaxation function. On using the FRW BGK solution (4.7), we get

$$\int_0^\infty P^2[P^2+m^2R^2]^{-1/2}\gamma(t,P)e^{-\Gamma(t,P)}H(P)dP=0 \tag{4.12a}$$

which implies (by $(4.3b)$)

$$\frac{\partial}{\partial t}\int_0^\infty P^2H(P)e^{-\Gamma(t,P)}dP=0. \tag{4.12b}$$

For any particular t, non-zero harmonic coefficients H may be found that give a zero integral in $(4.12a)$; however as t changes the condition will no longer be satisfied. In order to have conservation of particles for all t, $(4.12a)$ requires that H vanishes:

$$n^a{}_{;a}=0 \quad \text{for arbitrary } \gamma(t,P) \Rightarrow H(P)=0. \tag{4.12c}$$

The harmonic form for the conservation of energy is given by $(3.5c)$ for a general isotropic relaxation function. On using the distribution (4.7) we find the condition

$$\int_0^\infty \gamma(t,P)e^{-\Gamma(t,P)}H(P)P^2dP=0 \tag{4.13a}$$

which implies

$$\int_0^\infty H(P)P^2[P^2+m^2R^2]^{1/2}\left[\frac{\partial}{\partial t}e^{-\Gamma(t,P)}\right]dP=0. \tag{4.13b}$$

Consistent with the condition for number conservation (4.12), the condition for the conservation of energy also requires that H vanishes:

$$u_aT^{ab}{}_{;b}=0 \quad \text{for arbitrary } \gamma(t,P) \Rightarrow H(P)=0. \tag{4.13c}$$

Momentum conservation, by $(3.5d)$, gives directly using (4.7)

$$\int_0^\infty \gamma(t,P)e^{-\Gamma(t,P)}\, H_a(P)P^3[P^2+m^2R^2]^{-1/2}dP = 0 \tag{4.14a}$$

which implies

$$\frac{\partial}{\partial t}\int_0^\infty e^{-\Gamma(t,P)}\, H_a(P)P^3 dP = 0. \tag{4.14b}$$

Condition (4.14a) implies, consistent with the condition for a vanishing number flux (4.9), that for a general isotropic relaxation function H_a must vanish:

$$h_{ab}T^{bc}{}_{;c} = 0 \quad \text{for arbitrary } \gamma(t,P) \Rightarrow H_a(P) = 0. \tag{4.14c}$$

In summary:

For an arbitrary isotropic relaxation function $\gamma(t,P)$, the conservation equations place the following restrictions on the FRW BGK solution (4.7)*: particle number and energy conservation requires that H vanishes, and momentum conservation (and the condition for vanishing particle drift) requires that H_a vanishes. It is not possible to obtain non-equilibrium, isotropic FRW BGK solutions for an arbitrary, isotropic relaxation function (as H must vanish).*

4.5 Einstein-BGK Solution in Robertson-Walker Spacetime

The Einstein field equations for the Robertson-Walker spacetime are given by equations (2.10). In order to ensure that the structure of the kinetic energy momentum tensor (1.21) matches that of the geometrically determined Einstein tensor we restrict the distribution f by imposing conditions on the dynamic quantities determining the kinetic energy-momentum tensor (4.10). This is done by comparing the quantities (4.10) with the Einstein field equations (2.10). Using (2.10a) and (4.10c) we find that

$$\int_0^\infty P^3 H_a(P)e^{-\Gamma(t,P)}dP = 0. \tag{4.15a}$$

A vanishing energy flux is thus given by a distribution function that satisfies (4.15a). Also, for an arbitrary isotropic relaxation function, (4.15a) requires $H_a = 0$.

Similarly, (2.10b) and (4.10d) give

$$\int_0^\infty P^4[P^2+m^2R^2(t)]^{-1/2}H_{ab}(P)e^{-\Gamma(t,P)}dP=0 \ . \tag{4.15b}$$

By using the relation (4.3*b*), we can write (4.15*b*) as

$$\int_0^\infty \gamma^{-1}(t,P)P^4\,\frac{\partial e^{-\Gamma}}{\partial t}H_{ab}(P)dP=0. \tag{4.15c}$$

Thus, to obtain a vanishing anisotropic stress, the distribution function must satisfy (4.15*b*). Also, for an arbitrary isotropic relaxation function, (4.15*b*) requires $H_{ab}=0$.

As we know that the field equation (2.10*c*) is a first integral of equation (2.10*d*) (using the conservation equations) only one field equation remains to be investigated. Using (4.10*a*) in equation (2.10*c*) we can write it as

$$\dot{R}(t)=\left\{\frac{4\pi}{3R^2(t)}\int_0^\infty P^2[P^2+m^2R^2(t)]^{1/2}\bar{F}(P)dP+\right.$$
$$\left.\frac{4\pi}{3R^2(t)}\int_0^\infty P^2[P^2+m^2R^2(t)]^{1/2}H(P)e^{-\Gamma(t,P)}dP\right\}^{1/2} \tag{4.16a}$$

Making the simplifying assumption that the isotropic part of the distribution that is undergoing the relaxation is zero, $H(P)=0$ (see also (4.12*c*), (4.13*c*)), we can rearrange this equation and find an expression for t:

$$t=\frac{\sqrt{3}}{2}\int_0^{R^2}\left[4\pi\int_0^\infty P^2[P^2+m^2u]^{1/2}\bar{F}(P)dP\right]^{-1/2}du \ . \tag{4.16b}$$

Note that (4.16*b*) takes the same form as for the dynamic equilibrium Einstein solution (2.12). The assumption $H(P)=0$ is a weak restriction and is also required by the conservation of energy and particle number for an arbitrary isotropic relaxation function. *We have thus shown that if the cosmic time is given by* (4.16*b*) *and the distribution function* (4.4) *satisfies equations* (4.15*a*) *and* (4.15*b*), *then all the field equations will be satisfied. For an arbitrary isotropic relaxation function* $\gamma(t,P)$, *a vanishing energy flux requires that* H_a *vanishes, while a vanishing anisotropic stress requires that* H_{ab} *vanishes.* By specifying $\bar{F}$, $R(t)$ can be determined in principle, and the metric (2.3) will be known - completing the Einstein solution. This is analogous to specifying an equation of state in a fluid model.

4.6 Properties of the Solution for Arbitrary Isotropic Relaxation Functions

In Chapter 3 we have shown that, for the BGK model, it is necessary to consider the field equations as well as the conservation equations. In this section we investigate the implications of the restrictions imposed by the field and the conservation equations. In order to get a clear picture, these conditions are repeated below. The conditions are also used in Section 4.7 to investigate special solutions by restricting the relaxation function.

Conservation equations:

$$\int_0^\infty P^2[P^2+m^2R^2]^{-1/2}\gamma(t,P)e^{-\Gamma(t,P)}H(P)dP=0 \tag{4.12a}$$

$$\int_0^\infty \gamma(t,P)e^{-\Gamma(t,P)}H(P)P^2dP=0 \tag{4.13a}$$

$$\int_0^\infty \gamma(t,P)e^{-\Gamma(t,P)}\,H_a(P)P^3[P^2+m^2R^2]^{-1/2}dP=0 \tag{4.14a}$$

Field equations:

$$\int_0^\infty P^3H_a(P)e^{-\Gamma(t,P)}dP=0 \tag{4.15a}$$

$$\int_0^\infty P^4[P^2+m^2R^2]^{-1/2}H_{ab}(P)e^{-\Gamma(t,P)}dP=0 \tag{4.15b}$$

$$t=\frac{\sqrt{3}}{2}\int_0^{R^2}\left[4\pi\int_0^\infty P^2[P^2+m^2u]^{1/2}\bar{F}(P)dP\right]^{-1/2}du \quad \text{with } H(P)=0. \tag{4.16b}$$

These integral equations can be satisfied without restricting the relaxation function γ only if $H=H_a=H_{ab}=0$. We start our investigation with (4.12a) and (4.13a) which require that $H(P)=0$ for an arbitrary isotropic relaxation model γ. The implication of this is that (4.16b) holds and that the isotropic part of the Robertson-Walker solution (4.7) is the equilibrium distribution $\bar{F}(P)$. This result has the further implication that the particle number density N (compare the results of Section 3.4), the total energy density μ, and the isotropic pressure p take the same form as the equilibrium dynamically isotropic Liouville solutions (2.9a, b, c) i.e.

$$N=\bar{N},\ \mu=\bar{\mu}\,,\ p=\bar{p}\Rightarrow \Pi=0. \tag{4.17}$$

Thus $H(P)=0$, which follows from number or energy conservation for the general *isotropic γ, leads to vanishing bulk viscosity* $(m \geq 0)$. Furthermore, by symmetry, the energy flux and shear viscosity relative to $u^a = \delta^a{}_0$ vanish. The solution has vanishing non-equilibrium pressures and fluxes $(\Pi = j_a = q_a = \pi_{ab} = 0)$, but is still non-equilibrium as by (3.4), $C[f] \neq 0$. *Thus the distribution has an equilibrium energy-momentum tensor if the solution is to hold for all γ.* Note that special forms of γ will change these conclusions. However, for $m > 0$, there is no γ that can avoid $H(P) = 0$, since this is the only solution to the number conservation and energy conservation equations (4.12a), (4.13a), which implies that *isotropic non-equilibrium solutions are not possible for $m > 0$. Thus, any spatially homogeneous Einstein-BGK solution in FRW with $m > 0$ and particle number conserved, that is relaxing towards isotropic equilibrium, has zero bulk viscosity:*

$$m > 0 \Rightarrow \Pi = 0. \tag{4.18}$$

This raises questions about the kinetic basis for bulk viscous fluid solutions in FRW (see Pavon, Bafaluy, and Jou (1991) and references quoted therein). For $m = 0$ the approximation schemes of kinetic theory give $\Pi = 0$ (see for example Israel and Stewart (1979)). However, our treatment shows that it is in principle possible to have exact non-zero bulk viscosity for a massless gas. We need to find a non-zero H and a relaxation model γ such that (4.12a), (4.13a) are satisfied and $\Pi \neq 0$. Now by (4.15c) we have that (4.12a), (4.13a) imply

$$\frac{\partial}{\partial t}\int_0^\infty P^n H(P) e^{-\Gamma(t,P)} dP = 0 \quad n = 2, 3 \tag{4.19}$$

while (4.10e) gives

$$\Pi = \frac{4\pi}{3R^4(t)}\int_0^\infty P^3 H(P) e^{-\Gamma(t,P)} dP. \tag{4.20}$$

Then (4.19), (4.20), (2.9b,c) and (2.11a) show that

$$m = 0 \Rightarrow \Pi = \Pi_0 \mu \tag{4.21}$$

where Π_0 is a constant. In order to have $\Pi_0 \neq 0$ and (4.19) satisfied we try $e^{-\Gamma} = A(P) + h(t)$ so that by (4.15c), $\gamma = -P\dot{h}R^{-1}(A+h)^{-1}$. Choosing $A(P)$, $h(t)$ appropriately, we can find a solution. For example, with $A(P) = ae^{-P}$ and $h(t) = bR^{-1}$ (a, b constant) we get the following

$$\Pi = \tfrac{4\pi a}{3\mu_0}\int_0^\infty P^3 H(P) e^{-P} dP \tag{4.22}$$

from (4.20), where $\mu = \mu_0 R^{-4}$, and from (4.19) we see that H must satisfy

$$\int_0^\infty P^n H(P) dP = 0 \quad n = 2, 3. \tag{4.23}$$

It is certainly possible to get non-zero solutions of (4.23) such that $\Pi_0 \neq 0$. The relaxation function is

$$\gamma = \frac{bP\dot{R}}{R^3(bR^{-1} + ae^{-P})} \tag{4.24}$$

which shows that $\gamma \to 0$ as $t \to \infty$ and $\gamma \to \infty$ as $P \to \infty$. Of course we do not claim that (4.24) is a fully physical reasonable model - but it does at least have acceptable asymptotic behaviour, and produces *non-zero bulk viscosity for a massless gas.*

We next investigate the conditions placed on the distribution function by the first and second field equations $(4.15a, b)$. The only solution valid for an arbitrary isotropic relaxation model $\gamma(t, P)$ is

$$H_\nu(P) = 0, \quad H_{\nu\mu}(P) = 0. \tag{4.25}$$

In Section 4.7 we investigate special forms of the relaxation function γ that allow non-zero H_ν, $H_{\nu\mu}$. The conditions (4.25) imply that the first and second order harmonic coefficients of $f(t, p_\nu)$ (4.7) are zero: $F_a(t, P) = F_{ab}(t, P) = 0$. By $(4.8b)$, $H_\nu = 0$ implies $j_\nu = 0$, i.e. the kinematic average 4-velocity is the preferred velocity. Furthermore, by $(4.14a)$, $H_\nu = 0$ ensures that the momentum conservation equation is identically satisfied.

In fact for the case where f is a Einstein-Boltzmann solution in Robertson-Walker spacetime, the condition $F_a = 0$ is a sufficient condition to guarantee a perfect fluid form for the kinetic energy-momentum tensor. This can easily be confirmed as follows. Using the average 4-velocity $\bar{u}^a$ as determined by the particle distribution function and the 4-velocity u^a as determined by the Robertson-Walker symmetry we can write the following expressions

$$T^{ab} = \mu u^a u^b + p h^{ab} = \mu \bar{u}^a \bar{u}^b + p \bar{h}^{ab} + \bar{\pi}^{ab} + 2\bar{q}^{(a}\, \bar{u}^{b)} \tag{4.26}$$

$$n^a = Nu^a + j^a = N\bar{u}^a. \tag{4.27}$$

We have already shown in (4.9) that the condition $H_\nu = F_a = 0$ implies a zero number flux vector and thus a non-tilted kinematic average 4-velocity, $u^a = \bar{u}^a$, by (4.27). Using this result in (4.26) it is clear that $\bar{\pi}_{ab} = \bar{q}_a = 0$, which leads to perfect fluid form of the energy momentum tensor. Furthermore, from (1.47*b*), (4.26), (4.27) it is clear that this conclusion is valid for any collision term $C[f]$.

To show that the energy momentum tensor takes the perfect fluid form in FRW spacetime, it is therefore sufficient to show that the first order harmonic coefficient of the distribution function is zero.

The conditions for the conservation of energy-momentum and the field equations place no restrictions on the collision rate γ. The solution is therefore not forced to isotropise nor are collisions forced to die away. However, a requirement that the solution generates entropy $S^a{}_{;a} > 0$, may pick out special behaviour of γ (eg. isotropisation).

We have therefore found an exact Einstein-BGK solution (4.7) in Robertson-Walker spacetime with $F_\nu = F_{\nu\mu} = 0$ and t related to $R(t)$ by (4.16*b*). This solution is dynamically anisotropic and the energy-momentum tensor takes the perfect fluid form. An example of such a solution is given by Treciokas and Ellis (1971). However, in our case the solution is non-equilibrium as $C[f] \neq 0$. In summary:

The results hold for a spatially homogeneous, dynamically anisotropic BGK solution with arbitrary spatially homogeneous, dynamically isotropic relaxation mechanism, relaxing towards isotropic equilibrium. We have shown that if the solution holds for arbitrary γ, then $\Pi = 0$, $T^{ab} = \bar{T}^{ab}$, $n^a = \bar{n}^a$ for all $m \geq 0$. Particular forms of γ may allow us to get $n^a \neq \bar{n}^a$, $\Pi \neq 0$. However, for $m > 0$, there is no γ for which $\Pi \neq 0$.

4.7 Special Cases of the Solution

In this section we consider special forms of the relaxation model $\gamma(t, P)$. First, we investigate whether γ may be chosen such that the kinematic average 4-velocity $\bar{u}^a$ is tilted, i.e. $j_a \neq 0$. In this case, the energy-momentum tensor takes an imperfect fluid form with respect to $\bar{u}^a$. Now $j_a \neq 0$ requires $H_\nu \neq 0$ by (4.8*b*). Furthermore, we must show that it is possible to have $H_\nu \neq 0$ and still satisfy (4.15a). Solutions of this kind are possible if

$$\Gamma(t, P) = C(t) \tag{4.28}$$

where C is arbitrary. This restriction allows one to take Γ outside the integral in (4.15a) and therefore many solutions $H_\nu \neq 0$ are possible. This restriction on Γ restricts γ as follows. By (4.3),

$$\Gamma(t, P) = \int_0^t \gamma(t', P)[m^2 + R^{-2}(t')P^2]^{-1/2} dt',$$

the condition (4.28) implies

$$\gamma^{-1}(t, P)[m^2 + R^{-2}(t)P^2]^{1/2} = \tau(t)$$

where τ is arbitrary. Re-arranging this expression we find

$$\gamma(t, P) = E/\tau(t),$$

$$\Gamma(t) = \int_0^t dt'/\tau(t'). \tag{4.29}$$

It is remarkable that the form of γ required is exactly the Anderson-Witting form (3.3). It is for this reason that we have kept the relaxation model general so far in this chapter. The interesting conclusion is that *for the Robertson-Walker metric, the AW form of the BGK collision functional allows the existence of tilted solutions with imperfect fluid energy-momentum tensors* (with respect to the kinematic average 4-velocity). Of course it is also possible to have solutions with $j_a = 0$ with the AW BGK model. By (4.8b) we simply take $H_\nu = 0$ for $m > 0$ and

$$\int_0^\infty P^2 H_\nu(P) dP = 0 \text{ for } m = 0.$$

In fact we can prove the following general results about spatially homogeneous AW BGK solutions of the Einstein-Boltzmann equations in FRW. By (4.26), (4.8b), (4.12a), (4.13a), (4.14a), (4.15$a-b$):

for $m > 0$,

$$H = 0\ (\Rightarrow \Pi = 0),$$

$$j_a = 0 \Rightarrow H_a = 0,$$

$$H_{ab} = 0, \tag{4.30}$$

for $m = 0$,

$$\int P^n H dP = 0,\ n = 2,3\ \ (\Rightarrow \Pi = 0)$$

$$\int P^3 H_a dP = 0,$$

$$j_a = 0 \Rightarrow \int P^2 H_a dP = 0,$$

$$\int P^3 H_{ab} dP = 0, \tag{4.31}$$

(i.e., $\Pi = 0$ for $m \geq 0$).

Thus for non-tilted solutions ($j_a = 0$), we see that the $m > 0$ distribution is equilibrium up to the second order anisotropy. This means that *for truncated quadratic distributions* (as in Chapter 3)

$$f = F + F_a e^a + F_{ab} e^a e^b$$

there is no non-equilibrium AW BGK solution of the Einstein-Boltzmann equations when $m > 0$. *The AW BGK solution produces zero bulk viscosity for* $m \geq 0$ *and also permits an isotropic, non-equilibrium distribution for* $m = 0$ (taking $H \neq 0$, but $H_{a\ldots b} = 0$).

By restricting the collision rate it is possible to find solutions in which relaxation leads to isotropisation. Isotropisation is not in general required by the conservation or the field equations (such a requirement may result from the H-theorem). By (4.7), isotropisation requires $e^{-\Gamma} \rightarrow 0$, i.e., $\Gamma \rightarrow \infty$ as $t \rightarrow \infty$. For applications to an expanding universe, it also seems natural to have a collision rate that decreases with increasing expansion (i.e., $\tau \rightarrow \infty$ with expansion). The mean time between collisions could be associated with the natural time rate defined by the expansion, $\dot{R}/R$ (compare Hawking

(1966)), by for example taking

$$\tau = \tau_0 R/\dot{R} \ , \ \ \Gamma = \log\left[R/R_0\right]^{1/\tau_0} \tag{4.32}$$

with $R_0 = R(t_0)$, $R(t_0 = 0) = 1$. By inserting τ into (4.29) and taking the limit as t increases from t_0 to ∞, we find that isotropisation indeed occurs. Thus, if the relaxation time is related to the expansion as in (4.32), the expansion is accompanied by relaxation and isotropisation of the distribution function, the collisions disappear, and the number of collisions that have occurred increases.

For the AW collision model, an example where the distribution function does not isotropise with increasing t is

$$\tau = \tau_0 e^t \ . \tag{4.33}$$

In order for the distribution function to remain anisotropic, (4.7) requires that $e^{-\Gamma(t)}$ approaches some finite, non-zero value (i.e. $\Gamma \to$ some finite, non-zero value). In this instance, (4.29), $\Gamma \to 1/\tau_0$ as $t \to \infty$ and it applies to both $m = 0$ and $m \neq 0$. This example also leads to the disappearance of the collisions as $\tau \to \infty$ with $t \to \infty$.

Thus, by appropriately restricting the collision rate γ, it is possible to obtain tilted Einstein-BGK solutions, which could be considered as tilted analogues of the tilted fluid solutions of Coley and Tupper (1984a,b). (See Coley and O'Neill (1991) for further discussion.) Further restrictions on $\tau(t)$ determine whether the solution will isotropise or whether the collision rate will decrease with increasing t. The specific case will depend on the specific cosmological situation to which the solution is applied.

4.8 Application of the Non-Equilibrium Robertson-Walker Solution

Most modern cosmological models are based on the Robertson-Walker geometry (see for example Weinberg (1977)). The spacetime may be sliced into hypersurfaces of constant time which are homogeneous and isotropic, and the mean rest frame of the galaxies is assumed to agree with this definition [Shutz (1985)]. The particular significance of the Robertson-Walker model is that it can be used to describe an expanding universe.

In the 'Standard Model' for the evolution of the universe, it expands from an early radiation dominated era to the current matter dominated era [Weinberg (1977)]. During the radiation dominated era the universe is filled with a soup of matter and radiation particles. Because of the high density and temperature the particles collide very rapidly, are in thermal equilibrium and behave as radiation. The matter dominated era starts after the universe has expanded and cooled to such an extent that stable atoms can be formed. At this point the universe becomes transparent to radiation and the matter and the radiation decouple. The matter dominated era as it currently exists is essentially collision-free, and a collision-free equilibrium exists. The equilibrium kinetic theory Robertson-Walker models described in Chapter 2 could be used to describe the universe during these two types of equilibrium situations. During the radiation dominated era $(m = 0)$ collision dominated equilibrium exists and the Maxwell-Boltzmann distribution (1.35) can be used, while the collision-free equilibrium existing during the matter dominated $(m > 0)$ era implies $C[f] \equiv 0$.

During the evolution from the radiation era to the matter dominated era a number of non-equilibrium processes occur. In the first part of the evolution the universe is dominated by electrons, positrons, neutrinos and photons. These particles are in thermal equilibrium and above their threshold energies. The particle energy due to the high temperature is so high compared to the particle rest mass that all particles behave as radiation. The first non-equilibrium process that occurs is the decoupling of the neutrinos and the anti-neutrinos when the temperature decreases to about 10^{10} $^{\circ}K$. At about 3×10^{9} $^{\circ}K$ the threshold temperature for electrons and positrons is reached, and they begin to annihilate each other rapidly. This is the second main non-equilibrium process that occurs. A short time after the universe has cooled to 10^{9} $^{\circ}K$, deuterium nuclei can hold together. This allows the formation of tritium and helium through a series of two particle reactions and represents a further non-equilibrium process.

The need to model these non-equilibrium processes occurring in the Standard Model of the universe suggests that potential exists for the use of non-equilibrium BGK-Robertson-Walker models as the mathematical background for a cosmological model of the universe during some of its phases of expansion. Bernstein (1988) gives a comprehensive analysis, but his treatment uses approximations, whereas we are pursuing exact models.

In conclusion, the model has potential application for non-equilibrium processes in cosmology. If the model is extended to include inhomogeneous distributions and mixtures of particles, further areas of application could be found.

4.9 Summary

In this chapter we presented a non-equilibrium Einstein-BGK solution (4.4) in FRW spacetime. The solution is spatially homogeneous and dynamically anisotropic, and relaxes towards isotropic equilibrium. The relaxation mechanism was taken to be an arbitrary, spatially homogenous, isotropic function. We showed that for such a distribution function the conservation of particles and energy requires that the isotropic component H of the relaxing distribution f vanishes, implying that non-equilibrium isotropic solutions are not possible (unless $m = 0$, and γ restricted). The conservation of momentum and the condition for vanishing particle drift in turn requires that the first order anisotropy H_a of the distribution f vanishes. In order to obtain an Einstein solution, we showed that if the cosmic time is given by (4.16b) and the distribution function (4.4) satisfies equations (4.15a) and (4.15b), then all the field equations will be satisfied. For an arbitrary isotropic relaxation function $\gamma(t, P)$, a vanishing energy flux requires that dipole anisotropy H_a vanishes, while a vanishing anisotropic stress requires that quadropole anisotropy H_{ab} vanishes. Furthermore, we showed that for arbitrary isotropic relaxation mechanisms, the condition $H = 0$ leads to vanishing bulk viscosity and an equilibrium energy-momentum tensor. By restricting the relaxation mechanism, we found an example for a massless gas which produces non-zero bulk viscosity. However, for a massive gas, it is not possible to satisfy both number conservation and energy conservation and have non-vanishing bulk viscosity. Furthermore, we showed that for the FRW metric the AW form of the BGK collision functional allows the existence of tilted solutions with imperfect fluid energy-momentum tensors (with respect to the 4-velocity determined by the particle distribution), and that no non-equilibrium solutions with quadratic truncation (and $m > 0$) exist. Some special forms of the relaxation function giving isotropisation of the distribution function were also identified. We also found that to show that the energy momentum tensor takes the perfect fluid form in FRW spacetime, it is sufficient to show that the distribution function has vanishing first order anisotropy.

5.0 EXACT ISOTROPIC EINSTEIN-BGK SOLUTIONS IN ROBERTSON-WALKER SPACETIME

5.1 Introduction

In this chapter we extend the work of Chapter 4 and obtain an isotropic but non-equilibrium solution in Robertson-Walker spacetime. The solution represents an isotropic distribution function relaxing toward an equilibrium isotropic distribution function. It is expected that this solution may find application to the non-equilibrium conditions in the early universe. Although the solution could be viewed as a special case of the anisotropic model of Chapter 4, its simple nature allows us to derive its properties explicitly, and a separate treatment is justified. The isotropic solution has the further advantage that it is exact. The truncated solutions (see for example Bernstein (1988)) are limited in the sense that only small deviations from equilibrium are allowed. In the exact case the deviation from equilibrium may be large. The solution lacks the precision of the approximate theory [Bernstein(1988)], but its exact nature allows more flexibility and generality.

The approach taken here generally follows that of Chapter 4. We present the solution in Robertson-Walker spacetime, investigate the kinematics, dynamics and entropy properties, and finally obtain the Einstein solution satisfying the conditions for the conservation of energy and momentum. We have shown in Chapter 4 that the particle number and energy conservation equations require that $H(P)$ (i.e. the zero order term in the covariant harmonic expansion of $h(p_\nu)$) must be zero if γ is arbitrary. *A non-equilibrium isotropic solution requires $H \neq 0$ and thus will only be possible for a particular class of γ.* A related result from Chapter 4 is that the conservation of particle number and energy requires that $H(P) = 0$ for $m > 0$ (for any γ). It implies that *non-equilibrium, isotropic, spatially homogeneous Einstein-BGK solutions (with $m > 0$) are not possible in FRW spacetime.* This extends the result of Maartens and Wolvaardt (1994a), on the AW relaxation function, to hold for a general isotropic relaxation function. In our investigation of isotropic, non-equilibrium solutions, we therefore restrict ourselves to the case $m = 0$ with a particular γ (AW form). One of the key things done in Chapter 5 and not in Chapter 4 is a detailed investigation of the entropy production. In principle this could be applied to finding a model to account for the high entropy per baryon (i.e. the massive photon to baryon ratio). The $m = 0$ isotropic, non-equilibrium f provides a possible model for: photons from the radiation era through decoupling to the collision-free blackbody microwave background [Peebles and Yu (1970)]; neutrinos in the same scenario [Bernstein (1988)]; the non-equilibrium ultra-relativistic plasma (electrons, photons etc.) in the radiation era, up to the decoupling of the first species (note that $n^a{}_{;a} \neq 0$ here since photons are produced in various reactions) [Bernstein (1988)]; and also for gravitons [Hawking (1966)].

Some of the results of this chapter are based on Maartens and Wolvaardt (1994a), and Maartens (1994).

5.2 Isotropic BGK Solution in Robertson-Walker Spacetime

As for Chapter 4 the spacetime geometry is determined by the metric (2.3) and the preferred 4-velocity is given by (2.4). We are seeking a solution where an isotropic distribution function f is relaxing toward an isotropic equilibrium distribution function $\bar{f}$ as the number of collisions that have occurred increases. The formal BGK solution (3.2a) takes the following form for this application

$$f(t,P) = \bar{f}(P) + h(P)e^{-\Gamma(t,P)} \tag{5.1a}$$

where P is an isotropic constant of the motion defined by (2.8) and the number of collisions that have occurred is given by (4.3) as

$$\Gamma(t,P) = \int_0^t \gamma(t',P)R(t')P^{-1}dt', \tag{5.1b}$$

for $m = 0$. For the sake of easing the derivations that follow we define $g \equiv h/\bar{f}$ and write (5.1a) as

$$f(t,P) = \bar{f}(P)\Big[1 + g(P)e^{-\Gamma(t,P)}\Big] \ . \tag{5.1c}$$

Note that for an approximate treatment one could take $g \ll 1$. Because of the isotropic nature of the solution, only the zero order terms remain in a covariant harmonic decomposition of the solution (it is not strictly necessary to perform this decomposition in order to obtain the solution, but it helps in making conclusions about the kinematics and dynamics of the solution)

$$f(t,P) = F(P)\left[1 + G(P)e^{-\Gamma(t,P)}\right] . \tag{5.1d}$$

The solution (5.1a) requires that $\bar{f}$ is *dynamic equilibrium* (see (1.37) and (3.2a)). In the discussion following (1.37) we point out that dynamic equilibrium can occur due to the absence of collisions (Liouville equilibrium) or due to the detailed balancing of collisions (collision-dominated equilibrium). As we wish to develop a model that could be applied to both collision-dominated equilibrium conditions in the early universe and to collision-free equilibrium in the late universe (see discussion of Section 5.1), we take $\bar{f}$ to be the Maxwell-Boltzmann distribution (1.35). The Maxwell-Boltzmann distribution is the only valid collision-dominated equilibrium model for elastic binary point collisions and could for example be used in (5.1a) to model a non-equilibrium distribution of photons relaxing towards collision-dominated equilibrium in the high collision conditions of the early universe. The Maxwell-Boltzmann distribution is also one of many distributions that may arise in the case of collision-free equilibrium. It could be applied to a distribution of photons relaxing towards collision-free equilibrium in the late universe as the microwave background radiation is blackbody (i.e. Maxwell-Boltzmann).

Furthermore, in selecting the Maxwell-Boltzmann distribution the important conditions (1.38) must be considered. In particular, in a FRW spacetime the application of such a model is restricted to radiation ($m = 0$) since an expanding gas with $m > 0$ cannot be in collision-dominated equilibrium. Also, we have already shown in Section 5.1 that non-equilibrium, isotropic Einstein-BGK solutions are possible only for $m = 0$. We thus take

$$F = \exp[\alpha(x) + \beta_a(x)p^a].$$

For the case of the Robertson-Walker metric this reduces to

$$F(P) = \exp(\alpha - P) \tag{5.1e}$$

where we used $m = 0$, $E = PR^{-1}$, $\beta_a = \beta u_a$, and $\beta = R$.

5.3 Kinematics, Dynamics, and Conservation of Particles, Energy and Momentum

This section follows Chapter 4 and investigates the kinematics and dynamics of the solution as well as the conditions for the conservation of particles and energy-momentum. We postpone the discussion of entropy until after the Einstein solution is determined. Using equation (4.8a) it is easy to see that the particle number density is given by

$$N = \frac{4\pi}{R^3(t)} \int_0^\infty P^2 F(P)\left[1 + G(P)e^{-\Gamma(t,P)}\right] dP$$

$$= \bar{N} + \frac{4\pi}{R^3(t)} \int_0^\infty P^2 F(P) G(P) e^{-\Gamma(t,P)} dP. \tag{5.2}$$

Given that the harmonic decomposition only has the zero order terms it is clear from equation (4.8b) that the number flux is identically zero yielding a non-tilted average 4-velocity as required.

By applying the same argument to (4.10$a-d$) it is clear that the energy flux vector q_a and the trace free pressure tensor π_{ab} are zero while the energy density, the isotropic pressure, and bulk viscosity are given by

$$\mu = \frac{4\pi}{R^4(t)} \int_0^\infty P^3 F(P)\left[1 + G(P)e^{-\Gamma(t,P)}\right] dP$$

$$= \bar{\mu} + \frac{4\pi}{R^4(t)} \int_0^\infty P^3 F(P) G(P) e^{-\Gamma(t,P)}\, dP \tag{5.3a}$$

$$p = \tfrac{1}{3}\mu \tag{5.3b}$$

$$\Pi = \frac{4\pi}{3R^4(t)} \int_0^\infty P^3 F(P) G(P) e^{-\Gamma(t,P)}\, dP \tag{5.3c}$$

Using the conditions for the conservation of particles, energy and momentum (4.12), (4.13), and (4.14), we get

$$n^a{}_{;a} = 0 \;\Rightarrow\; \int_0^\infty P F(P) G(P) \gamma(t,P) e^{-\Gamma(t,P)} dP = 0 \tag{5.4a}$$

$$u_a T^{ab}{}_{;b} = 0 \;\Rightarrow\; \int_0^\infty P^2 F(P) G(P) \gamma(t,P) e^{-\Gamma(t,P)} dP = 0 \tag{5.4b}$$

$$h_{ab} T^{bc}{}_{;c} = 0 \;\; \text{identically satisfied as } h \text{ is isotropic.} \tag{5.4c}$$

The implications of the above conditions are discussed in Section 5.5. *We have shown that if the distribution function* (5.1) *satisfies the conditions* (5.4a) *and* (5.4b), *then particle number is conserved and energy-momentum is conserved.*

5.4 Einstein-BGK Solution

The Einstein field equations are given by equations $(4.15a - c)$. Again, as the harmonic decomposition only contains the zero order term it is obvious that equations $(4.15a, b)$ are identically satisfied. The field equation $(2.10c)$ together with the conservation equation $(2.11a)$ and $(5.3b)$ gives $R = R_0 t^{1/2}$ so that by $(5.3a)$ we get

$$\int_0^\infty P^3 F(P)\Big[1 + G(P) e^{-\Gamma(t,P)}\Big]\, dP = \frac{3R_0^4}{16\pi}\,. \tag{5.5}$$

Now by $(5.1b)$ it is clear that $\partial\Gamma/\partial t \neq 0$ so that (5.5) implies

$$\int_0^\infty P^3 F(P) G(P) e^{-\Gamma(t,P)}\, dP = 0 \tag{5.6}$$

and using $(5.3a)$, $(5.3c)$

$$\mu = \bar{\mu}\,, \ \ \Pi = 0. \tag{5.7}$$

We have thus shown that if the distribution function (5.1) *satisfies the condition* (5.6), *then all the Einstein field equations are satisfied, the energy density μ takes the equilibrium form and the bulk viscosity vanishes.*

5.5 Properties of the Isotropic BGK Solution

The isotropic Einstein-BGK solution is therefore subject to the conditions derived in Sections 5.3 and 5.4. We now investigate the implications of these conditions. In order to get a clear picture, the conditions are repeated below

$$\int_0^\infty P F(P) G(P) \gamma(t,P) e^{-\Gamma(t,P)} dP = 0 \Leftrightarrow \frac{\partial}{\partial t}\int_0^\infty P^2 F(P) G(P) e^{-\Gamma(t,P)} dP = 0 \tag{5.4a}$$

$$\int_0^\infty P^2 F(P) G(P) \gamma(t,P) e^{-\Gamma(t,P)} dP = 0 \Leftrightarrow \frac{\partial}{\partial t}\int_0^\infty P^3 F(P) G(P) e^{-\Gamma(t,P)} dP = 0 \tag{5.4b}$$

$$\int_0^\infty P^3 F(P) G(P) e^{-\Gamma(t,P)}\, dP = 0 \tag{5.6}$$

It is clear that (5.6) implies (5.4b). We can satisfy (5.4a) and (5.6) for non-zero G if

$$e^{-\Gamma} = A(P)B(t) \Rightarrow \gamma = -\frac{\dot{B}}{BR}P \tag{5.8}$$

where we have used $\gamma e^{-\Gamma} = -\frac{P}{R}\frac{\partial}{\partial t}e^{-\Gamma}$. The relaxation function in (5.8) is the AW form $\gamma = E/\tau(t)$ and using this in (5.1b) we find

$$\Gamma = \int_0^t \frac{dt'}{\tau(t')} \tag{5.9}$$

so that $A'(P) \equiv 0$. Then (5.4a,b), (5.6) will hold provided

$$\int_0^\infty P^n F(P)G(P)dP = 0 \qquad n = 2,3. \tag{5.10}$$

(Note that if the requirement for number conservation is dropped, only $n = 3$ remains). *It is remarkable how the AW model emerges as the most simple possibility.*

As pointed out in Section 5.4, (5.10) implies vanishing bulk viscosity Π (Note that this is true in general for the AW model - see (4.30), (4.31)). Thus the isotropic non-equilibrium distribution f has zero non-equilibrium pressures and fluxes ($\Pi = j_a = q_a = \pi_{ab} = 0$), but is still non-equilibrium, since $C[f] = -\frac{P}{\tau(t)R}FGe^{-\Gamma} \neq 0$. *This shows that* Π, j_a, q_a, π_{ab} *are not in general sufficient to characterise a non-equilibrium state* - it only holds approximately in the near-equilibrium theories of Eckart/Landau-Lifshitz and Israel-Stewart. In Section 5.6, we show that f also generates entropy.

Now, we know that $f(P) \geq 0$. This and (5.1d) implies that $G(P)e^{-\Gamma(t,P)} \geq -1$. This is consistent with the conditions (5.10) which in fact require that $G(P) < 0$ for some P for non-trivial solutions to exist. It is possible to define the solution further. We have

$$f(t,P) = F(P)\left[1 + G(P)\exp\left[-\int_0^t dt'/\tau(t')\right]\right] \tag{5.11}$$

where F is equilibrium (see (5.1e)), $\tau > 0$ gives the mean collision time (unrestricted) and G is subject to (5.10). If f models a high-collision non-equilibrium phase relaxing towards collision-free blackbody equilibrium (eg. photons from the early to late universe) then F is given by (5.1e) and τ should increase from 0 to ∞ with the expansion of the universe. The natural time rate defined by the expansion is $\dot{R}/R$, so that we could take (compare [Hawking (1966)])

$$\tau = \tau_0 R/\dot{R} = 2\tau_0 t \tag{5.12}$$

then

$$f(t,P) = e^{\alpha - P}\left[1 + R_0^{-1/\tau_0} G(P) t^{-1/2\tau_0}\right]. \tag{5.13}$$

Furthermore, by recognising that (5.10) defines orthogonality between F and G, we can find an explicit solution for $G(P)$. Using (5.1e), we can write (5.10) as

$$\int_0^\infty P^n e^{-P} G(P) dP = 0 \quad n = 2,3 \tag{5.14}$$

Now, a system of polynomials $f_m(x)$ (degree m) is orthogonal on the interval $a \leq x \leq b$, with respect to a weight function $w(x)$, if [Abramowitz and Stegun (1970, p773)]

$$\int_a^b w(x) f_k(x) f_m(x) dx = 0,$$

where $w(x) \geq 0$ (and $m \neq k$; $m,k = 0,1,2,\ldots$). Comparing (5.14) and defining $P = x$, $w(P) = P^n e^{-P}$, $f_k(P) = 1$, and $f_m(P) = G(P)$, we find that the weight function and the limits of integration define the orthogonality relation for which the system of polynomials f_m are generalised Laguerre polynomials i.e. $w(P) = P^n e^{-P}$ (with $n = 2$,3) and $0 \leq x \leq \infty$. We can therefore satisfy (5.14) if G is expanded as a generalised Laguerre polynomial

$$G(P) \equiv L_m^{(n)}(P) = \sum_{l=0}^{m} (-1)^l \binom{m+n}{m-l} \frac{1}{l!} P^l ,$$

[Abramowitz and Stegun (1970, p775)]. Note that the polynomials are different for $n = 2,3$ and thus care must be taken if both energy conservation and particle number conservation are imposed.

5.6 Entropy

For a physically reasonable non-equilibrium model, the solution must have positive entropy production. In order to confirm this we investigate the entropy production rate

$S^a{}_{;a}$ given by (4.11c). This expression can easily be re-written for the isotropic case, by setting the higher order harmonics to zero and using the solution (5.1):

$$S^a{}_{;a} = \tfrac{4\pi}{R^2}\int_0^\infty \gamma F(P)G(P)e^{-\Gamma}P \log\Bigl\{F(P)[1+G(P)e^{-\Gamma}]\Bigr\}\, dP$$

$$= \frac{4\pi}{\tau(t)R^3(t)}\, e^{-\int dt'/\tau(t')}\int_0^\infty F(P)G(P)P^2 \log\Bigl\{F(P)[1+G(P)e^{-\int dt'/\tau(t')}]\Bigr\}\, dP$$

Now, taking F as (5.1e) allows us to write

$$S^a{}_{;a} = \frac{4\pi}{\tau(t)R^3(t)}\,(I_1 + I_2) \tag{5.15}$$

where

$$I_1 = e^{-\int dt'/\tau(t')}\int_0^\infty F(P)G(P)P^2(\alpha - P)dP$$

$$I_2 = e^{\alpha}\int_0^\infty P^2 e^{-P} K \log(1+K)dP, \quad K = G(P)e^{-\int dt'/\tau(t')}.$$

Now $I_1 = 0$ by (5.10), and $I_2 \geq 0$ since $K \log(1+K) \geq 0$ for $K \geq -1$ (we have already shown that $Ge^{-\Gamma} \geq -1$ - see Section 5.5) and $P^2e^{-P} \geq 0$. We have thus shown that the isotropic non-equilibrium model can lead to positive entropy production. *In principle, the entropy production rate $S^a{}_{;a}$ can be very large, even for a $m = 0$ gas.* This is in contrast to the Eckart bulk viscosity model which gives an entropy production rate that is too small [Weinberg (1971)]. The isotropic solution f is neither in dynamic nor in kinematic equilibrium, but has vanishing non-equilibrium pressures and fluxes.

5.7 Summary

In this chapter we presented an isotropic non-equilibrium Einstein-BGK solution. This solution represents the relaxation of an isotropic distribution function to an isotropic, equilibrium distribution function which is taken to be Maxwell-Boltzmann. Using results from Chapter 4 we showed that such a solution is only possible for the radiation case ($m = 0$). This solution may find application in the early universe for modelling photons and neutrinos in the radiation era relaxing through decoupling to the collision free blackbody background radiation.

The particle number density for the solution was determined (5.2) and it was shown that the number flux is zero. Equations giving the energy density, the isotropic pressure, and the bulk viscosity were derived (5.3). The energy flux and the trace free pressure tensor were found to be identically zero. Applying the conditions for the conservation of energy-momentum and particles, resulted in the restrictions (5.4) on the distribution function. The Einstein field equations required condition (5.6) and this gave an equilibrium form for the energy density and vanishing bulk viscosity. We found that the AW relaxation model gives a solution that satisfies the conditions (5.4) and (5.6). These conditions yield a non-equilibrium solution with zero non-equilibrium fluxes and pressures ($\Pi = j_a = q_a = \pi_{ab} = 0$). We have thus shown that these non-equilibrium fluxes and pressures are not sufficient to characterise a non-equilibrium state. Furthermore, using the AW model we were able to associate the relaxation time with the natural time rate as defined by the expansion of the universe. It was also found that the distribution function can be expressed in the form of generalised Laguerre polynomials.

The particularly simple nature of the model allowed the derivation of an explicit expression for the entropy production rate (5.15). We found that the entropy production rate is ≥ 0 and that it can be very large (even for a $m = 0$ gas). This is consistent with the non-equilibrium nature of the model.

The solution is particularly significant as it is not restricted to be close to equilibrium, unlike the standard linearised approaches which consider small deviations from equilibrium.

6.0 EVOLUTION OF ANISOTROPIES IN ROBERTSON-WALKER COSMOLOGIES

6.1 Introduction

In this chapter we extend the work of Matravers and Ellis (1989) and Ellis and Matravers (1990, 1992) in which they demonstrated that a number of exact equilibrium solutions allow the situation where the universe started off being smooth and then developed inhomogeneity and anisotropy. This is in contrast with the work of Misner (1968) in which it is considered that the universe started off in a chaotic state and then evolved to a smooth state. The aim was to explain for example the isotropy of the microwave background radiation. Several smoothing mechanisms such as the vast expansion associated with an inflationary universe [Guth (1981)], or ideas from kinetic theory such as viscous dissipation of anisotropy during the lepton era [Caderni et al. (1978)], have been proposed. On the other hand Penrose (1989, 1979) has suggested, based on entropy arguments, that the universe may have started off in a smooth state. The approach of Ellis and Matravers utilises kinetic theory to show that mechanisms exist which allow the universe to evolve from a smooth state to an inhomogeneous or anisotropic state. This in turn could provide the seeds for galaxy formation.

The basis of the approach is the assumption that the universe is described at early times by an equilibrium solution to the Einstein-Boltzmann system of equations. Such solutions as presented in Chapter 2 of this thesis can be anisotropic or inhomogeneous. The idea is that models can be constructed where the universe initially has an isotropic and homogeneous geometry (i.e. a Robertson-Walker metric) but with anisotropic or inhomogeneous distribution of matter. Several mechanisms exist that could communicate the anisotropy or inhomogeneity of the matter distribution to the spacetime geometry, forcing the universe to evolve away from the smooth state. These mechanisms are described further in Section 6.3. To demonstrate the generation of anisotropy a Bianchi I geometry is used.

The contribution here is to extend the work to the case where the initial isotropic and homogeneous phase of the evolution may be described by a non-equilibrium Einstein-BGK solution. We then illustrate how this model can be used to describe the evolution of anisotropy. In order to achieve this the Einstein-BGK solution in Bianchi I geometry is utilised. We therefore start the discussion with a presentation of the formal non-equilibrium Bianchi I solution in Section 6.2, while the mechanisms for the evolution of shear are given in Section 6.4.

6.2 BGK Solution in Bianchi I Spacetime

In Section 2.4 we presented the equilibrium Einstein-Boltzmann solutions in Bianchi I geometry as developed by Matravers and Ellis (1989) and Maartens and Maharaj (1990). Taking an approach similar to the one used for the Robertson-Walker solutions of Chapter 4, these equilibrium solutions are used along with the formal solution of the BGK equation of Chapter 3 to determine an Einstein-BGK solution in Bianchi I geometry. The Bianchi I metric in a spacetime with standard coordinates $x^i = (t, x^1, x^2, x^3)$, is given by (2.15) as

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2 \ .$$

with preferred 4-velocity (2.16): $u^i = \delta^i{}_0$.

We are seeking a solution representative of the physical situation where a spatially homogeneous, dynamically anisotropic distribution function f is relaxing toward an equilibrium solution $\bar{f}$ (or the case where the relaxation is away from the equilibrium $\bar{f}$ toward an anisotropic f). To find it, the set of coupled Einstein-Boltzmann equations (3.17) must be solved for the metric given by (2.15). We start by finding a distribution function that satisfies the BGK equation (3.17a). In Chapter 3 it is shown that the solution of the BGK equation is given in any spacetime by (3.2a) as $f(v) = \bar{f}(v) + he^{-\Gamma(v)}$ with $\mathcal{L}(h) = \mathcal{L}(\bar{f}) = 0$ (i.e. $\bar{f}$ and h are Liouville solutions). We must thus find solutions $\bar{f}$ and h in Bianchi I and express Γ in its appropriate form. Now, we have also shown in Chapter 2 that the three Killing vectors, $\xi_\nu = \partial/\partial x^\nu$, of Bianchi I spacetime generate three constants of the motion $p_\nu = \xi_\nu \cdot \boldsymbol{p}$. Therefore, any function $f = F(p_1, p_2, p_3)$ then represents a solution to the Liouville equation for Bianchi I spacetimes. We therefore define $\bar{f}$ and h as

$$\bar{f} \equiv \bar{f}(P) \ , \qquad h \equiv h(p_\nu) \tag{6.1}$$

where P is a function of p_ν (and thus a constant of the motion) given by

$P^2 = X^4(t)(p^1)^2 + Y^4(t)(p^2)^2 + Z^4(t)(p^3)^2$, (see (2.20)). Note that

$$E = [m^2 + X^{-2}(t)(p_1)^2 + Y^{-2}(t)(p_2)^2 + Z^{-2}(t)(p_3)^2]^{1/2}$$

$$P = [(p_1)^2 + (p_2)^2 + (p_3)^2]^{1/2}$$

are both spatially homogeneous. However, there is no simple relation between the anisotropic constant of motion P and the isotropic non-constant energy E, unlike the FRW case. This happens since h_{ab} is anisotropic:

$$\lambda^2 = E^2 - m^2 = h^{\mu\nu} p_\mu p_\nu$$

$$P^2 = \delta^{\mu\nu} p_\mu p_\nu,$$

(in the FRW case $h^{\mu\nu} = R^{-2}(t)\delta^{\mu\nu}$).

Also, the equilibrium solution $\bar{f}$ is generally dynamically anisotropic except for the FRW case where $X = Y = Z$. This form of the equilibrium solution is selected to ensure that the Bianchi I solution matches with the Robertson-Walker solution at their interface - see Section 6.4 for more details.

Next, the form of the time-dependent relaxation term Γ must be determined. If the mean collision rate is spatially homogeneous and isotropic, i.e. $\gamma = \gamma(t, E)$, then (3.2b) gives

$$\Gamma(t, E) = \int_0^t \frac{\gamma(t', E)}{[m^2 + X^{-2}(t')(p_1)^2 + Y^{-2}(t')(p_2)^2 + Z^{-2}(t')(p_3)^2]^{1/2}} dt'. \tag{6.2}$$

In the integration, p_ν are constants.

Based on the discussion above a spatially homogeneous dynamically anisotropic BGK solution may therefore be given as

$$f(t, p_\nu) = \bar{f}(P) + h(p_\nu)e^{-\Gamma(t, E)}. \tag{6.3}$$

In order to find solutions to the Einstein-BGK problem and to investigate resulting conditions on the solution it is convenient to perform a covariant harmonic decomposition. Unlike the Robertson-Walker case of (4.6), $\bar{f}$ is also anisotropic. The decomposition is therefore

$$f(t,p_\nu) = F(t,E) + F_a(t,E)e^a + F_{ab}(t,E)e^ae^b + \dots$$

where

$$F_{a\dots b}(t,E) = \bar{F}_{a\dots b}(t,E) + e^{-\Gamma(t,E)}H_{a\dots b}(t,E). \tag{6.4}$$

Although the solution and its decomposition do not take the convenient form of the Robertson-Walker case, it gives us the necessary tools to investigate the conditions for an Einstein solution.

Ellis et al. (1983c) and Matravers and Ellis (1989) show that the first two Boltzmann harmonic equations for a homogeneous distribution function in a Bianchi I spacetime (for which $\dot{u}^a = 0 = \omega_a$), are

$$-\tfrac{2}{15}\lambda^{-1}\partial(\lambda^3\sigma^{ab}F_{ab})/\partial E + E\dot{F} - \tfrac{1}{3}\lambda^2\theta\partial F/\partial E = b$$

$$V_a + Eh_a{}^d\dot{F}_d - \tfrac{1}{3}\lambda^2\theta(\partial F_a/\partial E) - Q_a = b_a \tag{6.5}$$

where $V_a = \frac{6}{35}\lambda^{-2}\partial(\lambda^4\sigma^{bc}F_{abc})/\partial E$ and $Q_a = \frac{2}{5}\lambda^{1/2}\partial(\lambda^{3/2}F_d\sigma^d{}_a)/\partial E$. For the BGK collision term with isotropic γ, b and b_a are given by (3.4), so that (6.5) become

$$2\lambda^{-1}\partial(\lambda^3\sigma^{ab}F_{ab})/\partial E - 15E\dot{F} + 5\lambda^2\theta\partial F/\partial E = 15\gamma(F - \bar{F})$$

$$3V_a + 3Eh_a{}^d\dot{F}_d - \lambda^2\theta(\partial F_a/\partial E) - 3Q_a = 3\gamma(\bar{F}_a - F_a). \tag{6.6}$$

The higher order harmonics equations [Ellis et al. (1983c, p492)] together with (6.6) show that if F, F_a are specified, then the harmonic equations place a chain of restrictions on the second and higher harmonics. The important point is that *if the shear* σ^{ab} *is non-zero, the harmonics are no longer independent.*

The kinematics and dynamics of the solution are given by equations $(1.47a-g)$, using (6.4):

$$N = \bar{N} + 4\pi\int_m^\infty E\lambda e^{-\Gamma(t,E)}H(t,E)dE \tag{6.7a}$$

$$j_a = \bar{j}_a + \tfrac{4\pi}{3}\int_m^\infty \lambda^2 e^{-\Gamma(t,E)} H_a(t,E)dE \tag{6.7b}$$

$$\mu = \bar{\mu} + 4\pi \int_m^\infty E^2\lambda e^{-\Gamma(t,E)} H(t,E)dE \tag{6.7c}$$

$$p = \bar{p} + \tfrac{4\pi}{3}\int_m^\infty \lambda^3 e^{-\Gamma(t,E)} H(t,E)dE \tag{6.7d}$$

$$q_a = \bar{q}_a + \tfrac{4\pi}{3}\int_m^\infty E\lambda^2 e^{-\Gamma(t,E)} H_a(t,E)dE \tag{6.7e}$$

$$\pi_{ab} = \bar{\pi}_{ab} + \tfrac{8\pi}{15}\int_m^\infty \lambda^3 e^{-\Gamma(t,E)} H_{ab}(t,E)dE \tag{6.7f}$$

$$\Pi = \tfrac{4\pi}{3}\int_m^\infty \lambda^3 e^{-\Gamma(t,E)} H(t,E)dE \tag{6.7g}$$

The conservation of particles (3.5b), energy (3.5c) and momentum (3.5d) respectively place the following conditions on the solution (6.4):

$$\int_m^\infty \gamma(F - \bar{F})\lambda dE = 0 \tag{6.8a}$$

$$\int_m^\infty \gamma(F - \bar{F})\lambda E dE = 0 \tag{6.8b}$$

$$\int_m^\infty \gamma(F_a - \bar{F}_a)\lambda^2 dE = 0 \tag{6.8c}$$

Note that $\bar{F}_a \neq 0$ in general, since $\bar{f}$ is anisotropic. If we use the AW model ($\gamma = E/\tau(t)$), then (6.8) give, by (3.6)

$$N = \bar{N},\ \mu = \bar{\mu},\ q_a = \bar{q}_a. \tag{6.9}$$

The kinematics and dynamics of the non-equilibrium BGK solution in Bianchi I are given by (6.7) *while the conditions for the conservation of particles, energy and momentum are given by* (6.8). *For the AW collision model, the solution has equilibrium particle number density, energy density, and energy flux.*

6.3 Einstein-BGK Solution in Bianchi I Spacetime

The Einstein field equations for the Bianchi I metric are given by equations (2.19$a-f$), with the conservation equations yielding (2.19i) and are not repeated here. The conservation equation contains the non-zero shear σ^{ij} which is of importance for our discussion here. In turn the Einstein field equations determine the form of the trace-free pressure tensor as given by $(2.19g,h)$. It is important to keep in mind that the shear is non-zero in general and its evolution is determined by (2.17). The non-zero shear implies that the harmonic coefficients in the covariant harmonic decomposition are no longer independent of each other.

We next investigate the conditions placed on the solution by the field equations. The equation $(2.19b)$ along with $(1.47e)$, $(6.7e)$ requires that the following condition be placed on the first order harmonic coefficient of (6.4)

$$\int_m^\infty E\lambda^2 F_a(t,E)dE = 0\ , \tag{6.10}$$

while the equations $(2.19g)$, $(1.47f)$, and $(6.7f)$ require that the second order harmonic coefficient satisfies the following relation

$$\int_m^\infty \lambda^3 F_{ab}(t,E)dE = \tfrac{15}{8\pi}\ \mathrm{diag}\left\{0, \pi_{11}, \pi_{22}, -Z^2(X^{-2}\pi_{11} + Y^{-2}\pi_{22})\right\}. \tag{6.11}$$

The trace-free condition $(2.19h)$ is identically satisfied since $h^{ab}F_{ab} = 0$, which means that at most two of the $F_{\nu\nu}$ are independent. Then (6.5), (6.6), (6.8), (6.10) and (6.11) are the restrictions on the harmonics for an Einstein-BGK solution. Once F and F_{aa} are specified, the remaining field equations in principle yield a solution $\{X(t), Y(t), Z(t)\}$ (this is analogous to specifying equations of state for p and π_{ab} in fluid models).

The Einstein field equations are satisfied by a Bianchi I BGK distribution that satisfies conditions (6.10) *and* (6.11).

The restrictions (6.8), (6.10), and (6.11) on the harmonics may for example be satisfied by the choice

$$F = \bar{F},\ F_a = 0 = \bar{F}_a,\ \bar{F}_{ab} = 0,\ H_{ab} = \mathrm{diag}\left\{0, V_1, V_2, -Z^2(X^{-2}V_1 + Y^{-2}V_2)\right\} \tag{6.12}$$

where $V_I(t,E)$ are such that π_{ab} is non-zero. For this choice it follows from $(6.7b)$ and $(6.7g)$ that the number flux and bulk viscosity vanish:

$$j_a = 0,\ \Pi = 0.$$

Furthermore, by (6.7a), (6.7c) and (6.7e), the AW matching conditions will hold (even if γ is not of AW form) for this choice:

$$N = \bar{N}, \ \mu = \bar{\mu}, \ q_a = \bar{q}_a = 0.$$

The BGK harmonic equations (6.6) reduce to

$$2\partial(\lambda^3\sigma^{ab}F_{ab})/\partial E - 15\lambda E\dot{F} + 5\lambda^3\theta\partial F/\partial E = 0 \tag{6.13}$$

$$\partial(\lambda^4\sigma^{bc}F_{abc})/\partial E = 0. \tag{6.14}$$

The higher order BGK harmonic equations become conditions on the fourth and higher harmonics which may always be satisfied, since these harmonics are otherwise unrestricted. The condition (6.14) gives $\sigma^{bc}F_{abc} = 0$ while (6.13) is a constraint on F and F_{aa}. Once F, F_{aa} are specified subject to (6.13), the remaining field equations determine $g_{\nu\mu}(t)$ in principle, thus completing the Einstein-BGK solution.

6.4 Anisotropy Generation in Robertson-Walker Cosmologies

The approach we use to show that a universe could evolve away from an initial isotropic state to an anisotropic state is to assume that the universe has initially Robertson-Walker geometry, while the matter distribution is described by an anisotropic distribution function which is also compatible with Robertson-Walker geometry. It is then shown that mechanisms exist which can communicate the anisotropy of the particle distribution to the geometry of the universe, forcing it away from the isotropic state to a Bianchi I geometry. Ellis and Matravers (1992, 1990) and Matravers and Ellis (1989) propose two main mechanisms. They start with a universe that is initially Robertson-Walker with an equilibrium particle distribution that is inhomogeneous and anisotropic, but the anisotropy and inhomogeneity is not communicated to the spacetime geometry as the particle distribution is effectively collision-free, since the particles enjoy asymptotic freedom under high temperature conditions (such as would exist in the early universe). Now, as the universe expands, the temperature drops and the collisions become significant, allowing the anisotropy to be communicated to the geometry. Secondly, while the temperature is very high the particles effectively behave as if their rest mass is zero. As the universe expands and cools, and the threshold energies of different particles are reached, their rest mass becomes significant. Again, this effective change in the zero rest mass allows the inhomogeneity and anisotropy to be communicated. Ellis and Matravers show that both anisotropy and inhomogeneity can be generated. In this section we show that the generation of anisotropy can be modelled starting from an anisotropic, non-equilibrium BGK solution. We will first describe the model mathematically, and then discuss possible physical mechanisms. A non-equilibrium model allows us to overcome some of the drawbacks of the equilibrium model - for example, the difficulty in motivating a collision-free early-universe phase.

We assume that the universe initially ($t \leq t_0$) has Robertson-Walker geometry and that the particle distribution is given by (4.7) with AW collision model (4.29). This solution is dynamically anisotropic and has Bianchi I symmetry. For the high temperature conditions in the early universe, the particle rest mass becomes insignificant and an Einstein-BGK solution may be chosen that is consistent with the Bianchi I solution (6.12). By Chapter 4 we can take (see equations (4.31))

$$H = 0,\ H_a = 0,\ \int_0^\infty P^3 H_{ab}(P)dP = 0,\ (P = R(t)E). \tag{6.15}$$

Thus, for high temperature conditions for which $m = 0$, it is possible to find distribution functions with non-zero second order harmonic coefficients, but for which the condition $\pi_{ab} = 0$ is satisfied. We need non-zero $H_{\mu\nu}$ in order to 'switch on' π_{ab} (the mechanism for this will be described below). Then by (2.17) the shear anisotropy σ_{ab} will emerge and the geometry will evolve to a Bianchi I phase. By (6.12) and (6.15), we could take H_{ab} of the form

$$H_{ab}(P) = \text{diag}\left\{0, V_1(P), V_2(P), -V_1(P) - V_2(P)\right\} \tag{6.16}$$

where V_I are non-zero but satisfy

$$\int_0^\infty P^3 V_I(P)dP = 0.$$

We now have two solutions. For $t \leq t_0$ the universe is considered Robertson-Walker, while for $t \geq t_0$ it is Bianchi I. On the hypersurface $t = t_0$, the geometry and the distribution function symmetry must satisfy both the conditions for Robertson-Walker and Bianchi I spacetimes. Matravers and Ellis (1989) show that this is the case if the following matching conditions are satisfied

$$\dot{X}(t_0) = \dot{Y}(t_0) = \dot{Z}(t_0), \text{ and } \quad X(t_0) = Y(t_0) = Z(t_0). \tag{6.17}$$

These matching conditions become the initial conditions for the Bianchi I phase. At the interface $\sigma_{ab}(t_0) = 0$, and σ_{ab} remains zero unless π_{ab} becomes non-zero to force the universe to evolve to the Bianchi I geometry. In the Bianchi I phase $t \geq t_0$, the solution is described in Section 6.3 and it clearly reduces to the Robertson-Walker solution at $t = t_0$ as a result of the matching conditions (6.17). Because P becomes isotropic and reduces to the Robertson-Walker form with the application of the matching conditions, the Bianchi I equilibrium solution $\bar{f}$ reduces to the Robertson-Walker equilibrium solution at t_0. Also, the AW Γ for the Bianchi I case reduces to the AW Γ for the Robertson-Walker case.

Before investigating possible physical mechanisms which could trigger the generation of anisotropy, we provide a summary of the mathematical model presented so far. The collision model is AW, i.e.,

$$\begin{aligned} \gamma(t, E) &= E/\tau(t) && \text{for all } t \geq 0, \\ \Gamma(t) &= \int_0^t dt'/\tau(t') && \text{for all } t \geq 0. \end{aligned} \tag{6.18}$$

The particle 4-momentum is given by

$$E = -u_a p^a = p^0 = \begin{cases} R^{-1}(t)[(p_1)^2 + (p_2)^2 + (p_3)^2]^{1/2} \ (m = 0) & t \leq t_0 \\ [m^2 + X^{-2}(t)(p_1)^2 + Y^{-2}(t)(p_2)^2 + Z^{-2}(t)(p_3)^2]^{1/2} & t \geq t_0 \end{cases}$$

with $P^2 = (p_1)^2 + (p_2)^2 + (p_3)^2$ for all $t \geq 0$ (note that $P = R(t)E$ for $t \leq t_0$).

The particle distribution function is given by

$$f(t, p_\nu) = \bar{f}(P) + h(p_\nu)e^{-\Gamma(t)}$$

where

$$\bar{f} = \begin{cases} \bar{F}(R(t)E) & t \leq t_0 \\ \bar{F}(t, E) + \bar{F}_a(t, E)e^a + \ldots & t \geq t_0 \end{cases}$$

with $\bar{F}_{a_1\ldots a_r}(t_0,E)=0 \quad (r\geq 1)$, $\bar{F}(t_0,E)=\bar{F}(R(t_0)E)$, and

$$h(p_\nu)=H(t,E)+H_a(t,E)e^a+\ldots$$

with $H_{a_1\ldots a_r}(t,E)=H_{a_1\ldots a_r}(R(t)E))$ $(r\geq 0)$ for $t\leq t_0$. Note that in the Robertson-Walker phase $(t\leq t_0)$ each $\bar{F}_{a_1\ldots a_r}$ and $H_{a_1\ldots a_r}$ is a Liouville solution (hence functions of P only), because the harmonics decouple in $\mathbf{L}(f)=0$ as $\sigma_{ab}=0$. This is no longer true in the Bianchi I phase $(t\geq t_0)$ and hence $\bar{F}_{a_1\ldots a_r}$ and $H_{a_1\ldots a_r}$ are functions of t and E (unknown).

The distribution is specified to yield an Einstein-BGK solution for $t\geq 0$:

$$H(t,E)=H_a(t,E)=0 \qquad \text{for all } \; t\geq 0$$

$$\bar{F}_a(t,E)=\bar{F}_{ab}(t,E)=0 \qquad \text{for all } \; t\geq t_0$$

where $\bar{F}_a$, $\bar{F}_{ab}$ are identically zero for $t\leq t_0$. (This condition implies that the Liouville solution $\bar{f}$ in Bianchi I has $\pi_{ab}=0=q_a$). As a result the solution is

$$f(t,p_\nu)=\bar{F}(t,E)+e^{-\Gamma(t)}H_{ab}(t,E)e^ae^b+\ldots \tag{6.19}$$

(with $\bar{F}=\ \bar{F}(R(t)E)$, $H_{ab}=H_{ab}(R(t)E)$ for $t\leq t_0$), and H_{ab} obeys

$$H_{ab}(t,E)=\mathrm{diag}\left\{0,V_1(t,E),V_2(t,E),V_3(t,E)\right\}$$

where for $t\leq t_0$:

$$V_\nu(t,E)=V_\nu(R(t)E) \;\; \text{not all zero}$$

$$V_3=-V_1-V_2$$

$$\int_0^\infty P^3V_I(P)dP=0 \quad (P=R(t)E) \quad I=1,2. \tag{6.20}$$

and for $t\geq t_0$:

$$V_3=-Z^2(X^{-2}V_1+Y^{-2}V_2)$$

$$\pi_{II}(t) = \tfrac{8\pi}{15}\, e^{-\Gamma(t)} \int_m^\infty (E^2 - m^2)^{3/2}\, V_I(t,E) dE$$

$$\neq 0. \tag{6.21}$$

It is clearly possible to find V_I such that (6.20) and (6.21) are satisfied (and therefore all field and conservation equations will be satisfied). The solution has zero number flux, bulk viscosity and energy flux for all $t \geq 0$.

We now investigate a physical mechanism that would make $\pi_{ab} \neq 0$ at t_0. Suppose the universe with Robertson-Walker geometry is in an early phase and the temperature is so high that the particles have $m = 0$; and that the matter distribution is described by the distribution function (6.19) with Γ of AW form (6.18). The conditions (6.20) and (6.21) can be satisfied by appropriate $H_{ab} \neq 0$ and thus the field and conservation conditions are satisfied for both the FRW and Bianchi I phases. As the universe expands and cools below the threshold energy (at $t = t_0$) for the particles under consideration, the distribution is no longer effectively of zero rest mass particles and the condition (6.20) is no longer satisfied. Condition (6.21) is required to ensure vanishing anisotropic stress in the FRW phase (i.e. the field equation $\pi_{ab} = 0$ is satisfied). For the AW collision model (6.20) is satisfied by $H_{ab} \neq 0$ only if $m = 0$ (see (4.31)). Thus, as soon as the particle rest mass is no longer effectively zero, the anisotropic stress becomes non-zero ($\pi_{ab} \neq 0$) and is given by (6.20). As a result shear anisotropy emerges, forcing the universe to evolve away from the FRW to the Bianchi I phase. Notice that as the solution satisfies (6.21) also for $t \leq t_0$, condition (6.21) is satisfied for $t \geq t_0$ and hence the evolution is towards Bianchi I (this ensures that the conditions on the second order harmonic component match at the interface).

During the phase $t \leq t_0$ the collision rate is high and therefore $e^{-\Gamma}$ may become small (as the number of collisions that have occurred Γ becomes large). This forces the non-equilibrium, anisotropic distribution function $f(t, p_\nu)$ to approach the isotropic, collision-dominated equilibrium distribution $\bar{f}(P)$. However, as long as $e^{-\Gamma} > 0$ (even if it is very small) the model presented here works. The model therefore represents a high temperature situation where the matter distribution is nearly isotropic (forced by the high collision rate). The distribution function has Bianchi I symmetry but satisfies all the conditions for a Robertson-Walker universe. As the universe cools the particle threshold energy is reached and the Robertson-Walker condition is no longer satisfied. The remnant of the initial anisotropy therefore acts as the seed for the change to

Bianchi I geometry as the particle mass becomes significant, communicating the anisotropy of the distribution function to the spacetime geometry. The small anisotropy in the microwave background radiation could be a physical example of exactly the remnant anisotropy considered here (see Caderni et al. (1978) for a related discussion). Hence, this model suggests a mechanism by which anisotropy present in the radiation era (of which the anisotropy in the microwave background radiation may be a remnant) could act as the seed for anisotropy generation in the spacetime geometry.

6.5 Summary

In this chapter we presented a formal Einstein-BGK solution in Bianchi I spacetime. We determined the kinematics and dynamics of the solution (see (6.7)) as well as the conditions for the conservation of particles, energy and momentum (see (6.8)). We found that for the AW collision model, the solution has equilibrium particle number density $\bar{N}$, energy density $\bar{\mu}$, and energy flux $\bar{q}_a$. In the Bianchi I model the shear is non-zero, and the harmonics are not independent. The solution must therefore also satisfy the harmonics equations. Hence, we determined the first two harmonic equations for the Bianchi I BGK solution (see (6.6)). The conditions required to satisfy the Einstein field equations were determined (see (6.10), (6.11)), and an example of an Einstein-BGK solution that would satisfy all restrictions was given.

This solution, with an AW collision model, along with the anisotropic AW FRW solution, was used to demonstrate a model where the anisotropy of the distribution function is communicated to the spacetime geometry forcing it away from isotropy. We showed that it is possible to find a non-equilibrium anisotropic distribution function (with $m = 0$) that would satisfy the conservation and field equations in FRW, but has Bianchi I symmetry (see (6.20)). We showed that if non-zero anisotropic stress is generated (and thus non-zero shear), the solution would be forced to become Bianchi I (the solution also satisfies the conservation and field equations in Bianchi I geometry). An example of a physical mechanism that would generate non-zero anisotropic stress was presented. In this model the particles initially have effective zero restmass, but as the universe cools and the threshold energy is reached, the communication between the matter distribution and the geometry is triggered. Other mechanisms within the BGK framework presented here may exist and would merit further investigation. An example may be a change in the collision rate γ.

7.0 EXACT SPHERICALLY SYMMETRIC EINSTEIN-BGK SOLUTIONS

7.1 Introduction

In this chapter we consider BGK relaxation solutions in static spherically symmetric spacetimes. We seek to extend the work of Fackerell (1968), Ray (1982), and Maharaj and Maartens (1986) on static spherically symmetric Liouville solutions to the BGK case. The spherically symmetric Liouville solution given in Section 2.5 is used as a basis for the BGK solution. The solution we present consists of a dynamically anisotropic distribution function relaxing with changing radius towards a dynamically isotropic distribution function. In order to obtain a static relaxation solution, we introduce the concept of the relaxation length collision function, extending the AW relaxation time concept. The conditions imposed on the distribution function by the conservation of particles, energy and momentum are derived. In order to investigate the restrictions on the distribution function required to satisfy the Einstein field equations, the dynamic quantities determining the energy-momentum tensor are derived. These are used to show that if the isotropic part of the distribution function is specified along with the relaxation function, the spacetime metric function can be determined in principle. We conclude with a brief discussion of the possible physical applications of a non-equilibrium spherically symmetric relaxation model. In Appendix B, we give an alternative model using a more general form of the AW relaxation function.

7.2 Static Spherically Symmetric BGK Solution

In static spherically symmetric (SSS) spacetimes with standard coordinates $x^i=(t,r,\theta,\phi)$, the spacetime metric is given by (2.23)

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\psi(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

The SSS symmetry defines the preferred 4-velocity $u = e^{-\nu/2}\partial_t$, while an orthonormal tetrad basis can be defined as follows (2.25)

$$\omega^a = \{e^{\nu/2}dt,\ e^{\psi/2}dr,\ rd\theta,\ r\sin\theta\ d\phi\}.$$

In this section we obtain a BGK solution for a single-component gas in a SSS space-time. The solution is restricted to ensure the conservation of particles, energy and momentum. As for the FRW and Bianchi I examples presented previously, the solution is based on the Liouville solution for the SSS case as given in Chapter 2. The solution represents the situation where a SSS but dynamically anisotropic distribution function relaxes towards an equilibrium distribution function with increasing or decreasing radius. We build on the relaxation time approach of the nonstatic models by introducing the new concept of relaxation length for the SSS case. The relaxation mechanism is taken to be collisions between the particles of the gas. We further assume that the equilibrium distribution function is isotropic, implying that the effect of collisions is to isotropise the distribution of particles. The distribution function and the space-time remain constant in time while the effect of the collisions is reflected in an increased tendency towards isotropy with changing radius. At this point we do not specify whether anisotropy lessens with decreasing or increasing radius. We also do not specify the nature of the gas particles. In order to achieve a stationary solution in the case of a bound system, it is assumed that for each particle escaping the system, another enters [Misner, Thorne, and Wheeler (1973, p.680)]. These issues are discussed further in Section 7.6, where the possible physical applications of the solution are outlined.

In order to obtain the model described above, the BGK solution (3.2a) requires specialisation to the static spherically symmetric case. In terms of the standard coordinates (2.23) and the orthonormal one-form basis (2.25) for SSS spacetimes, a SSS distribution has the form

$$\begin{aligned} f &= K[r, e^{\nu}p^t, r^4((p^\theta)^2 + \sin^2\theta\ (p^\phi)^2)] \\ &= K[r, e^{\nu/2}p^0, r^2((p^2)^2 + (p^3)^2)]. \qquad (7.1) \end{aligned}$$

K is independent of t because it is static, and independent of spatial angles θ, ϕ, and momentum-space angle $\Theta = tan^{-1}(p^3/p^2)$ because of spherical symmetry. Note that the non-equilibrium nature of f leads to an explicit r-dependence unlike the equilibrium solution (2.28). The equilibrium distribution function $\bar{f}$ is taken as the dynamically isotropic case of (2.28):

$$\bar{f} = \bar{F}(e^{\nu}p^t) = \bar{F}(e^{\nu/2}p^0) \tag{7.2}$$

which is a Liouville solution since $e^{\nu}p^t \equiv \widetilde{E}$ (see (7.5) below) is a constant of the motion.

The dynamically anisotropic part h of the distribution function is given by

$$\begin{aligned} h &= Q[e^{\nu}p^t, r^4((p^{\theta})^2 + \sin^2\theta\ (p^{\phi})^2)] \\ &= Q[e^{\nu/2}p^0, r^2((p^2)^2 + (p^3)^2)]. \end{aligned} \tag{7.3}$$

As the arguments of Q are constants of the motion, Q is a Liouville solution. By (3.2a) the conditions for (7.1) to be a SSS BGK-solution are therefore satisfied if we take $\bar{f}$ and h of the forms (7.2), (7.3), and if the γ is a SSS function

$$\gamma = G[r, e^{\nu}p^t, r^4((p^{\theta})^2 + \sin^2\theta\ (p^{\phi})^2)]. \tag{7.4}$$

It is useful to follow (2.32) and define

$$\begin{aligned} \widetilde{E} &\equiv e^{\nu/2}p^0 \\ J^2 &\equiv r^2[(p^2)^2 + (p^3)^2] \end{aligned} \tag{7.5}$$

where $\widetilde{E}$ is proportional to the energy, $E = -u_a p^a = \widetilde{E}e^{-\nu/2}$, $(\widetilde{E} \geq me^{\nu/2})$ and J is the angular momentum of the particles as measured by static observers $(J \geq 0)$. $\widetilde{E}$ and J are constants of the motion generated by the static spherically symmetric Killing vectors (2.27) [Maharaj and Maartens (1986)]. An alternative approach to obtaining a solution is to first write the BGK equation explicitly in terms of the SSS metric and then to solve it.

As our aim is to find a general solution, no attempt has been made so far to restrict the variable dependence of the relaxation function γ. It is assumed only that γ is a general function of r, J, and $\widetilde{E}$. A new and more explicit form (the relaxation length form) of γ is discussed in Sections 7.3 where γ is restricted in order to satisfy the conditions for the conservation of energy and momentum and the Einstein field equations.

Using these variables and (3.2) we can write the BGK-solution (7.1-4) in the following form

$$f(r,\widetilde{E},J) = \bar{F}(\widetilde{E}) + h(\widetilde{E},J)\exp[-\Gamma(r,\widetilde{E},J)] \tag{7.6}$$

where

$$\Gamma \equiv \int^{v} \gamma(u)du\ .$$

Before we write down the final form of the solution we require an expression for the radial component of particle momentum in terms of particle rest mass. Using (1.1) and (2.23) the particle rest mass is given by

$$-m^2 = -e^{-\nu}\widetilde{E}^2 + (p^1)^2 + r^{-2}J^2.$$

It is easy to rewrite this as an expression for the radial component of momentum

$$p^1 = \pm\left[e^{-\nu}\widetilde{E}^2 - r^{-2}J^2 - m^2\right]^{1/2} = e^{\psi/2}p^r. \tag{7.7}$$

Furthermore, because $\widetilde{E}$ and J are constants of the motion, and γ is independent of t, we may use (7.7) to rewrite the integral along the phase flow in (7.6) as

$$\begin{aligned}\int^{v} \gamma du &= \int^{r} \gamma\, \frac{dv}{dr}dr \\ &= \int^{r} e^{-\psi(r')/2}\left[e^{-\nu(r')}\widetilde{E}^2 - r'^{-2}J^2 - m^2\right]^{-1/2} \gamma(r',\widetilde{E},J)dr'. \end{aligned} \tag{7.8}$$

Note that the $\pm$ sign in (7.7), which refers to out- or in-going particles is absorbed into the collision rate γ in (7.8).

Using (7.6) and (7.8) we can now write down the final form of the static spherically symmetric BGK-solution as proposed earlier

$$f(r,\widetilde{E},J) = \bar{F}(\widetilde{E}) + h(\widetilde{E},J)\exp[-\Gamma(r,\widetilde{E},J)]$$

where

$$\Gamma(r,\widetilde{E},J) = \int^{r} e^{\psi(r')/2}\gamma(r',\widetilde{E},J)\left[e^{-\nu}\widetilde{E}^2 - r'^{-2}J^2 - m^2\right]^{-1/2} dr'. \tag{7.9}$$

The BGK solution is not complete until the conditions for the conservation of particles, energy and momentum are determined. However, in the solution (7.9) Γ is dynamically anisotropic. The general case for which Γ is dynamically anisotropic is basically intractable. In order to investigate the conservation equations and to obtain exact Einstein-BGK solutions, we introduce the relaxation length collision function in the following section.

7.3 SSS BGK Relaxation Function and Solution

In order to use the SSS BGK solution (7.9) to obtain exact Einstein solutions that satisfy the conservation of particles, energy and momentum, a more explicit form of the relaxation function is required. We wish to obtain a static collision function that yields a dynamically isotropic Γ. Such a collision function is given by a modified form of the AW model, adapted to the static case. Since the relaxation function is SSS, it is no longer meaningful to use the AW form

$$\gamma = -u_a p^a / \tau(x).$$

Instead we will take

$$\gamma = c_a p^a / \delta(r) \tag{7.10}$$

where,

$$c_a = \text{unit radial vector}$$

$$\delta = \text{relaxation length},$$

so that the collision rate depends only on radial position and the radial component of momentum. This seems reasonable for a static relaxation process and introduces the concept of *relaxation length* $\delta(r)$, analogous to the mean time between collisions $\tau(x)$ (in fact, we show (see (7.13)) that $\delta(r)$ reduces to a relaxation length in Minkowski space). In Appendix B we present an alternative model using a more general AW type relaxation function $\gamma = -u_a p^a / \tau(r)$.

Using the $3+1$ decomposition of (1.10) the projection of the 4-momentum onto the unit radial vector c_a yields

$$c_a p^a = \lambda c_a e^a \equiv p^1. \tag{7.11}$$

Using (7.11) and (7.7) the relaxation function becomes

$$\gamma = \delta^{-1}(r)\left[e^{-\nu}\widetilde{E}^2 - r^{-2}J^2 - m^2\right]^{1/2}. \tag{7.12}$$

It means also that the collision rate depends explicitly on the angular momentum. In addition, the form of γ yields a Γ that is dynamically isotropic as required. Using (7.9) we find

$$\Gamma = \int^r e^{\psi(r')/2}\,\delta^{-1}(r')dr' = \int^l \delta^{-1}dl \tag{7.13}$$

where $l =$ proper length. It is clear that δ reduces to a relaxation length in Minkowski spacetime ($\delta =$ constant).

It is also useful to note that the angular momentum (7.5) can be written in terms of the unit radial vector as

$$J^2 = \tfrac{2}{3}r^2\lambda^2 + r^2\lambda^2(\tfrac{1}{3}h_{ab} - c_a c_b)e^a e^b\ .$$

Using γ and Γ as defined by (7.12) and (7.13), we can rewrite the BGK solution (7.9) as

$$f(r,\widetilde{E},J) = \bar{F}(\widetilde{E}) + h(\widetilde{E},J)\exp[-\Gamma(r)]\ . \tag{7.14}$$

One of the advantages of having a dynamically isotropic Γ is that the covariant harmonic decomposition of the solution (7.14) takes a convenient form similar to the case for the FRW spacetime of Chapter 4. Note that the model in Appendix B leads to anisotropic Γ, which complicates the harmonic form of f.

In order to find exact solutions to the Einstein-BGK problem and to investigate kinematics and dynamics of the solutions, it is necessary to decompose $f(r,\widetilde{E},J)$ using the covariant harmonic decomposition as follows

$$f(r,\widetilde{E},J) = \bar{F}(\widetilde{E}) + e^{-\Gamma(r)}[H(\widetilde{E}) + H_a(\widetilde{E})e^a + H_{ab}(\widetilde{E})e^a e^b + H_{abc}(\widetilde{E})e^a e^b e^c + \ldots] \tag{7.15}$$

where the $H_{a\ldots b}$ are chosen so that $h(\widetilde{E}\,,J)$ is non-negative for all $\widetilde{E}$. Notice that as $\bar{F}$ is taken to be isotropic its decomposition only contains the zero order harmonic.

7.4 Conservation of Particles, Energy and Momentum

We have shown in Chapter 3 that the BGK solution does not guarantee the conservation of particles, energy and momentum. It is therefore necessary to impose these conditions on the solution (7.15) to ensure that it is physical.

The conservation of particles is investigated first. Using the solution (7.15) and (1.47a, b) the particle number density and number flux are:

$$N = \bar{N} + 4\pi e^{-\Gamma(r)} \int_m^\infty E\lambda H dE$$

$$j_a = \tfrac{4\pi}{3} e^{-\Gamma(r)} \int_m^\infty E\lambda^2 H_a dE. \tag{7.16a}$$

Although we already know the condition resulting from the conservation of particles from (3.5b), we present a more complete derivation for the SSS case because a new collision function is introduced. We know from (1.29) and (1.44) that

$$\begin{aligned} n^a{}_{;a} &= \int\int c_c p^c \delta^{-1}(r)(\bar{f} - f)\lambda dE d\Omega \\ &= -c_c \delta^{-1}(r) \int\int p^c e^{-\Gamma(r)}[H(\widetilde{E}) + H_a(\widetilde{E})e^a + H_{ab}(\widetilde{E})e^a e^b + \ldots]\lambda dE d\Omega \\ &= -c_c \delta^{-1}(r) e^{-\Gamma(r)} \tfrac{4\pi}{3} \int_m^\infty H^c(\widetilde{E})\lambda^2 dE \\ &= 0 \end{aligned} \tag{7.16b}$$

where we used $c_a u^a = 0$. In general, this condition requires

$$\int_m^\infty c_a H^a(\widetilde{E})\lambda^2 dE = 0 \tag{7.16c}$$

of which $H^a = 0$ is a special case. By (7.16a), (7.16c) gives $c_a j^a = 0$, which in turn gives $j^a = 0$ by SSS. This condition implies a non-tilted kinematic average 4-velocity. Unlike the result (3.5b), the conservation of particles places a condition on the first order anisotropy of the distribution (justifying the more complete treatment followed here).

The condition for the conservation of particles (7.16c) for a SSS relaxation length model requires a distribution with a non-tilted kinematic average 4-velocity and a zero number flux.

Again, as we have introduced a new form of the collision function, we provide more complete derivations for the conditions resulting from the conservation of energy and momentum (instead of using the general results (3.5c, d)). Using the condition (3.5a) for the conservation of energy-momentum and the SSS solution (7.15) we get

$$
\begin{aligned}
T^{ab}{}_{;b} &= -\int p^a \gamma (f - \bar{f}) d\mathcal{P} \\
&= -c_c \delta^{-1}(r) e^{-\Gamma(r)} \int\int p^a e^c [H(\tilde{E}) + H_a(\tilde{E}) e^a + H_{ab}(\tilde{E}) e^a e^b + \ldots] \lambda^2 dE d\Omega \\
&= 0 \qquad (7.17)
\end{aligned}
$$

By projecting $T^{ab}{}_{;b}$ onto the preferred 4-velocity u^a, the condition for the conservation of energy can now be written using $u_a p^a = -E$:

$$
\begin{aligned}
u_a T^{ab}{}_{;b} &= c_c \delta^{-1}(r) e^{-\Gamma(r)} \int\int E e^c [H(\tilde{E}) + H_a(\tilde{E}) e^a + \ldots] \lambda^2 dE d\Omega \\
&= c_c \delta^{-1}(r) e^{-\Gamma(r)} \frac{4\pi}{3} \int_m^\infty E H^c(\tilde{E}) \lambda^2 dE \\
&= 0 \qquad (7.18a)
\end{aligned}
$$

The condition for the conservation of energy requires

$$\int_m^\infty c_a H^a(\tilde{E}) \lambda^2 E dE = 0 \qquad (7.18b)$$

with $H^a(\tilde{E}) = 0$ as a special case. In fact, both the conditions for conservation of particles and energy can be satisfied if we have $c_a H^a(\tilde{E}) = 0$ (or $H^a(\tilde{E}) = 0$). Unlike the result (3.5c), the conservation of energy restricts the first order anisotropy of the distribution.

By projecting $T^{ab}{}_{;b}$ onto the 3-surface orthogonal to the preferred 4-velocity using the projection tensor $h_a{}^d$, a condition for the conservation of momentum can be obtained:

$$
\begin{aligned}
h_a{}^d T^{ab}{}_{;b} &= -c_c\delta^{-1}(r)e^{-\Gamma(r)}\int\int e^d e^c[H(\widetilde{E})+H_a(\widetilde{E})e^a+\ldots]\lambda^3 dE d\Omega \\
&= -c_c\delta^{-1}(r)e^{-\Gamma(r)}\frac{4\pi}{3}\int_m^\infty h^{dc}H(\widetilde{E})\lambda^3 dE \\
&= -c^d\delta^{-1}(r)e^{-\Gamma(r)}\frac{4\pi}{3}\int_m^\infty H(\widetilde{E})\lambda^3 dE \\
&= 0
\end{aligned}
\tag{7.19a}
$$

The condition for the conservation of momentum requires

$$
\int_m^\infty H(\widetilde{E})\lambda^3 dE = 0 \tag{7.19b}
$$

with $H(\widetilde{E})=0$ *as a special case.* Unlike the general result (3.5d), conservation of momentum restricts the zero order harmonic of h. An alternative approach to follow when investigating the conditions for conservation of energy-momentum is to use

$$
c_a(T^{ab}-\bar{T}^{ab})=0\ . \tag{7.20}
$$

This relation can easily be obtained from (7.17). We next investigate the conditions imposed by the Einstein field equations.

7.5 Einstein-BGK Solution

In order to obtain an Einstein solution it is necessary for the energy-momentum tensor to satisfy the Einstein field equations $G^{ab}=T^{ab}$. The energy-momentum tensor in SSS geometry with BGK solution (7.14) is given by

$$
T^{ab} = \int p^a p^b \frac{1}{p^0}\left[\bar{F}(\widetilde{E})+h(\widetilde{E},J)\exp[-\Gamma(r)]\right]dp^{123}\ . \tag{7.21}
$$

The Einstein tensor G^{ab} is given by (2.26) in the orthonormal tetrad (2.25) as

$$
G^{ab}=\mathrm{diag}(G^{00},G^{11},G^{22},G^{33}) \tag{7.22}
$$

with $G^{00},G^{11},G^{22}=G^{33}$ determined by (2.1).

To find the conditions imposed by the Einstein field equations we use the decomposition (and resulting average dynamic quantities) of T^{ab} with respect to u^a (as given by (1.12)). The heat flow is investigated first. Note that as for the conservation of energy-momentum we provide a more complete derivation in order to ensure that the collision functional is properly considered. The heat flow is given by

$$
\begin{aligned}
q^t &= -h^t{}_r T^{rs} u_s \\
&= \int\int E\left[\bar{F}(\tilde{E}) + e^{-\Gamma(r)}[H(\tilde{E}) + H_a(\tilde{E})e^a + \ldots]\right]\lambda^2 e^t dE d\Omega \\
&= \tfrac{4\pi}{3}\int_m^\infty E H^t(\tilde{E})\lambda^2 dE \, . \qquad (7.23a)
\end{aligned}
$$

Using (7.23a) and the condition for the conservation of energy (7.18b), we find $c_a q^a = 0$. Since q^a is SSS this condition gives $q^a = 0$. *The condition for the conservation of energy for a SSS relaxation length solution requires that the heat flow vanishes.*

Note that the same conclusion can be obtained using (7.20); i.e., using $c_a u^a = 0$, $\bar{T}^{ab} c_b = \bar{p}\, c^a$ (as $\bar{f} = \bar{F}(\tilde{E})$), and (1.12) it is easy to show that (7.20) implies

$$
q_a = 0 \, . \qquad (7.23b)
$$

We continue to investigate the dynamics of the solution by calculating the energy density:

$$
\begin{aligned}
\mu &= T^{rs} u_r u_s \\
&= \int\int E^2\left[\bar{F}(\tilde{E}) + e^{-\Gamma(r)}[H(\tilde{E}) + H_a(\tilde{E})e^a + \ldots]\right]\lambda dE d\Omega, \\
&= \tfrac{4\pi}{3}\int_m^\infty E^2\left[\bar{F}(\tilde{E}) + e^{-\Gamma(r)} H(\tilde{E})\right]\lambda dE \\
&= \bar{\mu} + \tfrac{4\pi}{3}\, e^{-\Gamma(r)}\int_m^\infty E^2\, H(\tilde{E})\, \lambda dE \, . \qquad (7.24a)
\end{aligned}
$$

To obtain an equilibrium form of the energy density it is necessary that

$$
\int_m^\infty E^2\, H(\tilde{E})\, \lambda dE = 0 \, . \qquad (7.24b)
$$

This condition along with the condition for the conservation of momentum (7.19*b*) can be satisfied by taking $\bar{H} = 0$.

We continue our investigation of the dynamics of the solution by calculating the isotropic pressure:

$$p = \tfrac{1}{3}\, h_{rs} T^{rs}$$

$$= \tfrac{4\pi}{3} \int_m^\infty \lambda^3 \left[\bar{F}(\tilde{E}) + e^{-\Gamma(r)} H(\tilde{E})\right] dE. \tag{7.25}$$

From (7.25) it is clear that the bulk viscosity is given by

$$\Pi = p - \bar{p}$$

$$= \tfrac{4\pi}{3}\, e^{-\Gamma(r)} \int_m^\infty \lambda^3 H(\tilde{E})\; dE \tag{7.26}$$

and that taking $\bar{H} = 0$ (as suggested by (7.19*b*) and (7.24*b*)) yields zero bulk viscosity.

Instead of using the relation

$$\pi^{ab} = h^a{}_r h^b{}_s T^{rs} - \tfrac{1}{3} h_{rs} T^{rs} h^{ab}$$

to investigate the anisotropic pressure, we use (7.20) and (7.23*b*) to find

$$\pi_{ab} c^b = (\bar{p} - p) c_a \,. \tag{7.27}$$

Furthermore, SSS implies that the anisotropic pressure must take the form [Maharaj (1986, p74)]:

$$\pi_{ab} = Q(r)(h_{ab} - 3c_a c_b) \tag{7.28}$$

where Q is some function of r. Now, (7.27) and (7.28) give:

$$\pi_{ab} = \tfrac{1}{2}\Pi(h_{ab} - 3c_a c_b) \,. \tag{7.29}$$

From (7.29) it can be concluded that *in the SSS model under consideration, the anisotropic pressure comes from the bulk viscosity.* Zero bulk viscosity ($\bar{H} = 0$) implies zero anisotropic pressure. Note that taking $\bar{H}=0$ ensures that momentum conservation is satisfied, the energy density has equilibrium form, the bulk viscosity is zero, and that the anisotropic pressure vanishes.

Thus, in an orthonormal basis (2.25) and with $\pi_a{}^a = 0$, (7.29) implies

$$\pi_{ab} = \mathrm{diag}(0, -\Pi, \tfrac{1}{2}\Pi, \tfrac{1}{2}\Pi) \ . \tag{7.30}$$

Comparing to (1.47f), (7.30) shows that one may take

$$F_{ab} = \mathrm{diag}(0, F_{11}, -\tfrac{1}{2}F_{11}, -\tfrac{1}{2}F_{11}) \ . \tag{7.31}$$

Using (1.47f) and (1.47g) it is easy to show that (7.30) also leads to the condition

$$\int_m^\infty \lambda^3 \left[5(F-\bar{F}) + 2F_{11}\right] dE = 0 \ . \tag{7.32}$$

In terms of the SSS solution (7.15) the conditions (7.32) and (7.31) can be satisfied if

$$H_{ab} = \mathrm{diag}(0, -\tfrac{5}{2}H, \tfrac{5}{4}H, \tfrac{5}{4}H) \ . \tag{7.33}$$

The dynamic quantities discussed above allows us to write the energy-momentum tensor as follows:

$$T_{ab} = \mu u_a u_b + p h_{ab} + \tfrac{1}{2}\Pi(h_{ab} - 3c_a c_b) \tag{7.34}$$

with $T_{22} = T_{33}$ and $T_{ab} = 0$ for $a \neq b$.

Given the energy-momentum tensor, we can now investigate the Einstein-BGK solution by comparing (7.34) to the Einstein tensor (7.22), (2.26). Using $G^{ab} = T^{ab}$ we get:

$$G^{00} = \mu$$

$$G^{11} = p - \Pi \quad (= \bar{p})$$

$$G^{22} = G^{33} = p + \tfrac{1}{2}\Pi \ .$$

The equations for G^{00} and G^{11} contains five variables ν, ψ, μ, p, Π. By specifying

$\bar{F}(\widetilde{E})$, $\bar{H}(\widetilde{E})$, and $\delta(r)$ the variables p, μ, and Π can be written in terms of ν and ψ. Thus, in principle the equations for G^{00} and G^{11} (7.22) determine ν and ψ, and as a result the spacetime metric (2.23) for the SSS relaxation length model is known.

By specifying the isotropic part of the distribution function as well as the relaxation length, one can in principle determine the spacetime metric for the SSS relaxation model.

7.6 Application of the Spherically Symmetric Solution

In this section we briefly discuss three possible applications of the static spherically symmetric solution: the spherically symmetric accretion of a gas around a star, a spherically symmetric star cluster, and the formation of stars out of large clouds of gas particles. A critical component of the solution is that it is static. We are therefore limited to those physical situations that can approximately be described as static. In other words, the model requires that the number density in phase space describing the gas or cluster and the gravitational field of the system must be independent of time [Ipser and Thorne (1968), and Misner, Thorne, and Wheeler (1973, p680)]. For every particle that leaves the bound system another must enter. Unlike the Liouville solutions, which require the further idealisation that the collisions or interactions between the particles can be ignored, our model can be applied to non-equilibrium cases where collisions are significant, such as the example of the non-equilibrium situation described by Podurets (1970). In fact, the interactions between the particles is the mechanism responsible for the relaxation or thermalisation in the distribution function with changing radius. Examples of the application of the collisionless case or Vlasoff equations to spherical star clusters are presented in Misner et al. (1973, p679), while the Liouville equation is also applied to spherically symmetric accretion in Saslaw (1985, p121). The relaxation process could lead to an equilibrium distribution function taking the Boltzmann form [Misner et al. (1973, p685)], so that $\bar{F}$ can be taken to be the isotropic 'thermalised' Boltzmann distribution. However, the Boltzmann distribution assumes that an equal number of particles will escape the system due to high particle energy $E > E_{max}$ as the number of particles (with the same energies) entering the system. As such a system is not very realistic it may be necessary to use a truncated form of the Boltzmann distribution as described in Misner et al. (1973, p685) and the references therein.

One further idealisation to consider is the assumption that all the stars or gas particles have the same mass. This is, however, not a serious limitation as the solution could be extended to the more general case without changing the basic results. So far no distinction has been made between gas particles, atoms or molecules, or stars. Care must however be taken if stability calculations are performed, as the results for gas spheres cannot directly be applied to star clusters [Ipser and Thorne (1968), Kandrup and Morrison (1993)].

The first application that we discuss is the spherically symmetric accretion of a gas around a star. A number of authors have examined this form of accretion using the collisionless Liouville or Vlasoff equations - see Saslaw (1985) for example. Demiański (1985) describes the spherically symmetric accretion of matter into neutron stars or black holes. Neutron stars are formed during super-novae explosions where part of the stellar mass is blown into the space surrounding the star, while the central part contracts to form a neutron star or black hole. This explosión and contraction process results in a spherically symmetric situation where a dense central core is surrounded by a cloud of gas. Depending on the density of the gas the interactions between the particles may be significant and a treatment allowing for collisions may become important. Far from the centre the particles experience disordered thermal motion. Beyond a certain critical radius r_c the thermal energy of the particles exceed their potential energy and the escape velocity of the particles is the same as the thermal velocity. Inside r_c the particles are falling towards the centre. During this fall the particles then collide or interact with each other with greater frequency resulting from denser particle distributions. The interactions between the particles then result in relaxation in the velocity distribution of the particles, with particles closer to the centre having undergone more collisions and therefore a greater degree of relaxation. Furthermore, as particles fall into the centre from r_c other particles can freely enter from beyond r_c due to the random thermal motion and a potentially static situation can result. Under such circumstances it should be possible to find a relationship between the accretion rate and the relaxation or collision rate.

Further interesting cases can result where accretion of matter into a black hole or neutron star in a binary system occurs. In such cases the distribution of particles may take the form of a rotating disk. A distribution function that depends on the orientation of the particle's orbital plane (i.e. the magnitude and direction of the angular momentum) could be used [Misner et al. (1973, p681)]. It would then be necessary to specialise the SSS solution which depends only on the magnitude of the angular momentum.

The second example that we present is the SSS star cluster. Collision free examples of SSS star clusters have been described in a number of publications (see Misner et al. (1973, p679) and the references therein). Also, the interactions between stars would have the effect of relaxing or thermalising the distribution function [Misner et al. (1973, p685), Podurets (1970), Saslaw (1985)]. This implies that non-equilibrium situations can exist and that the collisionless treatment may not be sufficient. Different interactions between stars are possible. Saslaw (1985) describes gentle relaxation situations where the stars in a cluster are far enough apart to act as point masses and thus alter each others orbits through deflections or scatterings. Podurets (1970) on the other hand describes situations during the late evolutionary stage of stellar systems where inelastic two-body head-on collisions become significant. Different possibilities also exist for obtaining a static situation. As discussed above, Misner et al. (1973) describe the situation in which a star cluster relaxes toward a truncated Boltzmann distribution where stars with energies higher than a critical value are able to escape the system, while they are replaced with equal numbers of stars with the same energies from $r = \infty$. On the other hand the situation as described by Podurets (1970) is perhaps more applicable. During the late stage of stellar cluster evolution the situation exists where the number density of the stars is high and inelastic two-body head-on collisions are frequent. Under such circumstances the mean time between collisions is much shorter than the relaxation time of the system as a whole. The conditions for an equilibrium distribution function are no longer satisfied. However, the conditions for a quasistatic distribution function is satisfied almost throughout the system because the characteristic hydrodynamic time is considerably shorter than the mean time between collisions. Podurets presents a Kinetic theory approach to this situation. The possible application of the BGK-solution as described in this chapter could yield interesting results as the model here represents the physical situation closely.

The third possible application that we briefly mention is the non-equilibrium processes during the early life of stars. Stars condense under their own gravity from large molecular hydrogen cloud complexes. These isolated condensations are known as dense cores. A gradually outward moving region of collapse then develops around the dense cores. A protostar begins to form from the collisions of the gas. The object formed then behaves much like an ordinary star surrounded by an incoming accretion flow. This process then continues until the temperature in the protostar is high enough such that first, fusion of deuterium and later, fusion of hydrogen occurs. During this accretion process a quasistatic non-equilibrium situation may develop to which a BGK type of treatment may be applied. However, it must be pointed out that the processes occurring during the early life of stars are very complicated and still not well understood.

7.7 Summary

In this chapter we presented a non-equilibrium static spherically symmetric BGK solution. The solution is dynamically anisotropic and relaxes toward the equilibrium with changing radius and is given by (7.14). In order for the solution to be physical it is necessary to impose the conditions for the conservation of particles, energy and momentum. By performing a covariant harmonic decomposition of the distribution function it is possible to determine these constraints on the solution. The conservation of particles imposes the condition (7.16c), which implies (along with SSS) that the particle number flux vanishes and the kinematic average 4-velocity is non-tilted. The condition for the conservation of energy requires (7.18b), which along with the condition for the conservation of particles can be satisfied by taking $F_a = 0$. The condition for the conservation of momentum leads to (7.19b).

If the solution is applied to self gravitating situations, then it must also satisfy the Einstein field equations. Using the decomposed form of the energy-momentum tensor, we showed that the condition for the conservation of energy guarantees a zero heat flow and that anisotropic pressure comes from the bulk viscosity. Comparing the dynamic quantities to the Einstein field equations, we showed that the SSS metric may be determined in principle by specifying the isotropic part of the distribution function as well as the relaxation length component of the relaxation function.

Finally, we presented possible applications of the SSS BGK solution. The three cases briefly discussed are; the SS accretion around black holes; the early life of stars; and SS star clusters with particular emphasis on the conditions persisting during the late evolutionary stages of stellar clusters.

8.0 CONCLUDING REMARKS

8.1 General

We have achieved our objective of finding (and investigating the properties of) exact Einstein-Boltzmann solutions with a relaxation model of the collision term. We started with an overview of relativistic kinetic theory and a summary of the known exact Liouville solutions of the Einstein-Boltzmann equations. This was followed by a presentation of the solution of the general BGK problem and a discussion of its properties. The BGK solution and the known Liouville solutions were used to construct exact Einstein solutions in FRW, Bianchi I, and SSS spacetimes. For the FRW geometry, both anisotropic and isotropic solutions were obtained. The Bianchi I and FRW models allowed us to construct a non-equilibrium model for the evolution of anisotropy in FRW geometry.

In addition to the review of kinetic theory and the useful techniques (such as the covariant harmonic decomposition, the $3+1$ decomposition, and orthonormal tetrad approach) presented in Chapter 1, we have also given equations relating dynamic and kinematic quantities as measured by different observers u_a and $\tilde{u}_a$. These quantities appear in the decomposition of the energy-momentum tensor and the 4-current vector with respect to a chosen 4-velocity. Further contributions were the harmonic forms of the H-theorem, entropy density, and entropy flux.

8.2 BGK Collision Term

A comprehensive analysis of the relaxation-time model for collisions was presented. We provided an overview of the BGK relaxation function and introduced the Anderson-Witting form of the relaxation time. The solution of the BGK relaxation equation was obtained in general as well as in harmonic form. This established the foundation of the thesis. The general conditions for the conservation of energy-momentum and of particles were derived for the BGK solution. We also found the matching conditions imposed by the conservation equations for linear relaxation terms. One of these conditions allowed us to define a new local rest frame, which is neither Eckart nor Landau-Lifshitz. This analysis is complementary to the work in approximations of Anderson and Witting (1974).

Entropy production results were derived in general as well as for a first order truncated distribution function (with dipole anisotropy only), and the conditions were given for the H-theorem to be satisfied. Using the AW model we found that, to the lowest order, anisotropy in the distribution tends to increase the entropy production rate. Also, the isotropisation of the distribution function leads to a reduction in the entropy production rate. For the first order truncated AW model, we were able to derive closed form expressions for the entropy density and the entropy production rate which showed that the entropy production rate satisfies the H-theorem, isotropisation leads to increased entropy density, and isotropisation also decreases the entropy production rate.

Furthermore, an exact second order truncated solution (quadrupole anisotropy) with AW collision model was used to derive the harmonics of the Boltzmann equation. The resultant set of exact harmonic equations is particularly important as they allow comparisons with the results of other approximation methods, and are not restricted to near equilibrium conditions. The matching conditions imposed by the conservation equations on this model were determined. The harmonic equations were also integrated to obtain equations in the moments of the distribution, some of which take a form similar to the Israel-Stewart thermodynamics laws. The consistency conditions following from the truncation of the distribution were found and include shear free flow and a condition on the anisotropic stress. The consequences of these results were investigated for an exact truncated solution on a FRW $(k = 0)$ background. We found that no integrability conditions are imposed on the solution, and that the bulk viscosity vanishes for all $m \geq 0$.

The Einstein-Boltzmann problem was presented for the BGK relaxation function. The conditions placed on the distribution function by the field equations and the integrability conditions were derived in harmonic form using the decomposed Einstein field equations.

It is exactly the AW relaxation function that allowed us later in the thesis to find a tilted solution with imperfect fluid energy-momentum tensor in a FRW geometry, made the isotropic, non-equilibrium FRW solution possible, allowed us to find a non-equilibrium model for the evolution of shear, and guided us in finding SSS solutions.

8.3 Einstein-BGK Solutions

In the remainder of the thesis we obtained exact Einstein-BGK solutions in FRW, Bianchi I, and SSS geometries. We accomplished this by specifying the geometry (i.e. the form of the spacetime metric in terms of unknown metric functions) and then solving the BGK equation for the distribution function. The restrictions placed on the distribution function by the Einstein field equations and the conservation equations were then determined. We found that these conditions in general lead to a degree of arbitrariness in the distribution function because they are applied to an integrated average of the distribution function. In each case we showed that once the arbitrariness is specified (analogous to specifying an equation of state in a fluid model), the unknown metric functions could be obtained in principle, yielding the Einstein solution. For each case, we also determined the kinematics and dynamics of the solution, and investigated their properties.

For the FRW geometry we presented an isotropic as well as an anisotropic solution. We found the anisotropic solution using an arbitrary isotropic relaxation function and for the AW relaxation function. For the arbitrary isotropic relaxation function we showed that the conditions for the conservation of particles and energy lead to vanishing bulk viscosity and an equilibrium energy-momentum tensor, while the conditions for vanishing particle drift and conservation of momentum require vanishing dipole anisotropy in the distribution function. The condition for vanishing energy flux (resulting from the field equations) also requires vanishing dipole anisotropy. Similarly, we found that the condition for zero anisotropic stress requires vanishing quadrupole anisotropy. We showed that appropriately restricting the relaxation mechanism yields a solution with non-zero bulk viscosity for a massless gas and that no such solution exists for a massive gas. The AW relaxation model allowed us to obtain tilted solutions with imperfect fluid energy-momentum tensors.

Our investigation of the anisotropic FRW solution also showed that an arbitrary isotropic relaxation model as well as the conditions for the conservation of particle number and energy, do not permit isotropic non-equilibrium solutions for a massive gas. We used this conclusion to find a non-equilibrium isotropic solution with an AW relaxation model for a massless gas. The solution was found to have vanishing bulk viscosity, number flux, energy flux and anisotropic stress. The solution also has an equilibrium energy density. This showed that the non-equilibrium fluxes and pressures are not sufficient to characterise a non-equilibrium state. For this solution, we were able to find an explicit expression for the entropy production rate which satisfies the H-theorem (even for a massless gas).

We also presented a formal anisotropic BGK solution in Bianchi I geometry. We found that this solution with an AW relaxation model gives a solution with equilibrium particle number density, energy density, and energy flux. In addition to determining the restrictions resulting from the field and conservation equations, a particular solution that satisfies all conditions was presented. The Bianchi I geometry leads to non-zero shear and as a result the harmonic functions are no longer independent. The first two harmonic equations, illustrating this dependence, were presented. The solution with an AW collision model, along with the anisotropic AW FRW solution, was used to demonstrate a situation in which the anisotropy of the distribution function is communicated to the spacetime geometry forcing it away from isotropy. We showed that it is possible to find a non-equilibrium anisotropic distribution function (with $m = 0$) that would satisfy the conservation and field equations in FRW spacetimes but has Bianchi I symmetry and that if non-zero anisotropic stress is generated (and thus non-zero shear), the solution would be forced to become Bianchi I. An example of a physical mechanism that would generate non-zero anisotropic stress was presented. In this model the particles initially have effective zero rest mass, but as the universe cools and the threshold energy is reached, the communication between the matter distribution and the geometry is triggered and the spacetime is forced to evolve away from isotropy.

We also obtained an anisotropic, non-equilibrium SSS solution with relaxation of the distribution with changing radius. The concept of 'relaxation length' as opposed to 'relaxation time' was introduced for the static model. We found that the condition for the conservation of particles leads to a non-tilted solution, and along with the condition for the conservation of energy, can be satisfied by vanishing dipole anisotropy. An integral condition for the conservation of momentum was also derived. We showed that the field equations (along with the conservation of energy) require vanishing heat flow and that anisotropic pressure comes from the bulk viscosity.

For all of the above geometries, we also suggested physical situations to which the particular models may be applied. The FRW models and the model for evolution of shear may be useful for descriptions of the early universe and 'standard' model of universe evolution, whereas the SSS solution may find application to star clusters, spherically symmetric accretion, or the early life of stars.

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APPENDIX A

Conservation of Energy-Momentum in FRW BGK Model

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Conservation of Energy-Momentum in FRW BGK Model

As confirmation of the results of Chapters 3 and 4, we now obtain the restrictions placed on the anisotropic FRW BGK solution (4.7) by the conditions for the conservation of energy and momentum. We follow an alternative approach and start directly from the basic condition (1.33), instead of using the results of Chapter 3 (3.5$b-d$). Using the general expression for the 4-momentum production rate (1.30) and the solution (4.7) we obtain:

$$T^{ab}{}_{;b} = -\int \gamma(t,P)p^a e^{-\Gamma(t,P)}[H + H_c e^c + \ldots]P^2 R^{-2}(t)[P^2 + m^2R^2]^{-1/2}dP d\Omega$$

$$= 0. \qquad (A.1)$$

In order to obtain an expression for the conservation of energy, $T^{ab}{}_{;b}$ is projected onto the preferred 4-velocity

$$u_a T^{ab}{}_{;b} = -\int \gamma(t,P)e^{-\Gamma(t,P)}[H + H_c e^c + \ldots]u_a p^a P^2 R^{-2}(t)[P^2 + m^2R^2]^{-1/2}dP d\Omega$$

$$= R^{-3}(t)\int \gamma(t,P)e^{-\Gamma(t,P)}[H + H_c e^c + \ldots]\, P^2 dP d\Omega$$

$$= 0,$$

where the relation $u_a p^a = -E = -R^{-1}(t)[P^2 + m^2R^2(t)]^{1/2}$ is used.

Using (1.19) all the odd terms (except for the zero order term) in this expression for are zero. By using $H_{a\ldots cd}h^{cd} = 0$ and (1.19) it is evident that all the even terms are zero. The only term that remains is the zero order term

$$u_a T^{ab}{}_{;b} = \tfrac{4\pi}{R^3(t)}\int_0^\infty \gamma(t,P)e^{-\Gamma(t,P)}H(P)P^2 dP$$

$$= 0\ .$$

The condition for the conservation of energy is thus given by

$$\int_0^\infty \gamma(t,P)e^{-\Gamma(t,P)}H(P)P^2 dP = 0\ . \qquad (A.2)$$

Note that $(A.2)$ confirms the result (4.13).

Next, we investigate the conservation of momentum condition. This time $T^{ab}{}_{;b}$ $(A.1)$ is projected onto the rest space orthogonal to the 4-velocity using the projection tensor $h_a{}^c$ to obtain

$$\begin{aligned} h_a{}^c T^{ab}{}_{;b} &= -\int \gamma(t,P) e^{-\Gamma(t,P)} [H + H_d e^d + \ldots](h_a{}^c p^a) P^2 R^{-2}(t) [P^2 + m^2 R^2]^{-1/2} dP d\Omega \\ &= -\int \gamma(t,P) e^{-\Gamma(t,P)} [H + H_d e^d + \ldots] e^c P^3 R^{-3}(t) [P^2 + m^2 R^2]^{-1/2} dP d\Omega \\ &= 0 \, , \end{aligned}$$

where $h_a{}^c p^a = \lambda e^c = P R^{-1} e^c$ is used. Using (1.19) all the odd terms in this expression are zero. By using $H_{a\ldots de} h^{de} = 0$ and (1.19) it is evident that all the even terms are zero except for the term containing $H_d e^d e^c$

$$\begin{aligned} h_a{}^c T^{ab}{}_{;b} &= -h^{bc} \frac{4\pi}{R^3(t)} \int_0^\infty \gamma(t,P) e^{-\Gamma(t,P)} H_d(P) P^3 [P^2 + m^2 R^2]^{-1/2} dP \\ &= 0 \, . \end{aligned}$$

The condition for the conservation of momentum is therefore given by

$$\int_0^\infty \gamma(t,P) e^{-\Gamma(t,P)} H_a(P) P^3 [P^2 + m^2 R^2]^{-1/2} dP = 0 \, . \qquad (A.3)$$

Using the expression (4.15c) this equation can be written as

$$\frac{\partial}{\partial t} \int_0^\infty e^{-\Gamma} P^3 H_a(P) \, dP = 0 \, .$$

Note that $(A.3)$ confirms (4.14).

APPENDIX B

Alternative Static Spherically Symmetric BGK Solution

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Alternative Static Spherically Symmetric BGK Solution

We now investigate the properties of an alternative SSS BGK solution. In Chapter 7 we obtained the SSS BGK solution $f(r,\widetilde{E},J)$ (see (7.9)) using a general SSS relaxation function $\gamma(r,\widetilde{E},J)$ (see (7.4)). In order to obtain exact Einstein solutions that satisfy the conditions for the conservation of particles, energy and momentum, a more explicit form of the relaxation function is required. In Chapter 7 we used the convenient form $\gamma = c_a p^a/\delta(r)$ which gave an isotropic Γ. For the alternative approach we use a relaxation function that is more general, i.e.,

$$\gamma = -u_a p^a/\tau(r)\ , \tag{B.1}$$

which is by (7.4) compatible with the SSS geometry. This form leads to an anisotropic Γ, which complicates the harmonic decomposition of the distribution function f:

$$f(r,\widetilde{E},J) = \bar{F}(\widetilde{E}) + h(\widetilde{E},J)\exp[-\Gamma(r,\widetilde{E},J)]$$

where

$$\Gamma(r,\widetilde{E},J) = \int^{r} e^{\psi(r')/2}\gamma(r',\widetilde{E})\left[e^{-\nu}\widetilde{E}^2 - r'^{-2}J^2 - m^2\right]^{-1/2} dr'. \tag{B.2}$$

Next, the covariant harmonic decomposition of f is obtained. We decompose $h(\widetilde{E},J)$ as follows

$$h(\widetilde{E},J) = \bar{H}(\widetilde{E}) + H_a(\widetilde{E})e^a + H_{ab}(\widetilde{E})e^a e^b + H_{abc}(\widetilde{E})e^a e^b e^c + \ldots \tag{B.3}$$

where the $H_{a\ldots b}$ are chosen so that $h(\widetilde{E},J)$ is non-negative for all $\widetilde{E}$. Unlike the FRW case it is also necessary to decompose $e^{-\Gamma}$ as it is a function of the angular momentum J, as follows

$$\exp[-\Gamma(r,\widetilde{E},J)] = \bar{\Gamma}(r,\widetilde{E}) + \Gamma_a(r,\widetilde{E})e^a + \Gamma_{ab}(r,\widetilde{E})e^a e^b + \Gamma_{abc}(r,\widetilde{E})e^a e^b e^c + \ldots \tag{B.4}$$

The SSS BGK solution can therefore be written as

$$f(r,\widetilde{E},J) = \bar{F}(\widetilde{E}) + [\bar{H}(\widetilde{E}) + H_a(\widetilde{E})e^a + \ldots][\bar{\Gamma}(r,\widetilde{E}) + \Gamma_a(r,\widetilde{E})e^a + \ldots]. \tag{B.5}$$

Notice that as $\bar{F}$ is taken to be isotropic its decomposition only contains the zero order harmonic. In order to investigate the properties of the solution, it is necessary to expand the product of the two sums in $(B.5)$:

$$
\begin{aligned}
e^{-\Gamma}h = \bar{H}\bar{\Gamma} + \bar{H}\Gamma_k e^k + \bar{H}\Gamma_{kl}e^k e^l + \bar{H}\Gamma_{klm}e^k e^l e^m + \\
H_a\bar{\Gamma}e^a + H_a\Gamma_k e^a e^k + H_a\Gamma_{kl}e^a e^k e^l + \\
H_{ab}\bar{\Gamma}e^a e^b + H_{ab}\Gamma_k e^a e^b e^k + \\
H_{abc}\bar{\Gamma}e^a e^b e^c + \ldots \\
= \bar{H}\bar{\Gamma} + (\bar{H}\Gamma_a + H_a\bar{\Gamma})e^a + (\bar{H}\Gamma_{ab} + H_a\Gamma_b + H_{ab}\bar{\Gamma})e^a e^b + \ldots \\
\equiv K + K_a e^a + K_{ab}e^{ab} + K_{abc}e^{abc} + \ldots
\end{aligned}
\tag{B.6}
$$

Notice that the $K_{a\ldots b}$ as defined here are not trace-free. Next, we investigate the properties of the solution. Because of the more complex decomposition and the new relaxation function, we derive the properties instead of using the results from Chapter 3.

We determine the conditions imposed by conservation of particles, energy and momentum first. Using condition (3.6) for the conservation of energy-momentum, $(B.5)$, and $(B.6)$ we get

$$
\begin{aligned}
T^{ab}{}_{;b} &= -\int p^a\gamma(f-\bar{f})d\mathcal{P} \\
&= -\int\int p^a\,\frac{E}{\tau(r)}\,[K + K_a e^a + K_{ab}e^{ab} + K_{abc}e^{abc} + \ldots]\lambda dE d\Omega \\
&= 0
\end{aligned}
\tag{B.7}
$$

By projecting $T^{ab}{}_{;b}$ onto the preferred 4-velocity u^a and using $u_a p^a = -E$, the condition for the conservation of energy can now be written down:

$$
u_a T^{ab}{}_{;b} = \int\int \frac{E^2}{\tau(r)}[K + K_a e^a + K_{ab}e^{ab} + K_{abc}e^{abc} + \ldots]\lambda dE d\Omega = 0. \tag{B.8}
$$

Using (1.19) and $(B.6)$ it is easy to show that all terms odd in e^a are zero, while (1.19) and the relations $H_{a\ldots bc}h^{bc} = 0$ and $\Gamma_{a\ldots bc}h^{bc} = 0$ ensure that all terms even in e^a are zero except for the terms $\bar{H}\bar{\Gamma} + H_a\Gamma_k e^a e^k$. The condition for the conservation of energy can now be written as

$$\begin{aligned}
u_a T^{ab}{}_{;b} &= \int\int \frac{E^2}{\tau(r)}\left[\bar{H}\bar{\Gamma} + H_a\Gamma_k e^a e^k\right]\lambda dE d\Omega \\
&= \frac{4\pi}{3}\int \frac{E^2}{\tau(r)}\left[3\bar{H}\bar{\Gamma} + H_a\Gamma_k h^{ak}\right]\lambda dE \\
&= 0. \qquad (B.9)
\end{aligned}$$

In order to satisfy the conservation of energy condition, we therefore have that

$$\int E^2\left[3\bar{H}\bar{\Gamma} + H_a\Gamma_k h^{ak}\right]\lambda dE = 0. \qquad (B.10a)$$

Now, as $\Gamma_{a\ldots b} = \Gamma_{a\ldots b}(r, E)$, non-zero terms $3\bar{H}\bar{\Gamma} + H_a\Gamma^a$ can be found satisfying $(B.10a)$ at any one radius r_0. However, at a different radius $\bar{\Gamma}$, Γ^a will be different and these equations will no longer be satisfied. As r varies on an open interval, $(B.10a)$ can only be satisfied if the integrand vanishes identically; that is when

$$3\bar{H}\bar{\Gamma} + H_a\Gamma^a = 0. \qquad (B.10b)$$

By projecting $T^{ab}{}_{;b}$ onto the 3-surface orthogonal to the preferred 4-velocity using the projection tensor $h_a{}^d$, a condition for the conservation of momentum can be obtained

$$\begin{aligned}
h_a{}^d T^{ab}{}_{;b} &= -\int\int \frac{E}{\tau(r)}\, h_a{}^d p^a[K + K_a e^a + K_{ab}e^{ab} + K_{abc}e^{abc} + \ldots]\lambda dE d\Omega \\
&= -\int\int \frac{\tilde{E}}{\tau(r)}\,\lambda^2[K + K_a e^a + K_{ab}e^{ab} + K_{abc}e^{abc} + \ldots]e^d dE d\Omega \\
&= 0, \qquad (B.11)
\end{aligned}$$

where $h_a{}^d p^a = \lambda e^d$. Again, by using (1.19) and $(B.6)$, the terms containing odd numbers of e^a are zero, while (1.19), $H_{a\ldots bc}h^{bc} = 0$ and $\Gamma_{a\ldots bc}h^{bc} = 0$ ensure that all terms even in e^a are zero except for the terms $\bar{H}\Gamma_k e^k$ and $H_a\bar{\Gamma}e^a$. The condition for the conservation of momentum can now be written as

$$h_a{}^d T^{ab}{}_{;b} = -\int\int \frac{E}{\tau(r)} \left[\bar{H}\Gamma_k e^k + H_a \bar{\Gamma} e^a\right] \lambda^2 e^d dE d\Omega$$

$$= -\tfrac{4\pi}{3} \int \frac{E}{\tau(r)} \left[\bar{H}\Gamma_k h^{kd} + H_a \bar{\Gamma} h^{ad}\right] \lambda^2 dE$$

$$= 0, \qquad (B.12)$$

which implies the general condition

$$\int E\left[\bar{H}\Gamma_k h^{kd} + H_a \bar{\Gamma} h^{ad}\right] \lambda^2 dE = 0. \qquad (B.13a)$$

Now, as $\Gamma_{a\ldots b} = \Gamma_{a\ldots b}(r, E)$, non-zero terms $\bar{H}\Gamma^d + H^d\bar{\Gamma}$ can be found satisfying $(B.13a)$ at any one radius r_0. However, at a different radius $\bar{\Gamma}$, Γ^a will be different and these equations will no longer be satisfied. As r varies on an open interval, $(B.13a)$ can only be satisfied if the integrand vanishes identically; that is, when

$$\bar{H}\Gamma^d + H^d\bar{\Gamma} = 0. \qquad (B.13b)$$

Next, we investigate the condition for the conservation of particle number. We know from (1.29) and (1.44) that

$$n^a{}_{;a} = \int\int \frac{E}{\tau(r)} (\bar{f} - f)\lambda dE d\Omega$$

$$= -\int\int \frac{E}{\tau(r)} [K + K_a e^a + K_{ab} e^{ab} + K_{abc} e^{abc} + \ldots]\lambda dE d\Omega$$

$$= 0 \qquad (B.14)$$

Using (1.19) and $(B.6)$ it is easy to show that all terms odd in e^a are zero, while (1.19) and the relations $H_{a\ldots bc}h^{bc} = 0$ and $\Gamma_{a\ldots bc}h^{bc} = 0$ ensure that all terms even in e^a are zero except for the terms $\bar{H}\bar{\Gamma}$ and $H_a\Gamma_k e^a e^k$. The condition for the conservation of particles can now be written as

$$n^a{}_{;a} = \int\int \frac{E}{\tau(r)} \left[\bar{H}\bar{\Gamma} + H_a \Gamma_k e^a e^k\right] \lambda dE d\Omega$$

$$= \tfrac{4\pi}{3} \int \frac{E}{\tau(r)} \left[3\bar{H}\bar{\Gamma} + H_a \Gamma_k h^{ak}\right] \lambda dE$$

$$= 0. \qquad (B.15)$$

In order to satisfy the conservation of particles condition, we therefore have that

$$\int E\left[3\bar{H}\bar{\Gamma}+H_a\Gamma_k h^{ak}\right]\lambda dE=0. \tag{B.16}$$

Comparing (B.16) with the condition imposed by the conservation of energy (B.10), we find that both the conditions for the conservation of particles and energy require that $3\bar{H}\bar{\Gamma}+H_a\Gamma^a=0$. The three conditions ($B$.10), ($B$.13) and ($B$.16) can be satisfied by taking for example $\bar{\Gamma}=\Gamma_k=0=\bar{H}=H_b$. Such a SSS BGK solution satisfies the conditions for the conservation of particles, energy and momentum.

The conditions for the conservation of particle number and energy require that

$$3\bar{H}\bar{\Gamma}+H_a\Gamma^a=0\ .$$

The condition for the conservation of momentum requires that

$$\bar{H}\Gamma^d+H^d\bar{\Gamma}=0\ .$$

We now investigate the dynamics and kinematics of the solution. To accomplish this the decomposition of T^{ab} with respect to u^a, as given by (1.12), is utilised. We investigate the average dynamic quantities of the gas, as given by this decomposition, individually. The heat flow is investigated first and is given by

$$\begin{aligned} q^t &= -h^t{}_r T^{rs}u_s \\ &= \int\int E\left[\bar{F}+K+K_a e^a+\ldots\right]\lambda^2 e^t dE d\Omega \\ &= \frac{4\pi}{3}\int E\left[\ \bar{H}\Gamma^t+H^t\bar{\Gamma}\right]\lambda^2 dE, \end{aligned} \tag{B.17}$$

where $h^t{}_r p^r=\lambda e^t$, $p^s u_s=-E$. By the condition for the conservation of momentum (B.13) it is clear that $q^t=0$.

We continue to investigate the dynamics of the solution by calculating the energy density:

$$\begin{aligned}\mu &= T^{rs}u_r u_s \\ &= \int\int E^2\left[\bar{F} + K + K_a e^a + \ldots\right]\lambda dE d\Omega \\ &= \frac{4\pi}{3}\int E^2\left[3\bar{F} + 3\bar{H}\bar{\Gamma} + H_a\Gamma_k h^{ak}\right]\lambda dE \\ &= 4\pi\int E^2\bar{F}\lambda dE, \end{aligned} \tag{B.18}$$

where $E = -p^r u_r$ and the condition for the conservation of energy $(B.10)$ is used. The expression for the energy density takes the equilibrium form.

Next, we find the isotropic pressure:

$$\begin{aligned} p &= \frac{1}{3}h_{rs}T^{rs} \\ &= \frac{1}{3}\int\int \lambda^3\left[\bar{F} + K + K_a e^a + \ldots\right] dE d\Omega \\ &= \frac{4\pi}{9}\int \lambda^3\left[3\bar{F} + 3\bar{H}\bar{\Gamma} + H_a\Gamma_k h^{ak}\right] dE, \end{aligned} \tag{B.19}$$

where the conditions for the conservation of energy and particles $(B.10b)$, $(B.16)$ can be used to simplify the expression to

$$p = \frac{4\pi}{3}\int \lambda^3\bar{F} dE. \tag{B.20}$$

In this case, the isotropic pressure takes the equilibrium form. As a result the distribution in general has vanishing bulk viscosity $\Pi = 0$.

To complete our investigation of the dynamics of the solution we calculate the trace-free stress tensor. This tensor is given by: *(next page)*

$$\pi^{ab} = h^a{}_r h^b{}_s T^{rs} - \tfrac{1}{3} h_{rs} T^{rs} h^{ab}$$

$$= \int\int \lambda^3 \left[\bar{F} + K + K_a e^a + \ldots\right] e^a e^b dE d\Omega -$$

$$\tfrac{1}{3}\int\int h^{ab}\lambda^3 \left[\bar{F} + K + K_a e^a + \ldots\right] dE d\Omega$$

$$= \int\int \lambda^3 \left[K + K_a e^a + \ldots\right] e^a e^b dE d\Omega -$$

$$\tfrac{1}{3}\int\int h^{ab}\lambda^3 \left[K + K_a e^a + \ldots\right] dE d\Omega$$

$$= \tfrac{4\pi}{5}\int \lambda^3 \left[\tfrac{2}{3}(\bar{H}\Gamma_{rs} + H_{rs}\bar{\Gamma})h^{ar}h^{bs} + H_r \Gamma_s h^{(ab}h^{rs)}\right] dE -$$

$$\tfrac{4\pi}{9}\int \lambda^3 H_r \Gamma_s h^{ab} h^{rs} dE$$

$$= \tfrac{8\pi}{15}\int \lambda^3 \left[(\bar{H}\Gamma_{rs} + H_{rs}\bar{\Gamma} + H_r\Gamma_s)h^{ar}h^{bs} - \tfrac{1}{3}H_r\Gamma^r h^{ab}\right] dE. \qquad (B.21a)$$

Furthermore, SSS in general implies that the anisotropic pressure must take the form

$$\pi_{ab} = Q(r)(c_a c_b - \tfrac{1}{3}h_{ab}) \qquad (B.21b)$$

where Q is some function of r and c_a is the unit radial vector [Maharaj (1986, p74)] (see (7.28)). The form of $(B.21a)$ is clearly consistent with this, so that

$$\pi^{00} = 0, \quad \pi^{11} = \tfrac{2}{3}Q(r), \quad \pi^{22} = \pi^{33} = -\tfrac{1}{3}Q(r), \qquad (B.21c)$$

place restrictions on $\bar{H}$, H_a, H_{ab}, $\bar{\Gamma}$, Γ_b, and Γ_{ab}. These symmetry conditions can be satisfied by taking for example $\bar{\Gamma} = \Gamma_a = \Gamma_{ab} = 0 = \bar{H} = H_b = H_{ab}$.

To determine the kinematics of the solution we utilise the decomposition of the 4-current vector (1.23). The particle number density is

$$N = -u_t n^t$$

$$= \int\int E\left[\bar{F} + K + K_a e^a + \ldots\right] \lambda dE d\Omega$$

$$= \tfrac{4\pi}{3}\int E\left[3\bar{F} + 3\bar{H}\bar{\Gamma} + H_a \Gamma_k h^{ak}\right] \lambda dE, \qquad (B.22)$$

where $E = -u_t p^t$. The conservation of particles $(B.16)$ leads to

$$N = 4\pi \int \mathrm{E}\bar{F}\lambda dE. \qquad (B.23)$$

The particle number density takes the equilibrium form.

Next, we turn our attention to the particle number flux, which is given by

$$\begin{aligned} j^t &= h^t{}_r n^r \\ &= \int \int e^t \left[\bar{F} + K + K_a e^a + \ldots\right] \lambda^2 dE d\Omega \\ &= \tfrac{4\pi}{3} \int \left[\bar{H}\Gamma_k h^{kt} + H_a \bar{\Gamma} h^{at}\right] \lambda^2 dE \\ &= \tfrac{4\pi}{3} \int \left[\bar{H}\Gamma^t + H^t \bar{\Gamma}\right] \lambda^2 dE, \end{aligned} \qquad (B.24)$$

where $h^t{}_r p^r = \lambda e^t$. The condition for the conservation of momentum $(B.13b)$ requires that the number flux $(B.24)$ vanishes. Hence, the distribution has a non-tilted kinematic average 4-velocity.

The condition for the conservation of momentum requires a distribution with a vanishing heat flux and a non-tilted kinematic average 4-velocity. Conservation of energy and particles give equilibrium energy density and vanishing bulk viscosity. The conservation of particles gives equilibrium particle number density.

In order to obtain an Einstein solution it is necessary for the energy-momentum tensor to satisfy the Einstein field equations, i.e. $G^{ab} = T^{ab}$, where the Einstein tensor G^{ab} is given by (2.26) in the orthonormal tetrad (2.25) and takes the form

$$G^{ab} = \mathrm{diag}(G^{00}, G^{11}, G^{22}, G^{33}), \quad \text{with } G^{22} = G^{33}. \qquad (B.25)$$

As Γ is anisotropic, we take a different approach than in Section 7.5. We first verify that T^{ab} is consistent with the general form of $(B.25)$. First, it is necessary to determine T^{ab} using $(B.2)$ and (2.30):

$$T^{ab} = \int p^a p^b \frac{1}{p^0} f[r, e^{\nu/2} p^0, r^2((p^2)^2 + (p^3)^2)] dp^{123}$$
$$= \int p^a p^b \frac{1}{p^0} \left\{ \bar{F}(\tilde{E}) + h(\tilde{E}, J) \exp[-\Gamma(r, \tilde{E}, J)] \right\} dp^{123}$$
$$= \bar{T}^{ab} + T'^{ab}. \qquad (B.26)$$

Since the integrand of $\bar{T}^{ab}$ is odd in p^1, p^2, and p^3 for all $a \neq b$, even in p^1, p^2, and p^3 for all $a = b$, with $\bar{T}^{11} = \bar{T}^{22} = \bar{T}^{33}$, it follows that

$$\bar{T}^{ab} = \mathrm{diag}(\bar{T}^{00}, \bar{T}^{11}, \bar{T}^{22}, \bar{T}^{33}). \qquad (B.27)$$

The equilibrium part $\bar{T}^{ab}$ of T^{ab} is thus consistent with the Einstein field equations and therefore imposes no additional restrictions on the functional form of f.

On the other hand, the integrand of T'^{ab} is odd and symmetric in p^2 and p^3 for $a \neq b$ and even and symmetric in p^2 and p^3 for $a = b$. This implies that $T^{02} = T^{03} = T^{12} = T^{13} = T^{23} = 0$ and $T^{22} = T^{33} \neq 0$. Because of the dependence of Γ on p^1 (see (7.7), $(B.2)$), the off-diagonal term T'^{01} is therefore not identically zero. However, since $G^{01} = 0$ for SSS spacetimes, we need a zero T^{01} term in order to satisfy the Einstein field equations. Now, by spherical symmetry, a zero heat flow $q^a = 0$ implies $T^{01} = 0$. We have already shown that the condition for conservation of momentum gives a vanishing heat flux. The energy-momentum tensor is therefore consistent with the functional form of the Einstein tensor.

Using the dynamic quantities $(B.17)$, $(B.18)$, $(B.20)$, and $(B.21)$, the energy-momentum tensor is:

$$T_{ab} = \bar{\mu} u_a u_b + \bar{p} h_{ab} + \pi_{ab} \qquad (B.28)$$

where $\bar{\mu}, \bar{p}$ indicate the equilibrium energy density and isotropic pressure. The Einstein field equations can be obtained using $(B.28)$:

$$G^{00} = \bar{\mu}$$
$$G^{11} = \bar{p} + \tfrac{2}{3} Q(r)$$
$$G^{22} = G^{33} = \bar{p} - \tfrac{1}{3} Q(r). \qquad (B.29)$$

Taking G^{00} and G^{11}, we have two equations in five variables ν, ψ, $\bar{\mu}$, $\bar{p}$, Q. By specifying $\bar{F}$, $\bar{H}$, H_a, H_{ab}, $\bar{\Gamma}$, Γ_b, and Γ_{ab} (according to the conservation conditions (B.10), (B.13), (B.16) and the symmetry condition (B.21c)), the variables $\bar{p}$, $\bar{\mu}$, and Q can be written in terms of ν and ψ. Thus, in principle the equations for G^{00} and G^{11} determine ν and ψ, and as a result the spacetime metric (2.23) for the SSS relaxation model is known, completing the SSS Einstein-BGK solution. Such a solution can be obtained by taking for example $\bar{\Gamma} = \Gamma_a = \Gamma_{ab} = 0 = \bar{H} = H_b = H_{ab}$. Specifying these quantities is analogous to specifying an equation of state in a fluid model.

By specifying the zero, first, and second order anisotropy of the distribution function (according to the conservation and symmetry conditions), the spacetime metric can be obtained - completing an Einstein solution.