Modelling the South African Inter-Bank Interest Rate Market using a Log-Normal Rational Pricing Kernel Model

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A dissertation submitted to the Faculty of Commerce, University of Cape Town, in partial fulfilment of the requirements for the degree of Master of Philosophy.

October 1, 2019

MPhil in Mathematical Finance,
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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy to the University of Cape Town. It has not before been submitted for any degree or examination.

Signed by candidate

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Abstract

This dissertation examines the performance of two log-normal rational pricing kernel models and their calibration to the South African Inter-bank interest rate market. We investigate using Monte-Carlo simulation to price caps, floors and swaptions. Model-performance for both models was tested on single-strikes and entire volatility surfaces. Our results show that a one-factor model cannot reproduce the volatility smile present in the caps/floor market but can reproduce the at-the-money swaption volatility surface. The two-factor model produces a better calibration to the volatility smile and captures most of the characteristics of the volatility surface.

Acknowledgements

I would like to thank all those who have helped me along this journey. A special mention goes to Prof David Taylor and Obeid Mahomed for their supervision.

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Chapter 1

Introduction

Globally the inter-bank interest rate market is important for all market participants as it provides an indication of deposit rates and is essential for banks to meet reserve requirements and liquidity requirements. Macrina and Mahomed (2018) note that the spot inter-bank offer rate (IBOR) is considered a suitable proxy for the risk-free rate in a single-curve framework. Under this single-curve frame-work the IBOR can be replicated using zero coupon bonds. The following paragraph describes the inter-bank market which is the market that includes most interest rate models. The South African market differs to the global market as it has not developed a number of components seen in the more developed global markets. One of the main differences between the local market and the global market are the interest rate curves present. These are described in the sections that follow.

The South African inter-bank interest rate market is defined by the JIBAR-linked (The Johannesburg Interbank Agreed Rate) curve and the reference rate is the three-month JIBAR. Jakarasi, Labuschagne and Mahomed (2015) identify a lack of consensus amongst experts in the interest rate markets as having hindered the development of an overnight indexed swap (OIS) market in South Africa and this is the reason given by Jakarasi *et al.* (2015) for there being no OIS zero-coupon yield curve. The instruments that are modelled in this dissertation are introduced in the next paragraph.

The inter-bank market offers linear derivatives in the form of Forward Rate Agreements (FRAs) and Interest Rate Swaps (IRSs). It also provides non-linear derivatives in the form of Caps, Floors and Swaptions. Grbac and Runggaldier (2015) cites that prior to the economic crisis of 2009, it was possible to price interest rate derivatives using the IBOR alone. Since the crisis, several different curves have been introduced to account for the price of default. This has made the pricing of these derivatives more complex which has resulted in the need for more expensive calculations to price interest rate derivatives such as caplets, floorlets and swaptions.

In this dissertation the South African Inter-bank interest rate market is modelled which includes the cap/floor market and the swaption market. The volatility smiles present in the cap/floor market and the ATM swaption volatility surface is incorporated. Only the single curve framework is considered in this dissertation, which means multi-interest rate curve scenarios are not explored. Consequently, there will be no need to consider any other curves but the three-month JIBAR. The models that have been implemented in this dissertation are derived from and developed from models described in Macrina (2014).

Reasonable restrictions on the model are explored, such as enforcing positive interest rates, and how these restrictions affect calibration. It is important to note that the calibration of the one and two-factor models has not been done in the South African markets before and similar models have been calibrated under a multi-curve framework. A parametric form is fitted to the two models which consists of deriving a pre-determined structure for the models that is constant over all time periods. A parametrised model is more efficient as it consists of fewer parameters and thus requires less computational power to calibrate. The models used in this dissertation as well as the structure breakdown are highlighted in the next paragraph.

The two models used in this dissertation, the one-factor model, which consists of a single factor and the two-factor model, which has two factors are described in detail in Chapter 2 and Chapter 3. These are described in Brigo and Mercurio (2006). Chapter 2 builds the foundations needed to introduce the one and two-factor model and will provide some background to similar models in the markets. An emphasis is placed on explaining pricing kernels, since the two models belong to the pricing kernel class. Interest rate modelling often consists of complex models that can be challenging to calibrate and run. These complex models are expensive to run as they often consist of more than two factors and this is highlighted by Macrina and Mahomed (2018). The one-factor and two-factor models can be used instead of complex multi-factored models if the one and two-factor model can be adequately calibrated to the caps/floor and swaption markets. Hedging, the process of offsetting exposures can be carried out with well calibrated models. Chapter 3 considers how to derive a closed-form price for the one-factor model. In addition, the required techniques used in the derivation are introduced. The inability to derive a closed-form solution when the two-factor model is used is shown in the chapter. In Chapter 4, an introduction to a variety of techniques for numerical pricing under the two-factor model is given and the possible methods to price are explored. The method that provides the most accurate prices in the least amount of time is chosen. The method chosen is a class of Quasi Monte-Carlo pricing and is shown to be efficient and accurate.

In Chapter 5, the calibration of the one and two-factor model to the market data is carried out and different possible fits and calibrations are explored. The Quasi-Monte Carlo method used is efficient enough for the calibration in chapter 5 to run and it ensures the numerical prices are sufficiently close to the true values. The Quasi-Monte Carlo method is used to calibrate the models to both the caps/floor and the swaption markets. The ability of the one and two-factor models to reproduce the market volatility surfaces is examined and the effect of different forms of the models in producing different surfaces with some more accurate than others is also examined. In conclusion, the results of the calibration are shown to be consistent in value and distribution over a five-year historic period which ensures the models are robust enough to handle different data sets. The prior expectation from Macrina and Mahomed (2018) is that the one-factor model performs poorly, while the two-factor model performs significantly better in the caps/floor and swaption markets which is shown to be the case in this dissertation.

Chapter 2

Interest Rate Models

This chapter will focus on introducing the four main categories of interest rate models commonly used in the markets globally. A brief use for each category is stated and greater emphasis is placed on pricing kernel models as this is the category to which the one and two-factor model belong. The one and two-factor model have been calibrated in overseas markets before, but no calibration has been done using the South African inter-bank market. This dissertation looks at how well the one and two-factor model calibrate to the markets.

2.1 Interest rate models

Interest rate models can be classified into four different categories with the first being short-rate models. In this model, future interest rates are described through using the short rate. The short rate can be described as the amount an individual can borrow over a very small time. An underlying diffusion process drives the short rate and the model is classified as a one-factor model if the diffusion process is one-dimensional. A single short rate cannot be used to determine the whole yield curve. Brigo and Mercurio (2006) show that the one-factor model can be calibrated to the caps volatility curve and swaptions volatility curve. Cox Ingersoll Ross, Vasicek, Ho and Lee and the Exponential Vasicek model are one-factor models as given in Brigo and Mercurio (2006). If the diffusion process is multi-factored, then the short rate interest model is known as a multi-factor model. Models such as the two-factor Hull and White model are examples of multi-factor models.

The second category of models are known as Heath-Jarrow-Morton (HJM) models. In these models the forward rate is modelled, and a yield curve can be determined from these forward rates. This differs, as noted by Brigo and Mercurio (2006) to the short rate model where the short rate cannot characterise the entire interest-rate model.

LIBOR Market Models (LMMs) form the third class of models known as the market models. The lognormal forward-LIBOR model (LFM) and the forward-swap model (LSM) are both market models. Brigo and Mercurio (2006) note that the Libor Market Models are useful as the Libor Market Models can be used with Blacks market formulas to price caps and swaptions. Calibration can be achieved using market data, but Brigo and Mercurio (2006) suggest greater care is needed in calibrating swaption models as the models are more complex. The next paragraph introduces the pricing kernel models which are the fourth category of interest rate models.

2.2 Pricing kernels

A pricing kernel is also known as a stochastic discount factor and works as a tool in discounting the value of an instrument under a particular measure. If the bond prices and the short rate consist of a rational combination of Markov processes, then the model is called a rational model as highlighted in Grbac and Runggaldier (2015). Under the pricing kernel framework, the discount bond price can be represented as

$$P_{tT} = \frac{1}{\pi_t}\mathbb{E}^{\mathbb{P}}[\pi_T|\mathcal{F}_t].$$

Where π_t is the pricing kernel to be used in the model. A simple stochastic process is usually used to model the underlying noise. Weigel (2003) shows that the kernel will then be defined as some strictly positive function of the underlying stochastic process. One form of pricing kernel given by Weigel (2003), is for the stochastic process to be an Ornstein-Uhlenbeck process and the function relating the kernel to stochastic process to be exponential. A rational log-normal model with a kernel of the form below can be used

$$\pi_t = f_t + exp(c_t + X_t).$$

Weigel (2003) proposes other possible forms including the exponential-linear kernel, the exponential-quadratic kernel and the quadratic kernel. In this dissertation a specific form of pricing kernel is used, and the effectiveness of the model is explored. The short rate model that arises from the pricing kernel process, as discussed in Crépey, Macrina, Nguyen and Skovmand (2016), could be used as a proxy model for the overnight indexed swap (OIS) rate and could form the OIS model. The OIS short rate (r_t), as defined in Brigo and Mercurio (2006), is obtained using the following equation

$$r_t = -(\delta_T \ln P_{tT})|_{T=t}.$$

A particular type of pricing kernel models is defined below and is closely linked to the one and two-factor model used in this dissertation.

2.3 The Flesaker-Hughston Model

The Flesaker-Hughston Model (FH) Model described in Brigo and Mercurio (2006), was one of the first instances of interest rate models that made use of the pricing kernel methodology. The FH Model considers a family of positive martingales $(M(t,T))_{T\geq t}$ and

$$M(t,T) = \frac{P_{tT}}{B(t)},$$

where $B(t)$ is the standard bank account numeraire as defined in Brigo and Mercurio (2006). By construction under the FH Model $P_{tT} = B(t)M(t,T)$ is decreasing in T and thus the interest rates are positive. The possibility of negative interest rates and how negative rates can be prevented are explored in this dissertation. It is important to note that the European market currently experiences negative rates and as such negative rates may be exceptable under those conditions. The FH model is capable of pricing caps and floors with an analytical solution like the market Black formula. In addition, the FH can be used with different exchange rates and thus different currency interest rate curves. The one-factor model used in this dissertation has the same structure as the FH model and it shares the same characteristics of the FH model.

Chapter 3

Log-Normal Rational Pricing Kernel Model

This dissertation explores the calibration of the one and two-factor model to the caps/floor and swaption markets. The purpose of the calibration is to determine if the one and two-factor model can be calibrated so that these models can be used to price interest rate derivatives in the South African market. In order to have priced the caps, floors and swaptions a pricing system had to be determined that made use of the one and two-factor models described previously. In this chapter, closed form prices are derived for the one-factor model and the inability to derive a closed form price for the two-factor model is highlighted. To begin with the one and two-factor model are described in detail and then the method for deriving closed form prices for the one-factor model is explained which requires the use of the change-of-numeraire technique and lognormal pricing formula. Firstly, the one and two-factor model are introduced below.

3.1 Log-Normal Rational Pricing Kernel Model

Considering a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with real world measure $\mathbb{P}$ and market filtration $(\mathcal{F}_t)_{t\geq 0}$ as a model for the financial market then the one default free bond price system is given by the following equation

$$P_{tT} = \frac{1}{\pi_t}\mathbb{E}[\pi_T P_{TT}|\mathcal{F}_t].$$

The bond price system is used to determine the price of a bond at time t, P_{tT}, with the pricing kernel π as noted in Crépey *et al.* (2016). For interest rate modelling only Zero Coupon Bonds (ZCB) are considered and $P_{TT} = 1$ simplifying the equation above.

The general form of the pricing kernel, as shown by Crépey *et al.* (2016), is given by

$$\pi_t = D_t M_t,$$

where $D_t = e^{-\int_0^t r_s ds}$ is the standard stochastic discount factor and M_t is a martingale that induces a change of measure from the real-world measure $\mathbb{P}$ to an equivalent measure $\mathbb{M}$. Under the risk-neutral measure $\mathbb{Q}$ the bond price system (in the general form) is given by the following equation

$$P_{tT} = \frac{1}{D_t}\mathbb{E}^{\mathbb{Q}}[D_T P_{TT}|\mathcal{F}_t].$$

Two different forms of pricing kernel models are considered. The one-factor model and the two-factor model. The form of the pricing kernel models is given by

$$\pi_t = \frac{\pi_0}{M_0}[P_{0t} + \sum_{i=1}^{n} b_i(t)A_t^{(i)}]M_t$$

where for the one-factor model n=1 and for the two-factor model n=2. The $b_i(t)$ are deterministic functions and the form of the $b_i(t)$ functions are determined through calibration. The log-normal processes

$$A_t^{(i)} = exp(a_i X_{(t)}^{(i)} - 0.5a_i^2 t) - 1$$

for i=1,2 are martingales under the measure $\mathbb{M}$. The a_i are constants while the $X_{(t)}^{(i)}$ are standard Brownian motions and this form for the $A_t^{(i)}$ ensures that they are martingales as shown in Macrina and Mahomed (2018). Levys characterization, a fundamental theorem for describing the relationship between Browian motion and martingales is applied in this instance. The form of the above models makes returning the initial term structure simple, P_{0T}, and relies on setting t=0.

3.2 The Change-of-Numeraire Technique

An important proposition provided by Brigo and Mercurio (2006) states that given a numeraire N with a probability measure Q^N that is equivalent to an initial probability measure Q_0 such that the price of a tradable asset X_t is given by

$$\frac{X_t}{N_t} = \mathbb{E}^{\mathbb{N}}[\frac{X_T}{N_T}|\mathcal{F}_t]$$

for $0 \leq t \leq T$ and is also a martingale, then

$$\frac{Y_t}{U_t} = \mathbb{E}^{\mathbb{U}}[\frac{Y_T}{U_T}|\mathcal{F}_t]$$

for $0 \leq t \leq T$ for an arbitrary numeraire U and the existence of a probability measure Q^U equivalent to Q_0. Y_T is the price of any attainable claim.

Considering the Radon-Nikodym derivative of Q^U as

$$\frac{dQ^U}{dQ^N} = \frac{U_T N_0}{U_0 N_T},$$

then for a tradable asset Z it follows that

$$\mathbb{E}^{\mathbb{N}}[\frac{Z_T}{N_T}] = \mathbb{E}^{\mathbb{U}}[\frac{U_0 Z_T}{N_0 U_T}]$$

and

$$\mathbb{E}^{\mathbb{N}}[\frac{Z_T}{N_T}] = \mathbb{E}^{\mathbb{U}}[\frac{Z_T}{N_T}\frac{dQ^N}{dQ^U}].$$

The next result required for the derivation of the closed form price is the use of lognormal pricing.

3.3 Lognormal Pricing

Brigo and Mercurio (2006) state that if $lnX \sim N(u, s^2)$, then $\mathbb{E}[X] = e^{u+0.5s^2}$, and

$$\mathbb{E}[(X-K)^+] = \mathbb{E}[X]N(d_+) - KN(d_-)$$

where

$$d_{\pm} = \frac{ln\frac{\mathbb{E}[X]}{K} \pm 0.5s^2}{s}$$

where $N(x)$ is the standard normal cumulative distribution function. The results from 3.2 and 3.3, together with the prices given in the next paragraph, can now be used with the pricing kernel models to derive the closed form prices.

3.4 Interest Rate Derivative Pricing

A caplet is call option on the spot LIBOR rate $L(t,T)$ with a strike K, nominal amount N, maturity T and has the following payoff

$$N(L(t,T) - K)^+$$

over the time interval $[t, T]$. Under the real world measure $\mathbb{P}$ the price of a bond put option at time 0 with the option expiring at time t, strike K and bond expiry of time T is given by,

$$p_{0t} = \frac{1}{\pi_0}\mathbb{E}^{\mathbb{P}}[\pi_t(K - P_{tT})^+].$$

Brigo and Mercurio (2006) show that in order to relate the put bond price to the price of a caplet the following relationship must be used and states that the price of a caplet is given by

$$Cpl_{0t} = \frac{(1 + K(T - t))}{\pi_0}\mathbb{E}^{\mathbb{P}}[\pi_t(\frac{1}{1 + K(T - t)} - P_{tT})^+].$$

In the same method the price of a floorlet can be determined using a call option on a bond

$$Flt_{0t} = \frac{1}{\pi_0}\mathbb{E}^{\mathbb{P}}[\pi_t(P_{tT} - K)^+].$$

Lastly, the price of a swaption with an underlying interest rate swap with yearly tenors is given by

$$Swp_{0t} = \frac{1}{\pi_0}\mathbb{E}^{\mathbb{P}}[\pi_t(1 - P_{tT} - K\sum_{i=1}^{n} P_{tT_i})^+]$$

for $0 \leq t \leq T_1 \leq T_2 ... \leq T_n = T$ and $T_0 = t$. The techniques discussed in section 3.2 and 3.3 together with the prices given above for the interest rate derivatives are now used to derive the closed form prices for the one-factor model.

3.5 One-factor Model

The closed form prices for caplets, floorlets and swaptions are derived using the one-factor pricing kernel model described earlier. Next, using the change of measure martingale M_t from Macrina (2014),

$$\frac{\mathrm{d}\mathbb{M}}{\mathrm{d}\mathbb{P}} = \frac{M_t}{M_0}$$

and ensuring the A_t is generated using a $\mathbb{M}$ Brownian Motion the closed form price can be derived. Starting with the pricing kernel,

$$\pi_t = \frac{\pi_0}{M_0}[P_{0t} + b(t)A_t]$$

which is positive by construction, the bond price process P_{tT} can be derived where P_{0t} is the initial term structure of the discount bond system.

$$P_{tT} = \mathbb{E}^{\mathbb{P}}[\frac{\pi_T}{\pi_t}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{P_{0T} + b(T)A_T}{P_{0t} + b(t)A_t}\frac{M_T}{M_t}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{M}}[\frac{P_{0T} + b(T)A_T}{P_{0t} + b(t)A_t}|\mathcal{F}_t]$$

$$= \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t}$$

which follows from the Change-of-Numeraire Technique and the fact the $b(T)$ are deterministic functions and that A_t is a martingale. It is important to note interest rates will be positive provided the below holds

$$b(T) \leq 1 - P_{0T}.$$

The above bond pricing process is now used to price caplets, floorlets and swaptions.

Caplet Price

$$Cpl_{0t} = \frac{1}{\pi_0}\mathbb{E}^{\mathbb{P}}[\pi_t(K - P_{tT})^+] = \mathbb{E}^{\mathbb{P}}[(P_{0t} + b(t)A_t)(K - \frac{P_{0T} + b(T)A_t}{P_{0t} + b(t)A_t})^+ \frac{M_T}{M_t}]$$
$$= \mathbb{E}^{\mathbb{M}}[(KP_{0t} - P_{0T} + (Kb(t) - b(T))(e^{(aW_t - 0.5a^2t)} - 1))^+] = \mathbb{E}^{\mathbb{M}}[(K_1 + K_2 e^{(aW_t - 0.5a^2t)})^+]$$

Where $K_2 = Kb(t) - b(T)$ and $K_1 = KP_{0t} - P_{0T} - K_2$ and W_t is a $\mathbb{M}$ Brownian Motion. Next, the cases in which the caplet is in the money is considered.

Case 1
If $K_1 > 0, K_2 < 0$ then

$$Cpl_{0t} = \mathbb{E}^{\mathbb{M}}[(K_1 - e^{Ln|K_2| + aW_t - 0.5a^2t})^+].$$

Case 2
If $K_1 > 0, K_2 > 0$ then

$$Cpl_{0t} = K_1 + K_2.$$

Case 3
If $K_1 < 0, K_2 > 0$ then

$$Cpl_{0t} = \mathbb{E}^{\mathbb{M}}[(e^{LnK_2 + aW_t - 0.5a^2t} - |K_1|)^+].$$

Case 4
If $K_1 < 0, K_2 < 0$ then

$$Cpl_{0t} = 0.$$

The expressions for case 1 and case 3 are simplified further. In case 1,

$$e^{Ln|K_2| + aW_t - 0.5a^2t} \sim LN(Ln|K_2| + aW_t - 0.5a^2t, a^2t)$$

by $L\acute{e}vy's\ Characterization$ and thus using the log-normal pricing formula

$$Cpl_{0t} = K_1 N(-d_2) - \mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}]N(-d_1)$$

where N is the standard normal cumulative distribution function and

$$d_1 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}])}{K_1} + 0.5a^2t}{a\sqrt{t}}, d_2 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}])}{K_1} - 0.5a^2t}{a\sqrt{t}}$$

and $\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}] = |K_2|$. Similarly, for case 3,

$$e^{Ln(K_2)+aW_t-0.5a^2t} \sim LN(Ln(K_2) + aW_t - 0.5a^2t, a^2t)$$

and thus using the log-normal pricing formula

$$Cpl_{0t} = \mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}]N(d_1) - |K_1|N(d_2)$$

and

$$d_1 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}])}{|K_1|} + 0.5a^2t}{a\sqrt{t}}, d_2 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}])}{|K_1|} - 0.5a^2t}{a\sqrt{t}}$$

and $\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}] = K_2$.

Floorlet Price

Using the same steps as with the caplet to obtain,

$$Flt_{0t} = \mathbb{E}^{\mathbb{M}}[(-(K_1) + (-K_2)e^{(aW_t-0.5a^2t)})^+]$$

where $K_2 = Kb(t) - b(T)$, $K_1 = KP_{0t} - P_{0T} - K_2$ and W_t is a $\mathbb{M}$ Brownian Motion. Next, the cases in which the floorlet is in the money is considered.

Case 1
If $K_1 > 0, K_2 < 0$ then

$$Flt_{0t} = \mathbb{E}^{\mathbb{M}}[(e^{Ln|K_2|+aW_t-0.5a^2t} - K_1)^+].$$

Case 2
If $K_1 > 0, K_2 > 0$ then

$$Flt_{0t} = 0.$$

Case 3
If $K_1 < 0, K_2 > 0$ then

$$Flt_{0t} = \mathbb{E}^{\mathbb{M}}[(|K_1| - e^{Ln(K_2)+aW_t-0.5a^2t})^+].$$

Case 4
If $K_1 < 0, K_2 < 0$ then

$$Flt_{0t} = K_1 + K_2.$$

The expressions for case 1 and case 3 are now simplified. In case 1,

$$e^{Ln|K_2|+aW_t-0.5a^2t} \sim LN(Ln|K_2| + aW_t - 0.5a^2t, a^2t)$$

and thus using the log-normal pricing formula,

$$Flt_{0t} = \mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}]N(d_1) - K_1N(d_2)$$

where N is the standard normal cumulative distribution function and

$$d_1 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}])}{K_1} + 0.5a^2t}{a\sqrt{t}}, d_2 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}])}{K_1} - 0.5a^2t}{a\sqrt{t}}$$

and $\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}] = |K_2|$. Similarly for case 3,

$$e^{Ln(K_2)+aW_t-0.5a^2t} \sim LN(Ln(K_2) + aW_t - 0.5a^2t, a^2t)$$

and thus using the log-normal pricing formula,

$$F_{0t} = |K_1|N(-d_2) - \mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}]N(-d_1)$$

and

$$d_1 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}])}{|K_1|} + 0.5a^2t}{a\sqrt{t}}, d_2 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}])}{|K_1|} - 0.5a^2t}{a\sqrt{t}}$$

and $\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}] = K_2$.

Swaption Price

A similar procedure to the caplet and floorlet pricing is considered but more terms in the pricing formula are used. The T_i are the reset dates of the underlying swap and a yearly reset period is assumed. The first reset date T_0 corresponds to the maturity date of the option.

$$Swp_{0t} = \frac{1}{\pi_0}\mathbb{E}^{\mathbb{P}}[\pi_t(1-P_{tT_n}-K\sum_{i=1}^{n} P_{tT_i})^+] = \mathbb{E}^{\mathbb{P}}[(P_{0t}+b(t)A_t)(1-P_{tT_n}-K\sum_{i=1}^{n} P_{tT_i})^+]$$

$$= \mathbb{E}^{\mathbb{P}}[(P_{0t} + b(t)A_t)(1 - \frac{P_{0T_n} + b(T_n)A_t}{P_{0t} + b(t)A_t} - K\sum_{i=1}^{n}\frac{P_{0T_i} + b(T_i)A_t}{P_{0t} + b(t)A_t})^{+}\frac{M_{T_n}}{M_t}]$$

$$= \mathbb{E}^{\mathbb{M}}[(P_{0t} + b(t)A_t - (P_{0T_n} + b(T_n)A_t) - (K\sum_{i=1}^{n}P_{0T_i} + b(T_i)A_t))^{+}]$$

$$= \mathbb{E}^{\mathbb{M}}[(P_{0t} - P_{0T_n} - K\sum_{i=1}^{n}P_{0T_i} + (b(t) - b(T_n) - K\sum_{i=1}^{n}b(T_i))A_t)^{+}] = \mathbb{E}^{\mathbb{M}}[(K_1 + K_2 e^{(aW_t - 0.5a^2t)})^{+}]$$

where $K_2 = b(t) - b(T_n) - K\sum_{i=1}^{n}b(T_i)$, $K_1 = P_{0t} - P_{0T_n} - K\sum_{i=1}^{n}P_{0T_i} - K_2$ and W_t is a $\mathbb{M}$ Brownian Motion. Next, the cases in which the swaption is in the money is considered.

Case 1

If $K_1 > 0, K_2 < 0$ then

$$Swp_{0t} = \mathbb{E}^{\mathbb{M}}[(K_1 - e^{Ln|K_2| + aW_t - 0.5a^2t})^{+}].$$

Case 2

If $K_1 > 0, K_2 > 0$ then

$$Swp_{0t} = K_1 + K_2.$$

Case 3

If $K_1 < 0, K_2 > 0$ then

$$Swp_{0t} = \mathbb{E}^{\mathbb{M}}[(e^{LnK_2 + aW_t - 0.5a^2t} - |K_1|)^{+}].$$

Case 4

If $K_1 < 0, K_2 < 0$ then

$$Swp_{0t} = 0.$$

Again, the expressions for case 1 and case 3 are simplified. In case 1,

$$e^{Ln|K_2| + aW_t - 0.5a^2t} \sim LN(Ln|K_2| + aW_t - 0.5a^2t, a^2t)$$

and thus using the log-normal pricing formula,

$$Swp_{0t} = K_1N(-d_2) - \mathbb{E}^{\mathbb{M}}[e^{Ln|K_2| + aW_t - 0.5a^2t}]N(-d_1)$$

where N is the standard normal cumulative distribution function and

$$d_1 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2| + aW_t - 0.5a^2t}])}{K_1} + 0.5a^2t}{a\sqrt{t}}, d_2 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2| + aW_t - 0.5a^2t}])}{K_1} - 0.5a^2t}{a\sqrt{t}}$$

and $\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}] = |K_2|$ while for case 3,

$$e^{Ln(K_2)+aW_t-0.5a^2t} \sim LN(Ln(K_2)+aW_t-0.5a^2t, a^2t)$$

and thus using the log-normal pricing formula,

$$Swp_{0t} = \mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}]N(d_1) - |K_1|N(d_2)$$

and

$$d_1 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}])}{|K_1|} + 0.5a^2t}{a\sqrt{t}}, d_2 = \frac{\frac{Ln(\mathbb{E}^{\mathbb{M}}[e^{Ln|K_2|+aW_t-0.5a^2t}])}{|K_1|} - 0.5a^2t}{a\sqrt{t}}$$

and $\mathbb{E}^{\mathbb{M}}[e^{LnK_2+aW_t-0.5a^2t}] = K_2$.
Case 2 and 4 are not simplified. The derivation in the next section highlights the fact that no closed form solution exists under the two-factor model.

3.6 Two-factor model

Using the two-factor model it is not possible to derive closed form solutions for caplets, floorlets and swaptions. The derivation below shows why it is not possible to derive a closed form solution for caplets, floorlets and swaptions. By way of example, the caplet price is used to highlight the lack of a closed form price and this will apply to the other two interest rate derivatives.

$$\pi_t = \frac{\pi_0}{M_0}[P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}]$$

which is positive by construction and can be used to determine the pricing process P_{tT} where P_{0t} is the initial term structure of the discount bond system.

$$P_{tT} = \mathbb{E}^{\mathbb{P}}[\frac{\pi_T}{\pi_t}|\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}[\frac{P_{0T} + b_1(T)A_T^{(1)} + b_2(T)A_T^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}}\frac{M_T}{M_t}|\mathcal{F}_t]$$

$$= \mathbb{E}^{\mathbb{M}}[\frac{P_{0T} + b_1(T)A_T^{(1)} + b_2(T)A_T^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}}|\mathcal{F}_t] = \frac{P_{0T} + b_1(T)A_t^{(1)} + b_2(T)A_t^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}}$$

which follows from the change of measure and the fact the $b(T)$ are deterministic functions while the $A_t^{(i)}$ are independent martingales. Interest rates will be positive provided the below holds

$$b_1(T) + b_2(T) \leq 1 - P_{0T}.$$

Now using the same method as before and starting with the price of a caplet,

$$Cpl_{0t} = \mathbb{E}^{\mathbb{P}}[(P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)})(K - \frac{P_{0T} + b_1(T)A_t^{(1)} + b_2(T)A_t^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}})^+\frac{M_T}{M_t}]$$

$$= \mathbb{E}^{\mathbb{M}}[(KP_{0t} - P_{0T} + (Kb_1(t) - b_1(T))(e^{(a_1 W_t^1 - 0.5a_1^2 t)} - 1) + (Kb_2(t) - b_2(T))(e^{(a_2 W_t^2 - 0.5a_2^2 t)} - 1))^+]$$
$$= \mathbb{E}^{\mathbb{M}}[(K_1 + K_2 e^{(a_1 W_t^1 - 0.5a_1^2 t)} + K_3 e^{(a_2 W_t^2 - 0.5a_2^2 t)})^+]$$

where $K_2 = Kb_1(t) - b_1(T)$, $K_3 = Kb_2(t) - b_2(T)$ and $K_1 = KP_{0t} - P_{0T} - K_2 - K_3$. In the one-factor model case it was possible to proceed as the distribution of a single log-normal model was known. Under the two-factor model the price has two log-normal processes but the distribution of the sum of two log-normal processes cannot be determined and thus the log-normal pricing process cannot be used as before in the one-factor model.

Chapter 4

Monte Carlo Pricing

In chapter 3 it was shown that there are no closed form solutions for the prices of the interest rate derivatives, caplets, floorlets and swaptions, when using the two-factor model. Thus, to price caplets/floorlets and swaptions, numerical pricing techniques are used. The following chapter explores three different methods of pricing under the two-factor model. Monte Carlo simulations as well as quasi-Monte Carlo methods can be used to price the above three interest rate derivatives. Quasi-Monte Carlo simulations are determined as the best method for pricing. Initially, the crude Monte Carlo technique is introduced below.

4.1 Monte Carlo

Monte Carlo simulation can be used to evaluate integrals of the following form

$$I(f) = \int_A f(x)w(x)dx$$

where $A \subseteq \mathbb{R}^{\mathbb{K}}$ and w is a probability density function. Two important results from probability theory, shown in Hulley *et al.* (2018), are needed to analyse the Monte Carlo integrations, Strong Law of Large Numbers and the Central Limit Theorem. Using these two results, Strong Law of Large Numbers and the Central Limit Theorem, the approximation is given by the formula below.

$$\hat{I}_{A,n}(fw) = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

The above estimate, highlighted by Hulley *et al.* (2018) is also an unbiased estimate of the original integral. For both the one and two-factor model the M Brownian Motion can be simulated using randomly generated standard normal values and the mean can be taken of the simulated values leading to the crude Monte Carlo estimate. The Brownian Motions W_t are simulated using $\sqrt{t}N(0,1)$ where $N(0,1)$ is a randomly generated standard normal value. In all of the graphs produced the error bounds are calculated as a two standard deviation error bound of the simulations. Crude Monte Carlo simulation is used as a base measure for the performance of the two alternative methods discussed in 4.2 and 4.3, starting with Antithetic Variates and then Quasi-Monte Carlo methods.

4.2 Antithetic Variates

Antithetic Variates, described in Hulley *et al.* (2018), is a Monte Carlo technique that incorporates randomly generated standard normal numbers twice, n sample size, to provide an effective sample size $2n$ of simulations. The Antithetic Monte Carlo estimate is given by

$$\begin{aligned}\hat{I}_{A_{\pm},n}(fw) &= 0.5(\hat{I}_{A,n}(fw)) + \hat{I}_{\bar{A},n}(fw)) \\ &= \frac{1}{2n}\sum_{i=1}^{n}(f(x_i) + f(2\mathbb{E}(X) - X_i))\end{aligned}$$

where $\hat{I}_{\bar{A},n}(fw))$ is the Antithetic Variate estimate. The estimate of the price can then be calculated as

$$\frac{1}{0.5n}\sum_{i=1}^{0.5n}(f(Z_i) + f(-Z_i))$$

where Z are standard normal variables and f is the payoff function of the interest rate derivative to be priced. In the following paragraph, Quasi-Monte Carlo methods are discussed.

4.3 Quasi-Monte Carlo

Hulley *et al.* (2018) explains that Quasi-Monte Carlo integration involves integrals of the following form

$$I_{[0,1)^s}(f) = \int_{[0,1)^s} f(x)dx$$

about the s-dimensional unit cube. This expression is equivalent to

$$I_{[0,1)^s}(f) = E[f(U)]$$

where $U \sim U[0,1)^s$ and one could estimate this using the crude Monte Carlo estimator.

$$\hat{I}_{[0,1)^s,n}(f) = \frac{1}{n}\sum_{i=1}^{n} f(u_i)$$

where the u_i can be randomly generated from the $U[0,1)^s$. Quasi-Monte Carlo integration uses deterministic rules to generate the u_i points instead of randomly generating them. This method, noted in Hulley *et al.* (2018), provides a better rate of convergence than crude Monte-Carlo. To price the caplets, floorlets and swaptions, an open rule Quasi-Monte Carlo technique that makes use of Van der Corput Sequences is used. The paragraph below explains how Van der Corput Sequences are determined.

Van der Corput Sequences

The definition of Van der Corput Sequences as given by Hulley *et al.*, 2018 is as follows. Let $r > 1$ be an integer. Any $m \in \mathbb{N}$ can be expressed uniquely in base r as

$$m = a_0 + a_1 r + .. + a_l r^l$$

where $0 \le a_i \le r-1, a_l \neq 0$, and $r^l \le m \le r^{l+1}$. Now, the base r radical inverse function $\phi_r : \mathbb{N} \to [0,1)$ is defined by

$$\phi_r := a_0 r^{-1} + a_1 r^{-2} + .. + a_l r^{-l-1},$$

The sequence

$$\phi_r(0), \phi_r(1), \phi_r(2), \phi_r(3), ...,$$

is called a Van der Corput Sequence. The next section describes how numerical prices for the one-factor model are determined.

4.4 One-Factor Model

To determine the method that is more accurate and efficient, Crude Monte Carlo, Antithetic Variates and Quasi-Monte Carlo techniques were applied to the interest rate derivatives and compared.

Caplet

The expression used to calculate the simulated price is given by

$$\mathbb{E}^{\mathbb{M}}[(KP_{0t} - P_{0T} + (Kb(t) - b(T))(e^{(aW_t - 0.5a^2 t)} - 1))^+]$$

and figure 4.1 shows the results of the simulation.

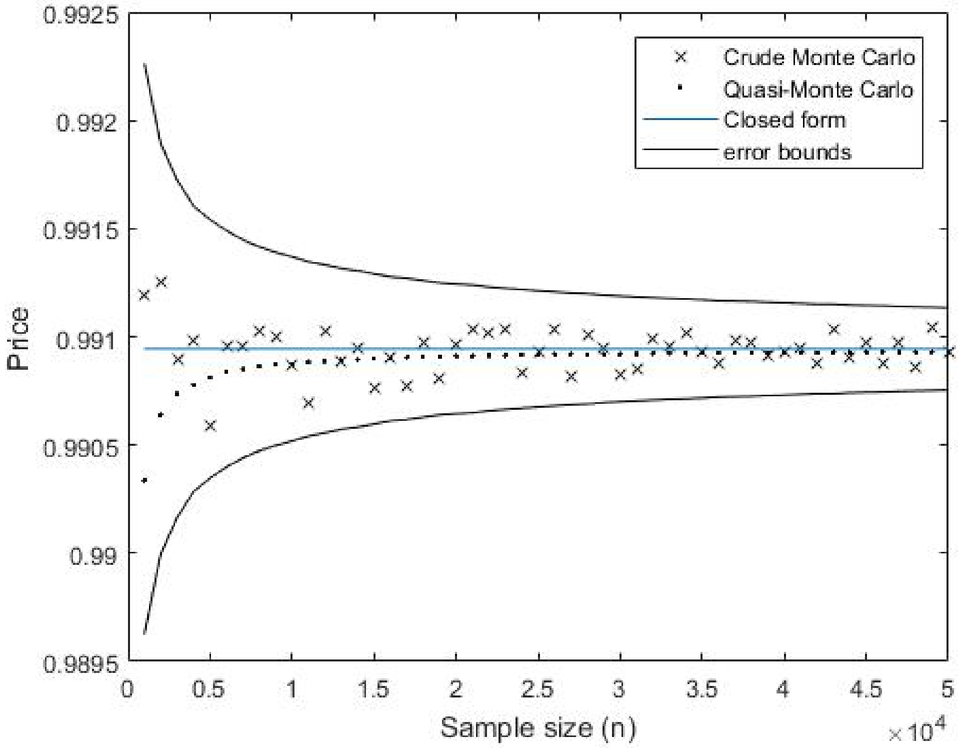


Fig. 4.1: Caplet numerical price

Floorlet

The expression used to calculate the simulated price is given by

$$\mathbb{E}^{\mathbb{M}}[(-KP_{0t} + P_{0T} - (Kb(t) - b(T))(e^{(aW_t - 0.5a^2 t)} - 1))^+]$$

and figure 4.2 shows the results of the simulation.

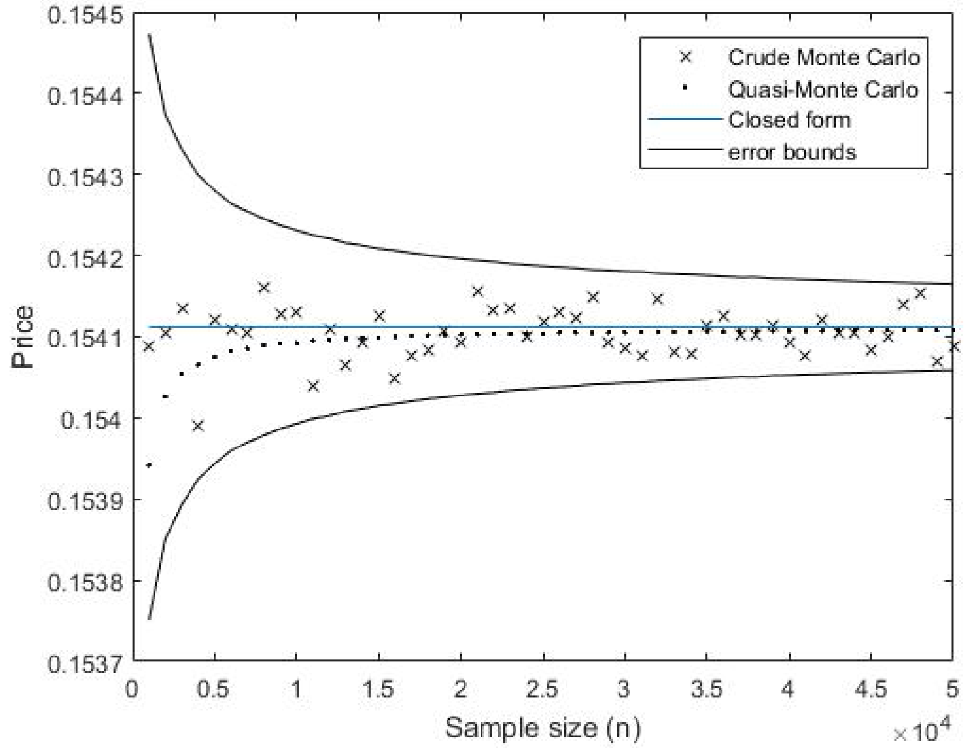


Fig. 4.2: Floorlet numerical price

Swaption

The expression used to calculate the simulated price is given by

$$\mathbb{E}^{\mathbb{M}}[(-KP_{0t} + P_{0T} - (Kb(t) - b(T))(e^{(aW_t - 0.5a^2 t)} - 1))^+]$$

and figure 4.3 shows the results of the simulation.

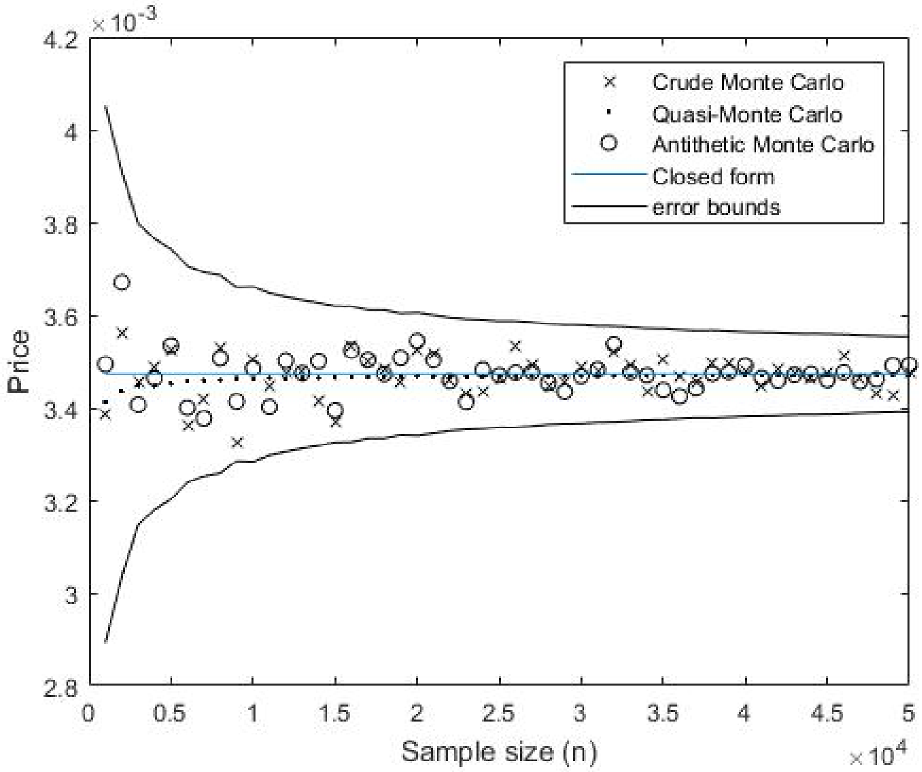


Fig. 4.3: Swaption numerical price

In all three cases, caplets, floorlets and swaptions, the Quasi-Monte Carlo simulation method provides a more efficient way to price the interest rate derivatives. Both the crude and the Antithetic Monte Carlo estimates only appear to converge after a sample size larger than $100,000$ random simulations is used, whereas the Quasi-Monte Carlo estimates converge with a sample size around $30,000$ random simulations. In the figures that follow only the swaption will have the Antithetic Monte Carlo estimate included as it is harder to observe in the case of the caplet and floorlet. The more efficient Quasi-Monte Carlo estimation is important for the calibration of the models to the market data as it speeds the process of calibration up and provides more accurate results. This is significantly important in the case of the two-factor models where there is no closed form solution. The jumps between simulated values when using Crude Monte Carlo simulation were too large for the minimization process in the calibration and thus proving the need for the Quasi-Monte Carlo method to be used. In section 4.5 figures 4.4, 4.5 and 4.6 the increased efficiency of Quasi-Monte Carlo simulation is highlighted.

4.5 Two-factor Model

Caplet

In order to compute a Monte-Carlo solution for the two-factor model interest rate derivatives, a simplified expression for derivatives prices under the two-factor model assumptions is required. The caplet price has already been simplified previously and the following expression for the Monte-Carlo estimation is used.

$$\mathbb{E}^{\mathbb{M}}[(KP_{0t}-P_{0T}+(Kb_1(t)-b_1(T))(e^{(a_1W_t^1-0.5a_1^2t)}-1)+(Kb_2(t)-b_2(T))(e^{(a_2W_t^2-0.5a_2^2t)}-1))^+]$$

Using the above expression W_t^1 and W_t^2 can be used to price the caplet.

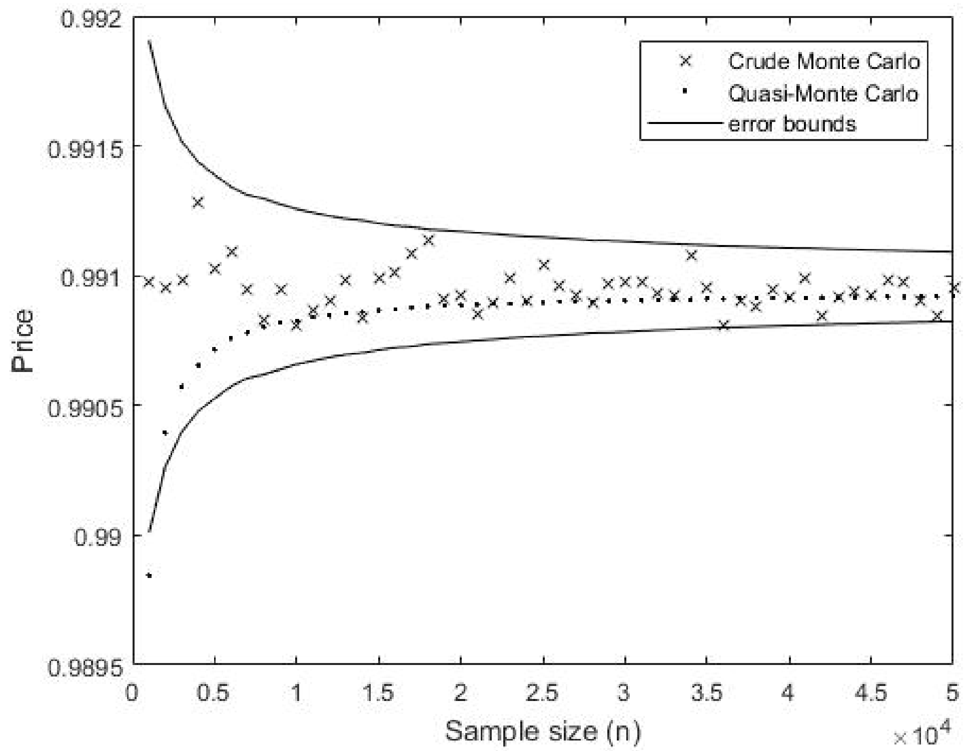


Fig. 4.4: Caplet numerical price

Floorlet

To derive an expression that can be used for the Monte Carlo estimation of the floorlet price the same technique as used in the one-factor model case is used.

$$P_{tT} = \frac{P_{0T} + b_1(T)A_t^{(1)} + b_2(T)A_t^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}}$$

while

$$\mathbb{E}^{\mathbb{P}}[(P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)})(\frac{P_{0T} + b_1(T)A_t^{(1)} + b_2(T)A_t^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}} - K)^+ \frac{M_T}{M_t}]$$

$$= \mathbb{E}^{\mathbb{M}}[(P_{0T} - KP_{0t} - (Kb_1(t) - b_1(T))(e^{(a_1 W_t^1 - 0.5a_1^2 t)} - 1) - (Kb_2(t) - b_2(T))(e^{(a_2 W_t^2 - 0.5a_2^2 t)} - 1))^+].$$

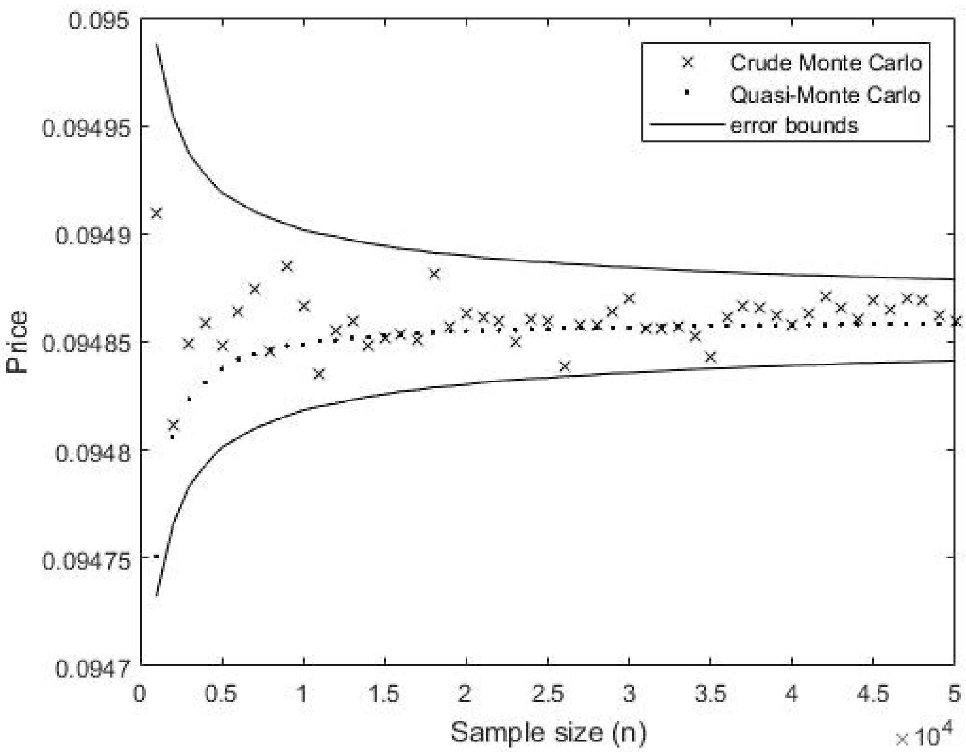


Fig. 4.5: Floorlet numerical price

Swaption

Similarly for the swaption,

$$\mathbb{E}^{\mathbb{P}}[(P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)})(1 - P_{tT_n} - K\sum_{i=1}^{n} P_{tT_i})^+]$$

$$= \mathbb{E}^{\mathbb{P}}[(P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)})(1 - \frac{P_{0T_n} + b_1(T_n)A_t^{(1)} + b_2(T_n)A_t^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}}$$

$$-K\sum_{i=1}^{n} - \frac{P_{0T_i} + b_1(T_i)A_t^{(1)} + b_2(T_i)A_t^{(2)}}{P_{0t} + b_1(t)A_t^{(1)} + b_2(t)A_t^{(2)}})^+ \frac{M_T}{M_t}]$$

$$= \mathbb{E}^{\mathbb{M}}[(P_{0t} - P_{0T_n} - K\sum_{i=1}^{n} P_{0T_i} + (b_1(t) - b_1(T_n)$$

$$-K\sum_{i=1}^{n} b_1(T_i))A_t^{(1)} + (b_2(t) - b_2(T_n) - K\sum_{i=1}^{n} b_2(T_i))A_t^{(2)})^+].$$

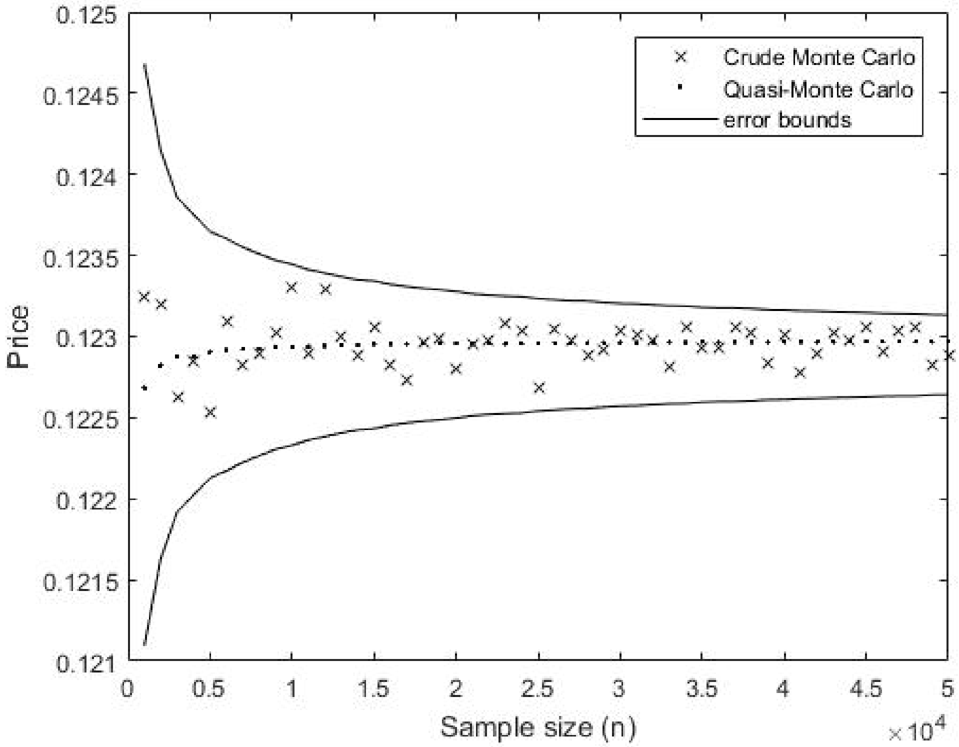


Fig. 4.6: Swaption numerical price

It is important to note that the two Brownian Motions used in the two-factor models are simulated using independent standard normal random variables as by assumption the $A_t^{(i)}$ are independent.

By comparison in figure 4.4, 4.5 and 4.6 a similar pattern to the one-factor model case can be seen in which the Quasi-Monte Carlo method achieves convergence more efficiently than both the crude and Antithetic Monte Carlo estimates. For this reason, Quasi-Monte Carlo methods are used in the pricing and calibration of the model for the two-factor model. There is no closed form solution to compare the estimations to, however the figures highlight the more efficient estimations using the Quasi-Monte Carlo method.

Chapter 5

Data and Calibration

Chapter 5 explores the calibration of the one-factor and two-factor model to the market data which consists of caps/floor volatility prices and swaption volatility prices. The cap and floor options make use of the same market prices as the options only vary depending on which party is buying or selling. Thus, only a calibration to the caps market is required and this calibration to the caps market results in the same parameters for the floor market. The first section below introduces difference caps and how difference caps are calculated.

5.1 Difference Caps

Before calibrating the caplet/floorlet models the market cap prices are converted into caplet prices. To do this, Black's formula for cap prices is used as the market data is priced using Black's formula. Black's formula is introduced below and is given in Brigo and Mercurio (2006).

Black's formula

$$BL(K, F, v) = F\Phi(d_1(K, F, v)) - K\Phi(d_2(K, F, v))$$

where

$$d_1(K, F, v) = \frac{ln(F/K) + 0.5v^2}{v}$$
$$d_2(K, F, v) = \frac{ln(F/K) - 0.5v^2}{v}$$

and

$$v = \sigma\sqrt{T}.$$

The rate F is the forward rate applicable for the period. The price of a cap is then given by

$$Cap = N \sum_{i=\alpha+1}^{\beta} P_{0,T_i}\tau_i BL(K, F(0, T_{i-1}, T_i), v_i)$$

with T_α the start date, τ_i the period of time between dates and T_β the expiry date. The implied caplet volatility can be backed out from the caplet price and is the value $v_{t,T}$ satisfying

$$Cpl(0, t, T, N, K) = NP_{o,T}(T - t)BL(K, F(0, t, T), v_{t,T}\sqrt{T}).$$

The next step makes use of Black's formula to calculate the difference caps for the ten caps over the data range. The market data spans over ten years and has one-year caps through to ten-year caps for various strikes. The difference caps are calculated by subtracting the price of the nine-year cap from the ten-year cap and repeating this for all the caps moving one year backwards. This ensures that the resulting difference cap prices consist of the four caplets contained in each year long period. These single year difference caps are independent of the previous year caplets. The chosen date for the calibration is the most recent data date which is the 29^{th} of March 2018 and the chosen strike price initially is 7%. For the initial calibration only, a single strike is used, and the volatility skew is not considered in the preliminary calibrations as it is important to explore how the one and two-factor model recover simple volatility surfaces. In the next section the conversion of swaption volatilities to market prices is shown.

Tab. 5.1: Caplet stripping

Term	Volatilities (%)	Price	Difference caps
1	8.75	0.0005	0.0005
2	12.89	0.0039	0.0034
3	14.79	0.0107	0.0068
4	15.62	0.0195	0.0088
5	16.00	0.0299	0.0103
6	16,25	0.0417	0.0118
7	16,68	0.0546	0.0129
8	17,1	0.0684	0.0138
9	17,5	0.0826	0.0142
10	17,91	0.0962	0.0136

5.2 Swaption Pricing

Similarly, to the caplet pricing the swaptions are quoted in terms of volatilities and must be converted into prices. This section explores how the swaption prices are calculated from the market data. Firstly, the at the money strikes of the swaptions that are used in Black's price for swaptions are determined as highlighted in Brigo and Mercurio (2006) and the strike is calculated below.

$$K = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)}$$

This is equivalent to the forward swap rate $(S(s,T_\alpha,T_n))$ as the swaption is ATM. The ATM strike can then be used to price the swap using the formula below given in Brigo and Mercurio (2006).

$$Swp(0,T_\alpha,T_n) = NBL(K, S(0,T_\alpha,T_n), \sigma_{\alpha,\beta}\sqrt{T_\alpha}) \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i).$$

While the implied swaption volatility is the value of v_{T_α,T_n} satisfying

$$Swp(0,T_\alpha,T_n) = NBL(K, S(0,T_\alpha,T_n), v_{T_\alpha,T_n}\sqrt{T_\alpha}) \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i).$$

For initial investigations the most recent date in the data 29^{th} of March 2018 is used, as well as a maturity of 5 years. The table below summarises these details.

Tab. 5.2: Swaption prices

Term	ATM Strike	Volatilities (%)	Price
1	0.0719	13,77	0.0081
2	0.0725	14,94	0.0123
3	0.0731	15,77	0.0158
4	0.0738	16,42	0.0188
5	0.0792	16,73	0.0247
6	0.0929	16,90	0.0264
7	0.0961	17,15	0.0313
8	0.0992	17,31	0.0347
9	0.1015	17,52	0.0369
10	0.1037	18,08	0.0391
11	0.1057	18,83	0.0411
12	0.1072	19,88	0.0430
13	0.1083	20,64	0.0436
14	0.1091	20,77	0.0426

The next section introduces the calibration technique used on both the one and two-factor model.

5.3 Calibration Technique

All the calibrations in this paper make use of the lsqnonlin function in MATLAB which solves nonlinear least-squares curve fitting problems. lsqnonlin minimizes the following expression

$$min_x||f(x)||_2^2$$

where $f(x)$ is a vector of price differences between the market prices and the models price given the parameter inputs.

To begin with both the one-factor and the two-factor model are calibrated to the difference caps and market swaptions separately. The parameters calibrated from the one-factor model are the constant a from the martingale process A_t and the deterministic function $b(t)$.

For this initial investigation it is assumed that negative interest rates are possible.

Difference Cap Calibration

The $b(t)$ function must be deterministic and $b(t)$ must be decreasing in time. The following form, shown below, for $b(t)$ is used suitable values for $b0$ and $b1$ are found.

$$b(t) = b0e^{-b1t}$$

The calibration settles on the following parameter values $a = 0.2241$, $b0 = 1.4629$ and $b1 = 0.0386$ while the performance of the model is shown in the figure below. For the two-factor model two $b(t)$ functions of the form

$$b_1(t) = b0e^{-b1t}, b_2(t) = b2e^{-b3t}$$

are chosen as well as a_1 and a_2 from the two log-normal processes. The calibration settles on the following values for the parameters, $a_1 = 0.1074$, $a_2 = 0.0003$, $b0 = 2.3739$, $b1 = 0.0305$,$b2 = 2.5063$ and $b3 = 0.1129$. For both models an initial calibration is used where all parameters are calibrated. The parameters corresponding to the one-factor model are held as constants and the model is re-calibrated to ensure the best fit is achieved. This is done for the two-factor model.

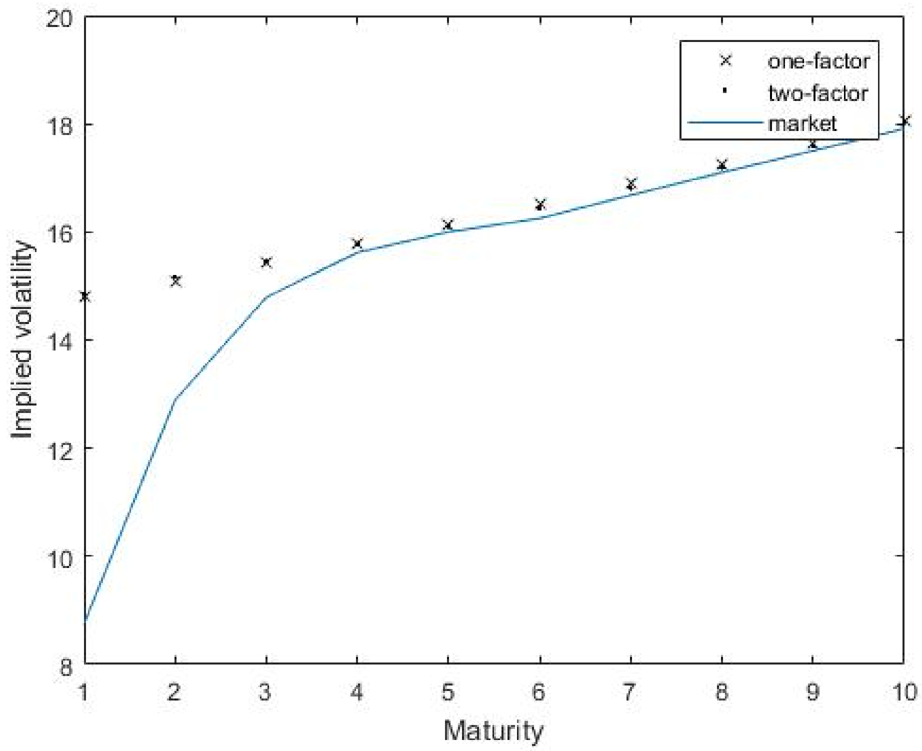


Fig. 5.1: Simple cap calibration

As seen in the figure above the models provide a good fit to the difference caps and both have slight trouble with the first cap price. This is most likely due to the small difference in the strike of the market cap, 7%, and the yield curve values around that period. However, both models generally reproduce the market difference caps for a single strike of 7%.

Swaption Calibration

The one and two-factor models are calibrated to ATM swaptions to determine how well the one and two-factor model calibrate to the swaption market. The calibration settles on parameter values $a = 1.0275$, $b_0 = 0.2573$ and $b_1 = 0.0331$ for the one-factor model while the two-factor model has the following parameters values, $a_1 = 11.9165$, $a_2 = 0.4101$, $b0 = 0.5950$, $b1 = 0.0558$,$b2 = 0.8080$ and $b3 = 0.0099$. Notably, these values differ to the difference cap calibration. An observable difference is the value for the a parameters between the two cases of calibration. When calibrating to the difference caps the value for the a parameter is less than one whereas in the case of this calibration to the swaption market the values of the

a's are greater than one. The graph below shows the relative performance of the calibration.

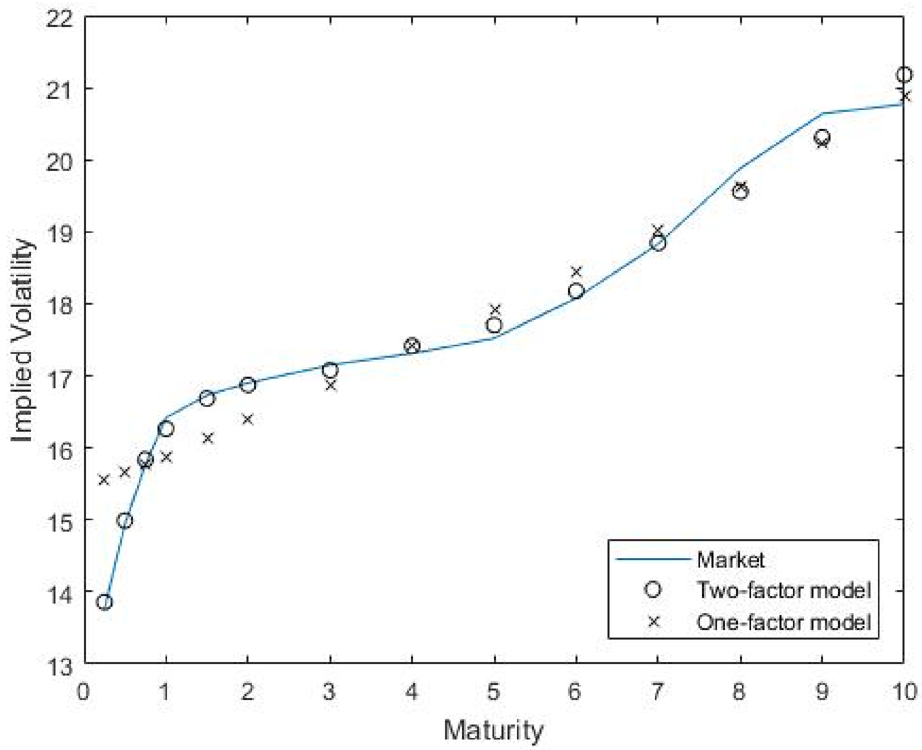


Fig. 5.2: Simple swaption calibration

Unlike the calibration to the cap market, the two-factor model appears to have a significantly better fit. This model performs well achieving similar values to those observed in the market. However, there is a 'lag' in the one-factor model behind the market volatilities as it attempts to match the prices of the market. This suggests a weakness to the one-factor model in its ability to model the swaption market.

Difference Cap and Swaption Calibration

A calibration to both the cap and swaption market is carried out to determine if the one and two-factor model can capture both the cap and swaption market simultaneously. The calibrated parameters $a = 1.1302$, $b_0 = 0.2385$ and $b_1 = 0.0328$ are closer to the values observed when only the swaption market was considered for the one-factor case. The value for a is in between the value settled under the cap only market and the swaption only market. Nevertheless, a reasonable fit is still achieved suggesting the one-factor model could be enough to model both caps and swaptions together in this simple scenario. The fit to the market swaptions is off for the initial and end maturity swaptions highlighting the limitations of the one-factor model. The two-factor model achieves a significantly better fit settling on the following parameter values $a_1 = 0.0830$, $a_2 = 0.0370$, $b0 = 1.6784$, $b1 = 0.0124$,$b2 = 2.7882$ and $b3 = 0.0099$. The conflicting parameter values between the two models shows that the individual markets should be calibrated to separately.

The next two sections describe the volatility surfaces of the caps and swaption markets and how the one and two-factor model are calibrated to the caps and swaption volatility surfaces.

5.4 Volatility Smile

Thus far, only the performance of the one and two-factor model in specific cases has been explored. It has been shown that the two-factor model outperforms the one-factor model. This section examines the one and two-factor models ability to handle the volatility smile present in the market.

Volatility Smile

As shown at the start of this chapter Black's formula is used in the caps market to price caplets. To illustrate the volatility smile, consider the market prices of two different caplets priced using Black's formula with the caplets having the same underlying forward rates and maturity but a different strike. Brigo and Mercurio (2006) state that under Black's formula these two caplets should have the same volatility parameter. That is v satisfies both

$$CPL^{MKT}(0, T_1, T_2, K_1) = P_{0T_2}(T_2 - T_1)BL(K_1, F, v)$$

and

$$CPL^{MKT}(0, T_1, T_2, K_2) = P_{0T_2}(T_2 - T_1)BL(K_2, F, v),$$

however, in reality two different volatilities are required for these two expressions to hold. That is a v_1 and a v_2, such that

$$CPL^{MKT}(0, T_1, T_2, K_1) = P_{0T_2}(T_2 - T_1)BL(K_1, F, v_1)$$

and

$$CPL^{MKT}(0, T_1, T_2, K_2) = P_{0T_2}(T_2 - T_1)BL(K_2, F, v_2)$$

holds. This shows that each caplet market price requires a unique Black's volatility dependent on the caplets strike. The figure below highlights the ”smile” observed and how there is a bend in the implied volatility surface.

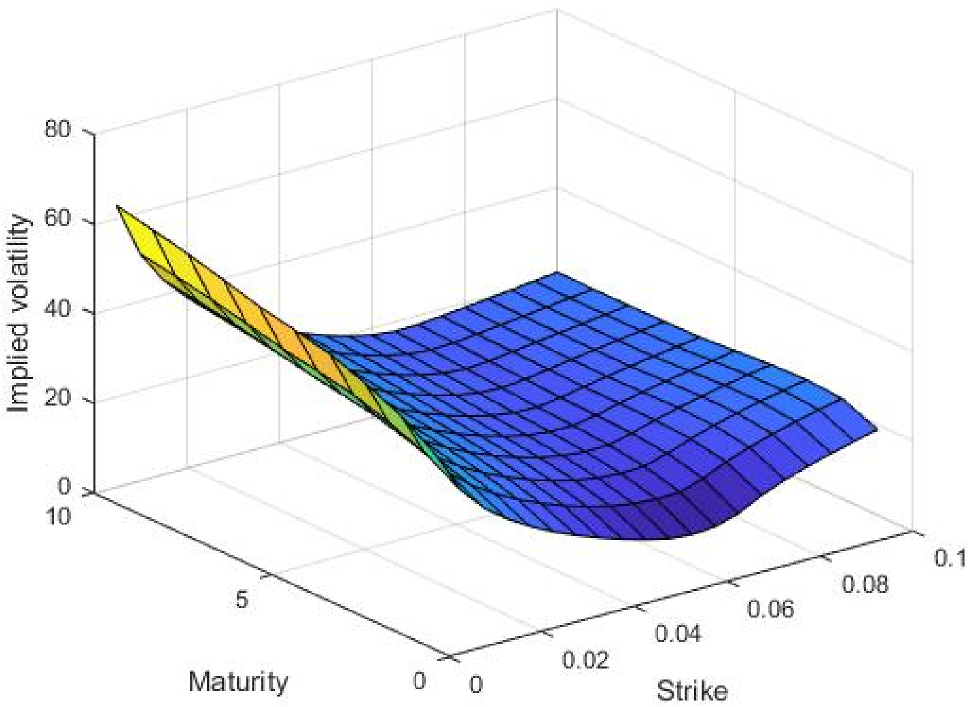


Fig. 5.3: Volatility smile

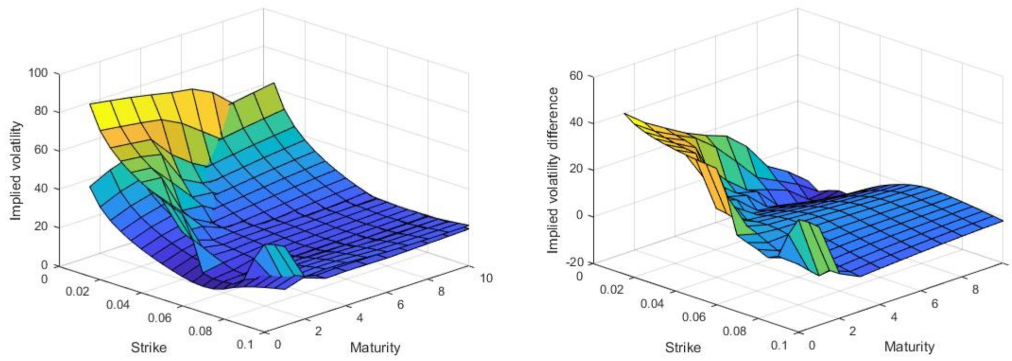
In the next step the ability of the one and two-factor models to reproduce the volatility smile observed in the caps/floor market is investigated. The calibration technique used is the same as discussed in section 5.4. The difference caps for each maturity and across all the given strike prices are calculated and it is this data that are used in the calibration. The table below shows the difference caps used in the calibration.

Tab. 5.3: Difference caps (20 Strikes)

Term	0.005	0.01	0.015	0.02	0.025	0.03	0.035	0.04	0.045	0.05
1	0,0605	0,0557	0,0509	0,0460	0,0412	0,0364	0,0316	0,0268	0,0220	0,0172
2	0,0565	0,0521	0,0476	0,0431	0,0386	0,0341	0,0296	0,0252	0,0207	0,0162
3	0,0557	0,0515	0,0474	0,0432	0,0390	0,0348	0,0307	0,0265	0,0224	0,0184
4	0,0538	0,0499	0,0461	0,0422	0,0383	0,0345	0,0307	0,0269	0,0232	0,0196
5	0,0520	0,0485	0,0449	0,0413	0,0378	0,0343	0,0308	0,0273	0,0240	0,0207
6	0,0503	0,0471	0,0438	0,0406	0,0373	0,0341	0,0310	0,0278	0,0247	0,0218
7	0,0476	0,0447	0,0418	0,0388	0,0359	0,0330	0,0301	0,0273	0,0246	0,0219
8	0,0453	0,0427	0,0400	0,0374	0,0348	0,0322	0,0296	0,0270	0,0245	0,0221
9	0,0428	0,0405	0,0381	0,0357	0,0334	0,0310	0,0287	0,0264	0,0241	0,0219
10	0,0391	0,0371	0,0350	0,0329	0,0307	0,0286	0,0265	0,0245	0,0224	0,0204

Term	0.055	0.06	0.065	0.07	0.075	0.08	0.085	0.09	0.095	0.1
1	0,0123	0,0075	0,0030	0,0005	0,0002	0,0001	0,0000	0,0000	0,0000	0,0000
2	0,0119	0,0080	0,0051	0,0034	0,0024	0,0017	0,0013	0,0010	0,0008	0,0006
3	0,0146	0,0113	0,0087	0,0068	0,0055	0,0045	0,0037	0,0031	0,0027	0,0023
4	0,0162	0,0132	0,0107	0,0088	0,0073	0,0062	0,0053	0,0046	0,0041	0,0037
5	0,0176	0,0148	0,0124	0,0103	0,0087	0,0075	0,0065	0,0057	0,0051	0,0046
6	0,0189	0,0163	0,0139	0,0118	0,0101	0,0087	0,0076	0,0068	0,0061	0,0055
7	0,0194	0,0170	0,0148	0,0129	0,0113	0,0100	0,0089	0,0081	0,0074	0,0068
8	0,0198	0,0176	0,0156	0,0138	0,0122	0,0109	0,0098	0,0090	0,0083	0,0077
9	0,0198	0,0177	0,0159	0,0142	0,0127	0,0114	0,0104	0,0095	0,0089	0,0083
10	0,0186	0,0168	0,0151	0,0136	0,0124	0,0113	0,0104	0,0096	0,0090	0,0086

Initially it is assumed the b functions are constant over each of the maturity years i.e. from years one till ten. This is because the calibration is calibrating to yearly difference caps and not quarterly caplets. To illustrate the ability of the models to capture the volatility smile shown in figure 5.3 the implied volatility is compared to the market implied volatility. Where there is a good fit the difference between the market volatility surface and the models implied volatility, surface is plotted. No constraints are forced on the b functions for the initial calibration.



(a) One-factor model　　**(b)** Volatilty surface difference

Fig. 5.4

One-factor model

The attempted calibration shows the one-factor model fails to recover the volatility smile and only manages to recover the smooth areas of the smile. These are the regions where maturity is greater than four years and where strikes are greater than 0.05. The limitations of the one-factor model are evident as a very poor fit is achieved over regions of the smile where there is significant volatility.

Thereafter the effect of forcing positive interest rates in the model is explored.

Figure 5.5 shows how the b function values are initially restricted under the desired calibration values and this results in a worse off fit. Figure 5.6 reaffirms the poor fit achieved using the one-factor model to model the volatility smile.

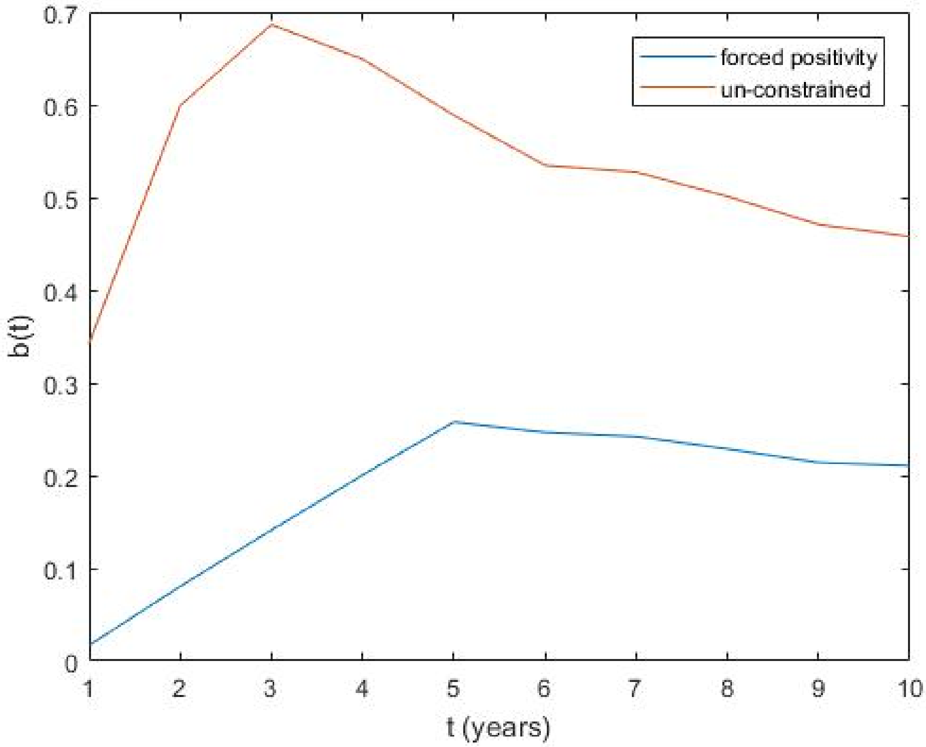


Fig. 5.5: $b(t)$ one-factor model

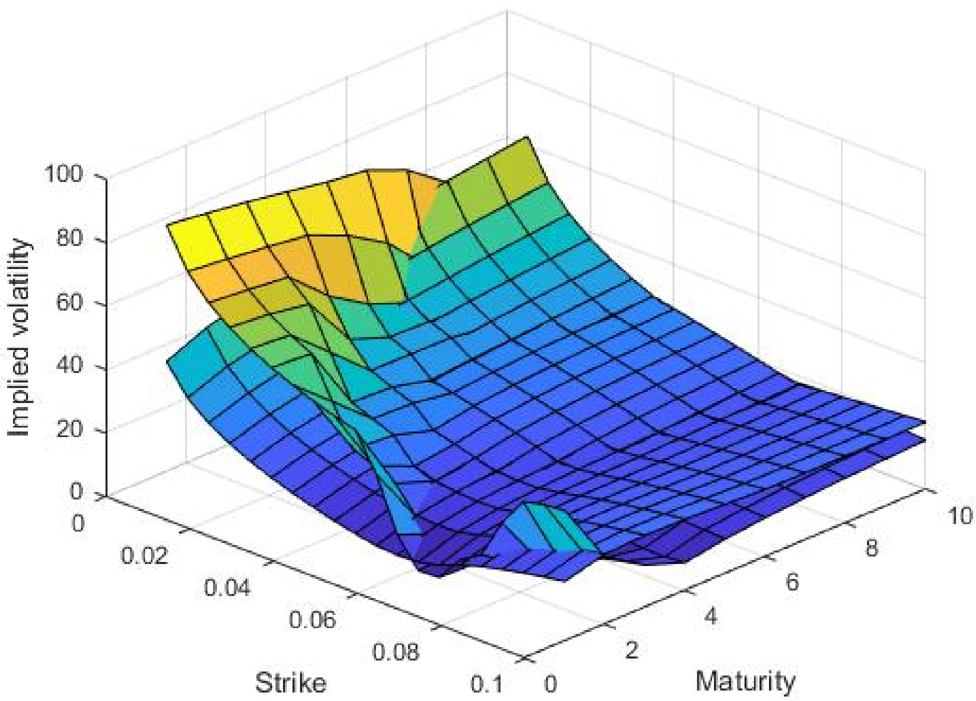
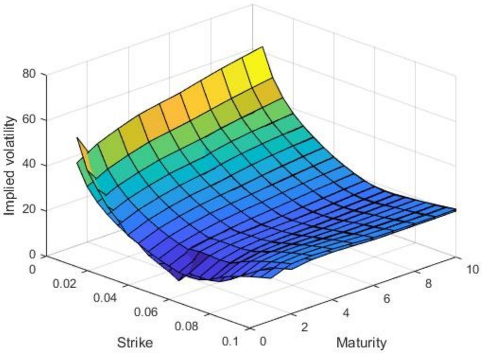
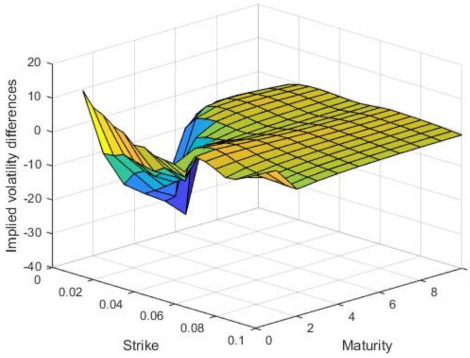


Fig. 5.6: Volatility smile one-factor model constrained



(a) Volatility smile two-factor model



(b) Volatilty surface difference

Fig. 5.7

The above figures show the next step which is an attempt to fit the two-factor model to the market as it is clear the one-factor model cannot recover the smile.

Two-factor model

The two-factor model achieves a significantly better fit with only the very extreme regions of the smile failing to be recovered. This region covers the smaller strikes and one-year maturity caps. The failure of the model to capture this region may be attributed to using b's over yearly periods and not quarterly. When the positive interest rate constraint was considered the form of the b function was changed. This resulted in a worse off calibration and fit, compared to the un-constrained model. Figure 5.8 does not include the un-constrained b_2 values because they are too large to be shown on the same axis as the other values. Notably they follow a similar shape as the one-factor b values.

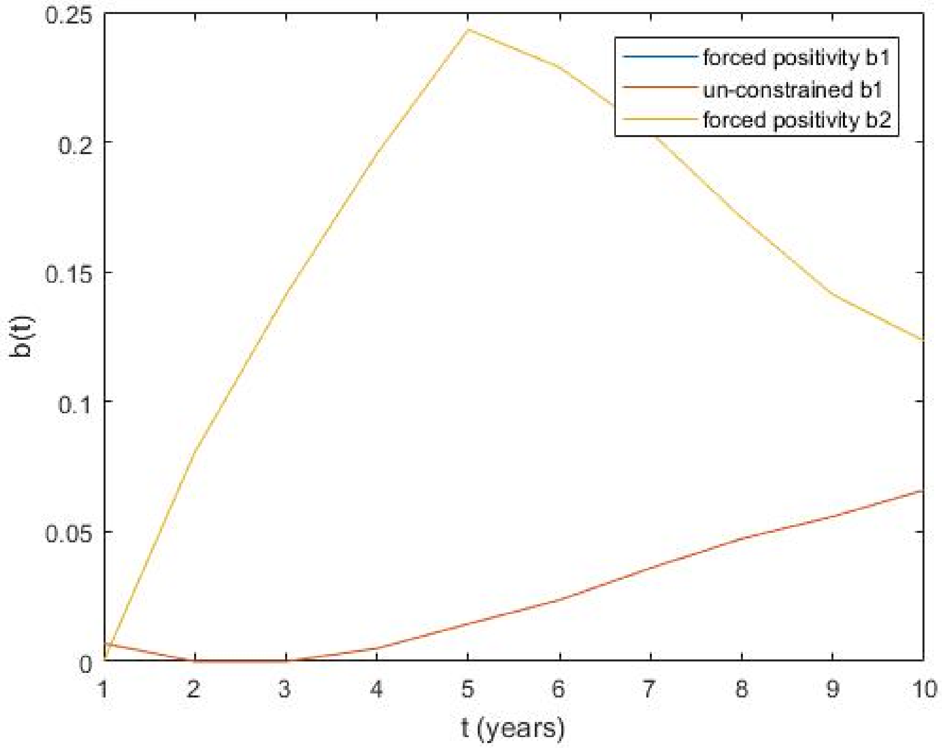


Fig. 5.8: $b(t)$ two-factor model

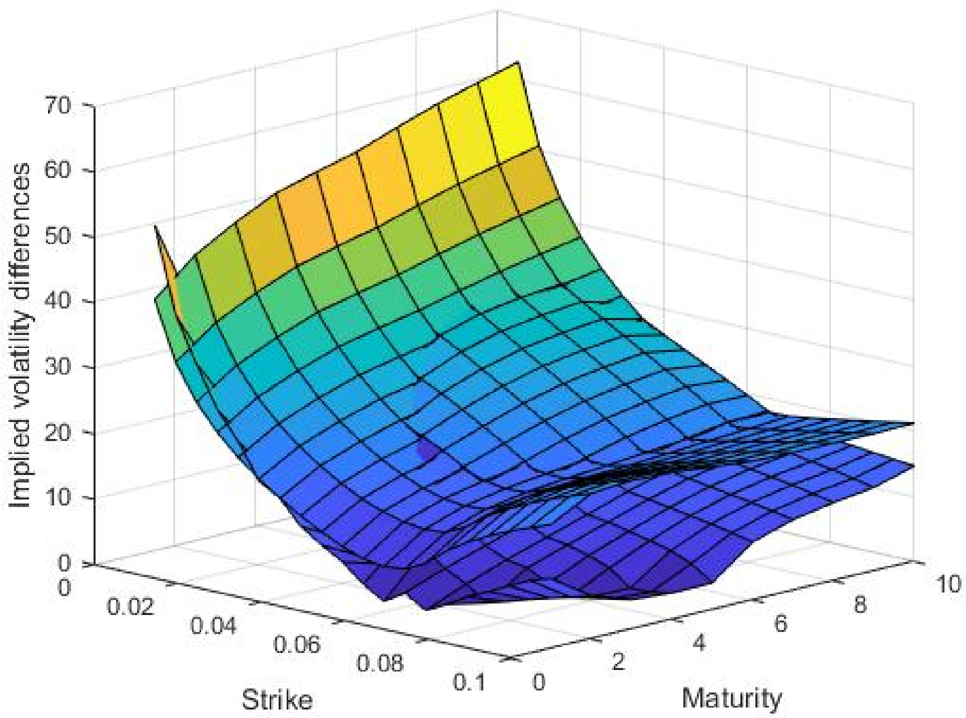
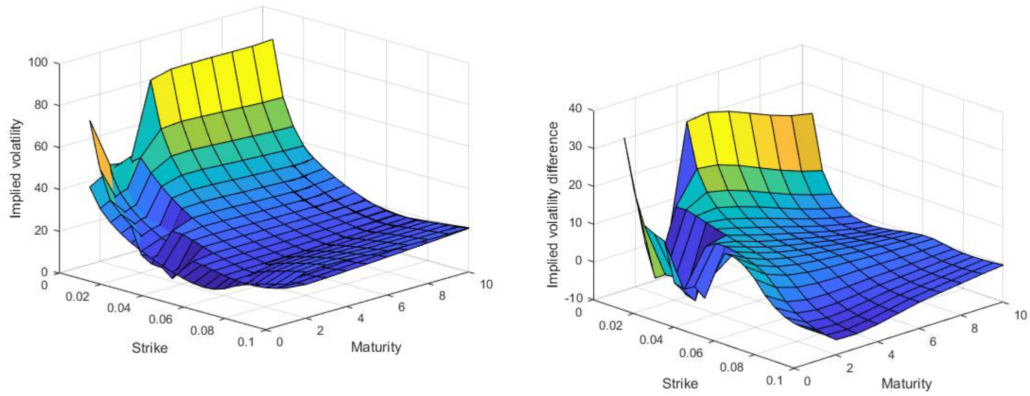


Fig. 5.9: Volatility smile two-factor model constrained



(a) Volatility smile two-factor model parametrised

(b) Volatilty surface difference

Fig. 5.10

Parametrized $b(t)$ function

For the final calibration a simple parametric form of the b function is fitted to determine if a decent fit of the volatility smile is achieved. The possibility of negative interest rates is allowed in this calibration. Looking at the shape of the b functions the same decreasing exponential function as used in section 5.3, is fitted. The one-factor model is not considered as it is clear the model cannot recover the smile. The resulting calibration shows a similar result as that achieved under the constant b calibration. The fit is not as close, but the smile is still recovered for most of the surface. The two-factor models inability to recover the smile for small maturities and strikes is evident again.

The table below summarises the values of the a parameters in the above calibrations. The value of a appears to control the smile and may have an important role to play in the model.

Tab. 5.4: Parameter values

Model	a_1	a_2
One-factor	0.2297	0
One-factor constrained	0.5198	0
Two-factor	2.1385	0.0025
Two-factor constrained	0.6764	16.4977
Two-factor parametrized	0.0013	0.0121

5.5 Swaption Volatility Surface

The last calibration that is considered is the at-the-money (ATM) swaption volatility surface. The smile present in the swaption market is not considered but rather swaptions over different maturities and different underlying swap tenors. The same procedures as the cap market calibration is followed to show the performance of the one and two-factor models. The figure below shows the swaption market surface.

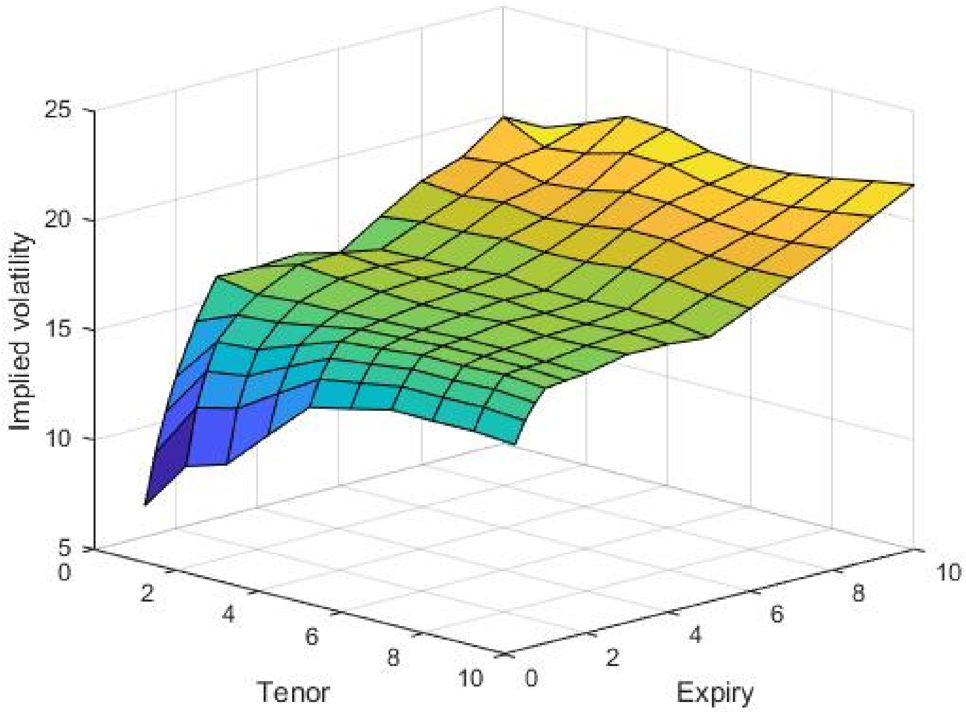


Fig. 5.11: Swaption volatility surface

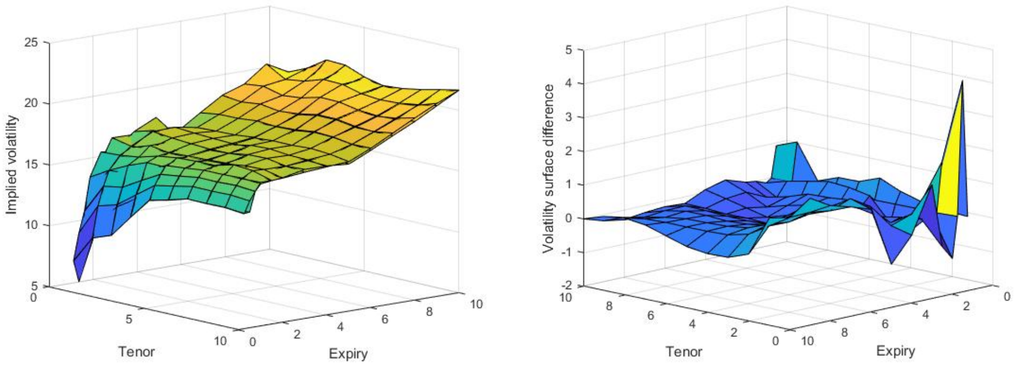
The ATM swaption prices for all the market data points are calculated using the same method as in section 5.2 and obtain the following ATM swaption prices.

Tab. 5.5: Swaption prices (1 to 10-year tenors)

Maturity	1	2	3	4	5	6	7	8	9	10
0.25	0,0009	0,0024	0,0036	0,0055	0,0076	0,0091	0,0106	0,0116	0,0127	0,0134
0.5	0,0017	0,0041	0,0062	0,0088	0,0115	0,0135	0,0156	0,0172	0,0187	0,0198
0.75	0,0024	0,0056	0,0083	0,0114	0,0146	0,0171	0,0196	0,0216	0,0235	0,0250
1	0,0031	0,0070	0,0103	0,0138	0,0174	0,0203	0,0231	0,0254	0,0277	0,0295
1.5	0,0045	0,0091	0,0133	0,0174	0,0214	0,0249	0,0282	0,0311	0,0337	0,0358
2	0,0057	0,0108	0,0158	0,0204	0,0246	0,0286	0,0323	0,0356	0,0384	0,0410
3	0,0067	0,0134	0,0192	0,0245	0,0293	0,0339	0,0384	0,0419	0,0455	0,0490
4	0,0075	0,0149	0,0214	0,0273	0,0327	0,0379	0,0423	0,0464	0,0506	0,0540
5	0,0079	0,0158	0,0228	0,0293	0,0355	0,0405	0,0455	0,0503	0,0542	0,0572
6	0,0087	0,0170	0,0245	0,0316	0,0377	0,0435	0,0494	0,0541	0,0579	0,0615
7	0,0095	0,0180	0,0261	0,0330	0,0399	0,0467	0,0524	0,0568	0,0612	0,0657
8	0,0098	0,0190	0,0267	0,0345	0,0422	0,0485	0,0536	0,0585	0,0636	0,0687
9	0,0101	0,0185	0,0268	0,0351	0,0420	0,0475	0,0529	0,0584	0,0640	0,0695
10	0,0096	0,0190	0,0284	0,0361	0,0423	0,0483	0,0545	0,0607	0,0669	0,0728

One-factor model

Calibration to the swaption market requires a slightly different approach to the one used for the cap/floor market. There are different underlying swap tenors present in the market prices and thus the b function values overlap. As an example, a one and a half year swaption written on a one-year swap uses the same b values as a half year swaption written on a two-year swap. However, these two derivatives have different ATM prices, and this is where the calibration will have to consider this result. The calibration method needs to take this into account and ensure the best b values are chosen. Swap prices at each quarterly time point are calibrated over the range of market data and a constant b value at each of these time periods is assumed. This results in 80 b values spanning the volatility curve. The figure below shows that the one-factor model can recover the surface almost identically. This is likely due to the lack of consideration of the skew in the swaption market.



(a) Volatility surface one-factor model **(b)** Volatility surface difference

Fig. 5.12

In the same manner as the cap market calibration a restriction to ensure only positive interest rates arise is imposed. Figure 5.13 shows the values of the b both constrained and un-constrained. Figure 5.14 highlights that the b values are affected by the positive interest rate constraint, but a suitable fit is still achieved under the one-factor model parameters. Only a few of the outer regions of the volatility surface are not captured accurately by the constrained one-factor model. An exact calibration of the two-factor model is not carried out as there are more parameters to estimate than swaptions prices available in the data.

The performance of a parametrized function for b function is examined for both the one and two-factor models, allowing for negative interest rates.

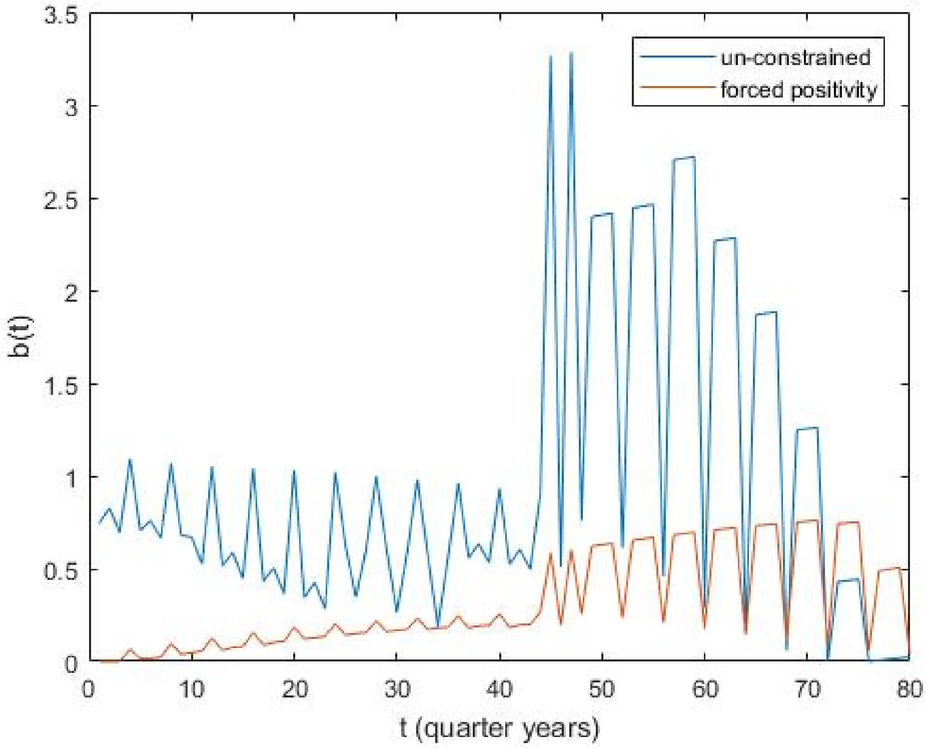
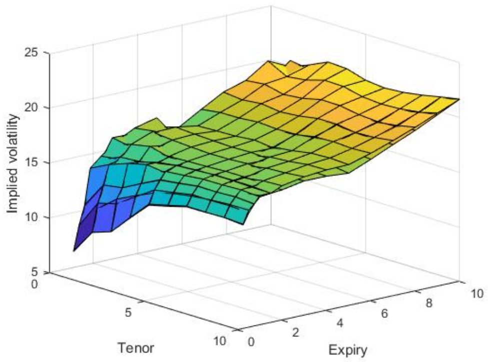
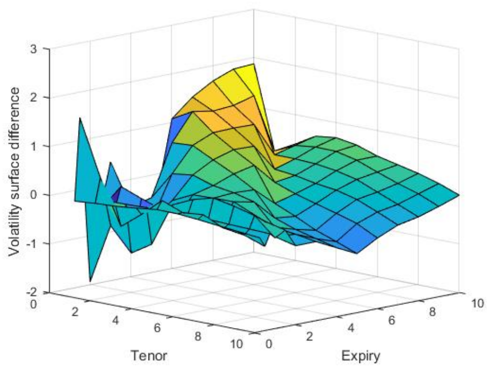


Fig. 5.13: $b(t)$ one-factor model



(a) Volatility surface one-factor model constrained



(b) Volatility surface difference

Fig. 5.14

Parametrized $b(t)$ function

Observing the form of the b values a fifth order polynomial function is fitted to recover the double humped shape observed in the b values. The following form for the b function is assumed. This form is consistent with the shape of the observed b values and is chosen to try reproduce the shape of the $b(t)$ function.

$$b(t) = b_1 t^5 + b_2 t^4 + b_3 t^2 + b_4 t + b_5$$

One-factor model

In the one-factor model case a reasonable fit is achieved but there is still not enough flexibility in the model to recover the exact structure. The figure below shows the fit of this model.

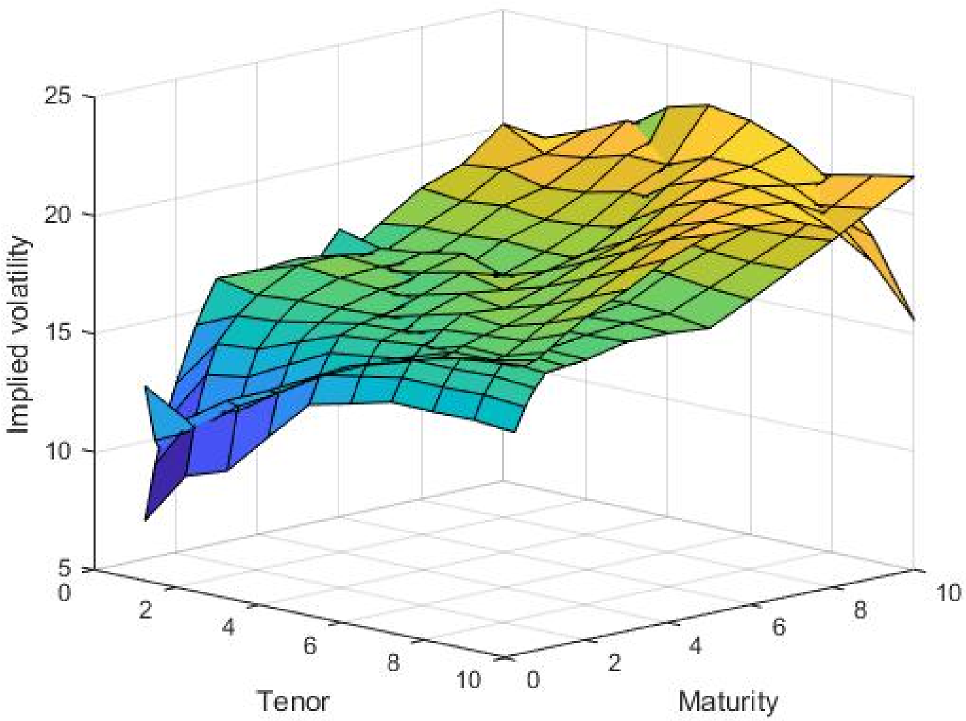
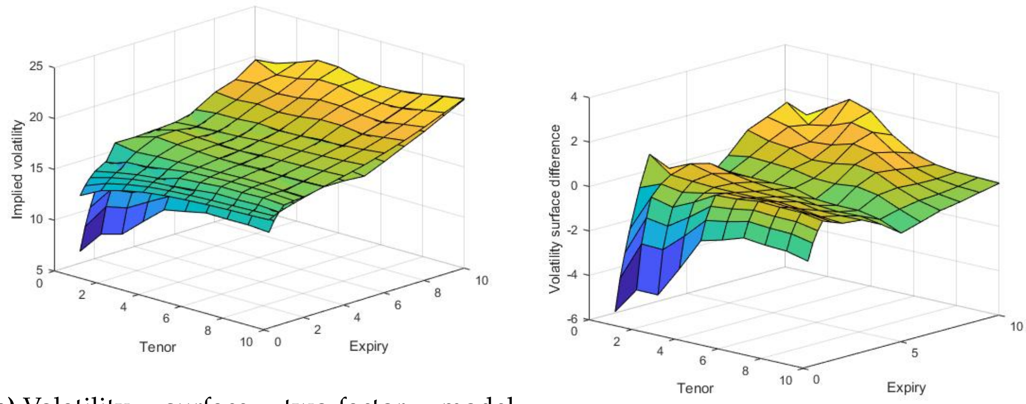


Fig. 5.15: Volatility surface one-factor model parametrised



(a) Volatility surface two-factor model parametrised

(b) Volatility surface difference

Fig. 5.16

<u>Two-factor model</u>

For the two-factor model the fifth order polynomial parametrization produces a very poor fit to the volatility surface. Thus, the model from section 5.3 is used, a decreasing exponential function for the b functions. This decreasing exponential parametrization produced a good fit under the simplified calibration surface and therefore is a good consideration.
In the above figure it is clear that most of the surface is reproduced except for outlier regions on the surface. This suggests a more robust parametrization may be required.

lastly, the table below will show the values of the a constants in each of the models.

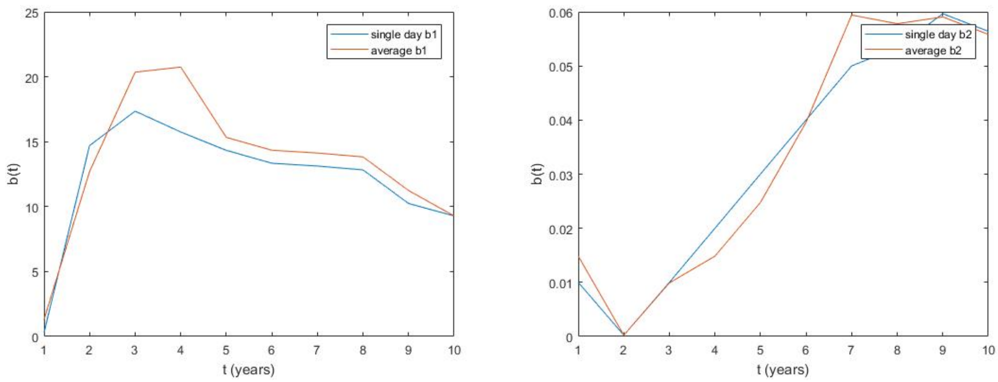
Tab. 5.6: Parameter values

Model	a_1	a_2
One-factor	0.2732	0
One-factor constrained	0.2738	0
One-factor parametrized	1	0
Two-factor parametrized	0.2183	0.3617

5.6 Historical distribution of $b(t)$ parameters

In this last section the shape of the $b(t)$ function over the last five years is considered. Quarterly dates, going back to the year 2013 are used to check the b functions are consistent over longer periods of time. Monthly end days are chosen, and the calibration is carried out on 20 data points each consisting of a quarterly date. Because it was shown that the one-factor model provides a poor fit of the caps/floor market only the two-factor model is considered for the caps/floor market over the five-year period. This is shown in figure 5.17. Similarly, only the one-factor model for the swaption market is used, as it was shown there was a good fit to the market data. This is highlighted in figure 5.18 below.

Two-factor model caps/floor market



(a) b_1 **(b)** b_2

Fig. 5.17

One-factor model swaption market

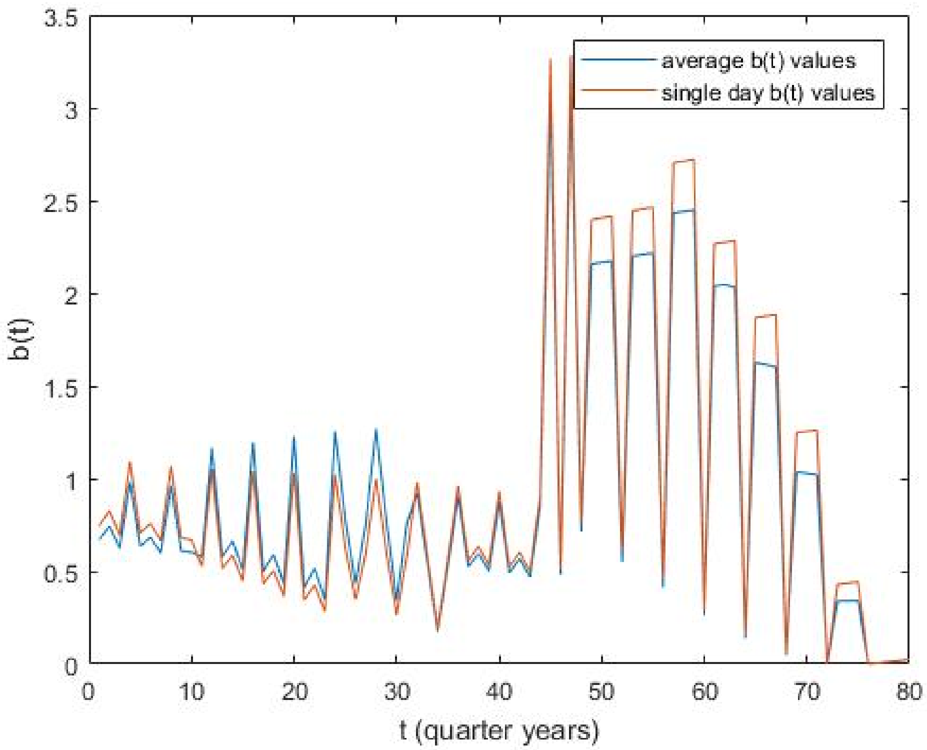


Fig. 5.18: b function

These results show that the calibrations that were carried out would hold on other days, as the $b(t)$ functions follow, on average, similar forms to the $b(t)$ functions in the previous sections historically over the last five years. Consequently, the parametrised forms that have been calibrated in this chapter, would hold on other days of data.

Chapter 6

Conclusion

The one-factor and two-factor model had varied degrees of success in reproducing the volatility surfaces. The one-factor model performed poorly in the caps/floor market while the two-factor model had a significantly improved fit to the market. The swaption volatility surface was recovered almost perfectly by the one-factor model as the smile present in the market was not considered. Further research could explore the performance of the two-factor model in hedging.

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