APPLICATIONS

OF THE

GAUGE THEORY/GRAVITY CORRESPONDENCE

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Abstract

The gauge theory/gravity correspondence encompasses a variety of different specific dualities. We study examples of both Super Yang-Mills/type IIB string theory and Super Chern-Simons-matter/type IIA string theory dualities. We focus on the recent ABJM correspondence as an example of the latter.

We conduct a detailed investigation into the properties of D-branes and their operator duals. The D2-brane dual giant graviton on $AdS_4 \times \mathbb{CP}^3$ is initially studied: we calculate its spectrum of small fluctuations and consider open string excitations in both the short pp-wave and long semiclassical string limits.

We extend Mikhailov's holomorphic curve construction to build a giant graviton on $AdS_5 \times T^{1,1}$. This is a non-spherical D3-brane configuration, which factorizes at maximal size into two dibaryons on the base manifold $T^{1,1}$. We present a fluctuation analysis and also consider attaching open strings to the giant's worldvolume. We finally propose an ansatz for the D4-brane giant graviton on $AdS_4 \times \mathbb{CP}^3$, which is embedded in the complex projective space.

The maximal D4-brane giant factorizes into two $\mathbb{CP}^2$ dibaryons. A comparison is made between the spectrum of small fluctuations about one such $\mathbb{CP}^2$ dibaryon and the conformal dimensions of BPS excitations of the dual determinant operator in ABJM theory.

We conclude with a study of the thermal properties of an ensemble of pp-wave strings under a Lunin-Maldacena deformation. We investigate the possibility that the Hagedorn temperature - dual to the temperature of the confinement/deconfinement transition in planar SYM theory - may be a universal quantity, at least under a partial breaking of supersymmetry.

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Chapter 3: A. Hamilton, J. Murugan, A. Prinsloo and M. Strydom, "*A note on dual giant gravitons in* $AdS_4 \times \mathbb{CP}^3$", J. High Energy Phys. **0409**, 132 (2009), `0901.0009 [hep-th]`.

Chapter 4: J. Murugan and A. Prinsloo, "*ABJM dibaryon spectroscopy*", in preparation.

Chapter 5: A. Hamilton, J. Murugan and A. Prinsloo, "*Lessons from giant gravitons on* $AdS_5 \times T^{1,1}$", to appear in the J. High Energy Phys., `1001.2306 [hep-th]`.

Chapter 6: A. Hamilton, J. Murugan and A. Prinsloo, "*A note on the universality of the Hagedorn behavior of pp-wave strings*", J. High Energy Phys. **0108**, 008 (2008), `0712.3059 [hep-th]`.

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other university for a degree and that it represents my own work.

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Contents

List of Figures

Part I

Introduction and Overview

Introduction

Over the last decade, a substantial body of evidence has emerged that various string theories on anti-de Sitter (AdS) spacetimes are dual to conformal field theories (CFTs) in flat space. Somehow it seems that the degrees of freedom encoded in the gauge theories (albeit mainly superconformal theories, unlike those observed in nature) rearrange themselves into stringy degrees of freedom in the strong coupling regime. This surprising phenomenon points to a fundamental link between gravity and modern particle physics, which is, as yet, improperly understood.

The original idea is owing to t'Hooft [1]: $U(N)$ gauge theories, with N large, admit a $\frac{1}{N}$ perturbative expansion in which the Feynman diagrams are organized in terms of the genus of the surface upon which they are drawn (to leading order, only planar diagrams survive). This resembles the genus expansion in g_s of an interacting string. The effective coupling constant at large N - associated with a loop expansion - is the 't Hooft coupling $\lambda \equiv g^2 N$, where g is the original coupling constant in the gauge theory. It is also possible [2] to extend the 't Hooft limit to include $U(N) \times U(N)$ gauge theories, with the perturbative expansion in $\frac{1}{N^2}$.

In 1997, Maldacena [3] proposed a concrete example of a gauge theory/gravity duality. This AdS/CFT correspondence links $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory with type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Various other $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondences between 4-dimensional CFTs and type IIB string theories on AdS_{d+1} spacetimes were later suggested [4] - [8]. Being strong/weak coupling dualities, these are difficult to test, but allow for the possibility of solving otherwise intractable problems by mapping them onto the appropriate dual. For example, applications include the computation of gluon scattering amplitudes in $\mathcal{N} = 4$ SYM theory at strong coupling [9], a conjectured universal lower bound on the shear-viscosity to entropy density ratio of the strongly coupled quark-gluon plasma [10] and a dual description of condensed matter systems in the vicinity of a quantum critical point [11]. Despite differing in specific details, however, these $\mathrm{AdS}_5/\mathrm{CFT}_4$ dualities are of the same general form.

There was widespread interest and, indeed, much excitement when, in 2008, Aharony, Bergman, Jafferis and Maldacena (ABJM) proposed a very different example of an AdS_4/CFT_3 correspondence [2] between an $\mathcal{N} = 6$ Super Chern-Simons (SCS)-matter theory and M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, which becomes type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ upon compactification. Not only does this provide us with a new, interesting SCS-matter/type IIA string theory duality, but for the first time directly links a CFT with an M-theory. It might therefore prove feasible to study non-perturbative membranes from a gauge theoretic perspective.

Let us consider a generic AdS_{d+1}/CFT_d correspondence. The CFT in the Euclidean space $\mathbb{R}^d$ (after a Wick rotation) can be reformulated as a CFT on the boundary $S^{d-1} \times \mathbb{R}$ of the AdS_{d+1} spacetime[1] [12]. The radial coordinate in $\mathbb{R}^d$ is set to $r \equiv e^\tau$, so that the metric on $\mathbb{R}^d$ can be written as follows:

$$ds^2_{\mathbb{R}^d} = dr^2 + r^2 d\Omega^2_{d-1} = e^{2\tau}\left(d\tau^2 + d\Omega^2_{d-1}\right),$$

which, up to the conformal factor $e^{2\tau}$, is the metric on $S^{d-1} \times \mathbb{R}$. Notice that dilatations $r \to e^\alpha r$ become time translations $\tau \to \tau + \alpha$, so that scaling dimensions are mapped to energies. Correlators $\langle \mathcal{O}_1^\dagger \mathcal{O}_2 \rangle$ of local operators in the original CFT_d become overlaps $\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle$ of physical states at spatial infinity by the operator-state correspondence. The AdS_{d+1}/CFT_d duality allows us to associate physical states in the CFT on the boundary $S^{d-1} \times \mathbb{R}$ with string states in the bulk AdS_{d+1} spacetime. The following dictionary should therefore exist between the original CFT_d and the string theory on an AdS_{d+1} spacetime: local operators are dual to string states in such a way that anomalous dimensions map to string excitation energies. Moreover, these operators carry $\mathcal{R}$-charge under global supersymmetry transformations, which maps to the string angular momentum in the compact space.

More recently, it has become apparent that organizing these operators according to their $\mathcal{R}$-charge[2] allows us to distinguish, in the gauge theory, between gravitons [14], strings [15], membranes [16] - [19] and even whole new geometries [14, 20]. A suitable operator basis for single particle states in the CFT must be orthogonal with respect to the two-point correlation function. When the $\mathcal{R}$-charge $J \ll N$, a single trace operator basis will suffice - the correlators involve terms of $O\left(\frac{1}{N}\right)$, which are suppressed. However, when the $\mathcal{R}$-charge becomes comparable with the rank of the gauge group, a calculation of the two-point correlation function involves combinatoric factors of $O(N)$, canceling the $\frac{1}{N}$ suppression. A new operator basis must then be constructed - this takes the form of Schur polynomials of fields [17]. Therefore, to summarize, single trace

[1] The spacetime takes the form $AdS_{d+1} \times X^{9-d}$, but the specifics of the compact space X^{9-d} are unnecessary here.

[2] For a recent concise review, see [13].

operators with $\mathcal{R}$-charge of $O(1)$ and $O(\sqrt{N})$ map to gravitons and strings respectively, while Schur polynomial operators with $\mathcal{R}$-charge of $O(N)$ and $O(N^2)$ are associated with membranes and geometries.

An investigation of the string degrees of freedom from the perspective of the dual gauge theory therefore involves a study of single trace operators. Minahan and Zarembo noticed [21] that a single trace operator composed of scalar fields in $\mathcal{N}=4$ SYM theory can be mapped to an integrable $SO(6)$ spin chain with nearest-neighbour interactions. The one-loop matrix of anomalous dimensions is associated with the spin chain Hamiltonian. This result was extended to include other sectors of the gauge theory [22]. Similarly, single trace operators composed of scalar fields in $\mathcal{N}=6$ SCS-matter theory can be mapped to an integrable $SU(4)$ spin chain [23]. This spin chain technology greatly simplifies computations in the CFT, as well as providing insight into the integrable structures contained therein.

Shortly after the original Maldacena conjecture, the quantum numbers of $\frac{1}{2}$-BPS operators, the dimensions of which are protected by supersymmetry, were matched to those of type IIB supergravity states [24]. Berenstein, Maldacena and Nastase (BMN) then studied a class of 'near-BPS' operators [15], which can be mapped to long spin chains with relatively few excitations. It turns out that the effective coupling $\tilde{\lambda}=\frac{\lambda}{J^2}$ in this sector of the gauge theory depends also on the length J of the operator (in this case, equal to its $\mathcal{R}$-charge). The BMN double scaling limit λ, $J\to\infty$, with $\tilde{\lambda}=\frac{\lambda}{J^2}\ll 1$ held fixed, allowed [15] to circumvent the strong/weak coupling problem and match the anomalous dimensions of near-BPS operators with the excitation energies of type IIB closed pp-wave strings. This proved a major success for the AdS/CFT correspondence. In addition, Hofman and Maldacena constructed long 'giant magnon' string configurations dual to magnon excitations of the $\mathcal{N}=4$ SYM spin chain [25]. The near-flat space limit of [26] interpolates between the pp-wave and giant magnon sectors of type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. Type IIB string theory on $\mathrm{AdS}_5\times\mathrm{T}^{1,1}$ has also been extensively studied: the associated pp-wave [27, 28] and near-flat space [29] geometries were obtained, and giant magnons constructed [30].

Recent studies [31] - [38] of the closed string sector of type IIA string theory on $\mathrm{AdS}_4\times\mathbb{CP}^3$ suggest that the ABJM duality is somewhat more subtle. It is possible to take a Penrose limit about a null geodesic in the $\mathrm{AdS}_4\times\mathbb{CP}^3$ spacetime and study closed strings on the resulting pp-wave background. The pp-wave string excitation energies were compared with the anomalous dimensions of long near-BPS operators [31, 32], but a mismatch was found. Long semiclassical strings and giant magnons were also investigated [34] - [36], together with strings on the near-flat space geometry [37]. These results seem to indicate that BMN scaling is violated in ABJM theory: long near-BPS operators scale like $\frac{1}{J^2} f(\lambda)$, where $f(\lambda) \sim \lambda$ when $\lambda \gg 1$ (the string theory is weakly coupled) and $f(\lambda) \sim \lambda^2$ when $\lambda \ll 1$ (the perturbative regime of the gauge theory).

This discussion would be incomplete without some mention of the open string sectors of the type IIB and type IIA string theories. In studying open string configurations, an understanding of D-branes is vital - these are non-perturbative dynamic membranes embedded in the background spacetime. Open string excitations of dual and sphere D3-brane giant gravitons on $\mathrm{AdS}_5 \times \mathrm{S}^5$ were considered in [39] - [41]. The dual $\mathcal{N} = 4$ SYM operators involve words attached to Schur polynomials. In the BMN limit, the anomalous dimensions of these words were shown to match the open pp-wave string excitation energies. The open string sector of $\mathrm{AdS}_4 \times \mathbb{CP}^3$ remains largely unexplored, partly due to our current lack of knowledge concerning D-brane configurations on this background, as well as the more complex nature of the SCS-matter open spin chain.

In seeking to better understand the manner in which the gauge theory degrees of freedom are encoded in the string theory, it is crucial to investigate the gauge theory/gravity correspondence in as many different laboratories as possible. Towards this end, we study applications of both $\mathrm{AdS}_5/\mathrm{CFT}_4$ and $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities.

Overview of thesis

This thesis is organized in parts, which contain related chapters, as outlined below:

Part I contains introductory information concerning the SYM/type IIB string theory and SCS-matter/type IIA string theory dualities. In chapter 1, we discuss both the original Maldacena conjecture, as well as the correspondence suggested by Klebanov and Witten between an $\mathcal{N} = 1$ SYM theory and type IIB string theory on $AdS_5 \times T^{1,1}$. The new ABJM duality is described in chapter 2.

Part II involves an investigation of the properties of D-branes and giant gravitons, as well as their gauge theory counterparts. This is the primary focus of this thesis and several new results are presented in chapters 3, 4 and 5 relating to giant gravitons and dibaryons in type IIB string theory on $AdS_5 \times T^{1,1}$ and type IIA string theory on $AdS_4 \times \mathbb{CP}^3$.

Marginal deformations of $\mathcal{N} = 4$ SYM theory are discussed in part III. In the special case of a γ-deformation, we describe the construction of the gravitational dual - type IIB string theory on a Lunin-Maldacena background. In chapter 6, we study the thermodynamics of an ensemble of pp-waves strings on the Lunin-Maldacena background, with a view towards establishing whether the Hagedorn temperature is a universal quantity, invariant across a class of type IIB string theories on AdS_5 spacetimes.

A summary of results, together with concluding remarks, are presented in part IV. We discuss extensions of this work and describe various avenues for future research in the field of gauge theory/gravity duality.

Chapter 1

Super Yang-Mills/type IIB string theories

Type IIB string theory is an $\mathcal{N} = 2$ supersymmetric chiral[1] theory in 10 dimensions. Its associated supergravity theory - involving the low energy limit $\alpha' \to 0$, in which the string tension becomes large and only massless modes survive - is type IIB 10D supergravity. Here the bosonic degrees of freedom are the metric $g_{\mu\nu}$, dilaton ϕ and 2-form Neveu-Schwarz (NS) B-field b_2, as well as the even dimensional Ramond-Ramond (RR) potential forms c_0, c_2 and c_4, which couple to odd dimensional branes. The fermionic field content consists of a Weyl gravitino $\psi_{\mu,\alpha}$ and dilatino λ_α [42]. We often speak of string theories on fixed background spacetimes (at low energies, backreaction may be neglected). These can be thought of as coherent states of a large number of gravitons, dilatons, etc. The fixed background fields G, Φ, B and C_n (together with vanishing fermionic superpartners, for bosonic solutions) must hence solve the type IIB 10D supergravity equations of motion.

Let us now consider two possible low energy descriptions of N coincident D3-branes in type IIB string theory on some, as yet unspecified, background spacetime [3, 43]:

The lowest energy, massless modes of open strings ending on these D3-branes are described by a 4-dimensional non-abelian SYM theory with an $SU(N)$ gauge group[2]. Scalar fields describe the six transverse coordinates, while the gauge fields correspond to the four worldvolume degrees of freedom. There are also spinor fields, which descend

[1] In the Green-Schwarz formalism, each fermionic field consists of two 16-component Weyl-Majorana spinor coordinates with the same chirality, which can be combined into a single 32-component Weyl spinor.

[2] The gauge group is actually $U(N) = SU(N) \times U(1)$, but the $U(1)$ subgroup decouples.

from the spinor coordinates in the bulk string theory.

The D3-branes are massive objects, which deform the geometry around them. A type IIB superstring in the vicinity of the membranes propagates on the near-horizon spacetime, which takes the form $\mathrm{AdS}_5 \times \mathrm{X}^5$, with X^5 some 5-dimensional compact space. From the point of view of an observer at infinity, all the string modes are gravitationally redshifted to low energies, yielding the full type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{X}^5$.

Since these are dual descriptions of the same system, we obtain various $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondences between 4-dimensional SYM theories and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{X}^5$ spacetimes. We shall focus on two examples:

N **coincident D3-branes in 10-dimensional Minkowski spacetime** $\mathcal{M}_{10}$**:** This yields the original AdS/CFT correspondence [3] (or Maldacena conjecture) between $\mathcal{N} = 4$ SYM theory, with an $SU(N)$ gauge group, and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

N **coincident D3-branes in** $\mathcal{M}_4 \times \mathcal{C}$ **at the conic singularity of** $\mathcal{C}$ **in** $\mathbb{C}^4$**:** The duality which arises is between an $\mathcal{N} = 1$ SYM theory (known as Klebanov-Witten theory [4]) with an $SU(N) \times SU(N)$ gauge group, and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$.

We shall discuss both sides of these conjectured gauge theory/gravity correspondences, as well as the dictionary between them.

1.1 The Maldacena conjecture

1.1.1 $\mathcal{N} = 4$ Super Yang-Mills theory

$\mathcal{N} = 4$ SYM theory is a maximally supersymmetric 4-dimensional field theory with an $SU(N)$ gauge group. It was originally constructed [44] from a 10-dimensional $\mathcal{N} = 1$ SYM theory containing a single gauge superfield (composed of a gauge field and a Weyl-Majorana spinor). Dimensional reduction then yields the $\mathcal{N} = 4$ gauge multiplet[3] $(A_\mu, \chi^a_\alpha, \phi^k)$ consisting of the gauge field A_μ, four 4-component Majorana spinors χ^a_α and six real scalar fields ϕ^k. These fields all transform in the adjoint representation of $SU(N)$, and have conformal dimensions $[A_\mu] = [\phi^k] = 1$ and $[\chi^a_\alpha] = \frac{3}{2}$.

[3] The scalar fields ϕ^k are the additional components of the original gauge field corresponding to the six extra dimensions, which are reduced. The original Weyl-Majorana spinor can be written in terms of the four Majorana spinors χ^a_α.

The $\mathcal{N} = 4$ SYM action takes the form [43, 45]

$$S = \frac{2}{g_{\rm YM}^2} \int d^4x \ \mathrm{tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \sum_k D_\mu \phi^k D^\mu \phi^k + \frac{1}{4} \sum_{j,k} \left[\phi^j, \phi^k \right]^2 + \text{fermions} \right\}, \tag{1.1}$$

with field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu]$. Here $D_\mu \phi^k \equiv \partial_\mu \phi^k + i \left[A_\mu, \phi^k \right]$ is the covariant derivative and $g_{\rm YM}$ is the Yang-Mills coupling constant. There is also an instanton term proportional to the topological charge [43].

Let us define the complex scalar fields

$$X \equiv \phi^1 + i\phi^2, \qquad Y \equiv \phi^3 + i\phi^4 \qquad \text{and} \qquad Z \equiv \phi^5 + i\phi^6. \tag{1.2}$$

When written in $\mathcal{N} = 1$ superspace, $\mathcal{N} = 4$ SYM theory contains a gauge superfield $\mathcal{V}$ and three chiral superfields $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ associated with the above scalar fields. The superpotential is then given by [43]

$$\mathcal{W} = \tfrac{1}{2}\, g_{\rm YM}\, \mathrm{tr}\,(\mathcal{XYZ} - \mathcal{XZY})\,. \tag{1.3}$$

There is an $SU(4) \cong SO(6)$ $\mathcal{R}$-symmetry group and an $SO(2,4)$ group of conformal transformations (consisting of translations, Lorentz transformations, scalings and special conformal transformations). This conformal symmetry remains unbroken at the quantum level. Due to its maximally supersymmetric nature, $\mathcal{N} = 4$ SYM theory is finite - the renormalization group β-functions, which describe the dependence of the coupling $g_{\rm YM}$ on the mass scale, vanish identically [43, 46].

Gauge invariant single trace operators can be constructed by tracing over the $SU(N)$ gauge group indices of a product of $\mathcal{N} = 4$ SYM scalar fields. For example, the chiral primary[4] single trace operators, which belong to shortened $\frac{1}{2}$-BPS multiplets of operators, are given by

$$\mathcal{O}^{i_1 i_2 \ldots i_n} = \mathrm{str}\left(\phi^{i_1} \phi^{i_2} \ldots\ \phi^{i_n} \right), \tag{1.4}$$

which denotes a symmetrized trace over n scalar fields. The dimensions of these chiral primaries are protected from quantum corrections by supersymmetry [42, 43].

1.1.2 Type IIB string theory on $\mathbf{AdS_5 \times S^5}$

A maximally supersymmetric solution of the type IIB 10D supergravity equations of motion is $\mathrm{AdS}_5 \times \mathrm{S}^5$ [43, 47]. The background metric is given by

$$ds^2 = R^2 \left\{ ds^2_{\mathrm{AdS}_5} + ds^2_{\mathrm{S}^5} \right\}, \tag{1.5}$$

[4]The primary field/operator in a multiplet has the lowest dimension and is annihilated by all the supercharges. It can be used to build up the other descendent fields/operators by acting with the conjugate supercharges [42].

where R is the radius of the anti-de Sitter and 5-sphere spaces, with individual metrics

$$ds^2_{\mathrm{AdS}_5} = -\left(1+r^2\right)dt^2 + \frac{dr^2}{(1+r^2)} + r^2\left(d\alpha_1^2 + \cos^2\alpha_1\, d\beta_1^2 + \sin^2\alpha_1\, d\beta_2^2\right) \tag{1.6}$$

$$ds^2_{\mathrm{S}^5} = d\theta_1^2 + \cos^2\theta_1\, d\phi_1^2 + \sin^2\theta_1\left(d\theta_2^2 + \cos^2\theta_2\, d\phi_2^2 + \sin^2\theta_2\, d\phi_3^2\right). \tag{1.7}$$

Note that there is a 3-sphere embedded in both S^5 and AdS_5. The dilaton Φ vanishes, together with the B-field B_2. The only non-trivial field strength is the self-dual 5-form $F_5 = \mathcal{F} + *\mathcal{F} = dC_4$, with

$$\mathcal{F} \equiv 4R^4\,\mathrm{vol}(S^5) = 4R^4\ \cos\theta_1\sin^3\theta_1\cos\theta_2\sin\theta_2\, d\theta_1\wedge d\theta_2\wedge d\phi_1\wedge d\phi_2\wedge d\phi_3. \tag{1.8}$$

1.1.3 Dictionary

The original gauge theory/gravity correspondence between the $\mathcal{N}=4$ SYM theory and type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ is a strong/weak coupling duality. The 't Hooft coupling in the gauge theory is related[5] to the radii of the AdS_5 and S^5 spaces [3, 43]:

$$\lambda \equiv g^2_{\mathrm{YM}}N = R^4. \tag{1.9}$$

A strongly (weakly) coupled SYM theory corresponds to small (large) spacetime curvature. When the string length is small by comparison to the size of the space in which it lives, the gauge theory is strongly coupled. On the other hand, allowing for perturbative expansions in λ in the SYM theory takes us far from the supergravity regime of the type IIB string theory.

In Maldacena's D3-brane construction [3], the number of coincident branes simply corresponds to the rank N of the SYM gauge group. Now, each D3-brane carries one unit of charge with respect to the 4-form potential C_4, so N must be related to the flux of the 5-form field strength $F_5 = dC_4$ through the 5-sphere as follows [42]:

$$N = \frac{1}{(2\pi)^4}\int_{S^5} F_5 = \frac{R^4}{4\pi}. \tag{1.10}$$

Isometries in $\mathrm{AdS}_5\times\mathrm{S}^5$ correspond to superconformal symmetries in the gauge theory. The $SO(2,4)$ group of AdS_5 isometries matches the conformal group, whereas the $SO(6)$ rotational symmetry of S^5 is associated with the $\mathcal{R}$-symmetry. The three $U(1)$ charges of this $SU(4)_{\mathcal{R}}$ are dual to the angular momenta $J_i \equiv -i\frac{\partial}{\partial\phi_i}$ on the 5-sphere [42, 43].

[5] Here we make use of units in which $\alpha' = 1$.

1.2 The Klebanov-Witten duality

1.2.1 Klebanov-Witten theory

We begin by considering an $\mathcal{N} = 1$ SYM theory in 4-dimensions containing two gauge superfields, and two sets of two left-handed chiral superfields $\mathcal{A}_i$ and $\mathcal{B}_i$ in the $(N, \bar{N})$ and $(\bar{N}, N)$ bifundamental representations of the $SU(N) \times SU(N)$ gauge group respectively. Flowing to the IR fixed point, we then perturb by the non-renormalizable marginal superpotential

$$\mathcal{W} = \tfrac{1}{2}\lambda\, \epsilon^{ij}\epsilon^{kl}\, \mathrm{tr}\left(\mathcal{A}_i\mathcal{B}_k\mathcal{A}_j\mathcal{B}_l\right). \tag{1.11}$$

This is Klebanov-Witten theory [4]. There is an $SU(2)_A \times SU(2)_B$ symmetry group - with one $SU(2)$ acting on the $\mathcal{A}_i$'s and the other on the $\mathcal{B}_i$'s - as well as a $U(1)$ $\mathcal{R}$-symmetry group under which the chiral superfields have $\mathcal{R}$-charge $\frac{1}{2}$. The scalar field components of these chiral multiplets have conformal dimension $[A_i] = [B_i] = \frac{3}{4}$, and carry baryon number 1 and -1 with respect to the global $U(1)$ symmetry group.

Single trace operators in Klebanov-Witten theory must be constructed from *composite* fields with gauge group indices in only *one* $SU(N)$. The primary single trace operators, with the lowest dimensions for a given $\mathcal{R}$-charge, involve the composite scalar fields A_iB_j. Symmetrizing over the i and j indices separately, we obtain [4, 48]

$$\mathcal{O}^{i_1 i_2 \dots i_n,\; j_1 j_2 \dots j_n} = \mathrm{str}\left\{\left(A^{i_1}B^{j_1}\right)\left(A^{i_2}B^{j_2}\right)\dots\left(A^{i_n}B^{j_n}\right)\right\}, \tag{1.12}$$

which are primary single trace operators, protected by supersymmetry, with vanishing anomalous dimensions.

1.2.2 Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$

Another solution of type IIB 10D supergravity is $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$, where $\mathrm{T}^{1,1}$ is the base manifold of a cone in $\mathbb{C}^4$ [4, 48]. The background metric is given by

$$ds^2 = R^2\left\{ds^2_{\mathrm{AdS}_5} + ds^2_{\mathrm{T}^{1,1}}\right\}, \tag{1.13}$$

with R the radius of the anti-de Sitter and $\mathrm{T}^{1,1}$ spaces. The dilaton Φ and B-field B_2 are again zero, and the self-dual 5-form $F_5 = \mathcal{F} + *\mathcal{F} = dC_4$ is the only non-vanishing field strength, where now

$$\mathcal{F} \equiv 4R^4\, \mathrm{vol}(\mathrm{T}^{1,1}) = \tfrac{1}{27}R^4 \sin\theta_1 \sin\theta_2\, d\theta_1 \wedge d\theta_2 \wedge d\psi \wedge d\phi_1 \wedge d\phi_2. \tag{1.14}$$

This $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ background solution preserves $\frac{1}{4}$ of the supersymmetries [27] of the type IIB supergravity effective action[6].

The base manifold $\mathrm{T}^{1,1}$

The cone $\mathcal{C}$, which is embedded in $\mathbb{C}^4$, is described by the four complex coordinates z^A satisfying $z^1 z^2 = z^3 z^4$. These may be parameterized as follows [49]:

$$z^1 = r_{\mathcal{C}}^{\frac{3}{2}} \sin\tfrac{\theta_1}{2} \sin\tfrac{\theta_2}{2}\; e^{\frac{1}{2}i(\psi-\phi_1-\phi_2)} \qquad z^2 = r_{\mathcal{C}}^{\frac{3}{2}} \cos\tfrac{\theta_1}{2} \cos\tfrac{\theta_2}{2}\; e^{\frac{1}{2}i(\psi+\phi_1+\phi_2)}$$
$$z^3 = r_{\mathcal{C}}^{\frac{3}{2}} \cos\tfrac{\theta_1}{2} \sin\tfrac{\theta_2}{2}\; e^{\frac{1}{2}i(\psi+\phi_1-\phi_2)} \qquad z^4 = r_{\mathcal{C}}^{\frac{3}{2}} \sin\tfrac{\theta_1}{2} \cos\tfrac{\theta_2}{2}\; e^{\frac{1}{2}i(\psi-\phi_1+\phi_2)} \tag{1.15}$$

in terms of the overall radius $r_{\mathcal{C}}$, and angles $\theta_i \,\epsilon\, [0,\pi]$, $\phi_i \,\epsilon\, [0,2\pi]$ and $\psi \,\epsilon\, [0,4\pi]$. The base manifold $\mathrm{T}^{1,1}$ is obtained by setting $r_{\mathcal{C}}$ to a constant, chosen to be unity.

Remarkably, it is possible to construct [49] a Kähler, Ricci-flat metric on $\mathcal{C}$, corresponding to the Kähler potential $\mathcal{F}(r_{\mathcal{C}}^2) = r_{\mathcal{C}}^2$, which is given by

$$ds_{\mathcal{C}}^2 = dr_{\mathcal{C}}^2 + r_{\mathcal{C}}^2\, ds_{T^{1,1}}^2, \tag{1.16}$$

where the metric on the base manifold $\mathrm{T}^{1,1}$ takes the form

$$ds_{\mathrm{T}^{1,1}}^2 = \tfrac{1}{9}\left[d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right]^2 + \tfrac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right) + \tfrac{1}{6}\left(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right). \tag{1.17}$$

This describes a $\frac{SU(2)\times SU(2)}{U(1)}$ manifold - the $U(1)$ identifies the coordinates $\psi_1 \equiv \psi_2 \equiv \frac{1}{2}\psi$ in each of the $SU(2)$ spaces parameterized by the Euler angles $(\theta_i, \phi_i, \psi_i)$. Alternatively, we may view $\mathrm{T}^{1,1}$ as two 2-spheres (θ_i, ϕ_i) and an additional non-trivial $U(1)$ fibre parameterized by ψ. In these coordinates, the volume element is given by

$$\mathrm{vol}(\mathrm{T}^{1,1}) = \tfrac{1}{108} \sin\theta_1 \sin\theta_2\; d\theta_1 \wedge d\theta_2 \wedge d\psi \wedge d\phi_1 \wedge d\phi_2 \tag{1.18}$$

and, integrating, we find that $\frac{16}{27}\pi^3$ is the volume of $\mathrm{T}^{1,1}$.

1.2.3 Dictionary

Due to the non-renormalizability of the $\mathcal{N} = 1$ SYM theory, less can be said about this example of an $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, which links Klebanov-Witten theory and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$. The rank N of the $SU(N) \times SU(N)$ gauge

[6] 8 of the 32 components in the Weyl gravitino spinor remain invariant under supersymmetry transformations.

group is related to the flux of the 5-form field strength through $T^{1,1}$ [48]:

$$N = \frac{1}{(2\pi)^4}\int_{T^{1,1}} F_5 = \frac{4R^4}{27\pi}. \tag{1.19}$$

The group $SO(2,4)$ of AdS_5 isometries again matches the conformal group. The isometry group of $T^{1,1}$ is $SO(3)\times SO(3)\times U(1)$, which corresponds to the $SU(2)\times SU(2)\times U(1)_{\mathcal{R}}$ symmetry group of Klebanov-Witten theory [4, 48]. The $\mathcal{R}$-charge of an operator in Klebanov-Witten theory is associated with the angular momentum $J \equiv -2i\frac{\partial}{\partial\psi}$ of the dual type IIB string state along the fibre direction.

Chapter 2

Super Chern-Simons-matter/type IIA string theories

Type IIA string theory is an $\mathcal{N} = 2$ supersymmetric non-chiral[1] theory in 10 dimensions related by a T-duality transformation to type IIB string theory. M-theory is the strong coupling limit of type IIA string theory and contains no strings, but rather M2- and M5-branes in 11 dimensions. Due to the higher dimensional nature of their worldvolume spaces, these M-branes - the fundamental objects in M-theory - cannot be quantized perturbatively, and hence far less is known about M-theory than about any of the five consistent superstring theories.

The effective field theory, which describes only the massless modes, associated with M-theory is maximally supersymmetric $\mathcal{N} = 8$ 11D supergravity. The field content consists of the metric $g_{\mu\nu}$, the 3-form potential a_3 (coupling electrically to M2-branes and magnetically to M5-branes) and the Majorana spinor $\psi_{\mu,\alpha}$. Type IIA 10D supergravity can be obtained from 11D supergravity via dimensional reduction. The bosonic sector contains the metric $g_{\mu\nu}$, dilaton ϕ and 2-form NS B-field b_2, as well as the odd dimensional RR potential forms c_1 and c_3, which couple to even dimensional branes. The fermionic degrees of freedom consist of a gravitino $\psi_{\mu,\alpha}$ and dilatino λ_α [42, 47, 50].

Until recently, the worldvolume theory of N coincident M2-branes, even in flat Minkowski spacetime, was unknown. A maximally supersymmetric[2] $\mathcal{N} = 8$ gauge theory in

[1] A fermionic field consists of two 16-component Weyl-Majorana spinor coordinates with opposite chirality, so the combined 32-component spinor is not Weyl.

[2] 16 supersymmetries - half the 32 supersymmetries of the original $\mathcal{N} = 8$ 11D supergravity background solution - should be preserved by the M2-branes, i.e. there should be eight 2-component spinor coordinates in 3 dimensions.

3 dimensions was expected. It should contain eight scalar fields (corresponding to the transverse directions) transforming under the $SO(8)$ $\mathcal{R}$-symmetry group. Bagger, Lambert and (independently) Gustavsson (BLG) constructed [51] a 3-algebra Super Chern-Simons (SCS)-matter theory with matter in the bifundamental representation of the $SU(2)_k \times SU(2)_{-k}$ gauge group, and opposite level numbers k and $-k$. The special case $k = 2$ yields the worldvolume theory of two M2-branes at a $\mathbb{R}^8/\mathbb{Z}_2$ singularity. Aharony, Bergman, Jafferis and Maldacena (ABJM) were able [2] to extend this construction to the worldvolume theory of N coincident M2-branes at a $\mathbb{C}^4/\mathbb{Z}_k$ singularity - an $\mathcal{N} = 6$ SCS-matter theory with matter in the bifundamental representation of the $U(N)_k \times U(N)_{-k}$ gauge group (known as ABJM theory).

The near-horizon geometry of N coincident M2-branes placed at the tip of the cone in $\mathcal{M}_3 \times \mathbb{C}^4/\mathbb{Z}_k$ is the orbifold $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$. Identifying the worldvolume theory of these M2-branes with M-theory in the near-horizon geometry, ABJM postulated [2]:

The M-theoretic version of an $\mathbf{AdS_4/CFT_3}$ correspondence: $\mathcal{N} = 6$ SCS-matter theory with a $U(N)_k \times U(N)_{-k}$ gauge group (with $k \ll N^{\frac{1}{5}}$) is dual to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$.

It is also possible to compactify M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7$ to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. (This is equivalent to taking the large k limit of M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$.) The duality can therefore be reformulated as follows [2]:

The string-theoretic version of an $\mathbf{AdS_4/CFT_3}$ correspondence: $\mathcal{N} = 6$ SCS-matter theory with a $U(N)_k \times U(N)_{-k}$ gauge group (with $N^{\frac{1}{5}} \ll k \ll N$) is dual to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

The M-theoretic version of this duality is the first gauge theory/gravity correspondence involving M-theory and we might therefore hope to gain insight into non-perturbative features, which are otherwise inaccessible. However, we shall concentrate on the second version of this $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, involving type IIA string theory, since all the technology developed to study $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondences becomes applicable.

2.1 ABJM theory

ABJM theory is an $\mathcal{N} = 6$ SCS-matter theory in 3 dimensions with a $U(N)_k \times U(N)_{-k}$ gauge group, and opposite level numbers k and $-k$. Aside from the gauge fields[3] A_μ and

[3] All the other fields in the gauge multiplets are auxiliary [52].

$\hat{A}_\mu$, there are two sets of two chiral multiplets $(A_i, \psi_\alpha^{A_i})$ and $(B_i, \psi_\alpha^{B_i})$, corresponding to the chiral superfields $\mathcal{A}_i$ and $\mathcal{B}_i$ in $\mathcal{N} = 2$ superspace, which transform in the $(N, \bar{N})$ and $(\bar{N}, N)$ bifundamental representations respectively. The scalar fields have conformal dimension $[A_i] = [B_i] = \frac{1}{2}$, whereas $[A_\mu] = [\hat{A}_\mu] = [\psi_\alpha^{A_i}] = [\psi_\alpha^{B_i}] = 1$ for the gauge and 2-component spinor fields [2, 52].

These scalar fields[4] can be arranged into the multiplet $Y^a = (A_1, A_2, B_1^\dagger, B_2^\dagger)$, with hermitian conjugate $Y_a^\dagger = (A_1^\dagger, A_2^\dagger, B_1, B_2)$, in terms of which the $N = 6$ SCS-matter action can be written as [52, 23]

$$\begin{aligned} S = \frac{k}{4\pi} \int d^3x \ \mathrm{tr} \Big\{ & \varepsilon^{\mu\nu\lambda} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \\ & + D_\mu^\dagger Y_a^\dagger D^\mu Y^a + \frac{1}{12} Y^a Y_a^\dagger Y^b Y_b^\dagger Y^c Y_c^\dagger + \frac{1}{12} Y^a Y_b^\dagger Y^b Y_c^\dagger Y^c Y_a^\dagger \\ & - \frac{1}{2} Y^a Y_a^\dagger Y^b Y_c^\dagger Y^c Y_b^\dagger + \frac{1}{3} Y^a Y_b^\dagger Y^c Y_a^\dagger Y^b Y_c^\dagger + \text{fermions} \Big\}, \end{aligned} \tag{2.1}$$

where the covariant derivatives are defined to be $D_\mu Y^a \equiv \partial_\mu Y^a + i A_\mu Y^a - i Y^a \hat{A}_\mu$ and $D_\mu^\dagger Y_a^\dagger \equiv \partial_\mu Y_a^\dagger - i A_\mu Y_a^\dagger + i Y_a^\dagger \hat{A}_\mu$. There are no kinetic terms associated with the gauge fields - they are dynamic degrees of freedom only by virtue of their coupling to matter.

When written in $\mathcal{N} = 2$ superspace [52], the ABJM superpotential takes the form

$$\mathcal{W} = \frac{2\pi}{k} \, \epsilon^{ij} \epsilon^{kl} \ \mathrm{tr} \left(\mathcal{A}_i \mathcal{B}_j \mathcal{A}_k \mathcal{B}_l \right), \tag{2.2}$$

which exhibits an explicit $SU(2)_A \times SU(2)_B$ $\mathcal{R}$-symmetry - the two $SU(2)$'s act on the doublets (A_1, A_2) and (B_1, B_2) respectively. There is also an additional $SU(2)_\mathcal{R}$ symmetry, under which $(A_1, B_1^\dagger)$ and $(A_2, B_2^\dagger)$ transform as doublets, which enhances the symmetry group to $SU(4)_\mathcal{R}$ - the multiplet Y^a transforms in the fundamental representation. The scalar fields A_i and B_i carry baryon number 1 and -1 respectively. The conformal group is $SO(2,3)$.

Single trace operators in ABJM theory must be built out of composite $\mathcal{N} = 6$ SCS-matter fields with gauge group indices in one $U(N)$. Primary single trace operators are constructed from $Y^a Y_b^\dagger$. For example, restricting to the chiral primaries [23],

$$\mathcal{O}^{a_1 a_2 \ldots a_n, \ b_1 b_2 \ldots b_n} = \mathrm{str} \left\{ \left(Y^{a_1} Y_{b_1}^\dagger \right) \left(Y^{a_2} Y_{b_2}^\dagger \right) \ldots \left(Y^{a_n} Y_{b_n}^\dagger \right) \right\}, \tag{2.3}$$

where we symmetrize over the a and b indices separately. The dimensions of these chiral primary single trace operators are protected by supersymmetry.

[4] The associated chiral and anti-chiral multiplets can be written as (Y^a, ψ_α^a) and $(Y_a^\dagger, \bar{\psi}_a^{\dot{\alpha}})$ as in [23].

2.2 M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$

A maximally supersymmetric solution of 11D supergravity is $\mathrm{AdS}_4 \times \mathrm{S}^7$ [47]. The background metric is given by

$$ds^2 = \tilde{R}^2 \left\{ ds^2_{\mathrm{AdS}_4} + 4\, ds^2_{\mathrm{S}^7} \right\}, \tag{2.4}$$

with $\tilde{R}$ and $2\tilde{R}$ the radii of the anti-de Sitter and 7-sphere spaces respectively. The metrics of these subspaces take the form

$$ds^2_{\mathrm{AdS}_4} = -\left(1+r^2\right) dt^2 + \frac{dr^2}{(1+r^2)} + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{2.5}$$

$$\begin{aligned} ds^2_{\mathrm{S}^7} &= d\alpha_1^2 + \cos^2\alpha_1\, d\beta_1^2 \\ &\quad + \sin^2\alpha_1 \left\{ d\alpha_2^2 + \cos^2\alpha_2\, d\beta_2^2 + \sin^2\alpha_2 \left(d\alpha_3^2 + \cos^2\alpha_3\, d\beta_3^2 + \sin^2\alpha_3\, d\beta_4^2 \right) \right\}, \end{aligned} \tag{2.6}$$

where the embedded 2-sphere and 5-sphere are clearly visible. The 4-form field strength $F_4 = dA_3$ is

$$F_4 = -3\tilde{R}^3\, \mathrm{vol}(\mathrm{AdS}_4) = -3\tilde{R}^3 r^2 \sin\theta\, dt\wedge dr \wedge d\theta \wedge d\varphi. \tag{2.7}$$

This is Hodge dual to the 7-form field strength $F_7 = *F_4 = dA_6$, which may be calculated to be

$$\begin{aligned} F_7 &= 3(128)\tilde{R}^6\, \mathrm{vol}(\mathrm{S}^7) \\ &= 384\, \tilde{R}^6 \cos\alpha_1 \sin^5\alpha_1 \cos\alpha_2 \sin^3\alpha_2 \cos\alpha_3 \sin\alpha_3\, d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \wedge d\alpha_4 \wedge d\alpha_5. \end{aligned} \tag{2.8}$$

Hopf fibration of S^7 over $\mathbb{CP}^3$

The complex coordinates z^A in $\mathbb{C}^4$, which are confined to S^7 by setting the overall magnitude to one, can be parameterized as follows:

$$\begin{aligned} z^1 &= \cos\zeta \sin\tfrac{\theta_1}{2}\; e^{i\left(y+\frac{1}{4}\psi-\frac{1}{2}\phi_1\right)} & z^2 &= \cos\zeta \cos\tfrac{\theta_1}{2}\; e^{i\left(y+\frac{1}{4}\psi+\frac{1}{2}\phi_1\right)} \\ z^3 &= \sin\zeta \sin\tfrac{\theta_2}{2}\; e^{i\left(y-\frac{1}{4}\psi+\frac{1}{2}\phi_2\right)} & z^4 &= \sin\zeta \cos\tfrac{\theta_2}{2}\; e^{i\left(y-\frac{1}{4}\psi-\frac{1}{2}\phi_2\right)} \end{aligned} \tag{2.9}$$

with radial coordinates $\zeta \in [0, \frac{\pi}{2}]$ and $\theta_i \in [0,\pi]$, and angular coordinates y, $\phi_i \in [0, 2\pi]$ and $\psi \in [0, 4\pi]$, with y the total phase.

In terms of these coordinates, the metric of the 7-sphere becomes

$$ds^2_{\mathrm{S}^7} = (dy+\omega)^2 + ds^2_{\mathbb{CP}^3}, \tag{2.10}$$

where the Fubini-Study metric of the complex projective space $\mathbb{CP}^3$ and the 1-form field ω are given by

$$\begin{aligned} ds^2_{\mathbb{CP}^3} = d\zeta^2 + \tfrac{1}{4}\cos^2\zeta\sin^2\zeta\left[d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right]^2 \\ +\tfrac{1}{4}\cos^2\zeta\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right) + \tfrac{1}{4}\sin^2\zeta\left(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right) \end{aligned} \tag{2.11}$$

$$\omega = \tfrac{1}{4}\cos(2\zeta)d\psi + \tfrac{1}{2}\cos^2\zeta\cos\theta_1 d\phi_1 - \tfrac{1}{2}\sin^2\zeta\cos\theta_2 d\phi_2, \tag{2.12}$$

which is twice the Kähler form on $\mathbb{CP}^3$. Here we have written the metric of the 7-sphere S^7 as a Hopf fibration of a circle S^1 over the complex projective space $\mathbb{CP}^3$. The volume element on the 7-sphere is $\mathrm{vol}(S^7) = dy \wedge \mathrm{vol}(\mathbb{CP}^3)$, with

$$\mathrm{vol}(\mathbb{CP}^3) = \tfrac{1}{32}\cos^3\zeta\sin^3\zeta\sin\theta_1\sin\theta_2\ d\zeta\wedge d\theta_1\wedge d\theta_2\wedge d\psi\wedge d\phi_1\wedge d\phi_2 \tag{2.13}$$

and, integrating, we find that the volume of the 7-sphere is $\frac{\pi^4}{3}$, while that of the complex projective space is $\frac{\pi^3}{6}$.

The metric of S^7 can be written as a Hopf fibration over the complex projective space $\mathbb{CP}^3$ (as described above). The orbifold $S^7/\mathbb{Z}_k$ is obtained by identifying the total phase (which is the fibre) $y \sim y + \frac{2\pi}{k}$ up to an angle of $\frac{2\pi}{k}$, with k some positive integer. This effectively shrinks this circle by a factor of $\frac{1}{k}$. We can then rewrite the metric (2.10) in terms of the new coordinate $\tilde{y} \equiv ky\ \epsilon\ [0, 2\pi)$.

Hence, we deduce that another solution of 11D supergravity is $AdS_4 \times S^7/\mathbb{Z}_k$, with the background metric [2, 53]

$$ds^2 = \tilde{R}^2\left\{ds^2_{AdS_4} + 4ds^2_{S^7/\mathbb{Z}_k}\right\}, \tag{2.14}$$

where the metric of the orbifold $S^7/\mathbb{Z}_k$ is given by

$$ds^2_{S^7/\mathbb{Z}_k} = \tfrac{1}{k^2}\left(d\tilde{y} + k\omega\right)^2 + ds^2_{\mathbb{CP}^3}. \tag{2.15}$$

The 4-form field strength (2.7) is now dual to the 7-form field strength

$$\begin{aligned} F_7 = *F_4 &= 3(128)\tilde{R}^6\ \mathrm{vol}(S^7/\mathbb{Z}_k) = 3(128)\tilde{R}^6\ \tfrac{1}{k}\ \mathrm{vol}(S^7) \\ &= 12\tilde{R}^6\ \tfrac{1}{k}\cos^3\zeta\sin^3\zeta\sin\theta_1\sin\theta_2\ dy\wedge d\zeta\wedge d\theta_1\wedge d\theta_2\wedge d\psi\wedge d\phi_1\wedge d\phi_2. \end{aligned} \tag{2.16}$$

Note that the volume of the compact space has been reduced by a factor of $\frac{1}{k}$, as expected. The orbifolding to $AdS_4 \times S^7/\mathbb{Z}_k$ breaks $\frac{1}{4}$ of the supersymmetries [53] of the maximally supersymmetric [54] $AdS_4 \times S^7$ background[5].

[5] 24 of the 32 components in the gravitino remain invariant under supersymmetry transformations after the orbifolding.

2.3 Type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

$\mathrm{AdS}_4 \times \mathbb{CP}^3$ is a solution of type IIA 10D supergravity and can be obtained [55] via a Kaluza-Klein dimensional reduction from $\mathrm{AdS}_4 \times \mathrm{S}^7$. This can be thought of as the large k limit of $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ in which the circle described by the total phase y shrinks to zero size.

Kaluza-Klein reduction of 11D supergravity to type IIA 10D supergravity

Consider a bosonic solution of 11D supergravity with a metric and 4-form field strength, which can be written in the form

$$d\hat{s}^2 = e^{-\frac{2}{3}\Phi} ds^2 + e^{\frac{4}{3}\Phi} \left[dy + A_1(x)\right]^2 \tag{2.17}$$

$$\text{and} \quad \hat{F}_4 = d\hat{A}_3, \qquad \text{with} \quad \hat{A}_3(x,y) = A_3(x) + A_2(x) \wedge dy, \tag{2.18}$$

where y parameterizes the circle upon which we reduce and the x^μ denote the other ten coordinates. There exists a bosonic solution of type IIA 10D supergravity with metric ds^2, dilaton Φ, B-field $B_2 = dA_2$, and 2-form and 4-form field strengths $F_2 = dA_1$ and $F_4 = dA_3 + A_1 \wedge dA_2$ [55].

We can apply the above Kaluza-Klein prescription to the metric (2.4) and 4-form field strength (2.7) of $\mathrm{AdS}_4 \times \mathrm{S}^7$, with S^7 written as the Hopf fibration (2.10) over $\mathbb{CP}^3$. Redefining $R^2 = \frac{1}{k}\tilde{R}^3$, we obtain an $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background [2, 53], which has the metric

$$ds^2 = R^2 \left\{ ds^2_{\mathrm{AdS}_4} + 4 ds^2_{\mathbb{CP}^3} \right\}. \tag{2.19}$$

The dilaton Φ satisfies $e^{2\Phi} = \frac{4R^2}{k^2}$, while the B-field B_2 still vanishes. The 2-form field strength $F_2 = dC_1$ and 4-form field strength $F_4 = dC_3$ are given by

$$\begin{aligned} F_2 = -\tfrac{1}{2}k \{ &\sin(2\zeta)\, d\zeta \wedge (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \\ &+ \cos^2\zeta \sin\theta_1\, d\theta_1 \wedge d\phi_1 - \sin^2\zeta \sin\theta_2\, d\theta_2 \wedge d\phi_2 \} \end{aligned} \tag{2.20}$$

$$F_4 = -\tfrac{3}{2}kR^2\, \mathrm{vol}(\mathrm{AdS}_4) = -\tfrac{3}{2}kR^2 r^2 \sin\theta\, dt \wedge dr \wedge d\theta \wedge d\varphi, \tag{2.21}$$

with Hodge duals $F_6 = *F_4$ and $F_8 = *F_2$. In particular, the 6-form field strength can be determined to be

$$\begin{aligned} F_6 &= \tfrac{3}{2}(64)kR^4\, \mathrm{vol}(\mathbb{CP}^3) \\ &= 3kR^4 \cos^3\zeta \sin^3\zeta \sin\theta_1 \sin\theta_2\, d\zeta \wedge d\theta_1 \wedge d\theta_2 \wedge d\psi \wedge d\phi_1 \wedge d\phi_2. \end{aligned} \tag{2.22}$$

2.4 Dictionary

Both versions of the ABJM gauge theory/gravity correspondence are strong/weak coupling dualities. The M-theoretic version, which links $\mathcal{N} = 6$ SCS-matter theory with M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7$, relates the 't Hooft coupling to the radius $\tilde{R}$ as follows [2]:

$$\lambda \equiv \frac{N}{k} = \frac{2}{\pi^2}\,\frac{\tilde{R}^6}{k^2}, \tag{2.23}$$

where the rank N of the $U(N) \times U(N)$ gauge group may be determined from the flux of the 7-form field strength F_7 through the orbifold $S^7/\mathbb{Z}_k$ to be

$$N = \frac{1}{(2\pi)^6}\int_{S^7/\mathbb{Z}_k} F_7 = \frac{2}{\pi^2}\,\frac{\tilde{R}^6}{k}. \tag{2.24}$$

In the string theoretic version of the ABJM duality, involving $\mathcal{N} = 6$ SCS-matter theory and type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, the 't Hooft coupling can be written as [2, 53]

$$\lambda \equiv \frac{N}{k} = \frac{R^4}{2\pi^2}. \tag{2.25}$$

The rank N of the gauge group is related to the flux of the 6-form field strength F_6 through the complex projective space $\mathbb{CP}^3$ via

$$N = \frac{1}{(2\pi)^5}\int_{\mathbb{CP}^3} F_6 = \frac{kR^4}{2\pi^2}. \tag{2.26}$$

Notice that the radius of the circle in $\mathrm{S}^7/\mathbb{Z}_k$, upon which we perform the orbifolding, is given by

$$\frac{R}{k} = \left(\frac{\pi^2}{2}\,\frac{N}{k^5}\right)^{\frac{1}{6}}. \tag{2.27}$$

This becomes small when $k \ll N^{\frac{1}{5}}$ - the regime in which the compactified type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is the valid gravitational description. M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ is the gravitational dual of ABJM theory when $N^{\frac{1}{5}} \ll k \ll N$ [2].

Isometries in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ (and $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$) match superconformal symmetries in ABJM theory. The group of AdS_4 isometries/conformal transformations is $SO(2,3)$, while the $SU(4)$ $\mathcal{R}$-symmetry group maps to the group of rotations on the compact space. The $\mathcal{R}$-charge of a local ABJM operator therefore corresponds to the angular momentum of the dual type IIA string state in $\mathbb{CP}^3$. The $U(1)$ charges of the $SU(4)_{\mathcal{R}}$ are given by [31]

$$J_1 \equiv -i\left(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\psi}\right), \qquad J_2 \equiv -i\left(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\psi}\right) \qquad \text{and} \qquad J_3 \equiv -2i\,\frac{\partial}{\partial\psi}. \tag{2.28}$$

Here J_1 and J_2 are associated with motion on each of the 2-spheres, and J_3 with motion along the (shifted) fibre direction.

Part II

D-branes and Giant Gravitons

Introduction:
D-branes and giant gravitons

A Dp-brane is a $(p+1)$-dimensional hypersurface in d dimensions, which describes the end-points of open strings. These membranes are themselves dynamical objects, moving under the influence of the background spacetime. The open string modes are described by a supersymmetric $U(1)$ gauge theory on the worldvolume of the Dp-brane, which contains a gauge field A_a and $d-p-1$ scalar fields X^i (corresponding to the transverse coordinates), as well as fermionic superpartners. The Dp-brane action $S_{\mathrm{D}p} = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}$ consists of the following Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) terms [56]:

$$S_{\mathrm{DBI}} = -T_p \int_\Sigma d^{p+1}\sigma \, e^{-\Phi} \sqrt{-\det\left(\mathcal{P}\left[G+B\right]_{ab} + 2\pi F_{ab}\right)}$$

$$S_{\mathrm{WZ}} = \pm T_p \int_\Sigma \left(\sum_l \mathcal{P}\left[C_l\right] \right) \wedge e^{2\pi F + \mathcal{P}[B]},$$

with tension $T_p = \frac{1}{(2\pi)^p}$ and worldvolume field strength $F = dA$. Here $\mathcal{P}$ denotes the pullback to the worldvolume Σ, described by the coordinates σ^a. A Dp-brane is therefore naturally charged under the RR $(p+1)$-form potential C_{p+1} [57], but may also couple to the lower dimensional RR potentials C_l, with $l < p$.

A non-abelian extension of the Dp-brane action to describe a system of N coincident Dp-branes was proposed by Myers [58]. The gauge symmetry is augmented from $U(1)^N$ to $U(N)$, with the scalar fields in the adjoint representation. One significant finding was that the non-abelian WZ action involves a coupling to RR potentials C_l, with $l > p$. It was hence shown that D0-branes in an external 4-form field are polarized and expand into a non-commutative or 'fuzzy' 2-sphere, which can be interpreted as a (D2,D0)-brane bound state (known as the 'Myers effect'). This system can alternatively be viewed as a single spherical D2-brane with a non-trivial worldvolume gauge field. McGreevy, Susskind and Toumbas [59] constructed a similar D2-brane, the extension of which is supported, not by worldvolume flux, but by its angular momentum in the compact space - the coupling of the D2-brane to the 4-form field strength produces a Lorentz-like force that balances the brane tension. These (classically) stable D2-brane configurations were dubbed 'giant gravitons'. D3-brane giants in type IIB string theory on $AdS_5 \times S^5$, and M2-brane and M5-brane giants in M-theory on $AdS_4 \times S^7$ were studied in [54]. These were shown to be $\frac{1}{2}$-BPS objects, preserving half the supersymmetries of the maximally supersymmetric background spacetimes.

In $\mathcal{N} = 4$ SYM theory, giant gravitons on $AdS_5 \times S^5$ are dual to operators with $\mathcal{R}$-charge of $O(N)$, constructed as a Schur polynomial of the complex scalar fields X, Y and Z. For example, restricting to the single matrix (Z) model [16] - [18]:

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \,\epsilon\, S_n} \chi_R(\sigma)\, Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \ldots Z^{i_n}_{i_{\sigma(n)}}.$$

Here σ is an element of the permutation group S_n, with character $\chi_R(\sigma)$, in the representation R. Each such Schur polynomial has conformal dimension Δ equal to its $\mathcal{R}$-charge n and is labeled by a Young diagram with n boxes [19]. Not only do these gauge invariant Schur polynomials diagonalize the free two-point correlation function, thereby providing a suitable basis for the $\frac{1}{2}$-BPS sector of $\mathcal{N} = 4$ SYM theory, but they also realize quite explicitly some of the characteristic properties of the dual D-branes. For example, depending on which 3-cycle is wrapped by the D3-brane, giant gravitons on $AdS_5 \times S^5$ come in two flavours: AdS and sphere giants [54, 60]. These correspond to Schur polynomials in the totally symmetric and totally antisymmetric representations respectively. The latter can be written equivalently as the subdeterminant [16]

$$\mathcal{O}_n = \frac{1}{n!}\, \epsilon_{\alpha_1 \ldots \alpha_n \alpha_{n+1} \ldots \alpha_N}\, \epsilon^{\beta_1 \ldots \beta_n \alpha_{n+1} \ldots \alpha_N}\, Z^{\alpha_1}_{\beta_1} \ldots Z^{\alpha_n}_{\beta_n}.$$

Notice the upper bound $n \leq N$ on the $\mathcal{R}$-charge - an interpretation [59] of this so-called 'stringy exclusion principle' [61] is that the size of the sphere giant (which depends on its angular momentum J) is limited by the radius of the 5-sphere.

Open string excitations of giant gravitons on $AdS_5 \times S^5$ were studied in [39] - [41]. These are dual to 'words' (built out of both Z's and other SYM fields) attached to the Schur polynomial operator. The combinatorics of attaching such words encodes the Gauss law constraint satisfied by the spherical D3-brane [62]. A comparison between open string excitation energies and the anomalous dimensions of attached words is assisted by a map from any such word to an open spin chain [39].

Prior to this work, much less was known about giant gravitons on $AdS_5 \times T^{1,1}$. Although Mikhailov proposed an ansatz [63], in terms of holomorphic curves on the cone, for D3-branes embedded in $T^{1,1}$, these configurations were otherwise unstudied. In Klebanov-Witten theory, operators dual to giant gravitons are Schur polynomials of composite scalar fields A_iB_j. For example, one such $\frac{1}{2}$-BPS operator $\chi_R(A_1B_1)$, with conformal dimension $\Delta = \frac{3}{2}n$ and $\mathcal{R}$-charge n, is obtained by replacing Z with the combination A_1B_1. This has at least one interesting consequence when R is completely antisymmetric: at maximum size $n = N$, the subdeterminant operator $\mathcal{O}_n(A_1B_1)$ factorizes into the product of two determinant operators:

$$\begin{aligned}\mathcal{O}_n\left(A_1B_1\right) &= \frac{1}{N!}\,\epsilon_{\alpha_1\ldots\alpha_N}\left(A_1\right)^{\alpha_1}{}_{\gamma_1}\ldots\left(A_1\right)^{\alpha_N}{}_{\gamma_N}\ \epsilon^{\beta_1\ldots\beta_N}\left(B_1\right)_{\alpha_1}{}^{\gamma_1}\ldots\left(B_1\right)_{\alpha_N}{}^{\gamma_N}\\ &= \det A_1 \det B_1.\end{aligned}$$

These dibaryon operators $\det A_1$ and $\det B_1$ are dual to topologically stable D3-branes wrapped on non-contractible 3-cycles in $\mathrm{T}^{1,1}$ [64] - [66]. Among other things, we would like to know how the transition from the giant graviton to two dibaryons happens.

The open string sector of type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, which encodes all the information about D-branes and their dynamics, has not been much studied. What we do know about open strings on this background reveals a remarkably rich structure [53], [67] - [69], including new spinning M2-brane solutions and giant tori. This last configuration is particularly interesting, since the operator dual to this torus should reproduce not only Gauss' law for a compact object, but also its non-zero genus. This would be a decidedly non-trivial test of the idea that topology is an emergent property in the gauge theory. Initial steps towards developing this idea have been taken [69, 70].

There should be D4-brane giant gravitons on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, which wrap non-trivial 4-cycles in the complex projective space, as well as the 'dual' spherical D2-brane giants [53] in the anti-de Sitter space. These descend from the M5-brane and M2-brane giant gravitons on $\mathrm{AdS}_4 \times \mathrm{S}^7$, which were described by [54], upon compactification. The operators in ABJM theory dual to giant gravitons are structurally identical to those in Klebanov-Witten theory. Again choosing the composite scalar field A_1B_1, the Schur polynomial $\chi_R(A_1B_1)$, with conformal dimension Δ the same as its $\mathcal{R}$-charge n, describes a D2- and D4-brane giant graviton, when R is totally symmetric and totally antisymmetric respectively. The latter, being equivalent to the subdeterminant $\mathcal{O}_n(A_1B_1)$, again factorizes at maximum size into two dibaryons $\det A_1$ and $\det B_1$. These should be dual to topologically stable D4-branes wrapped on non-contractible $\mathbb{CP}^2$ cycles in $\mathbb{CP}^3$.

This part involves an extensive study of D-branes, mainly giant gravitons, in type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ and type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. In chapter 3, we begin by investigating the dual giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ constructed in [53]. Chapter 4 involves a review of [66] concerning dibaryons on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$, together with an investigation of the spectrum of $\mathbb{CP}^2$ dibaryons on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. A detailed study of a D3-brane giant graviton on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$, obtained using Mikhailov's holomorphic curve construction [63], is presented in chapter 5 and we adapt this ansatz to describe a similar D4-brane giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

Chapter 3

Dual giant gravitons

We shall now focus on the dual giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. A review of its construction by [53] is presented. We perturb about this D2-brane solution - taking into account both scalar and worldvolume fluctuations - and solve for its spectrum of small fluctuations. Open string excitations are then considered. We attach short open strings to the worldvolume of the dual giant and compute the bosonic spectrum in a pp-wave limit. We also write down the Polyakov action for fast moving semiclassical long strings, which should correspond to the Landau-Lifshitz action of an open ABJM spin chain (mapped from long words attached to the ABJM Schur polynomial operator).

3.1 Dual giant gravitons on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

Our construction of the dual giant graviton closely follows that of [53]. This dual giant on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is a D2-brane wrapping an $S^2 \subset \mathrm{AdS}_4$ with angular momentum in $\mathbb{CP}^3$. It is dual to the Schur polynomial $\chi_R(A_1 B_1)$ in the totally symmetric representation of the permutation group.

3.1.1 Ansatz for the dual giant graviton

Type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is described in section 2.3. In order to simplify the dual giant graviton ansatz, it is convenient to make the coordinate change

$$\chi \equiv \tfrac{1}{2}\left(\psi - \phi_1 - \phi_2\right), \qquad \varphi_1 \equiv \phi_1 \qquad \text{and} \qquad \varphi_2 \equiv \phi_2, \tag{3.1}$$

thereby shifting the fibre direction (visible in the metric (2.11) of the complex projective space). The $AdS_4 \times \mathbb{CP}^3$ background metric (2.19) is now given by

$$\begin{aligned} R^{-2}ds^2 = &-\left(1+r^2\right)dt^2 + \frac{dr^2}{(1+r^2)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\ &+ 4\,d\zeta^2 + 4\cos^2\zeta\sin^2\zeta\left(d\chi + \cos^2\tfrac{\theta_1}{2}\,d\varphi_1 + \cos^2\tfrac{\theta_2}{2}\,d\varphi_2\right)^2 \\ &+ \cos^2\zeta\left(d\theta_1^2 + \sin^2\theta_1\,d\varphi_1^2\right) + \sin^2\zeta\left(d\theta_2^2 + \sin^2\theta_2\,d\varphi_2^2\right), \end{aligned} \tag{3.2}$$

while the constant dilaton Φ satisfies $e^{2\Phi} = \frac{4R^2}{k^2}$. The 2-form field strength $F_2 = dC_1$, shown in (2.20), becomes

$$\begin{aligned} F_2 = -k\,\big\{&\sin(2\zeta)\;d\zeta\wedge\left(d\chi + \cos^2\tfrac{\theta_1}{2}\,d\varphi_1 + \cos^2\tfrac{\theta_2}{2}\,d\varphi_2\right) \\ &+\tfrac{1}{2}\cos^2\zeta\sin\theta_1\,d\theta_1\wedge d\varphi_1 - \tfrac{1}{2}\sin^2\zeta\sin\theta_2\,d\theta_2\wedge d\varphi_2\big\} \end{aligned} \tag{3.3}$$

and corresponds to a 1-form potential

$$C_1 = \tfrac{1}{2}\,k\left\{\cos(2\zeta)\left(d\chi + \cos^2\tfrac{\theta_1}{2}\,d\varphi_1 + \cos^2\tfrac{\theta_2}{2}\,d\varphi_2\right) + \tfrac{1}{2}\cos\theta_1\,d\varphi_1 - \tfrac{1}{2}\cos\theta_2\,d\varphi_2\right\}. \tag{3.4}$$

The 4-form field strength (2.22) can be written as $F_4 = dC_3$, with 3-form potential

$$C_3 = \tfrac{1}{2}\,kR^2r^3\sin\theta\,dt\wedge d\theta\wedge d\varphi. \tag{3.5}$$

Let us consider a D2-brane, with worldvolume coordinates $\sigma^a = (t,\theta,\varphi)$, which wraps the 2-sphere (θ,φ) with constant radius r in AdS_4. It moves along the circle parameterized by $\chi(t)$, which is situated at $\zeta = \frac{\pi}{4}$ and $\theta_1 = \theta_2 = \pi$ (the south pole of each of the 2-spheres - set to the same size - in the complex projective space). We shall also assume that the worldvolume field strength $F = dA$ vanishes.

3.1.2 D2-brane action

A D2-brane is described by the action

$$S_{\rm D2} = -T_2\int_\Sigma d^3\sigma\,e^{-\Phi}\sqrt{-\det\left(\mathcal{P}[g] + 2\pi F\right)} + T_2\int_\Sigma \mathcal{P}[C_3] + 2\pi T_2\int_\Sigma \mathcal{P}[C_1]\wedge F, \tag{3.6}$$

with tension $T_2 = \frac{1}{(2\pi)^2}$. Here $\mathcal{P}$ denotes the pullback to the worldvolume Σ. The dual giant graviton ansatz allows us to simplify this action:

$$S_{\rm D2} = -\frac{kR^2}{2\pi}\int dt\;\left\{r^2\sqrt{1+r^2-\dot\chi^2} - r^3\right\}. \tag{3.7}$$

Now, the conserved momentum P_χ conjugate to χ satisfies

$$p \equiv \frac{2\pi}{kR^2}\;P_\chi = \frac{r^2\;\dot\chi}{\sqrt{1+r^2-\dot\chi}}, \tag{3.8}$$

from which it follows that $\chi = \omega_0 t$ is a solution to the equation of motion. The Hamiltonian $H = P_\chi \dot{\chi} - L$ is easily computed to be

$$H = \frac{kR^2}{2\pi}\left\{\sqrt{1+r^2}\sqrt{r^4+p^2} - r^3\right\}. \tag{3.9}$$

This energy functional has two minima, one at $r = 0$ associated with the point graviton[1] and the other at $r = p \equiv r_0$ (when $\dot{\chi}^2 = \omega_0^2 = 1$), and attains a maximum at $r = \frac{1}{3}(4-3p^2)^{\frac{1}{2}} - \frac{2}{3} \equiv r_{\text{max}}$ in between (see figure 3.1). It is the second minimum that we shall call the dual giant graviton configuration in $\text{AdS}_4 \times \mathbb{CP}^3$. Note that its extension to a size $r_0 = p$ in the AdS_4 space is a direct result of its angular momentum in $\mathbb{CP}^3$ - the dual giant carries angular momenta $J_1 = J_2 = 0$ and $J_3 = P_\chi$. This solution satisfies the BPS bound $H = \mathcal{P}_\chi$, which is an indication that the dual giant is supersymmetric.

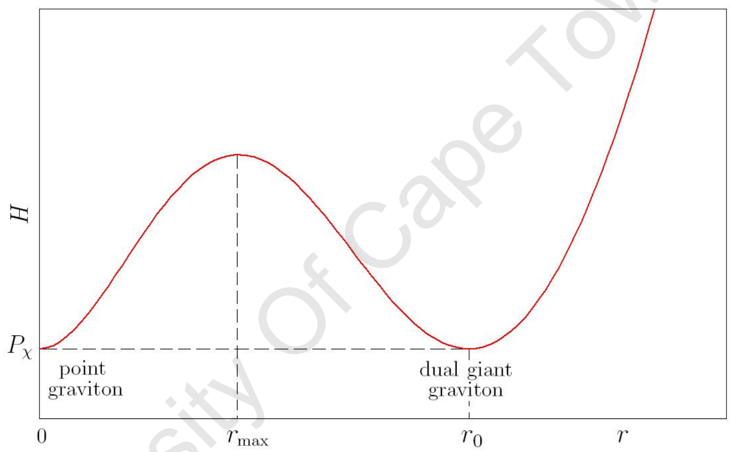

Figure 3.1: A sketch of the energy H as a function of the radius r at fixed momentum P_χ. The dual giant graviton is energetically degenerate with the point graviton.

3.2 Fluctuation analysis

Having written down the classical brane configuration, we shall now analyze its stability. This information is encoded in the spectrum of small fluctuations about the dual giant graviton [71] - [74]. However, it is crucial to note[2] that fluctuations of the worldvolume gauge field must also be taken into account.

[1] Care must be taken in setting $r = 0$, since (3.8) is singular when $r = 0$ and $\omega_0^2 = 1$. However, it can be shown [59] - [60] that this is a sensible limit.

[2] We thank Kostas Skenderis for pointing this out.

In 3 worldvolume dimensions, the analysis of the gauge field simplifies considerably because of its Hodge duality with a massless scalar field. To see this, note that, in the limit of small worldvolume flux, the relevant part of the action is

$$S_A = -(2\pi)^2\, T_2 \int \tfrac{1}{2}\, e^{-\Phi}\, dA \wedge *dA + 2\pi T_2 \int C_1 \wedge dA, \tag{3.10}$$

where A is the gauge field and $F = dA$ the associated field strength. To dualize to the scalar field, we must first change the degree of freedom to F, while at the same time adding a Lagrange multiplier ϕ in order to enforce the Bianchi identity $dF = 0$:

$$S_F = -(2\pi)^2\, T_2 \int e^{-\Phi} \left\{ \tfrac{1}{2}\, F \wedge *F + \phi\, dF \right\} + 2\pi T_2 \int C_1 \wedge F. \tag{3.11}$$

Integrating out $F = -*\left(d\phi + \frac{1}{2\pi}\, e^{\Phi}\, C_1\right)$, the dualized action is

$$S_\phi = -(2\pi)^2\, T_2 \int d^3\sigma\, e^{-\Phi} \sqrt{-\det \mathcal{P}[g]}\, \left\{ \frac{1}{2}\, \partial_a \phi\, \partial^a \phi + \frac{e^{\Phi}}{2\pi}\, C_{1a}\, \partial^a \phi + \frac{1}{2} \left(\frac{e^{\Phi}}{2\pi} \right)^2 C_{1a}\, C_1^a \right\}. \tag{3.12}$$

For the transverse fluctuations, it is convenient not only to make use of the new angular coordinates χ, ϕ_1 and ϕ_2, but also to write each of two 2-spheres in $\mathbb{CP}^3$ in terms of the Euclidean coordinates

$$u_1 = \sin\theta_1 \cos\varphi_1 \qquad u_2 = \sin\theta_1 \sin\varphi_1 \qquad u_3 = \cos\theta_1 \tag{3.13}$$

$$v_1 = \sin\theta_2 \cos\varphi_2 \qquad v_2 = \sin\theta_2 \sin\varphi_2 \qquad v_3 = \cos\theta_2. \tag{3.14}$$

Here $u_1^2 + u_2^2 + u_3^2 = 1$ and $v_1^2 + v_2^2 + v_3^2 = 1$, so that we can eliminate u_3 and v_3 in favour of the other coordinates. The $\mathrm{AdS}_4 \times \mathbb{CP}^3$ metric is hence given by

$$\begin{aligned} R^{-2} ds^2 = &- \left(1 + r^2\right) dt^2 + \frac{dr^2}{(1 + r^2)} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) + 4\, d\zeta^2 \\ &+ \cos^2\zeta \left[du_1^2 + du_2^2 + \frac{(u_1 du_1 + u_2 du_2)^2}{1 - (u_1^2 + u_2^2)} \right] + \sin^2\zeta \left[dv_1^2 + dv_2^2 + \frac{(v_1 dv_1 + v_2 dv_2)^2}{1 - (v_1^2 + v_2^2)} \right] \\ &+ 4 \cos^2\zeta \sin^2\zeta \left[d\chi + \frac{1}{2} \left(\left[1 - (u_1^2 + u_2^2)\right]^{\frac{1}{2}} - 1 \right) \frac{(u_1 du_2 - u_2 du_1)}{(u_1^2 + u_2^2)} \right. \\ &\left. \qquad\qquad - \frac{1}{2} \left(\left[1 - (v_1^2 + v_2^2)\right]^{\frac{1}{2}} - 1 \right) \frac{(v_1 dv_2 - v_2 dv_1)}{(v_1^2 + v_2^2)} \right]^2 . \end{aligned} \tag{3.15}$$

The fluctuations about the dual giant graviton - including both scalar and worldvolume degrees of freedom - can be written as

$$\begin{aligned} &r = r_0 + \varepsilon\, \delta r(\sigma^a) \qquad &&\chi = \omega_0\, t + \varepsilon\, \delta\chi(\sigma^a) \qquad &&\zeta = \tfrac{\pi}{4} + \varepsilon\, \delta\zeta(\sigma^a) \\ &u_i = \varepsilon\, \delta u_i(\sigma^a) &&v_i = \varepsilon\, \delta v_i(\sigma^a) &&\phi = \tfrac{1}{2\pi} R^2\, \varepsilon\, \delta\phi(\sigma^a), \end{aligned} \tag{3.16}$$

where $\sigma^a = (t, \theta, \varphi)$ are the worldvolume coordinates and ε is a small parameter. We shall expand the D2-brane action (3.6) to second order in ε as follows:

$$S_{\rm D2} \approx -\frac{kR^2}{8\pi^2}\int d^3\sigma \left\{R^{-3}\sqrt{-\det\mathcal{P}[g]} - \left(r_0^3 + 3\varepsilon r_0^2\,\delta r + 3\varepsilon^2 r_0\,\delta r^2\right)\sin\theta\right\} + S_\phi. \tag{3.17}$$

Here we write the pullback of the metric to the worldvolume of the perturbed D2-brane as $\det\mathcal{P}[g] = -R^6\,(b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \ldots)$, so that

$$R^{-3}\sqrt{-\det\mathcal{P}[g]} \approx \sqrt{b_0} + \varepsilon\left(\frac{1}{2}\frac{b_1}{\sqrt{b_0}}\right) + \varepsilon^2\left(\frac{1}{2}\frac{b_2}{\sqrt{b_0}} - \frac{1}{8}\frac{b_1^2}{b_0\sqrt{b_0}}\right), \tag{3.18}$$

in terms of

$$b_0 \equiv \left(1 + r_0^2 - \omega_0^2\right) r_0^4\,\sin^2\theta \tag{3.19}$$

$$b_1 \equiv 2\left[\left(2 + 3r_0^2 - 2\omega_0^2\right) r_0^3\,\delta r - \omega_0 r_0^4\,\dot{\delta\chi}\right]\sin^2\theta \tag{3.20}$$

$$\begin{aligned}
b_2 \equiv\ & \frac{1}{(1+r_0^2)}\left[-r_0^4\,\dot{\delta r}^2 + r_0^2\left(1 + r_0^2 - \omega_0^2\right)(\nabla\delta r)^2\right]\sin^2\theta \\
& + \left[9r_0^4 + 6r_0^2\left(1 + r_0^2 - \omega_0^2\right)\right]\delta r^2\sin^2\theta - 8\omega_0 r_0^3\,\dot{\delta\chi}\delta r\ \sin^2\theta \\
& + \left[-r_0^4\,\dot{\delta\chi}^2 + r_0^2\left(1 + r_0^2\right)(\nabla\delta\chi)^2\right]\sin^2\theta \\
& + 4\left[-r_0^4\,\dot{\delta\zeta}^2 + \omega_0^2 r_0^4\,\delta\zeta^2 + r_0^2\left(1 + r_0^2 - \omega_0^2\right)(\nabla\delta\zeta)^2\right]\sin^2\theta \\
& + \frac{1}{2}\sum_i\left[-r_0^4\,\dot{\delta u_i}^2 + r_0^2\left(1 + r_0^2 - \omega_0^2\right)(\nabla\delta u_i)^2\right]\sin^2\theta \\
& + \frac{1}{2}\sum_i\left[-r_0^4\,\dot{\delta v_i}^2 + r_0^2\left(1 + r_0^2 - \omega_0^2\right)(\nabla\delta v_i)^2\right]\sin^2\theta \\
& + \frac{1}{2}\,\omega_0 r_0^4\left(\delta u_1\dot{\delta u_2} - \delta u_2\dot{\delta u_1}\right)\sin^2\theta + \frac{1}{2}\,\omega_0 r_0^4\left(\delta v_1\dot{\delta v_2} - \delta v_2\dot{\delta v_1}\right)\sin^2\theta,
\end{aligned} \tag{3.21}$$

while the action of the scalar field becomes

$$S_\phi = -\frac{kR^2}{16\pi^2}\,\varepsilon^2 r_0\int d^3\sigma\ \sin\theta\left\{\left[-\dot{\delta\phi}^2 + (\nabla\delta\phi)^2\right] + 4\omega_0\,\delta\zeta\dot{\delta\phi} - 4\omega_0^2\,\delta\zeta^2\right\}. \tag{3.22}$$

The gradient squared of any function $f(\theta,\varphi)$ on the 2-sphere (θ,φ), on which the dual giant is wrapped, is defined as follows:

$$(\nabla f)^2 \equiv (\partial_\theta f)^2 + \frac{1}{\sin^2\theta}\,(\partial_\varphi f)^2\,. \tag{3.23}$$

Let us now organize the expansion of this D2-brane action $S_{\rm D2} = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \ldots$ in powers of ε. At zeroth order, we obtain simply the D2-brane action (3.7) of the dual giant graviton evaluated at $r = r_0$. The first order action

$$S_1 = -\frac{kR^2}{8\pi^2}\int d^3\sigma\ \sin\theta\left\{\left[\frac{\left(2 + 3r_0^2 - 2\omega_0^2\right) r_0}{\sqrt{1 + r_0^2 - \omega_0^2}} - 3r_0^2\right]\delta r - \frac{\omega_0 r_0^2}{\sqrt{1 + r_0^2 - \omega_0^2}}\,\dot{\delta\chi}\right\}, \tag{3.24}$$

is less trivial. The second term in S_1 is easily recognized as a total derivative and can be dropped. Setting $\omega_0 = 1$, we see that the first term also vanishes. This is an indication that the dual giant graviton configuration, with radius $r_0 = p$ and angular velocity $\omega_0 = 1$, is, indeed, a solution to the equations of motion.

After integrating by parts, the second order action is given by

$$\begin{aligned} S_2 = \frac{kR^2}{16\pi^2}\, r_0 \int d^3\sigma\, \sin\theta \Big\{ & \frac{\delta r}{(1+r_0^2)} \left[-\ddot{\delta r} + \nabla^2 \delta r \right] + \frac{(1+r_0^2)}{r_0^2}\, \delta\chi \left[-\ddot{\delta \chi} + \nabla^2 \delta\chi \right] + 2\, \dot{\delta\chi} \delta r \\ & + \frac{1}{2} \sum_i \delta u_i \left[-\ddot{\delta u_i} + \nabla^2 \delta u_i \right] + \frac{1}{2} \left(\dot{\delta u_1} \delta u_2 - \dot{\delta u_2} \delta u_1 \right) \\ & + \frac{1}{2} \sum_i \delta v_i \left[-\ddot{\delta v_i} + \nabla^2 \delta v_i \right] + \frac{1}{2} \left(\dot{\delta v_1} \delta v_2 - \dot{\delta v_2} \delta v_1 \right) \\ & + 4\,\delta\zeta \left[-\ddot{\delta\zeta} + \nabla^2 \delta\zeta \right] + \delta\phi \left[-\ddot{\delta\phi} + \nabla^2 \delta\phi \right] - 4\,\delta\zeta \dot{\delta\phi} \Big\}. \end{aligned} \tag{3.25}$$

The Laplacian on the 2-sphere

$$\nabla^2 \equiv \frac{1}{\sin\theta}\, \partial_\theta \left(\sin\theta\, \partial_\theta \right) + \frac{1}{\sin^2\theta}\, \partial_\varphi^2 \tag{3.26}$$

is associated with the usual spherical harmonics $Y_{lm}(\theta, \varphi)$, which satisfy the eigenvalue equation $\nabla^2 Y_{lm} = -l(l+1) Y_{lm}$.

Varying this second order D2-brane action allows us to calculate the equations of motion for the fluctuations $(\delta u_1, \delta u_2)$, $(\delta v_1, \delta v_2)$, $(\delta r, \delta\chi)$ and $(\delta\zeta, \delta\phi)$, which are coupled in pairs. Defining

$$\delta x_\pm^a = \left(\delta u_1 + i \delta u_2,\ \delta u_1 + i\delta u_2,\ \delta r \mp \tfrac{i(1+r_0^2)}{r_0^2}\, \delta\chi,\ \delta\zeta \pm \tfrac{i}{2}\, \delta\phi \right), \tag{3.27}$$

these equations of motion decouple as follows:

$$-\ddot{\delta x}_\pm^a + \nabla^2 \delta x_\pm^a \pm i \dot{\delta x}_\pm^a = 0. \tag{3.28}$$

To proceed further, we decompose the perturbations into Fourier components:

$$\delta x_\pm^a (t, \theta, \varphi) = \sum_{l,m} C_\pm^a\, e^{-i\omega_{lm}^{a\pm} t}\, Y_{lm}(\theta, \varphi). \tag{3.29}$$

This expansion satisfies the above equations of motion when the frequencies are given by $\omega_{lm}^{a\pm} = \pm l$ or $\mp(l+1)$. The spectrum of small fluctuations is entirely real (there are no tachyonic modes), from which we conclude that, like its D3-brane counterpart [71] in $AdS_5 \times S^5$, the dual giant graviton on $AdS_4 \times \mathbb{CP}^3$ is perturbatively stable.

To conclude our fluctuation analysis, there are several points worth noting:

- Any dependence of the spectrum on the size of the giant would be a trace signature of the geometry that could be probed in the dual field theory. However, we see here that the frequencies are independent of the radius of the dual giant graviton. Physically, this can be attributed to the exact cancelation of two competing effects: (1) blue-shifting of frequency from rising within an AdS gravitational well, and (2) the increased wavelength on a larger worldvolume[3].

- Since there are frequencies $\pm l$, the spectrum contains massless goldstone modes from the breaking of a number of continuous symmetries. The radial part of the $SO(2,3)$ AdS symmetry is broken by our choice of the radius/momentum $r_0 = p$ of the dual giant graviton, leading to a goldstone mode associated with the $(\delta r, \delta\chi)$ fluctuations. There is also a broken $SU(2) \times SU(2) \subset SU(4)$ symmetry, corresponding to the two 2-spheres contained in $\mathbb{CP}^3$ - the goldstone bosons are associated with normal modes of the $(\delta u_1, \delta u_2)$ and $(\delta v_1, \delta v_2)$ fluctuations.

- Perhaps most intriguingly, there is also a massless mode which arises from the coupling of the gauge field and the radius of the S^1 direction of motion (dependent on the coordinate ζ). This implies the existence of giant graviton solutions (with the same energy) involving non-trivial gauge fields. Such solutions can be thought of as D0-brane charge dissolved in the giant worldvolume. In fact, the infinitesimal worldvolume gauge flux associated with the zero mode fluctuation ($\delta\zeta$ and $\delta\phi$ constant) is given by

$$F = - * \left(d\phi + \tfrac{1}{2\pi}\, e^{\Phi}\, C_1\right) = -\varepsilon\, \tfrac{1}{\pi} R^2 r_0 \delta\zeta\, \sin\theta\, d\theta \wedge d\varphi, \qquad (3.30)$$

  which is the flux of a Dirac monopole of charge $-\varepsilon\, \frac{1}{\pi} R\delta\zeta$. Interestingly, such solutions would seem to have maximum D0-brane charge when the radius of the S^1 direction of motion shrinks to zero size. Similar solutions were found in [53], although without $\mathbb{CP}^3$ momentum. It would be interesting to find the relevant charged dual giant gravitons in this case.

3.3 Open string excitations

Transverse fluctuations of D-branes are encoded in open strings attached to and moving on the brane. In this section, we shall explicitly study these open string excitations from the worldsheet perspective. Although a full quantum treatment of the worldsheet sigma model is sorely lacking, several interesting and instructive limits exist. Two, in particular, will be of interest to us: short strings and long semiclassical stings. We shall extend the results of [40, 41] and present a systematic treatment of both these limits for the dual giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

[3]We thank Robert de Mello Koch for pointing this argument out to us.

3.3.1 Short strings in a pp-wave geometry

To study short strings attached to the dual giant graviton, we need to take a limit in which the size of the string is 'amplified' with respect to the geometry that it probes. It was argued in [40] that a good description of similar short strings attached to a submaximal sphere giant on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is of open strings attached to a flat D3-brane on a pp-wave background. This was later extended to short open string excitations of the AdS giant on $\mathrm{AdS}_5 \times \mathrm{S}^5$ with qualitatively similar results [41]. The pp-wave limit is particularly useful, since the string action becomes quadratic in the lightcone gauge and consequently solvable [75, 76]. In what follows, we quantize the short string sigma model on the pp-wave background associated with a null geodesic on the worldvolume of the D2-brane giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ and compute its bosonic spectrum exactly.

To take the Penrose limit, it is convenient to redefine the radial coordinate $r \equiv \sinh\rho$ in the anti-de Sitter space, so that the metric of $\mathrm{AdS}_4 \times \mathbb{CP}^3$ reads

$$\begin{aligned} R^{-2}ds^2 = & -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\ & + 4\, d\zeta^2 + 4\cos^2\zeta\sin^2\zeta\left(d\chi + \cos^2\tfrac{\theta_1}{2}\, d\varphi_1 + \cos^2\tfrac{\theta_2}{2}\, d\varphi_2\right)^2 \\ & + \cos^2\zeta\left(d\theta_1^2 + \sin^2\theta_1\, d\varphi_1^2\right) + \sin^2\zeta\left(d\theta_2^2 + \sin^2\theta_2\, d\varphi_2^2\right). \end{aligned} \tag{3.31}$$

The null geodesic

$$t = \chi = \varphi = u, \qquad \theta = \tfrac{\pi}{2}, \qquad \rho = \rho_0, \qquad \zeta = \tfrac{\pi}{4} \qquad \text{and} \qquad \theta_i = \pi \tag{3.32}$$

describes a trajectory parameterized by $\varphi = u$ on the dual giant graviton. To construct the pp-wave geometry associated with this null geodesic, we take the ansatz

$$\begin{aligned} t &= u + \frac{v}{R^2\cosh^2\rho_0} \\ \chi &= u - \frac{v}{R^2\cosh^2\rho_0} - \frac{\tanh\rho_0\, y_1}{R} \\ \varphi &= u - \frac{v}{R^2\cosh^2\rho_0} + \frac{y_1}{R\cosh\rho_0\sinh\rho_0} \end{aligned} \tag{3.33}$$

and expand the coordinates, which are fixed on the geodesic, as follows:

$$\rho = \rho_0 + \frac{y_2}{R} \qquad \theta = \frac{\pi}{2} + \frac{z_1}{R\sinh\rho_0} \qquad \zeta = \frac{\pi}{4} + \frac{z_2}{2R} \qquad \theta_i = \pi - \frac{\sqrt{2}\, r_i}{R}, \tag{3.34}$$

with φ_i unspecified. We can now take the Penrose limit, in which R becomes large and we zoom in on the null geodesic. When $r_0 \equiv \sinh\rho_0$ is fixed, the radius of the dual giant diverges like R. Therefore, as in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, this short string limit is effectively just a treatment of open strings attached to a flat D2-brane and propagating on the pp-wave background

$$\begin{aligned} ds^2 = &- 4du dv - \left(\sum_{i=1}^{2} z_i^2\right) du^2 + \sum_{i=1}^{4} dx_i^2 + \sum_{i=1}^{2} dy_i^2 + \sum_{i=1}^{2} dz_i^2 \\ &+ 4y_2 dy_1 du + \left(x_2 dx_1 - x_1 dx_2\right) du + \left(x_4 dx_3 - x_3 dx_4\right) du, \end{aligned} \tag{3.35}$$

where we make use of the Cartesian coordinates $x_{2k-1} = r_k \cos\phi_k$ and $x_{2k} = r_k \sin\phi_k$, with $k \,\epsilon\, \{1,2\}$. When written in this form, there is a manifest similarity between this pp-wave geometry and the homogenous plane wave background of [77], so that the techniques developed there are easily adapted to this case[4].

In lightcone gauge $u = 2p^u\tau$, the bosonic part of the Polyakov action is

$$\begin{aligned} S = \int d\tau \int_0^{\pi} \frac{d\sigma}{2\pi} \Bigg\{ &\sum_{I=1}^{4} \left(\tfrac{1}{2}\dot{X}_I^2 - \tfrac{1}{2}X_I'^2\right) + \tfrac{1}{2}m\left(X_2\dot{X}_1 - X_1\dot{X}_2\right) + \tfrac{1}{2}m\left(X_4\dot{X}_3 - X_3\dot{X}_4\right) \\ &+ \sum_{I=1}^{2} \left(\tfrac{1}{2}\dot{Y}_I^2 - \tfrac{1}{2}Y_I'^2\right) + 2mY_2\dot{Y}_1 + \sum_{I=1}^{2}\left(\tfrac{1}{2}\dot{Z}_I^2 - \tfrac{1}{2}Z_I'^2 + \tfrac{1}{2}m^2 Z_I^2\right) \Bigg\}, \end{aligned} \tag{3.36}$$

with $m \equiv 2p^u$. Note that the string embedding coordinates X_I, associated with the two 2-spheres, are coupled in pairs,

$$X_{K\pm} = X_{2K-1} \pm iX_{2K}, \qquad \text{with} \quad K \,\epsilon\, \{1,2\}, \tag{3.37}$$

as are the Y_I, which descend from ρ and χ,

$$Y_\pm = Y_1 \pm iY_2. \tag{3.38}$$

In the large R limit, the open string boundary conditions associated with the dual giant graviton imply Dirichlet boundary conditions on X_I, Y_I and Z_2, and Neumann boundary conditions[5] on the lightcone coordinates U and V, as well as Z_1. Solving the open string equations of motion - subject to the appropriate boundary conditions - and quantizing the bosonic sector, we obtain the following expressions for the Neumann embedding coordinate

$$Z_1(\tau,\sigma) = \sqrt{\frac{1}{m}}\left[\xi_0^1\, e^{-im\tau} + \left(\xi_0^1\right)^\dagger e^{im\tau}\right] + \sum_{n=1}^{\infty}\sqrt{\frac{2}{\omega_n}}\left[\xi_n^1\, e^{-i\omega_n\tau} + \left(\xi_n^1\right)^\dagger e^{i\omega_n\tau}\right]\cos(n\sigma) \tag{3.39}$$

and the Dirichlet coordinates

$$X_{K\pm}(\tau,\sigma) = \sum_{n=1}^{\infty}\sqrt{\frac{4}{\tilde{\omega}_n}}\left[\alpha_n^{K\mp}\, e^{-i\tilde{\omega}_n^{\mp}\tau} + \left(\alpha_n^{K\pm}\right)^\dagger e^{i\tilde{\omega}_n^{\pm}\tau}\right]\sin(n\sigma) \tag{3.40}$$

[4]See also [73] for a further discussion of the relation between the homogeneous plane wave and the standard pp-wave in magnetic coordinates.

[5]Note that the lightcone gauge choice is consistent with Neumann boundary conditions on U.

$$Y_+(\tau,\sigma) = \sum_{n=1}^{\infty} \sqrt{\frac{4}{\omega_n}} \left[\beta_n^{\mp}\, e^{-i\omega_n^{\mp}\tau} + \left(\beta_n^{\pm}\right)^{\dagger} e^{i\omega_n^{\pm}\tau}\right] \sin(n\sigma) \tag{3.41}$$

$$Z_2(\tau,\sigma) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\omega_n}} \left[\xi_n^{2}\, e^{-i\omega_n\tau} + \left(\xi_n^{2}\right)^{\dagger} e^{i\omega_n\tau}\right] \sin(n\sigma). \tag{3.42}$$

Here the string frequencies are

$$\omega_n \equiv \sqrt{m^2+n^2}, \quad \tilde{\omega}_n \equiv \sqrt{\tfrac{1}{4}m^2+n^2}, \quad \omega_n^{\pm} \equiv \omega_n \pm m \quad \text{and} \quad \tilde{\omega}_n^{\pm} \equiv \tilde{\omega}_n \pm \tfrac{1}{2}m \tag{3.43}$$

and the creation and annihilation operators satisfy the usual commutation relations

$$\left[\alpha_m^{K+}, \left(\alpha_n^{L+}\right)^{\dagger}\right] = \left[\alpha_m^{K-}, \left(\alpha_n^{L-}\right)^{\dagger}\right] = \left[\xi_m^{K}, \left(\xi_n^{L}\right)^{\dagger}\right] = \delta^{KL}\delta_{mn}, \quad \text{for } \; K, L \,\epsilon\, \{1,2\}$$

$$\left[\beta_m^{+}, \left(\beta_n^{+}\right)^{\dagger}\right] = \left[\beta_m^{-}, \left(\beta_n^{-}\right)^{\dagger}\right] = \delta_{mn}. \tag{3.44}$$

To determine the string spectrum, we express the lightcone Hamiltonian

$$H_{lc} = \frac{1}{m}\int_0^{\pi} \frac{d\sigma}{2\pi} \left\{ \sum_{I=1}^{4} \left(\tfrac{1}{2}\dot{X}_I^2 + \tfrac{1}{2}X_I'^2\right) + \sum_{I=1}^{2} \left(\tfrac{1}{2}\dot{Y}_I^2 + \tfrac{1}{2}Y_I'^2\right) + \sum_{I=1}^{2} \left(\tfrac{1}{2}\dot{Z}_I^2 + \tfrac{1}{2}Z_I'^2 + \tfrac{1}{2}m^2 Z_I^2\right) \right\}, \tag{3.45}$$

in the normal ordered harmonic oscillator basis as follows:

$$\begin{aligned} H_{lc} = \left(\xi_0^1\right)^{\dagger} \xi_0^1 &+ \sum_{n=1}^{\infty} \frac{\omega_n}{m} \left[\left(\xi_n^1\right)^{\dagger} \xi_n^1 + \left(\xi_n^2\right)^{\dagger} \xi_n^2\right] + \sum_{n=1}^{\infty} \left[\frac{\omega_n^{+}}{m} \left(\beta_n^{+}\right)^{\dagger} \beta_n^{+} + \frac{\omega_n^{-}}{m} \left(\beta_n^{-}\right)^{\dagger} \beta_n^{-}\right] \\ &+ \sum_{n=1}^{\infty} \sum_{K=1}^{2} \left[\frac{\tilde{\omega}_n^{+}}{m} \left(\alpha_n^{K+}\right)^{\dagger} \alpha_n^{K+} + \frac{\tilde{\omega}_n^{-}}{m} \left(\alpha_n^{K-}\right)^{\dagger} \alpha_n^{K-}\right]. \end{aligned} \tag{3.46}$$

Let us now relate the mass m of these short strings to parameters in the original $\mathrm{AdS}_4 \times \mathbb{CP}^3$ spacetime. In the Penrose limit, the energy and momenta, E, J_χ and J_φ, of the string translate to the lightcone momenta

$$H_{lc} = -p_u = E - (J_\chi + J_\varphi) \qquad \text{and} \qquad -p_v = \frac{E + (J_\chi + J_\varphi)}{R^2 \cosh^2 \rho_0} = m, \tag{3.47}$$

and the spatial momentum in the y_1 direction

$$p_{y_1} = -\frac{1}{R} \tanh \rho_0 \left(J_\chi - \frac{J_\varphi}{\sinh^2 \rho_0}\right). \tag{3.48}$$

To keep the lightcone Hamiltonian finite, we must require that $E = J_\chi + J_\varphi + O(1)$, whereas $J_\chi = \frac{1}{\sinh^2 \rho_0} J_\varphi + O(R)$ for finite p_{y_1}. Hence the original charges associated with the short string,

$$J_\varphi \equiv L, \quad J_\chi = \frac{L}{\sinh^2 \rho_0} \left(1 + O\left(\tfrac{R}{L}\right)\right) \quad \text{and} \quad E = L \coth^2 \rho_0 \left(1 + O\left(\tfrac{R}{L}\right)\right). \tag{3.49}$$

are specified (to zeroth order) by a single charge L, chosen to be the momentum J_φ along the circle $\theta = \frac{\pi}{2}$, parameterized by φ, on the dual giant graviton. This gives the inverse mass squared of the short open string excitations

$$\frac{1}{m^2} = \frac{1}{p_v^2} = \frac{R^4 \cosh^4 \rho_0}{(E + J_\chi + J_\varphi)^2} = \frac{\pi^2 \lambda}{2L^2} \sinh^4 \rho_0 \left(1 + O\left(\tfrac{R}{L}\right)\right), \tag{3.50}$$

with t'Hooft coupling $\lambda = \frac{R^4}{2\pi^2}$, as discussed in section 2.4. In the limit in which $\tilde{\lambda} \equiv \frac{\lambda}{L^2}$ is fixed and small, the energy eigenvalues of the lightcone Hamiltonian can be approximated as follows:

$$\alpha_n^{K\pm}: \quad \frac{\tilde{\omega}_n^{\pm}}{m} = \sqrt{\tfrac{1}{4} + \tfrac{n^2}{m^2}} \pm \tfrac{1}{2} \approx 1 + \frac{n^2}{m^2} \text{ or } \frac{n^2}{m^2}, \quad \text{with } n \,\epsilon\, \{1, 2, \ldots\} \tag{3.51}$$

$$\beta_n^{\pm}: \quad \frac{\omega_n^{\pm}}{m} = \sqrt{1 + \tfrac{n^2}{m^2}} \pm 1 \approx 2 + \frac{n^2}{2m^2} \text{ or } \frac{n^2}{2m^2}, \quad \text{with } n \,\epsilon\, \{1, 2, \ldots\} \tag{3.52}$$

$$\xi_n^{K}: \quad \frac{\omega_n}{m} = \sqrt{1 + \tfrac{n^2}{m^2}} \approx 1 + \frac{n^2}{2m^2}, \quad \text{with } n \,\epsilon\, \{0, 1, 2, \ldots\}, \tag{3.53}$$

which can clearly be organized in powers of $\tilde{\lambda}$, including states which are nearly massless, since $\tilde{\lambda} \ll 1$.

In the canonical $AdS_5 \times S^5$ background, the existence of a finite $\tilde{\lambda} \equiv \frac{\lambda}{L}$ scaling limit is tied to the BMN scaling in $\mathcal{N} = 4$ SYM theory [15]. In this case, the anomalous dimensions of near-BPS words attached to the Schur polynomial operator $\chi_R(Z)$ were matched to similar open string excitation energies [40, 41]. It would be interesting to see whether the mismatch observed [31, 32, 34] in the closed string/spin chain sector of the ABJM duality persists for open strings by comparing the above energy spectrum with the anomalous dimensions of the dual open SCS-matter spin chain.

3.3.2 Long semiclassical strings

Zooming back out to the full $AdS_4 \times \mathbb{CP}^3$ spacetime, we shall now consider long strings ending on the dual giant graviton. Even though the string worldsheet on $AdS_4 \times \mathbb{CP}^3$ is just as difficult to quantize as the usual $AdS_5 \times S^5$ case, we can still look at the worldsheet action of a subspace of these string states, which facilitates comparison with a semiclassical analysis of the ABJM spin chain. For long open strings attached to the D2-brane giant, the analysis (at least on the gravity side) is identical to the case of a D3-brane giant on $AdS_5 \times S^5$, the details of which are provided in [41].

The idea is to restrict to a string propagating on an $AdS_3 \times S^1 \subset AdS_4 \times \mathbb{CP}^3$ by setting $\theta = \frac{\pi}{2}$, $\zeta = \frac{\pi}{4}$ and $\theta_i = \pi$. The metric of this subspace is given by

$$R^{-2} ds^2 = -\left(1 + r^2\right)\, dt^2 + \frac{dr^2}{(1 + r^2)} + r^2\, d\varphi^2 + d\chi^2. \tag{3.54}$$

The D2-brane is located at $r = p$ and $\chi = t$, leading to the appropriate Dirichlet boundary conditions. The key to simplifying the action lies in choosing the correct worldsheet gauge [78]. Keeping in mind an eventual comparison with the gauge theory, we shall work in the static gauge $t = \tau$, so that the worldsheet and spacetime energies coincide (being dual to the anomalous dimensions of the ABJM operators). We also choose a gauge $p_\varphi = 2L$, in which the momentum along the trajectory on the dual giant graviton is constant along the string. Here $L = \frac{1}{2\pi}\int_0^\pi d\sigma\, p_\varphi$ is dual to the spin of the ABJM operator. This choice of gauge has a subtle interpretation in the canonical $\mathrm{AdS}_5 \times \mathrm{S}^5$ case: in deriving the spin chain from the $\mathcal{N} = 4$ SYM operator, a choice must be made regarding the separation of the operator into 'sites' of the spin chain, and choosing to spread the AdS_4 momentum density evenly along the string leads to a spin chain for which the sites are organized according to worldvolume spin (i.e. each covariant derivative in the operator corresponds to a site in the spin chain). We expect a similar interpretation in the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ case.

Let us define $\eta \equiv \cosh\rho = \sqrt{1+r^2}$ and $\phi \equiv \chi - t$, which vanishes on the dual giant graviton. In the 'fast motion' limit, in which we assume that the momentum L along the trajectory parameterized by φ is large, so that $\partial_\tau \sim \frac{\lambda}{L^2} \ll 1$, the action reduces to

$$S = -L\int d\tau d\sigma\,\left\{\frac{\dot{\phi}}{(\eta^2-1)} - \frac{\lambda}{4L^2}\left[(\eta')^2 + \eta^2(\phi')^2\right] + O\left(\tfrac{\lambda^2}{L^4}\right)\right\}. \tag{3.55}$$

In the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, this action has been matched to the semiclassical Landau-Lifshitz action derived in the coherent state basis of the dual $sl(2)$ spin chain [41], and we would expect a similar matching to occur for the analogous limit in our case, modulo one important caveat[6]: the $sl(2)$ sector of ABJM theory, unlike $\mathcal{N} = 4$ SYM theory, is not closed. Indeed, at the level of the closed string $OSp(2,2|6)$ spin chain found in [23], it was reported that operators involving the combination $D_\mu Y_a^\dagger Y^b$ mix with fermionic operators containing $\bar{\psi}^b\gamma_\mu\psi_a$. One interpretation of this mixing is that covariant derivative excitations do not correspond to elementary magnons on the closed string spin chain, but should instead be thought of as bound states of fermionic magnons. It was shown that only in the strictly infinite strong coupling limit do these excitations look independent [79], so it is expected that, when string corrections are accounted for, they will dissolve into fermions. Understanding this phenomenon in the open string sector would be an interesting extension of the above analysis.

[6] We would like to thank the anonymous referee of `0901.0009 [hep-th]` for bringing this point to our attention.

Chapter 4

Dibaryon spectroscopy

This chapter concerns dibaryon operators in the Klebanov-Witten and ABJM theories. The conformal dimensions of BPS excitations of determinant operators (special cases of dibaryons) are compared with the spectrum of small fluctuations about the dual D3-brane and D4-brane configurations on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$ respectively.

4.1 Klebanov-Witten dibaryon spectroscopy

We begin by reviewing the comparison [66] of the spectrum of BPS excitations of dibaryon operators in Klebanov-Witten theory with the spectrum of small fluctuations about the dual D3-brane configurations in type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$.

4.1.1 Dibaryons in Klebanov-Witten theory

Dibaryon operators in Klebanov-Witten theory are constructed as follows [65, 66]:

$$\mathcal{D}_{1l} = \epsilon_{\alpha_1 \ldots \alpha_N}\, \epsilon^{\beta_1 \ldots \beta_N} \left\{ D_l^{k_1 \ldots k_N} \left(A_{k_1}\right)^{\alpha_1}{}_{\beta_1} \ldots \left(A_{k_N}\right)^{\alpha_N}{}_{\beta_N} \right\} \tag{4.1}$$

$$\mathcal{D}_{2l} = \epsilon^{\alpha_1 \ldots \alpha_N}\, \epsilon_{\beta_1 \ldots \beta_N} \left\{ D_l^{k_1 \ldots k_N} \left(B_{k_1}\right)_{\alpha_1}{}^{\beta_1} \ldots \left(B_{k_N}\right)_{\alpha_N}{}^{\beta_N} \right\}, \tag{4.2}$$

which involve symmetric combinations of the scalar fields A_k and B_k, and carry positive and negative baryon number respectively. These dibaryons have conformal dimension $\Delta = \frac{3}{4}N$ and $\mathcal{R}$-charge $\frac{1}{2}N$. We focus on the special case of the determinant operators $\det A_1$ and $\det B_1$.

Let us now replace one scalar field in the determinant operator $\det A_1$ with a word (composed of the same or other scalar fields) in the same representation of the gauge group. First consider a word beginning or ending with an A_1; for example

$$\begin{aligned} A_1 &\to A_1 \left(B_{i_1} A_{j_1}\right) \ldots \left(B_{i_n} A_{j_n}\right) \\ &= \left(A_1\right)^{\alpha}{}_{\gamma_1} \left[\left(B_{i_1}\right)_{\delta_1}{}^{\gamma_1} \left(A_{j_1}\right)^{\delta_1}{}_{\gamma_1}\right] \ldots \left[\left(B_{i_n}\right)_{\delta_n}{}^{\gamma_n} \left(A_{j_n}\right)^{\delta_n}{}_{\beta}\right], \end{aligned} \tag{4.3}$$

where the indices have been explicitly shown. It was argued in [66] that there is no BPS excitation of this form - the BPS operator with these quantum numbers is the multiparticle state

$$\left(\det A_1\right) \mathrm{tr}\left\{\left(B_{i_1} A_{j_1}\right) \ldots \left(B_{i_n} A_{j_n}\right)\right\}, \tag{4.4}$$

which is dual to a graviton or closed string state in the presence of the original dibaryon.

However, it turns out [66] that words that do not begin or end in this scalar field A_1 do not factorize into the product of any other states (at the level of the chiral ring). Let us consider replacing A_1 as follows:

$$A_1 \to A_2 \left(B_{i_1} A_2\right) \ldots \left(B_{i_n} A_2\right). \tag{4.5}$$

The new operator has conformal dimension $\Delta = \frac{3}{4}N + \frac{3}{2}n$, and $U(1)$ charges $\frac{1}{2}N + \frac{1}{2}n$ and $\frac{1}{2}n$ with respect to the $SU(2)_A$ and $SU(2)_B$ global symmetry groups. For a given n, there will exist some combination of these single particle states that is BPS and corresponds to the lowest energy open string excitation (with angular momentum $\frac{1}{2}n$ on both 2-spheres) of the first dibaryon, which is dual to $\det A_1$, on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$.

Similar results hold for the determinant operator $\det B_1$. We shall consider replacing

$$B_1 \to B_2 \left(A_{i_1} B_2\right) \ldots \left(A_{i_n} B_2\right), \tag{4.6}$$

to obtain an operator with conformal dimension $\Delta = \frac{3}{4}N + \frac{3}{2}n$, and $U(1)$ charges $\frac{1}{2}n$ and $\frac{1}{2}N + \frac{1}{2}n$ with respect to the $SU(2)_A$ and $SU(2)_B$. The BPS operator of this form is dual to the lowest energy open string excitation of the second dibaryon on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$.

4.1.2 Dibaryons on $\mathbf{AdS}_5 \times \mathbf{T}^{1,1}$

We shall now construct (based on [66]) the two dibaryons, which are dual to $\det A_1$ and $\det B_1$, in type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$. We calculate the spectrum of small scalar fluctuations, with emphasis on those associated with the transverse $\mathrm{T}^{1,1}$ degrees of freedom.

Ansätze for the dibaryons

Type IIB string theory on $AdS_5 \times T^{1,1}$ is discussed in section 1.2.2. It is convenient [71] to define cartesian coordinates v_k to describe the spatial extent of the AdS_5 spacetime as follows:

$$\begin{aligned} v_1 &= r \cos\alpha_1 \cos\beta_1 \qquad & v_2 &= r\cos\alpha_1 \sin\beta_1 \\ v_3 &= r\sin\alpha_1\cos\beta_2 \qquad & v_4 &= r\sin\alpha_1 \sin\beta_2. \end{aligned} \tag{4.7}$$

The ansätze for the two dibaryons, which wrap different 2-spheres and the fibre direction in $T^{1,1}$, and are dual to the determinant operators $\det A_1$ and $\det B_1$ respectively, are given below.

1st dibaryon ($\theta_2 = 0$) ansatz	**2nd dibaryon ($\theta_1 = 0$) ansatz**
$v_k = 0$ $\theta \equiv \theta_2 = 0$ $\varphi(\sigma^a) \equiv \phi_2(\sigma^a)$ unspecified with worldvolume coordinates $\sigma^0 \equiv \tau = t$ $\sigma^1 = z \equiv \cos^2\frac{\theta_1}{2}$ $\sigma^2 = \xi \equiv \psi + \phi_2$ $\sigma^3 = \phi \equiv \phi_1$	$v_k = 0$ $\theta \equiv \theta_1 = 0$ $\varphi(\sigma^a) \equiv \phi_1(\sigma^a)$ unspecified with worldvolume coordinates $\sigma^0 \equiv \tau = t$ $\sigma^1 = z \equiv \cos^2\frac{\theta_2}{2}$ $\sigma^2 = \xi \equiv \psi + \phi_1$ $\sigma^3 = \phi \equiv \phi_2$

We also assume that the worldvolume field strength $F = dA = 0$, so that the worldvolume gauge field A is trivial.

In terms of the coordinates (t, v_k) and $(z, \theta, \xi, \varphi, \phi)$, the $AdS_5 \times T^{1,1}$ background metric is given by

$$\begin{aligned} R^{-2} ds^2 = &-\left(1 + \sum_k v_k^2\right) dt^2 + \sum_{i,j}\left(\delta_{ij} - \frac{v_i v_j}{(1+\sum_k v_k^2)}\right) dv_i dv_j + \frac{dz^2}{6z\,(1-z)} \\ &+ \tfrac{1}{9}\left[d\xi + (2z-1)\,d\phi - (1-\cos\theta)\,d\varphi\right]^2 + \tfrac{2}{3}\,z\,(1-z)\,d\phi^2 + \tfrac{1}{6}\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \end{aligned} \tag{4.8}$$

The 5-form field strength $F_5 = \mathcal{F} + *\mathcal{F}$ is associated with a 4-form potential C_4, which couples to D3-branes. The only contribution relevant for branes (and small fluctuations thereof) extended entirely in $T^{1,1}$ is

$$\mathcal{F} = -\tfrac{2}{27} R^4 \sin\theta \, d\theta \wedge d\varphi \wedge dz \wedge d\xi \wedge d\phi, \tag{4.9}$$

with the corresponding potential

$$C = \tfrac{2}{27} R^4\,(1+\cos\theta)\, d\varphi \wedge dz \wedge d\xi \wedge d\phi. \tag{4.10}$$

D3-brane action

A D3-brane in type IIB string theory on $AdS_5 \times T^{1,1}$ is described by the action

$$S_{\rm D3} = -T_3 \int_\Sigma d^3\sigma \, \sqrt{-\det\left(\mathcal{P}\left[g\right] + 2\pi F\right)} + T_3 \int_\Sigma \mathcal{P}\left[C_4\right], \tag{4.11}$$

with tension $T_3 = \frac{1}{(2\pi)^3}$. The action associated with one of these dibaryons is therefore given by

$$S_{\rm D3} = -\frac{R^4}{72\pi^3} \int_\Sigma d\sigma^4 \left\{1 - \frac{4}{3}\,\dot{\varphi}(\sigma^a)\right\}. \tag{4.12}$$

The momentum conjugate to φ and the Hamiltonian $H = P_\varphi \dot{\varphi} - L$ are

$$P_\varphi = \frac{4R^4}{27\pi} = N \qquad \text{and} \qquad H = \frac{R^4}{9\pi} = \frac{3}{4}N, \tag{4.13}$$

where we make use of the correspondence (1.19) between the rank N of the gauge group in Klebanov-Witten theory and the flux through the base manifold $T^{1,1}$. Notice that the energy $\frac{3}{4}N$ matches the conformal dimension of either of the determinant operators $\det A_1$ or $\det B_1$.

Fluctuation analysis

Let us consider small fluctuations about a dibaryon. It is sufficient to take into account only scalar fluctuations, as the worldvolume fluctuations decouple. We shall specify

$$v_k = \varepsilon\, \delta v_k(\sigma^a), \qquad \theta = \varepsilon\, \delta\theta(\sigma^a) \qquad \text{and} \qquad \varphi(\sigma^a) \ \text{unspecified}, \tag{4.14}$$

and define $\delta y_1 \equiv \delta\theta \cos\varphi$ and $\delta y_1 \equiv \delta\theta \sin\varphi$. Here $\sigma^a = (t, z, \xi, \phi)$ are the worldvolume coordinates and ε is a small parameter.

The D3-brane action (4.14) may then be approximated by $S_{\rm D3} = S_0 + \varepsilon^2 S_2 + \ldots$, with S_0 the original action for the dibaryon and

$$S_2 = -\frac{R^4}{72\pi^3} \int_\Sigma d^4\sigma \left\{ \sum_k \left[\delta v_k^2 - \delta\dot{v}_k^2 + (\nabla \delta v_k)^2\right] + \frac{1}{6}\sum_i \left[-\delta\dot{y}_i^{\,2} + (\nabla\delta y_i)^2\right] \right.$$
$$\left. -\frac{2}{3}\left[\delta y_2 \delta\dot{y}_1 - \delta y_1 \delta\dot{y}_2\right] + \left[\delta y_2\left(\partial_\xi \delta y_1\right) - \delta y_1\left(\partial_\xi \delta y_2\right)\right] \right\} \tag{4.15}$$

the second order corrections. The gradient squared of any function $f(z, \xi, \phi)$ on the worldvolume of the dibaryon is given by

$$(\nabla f)^2 \equiv 6z\,(1-z)\,(\partial_z f)^2 \tag{4.16}$$
$$+ \frac{3}{2z\,(1-z)}\left\{\left[2z\,(1-z) + 1\right](\partial_\xi f)^2 + (\partial_\phi f)^2 - 2\,(2z-1)\,(\partial_\xi f)\,(\partial_\phi f)\right\}.$$

The equations of motion take the form

$$\delta v_k + \ddot{\delta v}_k - \nabla^2 \delta v_k = 0 \tag{4.17}$$

$$\ddot{\delta y}_\pm - \nabla^2 \delta y_\pm \mp 4i\dot{\delta y}_\pm \pm 6i\left(\partial_\xi \delta y_\pm\right) = 0, \tag{4.18}$$

where we define the new transverse coordinates $\delta y_\pm \equiv \delta y_1 \pm i\delta y_2 = \delta\theta\, e^{\pm i\varphi}$, in terms of which the equations of motion decouple, and the Laplacian

$$\nabla^2 \equiv 6\partial_z \left\{z(1-z)\partial_z\right\} + \frac{3}{2z(1-z)}\left\{\left[2z(1-z)+1\right]\partial_\xi^2 + \partial_\phi^2 - 2(2z-1)\partial_\xi\partial_\phi\right\}. \tag{4.19}$$

Expanding the fluctuations in terms of their Fourier modes

$$\delta v_k(t,z,\xi,\phi) = \sum_{m,n,s} C_{smn} e^{-i\omega^k_{smn}t}\Phi_{smn}(z,\xi,\phi) \tag{4.20}$$

$$\delta y_\pm(t,z,\xi,\phi) = \sum_{m,n,s} \tilde{C}_{smn} e^{-i\omega^\pm_{smn}t}\Phi_{smn}(z,\xi,\phi), \tag{4.21}$$

where $\Phi_{smn}(z,\xi,\phi)$ are the eigenfunctions (A.5) of the stationary eigenvalue problem, we obtain the following spectrum for one of the dibaryons

$$\left(\omega^k_{smn}\right)^2 = 6l(l+1) + 3m^2 + 1 \tag{4.22}$$

$$\left(\omega^\pm_{smn} \pm 2\right)^2 = 6l(l+1) + 3(m \mp 1)^2 + 1, \tag{4.23}$$

with $l = s + \max\{|m|,|n|\}$. Here s and n are integers, with s non-negative, and m is an integer or half-integer.

The lowest frequency $s = 0$ mode, with $|m| \geq |n|$, is $\omega^+ = 3m$ or $\omega^- = 3|m|$ for a given m positive or negative respectively. Since m takes on integer or half-integer values, we see that these frequencies increase in steps of $\frac{3}{2}$, matching the conformal dimensions of the BPS excitations of $\det A_1$ and $\det B_1$, which are constructed using words of the form (4.5) and (4.6) respectively. The $U(1)_A$ and $U(1)_B$ charges $\frac{1}{2}n$ of these words match the spin m of the open string excitations on the first and second 2-spheres respectively.

4.2 ABJM dibaryon spectroscopy

We shall now extend this comparison to dibaryon operators in ABJM theory. BPS excitations of these states should map to open string excitations of the dual D4-brane configurations in type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. A comparison is made at the level of the fluctuation spectrum.

4.2.1 Dibaryons in ABJM theory

Dibaryons in ABJM theory may naïvely be constructed in a similar manner as those in Klebanov-Witten theory. The two classes of dibaryons take the form

$$\mathcal{D}_{1l} = \epsilon_{\alpha_1 \ldots \alpha_N}\, \epsilon^{\beta_1 \ldots \beta_N} \left\{ D_{l,a_1 \ldots a_N} \left(Y^{a_1}\right)^{\alpha_1}{}_{\beta_1} \ldots \left(Y^{a_N}\right)^{\alpha_N}{}_{\beta_N} \right\} \tag{4.24}$$

$$\mathcal{D}_{2l} = \epsilon^{\alpha_1 \ldots \alpha_N}\, \epsilon_{\beta_1 \ldots \beta_N} \left\{ D_l^{a_1 \ldots a_N} \left(Y^\dagger_{a_1}\right)_{\alpha_1}{}^{\beta_1} \ldots \left(Y^\dagger_{a_N}\right)_{\alpha_N}{}^{\beta_N} \right\}, \tag{4.25}$$

which carry positive and negative baryon number respectively, and have conformal dimension $\Delta = \frac{1}{2}N$. For simplicity, we shall choose to consider the representatives $\det Y^1 = \det A_1$ and $\det Y_3^\dagger = \det B_1$.

Let us now replace one of the scalar fields in the determinant $\det A_1$ or $\det B_1$ as follows:

$$A_1 \to A_2\,(B_{i_1} A_2) \ldots (B_{i_n} A_2) \qquad \text{or} \qquad B_1 \to B_2\,(A_{i_1} B_2) \ldots (A_{i_n} B_2)\,. \tag{4.26}$$

The new operators have conformal dimension $\Delta = \frac{1}{2}N + n$, and $U(1)$ $\mathcal{R}$-charges $\frac{1}{2}N + \frac{1}{2}n$ and $\frac{1}{2}n$ (or vice versa) under the $SU(2)_A$ and $SU(2)_B$ subgroups respectively. They are dual to open string excitations of the two $\mathbb{CP}^2$ dibaryons on $\text{AdS}_4 \times \mathbb{CP}^3$. Although a general operator of this form will pick up an anomalous dimension, as was argued in [66] for similar words attached to Klebanov-Witten dibaryons, we expect that, for a given n, there will exist a combination that remains BPS. The dimensions of these BPS excitations are protected and should match the lowest energy open string modes.

A significant difference between the ABJM and Klebanov-Witten theories is that, in the first case, the gauge group is $U(N) \times U(N)$ rather than $SU(N) \times SU(N)$. Each $U(N)$ contains an additional local $U(1)$ symmetry, but the current associated with the second $U(1)$ couples only to that of the first $U(1)$ and is hence trivial [2]. Now, the dibaryons (4.24) and (4.25) are charged with respect to the extra local $U(1)$ symmetry in ABJM theory, and are therefore not gauge invariant operators. It is, however, possible to attach Wilson lines - exponentials of integrals over gauge fields - to the dibaryons to make them gauge invariant. It was argued in [2] that these operators remain local and that the Wilson lines are unobservable. This modification should therefore not effect the conformal dimensions of the determinant operators or their BPS excitations [2, 69].

4.2.2 $\mathbb{CP}^2$ dibaryons on $\text{AdS}_4 \times \mathbb{CP}^3$

We shall now construct two $\mathbb{CP}^2$ dibaryons, which are dual to $\det A_1$ and $\det B_1$, in type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$. The spectrum of small fluctuations is calculated. We compare the lowest frequency modes associated with the transverse $\mathbb{CP}^3$ coordinates with the conformal dimensions of BPS excitations of the determinants in ABJM theory.

Ansätze for the $\mathbb{CP}^2$ dibaryons

Type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is described in section 2.3. Again, we make use of the following cartesian coordinates to reparameterize the AdS_4 spacetime:

$$v_1 = r\cos\theta \qquad v_2 = r\sin\theta\cos\varphi \qquad v_3 = r\sin\theta\sin\varphi. \tag{4.27}$$

The dibaryons are D4-branes wrapped on the two natural $\mathbb{CP}^2$ subspaces $(\zeta, \xi, \theta_1, \phi_1)$ and $(\zeta, \xi, \theta_2, \phi_2)$ of the complex projective space $\mathbb{CP}^3$. These are non-contractible cycles - the D4-branes are therefore topologically stable configurations. The ansätze for the two $\mathbb{CP}^2$ dibaryons, which are dual to $\det A_1$ and $\det B_1$ respectively, are given below.

1st $\mathbb{CP}^2$ dibaryon ($\theta_2 = 0$) ansatz	**2nd $\mathbb{CP}^2$ dibaryon ($\theta_1 = 0$) ansatz**
$v_k = 0$	$v_k = 0$
$\theta \equiv \theta_2 = 0$	$\theta \equiv \theta_1 = 0$
$\varphi(\sigma^a) \equiv \phi_2(\sigma^a)$ unspecified	$\varphi(\sigma^a) \equiv \phi_1(\sigma^a)$ unspecified
with worldvolume coordinates	with worldvolume coordinates
$\sigma^0 \equiv \tau = t$	$\sigma^0 \equiv \tau = t$
$\sigma^1 = x \equiv \sin^2\zeta$	$\sigma^1 = x \equiv \cos^2\zeta$
$\sigma^2 = z \equiv \cos^2\frac{\theta_1}{2}$	$\sigma^2 = z \equiv \cos^2\frac{\theta_2}{2}$
$\sigma^3 = \xi \equiv \psi + \phi_2$	$\sigma^3 = \xi \equiv \psi + \phi_1$
$\sigma^4 = \phi \equiv \phi_1$	$\sigma^4 = \phi \equiv \phi_2$

The worldvolume field strength $F = dA = 0$ is taken to vanish.

In terms of the coordinates (t, v_k) and $(x, z, \theta, \xi, \phi, \varphi)$, the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background metric is given by

$$\begin{aligned} R^{-2}ds^2 = &-\left(1 + \sum_k v_k^2\right)dt^2 + \sum_{i,j}\left(\delta_{ij} - \frac{v_i v_j}{(1+\sum_k v_k^2)}\right)dv_i dv_j \\ &+ \frac{dx^2}{x(1-x)} + x(1-x)\left[d\xi + (2z-1)\,d\phi - (1-\cos\theta)\,d\varphi\right]^2 \\ &+ (1-x)\left[\frac{dz^2}{z(1-z)} + 4z(1-z)\,d\phi^2\right] + x\left[d\theta^2 + \sin^2\theta d\varphi^2\right], \end{aligned} \tag{4.28}$$

while the field strengths[1] can be written as

$$\begin{aligned} F_2 = &-\tfrac{1}{2}k\left\{dx \wedge \left[d\xi + (2z-1)\,d\phi - (1-\cos\theta)\,d\varphi\right]\right. \\ &\left. + 2(1-x)\ dz \wedge d\phi + x\sin\theta\, d\theta \wedge d\varphi\right\} \end{aligned} \tag{4.29}$$

[1]Note that the ansatz for the second $\mathbb{CP}^2$ dibaryon involves a change in orientation and the 2-form and 6-form field strengths, as well as the associated 1-form and 5-form potentials, pick up an additional minus sign as a result.

$$F_4 = -\tfrac{3}{2}\,kR^2\; dt \wedge dv_1 \wedge dv_2 \wedge dv_3 \tag{4.30}$$

$$F_6 = -\,3kR^4\; x\,(1-x)\sin\theta\; d\theta \wedge d\varphi \wedge dx \wedge dz \wedge d\xi \wedge d\phi, \tag{4.31}$$

with associated potentials

$$C_1 = -\,\tfrac{1}{2}\,k\,\{2\,(1-x)\,[d\xi + (2z-1)\,d\phi - (1-\cos\theta)\,d\varphi] - (1+\cos\theta)\,d\varphi\} \tag{4.32}$$

$$C_3 = \tfrac{1}{2}\,kR^2\; dt \wedge (v_1 dv_2 \wedge dv_3 + v_2 dv_3 \wedge dv_1 + v_3 dv_1 \wedge dv_2) \tag{4.33}$$

$$C_5 = 3kR^4\; x\,(1-x)\,(1+\cos\theta)\;\; d\varphi \wedge dx \wedge dz \wedge d\xi \wedge d\phi. \tag{4.34}$$

D4-brane action

A D4-brane on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is described by the action $S_{\mathrm{D4}} = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}$, which consists of the Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) terms

$$S_{\mathrm{DBI}} = -T_4 \int_\Sigma d^5\sigma\;\; e^{-\Phi}\sqrt{-\det\left(\mathcal{P}[g] + 2\pi F\right)} \tag{4.35}$$

$$S_{\mathrm{WZ}} = T_4 \int_\Sigma \left\{\mathcal{P}[C_5] + 2\pi\; \mathcal{P}[C_3] \wedge F + \frac{1}{2}\,(2\pi)^2\; \mathcal{P}[C_1] \wedge F \wedge F\right\}, \tag{4.36}$$

with tension $T_4 \equiv \frac{1}{(2\pi)^4}$. The action for a $\mathbb{CP}^2$ dibaryon may hence be calculated as follows:

$$S_{\mathrm{D4}} = -\frac{kR^4}{16\pi^4}\int_\Sigma d^5\sigma\;\; (1-x)\,\{1 - 6x\;\dot{\varphi}(\sigma^a)\}\,. \tag{4.37}$$

The momentum P_φ conjugate to φ and the Hamiltonian $H = P_\varphi\,\dot{\varphi} - L$ are given by

$$P_\varphi = \frac{kR^4}{2\pi^2} = N \qquad\qquad \text{and} \qquad\qquad H = \frac{kR^4}{4\pi^2} = \frac{1}{2}\,N. \tag{4.38}$$

Here we make use of the correspondence (2.26) to express the above quantities in terms of the rank N of the gauge group in ABJM theory. Notice that the energy of the $\mathbb{CP}^2$ dibaryon matches the conformal dimension $\frac{1}{2}N$ of the associated determinant operator.

Fluctuation analysis

Let us consider small fluctuations about a $\mathbb{CP}^2$ dibaryon. Both scalar and worldvolume fluctuations will be taken into account, since it is not immediately obvious that the latter decouple (although it turns out that they do). The ansatz for the scalar fluctuations is

$$v_k = \varepsilon\;\delta v_k(\sigma^a) \qquad\qquad \text{and} \qquad\qquad \theta = \varepsilon\;\delta\theta(\sigma^a), \tag{4.39}$$

with $\varphi(\sigma^a)$ unspecified. It is convenient to rather make use of the transverse $\mathbb{CP}^3$ coordinates $y_1 = \sin\theta\cos\varphi$ and $y_2 = \sin\theta\sin\varphi$, which vanish on the worldvolume of the $\mathbb{CP}^2$ dibaryon and are perturbed as follows:

$$y_i = \varepsilon\, \delta y_i(\sigma^a), \qquad \text{with} \quad \delta y_1 = \delta\theta \cos\varphi \quad \text{and} \quad \delta y_2 = \delta\theta \sin\varphi. \tag{4.40}$$

The worldvolume fluctuation ansatz takes the form

$$2\pi F = \varepsilon\, \delta F(\sigma^a), \qquad \text{with} \quad \delta F(\sigma^a) = d\, \delta A(\sigma^a). \tag{4.41}$$

Here $\sigma^a = (t, x, z, \xi, \phi)$ are the worldvolume coordinates and ε is a small parameter.

We shall now calculate the D4-brane action, which describes these small fluctuations about the $\mathbb{CP}^2$ dibaryon, to second order in ε. The DBI action (4.35) is

$$S_{\text{DBI}} = -\frac{1}{16\pi^4}\frac{k}{2R}\left\{\int_\Sigma d^5\sigma\, \sqrt{-\det \mathcal{P}[g]} + \frac{1}{2}\varepsilon^2 \int_\Sigma \delta F \wedge * \delta F\right\}, \tag{4.42}$$

with

$$\begin{aligned}\sqrt{-\det \mathcal{P}[g]} \approx 2\,(1-x)\Bigg\{ & 1 + \frac{1}{2}\varepsilon^2 \sum_k \left[\delta v_k^2 - \dot{\delta v_k}^2 + (\nabla \delta v_k)^2\right] \\ & + \frac{1}{2}\varepsilon^2 x \sum_i \left[-\dot{\delta y_i}^2 + (\nabla \delta y_i)^2\right] + \frac{1}{2}\varepsilon^2 \left[\delta y_2\, (\partial_\xi \delta y_1) - \delta y_1\, (\partial_\xi \delta y_2)\right]\Bigg\}\end{aligned} \tag{4.43}$$

and $*$ the Hodge star operator on the worldvolume of the $\mathbb{CP}^2$ dibaryon. The gradient squared of any function $f(x, z, \xi, \phi)$ on the $\mathbb{CP}^2$ subspace is defined as follows:

$$\begin{aligned}(\nabla f)^2 \equiv{}& x\,(1-x)\,(\partial_x f)^2 + \frac{1}{x\,(1-x)}\,(\partial_\xi f)^2 \\ & + \frac{1}{(1-x)}\left\{ z\,(1-z)\,(\partial_z f)^2 + \frac{1}{4z\,(1-z)}\left[(2z-1)\,(\partial_\xi f) - (\partial_\phi f)\right]^2\right\}.\end{aligned} \tag{4.44}$$

The WZ action takes the form[2]

$$S_{WZ} = \frac{1}{16\pi^4}\int_\Sigma \left\{\mathcal{P}\,[C_5] + \varepsilon R^2\, \mathcal{P}\,[C_3] \wedge \delta F + \frac{1}{2}\varepsilon^2 R^4\, \mathcal{P}\,[C_1] \wedge \delta F \wedge \delta F\right\}, \tag{4.45}$$

where the potentials, pulled back to the worldvolume of the D4-brane, are given by

$$\begin{aligned}\mathcal{P}\,[C_5] &= 6kR^4\, x\,(1-x)\left[\dot\varphi - \frac{1}{4}\varepsilon^2 \left(\delta y_2 \dot{\delta y_1} - \delta y_1 \dot{\delta y_2}\right) + O(\varepsilon^4)\right] dt \wedge dx \wedge dz \wedge d\xi \wedge d\phi \\ \mathcal{P}\,[C_3] &= O(\varepsilon^3) \\ \mathcal{P}\,[C_1] &= -\frac{1}{2}k\,(1-x)\,[d\xi + (2z-1)\,d\phi] + k\,(\partial_a \varphi)\, d\sigma^a + O(\varepsilon^2).\end{aligned} \tag{4.46}$$

[2]Note that, in this WZ action, we need to subtract off similar expressions evaluated at $\theta = \pi$ (since these terms come from an integral over $\mathbb{CP}^3$ of the corresponding field strength forms and the θ integral runs from π to 0). This makes no difference to the term involving the 5-form potential, which has been chosen to vanish when $\theta = \pi$, but does result in an additional subtraction from the integral over the 1-form potential. With this taken into account, the last term results only in a total derivative in the WZ action.

Combining the DBI and WZ terms, the D4-brane action can be approximated by $S_{D4} = S_0 + \varepsilon^2 S_2 + \ldots$, with S_0 the original action (4.37) for the $\mathbb{CP}^2$ dibaryon and

$$S_2 = -\frac{kR^4}{32\pi^4}\int_\Sigma d^5\sigma\ (1-x)\left\{\sum_k\left[\delta v_k^2 - \delta\dot{v}_k{}^2 + (\nabla\delta v_k)^2\right] + x\sum_i\left[-\delta\dot{y}_i{}^2 + (\nabla\delta y_i)^2\right]\right.$$
$$\left. -3x\left(\delta y_2\delta\dot{y}_1 - \delta y_1\delta\dot{y}_2\right) + \left[\delta y_2\left(\partial_\xi\delta y_1\right) - \delta y_1\left(\partial_\xi\delta y_2\right)\right]\right\}$$
$$-\frac{1}{16\pi^4}\frac{k}{2R}\int_\Sigma \delta F\wedge *\delta F, \tag{4.47}$$

the second order corrections. Notice that the worldvolume fluctuations decouple.

The equations of motion for δv_k and $\delta y_\pm$, which are the combinations $\delta y_\pm \equiv \delta y_1 \pm i\delta y_2$ of the transverse $\mathbb{CP}^3$ fluctuations, are

$$\delta\ddot{v}_k - \nabla^2\delta v_k + \delta v_k = 0, \tag{4.48}$$
$$\delta\ddot{y}_\pm \mp 3i\delta\dot{y}_\pm - \nabla^2\delta y_\pm - (1-x)\left(\partial_x\delta y_\pm\right) \pm \frac{i}{x}\left(\partial_\xi\delta y_\pm\right) = 0, \tag{4.49}$$

with the $\mathbb{CP}^2$ Laplacian

$$\nabla^2 \equiv \partial_x\left[x(1-x)\,\partial_x\right] - x\,\partial_x + \frac{1}{x(1-x)}\,\partial_\xi^2$$
$$+\frac{1}{(1-x)}\left\{\partial_z\left[z(1-z)\,\partial_z\right] + \frac{1}{4z(1-z)}\left[(2z-1)\,\partial_\xi - \partial_\phi\right]^2\right\}. \tag{4.50}$$

Let us expand the AdS_4 fluctuations in terms of the complete set of chiral primaries χ_l defined in (4.56). Solutions to the equations of motion (4.48) take the form

$$\delta v_k = \sum_l e^{-i\omega_l^k t}\ \chi_l\left(z^A, \bar{z}_B\right), \tag{4.51}$$

with frequencies

$$\left(\omega_l^k\right)^2 = l(l+2)+1, \tag{4.52}$$

where l is a non-negative integer.

Laplacian and chiral primaries on $\mathbb{CP}^2$

The homogenous coordinates of $\mathbb{CP}^2 \subset \mathbb{CP}^3$ subspace take the form

$$z^1 = x\ e^{\frac{1}{2}i\xi}, \qquad z^2 = \sqrt{(1-x)(1-z)}\ e^{\frac{1}{2}i\phi} \quad \text{and} \quad z^3 = \sqrt{(1-x)\,z}\ e^{-\frac{1}{2}i\phi}, \tag{4.53}$$

where $x, z \,\epsilon\, [0,1]$, $\xi \,\epsilon\, [0,4\pi]$ and $\phi \,\epsilon\, [0,2\pi]$. These can be obtained (up to an overall phase and an interchange of the z^A's) from the homogenous coordinates (2.9) of $\mathbb{CP}^3$.

Following [23], we shall write down the Laplacian and chiral primaries in terms of these homogenous coordinates of $\mathbb{CP}^2$. Let us first define the Laplace-Beltrami operator

$$L^A{}_B \equiv z^A \frac{\partial}{\partial z^B} - \bar{z}_B \frac{\partial}{\partial \bar{z}_A}, \tag{4.54}$$

in terms of which the Laplacian can be written as

$$\nabla^2 \equiv -\frac{1}{2}\sum_{A,B} L^A{}_B\, L^B{}_A = -\frac{1}{2}\sum_{A,B}\left\{z^A z^B \frac{\partial}{\partial z^A}\frac{\partial}{\partial z^B} + \bar{z}_A \bar{z}_B \frac{\partial}{\partial \bar{z}_A}\frac{\partial}{\partial \bar{z}_B}\right\} + \sum_A \frac{\partial}{\partial z^A}\frac{\partial}{\partial \bar{z}_A} - \frac{3}{2}\sum_A \left\{z^A \frac{\partial}{\partial z^A} + \bar{z}_A \frac{\partial}{\partial \bar{z}_A}\right\}. \tag{4.55}$$

Any function on $\mathbb{CP}^2$ can be expanded in terms of the chiral primaries

$$\chi_l\left(z^A, \bar{z}_B\right) = \sum_{A_i, B_i} \chi^{B_1 \ldots B_l}_{A_1 \ldots A_l}\, z^{A_1} \ldots z^{A_l}\, \bar{z}_{B_1} \ldots \bar{z}_{B_l}, \tag{4.56}$$

with $\chi^{B_1 \ldots B_l}_{A_1 \ldots A_l}$ symmetric (under interchange of any two A_i or B_i) and traceless. These are eigenfunctions of the Laplacian on $\mathbb{CP}^2$:

$$\nabla^2 \chi_l = -l\,(l+2)\,\chi_l, \tag{4.57}$$

where the eigenvalues are dependent only on the length l.

It turns out, however, that the chiral primaries are not a suitable set of functions over which to expand the transverse $\mathbb{CP}^3$ fluctuations. We rather make use of the eigenfunctions $\Phi^{\pm}_{smn}(z, x, \xi, \phi)$, written down explicitly in (A.32), of modified operators $\mathcal{O}_{\pm}$. The transverse fluctuations are then given by

$$\delta y_{\pm} = \sum_{s,m,n} e^{-i\omega^{\pm}_{smn}\, t}\, \Phi^{\pm}_{smn}\left(x, z, \xi, \phi\right), \tag{4.58}$$

which solve the equations of motion (4.49), if the frequencies satisfy

$$\omega^{+}_{smn}\left(\omega^{+}_{smn} + 3\right) = l\,(l+3) \qquad \text{and} \qquad \omega^{-}_{smn}\left(\omega^{-}_{smn} + 3\right) = l\,(l+3) + 2, \tag{4.59}$$

with $l = s + 2m$, where s is an non-negative integer and m an integer or half-integer. Here we have assumed that m is positive. The lowest frequency mode with $s = 0$ has $\omega^{+} = 2m$, simply increasing in integer steps. (A similar result applies when m is negative - the lowest frequency mode is then $\omega^{-} = 2|m|$). These frequencies match the conformal dimensions of BPS excitations of the form (4.26) of the determinant operators $\det A_1$ and $\det B_1$ in ABJM theory. Again the $U(1)_A$ and $U(1)_B$ $\mathcal{R}$-charges correspond to the spin m. An interpretation of the $m \geq n$ bound is simply that these words are in the singlet highest spin state of the $SU(2)_A$ and $SU(2)_B$ subgroups respectively.

Chapter 5

Giant gravitons

We shall now make use of Mikhailov's holomorphic curve construction to build giant gravitons in type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$. We analyze small fluctuations about this D3-brane configuration, determining the spectrum explicitly for the small submaximal and maximal giants. Open string excitations are also considered. We close with an ansatz for a giant graviton in type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

5.1 Giant gravitons on $\mathbf{AdS}_5 \times \mathbf{T}^{1,1}$

We shall consider a giant graviton on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ dual to the subdeterminant operator $\mathcal{O}_n(A_1B_1)$ in Klebanov-Witten theory. This is a D3-brane, with angular momentum in $\mathrm{T}^{1,1}$, wrapped on a contractible cycle in this compact space. Our construction of this giant graviton is based on the ansatz of [63].

5.1.1 Giant graviton ansatz via a holomorphic curve

Type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ is discussed in section 1.2.2. The base manifold $\mathrm{T}^{1,1}$ is embedded in a cone $\mathcal{C}$ described by four complex coordinates z^A, which satisfy $z^1z^2 = z^3z^4$. This constraint suggests that we associate these four complex directions with the scalar fields A_i and B_i in Klebanov-Witten theory as follows [48]:

$$z^1 \to A_1B_1 \qquad z^2 \to A_2B_2 \qquad z^3 \to A_2B_1 \qquad z^4 \to A_1B_2. \tag{5.1}$$

Let us choose the holomorphic curve on the cone $\mathcal{C}$ to be

$$F(z^A) = z^1 = \rho, \tag{5.2}$$

where ρ is a real constant. The surface of the giant graviton is the intersection of the holomorphic curve with the base manifold $\mathrm{T}^{1,1}$. For this intersection to be non-empty, ρ must be confined to the unit interval, so that we may define $\rho \equiv \sqrt{1-\alpha^2}$, with $\alpha \,\epsilon\, [0,1]$. Here α may be thought of as the 'size' of the giant graviton.

To introduce motion, take $z^A \to z^A e^{-i\varphi(t)}$ in the holomorphic function $F(z^A)$, with $\varphi(t)$ an overall time-dependent phase[1]. Hence

$$F(z^A e^{-i\varphi(t)}) \equiv z^1 e^{-i\varphi(t)} = \sqrt{1-\alpha^2} \qquad \Rightarrow \quad z^1 = \sqrt{1-\alpha^2}\; e^{i\varphi(t)}, \tag{5.3}$$

where we hold the other *independent* coordinates fixed. There is a subtlety involved, however, in choosing which two complex coordinates on the cone, aside from z^1, to consider as independent. These should correspond to exactly those angular directions along which the giant graviton does *not* rotate. Since the dual operators are constructed out of equal numbers of A_1's and B_1's, we see that z^1, z^2 and z^3/z^4 (or z^4/z^3) are the correct independent coordinates to use. Therefore, we rotate around the $\frac{1}{2}(\psi-\phi_1-\phi_2)$ direction, while holding $\frac{1}{2}(\psi+\phi_1+\phi_2)$ and $\phi_1-\phi_2$ fixed.

We shall now define

$$\chi_1 \equiv \tfrac{1}{3}\left(\psi - \phi_1 - \phi_2\right) \tag{5.4}$$

$$\chi_2 \equiv \tfrac{1}{3}\left(\psi + 3\phi_1 - \phi_2\right) = \tfrac{2}{3}\left[\tfrac{1}{2}\left(\psi+\phi_1+\phi_2\right) + \left(\phi_1-\phi_2\right)\right] \tag{5.5}$$

$$\chi_3 \equiv \tfrac{1}{3}\left(\psi - \phi_1 + 3\phi_2\right) = \tfrac{2}{3}\left[\tfrac{1}{2}\left(\psi+\phi_1+\phi_2\right) - \left(\phi_1-\phi_2\right)\right], \tag{5.6}$$

where[2] $\chi_1 \,\epsilon\, [0, \frac{4\pi}{3}]$ and χ_2, $\chi_3 \,\epsilon\, [0, \frac{8\pi}{3}]$. Note that χ_2 and χ_3 are combinations of the phases of our independent coordinates z^2 and z^3/z^4. The complex coordinates z^A, confined to the base manifold $T^{1,1}$, can be written as

$$z^1 = \sin\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2}\; e^{\frac{3}{2}i\chi_1} \qquad\qquad z^2 = \cos\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2}\; e^{\frac{3}{4}i(\chi_2+\chi_3)}$$
$$z^3 = \cos\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2}\; e^{\frac{3}{4}i(\chi_1+\chi_2)} \qquad\qquad z^4 = \sin\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2}\; e^{\frac{3}{4}i(\chi_1+\chi_3)}. \tag{5.7}$$

The giant graviton ansatz then translates into setting

$$\sin\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2} = \sqrt{1-\alpha^2}, \tag{5.8}$$

and considering the angular direction of motion $\chi_1(t)$.

[1]The preferred direction of [63], induced by the embedding of $\mathrm{T}^{1,1}$ into the cone $\mathcal{C}$, is along the fibre ψ. This is independent of which holomorphic function is chosen to construct a particular giant graviton and should not be confused with the direction of motion, which is the component of the preferred direction perpendicular to the giant graviton's surface.

[2]These angular coordinate ranges are not immediately obvious from the definitions (5.4) - (5.6), but can be obtained from a consideration of the volume form.

Point graviton ($\alpha = 0$)

When $\alpha = 0$, we obtain the point graviton. Here $z^1 = e^{\frac{3}{2}i\chi_1(t)}$ and $z^2 = z^3 = z^4 = 0$, which describes the motion of a point along a circle of maximum radius.

Maximal giant graviton ($\alpha = 1$)

The maximal giant graviton is obtained by setting $\alpha = 1$. The polar coordinates θ_1 and θ_2 on each of the 2-spheres decouple and we find two distinct solutions

$$\sin\tfrac{\theta_2}{2} = 0 \qquad \text{or} \qquad \sin\tfrac{\theta_1}{2} = 0, \tag{5.9}$$

which describes the union of the two spaces $\theta_2 = 0$ or $\theta_1 = 0$. These are the dibaryons of [66] - two D3-branes wrapped on different 2-spheres and the $U(1)$ fibre - corresponding to the determinant operators $\det A_1$ and $\det B_1$ respectively.

Submaximal giant graviton

We would like to understand this factorization into two dibaryons as some submaximal giant graviton configuration in the limit as $\alpha \to 1$. Key in this endeavour is our choice of worldvolume coordinates: the obvious independent angles are χ_2 and χ_3, but how do we choose a radial parameter describing the giant graviton worldvolume? To obtain the maximal giant as a limiting case, we cannot choose either θ_1 or θ_2, as this choice would eliminate half the maximal giant a priori. Let us rather consider the combination

$$u = \cos\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2}, \tag{5.10}$$

which is the magnitude of the complex coordinate z^2. Using the relation (5.8) between θ_1 and θ_2 on the giant graviton worldvolume, we can rewrite

$$u(\theta_i) = \cot\tfrac{\theta_i}{2}\sqrt{\alpha^2 - \cos^2\tfrac{\theta_i}{2}}. \tag{5.11}$$

Note that θ_i is only defined on the interval $[2\arccos\alpha, \pi]$. The function $u(\theta_i)$ vanishes at both ends of this interval and attains a maximum value $u_{\max} = 1 - \sqrt{1-\alpha^2}$ at the polar angle $\theta_{i,\max} = 2\arcsin(1-\alpha^2)^{1/4}$.

Now, we observe that the worldvolume of the giant graviton is a double-covering of u. The θ_1 interval naturally splits into two pieces $[2\arccos\alpha, 2\arcsin(1-\alpha^2)^{1/4}]$ and $[2\arcsin(1-\alpha^2)^{1/4}, \pi]$, which, since the $u(\theta_1)$ maximum occurs when $\theta_1 = \theta_2$, simply correspond to $\theta_1 \leq \theta_2$ and $\theta_1 \geq \theta_2$. Leaving the second interval in terms of θ_1, one could choose to describe the first region in terms of θ_2, mapping[3] it onto the second interval in the analogous $u(\theta_2)$ diagram (see figure 5.1).

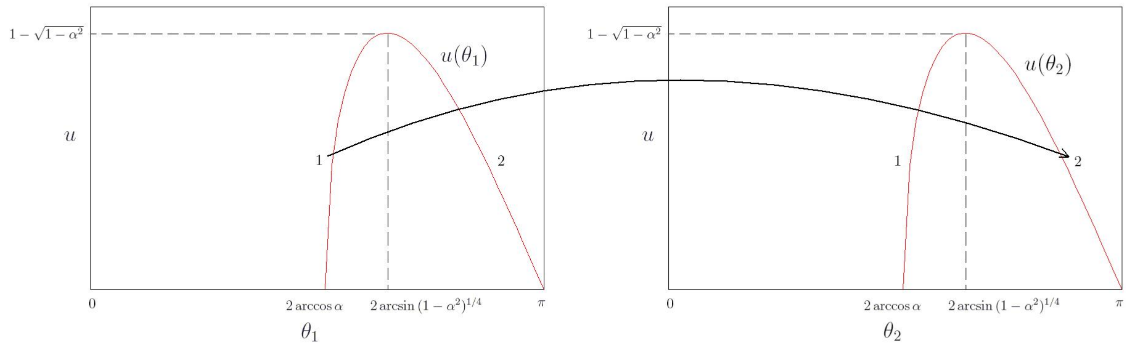

Figure 5.1: The radial coordinate u, confined to the worldvolume of the giant graviton on which α is constant, plotted as a function of the polar angles θ_1 and θ_2 respectively. The mapping between the $u(\theta_1)$ and $u(\theta_2)$ diagrams is shown.

The D3-brane action for the submaximal giant graviton will consist of two identical parts, involving integrals over θ_1 and θ_2 respectively, which run from $2\arcsin(1-\alpha^2)^{1/4}$ to π (although we shall find it more convenient to simply double the integral over u from 0 to $1-\sqrt{1-\alpha^2}$). Note that this action still describes a single D3-brane extended on both 2-spheres. In the limit $\alpha \to 1$, each of the second θ_i regions covers an entire 2-sphere, whilst the first completely vanishes. In this way, we recover both halves of the maximal giant graviton.

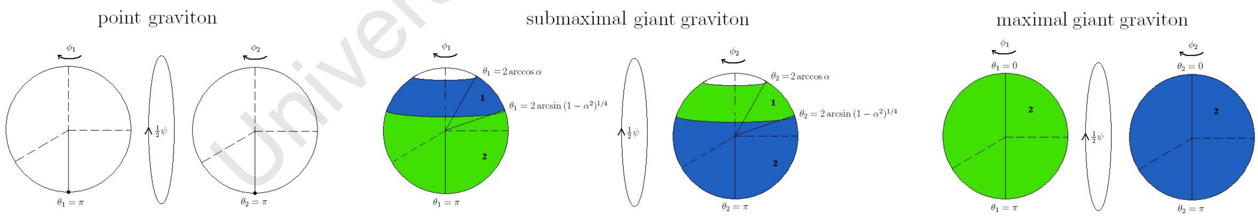

Figure 5.2: Pictorial representation of the expansion of a point graviton, via a submaximal giant graviton intermediate state, into the maximal giant graviton on $AdS_5 \times T^{1,1}$. Regions identically shaded (either blue or green) are mapped onto each other by the constraint $\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \sqrt{1-\alpha^2}$, which describes the worldvolume of the giant graviton. The factorization of the maximal giant into two dibaryons is clearly visible.

This construction allows us to see the intermediate state - the submaximal giant graviton - between the point graviton and the maximal giant graviton. We observe the manner in which the maximal giant factorizes - the relation between θ_1 and θ_2, and

[3]Note that there is a change in orientation under this map.

the mapping between the different regions (shaded the same colour in figure 5.2) of the two 2-spheres disappears.

5.1.2 Angular and radial coordinate changes

The giant graviton is situated at the center of the AdS_5 spacetime (moving only in time t). We may therefore restrict ourselves to the background $\mathbb{R} \times T^{1,1}$, which has the following metric:

$$R^{-2}ds^2 = -dt^2 + ds^2_{\text{radial}} + ds^2_{\text{angular}}, \tag{5.12}$$

with

$$ds^2_{\text{radial}} = \tfrac{1}{6}\left\{d\theta_1^2 + d\theta_2^2\right\} \tag{5.13}$$
$$ds^2_{\text{angular}} = \tfrac{1}{18}\left\{2\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 + 3\sin^2\theta_1 d\phi_1^2 + 3\sin^2\theta_2 d\phi_2^2\right\}, \tag{5.14}$$

describing the radial and angular parts of the metric (separated for later convenience) associated with the magnitudes and phases of the complex coordinates z^A. The giant graviton couples to the 4-form potential with corresponding $T^{1,1}$ field strength (1.14).

Angular coordinates χ_i

We shall now change to the angular coordinates χ_i, defined in (5.4) - (5.6), which most conveniently parameterize the direction of motion and angular extension of the giant graviton. The components of the angular metric $(g_\chi)_{ij}$ are stated in (5.19) - (5.24).

The determinant of the angular metric is given by

$$\det g_\chi = \left(\tfrac{3}{32}\right)^2 \sin^2\theta_1 \sin^2\theta_2. \tag{5.15}$$

We shall also need to know the determinant of the angular metric restricted to the worldvolume coordinates χ_2 and χ_3:

$$\left(C_\chi\right)_{11} = 3\left(\tfrac{1}{32}\right)^2\left\{2\sin^2\theta_1\left(1+\cos\theta_2\right)^2 + 2\sin^2\theta_2\left(1+\cos\theta_1\right)^2 + 3\sin^2\theta_1\sin^2\theta_2\right\}, \tag{5.16}$$

which is also the cofactor of the element $\left(g_\chi\right)_{11}$. The 5-form field strength in terms of these angular coordinates χ_i and the original radial coordinates θ_i is given by

$$F_5 = \tfrac{1}{16}R^4 \sin\theta_1 \sin\theta_2 \; d\theta_1 \wedge d\theta_2 \wedge d\chi_1 \wedge d\chi_2 \wedge d\chi_3. \tag{5.17}$$

Angular metric in the coordinates (θ_i, χ_i)

The angular metric (5.14) of the base manifold $T^{1,1}$, which depends on the phases χ_i defined in (5.4) - (5.6), is given by

$$ds^2_{\text{angular}} = \sum_{i,j} (g_\chi)_{ij}\, d\chi_i\, d\chi_j, \tag{5.18}$$

where the components can explicitly be written as follows:

$$(g_\chi)_{11} = \tfrac{1}{32}\left[2\left(2-\cos\theta_1-\cos\theta_2\right)^2 + 3\sin^2\theta_1 + 3\sin^2\theta_2\right] \tag{5.19}$$
$$(g_\chi)_{22} = \tfrac{1}{32}\left[2\left(1+\cos\theta_1\right)^2 + 3\sin^2\theta_1\right] \tag{5.20}$$
$$(g_\chi)_{33} = \tfrac{1}{32}\left[2\left(1+\cos\theta_2\right)^2 + 3\sin^2\theta_2\right] \tag{5.21}$$
$$(g_\chi)_{12} = \tfrac{1}{32}\left[2\left(2-\cos\theta_1-\cos\theta_2\right)\left(1+\cos\theta_1\right) - 3\sin^2\theta_1\right] \tag{5.22}$$
$$(g_\chi)_{13} = \tfrac{1}{32}\left[2\left(2-\cos\theta_1-\cos\theta_2\right)\left(1+\cos\theta_2\right) - 3\sin^2\theta_2\right] \tag{5.23}$$
$$(g_\chi)_{23} = \tfrac{1}{16}\left(1+\cos\theta_1\right)\left(1+\cos\theta_2\right) \tag{5.24}$$

in terms of the radial coordinates θ_i.

Orthogonal radial coordinates α and v

The coordinates α and u, which are defined by

$$\sqrt{1-\alpha^2} \equiv \sin\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2} \qquad \text{and} \qquad u \equiv \cos\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2}, \tag{5.25}$$

are well suited to describe the size and radial extension of the giant graviton, but turn out to be non-orthogonal. We shall hence rather choose to describe the radial space in terms of the orthogonal radial coordinates α and v, with the latter defined as follows:

$$v \equiv \frac{2u}{\alpha^2+u^2}. \tag{5.26}$$

The radial metric has components g_α and g_v in these coordinates, which are given in (5.32) - (5.33) below.

We can rewrite the determinant of the angular metric in terms of the orthogonal radial coordinates α and v as follows:

$$\det g_\chi = \frac{9}{64}\left(1-\alpha^2\right)\frac{1}{v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)^2, \tag{5.27}$$

and the cofactor associated with $(g_\chi)_{11}$ becomes

$$(C_\chi)_{11} = \frac{3}{64}\frac{1}{v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)^2\left\{\frac{4}{v^2}\sqrt{1-\alpha^2v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)+3\left(1-\alpha^2\right)\right\}. \tag{5.28}$$

The 5-form field strength is

$$F_5 = \mp \frac{R^4}{2} \frac{\alpha\left(1-\sqrt{1-\alpha^2 v^2}\right)}{v\sqrt{1-v^2}\sqrt{1-\alpha^2 v^2}}\ d\alpha \wedge dv \wedge d\chi_1 \wedge d\chi_2 \wedge d\chi_3 \tag{5.29}$$

and the associated 4-form potential is given by

$$C_4 = \mp \frac{R^4}{4} \frac{\left(1-\sqrt{1-\alpha^2 v^2}\right)^2}{v^3\sqrt{1-v^2}}\ dv \wedge d\chi_1 \wedge d\chi_2 \wedge d\chi_3, \tag{5.30}$$

where the $\mp$ distinguishes between the intervals $\theta_1 \geq \theta_2$ and $\theta_1 \leq \theta_2$. Notice that we have chosen a gauge in which C_4 is non-singular at $\alpha = 0$.

Radial metric in the coordinates (α, v)

The radial metric (5.13) can be rewritten in terms of the orthogonal radial coordinates α and v, which parameterize the size and radial direction on the worldvolume of the giant graviton, as follows:

$$ds^2_{\text{radial}} = g_\alpha d\alpha^2 + g_v dv^2, \tag{5.31}$$

with components

$$g_\alpha = \frac{\alpha^2 v^2}{3\left(1-\alpha^2\right)\sqrt{1-\alpha^2 v^2}\left(1-\sqrt{1-\alpha^2 v^2}\right)} \tag{5.32}$$

$$g_v = \frac{\left(1-\sqrt{1-\alpha^2 v^2}\right)}{3v^2\left(1-v^2\right)\sqrt{1-\alpha^2 v^2}}. \tag{5.33}$$

5.1.3 D3-brane action

As a dynamical object in type IIB string theory, this giant graviton on $\text{AdS}_5 \times \text{T}^{1,1}$ is described by the D3-brane action $S_{\text{D3}} = S_{\text{DBI}} + S_{\text{WZ}}$, with DBI and WZ terms

$$S_{\text{DBI}} = -T_3 \int_\Sigma d^4\sigma\ \sqrt{-\det \mathcal{P}\left[g\right]} \qquad \text{and} \qquad S_{\text{WZ}} = T_3 \int_\Sigma \mathcal{P}\left[C_4\right], \tag{5.34}$$

where $T_3 = \frac{1}{(2\pi)^3}$ is the tension. We shall choose $\sigma^a = (t, v, \chi_2, \chi_3)$ to be the worldvolume coordinates, with v double-covering the $[0,1]$ interval.

Let us consider the DBI action. The determinant of the pull-back of the metric satisfies

$$-R^{-8} \det \mathcal{P}\left[g\right] = g_v \left[\left(C_\chi\right)_{11} - \dot{\chi}_1^2 \left(\det g_\chi\right)\right], \tag{5.35}$$

and, using the expressions (5.28) - (5.27) and(5.33) for the relevant components and determinants associated with the angular and radial parts of the metric, we obtain

$$-\det \mathcal{P}\left[g\right] = \frac{R^8}{16} \frac{\left(1-\sqrt{1-\alpha^2 v^2}\right)^4}{v^6\left(1-v^2\right)} \left\{1 + \frac{3}{4} \frac{\left(1-\alpha^2\right) v^2 \left(1-\dot{\chi}_1^2\right)}{\sqrt{1-\alpha^2 v^2}\left(1-\sqrt{1-\alpha^2 v^2}\right)}\right\}. \tag{5.36}$$

To determine the WZ term, we require the pull-back of the 4-form potential (5.30), which is given by

$$\mathcal{P}\left[C_4\right] = \frac{R^4}{4}\frac{\left(1-\sqrt{1-\alpha^2 v^2}\right)^2}{v^3\sqrt{1-v^2}}\ \dot{\chi}_1\ dt\wedge dv\wedge d\chi_2\wedge d\chi_3. \tag{5.37}$$

Combining the above results, we obtain the D3-brane action, which describes a giant graviton extended and moving in $\mathbb{R}\times T^{1,1}$, as follows:

$$S = \frac{2R^4}{9\pi}\int dt\ L, \tag{5.38}$$

with the Lagrangian

$$L = \int_0^1 dv\ \frac{2\left(1-\sqrt{1-\alpha^2 v^2}\right)^2}{v^3\sqrt{1-v^2}}\ \left\{\dot{\chi}_1 - \sqrt{1+\frac{3}{4}\frac{\left(1-\alpha^2\right)v^2\left(1-\dot{\chi}_1^2\right)}{\sqrt{1-\alpha^2 v^2}\left(1-\sqrt{1-\alpha^2 v^2}\right)}}\right\}, \tag{5.39}$$

where we have integrated over χ_2 and χ_3.

The conserved momentum $P_{\chi_1}\equiv\frac{\partial L}{\partial\dot{\chi}_1}$ conjugate to the angular coordinate χ_1, which describes the direction of motion, is

$$P_{\chi_1} = \int_0^1 dv\ \frac{2\left(1-\sqrt{1-\alpha^2 v^2}\right)^2}{v^3\sqrt{1-v^2}}\left\{1+\dot{\chi}_1\frac{\frac{3}{4}\frac{\left(1-\alpha^2\right)v^2}{\sqrt{1-\alpha^2v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)}}{\sqrt{1+\frac{3}{4}\frac{\left(1-\alpha^2\right)v^2\left(1-\dot{\chi}_1^2\right)}{\sqrt{1-\alpha^2v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)}}}\right\}. \tag{5.40}$$

The Hamiltonian $H=\dot{\chi}_1 P_{\chi_1}-L$ can be explicitly written as

$$H = \int_0^1 dv\ \frac{2\left(1-\sqrt{1-\alpha^2 v^2}\right)^2}{v^3\sqrt{1-v^2}}\ \frac{\left[1+\frac{3}{4}\frac{\left(1-\alpha^2\right)v^2}{\sqrt{1-\alpha^2v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)}\right]}{\sqrt{1+\frac{3}{4}\frac{\left(1-\alpha^2\right)v^2\left(1-\dot{\chi}_1^2\right)}{\sqrt{1-\alpha^2v^2}\left(1-\sqrt{1-\alpha^2v^2}\right)}}}. \tag{5.41}$$

These expressions describe the momentum and energy in units of $\frac{2R^4}{9\pi}$. We would now like to minimize $H(\alpha, P_{\chi_1})$ with respect to α for fixed momentum P_{χ_1}. However, since it is not immediately obvious how to invert $P_{\chi_1}(\dot{\chi}_1)$ analytically, we shall first consider certain special cases.

Maximal giant graviton ($\alpha = 1$)

When $\alpha = 1$, it is possible to evaluate the integrals over v analytically. The Lagrangian becomes $L = \dot{\chi}_1 - 1$ and $H = P_{\chi_1} = 1$. All dependence on $\dot{\chi}_1$ disappears from the Hamiltonian H and the momentum P_{χ_1}, which is now due entirely to the extension of the D3-brane rather than to its motion along χ_1.

Small submaximal giant graviton ($\alpha \ll 1$)

Let us assume that $\alpha \ll 1$, so that we are considering a small submaximal giant graviton, and expand the Lagrangian in orders of α:

$$L \approx \frac{\alpha^3}{2}\left\{\alpha\dot{\chi}_1 - \sqrt{\alpha^2 + \tfrac{3}{2}\left(1-\dot{\chi}_1^2\right)}\right\}. \tag{5.42}$$

Here we must be careful to allow for the possibility that $1-\dot{\chi}_1^2$ might be small, which is why we cannot further simplify the square root. The momentum conjugate to χ_1 is thus

$$P_{\chi_1} \approx \frac{\alpha^3}{2}\left\{\alpha + \frac{\frac{3}{2}\dot{\chi}_1}{\sqrt{\alpha^2 + \frac{3}{2}\left(1-\dot{\chi}_1^2\right)}}\right\} \tag{5.43}$$

and the Hamiltonian is given by

$$H \approx \frac{\alpha^3}{2}\frac{\left(\alpha^2+\frac{3}{2}\right)}{\sqrt{\alpha^2 + \frac{3}{2}\left(1-\dot{\chi}_1^2\right)}}. \tag{5.44}$$

In this approximation of small α, it is possible to isolate all dependence on $\dot{\chi}_1$ and invert the momentum. We obtain

$$\dot{\chi}_1^2 = \frac{\left(\alpha^2+\frac{3}{2}\right)\left(P_{\chi_1}-\frac{1}{2}\alpha^4\right)^2}{\left[\frac{9}{16}\alpha^6 + \frac{3}{2}\left(P_{\chi_1}-\frac{1}{2}\alpha^4\right)^2\right]}. \tag{5.45}$$

We can write the Hamiltonian as a function of the size of the giant graviton α and its momentum P_{χ_1} as follows:

$$H \approx \sqrt{\tfrac{2}{3}\alpha^2+1}\sqrt{\tfrac{3}{8}\alpha^6 + \left(P_{\chi_1}-\tfrac{1}{2}\alpha^4\right)^2}. \tag{5.46}$$

Now, it is possible to solve for the maxima and minima. When the momentum P_{χ_1} is positive, this approximate energy is minimum at $\alpha = 0$ and $\alpha = \alpha_0$, and maximum at $\alpha = \alpha_{\text{max}}$ in between (see figure 5.3). Here we define

$$\alpha_0 \equiv \sqrt{\sqrt{\left(\tfrac{3}{4}\right)^2 + 2P_{\chi_1}} - \tfrac{3}{4}} \qquad \text{and} \qquad \alpha_{\text{max}} \equiv \sqrt{\sqrt{\left(\tfrac{9}{20}\right)^2 + \tfrac{2}{5}P_{\chi_1}} - \tfrac{9}{20}}. \tag{5.47}$$

These minima are energetically degenerate with $H(\alpha_0, P_{\chi_1}) \approx H(0, P_{\chi_1}) = P_{\chi_1}$. The non-trivial minimum at α_0 is associated with the submaximal giant graviton. Although the expression for the Hamiltonian is approximate, the energy of the point graviton solution at $\alpha = 0$ is exact. Furthermore, we shall later argue that the submaximal giant graviton remains degenerate with the point graviton even when its size α_0 is large. When the momentum P_{χ_1} is negative, only the trivial minimum at $\alpha = 0$, corresponding to a point graviton with energy $H(0, P_{\chi_1}) = -P_{\chi_1}$, exists.

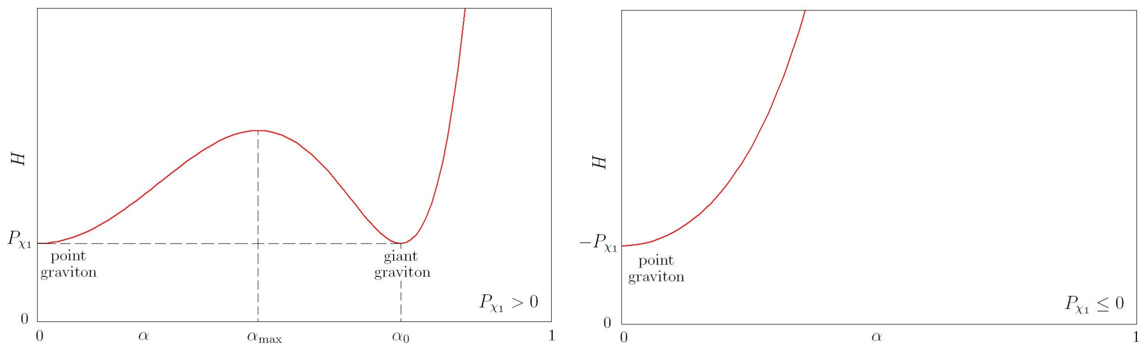

Figure 5.3: A generic sketch of the approximate energy $H(\alpha, P_{\chi_1})$, when $\alpha \ll 1$, as a function of α at fixed momentum P_{χ_1}. Note that the submaximal giant graviton only exists when P_{χ_1} is positive and is energetically degenerate with the point graviton.

Submaximal giant graviton

It turns out that, for all values of α, the Lagrangian (5.39) vanishes when $\dot{\chi}_1 = 1$. This implies that $H = P_{\chi_1}$, which remains a minimum[4] and corresponds to the submaximal giant graviton solution, described in the previous section when $\alpha \ll 1$. The size of the giant α_0 is then determined from the momentum P_{χ_1} via the following relation:

$$P_{\chi_1}(\alpha_0) = I_1\left(\alpha_0^2\right) + \frac{3}{4}\left(1-\alpha_0^2\right) I_2\left(\alpha_0^2\right), \tag{5.48}$$

where

$$I_1\left(\alpha_0^2\right) \equiv \int_0^1 dv\, \frac{2\left(1-\sqrt{1-\alpha_0^2 v^2}\right)^2}{v^3\sqrt{1-v^2}} = \left(1-\alpha_0^2\right)\ln\left(1-\alpha_0^2\right) + \alpha_0^2 \tag{5.49}$$

$$I_2\left(\alpha_0^2\right) \equiv \int_0^1 dv\, \frac{2\left(1-\sqrt{1-\alpha_0^2 v^2}\right)}{v\sqrt{1-v^2}\sqrt{1-\alpha_0^2 v^2}} = \frac{\partial I_1\left(\alpha_0^2\right)}{\partial\left(\alpha_0^2\right)} = -\ln\left(1-\alpha_0^2\right). \tag{5.50}$$

Simplifying, the exact expression for the energy and momentum of a submaximal giant graviton with size α_0 is

$$H(\alpha_0) = P_{\chi_1}(\alpha_0) = 1 - \left(1-\alpha_0^2\right)\left[1 - \frac{1}{4}\ln\left(1-\alpha_0^2\right)\right], \tag{5.51}$$

which is shown in figure 5.4.

This giant graviton is a BPS configuration, energetically degenerate with the point graviton, and exists by virtue of its motion along the χ_1 angular direction with conjugate momentum P_{χ_1}.

[4]This may be deduced by expanding $H - P_{\chi_1}$ in the vicinity of $\dot{\chi}_1 = 1$ and noticing that it is always non-negative.

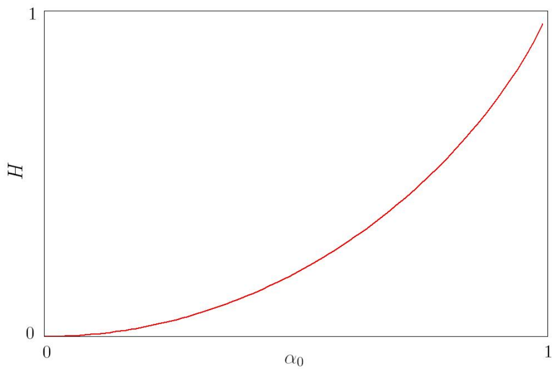

Figure 5.4: The energy $H(\alpha_0)$ of the submaximal giant graviton in units of $\frac{2R^4}{9\pi} = \frac{3}{2}N$ (which is twice the energy of a dibaryon [66]) as a function of its size α_0.

5.2 Fluctuation analysis

For the purposes of the fluctuation analysis, it is convenient to describe AdS_5 in terms of the coordinates (t, v_k), where the cartesian coordinates v_k are specified in (4.7). We shall also define $z_i \equiv \cos^2 \frac{\theta_i}{2}$ in terms of which the orthogonal radial coordinates can be written as

$$\alpha = \sqrt{z_1 + z_2 - z_1 z_2} \qquad \text{and} \qquad v = \frac{2\sqrt{z_1 z_2}}{z_1 + z_2} \equiv \sin\beta, \qquad \text{with} \quad \beta \epsilon \left[0, \tfrac{\pi}{2}\right]. \tag{5.52}$$

Radial metric in the coordinates z_i and (α, β)

The radial metric (5.13) in the alternative radial coordinates $z_i \equiv \cos^2 \frac{\theta_i}{2}$ is given by

$$ds^2_{\text{radial}} = \frac{1}{6}\left\{\frac{dz_1^2}{z_1(1-z_1)} + \frac{dz_2^2}{z_2(1-z_2)}\right\}. \tag{5.53}$$

We may also rewrite (5.31), using $v \equiv \sin\beta$, in terms of the alternative orthogonal radial coordinates α and β:

$$ds^2_{\text{radial}} = g_\alpha d\alpha^2 + g_\beta d\beta^2, \tag{5.54}$$

with components

$$g_\alpha = \frac{\alpha^2 \sin^2\beta}{3\left(1-\alpha^2\right)\sqrt{1-\alpha^2\sin^2\beta}\left(1-\sqrt{1-\alpha^2\sin^2\beta}\right)} \tag{5.55}$$

$$g_\beta = \frac{\left(1-\sqrt{1-\alpha^2\sin^2\beta}\right)}{3\sin^2\beta\sqrt{1-\alpha^2\sin^2\beta}}. \tag{5.56}$$

The surface of the giant graviton is described by $\alpha = \alpha_0$ constant, which is a shifted hyperbola in the (z_1, z_2)-plane:

$$(1 - z_1)(1 - z_2) = 1 - \alpha_0^2. \tag{5.57}$$

The giant graviton solution can therefore be represented more simply in the coordinates z_i (see figure 5.5 below), which shall also prove useful in our fluctuation analysis.

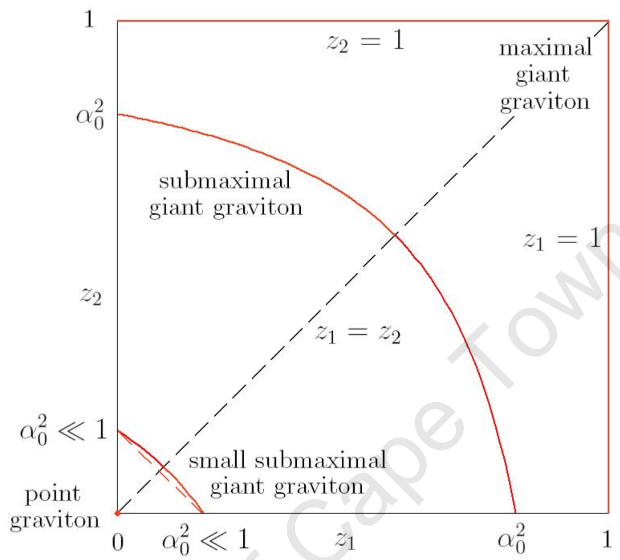

Figure 5.5: The giant graviton on the (z_1, z_2)-plane: the point graviton, small submaximal giant graviton (with approximate solution), submaximal giant graviton and maximal giant graviton (two dibaryons) are indicated in the sketch. The line $z_1 = z_2$ (so $v = 1$) separates the two regions which double-cover the $v \,\epsilon\, [0, 1]$ interval.

We can invert these relations for z_1 and z_2 as follows:

$$z_1 \equiv \cos^2 \tfrac{\theta_1}{2} = \frac{1}{v^2}\left(1 - \sqrt{1 - \alpha^2 v^2}\right)\left(1 \mp \sqrt{1 - v^2}\right) \tag{5.58}$$

$$z_2 \equiv \cos^2 \tfrac{\theta_2}{2} = \frac{1}{v^2}\left(1 - \sqrt{1 - \alpha^2 v^2}\right)\left(1 \pm \sqrt{1 - v^2}\right), \tag{5.59}$$

where the $\mp$ and $\pm$ distinguish between the regions $z_1 \leq z_2$ and $z_1 \geq z_2$. Note that, on the first covering the z_i may also be expressed in terms of the alternative radial worldvolume coordinate β:

$$z_1 = \frac{1}{\sin^2 \beta}\left(1 - \sqrt{1 - \alpha^2 \sin^2 \beta}\right) \sin^2 \tfrac{\beta}{2} \qquad z_2 = \frac{1}{\sin^2 \beta}\left(1 - \sqrt{1 - \alpha^2 \sin^2 \beta}\right) \cos^2 \tfrac{\beta}{2}, \tag{5.60}$$

whereas we must interchange $\sin^2 \frac{\beta}{2}$ and $\cos^2 \frac{\beta}{2}$ on the second covering. However, this is equivalent to taking $\beta \to \pi - \beta$, so both coverings may be parameterized by simply extending the range of β to $[0, \pi]$.

5.2.1 General fluctuation analysis

Let us consider the general fluctuation ansatz

$$v_k = \varepsilon\, \delta v_k(\sigma^a), \qquad \alpha = \alpha_0 + \varepsilon\, \delta\alpha(\sigma^a) \qquad \text{and} \qquad \chi_1 = t + \varepsilon\, \delta\chi_1(\sigma^a), \tag{5.61}$$

with worldvolume coordinates $\sigma^a = (t, \beta, \chi_2, \chi_3)$. Here α_0 is the size of the giant graviton, about which we are perturbing, and ε is a small parameter.

We shall now approximate the D3-brane action to second order in ε. The components of the metric are left in terms of $\alpha = \alpha_0 + \varepsilon\, \delta\alpha$ (we only need to expand certain combinations at the end of the calculation). Hence

$$\begin{aligned} S_{\mathrm{D3}} \approx \frac{R^4}{8\pi^3} \int dt\, d\beta\, d\chi_2\, d\chi_3\, \sqrt{g_\beta\left[(C_\chi)_{11} - \det g_\chi\right]} \\ \times \Bigg\{ \frac{\varepsilon}{\left[(C_\chi)_{11} - \det g_\chi\right]} \left[(C_\chi)_{11}\, \delta\dot{\chi}_1 + (C_\chi)_{12}\, (\partial_{\chi_2}\delta\chi_1) + (C_\chi)_{13}\, (\partial_{\chi_3}\delta\chi_1)\right] \\ + \frac{1}{2}\varepsilon^2 \left[\sum_k \left\{ - \frac{(C_\chi)_{11}\, \delta v_k^2}{\left[(C_\chi)_{11} - \det g_\chi\right]} - (\partial\, \delta v_k)^2 \right\} - g_\alpha\, (\partial\, \delta\alpha)^2 - \frac{(\det g_\chi)\, (\partial\, \delta\chi_1)^2}{\left[(C_\chi)_{11} - (\det g_\chi)\right]} \right] \Bigg\}, \end{aligned} \tag{5.62}$$

where the components g_α and g_β of the radial metric in the alternative orthogonal radial coordinates α and β are given in (5.55) and (5.56). The expressions involving the cofactors $(C_\chi)_{ij}$ of the angular metric components $(g_\chi)_{ij}$ are both stated explicitly in (5.64) - (5.69) and (5.71) - (5.76) in terms of the z_i, which are the functions (5.60) of α and β.

The terms involving a single ε coefficient should be expanded to first order in $\varepsilon\, \delta\alpha$. The zeroth order terms in these expansions - corresponding to $O(\varepsilon)$ terms in the D3-brane action - yield total derivatives, which is to be expected, since the giant graviton is a solution to the equations of motion. The first order terms contribute addition $O(\varepsilon^2)$ terms to the D3-brane action.

The (spacetime) gradient squared on the worldvolume of the giant graviton is defined as follows:

$$\begin{aligned} (\partial f)^2 \equiv \frac{(\partial_\beta f)^2}{g_\beta} - \frac{1}{\left[(C_\chi)_{11} - \det g_\chi\right]} \Big\{ (C_\chi)_{11}\, \dot{f}^2 + \left[(C_\chi)_{22} - (g_\chi)_{33}\right] (\partial_{\chi_2} f)^2 \\ + \left[(C_\chi)_{33} - (g_\chi)_{22}\right] (\partial_{\chi_3} f)^2 + 2\,(C_\chi)_{12}\, \dot{f}\, (\partial_{\chi_2} f) \\ + 2\,(C_\chi)_{13}\, \dot{f}\, (\partial_{\chi_3} f) + 2\left[(C_\chi)_{23} + (g_\chi)_{23}\right] (\partial_{\chi_2} f)\, (\partial_{\chi_3} f) \Big\}, \end{aligned} \tag{5.63}$$

with $f(t, \beta, \chi_2, \chi_3)$ any function of the worldvolume coordinates. Here all the cofactors $(C_\chi)_{ij}$, and metric components $(g_\chi)_{ij}$ and g_β, as well as the determinant $\det g_\chi$ are now evaluated at $\alpha = \alpha_0$.

Angular metric in the coordinates (z_i, χ_i)

The components of the angular metric (5.18), which are (5.19) - (5.24) in terms of the radial coordinates θ_i, can be written as

$$(g_\chi)_{11} = \tfrac{1}{8}\left[2\left(2 - z_1 - z_2\right)^2 + 3z_1\left(1 - z_1\right) + 3z_2\left(1 - z_2\right)\right] \tag{5.64}$$
$$(g_\chi)_{22} = \tfrac{1}{8}z_1\left(3 - z_1\right) \tag{5.65}$$
$$(g_\chi)_{33} = \tfrac{1}{8}z_2\left(3 - z_2\right) \tag{5.66}$$
$$(g_\chi)_{12} = \tfrac{1}{8}z_1\left(1 + z_1 - 2z_2\right) \tag{5.67}$$
$$(g_\chi)_{13} = \tfrac{1}{8}z_2\left(1 - 2z_1 + z_2\right) \tag{5.68}$$
$$(g_\chi)_{23} = \tfrac{1}{4}z_1 z_2 \tag{5.69}$$

and the determinant is then given by

$$\det g_\chi = \tfrac{9}{64}\, z_1 z_2\left(1 - z_1\right)\left(1 - z_2\right), \tag{5.70}$$

in terms of the radial coordinates z_i.

We shall also require expressions for various combinations of the cofactors of this angular metric as follows:

$$(C_\chi)_{11} = \tfrac{3}{64}z_1 z_2\left(3 - z_1 - z_2 - z_1 z_2\right) \tag{5.71}$$
$$(C_\chi)_{12} = -\tfrac{3}{64}z_1 z_2\left(1 + z_1 - 3z_2 + z_1 z_2\right) \tag{5.72}$$
$$(C_\chi)_{13} = -\tfrac{3}{64}z_1 z_2\left(1 - 3z_1 + z_2 + z_1 z_2\right) \tag{5.73}$$
$$(C_\chi)_{22} - (g_\chi)_{33} = -\tfrac{3}{64}z_1 z_2\left(5 + z_1 - 7z_2 + z_1 z_2\right) - \tfrac{1}{4}z_2^2 \tag{5.74}$$
$$(C_\chi)_{33} - (g_\chi)_{22} = -\tfrac{3}{64}z_1 z_2\left(5 - 7z_2 + z_2 + z_1 z_2\right) - \tfrac{1}{4}z_1^2 \tag{5.75}$$
$$(C_\chi)_{23} + (g_\chi)_{23} = \tfrac{1}{64}z_1 z_2\left(1 + 9z_1 + 9z_2 - 11z_1 z_2\right) \tag{5.76}$$

and

$$(C_\chi)_{11} - \det g_\chi = \tfrac{3}{32}z_1 z_2\left(z_1 + z_2 - 2z_1 z_2\right). \tag{5.77}$$

5.2.2 Spectrum for the small submaximal giant graviton

In order to solve the equations of motion resulting from the second order D3-brane action and obtain the fluctuation spectrum, we shall consider a small submaximal giant graviton with $\alpha_0 \ll 1$.

It now becomes possible to approximate

$$z_1 \approx \alpha^2 \sin^2 \tfrac{\beta}{2} = \alpha^2(1-z) \qquad \text{and} \qquad z_2 \approx \alpha^2 \cos^2 \tfrac{\beta}{2} = \alpha^2 z, \tag{5.78}$$

where $z \equiv \cos^2 \frac{\beta}{2}$ runs over the unit interval. The D3-brane action (5.62) to second order in ε then simplifies as follows:

$$\begin{aligned} S_2 \approx \frac{R^4\alpha_0^2\varepsilon^2}{128\pi^3} \int dt\, dz\, d\chi_2\, d\chi_3 & \left\{ \sum_k \left[-\frac{3}{2}\delta v_k^2 - \left(\tilde{\partial}\,\delta v_k\right)^2 \right] \right. \\ & \left. -\frac{2}{3}\left(\tilde{\partial}\,\delta\alpha\right)^2 - \frac{2}{3}\left(\tilde{\partial}\,\tilde{\delta\chi_1}\right)^2 + \frac{4}{3}\left[3\dot{\tilde{\delta\chi_1}} - \left(\partial_{\chi_2}\tilde{\delta\chi_1}\right) - \left(\partial_{\chi_3}\tilde{\delta\chi_1}\right)\right]\delta\alpha \right\}, \end{aligned} \tag{5.79}$$

with $\tilde{\delta\chi_1} \equiv \frac{1}{\alpha_0}\delta\chi_1$ and $\tilde{\partial} \equiv \alpha_0\,\partial$ rescaled for convenience. The (spacetime) gradient squared is given by

$$\begin{aligned} \left(\tilde{\partial} f\right)^2 \equiv\ & 6\,z(1-z)\,(\partial_z f)^2 - \frac{3}{2}\dot{f}^2 + \frac{5}{2}\,(\partial_{\chi_2} f)^2 + \frac{5}{2}\,(\partial_{\chi_3} f)^2 + \dot{f}\,(\partial_{\chi_2} f) + \dot{f}\,(\partial_{\chi_3} f) \\ & -\frac{1}{3}\,(\partial_{\chi_2} f)\,(\partial_{\chi_3} f) + \frac{8z}{3\,(1-z)}\,(\partial_{\chi_2} f)^2 + \frac{8\,(1-z)}{3z}\,(\partial_{\chi_3} f)^2 . \end{aligned} \tag{5.80}$$

The equations of motion are

$$\tilde{\Box}\,\delta v_k - \tfrac{3}{2}\,\delta v_k = 0 \tag{5.81}$$

$$\tilde{\Box}\,\delta y_\pm \mp 3i\,\delta\dot{y}_\pm \pm i\,(\partial_{\chi_2}\delta y_\pm) \pm i\,(\partial_{\chi_3}\delta y_\pm) = 0, \tag{5.82}$$

where we define $\delta y_\pm \equiv \delta\alpha \pm i\tilde{\delta\chi_1}$. The (rescaled) d'Alembertian on the worldvolume of the giant graviton is

$$\begin{aligned} \tilde{\Box} \equiv\ & 6\partial_z \left\{ z\,(1-z)\,\partial_z \right\} \\ & -\frac{3}{2}\,\partial_t^2 + \frac{5}{2}\,\partial_{\chi_2}^2 + \frac{5}{2}\,\partial_{\chi_3}^2 + \partial_t\partial_{\chi_2} + \partial_t\partial_{\chi_3} - \frac{1}{3}\,\partial_{\chi_2}\partial_{\chi_3} + \frac{8z}{3\,(1-z)}\,\partial_{\chi_2}^2 + \frac{8\,(1-z)}{3z}\,\partial_{\chi_3}^2 . \end{aligned} \tag{5.83}$$

We shall now expand these fluctuations as

$$\delta v_k(t, z, \chi_2, \chi_3) = \sum_{m,n,s} C_{smn}\Psi_{smn}(t, z, \chi_2, \chi_3) \tag{5.84}$$

$$\delta y_\pm(t, z, \chi_2, \chi_3) = \sum_{m,n,s} \tilde{C}_{smn}\Psi_{smn}(t, z, \chi_2, \chi_3), \tag{5.85}$$

in terms of the eigenfunctions $\Psi_{smn}(t, z, \chi_2, \chi_3)$ described in appendix A.3. Insisting that the equations of motion must be satisfied places the following constraints on the frequencies ω_{smn} (already contained in the definition (A.38) of these eigenfunctions):

$$\left[\omega_{smn}^k + \tfrac{1}{4}\,(m+n)\right]^2 = 4l\,(l+1) + 1 \tag{5.86}$$

$$\left[\omega_{smn}^\pm + \tfrac{1}{4}\,(m+n) \mp 1\right]^2 = 4l\,(l+1) + 1, \tag{5.87}$$

with $l \equiv s + \max\left\{\frac{1}{2}|m+n|, \frac{1}{2}|m-n|\right\}$. Here $s \leq 0$, m and n are integers.

Notice that the above expressions are always positive, indicating that the frequencies ω^k_{smn} and $\omega^{\pm}_{smn}$ are real. Hence the small submaximal giant graviton is a stable configuration. Furthermore, in this approximation $\alpha_0 \ll 1$, the fluctuation spectrum is independent of the size of the giant graviton α_0. This appears not to be true in general, however, as we shall now observe by comparing these results with the fluctuation spectrum of the maximal giant graviton.

5.2.3 Spectrum for the maximal giant graviton

Small fluctuations about the maximal giant graviton cannot be described by our general ansatz (5.61) with $\alpha_0 = 1$. We must now require

$$\sin^2 \tfrac{\theta_1}{2} \sin^2 \tfrac{\theta_2}{2} = (1 - z_1)(1 - z_2) = 1 - \alpha^2 = \rho^2 \tag{5.88}$$

to be of $O\left(\varepsilon^2\right)$. Hence we shall modify our ansatz as follows:

$$v_k = \varepsilon\, \delta v_k(\sigma^a) \qquad \text{and} \qquad \rho = \varepsilon\, \delta\rho(\sigma^a), \tag{5.89}$$

with worldvolume coordinates $\sigma = (t, z, \xi, \phi)$ covering both halves of the maximal giant graviton (dibaryons) on which $\phi_1(\sigma^a)$ and $\phi_2(\sigma^a)$ respectively remain unspecified.

The fluctuations of the AdS_5 coordinates are simply the sum of these fluctuation about each dibaryon:

$$\delta v_k = \sum_{s,m,n} e^{-i\omega^k_{smn} t} \left\{ C^{(1)}_{smn} \Phi_{smn}(z_1, \psi + \phi_2, \phi_1) + C^{(2)}_{smn} \Phi_{smn}(z_2, \psi + \phi_1, \phi_2) \right\}, \tag{5.90}$$

where Φ_{smn} are the eigenfunctions (A.5) of the Laplacian on a dibaryon and the frequencies satisfy (4.22). We must impose the condition $C^{(1)}_{smm} = C^{(2)}_{smm}$ for the fluctuations which do not vanish at $z_1 = z_2 = 1$ to match up.

Now, the fluctuations of the radial $T^{1,1}$ coordinate ρ can be written as

$$\delta\rho = \tfrac{1}{2} \left(1 - z_2\right)^{\frac{1}{2}} \delta\theta_1(t, z_2, \psi + \phi_1, \phi_2) + \tfrac{1}{2} \left(1 - z_1\right)^{\frac{1}{2}} \delta\theta_2(t, z_1, \psi + \phi_2, \phi_1). \tag{5.91}$$

However, we must allow, not only for the usual $\delta\theta$ fluctuations of the dibaryon, but also for the possibility that $\delta\theta$ diverges like $(1 - z)^{-\frac{1}{2}}$ as z goes to 1 (see appendix A.1 for details). These yield non-vanishing, but finite, contributions to $\delta\rho$ at $z_1 = z_2 = 1$ (which must match). These additional fluctuations correspond to open strings stretched between the two halves of the maximal giant graviton.

The fluctuations $\delta y_\pm^{(1)} \equiv \delta\theta_2\ e^{\pm i\phi_2}$ and $\delta y_\pm^{(2)} \equiv \delta\theta_1\ e^{\pm i\phi_1}$ of the $\mathrm{T}^{1,1}$ coordinates transverse to the different dibaryons, which both contribute to $\delta\rho$, are then

$$\delta y_\pm^{(1)} = \sum_{s,m,n} \tilde{C}_{smn}^{(1)}\ e^{-i\omega_{smn}^\pm t}\ \Phi_{smn}(z_1, \psi+\phi_2, \phi_1) + \sum_{\substack{s,m,\\ n=m\pm1}} \tilde{C}_{smn}^{\mathrm{mod}(1)}\ e^{-i\omega_{smn}^{\mathrm{mod},\pm} t}\ \Phi_{smn}^{\mathrm{mod}}(z_1, \psi+\phi_2, \phi_1) \tag{5.92}$$

$$\delta y_\pm^{(2)} = \sum_{s,m,n} \tilde{C}_{smn}^{(2)}\ e^{-i\omega_{smn}^\pm t}\ \Phi_{smn}(z_2, \psi+\phi_1, \phi_2) + \sum_{\substack{s,m,\\ n=m\pm1}} \tilde{C}_{smn}^{\mathrm{mod}(2)}\ e^{-i\omega_{smn}^{\mathrm{mod},\pm} t}\ \Phi_{smn}^{\mathrm{mod}}(z_2, \psi+\phi_1, \phi_2), \tag{5.93}$$

where $\Phi_{smn}^{\mathrm{mod}}$ are the modified eigenfunctions (A.11) and we impose the matching condition $\tilde{C}_{smn}^{\mathrm{mod}(1)} = \tilde{C}_{smn}^{\mathrm{mod}(2)}$. The frequencies $\omega_{smn}^\pm$ of the original contributions still satisfy (4.23), while the modified frequencies $\omega_{smn}^{\mathrm{mod},\pm}$ satisfy an identical condition, but with $l^{\mathrm{mod}} = s + \frac{1}{2}(|m+n|-1)$. Since these are non-negative integers (the unmodified l values could be integer or half-integer), the spectrum of the maximal giant graviton is entirely contained within the original spectrum of the separate dibaryons. The frequencies are therefore still real, indicating stability.

5.3 Open strings excitations

We shall now turn our attention to open string excitations of the submaximal giant graviton. Since a full quantum description of strings in $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ remains unknown, we shall consider a simplifying limit [40]: short open strings moving on the pp-wave geometry associated with a null geodesic on the worldvolume of the giant graviton. (Note that different null geodesics produce distinct results, due to the non-spherical nature of the submaximal giant, and we shall discuss two possibilities.)

5.3.1 Short pp-wave strings: the null geodesic $t = \chi_1 = \chi_2 \equiv u$

Let us consider the null geodesic

$$t = \chi_1 = \chi_2 = u, \qquad v_k = 0, \qquad \alpha = \alpha_0 \qquad \text{and} \qquad v = 0 \tag{5.94}$$

on the worldvolume of the submaximal giant graviton with size α_0. We observe that $\theta_1 = 2\arccos\alpha_0$ and $\theta_2 = \pi$ then specifies the trajectory on the two 2-spheres.

To construct the pp-wave background, we choose the ansatz

$$
\begin{aligned}
t &= u + \frac{\xi}{R^2} & v_k &= \frac{x_k}{R} \\
\chi_1 &= u - \frac{\xi}{R^2} - \sqrt{\frac{2}{3}}\frac{\alpha_0}{\sqrt{1-\alpha_0^2}}\,\frac{y_1}{R} & \alpha &= \alpha_0 + \sqrt{\frac{3}{2}}\sqrt{1-\alpha_0^2}\,\frac{y_2}{R} \\
\chi_2 &= u - \frac{\xi}{R^2} + \sqrt{\frac{2}{3}}\frac{(2-\alpha_0^2)}{\alpha_0\sqrt{1-\alpha_0^2}}\,\frac{y_1}{R} & v &= \frac{\sqrt{6}}{\alpha_0}\frac{\tilde{y}}{R} \\
\chi_3 &= \frac{4}{3}\chi,
\end{aligned}
\tag{5.95}
$$

which corresponds to setting

$$
\theta_1 = 2\arccos\alpha_0 - \sqrt{6}\,\frac{y_2}{R} \qquad \text{and} \qquad \theta_2 = \pi - \sqrt{6}\,\frac{\tilde{y}}{R}, \tag{5.96}
$$

with $\tilde{y}_1 = \tilde{y}\cos\chi$ and $\tilde{y}_2 = \tilde{y}\sin\chi$. Now, taking the Penrose limit, in which R becomes large and we zoom in on this null geodesic, we obtain the pp-wave metric

$$
\begin{aligned}
ds^2 = &-4\,du d\xi - \left\{\sum_{k=1}^{4} x_k^2 + \frac{15}{16}\sum_{i=1}^{2}\tilde{y}_i^2\right\} du^2 + \sum_{k=1}^{4} dx_k^2 + \sum_{i=1}^{2} dy_i^2 + \sum_{i=1}^{2} d\tilde{y}_i^2 \\
&+ 4\,y_2 dy_1 du + \tfrac{1}{2}\left(\tilde{y}_1 d\tilde{y}_2 - \tilde{y}_2 d\tilde{y}_1\right) du,
\end{aligned}
\tag{5.97}
$$

which has a flat direction y_1. In this limit, the 5-form field strength becomes constant:

$$
F_5 = 4\,du \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + 8\,du \wedge dy_1 \wedge dy_2 \wedge d\tilde{y}_1 \wedge d\tilde{y}_2. \tag{5.98}
$$

In the lightcone gauge $u = 2p^u\tau$, the bosonic part of the Polyakov action for open strings moving in this pp-wave geometry is

$$
\begin{aligned}
S = \int d\tau \int \frac{d\sigma}{2\pi}\Bigg\{&\sum_{K=1}^{4}\left(\tfrac{1}{2}\dot{X}_K^2 - \tfrac{1}{2}\left(X_K'\right)^2 - \tfrac{1}{2}m^2 X_K^2\right) + \sum_{I=1}^{2}\left(\tfrac{1}{2}\dot{Y}_I^2 - \tfrac{1}{2}\left(Y_I'\right)^2\right) + 2mY_2\dot{Y}_1 \\
&+ \sum_{I=1}^{2}\left(\tfrac{1}{2}\dot{\tilde{Y}}_I^2 - \tfrac{1}{2}\left(\tilde{Y}_I'\right)^2 - \tfrac{15}{32}m^2\tilde{Y}_I^2\right) + \tfrac{1}{4}m\left(\tilde{Y}_1\dot{\tilde{Y}}_2 - \tilde{Y}_2\dot{\tilde{Y}}_1\right)\Bigg\},
\end{aligned}
\tag{5.99}
$$

with $m \equiv 2p^u$. Notice that the equations of motion for each pair of embedding coordinates Y_I and $\tilde{Y}_I$ decouple, if we define

$$
Y_\pm \equiv \frac{1}{\sqrt{2}}\left(Y_1 \pm iY_2\right) \qquad \text{and} \qquad \tilde{Y}_\pm \equiv \frac{1}{\sqrt{2}}\left(\tilde{Y}_1 \pm i\tilde{Y}_2\right). \tag{5.100}
$$

The assumption that the open pp-wave string ends on the submaximal giant graviton (which becomes a flat D3-brane in the large R limit) implies that the X_K and Y_I satisfy Dirichlet boundary conditions, whereas the $\tilde{Y}_I$ must obey Neumann boundary conditions. Quantizing the open string embedding coordinates, we then obtain

$$
X_K(\tau,\sigma) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{\omega_n}}\left\{\alpha_n^K e^{-i\omega_n\tau} + \left(\alpha_n^K\right)^\dagger e^{i\omega_n\tau}\right\}\sin(n\sigma) \tag{5.101}
$$

$$Y_{\pm}(\tau,\sigma) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{\omega_n}}\left\{\beta_n^{\mp}e^{-i\omega_n^{\mp}\tau} + \left(\beta_n^{\pm}\right)^{\dagger}e^{i\omega_n^{\pm}\tau}\right\}\sin(n\sigma) \tag{5.102}$$

$$\tilde{Y}_{\pm}(\tau,\sigma) = \frac{1}{\sqrt{m}}\left\{\tilde{\beta}_n^{\pm}e^{-i\tilde{\omega}_0^{\pm}\tau} + \left(\tilde{\beta}_n^{\mp}\right)^{\dagger}e^{i\tilde{\omega}_0^{\mp}\tau}\right\} + \sum_{n=1}^{\infty}\sqrt{\frac{2}{\omega_n}}\left\{\tilde{\beta}_n^{\pm}e^{-i\tilde{\omega}_n^{\pm}\tau} + \left(\tilde{\beta}_n^{\mp}\right)^{\dagger}e^{i\tilde{\omega}_n^{\mp}\tau}\right\}\cos(n\sigma), \tag{5.103}$$

where we define $\omega_n^{\pm} \equiv \omega_n \pm m$ and $\tilde{\omega}_n^{\pm} \equiv \omega_n \pm \frac{1}{4}m$ in terms of $\omega_n \equiv \sqrt{m^2+n^2}$. The creation and annihilation operators satisfy the following commutation relations:

$$\left[\alpha_n^{K_1}, \left(\alpha_l^{K_2}\right)^{\dagger}\right] = \delta^{K_1K_2}\delta_{nl} \quad \text{and} \quad \left[\beta_n^{\pm}, \left(\beta_l^{\pm}\right)^{\dagger}\right] = \left[\tilde{\beta}_n^{\pm}, \left(\tilde{\beta}_l^{\pm}\right)^{\dagger}\right] = \delta_{nl}, \tag{5.104}$$

with all others zero. The lightcone Hamiltonian $H_{lc} = \frac{1}{m}H$ is quadratic in the embedding coordinates, and can be written in terms of these (normal ordered) harmonic oscillators:

$$H_{lc} = \sum_{n=1}^{\infty}\sum_{K=1}^{4}\frac{\omega_n}{m}\left(\alpha_n^K\right)^{\dagger}\alpha_n^K + \sum_{n=1}^{\infty}\left\{\frac{\omega_n^+}{m}\left(\beta_n^+\right)^{\dagger}\beta_n^+ + \frac{\omega_n^-}{m}\left(\beta_n^-\right)^{\dagger}\beta_n^-\right\}$$
$$+ \sum_{n=0}^{\infty}\left\{\frac{\tilde{\omega}_n^+}{m}\left(\tilde{\beta}_n^+\right)^{\dagger}\tilde{\beta}_n^+ + \frac{\tilde{\omega}_n^-}{m}\left(\tilde{\beta}_n^-\right)^{\dagger}\tilde{\beta}_n^-\right\}. \tag{5.105}$$

Let us now consider the interpretation of these results: The momentum associated with lightcone time

$$p_{\xi} = -\frac{1}{R^2}\left(E + J_{\chi_1} + J_{\chi_2}\right) = -m \tag{5.106}$$

should be non-vanishing in the large R limit. Also, we choose the lightcone Hamiltonian

$$H_{lc} = -p_u = E - J_{\chi_1} - J_{\chi_2}, \tag{5.107}$$

and the momenta

$$p_{y_1} = \sqrt{\frac{2}{3}}\frac{1}{R}\frac{\left[-\alpha_0^2 J_{\chi_1} + \left(2-\alpha_0^2\right)J_{\chi_2}\right]}{\alpha_0\sqrt{1-\alpha_0^2}} \quad \text{and} \quad p_{\chi} = \frac{4}{3}J_{\chi_3}, \tag{5.108}$$

to remain finite. Hence we observe that

$$E = \tfrac{3}{2}L + O(R), \qquad J_{\chi_1} = \tfrac{3}{4}\left(2-\alpha_0^2\right)L + O(R) \quad \text{and} \quad J_{\chi_3} = O(1), \tag{5.109}$$

where L is defined through

$$J_{\chi_2} \equiv \tfrac{3}{4}\alpha_0^2 L \tag{5.110}$$

and must be of $O(R^2)$. This leads to an inverse mass squared

$$\frac{1}{m^2} = \frac{R^4}{9L^2}\left(1 + O\left(\tfrac{R}{L}\right)\right). \tag{5.111}$$

The energy eigenvalues of the lightcone Hamiltonian, in the limit $\frac{1}{m^2} \ll 1$, become

$$\alpha_n^K: \quad \frac{\omega_n}{m} = \sqrt{1+\frac{n^2}{m^2}} \approx 1 + \frac{n^2}{2m^2}, \qquad \text{with } n \in \{1, 2, \ldots\} \tag{5.112}$$

$$\beta_n^\pm: \quad \frac{\omega_n^\pm}{m} = \sqrt{1+\frac{n^2}{m^2}} \pm 1 \approx 2 + \frac{n^2}{2m^2} \text{ or } \frac{n^2}{2m^2}, \qquad \text{with } n \in \{1, 2, \ldots\} \tag{5.113}$$

$$\tilde{\beta}_n^\pm: \quad \frac{\tilde{\omega}_n^\pm}{m} = \sqrt{1+\frac{n^2}{m^2}} \pm \frac{1}{4} \approx \frac{3}{4} + \frac{n^2}{2m^2} \text{ or } \frac{5}{4} + \frac{n^2}{2m^2}, \qquad \text{with } n \in \{0, 1, 2, \ldots\}. \tag{5.114}$$

In other examples of gauge theory/gravity dualities (with renormalizable CFTs), this would be related to the BMN scaling limit [15] of the corresponding operators.

In the Klebanov-Witten theory, these open string excitations correspond to words attached to the subdeterminant operator $\mathcal{O}_n(A_1B_1)$. The above results can be linked to the number of different types of composite fields in these words. Let us define

$$\tilde{\phi}_1 \equiv \tfrac{3}{2}\chi_1 \qquad \rightarrow \quad A_1B_1 \tag{5.115}$$

$$\tilde{\phi}_2 \equiv \tfrac{3}{4}(\chi_2+\chi_3) \qquad \rightarrow \quad A_2B_2 \tag{5.116}$$

$$\tilde{\phi}_3 \equiv \tfrac{3}{4}(\chi_1+\chi_2) \qquad \rightarrow \quad A_2B_1, \tag{5.117}$$

whereas $\tilde{\phi}_4 \equiv \frac{3}{4}(\chi_1+\chi_3) = \tilde{\phi}_1 + \tilde{\phi}_2 - \tilde{\phi}_3 \rightarrow A_1B_2$ is *not* independent. The associated angular momenta are then

$$J_{\tilde{\phi}_1} = \left(1-\alpha_0^2\right)L + O(R), \qquad J_{\tilde{\phi}_2} = O(R) \qquad \text{and} \qquad J_{\tilde{\phi}_3} = \alpha_0^2 L + O(R), \tag{5.118}$$

so our word is made up of a large number of A_1B_1 and A_2B_1 composite scalar fields[5] and comparatively few of the combinations A_1B_2 and A_2B_2. Note that L may be defined as the total number of composite scalar fields in the word.

5.3.2 Short pp-wave strings: the null geodesic $t = \chi_1 = \chi_+ \equiv u$

We shall now consider the null geodesic[6]

$$t = \chi_1 = \chi_+ = u, \qquad v_k = 0, \qquad \alpha = \alpha_0, \qquad v = 1 \qquad \text{and} \qquad \chi_- = 0, \tag{5.119}$$

with $\chi_\pm \equiv \frac{1}{2}(\chi_2+\chi_3)$. Here also $\theta_1 = \theta_2 = 2\arcsin\left(1-\alpha_0^2\right)^{1/4} \equiv \theta_0$ and we notice that this setup is symmetric under interchange of the 2-sphere coordinates.

[5] The open string boundary conditions require *some* A_2B_1 composite scalar fields, but, since the number of these only goes to zero when the giant graviton becomes point-like, there is no inconsistency.

[6] Note that the choice $v = 1$ is not necessary to obtain a null geodesic, but is required for the associated pp-wave background to be consistent.

The ansatz for the pp-wave background is then given by

$$
\begin{aligned}
t &= u + \frac{\xi}{R^2} & v_k &= \frac{x_k}{R} \\
\chi_1 &= u - \frac{\xi}{R^2} - \frac{2}{\sqrt{3}} \frac{\sqrt{1-\sqrt{1-\alpha_0^2}}}{\left(1-\alpha_0^2\right)^{1/4}} \frac{y_1}{R} & \alpha &= \alpha_0 + \frac{\sqrt{3}}{\alpha_0} \left(1-\alpha_0^2\right)^{3/4} \sqrt{1-\sqrt{1-\alpha_0^2}}\, \frac{y_2}{R} \\
\chi_+ &= u - \frac{\xi}{R^2} + \frac{2}{\sqrt{3}} \frac{\left(1-\alpha_0^2\right)^{1/4}}{\sqrt{1-\sqrt{1-\alpha_0^2}}} \frac{y_1}{R} & v &= 1 - \frac{3}{2} \frac{\sqrt{1-\alpha_0^2}}{\left(1-\sqrt{1-\alpha_0^2}\right)} \left(\frac{\tilde{y}_2}{R}\right)^2 \\
\chi_- &= \frac{2}{\sqrt{3}} \frac{1}{\left(1-\alpha_0^2\right)^{1/4} \sqrt{1-\sqrt{1-\alpha_0^2}}} \frac{\tilde{y}_1}{R}, & &
\end{aligned}
\tag{5.120}
$$

which corresponds to choosing

$$
\theta_1 = \theta_0 - \sqrt{3}\left(\frac{y_2}{R} + \frac{\tilde{y}_2}{R}\right) \qquad \text{and} \qquad \theta_2 = \theta_0 - \sqrt{3}\left(\frac{y_2}{R} - \frac{\tilde{y}_2}{R}\right). \tag{5.121}
$$

Again, we take the large R Penrose limit to obtain the pp-wave geometry

$$
ds^2 = -4du\,d\xi - \left(\sum_{k=1}^{4} x_k^2\right) du^2 + \sum_{k=1}^{4} dx_k^2 + \sum_{i=1}^{2} dy_i^2 + \sum_{i=1}^{2} d\tilde{y}_i^2 + 4y_2 dy_1 du + 4\tilde{y}_2 d\tilde{y}_1 du, \tag{5.122}
$$

with flat directions y_1 and $\tilde{y}_1$. The 5-form field strength becomes

$$
F_5 = 4du \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - 4du \wedge dy_1 \wedge dy_2 \wedge d\tilde{y}_1 \wedge d\tilde{y}_2. \tag{5.123}
$$

The bosonic part of the Polyakov action for open strings in the lightcone gauge is

$$
\begin{aligned}
S = \int d\tau \int \frac{d\sigma}{2\pi} \Bigg\{ &\sum_{K=1}^{4} \left(\tfrac{1}{2}\dot{X}_K^2 - \tfrac{1}{2}\left(X_K'\right)^2 - \tfrac{1}{2}m^2 X_K^2\right) + \sum_{I=1}^{2} \left(\tfrac{1}{2}\dot{Y}_I^2 - \tfrac{1}{2}\left(Y_I'\right)^2\right) + 2mY_2\dot{Y}_1 \\
&+ \sum_{I=1}^{2} \left(\tfrac{1}{2}\dot{\tilde{Y}}_I^2 - \tfrac{1}{2}\left(\tilde{Y}_I'\right)^2\right) + 2m\tilde{Y}_2\dot{\tilde{Y}}_1 \Bigg\},
\end{aligned}
\tag{5.124}
$$

for this pp-wave geometry. The X_K and Y_I are subject to Dirichlet boundary conditions, while the $\tilde{Y}_I$ satisfy Neumann boundary conditions. Quantizing the open string, we obtain the embedding coordinates

$$
X_K(\tau,\sigma) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\omega_n}} \left\{ \alpha_n^K e^{-i\omega_n \tau} + \left(\alpha_n^K\right)^\dagger e^{i\omega_n \tau} \right\} \sin(n\sigma) \tag{5.125}
$$

$$
Y_\pm(\tau,\sigma) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\omega_n}} \left\{ \beta_n^\mp e^{-i\omega_n^\mp \tau} + \left(\beta_n^\pm\right)^\dagger e^{i\omega_n^\pm \tau} \right\} \sin(n\sigma) \tag{5.126}
$$

$$
\tilde{Y}_\pm(\tau,\sigma) = \frac{1}{\sqrt{m}} \left\{ \tilde{\beta}_0^\mp e^{-i\omega_0^\mp \tau} + \left(\tilde{\beta}_0^\pm\right)^\dagger e^{i\omega_0^\pm \tau} \right\} + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\omega_n}} \left\{ \tilde{\beta}_n^\mp e^{-i\omega_n^\mp \tau} + \left(\tilde{\beta}_n^\pm\right)^\dagger e^{i\omega_n^\pm \tau} \right\} \cos(n\sigma), \tag{5.127}
$$

which, as before, satisfy the usual commutation relations (5.104). The lightcone Hamiltonian, in terms of the (normal ordered) creation and annihilation operators, is then

$$\begin{aligned} H_{lc} = &\sum_{n=1}^{\infty}\sum_{K=1}^{4}\frac{\omega_n}{m}\left(\alpha_n^K\right)^\dagger\alpha_n^K + \sum_{n=1}^{\infty}\left\{\frac{\omega_n^+}{m}\left(\beta_n^+\right)^\dagger\beta_n^+ + \frac{\omega_n^-}{m}\left(\beta_n^-\right)^\dagger\beta_n^-\right\} \\ &+ \sum_{n=0}^{\infty}\left\{\frac{\omega_n^+}{m}\left(\tilde{\beta}_n^+\right)^\dagger\tilde{\beta}_n^+ + \frac{\omega_n^-}{m}\left(\tilde{\beta}_n^-\right)^\dagger\tilde{\beta}_n^-\right\}. \end{aligned} \tag{5.128}$$

We can interpret the above results as follows:

$$p_\xi = -\frac{1}{R^2}\left(E + J_{\chi_1} + J_{\chi_+}\right) = -m \tag{5.129}$$

may not vanish, whereas

$$p_u = -E + J_{\chi_1} + J_{\chi_+} = -H_{lc}, \tag{5.130}$$

$$p_{y_1} = \frac{2}{\sqrt{3}}\frac{1}{R}\frac{\left[-\left(1-\sqrt{1-\alpha_0^2}\right)J_{\chi_1}+\sqrt{1-\alpha_0^2}J_{\chi_+}\right]}{(1-\alpha_0^2)^{1/4}\sqrt{1-\sqrt{1-\alpha_0^2}}} \quad \text{and} \quad p_{z_1} = \frac{2}{\sqrt{3}}\frac{1}{R}\frac{J_{\chi_-}}{(1-\alpha_0^2)^{1/4}\sqrt{1-\sqrt{1-\alpha_0^2}}}$$

must remain finite. Hence

$$E = \frac{3}{2}L + O(R), \qquad J_{\chi_1} = \frac{3}{2}\sqrt{1-\alpha_0^2}\,L + O(R) \qquad \text{and} \qquad J_{\chi_-} = O(R), \tag{5.131}$$

where we define

$$J_{\chi_+} \equiv \frac{3}{2}\left(1 - \sqrt{1-\alpha_0^2}\right)L, \tag{5.132}$$

which is of $O(R^2)$. As before, the inverse mass squared is given by (5.111). In the limit in which $\frac{1}{m^2} \ll 1$, the energy eigenvalues of the lightcone Hamiltonian become

$$\alpha_n^K: \quad \frac{\omega_n}{m} = \sqrt{1+\frac{n^2}{m^2}} \approx 1 + \frac{n^2}{2m^2}, \qquad \text{with } n \,\epsilon\, \{1, 2, \ldots\} \tag{5.133}$$

$$\beta_n^\pm: \quad \frac{\omega_n^\pm}{m} = \sqrt{1+\frac{n^2}{m^2}} \pm 1 \approx 2 + \frac{n^2}{2m^2} \text{ or } \frac{n^2}{2m^2}, \qquad \text{with } n \,\epsilon\, \{1, 2, \ldots\} \tag{5.134}$$

$$\tilde{\beta}_n^\pm: \quad \frac{\omega_n^\pm}{m} = \sqrt{1+\frac{n^2}{m^2}} \pm 1 \approx 2 + \frac{n^2}{2m^2} \text{ or } \frac{n^2}{2m^2}, \qquad \text{with } n \,\epsilon\, \{0, 1, 2, \ldots\}. \tag{5.135}$$

We can, again, link these results to the number of composite scalar fields in the word attached to the subdeterminant operator $\mathcal{O}_n(A_1B_1)$. In this case

$$J_{\tilde{\phi}_1} = \sqrt{1-\alpha_0^2}\,L + O(R), \qquad J_{\tilde{\phi}_2} = \left(1 - \sqrt{1-\alpha_0^2}\right)L + O(R) \qquad \text{and} \qquad J_{\tilde{\phi}_3} = O(R), \tag{5.136}$$

with L the total number of composite scalar fields. There are now a large number of the combinations A_1B_1 and A_2B_2 in this word.

5.4 Giant gravitons on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

We finally propose an ansatz for the D4-brane giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, which is dual to the subdeterminant operator $\mathcal{O}_n(A_1 B_1)$ in ABJM theory and factorizes into two $\mathbb{CP}^2$ dibaryons at maximal size.

We may associate the four homogeneous coordinates of the complex projective space $\mathbb{CP}^3$ with the ABJM scalar fields A_1, A_2, $\bar{B}_1$ and $\bar{B}_2$ in the multiplet Y^a. Hence

$$z_1\bar{z}_3 = \tfrac{1}{2}\sin(2\zeta)\sin\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2}\, e^{\frac{1}{2}i(\psi-\phi_1-\phi_2)} \longrightarrow A_1B_1 \tag{5.137}$$

$$z_2\bar{z}_4 = \tfrac{1}{2}\sin(2\zeta)\cos\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2}\, e^{\frac{1}{2}i(\psi+\phi_1+\phi_2)} \longrightarrow A_2B_2 \tag{5.138}$$

$$z_2\bar{z}_3 = \tfrac{1}{2}\sin(2\zeta)\cos\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2}\, e^{\frac{1}{2}i(\psi+\phi_1-\phi_2)} \longrightarrow A_2B_1 \tag{5.139}$$

$$z_1\bar{z}_4 = \tfrac{1}{2}\sin(2\zeta)\sin\tfrac{\theta_1}{2}\cos\tfrac{\theta_2}{2}\, e^{\frac{1}{2}i(\psi-\phi_1+\phi_2)} \longrightarrow A_1B_2. \tag{5.140}$$

Aside from the additional factor of $\frac{1}{2}\sin(2\zeta)$, these coordinates bear an obvious resemblance to the parametrization (1.15) of the base manifold $\mathrm{T}^{1,1}$.

Our ansatz for the giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ then takes the form

$$\sin(2\zeta)\sin\tfrac{\theta_1}{2}\sin\tfrac{\theta_2}{2} = \sqrt{1-\alpha^2}, \tag{5.141}$$

where the constant $\alpha \,\epsilon\, [0,1]$ describes the size of the giant. Motion is along the angular direction $\chi \equiv \frac{1}{2}(\psi - \phi_1 - \phi_2)$, as in the case of the D2-brane dual giant graviton of chapter 3. This is the same as the direction of motion of the giant graviton on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$, up to a constant multiple, which we have included to account for the difference in conformal dimensions between the scalar fields in the Klebanov-Witten and ABJM theories.

Conclusion:
D-branes and giant gravitons

In this part, we catalogued some features of the giant graviton phenomenon in type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ and type IIB string theory on $AdS_5 \times T^{1,1}$. We also studied D-brane configurations dual to dibaryons - these may be thought of as half of a maximal giant graviton.

In chapter 3, we focused on the D2-brane dual giant graviton on $AdS_4 \times \mathbb{CP}^3$ and its excitations. We found that, just as for the canonical case of the D3-brane dual giant on $AdS_5 \times S^5$ [71], all the fluctuation modes have real, positive frequencies squared that are manifestly independent of its size. The absence of any tachyonic modes in the spectrum again means that there are no perturbative instabilities. A particularly interesting aspect of the fluctuation spectrum is the existence of a coupling between the worldvolume gauge field and transverse fluctuations of the brane - this differs from the usual result. Similar couplings were found in [74], but here the coupling results in a massless goldstone mode, indicating the existence of a new type of D2-brane with both momentum and D0-brane charge.

Motivated by the remarkable insights yielded by similar studies [19, 40, 41] of giant gravitons on $AdS_5 \times S^5$, we presented an analysis of open strings attached to the D2-brane giant in the limit of short pp-wave and long semiclassical strings. In the first case, we confirmed that the families of null geodesics on the worldvolume of the giant, which were found in [40, 41] for canonical D3-brane giants, once more exist for the dual giant graviton on $AdS_4 \times \mathbb{CP}^3$. Consequently, we were able to take a Penrose limit about one such geodesic and quantize short open strings on the pp-wave background. Our findings are in agreement with the reported D3-brane results: the spectral structure of these open string excitations has a $\tilde{\lambda} = \frac{\lambda}{L^2}$ perturbative expansion for large angular momentum L. In the dual ABJM theory, this suggests a potential simplification in the BMN or thermodynamic limit of the open spin chain. However, evidence from the closed string sector implies a breaking of BMN scaling [31, 32, 34]. To study long open strings attached to the dual giant graviton, we took a semiclassical limit and computed the leading order Polyakov action. We found complete agreement with similar results [41] for the more familiar D3-brane giant.

Although these similarities between the dual D3-brane giant and its D2-brane counterpart might seem mundane, it is worth remembering two important points:

- The 3-dimensional $\mathcal{N} = 6$ SCS-matter theory conjectured to be dual to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is decidedly different from $\mathcal{N} = 4$ SYM theory in field content, number of supersymmetries and degrees of freedom. This precise agreement was therefore by no means clear a priori.

- The D2-brane studied in this note is a descendent of an M2-brane upon compactification to the type IIA string theory. It would seem that the dynamics of the dual giant graviton encodes, in a non-trivial way, membrane dynamics in M-theory. The similarity between the D2-brane and D3-brane dual giants would therefore appear even more remarkable.

These open string excitations of the dual giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ are dual to words attached to the Schur polynomial $\chi_R(A_1 B_1)$ in the totally symmetric representation. A map to an open spin chain with boundaries is possible. The most concrete extension of this work would be to match the spectrum of open string excitations with the anomalous dimensions of the dual operators in ABJM theory - interpreted as the energy eigenvalues of the spin chain Hamiltonian.

A study of dibaryons on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$ was presented in chapter 4. The comparison of [66] between the spectra of BPS excitations of determinant operators in Klebanov-Witten theory and open string excitations of the dual D3-brane configurations in type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ was extended to the ABJM/type IIA string theory duality. The $\mathbb{CP}^2$ dibaryons - D4-branes wrapped on $\mathbb{CP}^2$ subspaces of the compact $\mathbb{CP}^3$ space - were constructed, together with their spectrum of small fluctuations. The lowest frequency modes associated with the transverse $\mathbb{CP}^3$ directions were shown to match the conformal dimensions of the ABJM determinant operators $\det A_1$ and $\det B_1$. A slight complication, in this case, is that Wilson lines must be appended to the usual dibaryons to ensure that they are gauge invariant physically meaningful states. However, this alteration does not effect the conformal dimensions of BPS excitations. Although the existence of such BPS configurations is all that is necessary for the above comparison, it would be interesting to determine the complete set of quantum numbers which describe these states.

In chapter 5, we translated Mikhailov's elegant construction [63] of giant gravitons in terms of holomorphic curves into the more familiar DBI construction for giants on $AdS_5 \times T^{1,1}$. This solution is dual to the subdeterminant operator $\mathcal{O}_n(A_1B_1)$. Its factorization, at maximal size $\alpha_0 = 1$, into the two dibaryons of [66], described in chapter 4 and dual to the determinant operators $\det A_1$ and $\det B_1$, may be thought of as the disappearance of the map between the polar coordinates of the two 2-spheres embedded in $T^{1,1}$. This results from the constraint $\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} = \sqrt{1-\alpha_0^2}$ in the definition of the giant graviton and may be visualized by colouring its worldvolume. The D3-brane then independently wraps both 2-spheres.

We also presented an analysis of small fluctuations about this giant graviton. The spectrum was calculated in two special cases: the small submaximal giant graviton with $\alpha_0 \ll 1$ and the maximal giant graviton. The latter turns out to be the same as the spectrum [66] obtained for separate dibaryons, even taking into account excitations between the two halves of the maximal giant. A comparison between these results indicates that the frequencies are dependent on the size of the giant graviton[7]. This is a curious new phenomenon never before observed - probably because all the previous giant gravitons have been spherical or nearly spherical configurations. The reason that the fluctuation spectrum is independent of the size of the giant is quite clear in the $AdS_5 \times S^5$ case (and for our D2-brane dual giant of chapter 3) - when the brane's radius is increased, the blueshift of the geometry exactly cancels the increase in wavelength of the modes. We do not yet understand the physics behind the spectrum of the giant graviton on $AdS_5 \times T^{1,1}$.

We completed this study by attaching open strings to this giant graviton. We were able to quantize short open strings in pp-wave geometries associated with two distinct worldvolume null geodesics and obtain their energy spectra. These open strings ending on the giant graviton should correspond to certain words (composed of combinations A_iB_j of scalar fields in Klebanov-Witten theory) with $\mathcal{R}$-charge of $O(\sqrt{N})$ attached to the subdeterminant operator $\mathcal{O}_n(A_1B_1)$. However, the non-renormalizability of Klebanov-Witten theory makes a comparison of the anomalous dimensions of these near-BPS operators with the corresponding energies problematic.

We close the chapter with an ansatz for the D4-brane giant graviton on $AdS_4 \times \mathbb{CP}^3$, which is dual to the subdeterminant operator $\mathcal{O}_n(A_1B_1)$ in ABJM theory. We make use of the similarity between $T^{1,1}$ and $\mathbb{CP}^3$, but find that we must take into account the additional radial coordinate ζ in the complex projective space $\mathbb{CP}$, which yields the extra worldvolume degree of freedom required by the D4-brane. The identical structure of the operators in the Klebanov-Witten and ABJM theories leads us to believe that

[7] This is not true for the approximate spectrum of the small submaximal giant graviton, when taken on its own.

the factorization of the D4-brane maximal giant graviton on $AdS_4 \times \mathbb{CP}^3$ into two $\mathbb{CP}^2$ dibaryons should take place in a qualitatively similar way. ABJM theory has the same superpotential as Klebanov-Witten theory, but, since it is a 3-dimensional CFT, is renormalizable. Consequently, it should now be possible to match the energies of open string excitations to the anomalous dimensions of near-BPS ABJM operators. We leave this as a topic for future research.

Part III

Marginal Deformations

Introduction: marginal deformations

As a long-term goal, it is hoped that the AdS/CFT correspondence will eventually lead to a complete understanding of Quantum Chromodynamics (QCD) - a non-conformal, non-abelian Yang-Mills theory with finite N. It is therefore crucial to identify, among the plethora of remarkable results for $\mathcal{N} = 4$ SYM theory, which are a consequence of the large amounts of symmetry and which are truly universal. With this in mind, we investigate the possibility that the Hagedorn temperature in type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ - related to the temperature of the confinement/deconfinement transition in planar $\mathcal{N} = 4$ SYM theory [80] - is a universal property under $\mathcal{N} = 1$ marginal deformations of the $\mathcal{N} = 4$ SYM superpotential.

A direct computation of the Hagedorn temperature (as well as the behaviour of strings near the Hagedorn point) is hampered by the need for an explicit, quantized string spectrum - something lacking for the type IIB superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Most of the literature is therefore restricted to flat backgrounds or toroidal compactifications (see, for example, [81]). Fortunately, there is another entirely non-trivial arena in which both sides of the AdS/CFT correspondence are explicitly known and the string theory is exactly soluble: type IIB string theory on the maximally supersymmetric pp-wave background [75], obtained from $\mathrm{AdS}_5 \times \mathrm{S}^5$ by applying a suitable Penrose limit, and $\mathcal{N} = 4$ SYM theory in the BMN double scaling limit of [15]. The study of thermal strings on this background [82] proved extremely fruitful, demonstrating the existence of a Hagedorn temperature (and the accompanying exponential growth of states), which is an indication of a phase transition rather than a limiting temperature [82, 83]. This would seem to mesh neatly with the confinement/deconfinement transition observed in planar $\mathcal{N} = 4$ SYM theory, except for the fact that, in the BMN limit, only a subset of states survive - the near-BPS states, the anomalous dimensions of which are systematically close to those of the chiral primaries.

The problem is one of the compatibility of regimes: where we are finally able to compute the exact string spectrum and Hagedorn temperature on the gravity side of the duality, there are an insufficient number of states left in the gauge theory to account for the required exponential growth. Circumventing this difficulty is non-trivial and was only recently accomplished [84] by identifying a different decoupling limit of $\mathcal{N} = 4$ SYM theory, in which the physics is captured by a ferromagnetic $\mathrm{XXX}_{\frac{1}{2}}$ Heisenberg spin chain. The Hagedorn temperature was then computed from well-known thermodynamic properties of the spin chain and matches excellently with the string theory result.

We are interested in the Hagedorn behaviour of pp-wave strings on the Lunin-Maldacena background. The dual gauge theory is γ-deformed $\mathcal{N} = 1$ SYM theory - the supersymmetry has been partially broken by a marginal deformation as explained below:

A marginal deformation of a superconformal field theory is constructed by adding an exactly marginal operator to the superpotential to generate a continuous set of new fixed points. Leigh and Strassler [85] found an $\mathcal{N} = 1$ marginal deformation of $\mathcal{N} = 4$ SYM theory with the superpotential

$$\mathcal{W} = \tfrac{1}{2}\,\mathrm{tr}\left\{h_1\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\right) + h_2\left(\mathcal{X}\mathcal{Z}\mathcal{Y}\right) + h_3\left(\mathcal{X}^3 + \mathcal{Y}^3 + \mathcal{Z}^3\right)\right\},$$

depending on three parameters h_i and invariant under a $\mathbb{Z}_3$ discrete symmetry transformation $\mathcal{X} \to \mathcal{Y}$, $\mathcal{Y} \to \mathcal{Z}$ and $\mathcal{Z} \to \mathcal{X}$. The fixed line $h_1 = -h_2 = g_{\mathrm{YM}}$ and $h_3 = 0$ yields the original $\mathcal{N} = 4$ SYM theory.

The special class of deformations for which $h_1 = g_{\mathrm{YM}}\, e^{i\pi\beta}$, $h_2 = -g_{\mathrm{YM}}\, e^{-i\pi\beta}$ and $h_3 = 0$, with β some complex parameter, are known as β-deformations. The superpotential then takes the form

$$\mathcal{W} = \tfrac{1}{2}\, g_{\mathrm{YM}}\,\mathrm{tr}\left\{e^{i\pi\beta}\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\right) - e^{-i\pi\beta}\left(\mathcal{X}\mathcal{Z}\mathcal{Y}\right)\right\}$$

and is invariant under two global $U(1)$ symmetry transformations as follows:

$$\begin{array}{llll} U(1)_1: & \mathcal{X} \to e^{i\alpha_1}\,\mathcal{X} & \mathcal{Y} \to e^{i\alpha_1}\,\mathcal{Y} & \mathcal{Y} \to e^{-2i\alpha_1}\,\mathcal{Z} \\ U(1)_2: & \mathcal{X} \to e^{-2i\alpha_2}\,\mathcal{X} & \mathcal{Y} \to e^{i\alpha_2}\,\mathcal{Y} & \mathcal{Y} \to e^{i\alpha_2}\,\mathcal{Z}. \end{array}$$

Therefore β-deformed $\mathcal{N} = 1$ SYM theory has a global $U(1) \times U(1) \times U(1)_{\mathcal{R}}$ symmetry group. We shall focus on $\beta \equiv \gamma$ real, which are known as γ-deformations.

In 2005, Lunin and Maldacena [7] were able to construct a gravitation dual of $\mathcal{N} = 1$ β-deformed SYM theory - type IIB string theory on a Lunin-Maldacena background. Again, let us choose $\beta \equiv \gamma$ to be real. The background spacetime is then $\mathrm{AdS}_5 \times S^5_\gamma$, where the 5-sphere space has undergone a γ-deformation.

They reasoned as follows: The two remaining $U(1)$ symmetries correspond to isometries $\varphi_1 \to \varphi_1 + \alpha_1$ and $\varphi_2 \to \varphi_2 + \alpha_2$ on the special torus (φ_1, φ_2), where these angular coordinates are the combinations

$$\varphi_1 = \tfrac{1}{3}\left(\phi_1 + \phi_2 - 2\phi_3\right) \qquad \text{and} \qquad \varphi_2 = \tfrac{1}{3}\left(-2\phi_1 + \phi_2 + \phi_3\right),$$

of the original phases ϕ_i on the 5-sphere space (1.7). These symmetries should be preserved after the γ-deformation. Let us define the parameter $\tau = B_{12} + i\sqrt{g}$, which depends on the volume of the (φ_1, φ_2) torus, as well as the component on the NS B-field associated with these angular directions. Lunin and Maldacena found a suitable γ-deformation of the $AdS_5 \times S^5$ background by taking

$$\tau \to \frac{\tau}{1 + \hat{\gamma}\,\tau}, \qquad \text{where} \quad \hat{\gamma} = R^2\,\gamma$$

relates to the deformation parameter γ in the dual gauge theory. An alternative construction due to [8] involves a TsT transformation on the torus (φ_1, φ_2), with a shift dependent on $\hat{\gamma}$.

Type IIB string theory on a Lunin-Maldacena $AdS_5 \times S^5_\gamma$ background has the metric

$$\begin{aligned} ds^2 = R^2 \big\{ & -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho \left(d\alpha^2 + \cos^2\alpha\, d\beta_1^2 + \sin^2\alpha\, d\beta_2^2\right) \\ & + d\theta_1^2 + G\cos^2\theta_1\, d\phi_1^2 + \sin^2\theta_1 \left(d\theta^2 + G\cos^2\theta_1\, d\phi_2^2 + G\sin^2\theta_2\, d\phi_3^2\right) \\ & + \hat{\gamma}^2\, G\cos^2\alpha\,\sin^4\alpha\,\cos^2\theta\,\sin^2\theta\,(d\phi_1 + d\phi_2 + d\phi_3)^2 \big\}, \end{aligned}$$

with $r = \sinh\rho$ the AdS_5 radial coordinate. Here

$$G^{-1} \equiv 1 + \hat{\gamma}^2 \left(\cos^2\theta_1\sin^2\theta_1\cos^2\theta_2 + \cos^2\theta_1\sin^2\theta_1\sin^2\theta_2 + \sin^4\theta_1\cos^2\theta_2\sin^2\theta_2\right).$$

The γ-deformation turns on an NS B-field

$$\begin{aligned} B_2 = \hat{\gamma}\, R^2 G \big\{ & \cos^2\theta_1\sin^2\theta_1\cos^2\theta_2\, d\phi_1 \wedge d\phi_2 + \cos^2\theta_1\sin^2\theta_1\sin^2\theta_2\, d\phi_3 \wedge d\phi_1 \\ & + \sin^4\theta_1\cos^2\theta_2\sin^2\theta_2\, d\phi_2 \wedge d\phi_3 \big\}, \end{aligned}$$

while the 3-form and 5-form field strengths are given by

$$F_3 = -4\hat{\gamma} R^2 e^{-\Phi_0}\, \cos^2\theta_1\sin^3\theta_1\cos\theta_2\sin\theta_2\, d\theta_1 \wedge d\theta_2 \wedge (d\phi_1 + d\phi_2 + d\phi_3)$$

$$\begin{aligned} F_5 = 4R^4 e^{-\Phi_0}\, \big\{ & \cosh\rho\sinh^3\rho\cos\alpha_1\sin\alpha_1\, dt \wedge d\rho \wedge d\alpha_1 \wedge d\alpha_2 \wedge d\alpha_3 \\ & + G\cos\theta_1\sin^3\theta_1\cos\theta_2\sin\theta_2\, d\theta_1 \wedge d\theta_2 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 \big\}, \end{aligned}$$

with a dilaton Φ, which satisfies $e^{2\Phi} = G\, e^{2\Phi_0}$. Here Φ_0 is the constant dilaton in the original $AdS_5 \times S^5$ background - usually this is chosen to vanish. More generally, the AdS/CFT dictionary, discussed in section 1.1.3, can be modified so that $\lambda = R^4$ and

$R^4 = 4\pi g_s N$, with $g_s = e^{\Phi_0}$ the string coupling. Since strings - the objects of interest in this part - couple only to the background geometry and the B-field, we can, and will, ignore the RR-sector in what follows.

This $\mathrm{AdS}_5 \times \mathrm{S}^5_\gamma$ Lunin-Maldacena background supports two classes of BPS pp-wave geometries, distinguished by the choice of null geodesics about which the Penrose limit is taken. Type IIB string theory on both pp-wave backgrounds has been studied in some detail with somewhat remarkable results. Taking the Penrose limit about any one of the $(J, 0, 0)$, $(0, J, 0)$ and $(0, 0, J)$ single charge null geodesics yields a conventional pp-wave background on which the string spectrum exhibits a $\hat{\gamma}$-dependence [86]. The second class of BPS geometries, obtained by taking a Penrose limit about the null geodesic (J, J, J), is a set of homogenous plane waves whose metric lies in a different diffeomorphism class from that of the former [77]. Intriguingly, in this case, the string spectrum is independent of the deformation parameter - a result verified to one-loop from the spectrum of anomalous dimensions of near-BPS operators in the dual gauge theory [87, 88].

In this part, we investigate the Hagedorn behaviour of γ-deformed pp-wave strings and calculate the Hagedorn temperature, which is again associated with a phase transition. We discuss matching these results with the dual $\mathcal{N} = 1$ SYM theory.

Chapter 6

Hagedorn behaviour of γ-deformed pp-wave strings

We begin this chapter with a detailed review of the construction [82] of the single string and multi-string partition functions for a canonical ensemble of type IIB strings on the undeformed maximally supersymmetric pp-wave background. We discuss the associated Hagedorn behaviour. With this foundation in place, we describe the two classes of Lunin-Maldacena BPS pp-wave geometries. Restricting our attention to the $(J,0,0)$ case, the γ-deformed multi-string partition function and Hagedorn temperature are derived. We discuss, using the prescription of [84], matching the temperatures of the Hagedorn transition of Lunin-Maldacena pp-wave strings and the confinement/deconfinement transition in the γ-deformed $\mathcal{N}=1$ SYM theory.

6.1 Thermodynamics of pp-wave strings

In this section, we present a detailed review of the thermodynamics of strings moving on a maximally supersymmetric pp-wave background, closely following [82].

Let us consider the $(J,0,0)$ null geodesic on $\mathrm{AdS}_5\times \mathrm{S}^5$, which is described by

$$t=\phi_1=\mu x^+ \qquad \text{and} \qquad \rho=\theta_1=0, \tag{6.1}$$

where we have set $r=\sinh\rho$ for convenience. Substituting the ansatz

$$t=\mu x^+ + \frac{x^-}{2\mu R^2} \qquad \phi_1=\mu x^+ + \frac{x^-}{2\mu R^2} \qquad \theta_1=\frac{x}{R} \qquad \rho=\frac{y}{R} \tag{6.2}$$

into the metric (1.5), we take the large R Penrose limit in which we zoom in on this null geodesic. We make use of the cartesian coordinates

$$\begin{aligned} x_1 &\equiv x\cos\theta_2\cos\phi_2 \qquad & x_2 &\equiv x\cos\theta_2\sin\phi_2 \\ x_3 &\equiv x\sin\theta_2\cos\phi_3 \qquad & x_4 &\equiv x\sin\theta_2\sin\phi_3 \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} x_5 &\equiv y\cos\alpha_1\cos\beta_1 \qquad & x_6 &\equiv y\cos\alpha_1\sin\beta_1 \\ x_7 &\equiv y\sin\alpha_1\cos\beta_2 \qquad & x_8 &\equiv y\sin\alpha_1\sin\beta_2 \end{aligned} \tag{6.4}$$

associated with the two 3-spheres $(\theta_2, \phi_2, \phi_3)$ and $(\alpha_1, \beta_1, \beta_2)$, with radii $\sin\theta_1 \approx \frac{x}{R}$ and $\sinh\rho \approx \frac{y}{R}$, which are embedded in S^5 and AdS_5 respectively. The metric of the maximally supersymmetric pp-wave background is given by

$$ds^2 = -2dx^+dx^- - \mu^2 \sum_{i=1}^{8} \left(x^i\right)^2 \left(dx^+\right)^2 + \sum_{i=1}^{8} \left(dx^i\right)^2 . \tag{6.5}$$

Here $x^\pm$ are the lightcone directions and the x^i describe eight transverse directions. Note that pp-wave geometries associated with similar null geodesics in $AdS_5 \times S^5$, which are parameterized by different angular coordinates, are equivalent up to lightcone time-dependent coordinate transformations.

Among the numerous symmetries contained in this metric are an $SO(8)$ rotational symmetry in the transverse coordinates (broken to $SO(4) \times SU(4)$ by the 5-form field strength), 16 boost-like symmetries in (x^i, x^-)-planes and two translational isometries in the lightcone $x^\pm$ directions - these are essential for the construction of the single pp-wave string partition function.

6.1.1 Single string partition function

The partition function describing a single pp-wave string (in the canonical ensemble) moving in a heat bath at temperature T can be constructed using a combination of the two translational isometries:

$$Z_1(a,b) = \mathrm{tr}_{\mathcal{H}} \left(e^{ap_+ + bp_-}\right) . \tag{6.6}$$

Here $p_\pm \sim -i\partial_\pm$, while the two variables a and b determine the heat bath temperature T, which satisfies

$$T^{-2} = ab + a^2\mu^2 \sum_{i=1}^{8} \left(x^i\right)^2 . \tag{6.7}$$

We shall now consider a pp-wave string in the lightcone gauge $X^+(\tau,\sigma) = 2p^+\tau$. Setting $m \equiv 2\mu p^+$, the lightcone Hamiltonian $H_{lc} = -p_+ = p^-$ is given by

$$H_{\rm lc} = \frac{\mu}{m}\left[\omega_0\left(N_0^B + N_0^F\right) + \sum_{n=1}^{\infty}\omega_n\left(N_n^B + N_n^F + \tilde{N}_n^B + \tilde{N}_n^F\right)\right], \tag{6.8}$$

where $\omega_n \equiv \operatorname{sign}(n)\sqrt{n^2+m^2}$, and $N_n^{B,F}$ and $\tilde{N}_n^{B,F}$ are the right- and left-moving number operators describing the eight bosonic and eight fermionic modes. (The right- and left-moving zero modes have been identified.) The zero point energy cancels out due to supersymmetry. The level-matching constraint

$$\mathcal{P} = \sum_{n=1}^{\infty} n\left(N_n^B + N_n^F - \tilde{N}_n^B - \tilde{N}_n^F\right) = 0, \tag{6.9}$$

which arises as a result of worldsheet translation invariance, must also be satisfied [76].

The single string partition function may now be written in the form

$$Z_1(a,b,\mu) = \int_0^{\infty} dp^+ \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\tau_1\; e^{-bp^+}\; z_{lc}\left(\tau_1, \frac{a}{4\pi p^+}; m \equiv 2\mu p^+\right), \tag{6.10}$$

with

$$z_{lc}(\tau_1,\tau_2,m) \equiv \operatorname{tr}_{\rm states}\left(e^{-2\pi\tau_2 H + 2\pi i\tau_1\mathcal{P}}\right). \tag{6.11}$$

The trace runs over all the eigenstates of the worldsheet Hamiltonian $H = 2p^+H_{lc}$ and the level-matching constraint is imposed using the delta function, which arises from the integral over τ_1.

Finally, it is known that this single string partition function may be written in terms of building blocks $\Theta_{\alpha,\delta}$. More specifically, we find that[1]

$$z_{lc}(\tau_1,\tau_2,m) = \left[\frac{\Theta_{\frac{1}{2},0}(\tau_1,\tau_2,m)}{\Theta_{0,0}(\tau_1,\tau_2,m)}\right]^4, \tag{6.12}$$

with[2]

$$\Theta_{\alpha,\delta}(\tau_1,\tau_2,m) \equiv e^{4\pi\tau_2 E_\delta(m)} \prod_{n=-\infty}^{\infty}\left(1 - e^{-2\pi\tau_2|\omega_{n+\delta}| + 2\pi i\tau_1(n+\delta) + 2\pi i\alpha}\right) \times \left(1 - e^{-2\pi\tau_2|\omega_{n-\delta}| + 2\pi i\tau_1(n-\delta) - 2\pi i\alpha}\right). \tag{6.13}$$

Here $E_\delta(m)$ is the casimir energy of a complex boson of mass m with boundary conditions $\phi(\sigma+2\pi,\tau) = e^{2\pi i\delta}\phi(\sigma,\tau)$ [89]. This casimir energy cancels out of the relevant ratio due to supersymmetry.

[1] The numerator and denominator of this ratio of building blocks describe the contributions from the fermionic and bosonic modes respectively.

[2] The two terms in the product describe two fields, which are complex conjugates, while the left- and right-moving modes are captured by negative and positive values of n respectively.

6.1.2 Multi-string partition function

The multi-string partition function, which describes an ideal gas of pp-wave strings, can be written in terms of the single string partition function of the bosonic modes (Z_1^B) and fermionic modes (Z_1^F) as follows:

$$\ln Z(a,b,\mu) = \sum_{r=1}^{\infty} \frac{1}{r} \left\{ Z_1^B(ar,br,\mu) - (-1)^r\, Z_1^F(ar,br,\mu) \right\}. \tag{6.14}$$

For the superstring, these partition functions for the two modes differ only by a finite number - the number of bosonic minus fermionic zero modes. This gives a small, constant contribution to the free energy, which, at high temperatures, will be negligible [82, 90, 91]. Thus

$$\ln Z(a,b,\mu) = \sum_{\substack{r=1 \\ r\ \mathrm{odd}}}^{\infty} \frac{1}{r}\, Z_1(ar,br,\mu). \tag{6.15}$$

Substituting (6.10) and (6.12) into the above expression, and changing the variable of integration from p^+ to $\tau_2 = \frac{ar}{4\pi p^+}$ now yields

$$\ln Z(a,b,\mu) = \frac{a}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{(\tau_2)^2} \sum_{\substack{r=1 \\ r\ \mathrm{odd}}}^{\infty} \left[\frac{\Theta_{\frac{1}{2},0}(\tau_1,\tau_2,\frac{\mu a r}{2\pi\tau_2})}{\Theta_{0,0}(\tau_1,\tau_2,\frac{\mu a r}{2\pi\tau_2})} \right]^4 e^{-\frac{abr^2}{4\pi\tau_2}}, \tag{6.16}$$

which is proportional to the Helmholtz free energy.

6.1.3 Hagedorn behaviour

We intend to investigate the Hagedorn behaviour of this gas of pp-wave strings and so, following [82], we begin by searching for an exponential divergence of the density of states. Towards this end, let us consider the building blocks $\Theta_{\alpha,\delta}$ in the high energy limit $p^+ \to \infty$ (or $\tau_2 \to 0$), with $\tilde{\mu} = m\tau_2 = \frac{\mu a r}{2\pi}$ held fixed. The definition (6.13) gives

$$\ln \Theta_{\alpha,\delta}(\tau_1,\tau_2,\tfrac{\tilde{\mu}}{\tau_2}) = 4\pi\tau_2 E_\delta\left(\tfrac{\tilde{\mu}}{\tau_2}\right) + \sum_{n=-\infty}^{\infty} \ln\left(1 - e^{-2\pi\tau_2|\omega_{n+\delta}| + 2\pi i \tau_1(n+\delta) + 2\pi i\alpha}\right) + \mathrm{c.c.} \tag{6.17}$$

and, setting $x \equiv \frac{\tau_2}{\tilde{\mu}}(n+\delta)$ and $\theta \equiv \frac{\tau_1}{\tau_2}$, we see that $\Delta x = \frac{\tau_2}{\tilde{\mu}}\Delta n \to dx$ in the high energy limit and x becomes a continuous variable over which we can integrate. Hence, since $\tau_2|\omega_{n+\delta}| = \tilde{\mu}\sqrt{1+x^2}$, we obtain

$$\begin{aligned} \ln \Theta_{\alpha,\delta}(\tau_1,\tau_2,\tfrac{\tilde{\mu}}{\tau_2}) &\longrightarrow \frac{\tilde{\mu}}{\tau_2} \int_{-\infty}^{\infty} dx\ \ln\left(1 - e^{-2\pi\tilde{\mu}\sqrt{1+x^2} + 2\pi i\tilde{\mu}\theta x + 2\pi i\alpha}\right) + \mathrm{c.c.} \\ &\equiv -\frac{\tilde{\mu}}{\sqrt{1+\theta^2}\,\tau_2} \left[f(\tilde{\mu},\theta,\alpha) + \mathrm{c.c.} \right]. \end{aligned} \tag{6.18}$$

Expanding out the logarithm,

$$\begin{aligned} f(\tilde{\mu},\theta,\alpha) &= \sqrt{1+\theta^2}\int_{-\infty}^{\infty} dx \sum_{l=1}^{\infty}\frac{1}{l}\, e^{-2\pi\tilde{\mu}\sqrt{1+x^2}+2\pi i\tilde{\mu}\theta x+2\pi i l\alpha} + \text{c.c.} \\ &= 2\sqrt{1+\theta^2}\sum_{l=1}^{\infty}\frac{1}{l}\, e^{2\pi i l\alpha}\int_0^{\infty} dx\, e^{-2\pi l\tilde{\mu}\sqrt{1+x^2}}\cos\left(2\pi l\tilde{\mu}\theta x\right) + \text{c.c.} \\ &= 2\sum_{l=1}^{\infty}\frac{1}{l}\, e^{2\pi i l\alpha}\, K_1(2\pi\tilde{\mu}l\sqrt{1+\theta^2}) + \text{c.c.} \end{aligned} \tag{6.19}$$

where $K_1(x)$ is a modified Bessel function of the second kind [92], which is a real positive monotonically decreasing function tending to zero quickly as $x \to \infty$.

Now, noticing that $\bar{f}(\tilde{\mu},\theta,\alpha) = f(\tilde{\mu},\theta,\alpha)$ when $\alpha = 0$ or $\alpha = \frac{1}{2}$, we may simplify

$$\begin{aligned} \ln\Theta_{\frac{1}{2},0}(\tau_1,\tau_2,\tfrac{\tilde{\mu}}{\tau_2}) - \ln\Theta_{0,0}(\tau_1,\tau_2,\tfrac{\tilde{\mu}}{\tau_2}) &\longrightarrow -\frac{2\tilde{\mu}}{\sqrt{1+\theta^2}\tau_2}\left[f(\tilde{\mu},\theta,\tfrac{1}{2}) - f(\tilde{\mu},\theta,0)\right] \\ &= \frac{8\tilde{\mu}}{\sqrt{1+\theta^2}\tau_2}\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}\frac{1}{l}\,K_1(2\pi l\tilde{\mu}\sqrt{1+\theta^2}). \end{aligned} \tag{6.20}$$

Thus the high energy behaviour of the multi-string partition function, in the limit as $\tau_2 \to 0$, is given by

$$\begin{aligned} \ln Z(a,b,\mu) \longrightarrow \frac{a}{4\pi}\sum_{\substack{r=1\\ r\text{ odd}}}^{\infty}\int_0^{\infty}\frac{d\tau_2}{\tau_2}\int_{-\frac{1}{2\tau_2}}^{+\frac{1}{2\tau_2}} d\theta \\ \times\exp\left\{-\frac{abr^2}{4\pi\tau_2} + \frac{16\mu ar}{\pi\tau_2}\frac{1}{\sqrt{1+\theta^2}}\left[\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}\frac{1}{l}\,K_1\left(\mu alr\sqrt{1+\theta^2}\right)\right]\right\}, \end{aligned} \tag{6.21}$$

where we have changed the integral over τ_1 into an integral over $\theta = \frac{\tau_1}{\tau_2}$.

We now wish to determine for which temperatures (values of a and b) this partition function converges. Only the $r = 1$ term need be considered[3]. The convergent/divergent nature of the integral over τ_2 depends critically on the sign of the expression in the exponential. The integral converges if

$$ab < \frac{64\,a\mu}{\sqrt{1+\theta^2}}\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}\frac{1}{l}\,K_1\left(\mu al\sqrt{1+\theta^2}\right) \le 64\,a\mu\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}\frac{1}{l}\,K_1\left(\mu al\right) \equiv \beta_H, \tag{6.22}$$

for all θ. This critical point $ab = \beta_H$ corresponds to the Hagedorn temperature T_H, which is defined by

$$T_H^{-2} = \beta_H + a^2\mu^2\sum_{i=1}^{8}\left(x^i\right)^2 = 64\,\mu a\sum_{l=1}^{\infty}\frac{1}{l}\,K_1(\mu al) + a^2\mu^2\sum_{i=1}^{8}\left(x^i\right)^2. \tag{6.23}$$

[3]The modified Bessel function K_1 is monotonically decreasing, so that all the $r > 1$ terms are much smaller (exponentially so) than the $r = 1$ term. Therefore, if the $r = 1$ term converges, then all the other terms are also convergent.

The nature of the partition function at $ab = \beta_H$ was considered in [83] and is related to the behaviour of thermodynamic quantities as they approach the critical point. A phase transition requires a finite free energy, although derived quantities, such as internal energy ($E = -(\ln Z)'$) or specific heat ($C = \beta^2(\ln Z)''$) may diverge. The hallmark of a limiting temperature, on the other hand, is a free energy which blows up near β_H. However, even in this case, it has been argued [93] that string interactions can turn this into a first order transition, with a critical temperature below the Hagedorn temperature. In the high energy limit $\tau_2 \to 0$, the integral over θ is dominated by the saddle point at $\theta = 0$, with $e^{-c\,\theta^2/\tau_2}$ fluctuations (c being a relatively unimportant constant). Integrating over these fluctuations will produce a factor of $\sqrt{\tau_2}$. After performing the integral over τ_2, the free energy and multi-string partition function go like

$$\beta F = -\ln Z(a,b,\mu) \propto \sqrt{\beta^2 - \beta_H^2} \; + \text{ regular}, \qquad \text{with} \quad \beta^2 = ab, \tag{6.24}$$

which remains finite at $\beta = \beta_H$, signaling a phase transition.

6.2 The deformation

There are two classes of BPS γ-deformations of this maximally supersymmetric pp-wave background, both arrived at by taking a Penrose limit about the appropriate null geodesic in the $\mathrm{AdS}_5 \times \mathrm{S}^5_\gamma$ Lunin-Maldacena background. We shall extend the above analysis to strings moving on these γ-deformed pp-wave backgrounds.

6.2.1 The pp-wave limit about a single-charge null geodesic

The first γ-deformed pp-wave geometry arises from taking a Penrose limit about the $(J,0,0)$ null geodesic - we substitute the ansatz (6.2) into the $\mathrm{AdS}_5 \times \mathrm{S}^5_\gamma$ metric and scale $R \to \infty$. The resulting background fields in the NS sector are

$$ds^2_\gamma = -2\,dx^+dx^- - \mu^2\left[\left(1+\hat{\gamma}^2\right)\sum_{i=1}^{4}\left(x^i\right)^2 + \sum_{i=5}^{8}\left(x^i\right)^2\right]\left(dx^+\right)^2 + \sum_{i=1}^{8}\left(dx^i\right)^2 \tag{6.25}$$

$$B_2 = \mu\hat{\gamma}\left(x^1dx^+\wedge dx^2 - x^2dx^+\wedge dx^1 + x^3dx^+\wedge dx^4 - x^4dx^+\wedge dx^3\right), \tag{6.26}$$

with a constant dilaton $\Phi = \Phi_0$.

The transverse coordinates in the original pp-wave background are naturally split into two sets of four coordinates (x_1, x_2, x_3, x_4) and (x_5, x_6, x_7, x_8) by the self-dual 5-form field strength. The effect of the deformation is to alter the effective string mass for oscillations in the first set of transverse directions, consequently breaking the $SO(8)$ degeneracy of the oscillation spectrum. Quantization of the closed string sigma model on this background yields the following oscillation spectrum:

$$\omega_n^{\pm} \equiv \text{sign}(n)\sqrt{m^2 + (n \pm \hat{\gamma} m)^2} \qquad \text{and} \qquad \omega_n \equiv \text{sign}(n)\sqrt{m^2 + n^2}, \tag{6.27}$$

for the first and second sets of transverse coordinates respectively. Here $m \equiv 2\mu p^+$ and the $\pm$ indicates the spin in the (x^1, x^2) and (x^3, x^4)-planes.

6.2.2 A homogeneous plane wave limit

Let us now focus on an inequivalent γ-deformed pp-wave background associated with the (J, J, J) null geodesic, which is parameterized by

$$t = \mu x^+ \qquad \text{and} \qquad \psi \equiv \tfrac{1}{3}(\phi_1 + \phi_2 + \phi_3) = -\mu x^+, \tag{6.28}$$

with the further specification that

$$\theta_1 = \theta_{10} \equiv \cos^{-1}\left(\tfrac{1}{\sqrt{3}}\right), \qquad \theta_2 = \tfrac{\pi}{4} \qquad \text{and} \qquad \rho = \varphi_1 = \varphi_2 = 0. \tag{6.29}$$

Here φ_1 and φ_2 are the two angular directions used in the construction [7] of the Lunin-Maldacena background (associated with the two $U(1)$ symmetries) and ψ is proportional to the total phase. Close to this null geodesic

$$t = \mu x^+ + \frac{x^-}{2\mu R} \qquad \rho = \frac{y}{R} \qquad \psi = -\mu x^+ + \frac{x^-}{2\mu R} \qquad \varphi_1 = \frac{\tilde{x}^3}{R} \qquad \varphi_2 = \frac{\tilde{x}^4}{R}$$
$$\theta_1 = \theta_{10} - \frac{x^2}{R} \qquad \theta_2 = \frac{\pi}{4} + \sqrt{\frac{2}{3}}\,\frac{x^1}{R}. \tag{6.30}$$

We shall now redefine

$$x^3 = \sqrt{\frac{2}{(3+\hat{\gamma}^2)}}\left(\tilde{x}^3 + \frac{1}{2}\tilde{x}^4\right) \qquad \text{and} \qquad x^4 = \sqrt{\frac{3}{2\,(3+\hat{\gamma}^2)}}\;\tilde{x}^4, \tag{6.31}$$

and take the $R \to \infty$ scaling limit. The resulting pp-wave geometry is described by the metric

$$ds^2 = -2\,dx^+dx^- - \mu^2\left[\frac{4\hat{\gamma}^2}{(3+\hat{\gamma}^2)}\sum_{i=1}^{2}\left(x^i\right)^2 + \sum_{i=5}^{8}\left(x^i\right)^2\right]\left(dx^+\right)^2 + \sum_{i=1}^{8}\left(dx^i\right)^2$$
$$+\frac{4\sqrt{3}\,\mu}{\sqrt{3+\hat{\gamma}^2}}\left(x^1dx^3 + x^2dx^4\right)dx^+. \tag{6.32}$$

The remaining fields in the NS sector of this type IIB multiplet are

$$B_2 = \frac{\hat{\gamma}}{\sqrt{3}}\,dx^3 \wedge dx^4 + \frac{2\mu\hat{\gamma}}{\sqrt{3+\hat{\gamma}^2}}\,dx^+ \wedge \left(x^1dx^4 - x^2dx^3\right) \tag{6.33}$$

$$e^{2\Phi} = \frac{1}{(1+\hat{\gamma}^2)}\, e^{2\Phi_0}. \tag{6.34}$$

Additionally, this background supports non-vanishing RR 2-form and 4-form field strengths which, while they result in rather sophisticated D-brane dynamics [73], are irrelevant for our analysis.

Closed strings in this background, their supersymmetries and dual gauge theory operators, were first studied in [87, 88]. It was noticed that a change of coordinates

$$x^- \longrightarrow x^- + \sqrt{\frac{3}{(3+\hat{\gamma}^2)}}\,\left(x^1x^3 + x^2x^4\right), \tag{6.35}$$

brings the (J, J, J) pp-wave metric into the homogenous plane wave form [77]

$$ds^2 = -2\,dx^+dx^- + \sum_{i,j=1}^{8} k_{ij}\,x^i x^j \left(dx^+\right)^2 + 2\sum_{i,j=1}^{8} f_{ij}\,x^i\,dx^j dx^+ + \sum_{i=1}^{8}\left(dx^i\right)^2, \tag{6.36}$$

where the matrices k_{ij} and f_{ij} are given by

$$k_{ij} = \mu^2\,\mathrm{diag}\left[\begin{array}{cccccccc} \frac{4\hat{\gamma}^2}{(3+\hat{\gamma}^2)} & \frac{4\hat{\gamma}^2}{(3+\hat{\gamma}^2)} & 0 & 0 & 1 & 1 & 1 & 1\end{array}\right] \tag{6.37}$$

$$f_{ij} = \sqrt{\frac{3\mu^2}{(3+\hat{\gamma}^2)}}\left[\begin{array}{cccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{6.38}$$

Remarkably, even though the background and associated string equations of motion depend on $\hat{\gamma}$ in a fairly non-trivial way, the quantum closed string spectrum

$$\omega_n = 1 \pm \sqrt{1+4n^2}, \tag{6.39}$$

determined by the frequency base ansatz of [77] for strings on homogeneous plane waves, exhibits no dependence on the deformation parameter [87, 88]. Consequently, we expect that the high temperature behaviour of an ensemble of strings on this particular γ-deformation of the maximally symmetric pp-wave background should be identical to that of homogeneous plane wave strings (see for example [91]). While the Hagedorn behaviour of strings on this particular class of homogeneous plane waves (i.e. non-trivial k_{ij} and f_{ij}) has not yet, to the best of our knowledge, been studied, it is clear that it will be independent of the deformation.

6.2.3 γ-deformed $(J,0,0)$ Hagedorn temperature

Returning again to the $(J,0,0)$ Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5_\gamma$, we shall now construct the partition function for an ideal gas of strings on this γ-deformed pp-wave background. In computing the partition function, much of the analysis is identical to the undeformed case. The difference comes from the modified string spectrum - there are four real oscillators with $|\omega_n| = \sqrt{m^2+n^2}$, and two each with $|\omega_n^\pm| = \sqrt{m^2+(n\pm\hat{\gamma}m)^2}$. This does, in fact, lead to a partition function with non-trivial $\hat{\gamma}$-dependence. Remarkably though, we shall see that, in the high temperature limit ($\tau_2 \to 0$), this difference disappears and the actual Hagedorn temperature itself is undeformed.

At the level of the building blocks, the effect of the γ-deformation is to change two of the $\Theta_{\alpha,\delta}$ into

$$\Theta^\pm_{\alpha,\delta}(\tau_1,\tau_2,m) \equiv e^{4\pi\tau_2 E^\pm_\delta(m)} \prod_{n=-\infty}^{\infty} \left(1 - e^{-2\pi\tau_2|\omega^\pm_{n+\delta}|+2\pi i\tau_1(n+\delta)+2\pi i\alpha}\right) \times \left(1 - e^{-2\pi\tau_2|\omega^\pm_{n-\delta}|+2\pi i\tau_1(n-\delta)-2\pi i\alpha}\right). \tag{6.40}$$

The exact form of the energy $E^\pm_\delta(m)$ is unimportant in our present discussion (as long as it is still independent of α), since it cancels out of the relevant ratio of building blocks due to the residual supersymmetry.

The γ-deformed $(J,0,0)$ multi-string partition function can now be written as follows:

$$\ln Z^\gamma(a,b,\mu) = \frac{a}{4\pi\alpha'}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^\infty \frac{d\tau_2}{(\tau_2)^2} \sum_{\substack{r=1\\ r\text{ odd}}}^{\infty} e^{-\frac{abr^2}{4\pi\alpha'\tau_2}} \left(\frac{\Theta_{\frac{1}{2},0}}{\Theta_{0,0}}\right)^2 \left(\frac{\Theta^+_{\frac{1}{2},0}}{\Theta^+_{0,0}}\right)\left(\frac{\Theta^-_{\frac{1}{2},0}}{\Theta^-_{0,0}}\right), \tag{6.41}$$

where each Θ is an implicit function $\Theta_{\alpha,\delta}\left(\tau_1,\tau_2,\frac{\mu a r}{2\pi\tau_2}\right)$.

Following the steps in section 2, it is not hard to show that, in the high-energy limit, the deformed oscillators lead to the replacement of the function f in (6.19) with

$$f^\pm_\gamma(\tilde{\mu},\theta,\alpha) = 2\sum_{l=1}^{\infty} \frac{e^{2\pi i l(\alpha\mp\hat{\gamma}\tilde{\mu}\theta)}}{l} K_1\left(2\pi\tilde{\mu}l\sqrt{1+\theta^2}\right) \equiv f\left(2\pi\tilde{\mu}l\sqrt{1+\theta^2}, \alpha\mp\hat{\gamma}\tilde{\mu}\theta\right). \tag{6.42}$$

In contrast to the undeformed case, we must now set $\bar{f}\,(x,\alpha\mp\hat{\gamma}\tilde{\mu}\theta) = f\,(x,\alpha\pm\hat{\gamma}\tilde{\mu}\theta)$, for $\alpha=0$ or $\alpha=\frac{1}{2}$. Consequently,

$$\left[\ln\Theta^\gamma_{\frac{1}{2},0} - \ln\Theta^\gamma_{0,0}\right] \longrightarrow -\frac{2\tilde{\mu}}{\sqrt{1+\theta^2}\tau_2}\left[\tfrac{1}{2}f\left(\tilde{\mu},\theta,\tfrac{1}{2}\right) + \tfrac{1}{4}f^\gamma_+\left(\tilde{\mu},\theta,\tfrac{1}{2}\right) + \tfrac{1}{4}f^\gamma_-\left(\tilde{\mu},\theta,\tfrac{1}{2}\right)\right. \tag{6.43}$$
$$\left. -\tfrac{1}{2}f\left(\tilde{\mu},\theta,0\right) - \tfrac{1}{4}f^\gamma_+\left(\tilde{\mu},\theta,0\right) - \tfrac{1}{4}f^\gamma_-\left(\tilde{\mu},\theta,0\right)\right]$$
$$= -\frac{4\tilde{\mu}}{\sqrt{1+\theta^2}\tau_2}\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}\frac{1}{l}\left[1+\cos\left(2\pi l\hat{\gamma}\tilde{\mu}\theta\right)\right]K_1(2\pi l\tilde{\mu}\sqrt{1+\theta^2}),$$

where these building blocks are now the functions $\ln\Theta^{\gamma}_{\alpha,\delta}\left(\tau_1,\tau_2,\frac{\tilde{\mu}}{\tau_2}\right)$ and the divergent part of the partition function (dominated by high energy modes) becomes

$$\ln Z^{\gamma}(a,b,\mu) \longrightarrow \frac{a}{4\pi}\sum_{\substack{r=1\\ r\text{ odd}}}^{\infty}\int_0^{\infty}\frac{d\tau_2}{\tau_2}\int_{-\frac{1}{2\tau_2}}^{+\frac{1}{2\tau_2}}d\theta \tag{6.44}$$
$$\times\exp\left\{-\frac{abr^2}{4\pi\alpha'\tau_2}+\frac{8\mu ar}{\pi\tau_2}\frac{1}{\sqrt{1+\theta^2}}\left[\sum_{\substack{l=1\\ l\text{ odd}}}^{\infty}\frac{1}{l}\left[1+\cos\left(\mu arl\theta\hat{\gamma}\right)\right]K_1\left(\mu alr\sqrt{1+\theta^2}\right)\right]\right\}.$$

Despite these changes to the partition function, when we evaluate its high energy behaviour, the θ integral is dominated by a gaussian which picks out $\theta=0$. All the $\hat{\gamma}$-dependence then vanishes. The behaviour of the free energy[4] is still given by (6.24), so that the Hagedorn temperature (6.23) once more describes a phase transition. To summarize: a key feature of this computation is that, in the high temperature limit, we find a continuum of states for which $x\equiv(n+\delta)/\tilde{\mu}$ is effectively continuous. The spacetime deformation is visible in the partition function only in that this continuous variable is changed from $x\rightarrow x\mp\hat{\gamma}$. Since the Hagedorn temperature is given by the density of states $\rho(w)=(dw(n)/dn)^{-1}$, it must remain unaltered, even though the spectrum of strings and the partition function on this background depend rather non-trivially on $\hat{\gamma}$.

6.3 Matching the γ-deformed gauge/string theories

A direct comparison between the thermodynamic properties of pp-wave strings (deformed or otherwise) and the corresponding SYM operators is non-trivial, largely because the pp-wave background is constructed by taking a Penrose limit in which the radius R, and hence also the t'Hooft coupling $\lambda=R^4$, becomes large. More precisely, the correspondence identifies the lightcone momenta $p^{\pm}$ of pp-wave strings with the conformal dimension Δ and $U(1)$ $\mathcal{R}$-charge J of SYM operators via

$$\frac{2p^+}{\mu}=\Delta-J \qquad\text{and}\qquad 2\mu p^-=\frac{\Delta+J}{\sqrt{\lambda}}, \tag{6.45}$$

so that, in the λ, $N\rightarrow\infty$ limit with $p^{\pm}$ finite, the only states which survive are those with conformal dimension and $\mathcal{R}$-charge that scale like $\sqrt{N}$. These are precisely the gauge theory states conjectured to be dual to pp-wave strings [15]. The problem with matching the Hagedorn/deconfinement temperature of the gauge theory to the

[4]In evaluating the gaussian, only the width changes - this affects the proportionality constant for F, but not the location of the singularity.

Hagedorn temperature of the string theory is now evident: BMN states form only a small subset of the set of all possible states in the SYM theory and, at small 't Hooft coupling, all these states should be taken into account. Including only the BMN sector results in an apparent gross mismatch with a state counting on the string theory side - here the number of states grows exponentially as the Hagedorn temperature is approached.

6.3.1 A novel decoupling limit

It was suggested in the series of works [84] that this problem may be (at least partially) resolved by a new decoupling limit of the AdS/CFT correspondence. In the gauge theory, this decoupling takes place at low temperatures and near-critical chemical potentials, while, in the string theory, we take the large μ limit of a particular pp-wave background with a flat direction. We shall now summarize the main results in this argument.

The $\mathcal{N} = 4$ SYM partition function is a sum over all multi-trace operators constructed from scalars, spinors and covariant derivatives:

$$Z(\beta, \Omega_i) = \mathrm{tr}\left(e^{-\beta D + \beta \sum_{i=1}^{3} R_i \Omega_i}\right), \tag{6.46}$$

where D is the dilatation operator and Ω_i are the three chemical potentials associated with the $\mathcal{R}$-charges J_i. Let us choose $(\Omega_1, \Omega_2, \Omega_3) = (\Omega, 0, 0)$, with $\epsilon \equiv 1 - \Omega$. The Harmark-Orselli limit [84] then sends T, λ, $\epsilon \to 0$, while keeping $\tilde{T} = \frac{T}{\epsilon}$ (or $\tilde{\beta} \equiv \epsilon\beta$) and $\tilde{\lambda} = \frac{\lambda}{\epsilon}$ fixed. Most of the $\mathcal{N} = 4$ SYM states decouple - only those with bare dimension equal to their $\mathcal{R}$-charge survive - and the system reduces to one of thermal quantum mechanics, with the partition function

$$Z(\tilde{\beta}) = \mathrm{tr}\left[e^{-\tilde{\beta}(D_0 + \tilde{\lambda} D_2)}\right]. \tag{6.47}$$

Here D_0 and D_2 are the tree and one-loop contributions to the dilatation operator. This decoupling limit leaves behind the well-known $SU(2)$ sector in which only two scalar fields, Z and X, contribute to multi-trace operators. In the planar limit $N \to \infty$, the single trace operators dominate and D_2 maps to the Hamiltonian of a spin-$\frac{1}{2}$ XXX-Heisenberg spin chain. The partition function is then given by [84]

$$Z(\tilde{\beta}) = \exp\left[\sum_{n=1}^{\infty}\sum_{l=1}^{\infty} \frac{1}{n}\, e^{-\tilde{\beta} l}\, Z_l^{XXX}(n\tilde{\beta})\right], \tag{6.48}$$

where Z_l^{XXX} denotes the partition function of a ferromagnetic spin chain of length l.

Key to matching this prescription in the dual string theory is to choose a pp-wave background with a flat direction (or spatial isometry). One such is associated with the $(J, J, 0)$ null geodesic, which is described by

$$t = \phi^+ \equiv \tfrac{1}{2}(\phi_1 + \phi_2) = \mu x^+, \quad \theta_1 = \tfrac{\pi}{2}, \quad \theta_2 = \tfrac{\pi}{4} \quad \text{and} \quad \rho = \phi_- \equiv \tfrac{1}{2}(\phi_1 - \phi_2) = 0. \tag{6.49}$$

Expanding about this null geodesic, we take the ansatz

$$\begin{aligned} t &= \mu x^+ + \frac{x^-}{2\mu R^2} \qquad \rho = \frac{y}{R} \qquad \phi^+ = \mu x^+ - \frac{x^-}{2\mu R^2} \qquad \phi^- = \frac{x^1}{R} \\ \theta_1 &= \frac{\pi}{2} + \frac{x}{R}, \qquad \theta_2 = \frac{\pi}{4} + \frac{x^2}{R}, \end{aligned} \tag{6.50}$$

with $x^3 = x\cos\phi_3$ and $x^4 = x\sin\phi_3$, and (x^5, x^6, x^7, x^8) the usual transverse coordinates (6.4). The scaling $R \to \infty$ produces the $(J, J, 0)$ pp-wave background

$$ds^2 = -2\,dx^+dx^- + \sum_{i=1}^{8}\left(dx^i\right)^2 - \mu^2\sum_{i=3}^{8}\left(x^i\right)^2\left(dx^+\right)^2 - 4\,\mu x^2\,dx^1dx^+. \tag{6.51}$$

Although this background is related to the $(J, 0, 0)$ one by a lightcone time-dependent coordinate rotation

$$\begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} = \begin{bmatrix} \cos(\mu x^+) & -\sin(\mu x^+) \\ \sin(\mu x^+) & \cos(\mu x^+) \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \tag{6.52}$$

in the (x^1, x^2)-plane, the physics is rather different. In particular, there is one vacuum state for each value of the momentum along the flat direction x^1. A modified Penrose limit was considered in [84], in which $\tilde{R} \to \infty$ with $\tilde{R}^4 \equiv \frac{R^4}{\epsilon}$, but $\lambda = R^4$ still remains small. When $\epsilon \to 0$, with

$$\tilde{\mu} \equiv \mu\sqrt{\epsilon}, \qquad \tilde{H}_{lc} \equiv \frac{H_{lc}}{\epsilon}, \qquad \tilde{g}_s \equiv \frac{g_s}{\epsilon} \qquad \text{and} \qquad p^+ \tag{6.53}$$

all held fixed, the pp-wave spectrum - and consequently the Hagedorn behaviour - exactly matches the weakly coupled gauge theory.

At this point, everything we have said so far applies specifically to maximally supersymmetric $\mathcal{N} = 4$ SYM theory. How then is this matching prescription affected by a systematic deformation - such as the $\mathcal{N} = 1$ γ-deformation - away from maximal supersymmetry? The γ-deformed superpotential can be resummed as a Moyal-like $*$-product deformation

$$\mathcal{X} * \mathcal{Y} = e^{i\pi\gamma\left(Q^1_{\mathcal{X}}Q^2_{\mathcal{Y}} - Q^2_{\mathcal{X}}Q^1_{\mathcal{Y}}\right)}\,\mathcal{X}\mathcal{Y} \tag{6.54}$$

and similarly for the other superfields, where $(Q^1_{\mathcal{X}}, Q^2_{\mathcal{X}})$, $\left(Q^1_{\mathcal{Y}}, Q^2_{\mathcal{Y}}\right)$ and $(Q^1_{\mathcal{Z}}, Q^2_{\mathcal{Z}})$ are the charges of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ under the $U(1)_1 \times U(1)_2$ global symmetry [7]. Consequently, not only is the Feynman diagram structure (at the planar level) unchanged by the γ-deformation, but, since this deformation preserves the three Cartan generators of the

$SO(6)$ $\mathcal{R}$-symmetry, any closed subset of single trace operators remains closed under renormalization group flow. Specifically, this is true of the $SU(2)$ and $SU(3)$ sectors[5] consisting of single trace operators built out of two and three complex scalar fields.

Like its undeformed counterpart, the dilatation operator of γ-deformed $\mathcal{N}=1$ SYM theory can be represented as the Hamiltonian of a spin chain acting on a spin-chain Hilbert space[6]. Under the γ-deformation, since the commutators $[X,Y] \to [X,Y]_\gamma \equiv e^{i\pi\gamma}XY - e^{-i\pi\gamma}YX$, interchanging any two differently charged fields in the single trace operator $\mathrm{tr}\left(X^{J_1}Y^{J_2}\right)$ comes with a γ-dependent phase. At the level of the spin-chain Hamiltonian, this deformation can be realized [94, 95] by either a parity-preserving ferromagnetic XXZ-spin chain with γ-twisted boundary conditions or as the following XXZ-spin chain with broken parity and periodic boundary conditions:

$$\begin{aligned} H_\gamma = \frac{\lambda}{(4\pi)^2}\sum_{l=1}^{J} \Big\{ & \mathbb{I}_l \otimes \mathbb{I}_{l+1} - \left(\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y + \sigma_l^z \otimes \sigma_{l+1}^z\right) \\ & + \left[1-\cos\left(2\pi\gamma\right)\right]\left(\sigma_l^x \otimes \sigma_{l+1}^x + \sigma_l^y \otimes \sigma_{l+1}^y\right) \\ & + \sin\left(2\pi\gamma\right)\left(\sigma_l^x \otimes \sigma_{l+1}^y - \sigma_l^y \otimes \sigma_{l+1}^x\right)\Big\}\,. \end{aligned} \tag{6.55}$$

Either way, the resulting spin chain lends itself to a Bethe ansatz-type solution [94] from which the energy spectrum may be extracted and, following [84], the Hagedorn temperature determined. In principle then, we should be able to match the temperature of the Hagedorn transition in the gauge theory with the Hagedorn temperature of the dual string theory. Or should we?

The problem is that the $SU(2)_\gamma$ gauge sector - the first non-trivial sector in which the matching prescription works - corresponds to the γ-deformed pp-wave background associated with the $(J,J,0)$ null geodesic, which is now parameterized by

$$\phi_+ \equiv \tfrac{1}{2}\left(\phi_1+\phi_2\right) = \mu x^+ \qquad \text{and} \qquad \phi^+ = \tfrac{1}{2}\sqrt{4+\hat{\gamma}^2}\,\mu x^+, \tag{6.56}$$

where we also set

$$\theta_1 = \tfrac{\pi}{2}, \qquad \theta_2 = \tfrac{\pi}{4} \qquad \text{and} \qquad \rho = \phi_- \equiv \tfrac{1}{2}\left(\phi_1-\phi_2\right) = 0. \tag{6.57}$$

We hence choose the modified ansatz

$$\begin{aligned} & t = \mu x^+ + \frac{x^-}{2\mu R^2} \qquad \rho = \frac{y}{R} \qquad \phi^+ = \tfrac{1}{2}\sqrt{4+\hat{\gamma}^2}\left(\mu x^+ - \frac{x^-}{2\mu R^2}\right) \qquad \phi^- = \frac{x^1}{R} \\ & \theta_1 = \frac{\pi}{2} + \frac{x}{R} \qquad \theta_2 = \frac{\pi}{4} + \frac{x^2}{R}. \end{aligned} \tag{6.58}$$

[5]Since the $U(1)$ sector of the theory is spanned by single trace operators constructed from just one of the complex SYM scalars, it is a straightforward consequence of the holomorphicity of these operators that this sector remains unaffected by the deformation.

[6]For the sake of definiteness and to facilitate a comparison with the (undeformed) Hagedorn/phase transition analysis of [84], we shall restrict ourselves to the $SU(2)_\gamma$ sector of the $\mathcal{N}=1$ SYM theory and content ourselves with comments on the $U(1)_\gamma$ and $SU(3)_\gamma$ sectors at the end of this section.

and take the large R Penrose limit. The metric which describes this γ-deformed $(J,J,0)$ pp-wave geometry takes the form

$$ds^2 = -2\,dx^+dx^- - \mu^2\left[-\frac{4\hat{\gamma}^2}{4+\hat{\gamma}^2}\left(x^2\right)^2 + \left(\frac{4-\hat{\gamma}^2-\hat{\gamma}^4}{4+\hat{\gamma}^2}\right)\sum_{i=3}^{4}\left(x^i\right)^2 + \sum_{i=5}^{8}\left(x^i\right)^2\right]\left(dx^+\right)^2 + \sum_{i=1}^{8}\left(dx^i\right)^2 - 4\mu x^2\,dx^1dx^+ + \frac{2\hat{\gamma}^2}{\sqrt{4+\hat{\gamma}^2}}\mu\left(x^3dx^4 - x^4dx^3\right). \qquad (6.59)$$

Notice the flat direction x^1. Like the (J,J,J) case, this pp-wave is rotationally disconnected from the γ-deformed $(J,0,0)$ pp-wave. In fact, the situation here is slightly worse - although the Penrose limit is well-defined at the level of the metric, this γ-deformed $(J,J,0)$ pp-wave background is actually non-BPS. The first manifestation of this fact arises when we try to apply the Penrose limit to the NS B-field. We find that, to leading order,

$$B_2 = \tfrac{1}{2}\,\hat{\gamma}\mu R\;dx^1\wedge dx^+, \qquad (6.60)$$

which diverges. The consequences are clear: if any comparison with the $SU(2)$ sector of γ-deformed $\mathcal{N}=1$ SYM theory is to be made, another way must be found which does not involve a direct comparison with the Hagedorn temperature of strings propagating on the γ-deformed $(J,J,0)$ pp-wave background. To date, we have not managed to do so, but, given the success of the program advanced in [84], it would be disappointing indeed if this were not possible for the $\mathcal{N}=1$ theory!

6.3.2 Decoupling the $U(1)_\gamma$ sector/γ-deformed $(J,0,0)$ pp-wave

To conclude this section, we make a few brief comments about the $U(1)$ sector of γ-deformed $\mathcal{N}=1$ SYM theory. The partition function takes the form (6.46), but now involves the γ-deformed dilatation operator D^γ. At weak 't Hooft coupling $\lambda \ll 1$, this becomes $D^\gamma = D_0 + \lambda D_2^\gamma$ to linear order in λ. The γ-deformation affects only interactions, so D_0 yields simply the bare scaling dimension. The $(J,0,0)$ decoupling limit corresponds to the choice $(\Omega_1,\Omega_2,\Omega_3) = (\Omega,0,0)$ of chemical potentials, with $\epsilon \equiv 1-\Omega \to 0$, while we hold fixed $\tilde{\beta} \equiv \epsilon\beta$ and $\tilde{\lambda} \equiv \frac{\lambda}{\epsilon}$. This results in small temperatures and couplings. The partition function is then given by

$$Z(\tilde{\beta}) = \mathrm{tr}\left[e^{-\tilde{\beta}\left(D_0+\tilde{\lambda}D_2^\gamma\right)}\right], \qquad (6.61)$$

where the trace now runs over only those multi-trace operators with $D_0 = J_1$.

The only surviving states in the Hilbert space are built out of a Fock space of single trace operators of the form $\mathrm{tr}\left(\Phi_1^L\right)$. Clearly holomorphic, these single trace $\frac{1}{2}$-BPS operators are protected by supersymmetry and therefore vanish under the action of D_2^γ

(as well as all higher order terms). Hence, for single trace operators, the γ-deformed partition function in the $(J, 0, 0)$ decoupling limit is given by

$$Z_1(\tilde{\beta}) = \mathrm{tr}\left(e^{-\tilde{\beta}J}\right) = \sum_{J=1}^{\infty} e^{-\tilde{\beta}J} = \frac{1}{1-e^{-\tilde{\beta}}}, \tag{6.62}$$

which is obviously independent of the deformation parameter γ.

To complete the study, we still need to show that the matching prescription of [84] goes through for the γ-deformed string theory on a $(J, 0, 0)$ pp-wave background. Taking the modified Penrose limit $\tilde{R} \to \infty$, with $R^4 \equiv \epsilon\,\tilde{R}^4$ fixed and small, results in a γ-deformed pp-wave metric identical to (6.5) up to an overall factor of $\sqrt{\epsilon}$. Here again (6.53) are held fixed as $\epsilon \to 0$, so that the mass parameter becomes large. We can deduce the rescaled spectrum for strings polarized in the two sets of four transverse directions as follows:

$$\omega_n^{\pm} = \frac{1}{\epsilon}\,\mathrm{sign}(n)\sqrt{\tilde{m}^2 + (\sqrt{\epsilon}n \pm \hat{\gamma}\tilde{m})^2} \qquad \text{and} \qquad \omega_n = \frac{1}{\epsilon}\,\mathrm{sign}(n)\sqrt{\tilde{m}^2 + \epsilon n^2}, \tag{6.63}$$

with $\tilde{m} \equiv m\sqrt{\epsilon}$ fixed. Notice that all these modes go like $\frac{1}{\epsilon}$ as $\epsilon \to 0$. Thus, using an argument similar to that of [84], we conclude that it is not possible to excite any of the transverse modes in the decoupling limit, as they correspond to states of infinite energy.

The γ-deformed single string partition function in this $(J, 0, 0)$ decoupling limit must hence be simply a function of the lightcone momentum p^+:

$$Z_1(b) = \int_0^{\infty} dp^+\, e^{-bp^+} = \frac{1}{b}. \tag{6.64}$$

This results in an expression

$$b = 1 - e^{-\tilde{\beta}}, \qquad \text{with} \qquad \tilde{\beta} \equiv \epsilon\beta, \tag{6.65}$$

for the variable b as a function of the inverse temperature β.

Conclusion: marginal deformations

The suggestion that the Hagedorn temperature of strings on a particular background may be invariant under a systematic breaking of supersymmetry (and possibly even conformal symmetry) is intriguing. As far as we are aware, the first study of the universality of the Hagedorn behaviour of strings on pp-wave geometries was carried out in [96]. There it was demonstrated that the Hagedorn temperature of pp-wave strings on a Lunin-Maldacena deformation of the Maldacena-Nuñez background [6] is independent of $\hat{\gamma}$. However, this is only a necessary condition for universality, which requires to be supplemented by additional arguments. In this part, we have pursued the line of reasoning initiated in [96]. The Hagedorn temperature of pp-wave strings on the $\mathrm{AdS}_5 \times \mathrm{S}^5_\gamma$ Lunin-Maldacena background was found to be independent of the deformation parameter, despite the complicated $\hat{\gamma}$-dependence (at least for the $(J, 0, 0)$ case) of the multi-string partition function.

On the gauge theory side, utilizing technology developed in [84], we explored the possibility of matching this Hagedorn temperature with that of the confinement/ deconfinement transition in planar γ-deformed $\mathcal{N} = 1$ SYM theory - with limited success. The $U(1)$ sector, composed of holomorphic $\frac{1}{2}$-BPS operators is unchanged by the deformation, so the matching of the Hagedorn behaviour goes through unaffected. The $SU(2)$ sector, on the other hand, is far from trivial. Under the $\mathcal{N} = 1$ marginal deformation, the XXX Heisenberg spin chain associated with single trace operators in this sector is mapped to an XXZ spin chain, the Hamiltonian of which may be diagonalized by an appropriate Bethe ansatz. Even though the temperature of the confinement/deconfinement transition may then be computed, we argue that no matching with the string theory is possible - at least not using the prescription of [84] - as the corresponding γ-deformed $(J, J, 0)$ pp-wave geometry is ill-defined.

It is clear that this study of the thermal properties of strings on γ-deformed pp-wave backgrounds has generated a number of possible lines of enquiry:

- The matching prescription of [84] might be adapted to the γ-deformed (J, J, J) pp-wave background. The Hagedorn temperature could then be compared with the temperature of the confinement/deconfinement transition in the $PSU(2|3)$ sector of the gauge theory. Since the completion of this work, substantial progress has been made in this direction: [97] were able to apply the matching prescription to several different sectors of $\mathcal{N} = 4$ SYM theory. In particular, the $PSU(2|3)$ sector is associated with a modified Penrose limit which results in a pp-wave background with two isometries. It should now be possible to extend this analysis to γ-deformed $\mathcal{N} = 1$ SYM theory.

- While our focus has been on γ-deformations of $\mathrm{AdS}_5 \times \mathrm{S}^5$, which affect only the 5-sphere space, it should be noted that a number of other deformations exist. In addition to the complex β-deformations [7] and the 3-parameter family of γ_i-deformations of [8, 95], the TsT transformation of [8] has also been applied to the global toroidal isometries of the AdS_5 spacetime [98]. It would be interesting to examine the Hagedorn behaviour of pp-wave strings on these backgrounds for signs of universality.

Gaining insight into the nature of gauge theories at strong coupling is particularly important, at this current point in time, given our proximity to the release of LHC results. It is to be hoped that studying quantities which are universal across a large class of such theories (with gravity duals) may allow us to make some tentative predictions as to the behaviour of the strongly coupled quark-gluon plasma.

Part IV

Summary and Conclusion

Conclusion

Applications of the gauge theory/gravity correspondence are varied and diverse. Rather than confining our attention to one aspect of a particular duality, we chose to consider two broad topics involving several different AdS/CFT correspondences - both of the original SYM/type IIB string theory, as well as the more recent SCS-matter/type IIA string theory variety.

The primary focus of this thesis is on D-branes, their open string excitations and (to a lesser extent) the dual local operators. Although we began with the intention of studying giant gravitons in type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ - and, indeed, conducted a preliminary investigation of the spherical D2-brane dual giant - we found it best, in the end, to take a more scenic route. The D4-brane giant graviton, embedded in the complex projective space, is a highly non-trivial object, quite unlike the usual spherical configurations. However, the existence of a known ansatz [63] for a similar D3-brane giant in type IIB string theory on $AdS_5 \times T^{1,1}$ suggested that an explicit construction of this non-spherical object might not only prove profitable, but also lend itself to an extension to the D4-brane giant graviton on $AdS_4 \times \mathbb{CP}^3$.

The nature of the maximal D4-brane giant graviton was initially more apparent. The subdeterminant operator in ABJM theory factorizes, at maximum size, into the product of two determinants (a special case of dibaryon operators). The maximal giant should therefore comprise the union of two $\mathbb{CP}^2$ dibaryons - D4-branes wrapped on different non-contractible $\mathbb{CP}^2$ cycles in $\mathbb{CP}^3$. With reference to a similar analysis [66] of dibaryons on $AdS_5 \times T^{1,1}$, we matched the spectrum of small fluctuations about a $\mathbb{CP}^2$ dibaryon (particularly, those associated with the transverse $\mathbb{CP}^3$ directions) with the conformal dimensions of BPS excitations of the ABJM determinant operators.

The D3-brane giant graviton on $AdS_5 \times T^{1,1}$ turned out to be exceedingly interesting in its own right. The construction involves a map between the two 2-spheres embedded in $T^{1,1}$, which disappears at maximal size - this is related to the factorization of the dual subdeterminant operator in Klebanov-Witten theory into two dibaryons. In fact, here what we are really viewing, from the gravitation perspective, is an indication of the bifundamental nature of the gauge group. The fluctuation spectrum of the giant graviton on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ was shown to be dependent on its size. This phenomenon is most unusual and probably relates to the fact that this configuration is far from spherical. Our study of the D3-brane giant graviton was extensive, but we were also able to conclude with an ansatz - thus far unverified - for the D4-brane giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

These results clearly suggest a wide range of possibilities for future research. An investigation of the open string excitations of the D4-brane giant graviton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ would be a fascinating enterprize. It should prove possible to match the open string excitation energies to the anomalous dimensions of words attached to the ABJM subdeterminant operator, which map to alternating open spin chains with fixed boundary conditions. It might even be possible to find an interpretation in the gauge theory for the dependence on size (which we expect to also observe for the D4-brane giant) of the spectrum of small fluctuations.

The second topic chosen for consideration involves a study of the thermal properties of an ensemble of closed strings. We tested the conjecture, initially motivated in [96], that the Hagedorn temperature is a universal quantity - this was verified for pp-wave strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ under a Lunin-Maldacena deformation. We discussed matching this temperature with that of the confinement/deconfinement transition in the dual gauge theory, which is an $\mathcal{N} = 1$ marginal γ-deformation of $\mathcal{N} = 4$ SYM theory. However, our results were unfortunately limited to the $U(1)$ sector, since the $SU(2)$ decoupling limit of [84], on the gravity side, does not admit a generalization to the γ-deformed case. Further studies of universal quantities might prove useful in gaining insight into generic properties of strongly coupled gauge theories. For example, this might lead to a better understanding of the strongly coupled quark-gluon plasma, which we expect to observe at the LHC, as well as the phase of matter inside neutron stars.

Recently, the AdS/CFT correspondence has been applied to a wider and wider range of systems. There has been much interest, in the last few years, in using this approach to study condensed matter physics - in the vicinity of a quantum critical point, conformal invariance is restored and the field theory lends itself to a dual gravitational description. AdS/CFT then offers a novel interpretation of a number of phenomena [11]. A partial understanding of the dictionary between both sides of the correspondence has also offered great insight into the fundamental nature of spacetime itself - this appears to be an emergent property encoded in the N^2 degrees of freedom of the matrices in the dual gauge theory. The notion of an emergent spacetime [13] is a fascinating subject, which will undoubtedly receive much attention in the future.

Part V

Appendices

Appendix A

Eigenvalue problems

This appendix contains the solutions of a number of eigenvalue problems (EVPs) associated with small fluctuations about dibaryons on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$, as well as giant gravitons on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$. They involve hypergeometric differential equations and are similar to problems discussed in [66, 99].

A.1 Stationary EVPs for a dibaryon and maximal giant graviton

We shall first consider (based on [66]) the stationary EVP $\nabla^2\Phi = -E\Phi$, with ∇^2 the Laplacian (4.19) on the spatial extension of a dibaryon on $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$. We then turn our attention to the maximal giant graviton, consisting of two dibaryons, which has fluctuations described in section 5.2.3. Here additional singular solutions of the original problem must be included.

A.1.1 Standard EVP for a dibaryon

Let us look for solutions of the form

$$\Phi(z,\xi,\phi) = f(z)\, e^{im\xi}\, e^{in\phi}, \qquad \text{with} \quad f(z) = z^{\frac{1}{2}|m+n|}\,(1-z)^{\frac{1}{2}|m-n|}\,h(z). \tag{A.1}$$

We find that $h(z)$ must satisfy the hypergeometric differential equation

$$\begin{aligned} &z\,(1-z)\,\partial_z^2 h(z) + \left[(|m+n|+1) - (|m+n|+|m-n|+2)\,z\right]\partial_z h(z) \\ &-\left\{\tfrac{1}{2}|m^2-n^2| + \tfrac{1}{2}|m+n| + \tfrac{1}{2}|m-n| + \tfrac{1}{2}(2m^2+n^2) - \tfrac{1}{6}E\right\} h(z) = 0, \end{aligned} \tag{A.2}$$

which has solutions $h(z) = F(a, b, c; z)$. This hypergeometric function is dependent on the parameters

$$a, b \equiv \tfrac{1}{2}\left(|m+n| + \tfrac{1}{2}|m-n| + 1\right) \pm \sqrt{\tfrac{1}{6}E - \tfrac{1}{2}m^2 + \tfrac{1}{4}} \quad \text{and} \quad c \equiv |m+n| + 1, \tag{A.3}$$

where a and b are associated with different signs in the $\pm$. For regularity at $z = 1$, either a or b must be a negative integer, so that

$$\tfrac{1}{2}\left(|m+n| + \tfrac{1}{2}|m-n| + 1\right) - \sqrt{\tfrac{1}{6}E - \tfrac{1}{2}m^2 + \tfrac{1}{4}} \equiv -s, \quad \text{with} \quad s \,\epsilon\, \{0, 1, 2, \ldots\}\,. \tag{A.4}$$

Hence the eigenfunctions of the Laplacian can be written as follows:

$$\Phi_{smn}(z, \xi, \phi) = z^{\frac{1}{2}|m+n|}\,(1-z)^{\frac{1}{2}|m-n|}\,F_{smn}(z)\,e^{im\xi}\,e^{in\phi}, \tag{A.5}$$

which correspond to the eigenvalues

$$E_{smn} = 6l(l+1) + 3m^2, \qquad \text{with} \quad l \equiv s + \max\{|m|, |n|\}. \tag{A.6}$$

Here $s \geq 0$ and n are integers, and m is an integer or half-integer. The hypergeometric functions $F_{smn}(z) \equiv F(a, b, c; z)$ previously described are dependent on s, m and n through the parameters a, b and c.

A.1.2 EVP for the maximal giant graviton

We shall now look for additional solutions, which behave like $\Phi \sim (1-z)^{-\frac{1}{2}}$ as $z \to 1$. These are physically meaningful non-singular contributions when both halves of the maximal giant graviton are taken into account. Setting

$$f(z) = z^{\frac{1}{2}|m+n|}(1-z)^{-\frac{1}{2}}\,h(z), \qquad \text{with} \quad n = m \pm 1, \tag{A.7}$$

we obtain the following hypergeometric differential equation:

$$z\,(1-z)\,\partial_z^2 h(z) + (|m+n|+1)\,(1-z)\,\partial_z h(z) - \left\{-\tfrac{1}{2} + \tfrac{1}{2}(2m^2+n^2) - \tfrac{1}{6}E\right\} h(z) = 0. \tag{A.8}$$

The solutions $h(z) = F(a, b, c; z)$ depend on the parameters

$$a, b \equiv \tfrac{1}{2}|m+n| \pm \sqrt{\tfrac{1}{6}E - \tfrac{1}{2}m^2 + \tfrac{1}{4}} \quad \text{and} \quad c \equiv |m+n| + 1, \tag{A.9}$$

where, for a or b to be a negative integer,

$$\tfrac{1}{2}|m+n| - \sqrt{\tfrac{1}{6}E - \tfrac{1}{2}m^2 + \tfrac{1}{4}} = -s, \qquad \text{with} \quad s \,\epsilon\, \{0, 1, 2, \ldots\}\,. \tag{A.10}$$

The modified eigenfunctions of the Laplacian, which are additional eigenfunctions with this particular behaviour at $z = 1$, therefore take the form

$$\Phi^{\text{mod}}_{smn}(z, \xi, \phi) = z^{\frac{1}{2}|m+n|}\,(1-z)^{-\frac{1}{2}}\,F^{\text{mod}}_{smn}(z)\,e^{im\xi}\,e^{in\phi}, \tag{A.11}$$

with associated eigenvalues

$$E^{\text{mod}}_{smn} = 6l^{\text{mod}}(l^{\text{mod}}+1) + 3m^2, \qquad \text{where} \quad l^{\text{mod}} \equiv s + \tfrac{1}{2}\left(|m+n| - 1\right). \tag{A.12}$$

The hypergeometric function $F^{\text{mod}}_{smn}(z) \equiv F(a, b, c; z)$ depends on $s \geq 0$, $n = m \pm 1$ and m, now all integers.

A.2 Stationary EVPs for a $\mathbb{CP}^2$ dibaryon

The standard stationary EVP $\nabla^2\Phi = -E\Phi$ on the complex projective space $\mathbb{CP}^2$ can be solved using the chiral primaries (4.56) with eigenvalues (4.57). However, we shall rather describe the $\mathbb{CP}^2$ subspace using the coordinates (x, z, ξ, ϕ), in terms of which the Laplacian can be written as (4.50), and look for separable solutions. This method is then applied to the modified EVP $\mathcal{O}_\pm\Phi = -E\Phi$, where we define

$$\mathcal{O}_\pm \equiv \nabla^2 + (1-x)\,\partial_x \mp \frac{i}{x}\,\partial_\xi, \tag{A.13}$$

which is associated with transverse $\mathbb{CP}^3$ fluctuations.

A.2.1 Standard EVP

Let us consider a separable solution of the form

$$\Phi(x, z, \xi, \phi) = f(z)\,g(z)\,e^{im\xi}\,e^{in\phi}, \tag{A.14}$$

with[1] $|m| \geq |n|$. We must now solve two related EVPs associated with $g(z)$ and $f(x)$, the first of which is given by

$$\partial_z\left[z\,(1-z)\,(\partial_z g)\right] - \left\{\frac{(m+n)^2}{4}\,\frac{(1-z)}{z} + \frac{(m-n)^2}{4}\,\frac{z}{(1-z)} + \frac{(n^2-m^2)}{2} - \lambda\right\} g = 0, \tag{A.15}$$

with λ some constant eigenvalue. Setting

$$g(z) = z^{\frac{1}{2}|m+n|}\,(1-z)^{\frac{1}{2}|m-n|}\,h_1(z), \tag{A.16}$$

[1]Functions in $\mathbb{CP}^2$ are built out of equal numbers of z's and $\bar{z}$'s - an excess of z^1's must be accounted for by no more $\bar{z}_2$ or $\bar{z}_3$'s (and similarly for an excess of $\bar{z}_1$'s).

we obtain the hypergeometric differential equation

$$\begin{aligned} &z\,(1-z)\,\partial_z^2 h_1 + \left[(|m+n|+1) - (|m+n|+|m-n|+2)\,z\right]\partial_z h_1 \\ &-\left\{\tfrac{1}{2}|m^2-n^2| + \tfrac{1}{2}|m+n| + \tfrac{1}{2}|m-n| + \tfrac{1}{2}\left(m^2-n^2\right) - \lambda\right\} h_1 = 0. \end{aligned} \tag{A.17}$$

Solutions take the form of hypergeometric functions $h_1(z) = F(a_1, b_1, c_1; z)$, which are dependent on the parameters

$$a_1,\, b_1 \equiv \tfrac{1}{2}|m+n| + \tfrac{1}{2}|m-n| + \tfrac{1}{2} \pm \sqrt{\lambda + m^2 + \tfrac{1}{4}} \quad \text{and} \quad c_1 \equiv |m+n|+1, \tag{A.18}$$

where a_1 and b_1 are associated with different signs in the $\pm$. For regularity at $z=1$, either a_1 or b_1 should be a negative integer. Hence

$$\tfrac{1}{2}|m+n| + \tfrac{1}{2}|m-n| + \tfrac{1}{2} - \sqrt{\lambda + m^2 + \tfrac{1}{4}} = -s_1, \qquad \text{with} \quad s_1 \,\epsilon\, \{0,\,1,\,2,\,\ldots\}\,, \tag{A.19}$$

so that $\lambda = k\,(k+1) - m^2$, where we define $k \equiv s_1 + |m|$.

The second EVP then becomes

$$\partial_x\left[x\,(1-x)\,(\partial_x f)\right] - x\,(\partial_x f) - \left\{\frac{m^2\,(1-x)}{x} + \frac{k\,(k+1)\,x}{(1-x)} + m^2 + k\,(k+1) - E\right\} f = 0. \tag{A.20}$$

We shall now take

$$f(x) = x^{|m|}\,(1-x)^k\; h_2(x), \tag{A.21}$$

where $h_2(x)$ satisfies the hypergeometric differential equation

$$\begin{aligned} &x\,(1-x)\,\partial_x^2 h_2 + \left[(2|m|+1) - (2|m|+2k+3)\,x\right]\partial_x h_2 \\ &-\left\{2\,(|m|k + |m|) + k\,(k+1) + m^2 - E\right\} h_2 = 0. \end{aligned} \tag{A.22}$$

The solutions $h_2(x) = F(a_2, b_2, c_2; x)$ are hypergeometric functions dependent on the parameters

$$a_2,\, b_2 = |m| + k + 1 \pm \sqrt{E+1} \qquad \text{and} \qquad c_2 = 2|m|+1, \tag{A.23}$$

where $E = l\,(l+2)$, with $l \equiv s_2 + k + |m|$ and $s_2 \,\epsilon\, \{0,\,1,\,2,\,\ldots\}$, for regularity at $x=1$.

Hence the eigenfunctions of the $\mathbb{CP}^2$ Laplacian are

$$\Phi_{smn}\,(x, z, \xi, \phi) = z^{\frac{1}{2}|m+n|}\,(1-z)^{\frac{1}{2}|m-n|}\,x^{|m|}\,(1-x)^{s_1+|m|}\,F^z_{s_1mn}(z)\,F^x_{s_2mn}(x)\,e^{im\xi}\,e^{in\phi}, \tag{A.24}$$

which correspond to the eigenvalues

$$E = l\,(l+2)\,, \qquad \text{with} \quad l = s + 2|m|. \tag{A.25}$$

Here $s = s_1 + s_2 \geq 0$ and n are integers, and m is an integer or half-integer. The hypergeometric functions $F^z_{s_1mn}(z) = F(a_1, b_1, c_1; z)$ and $F^x_{s_2mn}(x) = F(a_2, b_2, c_2; x)$ depend on s_i, n and m through the parameters a_i, b_i and c_i. These eigenvalues are in agreement with (4.57).

A.2.2 Modified EVP

We shall now look for separable solutions (A.14) to the modified EVP. Notice that taking $m \to -m$ implies that $\mathcal{O}_\pm \Phi \to \mathcal{O}_\mp \Phi$. It is therefore sufficient to consider m positive, bearing in mind that the m negative solutions can then constructed by simply interchanging the eigenfunctions Φ^+ and Φ^-.

The resolution of the first EVP for $g(z)$ remains unaltered. The second EVP becomes

$$\partial_x \left[x\,(1-x)\,(\partial_x f)\right] + (1-2x)\,(\partial_x f) \\ -\left\{\frac{m\,(m\pm 1)\,(1-x)}{x} + \frac{k\,(k+1)\,x}{(1-x)} + m\,(m\pm 1) + k\,(k+1) - E\right\} f = 0. \tag{A.26}$$

We must now distinguish between the $\pm$ signs. We shall take

$$f(x) = x^{|m|}\,(1-x)^k\;h_2^+(x) \quad \text{and} \quad f(x) = x^{|m|-1}\,(1-x)^k\;h_2^-(x), \tag{A.27}$$

respectively. The hypergeometric differential equation for $h_2^+(x)$ is given by

$$x\,(1-x)\,\partial_x^2 h_2^+ + \left[(2|m|+2) - (2|m|+2k+4)\,x\right]\partial_x h_2^+ \\ -\left\{2\,(|m|k + |m| + k) + m\,(m+1) + k\,(k+1) - E\right\} h_2^+ = 0, \tag{A.28}$$

which has solutions $h_2^+(x) = F(a_2^+, b_2^+, c_2^+; x)$ dependent on the parameters

$$a_1^+,\, b_2^+ \equiv |m| + k + \tfrac{3}{2} \pm \sqrt{E + \tfrac{9}{4}} \qquad \text{and} \qquad c_1^+ \equiv 2|m| + 2. \tag{A.29}$$

Here $E = l\,(l+3)$, with $l \equiv s_2 + k + |m|$ and $s_2 \,\epsilon\, \{0,\, 1,\, 2,\, \ldots\}$, for regularity at $x = 1$. Similarly, the hypergeometric differential equation for $h_2^-(x)$ takes the form

$$x\,(1-x)\,\partial_x^2 h_2^- + \left[2|m| - (2|m|+2k+2)\,x\right]\partial_x h_2^- \\ -\left\{2\,(|m|k + |m| - 1) + m\,(m-1) + k\,(k+1) - E\right\} h_2^- = 0 \tag{A.30}$$

and has solutions $h_2^-(x) = F(a_2^-, b_2^-, c_2^-; x)$ depending on

$$a_1^-,\, b_2^- \equiv |m| + k + \tfrac{1}{2} \pm \sqrt{E + \tfrac{9}{4}} \qquad \text{and} \qquad c_1^- \equiv 2|m|, \tag{A.31}$$

with $E = l\,(l+3) + 2$ and l defined as before.

The eigenfunctions of the modified operator $\mathcal{O}_\pm$ can therefore be written as follows:

$$\Phi^\pm_{smn}\,(x, z, \xi, \phi) = z^{\frac{1}{2}|m+n|}\,(1-z)^{\frac{1}{2}|m-n|}\,x^{|m|-\frac{1}{2}\pm\frac{1}{2}}\,(1-x)^{s_1+|m|} F^{\,z}_{s_1 mn}(z)\,F^{\,x\,\pm}_{s_2 mn}(x)\,e^{im\xi}\,e^{in\phi}, \tag{A.32}$$

and are associated with the eigenvalues

$$E^+_{smn} = l\,(l+3) \quad \text{and} \quad E^-_{smn} = l\,(l+3) + 2, \quad \text{with} \quad l \equiv s + 2|m|. \tag{A.33}$$

Here $s = s_1 + s_2 \geq 0$ and n are integers, and m is a positive integer or half-integer. The hypergeometric functions $F^{z}_{s_1 mn}(z) = F(a_1, b_1, c_1; z)$ and $F^{x\,\pm}_{s_2 mn}(z) = F(a_2^{\pm}, b_2^{\pm}, c_2^{\pm}; x)$ were previously described. Interchanging Φ^{+}_{smn} and Φ^{-}_{smn}, together with the associated eigenvalues E^{+}_{smn} and E^{-}_{smn}, gives the m negative solutions[2].

A.3 EVP for the small submaximal giant graviton

Let us consider the EVP $\tilde{\Box}\Psi = -\lambda\Psi$, with $\tilde{\Box}$ the (rescaled) d'Alembertian (5.83) on the worldvolume of the small submaximal giant graviton.

If we take an ansatz

$$\Psi(t, z, \chi_2, \chi_3) \equiv f(z)\, e^{-i\omega t}\, e^{\frac{3}{4} i m \chi_2}\, e^{\frac{3}{4} i n \chi_3}, \quad \text{with} \quad f(z) \equiv z^{\frac{1}{2}|n|}\,(1-z)^{\frac{1}{2}|m|}\, h(z), \quad \text{(A.34)}$$

then the problem reduces to solving the hypergeometric differential equation

$$\begin{aligned} &z\,(1-z)\,\partial_z^2 h(z) + \left[(|n|+1) - (|m|+|n|+2)\right]\partial_z h(z) \\ &-\tfrac{1}{4}\left\{2\,(|mn|+|m|+|n|) + m^2 + n^2 - \left[\omega + \tfrac{1}{4}(m+n)\right]^2 - \tfrac{2}{3}\lambda\right\} h(z) = 0. \end{aligned} \quad \text{(A.35)}$$

Solutions $h(z) = F(a, b, c; z)$ are hypergeometric functions dependent on the following parameters:

$$a, b \equiv \tfrac{1}{2}\,(|m|+|n|+1) \mp \sqrt{\tfrac{1}{6}\lambda + \tfrac{1}{4}\left[\omega + \tfrac{1}{4}(m+n)\right]^2 + \tfrac{1}{4}} \quad \text{and} \quad c \equiv |n|+1, \quad \text{(A.36)}$$

where, for regularity at $z = 1$, either a or b must be a negative integer. Hence

$$\tfrac{1}{2}\,(|m|+|n|+1) - \sqrt{\tfrac{1}{6}\lambda + \tfrac{1}{4}\left[\omega + \tfrac{1}{4}(m+n)\right]^2 + \tfrac{1}{4}} \equiv -s, \quad \text{with} \quad s \in \{0, 1, 2, \ldots\}. \quad \text{(A.37)}$$

The eigenfunctions of the (rescaled) d'Alembertian are therefore given by

$$\Psi_{smn}(t, z, \chi_2, \chi_3) \equiv z^{\frac{1}{2}|n|}\,(1-z)^{\frac{1}{2}|m|}\, F_{smn}(z)\, e^{-i\omega t}\, e^{\frac{3}{4} i m \chi_2}\, e^{\frac{3}{4} i n \chi_3} \quad \text{(A.38)}$$

and correspond to the eigenvalues

$$\lambda_{smn}(\omega) = 6l(l+1) - \tfrac{3}{2}\left[\omega + \tfrac{1}{4}(m+n)\right]^2, \quad \text{with} \quad l \equiv s + \max\left\{\tfrac{1}{2}|m+n|, \tfrac{1}{2}|m-n|\right\}. \quad \text{(A.39)}$$

The hypergeometric functions $F_{smn}(z) \equiv F(a, b, c; z)$ are dependent on the integers $s \geq 0$, m and n through the parameters a, b and c.

[2]Note that we have continued to use $|m|$ in the $\Phi^{\pm}$ eigenfunctions, despite the fact that m is positive, so as to allow for this generalization to the m negative case.

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