

# Promoting Understanding in Mathematical Problem-Solving through Writing: A Piagetian Analysis

by

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A thesis presented for the degree of  
Doctor of Philosophy  
in the Department of Mathematics and Applied Mathematics  
University of Cape Town  
April 2007

# **Promoting Understanding in Mathematical Problem-Solving through Writing: A Piagetian Analysis**

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## **Abstract**

This thesis describes research which suggests that writing about mathematical problem solving processes increases understanding. Piaget's theory of learning, employed, apparently, for the first time in a mathematical problem-solving context, is used to model the process by which such improved understanding is achieved.

Mathematics graduates are expected to be competent problem solvers, yet mathematics undergraduate students are primarily taught algorithms, leaving problem solving to be learnt tacitly. Teaching problem solving explicitly can require significant restructuring of a mathematics course, in content or in resources. The aim of the research project was to investigate the possibilities of using writing as a tool for reflection in mathematical problem solving, without requiring changes to course content or to the physical context of the classes.

The research was carried out in the context of a large first year university mathematics course, at the University of Cape Town. The experiment was carried out in three tutorial groups, two experimental and one control, with the author as tutor. The two experimental groups, in addition to standard tutorial requirements, were required to write explanatory paragraphs on the problem solving processes. One experimental group was required to write about problem solving before carrying out calculations, and the second experimental group wrote about problem solving processes after carrying out calculations.

Data was collected in the form of interviews, submissions of written material and standard assessment tasks. The interviews drew attention to the issue of understanding, particularly to the forcing of understanding which might not have occurred in the absence of the writing activity. The students' written submissions supported the inference of deeper mathematical engagement.

Investigation was undertaken to ascertain whether the writing activities differentially advantaged different language groups, or students with different levels of mathematical preparedness. Results suggest that the writing exercises were similarly advantageous for all student groupings, although second language English speakers found the task more challenging than English main language speakers.

The equitable nature of writing and the suitability of writing assignments for inclusion in an existing course, without substantial changes to the course, strongly recommend the use of writing as an activity to promote mathematical understanding.

### **Acknowledgements**

I owe a debt of gratitude to my supervisors. Professor Tim Dunne, in the Department of Statistical Sciences, UCT, provided strong guidance and was a source of many fine ideas. Professor John Webb, in the Department of Mathematics and Applied Mathematics, provided me with access to the class within which the study took place, and was always ready to listen when I needed to talk.

I have to thank two Tanzanian students in my 2006 MAM1000W class, Max Mwasilu and Cecilia Augustine, for providing me with insight into the current situation of Swahili as a language of learning and teaching in Tanzania.

Acknowledgements are due to the Multilingualism Education Project at UCT for providing statistics on languages spoken at UCT.

I thank Dr Jessica Tiffin the English Department at UCT for our weekly discussions on my data analysis process during the period of the interview analysis and assessment data analysis. It was a fruitful period for my research.

Lastly, I thank my husband, Patrick Vine, for his continual support. We met just before I registered for this degree, before I had any idea of what the final product would look like. Now we are married, with the world's most wonderful little girl, and we all need to get used to Life Without The Thesis, whatever that may be.

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# 1 Introduction

The writing of explanatory strategies in the context of mathematical problem-solving increases students' understanding of mathematical topics and deepens their engagement with the problems. The act of writing about mathematics, incorporated into a mathematical activity, makes cognitive and metacognitive demands of the students, requiring them to engage more deeply with the mathematical content than they might otherwise be encouraged to do by the problem requirements, ultimately resulting in the probability of a more insightful problem-solving process and greater understanding of mathematics.

The research project described in this thesis began life as an attempt to improve students' problem-solving skills by encouraging the students to reflect, by writing, on their problem-solving processes. The first step in a successful problem-solving endeavour is to understand the question and to understand the mathematical underpinning of the problem. Results of the research carried out in the writing study project suggest that the writing had at least a self-perceived beneficial effect on students' mathematical understanding and caused them to engage more deeply with the problems than students who were not taking part.

Evidence of the writing exercises undertaken by the students having a prominent effect on the students' problem-solving processes, beyond increased understanding, was not observed, perhaps because the writing intervention was insufficiently incorporated into the introductory course which served as the context of study, and was not summatively assessed. The research project that began with the optimistic intention of analysing what effects writing had on problem-solving skills ended with an analysis of the process by which writing encourages deeper engagement with and understanding of a mathematical problem. The theory by which this process of engagement was modelled is Piaget's stage independent theory of learning, most especially his differentiation between alpha behaviour (the creation of unstable knowledge structures) and beta behaviour (the creation of robust knowledge structures).

## 1.1 Research question

- What effect does the writing of explanatory strategies have on mathematical problem-solving?

Subquestions include

- Are any observed effects of writing in problem-solving different for students with differing main languages?
- Are any observed effects of writing in problem-solving different for students with differing degrees of mathematical preparedness?

While many studies have been carried out on the usefulness of writing within mathematics, the use of writing explicitly as a tool within mathematical problem-solving has not been well explored. It was considered that the metacognitive activity required in the reflective act of writing would have a beneficial effect on problem-solving processes and abilities. In addition, the use of Piaget's stage independent theory of learning in modelling the observed effects of the writing exercise has not been observed in the literature of mathematical problem-solving, or indeed widely in the field of mathematics education at all.

## **1.2 Rationale**

The author is a lecturer of first and second year mathematics at university level. The rationale for the research question explored in this thesis arose jointly from educational practice and from educational theory of both problem-solving and writing. It was observed in the classroom and in assessment tasks that many students either did not possess much skill in problem-solving or possessed low levels of self confidence in their ability to solve problems. It was simultaneously observed that the first year course, the vehicle for the writing project, did not explicitly teach problem-solving, instead it taught many algorithms and mathematical recipes. In conflict with the content of the course, the lecturers occasionally set problems that demanded a high level of problem-solving ability from the students. It was the aim of this study to address the occasional imbalance between what was taught and what was assessed by creating a course activity which would improve problem-solving skills.

A mathematical problem can be defined as "a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one" (Schoenfeld, 1985, p. 74) and mathematical problem-solving simply as the solving of such problems. Problem-solving is distinct from using well-worn algorithms for solving exercises, the former being much more difficult to teach than the latter. Teaching mathematical problem-solving is a matter of some concern to

mathematics educators, and, while such teaching (and associated concern) will have been occurring throughout the history of mathematics (see, for example, Stanic and Kilpatrick, 1989), serious scrutiny of the processes of problem-solving first influenced the greater educational community with the publication of George Pólya's (1945) *How To Solve It*. In *How To Solve It*, Pólya breaks the problem-solving process down into the four steps of

- understand the problem,
- devise a plan,
- carry out the plan and
- look back;

four steps that have been quoted repeatedly in the problem-solving literature. In addition to the four step model of problem-solving, a model which has proven influential, Pólya provided a list of heuristic strategies, rough guides to how to respond to particular situations. Pólya's heuristic strategies, or simply heuristics, have proven possibly even more influential than the four step problem-solving process.

In the 1980s and early 1990s problem-solving became a regular topic amongst mathematics educators with major works such as Alan Schoenfeld's *Mathematical Problem Solving* (1985) laying firm groundwork in the domain of teaching problem-solving, likewise edited volumes such as Silver (1983) *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* and Charles and Silver (1989) *The Teaching and Assessing of Mathematical Problem Solving*. The US National Council of Teachers of Mathematics has also played a role in the increasing attention to problem-solving during the 1980s and early 1990s (see Schoenfeld, 2008, in press, for a good overview).

Specific areas within the field of problem-solving which have received focussed interest include metacognition and expert-novice distinctions, both issues of interest in this thesis. Metacognition can be defined both as knowledge about cognitive phenomena, and monitoring and regulation of cognitive phenomena (Brown et al, 1996; Schoenfeld, 1985; Garofalo and Lester, 1985) with rather less frequent attention being given to a further definition of metacognition as beliefs and affects and their effects on performance (Schoenfeld, 1992). The writing exercises used to develop problem-solving behaviour required the students to reflect on their problem-solving processes, and in so doing to invoke metacognitive processes in their manifestation as declarative

knowledge of cognitive processes. The initial intent of the required reflection on cognitive problem-solving processes was to improve metacognitive monitoring and control. The observed positive effect of the activity was that reflection compelled deeper engagement with the mathematical material than might have been the case without reflection. Metacognitive control was not enhanced so much as active cognitive engagement with mathematics deeper than simply carrying out algorithms.

Expert-novice studies which have been particularly influential on this thesis are located in the subject areas of both physics and mathematics. Physics expert-novice studies have found that experts are less likely to be swayed by surface features (although they can, in fact, be swayed), tending more to recognising underlying physics principles. In addition, physics experts are less likely than novices to begin calculation before deciding on a general solution schema (Leonard et al, 1996; Chi et al, 1981; Hardiman et al, 1989). Mathematics expert-novice distinctions tend to highlight the use of heuristic strategies by experts, the greater extent of metacognitive monitoring and regulation than novices and the greater tendency of experts to reflect on a solution and thereby learn from it (Lester, 1994; Schoenfeld, 1992). Expert-novice studies formed part of the grounding for the writing study project, drawing on both mathematics and physics studies. The physics studies influenced the study project by inspiring the activity of writing explanatory strategies, a device used to great effect by Leonard, Dufresne and Mestre (1996) and Leonard, Gerace and Dufresne (1999), while the mathematics studies suggested that the expert-like strategies to be encouraged in the students were use of heuristic strategies, metacognitive control and reflection. In effect it was Pólya's steps of *devise a plan* and *look back* which were the focus of the writing exercises.

The study project described and discussed in this thesis involved requiring students to write explanatory paragraphs on their problem-solving processes, preferably in English sentences, using as little mathematical symbolic notation as possible. Immediately there are two language issues which must be examined. The first is the concern of having designed a learning initiative which might benefit speakers of English as a main language over speakers of other languages, and the other is the problematic embedding of the symbolic and precise technical language of mathematics within the verbal language of English.

South Africa has eleven official languages, all of which, and more besides, are spoken at the University of Cape Town (Adler, 2001; Moodley, 2000). It was one of the intents of the study project to determine whether any effects of the writing exercises involved in the project had a differential effect (either positive or negative) on speakers of different main languages. The final conclusion was that students' main languages did not affect their learning from the writing exercises, which was most encouraging. One aspect of language which did have an effect on some of the students, however, was the technical nature of use of the mathematics register. In many ways, mathematics is a language in its own right, with its own vocabulary, grammar and logical rules (Ellerton and Clements, 1991; Ellerton and Clarkson, 1996; Pimm, 1995). Students of mathematics need to begin to learn the mathematics register and become proficient in its use, a task some students find easier than others, and one that is confused by the necessity of teaching the precise language of mathematics within the less precise language of learning and teaching, in this case English. The writing exercises necessarily required the students to talk about mathematical processes using mathematical terminology, a task some students found daunting, to the point of being unable to begin the process, and others enjoyed, for allowing them to improve their fluency in the register.

A thorough overview of the intertwined concerns of mathematics and language can be obtained from Ellerton and Clements (1991) *Mathematics in Language: A Review of Language Factors in Mathematics Learning* as well as the edited volume of Connolly and Vilardi (1989) *Writing to Learn Mathematics and Science*. Despite more than a decade since either of those books was written, the issues they discuss and the questions they raise are still valid and pertinent. There is widespread support for the usefulness of writing in mathematics in general and in mathematical problem-solving in particular, but the reasons given for the perceived usefulness are not unanimous. There is the school of thought that writing and problem-solving involve exactly the same steps, essentially Pólya' problem-solving steps, and therefore combining the two activities simultaneously brings advantage to both (Kenyon, 1989; Mendez and Taube, 1997). There is the constructivist viewpoint that writing about mathematics requires the writer to form associations and construct cognitive knowledge structures in order to communicate thought processes in an understandable form (Ellerton and Clements, 1992; Sierpinska, 1998). Then there are the supporters of the metacognitive advantages of writing through processes of reflection, monitoring and reaction (Pugalee, 2001;

Kenyon, 1989). All three viewpoints have support and it is entirely possible that all are correct, although Ellerton and Clements repeatedly call for more thorough research to support all the facets of the writing process and the phenomena it is alleged to support.

All three views played a role in the project described in this thesis. The writing under scrutiny in the study project aimed to enhance students' problem-solving abilities by simultaneously addressing cognitive issues such as sense making and metacognitive issues in the form of declarative knowledge of cognitive processes. The writing required the students to describe their solutions strategy in written form and to reflect on their problem-solving processes, directly tying the activities involved in the writing process to Pólya's problem-solving steps of *devise a plan* and *look back*. The results of the project indicated that the enforced deeper engagement with the mathematics involved in the questions about which the students had to write, brought about improved understanding of the mathematics, specifically understanding that classmates not taking part in the writing study project did not always achieve. The observed effect of deeper engagement and improved understanding has been fruitfully analysed using a Piagetian constructivist learning theory.

Over the last few decades, it has become increasingly apparent that university classes are becoming more diverse in language, culture, expectations and academic preparedness (Zevenbergen, 2001; Wood, 2001). Reasons for the increase in diversity abound, the most obvious being the simple increase in numbers of school leavers committing to higher education. One measure of diversity, that of preparedness, has not received much research attention, the primary reaction in universities being to create bridging courses for the less well-prepared students. Even allowing for bridging courses, it is apparent that large mainstream classes still exhibit a wide range of mathematical preparedness. Turning to the literature for ideas about how to equitably teach such a class one can find few suggestions, and those that are found vary widely in form and in applicability. Suggestions for teaching large, academically diverse classes include fully integrated use of multimedia (Cavender and Rutter, 1997), overarching changes to classroom philosophy and epistemology (Northedge, 2003) and development of specific teaching and learning activities, of which explanatory writing is an example (De La Paz, 2005). The study project aimed to determine whether the writing of explanatory strategies in mathematical problem-solving was a suitable activity for a

class diverse in mathematical preparedness or whether it advantaged one group of students over another.

Jean Piaget (1896 – 1980), best known for his theories on childhood developmental stages, proposed a constructivist theory of learning through which the epistemic subject learns by the growing and changing of knowledge structures. Suggested readings, among the wealth of Piaget's work, are Piaget (1972) *The Principles of Genetic Epistemology*, translated from *L'Épistémologie Génétique* (1970) and Beth and Piaget (1966) *Mathematical Epistemology and Psychology*, translated from *Épistémologie mathématique et psychologie* (1965). Later in Piaget's career, in *L'Équilibration des Structures Cognitives* (1975), with 1978 (*The Development of Thought – Equilibration of Cognitive Structures*) and 1985 (*The Equilibration of Cognitive Structures*) translations, he embedded his cyclic model of successful learning, involving such concepts as accommodation, assimilation and equilibration, in a three-pronged interpretation of potential learning modes. As the analysis of observations of the study project will reveal, Piaget's model of learning was found to be satisfactorily descriptive of the processes exhibited by student problem-solving practices.

Lev Vygotsky (1896 – 1934) is the major influence in sociohistoric and sociocultural epistemologies. Vygotsky's work was concerned at all times not on the individual learner, but with the social aspects of learning, and how knowledge is socially mediated and constructed. Communication, therefore, plays a central role in Vygotskian studies, and writing is a form of communication. Vygotsky's (1962) *Thought and Language* (edited and translated from the Russian original *Myshlenie i Rech*, 1934) presents Vygotsky's support of writing in the learning process, for its characteristics of abstraction, demand for constructive thought and as a mode of communication. Vygotsky's claim for the constructive use of writing in learning combines successfully with Piaget's constructivist theory of learning to support the place of writing in learning mathematical problem solving.

The course within which the writing experiment took place was a year long course, divided into two semesters. The writing initiative was carried out in the second semester. The class was large, approximately 500 students divided into two smaller classes of approximately 250 students. The class further divided into afternoon tutorial groups of approximately 30 students, which even so are larger than some groups used in

problem-solving teaching experiments (Schoenfeld, 1985). Decreasing class size was not an available option. Tutorial classes were limited by the number of available venues as well as the number of available tutors. The course was a preparatory course for second year mathematics, applied mathematics, first and second year physics, chemistry and economics. It was a compulsory course for all Bachelor of Science students as well as Actuarial Science students. Decreasing course content was also not an available option, as every topic covered in the course is important, and lecturers of other courses even sometimes express the wish that more could be taught in the course. It was the aim of the study to design an intervention which could be added to the existing course without a need for content changes, and which would in some way enhance students' problem-solving abilities.

It was determined that such an intervention could not easily be carried out in the lectures of 200 to 250 students. Instead, the writing exercises were introduced in the tutorial classes, of which three (out of approximately twelve) were run with the author as tutor. Two of the three tutorial classes were experimental groups, running two different versions of the writing experiment, and the third was the control, identically run to the remaining non-experimental tutorial groups. One of the experimental groups (the A, After, group) were required to write about the problem-solving processes after having carried out a problem calculation, and the other group (the B, Before, group) were required to write about the planned problem-solving process before carrying out problem calculations. The After group had one extra element, which was to make a brief statement of expectation (a few symbols, a short phrase) on what form the problem solution was anticipated to take. Data was collected on the students in the form of interviews, the written exercises submitted over the course of a semester, and quantitative analyses of the students' assessment tasks throughout the academic year.

As shall be elucidated, the data revealed that the facet of the problem-solving process which the writing initiative particularly supported was Pólya's first step of *understand the problem*; more specifically understand the mathematics underpinning the problem. The process by which the writing encouraged understanding has been fruitfully modelled using Piaget's stage-independent learning theory.

### 1.3 Chapter summaries

Chapter 2 covers the topic of problem-solving, defining both problems and problem-solving in mathematical terms and comparing those definitions to the corresponding terms in other fields, notably physics. The definitions of both terms are often omitted in the literature, to the detriment of the reader's understanding, since the terms are not always used unambiguously. Mathematical sense making and its associations with classroom epistemology are reviewed, as is the literature on metacognition. The expert-novice studies which have played such a significant role in the design of the writing study project are discussed, in the fields of mathematics and physics; the physics expert-novice studies having been influential far beyond the borders of physics and physics education. Some mention is made of heuristic strategies, and the issue of writing in problem-solving is touched upon, to be covered in more detail in Chapter 4.

In Chapter 3, the issue of language is explored. The contextual setting of the writing project was a course in which students embody a range of main languages, which has to be taken into consideration in a project where the students' use of non-symbolic language is being demanded. The history of language in education in South Africa is one permeated with tension. This history is outlined briefly and the current situation is described against a backdrop of the country's history. Language in education is one issue, but language in mathematics is another. The mathematics register and the simultaneous verbal and symbolic needs of mathematics make mathematics distinctive in any study of subject specific language demands. While the study found that main language played little if no role in students' experience of the writing exercises, it found that the technical requirements of mathematical language did play a role, in both positive and negative ways.

Chapter 4 describes the forms and purposes of writing in and about mathematics. The links between writing and problem-solving are discussed, such as the encouragement of constructive learning through writing, the metacognitive activities encouraged by writing, and the similarities of writing to problem-solving and hence the efficacy of combining the two. Writing in mathematics or writing about mathematics can take many forms and can serve many purposes. In Chapter 4, the various forms of writing (as opposed to purposes of writing) are discussed in brief, these forms being: writing about problem-solving processes, journals, essays, reports on journal papers, writing about mathematical concepts (in contrast to processes), problem posing, investigative projects

and projective techniques such as letter writing. The focus of this thesis is writing about problem-solving processes, although other aspects do exert influence, such as journal entry analysis being used usefully in the writing exercise analysis in this project.

The diversity of mathematical preparedness of first year university students is discussed in Chapter 5, a situation found all over the world and, arguably, increasingly in the last couple of decades. A variety of approaches to take when teaching large classes of academically diverse students is summarised, including using multimedia, fundamentally altering the classroom epistemology and philosophy of learning, and using tools such as writing to promote equitable learning.

Piaget's theory of learning is described in detail in Chapter 6, listing and explaining the potentially confusing terms associated with it, such as assimilation, accommodation, equilibration, perturbation and schemes. Piaget's three-pronged description of learning responses, within which the learning theory can be embedded, is further elucidated. With a brief review of the comparisons between Piaget and Vygotsky, Vygotsky's views on the utility of writing as a tool for learning are given. As an acknowledgement of the presence of radical constructivism in the modern literature on Piagetian theories, a short outline of the theory is given, although it is not directly a topic of this thesis.

The research design of the writing project is described in Chapter 7. The population and sample population are described, as well as the process by which the different experimental groups were defined. The chapter describes what was required of the experimental groups as well as what data was collected. The ethical considerations, and subsequent consent requirements, of the experiment are discussed.

The data analysis of the three forms of data (interviews, writing exercises and problem-solving assessment tasks) is described in Chapter 8. The primary form of informative data was the interviews, with observations drawn from the interviews backed up by observations in the other data types, primarily the writing exercises themselves. A journal was kept throughout the duration of the project, and journal entries made at the time, or shortly after, the writing exercises took place are drawn on to deepen observational findings.

The appropriateness of Piaget's stage independent theory of learning to model the observations described in Chapter 7 is argued for in Chapter 9. Piaget's model is three-pronged, of which one prong (beta behaviour) is often analysed in isolation from the other two in the literature. It is vital to the application of Piaget's theory in this thesis that all three prongs be present. It is an argument of this thesis that writing reflectively on problem-solving processes causes, perhaps forces, beta behaviour when alpha behaviour is more likely to be the outcome in the absence of forced deeper engagement. In the conclusions drawn in Chapter 9, the findings of the study are summarised, constraints of the study are clarified and potential for further study is suggested.

The Appendices include

- an anonymous list of the students involved with all data pertaining to them,
- all the tutorial questions about which students were asked to write explanatory strategies,
- the ethical code and consent form,
- the test and examination questions used as indicators of problem-solving skills,
- the interview questions and the individual analyses and selection of students' original written exercises used as illustrations of observations discussed in the thesis,
- examples of actual student written submissions, and
- the quantitative assessment data and data analysis.

## 2 Problem-solving

Interest in solving mathematical problems, given in a variety of forms, has a long history. Stanic and Kilpatrick (1989) give examples from ancient Chinese and Egyptian texts of what amount to both algebraic and word problems. Schoenfeld (1989) notes that the view that “mathematics helps you think” (p. 84) is one that is widely held, and it certainly goes back a long way, at least as far as Plato: “those who are by nature good at calculation are, as one might say, naturally sharp in every other study” (Grube, 1974, cited in Stanic and Kilpatrick, 1989). In the 19<sup>th</sup> Century it was part of “mental discipline theory” (Stanic and Kilpatrick, 1989) that mathematics and the classic languages were the best vehicles for developing the mental faculties of memory, understanding and reasoning, among others. Schoenfeld (1992) considers that the Platonic view that training in mathematics helps you to be a good thinker in general was only challenged in the early 20<sup>th</sup> Century. While one can find strong signs of interest in theories of thinking and learning in the work of Descartes, Leibnitz, and Bolzano (Schoenfeld, 1992; Stanic and Kilpatrick, 1989) such theories did not develop into an empirical discipline until the late 19<sup>th</sup> Century and the development of a true field in mathematics education in the 20<sup>th</sup> Century. Stanic and Kilpatrick (1989) list a variety of texts from the late 19<sup>th</sup> and early 20<sup>th</sup> Centuries which present problem-solving in the form of supplying the learner with lists of problems to work through, with the solutions available elsewhere, an interpretation of problem-solving still around in 2007, to the exasperation of researchers of mathematical problem-solving. In 1945 there was a surge of interest in studies of thought and learning, especially in the context of mathematics, and several influential books were published in that year, notably Pólya’s *How To Solve It*, but also works by Hadamard, Duncker and Wertheimer. The constructivist epistemology promulgated by Piaget was catching the interest of theorists in the field and interest in problem-solving and theories of learning started to take off.

A major part of Pólya’s importance in the problem-solving field is the fact that he was the first problem-solving theorist whose work influenced school curricula (Stanic and Kilpatrick, 1989). Pólya’s list of “heuristics” or “heuristic strategies” for the problem-solving process is widely quoted in the problem-solving literature. The term *heuristic* is usually used as an adjective (Oxford English Dictionary), such as in *heuristic process*, *heuristic technique*, and so on, to the point where the adjective form is used to define the noun form. *Heuristic (adjective): serving to find out or discover. Heuristic (noun): A*

*heuristic process or method for attempting the solution of a problem; a rule or item of information used in such a process* (online Oxford English Dictionary, consulted 12 December 2006). Pólya's (1945) broad brush heuristics for the problem-solving process are

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back

Pólya's work on problem-solving forms the firm foundation for much of the problem-solving research carried out in the last few decades, notably the body of research carried out by Schoenfeld (1985 among others). Recent collaboration between mathematics educators, educational psychologists and cognitive psychologists to develop programs of study dealing with cognitive learning skills, has broadened the field far beyond a focus on mathematical problems in isolation (Kenyon, 1989). The 1980s in particular were fruitful years for the study of problem-solving (the "problem-solving bandwagon": Ellerton and Clements, 1991, p. 57), with much research being done in the United States, an interest that has waned somewhat over the past decade due to multiple factors. The history of problem-solving research in the US is an interesting one which has been covered fully elsewhere (Lester, 1994; Schoenfeld, 2008 in press).

In the last half century, particularly since the 1980s, much progress has been made on the theory of mathematical problem-solving. Schoenfeld (2008, in press) concludes that problem-solving research over the last 3 decades has achieved the following: recognising the importance of problem-solving, recognising the usefulness of heuristics and associated substrategies, determining that the use of heuristic strategies can be taught, great advances in research on metacognition as well as belief systems and on experience as a shaper of beliefs. Schoenfeld (2008, in press) and Lester (1994) feel there is still much work to be done, such as how much and what kind of practice is needed for students to learn a wide range of problem-solving strategies, as well as a focus on issues such as assessment and transfer of learning. Lester (1994) expressed concern over the waning of interest in problem-solving which he blames on one or more of (a) other issues taking attention away, (b) we think we know all about problem-solving, (c) constructivism has replaced problem-solving as the mathematical ideology, sometimes erroneously being used as a synonym for problem-solving, and (d) problem-solving is even more complex than previously thought. Schoenfeld (2008, in press)

agrees with (a), suggesting that the surge of interest in socio-cultural influence on mathematics and mathematics learning has drawn much attention away from problem-solving.

Is mathematical problem-solving important? Stanic and Kilpatrick (1989) report contrasting views in the literature as to why problem-solving is considered to be important, although it is widely regarded as being so. They stress the importance of Dewey's view (1920, 1933, 1963) that reflective thinking (problem-solving) is crucial to everyone's education, not only those destined to become mathematicians. Despite the development of the field of mathematics in the 20<sup>th</sup> Century there remains conflict among educators and the population in general as to what uses mathematics serves, what uses problem-solving serves and what kinds of mathematics should be in the school syllabus and for which pupils (Stanic and Kilpatrick, 1989). Resnick and Glaser (1976) argue that "a major aspect of intelligence is the ability to solve problems, and that careful analysis of problem-solving behavior constitutes a means of specifying many of the psychological processes that intelligence comprises" (p. 205), where they define intelligence as "the ability to learn" (p. 205). Carlson (1999) reports that being a successful mathematics graduate is considered as synonymous with being a good problem solver, a perception which is unfortunately not backed up by observations.

In the author's experience, problem-solving is not a skill customarily taught in the mathematics classroom. However, if problem-solving is as important as many people believe, and if a mathematics graduate is expected to be a good problem solver, then why isn't problem-solving taught explicitly? The answer is as simple as the subject is complicated: teaching mathematical problem-solving is very difficult, in fact, it is not truly understood how to teach it successfully at all. By its nature, problem-solving involves learning a variety of vaguely defined rules of thumb, for application to unspecified and unspecifiable problems. Teaching algorithms (You see Problem *X*, you use Algorithm *Y* to solve it) is so much easier, gets through so much more syllabus in a given period, and (this is important) fits so much more easily into most teachers' and students' views and beliefs of classroom culture that algorithms form the bulk, if not the entirety, of any traditional mathematics course. To complicate matters, the terms *problem* and *problem-solving* are encountered with multiple definitions, often given only implicitly. Before embarking on any study related to problem-solving it is important that the terms be defined clearly (Schoenfeld, 1992).

## 2.1 Problems, defined

Schoenfeld (1985) defines the term *problem*:

Problem: A task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one. . . . To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem.

(Schoenfeld, 1985, p. 74)

To similarly define a problem, Kenyon (1989) cites Bell (1978) as describing a problem by four criteria:

- a person must be aware of the situation
- the person must recognize that the situation requires action
- the person must want or need to act, and must actually take action
- the resolution must not be immediately obvious to the person.

(Kenyon, 1989, p. 75)

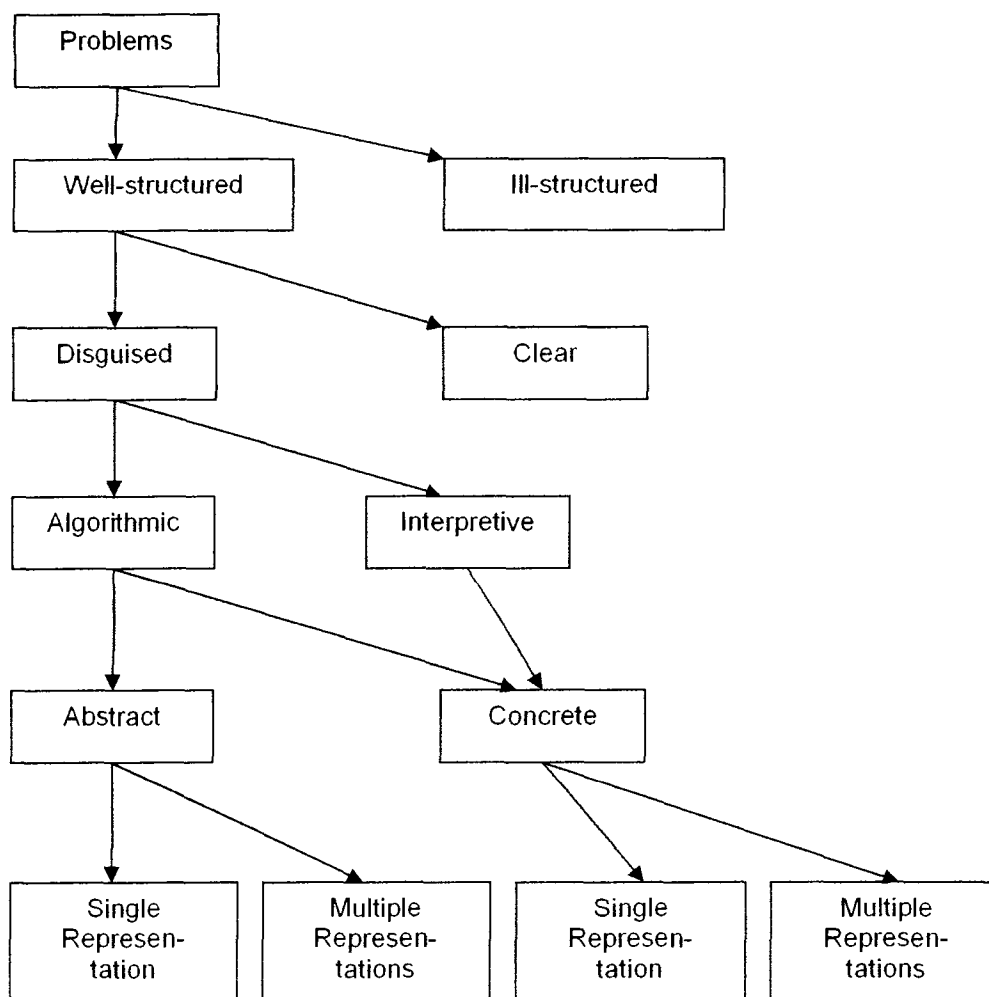
To illustrate with an example, suppose Newton's Method of locating the roots of a function is familiar to the student, in that it has been covered in class. A question in a test, requiring the student to find a root of a non-factorisable polynomial is not a problem for the student, merely an exercise, even if the student is not proficient with Newton's Method, and makes errors when attempting it. In the case where a student has never heard of Newton's Method, however, but has the tools necessary to find the root (with a hint, perhaps, depending on the grade of the student) the question becomes a problem, not an exercise.

Schoenfeld continues (1991) to define "good" problems as being relatively accessible, which does not necessarily mean easily solvable. Good problems have a number of different solution routes. The problems and their solutions should relate to important mathematical ideas and should lead into mathematical exploration. Carlson (1999) finds that repeated exposure to challenging and complex problems is one aspect, among several, encouraging students to enjoy mathematics and continue tertiary and postgraduate studies in mathematics. Pólya observed that "although routine problems can be used to fulfil certain pedagogical functions of teaching students to follow a specific procedure or use a definition correctly, only through the judicious use of nonroutine problems can students develop their problem solving ability" (Stanic and Kilpatrick, 1989, p. 16).

The use of the term *problem* often brings to mind disguised well-structured problems, as illustrated by the schematic diagram found in Craig (2002). The diagram is reproduced

below and suggests a framework within which different types of problems can be located. Any problem set in a typical school textbook or examination is a well-structured problem in that it is known to have a single solution, accessible by the solution techniques known to the person attempting the problem (or exercise). Disguised problems are ones which are not, at first glance, set in an algebraic format, typically being represented by a paragraph of non-algebraic text. As such, “word” problems are disguised well-structured problems, and, it may be suggested, tend to be the problems people think of when considering problem-solving. In particular, concrete problems are often the only problems considered, that is, ones that are set in a realistic physical context. In fact, problem-solving, as a mathematical skill, can be applied to problems on any branch of this framework. In particular this thesis is concerned with clear problems, not disguised ones; problems that are presented in an algebraic format, albeit in a way that is possibly strange to the student. For a detailed discussion of this framework see Craig (2001).

**Figure 2.1 Problem classification framework**



A brief illustration of a problem/exercise that can be located within this structure follows:

Disguised, Algorithmic, Concrete, Simply represented

A cooldrink container is made in the shape of a cylinder. The cylinder is made from a rectangle wrapped to form the side of the cylinder and two circles to form the top and bottom. The two circles are cut from two squares, with the cut away bits being wasted. The cost of the metal is  $10\text{c}/\text{cm}^2$ . If the volume of the container is to be 340ml, calculate the dimensions of the container that will minimize the cost of the metal used, including wastage.

This problem is *disguised* in that no clear algebraic formulae are given, instead they need to be developed by the student. The problem is *algorithmic* since it requires a calculation to be carried out and a quantitative answer to be given. The context of the problem is *concrete* since it is expressed in physical, realistic terms, and it is *simply represented* since it is represented solely in words, with no diagram. The “identical” problem could be altered, to be *clear* by giving the equations involved, to be *abstract* by making no reference to any realistic connection to the real world, and to be *multiply represented* by including a diagram.

The definition of a mathematical problem as a mathematical question which is accessible to the problem solver but whose method of solution is not immediately clarified, is the one that will be of use in this thesis. However, it must be noted that the definition of a problem in the physics education literature is quite different to that of a mathematical problem. The reasons for making special note of the physics definition of a problem are the impact that expert/novice studies (Larkin et al, 1980; Chi et al, 1981; Hardiman et al, 1989) have had in the education literature, as well as the impact that physics education literature has had on the concept of writing explanatory strategies as part of the problem-solving process (Leonard et al, 1996). In the physics education literature, a problem is invariably a word problem describing a physical situation where the problem solver is expected to identify appropriate governing principles, set up the associated equations and solve the problem with a calculation (Boote, 1998; Leonard et al, 1996). The difference in definition between the two kinds of problem (mathematical and physical) has far reaching consequences, especially for the writing study project described in this thesis. When attempting a mathematical problem, by definition the problem solver does not know how she is going to solve the problem until some work has been done on it. A physics problem, on the other hand, is necessarily one where, once the underlying physics principles and concepts have been determined, a plan of action is available to the problem solver before problem exploration is embarked on. An additional difference between mathematical and physical problems is the existence of a definable concept underlying the physics problem (for example: conservation of

momentum) while mathematical problems rarely have such a characterisable underlying principle. As will be made clear, the writing study project involved having students write about their problem-solving processes both before and after carrying out problem calculations. The study of writing *before* calculation is comparable to some extremely fruitful work which has been carried out in physics (Leonard, Dufresne and Mestre, 1996; Leonard, Gerace and Dufresne, 1999), while disallowing the possibility of problems being true *mathematical problems*. The study of students writing about problem-solving *after* carrying out calculations shifted the work away from the methodology of the physics research, while allowing for the possibility of true mathematical problems.

While problems are often not explicitly defined and their characteristics have to be inferred from the definition of problem-solving, it can also occur that problem-solving itself is not explicitly defined. It is often taken as given that the reader will understand what is meant by *problem-solving*, an assumption which is not a safe one to make, given the variety of definitions in use.

## **2.2 Problem-solving, defined**

A short, to-the-point, definition of problem-solving is difficult to achieve; as Schoenfeld (1991) points out “ask seven mathematics educators to define problem solving for you, and you're likely to get at least nine opinions” (p. 4). Other terminology is used to refer to the same, or related, skills, such as “thinking mathematically” (Schoenfeld, 1985, 1991) and “reflective thinking” (Dewey, 1910, cited in Stanic and Kilpatrick 1989). Often the definitions of both problems and problem-solving are left implicit, which can be problematic. Stanic and Kilpatrick (1989) divide views on problem-solving into three separate schools: problem-solving as context, problem-solving as skill and problem-solving as art, a trio of characterisations that can be used for locating a number of definitions or approaches in the literature. It is the view of the author, however, that the three categories need not be regarded as going from bad to better, but that each has a legitimate place in the teaching of mathematics. Indeed, it can be the case that a successful attempt to teach problem-solving in the classroom combines aspects from all three categorisations, illustrated, in the author's view, in Schoenfeld's (1985, and elsewhere) problem-solving course.

Within *problem-solving as context* are the subcategories

- problem-solving as justification for teaching mathematics at all,
- problem-solving as motivation to gain student interest in a greater mathematical topic,
- problem-solving as recreation,
- problem-solving as a vehicle through which new concepts or skills are learnt, and
- problem-solving as practice to reinforce skills and concepts taught more directly.

Word problems, in particular, are used as *justification* for teaching mathematics, in the sense that there is often a quest for relevance in a mathematics curriculum, and word problems allow the teacher or textbook writer to bring in problems that are ostensibly illustrations of real life problems. This view of the relevance of word problems is widely challenged (Gerofsky, 1996; Thomas and Gerofsky, 1997). For problem-solving as *motivation*, Stanic and Kilpatrick (1989) use an example, in a primary school class, of a problem involving adding by regrouping as an introduction leading to a lesson on algorithms for adding. Here the problem (possibly using manipulatives) is used as a means of attracting the students' interest in the concept of addition. Problem-solving as *recreation* considers the primary content of the curriculum as being algorithms, and the problems being ways to have fun with application of the algorithms.

The use of problem-solving as a *means through which various mathematical skills or concepts are taught* has ties with viewing problem-solving as an art. If problem-solving is viewed as holistic mathematical learning and a way of thinking mathematically, then within the context of problem-solving as the very nature of mathematics, problems can be used as the vehicles for introducing concepts, skills and algorithms that the curriculum requires taught (Santos-Trigo, 1998). After all, while it is an impoverished mathematics curriculum that teaches only algorithms, those algorithms still need to be taught, one way or another. Problem-solving as *practice* is the most common view of problem-solving: various concepts and skills are taught algorithmically and then various problems (often word problems, again) are given as the testing ground within which those algorithmic skills are applied. While some definitions of problem-solving as practice (Perkins, 1992) refer explicitly to standard textbook problems, problem-solving as practice need not be restricted to routine problems, but can, with great advantage, be extended to non-routine problems.

It is fitting in any discussion of problem-solving, especially problem-solving as context, that word problems be given special mention. In physics, a *problem* almost invariably refers to a word problem, simply because “problem” seems to be short for “verbally stated problem describing a realistic physical situation which students need to solve mathematically after identifying the governing physical concepts and principles” (inferred from Boote, 1998; Leonard et al, 1996, among others). Fruitful research has been done on physics problem-solving and associated expert-novice distinctions, research it is tempting (and hopefully possible) to extend to mathematical problem-solving, but the difference in problem definitions underscores the importance of making definitions explicit, rather than implicit. Kilpatrick (1983) reviews the study of mathematical problem-solving from 1958 to 1983. In his exploration of the techniques used in problem-solving and what we have learned from previous research, he keeps discussion fairly abstract and in general does not distinguish types of problems. However, he refers occasionally to solving word problems as synonymous with problem-solving. Indeed, he describes his own dissertation as being about problem-solving, yet the title of the dissertation was *Analysing the Solution of Word Problems in Mathematics. An Exploratory Study* (similarly Bell and Bell, 1985). The view that *problem-solving* and *solving word problems* are synonymous is widely held, a view that Schoenfeld (2008, in press) feels is partially the doing of textbook publishers, who are inclined against large changes in their publications, changes which would be called for if problem-solving were to be an overarching ethos in a textbook, rather than an inserted section of word problems at the end of each chapter. It is a narrow view of problem-solving that considers only word problems (Kenyon, 1989), and to restrict ourselves to such a view is to deny the potential richness of the subject.

*Problem-solving as skill* Stanic and Kilpatrick (1989) define as being a curriculum end in itself, rather than a means to some other end. In order to teach problem-solving as a skill it is necessary that a distinction be made between routine and non-routine problems, with the latter being regarded as requiring higher order skills than the former (Goos et al, 2002). Since, in this view, solving non-routine problems comes after having learnt to solve routine ones, a state which is reached after learning basic mathematical techniques and skills, it ends up being only the more capable students who get that far (Stanic and Kilpatrick, 1989); “the procedure by which a person uses *previously acquired knowledge and skills* to attempt to find a resolution, not immediately apparent, to a situation (problem) that confronts him or her” (Kenyon, 1989, pp. 75 – 76,

emphasis added). Leonard et al (1996) express concern that the view of problem-solving (in physics) as learning major principles and concepts and applying them to problems is based on the assumption that solving problems will bring about understanding of principles and concepts. This assumption is false; deep understanding does not come, even for proficient problem solvers (*ibid.*) in the physics sense.

If problem-solving is defined as “student performance ... on mathematical tasks where the solution or goal is not immediately attainable, and there is no obvious algorithm for the student to use” (McLeod, 1983, p. 267), that is as the *solution of problems* where problems themselves are defined as something non-routine or lacking direction as to best solution method, then is that problem-solving as skill or problem-solving as art? The answer is: it depends. It depends on how the problems are being approached as well as on the classroom culture and epistemology of mathematics being developed by the teacher or the class.

*Problem-solving as art* is, according to Stanic and Kilpatrick (1989) exemplified in Pólya’s heuristic strategies, but, more than that, his “deeper, more comprehensive view” of problem-solving (p. 15). Heuristic meant, to Pólya, “the art of discovery” (p. 15). Chapman (1997) lists a variety of problem-solving definitions, including “a method of inquiry” (citing Charles et al, 1987), “mathematical thinking” (citing Baroody, 1993) and “a description of mathematics” (citing the NCTM Standards, 1989) which all resonate with the idea of problem-solving as an overarching culture of the classroom, teaching a way of thinking, rather than a skill distinct from other skills. Chang and Weng (2002) similarly define problem-solving as “the ability to think critically, to reason analytically, and to create productively”. Schoenfeld (1992) is concerned that it is easy to simply repeat Pólya’s heuristic strategies, respect or even revere his work, yet ultimately end up ignoring his philosophy, thereby missing the point. It is very easy to reduce Pólya’s heuristics to algorithms and rules, and reduce problem-solving as art to problem-solving as skill (Stanic and Kilpatrick, 1989; Chapman, 1997).

### **2.3 Classroom culture, epistemology and sense-making**

The view of the nature of mathematics held by students is one which is acquired over the years through their experience of mathematics in the classroom (Schoenfeld, 1989). If the mathematics is taught as a set of procedures and rules for solving problems computationally and in little time, then it follows that the students will consider that

mathematics is a static body of knowledge which they can only replicate not create. They will not feel compelled to make judgements on their strategies or solutions to problems, and engaging them in a discussion on mathematical thinking will be difficult (Franke and Carey, 1997). With the culture of day to day rituals of learned algorithms comes a lack of sense-making (Schoenfeld, 1989).

For many undergraduates, problem-solving means learning the contents of a set of lecture notes and applying this knowledge to specific problems clearly related to the material taught. For research mathematicians, problem-solving is a more creative activity, which includes the formulation of a likely conjecture, a sequence of activities, testing, modifying and refining until it is possible to produce a formal proof of a well-specified theorem.  
(Tall, 1991, p. 18)

In order to encourage sense-making in mathematics, it is necessary to somehow make the culture of the classroom closer to that of the research mathematician, a task with obvious challenges and constraints. A sense of the discipline, of its Discourse (Gee, 1996, 1999), if you will, should be created in the classroom, rather than simply teaching the tools of the trade, in the form of (entirely necessary) algorithms and rules (Schoenfeld, 1989). Students should be encouraged to internalise that sense making, the “predilection to analyze and understand” (*ibid.*, p. 87) which is necessary for a mathematician. “Doing mathematics can be and should be an act of sense-making – and moreover ... the facts and procedures students learn in mathematical instruction should be a means to the end, rather than an end in themselves” (*ibid.*, p. 82). Bonotto (2002) is concerned that problem-solving as “solving word problems” adds to the lack of sense making as the supposed relevance of the word problems is sufficiently distanced from reality that the student is almost discouraged from making sense of them or their solutions. Bonotto suggests, as does Schoenfeld, that encouraging sense making requires far reaching changes in the teaching and learning environment. One of the changes he suggests (among several) is encouraging students to provide written descriptions of methods they use, providing an opportunity to induce reflection, as well as cognitive and metacognitive changes. Mathematical sense making as “modelling and symbolizing, communicating, analyzing, exploring, conjecturing and proving” (Schoenfeld, 1991, p. 7) remains a serious concern for mathematics educators (Schoenfeld, 2008, in press)

## 2.4 Metacognition

Metacognition can be defined in various ways, with potential confusion. Schoenfeld (1992) divides metacognitive processes into three categories: “(1) individuals’ declarative knowledge about their cognitive processes, (2) self-regulatory procedures, including monitoring and “on-line” decision making, and (3) beliefs and affects and their effects on performance” (p. 347). Often it is only the first two categories, that is, individuals’ awareness of their own cognitive processes, and the monitoring and regulation of these processes which are referred to in discussions on metacognition (Goos et al, 2002; Pugalee, 2001; Lester, 1994).

Garofalo and Lester (1985) consider it unfortunate that metacognitive processes are not made explicit in Pólya’s problem-solving framework. Successful problem-solving involves interplay between both cognitive and metacognitive strategies (Pugalee, 2001; Artzt and Armour-Thomas, 1992), although it can be difficult to tell one from the other. Artzt and Armour-Thomas (1992), acknowledging this difficulty of separating cognitive from metacognitive, divide the problem-solving process into episodes characterised as either cognitive or metacognitive or both. Lester (1994), in charting the progress of mathematical problem-solving research for the 25 years prior to 1994, recognises a growing trend to research metacognition and consider it an important facet of the problem-solving process: “By the end of the 1980s, metacognition not only was regarded as a force driving cognitive behaviors, but also was linked to a wide range of noncognitive factors – in particular beliefs and attitudes” (p. 666). Lester goes on to summarise what he considers to be three generally accepted metacognitive results:

- Effective metacognitive activity during problem-solving requires knowing what, when and how to monitor. The last aspect is very hard to teach.
- Teaching students to be more aware of their own cognitive and metacognitive activity should take place in a particular mathematical context, rather than generalising.
- The development of healthy metacognitive skills is difficult and often requires unlearning inappropriate metacognitive behaviours developed through previous experience.

In order to be a successful problem solver a list of heuristic strategies (or similar tools) and how to apply them is insufficient. Good executive skills and decision making ability is necessary in the form of analysing one’s own progress at solving the problem at

suitable moments and deciding whether there is a possibility of success or whether one should start anew with a different technique. This practice is variously referred to as metacognition (Brown et al, 1996), executive control (Schoenfeld, 1985), meta-level processes (Silver, 1987), reflective intelligence (Skemp, 1979 in Garofalo and Lester, 1985), cognitive monitoring and executive processes (Kitchener, 1983). Garofalo and Lester (1985) suggest that Piaget's *reflective abstraction* is "not unlike a metacognitive process" in that it is "a mechanism for extracting, reorganising and consolidating knowledge" (p. 164). Early definitions of metacognition tended to amalgamate all metacognitive aspects in one broad definition, rather than several more specific definitions such as Schoenfeld's, above, and those of Brown et al (1996), for example "Metacognition" refers to one's knowledge concerning one's own cognitive processes and products or anything related to them" (Flavell, 1976, p. 232) and "knowledge or cognition that takes as its object or regulates any aspect of cognitive endeavour" (Flavell, 1981, in Artzt and Armour-Thomas, 1992, p. 139). Garofalo and Lester (1985) find such definitions confusing, as it can be unclear whether any statement referring to metacognition is referring to knowledge of cognitive phenomena or regulation of such phenomena. The writing exercises described in this thesis are concerned with declarative knowledge of cognitive processes in contrast to the monitoring and regulation of cognitive processes.

## **2.5 Expert novice studies and distinctions**

Experts categorise problems differently to novices (Touger et al, 1995; Hardiman et al; 1989; Chi et al, 1981; Larkin et al, 1980), experts have different approaches to problem-solving (Schoenfeld, 1985) and have different conceptual understanding of their field to novices (Tall, 1991; DiSessa, 1987). Identifying contrasts between experts and novices, then attempting to encourage expert-like behaviour in novices, is a potential, although not guaranteed, way of bringing about change in problem-solving behaviour.

Two primary differences exist between the problem-solving processes of physics experts and physics novices. The first difference relates to categorisation. Experts tend to categorise problems by their underlying concepts, or by their deep structure. When provided with a collection of problems, experts group the problems by physical laws and principles. In contrast, novices tend to be swayed by surface features, and group all block-on-slope problems together, whatever the underlying principle might be. Hardiman et al (1989) required both novices and experts to categorise problems

according to which were most “similar”. They found that novices were more likely to be affected by surface features than by deep structure, but that surface features did sway experts occasionally too. They found that the inclination to be adversely influenced by surface features was positively correlated with performance, indicating that it might be beneficial to particularly focus on this influence as a stumbling block to achievement. Chi et al (1981) also required experts and novices to match physics problems according to similarity. They found that experts tended to work to a schema, such as energy conservation principles or Newton’s second law, which they identified by an initial sweep for cues and then a brief mental test for appropriateness. Novices tended to identify problems by the entities used in the problem description and lacked “abstracted solution methods” (p. 151).

A second difference between physics experts and novices is the tendency of novices to jump straight into manipulating equations and substituting values before having a sense of what solution strategy might be best (Leonard et al, 1996). Experts, on the other hand, first identify the physical laws affecting the problem and develop a general solution schema before they reach for the equations. It is as if novices treat the calculations as an end in themselves, whereas experts treat equations as a means to an end. With these two differences in mind, namely identification of concepts and making use of calculations, the qualitative strategies used to such good effect by Leonard, Dufresne and Mestre (1999), Leonard, Gerace and Dufresne (1999) required the students to spend a few minutes before indulging in calculation, naming the physics concepts and principles involved in the problem, how they could determine them, and how they were going to make use of them. Only then did the students continue with manipulation of equations. To physics novices, problem-solving means memorising facts and manipulating equations; to experts problem-solving means applying a small number of central ideas across a wide range of contexts. The writing exercises at the focus of the writing study project were influenced by the qualitative strategies employed by Leonard et al, however the transfer from physics to mathematics was challenging due to the difference in definitions of problem-solving, as well as the difference in expert-novice comparisons.

Expert novice contrasts in mathematics tend to be less concept driven, and more driven by metacognitive control over the problem-solving process (Schoenfeld, 1992). Access

to heuristic strategies, self regulation, sense-making and reflection are the major characteristics of distinction between experts and novices in mathematics.

Experts have ready access to a pool of heuristic strategies, “methods and rules of discovery and invention” (Pólya, 1945, p. 112), which they have developed through their experience in solving problems. Novices not only do not have as large a selection of heuristic strategies at their disposal, but are also less likely to draw upon them when necessary. As in physics, students tend to focus more on calculation and less on overarching problem structure and the concepts underlying the problem. In general, students’ conceptual problem-solving ability lags far behind their algorithmic problem-solving ability (Lin et al, 2002). Applebee (1984), in reference to expert and novice *writers* (making an interesting connection to the writing study project), avers that novice writers tend to exhibit less of the problem-solving behaviour discernible in the writing of expert writers, a difference which more writing practice can diminish.

Goos et al (2002) found that unsuccessful problem-solving was characterised by poor metacognitive decisions, and successful problem-solving was characterised by challenging unsuccessful ideas and actively endorsing useful strategies, which amounts to monitoring and regulating online decision making during the problem-solving process. Leonard et al (1999) found that experts can think about problem-solving while solving problems, while novices use all mental resources while solving problems. In addition to self-regulation “executive skills” (Schoenfeld, 1992), the act of reflection over a completed solution is one that is lacking in most novice problem-solving. Pólya (1945) insisted that “looking back” is a vital part of problem-solving.

By looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, [the students] could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.  
(Pólya, 1945, p. 15)

Traditional classroom processes of teaching by transmission rarely include looking back on any completed problem. Even when such activities are brought into the classroom, there is a tendency for some students to assume that such a process is for the struggling students in the class, not for them, and to ignore the procedure (Kantowski, 1977).

Experts, when looking back over a problem can check their answer by approaching the

problem with an alternative method, while novices tend to have only a single solution method available to them (Leonard et al, 1999).

Lester (1994) sums up the expert novice distinctions as regards mathematical problem-solving: experts

- know more, and what they know they know differently, their knowledge is well connected and composed of rich schemata (related to heuristic strategies),
- focus attention on structural features, not surface features (related to conceptual understanding, as in physics. Leonard et al (1999) consider experts to have forward looking concept based strategies while novices have backward looking means-end techniques),
- are more aware of their own strengths and weaknesses (related to metacognition in the sense of epistemic cognition, Kitchener, 1983),
- are better at monitoring and self-regulating their problem-solving efforts (metacognition) and
- are more concerned about obtaining elegant solutions (related to reflection, sense making and the mathematical Discourse, Gee, 1996, 1999).

The writing exercises carried out in the project described in this thesis aimed to bring about changes to problem-solving behaviour and processes by an amalgamation of reflection over the solution process, sense making in the appropriate mathematical context and metacognition in the form of declarative knowledge of cognitive activities. While it would have been a very pleasing result for the writing exercises to have brought about a transformation in the entire problem-solving process, what was observed was a marked improvement in understanding the problems and the underlying mathematics. Recalling that understanding the problem is the first important step in Pólya's problem-solving framework, the writing exercises did have an advantageous effect on the students' problem-solving behaviour.

## 2.6 Heuristic strategies

Pólya, in *How To Solve It*, promoted the use of heuristic strategies to solve problems. His suggested heuristic strategies, or simply heuristics, included *draw a diagram*, *solve a related problem*, *think of a theorem with a similar conclusion*, and several more. He refers to the questions he encourages problem solvers to ask themselves as “heuristics” the aim of which he defines as “to study the methods and rules of discovery and

invention” (Pólya, 1945, p. 112). Heuristics, in this sense, can be understood as being rules of thumb, or rough guides to how to respond to particular situations. Pólya’s work on heuristics is quoted dutifully in every work on problem-solving encountered, although Silver (1983) criticises this attitude, pointing out that the heuristics are good ideas, but little research has been done on how to teach them. Silver’s observation was made in 1983, however, and more research has been done since then.

Schoenfeld (1985) broke many of Pólya’s strategies into substrategies, specific to particular types of problems, and experienced considerable success in teaching a problem-solving course based on those substrategies. By giving a class a list of heuristic strategies and substrategies, and teaching them how to use them, one is teaching problem-solving as a skill, something that at least Stanic and Kilpatrick (1989) feel that Pólya was trying to avoid, aiming more for problem-solving as an art, as a way of thinking about mathematics. Schoenfeld, by adding to his heuristic approach to problem-solving an emphasised and explicit altering of the classroom culture, calling for whole class interaction in active creation of mathematics and a stress on the students taking responsibility for their mathematical actions, turns his problem-solving course into problem-solving as art, rather than skill (see also Flavell, 1976). It is the view of the author, however, that teaching problem-solving as a skill is no mean feat and greatly to be desired over only teaching mathematics as a collection of algorithms. Schoenfeld (1985) had the advantage of not being constrained by a packed curriculum (Arcavi et al, 1998), a circumstance not every mathematics course and educator is fortunate enough to experience. Schoenfeld (1992) concludes that heuristic strategies are useful in teaching problem-solving, but the effects can be subtle, it is difficult to do, and easy to not succeed at all (see also Tall, 1991). For much detail on this successful problem-solving teaching endeavour, see Schoenfeld (1985), Arcavi et al (1998), Schoenfeld (1998), Schoenfeld (1989), Santos-Trigo (1998).

Pólya’s heuristics can be used in a different sense to that employed by Schoenfeld. For example, Buxkemper and Hartfiel (1995) note that when a teacher presents a solution to a mathematical problem in the class, it is given in a polished, final form. They suggest that one reason is the lack of a model available to the teacher to demonstrate what occurs in the “discovery stage” of the problem, and suggest what amounts to Pólya’s heuristics as a model for the teacher to demonstrate mathematical discovery, instead of giving a polished solution. Using such a “discover stage model” would involve the

teacher demonstrating problems on the board in much the same way as in a traditional classroom, with teaching by transmission, but at least would allow the students some insight into the problem-solving process as opposed to making it appear as if the teacher has magical insight which the students lack. Berlinghoff (1989) has successfully demonstrated student involvement in the creation of locally original mathematics in combination with the students having access to a list of twelve heuristic strategies, the encouragement to use the strategies, and the requirement, after the fact, of identifying which heuristic strategies they employed.

## **2.7 Writing in problem-solving**

Spurred by the studies on expert-novice distinctions in physics problem-solving, Leonard, Gerace and Dufresne (1996) and Leonard, Dufresne and Mestre (1999) required students to write explanatory strategies before embarking on problem calculations in order to encourage the students to identify and motivate the underlying concepts. The success of the physics explanatory strategy initiative was marked, with greatly increased conceptual understanding and later recall of the physics principles taught. “By having students plan their approach (without actually solving the problem), justify their approach, or develop a strategy, they learn the value of concepts and conceptual analysis for problem-solving” (Leonard et al, 1999, p. 11) and “The more we can encourage students to perform qualitative analyses before solving problems, the more they will improve their problem-solving proficiency” (*ibid.*, p. 10). The reasons for supporting writing as a means of improving skills associated with problem-solving include

- writing is one method among many by which cognitive learning skills can be acquired (Kenyon, 1989)
- explaining one’s reasoning develops critical thinking skills (Leonard et al, 1999)
- writing enhances metacognitive skills in the form of monitoring one’s mental activities (Pugalee, 2001; Bonotto, 2002; Garofalo and Lester, 1985)
- breaking a problem down into steps and thereby explaining the reasoning process encourages understanding of the problem (Kenyon, 1989)
- writing encourages the student to be an active participant in mathematics creation, actively looking for problem solutions, instead of passively participating and giving up if a solution is not immediately at hand (Kenyon, 1989)

- steps of the writing process are akin to steps of the problem-solving process (Mendez and Taube, 1997; Kenyon, 1989)

Leonard, Gerace and Dufresne (1996) and Leonard, Defresne and Mestre (1999) stress that writing should not be invoked alone in the attempt to get students to become better problem solvers. Writing should form part of a suite of learning experiences designed to develop both problem-solving skills and conceptual analysis skills, such as exploring prior knowledge, interrelation of concepts, using concepts to analyse situations, problem-solving with expert-like principles, organising and prioritising knowledge (Leonard et al, 1999, in the physics context). Whatever writing initiative is employed should be fully integrated into the greater course and should form part of high stakes assessment (Leonard et al, 1996). As with teaching by heuristic strategies, using writing to promote problem-solving skills takes time and effort to achieve results (Kenyon, 1989; Leonard et al, 1999).

## **2.8 Conclusions**

The important points gleaned from the last few decades of problem-solving research can be summed up in a few sentences. It is a necessary although not sufficient condition of improving problem-solving ability that many problems must be solved; practice is important (Lester, 1994). It is extremely important to note, however, that simply solving many problems is no guarantee of developing good problem-solving skills, without embedding that practice in a larger teaching scheme, and drill in solution processes tends to encourage superficial learning and algorithmic approaches (Leonard et al, 1999). Problem-solving ability develops slowly over a long period of time (Kenyon, 1989; Lester, 1994; Tall, 1991). In order for students to benefit from problem-solving instruction, they must believe that the teacher thinks problem-solving is important (Schoenfeld, 1989; Schoenfeld, 1991; Tall, 1991; Franke and Carey, 1997). Most students benefit greatly from systematically planned problem-solving instruction (Leonard, Dufresne and Mestre, 1996; Leonard, Gerace and Dufresne, 1999; Schoenfeld, 1985; Lester, 1994). Teaching students directly about problem-solving strategies does little to improve their problem-solving ability in general (Lester, 1994).

### 3 Language

*If it is through language forms adopted in mathematics classrooms that mismatches between the goals and views of teachers and students are created, then it is also the case that, by coming to understand these language forms, the mismatches can be identified, observed, and ultimately resolved in ways which facilitate richer modes of communication between all involved in the creation of mathematical meaning.*

Ellerton and Clarkson, 1996, p. 1022

The object of the writing study project was to ascertain whether the act of writing about problem solving could have any positive effects on a student's problem solving abilities or skills. The writing exercises employed as a problem-solving tool insisted that the students attempt to write about their problem solving processes in English sentences, and avoid symbolic notation where possible. Mathematics is at least partially a language of symbols, and what non-symbolic language there is includes a very context-specific register. The issue of mathematics register, combined with the fact that many students in South African institutions do not speak English as a main language, means that the issue of language cannot be ignored. Questions that have to be asked are

- Does the fact that the exercises involved the writing of English descriptive sentences mean that students with or without English as a main language experienced the tasks differently?
- Were any results, either good or bad, different for students with different main languages?
- Were any general language challenges experienced by any of the students, whatever their main language?

These questions, in turn, require an awareness of language issues at the university, in the country, and in the fields of mathematics and mathematics education. The history of languages in South Africa is one fraught with tension and conflict. Today, in a new political climate, the politics of language have merely changed, rather than disappeared.

#### 3.1 Language Diversity

A characteristic of South Africa contributing to its diversity is the large number of languages spoken in the country, both indigenous and otherwise. South Africa has eleven official languages, all of which and many more are spoken at the University of Cape Town. The practical meaning of the official languages is that government services

should be available in all languages on request, and government documentation should be produced in all languages, in rotation. The teaching of the official languages at schools is encouraged, with the explicit aim of creating a multilingual, multicultural population (Advisory Panel on Language Policy, 2000). The language of learning and teaching at UCT is English, and a total of six languages was spoken in the experimental groups in the writing exercise, three of them not South African official languages (Shona, Portuguese and Mandarin), the other three being English, Afrikaans and Setswana. The two indigenous South African languages with the greatest number of main language speakers are isiXhosa and isiZulu. It was disappointing, but logistically unavoidable, that there were no students speaking isiXhosa or isiZulu as a main language in the experimental groups.

## **3.2 Language in South Africa**

### **3.2.1 A brief history of language, and language in education, in SA**

South Africa has eleven official languages, namely Afrikaans, English, isiNdebele, Sepedi, Sesotho, Setswana, siSwati, Tshivenda, Xitsonga, isiXhosa and isiZulu. All of these languages are spoken at UCT as main languages, in varying proportions, along with a variety of other languages, ranging widely through French, Arabic, Hebrew, Portuguese, Shona, Urdu, Cantonese, Hindi and German, to name just a few.

Educational literature focussing on multiple languages within a classroom tends to assume that the students who speak a language different to the language of learning and teaching (LoLT<sup>1</sup>, Adler, 2001) are a minority population group (Stoddart et al, 2002; Cummins, 1981). This parallel between language of instruction and majority population group is certainly the case in countries such as the United States of America, the United Kingdom, Australia and New Zealand, where English is the language of learning and teaching as well as the main language of the majority of the population. South Africa, however, experiences the opposite situation. The figures in Table 3.1 (Statistics South Africa, 1996; Statistics South Africa, 2001c) show clearly that the number of main language speakers of English is exceeded by isiZulu and Afrikaans in urban areas and is only slightly higher than that for isiXhosa. In the rural areas English is the least likely main language of all the official languages. Maintaining the use of the word “minority” as used in much of the language literature, Moodley (2000) refers to African languages being “minority languages in spite of ... African majority government” (p. 103). Even

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<sup>1</sup> Many language related terms and commonly used abbreviations are defined in the Glossary

in the context of the United States, where non English speakers are a population minority overall, Lee and Fradd (1998) argue against using the term “language minority” since classes vary so widely in respect of the natural proportion of the class speaking English as a main language.

**Table 3.1**

**Population breakdown by main language**

Language	Urban		Non-Urban		Total	
	%	Number	%	Number	%	Number
isiZulu	18.9	4 125 981	27.0	5 074 164	22.7	9 200 145
isiXhosa	14.7	3 201 998	21.2	3 994 122	17.7	7 196 120
Afrikaans	22.4	4 880 923	4.9	930 623	14.3	5811546
Sepedi	4.6	1 007 444	14.3	2 688 403	9.1	3 695847
English	15.3	3 329 501	0.7	127 965	8.5	3 457 466
Setswana	6.9	1 508 360	9.5	1 793 412	8.1	3 301 972
Sesotho	9.9	2 152 854	5.1	951 344	7.6	3 104 198
Xitsonga	2.4	527 878	6.5	1 228 227	4.3	1 756 105
siSwati	1.3	283 172	3.9	730 020	2.5	1 013 192
Tshivenda	0.7	159 427	3.8	716 979	2.2	876 406
isiNdebele	1.0	215 964	2.0	370 998	1.4	586 962
Other	1.8	388 308	1.1	195 506	1.4	583 814
Total	99.9*	21 782 010	100.0	18 801 763	99.8*	40 583 773

\*slight rounding of the percentages resulted in a total less than 100.0%

The statistics in Table 3.1 are from the South African 1996 census (Statistics South Africa Report 03-01-19). Total statistics are available for the 2001 census (Statistics South Africa Report 03-02-04 (2001)), but are not partitioned into urban and non-urban, due to the difficulty of defining communities as urban or otherwise (Statistics South Africa Report 03-02-20 (2001)). The largest difference between the 1996 statistics and the 2001 statistics is the proportion of main language isiZulu speakers changing from 22.7% to 23.8% of the total population, and the relative rankings of population size did not alter.

The language situation in South Africa is such that, more often than not, English is not only not the main language of some students, but is not the main language of *any* of the

students in a class. The absence of English as a main language for all students in a class is not without parallels in other English speaking countries, since, for example, there are some classes in the United States that are entirely Hispanic (Khisty, 1995), and classes in Australia with a majority of immigrant students (MacGregor, 1993). The difference lies in the fact that such classes are the majority in South Africa, not the minority, as elsewhere, making the issue of second or additional language factors in teaching and learning relevant to a greater percentage of South Africa's teachers, indeed, making it relevant to all of South Africa's teachers.

Prior to the coming to power of the Nationalist Party in 1948, there was a "loose policy" (Adler, 2001) of instruction in the students' "mother tongue" in primary schools (Hartshorne, 1992; 2001; Moodley, 2004). Shortly after the Nationalist Party assumed power in South Africa, African schools were placed under the National Department of Bantu Education, which stipulated in 1953 that "mother tongue" instruction be carried out in primary school, with both English and Afrikaans taken as second languages. In secondary school both English and Afrikaans were to be used on a 50-50 basis (Adler, 2001). Students at other schools, not under the Bantu Education Department and all racially segregated, had to take one of English or Afrikaans as a first language throughout schooling, and the other as a second language. Moodley (2000) compared and found similarity between the state support of Afrikaans (a minority language) with the support of French in Quebec. In both cases the state supported a minority language against the perceived economic power of a competing language (English in both cases, although still a minority language in South Africa) resulting in the overriding of individual rights by collective rights. Such support can be argued for in the light of cultural survival.

In 1976, the Soweto student uprising took place, in protest at the compulsory teaching of Afrikaans in schools, a last straw on top of an inferior education system, high student to teacher ratios and under qualified teachers. In partial reaction to the uprising, a new act was passed in 1979, stipulating "mother tongue" instruction up to Grade 4, with the wishes of the parents to be considered after that point. 1990 saw an amendment to the 1979 act, allowing parents to choose between having English (or another "second" language) as sole language of learning and teaching, or a transfer to English, either sudden or gradual. English as the sole language of learning and teaching was the choice

of many parents (Adler, 2001), who saw the use of “mother tongue” instruction as a means of oppression and a barrier to socio-economic development.

In 1990 the African National Congress was unbanned, Nelson Mandela and other political prisoners were freed, and the trends in South African education began to change. The first democratic elections were held in 1994, and a new constitution was drawn up in 1996. The constitution allows any of the eleven official languages to be chosen for the language of learning and teaching in South African schools (Advisory Panel on Language Policy, 2000), in a “policy environment supportive of the use of languages other than the one favoured language of learning and teaching in school” (Adler, 2001, p. 25).

Despite the freedom to choose any of the eleven official languages, in practice most schools have chosen English (Adler, 2001; Moodley, 2004) as the language of learning and teaching. There are two primary reasons for this pattern. One reason is that English is seen as a “language of power” (Adler, 2001, p. 14), or “language of access” (Setati and Adler, 2000, p. 247), being the language of tertiary instruction, a language spoken widely throughout the world, and the language of technology, commerce and science, at least in South Africa. There is a widespread belief that speaking English fluently will open more doors of opportunity in education and business, both in South Africa and elsewhere in the world (Moodley, 2000). To a certain extent, English is seen in a similar light in Tanzania (Rubagumya, 1990a) and Botswana (Kasule and Mapolelo, 2005). English is also seen as a “high prestige language” (Rutherford and Nkopodi, 1990, p. 445), spoken as it is by people of an already high socio-economic level in South Africa. French is similarly regarded in Burundi (Ndayipfukamiye, 1994).

The second reason to use English in the classroom is that, due to the educational history of the country, all students speak English at some level of competence, whereas it is quite possibly the case, particularly in urban schools, that there is no other language held in common by all the students (Setati, 1998), a situation extending to other African countries as well (Kasule and Mapolelo, 2005). In fact, the commonality of English alone can extend to tertiary education as well, as Jiya (1993) states in reference to the University of Fort Hare. Adler (1998) reports on a class in which the students themselves insisted on the use of English, despite none of the students speaking it as a main language, as it was the only language held in common, as well as the language

giving “the usual access/power rationalisations” (p. 29). Moodley (2000) quotes fully a letter to the editor of the Cape Times, 16 March 1999, in which a main language Xhosa speaker shows exasperation for the proposal of indigenous language education and insists on English as the door out of the prison of lack of access to information. “The valid pedagogical arguments for mother-tongue education notwithstanding, the historical experience of deliberate racial under-education has left lasting fears of marginalization” (Moodley, 2000, p. 113). The emphasis on English as a “tool to combat divisive Bantu Education” (Moodley, 2000, p. 108) and to unify the population of course decreases the use of indigenous African languages in schools.

In a rapid turn about, the relegation of Afrikaans, a previously dominant language, to simply one of eleven official languages has left Afrikaans speakers justifiably concerned for the continuance of their language and associated culture, prompting a desire for a few single medium schools and universities (Moodley, 2000). However, this single medium preference raises a concern about such institutions being able to obstruct the entry of previously disadvantaged Africans.

Table 3.2 lists all the universities in South Africa in existence as at 2006 (Department of Education, 2006). An investigation of the internet homepages of the universities revealed their official primary medium of instruction. In general, the universities show sensitivity towards the language situation in South Africa, and many suggest the possibility of teaching specific courses in other languages, should it be both useful and practically and economically feasible.

**Table 3.2****Language of teaching and learning at South African universities**

<b>Name of University</b>	<b>Languages of instruction</b>
University of Limpopo	English*
Nelson Mandela Metropolitan University	English
North West University	Setswana, English and Afrikaans
Rhodes University	English*
Stellenbosch University	Afrikaans, English and isiXhosa
Tshwane University of Technology	English
University of the Western Cape (UWC)	English*
University of Cape Town (UCT)	English
University of Fort Hare	English
University of Johannesburg	English*
University of Kwazulu-Natal	English
University of Pretoria (Tuks)	English and Afrikaans
University of South Africa (Unisa)	English
University of the Free State	English and Afrikaans
Walter Sisulu University	English*
University of Venda (Univen)	English*
University of the Witwatersrand (Wits)	English, phasing in Sesotho in addition to English
University of Zululand (Unizul)	English*

\* inferred from the university website, no official language policy being found

An opposing view, for secondary school at least, is that “there is a need to develop and promote African languages” (Setati, 1998, p. 34; Setati and Adler, 2000) and therefore it is appropriate that African languages be used in schools. Moodley (2000) compares the loss of African languages to the loss of natural species and suggests that similar concern should be shown for the loss of either. Dawe (1983), taking a pragmatic view of linguistic maintenance, strongly advises that use of main languages be supported and maintained, in order to support mathematical logical reasoning, which his research links to main language competence. In the case of subjects such as mathematics and science, this insistence creates the problem of having to develop appropriate registers in the indigenous African languages. Such register creation projects are underway in New Zealand and Australia (Barton et al, 1998; Roberts, 1998). A significant difference

between South Africa and New Zealand, however, is the existence of 9 languages that require mathematical registers, as opposed to merely one. The use of English at educational institutions in South Africa is decried by some (Goduka, 1998) and supported as logical by others (Starfield, 1996). Be that as it may, English is currently the predominant language at most tertiary institutions in South Africa, which results in many students studying in English as an additional language to their main language.

### 3.2.2 Language at the University of Cape Town

The language of learning and teaching at UCT is English. The majority of students at UCT speak English as a main language, yet as much as 29% (some 5530 students) of the student body does not.

**Table 3.3**

**Main languages spoken by UCT students**

Main Language	% of students
English	69
Xhosa	7
Afrikaans	5
Zulu	3
Setswana	3
South Sotho	2
Bilingual Eng./Afr.	2
German	1
Chinese	1
North Sotho	1
Other	6

(Statistics courtesy of the Multilingual Education Project at UCT's Centre for Higher Education Development, 2006)

Six main languages were spoken in the experimental groups taking part in the writing study project. These languages were English, Afrikaans, Setswana, Shona (a language spoken widely in Zimbabwe), Portuguese (spoken widely in Mozambique), and Mandarin. Every effort was made, within the existing structure of the tutorial system, to arrange matters so that the experimental groups would incorporate several languages. It was disappointing, but unavoidable, that the language spread in the classes settled

down, a few weeks into the semester, into a set of classes that were largely English. Designating which students should attend which tutorial classes, to create a greater language spread, would have involved a level of coercion which the author was determined to avoid. In addition, it was the continual intention of the study project that its processes be insertable into the existing course, and class structure, without having to make any changes. It would have been counter to the intention of the study project to begin the contrasting interventions with obvious manipulation of student groups.

### **3.3 Is language a concern?**

Is the issue of language really something with which we need to concern ourselves?

Yes, it is, for several reasons. Mathematics itself can be recognised as a language in its own right, a language that is meant to be written rather than spoken, and having its own vocabulary, grammar and punctuation (Griggs<sup>2</sup>, in Ellerton and Clarkson, 1996). In fact, mathematics is more than a language, or, rather, any language is more than simply its verbal and written utterances. Languages, including mathematics, are discourses; they are ways of seeing, understanding and thinking about the world (Sierpiska, 1998). Gee (1996) uses the capitalised term Discourse to refer to “ways of being in the world” (p. 127), which includes, along with all forms of associated language use, “non-linguistic symbol systems” (Gee, 1999, p. 11), values, beliefs and ways of thinking and acting that are socially accepted within a particular socially meaningful group. Mathematics is a Discourse in this sense (“language plus ‘other stuff’ ” Gee, 1999, p. 17, quotation marks in original), one aspect of which is the vocabulary and grammar of symbols and technical terms. The teaching of mathematics, however, takes place within a spoken language, such as English (Zevenbergen, 2001). While the symbology of mathematics is a worldwide language, classes take place with the symbols embedded in a verbal language, not merely with the lecturer writing wordless equations on the blackboard, accompanied by no explanatory text. At UCT that explanatory language is English, and not all the students speak English as their main language, or with a great deal of fluency.

The question arises: does it make it harder to learn mathematics when it is presented to the student in a language that is not his or her main language? Or, seeing as the students are all learning this new language of mathematics, with all its attendant difficulties, do

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<sup>2</sup> No date given, but earlier than 1936

English and non English main language students experience similar levels of difficulty in learning mathematics?

It remains controversial whether language is an issue in the mathematics classroom at all. On the one hand there are educators that believe firmly that language plays a pivotal role for students who do not speak the language of learning and teaching as a main language (Khisty, 1995; Coutis and Wood, 2002). This role could be simply as an impenetrable barrier to understanding, or as a skill that can be developed through explicit teaching. On the other hand there are educators (teachers in Adler, 1998; teachers in Kasule and Mapolelo, 2005) who believe that language plays a role in mathematics only in the form of word problems, and technical words such as “hypothesis” and “proposition”, and that speakers of any language have equal difficulties in learning the mathematics register, the “semantic structures, the vocabulary, and the symbolism of mathematics” (Ellerton and Clarkson, 1996, p. 1001; Zevenbergen, 2001).

Among the supporters of the claim that learning mathematics is more difficult for speakers of the language of learning and teaching as an additional language, Khisty (1995) suggested, for example, that LEP students could easily have problems with the words “sum” and “whole”, which have nonmathematical homonyms, whereas speakers of English as a main language would have greater ease in identifying subtleties between meanings (see also Zevenbergen, 2001). These students are not struggling with basic comprehension at the same time. Furthermore, Khisty continued, not only can lack of fluency in a language create difficulties of understanding, but dialogue is necessary for any understanding at all. “Without participatory dialogue, learning remains outside the person; it is something that is removed from personal experiences and mental connections, and it is elusive and difficult to hold onto” (Khisty, 1995, p. 290).

### **3.4 What is to be done?**

If it is the case that a student’s English language proficiency affects her ability to learn mathematics presented to her in an English speaking environment, then what is to be done about teaching mathematics in such a way as to be best understood and studied? There are long term and short term answers to that question. A long term answer is to develop mathematical registers in the students’ main languages and teach mathematics in those languages (Barton et al, 1998). Such an approach has limited practicality at a

large university, but could be helpful at schools. An answer that is immediate, and ongoing, is to create an awareness of and sensitivity towards potential language difficulties that the students may experience. One way of structuring such an awareness is by way of Adler's three teaching dilemmas of code-switching, mediation and transparency, expanded below. The long term solution is less easy to achieve, in the context of South Africa, where creating mathematical registers is not as uncomplicated as it was in, for example, New Zealand. Even there the task took a decade (Barton et al, 1998).

### **3.5 Creating mathematical registers**

Many languages do not have mathematics registers, such as many indigenous African languages (Kasule and Mapolelo, 2005), as well as Australian Aboriginal languages and Maori. When considering mathematics tuition, a decision needs to be made about the language in which to teach it. If one is to use a language in which a mathematics register already exists, such as English, then students with main languages other than English will be taught in an additional language. If, however, one insists that the students be taught in their main language, then a mathematics register needs to be constructed for that language. One such case where this construction has been carried out is Maori (Barton et al, 1998; Ellerton and Clarkson, 1996). Beginning in the mid-1980s the process of creating a Maori mathematics register was begun, and the process resulted in a school curriculum and mathematics dictionary being produced by the mid-1990s. The Maori Language Commission stipulated four conditions:

1. The terms should be consistent with one another.
2. The terms should be as short as possible without oversimplification.
3. The word should sound correct to a native speaker of Maori, both in itself and in context.
4. The usage should be grammatically correct.

(Barton et al, 1998, p. 5)

Various grammatical properties of Maori were used, specifically *gerunding*, to create a number of the required terms. However, there is a disadvantage in creating a mathematics register in this way, that is essentially creating a translation from English to Maori. The disadvantage is that, when mathematics develops naturally in the context of a particular language, the structure of that language tends to emphasize some aspects over others, hence "speakers of a particular language may 'see' mathematical ideas in different ways" (Barton et al, 1998, p. 8, emphasis in original). Developing a register by creating a one-to-one relationship between English terms and Maori terms does not allow for this difference in perspective.

Various Australian Aboriginal languages are informally developing mathematical registers as speakers of these languages choose to conduct discussions within the community in their main language. Roberts (1998) mentions, as examples, conversations on budgets, and fuel considerations for a football trip. If a true register is to exist in an Aboriginal language, such as Pitjantjatjara, one of the more widely spoken Aboriginal languages, it has to be consciously developed or English terms will be incorporated into the language due to the speakers having previously learnt mathematics in English. Roberts outlines three distinct ways in which mathematics might be discussed:

1. to only use English to discuss Western mathematics;
2. to elaborate Aboriginal languages, so that they incorporate some English grammatical structures, in order that some mathematical concepts are able to be more easily discussed;
3. to integrate the Western mathematical concepts into Aboriginal languages, so that traditional Aboriginal grammatical constructs are used to discuss these Western mathematical concepts.

(Roberts, 1998, p. 11)

The first option is the easiest to implement but might have far reaching detrimental effects on the culture and the use of the Aboriginal languages in everyday speech. The second two options would require a large expenditure of time and effort to construct registers; Aboriginal teachers admit that some Western mathematical ideas are very difficult to translate into their main languages (Roberts, 1998, p. 12). Roberts insists that if coherent mathematical registers are to be developed, then they need to be developed immediately. Mathematical terms are already being informally created in Aboriginal languages in some communities and soon explicit decisions in language engineering would no longer be possible.

Ellerton and Clarkson (1996) discuss two conflicting viewpoints related to using cultural concepts of Australian Aboriginal students in the classroom. One viewpoint is that Western concepts are largely foreign to Aboriginal students, particularly those from a rural background. If aspects of their personal worlds, particularly use of their first language were used in the mathematics classroom, then the classes would be more applicable and more accessible to the students. The opposing view is that a Western teacher using an Aboriginal language is more likely to confuse than to elucidate. Moreover to attempt to use an Aboriginal concept, such as geometry within a dance, in the class could, at worst, be sacrilegious. Ellerton and Clarkson admit that the latter view has merit, but that the practicalities of the education system are such that many

non-Western students are being taught by Western teachers, and any links that can be investigated should be explored. The “practicalities” referred to, here with reference to Australia, but easily extendable to New Zealand and the United States, include the fact that the speakers of the language of learning and teaching also make up the majority of the population. In South Africa this population-language correspondence is not the case. The practicalities in South Africa refer more to historical consequences (everyone speaking English to some degree) and the existence of teaching materials in English.

A third example of a country in which technical registers are being consciously developed in a language that did not previously have them, is Tanzania, for the language Kiswahili (or Swahili) (Rugemalira et al, 1990). Tanzania, in the 1980s, was in a similar situation to South Africa’s current dilemma, as regards language of learning and teaching and majority language in the country, in that English was then the language of learning and teaching while Kiswahili was the language spoken by the majority of the population. A striking difference, however, between Tanzania and South Africa, is that Kiswahili, spoken by 90% of the population while being the main language of 10% of Tanzanians (Rubagumya, 1990b), is an international language (Yahya-Othman, 1990), spoken in many African countries (for example Kenya, Merritt et al, 1988) and taught in many universities scattered throughout the world. A brief internet search (July 2006) reveals Swahili courses, or online materials, at Oxford University (United Kingdom), Brock University (Canada), the University of Amsterdam (the Netherlands), the University of Frankfurt (Germany) and the University of Illinois (the United States). No indigenous South African language has the distinction of being taught at so many international institutions (internet search, July 2006). In addition, since English is not spoken as a main language by anything other than a minuscule percentage of the population of Tanzania, it is slowly being taken over by Kiswahili in almost every sphere of life, except higher education, the high courts, and diplomacy. English is a foreign language in Tanzania (Rugemalira, 1990, p. 29), not an additional language, as it is in most of South Africa. The construction of technical registers in Kiswahili is taking place in rather less of an organised fashion than in Maori, but more systematically than in Aboriginal Australian languages. There is a commission to develop such registers, but there is allegedly little structural logic behind the construction of related or contrasting words (Mwansoko, 1990) with words such as *kilele* for climax, and *mpomoko* for anticlimax. Rugemalira et al (1990) shrug off the difficulties of lack of technical registers, firmly believing that any human language can

express whatever any other language does, it just has to be given the opportunity to do so.

The author discussed the issue of Kiswahili as the language of tertiary education with two Tanzanian mathematics students at UCT. One of the students wistfully wished that Kiswahili might become the language of tertiary learning and teaching in Tanzania, but both students were convinced that it would never occur, viewing the idea as too idealistic and impractical to ever come about. Too many teaching materials are available in English to make Kiswahili a viable option at university. Instead, the students, providing some information more up to date than the literature had provided, reported that the teaching of English in schools was to be taken more seriously, and embedded more deeply, thereby increasing the facility in English of students entering universities, and diminishing the need for a debate on the requirement of Kiswahili as a tertiary language of instruction.

### **3.6 Language dilemmas in the teaching of mathematics**

Adler (2001) isolated three dilemmas faced by teachers in a multilingual classroom. The first is the dilemma of code-switching, that is, switching between languages during a class, either for parts of the class or within sentences. The dilemma exists in deciding when to use a learner's main language and when to use English. This switching of medium is associated with related switching of intent between a focus upon developing English and developing mathematical meaning (Adler, 2001, p. 2, p. 68). Secondly, there is the dilemma of mediation, that is, choosing when to intervene with a group of learners that is struggling to develop mathematical meaning; helping the students express themselves in English versus allowing them to develop mathematical meaning by themselves without the possible discouragement of intervention (p. 3, p. 68). Lastly there is the dilemma of transparency. Transparency refers in this context to explicit teaching of mathematical language, with the associated danger of boring the class and focussing too much on how things are said versus the danger of learners not understanding the mathematical language and hence obscuring the entire topic under discussion (p. 4, p. 69). Sierpinska (1998) refers to the focus on new technical terms when a new mathematical topic is introduced in class, "symbolic carriers for the knowledge to be transported" (Steinbring, 1994, p. 97), where there is the danger that students might "focus on the concrete aspects of notation, although the mathematical ideas behind the notation may elude them" (Sierpinska, 1998, p. 55).

Although Adler categorised these dilemmas in relation to secondary teaching of mathematics, they are equally applicable to tertiary teaching. Transparency, although not necessarily labelled as such, is a topic of interest to many mathematics educators (Coutis and Wood, 2002; Adler, 1998; Sierpinska, 1998), mediation is unfortunately not a topic addressed by many (Moschkovich, 1999), and code-switching is a subject of much concern to researchers across the field (Setati et al, 2002; Khisty, 1995; Ndayipfukamiye, 1994).

### **3.6.1 Code-switching**

Code-switching refers to the teacher and/or students using more than one language in a discussion. It can refer to switching between languages within a single sentence, or using different languages for different sorts of activity within the classroom. It is a problematic issue, since, if English is an additional language for the students, too much use of the main language could be to the students' detriment in learning mathematical English. However, there is the potential problem of using English too much, and losing the understanding of the students. The term "code-switching" (Adler, 2001) is widely used, in various incarnations, such as "code switching" (Khisty, 1995, p. 285) and "code-mixing" (Roberts, 1998, p. 11), among others. While Secada (1992) does not use a specific term in reference to code-switching, it clearly appears in his list of effective strategies for teaching in bilingual classrooms:

(1) the active use of teaching strategies; (2) the mediation of instruction through both languages, including alternating between languages when necessary to ensure student understanding; (3) the integration of English-language skills development with content; (4) the use of students' home cultural norms for communication and for behaviour in mediating active teaching; and (5) congruence from intent to the organization and delivery of instruction.

(Secada, 1992, p. 643)

Code-switching was not practised in the writing study project described in this thesis. Indeed, it could be said that it was explicitly disallowed through the exercise's requirement for explanatory strategies written in English alone. As such, it is not within the scope of this thesis to dwell on the vast body of research in the area of code-switching, and further reading is merely suggested in Setati et al (2002), Adler (2001), Setati and Adler (2000), Setati (1998) ( all South Africa), Kasule and Mapolelo (2005) (Botswana), Ndayipfukamiye (1994) (Burundi), Merritt et al (1988) (Kenya), Lee and Fradd (1998), Khisty (1995) (both United States) and Roberts (1998) (Australia).

If a formal policy on code-switching were to be developed, decisions would have to be made on how much and when it would occur. Acknowledging that students are more likely to be familiar with the language of learning and teaching as they mature, one policy might be to suggest extensive use of code-switching in primary school, with a gradual decrease with increasing grade (Rutherford and Nkopodi, 1990). Alternatively the policy might be to encourage code-switching in some subjects and discourage it in others (Ndayipfukamiye, 1994). A consequence of a flexible policy in the mixing of languages would be to allow students to submit written examination solutions in a mixture of languages. Altogether, code-switching has obvious beneficial influences on teaching and learning (Setati et al, 2002; Lee and Fradd, 1998), but is not a teaching strategy to be embarked on without serious thought and planning.

### **3.6.2 Mediation**

The dilemma of mediation involves the difficulty in knowing when to look past the difficulties that students are experiencing with mathematical language to concentrate on their acquisition of mathematical meaning, and when to focus on the language difficulties, or the students' mathematical communicative competence, as a means of explaining terms more clearly. If one focuses on the language too much, one risks obscuring mathematical meaning, whereas if one ignores language errors, one risks the student making mathematical errors based on language difficulties at some point in the future.

Moschkovich (1999) supports the discourse approach to teaching mathematics to English additional language learners. The discourse perspective concentrates on the teacher extracting the mathematical meaning of students' statements, building on their responses and revoicing their statements using technical terms. The discourse approach contrasts with a more traditional approach of working on word problems and translating from English to symbols. Moschkovich suggests that the discourse approach allows the students to concentrate on the mathematical content of the lesson, rather than on language development. By concentrating on understanding and revoicing the mathematical content of the students' responses, and avoiding explicitly correcting errors in grammar and vocabulary, the teacher can support the students' participation in mathematical discussion.

In the writing exercises, mediation was practised in this study, in the sense that the author was looking for mathematical meaning and understanding in the students' writings, and not for perfectly phrased mathematical statements. The comments returned to the students responded to their mathematical meaning, rather than the way they had phrased themselves. In cases where it was deemed useful, some statements were revoiced using more correct or more technical language, so that the students could learn from that presentation, without any indication being given that original statements were wrong. The concentration on mathematical meaning, rather than grammar or spelling, gradually allows second language English speakers to write more freely about mathematical problems (Cooley, 2002).

### **3.6.3 Transparency**

Transparency concerns the explicit teaching of mathematical technical language. By making the meanings of terms clear in explicit lessons, the terms are able to become transparent and the students can concentrate on the mathematical meanings of problems without getting bogged down in language difficulties. These language difficulties affect all students, not just those who do not have the language of learning and teaching as their main language (Adler, 1998). Fluency in the mathematics register can be a skill that many students struggle to acquire. Admitting that the English mathematics register is troublesome for even English speakers to learn, it surely follows that speakers of English as an additional or foreign language must experience even greater difficulty (Rutherford and Nkopodi, 1990, among others), although conversations between the author and mathematical colleagues indicate that not all mathematicians believe this increased difficulty to be the case, a point of view also encountered by Ellerton and Clements (1989) cited in Ellerton and Clements (1991).

In any language, everyday and scientific meanings of a term might be different (Gee, 1999). This difficulty is exacerbated for students learning in a language not their own, as concepts could have different meanings in different languages, resulting in misconceptions. The example of "wind" is used by Sutherland and Dennick (2002); in the Maliceet language among Aboriginal Canadians, there is no noun for wind. Rather, there are verbs meaning "to blow" or "to be windy". As a result of this fundamental grammatical difference between matching concepts in two different languages, students can struggle with sentences that, to an English speaker, would seem easy to understand. Similarly Jiya (1993) refers to the confusion of "force", "energy" and "power" which might all be interpreted as "amandla" in some indigenous South African languages (p.

80). Another similar case is reported in Case (1968), referring to confusion in Malawi during science learning in English due to the absence, in the local language, of articles, that is “a” and “the”. The sentences “carbon dioxide is the gas which does not support combustion” and “carbon dioxide is a gas which does not support combustion” (p. 21) are identical for someone who does not understand the difference in the articles, due to the non-existence of such terms in his or her own language. Many further examples can be found of clashes between different languages’ vocabulary formations (see for example Merritt et al, 1988 and Case, 1968).

Coutis and Wood (2002) believed that teaching language skills related to statistics, in their case, benefits all students, not merely NESB students (see also Zevenbergen, 2001). The skills taught in the subject Mathematical Practice, discussed in their paper, are embedded in structuring a mini conference, including writing papers, having them peer-reviewed, producing a book of proceedings and making presentations. Students improve in reading, writing and comprehension, and themselves perceive their skills to have improved. Despite improvement however, Coutis and Wood conclude that students are not reaching a level 3 (Petocz and Reid, 2001) skill in engaging with the text. Level 3 is associated with a professional level of engagement and is expected in graduate students, but not necessarily in first year students, such as those involved in the study here. There are limits to how much one can increase the mathematical reading and writing capabilities of novice mathematicians, but an explicit focus on mathematical language aids both English and non English speakers.

The writing exercise did not focus explicitly on language or vocabulary, yet it required that the students use written language to describe mathematical processes, leaving it up to the student to choose whether to use technical language or not. Mathematical terms were used with varying degrees of facility by different students. Some of the students explicitly stated, after the end of the exercise, that their mathematical terminology had improved by the practice that resulted from taking part in the exercise.

### **3.7 The relationship between language and mathematical ability**

As something of an aside from the main themes of this thesis, there is some varied evidence that linguistic ability and mathematical ability might be linked. Studies have shown (Secada, 1992; Lambert, 1975, cited in MacGregor and Price, 1999) that bilingualism can be associated with high achievement in mathematics. However, Dawe

(1983) has found that some students learn an additional language before having a sound foundation in their main language, ultimately resulting in low levels of proficiency in both languages, and associated low academic achievement (see also Moodley, 2000). Dawe calls these students semilingual, rather than bilingual, raising a concern of Ellerton and Clarkson's (1996) that some bilingual (or perhaps semilingual) students have poor facility in both their main languages. Research shows a positive correlation between first language competence and ability to reason mathematically in a second language (Dawe, 1983). In addition, Ellerton and Clarkson (1996) cite Secada (1988) and Duran (1988) as suggesting a positively correlated relationship between degree of bilingualism and logical reasoning. Cummins (1981) reports on studies indicating that full bilingualism can, subtly yet consistently, enhance educational and intellectual abilities, and attributes this outcome to the quantity of language input that the child has had to decipher while learning the languages, and therefore the increased practice in analysing meaning. "Bilingualism and biliteracy appear to confer intellectual and academic advantages on the individual when proficiency in both languages continues to develop." (Cummins, 1981, p. 29)

MacGregor and Price (1999) were concerned that projects testing for relationships between linguistic and mathematical ability were unclear due to multiple factors such as bilingualism, and various social and cultural variables. A similar concern was also voiced by Hakuta (1986) cited in Secada (1992). Setati and Adler (2000) also acknowledge that there are many interrelated factors, and simple correlations between bilingualism and mathematical proficiency cannot be drawn. MacGregor and Price tested for a relationship between metalinguistic awareness and mathematical ability, among students with English as a main language. Metalinguistic awareness is defined as the linguistic ability that enables one to "reflect on and analyze spoken or written language" (MacGregor and Price, 1999, p. 451). The characteristics of metalinguistic awareness in manipulating objects and reflecting on structural and functional features are related to similar characteristics of mathematics, particularly algebra, in the opinion of the researchers. A test instrument was drawn up that tested for both mathematical ability and metalinguistic awareness, involving interspersed mathematics problems and English questions related to ambiguity, poetic interpretation and rephrasing badly phrased sentences. The results indicate numerical evidence of a positive correlation between these two abilities, in that few students with low scores in the metalinguistic test achieved high scores in the algebra test. It is postulated that time spent working

with these students to increase their metalinguistic skills might be better spent than time spent teaching them algebra that they are not yet at a level to understand.

Ellerton and Clarkson (1996), citing the National Statistics on Mathematics for Australian Schools (1991), are concerned about students who have a superficial capability in English creating an appearance of greater facility in English than is accurately the case, making it appear as if students experience mathematical difficulties, when it is really formal English difficulty (see also Lee and Fradd, 1998; MacGregor, 1993). Cummins (1996) differentiates between conversational and academic language competence and argues that a high level of the former often masks inadequate academic language competence.

Secada (1992) also refers (p. 637) to a correlation between language proficiency and mathematical achievement in that language, citing studies by Fernandez and Nielsen (1986), Duran (1981) and De Avila and Duncan (1981) among others. He concludes that although a relationship between language proficiency and mathematical achievement appears to exist, it is a complex relationship and is not simply based on fluency. Many of the studies carried out, however, such as the works cited by Secada (1992), test for the relationship between fluency in English and mathematical achievement, rather than fluency in the students' main language and mathematical achievement. Significant correlations are perhaps not surprising.

### **3.8 Discussion**

Short answers to the three questions about the study project of this thesis posed at the beginning of the chapter are probably no, no, and yes.

- Does the fact that the exercise involves the writing of English descriptive sentences mean that students with or without English as a main language experienced the tasks differently?
- Were any results, either good or bad, different for students with different main languages?
- Were any general language challenges experienced by any of the students, whatever their main language?

With somewhat greater clarity, the writing project found that there were probably no differences in the experiences of different language speakers, that the apparent results of

the project did not differ across language groups, and that some language challenges were experienced by students.

While detailed discussion of the language findings of the project can be found in the Data Analysis chapter, the overall findings can be covered briefly here. The number of students taking part in the writing project who did not speak English as a main language was rather small (11), making it difficult to infer whether any differences in experience were language based or not. The results suggest that the main language of the students did not play a role in how they experienced the writing exercises, which can perhaps be at least partially based on the fact that no grammar or spelling corrections were ever made, and facility in English was never remarked upon. The primary result of the writing exercises was the increased understanding experienced by the students, which was not an effect experienced and reported more by one language group than another. The only language issue that was apparent during and after the writing project was that of the mathematical register, which was apparent in both negative and positive ways. Some of the students were pleased at the effect that the writing exercises had had upon their own use of mathematical technical terms. Other students were intimidated at the idea of having to write anything mathematical, however informal, as they felt that a greater ability in the use of technical terms was being demanded of them than they felt able to provide. While language concerns were expected in the carrying out of the writing exercises, the apparent focus of the students on technical language, rather than English/not English difficulties was very interesting.

## 4 Writing

Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, the usefulness, and the supreme value of mathematics.

(Courant and Robbins, 1941, p. xv)

The quotation is given in answer to the question “What is mathematics?” The authors go on to say that “the element of constructive invention, of directing and motivating intuition ... remains the core of any mathematical achievement” (p. xvii) and “the only relevant assertions concerning [mathematical objects] do not refer to substantial reality; they state only the interrelations between [them]”. The literature supporting the efficacy of writing as a tool for mathematical learning echoes many of the same terms and concepts used by Courant and Robbins: active, analysis, construction, individuality, interplay, synthesis and constructive invention. Support for writing in mathematics is widespread, but has not penetrated very deeply into the mathematics community, perhaps, as many believe, because the enthusiasm is not backed up by firm theoretical grounding.

It is widely believed that writing in mathematics supports and develops understanding (Applebee, 1984; Shield and Galbraith, 1998; Habre, 2002; Borasi and Rose, 1989), awareness of the problem solver’s own thought processes (Bell and Bell, 1985; Borasi and Rose, 1989), mathematical reasoning skills (Applebee, 1984) and possibly even problem-solving skill in particular (Bell and Bell, 1985; Borasi and Rose, 1989). The writing involved in such admirable development cannot simply take any form, however. It has to be carefully chosen to support the desired cognitive phenomenon being developed, and must be “good writing” (Applebee, 1984, p. 577). “Expository writing” Bell and Bell (1985, p. 212) suggest as a useful tool in teaching mathematical problem-solving, expository defined as “writing which is intended to describe and explain mathematical ideas” (Shield and Galbraith, 1998, p. 29). Shepard (1993), cited in Shield and Galbraith (1998), suggests that the writing tasks become more demanding for the students.

## 4.1 Writing and learning

“Writing represents a unique mode of learning – not merely valuable, not merely special, but unique. ... Writing serves learning uniquely because writing as process-and-product possesses a cluster of attributes that correspond uniquely to certain powerful learning strategies” (Emig, 1977, p. 122). Why should writing be such a useful tool in mathematical learning? What characteristics of writing support cognitive development? The characteristics cited most frequently for the utility of writing are its permanence, its active nature and its pace, all characteristics which are innately part of the writing process, and all interact to make writing a powerful tool for the learning process in general.

Applebee (1984) considers that the *permanence* of the written word, “concrete language” (Bell and Bell, 1985), allows the writer to rethink and revise over an extended period. Emig (1977) agrees that the opportunity for review and re-evaluation is an important aspect of the written work, but also stresses that such review can occur immediately, rather than over an extended period, “because information from the *process* is immediately and visibly available as that portion of the *product* already written” (p. 125, emphasis in original). Secondly, Applebee (1984) describes the *active* nature of writing as “providing a medium for exploring implications entailed within otherwise unexamined assumptions.” (p. 577). Emig (1977) and Ishii (2003) both agree that writing involves active participation in the learning process, and Gaskins et al (1994) also refer to the “dynamic process” (p. 1040) of writing, drawing a parallel with the active nature of science learning. Thirdly, the *pace* of writing is slower than that of speaking, or of reading. Emig is a firm supporter of the slower pace of writing contributing to learning: “One writes best as one learns best, at one’s own pace. Or to connect the two processes, writing can sponsor learning because it can match its pace” (p. 126). Emig refers to writing as “self-rhythmed” (p.126), while Bell and Bell (1985, p.213) prefer the term “self-paced”. Emig continues that the slow pace of writing encourages “shuttling among past, present and future ... and the two major modes by which these three aspects are united are the processes of analysis and synthesis” (p. 127)

The permanence, active nature and slow pace of writing are characteristics which are unarguably innate to the process. In addition to these, there are characteristics which are arguable, but which tend to be taken as similarly incontestable, such as the necessity of making connections, of being clear and explicit, and of the personal involvement in the

process. At what could be considered a foundational level of learning, writing is thought to bring about both the creation and the organisation of cognitive structures. Applebee (1984) suggests that a factor in support of writing as a tool for learning is “the resources provided by the conventional forms of discourse for organizing and thinking through new ideas or experiences and for explicating the relationships between them” (p. 577). He is supported in his view of writing being connective by Emig (1977): “The medium then of written verbal language requires the establishment of systematic connections and relationships” (p. 126). Keys (1994) too, would agree that “the task of creating a written product can be a powerful tool for developing science understandings because it requires the writer to retrieve, synthesize and organize information” (p. 1003). Pugalee (2001) is convinced that the form of communication, defined as “transmission of thoughts mediated by language” (Sierpiska, 1998, p. 33), provided by the written word requires the writer “to maximally compact inner speech so that it is fully understandable, thus making necessary the deliberate structuring of a web of meaning forming associations between current and new knowledge” (Pugalee p. 236, drawing on Vygotsky, 1987). Similarly Connolly and Vilardi (1989) point out the requirement of making connections as part of the reasoning required by writing.

The premise of writing’s connective facility seems to be that the very characteristics of writing and the writing process place cognitive demands on the writer that encourage the formation of cognitive knowledge structures, and so enhance learning. It is further argued that not only is writing useful for the construction of cognitive structures, but also for organisation of those structures (Ellerton and Clements, 1992; Rivard, 1994). Listing several reasons for the usefulness of writing within a mathematical context, Ellerton and Clements (1992, drawing on Lovitt and Clarke, 1988) include organisation of thought. Kenyon (1989) describes the requirements of a writing assignment as having to gather, formulate and organise old knowledge, concepts and strategies and to synthesize this information into a new structure which becomes part of the writer’s own knowledge network. Does writing *necessarily* require making connections, however? Writing is necessarily active, the “enactive hand” (Emig, 1977, p. 124) holding the pencil and carrying out the task of making meaningful marks on paper, but is it necessarily connective or requiring synthesis and organisation? Perhaps writing, and the purpose to which that writing is being put in the classroom, is highly encouraging of connectivity, but it is arguable whether it is such an incontestably innate characteristic as some references would claim.

Bell and Bell (1985) consider that “by encouraging students to explain themselves in clear coherent prose, exposition allows them to become more aware of their thinking processes and more conscious of the choices they are making as they carry out the computation and analysis involved in solving math problems” (p. 220). Applebee (1984) agrees, considering explicitness to be a characteristic of writing “if meaning is to remain constant beyond the context in which it was originally written” (p. 577). Again, as with connectivity, it can be argued as to whether clarity is necessarily a characteristic of writing. It is no doubt a desirable feature of written material, but it is not necessarily present in every example of student writing.

Writing is considered a personal activity since writers need to use their own words to express their thoughts. “Restating concepts and rules in one’s own words can in fact facilitate their internalization” (Borasi and Rose, 1989, p. 355) and “students “have to create their own meaning for symbols in order to express them in words on paper”. Shield and Galbraith (1998) might disagree, as their study found that students tend to use the writing structure of textbooks when asked to write explanatory text, not to use their own words at all. Interestingly, the statement of Borasi and Rose that students have to create their own meaning for symbols can be reworded using the language of Chapter 6: in the event that students are required to write about a symbolic procedure in natural language and, in so doing, encounter the perturbing situation of realising that they have no cognitive structure indicative of such meaning, then they have to construct the requisite knowledge structures before, or in the process of, expressing themselves on paper.

A further facet of the writing which is used in support of its usefulness in the learning process is its role as a form of representation. Writing down one’s mathematical thoughts could, arguably, be regarded as translating information of knowledge from one representational form to another, that is, mental to verbal. Emig (1977) is perhaps supporting this view (claims Applebee, 1984) when she states that writing deploys three kinds of learning (active, iconic and symbolic) in that it involves “the symbolic transformation of experience through the specific symbol system of verbal language ... shaped into an icon (the graphic product) by the enactive hand. Writing reinforces learning through a combination of hand, eye and brain” (Emig, 1977, p. 124). Bell and

Bell (1985) support the use of writing simply as one representational form among many, as many as possible being used to facilitate learning.

A final, yet enthusiastic, line of argument in favour of writing in mathematics is its perceived similarity to problem-solving in particular, rather than to any other form of learning. In reference to problem-solving and expository writing, Bell and Bell (1985) suggest that if “their underlying processes are so very similar, practice in one area can reinforce competency in both by strengthening the student’s critical thinking ability” (p. 213). The ubiquity of sources claiming links between mathematical problem-solving and expository writing, and the relevance of such claims to the current research question, suggest that special attention be paid to the relationship between the two activities.

## **4.2 Writing and mathematical problem solving**

The constructivist model of epistemology (von Glasersfeld, 1995c; Steffe, 1995) draws an analogy between the gaining of knowledge and the physical creation of structures.

An analogy in chemistry that constructivism seems to suggest is that of crystal growth: in general, crystal growth occurs on a substrate of previously existing crystal structures. Once a first crystal is present in a solution, it is relatively easy for further crystal growth to occur, building on the structure already present. The acquisition of new knowledge first requires that a mental structure already exist in the mind that is in some way similar to or related to the new knowledge to be gained, whereupon the new knowledge brings about an alteration, growth or other adaptation of the existing cognitive structure.

Writing, it is argued, encourages such constructivist learning. The support of writing as a successful technique of teaching mathematics in general, and mathematical problem solving in particular, extend from the vague “the act of writing helps clarify both the process and content of problems” (Borasi and Rose, 1989) to the intense “writing *is* problem solving” (Kenyon, 1989, emphasis in original).

Writing encourages not only cognitive learning, but the acquisition of metacognitive skills (Kenyon, 1989). As the problem solver writes down, reflects, reacts to thoughts and ideas, the writing process enhances her problem solving abilities and the problem solving process becomes more effective (Kenyon, 1989). Construction and organisation of cognitive structures is an extremely important part of learning and practising mathematics, but for successful problem solving, there also has to be a level of

metacognitive control over the solution process. Writing in mathematics, by its nature, requires the problem solver to take a watchful stance over the process being described in the writing and as such develops metacognitive behaviour. Pugalee (2001) agrees that “metacognition, or the monitoring of one’s mental activities, is essential to employing the appropriate information and strategies during problem solving” (Pugalee 2001, p. 237) and that “among techniques that enhance metacognitive behaviors, writing appears to be a promising vehicle for providing the types of experiences necessary to promote the development of behaviours that are considered to be metacognitive” (*ibid.*, p. 237)

Why should writing in any way aid either cognitive construction or metacognitive control? Applebee (1984) suggests that the permanence of writing, the explicitness required for effective communication, the richness of discursive tools for refining ideas, and writing’s active nature all play a role in making writing a powerful tool for shaping thought. There is also support for the thesis that writing *is* problem solving, that thoughtful writing practice employs the identical cognitive processes as successful problem solving, and thus is an ideal tool for use within a problem solving context. Indeed, writing and problem-solving are felt to enhance one another, rather than the effect only going one way (Bell and Bell, 1985), with Shield and Galbraith (1998) citing Hamilton (1990) as linking the two activities through their shared “generative” nature”. Mendez and Taube (1997) insist that writing and mathematical problem solving have many parallels, although it has to be “real writing” and not “arrangement and presentation of pre-existing ideas in correct mechanical form” (p. 109). Mendez and Taube (1997) cite Wingersky, Boerner and Holguin-Balogh (1996) and Kirsznner and Mandell (1996) as respectively describing the writing process as *prewriting* – *organising* – *drafting* – *revising* – *editing*, and *getting bearings* – *using investigative strategies* – *selecting and arranging ideas* – *drafting* – *revising*. The writing steps are similar to Pólya’s (1945) problem solving steps of *understand the problem* – *devise a plan* – *carry out the plan* – *look back*. Writing and problem solving are complementary activities, they argue. Kenyon (1989) similarly defines a writing exercise as involving the following stages:

- the planning stage – attempting to understand, ideas are generated and organised
- the composition stage – ideas are translated into extended text
- the revising stage – text is reviewed, redundancies are removed, clarification is increased.

In support of their argument, Mendez and Taube quote the NCTM Standards of 1989 as declaring that “the writing process that emphasises brainstorming, clarifying and revising is directly related to solving a mathematical problem” (p. 142, quoted in Mendez and Taube, 1997, p. 108). Similarly Applebee (1984) refers to the “heuristic, problem-solving nature of writing about new material” (p. 582). In support of the assertion that writing aids understanding of the problem, Kenyon (1989) describes writing as requiring that the problem solver clarify her thoughts about approaching the problem, concepts have to be identified more clearly and sharply and the writing process becomes an integral part of the thought process. A further facet of the writing process that complements problem solving is the necessary action of *looking back* over the solution, which aids reflection. Writing down one’s thoughts and procedures for each step provides a problem solver with immediate feedback for reflection and review (Kenyon, 1989; Emig, 1977), and it is through reflection that we notice patterns, consider our handling of difficulties, and learn to structure our knowledge (Leonard et al, 1999).

Glynn and Muth (1994) describe the writing process in terms that are immediately reminiscent of competent problem solving behaviour.

... competent writers who have automated these [writing] skills ... can juggle several skills concurrently, interactively and fluidly ... Thus, while producing text, a competent writer also may be revising ideas or relations among ideas. It is important to note that all the writing skills interact with one another ... and are coordinated by the writer’s metacognition. The writer who recognises that the produced text, even after revision, still does not achieve his or her goals, might then revise the plan for meeting those goals ...  
(Glynn and Muth, 1994, p. 1065)

Reaction to the written word in mathematics is not all positive, however. Lee and Fradd (1998) are justifiably concerned that, among the many representations of scientific ideas and understandings, writing is a form that disadvantages NESB students more than, say a diagrammatic representation would. Analysis of the results of the writing study project would suggest that this initiative, at least, did not advantage English main language students over other students, in the sense of the writing providing a learning process. However, the non-English main language students did tend to find the writing tasks more difficult than the English students.

### 4.3 The uneasy position of writing in the curriculum

The history of writing in mathematics is erratic, with much having been expressed on the subject, from many different viewpoints, but very few, if any, writing initiatives having shown much longevity. Russell (1990) blames the fragility of the position of writing within any university discipline (other than languages) on the change in academic structure which took place at the close of the 19<sup>th</sup> Century, at least in the United States, the focus of Russell's interest. Before the advent of the modern university, Russell claims, there was linguistic and cultural homogeneity in the student body, and the intellectual "initiation" was carried out through "a series of highly language-dependent methods – the traditional recitation, disputation, debate, and oral examination of the old liberal curriculum" (pp. 54 - 55). The modern university, built on a German model, Russell explains, not only had a greatly expanded, non-homogeneous student body, but replaced the "active, personal, language-dependent" (p. 55) instruction with passive, less personal methods such as lectures and objective testing. Russell's (1990) careful study of the role of writing, specifically cross-disciplinary writing, in university curricula, reveals discomfort and instability of opinion on where writing belongs and who should teach it. The conflicting positions of rewarding students for being literate in a specific discourse community while not teaching such literacy within one's specific subject discipline leads to faculty relegating "illiterate" students to remedial linguistic courses, such as "writing labs" or of simply abandoning those students. There is a need for students to be literate in the discourse of a specific discipline, yet there is a view that writing is discipline independent, should have been taught at school, or at university should be taught by some other department, notably the English (or relevant language) department. It is in this environment of conflicting demands on the role of writing that various writing initiatives and courses have waxed and inevitably waned (for extensive details, see Russell, 1990) over the last century.

The evidence of support for writing in mathematics in the mathematics education literature of the past few decades echoes the waxing and waning of writing activity reported by Russell. In the late 1980s with publications such as Connolly and Vilardi (1989), Borasi and Rose (1989), Bell and Bell (1985), and the influential Applebee (1984), it appeared as if support for *writing to learn* was on the increase. The early 1990s saw some follow-on from the earlier enthusiasm, then interest faded once again, although never quite dying out. One thing that has remained constant, however, is the

continual call for more firmly grounded theoretical backing of the role of writing in learning mathematics.

#### **4.4 In search of theory**

Distinction must be drawn between reports of writing in the mathematics classroom which amount to “Here is my method and here is my reasoning. I tried it with my class and it worked very well”, and studies that attempt to provide a sound theoretical framework for the role of writing in mathematics. Applebee (1984) laments that little research has been carried out on how writing is related to mathematical reasoning, that there is belief that writing will lead to understanding, but that how this happens has been treated “superficially and anecdotally” (p. 577). Five years later, Borasi and Rose (1989) similarly describe available evidence as merely anecdotal. Ellerton and Clements (1992) suggested that “mathematics educators have yet to define appropriate research questions and associated research methodologies that will inform the “writing mathematics” movement” (p. 154) and refer again to “the dearth of appropriate research” (p. 154), which they attribute to the rapid growth of the writing mathematics movement. Russell (1990), however, would probably point out that the recent movement (1980s, early 1990s) is but one among many occurring over the last century.

Ellerton and Clements (1992) suggest five research issues concerning students:

If students participate regularly in a particular form of “writing mathematics”, are they likely

1. to perform as well as students who do not participate in this form of “writing mathematics” on standard tests of mathematical skills, concepts and principles?
2. to link more readily their mathematical understandings with their personal worlds?
3. to become more efficient and effective at monitoring their own mathematical thinking so that they improve their own problem-solving and problem-posing performances?
4. to develop more positive affective ... responses to mathematics and mathematical situations? ...
5. to become more aware of their own abilities, attitudes and preferences in mathematics, and to be prepared to modify these in response to their own reflection?

(Ellerton and Clements, 1992, p. 158)

In 1996, Ellerton and Clements repeat their call for further research, adding that some work has been done. They make particular reference to Clarke, Waywood and Stephen (1993) using the journal classification scheme also referred to in Waywood (1992). The author’s adaptation of this scheme is used in this thesis to code writing exercises. Shield and Galbraith (1998) continue to point out the

lack of evidence for the claim that writing enhances learning in mathematics, adding their scheme for coding students' written statements to the available research. Shield and Galbraith's findings were among some potentially negative results for writing in mathematics where they found that grade 8 students, when writing explanations of mathematical concepts and algorithms used textbook linguistic formats, instead of the individual creativity which writing is occasionally considered to encourage. Habre (2002) continues to consider that "research on writing in mathematics is not very extensive yet" (p. 2), and Kågesten and Engelbrecht (2006) admit that, despite research to date, "there is still a feeling that there is much to be done" (p. 2).

In addition to the schemes of Waywood (1992) and Shield and Galbraith (1998), some encouraging work has been done by Garofalo and Lester (1985) and Pugalee (2001) on the presence and role of metacognition in student writings on mathematical problem-solving. It is the claim of this research thesis that the writing study project has extended the theoretical support for the place of writing in mathematics, by using Piaget's theory of learning to model the deep engagement encouraged by writing and the circumstances of perturbation which have to occur for learning to occur.

The literature on writing to learn in mathematics takes on a variety of forms, not all drawing on data, whether quantitative or qualitative. It would be convenient were we able to divide the literature cleanly into those contributions based on data and those perhaps merely airing the author's opinion. The distinctions are not that clear, however. The perhaps weakest grounding in the literature for writing supporting learning in mathematics is found in papers that amount to lists of suggestions of types of writing to employ in the classroom, or recounting of apparently successful writing experiments with no data given in support (Berlinghof, 1989; Evans, 1984; Kenyon, 1989; Mendez and Taube, 1997; Mullin, 1989; Countryman, 1992; much of Sterrett, A., 1992). Even this weak form of support for writing has its uses, providing the reader with ideas and positive encouragement.

Examples in the literature of studies citing data tend heavily towards the qualitative (Baker and Saul, 1994; Borasi and Rose, 1989; Cooley, 2002; Fellows, 1994; Gaskins et al, 1994) rather than quantitative (Bell and Bell, 1985), although some studies successfully combine the two forms of analysis (Leonard et al, 1996; Kågesten and

Engelbrecht, 2006). The lack of quantitative analysis of writing projects in the context of mathematics is perhaps a cause for concern and can be viewed as an invitation to fill this gap.

A third clear category in the relevant literature is that providing, or attempting to provide, a theoretical viewpoint of the subject. Such examples can be accompanied with data from associated projects or can be free of data. Again, within this subset, there is wide variation, with some theoretical contributions providing coding or classification schemes for analysing written products, based perhaps on common sense or logic, or based on a deeper theoretical grounding. Examples include Garofalo and Lester (1985) providing a detailed set of categories of metacognitive activity, based on metacognitive theory, a theory also drawn on by Glynn and Muth (1994). Pugalee (2001) drew successfully on Garofalo and Lester's metacognitive framework to carry out a data driven analysis of student writing, finding positive indications of different forms of metacognitive activity present. Gopen and Smith (1989) suggest Reader Expectation Theory as a form of analysing student writing, with the aim of teaching the student to gain skill at expository writing. Keys (1994) provides a coding scheme for analysing written lab reports. Shield and Galbraith (1998), suggesting that a reason for the scattered nature of the writing branch of mathematics education research is the lack of coding schemes, provide a broad brush coding scheme for differentiating between expository writing and journal writing styles. Waywood's (1992) journal classification scheme, widely cited in the literature, was found to be highly applicable to the study project described in this thesis.

Other theoretical contributions are more general, drawing on a variety of epistemological frameworks (Sierpiska, 1998). Leonard et al (1996) drew on the expert-novice research to provide support for their successful writing intervention in physics, although they refer to their support as "motivation" rather than as a theoretical framework. In the references used in this literature review, Emig (1977) provides some early theoretical contributions, found valuable in the field, judging by the variety of texts in which it is cited. Emig's theoretical groundings for writing supporting learning in mathematics, however, are not as rigorous as some that follow in later decades, this perhaps being what Applebee (1984) meant with "her analyses seem impressionistic in comparison with more recent studies" (p. 582).

Finally, there are the review papers, of great value to the researcher attempting to get an overview of a field, but tending to blur the distinctions between data based studies and less formal contributions. Examples of review papers found useful in this study were Applebee (1984), Ellerton and Clarkson (1996), Ellerton and Clements (1991, 1996) and Rivard (1994).

The author would argue that theory and data are both necessary for a sound and convincing product. Data reporting a successful writing initiative are weakened by lack of theoretical grounding suggesting why the initiative was successful. Similarly any amount of theory is useless unless it can be applied in the classroom with a measurable result.

#### **4.5 Forms and purposes of writing activity**

In any literature review on the forms and uses of writing within a mathematical context, it will be found that there are many different writing formats, and many different objectives and purposes which the writing activities might serve. A number of different ways of categorising writing formats have been used (Ellerton and Clarkson, 1996; Rose, 1989; Kenyon, 1989; Shield and Galbraith, 1998), all quite different from one another. The categorisation used in this thesis expands on that of Ellerton and Clements (1992), and is governed by *form* rather than *purpose*. Different forms of writing assignments or activities include journal writing, problem posing, writing about problem solving processes and essays. Different purposes served by writing might be to practice using the mathematical register, to allow for expression of affective issues, to allow the teacher a window into student thinking and knowledge, and to encourage reflection and metacognition. The writing project discussed in this thesis was of a particular form (writing about problem solving processes) in order to achieve a specific purpose (improvement of problem solving skills). While the improvement of problem solving skills was certainly what was intended, and optimistically hoped for, with the practice of writing within mathematical problem solving, the form of the writing carried out in the tutorials was set and unambiguous. With this in mind, the categorisation of writing activities given below is by form rather than by purpose, allowing writing about problem solving processes to be considered as a category on its own, rather than merged with others (such as investigative projects and projective techniques) as serving the purpose of improving problem solving.

The categories used by Ellerton and Clarkson (1996) and Ellerton and Clements (1992) have been used, although expanded on to include additional writing forms those authors do not mention, such as writing about problem solving processes, and writing forms existing solely for expression of affective issues. Rose (1989) has a detailed breakdown of writing activities, categorising writing forms quite differently to Ellerton and Clements. For instance, Rose lists summaries, questions and explanations as different categories, which do not reconcile easily with the categories of Ellerton and Clements or those given below. A scrutiny of the differences between the categorisations suggests that Rose is defining the task by the intent of the student when writing, instead of the intent of the teacher in setting the task. Where possible, Rose's categories are incorporated into those below. Kenyon's (1989) breakdown of writing categories is roughly similar to Ellerton and Clements' (1992), dividing tasks into long term and short term writing tasks. Kenyon, however, includes word problems as a category, which is arguably not a writing task for the student so much as for the teacher setting the problems. Word problems are not included in the categorisation given here, as the student's response to word problems is rarely in the form of writing, and more often in the form of calculations.

There are different ways in which writing can be used within a mathematics course, and the different writing forms serve different purposes (Applebee, 1984). Sifting through the literature the following list of writing forms can be determined.

- Journal writing
- Essays on famous mathematicians, theorems or mathematical ideas
- Writing about problem solving processes
- Reports on journal articles
- Writing about mathematical concepts such as limits, or the derivative
- Problem posing
- Investigative mathematics project reports
- Projective techniques, such as writing explanations for a student who missed class

Purposes served by writing can be listed as

- Language issues
  - ❖ Encourage English usage (or whatever the appropriate language)
  - ❖ Encourage use and learning of mathematical terminology

- ❖ Teach writing skills
- Communication issues
  - ❖ Provide the teacher with a window into students' knowledge and thinking
  - ❖ Allow the student to take on the role of the teacher
  - ❖ Allow for the expression of affective issues
- Engagement with mathematics
  - ❖ Engage with mathematical concepts or facts
  - ❖ Engage with mathematical processes
  - ❖ Encourage reflection and metacognition
  - ❖ See the bigger picture of the mathematical world

Table 4.1 suggests a possible cross reference between writing forms and the purposes served.

**Table 4.1 Forms of writing employed in mathematics and possible purposes**

	Language			Communication			Mathematics			
	English	Maths	Writing	Window	Swap role	Affective	Concepts	Processes	Meta-cognition	Bigger Picture
Problem solving processes	yes	yes		yes				yes	yes	
Journal writing	yes		yes	yes		yes			yes	
Essays	yes		yes							yes
Reports on articles	yes		yes				yes			yes
Writing about concepts	yes	yes		yes			yes		yes	
Problem posing		yes		yes	yes		yes			
Investigative reports	yes	yes	yes	yes	yes		yes	yes		yes
Projective techniques	yes	yes	yes	yes	yes		yes	yes	yes	

The table above notes the purposes that a particular writing form almost certainly supports, rather than all possible purposes. For instance, journal writing necessarily supports the use of verbal language, but does not necessarily support the use of mathematical terminology. It could potentially encourage use of the mathematics register, if the journal writing instructions or requirements prompted for the register.

Similarly essays on famous mathematicians and their contributions to mathematics necessarily allow the student a glimpse into the larger world of mathematics and its history, but do not necessarily provide the teacher with a window into the students' thinking processes or knowledge. The essay task could be constructed to report such information, but it is not innate to the task. In choosing a writing task to be integrated into a mathematics class, it should be chosen carefully so as to support the desired aims.

#### **4.5.1 Writing about problem solving processes**

The Oxford English Dictionary provides the following definitions of process:

“Something that goes on or is carried on; a continuous action, or series of actions or events; a course or method of action”, “a systematic series of actions, physical or mental, directed to some end”, “A continuous and regular action or succession of actions, taking place or carried on in a definite manner, and leading to the accomplishment of some result” (Online OED, accessed 1 August 2006). When writing about problem solving procedures, the key is to describe the actions being carried out during the problem solving process, rather than focussing on the objects involved in the actions, or the concepts illustrated by or underpinning the actions.

There is no single way to carry out writing about problem solving. Berlinghof (1989) describes giving his students a list of twelve heuristic strategies (including restate the problem, draw a diagram, look for a pattern, argue by analogy, reason backwards from desired conclusion), drawn from Pólya (1945), and requiring the students to describe their problem solving processes in the context of the given list. Advantages of such an approach are that the students can find a way of discussing what they have done and why, without having succeeded in solving the problem, and the students sometimes succeed in doing more mathematics than they thought they could, having such a list of strategies available to them. Schoenfeld (1985) uses a similar heuristic strategy approach in teaching problem solving. Berlinghof has found that the best results are obtained when each student is working on his own investigative work, rather than all students working through the same set of problems. Kenyon (1989) has a number of different strategies in getting students to write about problem solving processes. Two column proofs, such as typical school geometry proofs featuring a mathematical expression on the left and the geometric reasoning on the right, can be replaced with “prose” descriptions; “explain how” and “explain why” descriptions can be required for a given problem, either short or long. The illustration of the technique that Kenyon

(1989) gives of “explain how” is of a standard algebraic manipulation, not what could be regarded as a true problem, however the technique obviously still applies. Kenyon suggests that replacing two column proofs with prose proofs helps with knowledge organisation as well as with anxiety about maths. Evans (1984) suggests requiring students to give explanations and corrections for errors they might have made in tests or assignments. Depending on the errors, these might or might not be associated with process. Several of Rose’s (1989) writing categories allow the opportunity to discuss problem solving processes, such as summaries, explanations and reports, although these writing activities would have to be structured with problem solving in mind.

Pugalee (2001) required students to carry out mathematical problems, writing down their every thought during the problem solving process. Schoenfeld (1985) and others have had some success with the practice of speaking the problem-solving process aloud, having students work in pairs on problems, talking to one another about what their thoughts are, and being audio taped. Pugalee’s strategy was to have each student working alone, with the observing teacher urging students to keep writing should they observe them pausing for more than 15 seconds. One of the arguments in favour of writing in mathematics, in multiple formats, is that writing encourages reflection. Pugalee argued that reflection would only be occurring were there a level of metacognitive activity during the writing process. Analysis of the writing generated and discussed in Pugalee (2001) revealed the presence of a metacognitive framework. Pugalee concluded that reflection is indeed potentially encouraged by writing in association with mathematical problem solving, and called for further research on writing in mathematics.

Gaskins et al (1994) describe a deeply integrated writing activity in a science course on machines. Problems were given that had more than one potential solution and students had to identify the exact problem, describe several solutions accompanied by diagrams, and describe the scientific principles involved. There was a focus on process rather than content in order to foster deep understanding and metacognitive skills. Embedded as the writing was in a science course, it was necessarily the case that the problems were of a physical nature and had associated principles and concepts (Leonard et al, 1996; Boote, 1998), but a primary focus was on the students describing their thought processes and problem solving processes. Writing was successfully integrated into the course, and

significant improvement was shown in conceptual understanding, although not in procedural understanding.

The form of writing within mathematics chosen for the study project was writing about problem-solving processes, for the purpose of improving the students' problem-solving skills. Writing within mathematics can take a variety of forms, however, a number of which are discussed below, with the aim of giving a thorough overview of this versatile field.

#### **4.5.2 Journal writing**

A writing assignment qualifies as a journal if it is the students' own words on mathematical matters and is a body of writing that is added to over a period of time, such as a school term, or the duration of a university course. The mathematical matters can vary through notes on the work covered in class, notes on reading outside of the course syllabus, or expression of emotional issues surrounding the work.

Journals are very flexible in their use and their purpose. Rose (1989), who divides writing tasks into "expressive" or "transactional" sees journals as belonging to both categories. Journals are transactional in that they can be used to "develop clear expression of ... mathematical understanding" (Rose, 1989, p. 17) through summaries or note taking, and expressive in that they can be used to "generate thoughts about mathematical concepts and processes" (Rose, 1989, p. 17) through the recording of feelings and questions. It is just this dual nature of journals that make them valuable. Kenyon (1989) considers that journals provide a place for problem descriptions and note taking during class, but also a place for thoughts and reflections about mathematics (Borasi and Rose, 1989). Rose (1989) places journals as midway between diaries (entirely personal thoughts) and class notebooks (impersonal facts), and considers them interdisciplinary and developmental. Journals can vary in structure, in content, in format and in how they are treated procedurally.

The transactional content of journals can take many forms. They can be used simply to take notes during class on the content covered each day (Kenyon, 1989; Rose, 1989). Waywood (1992) describes journals as being like workbooks, but requiring a greater degree of involvement from the student. They can be used to express emotional issues such as feelings towards mathematics, personal experiences with the mathematical work and questions that the student might feel awkward asking out loud in class (Kenyon,

1989). Shield and Galbraith (1998) divide writing tasks into journal writing (primarily affective issues, thoughts and feelings about mathematics) and expository writing (describing and explaining), although they cite Waywood's particular type of journal activity as being a combination of the two.

The primary use of journals seems to be to take notes on work covered in class, but in the student's own words, and with accompanying discussion of some kind. The student can simply explain to himself what the day's topic was about and how it works, or ask questions either to himself or the teacher on aspects he does not understand (Kenyon, 1989; Rose, 1989). Definitions and glossaries of mathematical terms can be recorded in the journals, to be referred back to when needed (Kenyon, 1989). By the student putting the work into his own words and discussing it (either with himself or with the teacher) in his writing, opportunity is provided for reflection on the work (Baker and Saul, 1994; Kenyon, 1989), and reflective abstraction (Cooley, 2002) is potentially encouraged. A sense of ownership of the mathematics is encouraged, rather than the student feeling disempowered by having work thrust upon him over which he has no control (Ellerton and Clarkson, 1996). Kenyon (1989) suggests an interesting use of journals, as an ongoing investigative project, to record thoughts and observations over a period of time with a chapter or book on a mathematical topic *not* covered in class. Some form of report or presentation might be called for at the end of the period.

Journals serve as a communication device between teachers and students (Borasi and Rose, 1989). Seeing the students' descriptions of mathematical content in their own words can give the teacher insight in the students' mathematical knowledge and beliefs, as well as insight into the students feelings about mathematics. Communication back to the students is provided by any commentary the teacher chooses to insert into the journals, assessment of the journals (Waywood, 1992), or answers to questions the student has posed (Kenyon, 1989; Ellerton and Clarkson, 1996; Cooley, 2002). A characteristic of journals that is valuable in a multilingual class is that the teacher can make it clear that mathematical content and thoughts are what is important, not degree of facility with English, or level of preparedness in mathematics, making it a non-discriminatory form of expression or of assessment (Waywood, 1992).

Waywood (1992) stresses that, in order for journals to have a beneficial effect on the students, to be more than "a diagnostic or alternative assessment tool" (Ellerton and

Clements, 1992, p. 156), the students must be clear on what the intent of the journal is, class time should be given over to the journals and the journals must be clearly valued, through assessment and being reported on. Waywood (1992) provides a careful description of how to assess journal entries, although Ellerton and Clements (1992) raise the question of whether it is ethical to assess something as potentially personal as a journal entry.

Ellerton and Clements (1992) suggest that a stronger theoretical basis for the benefits of journal writing be developed, and repeat that call in 1996. Waywood (1992) provides a carefully designed means of assessing journal writings (p. 39) but also calls for research projects that explore and define the connections between writing and learning, specifically learning mathematics. Ellerton and Clements (1996) support Clarke, Waywood and Stephens (1993) in their endeavours to provide a possible theoretical basis for analysing journal writing (with the journal entry classification scheme adapted for use in this study project), although such an analysis does not fill the need for finding a link between writing and learning.

### **4.5.3 Essays**

Essays can take a variety of forms, chief amongst which are essays on famous mathematicians or famous mathematical problems or events, and extend to essays on mathematical careers. While the use of essays to practise using the English language (or whatever the language of learning and teaching) and writing skills, opinions differ on the mathematical benefits of essays. Berlinghof (1989) has firm negative opinions, finding, in his experience, that they encourage plagiarism, and do not invite engagement with mathematics. Others have found benefit to essays, such as Mendez and Taube (1997), who were actively engaged in creating cross disciplinary tasks involving a mathematics course and a writing course. Mullin (1989) has found essays (with varying degrees of technicality) to be useful in encouraging heuristic and nonmathematical understanding of physics, which he infers is not necessarily developed through a purely mathematical approach to physics, and is necessary for intellectual development as a physicist.

Berlinghof's claims of plagiarism and lack of engagement could not be made by Rose (1989) whose essays are more than simply recitation of mathematical or historic facts. Rose's essays call for the students to explore mathematical relationships and explore mathematical ideas and are less accessible to plagiarism. Rose also refers to "essay

questions” in tests such as “what does it mean to solve an equation?” which could arguably not be called an essay. Kenyon (1989) extends the “famous mathematician” line of essays into essays on careers related to mathematics, although such essays would be as easy to plagiarise as those of which Berlinghof disapproves.

Essays completed by students could be simply handed in as assessment tasks, presented in some forum, or even collated into a conference proceedings form (Coutis and Wood, 2002; Ellerton and Clements, 1992). Ellerton and Clements (1992) criticise the “process/conference approach” (p. 154) as having too little evidence supporting its benefit to learning and suggest that students often do not find it enjoyable, when one of the benefits of writing is supposed to be that it is “an enjoyable, creative experience” (p. 154).

#### **4.5.4 Reports on journal articles or books**

Similarly to essay writing, researching a book or a journal article and writing a report on it is not only supposed to allow for the practice of English (or the language of learning and teaching) and the development of writing skills, but is also supposed to allow the student to see the application of the mathematical topic to the mathematical world or another related field (Coutis and Wood, 2002; Rose, 1989). Arguably journal reports would do the job of enlightening the student about the applicability of the topic better than an essay on, say, the life of Gauss would. However, some disagree on the benefit of such reports. Berlinghof (1989) does not consider journal article reports as beneficial to the students as the students’ submissions are simply several pages that boil down to “I liked it”, “I didn’t like it” or “I didn’t understand it” (Berlinghof, 1989, p. 89).

Berlinghof’s objection to journal reports is similar to his objection to essays, which is that they do not encourage engagement with the mathematics, a justifiable concern, and one that would have to be kept in mind by any teacher assigning such a task. An extension to reports on journal articles are book reviews (Coutis and Wood, 2002; Kenyon, 1989), and reports gleaned from multiple sources on the use of mathematics in various fields, such as insurance or computer technology (Kenyon, 1989)

#### **4.5.5 Writing about mathematical concepts**

Mathematical concepts can be investigated by students using a variety of writing formats, such as explanatory paragraphs (Gopen and Smith, 1989; Leonard et al, 1996), chapter outlines (Kenyon, 1989), thoroughly integrated problem design, description and solution (Gaskins et al, 1994), reworded definitions (Evans, 1984; Rose, 1989) and

summaries (Rose, 1989). Examples of concepts include mathematical ideas such as function, limits and derivatives (Cooley, 2002), and more physical terms such as boiling and dissolving (Fellows, 1994). Kenyon (1989) suggests requiring students to compare and contrast concepts such as circles and ellipses, functions and relations, and equations and graphs. Providing the students with triplets of examples, clearly representing a concept conformed to by two of the examples (circles, perhaps) and not conformed to by the third (an ellipse, perhaps) can be an encouragement for the student to develop a personal construct of the concept (Bannister and Fransella, 1986). It is important to note that writing activities are not necessarily designed such that process and concepts are distinguished from one another, and nor should they be. The purpose of the writing assignment should dictate its design, and that purpose need not distinguish between process and concepts/content.

The action of writing about mathematical concepts is thought to have many beneficial effects. Writing is said to bring about reflective abstraction (Cooley, 2002), which is defined as being a mechanism for the construction of new knowledge. It is widely believed that writing facilitates understanding (Cooley, 2002; Fellows, 1994; Gaskins et al, 1994) and dissuades misconceptions (Cooley, 2002). Fellows (1994) holds that that it is the looking back nature of writing that brings about changes in conceptual understanding, while Gaskins et al (1994) hold that it is the deep engagement brought about by writing that deepens understanding of concepts. Glynn and Muth (1994) consider writing beneficial in that it is a way of communicating mental representation of ideas and the relations between those ideas. It is clear that focussed research is needed to tease out what it is about writing that is encourages the growth of conceptual understanding, resulting in Evans' (1984) appealing description that the students result in *owning* the knowledge instead of *renting* the knowledge.

Physical principles and concepts play a large role in the teaching and understanding of physics and school science. The theory and study of writing about concepts is therefore relatively better represented in physics education literature than in mathematics education, although they can be closely related (Leonard, Gerace and Dufresne, 1999; Leonard, Dufresne and Mestre, 1996; Fellows, 1994; Glynn and Muth, 1994; Gaskins et al, 1994).

In common with many other writing forms (notably essays and journals), writing about mathematical concepts can encourage and improve English (or another language of learning and teaching) usage (Cooley, 2002), although simultaneously it could be regarded as a discriminatory practice, disadvantageous to NESB students (Lee and Fradd, 1998). Again in common with other writing forms (for instance journals and problem posing), writing about mathematical concepts can provide the teacher with a window into the students' thinking and knowledge, which can have beneficial effects on teaching.

#### **4.5.6 Problem posing**

Requiring the students to pose their own mathematical problems forms a small part of the practice of writing within mathematics. The problems can be posed simply for the student's own practice, for the students to give to one another to work on or even to put into a database to be drawn on in future assessment tasks. Rose (1989) refers only to the posing of word problems, however unless the course demands it there is no need to restrict students' problem posing to word problems alone. Problem posing is believed to simultaneously require good understanding of the material (Kenyon, 1989) and to encourage understanding of mathematical relationships (Ellerton and Clarkson, 1996; Ellerton and Clements, 1992; Bell and Bell, 1985). Kenyon (1989) adds that not only understanding is necessary, but the ability to organise and to synthesis information. Ellerton and Clarkson (1996) admit that problem posing appears well founded as a learning device, but suggest that more research is necessary in this area.

#### **4.5.7 Investigative mathematics project reports**

Berlinghoff (1989), who is somewhat critical of the usefulness of essays or journal article reviews, is an enthusiastic supporter of investigative projects, and has had success in bringing about student learning through the resulting mathematical engagement and the use of heuristic strategies. Berlinghof describes investigative projects involving pure mathematics, while Kenyon (1989) give a range of ideas that are more applied, such as drawing up a document outlining how to form a financial planning group to give financial advice to students, how maths is used to measure and evaluate intelligence or reading skills, and a discussion of a retired person's budget. Of course, there is the danger that Kenyon's suggestions could have very little mathematical content, so the project design would have to be carefully planned to encourage a maximum of mathematical content. Kenyon (1989) also suggests having the students slowly read through a chapter of a book on a topic not covered in the course

syllabus, keeping a journal throughout the process, and then writing up a report on the chapter at the end of the course. Rose (1989) refers to “projects” that are, in general, not investigative or even very long, like mathematical bumper sticker slogans or radio advertisements for some mathematical process.

Ellerton and Clements (1992), while not writing off investigative projects, are somewhat critical of the skills learnt (are the genre of writing, and research techniques learnt different from those learnt in other aspects of the course?) and the time spent (was it justified in terms of subsequent tertiary maths courses and learned ability to carry out research investigations?). In Ellerton and Clements (1996) they answer their own first question in the affirmative, in so far as they cite studies strongly suggesting that the genre of writing expected in reports is different from those expected in other types of mathematical inquiry, and students who satisfactorily complete investigative reports acquire research techniques not previously acquired

#### **4.5.8 Projective techniques**

Projective techniques are tasks that locate ignorance in someone other than the student and make the student, at least temporarily, the transmitter of mathematical knowledge (Adler and Davis, 2003). Evans (1984) suggests having the students write about mathematics to “an uninformed third party” (p. 831), while Kenyon (1989) provides a list of suggested tasks such as a letter to an intelligence “testing service” (p. 78), a letter to a friend on how to solve a particular problem, or, somewhat different in its location of knowledge, a letter from a professional on why the students need to learn mathematics. Shield and Galbraith (1998) similarly required students to write a letter to a friend who has missed a lesson, with the added characteristic there was, in fact, such a person. Rose (1989) has some variations on the theme, suggesting students set problems, notably word problems, for one another to answer, or for the students to draw up a book on a topic over the course of a term, such as one on matrix algebra. Ellerton and Clements (1992) refer briefly to students writing letters to (imagined) children who have missed a class. An anecdote that springs to mind from the writing project described in this thesis is of the tutor saying to a student who was reluctant to write about a particular problem: “Pretend we’re on the phone and I missed the lecture. What would you say to me?” “I’d say, don’t skip lectures!” responded the student.

#### **4.5.9 Forum for emotional expression**

While some writing formats allow for the expression of affective issues as one of their characteristics, notably journal writing (Borasi and Rose, 1989), other forms of writing exist solely for the expression of emotional issues of one kind or another. Rose (1989) mentions writing letters to the teacher on feelings about mathematics, submitting slips to the teacher with issues that can be dealt with anonymously either privately or with the whole class, and autobiographical writing about personal mathematical experiences. Kenyon (1989) refers to asking the students to write a letter as if to a friend on how they feel about mathematics. Clarke (1987) required students to respond in writing approximately once a fortnight on how they felt about mathematics, where they felt they needed help, their worries, and their suggestions for improvement to the mathematics class. The procedure was successful in making the students feel involved with the mathematics course and provided diagnostic information for teachers, although it did not appear to have much effect on the students' mathematical knowledge.

#### **4.6 The current predicament**

Ellerton and Clements (1992) list five research questions which they feel need to be addressed in relation to writing to learn in mathematics.

If students participate regularly in writing mathematics activities, are they likely

- to perform as well as non-participating students on standard assessment tasks?
- to link their mathematical understanding more easily with their personal worlds?
- to improve their monitoring of their own mathematical thinking so that they improve their problem solving performance?
- to develop more positive affective responses to mathematics, and to develop feelings of mathematical ownership?
- to become more aware of their own abilities, attitudes and preferences in mathematics and be prepared to modify these in response to their own reflection?

(Ellerton and Clements, 1992, p. 158)

Waywood (1992) joins in with Ellerton and Clements' general call for research projects that explore and define the connections between writing and learning, specifically learning mathematics. Pugalee (2001) raises the issue that, in order for writing to have a beneficial effect on reflection and metacognitive processes, as suggested in the third and fifth questions above, it must first be shown that such a metacognitive framework does exist during the process of writing mathematics. Pugalee's research strongly suggests that such a framework does exist, prompting him to call for more research in the area of metacognitive and reflective writing activities.

Problem solving, as convincingly defined by Pólya (1945), has *understanding the problem* as the first step. It is a not uncommon belief that writing aids understanding, although it is not clear what characteristic it is of the writing process that does so. Is it the reflective, and looking back nature of a thoughtful writing assignment (Fellows, 1994), or is it the deeper engagement that a thorough writing assignment encourages (Gaskins, 1994), or something else entirely?

While there is widespread belief and evidence that writing is a valuable learning device and activity, there is a need for greater research in what exactly it is about writing that encourages learning, in mathematics particularly. It is the suggestion of this thesis that writing increases mathematical understanding through the necessarily deep engagement with the subject matter, thereby encouraging a form of cognitive knowledge construction not encouraged by shallow engagement.

In closing, some support from an unexpected quarter:

*Reading maketh a full man, conference a ready man, and writing an exact man.*

Francis Bacon (1561 – 1626)

## 5 Mathematical Preparedness

A century ago, students enrolling for a university mathematics course tended to fit a particular profile. Certain skills could be assumed by the lecturer, and classes tended to be small (Zevenbergen, 2001). In recent times, this situation has been changing, with a greater number of students enrolling for courses, and those students having far more diverse backgrounds. “Recent times” for some means only the last decade or two (Cox, 2000), while others have noticed changes dating from the end of the Second World War (Wood, 2001). “Diversity” can take on different meanings. Student bodies are diverse in many respects, such as culture, language and academic preparedness. Classes, at tertiary level, tend to be large, often comprising several hundred students. The changes in the student body over the last several decades may perhaps require or demand changes in the modes of instruction, changes that often have not occurred. One must not fall into the trap of making it all sound simple, however. Should instructional forms change? Why should they change? How should they change? In response to what, exactly, should they change?

The author, from personal experience was concerned with the question “*How can I effectively teach problem-solving to large classes diverse in mathematical preparedness?*” While other issues addressed in this thesis, such as language, problem-solving, and theories of cognition, are backed up by a wealth of literature, the issue of preparedness finds little recognition in the available literature (Zevenbergen, 2001), leaving the lecturer of a large, academically diverse class at something of a loss when searching for strategies which may have worked well for others or that are well grounded in research.

### 5.1 Mismatch between school and university

The concern at tertiary level that student are not meeting the expectations of preparedness that once were met by their predecessors is not isolated. Hoyles et al (2001) cite a variety of studies expressing disquiet over the mismatch between university expectations and students’ levels of mathematical skill, as well as disquiet over the wide range of skill levels apparent. It is widely agreed that over a period of time, a divergence has come about between the mathematical skills developed by students at school level and the mathematical skills desired in a tertiary mathematics course. There is some disagreement over when this divergence started to appear, partly

because it is such a difficult phenomenon to measure, and partly because no one was attempting to measure it until recently. Wood (2001) describes the necessity for academic structures such as bridging courses for returning soldiers after the Second World War; Hoyles et al (2001) cite Thwaites (1972) referring to a view expressed in 1961 that the divergence was already clearly apparent; Cox (2001) considers the effect to have been primarily apparent in recent decades, and Hoyles et al (2001) cite Nishimori and Namikawa (1996) describing Japanese mathematicians as being more or less equally divided over whether the effect began in the 1980s or 1990s.

The reasons for the perceived changes are several, chief amongst them the great increase in the numbers of students committing to higher education. Not only is absolute population size increasing all over the world, but a greater proportion of that population is choosing to pursue education beyond secondary level (Wood, 2001; Cavender and Rutter, 1997; Hoyles et al, 2001; Northedge, 2003) at least in the countries whose phenomena dominate educational research and debate. The resulting increase in numbers of students at universities necessarily means increased diversity in the student body. One dimension, among many, along which diversity can be measured is mathematical preparedness. Terms used to describe the diversity of mathematical preparedness are numerous and include “academically diverse” (De La Paz, 2005), “diversity of preparedness” (Cavender and Rutter, 1997), “non-uniform degree of mathematical preparedness” (Steele, 2003), “variable skills profile” (Cox, 2000), “wide range of academic background” (Hoyles et al, 2001) and “mixed ability” (Agudelo-Valderrama, 1996). It must be noted that *preparedness* and *ability* are different concepts and should not be confused. When the students come from widely differing educational backgrounds, their levels of preparedness cannot be taken to indicate their levels of mathematical ability, and, in fact, it is this very disparity that is an issue of concern. If, in one’s class, one has students with different levels of preparedness, how can one teach so as to best nurture the mathematical abilities of the students? Perhaps the traditional teaching by transmission, “chalk and talk” and a positivist, traditional empiricist philosophy of teaching is no longer appropriate. Wood (2001) suggests that a proactive stance be taken with regard to the transition from school to university, the changes in higher education recently having been more reactive to student diversity and change: the view of tertiary mathematics remaining unchanged over the years and the changing student body being reacted to with bridging courses. The changing student profile has impacted higher education in various forms, such as the introduction of bridging

courses, changes in curriculum, changes in assessment strategies and provision of academic support programmes (Hoyles et al, 2001; Wood, 2001; Northedge, 2003).

## **5.2 Reactions to diversity**

One reaction to the changing student body is to recognise a mismatch between what students bring to their university studies and what the universities expect of the students and, as a result of that recognition, to change the curriculum or to provide a bridging course (Wood, 2001). For instance, Cox (2000, 2001) acknowledges that the students registering for a first year university mathematics course will almost certainly come in with widely differing mathematical knowledge and skill levels, and that such differences might not be immediately apparent in their school leaving symbols (in the UK). He has designed diagnostic tests to define the skills that might be lacking by some or all of the students and to target those skills specifically at the beginning of the class. Whether the “bridging” occurs at the beginning of a course, in order to bring all students to the same desired level of preparedness, or whether the bridging course is separate to any other courses, it still equates to the same effect of bringing about a desired level of (perceived) homogeneity in the students.

A second reaction is to create streaming (otherwise known as tracking or setting) in the course, where students of different levels of preparedness are taught in different classes (Cox, 2000). Depending on how the classes are treated, streaming might amount to creating a bridging course for one stream while the remainder of the students continue with the mainstream curriculum. However the classes are treated, streaming is a contentious issue and one not to be embarked on lightly (Rubin, 2003 among others). The biggest weakness with respect to preparedness, in the author’s view, of both streaming and bridging courses is that, unless classes are shrunk to the size of a handful of students, they will retain their diversity. Classes will perhaps have a smaller range of diversity of preparedness, thereby decreasing the difficulties inherent in teaching such a group, but those difficulties still exist and must be confronted by an educator intent on equity.

A third reaction is to create teaching materials, or adjust the mode of instruction, such that all students can participate in classroom activities at their own level, all students learning from the work with which they are engaging; a process easy to state, not easy to carry out. It is with respect to this third reaction that there is a dearth of literature,

perhaps not surprisingly due to its challenge. Answers to the question *How does one teach a large class of academically diverse students?* can be found scattered thinly in the literature, varying through use of classroom resources to changing the philosophy of teaching. The use of multimedia (Cavender and Rutter, 1997) involves changing the classroom format, preferably holding classes in classrooms that are at least partially computer laboratories or “smart” classrooms. Northedge (2003) suggests a rethinking of the entire teaching philosophy, viewing teaching as enabling participation by novice academics in the discourse of an unknown knowledge community; unknown in the sense of the novice being an incoming member of the community, yet superficially aware of the knowledge embodied by the community. Yet a third approach is to design activities to encourage particular modes of thinking, such as De La Paz (2005) using writing within a history course to encourage argument and historical reasoning. These three approaches, representing a broad cross-section of answers to the vexing question, are outlined in brief, but first the word *equity*, used so lightly, is differentiated from the word *equality*.

### **5.3 Equity and Equality**

Secada (1989) expresses concern that the terms *equity* and *equality* are often used synonymously. Often equity is meant when the word equality is used. Equity is “the use of processes, tools and mechanisms to promote equality of opportunity (both equality of access as well as equality of outcomes) in ensuring fair treatment of all” (Cassim, 2005, p. 653). Equity “refers to our judgements about whether or not a given state of affairs is just” (Secada, 1989, p. 68). Zevenbergen (2001) succinctly differentiates between equity and equality: unequal treatment to produce more equal outcome (equity) versus equal treatment with the potential of unequal outcomes (equality). Agudelo-Valderrama (1996) would agree that equality is not necessarily equitable: “Some may claim that by presenting each pupil with the same requirements, education is following an egalitarian line, but this claim is challenged by the sad fact that it does not allow each individual to his or her own potential, leading to failure in most cases” (p. 23). Cassim (2005) takes the view that “diversity is about acknowledging and managing differences” (p. 653), “providing for differentiation” (Agudelo-Valderrama, 1996, p. 20), to attain absence of discrimination.

When the seemingly unavoidable logistics of the classroom situate the teacher at the front of a classroom of 200 students representing a wide ranging diversity of

mathematical preparedness it is easy to slip into the traditional transmission form of teaching (Millett, 2001). However, aside from potential philosophical and pedagogical arguments against such a transmission form of teaching, there are equity arguments against such an approach. Each student, in such a scenario, receives equal treatment, but possibly not equitable treatment. In order to “deal with large classes on a whole class basis” (Cox, 2000, p. 7) rather than dividing or streaming the class into smaller groups, the teacher has to devise ways of treating the students equitably or risk either (perhaps both) boring the more prepared students or losing the less well prepared students.

## **5.4 A level experiences**

Studies in the United Kingdom on mathematical preparedness describe, in some detail, the changing nature of A levels and the variety of forms the examinations can take. Simply to have taken A level mathematics does not convey as clear a message as it might once have done, since there are different examination boards and different modules within some A level courses (Kitchen, 1999; Northedge, 2003; Steele, 2003). Since A level concerns are not a large issue in South Africa, the details will not be embarked on here, except to note that there is a nontrivial cohort of students studying first year mathematics at the University of Cape Town whose schools have offered A level mathematics, distinct from standard South African grade 12 mathematics. As a result, variation within the A level spectrum does have some effect on diversity of preparedness, albeit not as great an effect as socioeconomic and historic racial factors.

## **5.5 Dealing with diversity**

The reasons for the diversity of preparedness in a first year mathematics class in South Africa include the reasons pertinent elsewhere in the world, such as the simple one of increased numbers, and a variety of tertiary entry level examinations, but also socioeconomic factors and historical educational disadvantage. Details of how South Africa’s history impacted on education can be read elsewhere (Moodley, 2004; Howie and Plomp, 2002; this thesis Chapter 3), here we are concerned with the challenges inherent in teaching a large class where there is extreme diversity of preparedness. Three very different teaching approaches are outlined in brief.

### **5.5.1 Multimedia**

Multimedia, that is use of a large screen at the front of the classroom, displaying notes, animated graphics and video in various forms, can improve on the traditional large classroom in many ways, suggest Cavender and Rutter (1997). One change is the

straightforward physical change from a small overhead transparency screen or small blackboard to a large screen (if, of course, the lecture venue comes equipped with such a screen). The academic subject of concern for Cavender and Rutter is the biological sciences, where they find that increased enrolment, diversity of preparedness and a variety of learning preferences mean that the old fashioned teaching by transmission (which they call “straight lecturing” (p. 53)) does not work as well as it is perceived to have worked in the past, and new methodologies are called for. By using multimedia presentations, they suggest that multiple learning preferences can be met by providing audio, visual and text representations of information. In addition, in a science that experiences rapid changes, the internet provides a constant source of appealing multimedia material. Cavender and Rutter further go on to suggest that, not only do students exhibit a variety of learning preferences, but that student learning styles as a whole are changing, due to the influence of technology and commercial media.

The fascinating picture painted by Cavender and Rutter (1997), among many others, of the advantages and power of teaching through and with multimedia is, unfortunately, constrained by sometimes harsh reality. Not all classrooms have large screens. Not all classrooms are “smart”. “First, it is necessary to recognise that the variable skills profile that incoming students present due to wider access and changes in school curricula does require a significant rethink in the way we sometimes teach in universities, particularly in the first-year. But, equally, one has to recognize that our resources and options in this respect are limited” (Cox, 2000, p. 7). Switching from “talk and chalk” to multimedia is not necessarily as easy an option as Cavender and Rutter suggest.

### **5.5.2 A socio-cultural approach**

Northedge (2003) considers that “knowledge delivery” modes of teaching are insufficient in the context of diversity of backgrounds, expectations and levels of preparedness. He proposes a socio-cultural view of learning and teaching, with students participating in the discourse of an unknown knowledge community, and teaching understood as supporting that participation. Teaching, Northedge suggests, should be viewed as enabling participation in knowing, as opposed to the transmission of information or the construction of concepts.

Novice academics need to be able to participate in the discourse through speech and writing, in order to develop an identity as part of the community. He asks “How, then, can teachers construct discursive environments that allow students to participate at a

variety of levels? How can they support and assess students in ways that recognise the legitimacy of progress at different levels?” He recommends that an appropriate discursive environment includes

- an environment with multiple voices - Traditional courses have one lecturer, one voice, controlled, polished and authoritative. To be participants in knowledge, the students need dialogues in which they can take part. It is up to the lecturer, or course designer, to create the environment in which such conversations can legitimately take place.
- an appropriate target knowledge community – The target knowledge community is not necessarily the academic community, as is often tacitly assumed. Within what knowledge community are the students learning to participate?
- intermediate levels of discourse - Students are usually unprepared to engage in mainstream specialist discourse. “One of the primary functions of education is to construct intermediate levels of discourse” (p. 29) which allow for relatively unskilled participation. Such participation encourages the students to feel as if they are already true members of the community, which in turns encourages deeper participation and engagement with the knowledge embodied by the community.
- a sustained discursive environment - The environment should be coherent and consistent.

Northedge draws attention to the difference between vicarious and generative participation. Vicarious participation (distinct from passivity) is associated with listening, reading, and not directly contributing to the flow of information, such as listening to a lecturer in class or reading an academic paper, or even watching television. Vicarious participation is vital, as it allows the students to pick up discourse from more experienced discourse members, but generative participation is also important. Generative participation is associated with shaping shared meaning, taking part in verbal conversations and having a direct effect on information flow. Generative participation can be achieved through small group work as well as writing assignments. Writing assignments, Northedge stresses, have to be commented on or the student will not actually feel as if he is truly participating. The student has to be able to generate an identity as a discourse participant. In addition, if diverse students are engaging with work on a variety of levels, then the ability to engage creatively must be assessable, rather than assessing solely correct or incorrect answers.

The writing exercises employed in the study project did exhibit some of the attributes of Northedge's model, although the idea of discourse was not stressed and the sociocultural view of a community of shared learning was not present. The writing exercises did allow the students to engage with the work on multiple levels, and to take part in a conversation, albeit to a limited extent. The writing exercises were commented on, a fact which came out in student interviews as being an important part of the exercise. "It also shows that your teacher's really interested, because one thing that happens from school is you lose that teacher interest. All those writing exercises, and your comments, made me feel like I was actually being paid attention to, and that's very motivating" (AS1 student interview). The writing exercises and the comments allowed the student to take part in a mathematical conversation that was otherwise not available to them in the large, impersonal classroom (Wood, 2001).

### **5.5.3 Reasoning through writing**

De La Paz (2005) has successfully used writing in an academically diverse class to promote reasoning and argument in the subject of history, influenced by the distinctions between novice and expert historians. Experts view documents "as artefacts shaped by the events of the specific time periods, and representing interpretations of contextually bound events." (p. 139). When studying a topic or event, experts compare documents, weighing up contradictory evidence, using multiple methods of analysis to come to conclusions, and allow their knowledge of the time period to give a frame of reference to their reasoning. Novices want to gain knowledge on a topic and use documents as sources of knowledge, often ignoring contradictory evidence. Experts and novices differ when writing about historical events: experts construct an interpretation of an issue, relying on primary documentation, while novices struggle to integrate historical information into an argument.

Comparison between De La Paz (2005) and the literature on writing in mathematics suggests some interesting similarities between history and mathematics, for instance school teaching and textbooks are partly responsible for the novice attitude, portraying history as chunks of facts to be memorised and with no problem-solving approach. De La Paz found that requiring students to write arguments helped with integration of facts, improved historical reasoning, and promoted greater conceptual understanding of historical content. The class described in De La Paz was academically diverse, divided into talented writers, average writers, and students with disabilities. The initiative of

using writing to promote historical reasoning was successful for all students, roughly maintaining the academic hierarchy, but showing improvement for all students in persuasiveness, number of arguments and historical accuracy.

## 5.6 Conclusions

It is widely agreed that increased enrolment, the changing nature of degrees to meet employment demands, the changing nature of school leaving examinations and sundry other influences, some specific to particular countries, have increased the diversity of university student bodies all over the world. Increased diversity is observed in language, culture, expectations and preparedness, all of which need to be taken into account if equitable education is desired. While some aspects of diversity have received much attention (language, for example), others, such as preparedness, have not received as much attention, reactions primarily being to allow for less well prepared students by creating bridging courses or something similar. Answers to the question *How does one teach a large class of academically diverse students?* are not easily found, are not numerous, and are not even similar. Some answers are not easy to carry out, either personally (to change one's entire philosophy of teaching and learning) or practically (to change one's old fashioned lecture theatre into a smart classroom). Creating materials or teaching resources with which students can engage actively, at their own level is easier to carry out, demanding neither personal philosophy changes, nor razing buildings to the ground to rebuild from scratch, yet it is hardly an unproblematic option. It is the suggestion of this thesis that using writing as a tool within mathematical problem solving allows students of widely different levels of mathematical preparedness to actively engage with mathematics in a common classroom, to efficiently access learning and, in so doing, to experience mathematical motivation and support.

## 6 Learning theory: Piaget and Vygotsky

Jean Piaget (1896 – 1980) had a considerable influence on the field of psychology, notably child developmental psychology as well as epistemology. Arguably the best known theory of Piaget is his stage theory of development, which avers that a child's mind develops in cognitive stages. Particular elements simply cannot be learnt by a child at any specific time in its development if the child is not yet in the appropriate stage. Piaget suggested several stages, passing through four periods, called the period of sensori-motor activity, the period of pre-operational thought, the period of operational thought and the period of formal operations (Piaget, 1972). The period of formal operations includes the ability to think in abstract ways (Sutherland, 1992) and develop logico-mathematical reasoning. The original Piagetian stage theory suggests that by the age of approximately thirteen, a person has passed through all the stages of cognitive development. Criticisms of the stage theory range from utter rejection of the notion (Bryant, in Sutherland, 1992), to disagreeing on the ages during which the stages occur (Miller, 1993).

The static structuring of cognitive stages is insufficient in itself to describe development and Piaget also proposed a theory of learning, describing how knowledge is acquired and cognitive processes are formed. The learning theory, also called Piaget's stage-independent theory (Ozgun-Koca, 2006), is constructivist, in that it describes knowledge as constructed in the mind by the cognising subject as cognitive processes continually being created by restructuring of cognitive structures already present in the cognitive system. Modern constructivists subscribe to many different schools of constructivism, some of which are quite close to Piaget's early constructivism (for example, radical constructivism) (von Glasersfeld, 1995c; Steffe, 1995) and some which are quite different and quite critical of Piaget (for example, social constructivism) (Rosenthal and Zimmerman, 1978; Lave and Wenger, 1991).

While Piaget's studies centred on children and their development, his theory of learning is postulated to hold through adult life; it is not a theory designed only to apply to children (Ginsberg and Opper, 1988). The various types of "abstractions" outlined by Piaget are necessary for learning throughout the development during all the different periods and stages (Pascual-Leone, 1996), but they continue to be used as the means of learning throughout adult life (Ginsberg and Opper, 1988).

In the period of formal operations, the person learns to think in symbolic terms. Examples of symbolic thought include formulae in physics, the Holy Trinity in Christian religion, and allegory in literature and philosophy (Sutherland, 1992). Logico-mathematical experiences require a prior level of abstract thinking that is deemed to occur in this final stage of Piaget's theory of development.

Piaget's theories have come under heavy criticism, yet there are still schools of thought that support his theories. Even today, after decades of criticism (Sutherland, 1992), there exist supporters, ranging through fundamentalist Piagetians, neo-Piagetians and post-Piagetians, alongside the schools that completely disregard his theories.

There is more criticism for his stage theory of development than for his theory of learning, however, and such criticism of the learning theory is less a denial of its validity than its lack of underlying mechanism, being more descriptive than explanatory. The terms equilibrium, assimilation and accommodation are "so vague as to be irrefutable. The terms are polysyllabic descriptions rather than explanations of cognitive growth" (Boden, 1979, in Sutherland, 1992, p. 75). Resnick and Glaser (1976) also express dissatisfaction with Piaget's epistemology not providing an explicit plausible mechanism. Dubinsky and Lewin (1986) seem to be regretfully accepting the inaccessibility of mechanism when they say "it would seem one never has direct access to cognitive processes – thought *is* an unconscious activity of mind – but, at best, only to what an individual can articulate or demonstrate at the moment of insight itself. Precisely what occurs at that moment seems as inaccessible as it is essential." (p. 57, emphasis in original)

Vygotsky (1896 - 1934) has had enormous impact as a theorist of educational psychology and epistemology, focussed far more on the social influence on an individual than on the individual itself. Comparisons of the work of Piaget and Vygotsky can simultaneously find similarities between their theories and underline differences. Vygotsky and Piaget had different emphases on the importance and use of language in learning and knowledge development, Piaget suggesting language as being dependent on development and Vygotsky quite the opposite. Vygotsky's interest in language caused him to spend some time on the idea of writing within the epistemic

context and it is Vygotsky's thoughts on writing that are being addressed in the discussion below.

It is not the intention of this thesis to take a stand for or against any theoretical standpoint. There are theorists who adhere closely to Piaget's theories and reject much of Vygotsky, and vice versa. There are theorists who accept theories from both Piaget and Vygotsky, and still others who feel they were really saying the same things, each in his own way. This thesis does not attempt to position itself in relation to any of the standpoints, but rather recognises features of the learning process as observed in the writing study project to be describable using Piaget's learning theory and enriched by observations of Vygotsky on language and writing. The primary theoretical tool used for analysis is the three pronged alpha, beta and gamma behaviour (Piaget, 1978) demonstrated by the epistemic subject, with particular focus on the utility of a perturbation within that behavioural structure.

## **6.1 Abstraction**

Piaget's learning theory centres on abstraction (or its absence). He recognises three types: empirical abstraction, reflective abstraction and pseudo-empirical abstraction. Confrey (1994) points out that the Latin roots for the word "abstract" are *ab* (away from) + *trahere* (draw), that is, "withdrawn", and the word is also related to "absolve", meaning to free from material consideration. Confrey goes on to draw a more or less whimsical comparison between pure mathematics and religious or spiritual observance, in which mathematics is a "highly valued form of penance" (p. 40). Be that as it may, to abstract does still mean to draw something (an essence) away from something else (a source), to gather a fragment of information from an object or from some process or larger body of information.

### **6.1.1 Empirical abstraction**

Empirical abstraction is "abstraction from perceived objects" (Beth and Piaget, 1966, p. 188), and "consists merely of deriving the common characteristics from a class of objects" (*ibid*, p. 189). "Empirical abstraction consists in isolating the properties and relationships of external objects" (Vergnaud, 1990, p.18). In his early works, Piaget uses the term "simple abstraction" rather than empirical abstraction (Ginsberg and Opper, 1988; Confrey, 1995c). Empirical abstraction is associated with exogenic knowledge where "the source of knowledge is ... external to the person" (Ginsberg and

Opper, 1988, p. 218; Steffe, 1995). Dubinsky (1991a) holds that empirical abstraction has to do with experiences that are external to the subject; the properties involved in empirical abstraction are external to the subject, yet knowledge about them is constructed internally. However, von Glasersfeld (1991) insists that the objects involved in empirical abstraction are not external to the subject but merely external to the current mental process. Von Glasersfeld in another work (1995b) invokes physical objects as the source of empirical abstraction, applying Piaget's somewhat overused example of abstracting characteristics (such as colour) from apples. It could be argued that Piaget himself changed his definition of empirical abstraction as his 1966 work with Beth clearly seems to refer to objects external to the subject, while the references within von Glasersfeld (1991) to Piaget's 1977 work seems to permit them to be internal. Either way, it is not empirical abstraction that is of interest here.

A related phenomenon of observations of similarities in experiences is fundamental to Kelly's Personal Construct Psychology (Bannister and Fransella, 1986), developed in the 1950s. A personal construct is an explicit or implicit subjective set of contrasts within a triplet of objects, experiences or other constructs. A construct represents an imposition or extraction of a meaning, possibly one of many, which a person might draw from past experience or create anew. The notion of a comparative triplet need not be confined to physical objects or people, but can be extended to abstract entities, experiences or relationships. Triplets of objects, experiences or relationships, whether or not they are verbally labelled in the perception of the subject, give rise to higher order constructs, at further levels of abstraction, and constructs and construct systems are themselves hierarchical in nature. Constructs themselves, once elicited and subjectively explicit for the person, can serve as the basis for triplets of observation from which constructs at higher levels of abstraction can be formed. Evidence for the existence of at least some constructs with subjective primacy can be deduced by the use of a "repertory grid", which provides a means of finding correlations between elements and constructs. Subjects identify ways in which multiple observations, grouped in triplets, are composed of two similar observations and one that differs from the other two. The construction of a repertory grid, through tapping a subject's perceptions of similarity and contrast, provides a means of organising and accessing potentially unrealised meaning. The psychologist, requiring a subject to verbalise perceived similarities and contrasts between objects or people is merely accessing constructs the subject had already created internally, although possibly never having labelled them or verbalised

them. The bipolar nature of the personal constructs (same – different) leads to implicit anticipations and definitions.

Bannister and Fransella (1986), elucidating Kelly's psychology, state, as the fundamental postulate of Personal Construct Psychology "A person's processes are psychologically channelised by the ways in which they anticipate events" (p. 7). In this theory, people are scientists, having their own views of the world and their own expectations of it. Their behaviour is a continual experiment with life, "an impetus for seeking" (Piaget, 1985, p. 6). A person's experiences cause him or her to create bipolar (affirmation – negative) constructs such as good-bad and traditional-modern (Bannister and Fransella, 1986; Piaget, 1985). Repeated themes and perception of similarity and contrast create categorisation of constructs, and correlation and interrelation between them. Events are anticipated according to the personal constructs which are continually being modified or strengthened by construal of experience of those events. "A personal construct system is a theory being put to perpetual test" (*ibid.* p.14). Creating constructs is considered a basic activity innate to the human mind. The world is segmented by those constructs and events are unavoidably anticipated in terms of them, the person continually attempting to understand the world and to test that understanding.

The criticism expressed for Piaget's learning theory, that of it being descriptive, but lacking underlying mechanism, could be allayed by drawing on Kelly's personal constructs and the nature of perceived similarity and contrast by which the constructs have come to be created. The postulated mechanism admits increasing orders of abstraction from experience, increasing sophistication and complexity, and a coherent explanation for notions of surprise and perturbation resonant with Piagetian views of the structure and process of learning.

### **6.1.2 Reflective abstraction**

There is some confusion over Piaget's definition of reflective abstraction, due somewhat to the difficulty of translating his original French terms of *réfléchissement*, *réflexion* and *réfléchissante* (von Glasersfeld, 1991; von Glasersfeld, 1995c). In addition, he uses the term "pseudo-empirical" (Ginsberg and Opper, 1988) for what is essentially a type of reflective abstraction. Von Glasersfeld (1991) suggests reflective abstraction, reflected abstraction and reflective thought as possible translations, but what it all comes down to is the following. Reflective abstraction consists of two

inseparable parts. The one is a *reflecting*, as in “the sense of projecting on an upper level what is happening on a lower level”, the other is *reflection* as in “cognitive reconstruction or reorganization (more or less consciously) of what has ... been transferred” (Piaget, 1978, p. 35; Piaget, 1985, p. 29). Quoting Piaget (1975, the original French version of Piaget, 1978), von Glasersfeld (1991) uses the terms *réfléchissement* and *réflexion* for these two concepts. *Abstraction réfléchissante*, von Glasersfeld informs us, Piaget used as a generic term for both *réfléchissement* and *réflexion* together (von Glasersfeld, 1991, p. 59), and the distinction between these two elements tends to be lost in English translations. Reflective abstraction is “reflected” in two ways. The first consists of a projection of actions or existing knowledge onto a higher level, a higher plane of thought (reflecting), and the second consists of a reorganization, or reconstruction, of both the projected and previous actions into new structures and broader understanding (reflection) (Ginsberg and Opper, 1988; Dubinsky, 1991b). Hereafter, reflective abstraction can be understood to mean *abstraction réfléchissante*, that is, a combination of the two inseparable components of reflecting and reflection.

Reflective abstraction occurs due to “actions and operations” (Beth and Piaget, 1966, p. 188; Confrey, 1995c; Vergnaud, 1990) and “consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflection ... upon actions or operations of a higher level it guarantees” (*ibid*, p. 189). The source of reflective abstraction is “the subject, and it is completely internal” (Dubinsky, 1991a, p. 97). It is the process used to construct knowledge from understanding and being aware of the actions carried out on objects (Confrey, 1995c), a definition perhaps more reminiscent of pseudo-empirical abstraction. Reflective abstraction is associated with endogenic knowledge, “those structures which are developed by means of the regulations and operations of the subject” (Piaget, 1980, quoted in Steffe, 1995), that is, the source of the knowledge is found in the co-ordinations of the subject’s own actions (Ginsberg and Opper, 1988). An example used by Piaget, and rather over quoted in the subsequent literature, is the development of the concept of plurality or numerosity. Numerosity is abstracted from the act of counting, itself an abstraction based on the recognition of plurality. Plurality is abstracted from the repetition of observation of sufficiently similar items, thereby forming a collection of items. The concept of numerosity is developed by coordinating Piaget’s two schemas of classification (constructing a set of identical elements) and seriation (an interiorisation and coordination of multiple acts of pairing or tripling) (Dubinsky, 1991a; Piaget, 1952).

“The determining characteristic is not a sensory one, but one that arises from the particular operations of the mind that are carried out” (Vergnaud, 1990, p. 376; see also Bannister and Fransella, 1986).

Reflective abstraction, as a recursive process in the construction of knowledge, particularly logico-mathematical knowledge, is regarded by Steffe (1995) as the “lifeblood of constructivism” (p. 510). “For Piaget, logico-mathematical knowledge is derived from reflective abstraction” (Vergnaud, 1990, p. 18). Dubinsky (1991a) accepts reflective abstraction as “a description of the mechanism of the development of intellectual thought” (p. 99). Using slightly different terminology to refer to the same phenomenon, Brown et al (1996) describe one of four defined types of metacognition (mental experimentation) as being closely associated with reflective abstraction.

Dreyfus (1990) “tentatively” defines abstraction as “the replacement of concrete phenomena by concepts whose existence is confined to the human mind” (p. 114) by which he seems to be referring to empirical abstraction. However, the mathematical examples used, such as analysing, generalising and proving, use reflective abstraction, as the phenomena in question are not physical objects but abstract entities in themselves. In the language of personal construct theory, constructs interact to form construct system, which in turn create higher order construct systems, at increasingly higher levels of abstraction. Empirical abstraction and reflective abstraction would represent different levels in a continuum of construct systems, rather than distinct entities in themselves. Confusion of the types of abstraction can easily arise, particularly with the inclusion of pseudo-empirical abstraction conceived later in Piaget’s career (Ginsberg and Opper, 1988).

### **6.1.3 Pseudo-empirical abstraction**

In addition to “reflecting and reflection”, Piaget uses the term “pseudo-empirical abstraction” to refer to abstractions drawn from activities carried out on material objects, distinct from the objects themselves (Piaget et al, 1977, in von Glasersfeld, 1991), teasing out “properties like order or number that the subject’s operations have introduced into objects” (Piaget, 1985, p. 18). Ginsberg and Opper (1988) describe pseudo-empirical abstraction as occurring when one or more physical objects are used to represent something other than themselves, such as fingers used to represent numbers when counting. Von Glasersfeld (1991) agrees that “sensory-motor material” (p. 60) has

to be present in pseudo-empirical abstraction. This perhaps weak form of reflective abstraction is generally considered to occur in early developmental stages (Ginsberg and Opper, 1988), that is, in young children.

The experimental work which found the Piagetian learning theory to be a useful descriptor occurred in the context of tertiary mathematics education. The location of the learning material in the already abstract domain of mathematics suggests that empirical abstraction played a negligible role, if any at all, and that the ages and level of maturation (an important Piagetian concept) of the students should imply that pseudo-empirical abstraction also played a negligible role. In the realm of mathematics it is reflective abstraction which plays a role in learning and indeed in all logico-mathematical thought, which, for Piaget, meant almost all mature and rational thought processes. However, an exploration of the level of pseudo-empirical abstraction indicated by the actions of students when engaging in mathematics would make for an interesting study in the future, particularly if theoretically situated in personal construct theory with its hierarchical construct systems.

## **6.2 The process of learning**

The process of learning begins when a person (the epistemic subject, Dubinsky and Lewin, 1986; Lewin, 1991) encounters something new, a novel item (von Glasersfeld, 1991). Dubinsky and Lewin (1986) prefer to say that the subject experiences a novel cognitive *aliment*, a word the Oxford English Dictionary defines as “(*figurative*) that which supports or sustains the mind” (online OED, consulted 4 December 2006). This encounter triggers an activity, which, if it leads to an expected result, causes the person to not differentiate the item from those previously encountered. However, if the activity does not lead to an expected result, then that contrast results in a perturbation (von Glasersfeld, 1991) or disequilibrium (Dubinsky and Lewin, 1986; Piaget, 1985), and the subject discriminates the item from those previously encountered. The subject attempts to re-equilibrate. Simply put, the person realizes that the item is something different from anything encountered before and wants to come to some sort of understanding of the item. The subject attempts to assimilate the new item. The person, the epistemic subject, will have to modify his cognitive structures in order that the aliment no longer resists assimilation, that is, in order for the new item to “make sense”. This modification is called accommodation. It is in the process of accommodation that reflective abstraction occurs. The person now understands the new item. In short, when

something new is encountered, one tries to assimilate it. If it resists assimilation, one is forced to accommodate by modifying one's existing cognitive structures, thereby constructing new structures and finally understanding the novel item. In general, the process is not linear and a series of assimilations and accommodations will tend to occur, until there is no further disequilibrium (Sutherland, 1992). The process of repeated assimilation and accommodation is referred to as equilibration. Stepping slowly through the process of learning theory, each of the concepts of aliments, assimilation, accommodation and equilibration is explored in turn, as well as the concept of a scheme.

George Kelly (1905 - 1967) formulated Personal Construct Theory in the 1950s. Fundamental to Personal Construct Psychology is the view of human beings as scientists and as inquirers (Bannister and Fransella, 1986). The inquiring human mind is always theorising – modelling, anticipating and predicting the world. Piaget's theory of learning states that the learner, the epistemic subject, upon encountering a perturbation, and recognising the novelty of an item, will attempt to assimilate the item. Personal Construct Psychology enriches the view of that process, answering the question why? Why should a perturbation lead to an attempt at assimilation? Kelly's answer is that such a response is an unavoidable part of being human, with a scientific and inquiring mind, seeking to harness experience and draw from its explanatory and predictive power. "A personal construct system is a theory being put to perpetual test" (Bannister and Fransella, 1986, p. 14).

### **6.2.1 Aliment**

Piaget's learning theory, also called his "stage-independent" theory, begins with the learner (the cognizing subject, the epistemic subject) encountering an item of knowledge, an experience or an activity which he attempts to understand or match against existing knowledge and experience. Piaget himself (1978) used the terms *disturbance* or *disturbing element* for the new item or observation. Piaget gives a few examples of disturbances, such as "a perception which cannot be integrated in a seriation scheme sufficient until then" (1978, p. 66). Since the term "disturbance" is used by other theorists for the phenomenon otherwise referred to as a *perturbation*, Piaget's use of "disturbance" will not be used. Instead, von Glasersfeld's use of *novel item* (1991, p. 56) will be used or Dubinsky and Lewin's *aliment* (1986, p. 206).

The Oxford English Dictionary defines aliment as “that which supports or sustains the mind” and “the material or means of nourishing”. Dubinsky and Lewin (1986) define cognitive aliments as “that of which [the epistemic subject] becomes aware” (p. 58) and Lewin (1991) as “that which is epistemically novel” (p. 206). A point that is worth noting, particularly as it ties in with problem-solving and the distinction between problems and exercises, is that the novelty of an item is subjective to the observer (von Glasersfeld, 1991). Piaget’s learning theory describes a constructive process (Pascual-Leone, 1996; Duit, 1995) which is triggered by the epistemic subject (Lewin, 1991) encountering a novel item, or aliment, which needs to be assimilated into the existing cognitive system in order to be understood. This end product of understanding is achieved with varying degrees of success depending on the degree of maturation of the cognitive system and the depth of the processes of assimilation and accommodation that are a consequence of recognising an aliment as novel.

### **6.2.2 Assimilation**

The meaning of the word *assimilate* in the learning theory context has different shadings. cursory reading can lead to confusion as these different shades of meaning can seem to be quite different, while in fact they are all related ideas around the same concept. When the epistemic subject encounters a novel aliment he attempts to assimilate it into his existing cognitive system, in a similar manner to striving to attach a building block to a structure of blocks with a particular kind of attachment mechanism. Either the building block attaches easily (the aliment is assimilated) or it does not, in which case some sort of action is called for. Either the existing structure needs to change in order for the new aliment to be assimilated (this change is called accommodation), or the aliment is completely ignored (as happens when a child encounters something completely outside its experience, such as calculus), or the aliment itself is distorted in order to be assimilated. The distortion of aliments (Ginsberg and Opper, 1988) may sound very negative, a denial on the part of the epistemic subject of having observed one thing, convincing himself that he saw something different, and such denial can occur when there is a strong expectation of one kind of observation and something somewhat different is seen to occur. The “distortion” of the aliment can also simply be an imposition of structure on observations by the epistemic subject, such as a naturalist categorising plants or animals into species (Mays, in his translator’s note to Piaget, 1972, p. 6). The distortion of the aliment can be wilful, such as in a game of pretence where a child chooses to consider a stick to be a magic wand (Sutherland,

1992, p. 25). “An assimilatory scheme confers meaning on the objects it assimilates and assigns goals to the actions it organizes” (Piaget, 1985, p. 16)

Definitions for assimilation include “the incorporation of an external element ... into a sensorimotor or conceptual scheme” (Piaget, 1985, p. 5), “transforming experiences within the mind” (Sutherland, 1992, p. 25), “the process by which the subject applies his ... conceptual schemes to ... objects” (Mays, in his translator’s note to Piaget, 1972, p. 6), and the process of applying “to the aliment the set of cognitive operations which the knower has previously constructed” (Dubinsky and Lewin, 1986, p. 60). It is important to keep in mind that assimilation involves action on the aliment, rather than action on the cognitive system. Definitions such as “the process of assimilating experiences with the concepts already held” (Duit, 1995, p. 271), “the process of fitting reality into one’s current cognitive organisation” (Miller, 1993, p. 74), and “to become of the same substance; to become absorbed or incorporated into the system” (Online OED, consulted 4 December 2006) can mislead the casual reader into thinking that assimilation, by involving the absorption, as it were, of something new into a cognitive system, must imply change to that cognitive system. This perception of cognitive change would be incorrect; such a change falls under the definition of *accommodation*.

### **6.2.3 Accommodation**

Continuing the metaphor of attempting to attach a building block to a structure of blocks, accommodation is what occurs when the structure, or the attachment mechanism, has to change in order for the block to be attached. Accommodation is defined variously as “adjusting the mind to new experience” (Sutherland, 1992, p. 26), the knower modifying its cognitive structures (Dubinsky and Lewin, 1986), “adjustments in cognitive organization that result from the demands of reality” (Miller, 1993, p. 75) and “the way in which objects act on the subject so that he adapts his behaviour to these objects” (Mays, translator’s note to Piaget, 1972, p. 6). Von Glasersfeld (1991) has a subtly different definition of “the tightening of the criteria of assimilation that determine what can and what cannot be taken as a trigger for the particular scheme” (p. 56). This last definition is more easily understood once the concepts of scheme and perturbation are covered; it can be altered to “changing the definition for what is regarded as novel” and is not a variation of the definition of accommodation that will be used in this thesis.

It is in the process of accommodation that reflective abstraction occurs (von Glasersfeld, 1995b). Accommodation (and hence reflective abstraction) is not an activity that necessarily occurs when a novel item is encountered: “all accommodation is triggered by a perturbation – by the subject’s realization that something is amiss, does not work, or is in some way a surprising result” (von Glasersfeld, 1995b, p. 378), “accommodation is always secondary to assimilation” (Piaget, 1985, p. 6), and “people do not accommodate while assimilation is still reasonable” (Strike and Posner, 1992, p. 149). When encouraging students to learn mathematics, mathematics that is definitely new to the majority if not all of the students in the class, it is important that such perturbation is encountered by the students in order to trigger the process of accommodation. The reluctance of the cognizing subject to accommodate while assimilation is still reasonable has strong ties with the literature on the mismatch between concept image and concept definition (Strike and Posner, 1992; Tall and Vinner, 1981). Students, despite having been exposed to strict concept definitions in mathematics and science, can have concept images that are, in some circumstances, at odds with the definitions. If it is desired that the students internalise the formal definitions and discard or change their concept images, or their mental models (Ellerton and Clements, 1991), they have to encounter situations where the images are called into question (Strike and Posner, 1992). Resnick and Glaser (1976) somewhat wryly refer to processes of accommodation as being like “miniature scientific revolutions” (p. 206), in reference to the lack of insight into the process, rendering it as indivisible and unknowable; a black-box procedure. In this criticism, they are expressing the complaint that accommodation is merely a label for a process rather than a description of a mechanism.

Slightly different viewpoints exist on the relationship between assimilation and accommodation. Taking the view that the epistemic subject tests the aliment against the existing cognitive system, changes the system, tests again, perhaps changes the system again, assimilation (testing the aliment against the system) and accommodation (changing the system) can be seen as constituting a cycle. Ellerton and Clements (1991) refer to “the twin processes of assimilation and accommodation” (p. 90), although von Glasersfeld (1995c) warns against the tendency to consider assimilation and accommodation as reverse processes. If the concept of assimilation is understood to refer strictly to *successful* assimilation only, rather than the testing of the aliment against the cognitive system, such as Bishop (1988, p. 152) appears to, then the process

would not be a cycle of accommodation – assimilation resulting in equilibration. Instead, the cycle is accommodation – attempted assimilation, resulting finally in assimilation, when the accommodation is done successfully. In this thesis, the term *assimilation* will refer to the applying of cognitive operations to alimnts (Dubinsky and Lewin, 1986), whereas final successful assimilation of the alimnt into the (probably newly reconstructed) cognitive system would fall under the definition of *equilibration*.

Piaget (1985) underlines that accommodation and assimilation cannot be seen as processes independent of one another. Assimilation and accommodation are inseparable and one is necessary for the other to occur. “In other words, both activities are closely woven into an indissociable whole” (p. 33).

#### **6.2.4 Equilibration**

When the epistemic subject observes that “something is amiss, does not work, or is in some way a surprising result” (von Glasersfeld, 1995b, p. 378) there is said to be a *disequilibrium* between the observed result of applying cognitive schemes to the novel item and the expected result of the current scheme. In order to successfully assimilate the novel item, there has to be a *compensation* for the disequilibrium, and the return to a stable cognitive state is known as *equilibration*. Equilibration is variously defined as “a self regulated adjustment or progression of the current modes of thought” (Ginsberg and Opper, 1988, p. 212-3), “a state of balance or harmony between at least two elements which have previously been in a state of disequilibrium” (*ibid.*, p. 221-2), seeking a “relatively stable homeostasis between internal conceptions and new information in the environment” (Pintrich et al, 1993, p. 171) and “the elimination of perturbations” (von Glasersfeld, 1995c, p. 67).

The terms equilibrium and equilibration are borrowed from physics and biology (Ginsberg and Opper, 1988), Piaget having had a background in biology, but there is an important distinction which must be made. Ernest (1995) compares the use of the term *equilibration* to equilibrium in hydrodynamics, “like the adjustment of water levels toward equilibrium” (p. 474), where equilibrium in biology or physics or other sciences is usually taken as a stable and unvarying state of a system to which it returns after some disturbance. However, in Piaget’s learning theory, equilibrium does not imply static rest. Cognitive systems are always active and always changing (Ginsberg and Opper, 1988). Increase in knowledge, improved understanding and cognitive schemes

that have greater viability are achieved by the cognitive system always aiming for more understanding, a “better form of knowledge” (*ibid.*, p. 222). “Cognitive development consists of a succession of alternating equilibria and disequilibria” (*ibid.*, p. 222) where the cognitive system seeks to optimise equilibration, leading to successive improvements in equilibrium. Equilibrium is dynamic (Miller, 1993) leading to an increase in knowledge, not a return to a stable state, as equilibrium in the physical world implies. “The equilibration process is the backbone of mental growth” (Ginsberg and Opper, 1988, p. 221). Piaget himself stresses, however, that not all reequilibration results in growth; learning is achieved through creation of “new equilibrated forms” (1985, p.3).

It is known that a mathematical problem, distinct from an exercise, is a subjective description, one student’s problem being another’s exercise. A problem might be solved fairly quickly by one student, and be completely intractable for another. In the terminology of the learning theory, “disequilibrium is relative to the child’s [or student’s] developmental level” (Ginsberg and Opper, 1988, p. 229) and “the success of re-equilibration will depend, at least in part, on how adequate already existing cognitive structures are to accommodate the aliment” (Dubinsky and Lewin, 1986, p. 60).

Robust assimilation of an aliment does not necessarily occur upon the cognizing subject observing it. The aliment might be completely ignored, it might be distorted in order to be assimilated, or some level of assimilation and accommodation might occur without full success. What is more, the knower might fail to perceive that attempts to assimilate the aliment have not been successful. Thus reflective abstraction can occur and still not result in successful disequilibrium (Dubinsky and Lewin, 1986). Dubinsky and Lewin call such incomplete or unstable disequilibrium a “curious result” (p. 61) of the theory that knowers construct their own knowledge, but consider it to be clear from teaching students that active learning can occur, yet leave the student still short of full understanding. “Reequilibration often amounts to nothing more than returning to previous states of equilibrium without creating new equilibrated forms. By contrast, the reequilibrations more fundamental to development result in equilibria that not only are new but are also better than previous equilibria” (Piaget, 1985, p. 3).

Equilibration need not be seen as involving a lone state of equilibrium: Piaget (1985) refers to three types of equilibration, what Lewin (1991) refers to as three loci where

equilibration can occur, and these three equilibrations can be at different levels of compensation.

- 1) equilibration between assimilation and accommodation
- 2) interactions among the various subsystems of a total system
- 3) the hierarchical integration of lower level cognitive structures into a higher level component of the total cognitive system.

(Lewin, 1991)

These three loci themselves are not hierarchical nor in any kind of sequence. A student might exhibit a greater facility (Lewin provides several examples) in achieving equilibrium in one of these loci than in the others. Such a “grain size” of learning theory analysis will not be attempted here, although it does open up fascinating possibilities for further studies. In general, references to equilibration allude to locus (1), such as “in a state of equilibrium, neither assimilation nor accommodation dominates” (Miller, 1983, p. 75), although there is the occasional variation on the use of the terms such as “to equilibrate the subject ... and its environment” (Walkerdine, 1988, p. 3 - 4)

### **6.2.5 Scheme**

When the epistemic subject encounters an alimant bearing some resemblance to one already assimilated in the past, there is an attempt to assimilate the new alimant by applying a *scheme*. A scheme is a “package” of cognitive items and actions, “a cognitive system” (Piaget, 1985, p. 54), which is used to cognitively act on experiences in an attempt to assimilate them. “Schemes are composed of three elements: (1) an initial item or configuration ... (2) an activity the subject has associated with it, and (3) a subsequent experience associated with the activity as its outcome or result.” (von Glasersfeld, 1991, p. 55-6). It is when the expected result of the activity associated with a scheme does not occur that the subject experiences disequilibrium and attempts to re-equilibrate. Personal Construct Theory (Bannister and Fransella, 1986) suggests that the reason why schemes are applied at all is due to the innate human characteristics of inquiring and sense-seeking, forever anticipating and theorising about the nature of the world.

Confrey (1995c) conceives of radical constructivism, which draws heavily on Piagetian learning theory, as being formed of four “planks”, one of which is scheme theory. A scheme is “whatever is repeated or generalizable in an action” (Piaget, 1970, in Confrey, 1995c), and involves the anticipation or recognition of a situation. First there

is a perturbation, which calls for action; the perturbation is resolved internally by reflective abstraction. If the sequence of perturbation, action and reflective abstraction is repeated until action is stabilised, the sequence is a scheme. “For radical constructivist(s), the unit of analysis is a scheme and its genesis and modification” (Confrey, 1995c, p. 197). Piaget himself referred to schemes as *psychological units* (Pascual-Leone, 1996), which are “dynamic systems of organismic-situational processes” (*ibid.*, p. 85-6).

Von Glasersfeld might disagree that a scheme necessarily involves a perturbation. If the subject experiences something with which the cognitive system is entirely familiar, there will be no perturbation, and no increase in knowledge, but a scheme will still have been triggered in the cognitive system by the familiar experience, resulting in the expected result and no perturbation. Von Glasersfeld (1991) and Confrey (1995c) have very clear differences in their definitions of a scheme, von Glasersfeld locating a potential perturbation after the application of a scheme to a novel alimant, and Confrey considering a perturbation to be the triggering mechanism of an existing scheme. It is entirely possible that the two theorists agree on what a scheme is and are using the same word for slightly different concepts. In this thesis, it is von Glasersfeld’s definition of a scheme which will be used, that is a collection of cognitive objects and actions applied to an alimant, and resulting in a perturbation if an unexpected result occurs.

Just to stretch the possible terminology confusion, we note the use of the term *schema*, and its differing use by several theorists. In some cases, schema is synonymous with von Glasersfeld’s scheme, such as Dubinsky’s (1991b) definition “a schema ... is a more or less coherent collection of cognitive objects and internal processes for manipulating these objects. A schema includes various actions and constructions that can form new schemas” (p. 166). In other cases the term schema differs from the Piagetian scheme: “*Schema*: a conceptual structure. Can be derived from the idea of a cognitive map, if we regard a schema as analogous to a cognitive *atlas* in which (e.g.) a dot representing London or New York on a map of UK or USA can itself be expanded into a map. Not quite the same as Piaget’s “scheme” ” (Skemp, 1982, p. 189, punctuation as in original).

### 6.3 Type- $\alpha$ , Type- $\beta$ , Type- $\gamma$ behaviour

The epistemic subject can exhibit a number of different types of behaviour when encountering an alimnt. The novelty of the alimnt could be ignored, an incomplete process of equilibration could occur, leaving the understanding of the novel item in an unstable state, the novel item could be successfully assimilated through a robust process of assimilation and accommodation, or the cognitive structures could be sufficiently well developed that, despite the novelty of the item, it can still be assimilated without cognitive restructuring. Late in Piaget's career (1975, in *L'Equilibration des Structures Cognitives*) he codified these different behaviours into three categories, Type- $\alpha$ , Type- $\beta$  and Type- $\gamma$  behaviour (Piaget, 1978). These three behavioural categories are hierarchical in that they display increasing success in equilibration, and greater anticipation within the cognitive system of variations on experiences already encountered.

Alpha behaviour (Dubinsky and Lewin, 1986), also alpha compensation (Lewin, 1991), alpha reaction (Ginsberg and Opper, 1988), Type- $\alpha$  and type  $\alpha$  behaviour (Piaget, 1978; 1985), occurs either when the subject denies that the alimnt resists integration (does not realise that the item is something novel), or integrates it in an unstable way that cannot withstand criticism (thinks he understands it, but has not fully grasped the entire concept). The learner will "cancel the disturbance by neglecting it, or by simply avoiding it" (Piaget, 1978, p. 66), "the subject will ... pretend to consider it, but distort it in order to adjust it to the scheme retained for the discernment." (*ibid.*, p. 66).

Cognitive conflict is ignored and the cognitive system undergoes very little change (Ginsberg and Opper, 1988). "The equilibria that result from [type  $\alpha$  reactions] remain very unstable" (Piaget, 1985, p. 56). In denial of novelty, in ignoring of potential disequilibrium, there is no cognitive reconstruction, no accommodation and no reflective abstraction; in this all theorists seem to agree. There is argument, however as to whether the second classification of alpha behaviour, that of the creation of unstable knowledge structures, involves reflective abstraction or not, or perhaps it might be more accurate to say that there is argument about what constitutes alpha and what beta behaviour.

Dubinsky and Lewin (1986) refer to the possibility of incomplete or unstable disequilibrium occurring, involving the process of reflective abstraction, leaving the

learner unaware of the instability of the ensuing disequilibrium. A definition of alpha behaviour as including instances of unstable integration of novel items would seem to encompass Dubinsky and Lewin's unstable disequilibrium, yet Dubinsky and Lewin describe alpha behaviour explicitly as not involving reflective abstraction, "only the appearance, but not the reality, of cognitive reconstruction" (p. 63), which, they admit, occurs with "disheartening regularity" (p. 63). Lewin (1991) supports a definition of alpha behaviour not involving reflective abstraction:

It is important to note that while equilibration takes place more or less automatically, reflective abstraction does not. That is, a knower may re-equilibrate by denying, implicitly or explicitly, that the ailment offers an occasion for re-thinking, for cognitive re-construction. Re-equilibration without reflective abstraction offers a means of understanding the common yet counter-intuitive case in which students declare a new competence or understanding that is belied by their performance, or alternatively, when students deny the relevance of an example or idea explicitly intended to challenge their current understanding.

(Lewin, 1991, p. 213)

Alpha behaviour involves re-equilibration without reflective abstraction "with a potential proliferation of independent and unintegrated cognitive structures" (*ibid.*, p. 217)

Examples of students recognising a perturbation and considering themselves to have equilibrated, yet, from the mathematical observer's point of view, not having achieved reflective abstraction nor the creation of stable knowledge structures, can be found in the literature. Such an example in Dubinsky and Lewin (1986) is situated in the context of advanced mathematics, with a student struggling to define the negation of a compact set. The student proposes an incorrect definition, which the lecturer challenges, indicating in what ways the definition is at fault. The student recognises the flaw in the argument - "I see what you mean, ok" (p. 80) – yet continues with a similarly flawed definition. The student, while apparently recognising the conflict with his existing cognitive structures, does not refine those structures sufficiently to successfully compensate for the perturbation. Further examples can be found in Tall and Vinner (1981) in reference to the clash between concept definition and concept image. In a particularly striking example among the several cited, students took part in a project designed to engage explicitly with the definition of a mathematical limit and to actively cause conflict between the definition and their less rigorous concept image. The notable point is that these same students, some time later, were surprised with a limit question in class which 21 out of 22 answered incorrectly. Despite the explicit perturbations

encountered in the earlier project, those students had never equilibrated to the extent of accommodating their cognitive structures to allow for the subtlety of the limit definition.

While both of these examples have been used to illustrate the possibility of recognising a perturbation yet equilibrating without reflective abstraction and the resulting accommodation of cognitive structures, both examples are drawn from studies which were, in fact, successful in encouraging reflective abstraction. Dubinsky and Lewin's (1986) study, among other concerns, centres on the need to draw students into the period of formal operations, which many undergraduates have not reached and which is necessary for manipulating abstract objects. They go on to encourage the use of computers in associated pedagogy. Tall and Vinner (1981) did succeed in bringing about changes to concept images by forcing students to contend with apparent contradictions in their statements such as  $\lim_{n \rightarrow \infty} (1 + \frac{9}{10} + \frac{9}{100} + \dots + \frac{9}{10^n}) = 2$ , yet  $0.9 < 1$ .

While the previously stated example in which 21 out of 22 students returned to their informal and ultimately mathematically incorrect concept image the success rate of their intervention appears low, yet cognitive change was successfully observed in the study.

In beta behaviour an attempt is made to incorporate the conflicting event into the cognitive system. Again, we encounter different approaches to beta behaviour from several theorists. Piaget himself (1978) considered beta behaviour to be inferior to gamma behaviour, but only in that gamma behaviour involves simply an expanding of cognitive structures to absorb an item for which it was already prepared, whereas beta behaviour involves cognitive restructuring due to the unexpected novelty of the item. It is not apparent from Piaget's writings that he considered the actions carried out during beta behaviour to be partial or inadequate in any way. By "integrating into the system the disturbing element arising from without ... what was disturbing [becomes] a variation within a reorganized structure" (Piaget, 1978, p. 67). "By integrating or internalizing the disturbances at play in the cognitive system, these type  $\beta$  behaviors transform them into internal variations which are capable of being compensated [by equilibration], still partially, but nevertheless in a manner quite superior to that of type  $\alpha$  behaviours" (Piaget, 1978, p. 68). Ginsberg and Opper (1988) have chosen to interpret Piaget's beta behaviour to mean that the cognitive process codified in beta behaviour is in some way incomplete or cognitively immature, in that they describe beta

behaviour as making an attempt to incorporate the conflicting event into the cognitive system, but with only partial success, with limited variations upon a scheme (p. 228). At the other extreme, Dubinsky and Lewin (1986) consider “beta behaviour ... the paradigm case of successful learning.” (p. 64), involving as it does reflective abstraction enriching the cognitive system by reorganising it, and locating the novel alimant in the “re-constructed domain of concepts” (p. 64). It must not be forgotten however, that at least Dubinsky and Lewin assert that reflective abstraction, and some process of assimilation/accommodation, can occur and not result in stable equilibration. If such a process is not classified under alpha behaviour because of its inclusion of reflective abstraction, then it necessarily has to be classified under beta behaviour (if the alpha/beta/gamma structure is not to be altered), in turn necessitating a definition of beta behaviour that does not always result in successful understanding. Lewin (1991) describes beta behaviour as acknowledging perturbation by a cognitive alimant, the cognitive system undergoing accommodation and reflective abstraction, the disequilibrating alimant being integrated, and the assimilatory scheme being modified. Beta behaviour, in this thesis, will be considered to be that defined in Lewin (1991) and Dubinsky and Lewin (1986) with the point noted that the final result of the process is not necessarily stable understanding of the alimant. It must be noted that equilibration can have involved reflective abstraction or might not have. In the former case the cognitive activity is located in beta behaviour, and in the latter case it is located in alpha behaviour. Determination of the impact of the act of equilibration on the cognitive structures is indicated by the cognitive end product. If the perturbation and disequilibration has given way to the creation of new and stable knowledge structures then reflective abstraction has taken place and beta behaviour was engaged in. If the cognitive structures are incomplete or unstable, then it is possible that no reflective abstraction has occurred and, despite recognition of the initial perturbation, the student has engaged in alpha behaviour.

In illustration of the difference between alpha and beta behaviour, an example from the writing exercises of Group A is chosen. The students were given the equation of a family of nested ellipses, centred on the origin, and were asked to determine the associated family of orthogonal trajectories. The calculation involves setting up and solving an ordinary differential equation, a situation the students had already encountered on a few occasions. All students who attempted that question correctly set up the equation and solved it, every student leaving out an important pair of modulus

signs. As discussed elsewhere in this thesis, the only students who realised that their solution was incomplete were from among the students taking part in the writing study project. The example is used here to illustrate the ideas of a scheme and a perturbation. Setting up and solving, at least incompletely, the differential equation constitutes evidence of a *scheme*. Similar activities had been carried out many times by the students. Noticing that the solution implied a sign constraint when the concept behind the problem should have no sign constraint constituted the *perturbation*. The students who investigated further, and either came to some sort of realisation based on graphical symmetry or algebraic integration processes were undergoing *accommodation* with its associated *reflective abstraction*. The students who “solved” the equation and moved on, without inquiring as to the solution’s veracity, were exhibiting *alpha behaviour*, with no reflective abstraction, merely the application of a scheme. The students who questioned their solution were recognising a perturbation and experiencing *disequilibrium*, and those who compensated for the disequilibrium, coming to some conclusion which allowed them to relieve that sign constraint, were exhibiting *beta behaviour*.

The illustration does leave one thread rather loose, however. Surely, one might argue, if the scheme involved performing a mathematical calculation, with mathematical symbols, it involved reflective abstraction? A thorough reading of the definitions used widely in the literature would say no, reflective abstraction was not present in a rote calculation. Rote calculations involve playing games with symbols with no understanding of what the symbols mean (Bowie, 1998). Only in deep, thoughtful, questioning engagement with those symbols does reflective abstraction occur. This view does not assert that reflective abstraction never occurs unless a serious perturbation is encountered. An awareness of the meaning of the symbols one is manipulating, and a sensitivity to variation in meaning also involves reflective abstraction. It is unthinking manipulation of symbols, obeying rules and algorithms without engagement with the underlying mathematics that is involved in alpha behaviour.

If the cognitive system has, as part of its structure, the anticipation of further variation (Piaget, 1985), then the experience of novel aliments will not require a restructuring of the cognitive system, rather an enrichment and expansion of those structures already present (Lewin, 1991; Ginsberg and Oppen, 1988, Dubinsky and Lewin, 1986; Piaget, 1978). Such *gamma behaviour* occurs when “anticipations of possible perturbations

already exist as part of the cognitive structure, so that potentially disturbing aliments are compensated in advance ... As new aliments are assimilated, cognitive structures are enriched” (Lewin, 1991, p. 217). Despite the lack of significant restructuring, reflective abstraction is still considered to be present in gamma behaviour, in the extension of systems of concepts and the forming of new relations between the system and the novel (although anticipated) item (Dubinsky and Lewin, 1986; Piaget, 1978, 1985).

Lewin (1991) considers that, epistemically speaking, gamma behaviour is more desirable than either alpha or beta behaviour and represents “a more profound and deeper reorganization of cognitive structuring, which will therefore be more stable and resilient, than the other compensations [behaviours].” (p. 217). Indeed, Piaget (1978) referred to gamma behaviour as “superior behavior” (p. 68). However, it could be argued that, *pedagogically* speaking, what teachers are trying to achieve is beta behaviour. If the entire class consists of students whose cognitive systems are sufficiently well developed that all variations upon the mathematical topics encountered in the classroom have been anticipated, then they are not going to learn much that is new for them. The usual aim of the teacher is to teach students wholly new concepts and processes, for which the cognitive structures are, almost by definition, not yet sufficient to encompass the new material without accommodation. The first time students encounter complex numbers, for instance, as rich as their understanding of real numbers might be, there are going to be entirely new concepts for which they will have to do some cognitive restructuring. While gamma behaviour is admirable, it is not what the teacher of mathematics can fairly expect, in general, to see in her students. Instead, rich and deep beta behaviour indicates, as Dubinsky and Lewin have stated, successful learning, rather than the encountering of information for which the student was already prepared.

## 6.4 Perturbations

*Perturbation* refers to the cognitive conflict (Duit, 1995; Dreyfus, 1990; Ellerton and Clements, 1991) which occurs when objects or events are encountered that cannot be assimilated, and disequilibrium results (Ginsberg and Opper, 1988). Bickhard (1995) prefers the term *surprise*. Experience of the world causes one to anticipate further experience; violations of that anticipation (namely, surprises) constitute experiences which cannot be explained by the existing cognitive structures, requiring a

reconstruction before the mental structures once again correspond to the experienced reality (Bickhard, 1995; see also Bannister and Fransella, 1986).

The concept of a *perturbation* appears under numerous names in the literature. The term perturbation is fairly ubiquitous (Piaget, 1974, in von Glasersfeld, 1995c; Piaget, 1985; Steffe, 1995; Ginsberg and Opper, 1988), as is the term *cognitive conflict*, a theory which Ellerton and Clements (1991) consider to have its roots in Piagetian learning theory (also Ginsberg and Opper, 1988). The term *disturbance* emerges in slightly different contexts: Ginsberg and Opper (1988) use it as a synonym for perturbation, while Piaget (1978) appears to use it as a synonym for Dubinsky and Lewin's (1986) *alimant*. The term disturbance will not be used in this thesis due to the possibility for confusion. The theory of *conceptual change* is closely tied to the idea of perturbations and disequilibrium, however the related dysfunctional concepts, or conceptions (Brown et al, 1996), whose change is being sought through cognitive conflict and confrontation of misconceptions, tend to be at a large scale, such as changing scientific paradigms (Pintrich et al, 1993), in extreme cases akin to Kuhnian revolutions (Kuhn, 1962).

It is important to note that an experience only constitutes a perturbation if it is personally experienced as one by the individual student. The student cannot simply be told that, for instance, a concept image is at odds with a concept definition (Dreyfus, 1990; Tall and Vinner, 1981); she has to see and feel the concept conflict for herself. Surprise is not an experience about which one is told, it is something one feels, personally. "Unless a problem is seen and felt to be a problem by the student, it is unlikely to trigger reflective abstraction" (von Glasersfeld, 1995b, p. 378). This need for the students to encounter sufficiently unavoidable cognitive conflict to enable them to enter the process of beta behaviour with its associated accommodation activities, raises two questions for the teacher of mathematics:

- 1) How does one design situations in which the student encounters problems, contradictions to supposedly stable knowledge structures, or perturbations?
- 2) How does one structure such situations so that each student, with unique prior knowledge, and unique descriptions of what makes a problem a problem, has a chance of encountering such perturbations?

First of all, the student needs to be actively doing mathematics, rather than watching the teacher demonstrate something on the board. While some elements of mathematics can be learnt by observation, much has to be learnt by practice, and the student is more likely to stumble across a personal unstable knowledge structure if he is actively and explicitly using that part of his cognitive system, in engaging in mathematics. Duit (1995) discusses achieving cognitive conflict by requiring students to talk about their conceptions of physical processes, and by such voicing of their thoughts to encounter contradictions. Ellerton and Clements (1991) agree that students, finding themselves exposed to their own misconceptions, will “actively [resolve] inner cognitive conflicts” (p. 90). Writing about mathematics, in mathematics, while lacking Duit’s social context, plays a similar role of requiring students to give voice to their mathematical thoughts and processes, and by so doing to encounter weaknesses, gaps in knowledge, or possibly serious cognitive conflict. Duit also stresses the importance of repeated challenges to student preconceptions before conflict is resolved, a view which, continuing the comparison with the writing study project, supports the idea of tackling the writing exercises regularly and often.

Ellerton and Clements (1991) raise the difficulty of dealing with a large class of students exhibiting a wide variety of types and levels of prior knowledge. They interpret this large class as being in the order of 30 students, while at tertiary level, in the author’s experience, a large size is more likely to be 200 students. If all the students are being confronted with the same piece of work which the teacher hopes will bring about advantageous cognitive conflict, it is unlikely to achieve that goal in all the students. It is part of the idea behind the writing exercises that, requiring students to write about their own personal problem-solving processes, those differences in what each student is cognitively bringing to the classroom are not a topic of concern. Each student will approach the task at the level at which she is capable, whether that is with great technical expertise or not. If each student approaches the task thoughtfully and seriously, each student could potentially gain from the exercise, each in a different way.

Criticisms of Piaget’s theories abound (see Sutherland, 1992, for a coherent collection of criticisms; also Miller, 1993), among them criticism of the idea of perturbations as a stimulus for learning. Bryant (1972, in Sutherland, 1992) is firmly opposed to the notion of perturbations, denying any proof of conflict motivating learning and similarly denying any proof of the existence of such a cognitive activity as equilibration. Children

learn by agreement, not by contradiction or conflict, insists Bryant. Despite such criticism, support for perturbations and associated ideas such as equilibration is prevalent with such statements as “Children are seen as naturally motivated to learn when their experience is inconsistent with their current understanding or when they experience regularities in information that are not yet represented by their schemata” (Greeno, et al, 1996, p. 25)

## **6.5 Writing, and Vygotsky**

A major criticism of Piaget’s theories is their focus on the individual and the personal construction of knowledge and meaning, and a lack of focus on the social aspects of learning. Lev Vygotsky (1896 – 1934), with his sociohistorical (Sierpínska, 1998) theory, drew attention to the importance of the social aspects of learning, throwing Piaget’s much more individual theories into stark contrast. The differences, or differences in focus, of the theories of the two great influencers of modern pedagogy, Piaget and Vygotsky, are so easy to point out that the similarities often get lost or obscured. At a “rough slogan level” Bickhard (1995) summarises the differences between Vygotsky and Piaget as Vygotsky being concerned with social internalisation and Piaget being focussed on coordination of actions. Von Glasersfeld (1995a, 1995c) considers criticism of Piaget for not including social interaction in his theories as unfair and superficial (von Glasersfeld, 1995c). He avers that social interaction was implicitly considered important to Piaget as the social adaptation of the cognitive system to its environment (Brown et al, 1996), and hence it did not become a major focus of his analyses. Vygotsky’s theories and Piaget’s theories can be understood to complement one another, rather than contradict or contrast.

Pass (2004) takes an interesting view of the differences and similarities between Piaget and Vygotsky – not simply in their work but also in their lives and experiences. By charting Piaget’s and Vygotsky’s lives, major works and communications with one another, Pass can show how their respective ideas and theories influenced one another’s work, drawing them closer together. Pass concludes her monograph with a twelve point plan of pedagogy which she theorises might have been drawn up between the two influential theorists had they had time and opportunity to communicate for longer than was actually feasible (see also Confrey, 1995a). Communication between the two was stymied by Stalin’s restrictions on communications with the West, and the tragic death of Vygotsky at age 38. Piaget only had the opportunity to see several works by

Vygotsky decades after his death, despite Vygotsky's attempts to send them to Piaget in Switzerland, and Pass carefully delineates the tendency towards a more social viewpoint in Piaget's later works after having been exposed to more of Vygotsky's work. Pascual-Leone (1996) also carefully defines the points of agreement between Piaget and Vygotsky, namely the principles of praxis and modular organisation, and the principle of equilibrium or dynamic synthesis.

Vygotsky's interest in the social transmission of knowledge, and the learner as a social agent, caused him to consider language as the primary tool in the process of learning. It is in the interpretation of the role of language in learning that one can locate one of the points of difference between Piaget and Vygotsky (Ginsberg and Opper, 1988). In Vygotsky's theories, language influences development. The subject learns in a social context where transmission of knowledge occurs through a variety of means, one of which is language, and communication between learners and between teacher and learner. Piaget considered development to have occurred before being able to articulate and thus language is an indicator of development rather than a source of development (Sierpiska, 1998). The writing project discussed in this thesis involved tasks that were completed by the students working alone, not in groups. There were aspects of communication in that the writing was a form of communication between student and tutor, and the tutor commented on the written exercises in communicating back to the student, but the learning carried out in the process of writing is certainly not an exemplary case of learning through social interaction and group communication. The reason for inclusion of a discussion of Vygotsky's theories is for his views on the importance of writing. Since Vygotsky held that language is an important tool for learning, one way in which language can be utilised is in writing, not just in verbal communication in the classroom. "Vygotsky was claiming that writing can have an actual impact on development; Piaget would not say that the activity of communication can change the course of development. On the contrary, he would have claimed that development is a precondition for a person to express himself or herself clearly in writing" (Sierpiska, 1998, p. 45). The Piagetian view is that communication is central to the individual and language is moulded by thought, thus a student's language, as a symptom of this thought, can be used to build a model of what the student is thinking. The Vygotskian view is that communication is a cultural fact, thought and language interact and enhance one another, demanding standards within forms of expression and thereby increasing the level of logical thinking (Sierpiska, 1998).

The act of writing requires a level of abstraction that encourages cognitive development. Of writing, Vygotsky (1962) says “even its minimal development requires a high level of abstraction” (p. 98) and “it is the abstract quality of written language that is the main stumbling block” (p. 99) to writing. In this last statement, Vygotsky is referring to children learning to write and it could be argued that other features of writing, such as having to express oneself in an additional language, may create greater stumbling blocks for the tertiary student than writing’s abstract nature. “In written speech, we are obliged to create a situation, to present it to ourselves. This demands detachment from the actual situation” (*ibid.*, p. 99). In addition to the level of abstraction required, writing a coherent passage requires a degree of planning and deliberation: “Written speech ... must explain the situation fully in order to be intelligible. The change from maximally compact inner speech to maximally detailed written speech requires what might be called deliberate semantics – deliberate structuring of the web of meaning” (*ibid.*, p. 100). Talking about mathematics and writing about mathematics are different activities with different demands, with writing possibly requiring greater structure and a higher degree of planning: “Written language demands conscious work because its relationship to inner speech is different from that of oral speech” (*ibid.*, p. 99) “Written speech is a separate linguistic function, differing from oral speech in both structure and mode of functioning” (*ibid.*, p. 98). A pleasing comparison can be drawn between Pólya’s “devise a plan” and Vygotsky’s requirements of writing: “Planning has an important part in written speech, even when we do not actually write out a draft. Usually we say to ourselves what we are going to write; this is also a draft, though in thought only.” (*ibid.*, p. 144). Applebee’s (1984) interest in the active nature of writing finds an echo in Vygotsky’s insistence that writing requires “deliberate analytical action” (Vygotsky, 1962, p. 99) and is an “abstract, deliberate activity” (*ibid.*, p. 100). Sierpinska (1998) summarises: Vygotsky supports writing as development. Writing involves a different kind of thinking, is planned and conscious, but is not usually exercised spontaneously. Piaget would say development is needed before the writing is possible (hence the writing shows the evidence of the development), Vygotsky would say the writing causes development. For different reasons, it could be argued, Piaget and Vygotsky would find writing in mathematics a useful activity.

## 6.6 Radical constructivism

To investigate the precepts of radical constructivism is to wander slightly away from the main direction of Piagetian learning theory (with Vygotskian influence). However, when going in search of literature on Piaget's learning theory, it is impossible to avoid theory on radical constructivism for which the learning theory is a cornerstone. Since radical constructivism is not directly related to the theory being used to describe observations made during the writing study project, but is unavoidable when researching for such purpose, a brief outline of the theory is given here. This outline seeks to acknowledge the intertwined nature of the theories, without going into excessive detail. For further elucidation, von Glasersfeld (1995c) is recommended.

Ernst von Glasersfeld is the main proponent of radical constructivism, defining as he has its two much quoted principles:

1. Knowledge is not passively received either through the senses or by way of communication, but it is actively built up by the cognising subject.
2. The function of cognition is adaptive and serves the subject's organization of the experiential world, not the discovery of an objective ontological reality.

(von Glasersfeld, 1995c, p. 18)

It is in the first principle that Piaget's constructivist learning theory plays such a large role. Ernest (1995) considers Piaget to be a forerunner of radical constructivism. Piaget felt that unique mathematical knowledge structures emerge from the mind, and that is a view held by radical constructivists, who agree that common mathematical knowledge grows out of social processes and contexts involved in the epistemic process (Ernest, 1995, p. 476 - 477).

The second principle is concerned with the nature of reality as the cognizing subject can know it. Epistemologies such as traditional empiricism have a realist ontology, where the subject's knowledge is considered to be a "picture" of a true, objective reality. Greater experience of that reality causes the knower to have an evermore accurate picture of that reality. Radical constructivism, in contrast, insists that the subject can only ever know his experiential world. The subject cannot approach a true knowledge of reality but rather creates viable models of experiential reality; this reality is more or less subjective for each cognizing subject, tempered by experiences of the subject with others and the external world (Confrey, 1995c). Confrey considers radical constructivism to be based on four "planks":

1. Genetic epistemology – knowledge is constructed over time
2. Radical epistemology – knowledge does not reflect an objective ontological reality. A “picture” theory of knowledge is rejected. To know something is to act upon it; all knowledge consists of actions and reflection on those actions.
3. Scheme theory – A scheme is whatever is repeated or generalizable in an action, and involves the anticipation or recognition of a situation. First there is a perturbation, which calls for action; the perturbation is resolved internally by reflective abstraction. If the sequence of perturbation, action, reflective abstraction is repeated until action is stabilised, it constitutes a scheme. “For radical constructivist(s), the unit of analysis is a scheme and its genesis and modification” (p. 197)
4. Model building and construction of Others – the cognitive subject creates models of others (people as well as things), where understanding of an other is simply that the cognitive structures attributed to the model of the other have proven viable so far.

Such an experiential view of reality, rather than a view of reality as ontologically objective, means that a teacher is playing a socio-political role, presenting knowledge that is socially constructed, not objective at all (von Glasersfeld, 1995b). Teaching by transmission, telling students facts and items of objective knowledge will not encourage learning: “a fundamental tenet of the radical constructivist position is that mathematical concepts ... cannot be transmitted from one person to another by means of words alone” (Ellerton and Clements, 1991, p. 53).

Opinions differ as to whether Piaget himself believed in learning as approaching an accurate picture of true reality, or held a more radical constructivist position. On the one hand we find Walkerdine (1988) stating that “Piaget’s work is part of a ‘realism’ which treats the material world as knowable.” (p.1), Ginsberg and Opper (1988) describing learning in the Piagetian context as “an increasingly objective interpretation of the real world” (p. 218) and Miller (1993) getting “one step closer to reality” (p. 77), while on the other hand there are Dubinsky and Lewin (1986) firmly believing that Piaget was a radical constructivist and von Glasersfeld (1995c) describing Piaget’s genetic epistemology as rejecting “the traditional notion that knowledge should be a picture of reality” and defending Piaget’s “occasional passages that imply a realist stance” as “nothing but slips of mind” (p. 74).

## 6.7 Conclusions

As previously stated, it is not the intent of this thesis, nor of this theoretical chapter, to defend one of the many epistemological theories, nor argue one constructivist viewpoint against another. Observations made during the writing study project were found to be well described by Piaget's learning theory, in particular the differences between alpha and beta behaviour, and the phenomenon of perturbation. The confrontation of a student's cognitive system with a perturbation, thereby encouraging the deeper engagement of beta behaviour, compared to the shallow engagement of alpha behaviour, was not only observed to occur, but was found by the author to be a powerful pedagogical concept.

The question of an energising principle for the process of Piagetian learning was addressed by invoking the postulates of Kelly's personal construct theory. The tendency of the perceiving person to apply notions of same-different to triplets of experience in the formation of constructs, whether explicitly or implicitly, and the conscious decision to take that capacity under management, allow for an explanation of self-directed learning. The device of writing and its explicit application of language in a learning environment harness the learner's experience and at the same time provoke new perceptions for incorporation into the hierarchies or systems relating to the learning challenges.

For an educator of mathematics, beta behaviour is to be desired in student cognitive activity. Such behaviour has a greater probability of occurrence if perturbations and cognitive conflict are encountered by the student, encouraging him away from alpha behaviour and into beta behaviour with its associated reflective abstraction and cognitive reconstruction. Requiring students to write about their problem-solving processes and thereby challenging their expectations and misconceptions is one way of supplying such a perturbation. Better yet, it requires the students to supply their own perturbations, in that each student will engage with the mathematics at different levels of technical or algebraic expertise, yet, if they are engaging thoughtfully, there is the possibility that each student will encounter completely different cognitive conflicts.

Writing in mathematics, about mathematics, encourages engagement with the mathematics through its active nature. The level of planning required for writing coherent explanations has parallels with mathematical problem-solving. The level of abstraction required by the linguistic process of writing encourages beta behaviour at least in the writing activity, possibly promoting beta behaviour with the mathematical content. Writing about mathematical expectations and mathematical problem-solving processes increases the chances of students encountering perturbations and disequilibria, for which the requisite compensation obliges students to engage more deeply with the mathematics and undergo the constructive process of beta behaviour.

## 7 Research Design

### 7.1 Research question

- What effect does the writing of explanatory strategies have on mathematical problem-solving?

Subquestions are

- Are any observed effects of writing in problem-solving different for students with differing main languages?
- Are any observed effects of writing in problem-solving different for students with differing degrees of mathematical preparedness?

### 7.2 Population and sample

The mathematics course within which the writing study project was carried out was MAM100W. MAM100W was a first year university mathematics course for science majors (mathematics, applied mathematics, physics, chemistry, microbiology, geology) and for business science and actuarial science students. A number of life science students (botany, zoology) also took MAM100W by choice, although there was a half year course which they could have taken instead. A few (usually about 20) engineering students also took the course, largely as a result of a combination of having failed their engineering maths course the year before and having second year timetable clashes not allowing them to repeat their engineering maths course. The course was primarily a calculus course, including differential and integral calculus, partial differentiation and elementary differential equations, but also includes vector geometry, infinite series, complex numbers and linear algebra. The students who took part in the exercise were a primarily science and engineering students.

The structure of the course consisted of morning lectures and afternoon tutorials. The lectures were 45 minutes long, were held in large raked lecture theatres, and catered to groups of 180 to 250 students. Lecturing took the form of the standard “chalk and talk” transfer method. There were five lectures per week. In contrast to the lectures, the tutorials were smaller, and interactive. Students attended one two hour afternoon tutorial per week, with a ratio of about 27 students per tutor. Some tutorials were large with several tutors, and some were small with a single tutor. The tutorials involved in this exercise were all single tutor tutorials, with the author as the tutor. Students were

presented with a sheet of questions of varying levels of complexity and worked in groups of three to five seated around tables to solve them. The tutor was present to help with any difficulties which may have arisen, and help consisted of assisting the student discover the correct answer or technique rather than simply presenting the students with a correct solution. All tutorials, run by all the tutors, were designed to run the same way. In the particular semester of the study project, two of the tutorial groups differed, being two of the author's tutorial groups, running two versions of the writing initiative. It was an important aspect of the study project that the writing exercises not require any restructuring of the existing course, either in curriculum content or logistics such as tutorial organisation.

Three tutorial groups were involved in the exercise. The After group (A), the Before group (B) and the Control group (C). C was a completely usual tutorial group, differing only from all the other (non experimental) afternoon tutorials by having the author as the tutor. The A and B groups were usual tutorial groups with the addition of having writing exercises included with the standard sheet of questions. Not all attendees of the A and B tutorials took part in the writing exercises.

**Table 7.1 Student numbers in tutorial groups**

	Number of students regularly attending the tutorial	Number of students taking part in writing exercises
After (A)	28	18
Before (B)	23	21
Control (C)	29	NA

The definition of “taking part in the writing exercises” includes having signed the consent form in the affirmative, and having handed in at least three writing exercises. Students in similar degree programmes tend to have similar timetables and hence attend the same tutorials. As a result, there tends to be clumping of particular degree programmes in particular tutorials. While a range of degree programmes in the experimental groups was desired, it turned out to be the case that in both the A and B groups there were few business students, with a majority of science students. Each experimental group also included some engineering students. There was no apparent pattern to the students who did not take part in the exercises, for example it was not the

case that commerce students consented and science students did not, nor any other similar pattern.

**Table 7.2 Student profile by degree programme**

	Degree Programmes	Numbers of students
A – not taking part	COMB03	1
	ENGB19	1
	SCIB06	1
	SCIB13	5
	SCIB14	1
	SSHB27	1
A – taking part	COMB12	1
	ENGB19	2
	SCIB06	1
	SCIB12	1
	SCIB13	10
	SCIB14	3
B – not taking part	COMB03	1
	SCIB06	2
B – taking part	ENGB01	1
	ENGB09	2
	SCIB06	7
	SCIB13	3
	SCIB14	7
C	COMB03	11
	COMB04	4
	ENGB16	1
	ENGB23	1
	SCIB06	3
	SCIB11	1
	SCIB13	2
	SCIB14	6

Key to Degree Programme Codes:

COMB03	Bachelor of Business Science in Actuarial Science
COMB04	Bachelor of Business Science
COMB12	Bachelor of Commerce in Philosophy, Politics and Economics
ENGB01	Bachelor of Science in Engineering in Chemical Engineering
ENGB09	Bachelor of Science in Engineering in Electrical Engineering
ENGB16	Bachelor of Science in Engineering Foundation Programme
ENGB19	Bachelor of Science in Engineering in Geomatics
ENGB23	Bachelor of Science in Engineering in Electromechanical Engineering
SCIB06	Bachelor of Science in Information Technology
SCIB11	General Entry for Programmes in Science
SCIB12	Bachelor of Science in Biology, Earth and Environmental Sciences
SCIB13	Bachelor of Science in Chemical, Molecular and Cellular Sciences
SCIB14	Bachelor of Science in Mathematical, Physical and Statistical Sciences
SSHB27	Bachelor of Social Science in Politics, Philosophy & Economics

At the beginning of the semester, when the tutorial groups were allocated tutors and venues, there was some limited flexibility in the choices of students allocated to groups A, B and C. A major factor in the choices was the language distribution as it was desired that, at the end of the semester, analysis could validly address whether the writing exercises had a differential effect across language groups. Unfortunately, due to factors outside the author's control, the tutorial groups that finally settled down for the semester were predominantly English first language, leaving little comparative material to analyse with regard to language. Similarly, as with degree programme, there was no pattern to those refusing to take part in the exercise. In the A group 8 of the 10 students not participating had English as a main language, the other two speaking Afrikaans, and in the B group 2 of the 3 not participating had English as a main language, the third speaking an African language.

**Table 7.3.1 Main languages of participating students**

Group	English	Other	
A	15	3	Mandarin, Afrikaans, Portuguese
B	15	6	Shona x2, Zulu, Unknown*, Tswana x2
C	26	2	Xhosa, Zulu

\* AP was not an English main language speaker, however no information is available on his main language. He was not used in any analysis due to only submitting two writing exercises during the semester.

As discussed in greater detail in Chapter 8, the groups of students taking part in the writing study project consisted of students who were categorised as being either Prepared or Under-prepared for the university mathematics course, based on their school leaving mathematics result, their midyear course class record and their final end of year mathematics result. In the case of main language, the tutor was aware at the beginning of the experimental semester of the students' languages and was able to make (largely futile) efforts to increase the number of non-English speaking students within the tutorial groups. Unlike language, the assessment of students' level of preparedness was only made after the study project had been completed and no awareness of preparedness (except in cases where extreme proficiency or extreme ill-preparedness were obvious) was present during the study. As had previously been the case in the author's experience of the course and perceptions of the student body, a range of preparedness was represented in both experimental groups.

**Table 7.3.2 Mathematical background of participating students**

Group	Prepared	Under-prepared
A	8	10
B	7	14
C	14	15

The semester was 13 weeks long, but no writing exercises were required in the first week, to allow for students still settling down, nor in the 13<sup>th</sup> week as attendance was extremely low and there would be no opportunity for handing anything back to the students.

### **7.3 After and Before**

Two different approaches were taken with the writing exercises. One of the desired outcomes of the writing exercises was that they would positively affect the students' problem solving abilities, therefore the definition of a mathematical problem dictated how the writing exercises should be presented. A mathematical problem is defined as being a mathematical question for which the student has no immediate solution schema. A mathematical question for which the student knows what to do, even if he or she

struggles to do it, is an exercise and not a problem. Since the writing exercises required the students to explain their thought processes, it necessarily follows that for true problems, this explanation can only come after the solution has been attained. If the student were capable of carefully describing his or her thought processes before attempting the question, the question is necessarily not strictly a problem.

The practicalities of the tutorial question sheet, however, meant that there was not a true problem present every week, although there were generally one or more questions that were challenging and required some serious thought, rather than thoughtless application of a recipe. For this reason as well as for the inspiration supplied to the writing study project by the physics explanatory strategy initiative (Leonard, Dufresne and Mestre, 1996; Leonard, Gerace and Dufresne, 1999), which required students to write on problems before solving them, two experimental treatments were devised, one with a particular focus on true problems (when available) and the other on challenging problems that were accessible to solution before attempting calculations, as judged by the tutor. The Control group, with no writing exercises, was used as a comparison in the analysis of the quantitative data collected from assessment tasks.

The After group was that study segment on which the focus for problem-solving was of greatest interest. Each week, the tutor would choose the question on the tutorial sheet that was the best example of a problem. If there was no problem, in the strict sense adopted, the question that would best benefit from a reflective approach was selected. The tutorial question sheets were set by the course convenor, while the question chosen for the students to write about was chosen by the author as tutor.

An additional facet to the After writing exercises was a requirement that the student pause before beginning a solution and estimate what form the expected answer would take. No precision was required here, simply a recognition that the answer would be, say, positive, or a natural number, or a parabolic function. The estimate was to be presented in as few words or symbols as possible; no discussion was required at that point. After the solution was completed, the students were required to write an explanatory paragraph on what they did in their solution, and why. They were also required to look back at their solution estimate and comment on its accuracy. The aims of the After writing exercises were

- reflection on one's own thought processes

- an improved problem solving ability due to such reflection
- an understanding that a sensible solution space exists
- a decrease in nonsensical answers to problems due to the recognition of the solution space

The primary hope of the exercise was that problem solving ability would improve due to increased thoughtfulness, understanding of problem solution structures and improved metacognitive control due to reflection on one's own thought processes.

### Example of Group A exercise

Tutorial 23 (week 11)

Solve for  $n$ :  $\binom{n}{7} = \binom{n}{5}$

Before any calculations, can you estimate (roughly) what size you expect  $n$  to be? If you have no idea, please say so

After you have calculated the answer, explain what you did, in words. Was your answer roughly what you expected? If not, can you see how to improve such estimates in the future?

The Before group were required to write their explanatory paragraphs before attempting any solutions. As a result, strictly defined problems were not a suitable vehicle for the exercise. Instead, each week, a question on the tutorial sheet was chosen that would benefit from a “look before you leap” approach. Due to the sparsity of strictly defined problems on the regular weekly question sheets of MAM100W, the questions used in the A and B groups were often the same, chosen because they were layered, deep problems, with scope for the students to display some thoughtfulness.

### Example of Group B exercise

Let  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ .

Find  $A^{-1}$  and (if possible) matrices  $X$  and  $Y$  such that  $AX = B$  and  $YA = B$ .

Before calculating  $X$  and  $Y$ , explain how you are going to solve for them. In particular, explain how you can tell whether it is possible to solve for them.

A complete list of tutorial questions for which students were expected to complete writing assignments is included in Appendix 2. The way the writing exercises of both

groups were effected was the same. As the students arrived at the tutorial venue, they were handed the weekly question sheet as well as a separate piece of paper on which would be written something of the form

Writing exercise Tutorial 20

Question 3. Before attempting any calculations, write a paragraph on how you intend to solve this problem and why you have chosen this particular method.

Or

Writing exercise Tutorial 18

Question 2b. In a few words, say what form you expect the answer to take. After completing the calculations, write a paragraph on what you did in your solution and why you chose to do it. Was your expectation of the form of the answer correct?

The initial plan with the writing exercises was that the student responses would only be handed in for comment approximately every three weeks. However it was observed that the students only took their responses seriously on the days on which they were expected to submit them, so after week 6 (of 13 in the semester) the students were requested to submit responses each week. This change added vastly to the tutor's workload, as commenting constructively took a long time, but later, the interviews would indicate that the comments were greatly valued.

The Control group, of course, received no writing exercises, simply the weekly question sheet. A danger of which the author as tutor was aware was one of urging students in a particular group (say, the Control) to carry out the activity that defined one of the other groups (say, the Before group). The danger is grounded in the concern that if one is convinced that a particular technique is fruitful for learning (yet is without current evidence of efficacy), how can one stop oneself from urging all one's students to use it? The tutor was acutely aware of this danger of the imperative to intervene and only transgressed the boundaries on one single occasion, when urging a student in the Control group, once he had completed a solution which he found very complicated yet finally very rewarding, to write down what he done in careful steps so that it would be clear in his mind. One question, with one student out of a class of 28, in one tutorial out of, effectively, 11, was felt to be insufficient to affect any collected data.

A second danger, which could not be avoided, was that the experimental groups received more attention from the tutor than the Control group. There was no extra

attention during the actual tutorial, for any individual student (there simply was no time in the context), but the effort put into commenting on the writing exercises constituted extra attention, which was unavoidable. This extra attention has to be considered when comparing either experimental group with the Control group.

## **7.4 Ethics**

Any experiment involving human subjects is subject to ethical scrutiny. The Department of Psychology at the University of Cape Town follows the Ethical Code of Professional Conduct published by the Professional Board for Psychology in South Africa. The Code is available from the website of the Psychology department (<http://web.uct.ac.za/depts/psychology/research/resm.html>). The elements of the Ethical Code that pertained to the type of experiment being described in this thesis were selected from that source and an ethical code for this experiment was drawn up from them (see Appendix 3).

Addressing the requirements of the ethical code, permission was sought from the convenor of the MAM100W course and from the head of the Mathematics and Applied Mathematics Department to run the experiment, which was explained in depth to both decision makers. Such permission was received.

In addition, a consent form was drawn up (see Appendix 3) in which the experiment was briefly described. The students had the opportunity to agree or decline to participate as well as to give or deny permission to be quoted. Their understanding was specifically addressed to ensure that they knew they could withdraw from the exercise at any time, and that their participation or lack of participation would have no effect on their standing in the course. Two consent forms were supplied to each student so that they could keep a copy themselves to remember what it was that they had signed and review participation as they felt the need. Only students who signed the consent forms in the affirmative have been included in the study. One student initially agreed to take part, but later withdrew. Students were asked verbally whether they were over the age of eighteen. All students were over eighteen, so no guardians' permissions had to be sought. All consent forms were filed and preserved.

An outline of the experiment, as well as a draft of the consent form, was sent to the Ethics in Research Committee of the Faculty of Science, to allow that body opportunity

to question the experiment and to give or deny permission to carry it out. No such questioning or denial transpired.

## **7.5 Data**

Data was collected in three forms, interviews, the writing exercises themselves, and quantitative data from scrutiny of test and examination scripts. Not all students taking part in the exercise were interviewed due to time constraints, however the other two forms of data are available for all students taking part.

### **7.5.1 Interviews**

A cross section of the students was chosen for the interviews, with an attempt to make the numbers approximately equal across both groups (A and B), with equal spread across genders and as many non English main language speakers as possible. No objective measure of the students' levels of preparedness had been performed at the time of the interviews, however the author's observations of the students over the full year of the course provided accurate perceptions of degree of preparedness, resulting in post hoc evidence for a range of preparedness being represented in the students chosen for interviews.

The first student characteristic governing choice for interviews was main language. The author first targeted students who did not speak English as a main language, as language was a particular focus of the investigation. None of the non-English students were particularly high achievers in the course (subsequent analysis categorised them all as mathematically ill-prepared) so thereafter the author chose students who were high achievers in the course, there being a small number in each tutorial group. The resulting list was then scrutinised for gender, the author's perception of their level of preparedness and their involvement in the project. Students were then chosen to create a spread across gender, preparedness and language. In each case, students were taken aside privately after class, and requested to attend an interview. No monetary or other incentive was provided, all students asked agreed, although one (AN) backed out for reasons of ill health at the last minute.

**Table 7.4.1 Gender breakdown of students interviewed**

	Gender		Total
	Male	Female	
Group A	6	3	9
Group B	5	3	8
<b>Total</b>	<b>11</b>	<b>6</b>	<b>17</b>

**Table 7.4.2 Language breakdown of students interviewed**

	Language		Total
	English	Other	
Group A	6	3*	9
Group B	5	3**	8
<b>Total</b>	<b>11</b>	<b>6</b>	<b>17</b>

\* Portuguese, Mandarin, Afrikaans

\*\* Shona (x2), Tswana

**Table 7.4.3 Preparedness breakdown of students interviewed**

	Preparedness		Total
	Prepared	Under-Prepared	
Group A	3	6	9
Group B	3	5	8
<b>Total</b>	<b>6</b>	<b>11</b>	<b>17</b>

The students were presented with a list of the questions at the beginning of the interview. None of the students were interested in the list, however, and none read them through, although time was allowed for them to do so.

### List of interview questions

- I obviously know you're studying maths, what else are you studying this year? What are your majors? Are you intending to do postgraduate work? What is your favourite subject?
- What would you say MAM100W was about? Why was the course constructed as it was? What purpose does the maths department have for offering the course?
- Two different sorts of writing assignments this semester were organised in two tut classes. In one, writing about problems after the calculation was emphasised, with perhaps a brief mention of expectations before the calculation, and in the other writing about problems before attempting any

calculations was emphasised. You were in one of those groups. Can you identify which one?

- Did you find the writing exercises easy or difficult to do? In what way?
- During the tutorials, did you find that the writing exercises helped you think about the problems?
- Did you ever find that the writing exercises helped you outside the tutorials, like doing homework?
- How often did you read the comments that were made on the writing exercises when they were handed back? Always? Never? Sometimes? If yes, did you find them helpful in any way?
- Did you feel that the writing exercises took up a lot of time?
- Can you think of any ways in which the writing exercises could be made to be more useful to you?
- What would you describe as your major strengths in mathematics? Major weaknesses?
- There are two major types of mathematics that you encounter in this course. There are the rules, like “differentiate this”, “use the cross product”, “apply Newton’s Method”, and then there are the kinds of problems where you need to think for a bit before you know what to do. Like that one we did in the first test with the parabola and the circle that had to fit inside it. [During the interviews I generally gave a different example to the one just stated on the planned questions, one that was not a word problem.] Solving that sort of thing is called problem solving, and it is what the rules are made for, really. Problem solving involves applying rules, but it also involves solving a puzzle, although sometimes only a little one. If you compare problem solving with using rules (like differentiating, doing cross product, using Newton’s Method, etc.) which sort of maths do you prefer? Why?
- What would you say the aim of the writing exercises was? What do you think they were meant to achieve? Would you say that the writing exercises were a success in that regard?
- Do you think the writing exercises have made any difference to your actual marks in this course? To anybody else’s marks?
- Do you think you would have learnt as much as you have in this course if you had just done the tut and not been involved in the writing exercises?

- What languages do you speak? Which language do you regard as your main language, the one you speak at home, and are most fluent in/best at?
- How would you describe your fluency or ability in English? Speaking, reading and writing are all a bit different. Do you find any of these 3 skills harder than the other 2?
- Have you ever had any language problems in the mathematics course? Any kind at all?
- There are different schools of thought on mathematics and language. Some people say that mathematics is a whole new language and its difficulties are the same for everybody. Other people say that if it is taught in English and you don't speak English as a first language, then the new language difficulties of mathematics make it even harder for you than for English speakers. Do you have any views on that contrast of opinion?
- Do you feel you came into this course sufficiently mathematically prepared?
- How would you judge your progress in MAM100W (as the year has gone by)?
- What sort of skills would you say you have gained in this course?
- Over all, would you say the writing exercises were a good experience, a bad experience or made no difference to you at all?
- Would you recommend to a friend that they make an effort to come to a tut that involves writing exercises instead of a tut that does not?

The interviews were audio taped and were transcribed into written form later. Initials were used, instead of names, to protect the identities of the students. The interviews were carried out in the author's office, where furniture had been moved around to make the setting more conversational and less like someone in authority sitting behind a desk and carrying out an interrogation. The more comfortable chairs were assigned to the students.

### **7.5.2 Writing exercises**

Altogether the students were asked to hand in 7 writing exercises. In all 11 writing exercises were required of them during the semester, but regular submission was only required later, after it was observed that the students did not take the exercises seriously otherwise. Weeks 5, 7, 8, 9, 10, 11, 12 were the weeks of the semester during which submission was required. During the 24 hours after the tutorial, comments were written

on the writing exercises and then the exercises were ready to be collected by the students, an arrangement that was made clear to the students. The writing exercises were returned at the next week's tutorial if the students had not come to pick them up before. Only one student on one occasion came to collect a writing exercise, otherwise they were all returned the following week. Comments were all of a formative nature, no summative assessment (grading) was given. All comments were positive; a very serious effort was made to be encouraging and not say anything negative. In the case where something was written that was unarguably wrong the error was pointed out, but always with comments that showed that the tutor understood where the incorrect assumption was coming from and where there was logical thought that could be praised.

The students were of greatly varying degrees of mathematical ability and in consequence the comments differed. Students were addressed at their own personal level and were encouraged to stretch themselves a little further. In the second quarter of the semester, from week 8 onwards, a challenge was added to the comments and labelled as such. Each student was challenged to take their solution or explanation one step further. A challenge would be of the form

*(The student had solved the problem*

$$\text{Solve for } n: \binom{n}{7} = \binom{n}{5}$$

*algebraically, providing a clear description of her method of calculation. The writing exercise had been commented on and returned, with the following challenge).*

Challenge: Without doing any calculations, and using Pascal's Triangle, can you see how you could have still reached the solution  $n = 12$ ?

Some of the students would go so far as to treat a challenge as a serious question and even write an explanatory paragraph on it. Some would read the challenge and think about it briefly, some ignored it. Observations of this behaviour were backed up by the interviews, during which some students reported having found great value in the challenges, while others admitted to not paying them much attention.

### 7.5.3 Quantitative Data

Throughout the year the students wrote prescribed tests and examinations of the course within which they were asked questions which could be described as *problems*. In each such case the problem solving ability of the students in the study was recorded by the author. Four pieces of information were recorded for each student's approach to each of the problems.

First, an attempt was made to determine whether the student *understood* what the question was asking. Understanding the question might be signalled by a perfect answer just as much as by some preliminary sketches and scribbles.

Secondly, evidence was sought that the student was capable of *metacognitive control*. In many solutions there is no opportunity for any such evidence to be available. A perfect solution to a problem is inconclusive for metacognitive control. Such control may be revealed by starting a calculation or diagram that is incorrect, and abandoning that path to try something else, possibly correct, possibly not. Lack of control may be evidenced by doggedly pursuing a course which should be obvious to the student as incorrect. Calculators were not allowed in the tests, as a rule, so any calculation that would sensibly demand a calculator (multiplication of particularly large numbers, or calculations of logarithms, for instance) would be a good indicator of an error having crept in somewhere, an error that many students do not recognise.

The third piece of information recorded was whether the student had shown *flexibility* in her solution. The question was asked “does this student show the ability to throw ideas around?” in each case. Again, often the question was unanswerable, particularly for perfect solutions. For each of these three items the data was recorded as 2 (yes), 1 (no) or 0 (cannot tell).

The fourth data item recorded in each case was the actual *mark* (grade) that the student was assigned for the question.

In the first semester, not knowing which students would be in the second semester experimental groups, the entire class’s problem solving ability was analysed in this form. In the second semester only the papers of the students in the A, B and C groups were analysed. A considerable impact on the extent of the data, however, resulted from the second of the three assessment tasks in the second semester being practically excluded from this data set. During the year, three tests an hour and a half in length were written in the first semester, culminating in a three hour test at the end of the semester. In the second semester, two hour and a half tests were planned as well as two three hour examinations at the end of the semester. One of the examinations was not included in this data set as it was entirely of a multiple choice nature, leaving three

assessment exercises to be used for problem solving analysis, already fewer than in the first semester. The second test of the second semester, however, originally and accidentally coincided with Sukkot, a Jewish religious holiday, and had to be moved at the last minute. Practical problems including the approaching end of term and the lack of suitable venues for evening tests required that the test be written during the lecture periods. As a result the test was halved in length, two halves of the class wrote two different tests in different periods and the planned single written question was discarded in favour of purely multiple choice questions. The effect on the writing study project was that the students from the experimental and control groups did not answer the same questions (the students were split between the two tests) and no fully written solutions were available for scrutiny. The only analysis that could be performed was of the students' rough work on the questions, which is problematic enough without the added setback of different questions having been worked on by different students. Although such analysis was attempted, it was ultimately discarded as being completely without validity in this project.

The model of having at least one experimental group as well as a control group, and of making observations both before and after experimental treatment is based on one of the experimental designs considered by Cook and Campbell (1979). Cook and Campbell consider the cases in quasi-experimental design that involve non-equivalent groups due to using the pre-test results as the defining factor between groups. Though in this study the pre-test measures had no influence on the tutorial class choice, the groups were, inevitably, non-equivalent. Cook and Campbell (1979) list eleven non-equivalent control group research designs. They are, in brief,

1. one-group post-test only
2. post-test only with non-equivalent comparison groups
3. post-test only with predicted higher order interactions (that is, the treatment is expected to have different effects within the experimental group depending on an additional variable)
4. one-group pre-test-post-test
5. two-group pre-test-post-test using an untreated control group
6. non-equivalent dependent variables pre-test-post-test (A single group of persons is involved; tested on two scales, one of which is expected to change as a result of the treatment and the other is not.)
7. removed-treatment pre-test-post-test

8. repeated treatment
9. reversed-treatment pre-test-post-test non-equivalent comparison groups
10. cohort designs with cyclical turnover (a form of longitudinal study)
11. regression-discontinuity (that is, while change in the treatment variable has the same linear slope before and after treatment, there is a discontinuity immediately following the treatment.)

It is Cook and Campbell's fifth design that was in use during this teaching experiment. The first three designs, involving only a post-test, have very low validity as it would be impossible to determine what effects were due to the teaching techniques, or, indeed, whether the effects had even been positive. The fourth option Garson (2002) refers to as "common but flawed" (p. 146), and is also subject to validity problems since it is difficult to determine whether any improvements that happened to be observed might not have occurred in the absence of the specific teaching technique. The sixth research design listed, while providing reliable data, has the associated problems of selection of variables, and how to measure them. Options seven, eight and nine all involve removing, repeating or reversing the treatment in some way, a possibility not available in this study. Option ten is an option for a study that can be carried out over several years and is best suited for a primary or secondary school setting, and option eleven requires measures more accurate than those that were available in order to be both precise and accurate.

The pre-test-post-test structure was improved by running multiple tests, what Garson (2002) refers to as "interrupted time series with a non-equivalent non-treatment comparison group" (p. 148). Cook and Campbell (1979) employ a diagrammatic technique to summarise the experimental situation. The upper row denotes the experimental group, the lower the control and the  $X$  the treatment. Movement horizontally to the right is movement through time, and  $O_i$  is the  $i$ th test (observation). The dotted line suggests that the groups are non-equivalent. The simple case is depicted as

$A$	$O_1$	$X$	$O_2$
$B$	$O_1$		$O_2$

A more complicated situation of two experimental groups, with two different types of writing exercises and seven expected observations as envisaged in the design of this study, is depicted below.

<i>A</i>	$O_1$	$O_2$	$O_3$	$O_4$	$X_1$	$O_5$	$X_1$	$O_6$	$X_1$	$O_F$
<i>B</i>	$O_1$	$O_2$	$O_3$	$O_4$	$X_2$	$O_5$	$X_2$	$O_6$	$X_2$	$O_F$
<i>C</i>	$O_1$	$O_2$	$O_3$	$O_4$		$O_5$		$O_6$		$O_F$

The data from Observation 6 in this study, however, was abandoned before serious analysis.

**Table 7.5 Outline of assessment tasks throughout the year**

<b>First Semester</b>			
Assessment task	Length (hours)	Format	
Test 1	1.5	MCQ, with single fully written question	For purposes internal to the course, the students' progress had to be measured with some accuracy at the end of the first 6 weeks of the semester. To minimise required time, the first two tests of the year were MCQ.
Test 2	1.5	MCQ, with single fully written question.	
Test 3	1.5	Entire test fully written, not MCQ.	Two questions chosen for analysis.
Test 4	3	Half written, half MCQ. Test 4 is at the end of the semester.	A single question was chosen for analysis.
<b>Second Semester</b>			
Test 5	1.5	Fully written	A single question was chosen for analysis.
Test 6	1.5	The plan was to have the test be primarily MCQ with a single question requiring a fully written solution, as Tests 1 and 2.	The split of the group into two distinct tests at two distinct times, both tests MCQ, rendered Test 6 useless for the purposes of the study.
Paper 1	3	Fully written	Two questions were chosen for analysis, although one was subsequently discarded.
Paper 2	3	Entirely MCQ	Not analysed.

MCQ: Multiple choice questions

See Appendix 4 for greater detail on the questions chosen for analysis

One question initially chosen for analysis was discarded due to its poor evidence in favour of it being a problem from the students' point of view. In the final examination (Paper 1), the following two questions were initially chosen for analysis.

Final examination, first question

Evaluate the integral  $\int_0^1 x \ln x dx$ .

Final examination, second question

Prove that the lines

L1: through (0,1,5) and (7,-9,9)

L2: through (1,1,7) and (2,-1,7)

intersect and find the point of intersection.

While marking the tests and analysing the students' evident problem solving skills it was noted by the author that the second question chosen for analysis from the final examination was not a good choice. The majority of students clearly had access to a recipe, or algorithm for solving the question and showed no evidence of problem-solving behaviour. The fact that the question was algorithmic to the students and not literally a problem at all should have been expected before hand, but the reason for its initial choice was the danger in the response to the question of using the same parameter for each of the lines the equations of which the students were required to find, and thereby ending up with an inconsistent solution. The students had encountered this type of problem before, though, and very little problem solving behaviour was discernible. A difficulty with the final examination is that it was set without this research project in mind, and had very little scope for problem solving. The integral question, though, was quite a good question for determining problem solving skills as its identity as an improper integral was not made known to the students.

As covered elsewhere in this thesis, there are numerous reasons why writing about problem-solving might be expected to have an advantageous effect on problem-solving behaviour. A similar physics initiative (Leonard et al, 1996) achieved encouraging success in requiring students to write about problems before carrying out calculations. The definition of a true mathematics problem prohibits the problem solver from writing about a problem before making any solution attempt. Therefore, two experimental groups were designed, one of which attempted to seek out true problems and write about them after carrying out calculations, and the other tackled challenging problems

that were, nonetheless, accessible to solution before carrying out calculations, writing about them before calculations. A third, control, tutorial group did not carry out writing exercises, working through the same sets of questions as the two experimental groups, as indeed were several simultaneous tutorial groups with different tutors.

A constraint on the research design which had a great effect on the study project was imposed by the author, that of requiring the writing initiative to be designed as an incorporation into an existing course without changing the curriculum, the tutorial question sheets, or the logistics of lectures and tutorial classes. The greatest effect this self-imposed constraint had on the project was variation from one week to another in the tutorial questions, with no certainty that a true mathematical problem would be available.

A second constraint on the design, or perhaps constraint on the analysis, was the small number of non-English students in the experimental groups. By exerting a greater degree of coercion on allocation of students to tutorial groups, this imbalance could have been avoided, however it was felt that the freedom of the students to choose their tutorial group was more important than final student profile. Consider that, to increase the number of non-English students in the experimental groups, the tutor would have had to target non-English students in the tutorial groups occurring simultaneously and require them to change tutorial group, probably through threats of attendance requirements for the course, simply because of their main language. Such a situation was deemed unacceptably manipulative.

A third constraint on the research design was the course structure of large lectures and small tutorial groups. In order to deal with a manageable segment of student body, the writing initiative was carried out in three tutorial groups. Running anything in several tutorial groups has two effects, that of non-assessability since the entire class is not receiving the treatment, and that of shallow intervention since the tutorials (for any individual student) occur one afternoon per week compared to the lectures occurring every day. Running the initiative in the lectures would allow the entire class to be involved and thus make the writing exercises assessable, as well as deepen the intervention since the students could encounter the writing exercises as much as five times per week. However, the comments on the assignments would then inevitably have to fall away, as commenting on 500 written exercises would take prohibitive amounts of

staff time. The dilemma of shallow intervention versus lack of feedback appears insurmountable.

Three forms of data were collected from the students. Most informative were the interviews, providing unexpected insight into the students' processes of understanding mathematical material through deep engagement. The process of acquiring understanding has been modelled on Piaget's three-pronged constructivist theory of learning (Piaget, 1978). The students' submitted written exercises were analysed using a coding scheme designed by Waywood (1992) and adapted for this study project, revealing their increased creative engagement with the work throughout the semester, and their changing stance towards mathematical knowledge. The third form of data, and the only one with which the Control group played a role, proved the least informative. No significant change in the students' problem-solving behaviour in their tests and examinations throughout the year was observed, although there are slight suggestions that greater understanding was shown by the experimental groups, and that some factor (deeper understanding of mathematics, perhaps, or more efficient use of time through thoughtful problem-solving approaches) positively influenced the pass rates of the experimental groups.

## 8 Data Analysis

Three forms of data available for analysis were collected during the writing exercise project. The data took the form of interviews, the written work of the students and quantitative data taken from assessment of the students' tests and examinations during the year. In addition to the data collected from the students, the author kept a journal throughout the semester during which the project took place, making notes directly after each tutorial class.

The student utterances during the interviews draw attention to the difference between surface and deep approaches to mathematical engagement, and the resulting success or lack of success in understanding the topic in question. The interviews further reveal the non-spontaneity of taking a deep approach, in that the writing exercises "forced" the students to think about the problems when their natural inclination appeared not to do so. The two levels of engagement coupled with the unlikelihood of choosing deep engagement over a surface approach finds explanation in the Piagetian model of alpha and beta behaviour, with the requirement of a perturbation before beta behaviour is invoked. A third, and surprising, revelation of the interviews was the perception among several of the students that the writing exercises, far from taking time more profitably spent doing other work, saved time by causing them to solve problems quickly and efficiently due to the preliminary process of thinking deeply about the problems before beginning any calculation processes.

Categorising each student as English/Non English main language speaking and Prepared/Under-prepared mathematically, a few observations could be made, such as: not being main language English speakers does affect students' perceptions of the course, but not in ways which are immediately apparent to the students; and despite agreement with respect to the purpose of the mathematics course, the prepared and under-prepared students considered themselves to have gained different skills from the course.

Analysis of the written exercises suggests that, over the course of the writing project, the students showed an increased tendency towards investigating mathematical processes and away from simply obtaining answers, away from recounting facts and towards explaining phenomena. This explanatory change corresponds to a change in

stance towards knowledge: away from being a passive observer of objective knowledge and towards being an active participant in the creation of knowledge (Waywood, 1992). The quantitative data drawn from each student's assessment tasks proved the least informative data source. A suggestion of increased understanding can be observed in the data, although such a conclusion is not decisive. The suggestion of increased understanding is, however, consonant with the emphasis placed on understanding by the students during the interviews, and with the evidence of increasingly deep engagement with the mathematics revealed in analysis of the writing exercises themselves.

## **8.1 Language grouping**

At the beginning of the course, the students were required to fill in an enrolment form on which they recorded, among other information, their main language. In addition to the enrolment forms, lists were passed around in the experimental groups, on which the students were asked to record their main language as well as other languages in which they were proficient. A final source of language data was the interviews, during which the students interviewed were asked for their main language and other spoken languages. While not all students were represented in all three data sources, data were available on all students (except one (AP) who took part in the experiment too shallowly to be included in any analysis) and no contradictions were found in the majority of students on whom at least two sources of information were available.

**Table 8.1.1 Main languages of participating students****Group A**

Student	Main language
AJ	English
AN	English
AS2	Mandarin
DC	English
DK	English
DL	English
ET	English
IS	English
JL	English
MS2	English
MS1	English
NG	English
NW	Portuguese
OU	English
RG	English
RT	English
TH	English
WT	Afrikaans

**Table 8.1.2 Main languages of participating students**

**Group B**

Student	Main language
AP	Other*
AS1	English
BJ	English
BM	English
CM	Shona
CP	English
DB	English
GCP	English
IT	Tswana
JB	English
JE	English
JS	English
LM2	Zulu
LM1	English
MM	Tswana
MW	English
NH	English
RA	English
SW	English
TJ	Shona
TvH	English

\* For this one student (AP), no language data were available. However AP was not interviewed and he handed in too few writing exercises to be included in any analysis. Personal communication between the author and AP made it clear that the student was not a main language English speaker, although the student's main language is not known.

## **8.2 Preparedness grouping**

Three data items were chosen to designate each student as prepared or under-prepared for the mathematical requirements of the course, namely their school leaving (matric) mathematics result, their midyear result in MAM100W and their end of year result in MAM100W. Six categories were chosen, three of them summarising to Prepared, and three to Under-prepared. A typical trio of results (school leaving, mid year, end of year)

is shown after each description. The school leaving result is a symbol (A, B, C) and the latter two are percentages of a possible 100%. A table of results for each student is not given in order to protect the anonymity of the students.

#### Prepared

- consistently good performance from matric to the final exam (A, 81, 89)
- acceptable performance from matric to the final exam (A, 63, 85)
- good matric, weak midyear result, acceptable final result (A, 53, 70)

#### Under-prepared

- good matric, but weak performance at university (A, 32, 27)
- weak to begin with, but improving towards the end of the year (C, 39, 65)
- poor results from matric through to final exam (D, 46, 54)

Other combinations are theoretically possible, yet were not observed among the students under scrutiny, such as (hypothetically) a weak school leaving result (C), an excellent midyear result (95) and weak final result (25). The six categories given encompassed all students taking part in the study project and the coding scheme was not extended to include all other theoretical combinations.

The school leaving result was not available for five of the students, however their university results were sufficiently poor (39, 21) that the only categories which might apply to those students were Under-prepared categories. Where none of the categories was an immediately obvious choice, the student's other test assessment results during the year were scrutinised to determine an appropriate descriptor. While the term "preparedness" does refer to the students' level of preparedness upon entering university, school leaving results alone are an insufficient indicator of that level. Performance at university was used in addition to school leaving results to infer the students' entry level preparedness. For instance, a good school leaving result with consistently poor results at university (*good matric, but weak performance at university*) indicates that the student in question was not as prepared as the single data point of his school leaving mathematics result might have suggested. It was felt that scrutiny of university mathematics results, in addition to school leaving results, provided a more robust view of preparedness than school leaving results alone.

**Table 8.2.1 Preparedness of participating students: Group A**

Student	Prepared	Category of Preparedness
AJ	Yes	consistently good results from matric to the final exam
AN	No	either good matric and weak university OR weak throughout*
AS2	No	good matric, but weak performance at university
DC	Yes	acceptable performance from matric to the final exam
DK	Yes	good matric, weak midyear result, acceptable final result
DL	No	good matric, but weak performance at university
ET	No	either good matric and weak university OR weak throughout*
IS	No	poor results from matric through to final exam
JL	No	good matric, but weak performance at university
MS2	Yes	good matric, weak midyear result, acceptable final result
MS1	No	good matric, but weak performance at university
NG	Yes	good matric, weak midyear result, acceptable final result
NW	No	poor results from matric through to final exam
OU	No	either good matric and weak university OR weak throughout*
RG	Yes	consistently good results from matric to the final exam
RT	Yes	consistently good results from matric to the final exam
TH	Yes	good matric, weak midyear result, acceptable final result
WT	No	good matric, but weak performance at university

**Table 8.2.2 Preparedness of participating students: Group B**

Student	Prepared	Category of Preparedness
AP	No	either good matric and weak university OR weak throughout*
AS1	Yes	consistently good results from matric to the final exam
BJ	Yes	consistently good results from matric to the final exam
BM	No	either good matric and weak university OR weak throughout*
CM	No	weak to begin with, but improving towards end of year
CP	Yes	acceptable performance from matric to the final exam
DB	No	poor results from matric through to final exam
GCP	Yes	good matric, weak midyear result, acceptable final result
IT	No	poor results from matric through to final exam
JB	No	good matric, but weak performance at university
JE	No	good matric, but weak performance at university
JS	Yes	consistently good results from matric to the final exam
LM2	No	poor results from matric through to final exam
LM1	No	poor results from matric through to final exam
MM	No	weak to begin with, but improving towards end of year
MW	Yes	good matric, weak midyear result, acceptable final result
NH	No	poor results from matric through to final exam
RA	No	weak to begin with, but improving towards end of year
SW	No	good matric, but weak performance at university
TJ	No	weak to begin with, but improving towards end of year
TvH	Yes	consistently good results from matric to the final exam

\* School result is missing, yet university results are sufficiently poor that the students were categorised as Under-Prepared

### 8.3 Students involved in data analysis

A selection of students from each group (A and B) was interviewed: English, non-English, Prepared and Under-prepared. Intriguingly there were no prepared students who were non-English main language in the experimental groups, and hence among the students interviewed. Students who only handed in one or two written exercises were not included in the analysis of their written submissions. All students were included in the analysis of the quantitative data from the assessment tasks, except students who completed fewer than three writing assignments during the course of the semester.

Tables 8.3.1 and 8.3.2 show the breakdown of the students in each group (18 in A and 21 in B) taking part in the writing study project by language and preparedness. Tables 8.4.1 and 8.4.2 show the breakdown of students who were interviewed in each group (9 in A and 8 in B) by language and preparedness. Finally, Table 8.5 provides pertinent information on each student as regards language, preparedness and involvement in analysis. These tables also appear in Appendix 1.

**Table 8.3.1 Language/preparedness: Group A**

	Prepared	Under- Prepared	
English	7	8	15
Not English	0	3	3
	7	11	18

**Table 8.3.2 Language/preparedness: Group B**

	Prepared	Under- Prepared	
English	7	8	15
Not English	0	6	6
	7	14	21

**Table 8.4.1 Language/preparedness of interviewed students: Group A**

	Prepared	Under- Prepared	
English	3	3	6
Not English	0	3	3
	3	6	9

**Table 8.4.2 Language/preparedness of interviewed students: Group B**

	Prepared	Under- Prepared	
English	3	2	5
Not English	0	3	3
	3	5	8

**Table 8.5.1 Classification of participating students: Group A**

Student	Main Language	Prepared	Interviewed	Included in writing exercise analysis
AJ	English	Yes	Yes	Yes
AN	English	No	No	No
AS2	Mandarin	No	Yes	No
DC	English	Yes	Yes	Yes
DK	English	Yes	No	Yes
DL	English	No	Yes	Yes
ET	English	No	No	No
IS	English	No	Yes	Yes
JL	English	No	No	Yes
MS2	English	Yes	No	Yes
MS1	English	No	Yes	Yes
NG	English	Yes	No	Yes
NW	Portuguese	No	Yes	Yes
OU	English	No	No	No
RG	English	Yes	Yes	Yes
RT	English	Yes	No	Yes
TH	English	Yes	No	Yes
WT	Afrikaans	No	Yes	Yes

**Table 8.5.2 Classification of participating students: Group B**

Student	Main Language	Prepared	Interviewed	Included in writing exercise analysis
AP	Other	No	No	No
AS1	English	Yes	Yes	Yes
BJ	English	Yes	No	Yes
BM	English	No	No	No
CM	Shona	No	Yes	Yes
CP	English	Yes	Yes	Yes
DB	English	No	Yes*	No
GCP	English	Yes	No	Yes
IT	Tswana	No	Yes	No
JB	English	No	No	Yes
JE	English	No	No	Yes
JS	English	Yes	Yes	Yes
LM2	Zulu	No	No	Yes
LM1	English	No	Yes	Yes
MM	Tswana	No	No	Yes
MW	English	Yes	No	Yes
NH	English	No	No	Yes
RA	English	No	No	No
SW	English	No	No	Yes
TJ	Shona	No	Yes	Yes
TvH	English	Yes	No	Yes

\* The interview with DB was unplanned and unexpected. Audio taping equipment was not set up, and the interview was a reduced version of that carried out with the other interviewees, with field notes only. The field notes were typed up immediately after the interview, with the few recorded direct quotes carefully marked.

## 8.4 Interview Analysis

The interviews revealed two levels of engagement with the mathematical content of the tutorials, a surface approach and a deep approach (Chi et al, 1981, among others). A surface approach (as identified through the interviews and student responses) to the tutorial questions is characterised by being more likely to be invoked than the deep approach (that is, calculation without understanding requires less cognitive effort), not leading to understanding, often resulting in “dead ends” and providing only one view among many of a mathematical process. A deep approach is characterised by being more difficult to attain than a surface approach and is often non-spontaneous in that the writing exercises forced the students (so they reported in the interviews) into processes they would not otherwise have carried out. A surface approach is not necessarily deliberately chosen, over a deep approach, by the student (although it can be), but is often inadvertent. The writing exercises encouraged a deep approach where a surface approach might have occurred in the absence of the exercises.

In Piaget's learning theory, in the absence of a perturbation, alpha behaviour is more likely to be invoked than beta behaviour. Beta behaviour, with its reflective abstraction and accommodation, represents successful learning (Dubinsky and Lewin, 1986) while alpha behaviour represents either unstable or incomplete learning or no learning at all. The observations made by the students during the interviews support the observations made by the author during the study project of the existence of the two levels of engagement, shallow and deep, and their parallels with Piaget's alpha and beta behaviour. In addition, the requirement of a perturbation to invoke beta behaviour in the Piagetian model resonates with the aggressive language used by the students in reporting that the deeper engagement was "forced".

### **Interview excerpts**

While it is possible to provide a slew of examples illustrating all the observations and the richness observable in the interviews, constraints are necessary and one example of each observation (with supported exceptions) is given below:

A surface approach is easier than a deep approach

- [The aim of the writing exercises was] to force insight, I suppose. If you didn't have the writing, then people wouldn't care what's going on with the problem. They would just do it. (DC)

A surface approach does not lead to understanding, a deep approach does.

- It depends on what the person wants. Do you want to understand what you're doing, or do you just want to do it? If he wants to do it I'd tell him to go to a normal tutorial, but if you want to understand it more go to the writing. (DL)  
[All other students answering the question on whether they would recommend to a friend that s/he do a tutorial with writing exercises took it as given that understanding was the aim of working through mathematical problems. DL did not.]

A surface approach often leads to "dead ends" while a deep approach is less likely to.

- You see a question and write whatever comes to your mind, and, halfway through, you realise it is not right. Then you start again. So by having to think over the problem you know what you're supposed to do and you go straight to doing it. And you don't get scared of it, because you work it out and know I've got this one. (TJ)

A surface approach provides a single view of a mathematical idea, while a deep approach could provide many.

- If you're not used to [a difficult mathematical process], it helps to be forced to try and see it in another light. Because that can help you get through the block and then, getting the feedback, you can find another perspective, which, especially if you're finding one [problem] hard, if you came at it from another angle. I think it's more useful than if you already figured it out. (AJ)

A deep approach is often non-spontaneous, in that the student has to be "forced" by some requirement (such as the writing exercises) to take a deep approach. In this case, the use of the word "force" or "push" was so widely used that several excerpts from interviews have been chosen.

- If you have a challenged understanding of the work, then it pushes you to understand the processes, and see links between different ideas. (CP)
- It actually forces me to think. (IT)
- Well, I have to admit, if I didn't really understand it, I would probably have left it out. But because I knew I had to submit something, it forced me to think about, maybe read up from the textbook. I'd like to always produce an answer. (IS)
- It forced you to think about what you were doing. (DC)
- They [the writing exercises] kind of force you to [think about the problems]. (CP)
- but it really helps when you're looking at the harder stuff, to be forced to go through the process ... Because when the mathematics is easy then at least you can think through that; if you're not used to it, it helps to be forced to try and see it in another light. (AJ)
- [The aim of the writing exercises was] to force insight, I suppose. (DC)

A surface approach can be deliberately chosen ...

- Well, I have to admit, if I didn't really understand it, I would probably have left it out. (IS)

... or can be inadvertent, with the student not realising that the question has not been sufficiently thought through.

- With one or two of the writing exercises, because you had to think about it and take a guess, you could actually take an educated guess. And *that was surprising*, because you didn't think you would be able to know without working it out, but because you had to think about it, you can predict what will

happen, at least sort of. (DC, emphasis added. Context: DC was in the A group where an initial approximation of the solution was required before calculation) The writing exercises encouraged a deep approach, which in turn fosters understanding. (The two quotes given work well in tandem.)

- Well, for me, it helped in that you go a lot deeper into the actual question than you do in a normal tut where it is just do it, and get an answer. You had to sit and think about it (RG)
- If you say *explain*, then I really have to explain it, I can't make a mess of it, and if I'm going to explain something then I really have to understand it. And then I have to explain it clearly enough that it's clear enough for me and it will also be clear enough for someone else (NW)

The writing exercises can challenge faulty mathematical understanding.

- I remember on two occasions I had calculated the wrong argument. That was definitely a weakness, so it [the writing exercises] alerted me to my weakness, which was really crippling me. (AS1)

### **Contrast and perturbation**

The A group was instructed to make an initial statement of expectation of solution form and to refer back to that statement after writing the explanatory paragraph on the problem solving process. Such comparison of final solution with solution expectation explicitly invokes Kelly's notion of similarity and contrast. Observations made during the tutorials support the suggestion that such recognition of contrast creates or modifies construct systems, or draws the student's awareness to the existence of a construct, thereby encouraging learning through the subsequent perturbation. The B group was not required to explicitly explore comparisons. However the B group writing exercises did invoke tacit comparisons, as the students encountered contrasts with previously encountered problems or previously held concepts which proved to be incorrect or incomplete. Two examples are chosen to illustrate two "grain sizes" of construct modification or recognition: overarching level of engagement, and a problem specific example.

It is likely that the bipolar construct of deep engagement – shallow engagement was a construct already in existence in the students' personal construct systems. The writing exercises, however, drew the attention of the students to the deep engagement encouraged in the mathematical problems about which they were being required to

write. The students' awareness of the deeper level of engagement was not an awareness in isolation, it was paired with the awareness that the level of engagement in the tutorial questions was generally shallow. It was specifically the *contrast* of the two levels of engagement which was of interest to the students. ““We're used to, in normal tuts, we just kind of do it, get an answer, here it was, you have to sit down and think about it” (RG), ““so it was kind of difficult adjusting to thinking deeply about stuff” (IT) and “you can just sit and pretend you're working and you are not working. It will look like you are working but you weren't, so with writing exercises at least there is one example in the tut that you might think for. At least you're trying to think about something” (NW).

Examples of awareness of contrast are more common among the students in the A group, due to the writing exercise requirements, however an example from the B group is given due to the student's explicit reference to the example in the interviews. AS1 had been experiencing difficulties in complex number problems involving calculation of arguments. His understanding of how or when to calculate multiple arguments (such as  $\frac{\pi}{3} + 2\pi k$ ,  $k \in \mathbf{Z}$ ) was flawed. While he was aware of the existence of multiple arguments versus single arguments, only when he was required to write an explanation for a solution to a problem did he become aware of the flawed nature of his understanding. Once again, it was the contrast of his experience during the writing exercise with his experiences in similar problems prior to the writing exercises that alerted AS1 to the issue and caused him to modify his construct, or possibly construct system, related to argument calculation. “I remember on two occasions I had calculated the wrong argument. That was definitely a weakness, so it alerted me to my weakness, which was really crippling me” (AS1).

### **Non-participation**

In both experimental tutorial groups there were students who were not taking part in the writing exercises, one student in the B group having explicitly signed the consent form in the negative, and 8 in the A group. In addition there were two students in each group who signed the consent form in agreement to take part in the study project, yet did not complete any writing assignments during the semester.

**Table 8.6.1 Group A: Participation in complete tutorial group**

		Consented to take part?		
		Yes	No	
Effectively taking part?	Yes	18*	0	<b>18</b>
	No	2	8	<b>10</b>
		<b>20</b>	<b>8</b>	<b>28</b>

\* Student AP joined the group later in the semester and completed two writing assignments. He did not sign a consent form and has been used in no analysis.

**Table 8.6.2 Group B: Participation in complete tutorial group**

		Consented to take part?		
		Yes	No	
Effectively taking part?	Yes	21	0	<b>21</b>
	No	2	1	<b>3</b>
		<b>23</b>	<b>1</b>	<b>24</b>

While initially it was a disappointment that the entire tutorial group did not want to take part, the subgroups within the tutorial group provided a few interesting observations. On three occasions, the students taking part in the writing exercises made observations that the students not taking part did not, or delved for deeper understanding than the other students did, the most striking example of which was the question on orthogonal trajectories.

In week 7, where the students were called upon to solve for the orthogonal trajectories of a family of ellipses, all of the writing students drew a sketch, and all of the non writing students neither drew a sketch nor asked what their solution curve might look like. In addition, the calculation was tricky in that the solution required modulus signs that weren't automatically discovered or acquired through calculation, and five of the ten writing students who completed the problem inserted the modulus signs (or made an equivalent symmetry argument), while the seven non-participating students reached a (merely partially correct) algebraic solution and considered the problem complete, a paradigm case of alpha behaviour.

On two other occasions there were slightly less dramatic instances of differences between the two groups (week 8 (improper integrals) and week 10 (matrix transposes)) where the students taking part in the writing project asked searching questions about the forms of their solutions, or used new notation presented to them, while the students not taking part in the writing initiative did the minimum calculations necessary and moved on to the next question in the tutorial. Certainly from the tutor's point of view, it was very convincing that the act of writing, or more accurately the cognitive and

metacognitive demands the writing made of the students, resulted in much deeper engagement with the mathematical content of the questions and evidence for a more stable understanding of the mathematics involved.

### **Inferences from interviews**

Piaget's stage independent theory of learning describes alpha and beta (and gamma) behaviour upon the cognitive subject encountering a novel alimant (Piaget, 1978, Dubinsky and Lewin, 1986). In the event of the subject's knowledge structures being insufficient to accommodate the novel alimant<sup>1</sup>, either alpha or beta behaviour is practised. Alpha behaviour corresponds to a shallow approach to mathematical engagement and results in the production of incomplete or unstable knowledge structures which do not stand up to scrutiny. Beta behaviour corresponds to a deep approach to mathematical engagement, which potentially results in an active process of assimilation and accommodation, and ultimately the creation of stable knowledge structures. It is an argument of this thesis that the writing exercises encourage beta behaviour by simultaneously encouraging a deep approach to engagement (and hence encouraging beta behaviour from the outset) and providing a challenge, a perturbation, to the unstable knowledge structures created by or associated with alpha behaviour (and hence encouraging a switch to beta behaviour from initial alpha behaviour). While the writing exercises alone are neither a necessary nor sufficient condition for comprehensive understanding of any mathematical idea, their inclusion in a mathematical learning environment makes understanding a more likely outcome of engagement due to the associated beta behaviour.

A most surprising (to the interviewer) outcome of the interviews was the perception of several of the students that the writing exercises saved time over all, rather than taking time away from other activities. It was a concern that the time taken over the writing exercises might be resented by the students, when they could have been working through the standard set of tutorial problems instead of being involved in the writing project. During the interviews students were asked whether they felt that the writing exercises had taken a lot of time, and a few students responded that, on the contrary, they felt that the writing exercises, or more correctly, the deeper engagement demanded by the writing exercises, resulted in a saving of time, as incorrect solutions or "dead end" attempts were averted. "Before you got stuck into it, you'd save time, rather than

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<sup>1</sup> A novel item. "that which supports or sustains the mind" (OED). See Chapter 6. 6.2.1

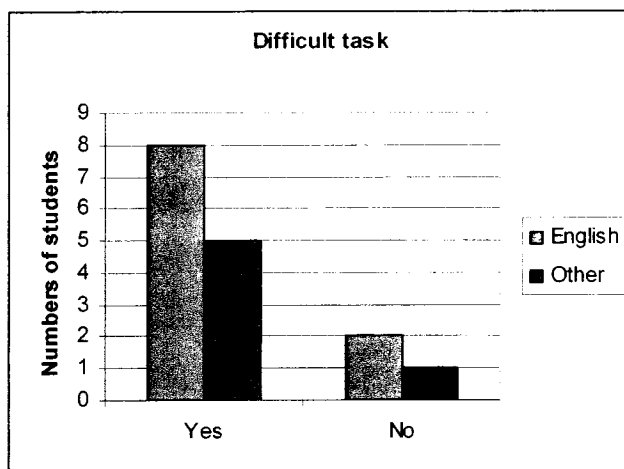
trying to solve it, and finding dead ends, starting again” (MS). “Well, without the exercises I would still have learnt about the topic, but it made me think more and realise that if you think more about doing it, it saves you time in the long run” (IT). “It also saves time, believe me. Sometimes, in a test, you panic. You see a question and write whatever comes to your mind, and, halfway through, you realise it is not right. Then you start again. So by having to think over the problem you know what you’re supposed to do and you go straight to doing it” (TJ).

## **8.5 Language and preparedness issues within interviews**

The student responses to each interview question were analysed by grouping similar responses together, creating a representative response for each group of similar responses, categorising the students by language and preparedness, and representing the response spread in column graphs. All such question analyses can be seen in Appendix 5. In many cases the responses were not particularly informative in that everyone answered similarly (Question 14: did you learn more because of the writing exercises?), or the range of responses was very wide (Question 10: strengths in mathematics), or suggestions of trends were not sufficiently pronounced to draw conclusions (Question 11: preferment of algorithms or problem-solving).

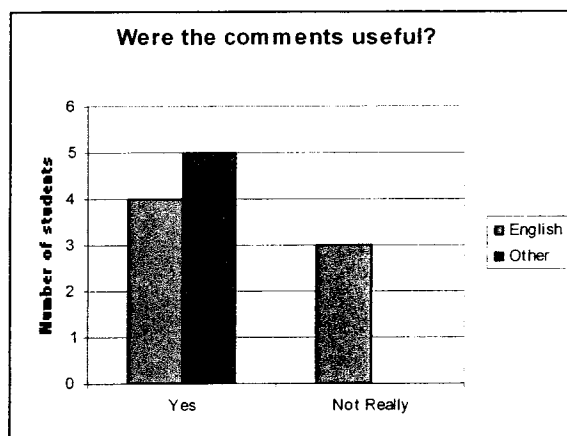
A question being asked in this study project is “*Are any observed effects of writing in problem-solving different for students with differing main languages?*” Analysis of any single interview question without reference to any other question does not produce a clear answer to that question. For instance, 8 English and 5 non-English students found the writing exercises difficult, while 2 English students and 1 non-English student did not find them difficult. Nothing much can be determined from such a spread of responses, and such is the case with most of the questions.

**Figure 8.7 (Question 4) Were the writing exercises difficult?**



The question “Did you find the comments useful?” received an interesting response with all of the non-English students replying in the affirmative, with only approximately half of the English students doing so.

**Figure 8.8 (Question 7) Were the comments useful?**



The importance of feedback and formative commentary cannot be overestimated, and it is a distinct possibility that the most important part of the exercise is the commentary (Northedge, 2003; Demaree, 2007). To run the writing initiative in a form where the commentary has to fall away (perhaps for reasons of time if the number of students is very large) would be detrimental to the exercise even in the absence of different language groups. The responses to this interview question suggest that dropping commentary and associated feedback would be particularly detrimental to non-English language students, further complicating the issue of extending the initiative to large classes.

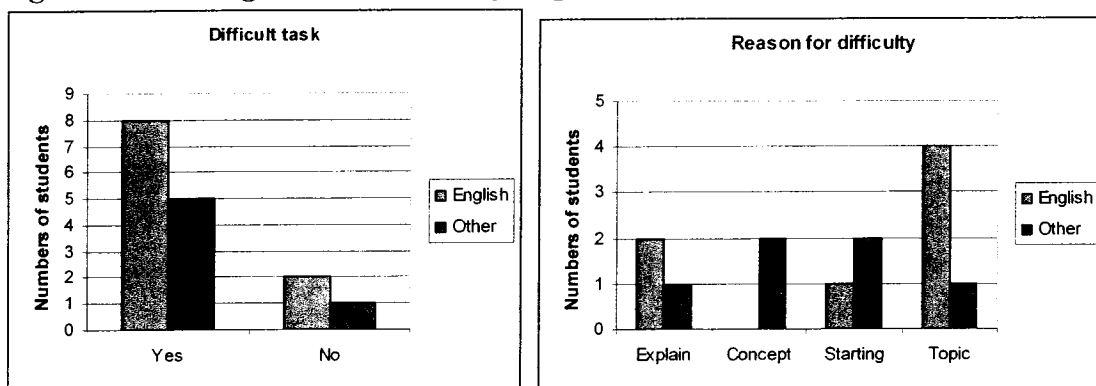
In a few cases, interesting conclusions could be drawn by examining responses, not just to single questions, but to collections of questions on related topics. Cross referencing

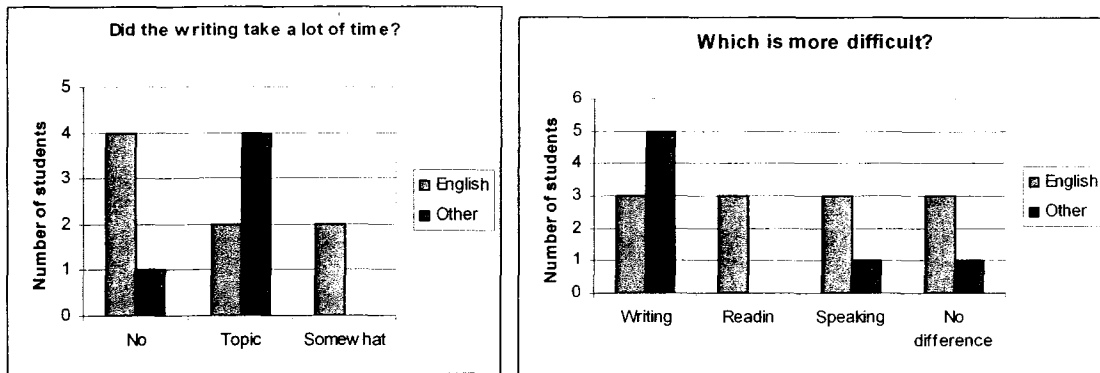
the responses to several different interview questions yielded information not discernible in the answers to any one of the questions alone. The breakdown by language group to the questions

- Did you find the writing exercises easy or difficult to do? (Question 4, Answer categories: *Yes* or *No*)
- In what way were the writing exercises difficult to do? (Question 4, Answer categories: *Explaining mathematical concepts in words; The concept of writing in mathematics is too strange; Getting started is difficult; The topic of the question determined the exercise's difficulty*)
- Did the writing exercises take a lot of time? (Question 8, Answer categories: *No; Depended on the topic; Somewhat/Quite a bit*)
- Which do you find more difficult: writing, speaking or reading? (Question 16, Answer categories: *Writing; Speaking; Reading; No difference*)

suggested that the writing exercises were more difficult for the non-English students than for the English students, in that the mathematical topic influenced the difficulty of the exercise as well as the time take to complete it, but that the non-English students had greater difficulty with the actual practice of writing than the English students experienced (see Emig, 1977, for a comparison of writing to speaking, not within the scope of this thesis).

**Figure 8.9 Writing exercise difficulty: topic and concept**





Most students, English and non-English felt that the writing exercises were fairly difficult, in agreement with a similar finding of Kågesten and Engelbrecht (2006). The English students tended to feel that the difficulty of the topic, primarily, influenced the difficulty of the writing exercises, whereas the non-English students were more general in their reasons, with only one choosing the topic as the primary source of difficulty. Interestingly, even though most of the English students felt that the writing exercises were difficult, they tended not to feel that they took a considerable amount of time. The non-English students, on the other hand, tended to feel that the writing exercises sometimes took a lot of time, depending on the topic of the question.

The comparison, including the writing/reading/speaking responses, suggests that the English students found the actual task of writing sufficiently unproblematic that any difficulty in the task was imposed by the mathematical content of what they were writing about and not the act of writing itself. The non-English students, finding writing in English more challenging than the English students, considered the mathematical content of the writing exercises *relatively* less problematic than the English students, and were more inclined to find the combination of mathematics and writing a strange and difficult concept (similarly reported in Kågesten and Engelbrecht, 2006). The non-English students did feel the influence of the mathematical content in the task, however, not as the major contributor to the difficulty, but as the major contributor to the time taken to complete the exercise. A possible conclusion is that increased difficulty of the mathematical focus of the writing exercise increases the difficulty and possibly the time taken to carry out the exercise, for all students, but the non-English students have the added difficulty of expressing themselves in writing, thereby diminishing the relative effect of the difficulty of the mathematical content. A suggested answer to the question “*Are any observed effects of writing in problem-solving different for students with differing main languages?*” is: Yes, students not speaking English as a main language

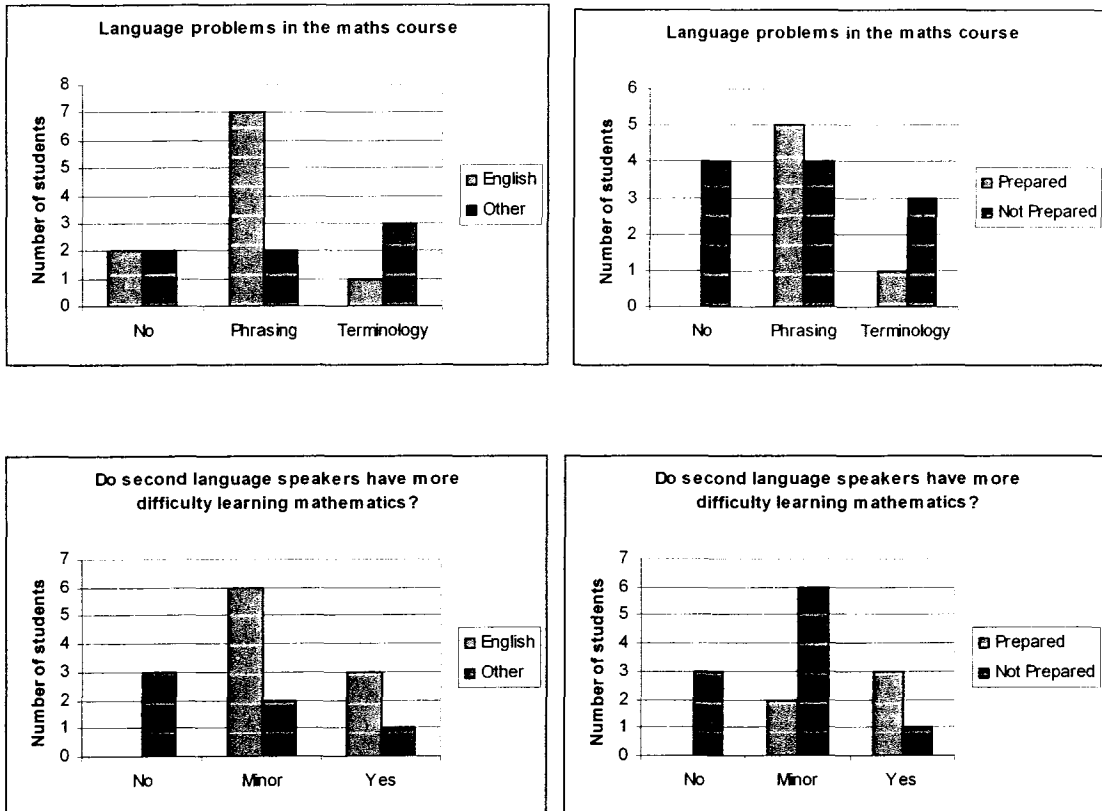
will have more difficulty completing the writing exercises than English students, since the mathematical topic of the question contributes to the difficulty for all the students, but the non-English students have the added difficulties of expressing themselves in English and struggling with the concept of writing in the context of mathematics. However, and this is important, this difference in difficulty does not necessarily mean that either language group gains more from the exercise than the other. Indeed, analysis of the submitted writing exercises themselves show a similar level of grappling with the mathematical content of the questions.

The perception of writing and mathematics as strange bedfellows was made very apparent during week 6 in the B group by LM2. LM2 had not been taking part in the writing exercises, even though she had agreed to do so. Querying her lack of involvement, it gradually became clear that her reluctance was due to her lack of confidence in being able to express herself in mathematical terminology. This reluctance was enhanced by the examples the students had been given at the beginning of the semester of the form of response that was expected by the project. The examples had been phrased in a manner judged informal by the tutor, well versed in the mathematics register, but judged very formal by (at least) LM2. This disparity of formal and informal mathematics registers was unfortunate and it took some convincing on the part of the tutor before LM2 was persuaded that ordinary everyday English was more than acceptable and no competence with a formal discourse specific register was being demanded.

Additional insight into difficulties experienced by non English students is evidenced by cross referencing the answers, broken down by language and preparedness, to the following questions:

- Have you ever experienced any language problems in the course?  
(Question 17, Answer categories: *No; Question phrasing can be ambiguous; Mathematical terminology can be difficult*)
- Do non English main language speakers have more difficulty learning mathematics taught in English? (Question 18, Answer categories: *No; There are minor difficulties; Yes*)

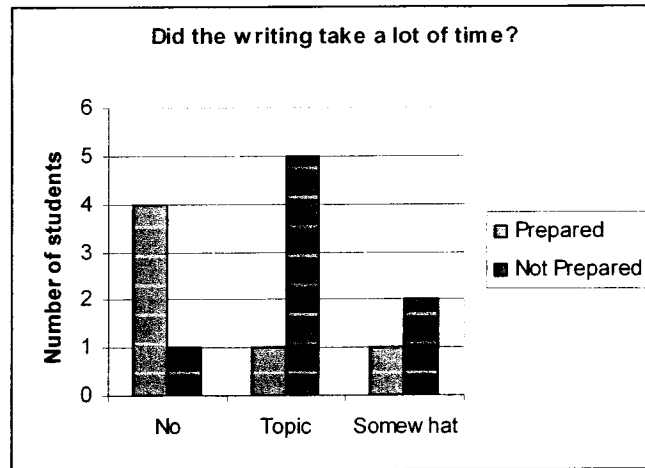
**Figure 8.10 Language difficulty and second language**



English students consider not being English to be a problem when learning mathematics in English. The non English students, who should be the ones experiencing those perceived problems, do not consider it much of a problem, if at all. Surprisingly, there is a perception of a difficulty that the students in question are either not experiencing, or not recognising (which may perhaps be likely). Note that it is particularly English students (heavily influenced by the cohort of prepared students who are all English) who report having experienced language problems, primarily in the form of confusing or ambiguous question wording. Now, if a question has been phrased ambiguously for a prepared, English student, then it seems strange that the under-prepared, non English students are not experiencing the same difficulty. It could be suggested that there are language difficulties for non English speakers, but some are sufficiently subtle (e.g. ambiguities in question wording) that they are not being recognised among other challenges, such as the mathematics register (which non English students consider the most challenging language problem) and the mathematical content of the problems (non English students all being under-prepared students). While such an observation is not a direct answer to any of the research questions posed in this thesis, it does indicate that language issues are potentially having impact on student learning and any writing initiatives should take care not to compound any associated problems.

The research question “*Are any observed effects of writing in problem-solving different for students with differing degrees of mathematical preparedness?*” has a less clear answer than the similar question on language differences. The interview question with perhaps the most striking response was “Did you find that the writing exercises took a lot of time?” with the prepared students replying in the negative, in the majority, and only one under-prepared student replying in the negative.

**Figure 8.11 (Question 8) Did the writing exercises take a lot of time?**



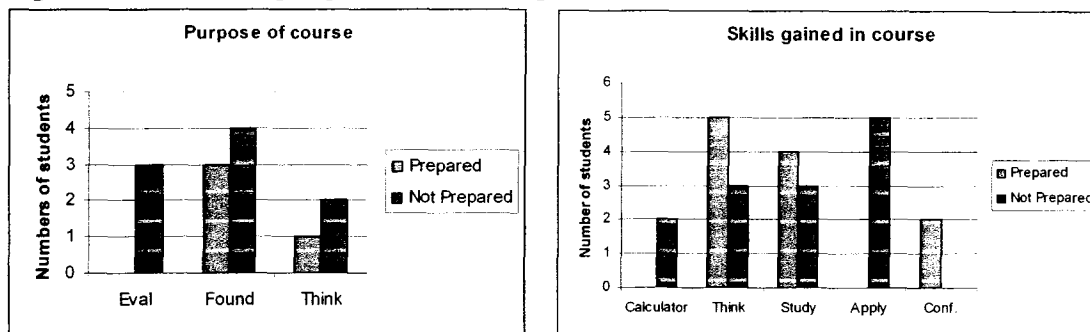
The difference in responses is unsurprising, particularly in light of the fact that all prepared students were also main language English students. Again, it must be stressed that difference in time taken on the writing exercises does not indicate differential gain from the involvement.

Cross referencing the responses to

- What would you say the purpose of the mathematics department was in offering this course? (Question 2b, Answer categories: *Evaluation of students' mathematical ability; Foundation for further mathematics and other subjects; To teach a way of thinking*)
- What skills would you say you have learned in this course? (Question 21, Answer categories: *Not using a calculator; Thinking mathematically; Study and work skills; Application of mathematics; Confidence*)

reveals an interesting distinction between prepared and under-prepared students.

**Figure 8.12 Course purpose and skills gained**



A common choice of course purpose for both prepared and under-prepared students was that the course served a foundational, grounding purpose, preparing the student for the use of mathematics in other subjects as well as further studies in mathematics. When asked for their perception of skills gained in the course, the under-prepared students were most likely to see themselves as having learned to apply mathematical techniques (correlating with the foundational view of the course), whereas the prepared students were more likely to see themselves as having learned to think mathematically (apparently correlating with the purpose of the course “to teach a way of thinking”). The numbers are sufficiently small, and the trend is insufficiently pronounced to draw any strong conclusion from the comparison, but it does suggest that prepared and under-prepared students might be benefiting quite differently from the course. There is also the possibility that the two different responses of *learning to apply mathematical techniques* and *learning to think mathematically* refer to something similar (to a novice), while a mathematician might regard them as being quite different. Taken altogether, there is no strong indication from the interview data that the Prepared and Under-prepared students experienced the writing exercises appreciably differently.

Recalling the two levels of engagement discernible in the interview responses, characterised as surface and deep, the interviews strongly suggest that the writing exercises encouraged Piaget’s beta behaviour and corresponding deep approach to mathematical engagement, both by encouraging a deep approach to the mathematical question from the outset and by challenging knowledge structures created through a surface approach, and alpha behaviour. In addition, there are convincing suggestions that non English main language speakers experience more difficulty in the mathematics course than English students, and that the writing exercises were experienced as a more difficult activity for non English speakers than for their English fellow students. While there are suggestions that prepared and under-prepared students develop different skills

during the course, the evidence is not strong, nor is there any evidence that prepared and under-prepared students experienced the writing exercises differently.

After the interviews, the written exercises themselves were the greater source of informative data, with the quantitative analysis of the assessment tasks appearing to be the least informative source of data. Three analysis procedures were applied to the written exercise analysis, of which Waywood's (1992) coding scheme of Recount, Summary and Dialogue proved the most efficacious, with robust categories comparable over a time period and satisfactorily measuring the desired problem-solving attributes.

## **8.6 The writing exercise data**

During the course of the writing project a total of 168 written exercises were collected, commented on and returned. These exercises were completed by 39 students, 18 in Group A and 21 in Group B. Students who had only handed in one or two exercises were eliminated from the subsequent analysis as they had not taken part in the project with sufficient commitment to be expected to show any change as a result of the writing. Since change to problem-solving behaviour over the time period of the study project was of primary interest, one or two submissions were decided to be too few for analysis. As a result, a total of 155 written exercises from 30 students, 14 from Group A and 16 from Group B, were analysed. While the students were expected to take part in the writing project in their tutorial every week, it was apparent in the first six weeks of the project that they were only taking the task seriously on the days when they were expected to submit their writing for commentary. The original plan had been to only ask participants to submit exercises every three weeks, so as to not create any negative feelings of pressure and coercion. The plan changed, after the difference in attitude was discerned, and from week seven the students submitted their writing exercises every week.

The first two weeks were orientation weeks in which the students were settling down to their tutorials. The writing project was explained to the participants, and consent forms were signed. The first assignment submitted (and therefore available for analysis) was completed in week 5, and thereafter participating students' written exercises are available for analysis continuously from week 7.

**Table 8.13** Numbers of writing exercises submitted for commentary

Week	Group A	Group B	Total	Question topic
5	10	12	22	Complex numbers
7	10	13	23	Orthogonal trajectories – ODEs
8	11	15	26	Improper integrals
9	9	13	22	Linear algebra
10	12	13	25	Linear algebra
11	11	14	25	Combinations
12	9	3	12	Maclaurin series
<b>Total</b>	<b>72</b>	<b>83</b>	<b>155</b>	

Week 13 was the final week of term, and classes were not held every day that week. The writing project stopped in Week 12, but, as is clear from Table 8.13 contributions had already decreased in number.

The single focussed aim of the writing exercises was to increase the students' problem-solving skills. The writing exercises aimed to encourage the students to be more reflective, to engage in sense making, and to engage more deeply with the mathematics in ways that would foster good problem-solving behaviour. Three different analysis methods were applied to the data, of which only one, that of Waywood (1992) was found to be appropriate, although all three were chosen for pertinent reasons. The metacognitive framework of Garofalo and Lester (1985) and Pugalee (2001) was chosen for its applicability to problem-solving, while the Reader Expectation Theory of Gopen and Smith (1989) was chosen for its focus on explanatory writing and the assessment of change over a time period. Waywood's journal entry classification of *recount*, *summary* and *dialogue* proved to be the most suitable instrument for assessing the writing exercises.

### 8.6.1 The metacognitive framework of Garofalo and Lester

Of the vast body of work on writing in mathematics there is little specifically on writing as it relates to problem-solving. A notable exception is Pugalee (2001) who carried out a careful study of whether or not students' writing in mathematical problem-solving demonstrated the existence of a metacognitive framework. The method of analysis used by Pugalee was drawn from Garofalo and Lester (1985) who designed a framework of four categories of activity involved in performing mathematical tasks. The four categories, orientation, organisation, execution and verification, are "related to, but are more broadly defined" (Garofalo and Lester, 1985, p. 171) than Pólya's (1945) four problem-solving stages of understand the problem, devise a plan, carry out the plan and

look back. Pugalee (2001) used the categories and their associated metacognitive behaviours to good effect and it was expected that the analysis technique would yield interesting results in the writing project under scrutiny in this thesis. Two serious difficulties became apparent, however, upon undertaking categorising the writing exercises, namely subjectivity of the categorisations and weak capacity for subsequent comparison.

The categories of writing activity and associated metacognitive behaviours are:

#### Orientation

- comprehension strategies
- analysis of information and conditions
- assessment of familiarity with the task
- initial and subsequent representation
- assessment of problem difficulty

#### Organisation

- identification of goals and subgoals
- making a global plan
- implementing a global plan
- drawing diagrams and organising data into other formats

#### Execution

- performance of local actions
- monitoring progress of local and global goals
- redirecting efforts/trade-off decisions
- interpretation of results [not present in either original source]

#### Verification

- evaluating orientation and organisation
  - ❖ adequacy of representations and decisions
  - ❖ consistency of global plans with goals
- evaluating execution
  - ❖ adequacy of actions
  - ❖ consistency of actions and local results with plans
  - ❖ consistency of local and final results with problem conditions

(Adapted from Garafolo and Lester, 1985, p. 171 and Pugalee, 2001, p. 245)

The Execution metacognitive action of “interpretation of results” is not present in either original source, and was inserted into the framework after time spent analysing the data showed a need for such a subcategory.

The first written exercise subjected to scrutiny was one on multiplication of complex numbers. The students had been provided with a picture of the complex plane with two points  $z$  and  $w$  on it, and were asked to calculate, by whatever means, where  $iw$ ,  $w - z$  and  $zw$  would lie on the diagram. One student had redrawn the diagram and had placed  $z$ ,  $w$ , and her answers on it. Did the drawing of the diagram constitute Orientation – initial and subsequent representation? Was it Organisation – implementing a global plan? Was it Organisation – drawing diagrams? Was it Execution – interpretation of result? Was it Orientation – analysis of information? The drawing was finally interpreted as representing two actions, that of Organisation – drawing diagrams, and Execution – interpretation of results, but such choices could be disputed. As more data items were categorised with varying levels of certainty and indecision, it became apparent that the choices had a component that might be substantially subjective, which weakened the conceptual robustness of any potential results. This subjectivity could be diminished, however, by a team of people working independently at first and thereafter collaboratively through the data and coming to negotiated decisions on the data.

As categorisation of a second student’s work began, however, the stark differences between the two students’ styles cast doubt on the possibilities of comparisons between different students’ work or even differences over time within a single student’s work. While Pugalee had been interested in discerning the existence of metacognitive behaviour, this study was more directed to investigating change over the duration of the project and therefore needed some observable quality or measurable quantity to be apparent so that features could be discerned to change (or not) over time. The first student wrote a great deal, and included her calculations as an integral part of her writing. The second student wrote tersely, with no calculations, but with more apparent insight and more thoughtfulness. If the *quantity* of a particular kind of utterance (say Organisation – drawing diagrams) was considered then those students who wrote more sparingly, but not less thoughtfully, would be inadequately described. If the *proportion* of a particular kind of utterance were to be considered then two writing exercises that were essentially the same as regards non-symbolic content would give different results if one were full of calculations (Execution – performance of local action) and the other

not. Again, this question of unclear results could have been resolved with further deliberation and some thoughtful decisions, but Garofalo and Lester's (1985) categorisation appeared to be less useful in the project context than had originally been supposed.

## 8.6.2 Reader Expectation Theory

A brief investigation was made on the appropriateness of using Gopen and Smith's (1989) Reader Expectation Theory. Gopen and Smith use Reader Expectation Theory to analyse students' mathematical writing over time, and it was such potential change over time that was of relevance to this writing exercise project. However the theory focuses on the structure and style of writing, which was of no concern here.

Reader expectation theory was born of the linguistic recognition that readers expect certain components of the substance of prose ... to appear in certain well-defined places in the structure of prose. Once consciously aware of these structural locations, a writer can know how to make rhetorical choices that maximise the probabilities that a reader will find in the prose precisely what the writer intended the reader to find.

(Gopen and Smith, 1989, p. 214)

Gopen and Smith stress that it is not just cosmetic consequences which are affected by the rules they suggest, but communicative competence. Encouraging students to better communicate their mathematical thoughts, while a worthy endeavour, was not at issue in this project, but rather the evidence of the mathematical thoughts and their nature was at issue. In a brief perusal of the writing exercises it was apparent that communicative competence was not strongly related to depth of mathematical engagement or exploratory, creative mathematical thinking in this group of students, and hence that Reader Expectation Theory would not measure the problem-solving attributes which were the focus of the study.

## 8.6.3 Waywood's journal entry classification system

Waywood (1992) describes a classification system for journal entries which allows for entries to be assessed for the presence of some interesting features. Journal entries are regarded as having the forms "recount", "summary" or "dialogue", each of which reflects a particular stance towards learning. Waywood's classification system worked satisfactorily for analysing the writing exercises, although it had to be adapted due to the difference in contexts. Waywood's original (1992) classification was concerned with the actions of summarising, collecting examples, asking questions and discussing. Original classification (drawn from Waywood, 1992):

**Recount**      Summary means record.

Examples show how to get answers.

Questions relate to how to do things.

Discussion means talking about what happened.

Recount mode is characterised by concrete things to be done, an emphasis on objective description and reporting of passive observation of objective knowledge.

**Summary**      Summaries are about stating and organising.

Examples show how a mathematical procedure is applied.

Questions are about misunderstandings, leading to discussion.

Discussion is about forming an overview.

Summary mode is characterised by utilitarian involvement, the recognising and ordering of important ideas and the integration of utilitarian knowledge.

**Dialogue**      Summaries are about integrating.

Examples are paradigms.

Questions are about analysing and directing.

Discussion is about formulating arguments.

Dialogue mode is characterised by the recognition of the requirement to generate mathematics, learning being shaped by inquiries and a creative stance towards knowledge generation.

Waywood (1992) describes journals being kept by mathematics students at Vaucluse College, Melbourne (since closed). The students kept mathematical journals throughout their schooling at the college, and were instructed to use the journals to summarise, collect examples, ask questions and discuss. Since the activities of summarising, questioning, and so on were expected and demanded from the students, those four activities could be assessed separately for interpretation and level of engagement.

The students taking part in the writing study project were instructed to explain and justify their actions in solving mathematical problems, a slightly different requirement to Waywood's. A rephrasing of Waywood's classification scheme allows for the description of the following three categories of writing exercise:

**Recount** – The student is reporting what has happened, there is evidence of passive observation and simple recording of events. There is a focus on obtaining answers, explaining how things are done and talking about what has happened. Knowledge is seen as objective, the observer has to simply receive it.

**Summary** – Content is codified and organised. External facts are integrated into an internal system of knowledge. There is an attempt at providing an overview. A good attempt is made to generalise observations. Technical terms are not simply used, but are explained. Knowledge is seen as functional, and has to be integrated with what is already known.

**Dialogue** – There is interaction between ideas. Content is integrated. Calculations are analysed and directed. Arguments are formulated, and there are attempts to explain phenomena or contradictions. There is evidence of creative use of knowledge. Student learning is shaped by their enquiries. Knowledge is seen as something to be created or recreated.

In contrast to Garofalo and Lester's (1985) framework, Waywood's classification system was easily applied to the data, with very little ambiguity about the category to which each writing exercise belonged. The categorisation of each of the writing exercises was independently verified by a colleague. Only one contrast of categorisation arose, with the interpretation of the Recount mode. A student, by recounting facts and steps taken in a mathematical procedure, can reveal that he has a clear and thorough understanding of how to carry out the procedure, suggesting that his writing should be not be categorised as Recount, which might suggest poor understanding of the technique. However, strict interpretation of the Recount mode does not differentiate between students who thoroughly recount facts and steps and students who carelessly recount facts and steps, if neither student takes their account any further. The verifying colleague felt that occasionally an imposed category of Recount was surprisingly strict, given the student's detailed report, however even thorough and detailed recounting of steps and facts cannot be regarded as Summary if taken no further. Agreement was reached with the colleague in question on interpretation of the Recount mode, and thereafter there were no further contrasts of categorisation.

In the assessment of depth of engagement with mathematical content, there was enough of a gap between the Recount and Summary categories that a fourth category was considered. That additional category was to have been called Attempt and would have had the description

**Attempt** - There are weakly formulated generalisations, and poorly grounded attempts at logical deduction or justification. There are attempts to find associations with other (possibly incorrect) information.

Again, positioning writing exercises in this category was usually not ambiguous, however, it was decided not to use the Attempt category as the data were not very dense and the smaller the number of categories, the clearer the patterns in the data. If future projects result in a large quantity of written data, the fourth category might be considered.

All examples from student writing exercises (although typed) use the students' original words, spelling and punctuation. Editorial comments are in square brackets. Details of the mathematical questions being responded to by the students are included in Appendix 2.

### Examples of Recount

“Well, I just kind of did what was in my notes. If the limit does not exist then the integral diverges. So if a limit can be solved and a value found then the integral does exist.” (RT, week 8)

“We have to divide through by  $x^2$  first to get  $y'$  on its own. We will then find the integrating factor and multiply through by it and then solve the linear differential equation” (LM2, week 7)

### Examples of Attempt (not a categorisation used in the final analysis)

“Before: I have no idea, but it must be  $> 7$  (factorial must be positive)”  
[calculation ensues, involving factorials, and algebraically solving for  $n = 12$ ]  
“After: I converted the two sides using the combinatoric formula, then got rid of the denominators by times by them on each side. The  $2n!$  cancelled out, because they are in essence the same number. I then divided by  $5!$  Which left  $7 \times 6$  on the left hand side. I then completed the sum by factorising the polynomials and saw that only 12 was a positive answer.  
This is an expected answer as it is greater than 7. It is seen that the sum of the 2  $r$ 's (7 & 5) given you 12, which is the same as the calculated answer.” (DK, week 11)

The categorisation as Attempt was due to the primary focus on recounting steps, with the generalisations being weak and not pursued. The entry was re-categorised as Summary, due to the student's attempts at generalisation, making it more than a simple recounting of steps.

“Divide by  $x^2$ , take the  $3x^5$  to the RHS and solve as a linear d.e.

Keep in mind that I might be eliminating the solution of  $x = 0$ .” (AS1, week 7)

The categorisation as Attempt was due to the thoughtfulness shown in the consideration of a lost solution. The entry was reclassified as Recount as, beyond that slight thoughtfulness, there were no explanations, no generalisations, and even the recounting of steps was terse.

### Examples of Summary

“The slope at any point that satisfies the ellipse-family equation is, by definition, the derivative of the equation.

Orthogonal trajectories are given by  $-1/\frac{dy}{dx}$  which yields the function  $|x|^{\frac{5}{2}}k^5$  which

looks quite similar to my guess.

Basically when the derivative of the ellipse family equals the inverse reciprocal of some functions derivative, that function gives the orthogonal trajectories map.” (DC, week 8)

“In order to have a 1<sup>st</sup> degree linear equ<sup>n</sup> we need  $y'$  to have coefficient 1  $\therefore$  must divide through by  $x^2$ .

We can take the  $3x^2$  to the other side of the equ<sup>n</sup> to make the process of finding a “reverse product rule” easier to see.

By finding the integrating factor ( $e^{\int P(x)dx}$ ) we can multiply through to simplify the LHS into a single  $\frac{d}{dx}(\quad)$  term.

It is then easy to integrate both sides with the RHS only in terms of  $x$ .” (CP, week 8)

## Examples of Dialogue

“1. Each pair of matrices contains the same numbers, just in different places. i and ii are transposed matrices of each other.”

[After calculations:]

“The inverses of i and ii are also transposed matrices of each other

$$\therefore (A^T)^{-1} = (A^{-1})^T$$

iii and iv's inverses have the same numbers, but in a different order.

4. They are transposed of each other. This will yield the same determinants, because products of diagonals are preserved.”

[After calculations:]

“My speculations were correct

$$\det(A) = \det(A^T) \text{ (DC, week 10)}$$

“I was not sure how to tackle this question, but I realised that the rows in the matrix are in fact plains in  $\mathbf{R}^3$ . When plains intersect, the intersection points form an infinite line along this intersection, thus proving an infinite number of points (providing the planes are not limited in any way). Thus, if I could find a value for  $k$  which would make the two planes equal (i.e. find their points of intersection), I might find a solution (providing my understanding of vectors is not flawed)

thus I put  $2 + k = 5 - k$  as subsequently solved for  $k$ . (desperate attempt)” (IS, week 9)

All spelling, grammar and punctuation in the examples above are as they were in the students' original writing. The second example of Dialogue was deliberately included as an example where the student did not correctly answer the question, or even attain an answer at all. The solution process does not have to be correct to merit the Dialogue description. The category of Dialogue was applied to that final example for the creativity shown, the way the student argues for his solution attempt, and the way the student is learning as he proceeds.

## 8.7 Results of writing exercise analysis

The results, reporting frequency and percentages of writing exercises each week in each of the three categories, show a noticeable trend away from Recount and towards Summary and Dialogue.

**Table 8.14.1 Number of written exercises submitted each week**

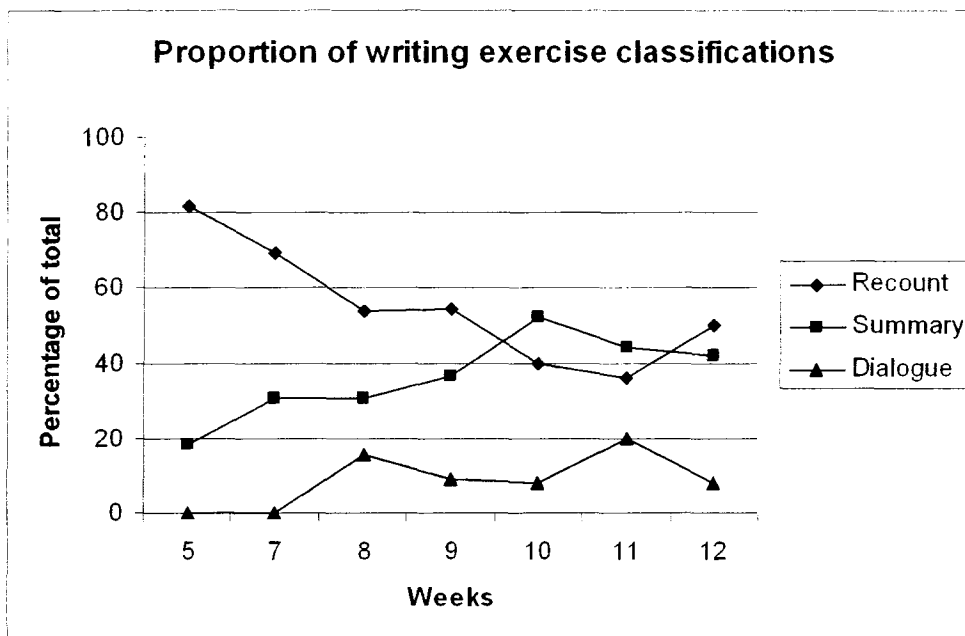
	Weeks						
	5	7	8	9	10	11	12
Recount	18	16	14	12	10	9	6
Summary	4	7	8	8	13	11	5
Dialogue	0	0	4	2	2	5	1
Total	22	23	26	22	25	25	12

**Table 8.14.2 Percentage of total of written exercises submitted each week**

	Weeks						
	5	7	8	9	10	11	12
Recount	82	70	54	55	40	36	50
Summary	18	30	31	36	52	44	42
Dialogue	0	0	15	9	8	20	8
Total	100	100	100	100	100	100	100

To read Tables 8.14.1 and 8.14.2: in week 5 there was a total of 22 writing exercises handed in, of which 18 (82%) were Recount, 4 (18%) were Summary and none (0%) was Dialogue. While the writing exercises continued to be carried out in week 12, only 12 were submitted that week, a drop of approximately fifty percent from the rest of the semester, and the Week 12 data does not show quite the same pattern as shown in previous weeks.

Figure 8.15 Proportion of Recount, Summary and Dialogue submissions



Over the course of the semester there was movement of relative frequencies away from simple recounting of facts and towards explanations. There was an inclination to think more about the processes involved in arriving at answers, rather than simply getting the answers. There was a change in the stance towards learning, away from the student as passive observer of objective knowledge and towards the student as active engager in the use or creation of knowledge. It is unfortunate that, by definition, no comparison can be made with the Control group, since the control group submitted no writing exercises. The lack of available comparison creates doubt as to whether the observed effect was due to the writing exercises or due to the course in general. It could be argued that, had the course had such an effect on the students, 80% of the submitted exercises being categorised as Recount is a rather high proportion, as the associated data point represents 5 weeks into the second semester of a year long course. A counterargument could be that, even had the course developed the sought-after mathematical thoughtfulness, the *writing* aspect was new and might have taken the students time to become accustomed to. Taken in combination, the author's experience of students' superficial participation in the course, the fact that the lecture mode of teaching encourages a passive stance towards knowledge (Borasi and Rose, 1989) and the evidence that writing encourages a deeper approach to learning, strongly suggest that the effect illustrated in Figure 8.15 was brought about by the writing exercises, and not by the course in general.

## 8.8 Quantitative data

Quantitative data in the context of this project refers to the frequency data accumulated from observations of problem-solving behaviour in students' assessment tasks, namely tests and examinations during the year. The data were collected for the full year, not only the experimental second semester, in order to be able to observe students' problem-solving progress (if any) both before and during the experiment. In the first semester the students wrote four tests, the fourth being a longer than normal midyear assessment, and in the second semester they wrote two tests and two final examinations. Each assessment task, except for the second final examination, which was entirely in multiple choice format, included a question which was considered a good indicator of problem-solving ability, as opposed to algorithmic manipulation. Unfortunately the second test in the experimental semester (Test 6) ran into some logistical problems, coinciding with a religious holiday, and had to be severely truncated, dropping the problem-solving question and being presented entirely in multiple choice format. Despite some attempts to salvage useable data from Test 6, it was discarded as not being useful for current analysis. The end result is that four tests were available for analysis for the period before the project, and only two (Test 5 and Final Examination 1) in the period during the project. This paucity of data, combined with the lack of striking, significant trends in the data, consequently means that inferred results can be interpreted as suggestions only, with a view to further research and investigation. No firm conclusions can be drawn from the quantitative data. Appendix 4 includes characteristic examples of the inconclusive analysis carried out on the quantitative data.

The quantitative data were not as informative as the interviews of the writing exercises themselves were. Three items of interest became apparent upon analysis of the data. The reasons for the lack of clarity emerging from the quantitative data are several. One reason is the regrettable need to discard Test 6 data from the sample, leaving the data with only two points within the experimental time period. Another is the difficulty in discerning from students' work what their thought processes were and whether they showed good problem-solving technique. For this reason several problem-solving studies, including Schoenfeld (1985), require the students to solve problems in pairs, speaking their thought processes out loud to one another. A further difficulty in collecting the data was that the choices were relatively subjective. One jotting in attempting to solve the problem might be regarded as showing good flexibility, or good metacognitive control, or, indeed, neither. As such, the analysis of the test and

examination data cannot be regarded in as convincing a light as that of the interviews and the writing exercises. The fallible nature of the data notwithstanding, there are three items of interest that can be suggested for scrutiny: evidence of increasing mathematical understanding, evidence of increased metacognitive control and possible evidence of increased passing of tests and exams (for a variety of possible reasons).

Four points of data were recorded for each problem analysed, namely the mark for the question, whether the student seemed to understand what the question was asking, whether there was evidence of metacognitive control during the problem-solving process, and whether the student showed flexibility in problem-solving. Both understanding and metacognitive control evidenced potential improvement, in comparison to the Control group. Table 8.16 displays the percentage in each group of students who clearly understood what the question was asking. The percentage is calculated relative to the number of students in the group answering the question, that is, removing from the group those who omitted the question.

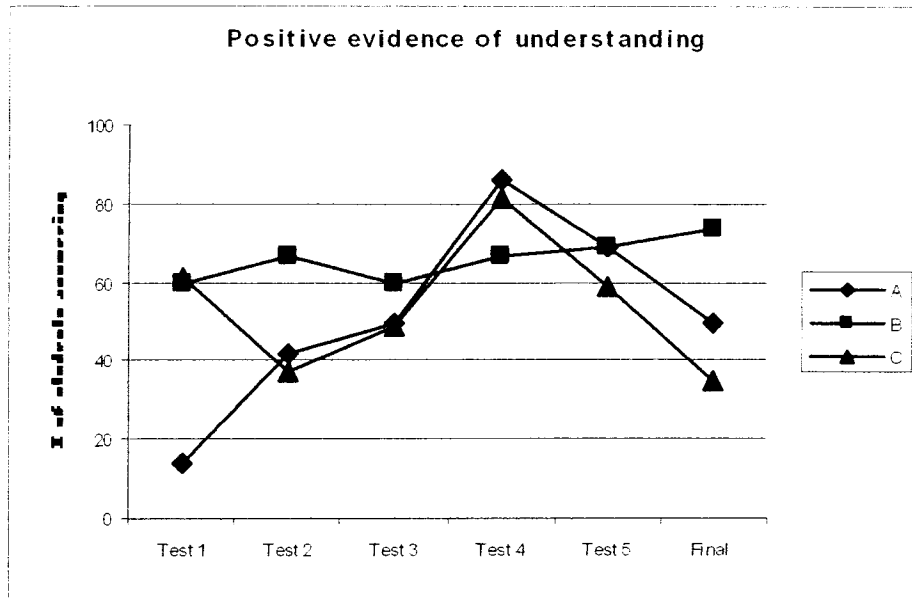
**Table 8.16 Percentage of group showing evidence of understanding**

	Group A	Group B	Group C
Test 1	14	60	62
Test 2	42	67	37
Test 3 *	50	60	49
Test 4 (midyear)	86	67	81
Test 5	69	69	59
Final exam **	50	73	35

\* Test 3 had two questions which were considered good indicators of problem-solving ability. In Table 8.16, the data for the two questions were amalgamated to have only one data point, otherwise the graph below might give the false impression of having more points along the time axis than is accurate.

\*\* While two questions in the final examination were originally chosen for analysis, the second one was discarded as it transpired to be a poor indicator of problem-solving behaviour. Too many questions of similar type had been encountered by the students and it was obviously approached mechanically by the students.

**Figure 8.17 Percentage of group showing evidence of understanding**



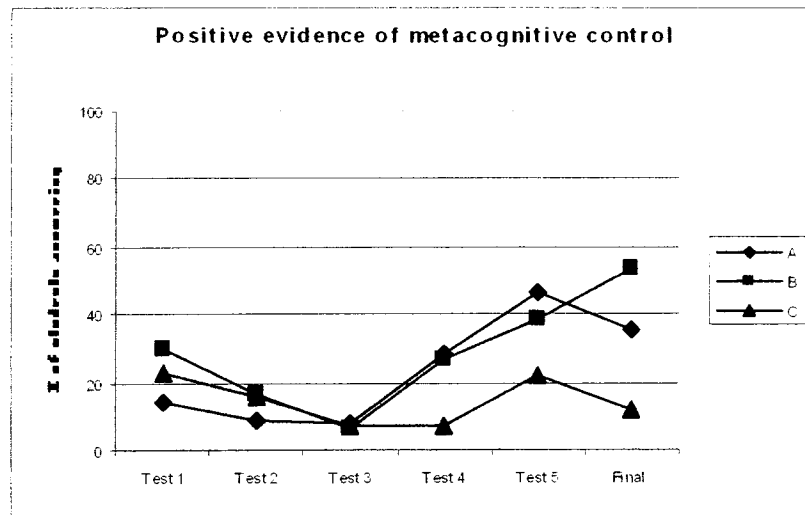
An argument that the graph in Figure 8.17 indicates clearly that writing exercises increase understanding is not being made here. The data are unclear and the data points in the experimental time period (Test 5 and Final) are too few to be used as the basis of a convincing argument. However, it is suggested that the data might indicate an improved ability to understand questions, and writing exercises therefore might be considered for use in a mathematics course as a tool for such improvement. The graph and accompanying data are interpreted as a suggestion and not an incontrovertible argument.

A similar situation applies in the analysis of positive evidence of metacognitive control.

**Table 8.18 Percentage evidencing metacognitive control**

	Group A	Group B	Group C
Test 1	14	30	23
Test 2	8	17	16
Test 3	8	7	7
Test 4 (midyear)	29	27	7
Test 5	46	38	22
Final exam	36	53	12

**Figure 8.19** Percentage evidencing metacognitive control



The dominant trend of groups A and B over C in figure 8.19 begins with Test 4, which occurred before the writing intervention began. Nevertheless, the continued relative performance, it is suggested, might be due to the positive effects of the writing exercises on the two experimental groups.

The third item of interest within the quantitative data analysis was not an immediate expectation. After some interview analysis had taken place and the insistence that the students were placing on their improved understanding, deeper thinking and saved time, the possibility of overall improvement across the syllabus was investigated. Different perceptions were reported by the students on whether the writing exercises had helped them across an entire section of mathematics, or just in the question on which the writing exercise was posed (as Applebee, 1984, suggests will be the case).

AS1, for instance, reported that the writing exercises had a broad effect. The writing exercises were quite challenging to do, because, in the absence of a writing exercise, the way to solve a problem is to try various calculations until it is solved. The writing exercises, however, involved having to think about and understand the problem in its entirety before attempting to do any calculations. “But therein lies the reward” [AS1], because by making that effort, “it’s not just that problem that you benefit from, it’s the whole section” [AS1]. “I think how it helped was, doing a sum then thinking about it, what I found is when I was faced with similar types of things it helped then rather than with the actual sum, because of having gone through the process and then clarified what I’d done.” [AJ] Other students felt that the writing exercises helped only with specific

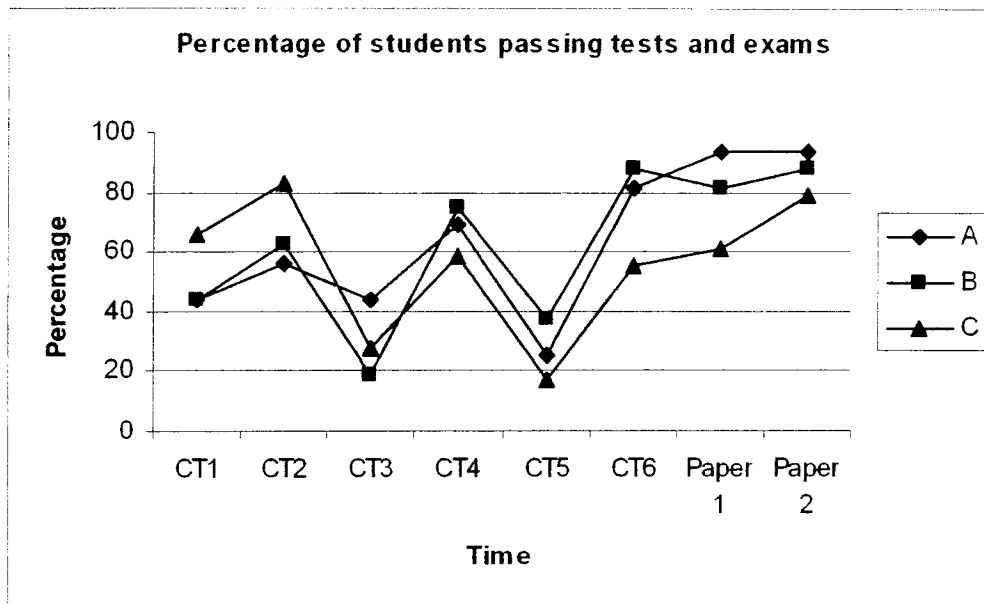
questions, such as DL: “Obviously in the specific question you’ve learnt more”, otherwise there isn’t much learning to be gained from the writing exercises, felt DL. If the writing exercises had had a broad effect in aiding understanding across entire sections of the syllabus, if they had encouraged deeper thinking about problems, and if they had the effect of saving time by laying out the entire problem-solving process at least in one’s mind if not on paper, then it might be the case that entire test results could have benefited, not just individual specific questions. The percentage of students simply passing and failing tests and examinations was investigated. For the purposes of this investigation of simple passes, test 6 was included in the sample as improved understanding and time saving devices could potentially have had an effect.

**Table 8.20 Percentage of groups attaining a passing grade**

	Group A	Group B	Group C
Test 1	44	44	66
Test 2	56	63	83
Test 3	44	19	28
Test 4 (midyear)	69	75	59
Midyear break – Intervention begins			
Test 5	25	38	17
Test 6	81	88	56
Final Paper 1	94	81	61
Final Paper 2	94	88	79

Final Paper 2 is in multiple choice format, which is why it does not appear in the analysis that required fully written solutions. It was also written two weeks after Final Paper 1.

**Figure 8.21 Percentage of groups attaining a passing grade**



Once again, no convincing argument is being attempted by the presentation of the graph above. There is simply the suggestion that the writing exercises encouraged a different way of thinking through problems which results in saved time in a test or examination, deeper understanding of sections of the syllabus than would otherwise have been attained, or a willingness to think more deeply about problems than might otherwise have been the case. The ascendancy of Groups A and B over C in the graph in Figure 8.19 begins at Test 4 which is before the writing exercise experiment began, potentially softening further any vague conclusion we might wish to draw from the graph. Once again, a suggestion is being made, an implication is being tentatively drawn and no compelling argument is being attempted.

## 8.9 Conclusions

In Piaget's stage independent theory of learning, deep understanding is attained through beta behaviour, and surface learning is encountered through alpha behaviour. In alpha behaviour, unstable or incomplete knowledge structures which will not stand up to scrutiny are created in an attempt to encompass some novel item of knowledge. Alpha behaviour can be said to be characterised by passive observation rather than active engagement, by the learner being an observer of processes rather than an active participant in the creation of knowledge. Beta behaviour, however, is characterised by active engagement in the knowledge creation process. Deep understanding is achieved through beta behaviour, yet alpha and beta behaviour are indistinguishable to the learner unless the unstable knowledge structures of alpha behaviour are subjected to scrutiny or

challenge, requiring the learner to commit to greater effort in attempting to understand the novel item. A challenge to the perceived accommodation of the novel alimnt constitutes disequilibrium, forcing the learner to enter the reflective cycle of assimilation and accommodation, which, if successful, results in equilibration. Two key features of the writing exercises promote beta behaviour over alpha behaviour, that is, their active nature and their reflective nature.

The active nature of writing an explanation of a problem's solution process is not by itself a sufficient condition for active engagement in the knowledge creation process, yet it is an encouraging factor. Applebee (1984) considers that the active nature of writing helps make writing a powerful tool for shaping thought. Gaskins et al (1994) agree, drawing a parallel between the dynamic process of writing and the active nature of science learning. Combined with the reflective demands that the writing exercises made on the students, the active nature of the writing encouraged active engagement with the mathematics and hence a deep approach to understanding modelled by Piaget's beta behaviour.

The requirement of the writing exercises was that participants describe and justify solution processes. In the After group, in which students wrote about problem-solving processes after carrying out calculation, the students were expected to look back (Pólya, 1945) over their solution, to reflect on what procedures had been carried out and why, and to reflect on what their solution expectations had been and whether their expectations had been fulfilled. In the Before group, in which students wrote about problem-solving processes before carrying out calculations, the students were expected to devise a plan (Pólya, 1945) reflecting on the problem requirements, possible procedures for solving the problem and why those procedures were justified. In both cases the demand to provide explanations and justifications required the students to engage more deeply with the mathematical requirements of the problems than might be expected through straightforward symbolic solutions of the problems. The demand for reflection on the problem solution processes encouraged the practice of reflective abstraction apparent in beta behaviour and not in alpha behaviour.

The tasks of looking back over a solution and devising a plan for a solution require different sorts of reflective processes. Both, if carried out thoughtfully, require paying attention to the mathematical nature of the problems, which, combined with the active

character of the writing process, strongly encourages the students to engage in beta behaviour and achieve deeper understanding of the problems and their mathematical underpinnings more than simply carrying out the solution processes would.

A surface approach to mathematical engagement is easier to achieve than a deep approach, with the student sometimes opting for a surface approach due to its relative lack of challenge, and sometimes not realising that there is a deeper approach to be taken. Understanding of the mathematical content of a question is more likely to be achieved if a deep approach is taken, that is if Piaget's beta behaviour is practised. The writing exercises encourage the students to take a deep approach both by encouraging beta behaviour from the first moment of working on a mathematical question and by challenging unstable knowledge structures created during alpha behaviour. Kelly's notion of contrast (similar-different) can be understood to be a different view of Piaget's notion of perturbation, or even to be a precursor to perturbation as the human mind perpetually seeks understanding of its experiential world, and continually and unavoidably tests hierarchical construct systems against construal of experiences. Recognition of contrast and perturbation as schemes result in unexpected outcomes in turn results in a beta behaviour and (at least partially) improved understanding. There is evidence that continual practice of the writing exercises gradually deepens students' engagement with mathematical content, corresponding to a changing stance towards knowledge as a creative process in which the student can be actively engaged.

There is some evidence to suggest that non English main language speakers experience greater difficulties with the mathematics course delivered in English than English main language speakers, and, in addition, experience greater difficulty with the writing exercises than the English speaking students. There is no significant evidence that a student's level of mathematical preparedness impacts on their performance in or on their perception of the writing exercises.

## 9 Discussion and Conclusions

The writing study project has concluded with several satisfactory findings and one particularly noteworthy feature. That feature is the use of the theoretical framework of Piaget's learning theory in a context in which it does not appear to have been used before, namely mathematical problem-solving, with respect to skills enhanced through writing. Piaget's model has been successfully used to model processes that result in an increase in understanding due to deeper engagement with the mathematical material, when in the absence of writing a surface approach to learning might have been undertaken.

Satisfactory findings include that writing can be used to boost problem-solving skills. In the perspective of Pólya's framework, the increases occur notably in the phase *understanding the problem*, but possibly also in other phases, such as *devising a plan*. Speakers of English as an additional language do appear to find studying mathematics in English more difficult than main language English speakers and hence experienced different difficulties in the writing exercises to the English students, and the writing exercises appeared to have similar effects for students with different levels of mathematical preparedness.

Requiring students to write explanatory paragraphs on their problem-solving processes demanded a deeper level of engagement with the material than was the case if they only carried out the calculation necessary to solve the problem. While the deeper level of engagement with the material went partway to enabling a richer understanding of the mathematics, the writing exercises often supplied a perturbation to the students' cognitive schemes by bringing the students face to face, as it were, with unstable elements in their own understanding. It is possible, when performing a calculation, to not realise that one's understanding is faulty or incomplete, or to not admit that instability to oneself. By writing about a solution process, one is more likely to encounter the cognitive instability or incompleteness and to rectify that deficient situation.

There was concern that the diversity of the student body in the two measures of language and preparedness would result in differential effects of the writing exercises, with, for instance, the better mathematically prepared students gaining more from the

writing exercises than the less well-prepared students (as observed by Swing and Peterson, 1988). In the case of first or main language those concerns were somewhat justified, although indirectly. A direct influence of language on the writing exercises was not associated with main language, but rather with mathematical terminology, the mathematics register. At least one student (LM2) struggled with the writing exercises due to her lack of confidence in her facility with the mathematics register even though it was stressed to the tutorial class that using informal language was entirely appropriate. In contrast, at least two students (DC and CP) appreciated the writing exercises for the positive effect exerted on their own use of mathematical terminology. Main language influence emerged more indirectly with the qualitative finding that speakers of English as an additional language experience greater challenges in learning mathematics in English than the English main language speakers do, resulting in the two groups experiencing different difficulties with the writing exercises. The English main language speakers tended to associate the difficulty of the writing exercises with the difficulty of the mathematical topic involved, whereas the additional language speakers, while acknowledging the effect of the topic difficulty on the exercises, found the challenge of expressing themselves through writing in English more pronounced.

The concern over differential effects for the differently prepared students was judged to be unfounded, as the level of preparedness of the students seemed to have no effect on the students' perceptions of the task, nor on any observations made during assessments or in the tutorial classes. This finding of egalitarian access to the task was encouraging, as the diversity of preparedness in large first year university classes adds an extra dimension to the already challenging task of teaching a large body of content, at a fast pace, to a large class. Prior to an in-depth discussion on the research findings, definitions and concepts covered in previous chapters are briefly revisited.

## 9.1 Research question

- What effect does the writing of explanatory strategies have on mathematical problem-solving?

Subquestions include

- Are any observed effects of writing in problem-solving different for students with differing main languages?
- Are any observed effects of writing in problem-solving different for students with differing degrees of mathematical preparedness?

Brief answers to the research questions are that explanatory writing improves understanding of the questions and the underlying mathematics, different main language students did experience the writing exercises differently, and students with different levels of mathematical preparedness did not appear to experience the writing exercises differently.

## **9.2 Rationale and Context**

The writing study project was born out of two sources. The first source was the author's experience of the combination of most students being poor problem solvers (problem-solving not being explicitly taught in class) and problem-solving being, at least partially, a target of assessment and certainly being a desired characteristic in a mathematics graduate. It is often tacitly assumed by mathematics lecturers that good problem-solving behaviour exhibited on the part of the lecturer will be observed and absorbed by the students, who will gradually become good problem solvers themselves. Indeed, that is how the behaviour was likely to have been acquired during the education of the lecturers themselves. The observations of the author, backed up by evidence in the literature suggested that problem-solving is a skill difficult to acquire and difficult to teach, although it can be taught (Schoenfeld, 1985).

The second impetus for the study project was the reported success of using writing explanatory strategies in physics to improve physics problem-solving (Leonard et al, 1996). The physics problem-solving initiative had met with such great reported success that the author was inspired to attempt something similar in mathematics. The differences between physics problems and mathematics problems, however, are nontrivial and these differences had to affect the design of the writing exercises. The two major differences between the physics initiative and that described in this thesis were the lack of easily named and defined principles and concepts in mathematical problems and the lack of a readily available solution schema for a mathematical problem.

Once the writing activity had been designed, the question arose of how to practically integrate the activity into the course in question, where approximately 430 students were taught in two lecture groups. One important aspect of the writing initiative which needs to be stressed is that the existing course had to be left unaltered. The course syllabus could not be changed, the logistics of lectures and tutorials could not be

altered, and the supplied tutorial exercise sheets were not to be modified. What was being sought was a teaching and learning activity which would encourage good problem-solving behaviour without having to radically decrease course content, which would not require access to computers, or otherwise substantially change course structure. This intention of creating a course activity focussed on problem-solving, without negatively impacting on course content or requiring logistic changes, was successfully realised in the writing study project.

The course was structured with large 45 minute morning lectures of approximately 200 students in large, raked lecture theatres, and two hour afternoon tutorials in groups of about 27 students to one tutor. The issue of large classes was avoided, in effect, by running the writing exercises in the afternoon tutorials. The author was the tutor for three different tutorial groups, two of which were experimental and one the control. The two experimental groups differed in one fundamental respect, which was in how problems were defined and hence how and when in the problem-solving process the problem solver could write about the solution process

### **9.3 Problems and Problem-solving**

As has been discussed, the chosen definition of a mathematical problem was

a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one. . . . To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem.

(Schoenfeld, 1985, p. 74)

This definition of a mathematical problem as a question to which one has no ready answer puts it in stark contrast to the physics problems discussed in the explanatory strategy study which had such success in improving physics problem-solving. That is, physics problems are, generally, word problems, for which the well-prepared student is expected to have a plan of approach, based on underlying physics principles such as conservation of momentum. It is this difference in definition of a problem that necessitated the two different kinds of experimental information in the design of the writing study project. One tutorial group was asked to write about problems before calculation, thereby mimicking the physics initiative, but requiring “problems” to which the student were expected to have an idea of attack; the second group, working, as much as was possible, with true mathematical problems required that the students write about the problem-solving process after having completed their thinking and calculations.

Problem-solving is most easily defined as the solution of problems, shifting the onus of strict definition onto the problems themselves. Such a definition of *problem-solving as a skill*, in contrast to *problem-solving as an art* (Stanic and Kilpatrick, 1989) could potentially be disappointing to a devotee of Pólya's problem-solving principles. However it is the author's view that teaching problem-solving as an art would require a restructuring of the course within which the study project was based, and problem-solving as a skill would be an admirable proficiency to develop.

Pólya (1945) segmented the problem-solving process into four distinct stages, that is

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back

It was the steps of *devise a plan* and *look back* that were the foci of the writing exercises, ultimately successful in bringing about greater facility in the first step of *understand the problem*. The fundamental grounding of the physics problem-solving writing activity was derived from the studies in expert-novice distinctions. In physics problem-solving, experts are more likely to base their problem solutions on the principles underpinning the problems, while novices are more likely to leap straight into ill-thought out calculations. The physics writing initiative therefore required students to practise expert-like behaviour in identifying physics principles and concepts before carrying out calculations. In using expert-novice distinctions in mathematics to design the mathematics writing initiative, metacognitive strategies of cognitive monitoring and knowledge of cognitive processes, as well as use of heuristic strategies, are more important than naming principles and concepts. As in the successful physics initiative, it was the aim of the study project to determine whether the encouragement of expert-like behaviour would enhance problem-solving skills, a result that would successfully emerge in the form of increased mathematical understanding, the necessary first step in a successful problem-solving undertaking. The process by which this increased understanding came about has been successfully modelled using Piaget's theory of learning.

## 9.4 The writing intervention

The tutorial groups each consisted of between 20 and 30 students, working in small groups on a supplied sheet of mathematical questions. The tutorials each lasted two hours and were held in the afternoon. The two different versions of the writing initiative were carried out in groups categorised After (A) and Before (B), with the author as the tutor, alongside an additional Control (C) tutorial group. Altogether in the course there were twelve tutorial groups running on different days as well as concurrently, run by nine tutors, all lecturers and postgraduate students in the mathematical and physical sciences. The only difference between the A and B groups and all other tutorial groups, including the C group, was the presence of the writing exercises. All twelve tutorial groups worked through the same set of mathematical questions.

The three groups, A, B and C, did not differ in any significant way from the rest of the students attending the course. The students in the three groups, experimental and control, were not chosen by the author, nor did any characteristic describe those students in contrast to the rest of the class. Precise language data are not available for the entire class, however the majority (perhaps 80%) of the entire class were English main language speaking. Investigation of the mathematical background of the students, inferred from school leaving results as well as progress in the course, suggests that the experimental groups had a slightly higher proportion of under-prepared students than the class as a whole.

**Table 9.1** Preparedness of entire class

	Prepared	Under-Prepared
Experimental groups	14	25
Entire class	233	180

The After group was required to write about the problem-solving processes after having completed a problem calculation. The After students were also asked to give a brief description before beginning the calculation of what sort of answer they were expecting. Part of the explanatory paragraph given after the calculation was to include a reflection on their answer expectation and whether that expectation had been met. The After group were supposed to be working on questions that were true *problems*, in that they would not be able to describe the problem-solving process before embarking on the problem, since true problems have no immediately accessible solution schema. However, not

every sheet of standard questions in the weeks of the course included true problems. For such sheets, a question was chosen that the tutor felt was the best analogue to a problem.

### Example of Group A exercise

Tutorial 23 (week 11)

Solve for  $n$ :  $\binom{n}{7} = \binom{n}{5}$

Before any calculations, can you estimate (roughly) what size you expect  $n$  to be? If you have no idea, please say so.

After you have calculated the answer, explain what you did, in words. Was your answer roughly what you expected? If not, can you see how to improve such estimates in the future?

The Before group was required to write about their problem-solving processes before carrying out any calculations. Clearly the Before group could not carry out such a requirement for truly defined mathematical problems, so questions were chosen that the tutor felt would best benefit from a *look before you leap* approach. Often the questions worked on by both groups were the same, due to the absence of a true problem but the presence of a challenging question, worthy of serious thought before doing calculations, yet still accessible to a solution schema.

### Example of Group B exercise

Let  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ .

Find  $A^{-1}$  and (if possible) matrices  $X$  and  $Y$  such that  $AX = B$  and  $YA = B$ .

Before calculating  $X$  and  $Y$ , explain how you are going to solve for them. In particular, explain how you can tell whether it is possible to solve for them.

The Before group was quite explicitly carrying out the *devise a plan* stage of Pólya's problem-solving categorisation. The After group were not so much devising a plan, as being required to *verbalise and describe the recently executed plan*. The After group were being asked to *look back* and reflect on their answer expectation. In the language of metacognition, both groups were being asked for evidence of declarative knowledge of cognitive phenomena. The presence or absence of sense-making in mathematics is a characteristic that often differentiates novices and experts – novices being inclined to

accept without question any answer to a calculation while an expert will consider the likelihood of the answer in light of the question context. Sense-making, as well as reflection, was an expert-like quality being encouraged in the After group.

## **9.5 Diversity of language and preparedness**

Two possible factors influencing the impact of the writing exercises on the student body were construed as diversity of language and mathematical preparedness. South Africa has eleven official languages, and many more besides are spoken in the country. The language situation at the University of Cape Town is similar. While English is the main language of the majority (69%) of the students, a large class can be expected to have as many as twenty main languages spoken by its students, both African languages as well as European and Asian. Since the teaching and learning activity being investigated in this study project involved writing in English, the possibility had to be considered that speakers of English as an additional language would experience greater challenges in the writing exercises than speakers of English as a main language.

Not only was there concern over the effect of different main languages on perceptions of the writing exercises, but there were the additional complications of mathematics as a language in its own right, embedded in the teaching language of English, to be considered. Mathematics has its own technical register within English, as do many disciplines, as well as its own extremely precise rules of logic and symbol manipulation. Overlaying both linguistic concerns there is the political sensitivity of the role of English as a language of learning and teaching in a country where English speakers are not a majority. The desire by some to have teaching conducted in a child's (or student's) main language is undermined by the absence of complete mathematics registers in indigenous African languages. Suffice it to say that the intertwined issues of mathematics and language are impossible to disentangle. Any classroom activities in a mathematics class which might potentially demand communicative competence, as well as use of the mathematics register, must be carried out with sensitivity towards language concerns, and, in consequence, any activities that do not discriminate between language groups are to be desired over those that do.

The University of Cape Town is not alone in experiencing an increasing range of preparedness among its students. All over the world there are efforts being made to change the way university classes have traditionally been taught in order to

accommodate academic diversity. The course involved in this study project, MAM100W, had an associated bridging course, or extended curriculum course, which absorbed the students who were the least prepared among the MAM100W student body. At the end of the first quarter of the year long course, some of the students who had registered for MAM100W changed to the extended curriculum course, yet MAM100W was still left, and is left every year, with a wide range of mathematical preparedness among its eventual set of students. It is not unusual to have, in the same lecture theatre, a student who has represented South Africa at the International Mathematics Olympiad (IMO) sitting next to a student who struggles to add fractions. In the traditional “teaching by transmission” it is difficult, if possible at all, to accommodate all students. The lecturer necessarily aims his or her lessons to some median majority of the class, trying valiantly not to lose the interest of the IMO student while not losing the poorly prepared students altogether.

If a teaching and learning activity could be designed that would be equally or equitably advantageous to the problem-solving skill development of all student preparedness levels, it could be used in conjunction with the unavoidable teaching by transmission, with desirable beneficial effects. The writing exercises used in the study project appeared to constitute such an activity, with students of all levels of preparedness finding them useful and each student being able to engage with the activity and the mathematical work at their own level.

## **9.6 Data**

Three forms of data were collected, namely student interviews, writing exercises carried out by the students, and data collected from the students’ assessment tasks during the year. The interviews were the most informative data source, suggesting points of interest that the author was then able to pick up in the writing exercises. The assessment task data, while optimistically collected, was the least informative data source, although suggesting several points for thought.

The first data collected were the quantitative assessments of the students’ problem-solving behaviour observed in solving test and examination questions throughout the year, recalling that it was only in the second semester that the writing initiative was run. The interviews were all run at the end of the second semester, most students being interviewed during the last week of lectures and in the week before the final

examinations. By the time the interviews were conducted it was becoming apparent that the assessment data might not be particularly informative, and by the time the interviews were transcribed and the assessment data had been analysed deeply, it was clear that there was little to be extracted from the assessment data. The analysis of the interviews began in an air of despondency, in the disappointment that no great improvement in the students' problem-solving skills had become apparent over the course of the initiative, as had been the hope. Slowly, as the transcripts were read and reread, a surprising general impression started to become apparent, that is, the students' stressing of their increased understanding of the mathematics involved in the writing exercises, and of mathematics in general. Once the phenomenon of increased understanding was isolated as a point of interest, the aggressive language that the students used in connection with the phenomenon started to emerge. The words "force" and "push" were used repeatedly by the students, as in "it pushes you to understand the processes" (CP), "It actually forces me to think" (IT) and "to force insight" (DC).

In the wake of the disappointment over the lack of data to support an inference that the writing exercises had improved the student's problem-solving to the point of it being visible in their assessment tasks, came the realisation that the writing exercises had, in fact, aided the students' problem-solving behaviour by enhancing that vital first step of the problem-solving process: *understand the problem*. It is easy to say that the increased work the students needed to undertake aided their understanding, but that is insufficient. Simply giving the students additional problems is unlikely to have the same effect. Similarly, it is easy to say that of course the deeper engagement required by having to verbalise, as it were, their thought processes, brought about deeper understanding. The statement that writing generates deeper engagement which generates greater understanding is itself not unproblematic as an inference, but there is more to it. Why were the students using such aggressive language? Why were they being "forced" to such engagement? Or rather, one might turn the inference around and say that if the writing "forced" engagement, then would the absence of writing have coincided with the absence of engagement? Why? Having focussed on this "forced" engagement with mathematics as worthy of further analysis, the submitted writing exercises themselves were investigated for further evidence, in conjunction with the journal kept by the author throughout the experimental semester. Evidence for the writing exercises having resulted in deeper engagement and increased understanding was found, contrasting to shallow engagement on the part of students not taking part in the writing exercises.

Piaget's theory of learning had been previously encountered by the author in Dubinsky (1991a) and Dubinsky and Lewin (1986) and that theory was found to be agreeably descriptive of the bound phenomena of "forced" engagement and increased understanding.

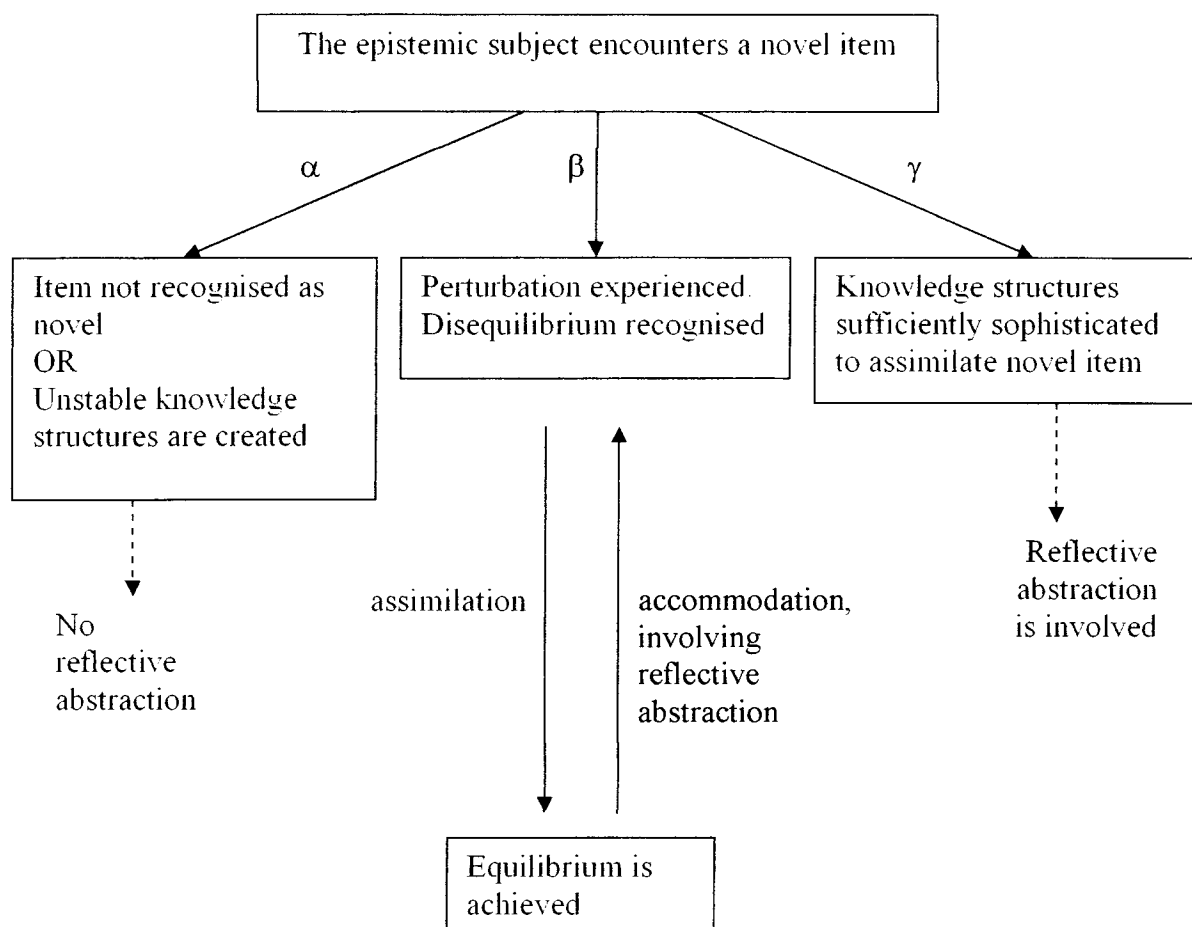
Piaget's theory of constructive learning begins with the learner, the epistemic subject, encountering a novel item, or *alimant*, "that which supports or sustains the mind" (online OED). The epistemic subject will exhibit one of three kinds of behaviour, categorised by Piaget (1978) as alpha, beta or gamma behaviour. Alpha behaviour encompasses both no learning and poor learning, with the subject either denying that the item is novel at all, or constructing unstable knowledge structures that will not stand up to scrutiny. If a learner (anyone, in everyday life, not necessarily a student in a classroom) has very shallow engagement with a subject, considers himself or herself to understand, but then, upon questioning, or having to use the unstable knowledge, comes to the realisation that he or she never understood in the first place, then what he or she is recognising is alpha behaviour. Alpha behaviour requires much less cognitive effort than beta behaviour (true learning) and can be undertaken both consciously or inadvertently (Piaget, 1978).

When the epistemic subject encounters a novel *alimant* there is an attempt to assimilate the *alimant* into the existing cognitive structures. The subject applies a scheme, a package of cognitive items and actions, to the *alimant*, chosen for the *alimant*'s similarity to items previously encountered. If the item is indeed novel then the scheme, properly applied, results in an unexpected outcome for the subject, termed a *perturbation*. The subject's cognitive structures are said to be in disequilibrium, and the process of reacquiring equilibrium is termed equilibration. Robust reaction to perturbation involves a cycle of assimilation and accommodation. Assimilation is the act of applying to the *alimant* the subject's already existing cognitive operations, and accommodation is the modification of cognitive structures. During the process of accommodation, if the *alimant* involves or impacts upon logico-mathematical reasoning (as mathematics does) a process occurs called reflective abstraction. The difference between alpha behaviour and beta behaviour particularly resides in the absence or presence of reflective abstraction.

In beta behaviour, the novelty of the item is recognised. A scheme has been applied to the item and an unexpected result has occurred; disequilibrium is recognised. The subject attempts to assimilate the item using existing cognitive structures and does not succeed. Therefore the subject is forced to accommodate by modifying the structures, thereafter attempting to assimilate once again. If assimilation is unsuccessful then accommodation occurs once again, repeated until assimilation does occur. This process is referred to as beta behaviour and represents learning such as that generally encouraged in the minds of the students by the mathematics teacher. Alpha behaviour occurs if the novelty of the item is never acknowledged, or if the assimilation process has not been carried out completely, no reflective abstraction has occurred, yet the subject does not register any need for it. It is important to note that beta behaviour can occur with reflective abstraction, and some level of accommodation, yet still not quite lead to complete equilibration of the disequilibrium brought to light by the original perturbation. In other words, even with deeper engagement (in contrast to alpha behaviour's shallow approach) true and complete understanding might not be reached. The observations of beta behaviour in the students' problem-solving processes, particularly in contrast to non-participating students' alpha behaviour, provided gratifying support for the place of the writing activity in the mathematics course.

Gamma behaviour is epistemically more desirable than beta behaviour, yet it is not the type of learning which is usually expected by the teacher in the classroom. Gamma behaviour occurs when the item is indeed novel, yet the existing cognitive structures are sufficiently well developed that they allow for the possibility of variation on items previously encountered. Reflective abstraction still occurs, as the cognitive structures expand to allow for the addition of the new item, but no substantive change has had to occur to the existing cognitive structures. While such a cognitive system is indeed desirable, and practitioners in a subject should well aim for such a system, the teacher of a subject cannot expect such a system in her students, nor should she. Beta behaviour is the behaviour sought by a teacher, in contrast to the oft observed alpha behaviour.

**Figure 9.2** Diagram representing Piaget's theory of learning



A particularly notable feature of Piaget's system, beyond the two pathways of alpha and beta, is the perturbation that is required to push, "force", one might say, the subject into exhibiting beta behaviour. In order for beta behaviour to occur, even before entering into the accommodation and reflective abstraction processes, there has to be a perturbation, a surprise, a disturbance or worrying occurrence, shunting the subject down the beta pathway instead of the much more easily accessed alpha pathway. This Piagetian model, particularly the alpha and beta prongs of the three-pronged model, was found to model the observations in the writing exercises very pertinently, backed up by the observations made by the students in the interviews as well as by the aggressive language they used.

The writing exercises submitted by the students for comment were not only used to provide support for the Piagetian model suggested by the interviews and the tutor observations during the semester, but were analysed according to a second model, that of Waywood's (1992) journal classification scheme, adapted for use in the writing

exercise context, somewhat different from the original journal context. The classification scheme strongly suggested that, over the course of the semester, the students were taking a more creative stance towards knowledge and moving away from a passive view of knowledge, with the learner as a participant in knowledge creation rather than as a passive recipient of objective knowledge. Over the course of the study project, the students were more inclined to think about and give explanations of the mathematical processes instead of recounting facts and steps with a focus only on arriving at an answer.

The Piagetian model suggests that, in the wake of a perturbation or a surprising result, the student is more inclined to engage deeply with the content, that is, in Piagetian terms, and that the student is more likely to enter a cycle of assimilation and accommodation with reflective abstraction. The student is more likely to equilibrate, that is, to resolve the surprise or perturbation, and come to a more insightful understanding of the mathematical content of the problem than might be the case in the absence of a perturbation or the ensuing reflective abstraction. The writing exercise analysis, using the adaptation of Waywood's scheme, deepens this Piagetian analysis by providing a closer look at that "deeper engagement", giving evidence that the students' approach to the problems and view of mathematics in general did change over the course of the study project.

The data collected from the students' assessment tasks throughout the year attempted to measure the students' problem-solving skills, in the anticipation of seeing an improvement over the course of the study project. The assessment process was designed to measure whether students understood the problems, whether they showed signs of metacognitive control and whether they showed flexibility in the solutions. The data were collected for the first semester (no study project) and for the second semester, during which the writing study project took place. No firm conclusions are attainable from the assessment data, unfortunately. The number of data points for the second semester was originally planned to be fewer than in the first semester (3 assessment tasks in the second semester as opposed to four in the first semester), a situation which worsened at the last minute with the effective cancellation of one of the measures. Reliance on two data points (one test and one examination) during the experimental period simply does not give enough evidence of any effects unless of a startling nature, and no startling results were apparent. Only three tentative suggestions can be made

from the quantitative data, one associated with understanding, one with metacognition and the third slightly more difficult to pin down.

The first suggestion from the assessment data is that of increased understanding. The control group C was only associated with the assessment data; by definition the writing exercises were not carried out by the Control group. Also no interviews were run for the Control group, since the interviews were largely about the writing exercises. In the case of the quantitative assessment data, there is some tentative evidence that the two experimental groups were more inclined than the Control group to show understanding of the problems in the assessment tasks during the study project in the second semester, a relationship not apparent during the first semester. Similarly the two experimental groups revealed tentative evidence that they practised greater metacognitive control than the Control group. These results are encouraging as they tend to support the findings from the analyses of the interviews and the writing exercises themselves.

The third finding from the quantitative assessment data was that the students in the two experimental groups showed a greater likelihood of passing tests and examinations in the second semester than the control, again a relationship not apparent in the first semester. Why should more students in the experimental groups be passing than in the control group? Multiple answers are possible, the least encouraging being that it was a fluke of the data, a scenario that might be viewed as not unlikely given the paucity of the data points. Another suggestion is that the extra attention those students had from the tutor helped them gain that advantage. The control group did have the same tutor, who tried to give the students close attention, but naturally they did not receive that extra attention constituted by the extra practice and the return commentary. A third suggestion relates to the increased understanding of mathematics in general that it is postulated was encouraged by the writing exercises in the students. Finally a fourth suggestion is associated with the time saving effect of the writing exercises reported by several students. To put the students' observations in terms Pólya might have used, the students perceived that the time taken to *devise a plan* for solving the problem saved time in actually working through the problem, as they did not waste time carrying out actions that were ultimately unfruitful. Neither of the first two explanations (fluke and extra attention) can be ruled out, however the last two explanations do suggest some support of the theory that the writing exercises encouraged good problem-solving behaviour in *understanding the problem*, and perhaps in *devising a plan*.

## 9.7 Language and preparedness effects

Analysis of the data led to several conclusions of the effects of language and preparedness diversity. In brief:

- The only single interview question exhibiting clear effects of language reveals that the non-English students found the comments more useful than the English students
- Combinations of interview questions revealed particular subtle language effects
  - Both main and additional English language speakers experience challenges with the writing exercises, but those challenges differ.
  - Speakers of English as an additional language do experience more difficulty with the course than speakers of English as a main language
  - Fluency in the mathematics register is simultaneously an obstacle to the activity as well as improved by the activity
- No clear effects of preparedness were apparent from the interview questions
- No clear effects of preparedness were apparent from writing exercise analysis

Both speakers of English as a main language (hereafter English) and speakers of English as an additional language (hereafter non-English) students were equally likely to find the writing exercises a difficult task. The English students tended to find that the mathematical content of any one writing exercise was the primary contributor to the difficulty of the task, while the non-English students reported that the concept of writing in mathematics and the difficulty of getting started were greater contributors to difficulty than the mathematical content. However, when asked whether the writing exercises had taken much time to complete, the English students were not inclined to think they were time consuming, while the non-English students found that time was dependent on the mathematical content about which they were being asked to write. The effect of the particular mathematical problem topic can be related to the responses about which of writing, reading and speaking English the students found more difficult – the English students reported all three modes equally easy or difficult, while the non-English students found writing to be much more challenging than either reading or speaking. A conclusion that could be drawn from the combination of these three interview question responses (questions 4, 8 and 16) is that the writing exercises are challenging to students from both language groups, with part of that challenge being the mathematical content of the question (some mathematics simply being experienced as more difficult by the students) and the additional challenge for the non-English students

of the act of writing in English. The burden of having to write about mathematics is enough of an obstacle for the non-English students that the mathematical topic of the question emerges as relatively less of a contributor to the difficulty of the exercise, although its effect is felt in the time taken to complete the task.

When the interviewed students were asked whether they felt that non-English student experienced greater difficulty than English students in learning mathematics in English, there was an interesting discrepancy. English students felt that non-English students were sure to experience greater challenges, yet the non-English students themselves were less inclined to think so. Similarly, the well-prepared, English students interviewed were quick to respond to the question on language difficulties experienced in the course, stating that they felt that questions were often ambiguously phrased, a perception that was not strongly apparent among the non-English students, nor the less well-prepared students. A conclusion which has been drawn from these responses suggests that there are subtle language difficulties associated with the course which are not being perceived by the non-English students. If a question is ambiguously phrased or found hard to understand by a well-prepared English student, it seems unlikely that the non-English students have a linguistic insight which the English students lack. Since the non-English students do not feel strongly that language is a major influence on their mathematics studies, but subtle language difficulties are being perceived by the English students, it is suggested that additional language speakers do have a more challenging time learning mathematics in English, yet it is a sufficiently understated effect relative to other challenges of under-preparedness that the students themselves might not even perceive it. Sensitivity towards language challenges seems a pertinent objective for the teacher of mathematics.

Two effects associated with the mathematics register were detected. The first, and negative, effect was the contribution to the difficulty of tasks arising from the need to express oneself in terms of a technical register. At least one student struggled to even begin the writing exercises due to her lack of confidence in her fluency with the mathematics register, and the non-English students considered mathematical terminology to be a relatively large factor contributing to difficulty. The less well-prepared students were also more likely than the well-prepared students to find mathematical terminology a contributor to difficulty. A second, and positive, effect was the increase in fluency in the mathematics register reported by some students regularly

taking part in the writing exercises. Competence in the use of the mathematics technical register is to be desired in a student of mathematics, suggesting that the writing exercises are a potential tool to use in such training, although the teacher using this sort of activity must be aware that less well-prepared students and non-English students might not only find the writing exercises challenging for register reasons, but possibly sufficiently challenging that some will be inclined not to take part in the activity at all.

## **9.8 Limitations of the study**

In all, only 39 students took part in the writing exercises, 18 in the After group and 21 in the Before group. While interesting and informative observations have been made on the progress of the students involved in the project, a greater number of students would have been more desirable and probably much more informative. The quantitative assessment data, in particular, would have benefited from greater numbers of students, since the conclusions suggested by the assessment data analysis are subtle and imprecise and cannot be construed as being strong evidence for definite effects without greater numbers of students to make the evidence of effects more statistically robust.

The students wrote four tests in the first semester, which was a satisfactory number. However they only wrote three tests (from which problem-solving data might have been collectable) in the second semester; that number including the hastily changed test which rendered data of one point useless for this study. At least four assessment tasks should have been available for the second (experimental) semester, and more in both semesters would have been preferable. The paucity of data in the experimental semester renders as tenuous any conclusions from the assessment data analysis alone.

The mathematical questions chosen for inclusion in the assessment tasks as good indicators of problem-solving behaviour were carefully chosen, but could perhaps have been better chosen, to allow for greater creativity on the part of the student problem solver and more flexibility in solution process. It was often not possible to tell from a student's solution whether he or she had shown metacognitive control or was capable of solution flexibility. Collecting data from assessment tasks might be impossible, irrespective of question choice, requiring (for measures of metacognitive control and flexibility) a completely different form of observation, such as having students work in pairs on problems, vocalising their thought processes to one another. Question choice could also be improved in the weekly tutorial sheets. Increasing the density of true

mathematical problems in the tutorial would not impact on course content and would remain true to the activity's intent to integrate into an existing course without changing it.

Two extremely important limitations to the study which should be addressed in any future research projects were the possible shallowness of the intervention and the absence of formal assessment of the student responses. The students had to complete one writing exercise per week, in one two hour tutorial. The semester had 11 effective weeks during which the students were involved in the writing study project. Not all participating students attended each week, nor did all students complete a writing exercise every week. It could perhaps be argued that the enaction of Polya's problem-solving steps, exhibited in this study project, not resulting in improved problem-solving behaviour in assessment tasks might bring into question the validity of Polya's problem-solving theory. It is the belief of the author, however, that the writing exercises might well have had a far greater effect on the students' problem-solving behaviour had the writing intervention been more integrated into the course, with the students having to complete more than one writing exercise per week. Questioning the validity of Polya's steps of successful problem-solving would have to be grounded in a problem-solving intervention far more integrated into the mathematics course than the one described in this thesis. Bell and Bell (1985), among others, insist that any writing initiative aiming to improve problem-solving skill must be an integral, carefully structured and regularly practised part of the teaching process, not merely an enrichment exercise. Of course, with only one tutorial scheduled per week that kind of regularity would require that the students either work on writing exercises during the lecture periods, or in their own time as homework, both possibilities having their own inherent challenges and constraints.

Integration into the course would be facilitated by having the writing exercises as an assessment requirement of the course. Assessing the writing exercises, however, would be difficult to perform within the current course structure. All marking or grading is undertaken by the set of tutors, which consists of the course lecturers and several postgraduate students. The writing exercises would have to be assessed qualitatively, quite differently from normal mathematics grading, a type of grading unfamiliar to most of the tutors both as science students themselves and in their role as tutors (Habre, 2002, agrees). For the writing exercises of the entire student body, approximately 400 to 500 students in MAM100W annually, to be assessed by a single tutor or lecturer would be

the most effective way to retain consistency, but helpful commentary would have to fall away. The time taken to provide commentary on the writing exercises of the 39 students taking part in this study project (approximately 8 hours per week) suggests it is impossible to extrapolate to 400 or 500 students if there is only one person responsible for the commenting and feedback.

Extending the writing exercises to the entire student body of the course and having the writing exercises assessed in some way almost certainly involves doing away with any form of feedback or commentary. Since the commentary is important for its informative content, motivational effect and epistemological role in allowing the student to participate in the discourse, this dilemma leaves the lecturer in a predicament. Integrate the activity more deeply into the course and cut off the commentary, or restrict the activity to a small number of students, non assessed, and make the activity more valuable through the commentary? An additional deterrent to omitting commentary is the evidence encountered in the interview analysis that the non-English students were unanimous in their agreement that the comments had been found very helpful, in contrast to merely approximately half of the English students. A partial solution might be a range of categorisations such as Waywood (1992) provides with his journal entry classification scheme. In the case of Waywood, journal entries were scrutinised for specifically sought types of statement (summary, examples, questions, discussion) which were then located on a scale. For example, in the case of “summary”, each entry could be located on the scale characterised by

- Able to regularly copy part or most of board notes into journal
- As above but also able to describe important aspects of what was done in class
- Able to record some of the main ideas of a lesson and able to write some thoughts about them
- Able to isolate and record in own words a sequence of connected ideas from a lesson, with an emphasis on expressing mathematical learning
- Able to formulate and state an overview of the material covered in a lesson, text or topic with appropriate use of formal language and vocabulary
- Able to extrapolate from material presented in class, or in a text, and reshape it in terms of own learning needs

(adapted from Waywood, 1992, p. 39)

Each journal entry would be assessed in this way for several characteristics, the summary being only one such characteristic. For writing exercises such as in this study project, the lecturer could assess the exercises according to scales such as provided by Waywood, which would give the students at least some feedback, if not detailed. It is encouraging, however, that effects of increased understanding were observed in the

study project described in this thesis, even though the writing exercises involved were not assessed.

## 9.9 Final inferences

Writing in and about mathematical problem-solving increases mathematical understanding, increases the student's sense of participation in the discourse and addresses affective issues such as motivation. In Pólya's language of problem-solving, writing about problem-solving in the form of *devising a plan* and *looking back* enhances problem-solving behaviour in the form of *understanding the problem* and its mathematical underpinnings, and possibly also at the stage of *devising a plan*.

In the language of Piaget, writing about problem-solving processes in the presence of unstable or incomplete cognitive structures potentially provides a perturbation that forces the problem solver into beta behaviour rather than alpha behaviour. The actions of assimilation and accommodation ensue, including the process of reflective abstraction so necessary for logico-mathematical reasoning, resulting in equilibration of the cognitive structures which were shown to be in disequilibrium by the perturbation. It is possible that the equilibration might not be complete, leaving the student with cognitive structures not as stable as might be desired, yet still substantially better off cognitively than would have been the case in alpha behaviour alone. The epistemic subject is left with more complex and sophisticated knowledge structures as well as with modified schemes for testing against any future novel items which might be encountered.

In the discourse of social constructivism (not employed much in this thesis, yet applicable to the study project) the writing exercises allowed the students to take part in a mathematical conversation within the greater discipline, where they were capable of taking ownership of their mathematical knowledge. The comments supplied by the tutor were an important part of the conversation, allowing the students to feel they were, in fact, participating in knowledge creation. This participation, as well as the affirmation expressed in the comments, had a motivating effect and increased the students' confidence in themselves as members of the mathematical discourse community, and allowed them to adopt legitimate peripheral participation (Lave and Wenger, 1991).

Legitimate peripheral participation is the participation necessary for new members of a discourse community to, over time, become core members of that community.

It can be concluded that writing about mathematical problem-solving processes enhances understanding of the mathematics underlying the problems and has a beneficial effect on students' engagement with mathematical content. The increased mathematical understanding observed as an effect of the writing exercises within the study project is a desirable outcome of any mathematical activity. Whether understanding is viewed as the necessary first stage of the problem-solving process, or whether understanding and insight are valued in themselves, distinct from any particular mathematical activity such as problem-solving, it is an essential attribute for a successful student of mathematics. The writing of explanatory paragraphs in the context of mathematical problem-solving was successful in creating an environment in which such understanding was encouraged. Piaget's theory of learning, particularly his differentiation between alpha and beta behaviour, was invoked successfully, in what appears to be a novel situation, in the environment of mathematical problem-solving.

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# Glossary

## Problem-solving terms

**Problem** - A task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one.... To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem. (Schoenfeld, 1985, p. 74)

## Language Terms

**Additional language** – An additional language (Adler, 2001; Setati et al, 2002; Kasule and Mapolelo, 2005) is a language that is not the speaker's **main language**, but is one that the speaker is learning or has learnt (probably at school), and is also encountered in a social context, such as in the media, in advertising or amongst friends. In South Africa, English is an additional language for any non-English speaker in an urban setting. English is ubiquitous in the media, in advertising and in shops. Kasule and Mapolelo (2005) refer to the “dominance of English in informal mathematics conversation in southern Africa” (p. 605). This term (additional language) is used in preference to “**second language**” or any other quantitative term, since many students in South Africa learn more than one language simultaneously while growing up, or, on the other extreme might consider a language to be their fourth or fifth language. Using the term “additional” steers clear of attempting to count the languages a student has acquired. (See **new language**)

**Foreign language** – A foreign language (Adler, 2001; Setati et al, 2002; Kasule and Mapolelo, 2005) is similar to an **additional language**, except that it is encountered solely in a formal context, such as in school, and not in any other context. Such a situation is typically encountered in an isolated South African rural context where any local media would be in the local language, and no people living in the area speak the language as a **main language**. In South Africa this prospect is realistic in a rural community. Trappes-Lomax (1990) defines a foreign language as being non-

indigenous to a particular speech community, not used outside the classroom for day to day communication within the speech community. (See **new language**)

**LoLT** – The Language of Learning and Teaching (Adler, 2001) refers to the language primarily used for teaching in the classroom. LoLT is used in preference to “**medium of instruction**” and other similar terms since it refers to both learning and teaching, and can easily be applied to textbooks, and other nonverbal material.

**Main language** – One’s main language (Adler, 2001) is the one in which the speaker is most proficient and uses most often. A synonymous term is **primary language** (Khisty, 1995; Adler, 2001). One’s main language is generally one’s “**native language**” (Khisty, 1995), that is the language of the family into which the speaker has been born. Other similar terms are “**background language**” (Goduka, 1998, p. 36), “**home language**” (Goduka, 1998, p. 37; Fradd and Lee, 1999), “**home tongue**” (Coutis and Wood, 2002, p. 2), “**parental home language**” (Moodley, 2000, p. 104), “**childhood language**” (Moodley, 2000, p. 108), “**mother tongue**” (Moodley, 2000, p. 105) and “**heritage language**” (Cummins, 1981, p. 9) where the main or home language is one indigenous to a country from which the person has emigrated.

**New language** – Fradd and Lee (1999) use the term “new language” with no differentiation between **additional** and **foreign languages**. In a South African context it is important to differentiate between the two categories. Similarly Stoddart et al (2002) use the term “English language learners” without specifying whether English is a foreign or an additional language for the students.

## **Abbreviations encountered in language literature**

ALLE – Additional language learning environment (Setati et al, 2002)

EIL – English as an international language (Schmied, 1990)

EL2 – English as a second language (Starfield, 1996)

ELD – English language development (Stoddart et al, 2002)

ENL – English as a native language (Schmied, 1990)

ESB – English-speaking background (Coutis and Wood, 2002)

ESL – English as a second language (Warren and Rosebery, 1995; Stoddart et al, 2002)

ESOL – English to speakers of other languages (Fradd and Lee, 1999)

FLLE – Foreign language learning environment (Setati et al, 2002)

L2, L3 – Second language, third language (Kasule and Mapolelo, 2005; Merritt et al, 1988; Dawe, 1983; Cummins, 1981)

LEP – Limited English proficiency (Secada, 1992; Khisty, 1995; Lee and Fradd, 1998)

NELB – non-English-language background (Lee and Fradd, 1998)

NESB – non-English-speaking background (Ellerton and Clarkson, 1996; Coutis and Wood, 2002)

NEP – non-English proficiency (Khisty, 1995)

SLA – Second language acquisition (Kasule and Mapolelo, 2005)

## **Piaget's constructivist theory of learning**

**Accommodation** – the modification of cognitive structures

**Aliment** – a novel item of knowledge or information, “(figurative) that which supports or sustains the mind”, “the material or means of nourishing” (online Oxford English Dictionary)

**Assimilation** - the process of applying to an aliment “the set of cognitive operations which the knower has previously constructed” (Dubinsky and Lewin, 1986, p. 60)

**Equilibration** – a dynamic “state of balance between or harmony between at least two elements which have previously been in a state of disequilibrium.

**Perturbation** – the cognitive conflict which occurs when objects or events are encountered which cannot be assimilated and disequilibrium results.

**Scheme** – a package of cognitive items and actions which is used to cognitively act on experiences in an attempt to assimilate them.

## Appendix 1 Tabular Data on Student Profiles

### Students taking part in the writing initiative

#### Students' groups, main languages, level of preparedness, and gender: Group A

Student	Degree Programme	Gender	Preparedness	Main language	# written submissions	Included in writing ex. analysis?	Interviewed
AJ	COMB12	Female	Yes	English	5	Yes	Yes
AN	SCIB13	Female	No	English	1	No	No
AS2	SCIB13	Female	No	Mandarin	3	No	Yes
DC	SCIB14	Male	Yes	English	6	Yes	Yes
DK	SCIB13	Male	Yes	English	7	Yes	No
DL	SCIB14	Male	No	English	3	Yes	Yes
ET	SCIB13	Male	No	English	1	No	No
IS	SCIB06	Male	No	English	6	Yes	Yes
JL	SCIB12	Female	No	English	4	Yes	No
MS2	SCIB13	Female	Yes	English	7	Yes	No
MS1	ENGB19	Male	No	English	5	Yes	Yes
NG	SCIB13	Female	Yes	English	6	Yes	No
NW	SCIB13	Female	No	Portuguese	7	Yes	Yes
OU	SCIB06	Female	No	English	1	No	No
RG	SCIB14	Male	Yes	English	5	Yes	Yes
RT	SCIB13	Female	Yes	English	4	Yes	No
TH	SCIB13	Female	Yes	English	7	Yes	No
WT	ENGB19	Male	No	Afrikaans	3	Yes	Yes

**Students' groups, main languages, level of preparedness, and gender: Group B**

Student	Degree Programme	Gender	Preparedness	Main language	# written submissions	Included in writing ex. analysis?	Interviewed
AP	SCIB06	Male	No	Other	2	No	No
AS1	SCIB14	Male	Yes	English	7	Yes	Yes
BJ	SCIB14	Male	Yes	English	4	Yes	No
BM	SCIB06	Male	No	English	1	No	No
CM	SCIB06	Female	No	Shona	4	Yes	Yes
CP	SCIB14	Male	Yes	English	6	Yes	Yes
DB	SCIB13	Female	No	English	1	No	Yes*
GCP	SCIB14	Male	Yes	English	5	Yes	No
IT	ENGB09	Male	No	Tswana	2	No	Yes
JB	SCIB06	Male	No	English	5	Yes	No
JE	SCIB06	Male	No	English	7	Yes	No
JS	SCIB14	Male	Yes	English	5	Yes	Yes
LM2	ENGB01	Female	No	Zulu	4	Yes	No
LM1	SCIB06	Male	No	English	7	Yes	Yes
MM	ENG09	Male	No	Tswana	4	Yes	No
MW	SCIB13	Male	Yes	English	6	Yes	No
NH	SCIB06	Male	No	English	6	Yes	No
RA	SCIB14	Male	N	English	2	No	No
SW	SCIB06	Male	No	English	4	Yes	No
TJ	SCIB13	Female	No	Shona	5	Yes	Yes
TvH	SCIB14	Male	Yes	English	5	Yes	No

## Language and Preparedness cross-reference of all participating students

(also in Chapter 8, see Tables 8.3.1 and 8.3.2)

### Group A language/preparedness cross-reference

	Prepared	Under- Prepared	
English	7	8	15
Not English	0	3	3
	7	11	18

### Group B language/preparedness cross-reference

	Prepared	Under- Prepared	
English	7	8	15
Not English	0	6	6
	7	14	21

## Language and Preparedness cross-reference of all students interviewed

(also in Chapter 8, see Tables 8.4.1 and 8.4.2)

### Group A language/preparedness cross-reference of interviewed students

	Prepared	Under- Prepared	
English	3	3	6
Not English	0	3	3
	3	6	9

### Group B language/preparedness cross-reference of interviewed students

	Prepared	Under- Prepared	
English	3	2	5
Not English	0	3	3
	3	5	8

**Students physically present in the tutorial group, but not taking part in the writing initiative**

**Students physically in the tutorial groups, but not taking part – having declined consent**

<b>Student</b>	<b>Degree</b>	<b>Language</b>	<b>Prepared</b>
<b>Group A</b>			
AB	ENGB19	Afrikaans	No
SD	SCIB13	English	No
TLF	SCIB14	English	Yes
JN	SSHB27	English	No
RS (withdrew)	SCIB13	English	Yes
LT	SCIB06	English	Yes
PvdH	SCIB13	English	Yes
CV	COMB03	Afrikaans	Yes
<b>Group B</b>			
JT	SCIB06	English	No

**Students physically in the tutorial groups, but not taking part – having given consent, but then simply not completing the writing exercises.**

<b>Student</b>	<b>Degree</b>	<b>Language</b>	<b>Prepared</b>
<b>Group A</b>			
DM	SCIB13	English	No
RSS	SCIB13	English	Yes
<b>Group B</b>			
OM	SCIB06	Not English	No
MM2	COMB03	English	No

## Appendix 2 Tutorial questions used in writing exercises

### Group A

Tutorial 15 (week 2) not submitted

A water trough has a vertical cross-section in the shape of a semicircle of radius 20cm. The trough is 1m long. Water is flowing into the trough at a rate of 1litre/second. How fast is the water level rising when the depth of water in the trough is 10cm?

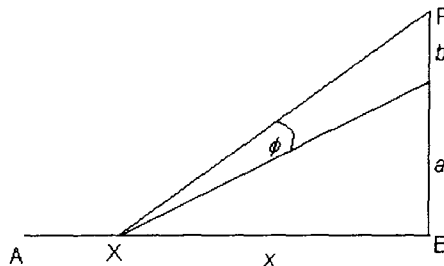
*First: make as intelligent a guess as you can as to your result. Keep it brief.*

*Next: Do the problem*

*Last: Write a short paragraph on your problem-solving process. Refer back to your guess and say whether your answer matches. If not, try to understand why.*

Tutorial 16 (week 3) not submitted

A rugby penalty must be kicked from a point on the line  $AB$ . The kicker chooses a point  $X$  on  $AB$  to maximise angle  $PXQ$  (where  $P$  and  $Q$  are the posts). Determine the value of  $x$ .



*Part 1: estimate your answer*

*Part 2: do the actual calculation*

*Part 3: Write a short paragraph on what you did and why.*

*Part 4: Comment on your original estimate*

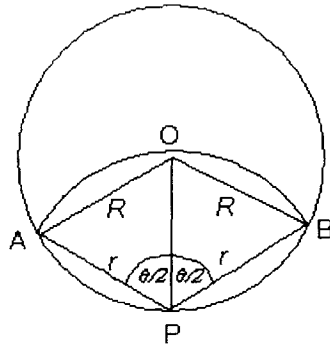
Tutorial 17 (week 4) not submitted

(There were no true problems, but, since the tutorial was on Newton's Method, there were several questions where estimation of answers could be made. This week's writing exercise concentrated on estimations, rather than written problem-solving strategies.)

2. Use Newton's Method to calculate  $\sqrt{7}$  correct to six decimal places.

3. Solve the equation  $\cos x = 2x$  correct to 5 decimal places.

6. A goat is in a circular field of radius  $R$ . It is tethered by a rope to a post on the boundary of the field. Find the length of the rope that will allow the goat to graze exactly half the area of the field.



In questions 2, 3 and 6:

First: estimate your solutions

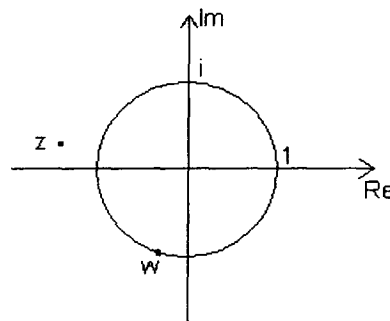
Next: Carry out the necessary calculations

Then: Compare your solutions to your estimates. Were your estimates accurate? If not, can you see why?

Tutorial 18 (week 5) submitted

Two complex numbers,  $z$  and  $w$ , are shown on the Argand diagram below. Use the position of these two numbers on the diagram to indicate where each of the following complex numbers will lie on the same diagram

... (e)  $iw$       (f)  $w - z$       (g)  $zw$



For questions 3 (e), (f) and (g), as soon as you have plotted your solution on the Argand diagram, write a sentence or two on how you figured out where to plot it. Pretend you are explaining to a fellow MAM100W student who finds complex numbers confusing. If anything strikes you as interesting, puzzling, or unexpected, make a note of it.

Tutorial 19 (week 6) not submitted

Does the polynomial  $p(z) = z^{91} + 3z^{34} + 5z - 6$  have an even number of real zeroes or an odd number of real zeroes? Carefully explain your answer.

*Q7 is already a writing question, so it will be the focus of today's writing exercise. Take special care with it, and write your explanation as if you were going to publish it a mathematics journal. Be clear. Be precise. Include everything that is needed.*

Tutorial 20 (week 7)

Find the orthogonal trajectories of the family of ellipses  $2x^2 + 5y^2 = C$ , and sketch several members of each family.

*Before you begin Q4, draw a sketch and guess what form you expect the orthogonal trajectories to take.*

*Then carry out the calculation*

*After you have finished, write a short paragraph on your solution process. Be sure to include what you did, why you did it, and whether your answer looked as expected.*

Revision tutorial (week 8) submitted

Which one of the following improper integrals converges?

$$(A) \int_0^1 \frac{1}{(x-1)^2} dx \quad (B) \int_{-1}^1 \frac{1}{(x+1)^2} dx \quad (C) \int_0^{\infty} \frac{1}{(x+1)^2} dx \quad (D) \int_0^{\infty} \frac{1}{(x-1)^2} dx \quad (E) \int_{-2}^2 \frac{1}{x^2} dx$$

*If you just glance through the available solutions, does one appear most likely? If so, which one?*

*Solve the problem, however you wish.*

*Pretend you are explaining the problem to a puzzled fellow student. Write out (in words, as much as possible) how you solved the problem.*

*Was your final answer the one you expected (if you did)? If not, can you explain why?*

Tut 21 (week 9) submitted

After Gauss reduction, the following augmented matrix was obtained.

$$\left( \begin{array}{ccc|c} 1 & 0 & 5 & 2+k \\ 0 & 1 & 3 & 5-k \\ 0 & 0 & k(k-1) & k \end{array} \right)$$

For which values of k does the system have

- i. no solution
- ii. infinitely many solutions?

*First, complete the problem.*

*Now, explain in words what your thought processes were while you were solving the problem. Try to include every detail, all the reasons for your final conclusions.*

Tut 22 (week 10) submitted

Find the inverses of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & 0 \\ 7 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

Find the determinants of the following matrices:

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 1 \\ 4 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 & 0 \\ 3 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

*Try the following steps in both question 1 and question 4:*

*Say what you notice about the given matrices.*

*Carry out the calculations.*

Describe what you notice, in words, and in a single equation. (You will need the notation  $A^T$ ; if you don't know what that means, ask the tutor.)

Tutorial 23 (week 11)      submitted

Solve for  $n$ :  $\binom{n}{7} = \binom{n}{5}$

Before any calculations, can you estimate (roughly) what size you expect  $n$  to be? If you have no idea, please say so.

After you have calculated the answer, explain what you did, in words. Was your answer roughly what you expected? If not, can you see how to improve any such estimates in the future?

Tutorial 24 (week 12)

Evaluate  $\int_0^{\frac{1}{10}} e^{-x^2} dx$  by using the first three non-zero terms in the Maclaurin expansion of  $e^{-x^2}$ .

Before calculating, estimate your expected answer. You can do this by drawing a rough sketch of  $e^{-x^2}$  and guessing the area that  $\int_0^{\frac{1}{10}} e^{-x^2} dx$  represents.

Carry out the calculation.

After obtaining a solution, explain in words what you did and why. Look back at your estimate and compare your answer to it. Were you almost right? If you were very wrong, can you figure out why?

## **Group B**

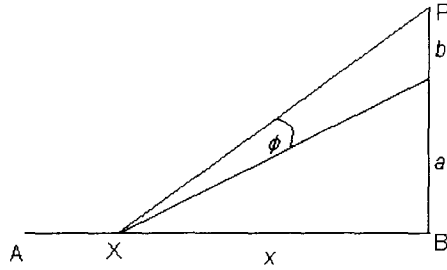
Tutorial 15 (week 2) not submitted

A water trough has a vertical cross-section in the shape of a semicircle of radius 20cm. The trough is 1m long. Water is flowing into the trough at a rate of 1 litre/second. How fast is the water level rising when the depth of water in the trough is 10cm?

Before carrying out any calculations, write an explanatory paragraph on question 3. Your paragraph should explain what you are going to do, and how you are going to do it. Use as few symbols as possible. The idea is to describe the process in words.

Tutorial 16 (week 3) not submitted

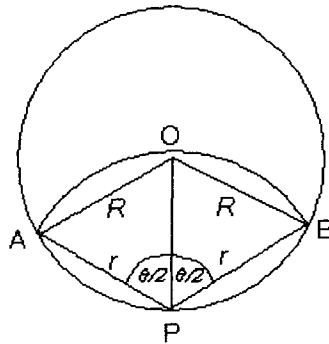
A rugby penalty must be kicked from a point on the line  $AB$ . The kicker chooses a point  $X$  on  $AB$  to maximise angle  $PXQ$  (where  $P$  and  $Q$  are the posts). Determine the value of  $x$ .



Write a short paragraph on the planned problem-solving process before attempting the question. Your paragraph should explain what you are going to do, as if to a fellow student who is not quite as good at this sort of a problem as you are.

Tutorial 17 (week 4) not submitted

A goat is in a circular field of radius  $R$ . It is tethered by a rope to a post on the boundary of the field. Find the length of the rope that will allow the goat to graze exactly half the area of the field.

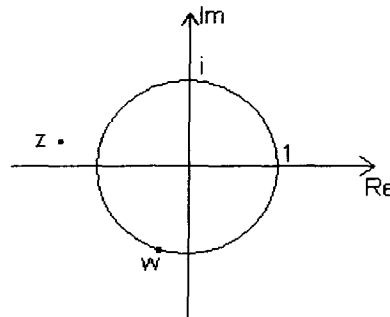


Before doing any calculations on question 6, write a short paragraph explaining what you are going to do and why you have chosen that particular method. Be clear, and try to express yourself in words, more than symbols.

Tutorial 18 (week 5) submitted

Two complex numbers,  $z$  and  $w$ , are shown on the Argand diagram below. Use the position of these two numbers on the diagram to indicate where each of the following complex numbers will lie on the same diagram

- ... (e)  $iw$       (f)  $w - z$       (g)  $zw$



In questions 3(e), (f) and (g) before attempting each of them, write down what you are going to do and why. It is important that you do the writing before you do the calculations.

Tutorial 19 (week 6) not submitted

Plot the fourth roots of  $-16$  on an Argand diagram and then write them in Cartesian form.

Write down what you are going to do and why, before actually carrying out the calculations. Think of questions a fellow student might be able to ask about your calculations and try to answer them.

Tutorial 20 (week 7) submitted

(There was no suitable question in this week's tutorial. The potential of forgetting to divide by  $x^2$  suggested that the following question might benefit from a "look before you leap" approach.)

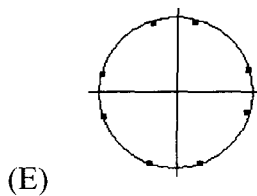
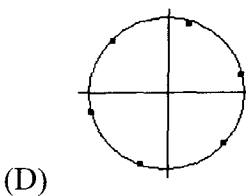
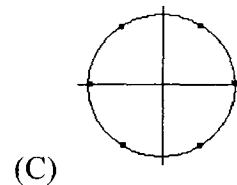
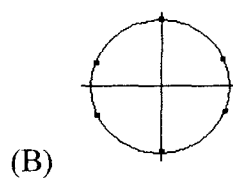
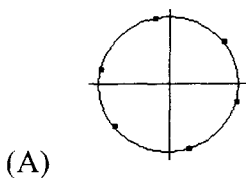
Solve the differential equation

$$x^2 y' + 5xy + 3x^5 = 0$$

Before attempting to solve the differential equation, write a paragraph to explain exactly what you are planning to do (and why, if appropriate). Be clear, be precise and don't leave anything out.

Revision tutorial (week 8) submitted

Which of the following diagrams best represents the 6<sup>th</sup> roots of  $-i$ ?



Before attempting to solve the problem, explain (as if to a puzzled fellow student) how you intend to solve the problem. As much as possible, try to use words instead of symbolic calculations.

Tutorial 21 (week 9) submitted

After Gauss reduction, the following augmented matrix was obtained.

$$\left( \begin{array}{ccc|c} 1 & 0 & 5 & 2+k \\ 0 & 1 & 3 & 5-k \\ 0 & 0 & k(k-1) & k \end{array} \right)$$

For which values of  $k$  does the system have

- (i) no solution
- (ii) infinitely many solutions?

*Before solving the problem, explain what reasoning you are going to apply. What are you going to do, and why?*

Tutorial 22 (week 10)                      submitted

Let  $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ .

Find  $A^{-1}$  and (if possible) matrices  $X$  and  $Y$  such that  $AX = B$  and  $YA = B$ .

*Before calculating  $X$  and  $Y$ , explain how you are going to solve for them. In particular, explain how you can tell whether it is possible to solve for them.*

Tutorial 23 (week 11)                      submitted

Solve for  $n$ :  $\binom{n}{7} = \binom{n}{5}$

*Without doing any calculations at all, estimate what  $n$  has to be for  $\binom{n}{7}$  to be equal to  $\binom{n}{5}$ . Explain your reasoning for the estimate. If you have no idea, ask me for a hint before giving up.*

Tutorial 24 (week 12)                      submitted

Let  $f(x) = \sin(x^2)$ . Determine  $f^{(10)}(0)$ .

*Before attempting any calculations, describe how you will find  $f^{(10)}(0)$ . It may help to realize that you don't need to do any differentiation.*

There were 13 weeks to the semester, writing exercises started in week 2. The writing exercises in weeks 2 and 3 were accompanied by an illustrative example. Week 4 was the first writing exercise carried out without any help from the tutor. Week 5 was the first one handed in. Week 13 was only a half week, attendance was very low, and no exercise was carried out in that week.

## **Appendix 3 - Documents related to ethical considerations**

### **Excerpts from The Ethical Code of Professional Conduct**

The Professional Board for Psychology

Health Professions Council of South Africa

(Obtainable from <http://web.uct.ac.za/depts/psychology/research/resm.html>, a webpage of the Department of Psychology, University of Cape Town.)

The code is “informed by international best practices and South African law, especially relevant sections of the Constitution of the Republic of South Africa” and deals with the ethics of collecting data on individuals, particularly psychometric data, and the rights to confidentiality of the participants. Paragraphs that are relevant (with appropriate rewording) to educational data gathering are given below, with numbering as in the original code, after which an ethical code for this thesis is laid out.

#### Excerpts of the code

- 1.3 Psychologists shall inform clients of the innovative nature and the known risks associated with the services or techniques, so that clients can exercise freedom of choice concerning the receipt of such services or the application of such techniques.
2. Psychologists shall respect the constitutional right of all to dignity and the right to have that dignity respected and protected and not to be discriminated against unfairly on any grounds, including age, belief, birth, colour, conscience, culture, disability, disease, ethnic and social origin, gender, language, marital status, pregnancy, race, religion, sexual orientation, or socio-economic status.
  - 2.2.1 When psychologists conduct research or provide assessment ... they obtain the informed consent of the individual, using language that is reasonably understandable to that person. The content of informed consent will vary depending on circumstances; however, informed consent ordinarily requires that the person (1) has the capacity to consent, (2) has been provided information concerning participation in the activity that reasonably might affect his or her willingness to participate including limits of confidentiality ... (3) is aware of the voluntary nature of participation and has freely and without undue influence expressed consent, and (4) has had the opportunity to ask questions and receive answers regarding these activities.
  - 2.2.4 Psychologists shall appropriately document written or oral consent, permission or assent.
  - 2.6 Psychologists shall take reasonable steps to avoid harming their ... students.
  - 2.10 Psychologists shall not exploit persons over whom they have supervisory ... authority.
3. Psychologists shall adhere to the constitutional right of all to privacy, confidentiality and the right to access to information directly affecting their lives... Psychologists shall safeguard the principle of confidentiality in all their dealings including the recording, storage, and dissemination of confidential information.
  - 3.1. Psychologists shall safeguard the confidential information obtained in the course of practice, teaching, research, or other professional duties.

- 3.2.1. Psychologists are obliged to discuss with persons and organisations with whom they establish a scientific or professional ... the relevant limitations on confidentiality.
- 3.2.2. Unless it is contraindicated, psychologists shall discuss confidentiality at the outset of the relationship and thereafter as new circumstances warrant its discussion.
- 3.3.1. Psychologists shall include in written and oral reports and consultations, only information germane to the purpose for which the communication is made.
- 3.3.2. Psychologists shall discuss confidential information obtained in their work only for appropriate scientific or professional purposes and only with persons concerned with such matters.
- 3.4.1. Psychologists may disclose confidential information only with the permission of the individual.
- 3.4.2. Psychologists may disclose confidential information with the appropriate consent of the client.
- 3.6.2. At the beginning of a professional relationship, psychologists shall inform clients who are below the age of 18 years or who have a legal guardian or who are otherwise legally dependent, of the limits the law imposes on the right of confidentiality with respect to their communications with psychologists.
- 3.9.2. When consulting with colleagues (1) psychologists shall not disclose confidential information that reasonably could lead to identification of a client, research participant, or other person or organisation with whom they have a confidential relationship unless they have obtained the prior consent of the person or organisation or the disclosure cannot be avoided, and (2) they disclose information only to the extent necessary to achieve the purposes of the consultation.
- 3.10. Psychologists shall not disclose in their writings, lectures, or other public media, confidential, personally identifiable information concerning their clients, organisations, research participants, supervisees, students, or other recipients of their services that they obtained during the course of their work, unless (1) they take reasonable steps to disguise the person or organisation, (2) the person or organisation has consented in writing, or (3) there is other ethical or legal authorisation for doing so.
- 3.11.1. Psychologists shall create, maintain, store, disseminate and retain records and data relating to their scientific and professional work in order to (1) enable efficacious provision of services by them or by other professionals, (2) allow for replication of research design and analysis, ... (5) facilitate subsequent professional intervention or inquiry, and (6) ensure compliance with law.
- 3.11.2. Psychologists shall maintain confidentiality in creating, storing accessing, transferring, and disposing of records under their control, whether these are written, automated, or in any other medium
- 5.1.1. Psychologists shall perform evaluations and diagnostic services only within the context of a defined professional relationship.
- 5.3.1. Psychologists shall obtain informed consent for assessments, evaluations, or diagnostic services, as described in Standard Informed Consent Forms, except when (1) testing is a legal requirement, (2) informed consent is implied because testing is conducted as a routine educational, institutional, or organisational activity.
- 5.5.2. Psychologists shall recognise limits to the certainty with which diagnoses, findings, or predictions can be made about individuals, especially where

linguistic, cultural and socio-economic variances exist. Psychologists shall make every effort to identify situations in which particular assessment methods or norms may not be applicable or may require adjustment in administration, scoring and interpretation because of factors such as age, belief, birth, colour, conscience, culture, disability, disease, ethnic and social origin, gender, language, marital status, pregnancy, race, religion, sexual orientation, or socio-economic status.

- 5.10.1. Psychologists who offer assessment or scoring procedures to other professionals shall accurately describe the purpose, norms, validity, reliability, and applications of the procedures and any special qualifications applicable to their use. They shall explicitly state the language, cultural and any other limitations of the norms.
- 9.6.1. In academic and supervisory relationships, psychologists shall establish an appropriate process for providing feedback to students, supervisees and trainees.
- 10.2. Psychologists shall obtain from host institutions or organisations appropriate approval prior to conducting research, and they shall provide accurate information about their research proposals. They shall conduct the research in accordance with the approved research protocol.
- 10.3. Prior to conducting, psychologists shall enter into an agreement with participants that clarifies the nature of the research and the responsibilities of each party.
- 10.4.1. Psychologists shall use language that is reasonably understandable to research participants in obtaining their appropriate informed consent. Such informed consent shall be appropriately documented.
- 10.4.2. Using language that is reasonably understandable to participants, psychologists shall inform participants of the nature of the research; they shall inform participants that they are free to participate or to decline to participate or to withdraw from the research; they shall explain the foreseeable consequences of declining or withdrawing; ... and they shall explain other aspects about which the prospective participants inquire.
- 10.8.3. Any ... deception that is an integral feature of the design and conduct of an experiment shall be explained to participants as early as is feasible, preferably at the conclusion of their participation, but no later than at the conclusion of the research.
- 10.9.1. Psychologists shall provide a prompt opportunity for participants to obtain appropriate information about the nature, results, and conclusions of the research, and psychologists shall attempt to correct any misconceptions that participants may have.
- 10.11.1. Psychologists shall not fabricate data or falsify results in their publications.
- 10.15. After research results are published, psychologists shall not withhold the data on which their conclusions are based from other competent professionals who seek to verify the substantive claims through re-analysis and who intend to use such data only for that purpose, provided that the confidentiality of the participants can be protected.

## **Ethics code for the writing study project**

1. Permission shall be obtained from the convenor of the course in which the teaching exercise will be held to collect data on the participating students. Permission will, in addition, be obtained from the head of the department. The research proposal will be outlined clearly and explained.
2. At the beginning of the teaching exercise, the purpose of the exercise will be explained to the students, as well as impact that the exercise might have on their tutorial work. Effort will be made to ensure that all students have understood the explanation.
3. The students in the two experimental groups will be given consent forms in which they will be asked for their consent to participate in the teaching experiment, and to have their work available for publication in research output, should that be an option. All students will be assured, and it will appear on the consent form, that aliases will be used, and the identity of all participants will be confidential, known only to the researcher, and destroyed after aliases have been substituted. Consent forms will be filed and retained.
4. Any students under the age of 18 will be asked to have their legal guardian approve their consent.
5. It will be made clear to the students that they are free to participate or not, and that it will not be detrimental for them to decline consent.
6. The means of assessment will be kept from the students until the exercise is completed, as knowledge of how they are being assessed might cause the students to take a different approach to the assessment tasks than they might otherwise do.
7. Assessment will be carried out by analysing chosen questions in the standard tests written by all students in the course. Permission will not be requested for such assessment since “informed consent is implied because testing is conducted as a routine educational, institutional, or organisational activity.” (5.3.1. Psychological code of ethics).
8. Since it is one of the aims of the exercise to determine whether it has differing levels of success due to the languages, cultures, or any other relevant factors, all assessment will be studied with such factors in mind, and any research output will make influential factors clear.
9. It shall be made clear to the students that any queries they have on the data and on how the experiment was carried out will be answered in full after the research is completed, that is, after the end of the second semester 2004. All participating students will be allowed access to the final data, if they so request. In the situation that they struggle to understand the data, it shall be explained to them, possibly by access to the completed thesis in the following year. The means of assessment will be explained to students after the research is completed, should they inquire.
10. No data will be falsified, and any contributions by other professionals or by participating students will be acknowledged.
11. All participating students will be acknowledged, although by their aliases.
12. At all times the constitutional rights and rights to confidentiality will be respected.

## Appendix 4 – Questions in assessment tasks analysed for evidence of problem-solving behaviour

### Test 1

Find, in two different ways, the equation of the plane that consists of points equidistant from  $(3,-1,5)$  and  $(-5,3,7)$ . Explain your reasoning. [5]

*Context: The test had 20 multiple choice questions of 5 marks each, as well as the fully written question given above, also for 5 marks. The time allocated to the entire test was approximately 100 minutes, suggesting roughly 5 minutes for the written question. In practice, since the multiple choice questions varied in difficulty, time allocation by level of difficulty might have suggested 8 to 10 minutes for the written question*

### Test 2

A circle of radius 1 has its centre on the positive y-axis, and touches the parabola  $y = x^2$  at two points. What are the coordinates of the centre of the circle? [5]

*Context: The test had 20 multiple choice questions of 5 marks each, as well as the fully written question given above, also for 5 marks. The time allocated to the entire test was approximately 100 minutes, suggesting roughly 5 minutes for the written question. In practice, since the multiple choice questions varied in difficulty, time allocation by level of difficulty might have suggested 8 to 10 minutes for the written question*

### Test 3, first question

Prove that the line through the points  $(1,0,4)$  and  $(7,6,-2)$  is tangent to the sphere  $x^2 + y^2 + z^2 + 2x - 4y - 2z = 8$ , and find the point of tangency. [8]

*Context: The total number of marks attainable was 50, and the time allocated in total was approximately 100 minutes, suggesting 16 minutes available for this question.*

## **Document accompanying students' consent forms**

### Writing as Problem Solving

I am an experienced lecturer and tutor of mathematics, having been teaching at UCT for 7 years and tutoring for 11 years. I have a Masters degree in Mathematics Education and have made in depth studies of many aspects of mathematics education. The current focus of my research is on whether the act of writing about mathematics helps students in being more successful mathematical problem solvers.

In the second semester 2004 I shall be collecting data for my PhD studies in maths education. In the tutorial classes for which I am the tutor, I shall be illustrating the usefulness of using writing as a problem solving tool within mathematics. The writing exercises will not take more than ten minutes in any tutorial period, and will not interfere with the normal running of the tutorial in any way. In all other respects the tutorial class will proceed as normal. At several points in the semester, I shall ask you to hand in your writing, which will not be marked (I shall comment on them constructively and return them to you), and will not affect your mark in MAM100W.

There is a possibility that I may want to use something you or other students have written in my thesis. If that is the case, I am required to have your permission and will guarantee a pseudonym, so that you are not identifiable. There is also a possibility that I may want to interview you.

If you wish to see the results of my data collection, I can make them available to you towards the end of the summer vacation, in February 2005. My completed thesis should be available for you to read after September 2005, although it will not yet have been passed by my examiners. At any point in the semester if you wish to discuss what use I am making of my data, you are welcome to come and talk to me about the data or the study. It is my desire that all matters pertaining to the writing exercise be completely transparent to you, while not affecting the validity of the data.

Tracy Craig

## Consent Form

I, ....., hereby accept/decline to take part in the writing exercises occurring in my tutorial group. In addition I give/deny permission to be quoted in resulting research literature, understanding that my identity will not be disclosed.

I understand that I may be requested to give an interview about my experiences of learning mathematics. I am aware that I may withdraw from the tutorial group at any time, and that my decision to participate or decline will not affect my standing in the course or the department

Signed ..... Date.....

Due to varying difficulty among the questions, such time allocation should not be interpreted strictly.

### **Test 3, second question**

Without calculating the integrals below, arrange them in increasing order of magnitude.

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \arctan x dx \quad \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x dx \quad \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cot x dx \quad \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2 \csc x dx$$

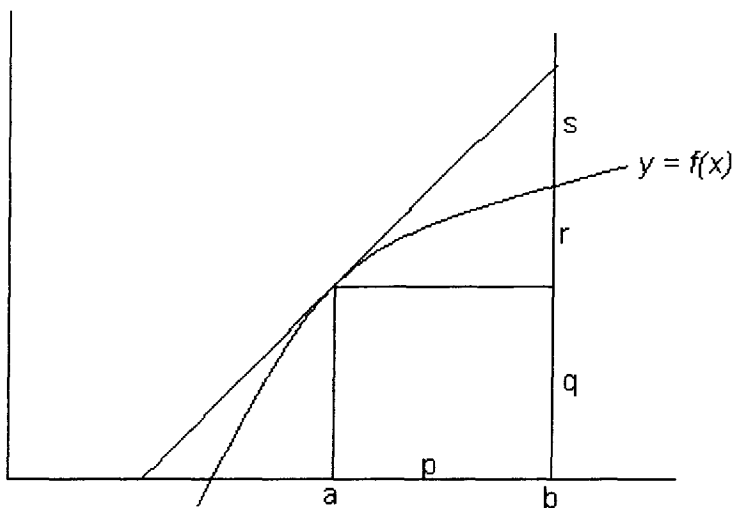
[7]

*Context: The total number of marks attainable was 50, and the time allocated in total was approximately 100 minutes, suggesting 14 minutes available for this question.*

*Due to varying difficulty among the questions, such time allocation should not be interpreted strictly.*

### **Test 4**

The figure shows the graph of a continuous function  $y = f(x)$  and the tangent to the curve at  $x = a$ . Lengths  $p$ ,  $q$ ,  $r$  and  $s$  are marked on the graph.



Express each of the following in terms of  $p$ ,  $q$ ,  $r$  and  $s$

(i)  $\lim_{x \rightarrow a} f(x)$

(ii)  $\int_a^b f(x) dx$

(iii)  $\int_a^b f'(x) dx$

(iv) Express  $s$  in terms of integrals and derivatives [7]

*Context: Test 4 consisted of 50 marks worth of multiple choice questions (20 questions at 2.5 marks each) and 50 marks worth of fully written questions. The total time allocated was 3 hours. Dividing time by marks, unaffected by difficulty of the question, would suggest approximately 14 minutes for the question above.*

Mid year break. The intervention began in the second semester

### **Test 5**

A solid iron sphere rests on the base of a cylindrical container with a circular base of radius 10cm and a height of 30cm. Water is poured into the cylinder until the sphere is covered. The size of the sphere is such that the amount of water is maximised. What is the largest amount of water needed? [6]

*Context: The total number of marks attainable was 50, and the time allocated in total was approximately 100 minutes, suggesting 12 minutes available for this question. Due to varying difficulty among the questions, such time allocation should not be interpreted strictly.*

A further test, Test 6, was scheduled, but no fully written question was able to be included

### **Final examination, first question**

Evaluate the integral  $\int_0^1 x \ln x dx$ . [8]

*Context: the entire examination totalled 100 possible marks, and the time available was 3 hours. Ignoring relative difficulty of questions, approximately 14 minutes is suggested for this question.*

### **Final examination, second question**

Prove that the lines

L1: through  $(0,1,5)$  and  $(7,-9,9)$

L2: through  $(1,1,7)$  and  $(2,-1,7)$

intersect and find the point of intersection.

[8]

*Context: the entire examination totalled 100 possible marks, and the time available was 3 hours. Ignoring relative difficulty of questions, approximately 14 minutes is suggested for this question.*

While marking the tests and analysing the students' evident problem solving skills, it was noted that the second question chosen for analysis from the final examination was not a good choice. The fact that it was algorithmic to the students and not literally a problem at all should have been expected before hand, but the reason for its choice was the danger of using the same parameter for each of the lines and ending up with an inconsistent solution. The students had encountered this type of problem before, though, and very little problem solving behaviour was discernable. A difficulty with the final examination is that it was set without this research project in mind, and had very little scope for problem solving. The integral question, though, was quite a good question for determining problem solving skills as its identity as an improper integral was not made known to the students.

## Appendix 5 – Interview question analysis

### Question 1

I obviously know you're studying maths, what else are you studying this year? What are your majors? Are you intending to do postgraduate work? What is your favourite subject?

Question 1 was an ice breaking question. There is nothing to analyse. Not enough of the students mentioned postgraduate studies to have anything to analyse there.

### Question 2(a)

What would you say MAM100W was about?

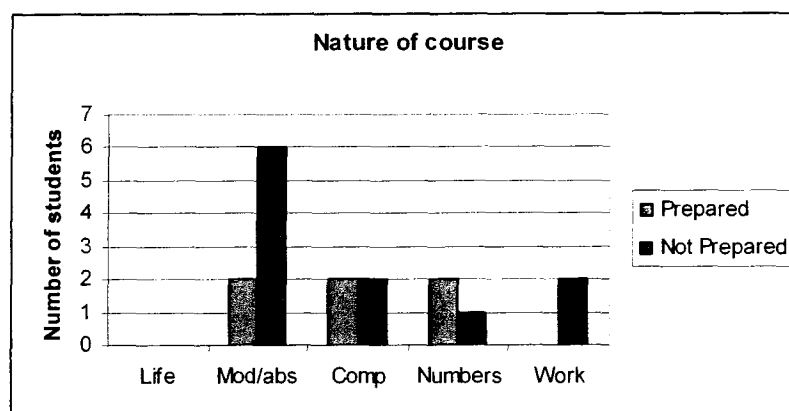
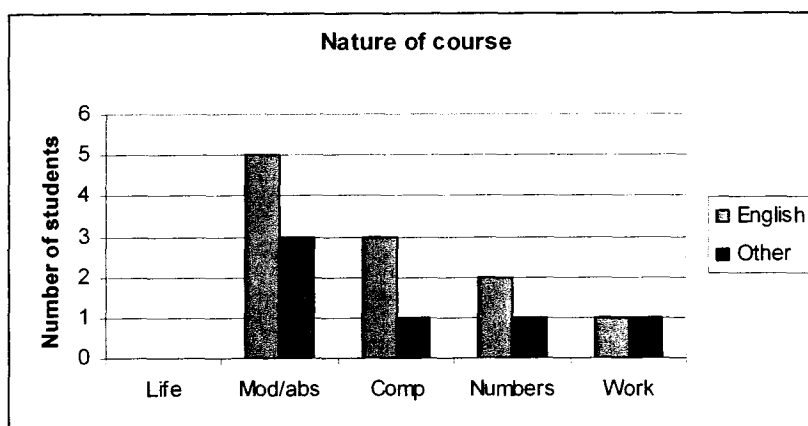
		Prepared	English
TJ	Number	No	No
CM	- "it is hectic"	No	No
IT	Modelling/abstract	No	No
IS	Modelling/abstract	No	Yes
DL	Modelling/abstract	No	Yes
DC	Components/toolbox	Yes	Yes
RG	Components/toolbox	Yes	Yes
CP	Numbers	Yes	Yes
JS	Modelling/abstract? "to give a grounding"	Yes	Yes
AS1	Numbers	Yes	Yes
DB	Components/toolbox	No	Yes
MS	Modelling/abstract (with help from 2b)	No	Yes
WT	Modelling/abstract (with help from 2b, but weak)	No	No
AJ	Modelling/abstract	Yes	Yes
NW	Components/toolbox	No	No
AS2	Modelling/abstract (with help from 2b)	No	No
LM	- "a lot of work"	No	Yes

Wood et al (2006) ask the question "What is mathematics?" to university students, and categorised their responses as

- Life
- Modelling/abstract
- Components/toolbox
- Number

These categories for the nature of mathematics fitted well with the responses given to the interview question "what was MAM100W about?" except that two students focussed on the work load of the course rather than its nature. To categorise responses to the interview question, Wood et al's categories were used, in addition to one on work load.

	English	Other	Prepared	Not Prepared
Life	0	0	0	0
Modelling/abstract	5	3	2	6
Components/toolbox	3	1	2	2
Numbers	2	1	2	1
Work aspect	1	1	0	2



### Question 2(b)

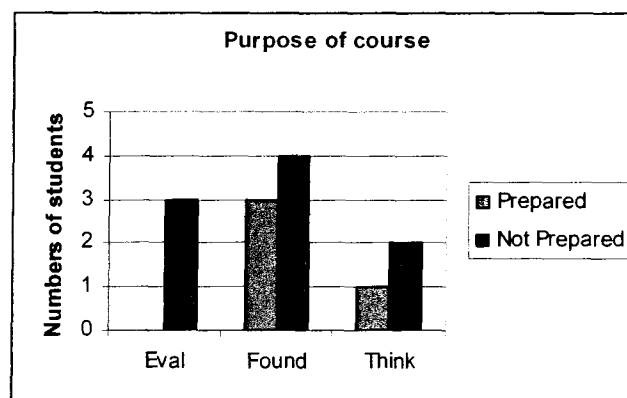
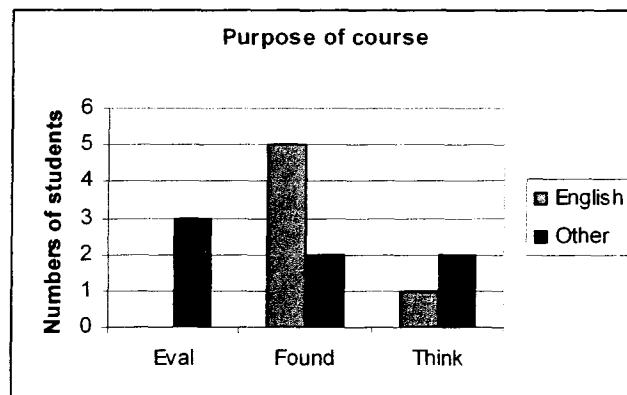
What purpose does the maths department have for offering the course?

Student responses took three forms

- Evaluative – the mathematics course is seen as a means of evaluating students' mathematical skills, to assess whether they should be attending university. Mathematics is seen as a “gatekeeper” subject
  - “to see what level you’re in maths, to see your background in maths. To make an evaluation, to whether you can make it at university maths” (CM)
  - “I think perhaps it’s designed to see if you’re good enough for university, to weed people out” (NW)

- Foundational – the mathematics course is seen as providing grounding for further studies in mathematics and for further work in other courses such as physics; the content is the focus.
  - “I think the purpose of mathematics is to get the applied side, the numerical side and the fundamental mathematics before going into the abstractness of it. Lay a foundation.” (DL)
  - “it’s important because most other courses require it, like physics” (MS)
- Way of thinking – the processes of mathematics and process of mathematical thinking are the focus of the “way of thinking” response.
  - “to encourage a way of thinking, of understanding, a way to approach mathematics.” (CP)
  - “To teach students to be able to think analytically, and to think hard about real life problems.” (IT)

	English	Other	Prepared	Not Prepared
Evaluative	0	3	0	3
Foundational	5	2	3	4
Way of thinking	1	2	1	2

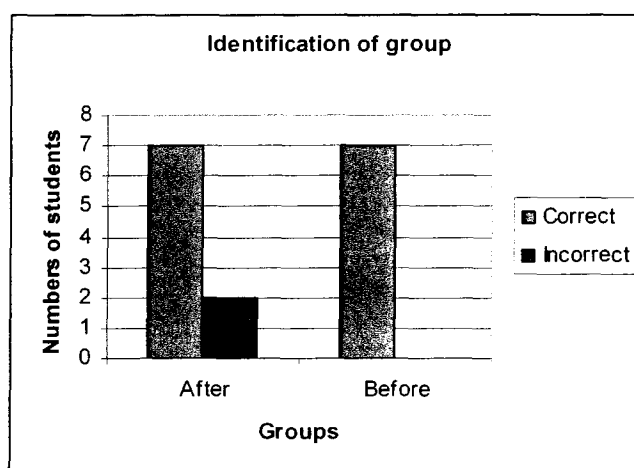


### Question 3

Two different sorts of writing assignments this semester were organised in two tut classes. In one, writing about problems after the calculation was emphasised, with perhaps a brief mention of expectations before the calculation, and in the other writing about problems before attempting any calculations was emphasised. You were in one of those groups. Can you identify which one?

All students in the “Before” group correctly identified their group. Two of the “After” group students did not identify their group correctly

	Correct	Incorrect
After	7	2
Before	7	0



### Question 4

Did you find the writing exercises easy or difficult to do? In what way?

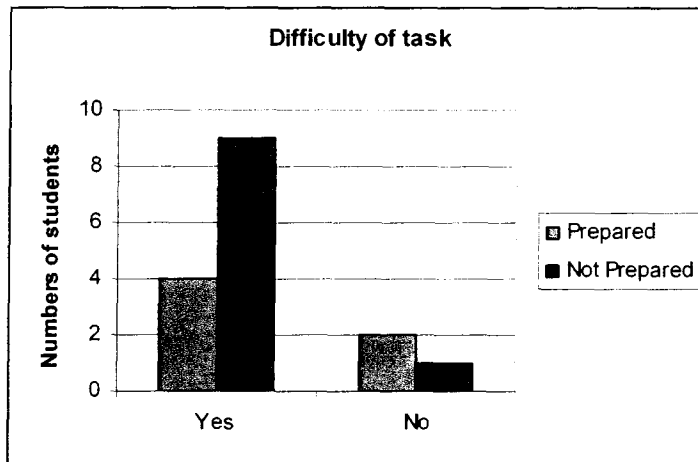
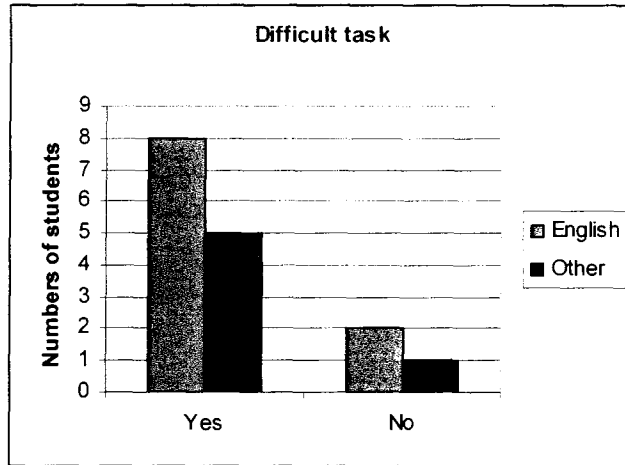
Student responses took four forms

- Yes, they were difficult
- Sometimes difficult, sometimes easy
- Only to begin with were they difficult
- No, they were not really difficult

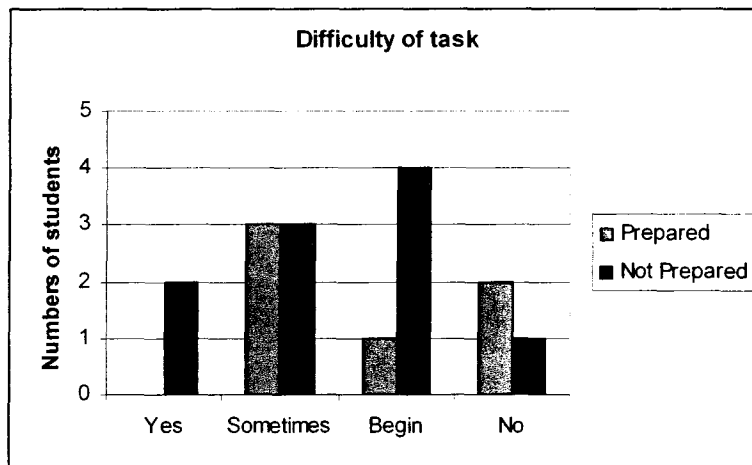
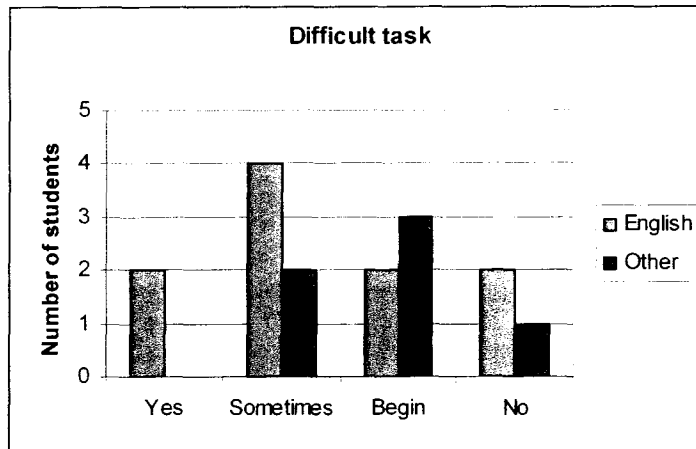
The student responses were tabled as the separate responses given above, but also grouping the first three together as “Yes” and only “not really” as “No”. Most of the students gave reasons for finding the writing exercises difficult, these reasons taking four forms

- communication, explaining mathematical concepts in words
- writing combined with mathematics is too strange a concept
- it is the thought required before beginning that is difficult, not the writing itself
- the difficulty of the writing exercises depended on the mathematical topic of the question, and how well the student knew the work

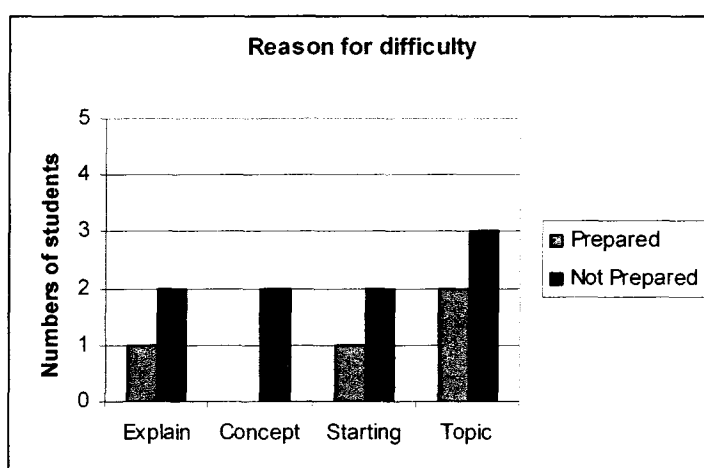
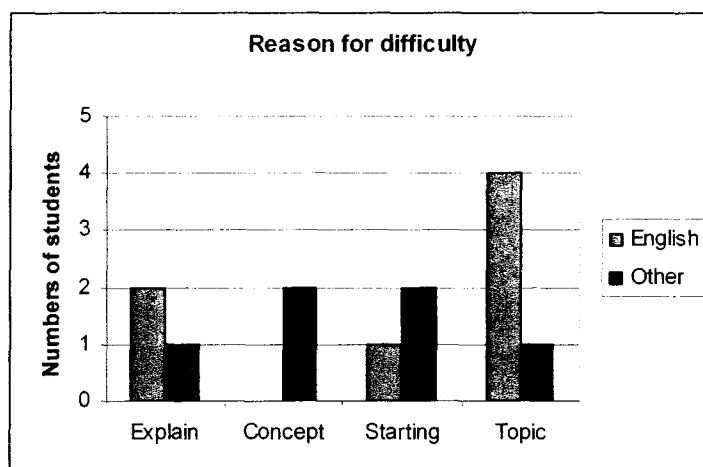
	English	Other		Prepared	Not Prepared
Yes	8	5		4	9
No	2	1		2	1



	English	Other		Prepared	Not Prepared
Yes	2	0		0	2
Sometimes	4	2		3	3
Yes, to begin with	2	3		1	4
No	2	1		2	1



	English	Other	Prepared	Not prepared
Communication, explaining in words	2	1	1	2
Too strange a concept	0	2	0	2
The thought required before starting	1	2	1	2
Depended on the topic	4	1	2	3



### Question 5

Did the writing exercises help you think about the problem?

The student responses were too varied to allow for grouping or significant analysis. Responses included

- the writing exercises save time in the long run
- the writing exercises help you not panic in a test situation
- the writing exercises help with revision, when you have your own explanations to read
- forceful terms were used by several students: “forces me to think” (IT)
- the writing exercises encourage deeper thinking, a deeper approach to problems
- the writing exercises help develop the mathematical register
- the practice of processes
- the communication process makes you understand them more deeply
- translating between different representations helps learning

The two aspects isolated for comment are the time saving aspect and the aggressive wording of “forcing you to think”.

	Various interesting ideas	Prepared	English
TJ	Saves time, don't panic	No	No
CM	helps with revision	No	No
IT	Saves time, "forces me to think"	No	No
IS	Forced more research "forced me to think"	No	Yes
DL	-	No	Yes
DC	"forced you to think"	Yes	Yes
RG	Go more deeply, terminology	Yes	Yes
CP	"they kind of force you to"	Yes	Yes
JS	Save time	Yes	Yes
AS1	-	Yes	Yes
DB	-	No	Yes
MS	Save time	No	Yes
WT	Save time	No	No
AJ	Practising the process	Yes	Yes
NW	Communication makes it clearer	No	No
AS2	Different representations	No	No
LM	revision	No	Yes

	English	Other	Prepared	Not Prepared
Saving time	2	3	1	4
Aggressive language	3	1	2	2

### Question 6

Did you ever find that the writing exercises helped you outside the tutorials, like doing homework?

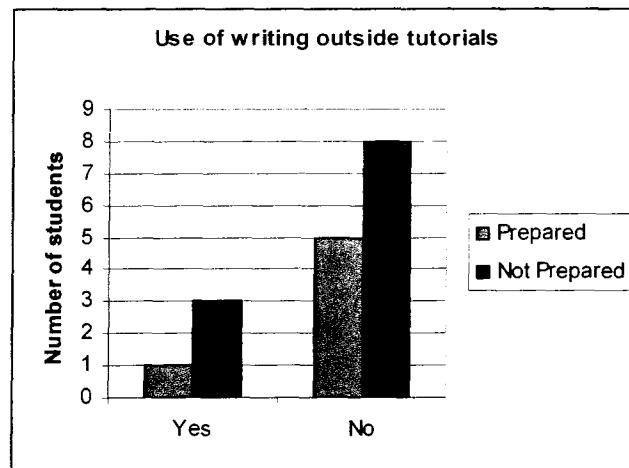
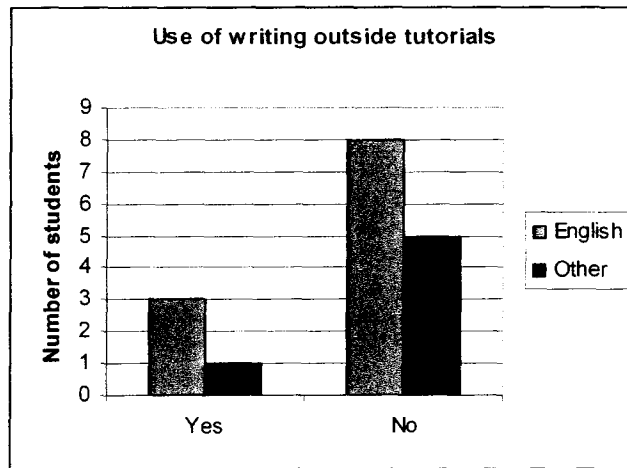
Student responses took two basic forms, simply No and Yes. There were four "Yes" responses, which, in more detail, were

- "Sometimes yes, especially with application of differentiation. I'd say "here's what I have", I'd draw my diagram to have something to remind me of where I am heading, "how am I going to solve it?", I'd write down things like formulas in point form. What I'm going to do and how I'm going to do it. It was helpful" (IT)
- "Yes, I was going through my tuts, and I got to a question where I was thinking "well, I have an idea, but not anything that's clear", so I wrote it out, to help my thoughts, and it just made everything clearer. It was all fuzzy in my mind before, in the back of my mind ... So I started writing. I thought "what do I want? ... I was writing out, direction of line, parametric equations. I visualised the plane, plug in the parametric equations, find the parameter and you've got your point. I just wrote all that stuff down and found I had everything sorted out" (AS1)
- "Yes, it really helps. Especially with abstract things like complex numbers and vectors. It helps to write about them on paper, to write down the

“characteristics of the thing”, the facts about them” (DB, not a direct quote, except bit in quotation marks, transcribed from field notes, not audio tape)

- “Yes, like now lately I’ve just (started to think about maybe) write about it. Before I wouldn’t” (LM, words in parentheses slightly blurred on audio tape)

	English	Other	Prepared	Not Prepared
Yes	3	1	1	3
No	8	5	5	8



Question 7

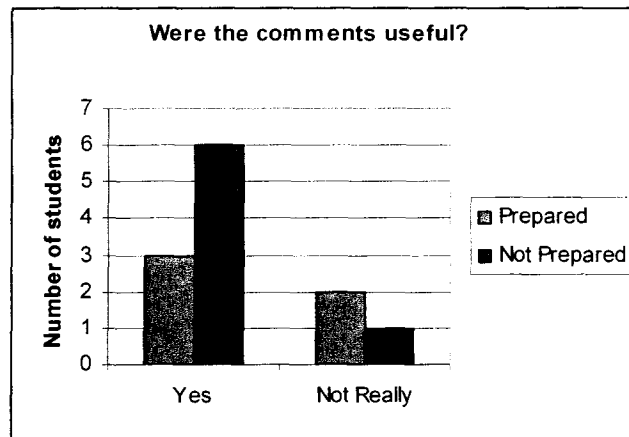
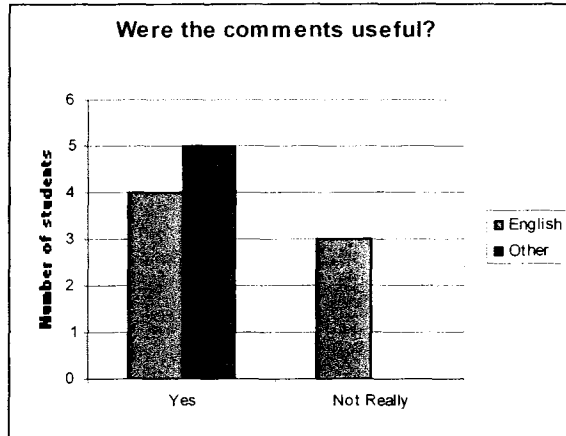
How often did you read the comments that were made on the writing exercises when they were handed back? Always? Never? Sometimes? (If yes) Did you find them helpful in any way?

All student responses were positive, either “yes” or “sometimes”. It is unclear whether the people answering “yes” meant that they read them all the time, or only sometimes, so no breakdown was done by yes/sometimes responses.

When asked whether the comments were useful, student responses took the two forms “yes” and “not really”. The reasons given were too few in number and too varied to allow for analysis, although the responses are tabulated below.

	English	Other		Prepared	Not Prepared
Provided motivation	1	1		1	1
Helped with the content	2	1		2	1
Helped later with exam revision	2	0		1	1
The challenge was helpful	1	0		1	0
The feedback provided was helpful	1	0		1	0
Helped with the writing process	0	1		0	1

	English	Other		Prepared	Not Prepared
Yes	4	5		3	6
Not Really	3	0		2	1



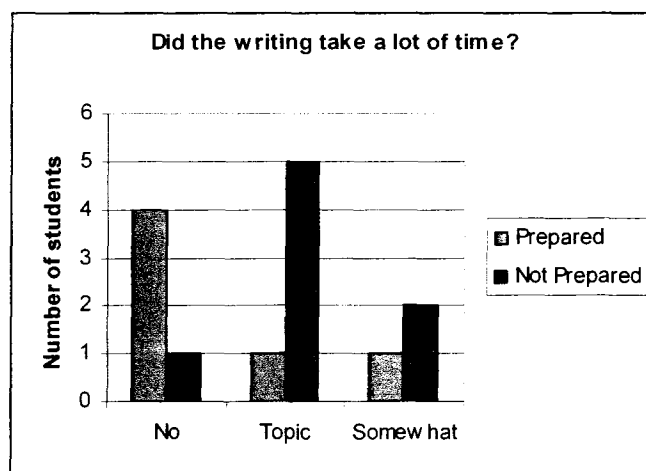
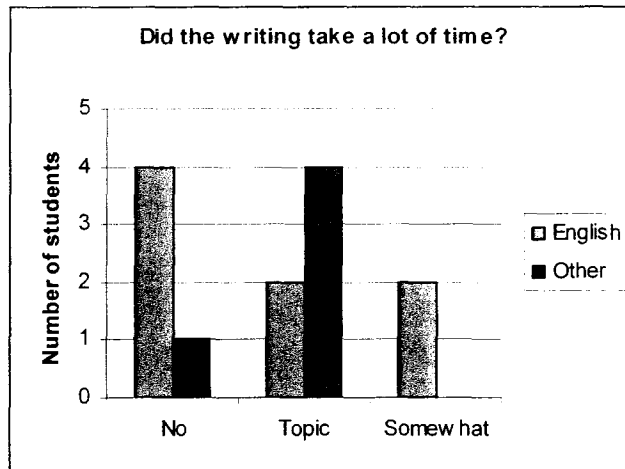
**Question 8**

Did you feel that the writing exercises took up a lot of time?

Student responses took three forms

- No, or not really
- They took "quite a bit of time" (DL)
- The time was dependent on the mathematical content of the question and how well the student knew the work.

	English	Other	Prepared	Not Prepared
No, not really	4	1	4	1
Topic dependent	2	4	1	5
Quite a bit	3	0	1	2



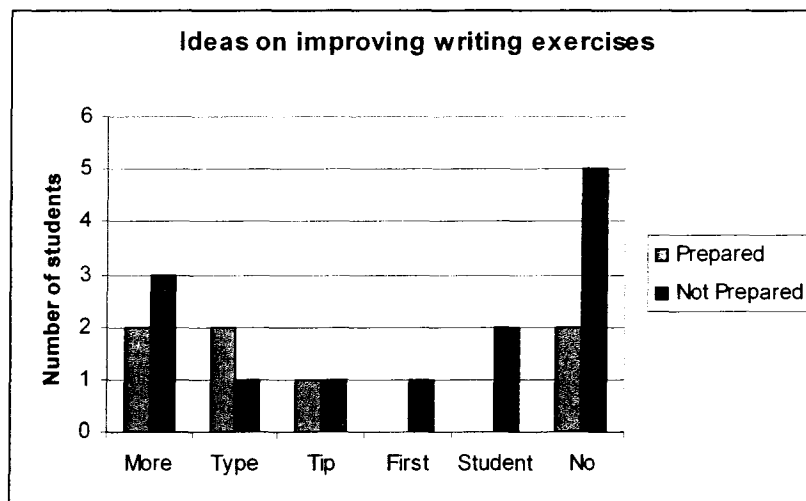
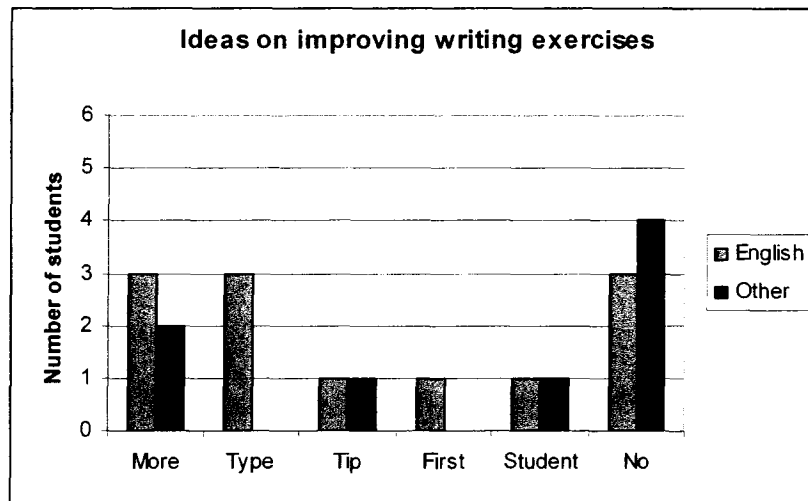
### Question 9

Can you think of any ways in which the writing exercises could be made to be more useful to you?

Student responses took six forms, rather too many to deduce anything from the results. Some students gave more than one suggestion, both of which were counted.

- There should be more writing exercises, either more frequently encountered, or more per tutorial session
- The writing exercises should be associated with a particular type of question: “understanding questions” (JS), “harder questions” (AJ), questions that “start you to think more” (LM)
- The writing exercises should give a tip as to how to approach the solution
- The writing exercises should be associated with the first question in the tutorial (only one student gave this response)
- For the writing exercises to be more effective, it is up to the students to change their attitudes, not up to the tutor to change the form of the writing exercises.
- No, cannot think of any way to make the writing exercises more effective

	English	Other	Prepared	Not Prepared
More of them	3	2	2	3
Type of question	3	0	2	1
Give a tip	1	1	1	1
First question	1	0	0	1
Up to student	1	1	0	2
No	2	2	1	3



**Question 10**

What would you describe as your major strengths in mathematics? Major weaknesses?

Student responses varied widely, almost a different response for every student interviewed, therefore there is no point in tabulating the responses. Responses included

- enjoying and conversely not enjoying problems of a visual nature

- particular content likes and dislikes, such as graphs, complex numbers, matrices, integration, vectors
- individual issues, such as being lazy, studying methods or time management
- cognition issues such as “thinking laterally” (DC), “improvising” (NW)

### Question 11

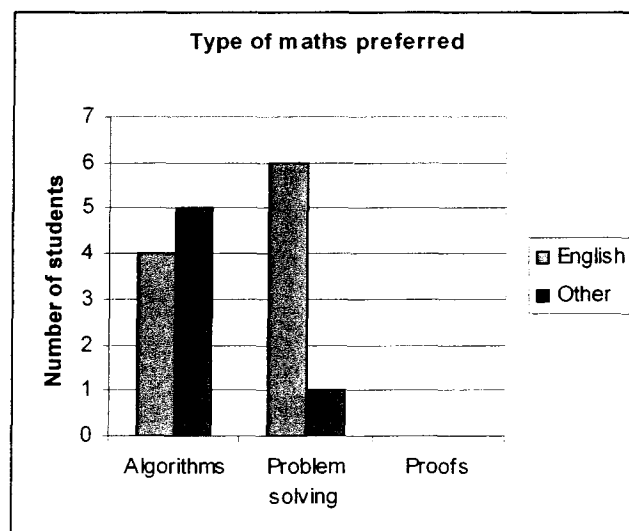
There are two major types of mathematics that you encounter in this course. There are the rules, like “differentiate this” “use the cross product” “apply Newton’s Method”, and then there are the kinds of problems where you need to think for a bit before you know what to do. Like that one we did in the first test with the parabola and the circle that had to fit inside it. Solving that sort of thing is called problem solving, and it is what the rules are made for, really. Problem solving involves applying rules, but it also involves solving a puzzle, although sometimes only a little one. If you compare problem solving with using rules (like differentiating, doing cross product, using Newton’s Method, etc.) which sort of maths do you prefer? Why? [In an early interview, a student mentioned proofs, so in subsequent interviews I included proofs as a 3<sup>rd</sup> section]

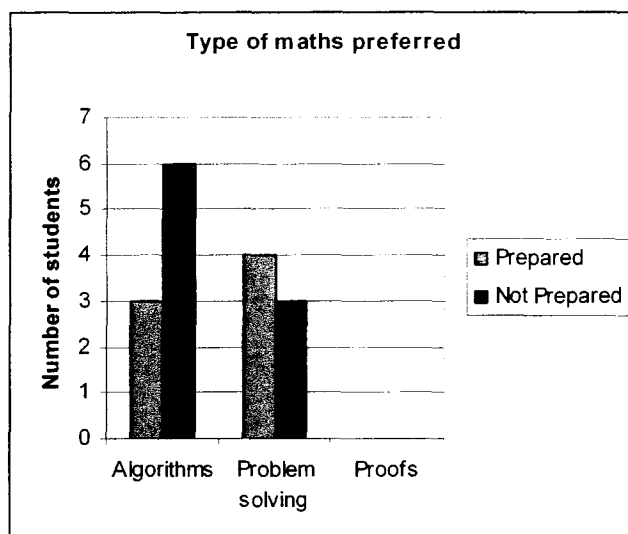
Student responses took three forms

- rules, or algorithms
- problem solving
- one student replied “both in good proportions” upon which he elaborated. This student (AS1) is tabulated as having replied positively to both algorithms and problem solving

No student chose proofs or theory as their preferred form of mathematics.

	English	Other	Prepared	Not Prepared
Algorithms	4	5	3	6
Problem solving	6	1	4	3
Theory, proofs	0	0	0	0





**Question 12**

What would you say the aim of the writing exercises was? What do you think they were meant to achieve? Would you say that the writing exercises were a success in that regard?

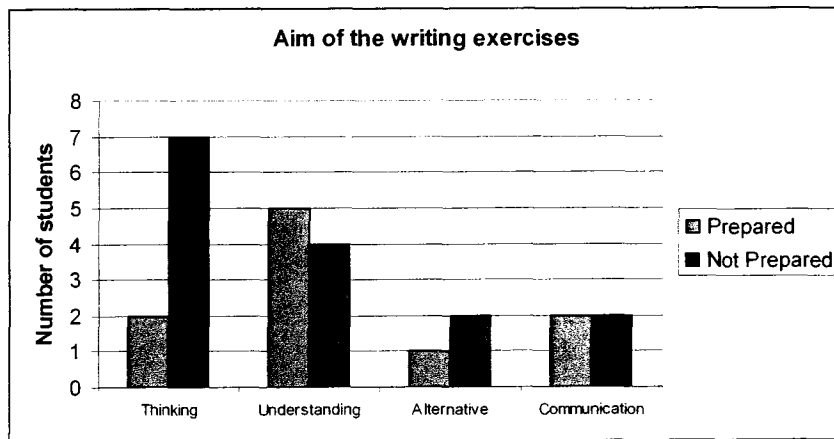
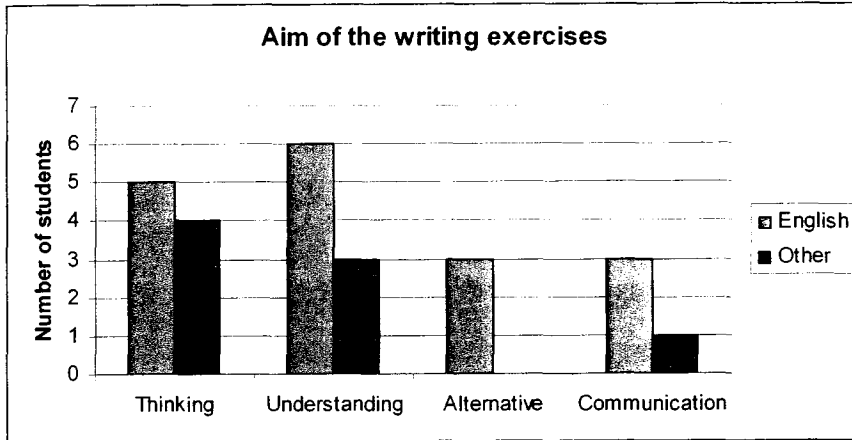
Student responses took four forms, two of which could be regarded as sufficiently similar to group together.

- Thinking: thinking about problems, organising thoughts, thinking clearly
- Understanding: to understand problems better
- Alternative: To find alternative or better ways to solve problems
- Communication: to serve as a form of communication between the student and the tutor, and to practise explaining one's thought processes clearly

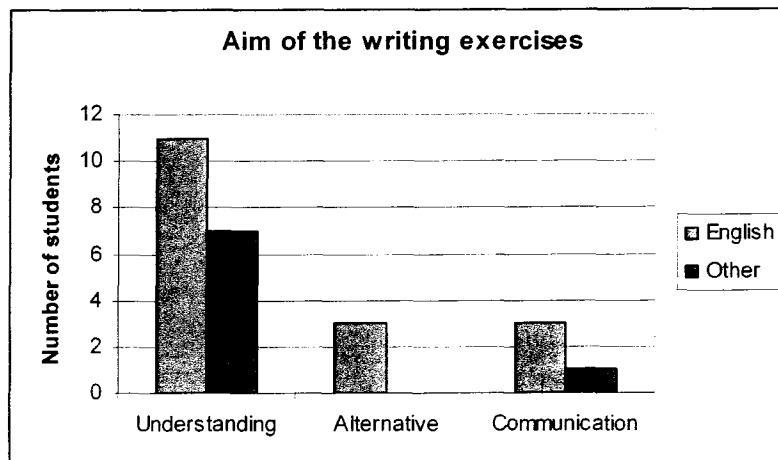
Thinking and understanding could be categorised together as “understanding” e.g. “they would help you formulate your thinking, and get a clearer idea of what you are doing as well as what the problem requires” (AJ)

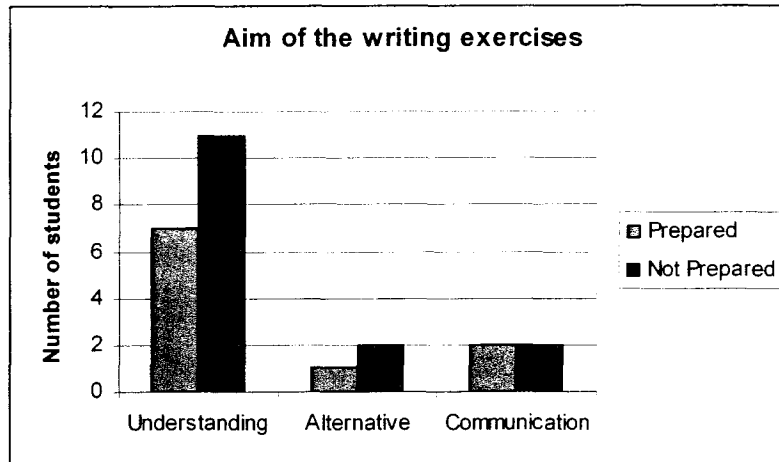
In the tabulations below, thinking and understanding are first considered separately, and then grouped together.

	English	Other	Prepared	Not Prepared
Thinking	5	4	2	7
Understanding	6	3	5	4
Alternative solutions	3	0	1	2
Communication	3	1	2	2



	English	Other	Prepared	Not Prepared
Understanding	11	7	7	11
Alternative solutions	3	0	1	2
Communication	3	1	2	2





### Question 13

Do you think the writing exercises have made any difference to your actual marks in this course? To anybody else's marks?

All students answered in the negative to this question, except for AJ, whose interview was directly after the multiple choice final examination, in which she found herself eliminating potential answers to a question using thought processes which she felt she had learnt through the writing exercises.

### Question 14

Do you think you would have learnt as much as you have in this course if you had just done the tut and not been involved in the writing exercises? Did you learn more because of the writing exercises?

The answers to this question were all positive, but vague. Responses were either of the form "I think it did make a difference" (DC), "It wasn't a huge leap of extra help, but it was useful" (JC) or problem specific, such as "I also learnt something, especially matrix multiplication" (CM), "I probably would still not know what an ellipse was" (AJ). No analysis was suggested by the responses.

### Question 15

What languages do you speak? Which language do you regard as your main language, the one you speak at home, and are most fluent in/best at?

Student responses to this question were used to determine main languages. Analysis of the interview questions involved two different ways of grouping the students, one by main language (English and Other) and the other by preparedness (Prepared and Not Prepared). Student responses to interview question 15 were used to determine the language grouping.

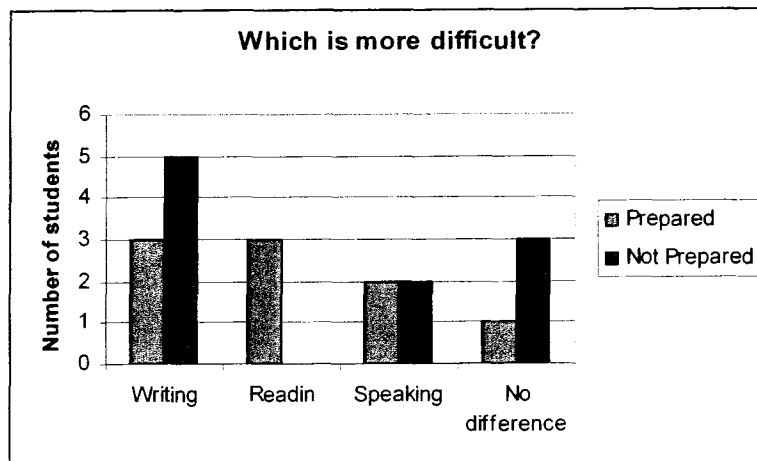
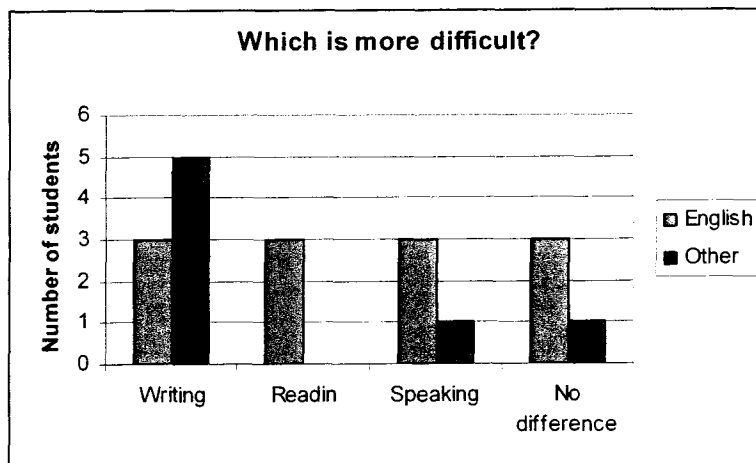
**Question 16**

Speaking, reading and writing are all a bit different. Do you find any of these 3 skills harder than the other 2?

Student responses took the forms

- writing
- reading
- speaking
- no difference
- combinations of writing, reading and speaking (tabulated multiply)

	English	Other	Prepared	Not Prepared
Writing	3	5	3	5
Reading	3	0	3	0
Speaking	3	1	2	2
No difference	3	1	1	3



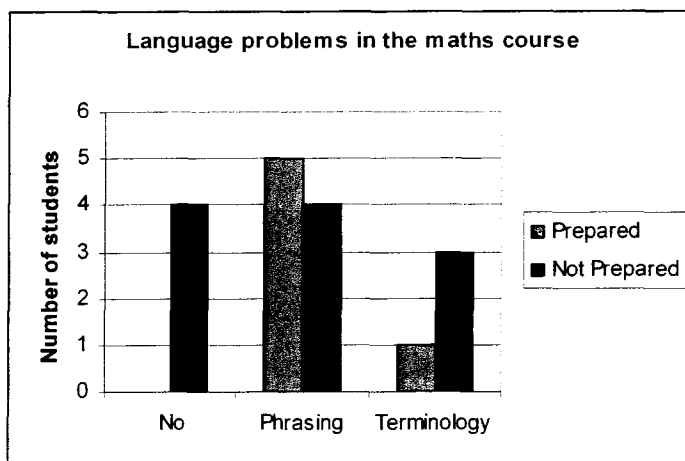
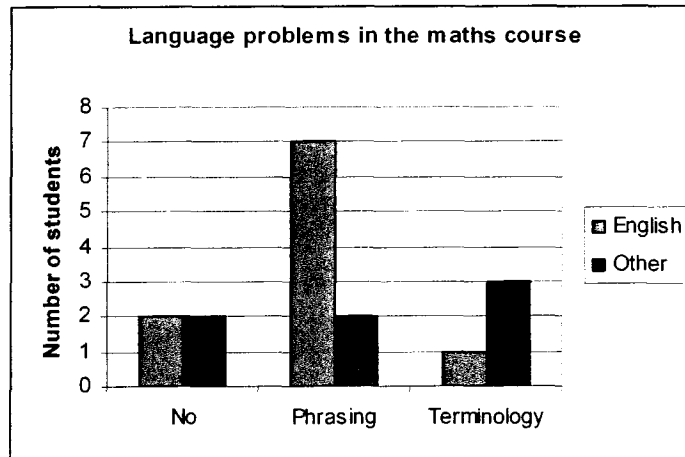
### Question 17

Have you ever had any language problems in the mathematics course? Any kind at all?

Student responses took three forms

- the phrasing of questions can sometimes be obscure, confusing or ambiguous
- mathematical terminology can be difficult
- no language problems experienced

	English	Other	Prepared	Not Prepared
No	2	2	0	4
Phrasing	7	2	5	4
Terminology	1	3	1	3



### Question 18

There are different schools of thought on mathematics and language. Some people say that mathematics is a whole new language and its difficulties are the same for everybody. Other people say that if it is taught in English and you don't speak English as a first language, then the new language difficulties of mathematics make it even

harder for you than for English speakers. Do you have any views on that contrast of opinion?

Student responses took three forms

- No, it is not a problem if you are not a main language English speaker
- There are minor difficulties if you are not a main language English speaker
- Yes, it makes the subject a lot more difficult if you are not a main language English speaker

The “No” responses took two forms

- it is unavoidable “well, to those people who say that maths is more difficult if not taught in your own language, in what language would maths have been taught to them? Maths has to be taught in English, I mean, really!” (CM, Shona speaker)
- the symbolic nature of mathematics is much more important “no, it doesn’t make a difference. You look at the formula and it’s the same in Afrikaans and English.” (WT, Afrikaans speaker)

The “Minor” responses took a variety of forms

- word problems are a difficulty “it’s not going to matter unless you have a long story sum or a word sum.” (IS, English speaker), “Like say we’re just going to do calculations, like just differentiate the questions, I don’t think English would be a problem, but then with a lot of a lot of word solving problems those people that don’t have English as a first language will find a bit more difficulty there.” (AS2, Mandarin speaker)
- explanations of otherwise symbolic content will be a problem “I think both 1<sup>st</sup> and 2<sup>nd</sup> language people could be ok. If they can see an equation or a proof and understand it just by looking at the maths, then fine. But if it needs to be explained first, then obviously it’s not going to be the same.” (DC, English speaker)
- There are different difficulties for both language groups, with main language English speakers actually speaking English more casually sometimes than second language speakers, and also potentially having predetermined, and incorrect, understanding of a word in the mathematics register.  
“AS1: I think, even if you’re learning maths in English and you are fluent in English, it can be a disadvantage, you can come to maths with the understanding of the word that is slightly different. I think, as long as no matter what language you come from you can learn what these words mean with regards to actual maths, then it would bridge the gap. But if real life problems have to be described in English, then that’s definitely a problem for a non-English speaking person.

TC: so there are different, quite distinct, difficulties for both groups.

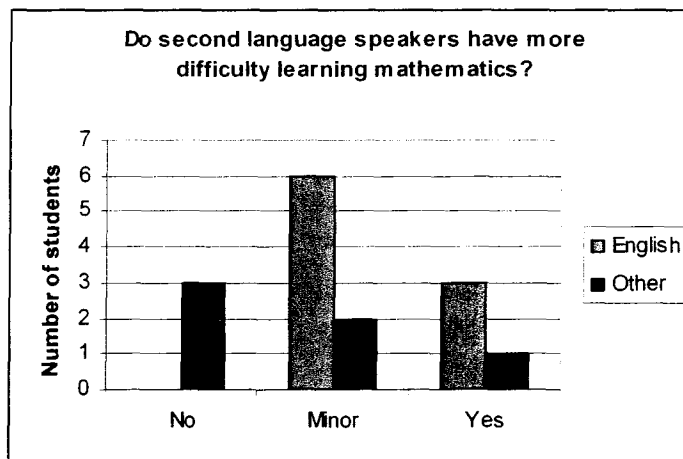
AS1: yes. I don’t think it’s too bad if you’re not that fluent in English. Any word if you take it out of context and put it into a different context will mean different things. You can learn those maths words, which to an English speaking person might be different to the other meanings and might be harder to learn. Also, when I think of some of my Xhosa friends, they use quite sophisticated words, but that’s just because when they’ve learnt those words they learnt what they actually mean and that’s the best word they can choose to describe that, whereas with me, I’ve learnt a whole set of words, so I just

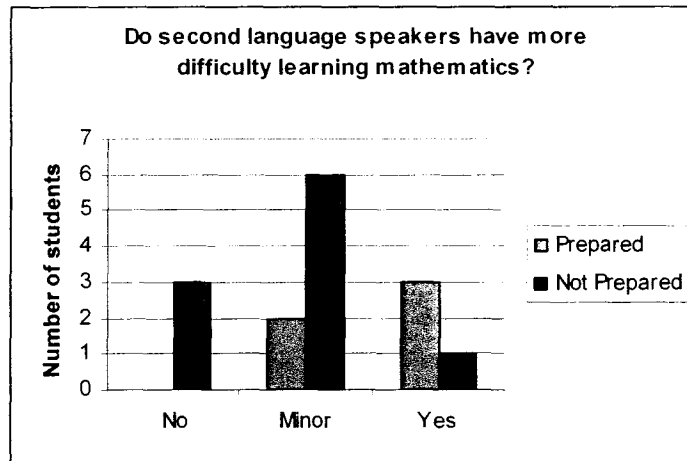
choose the easiest one, the most colloquial one. I grew up with easier, more slang sort of words. So, it might be easier for the person who hasn't had this meaning of the word drilled in their whole life."

- it would depend on the student's mathematical ability "I don't know, it depends on the person. I think if English is not your 1<sup>st</sup> language it will be a little bit of difficulty, but in maths there's not much *language language* ... I think it depends if the person is smart already then they would just need to read and then they'd know." (LM, English speaker)

The "Yes" responses were fairly simply of the form that explanations are a necessary part of learning, so not being fluent in English would necessarily make learning mathematics more difficult. "while maths is its own language and people of different languages can understand what you write down, concepts have to be explained to you, and that would be done in a language that isn't your home language. Mathematics has words that do cross language borders, but it doesn't mean they are understood first time around. So using it, it will be its own language, but learning it, you're going to have a problem somehow." (JS, English speaker)

	English	Other	Prepared	Not Prepared
No	0	3	0	3
Minor	6	2	2	6
Yes	3	1	3	1





**Question 19**

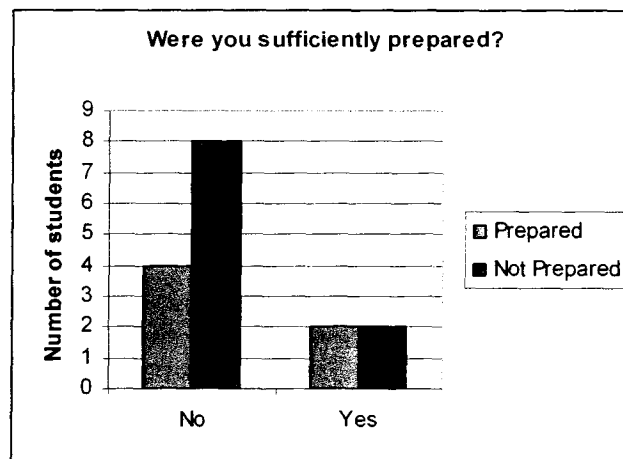
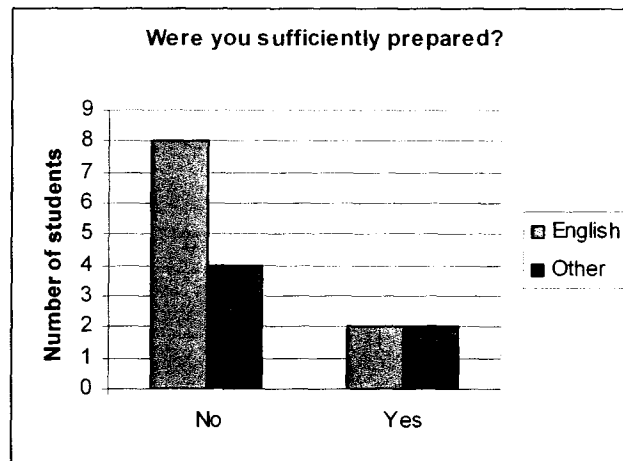
Do you feel you came into this course sufficiently mathematically prepared?

Student responses took the forms

- No
  - Vehement “definitely not” (IS)
  - No, but thought s/he was “I came to university thinking it was going to be a lot easier than before” (CM)
  - Not at first, but then later “The first part was fine, then it all came crashing down in the 2<sup>nd</sup> semester” (DC)
- Yes
  - Confident “Yes, I did, I came quite prepared for it” (NW)
  - Would have preferred better preparation “We weren’t badly prepared, but we could have been better prepared.” (RG)

Note that the grouping of the students as Prepared or Not Prepared was defined by factors not associated with these interviews, thus it is possible to cross reference the students’ perceptions of themselves as prepared against an independent description of their preparedness.

	English	Other		Prepared	Not Prepared
No	8	4		4	8
Yes	2	2		2	2



### Question 20

How would you judge your progress in MAM100W (as the year has gone by)?

The student responses could be categorised as

- steady progress (either steadily poor or steadily good)
- erratic progress
- improving over the year
- deteriorating over the year.

The responses are not very useful. An attempt was made to cross reference them against the preparedness descriptors of

- Consistently good results from matric to the final exam
- Acceptable performance from matric to the final exam
- Good matric, but weak performance at university
- Weak to begin with, but improving towards end of year
- Poor results from matric to the final exam

and to look for any correlations, tabulation given below. It was decided that the answers to the question, and the preparedness descriptors were too ill-matching to allow for comparison.

	This interview	Separate analysis	Prepared	English
TJ	Steady, but poor	Weak to begin with, but improving towards end of year	No	No
CM	Not steady	Weak to begin with, but improving towards end of year	No	No
IT	Not steady	Poor results from matric to the final exam	No	No
IS	Not steady	Poor results from matric to the final exam	No	Yes
DL	Steady, but poor	Good matric, but weak performance at university	No	Yes
DC	Steady	Acceptable performance from matric to the final exam	Yes	Yes
RG	Improved	Consistently good results from matric to the final exam	Yes	Yes
CP	Improved	Acceptable performance from matric to the final exam	Yes	Yes
JS	Steady, but good	Consistently good results from matric to the final exam	Yes	Yes
AS1	Improved	Consistently good results from matric to the final exam	Yes	Yes
DB	-		No	Yes
MS	Deteriorated	Good matric, but weak performance at university	No	Yes
WT	Steady	Good matric, but weak performance at university	No	No
AJ	Improved	Consistently good results from matric to the final exam	Yes	Yes
NW	-	Poor results from matric to the final exam	No	No
AS2	Improved	A good matric, but weak performance at university	No	No
LM	Improved	Poor results from matric to the final exam	No	Yes

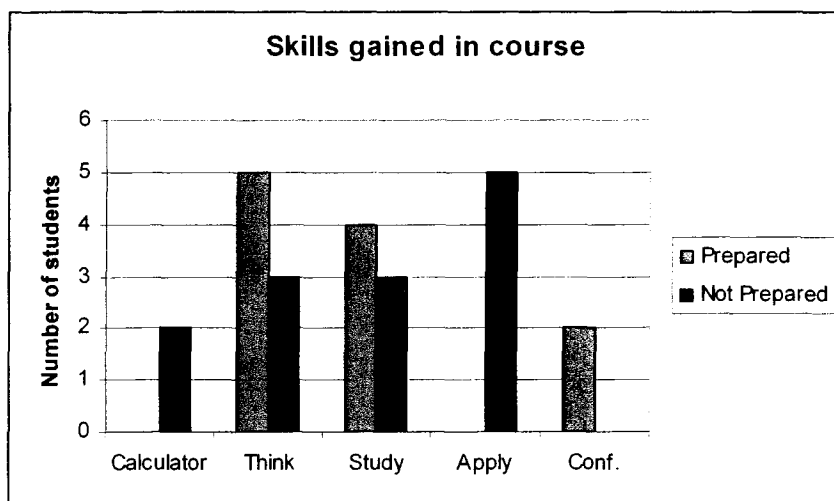
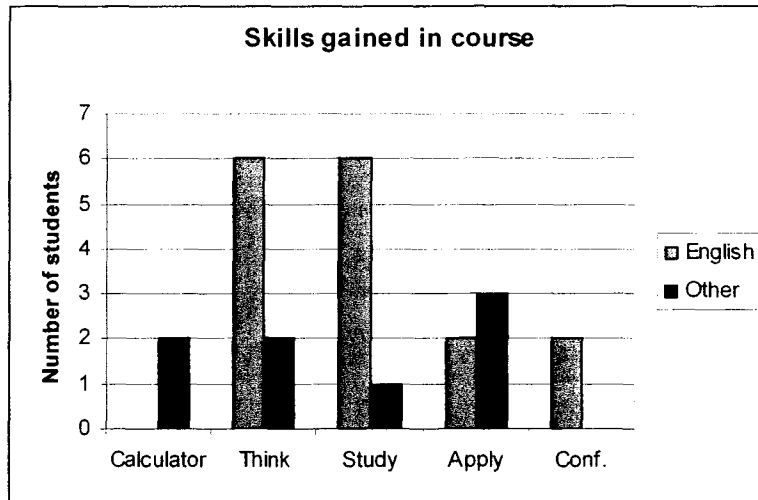
### Question 21

What sort of skills would you say you have gained in this course?

Student responses took five forms

- not using a calculator. The prohibition against calculators in tests and examinations was not popular with the majority of the students in the course. Learning to not use a calculator requires learning to do arithmetic and work with fractions, also possibly to have confidence in one's ability to do calculations. Even though confidence is a response form listed here, it was decided to list calculators as a separate response, as the issue was sufficiently different from all the other responses.
- Thinking mathematically, "thinking more about what the question wants than going straight to working on it" (TJ), "Maths teaches you a lot of problem solving skills, probably more problem solving skills than any other subject I know ... and creative problem solving skills." (DC)
  - The ability acquired during the year to take in information in lectures more efficiently was classified both as a "study skill" and as "learning to think mathematically". "being able to pick up what's going on in the lecture and understand. ... I remember how hard I worked at the beginning of the year, just to understand. Now I work hard to do well in the tests, not to understand. I understand things at the same speed as the lectures are going by." (CP)
- Study and work skills "You're having to do more work than you did in high school. Having to sit down and do stuff instead of staying on par. I feel that's a skill" (RG), "You've got to be able to pick things up first time, you can't put off learning things till the weekend." (JS), "Working. Like, I think it's more just trying to keep up. You have to do things on time. If you have time, do something" (LM)
- Application: mathematics is seen as something to be applied in other fields, such as physics, or chemistry "Genetics uses a lot of mathematics" (NW), "Problem solving in maths. I think that might be useful in biochemistry" (AS2)
- Confidence: "I got to the point where I've become so comfortable in what I could do, now I don't feel scared" (AJ), "I've gained a lot of confidence in myself, and I guess that will filter through to anything else" (AS1)

	English	Other	Prepared	Not Prepared
Calculator	0	2	0	2
Thinking mathematically	6	2	5	3
Study	6	1	4	3
Application	2	3	0	5
Confidence	2	0	2	0



Question 22

Over all, would you say the writing exercises were a good experience, a bad experience or made no difference to you at all?

Student responses took two forms

- It was a good experience
- It was an okay experience “mediocre” (DL)

No students responded that it had been a bad experience, but those who said it was an “okay” experience were quite clear about it:

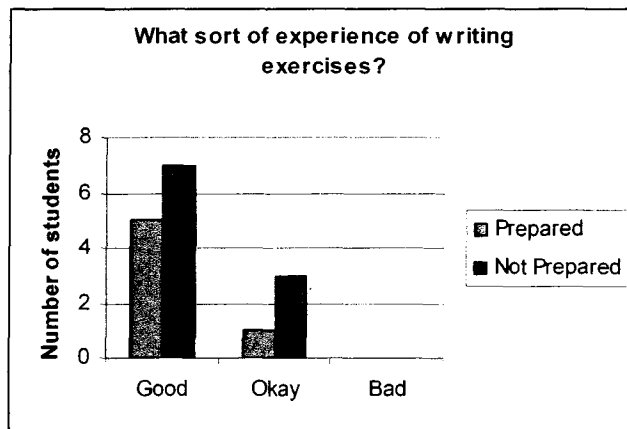
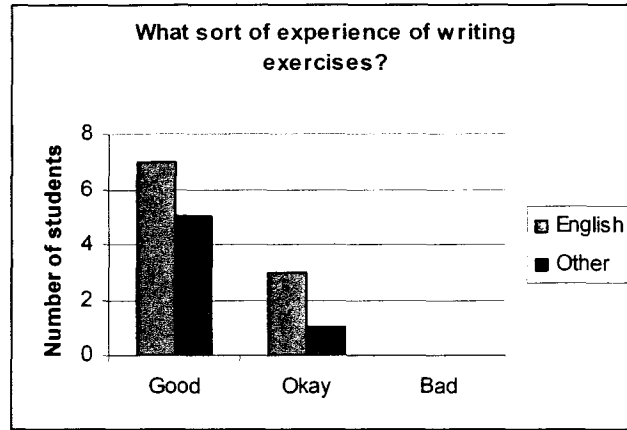
“I’d say mediocre, approaching good, but not quite good. Not great. It wasn’t bad, though.” (DL)

“Yes [they were a good experience]. They weren’t far off being nothing much at all, since we were only doing it for five minutes in a handful of tuts.” (CP)

“In some cases they helped, and in others not” (MS)

“It was an experience, not a bad experience, and not a good experience” (WT)

	English	Other	Prepared	Not Prepared
Good	7	5	5	7
Okay	3	1	1	3
Bad	0	0	0	0



### Question 23

Would you recommend to a friend that they make an effort to come to a tut that involves writing exercises instead of a tut that does not?

All students responded in the affirmative, except for one student who had the following to say

“It depends on what the person wants. Do you want to understand what you’re doing, or do you just want to do it? If he wants to do it I’d tell him to go to a normal tutorial, but if you want to understand it more go to the writing.”

Other students take it as given that understanding is what is being sought, “ja, ja, ja. Because you go deeper, rather than staying on the surface. You actually have to think about it more.” (RG), “Yes. Especially when you have a challenged understanding of the work, then it pushes you to understand the processes, and see links between different ideas.” (CP)

Too few students gave reasons for the worth of the writing exercises to allow for significant analysis.

		Reason?	Prepared	English
TJ	Yes		No	No
CM	Yes		No	No
IT	Yes		No	No
IS	Yes	Focus, structure to tut	No	Yes
DL	Depends on whether he wants to understand or just do	Understanding	No	Yes
DC	Yes	Understanding	Yes	Yes
RG	Yes	Go deeper	Yes	Yes
CP	Yes	Understanding	Yes	Yes
JS	Yes		Yes	Yes
AS1	Yes		Yes	Yes
DB	Yes		No	Yes
MS	Yes		No	Yes
WT	Yes		No	No
AJ	Yes	Different approach, confidence	Yes	Yes
NW	Yes	Makes you work	No	No
AS2	Yes	Helps you pass	No	No
LM	Yes		No	Yes

## Appendix 6 Examples of Student Submissions

### AS1 (Group B) Tutorial 23

3.  $n$  is 12.  $\binom{12}{7}$  means you are finding the max amount of different combinations of 7 when you have 12 different objects. So when you have a combination of 7 you have 5 left over. And when you have a combination of 5 you have 7 left over. So effectively a combination of 5 is a combination of 7.

$\binom{12}{5}$   $\binom{12}{7}$  our comb.

Since other side is chosen, i.e. ordered, it is the same as choosing 5.

If we choose 7, we are left with 5.

The overall combination is ordered, so  $\binom{12}{5} = \binom{12}{7}$ .

(Am I making sense? This is hard to get across in words).

## DK (Group A) Tutorial 23

3. BEFORE: I HAVE NO IDEA, BUT IT MUST BE  $> 7$ . (FACTORIAL MUST BE POSITIVE).

AFTER:

$$\frac{n!}{7!(n-7)!} = \frac{n!}{5!(n-5)!}$$

$$5!(n-5)!n^{\cancel{2}} = 7!(n-7)!n^{\cancel{2}}$$

$$(n-5)(n-6) = 7 \times 6$$

$$n^2 - 11n + 30 = 42$$

$$(n^2 - 14n + 12) = 0$$

$$(n-12)(n-1) = 0$$

$$\therefore n = 1 \quad \text{or} \quad \underline{12}$$

valid.

TEST:  $\binom{12}{7} = 792$

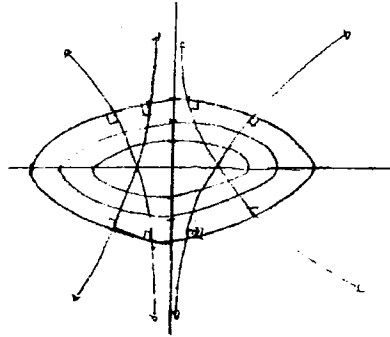
.....  $\binom{12}{5} = 792$

I converted the two sides using the combinatoric formula, then got rid of the denominators by times by them on each side. The 2  $n!$  cancelled out, because they are in essence the same number. I then divided by 5!, which left  $7 \times 6$  on the left hand side. I then completed the sum by factoring the polynomial and saw that only 12 was a possible answer.

This is an expected answer as it is greater than 7. It is seen that the sum of the 2  $r$ 's (7 & 5) given you 12, which is the same as the calculated answer.

# NG (Group A) Tutorial 20

Before:  $\frac{x^2}{25} + \frac{y^2}{10} = \frac{21}{10}$



The orthogonal trajectories should take the form of  $\ln(x)$  graphs.

Calc:  $2x^2 + 5y^2 = c$  — original curves

$$4x + 10y \frac{dy}{dx} = 0$$

$\frac{dy}{dx} = -\frac{2x}{5y}$  — gradient of original curves

required  $\frac{dy}{dx} = \frac{5y}{2x}$

$$\int \frac{5y}{2} dy = \int \frac{1}{2x} dx$$

$$\frac{5}{2} \ln y = \frac{1}{2} \ln x + B$$

$$\ln y = \frac{1}{5} \ln x + C$$

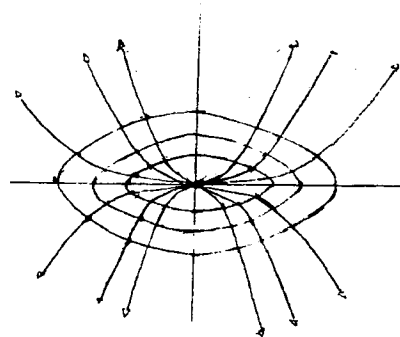
$$= \ln x^{1/5} + \ln |k|$$

$$= e^{\ln x^{1/5} + \ln |k|}$$

$$= e^{\ln x^{1/5}} \cdot e^c$$

$$= k e^{\ln x^{1/5}}$$

$$= k x^{1/5}$$



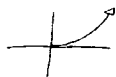
After: First I found the derivative of the original  $f^n$ .

I multiplied the derivative (which is the gradient) by  $-1$  & inverted it. This gives a new derivative, which <sup>a gradient</sup> is perpendicular to the gradient of the original function.

I have the gradient of the required  $f^n$ , I now solve the differential equation for the  $f^n$ .

The diagram:

$x^{5/2} = x^{2.5}$ ,  $x^2 \cdot x^3$  look like the diagram below (in the 1st quadrant)



Since this is  $\perp$  to the ellipses in the 1st quadrant, I should be able to extend the sol<sup>n</sup> to the other 3 quadrants

My answer did not look as expected! (But it seems logical to me now that my original "estimate" makes no sense ~~then~~).

## Appendix 7

### Data collected from students' assessment tasks throughout the year

Und: Evidence of understanding

Cont: Evidence of metacognitive control

Flex: Evidence of flexibility in solution process

2: Yes, 1: No, 0: Cannot tell

Group A	Class Test 1				Class Test 2				Class Test 3 Q2				Class Test 3 Q6				Class Test 4 Q6				Class Test 5 Q3				Final exam Q6iii						
	Mark	Und	Cont	Flex	Mark	Und	Cont	Flex	Mark	Und	Cont	Flex	Mark	Und	Cont	Flex	Mark	Und	Cont	Flex	Mark	Und	Cont	Flex	Mark	Und	Cont	Flex			
DC	3		2	0	2	1	2	0	0	7	2	2	2	*						5	2	1	2	3	2	2	1	5	2	2	0
RG	*				0	0	0	0	0	6	2	1	0	0	1	1	2	5	2	0	1	6	2	0	0	8	2	2	0		
NG	1	1	1	1	0	1	0	0	0	7	2	0	0	1	2	0	1	3	2	2	1	5	2	2	0	6	2	2	2		
TH	*				1	2	0	1	1	8	2	0	0	0	2	1	2	3	2	1	1	0	2	2	2	2	3	2	2	1	
AJ	0	1	2	2	1	0	0	1	1	*				1	2	1	1	7	2	2	2	2	2	1	1	0	2	1	1	0	
DK	*				1	1	2	1	1	7	2	0	0	0	1	0	0	3	2	0	2	3	2	1	2	3	1	1	0		
JL	*				0	2	0	0	0	2	2	1	1	2	2	1	1	0	0	0	1	4	2	2	1	0	2	1	1	1	
DL	0	1	0	2	*					0	1	0	0	0	1	0	0	1	2	2	2	2	0	1	0	2	1	1	1	0	
NW	0	1	0	1	0	1	1	2	2	3	1	0	0	0	1	0	0	1	2	1	1	0	0	2	1	0	3	1	1	0	
IS	*				0	0	0	0	0	2	2	2	1	2	1	0	0	0	1	1	1	1	4	2	2	1	3	1	1	0	
MS1	0	1	0	1	3	2	0	0	0	3	0	0	0	0	1	1	0	1	2	0	2	4	2	2	1	3	1	1	0		
MS2	*				*					2	1	0	0	0	0	0	0	3	2	0	1	0	1	0	1	4	2	2	1		
WT	*				0	1	0	0	0	4	2	1	1	1	1	0	0	3	2	2	2	3	1	1	2	5	2	1	2		
RT	0	1	0	1	0	2	0	0	0	2	1	0	2	2	2	1	0	3	2	0	1	*				3	1	1	0		
<b>Group B</b>																															
JB	*				*					0	2	1	2	0	2	0	1	4	2	0	1	1	2	0	0	0	0	2	1	0	
GCP	4	2	2	2	3	2	2	2	2	0	2	1	1	0	0	1	1	6	2	2	2	1	0	1	1	2	2	2	2		
JE	0	1	0	1	0	1	0	0	0	0	0	0	0	2	1	0	0	3	2	0	2	0	1	0	0	2	1	2	1		
NH	3	2	2	2	0	1	0	0	0	4	2	2	1	*				*				3	2	2	1	6	2	2	2		
TJ	*				*					7	2	0	0	0	1	0	0	0	0	0	1	3	2	1	1	6	2	2	2		
BJ	1	1	2	1	1	2	0	2	2	5	2	1	2	2	2	1	2	2	2	1	0	4	2	1	0	3	1	1	0		
CM	*				*					2	2	1	2	0	1	0	0	1	2	0	1	0	1	0	0	2	2	1	1		
LM2	*				1	1	1	1	1	4	2	1	1	0	2	1	1	3	2	1	2	*				3	2	1	0		
MM	*				0	2	0	0	0	4	2	1	1	0	1	0	0	0	0	0	0	*				7	2	2	2		
LM1	2	2	0	1	0	1	0	0	0	2	1	1	0	0	0	0	0	0	1	1	2	0	1	1	0	5	2	2	1		
CP	2	2	0	1	0	2	0	2	2	1	2	1	2	0	1	0	0	5	2	2	2	6	2	2	0	8	2	2	1		
AS1	0	0	0	1	0	2	0	0	0	8	2	0	0	0	0	0	1	1	2	1	1	6	2	2	2	4	2	2	1		
JS	3	2	0	1	4	2	2	2	2	2	2	1	1	1	1	0	0	5	2	2	2	5	2	2	2	6	2	2	1		
TvH	3	2	0	2	2	2	0	0	0	7	2	2	0	6	2	0	2	5	2	2	2	5	2	0	1	8	2	0	0		
MW	1	0	0	1	*					*				2	2	1	1	2	1	1	2	5	2	2	1	3	2	1	0		
SW	*				0	2	0	0	0	7	2	0	0	0	1	0	0	4	0	0	1	*				0	1	1	0		
<b>Group C</b>																															
Student 1	*				*					0	2	1	1	*				2	2	2	2	3	2	1	1	3	1	1	0		

Student 2	0	1	1	1	0	1	0	0	7	2	0	0	0	1	1	0	3	2	1	2	0	1	1	1	2	1	1	0	
Student 3	2		1	0	2	0	1	0	0	2	2	0	0	0	1	0	0	2	1	1	1	0	1	0	2	3	1	1	0
Student 4		*				0	0	0	0	3	0	1	0	0	1	0	0	0	2	0	1	0	1	0	2	2	1	1	0
Student 5		*				0	2	0	0	6	2	1	0	*			2	2	1	2	0	2	1	0	2	2	2	1	0
Student 6		*				0	2	1	2	7	2	0	0	1	2	1	0	2	2	1	1	6	2	0	0	3	2	1	1
Student 7	3		2	0	1	*				2	1	1	1	0	1	1	1	4	2	0	2	*			0	1	1	1	
Student 8	3		2	2	1	0	1	0	0	8	2	0	0	3	2	1	2	2	2	1	2	6	2	0	0	2	1	1	0
Student 9		*				0	1	0	0	3	2	1	2	0	1	0	0	*				1	1	0	2	0	1	1	0
Student 10	0		1	0	1	0	0	2	1	2	2	1	2	*			3	2	0	0	0	1	2	1					
Student 11	2		2	0	1	1	2	0	2	7	2	2	2	*			1	0	0	0	4	2	1	2	0	1	1	0	
Student 12		*				*				3	2	1	1	0	1	0	0	3	2	1	2	4	2	1	2	0	1	0	0
Student 13		*				*				3	2	1	1	*			0	2	1	1	6	2	0	0	3	1	1	0	
Student 14		*				*				4	2	1	1	0	1	0	0	1	2	0	1	1	2	2	1	0	1	1	1
Student 15	2		0	0	1	*				*				*			2	2	1	1	3	2	0	2	3	1	1	0	
Student 16	3		2	0	1	*				2	1	1	2	0	1	0	0	3	2	1	2	5	2	0	0	0	1	1	0
Student 17	2		1	0	1	*				*				0	1	0	0	1	1	0	0	4	2	1	1	0	1	1	0
Student 18	0		2	2	1	*				8	2	0	0	0	1	0	0	2	2	0	1	0	1	1	0	4	2	1	0
Student 19		*				*				*				*				0	2	0	1	2	1	2	1	3	2	1	2
Student 20		*				1	1	0	1	2	1	0	0	4	2	1	2	0	2	1	1	*			3	2	1	1	
Student 21		*				0	1	0	0	*				0	1	0	0	3	2	0	1	6	2	2	0	3	1	1	0
Student 22		*				0	1	0	0	*				*				6	2	0	2	5	2	2	1	*			
Student 23		*				1	0	0	0	*				*				*				0	1	1	1	8	2	2	1
Student 24	2		2	0	1	1	2	0	0	8	2	0	0	2	2	1	1	3	2	2	2	3	2	1	1	8	2	2	2
Student 25	5		2	2	2	1	2	2	1	3	2	1	2	1	0	0	0	4	2	0	2	3	2	2	1	8	2	2	2
Student 26		*				1	2	1	2	8	2	2	0	7	2	2	2	7	2	0	2	6	2	0	0	3	1	1	0
Student 27		*				1	2	0	0	4	0	1	0	0	0	0	0	0	0	0	2	0	1	1	1	6	2	1	0
Student 28	3		2	0	1	0	0	0	0	0	1	1	1	0	1	0	1	0	1	0	1	0	1	1	2	0	1	1	0
Student 29		*				0	0	2	0	6	2	1	2	*				5	2	0	2	0	1	0	0				

**Additional data used: students total results for each assessment task. The full table of data is not included here for reasons of anonymity**

**Excerpt:**

	CT1	CT2	CT3	CT4	CT5	CT6	FP1	FP2
Student a	76.8	72.3	38	64.5	30	52.5	78	70.6
Student b	80	56	68	89.4	58	87.5	93	96.9
Student c	53.5	53.8	60	46.3	42	70	82	71.3

CT1 refers to Class Test 1  
 FP1 refers to Final Paper 1

Groups A, B and C were included in the total result analysis

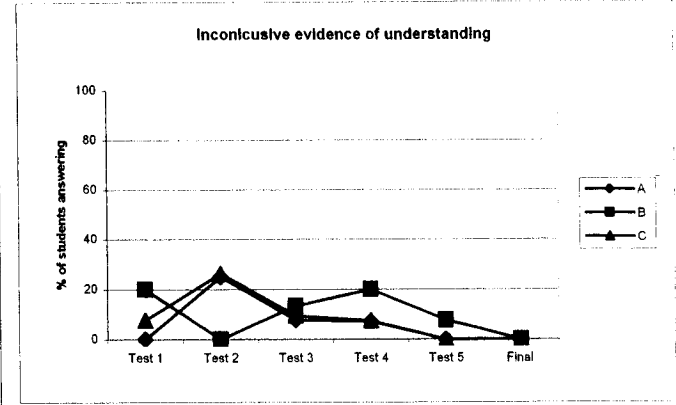
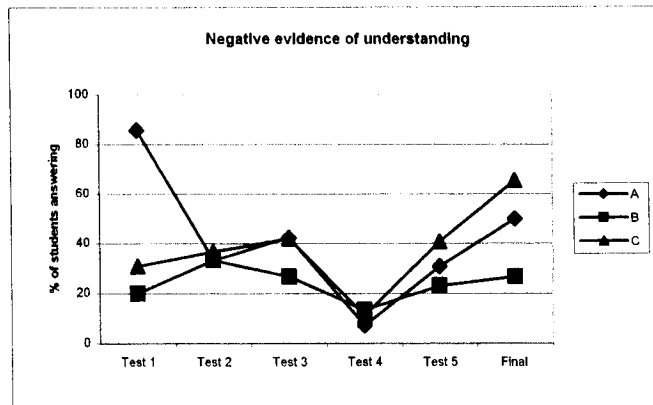
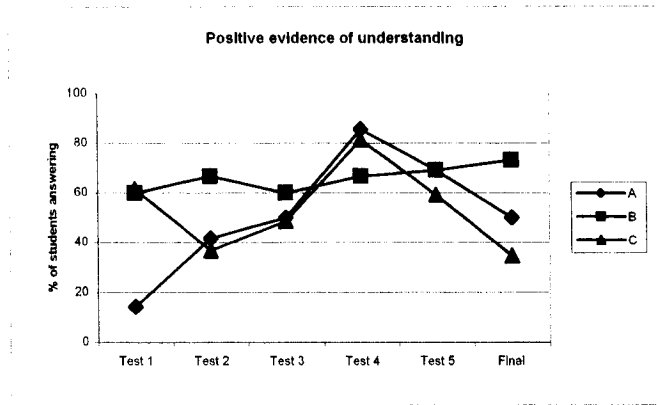
First phase of analysis: Comparison of A, B and C Groups on evidence of understanding, metacognitive control and flexibility in solutions

Understanding

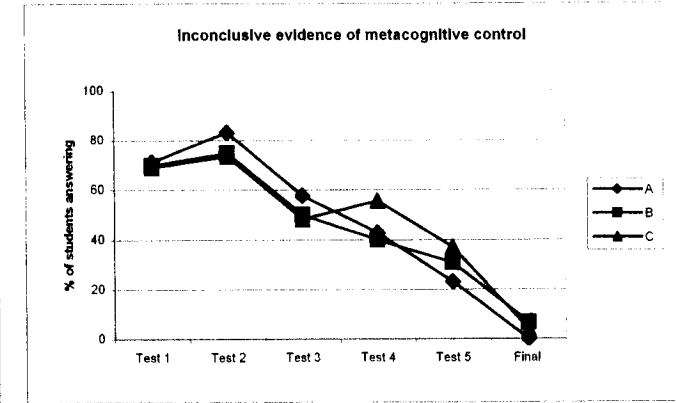
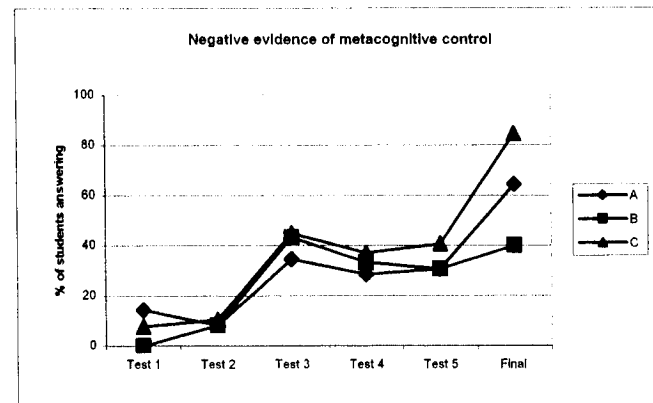
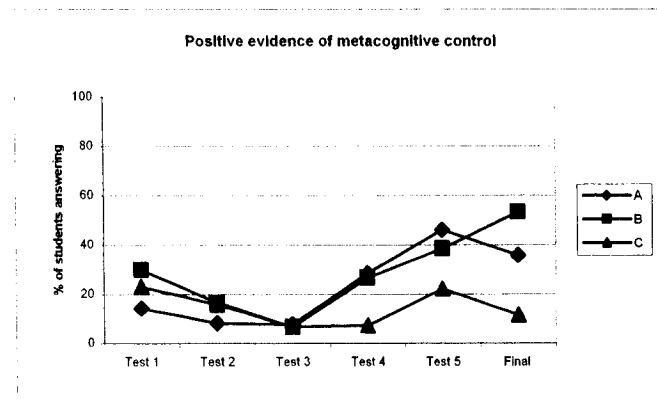
A: n=14

B: n=16

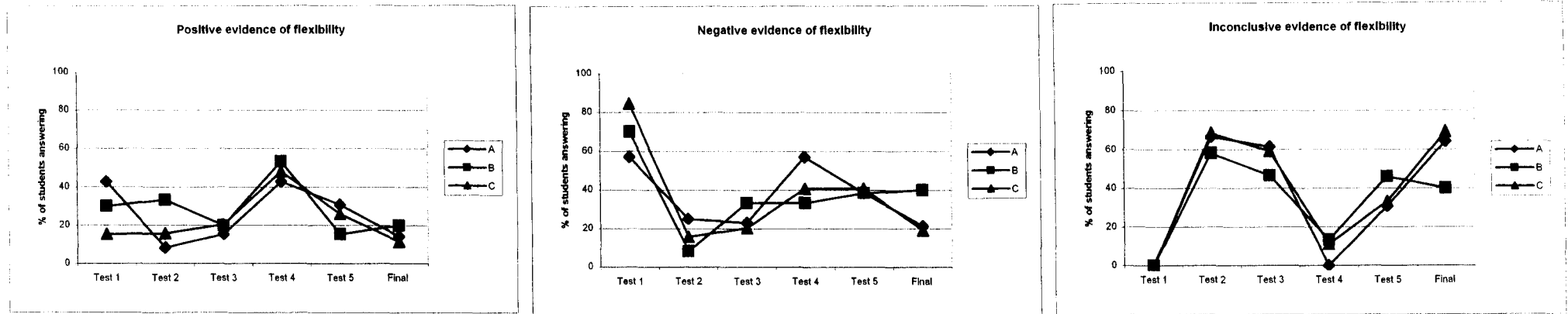
C: n=29



Metacognitive control



## Flexibility

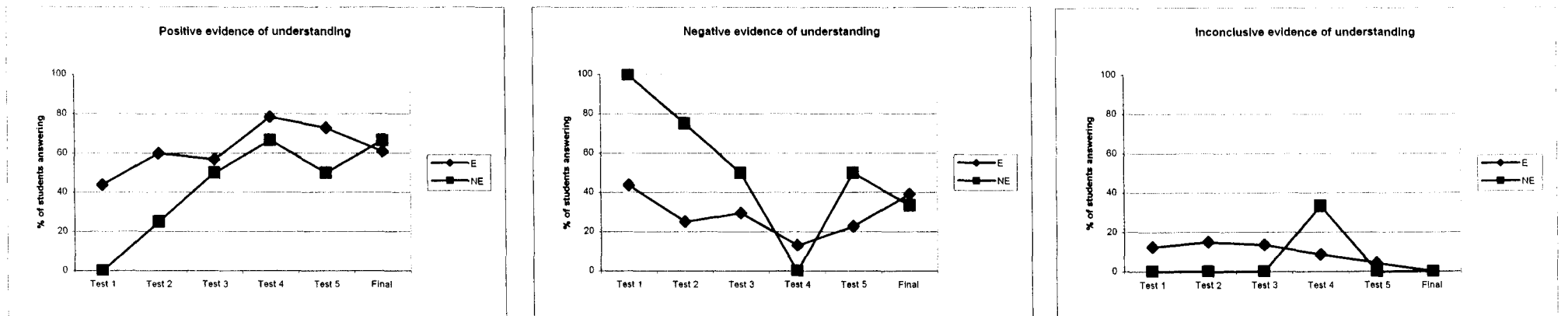


**Second phase of analysis: Merging groups A and B, dropping Group C, and categorising students as English or not English**  
 The objective is to determine whether the writing initiative benefitted either language group more than another.

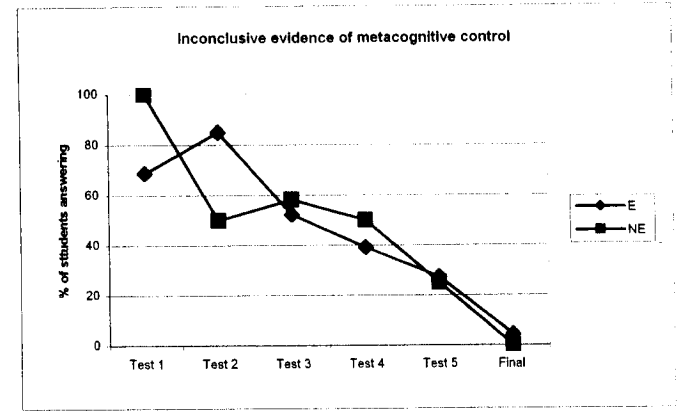
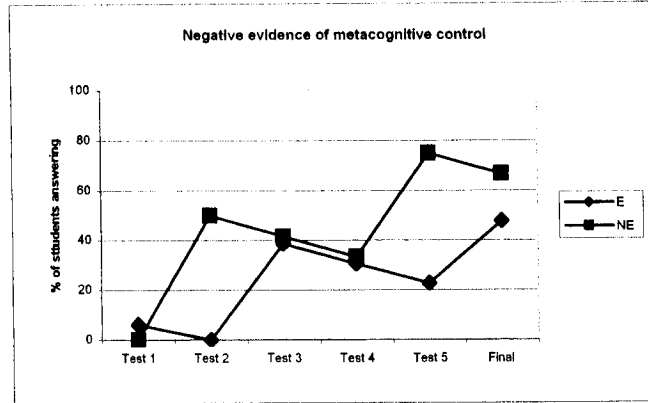
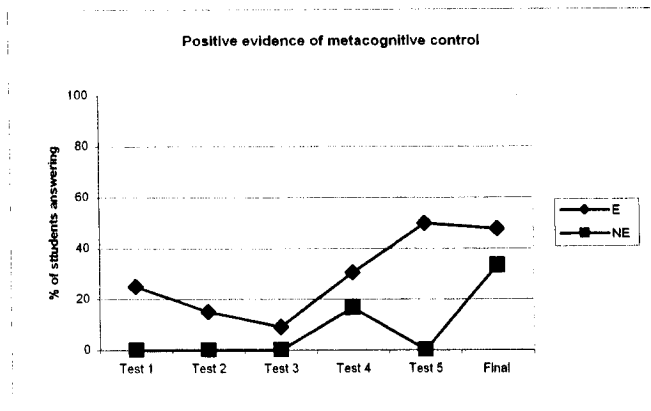
## Understanding

English (E): n = 24

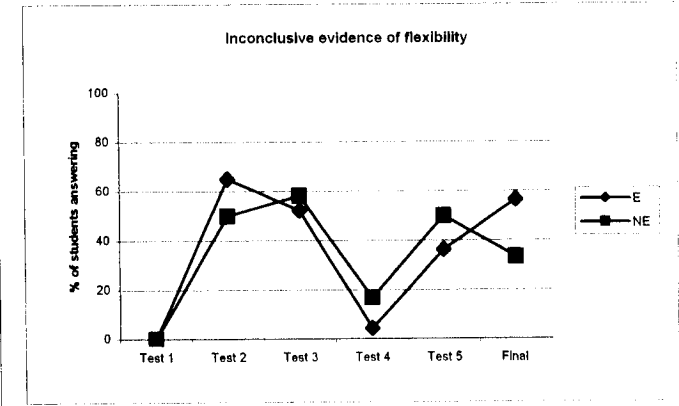
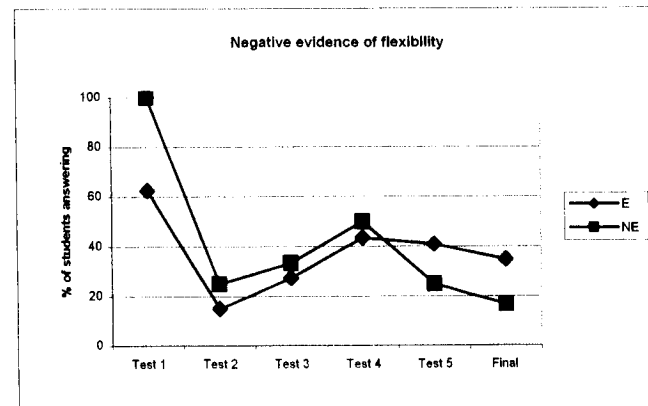
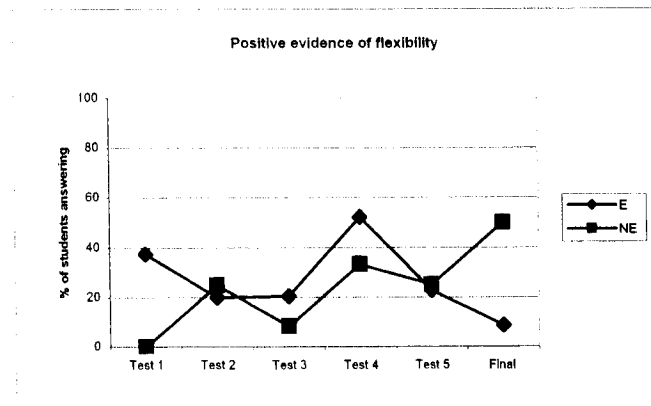
Not English (NE): n = 6



## Metacognitive control



## Flexibility

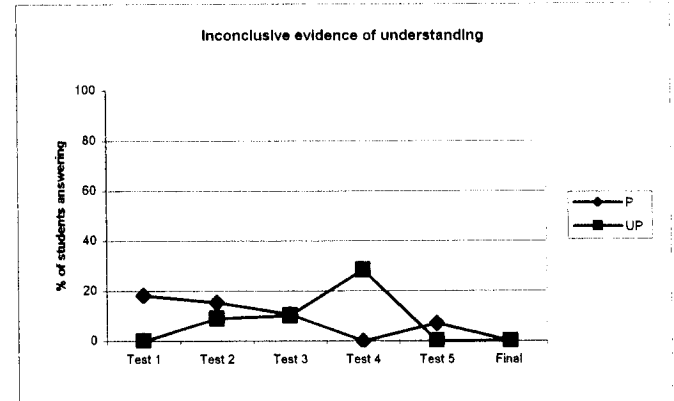
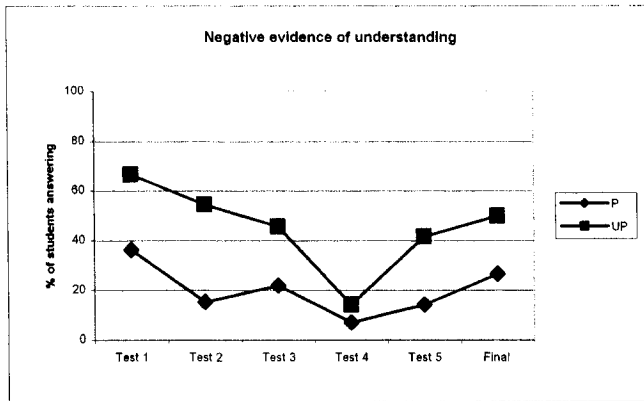
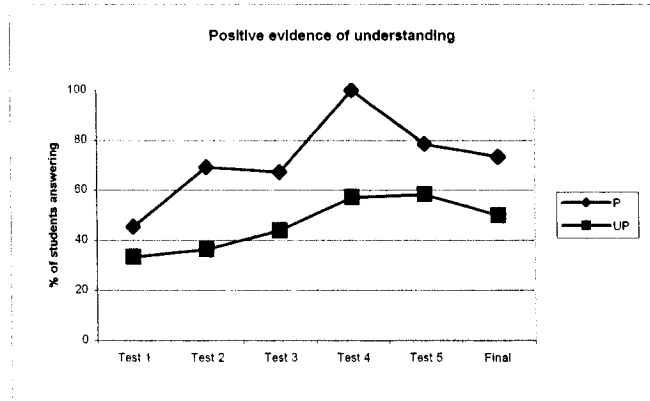


**Third phase of analysis: Merging groups A and B, dropping Group C, and categorising students as Prepared or Under Prepared**  
**The objective is to determine whether the writing initiative benefitted either group more than another.**

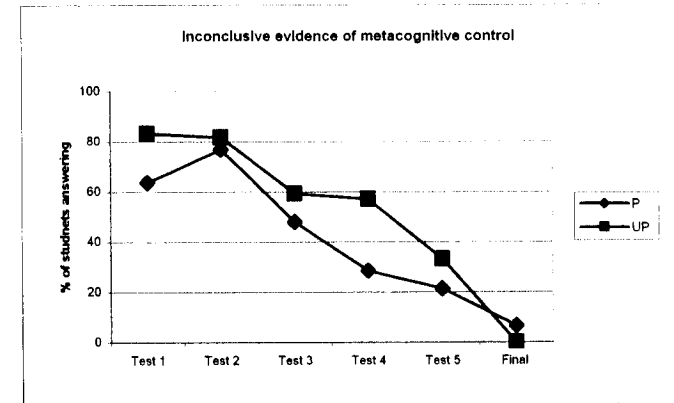
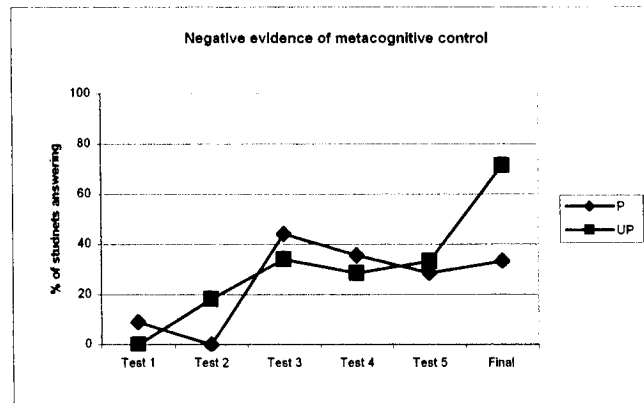
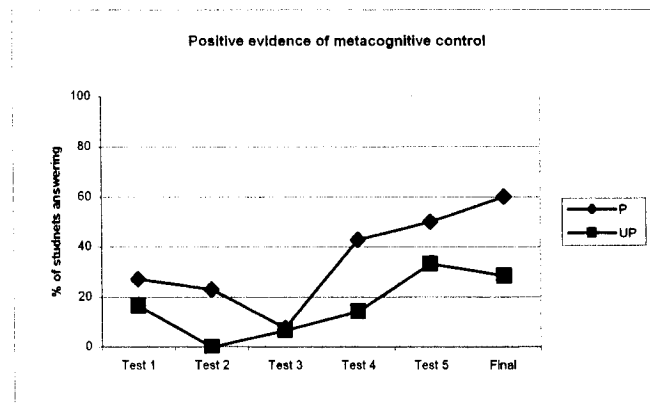
**Understanding**

Prepared (P): n = 15

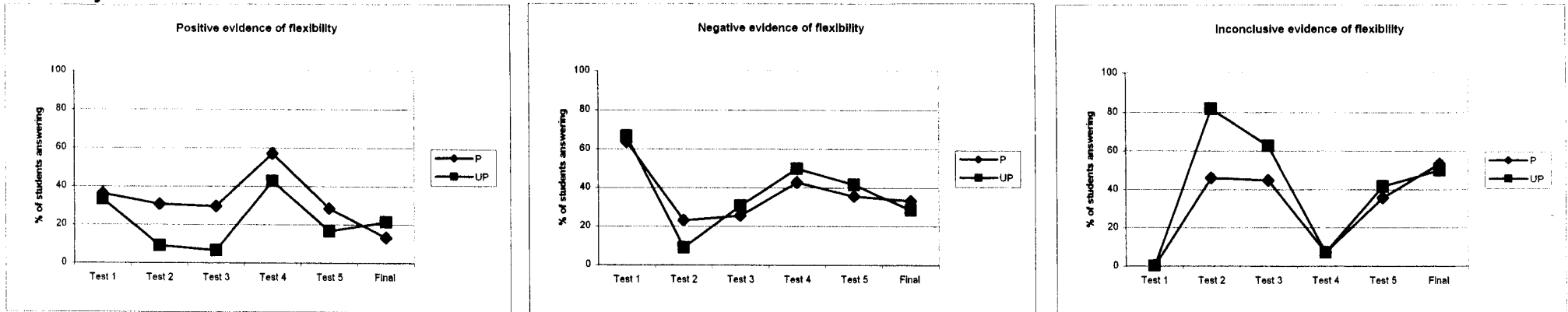
Under Prepared (UP): n = 15



**Metacognitive control**



## Flexibility



## Fourth phase of analysis: Comparing the total test marks of Groups A, B and C

Objective: to see whether any overall effects (as opposed to problem-specific effects) are discernible, and different for the different groups

