Model Misspecification and the Hedging of Exotic Options

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy at the University of Cape Town. It has not been submitted before for any degree or examination in any other University.

Signed by candidate

May 7, 2018

Acknowledgments

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Finally, I wish to thank my family and those closest to me. I appreciate the time and effort you invested in keeping me motivated and focussed.

Abstract

Asset pricing models are well established and have been used extensively by practitioners both for pricing options as well as for hedging them. Though Black-Scholes is the original and most commonly communicated asset pricing model, alternative asset pricing models which incorporate additional features have since been developed. We present three asset pricing models here - the Black-Scholes model, the Heston model and the Merton (1976) model. For each asset pricing model we test the hedge effectiveness of delta hedging, minimum variance hedging and static hedging, where appropriate. The options hedged under the aforementioned techniques and asset pricing models are down-and-out call options, lookback options and cliquet options. The hedges are performed over three strikes, which represent At-the-money, Out-the-money and In-the-money options. Stock prices are simulated under the stochastic-volatility double jump diffusion (SVJJ) model, which incorporates stochastic volatility as well as jumps in the stock and volatility process. Simulation is performed under two 'Worlds'. World 1 is set under normal market conditions, whereas World 2 represents stressed market conditions. Calibrating each asset pricing model to observed option prices is performed via the use of a least squares optimisation routine. We find that there is not an asset pricing model which consistently provides a better hedge in World 1. In World 2, however, the Heston model marginally outperforms the Black-Scholes model overall. This can be explained through the higher volatility under World 2, which the Heston model can more accurately describe given the stochastic volatility component. Calibration difficulties are experienced with the Merton model. These difficulties lead to larger errors when minimum variance hedging and alternative calibration techniques should be considered for future users of the optimiser.

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Chapter 1

Introduction

Asset pricing models have received detailed attention since the 1973 publication of *The Pricing of Options and Corporate Liabilities* (Black and Scholes, 1973). Essentially, an asset pricing model needs to provide a hedging strategy and must extrapolate the prices of non-traded (e.g. exotic) assets from the prices of traded ones (e.g. vanilla options). Exotic options have presented many opportunities and challenges for financial mathematicians. Their non-vanilla payoff profile provides additional complexities, increasing the effort required to hedge Tompkins (2002). Hedging involves an investment strategy to limit the risk of another investment. Many hedging strategies have been researched, including, but not limited to, delta hedging An and Suo (2009), minimum variance hedging An and Suo (2009) and static hedging Carr *et al.* (1998). While hedging has many applications, in this dissertation the use is to match the payoffs of exotic options. As with hedging, model misspecification is a broad term. Essentially, model misspecification refers to the degree to which a model doesn't account for the features that it should. From a financial mathematics perspective, this could be seen through not including e.g. jumps or stochastic volatility or mean-reversion parameters, where these features/parameters were necessary. When hedging exotic options, misspecification would likely lead to hedging errors as a result of not fully capturing the dynamics of the environment. Several studies have highlighted the significance of model risk in trading and hedging derivatives, with Green and Figlewski (1999) focussing on the importance of volatility estimates and Nalholm and Poulsen (2006) focussing on the static hedging of up and down-and-out call options. Nalholm and Poulsen (2006) note the importance of specifying and calibrating the model correctly when using Black-Scholes, since hedging barrier options relies on non-at-the-Money call options which Black-Scholes does not price accurately. The Black-Scholes model develops a partial differential equation (PDE), which provides a unique, arbitrage-free price and allows for perfect continuous delta-hedging (assuming the real world follows Black-Scholes dynamics) using just the underlying asset and the bank account.

Many have criticised the Black-Scholes model's ability to describe real world dynamics, since the stock price process does not account for jumps (Merton, 1976), nor does the process recognise the skewness and kurtosis of returns observed in real financial data (Geske and Torous, 1991). The assumption of constant volatility is contradicted by the observed phenomenon of a volatility smile or skew (Albuquerque, 2012), (Jondeau and Rockinger, 2003), (Sears and Wei, 1988) and (Friend and Westerfield, 1980). These distributional characteristics make it difficult to hedge in the Black-Scholes environment and leads to the potential for significant hedging errors, yet no model has been able to displace Black-Scholes from the number one spot. The fact that many practitioners still use Black-Scholes supports the view that a model which prices options poorly may still hedge relatively well.

1.1 Dissertation Structure

This dissertation is organised as follows:

Chapter 1: Introduction, where in addition to the discussion above, previous research is discussed along with the aims of this work.

Chapter 2: Models and Instruments, where the dynamics of the asset pricing models are presented and discussed.

Chapter 3: Generating the Data and Calibrating the models, involving Euler-Maruyama techniques in simulation and Fourier techniques in calibration.

Chapter 4: Hedging methods, where delta, minimum variance and static hedging is discussed.

Chapter 5: Results

Chapter 6: Conclusions and Discussions

1.2 Literature Review

Though no pricing model has taken centre-stage, there are models that have been developed to deal with the shortcomings listed above. There has been a substantial amount of empirical work to deduce whether these more complicated models improve hedging performance. However, the consistency of the results has been weak. Carr and Wu (2002) found that hedging using the Black-Scholes formula with implied volatility showed little deterioration to hedging using more complex models. However, Branger and Schlag (2003) showed that the performance of hedging strategies in the Black-Scholes environment differs significantly from the stochastic volatility environment.

Regarding jumps, Kim and Kim (2005) as well as Bakshi *et al.* (1997) argued that incorporating jumps does not improve hedging accuracy significantly while He *et al.* (2006) and Kennedy *et al.* (2009) improve hedging accuracy using the Merton model to limit jump risk. Kim and Kim (2005) argued that while pure jump processes improve hedging accuracy, this improvement falls away once stochastic volatility is already factored in (even for shorter-term options).

Tying the concepts to be used together, An and Suo (2009) compared the hedging effectiveness across various models of exotic options on currencies. The techniques they used were minimum variance hedging and delta-vega neutral hedging and they compared hedging performance between the Black-Scholes model, Merton model, Heston model and a Stochastic Volatility Jump process. They concluded that outperformance of a particular model depended on the exotic considered, with delta-vega hedging leading to Heston outperformance for up-and out call options. However, when they applied minimum variance hedging, out-performance was not as clear.

Elices and Giménez (2013) calculated the relative model risk of a double no-touch option, where dynamics were assumed to be Heston and the Black-Scholes model was compared with the Volga-Vanna model. The risk measure they used involved calculating the expected shortfall of the hedge portfolio at the lower 35% tail. They found that the Volga-Vanna model had significantly lower model risk than the Black-Scholes model (around 50%), but that model risk was still very high (over 80% of the price of the initial price of the option).

1.3 Aims

The key questions this paper aims to answer are:

1. Given that we do not fully know the true dynamics of the stock price, which model is the most accurate approximation, resulting in the smallest hedging errors? *This is carried out by deriving a distribution of hedging errors over many simulations of future stock movements.*
2. Is there a hedging technique that results in a consistent advantage in terms of mean and standard deviation of errors?
3. What are the potential dangers of using simplified programming techniques to perform these calculations?

There is an intuitive 5-step process which has been established by several other authors without formalisation, including Green and Figlewski (1999), Poklewski-

Koziell (2012) and An and Suo (2009). The methodology chosen to answer the research questions can broadly be understood as:

1. Simulate stock prices under a representative model of the real world. These values are assumed to be unknown to stakeholders that are trying to hedge.
2. Price each exotic option.
3. For each market model, calibrate the dynamics to observed call option prices over a range of strikes and terms to maturity.
4. Develop hedging strategies and calculate their values over many simulations.
5. Analyse and interpret the results.

This dissertation aims to evaluate the hedging performance of exotic options in the Black-Scholes model and compare the performance to two alternative models: 1) The Stochastic Volatility (Heston) model and 2) the Jump (Merton) model. This combination provides comprehensive coverage of the dynamics under the real world, with the Black-Scholes model providing a broad description of dynamics, the Heston model incorporating the additional feature of stochastic volatility and the Merton model incorporating a jump process. Three exotic options are considered, namely down-and-out call options, lookbacks and cliquets. This dissertation extends the work of An and Suo (2009) by considering lookback options and cliquets, and calibrating an SVJJ model to simulate many outcomes of the world. This dissertation also looks at static hedging of down-and-out call options. Beyond these differences, stock price dynamics are considered compared to currency prices. By analysing the average of the absolute hedge error, rather than at the average hedge error, the problem of positive and negative hedge errors cancelling out is prevented. The down-and-out call option was chosen due to the barrier feature where if the underlying stock value falls below the barrier level, the option expires worthless. The option, therefore, opens the door to further research using other barrier options. The lookback option was chosen due to the implicit memory feature, leading to the need to keep track of stock prices over the life of the option. Lastly, the cliquet option was chosen due the double-start feature of the option, since it effectively holds a vanilla call starting at the same time as the cliquet as well as a forward-starting call. Finally, hedge errors and their distributions are evaluated and discussed. Each model is calibrated weekly to cross-sectional vanilla call-option prices.

1.4 Key Concepts

Since practitioners do not fully understand the market they cannot perfectly describe it and therefore turn to the models available or generate their own model.

In this dissertation model misspecification refers to the extent to which the hedge practitioner's hedge portfolio mismatches the value of the option being hedged at maturity. Subtle changes to the definition of model misspecification have been used in other papers (Bakshi *et al.*, 1997), (Carr *et al.*, 1998), though the basic premise is the same. Model misspecification could lead to larger losses (or profits) than expected. The concern of model misspecification is particularly relevant when considering exotic options, since they are traded Over-the-Counter (OTC), offering little information after initiation. Another issue practitioners face when dealing with exotic options is that limited historical data is available, resulting in a heavy reliance on the chosen model. Exotic options also have a more complex payoff structure than their European counterparts and are therefore more likely to display significant hedging errors. Call-option misspecification is less relevant since the market provides significantly more information on the market price of the option over its lifetime. It is also important to note and consider the 'exoticness' of the option when hedging. For example, barrier options where the barrier is unlikely to be hit are more vanilla than barriers where the stock price is close to the barrier level. This is due to the fact that if the barrier is unlikely to be hit the option essentially behaves as it would without the presence of a barrier.

From the perspective of the researcher, the real world in which the hedge operates is a **simulated** real world, where the simulation uses a model more complicated than the models the entity uses to hedge. This adds the benefit of being able to simulate many paths over the same period, compared to observed data, which either provides only one realised path or introduces time inhomogeneity. From these simulated paths, one can calculate the means and variances of errors. The additional complexity of the simulated real world model compared to the other models prevents perfect calibrations and unrealistic hedging performance. In reality, practitioners act in the real world and thus cannot test perfectly beyond simulation and only face one true historic path. It is also assumed that the market is complete and therefore that a derivative's price is the discounted expected value of the future payoff under the risk-neutral measure. The complete market assumption implies that options can be replicated by traded assets and other options. In practice, exact replication isn't followed and hedging techniques such as delta hedging and minimum variance hedging have been developed which do not include holding options. Static hedging still leaves the practitioner will exposure due to discretisation of the hedge portfolio. If the market were assumed to be incomplete, this would render the risk-neutral probability measure non unique and would only allow the accuracy of the option price to be determined within a range.

Chapter 2

Models and Instruments

Step 1 of the process involves simulating future market data under a representative model of the real world. In order to ensure these values can be assumed to be unknown to stakeholders, a more complex asset pricing model than those used by hedge practitioners needs to be modelled. This dissertation made use of the stochastic-volatility double jump diffusion model henceforth referred to as the SVJJ model. This model was first studied by Duffie *et al.* (2000), and was used extensively by Poklewski-Koziell (2012). Step 1 allows us to compare asset prices to those calibrated and calculated by the practitioner's model. Broadie and Kaya (2006) review the SVJJ model using S&P 500 option futures data from 1987 to 2003 and advocate a model that incorporates jumps in both the stock and volatility process. The length of time considered in their study lends confidence in the results.

2.1 Models considered

2.1.1 The SVJJ Model

The risk-neutral dynamics of the SVJJ model are,

$$dS_t = (r - \lambda\mu_J)S_t dt + \sqrt{V_t} S_t dW_t^{(1)} + JS_t dN_t \tag{2.1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^{(2)} + Z dN_t \tag{2.2}$$

$$dW_t^{(1)} dW_t^{(2)} = \rho dt \tag{2.3}$$

where S_t is the stock price at time t, V_t is the variance, N_t is a Poisson process with jump intensity λ, r is the risk-neutral rate of return, θ is the long-run variance of the process, κ is the rate at which V_t reverts to θ and $W_t^{(1)}$, $W_t^{(2)}$ are correlated Brownian terms. The relevant jump-terms are:

$$Z \sim \text{Exponential}(\mu_v) \tag{2.4}$$

$$(1 + J)|Z \sim \text{log-normal}(\mu_s + \rho_J Z, \sigma_s^2) \tag{2.5}$$

Where ρ_J determines the skew of the return distribution
Finally

$$\mu_J = \frac{e^{\mu_s + \frac{\sigma_s^2}{2}}}{1 - \rho_J \mu_v} - 1 \tag{2.6}$$

The key features of this model are the jumps in both the stock and variance process, where the stock jumps are lognormally distributed, while the variance jumps are exponentially distributed. There are also two correlated Brownian terms; the first is in the stock price SDE and the second in the variance SDE. A benefit of the SVJJ model is that Black-Scholes, Heston and Merton are all nested inside its dynamics. When the jump parameters are set to zero and volatility is constant the Black-Scholes model is attained, while when just the jump parameters are set to zero the resulting dynamics are Heston.

2.1.2 The Black-Scholes Model

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{2.7}$$

The Black-Scholes model, where share price follows a general geometric Brownian motion, is the simplest of the models considered. It assumes a constant volatility and ignores jumps.

2.1.3 The Heston Model

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^{(1)} \tag{2.8}$$

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma_v \sqrt{\nu_t} dW_t^{(2)} \tag{2.9}$$

$$dW_t^{(1)} dW_t^{(2)} = \rho dt \tag{2.10}$$

The Heston model incorporates stochastic volatility into the dynamics of the stock price, where the volatility process follows an Cox-Ingersoll-Ross process with mean reversion parameter κ and long-term volatility θ. The stochastic volatility component allows testing of the performance of an asset pricing model with changing volatility, therefore providing a modelling framework that can accommodate many of the characteristics observed in the behavior of financial assets. In particular, the σ_v parameter controls the kurtosis of the underlying asset return distribution, while ρ sets its asymmetry. Stochastic volatility is not covered in the Black-Scholes model or the Merton model.

2.1.4 The Merton Model

$$dS_t = (r - \lambda\mu_J)S_t dt + \sigma S_t dW_t + JS_t dN_t \tag{2.11}$$

where

$$\mu_J = e^{\mu_s + 0.5\sigma_s^2} - 1 \tag{2.12}$$

The Merton model takes jumps in the stock price into account, though unlike Heston it assumes constant volatility. The Merton characteristic function was derived and checked with several sources, including a direct derivation in Matsuda (2004) and a modification of the Bates model characteristic function provided in Poklewski-Koziell (2012).

2.1.5 Monte Carlo Simulation

Crucial to this dissertation is the concept of Monte Carlo pricing. For simple options closed-form pricing formulas do exist, though for most exotics Monte Carlo pricing is required. The technique uses repeated sampling to obtain numerical results, which is especially powerful when closed-form solutions do not exist. By sampling enough times, the result obtained converges to the true underlying numerical results. Since three exotics were simulated over three models, with two or three hedging strategies over three strikes and for two models of the real world the number of pieces of information required is 12 (Cliquet) +54 (DOC) +36 (Lookback)=102, each of which requires simulation across 100 paths over 366 days. 100 paths were used due to run-time constraints. This must be borne in mind throughout this dissertation as it does increase the variablility of results.

A key assumption in this paper is that the real and risk-neutral environment are equivalent. This comes through when calibrating to a cross-section of option prices each week, where the dynamics used to calculate the call option prices were risk-neutral. Once the real world data is simulated, a change of measure is usually required to move to the risk-neutral world. This involves using a Girsanov transformation of measure. Mathematically, denoting $\tilde{\mathbb{P}}$ as the risk-neutral world and $\mathbb{P}$ as the real world, to ensure the dynamics are the same we need to assume that the expected return under the real world is equal to r and that all remaining parameters are unchanged when comparing stock dynamics from $\tilde{\mathbb{P}}$ to $\mathbb{P}$.

2.1.6 Call Pricing using the Fourier transform

Call option prices are crucial for Step 3 (calibration), as the practitioner will calibrate the chosen asset pricing model to the available option data. For many models, closed-form formulas for characteristic functions exist where closed-form formulas for the density functions do not. This led naturally to the use of Fourier transforms to determine call option prices since Fourier transforms rely on characteristic functions rather than closed-form density functions. Essentially, the Fourier transform allows call option pricing in terms of an integrated characteristic function, as opposed to a non-existent closed-form formula. The process of using Fourier transforms can be understood as follows:

Define $s_T = \log(S_T)$ where S_T is the value of the underlying at maturity time T of the option and let the risk-neutral density of s_T be given by $\tilde{p}(s_T)$. Also, let $k = \log(K)$, where K is the strike price of the option. Therefore the price of a call option, being the expected discounted payoff is given by

$$c_T(k) = \int_k^\infty e^{-rT}(e^{s_T} - e^k)\tilde{p}(s_T)ds_T$$

The limit of the above function as $k \to -\infty$ results in

$$\lim_{k\to-\infty} c_T(k) = S_0$$

The call pricing function is not square integrable and therefore the Fourier transform does not exist. Carr and Madan get around this problem by defining

$$C_T(k) := e^{\alpha k}c_T(k)$$

where α is a positive constant, leaving $C_T(k)$ square integrable.
The Fourier transform of $C_T(k)$ is then

$$\Psi_T(u) = \int_{-\infty}^{\infty} e^{iuk}C_T(k)dk \tag{2.13}$$

$$= \int_{-\infty}^{\infty} e^{-rT}e^{iuk}e^{\alpha k}\int_{-k}^{\infty}(e^{s_T} - e^k)\tilde{p}(s_T)ds_T\, dk \tag{2.14}$$

$$= \int_{-\infty}^{\infty} e^{-rT}e^{iuk}e^{\alpha k}\int_{-\infty}^{s_T}(e^{s_T+(\alpha+iu)k} - e^{(\alpha+1+iu)k})\tilde{p}(s_T)dk\, ds_T \tag{2.15}$$

$$\text{(change order of integration)} \tag{2.16}$$

$$= \int_{-\infty}^{\infty} e^{-rT}\tilde{p}(s_T)\left[\frac{e^{is_T(u-(\alpha+1)i)}}{(\alpha+iu)(\alpha+1+iu)}\right]ds_T \tag{2.17}$$

$$= \frac{e^{-rT}}{(\alpha+iu)(\alpha+1+iu)}\int_{-\infty}^{\infty} e^{i(u-(\alpha+1)i)s_T}\tilde{p}(s_T)ds_T \tag{2.18}$$

$$= \frac{e^{-rT}\phi_{s_T}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \tag{2.19}$$

where ϕ denotes the characteristic function of the log-stock price. Carr and Madan (1999) then use the inverse Fourier transform of $C_T(k)$ to derive $c_T(k)$ as follows:

$$e^{\alpha k} c_T(k) = C_T(k) \tag{2.20}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \Psi_T(u) du \tag{2.21}$$

Therefore

$$c_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \Psi_T(u) du \tag{2.22}$$
$$= \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} \text{Re}\left[e^{-iuk} \Psi_T(u)\right] du \tag{2.23}$$

The last move holds because $\text{Re}\left[e^{-iuk}\Psi_T(u)\right]$ is an even function (Carr and Madan, 1999). Therefore:

$$c_T(k) = \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} \frac{e^{-rT}\phi_{s_T}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} du \tag{2.24}$$

is the price of a European call option in terms of the characteristic function of the log-stock.

A technique to efficiently execute this result is credited to Cooley and Tukey (1965) and is known as the Fast Fourier transform (FFT). [1] Essentially the FFT provides a fast discretisation technique to compute the integral above. The results of this paper use Simpsons weightings, and the result is that European call option prices can be approximated as:

$$c_T(k) = \text{Re}\left[\frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(v-1)} e^{ibu_j} \Psi_T(u_j) \frac{\Delta}{3}(3 + (-1)^j - \mathbf{1}_{j=1})\right] \tag{2.25}$$

where v is the index of the rescaled log strike vector.

As seen above, characteristic functions are required to calculate vanilla European option prices. The SVJJ characteristic function represents the Fourier transform of the SVJJ probability density function, and allows for efficient calculation of vanilla European call and put prices. These are crucial for setting option prices to which asset pricing models will be calibrated. The SVJJ characteristic function under $\mathbb{P}$ is given by Duffie *et al.* (2000), Gatheral (2011):

$$\begin{aligned} \phi_{S_T}(u) &= \tilde{\mathbb{E}}[e^{iuS_T}] \\ &= \exp\{C(u,T)\theta + D(u,T)V_0 + P(u,T)\lambda + iu(\log(S_0) + rT)\} \end{aligned} \tag{2.26}$$

[1] Though the concepts date back as far as 1805, it was Cooley and Tukey who generalised the concept and described its use on computer.

where

$$P(u,T) = -T(1+iu\mu_J) + \exp\{ui\mu_s + \frac{\sigma_s^2(iu)^2}{2}\}\nu \tag{2.27}$$

$$C(u,T) = \kappa\left[rT - \frac{2}{\sigma_v^2}\log\left(\frac{1-g\exp^{-dT}}{1-g}\right)\right] \tag{2.28}$$

$$D(u,T) = r\left[\frac{1-\exp^{-dT}}{1-g\exp^{-dT}}\right] \tag{2.29}$$

$$\begin{aligned}\nu &= \frac{\beta+d}{(\beta+d)c-2\mu_V\alpha}T + \frac{4\mu_V\alpha}{(dc)^2-(2\mu_V\alpha-\beta c)^2} \\ &\times x\ \log\left[1 - \frac{(d-\beta)c+2\mu_v\alpha}{2dc}\left(1-\exp^{-dT}\right)\right]\end{aligned} \tag{2.30}$$

$$c = 1 - iu\rho_J\mu_V \tag{2.31}$$

$$d = \sqrt{\beta^2 - 4\alpha\gamma} \tag{2.32}$$

$$\beta = \kappa - \rho_J\sigma_v iu \tag{2.33}$$

$$\alpha = \frac{\left(-u^2 - iu\right)}{2} \tag{2.34}$$

$$\gamma = \frac{\sigma_v^2}{2} \tag{2.35}$$

The characteristic function of the Heston model is given by Kilin (2011), Heston (1993), Gatheral (2011), Duffie *et al.* (2000):

$$\begin{aligned}\phi_{S_T}(u) &= \tilde{\mathbb{E}}[e^{iuS_T}] \\ &= \exp\{C(u,T)\theta + D(u,T)V_0 + iu(\log(S_0) + rT)\}\end{aligned} \tag{2.36}$$

where V_0 is the initial value of the variance process, T is the expiration date of the option, and $C(u,T)$ and $D(u,T)$ are the same as for the SVJJ model.

Similarly, the characteristic function of the Merton model is given/implied by Matsuda (2004) and Poklewski-Koziell (2012) as:

$$\begin{aligned}\phi_{S_T}(u) &= \tilde{\mathbb{E}}[e^{iuS_T}] \\ &= \exp\{iu(\log(S_0) + (r - \lambda\mu_J - 0.5\sigma_v^2)T - 0.5\sigma_v^2Tu^2 + \\ &\lambda T(\exp(iu\mu_s - \sigma_s^2u^2/2) - 1)\}\end{aligned} \tag{2.37}$$

where μ_J is as defined in equation (2.6), setting ρ_J equal to zero.

2.2 Exotic Options

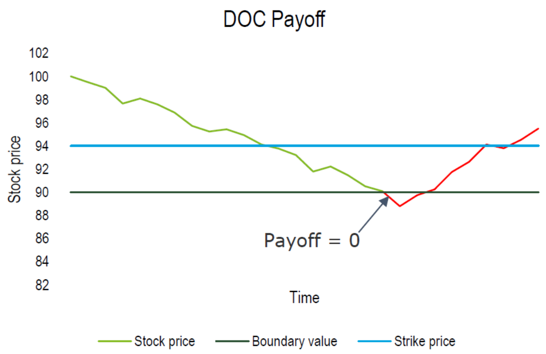
This falls into Steps 2 and 4 of the process. Exotic options provide more complex payoff structures than vanilla European call options, leading to more complex valuation methods. This section begins by discussing the payoffs of three exotic options, through down-and-out call options, lookback options and cliquet options.

2.2.1 Down-and-Out Call Options

The first exotic option considered is a down-and-out call (DOC) option, a specific form of the general class of barrier options. This has the usual payoff of a standard call option, conditional on the share price not touching/crossing a lower boundary during the life of the option, and can be hedged in a variety of ways. The popularity of the down-and-out option over a standard call is that it allows a confident bullish investor a lower premium since paths where the stock dips and then rises again are not taken into account. The DOC also allows for a more complex payoff compared to a standard call when paired with a down-and-in call if the down-and-in call has a different strike. A natural variation of this exotic is an up-and-out call option, where the barrier is some greater value than the strike of the option to ensure the possibility of positive payoff and the same benefits over standard vanilla options apply. In this paper the barrier is set at $L = 90$ and the initial share price is $S_0 = 100$.

$$\text{Payoff}_{\text{DOC}} = (S_T - K)^+ \mathbf{1}_{[\min\limits_{t \leq T} S_t > L]} \tag{2.38}$$

Fig. 2.1: Sample payoff of down-and-out call

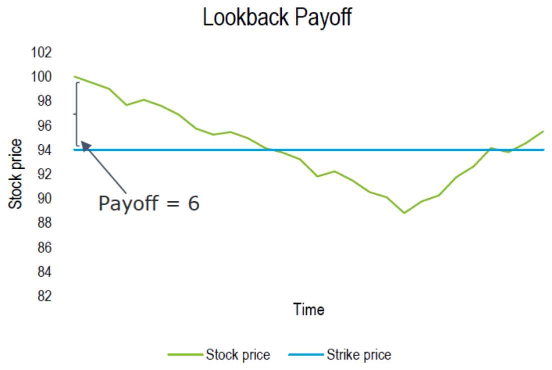


2.2.2 Lookback Options

A lookback option is a derivative contract on the maximum or minimum share price attained over the life of the option. For this dissertation the lookback involves the maximum share price attained. This suits a bullish investor who wants to make use of high volatility to lock into the highest stock price over the option life.

$$\text{Payoff}_{\text{Lookback}} = (\max_{t \leq T} S_t - K)^+ \tag{2.39}$$

Fig. 2.2: Sample payoff of lookback option



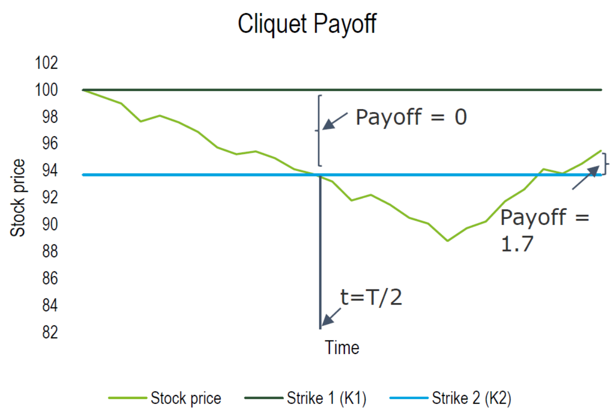
2.2.3 Cliquet Options

A cliquet option is a multiple-derivative contract where the payoff depends on many separate (and usually independent) time-frames. In this paper the cliquet will consist of two six-month call options. The first six-month call will be struck at the current stock price S_0, and the second will be struck at the prevailing stock price six-months into the life of the option. The option is more expensive than a standard call since the investor has a reset opportunity after six months. This suits an investor who is confident that the stock is likely to grow significantly in at least one of the periods, i.e. they are confident the stock will grow at some point in the year, though they are not sure if the stock value has troughed out yet.

$$\text{Payoff}_{\text{Cliquet}} = (e^{rT/2}(S_{T/2} - K_1)^+ + (S_T - K_2)^+) \tag{2.40}$$

where $K_1 = S_0$ and $K_2 = S_{T/2}$.

Fig. 2.3: Sample payoff of cliquet option



Chapter 3

Generating the Data and Calibrating the models

3.1 Data Generation

Monte Carlo requires the simulation of many paths, with Euler-Maruyama as the chosen discretisation method for this paper. For a stochastic process following the SDE:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{3.1}$$

where a,$b \in C^2(\mathbb{R})$ and X_0 constant, the Euler-Maruyama discretisation will approximate the sample path as

$$\hat{X}_i := \begin{cases} X_0, & \text{if } i = 0 \\ \hat{X}_{i-1} + a(\hat{X}_{i-1})\Delta t + b(\hat{X}_{i-1})\Delta W_t & \text{otherwise} \end{cases} \tag{3.2}$$

It is worth noting that if a process depends on another process, such as a stock with stochastic volatility, this volatility process will require its own Euler approximation.

The time period considered is one year with a step size of one day. One debatable topic around the SVJJ model is whether it makes sense to have jumps occuring at the same time in both the stock and volatility process, though intuitively one would expect that a jump in the stock would lead to greater uncertainty among investors and therefore higher volatility. Below is a plot of ten simulations of the stock-price process over one year with 366 time steps. The parameters used for illustration of the model in Figure 3.1 are the same as those used by Poklewski-Koziell (2012) in his initial calibrated parameter description of SVJJ calibrated to S&P 500 options data: $S_0 = 100$, $\kappa = 1.5$, $\theta = V_0 = 0.04$, $\sigma_v = 0.2$, $\rho = 0.8$, $\lambda = 3$, $\mu_s = -0.05$, $\sigma_s = 0.0001$, $\rho_J = -0.4$, $\mu_v = 0.01$. The paths were simulated using the

Euler-Maruyama approach with 366 time steps with the resulting recursion:

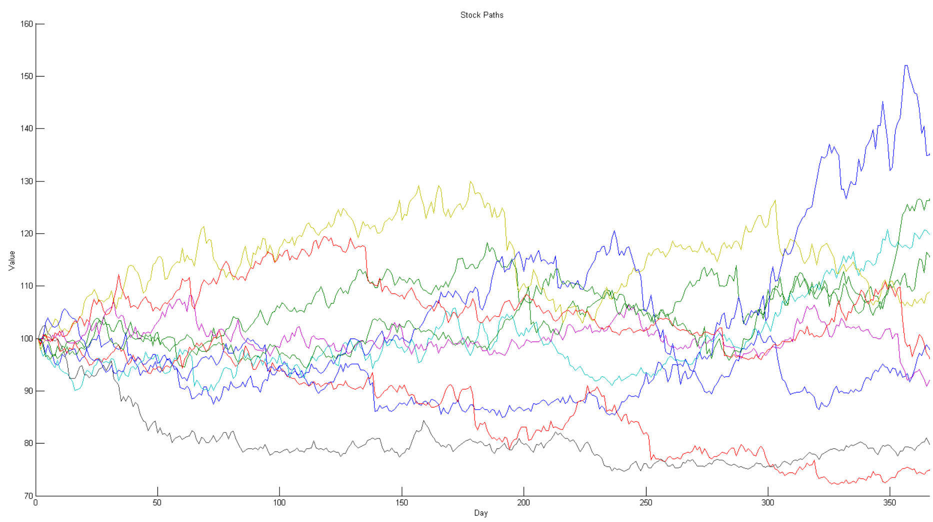
$$S_{t+\Delta t} = S_t + S_t(r - \lambda\mu_J)\Delta t + S_t\sqrt{V_t dt}Z_s + S_t J_1 dN_t \tag{3.3}$$

$$\text{where } V_{t+\Delta t} = V_t + \kappa(\theta - V_t)\Delta t + \sigma_v\sqrt{V_t\Delta t}Z_v + S_t J_2 dN_t \tag{3.4}$$

$$\text{and } Z_v \sim N(0,1), Z_s = \rho Zv + \sqrt{1-\rho^2}Z \tag{3.5}$$

while J_1, J_2 are simulated jump sizes occurring at the same time through dN_t. dN_t is simulated using an exponential distribution with mean parameter $1/\lambda$. The sizes of the jumps are simulated using exponential and log-normal random variables as per (2.4) and (2.5).

Fig. 3.1: 10 SVJJ Stock Price Paths



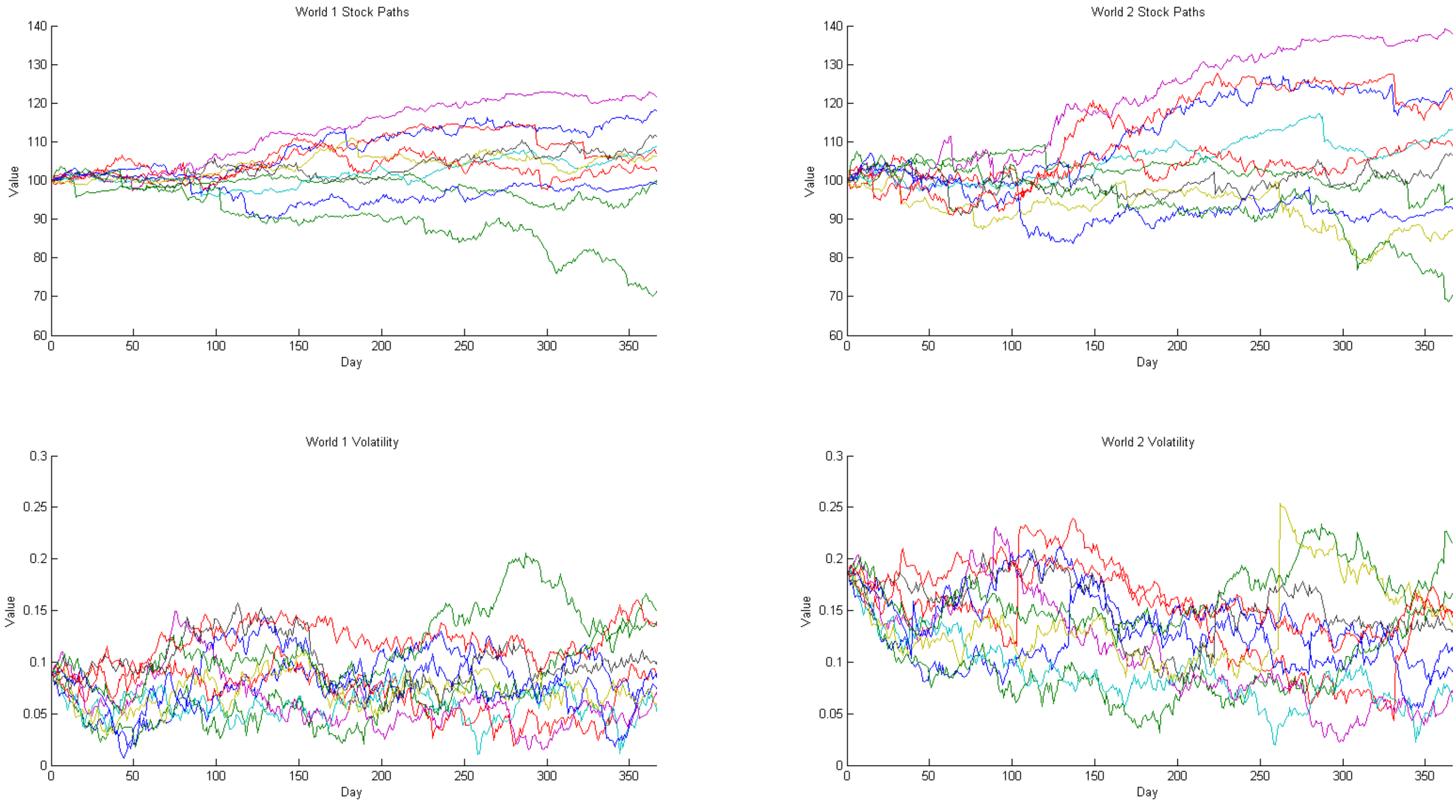
Two versions of the real world were created to be able to analyse hedging effectiveness under more than one state in which the real world is operating. The first World assumes a low volatility and no volatility jumps (μ_v = 0), while stock jumps occur 1.5 times a year on average. World 2 assumes a significantly higher initial volatility and volatility jump term, with the same long-term mean as World 1, which can be understood as recovering from a time of high market uncertainty such as a market crash. Table 3.1 summarises the parameters used in the simulation of each World.

Worlds		
Parameter	World 1 Value	World 2 Value
S_0	100	100
κ	3.5	3.5
θ	0.008	0.008
V_0	0.008	0.035
μ_v	0	0.01
σ_v	0.2	0.2
ρ	0	0
λ	1.5	2
μ_s	-0.04	-0.06
σ_s	0.0001	0.0001
ρ_J	0.04	0.04

Tab. 3.1: SVJJ Parameters in respective Worlds

Figure 3.2 shows ten simulated paths for each World. The jumps are clearly seen in World 2, and is especially visible in a path around day 225. Also visible is the higher volatility experienced in World 2, with a clear downward trend as the market improves until a jump lifts volatility again. One would expect that the Heston model would be relatively more effective in hedging under World 2 since the stochastic volatility term is more prominent and has a bigger influence. To improve interpretation it was decided to leave the correlation terms at zero.

Fig. 3.2: Stock Value Process (Top) and Corresponding Volatility (Bottom)



3.2 Calibration

Calibration falls within Step 3 of the 5-step process and involves assuming the role of the hedge practitioner. Each of the three asset pricing models used by the hedge practitioner requires calibration to prices of market observable instruments. Essentially, this involves determining the appropriate parameters given the call option prices, to minimise the summed square error between the prices the assumed model outputs and the actual prices in the market. These parameters are then used in the hedging of the the more complex, exotic options. The calibration is performed assuming the market instruments are call options with maturities up to one year in one-tenthly intervals and strikes range from 94 to 106 in steps of 2. The theoretical call prices are calculated using the characteristic functions under the Black-Scholes, Heston and Merton model and using the technique of the Fast Fourier transform (FFT) highlighted in Carr and Madan (1999) and discussed in

Chapter 2. The pricing of call options using characteristic functions is central to this paper, meriting inclusion. When calibrating, an efficient starting point for each parameter was the previous week's value since this led to quicker and more accurate convergence. The MATLAB optimizer used was *fmincon*, a non-linear gradient based optimiser, which estimates theoretical parameters to minimise the following equation:

$$\min \sum_i \sum_j (C(t_0, t_i, K_j, \textbf{theoretical}, S_0) - C(t_0, t_i, K_j, \textbf{market}, S_0))^2 \tag{3.6}$$

The theoretical option value is the calibrated call option price under the practitioners model, while the market option value is the call option price calculated using the SVJJ model. This dissertation uses the characteristic function for the Black-Scholes model. Though a closed-form pricing solution is available, well understood and an argument could be made for using it, the intention is to keep the method used for pricing the same between models. This allows like-for-like comparisons to be made and limits differences in results that are purely due to Monte Carlo error. The algorithm used was the *sqp* algorithm, which MATLAB boasts to have superior accuracy and speed than the default *Interior-Point* algorithm. With sqp, the function solves a quadratic programming subproblem at each iteration. The market prices were calculated using Fourier techniques, and represent the SVJJ prices in Worlds 1 and 2. An and Suo (2009) recommend using a market-price weighted minimisation for general model calibration, though when implemented with options near-the-money and near maturity it was found that calibration was less accurate so the least squares approach was kept unweighted. This is in line with the least squares formula used by Poklewski-Koziell (2012). Initial optimisation candidates also included *lsqnonlin*, a non-linear optimisation tool. Since calibration was frequent it was decided to continue with *fmincon* since it is converges faster in general. It is worth noting that *fmincon* is sensitive to the initial point given. To get around this issue (since the practitioner will only have an approximate idea of what the parameter should be) the model parameters were restricted to realistic, plausible values.

Figure 3.3 shows that model prices are very similar to the market call prices in World 1, which can be interpreted that all three models calibrate with a similar level of accuracy. The Merton model is not relatively rewarded for explaining jumps in the stock. This is surprising since there are more parameters over Black-Scholes and therefore one would expect a better fit. Interestingly, moving to World 2 in Figure 3.4 one observes that the Heston calibration significantly outperforms the Merton and Black-Scholes calibration as the calibrated call prices as much closer than the market prices.

Fig. 3.3: World 1 Calibrated call prices with varying terms to maturity and strikes vs Actual call prices.

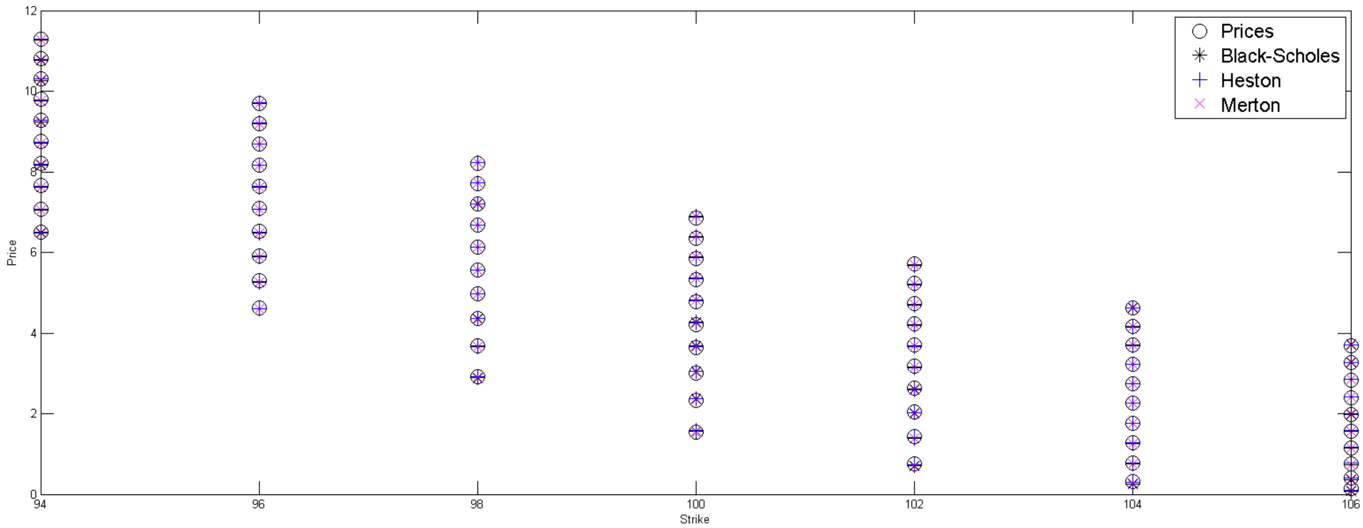


Fig. 3.4: World 2 Calibrated call prices with varying terms to maturity and strikes vs Actual call prices.

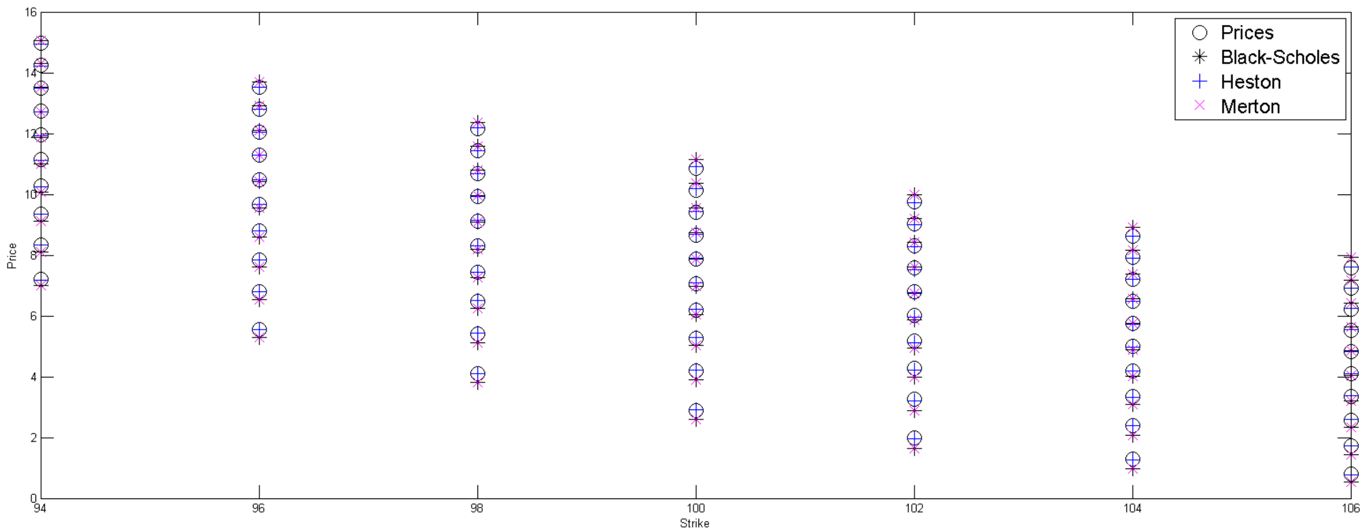


Table 3.2 below highlights the parameters for the three respective models for World 1 (W1) and World 2 (W2).

Heston	W1	W2	Merton	W1	W2	BS	W1	W2
κ	4	3.4569	λ	4.99	2.0873	σ	0.1025	0.1722
θ	0.0102	0.0223	σ	0.0794	0.1721			
σ_v	0.2421	0.3325	μ_s	0.0012	-0.001			
ρ	0.0659	-0.0337	σ_s	0.0294	0.003			
ν	0.0112	0.0432						

Tab. 3.2: Model parameters for estimation and Monte Carlo simulation.

3.2.1 Calibration Issues

An advantage of the Black-Scholes model to a practitioner is that, given a risk-free rate r, the only variable to calibrate to observed market prices is the volatility parameter. This aids in computational efficiency and allows for reliable, fast calibration in MATLAB. Calibration for the Heston and Merton model becomes tricky since multiple-minima may exist for different sets of parameters, some of which may be non-sensical. To attempt to avoid these issues, the following inequalities are placed on σ_v and ρ; $0 \leq \sigma_v \leq 1$ and $-1 \leq \rho \leq 1$, while κ involved inequalities of $2 \leq \kappa \leq 4.5$. Some issues were still arising in the Heston and Merton model after these constraints were set, so a multi-start approach was considered. The multi-start sets the initial point of calibration at many different places, returning a set of calibrated parameters (at a minimum) one at a time, after which the calibrated parameters which result in the the lowest minimum error are returned. This limited the issues involved with spurious minima in the Heston model since a global minimum is more likely to be achieved, though this is not guaranteed. The Merton model still gave errors in some of the calibrations. To overcome the Merton calibration issues, extensive effort was put into alternative constraint combinations and varying the number of options used for calibration. The best alternative constraint combination was borne about through setting the jump volatility parameter, σ_s to a fixed rate of 0.002 via trial and error. This limited the oscillations in both the volatility and jump arrival rate, however hedge errors and ratios were just as large. In addition, pursuing this method further was decided against since a trial and error approach is not available to those hedging.

An important consideration is that calibrated parameters become outdated as time moves forward, therefore updates to parameters are required where the prevailing stock price and market prices of call options are used to calibrate again. Recalibration was executed weekly when hedging these exotic options. Once the practitioner has calibrated the chosen model, it will be used to hedge the exotic contract.

Chapter 4

Hedging methods

Now that the exotics have been described, and the simulation and pricing techniques have been discussed, we continue by describing hedging techniques which practitioners can use to reduce their exposure to changes in stock values over time and therefore changes in the value of the underlying instrument. The purpose of a hedge is to create a self-financing portfolio that provides a payoff that is as close to that of the asset the portfolio is trying to hedge.

4.1 Preliminaries

This dissertation hedges exotic options using a portfolio of a bank account and shares. Where static hedging is considered, vanilla European call and put options are used. The self-financing condition is crucial to the hedging of exotic options. The assumption of self-financing implies that once a combination of cash and the underlying have been entered into at inception of the hedge, no further injections of money are allowed. The bank account will adjust to accommodate changes in the holding of the underlying. Mathematically, if a portfolio has value Val_t, X_t is the holding in stock S_t at time t to hedge and the bank account is worth B_t, then $\text{Val}_t = B_t + X_t S_t$ is self-financing if $d\text{Val}_t = dB_t + X_t dS_t$. To ensure the self-financing condition was not violated when delta and minimum-variance hedging, the bank account was adjusted so that at re-balancing time t, $B_t = B_{t-\Delta t}e^{r\Delta t} - (X_t - X_{t-\Delta t})S_t$. The aim is to construct a portfolio V such that $V_T = C_T$, where C is the derivative to be hedged and T is the option maturity date. We considered three hedging methods:

1. The first method considered is delta hedging, which works because the hedge portfolio holds enough shares to be just as sensitive to stock movements as the exotic itself, while the bank account ensures the portfolio is self-financing.
2. Secondly, minimum variance hedging is considered, which works by setting up the number of shares and bank account to minimise the variance of the

hedging error over each time interval.

3. Finally, static hedging is introduced, which sets up a hedge portfolio at inception comprising of simpler vanilla European call and put options. This works since the range of put option maturities entered into are chosen to make the hedge error of the exotic option as close to zero as possible at maturity of the option.

4.2 Delta Hedging

Delta hedging involves holding a combination of the underlying and the bank account to reduce the risk associated with changes in the share price. To achieve this, an estimate of the delta of the exotic option is required. A few estimating techniques considered were Pathwise, Maximum Likelihood and Central Differencing. Maximum Likelihood and Pathwise techniques potentially offer more accuracy, but they rely on continuity assumptions which barrier options may violate (Papatheodorou, 2005). To avoid these issues and make use of the intuitive appeal of Central Differencing, it was decided to go forward with Central Differencing. The technique relies on two price calculations, one where the option price is calculated by bumping up the share price and the other where the option price is calculated by bumping down the share price. The delta of an option is the change in the options price per unit change in the underlying stock. Mathematically:

$$\text{Delta} \approx \frac{C(t, S_t + \Delta, K, \Theta) - C(t, S_t - \Delta, K, \Theta)}{2\Delta} \tag{4.1}$$

Where C is the price of the option at time t, the strike is K, set of parameters Θ and Δ is the bump size.

Central Differencing was used since the technique leads to a cancellation of a significant number of terms in the Taylor expansion, whereas forward and backward estimation techniques do not. If the real world followed Black-Scholes dynamics and the delta hedged portfolio could be rebalanced continuously the hedge would be perfect. The same is not true for the Heston model, where the delta hedged portfolio still has a significant variance term that can only be reduced through market options where the underlying is the same, known as delta-sigma hedging (Kurpiel and Roncalli, 1998).

Figure 4.1 below plots the stock value for a randomly chosen path over the year of observation in both World 1 and World 2. Figure 4.2 then plots the delta estimates under the three exotic options chosen over the year. The strike of these exotics was equal to 92 with a boundary of $L = 90$ for the DOC.

Fig. 4.1: Stock price path corresponding to Deltas in World 1 (W1) and World 2 (W2)

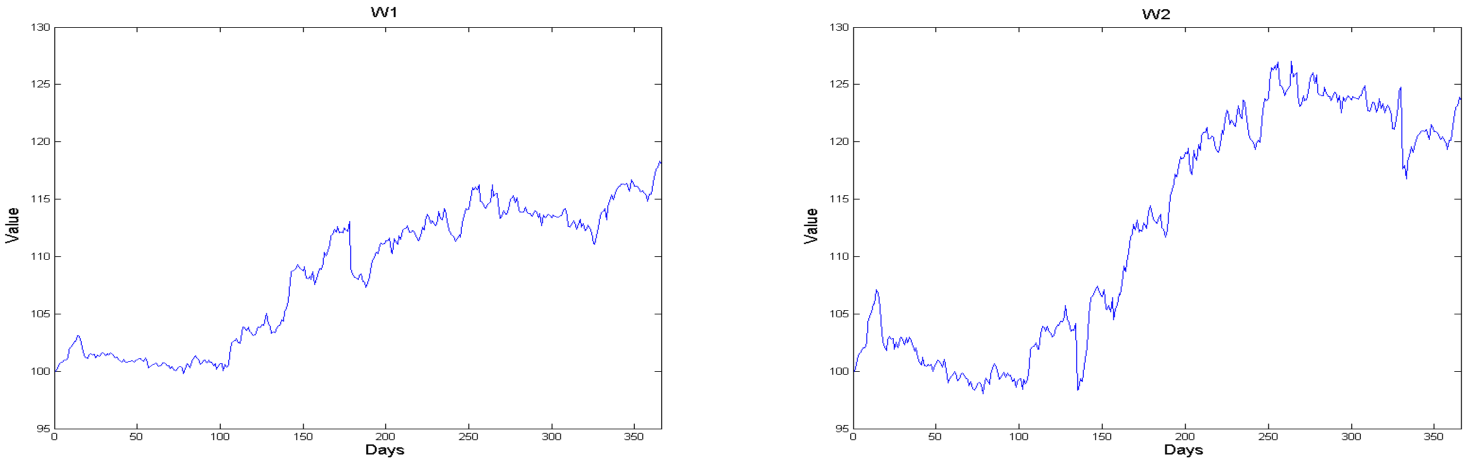
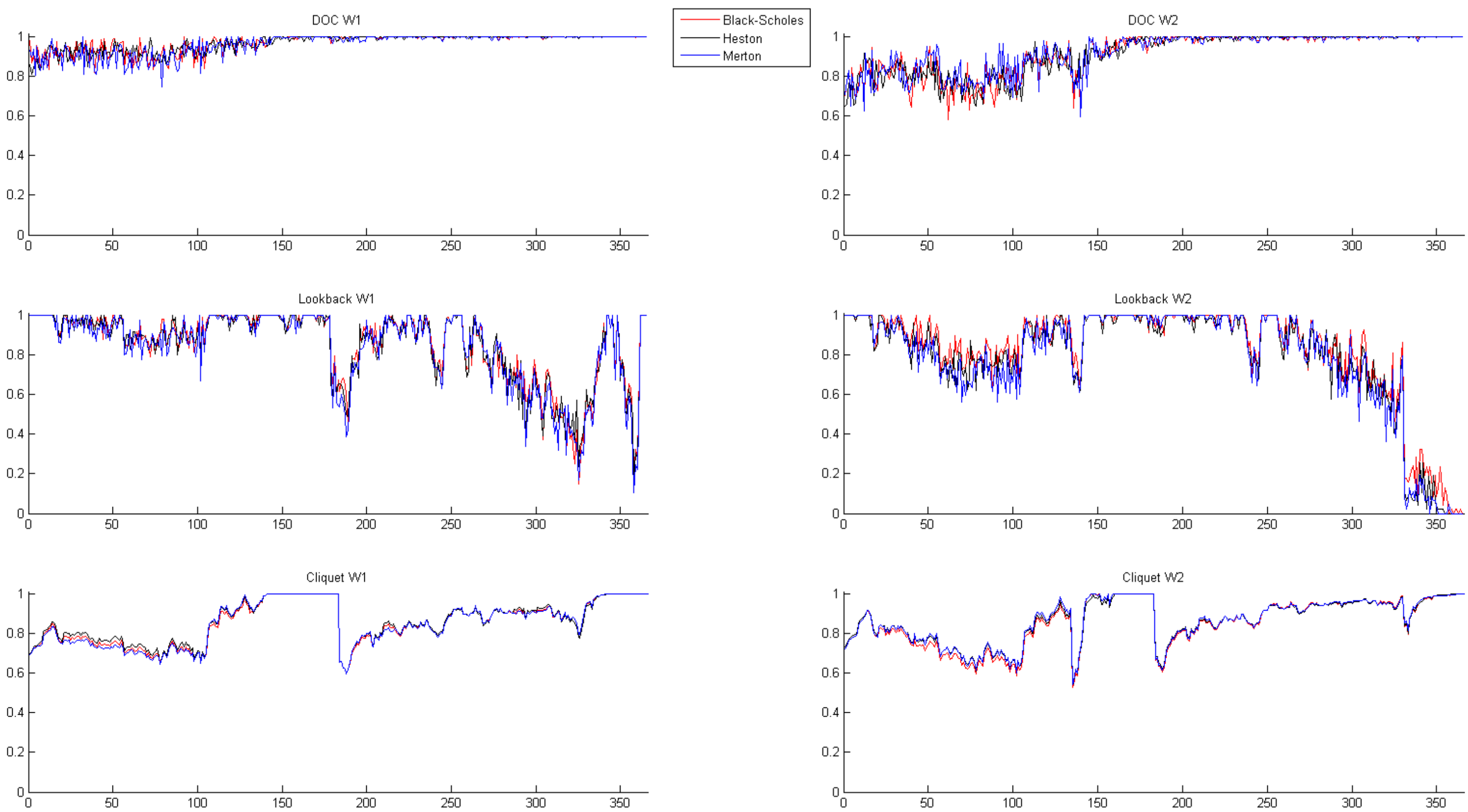


Fig. 4.2: Delta Estimates for W1 (left) and W2 (right), over three Exotics



From Figure 4.2, we see that the delta estimates track each other closely for each option. The delta of the lookback is particularly volatile due to its highly exotic nature. The spike in the cliquet delta is due to the recalculation of the strike half-way through the year. The only notable consistent difference between the deltas is in days 0-100 for the cliquet in World 1, though this difference is not unreasonable. Since central differencing is an approximation, the deltas crept slightly above a value of 1 in some cases. For these cases it was decided to truncate the value to 1.

4.3 Static Hedging

Static hedging was first developed by Derman *et al.* (1995), with Carr *et al.* (1998) developing and generalising the technique for exotic options. The technique involves setting up the hedge portfolio at the start of the contract in such a manner that the portfolio payoff closely matches the payoff of the option.

4.3.1 DOC Static Hedging

This dissertation semi-statically hedges Down-and-Out (DOC) call options, where the barrier level is L. This means that if the stock ever drops below L, the DOC will pay zero at expiry. The technique used in this dissertation involved holding a portfolio of standard vanilla puts and calls and follows the following logic: At each time step considered, calculate the relevant holding in vanilla calls/puts such that if the barrier is hit at that time, the hedge portfolio has zero value, matching the barrier option value. It is also necessary to ensure that if the barrier is not hit, then the hedge portfolio gives the same payoff as the exotic call. This involves holding a vanilla call option with the same strike as the exotic call. The hedge is semi-static in the sense that if the stock ever reaches or drops below L the portfolio will immediately be liquidated. We construct a linear combination of calls and puts such that

$$\Pi(t, S_t) = \sum_{n=1}^{N} \theta_n P(t, S_t, L, T_n) + C(t, S_t, K, T_N)$$

We need to ensure that Π satisfies $\Pi(T, S) = (S - K)^+$, $\Pi(L, T_n) = 0$ for $n = 1, ..., N-1$. A long vanilla call is required to match the payoff at expiry. To ensure $\Pi(T_{N-1}, L) = 0$, choose θ_N such that

$$\theta_N P(T_{N-1}, L, L, T_N) + C(T_{N-1}, L, K, T_N) = 0$$

Following this logic, keep working backwards. Given $\theta_{i+1}, \theta_{i+2}, ..., \theta_N$, add θ_i-many vanilla puts with strike L and expiry T_i so that the portfolio has zero value at T_{i-1}, therefore satisfying

$$\theta_i P(T_{i-1}, L, L, T_i) + \sum_{j=i+1}^{N} \theta_j P(T_{i-1}, L, L, T_j) + C(T_{i-1}, L, K, T_N) = 0$$

The errors when semi-statically hedging the DOC arise mainly due to discretisation. The finer the time partition of call and put maturities, the more immediate a liquidation is likely to be following a drop in the stock below level L. It is also assumed that, at the stage of calculating θ, the put option prices required to calculate θ_i at each time step are valued by the practitioner, where the practitioner uses their assumed asset pricing model. Once the practitioner has identified the put option holdings required, i.e. once θ is calculated, the options are either then found in the market or a market is created for them. These options are then bought at the actual (SVJJ) price. The market prices of the put options is now different to the prices assumed when calculting θ, as the SVJJ model is used to calculate market prices while the practitioner uses a more simplified model (e.g. Heston) in deriving θ. This allows the effect of model error to come through as the correct θ would likely be different under SVJJ. If it were assumed that option values at all strikes and maturities had readily available prices then model error wouldn't come through as the process of calculating θ wouldn't involve any asset pricing models.

4.3.2 Cliquet Static Hedging

There is limited documentation for the static hedging of cliquet options. While some have provided generic theoretical proofs for hedging exotic options, implementation has been centred around applying the concepts to vanilla European options (Carr and Wu, 2002). Allen and Padovani (2002) used a quasi-static approach where the hedge practitioner models the difference between the cliquet price at expiry of the first component (the rollover time, after six months) and the cliquet price at time $t = 0$. This price difference is calculated over a range of possible volatilities and stock prices. The result (after optimisation) is a portfolio of call options where the payoff best matches the differential required to boost the hedge portfolio to the value of the cliquet at the rollover time. The quasi-static hedge involved calculating three components:

- A, a matrix of call option prices at time $t = 0.5$ over a range of strikes, volatilities and stock values.
- b, a vector representing the difference between the cliquet option price at time $t = 0.5$ (over a range of possible stock values at $t = 0.5$) and the price of the cliquet at time t=0.
- x, which represents the weighting in each call option with a particular strike to be entered into at time 0. This is to be solved for.

The objective is to minimise the difference between the replicating portfolio payoff and the cliquet option itself. As mentioned before, this involves making sure that the call option portfolio closely matches the top-up/reduction required to purchase the second component of the cliquet option at time $t = 0.5$. This is written in a formula and the matrix descriptions are included below.

$$\min ||Ax - b||^2, \text{ where } Ax = b \text{ can be viewed as:}$$

$$\begin{bmatrix} C(S_{t,1}, K_1, \sigma_1) & C(S_{t,1}, K_2, \sigma_1) & \dots & C(S_{t,1}, K_k, \sigma_1) \\ C(S_{t,1}, K_1, \sigma_2) & C(S_{t,1}, K_2, \sigma_2) & \dots & C(S_{t,1}, K_k, \sigma_2) \\ \dots & \dots & \dots & \dots \\ C(S_{t,1}, K_1, \sigma_m) & C(S_{t,1}, K_2, \sigma_m) & \dots & C(S_{t,1}, K_k, \sigma_m) \\ C(S_{t,2}, K_1, \sigma_1) & C(S_{t,2}, K_2, \sigma_1) & \dots & C(S_{t,2}, K_k, \sigma_1) \\ \dots & \dots & \dots & \dots \\ C(S_{t,2}, K_1, \sigma_m) & C(S_{t,2}, K_2, \sigma_m) & \dots & C(S_{t,2}, K_k, \sigma_m) \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ .. \\ .. \\ .. \\ x_k \end{bmatrix} = \begin{bmatrix} Cl(S_{t,1}, K, \sigma_1) - V_0 \\ Cl(S_{t,1}, K, \sigma_2) - V_0 \\ \dots \\ Cl(S_{t,1}, K, \sigma_m) - V_0 \\ Cl(S_{t,2}, K, \sigma_1) - V_0 \\ \dots \\ Cl(S_{t,2}, K, \sigma_m) - V_0 \\ \dots \end{bmatrix}$$

$S_{t,1}$,S_{t_2},...,S_{t_k} and σ_1,σ_2,...,σ_m represent a grid of possible stock price and volatility values at time $T/2$. V_0 represents the rolled-forward time 0 value of the cliquet. There is an additional constraint that $v'_{t=0}x = 0$, where v' is a vector of call option prices used in the replicating portfolio. That is, the initial portfolio of replicating call options set up must have zero cost.

As an alternative to quasi-static hedging; semi-static hedging of the cliquet option can be achieved by:

- Entering into a half-year call struck at the current stock price.
- Entering into a half-year forward-starting call with a half-year term, with strike equal to the half-year forward price of the underlying stock.

However, the half-year forward price of the underlying stock will just be the stock accumulated at the risk-free rate for half a year, the since are operating in the risk-neutral environment. Therefore the only error that would come through would be marginal (0.001) due to calibration discrepancies rather than model outperformance. It was decided to use the technique of Allen and Padovani (2002) rather than the simple but non-informative method of using the forward-starting call.

4.4 Minimum Variance Hedging

This technique was explored in detail by (Benninga *et al.*, 1984), (An and Suo, 2009) and (Alexander and Nogueira, 2007), amongst others. Minimum variance hedging involves holding a calculated position in the underlying and the bank account to hedge a written option contract. Given that a practitioner is at time t, the value of the hedge portfolio is

$$H = -C + X_S S + B$$

where $B_t = C_t - X_S S_t$ at $t = 0$, and X_S is the hedge ratio calculated to minimise the variance of H. Clearly the hedge portfolio is self-financing and

$$dH = -dC + X_S dS + rBdt$$

With respect to the hedge ratio, X_s, we seek to minimise

$$\text{Var}(dH) = \text{Var}(dC) + X_S^2 \text{Var}(dS) - 2X_S \text{Cov}(dS, dC)$$

After differentiating with respect to X_s and setting to zero, we obtain a generic minimum variance ratio of

$$X_S = \frac{\text{Cov}(dS, dC)}{\text{Var}(dS)}$$

At time $t + \Delta$t, the value of the hedging portfolio and thus the hedging error is

$$H_{t+\Delta t} = -C_{t+\Delta t} + X_S S_{t+\Delta t} + B_t(1 + r\Delta t)$$

This procedure is repeated until one time step before maturity of the contract. The difficult step in minimum variance hedging is calculating the hedge ratio. In the Black-Scholes model this is simply the delta of the option since the market is complete and an option can be perfectly hedged using the underlying and the bank account. Bakshi *et al.* (1997) derived the hedge ratio for the generic SVJ model. The SVJ model incorporates stochastic volatility, as well as jumps in the stock process, but unlike the SVJJ model it does not include jumps in the volatility process. The hedge ratio for the SVJ model according to Bakshi *et al.* (1997) is:

$$X_S = \frac{\nu}{\nu + V}\frac{\partial C}{\partial S} + \rho\sigma\frac{\partial C}{\partial \nu}\frac{V_t}{S(\nu + V)} + \frac{\lambda}{S(\nu + V)}(\mathbb{E}[JC(t, S(1 + J))] - \mu_J C(t, S))$$

Following their logic one can derive the hedge ratio from the Merton model of:

$$X_S = \frac{\sigma^2}{\sigma^2 + V}\frac{dC}{dS} + \frac{\lambda}{S(\sigma^2 + V)}(\mathbb{E}[JC(t, S(1 + J))] - \mu_J C(t, S)) \tag{4.2}$$

$$\text{where } V = \lambda\mu_J^2 + \lambda(e^{\delta^2} - 1)(1 + \mu_J^2) \tag{4.3}$$

The Merton model experiences an additional source of Monte Carlo error through the estimation of the $\mathbb{E}[JC(t, S(1+J))]$ term when compared to the other two models which don't require the estimation of this term. This is a source of hedging error when minimum variance hedging using the Merton model.

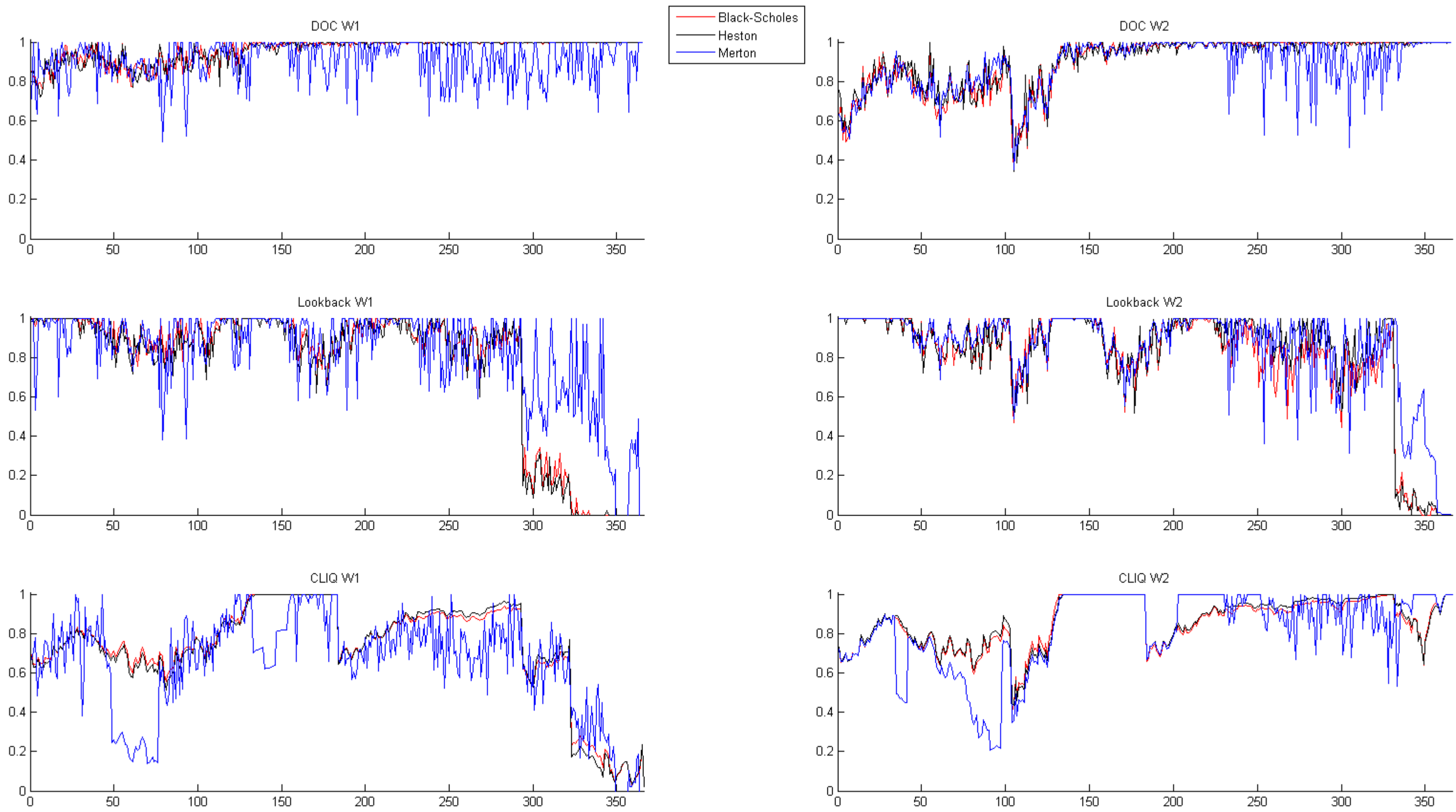
The hedge ratio for the Heston model can be deduced from the general SVJ equation by setting $\lambda = 0$, leading to

$$X_S = \frac{dC}{dS} + \frac{\rho\sigma}{S}\frac{dC}{d\nu}$$

Note that the Heston delta hedge requires the calculation of an extended vega, $\frac{\partial C}{\partial \nu}$, in the sense that ν is a variance in the Heston model. To do this, Central Differencing is used where ν is bumped up and down by Δ and divided by 2Δ.

$$\text{Vega} \approx \frac{C(t,\nu+\Delta,K,\Theta) - C(t,\nu-\Delta,K,\Theta)}{2\Delta} \tag{4.4}$$

Fig. 4.3: Hedge Ratio Estimates



Similar to the delta strategy, the Heston and Black-Scholes models track each other closely. A key observation from Figure 4.3 is the higher volatility experienced in the Merton hedge ratio in blue. Crucially, it seems that Merton is often calibrating correctly through the tracking of the hedge ratio over some portions of the option's lifetime. This is said as the minimum variance estimate is close to the Black-Scholes and Heston estimates through a significant portion of the projection period. For example, consider the World 2 DOC hedge ratio plot. From day 0 to day 225 the hedge ratio resulting from Merton is very close to that of Black-Scholes and Heston. Again, for the lookback in World 2 notice that the Merton hedge ratio is very similar to that of Black-Scholes and Heston all the way until day 330. This gives confidence to the calibration formulas used, and it is suspected that the large oscillations are a direct result of using *fmincon*. Calibration is largely trial and error, with computers able to do several thousand trials, and seems to be limited in its consistency when calibrating for Merton. Reasons for not exploring a completely new calibration method were largely time constraints (for running and developing the model) as well as the heterogeneity this would present when comparing results. For Merton, an array of alternative 'realistic' constraint combinations were tested and the resulting hedge ratios were more oscillatory than those seen above. Since the angle taken in this paper involves the use of MATLAB and non-linear optimisers available this result highlights a potential danger to practitioners that use MATLAB, and are considering the use of the Merton model when minimum variance hedging. An alternative calibration is the Genetic Algorithm, which was implemented by Poklewski-Koziell (2012) and may provide more stable solutions. A downside of his technique is the computational time required, and perhaps a less frequent recalibration could be used to account for this.

Chapter 5

Results

Since Steps 1 to 3 have been completed, Steps 4 and 5 can be carried out. Step 4 involves calculating both the hedge ratios over a range of simulations and the absolute hedge errors after one year. To make results more comparable this paper uses a proportionate approach where the errors are calculated as a proportion of the option price at time T=0. Therefore the results, including errors and standard deviations are calculated as a proportion of the prices of the exotic option at a specific strike (where applicable). So the proportionate error calculated at option maturity T is defined as:

$$\text{Proportionate Error} = \frac{\text{Portfolio Value} - \text{Option Payoff}}{\text{Exotic Price}}$$

Where the portfolio value represents the value of the hedge portfolio, the option payoff is the simulated option payoff and the option price is the observed market price of the exotic option.

Step 5 involves analysing the results and including insights and commentary. This is done in this chapter as well as in the conclusions in the next chapter. The table below summarises the market observable prices for the various options.

	W1			W2		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
DOC	12.95	6.78	2.56	14.35	9.51	5.50
Lookback	16.47	11.56	4.20	23.82	13.87	7.94
	Strike NA			Strike NA		
Cliquet	8.53			13.73		

Tab. 5.1: Exotic Option Prices and strikes for W1 and W2

W1 represents World 1 and W2 represents World 2. As defined earlier, the time to maturity is $T = 1$ year, $S_0 = 100$ and the barrier for the DOC is $L = 90$.

5.1 Summarised Proportionate Errors over Delta and Minimum Variance hedging

This section gives a graphical representation of the results obtained for the out-the-money options in World 1, regarding both their mean and standard deviation. Following this, the mean and standard deviation of the absolute proportionate errors are tabulated for each strike, where proportionate errors are defined at the beginning of the chapter. Therefore the graphical representation below is consistent with the tabulated errors for the out-the-money options. The results (means and variances) were calculated over 100 paths, with weekly recalibration of the practitioner's asset pricing models.

Fig. 5.1: Black- Scholes Proportionate Delta Errors

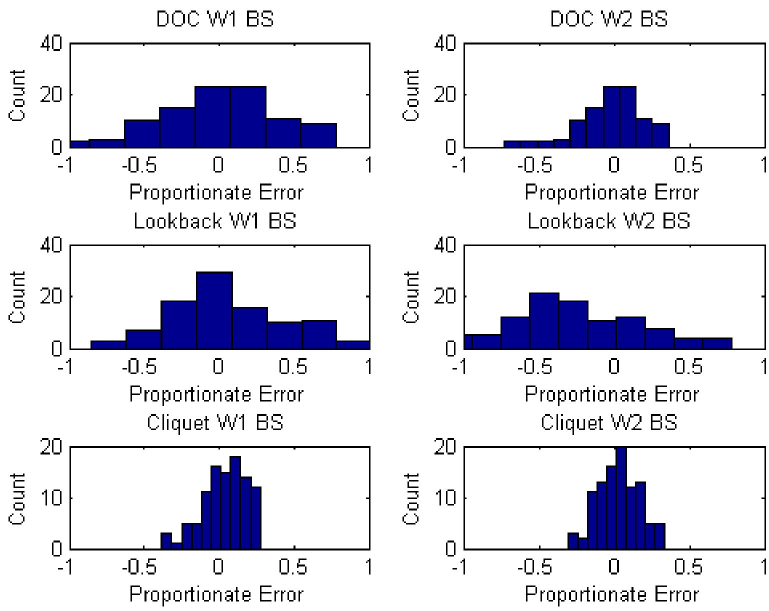


Fig. 5.2: Heston Proportionate Delta Errors

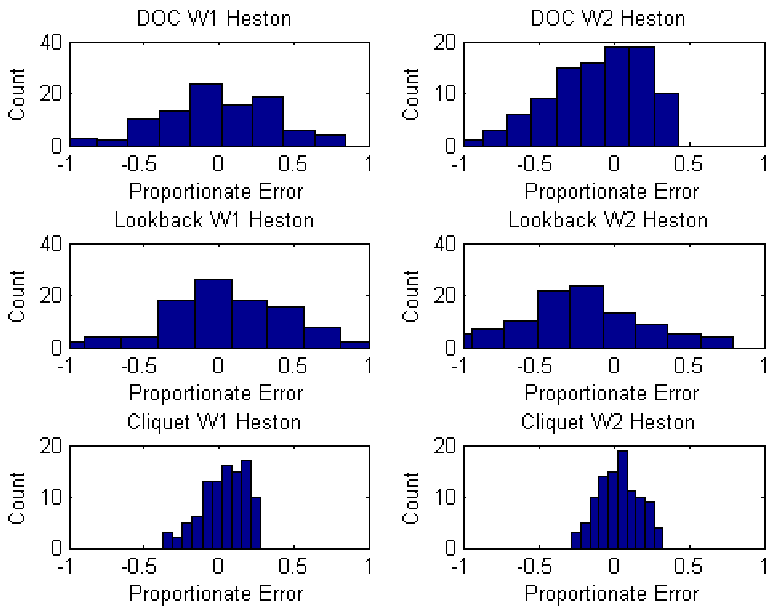


Fig. 5.3: Merton Proportionate Delta Errors

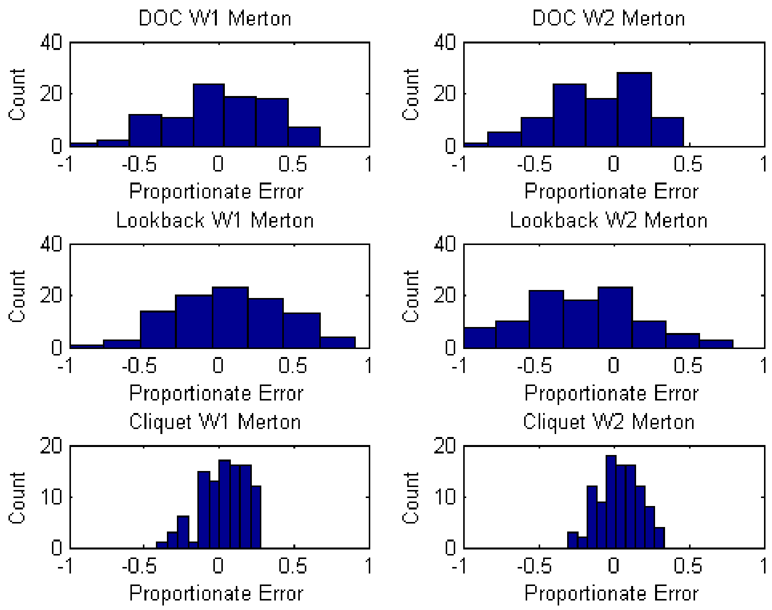


Fig. 5.4: Black- Scholes Proportionate Minimum Variance Errors

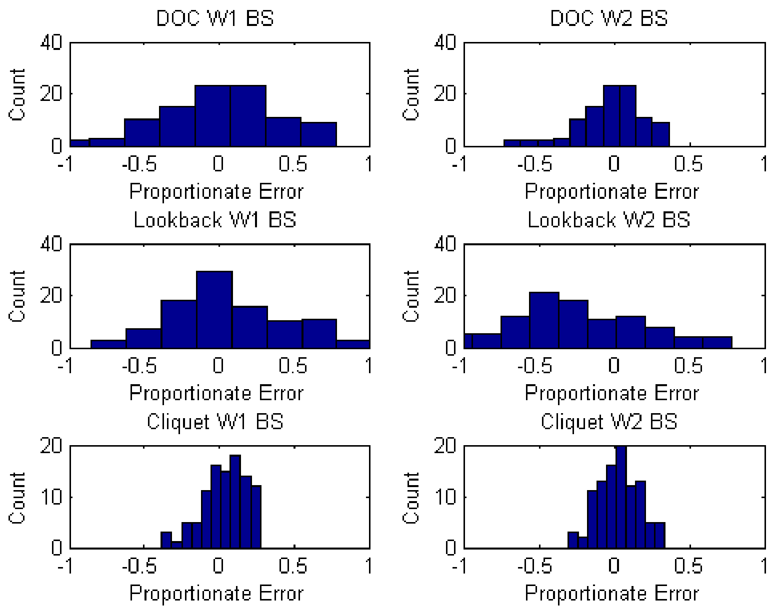


Fig. 5.5: Heston Proportionate Minimum Variance Errors

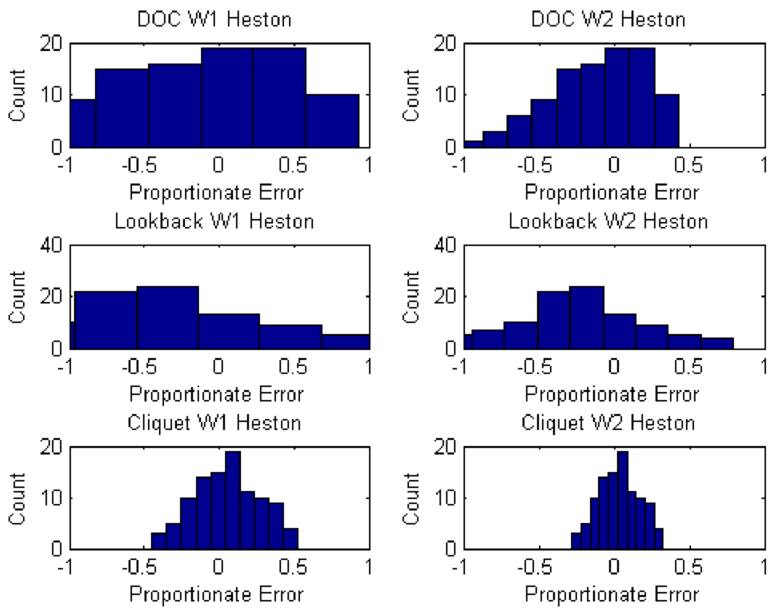
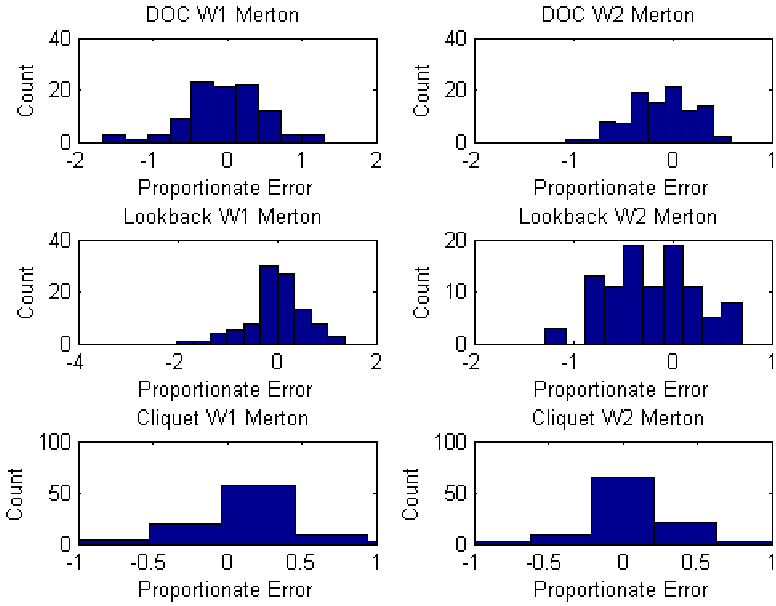


Fig. 5.6: Merton Proportionate Minimum Variance Errors



5.2 Down-and-Out Call Options

We will first present the down-and-out call option (DOC) results.

5.2.1 Static Hedging

The static hedge is less computationally expensive and therefore 2000 paths could be used. For paths which never crossed the barrier of 90 the error was zero since the call at maturity perfectly replicated the payoff of the DOC, which is then effectively a standard-call. The cost of the hedge was not equal to the value of the DOC, as the model used and assumed by the practitioner in deriving the hedge portfolio is not exactly the same as the true underlying model (SVJJ). The difference in value between the hedge portfolio and the DOC is given in the table below.

Given that the replicating portfolio weightings weren't the same as that under the SVJJ model, the resulting errors across the three asset pricing models considered are given below.

Value diff.	World 1			World 2		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
DOC Price	12.95	6.78	2.56	14.35	9.51	5.50
Black-Scholes	-0.5878	-0.1018	0.2712	-0.5761	-0.0683	0.4160
Heston	-0.5594	0.0594	0.4691	-0.5875	0.1521	0.7351
Merton	-0.5105	-0.0968	0.1728	-0.2556	0.8556	0.7780

Tab. 5.2: DOC Static Hedge cost less DOC value

World 1	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.0104	0.0051	0.0043	0.0394	0.0220	0.0123
Heston	0.0136	0.0063	0.0046	0.0449	0.0249	0.0167
Merton	0.0108	0.0153	0.0129	0.0335	0.0458	0.0371

Tab. 5.3: W1 Static Hedging absolute *proportionate* errors

From Table 5.3 there is no outstanding best model, both in mean and standard deviation. However, it is worth noting that for DOC's which are at-the money (ATM) or out-the-money (OTM) the Merton model appears to be less accurate than the others are. This might be due to the calibration issues presented in chapter 4. For in-the-money (ITM) options, the Merton model performs relatively well, achieving the lowest standard deviation.

World 2	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.0399	0.0442	0.0478	0.0818	0.085	0.0909
Heston	0.0444	0.0343	0.0309	0.0858	0.0721	0.0670
Merton	0.0362	0.0843	0.0785	0.0736	0.1375	0.1325

Tab. 5.4: W2 Static Hedging absolute *proportionate* errors

When the dynamics are changed and include more significant jumps in both the stock and volatility process, the Black-Scholes model begins to fail and the Heston model displays strength. In none of the scenarios does it perform best both in mean and in standard deviation. Again, Merton performs well for the ITM DOC but relatively poorly for ATM and OTM DOC's. Before concluding on the hedge performance, one must also factor in transaction and trading costs involved in building a replicating portfolio of options. These are significantly more than the costs associated with trading the stock and underlying; which is the strategy for delta and minimum variance hedging.

5.2.2 Delta Hedging

One DOC complexity involved identifying whether the stock had crossed the barrier, in which case the portfolio was liquidated and calculations ceased for the relevant path.

World 1	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.0726	0.1237	0.3547	0.1141	0.1091	0.2999
Heston	0.0653	0.1188	0.3261	0.1087	0.0908	0.2660
Merton	0.0713	0.1307	0.3250	0.1134	0.1111	0.2959

Tab. 5.5: DOC W1 Delta Hedging absolute *proportionate* errors

A quick glance reveals that the delta hedging strategy performs significantly worse than static hedging, with errors roughly six times larger. However, given that a delta hedging strategy has been employed in World 1 the Heston model performs best in both mean and variance terms, whist Merton performs better than Black-Scholes when OTM and ITM.

World 2	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.2979	0.2541	0.2852	0.1735	0.1801	0.2476
Heston	0.2804	0.2175	0.2860	0.1635	0.1681	0.2409
Merton	0.2826	0.2486	0.2912	0.1737	0.1751	0.2540

Tab. 5.6: DOC W2 Delta Hedging absolute *proportionate* errors

In World 2 Heston again displays its dominance, particularly when ATM. While Merton appears to have a slight edge over Black-Scholes.

5.2.3 Minimum Variance Hedging

The oscillations in the Merton model lead to large hedge errors under stable market conditions (World 1). When market condition are less stable, however, the Merton model hedges well for DOC's. This is despite the calibration difficulties and Monte Carlo techniques required when minimum variance hedging.

World 1	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.0649	0.1075	0.3093	0.1147	0.0883	0.2950
Heston	0.0630	0.1109	0.2942	0.1129	0.0956	0.2881
Merton	0.1027	0.1580	0.4191	0.0928	0.1287	0.3597
	Mean Daily Error			SD Daily Error		
Black-Scholes	0.0237	0.0382	0.0784	0.0172	0.0272	0.0565
Heston	0.0255	0.0404	0.0751	0.0246	0.0402	0.0841
Merton	0.0576	0.0925	0.1751	0.0547	0.0903	0.1865

Tab. 5.7: DOC W1 Minimum Variance Hedging absolute *proportionate* errors

In World 1 Black-Scholes displays the lowest mean daily error, which implies less significant variation in adjustments to the portfolio. Under the World 1 Heston model, mean errors are similar to those when delta hedging, and the only benefit over Black-Scholes arrives when OTM.

World 2	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.2908	0.2361	0.2947	0.1701	0.1702	0.2343
Heston	0.2788	0.2323	0.3225	0.1670	0.1606	0.2511
Merton	0.2380	0.2212	0.2700	0.1726	0.1490	0.2044
	Mean Daily Error			SD Daily Error		
Black-Scholes	0.0315	0.0403	0.0565	0.0362	0.0460	0.0680
Heston	0.0309	0.0393	0.0681	0.0359	0.0459	0.0681
Merton	0.0677	0.0891	0.1267	0.0761	0.1013	0.1510

Tab. 5.8: DOC W2 Minimum Variance Hedging absolute *proportionate* errors

When moving to World 2, mean errors generally increase significantly. Results are still similar between Black-Scholes and Heston, however Merton surprisingly outperforms across all three strikes. This could be due to the larger and more frequent jumps under the World 2 SVJJ model which Merton is better able to account for. Merton also outperforms when looking at mean daily errors and the standard deviation of daily errors.

5.3 Lookback Options

5.3.1 Delta Hedging

Delta hedging for the lookback option involved storing the largest value along time-steps for each path, and hedge errors were significantly lower than for DOC's. Again, results in World 1 were similar across all three models, with Merton outperforming Heston in all aspects at all strikes and holding a lower deviation of results compared to Black-Scholes.

World 1	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.1147	0.1746	0.3342	0.0890	0.1256	0.2938
Heston	0.1128	0.1871	0.3407	0.0976	0.1199	0.2962
Merton	0.1062	0.1744	0.3393	0.0906	0.118	0.2706

Tab. 5.9: Lookback W1 Delta Hedging absolute *proportionate* errors

Moving to World 2, it is easy to see that errors do not change significantly from World 1 when ITM. However, for ATM and ITM errors increase dramatically. Merton holds the lowest mean error when ATM and OTM with Heston outperforming Black-Scholes and holding the lowest error when ITM. Black-Scholes, however, holds the lowest deviation of errors deviation of errors for ATM and OTM, though the difference is not significant.

World 2	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.1157	0.2511	0.412	0.0868	0.1607	0.2666
Heston	0.1084	0.2388	0.3808	0.0880	0.1625	0.2870
Merton	0.1101	0.2210	0.3528	0.0824	0.1696	0.2864

Tab. 5.10: Lookback W2 Delta Hedging absolute *proportionate* errors

5.3.2 Minimum Variance Hedging

In World 1 results are similar between Black-Scholes and Heston, with similar deviations and means. In World 2, Black-Scholes holds the lowest deviation when

ATM and ITM. As seen with the DOC minimum variance hedging, Merton's relative hedging performance improves under World 2 compared to World 1. Unlike the DOC Merton still performs worst in World 1 and World 2.

World 1	Mean Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.1147	0.1746	0.3342	0.0890	0.1256	0.2938
Heston	0.1116	0.1790	0.3324	0.0942	0.1161	0.2737
Merton	0.1710	0.2267	0.4194	0.1713	0.1676	0.3846
	Mean Daily Error			SD Daily Error		
Black-Scholes	0.0411	0.0584	0.1322	0.0597	0.0848	0.1997
Heston	0.0458	0.0651	0.1521	0.0666	0.0949	0.2282
Merton	0.0344	0.0483	0.1122	0.0331	0.0469	0.1152

Tab. 5.11: Lookback W1 Minimum Variance Hedging absolute *proportionate* errors

World 2	Mean Absolute Error			Standard Deviation Error		
Model Used	Strike 92	Strike 100	Strike 108	Strike 92	Strike 100	Strike 108
Black-Scholes	0.1157	0.2511	0.412	0.0868	0.1607	0.2666
Heston	0.1189	0.2535	0.4068	0.0946	0.1741	0.2968
Merton	0.1584	0.2427	0.3996	0.1024	0.1883	0.2820
	Mean Daily Error			SD Daily Error		
Black-Scholes	0.0446	0.0771	0.1259	0.0663	0.1146	0.1902
Heston	0.0441	0.0760	0.1248	0.0508	0.0872	0.1461
Merton	0.0339	0.0576	0.0937	0.0348	0.0610	0.1022

Tab. 5.12: Lookback W2 Minimum Variance Hedging absolute *proportionate* errors

5.4 Cliquet

5.4.1 Static Hedging

The static hedge involved holding a representative portfolio of call options whose value at $t = 0.5$ would match the additional cost of entering into an option at $t = 0.5$ compared to $t = 0$. Two schools of thought can be applied to the Heston model. The first is that the results of the Heston model could be set equal to the results of the Black-Scholes model since volatility is assumed to be constant. The second is an experimental extension, and involves using Heston parameters with a changing volatility at $t = 0.5$. A third and more accurate extension would be to re-calibrate the remaining Heston model parameters to each of the m volatilities considered, $(\sigma_1, \sigma_2, \ldots, \sigma_m)$. For this dissertation, the remaining parameters were kept fixed over the m volatilities considered.

World 1	Mean Error	Standard Deviation Error
Model Used	Strike NA	Strike NA
Black-Scholes	0.1701	0.2871
Heston	0.5290	0.1309
Merton	0.2942	0.3280

Tab. 5.13: Cliquet W1 Static Hedging absolute *proportionate* errors

As expected, the Heston model hedge errors are quite large. This may be due to the remaining calibrated parameters staying fixed while different volatility levels are considered. Bear in mind that if the method of Allen and Padovani (2002) were to be strictly followed and volatility were assumed to be flat, the Heston errors would be exactly the same as the Black-Scholes errors. The experimental Heston errors seen above were calculated for extra knowledge.

World 2	Mean Error	Standard Deviation Error
Model Used	Strike NA	Strike NA
Black-Scholes	0.2133	0.1553
Heston	0.6602	0.2083
Merton	0.1850	0.0773

Tab. 5.14: Cliquet W2 Static Hedging absolute *proportionate* errors

In World 2 jump sizes and frequency are increased. Merton's ability to recognise these jumps leads to the lower hedging errors experienced.

5.4.2 Delta Hedging

Since the cliquet involves two strikes equal to the prevailing stock value at each reset date, the concept of a single *strike* value is not applicable. Cliquets are less sensitive to differences between World 1 and World 2 since the strike of the second option component is only known halfway through the year, partially flattening out half of the years variation. This is evidenced through the hedge errors. Results are very similar between the three models.

World 1	Mean Error	Standard Deviation Error
Model Used	Strike NA	Strike NA
Black-Scholes	0.1286	0.0867
Heston	0.1278	0.0866
Merton	0.1302	0.0889

Tab. 5.15: Cliquet W1 Delta Hedging absolute *proportionate* errors

In contrast to the other two exotics considered, World 2 actually led to lower hedge errors. The models again performed similarly, with Black-Scholes holding a very slight advantage in the means.

World 2	Mean Error	Standard Deviation Error
Model Used	Strike NA	Strike NA
Black-Scholes	0.1136	0.0822
Heston	0.1177	0.0827
Merton	0.1182	0.0823

Tab. 5.16: Cliquet W2 Delta Hedging absolute *proportionate* errors

5.4.3 Minimum Variance Hedging

Again, the Merton oscillations have left inflated error means and deviations. In World 1 Heston marginally outperforms in error and in deviations. However, when in World 2 Black-Scholes gains power and has a (relatively) large improvement in errors and deviations. As seen before, Merton actually improves most between World 1 hedging performance and World 2 hedging performance.

World 1	Mean Error	Standard Deviation Error
Model Used	Strike NA	Strike NA
Black-Scholes	0.1286	0.0867
Heston	0.1249	0.0858
Merton	0.3844	0.4628
	Mean Daily Error	SD Daily Error
Black-Scholes	0.0034	0.0082
Heston	0.0035	0.0085
Merton	0.0271	0.0775

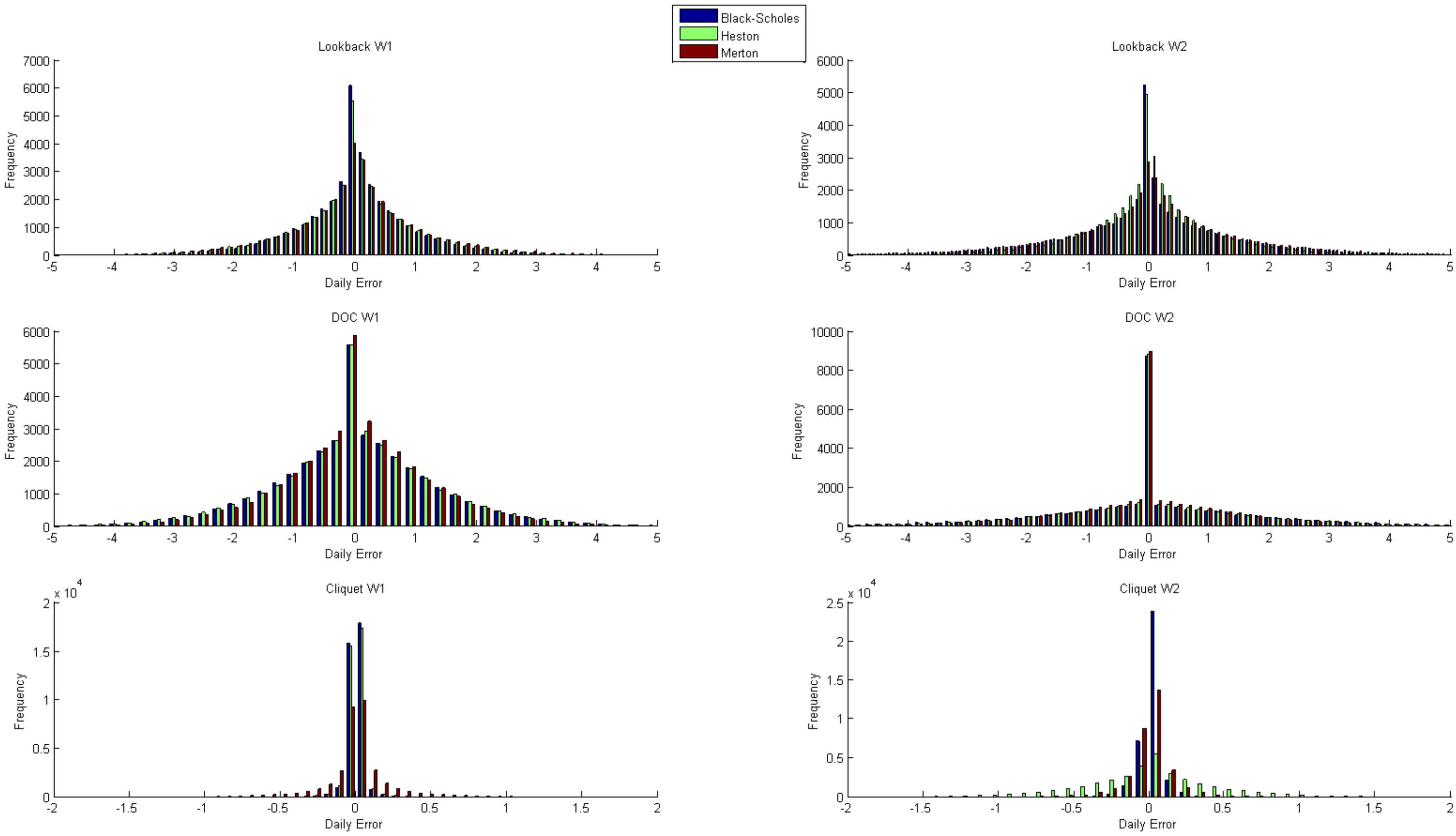
Tab. 5.17: Cliquet W1 Minimum Variance Hedging absolute *proportionate* errors

World 2	Mean Error	Standard Deviation Error
Model Used	Strike NA	Strike NA
Black-Scholes	0.1136	0.0822
Heston	0.1274	0.0892
Merton	0.2438	0.3565
	Mean Daily Error	SD Daily Error
Black-Scholes	0.0039	0.0084
Heston	0.0045	0.0091
Merton	0.0186	0.0610

Tab. 5.18: Cliquet W2 Minimum Variance Hedging absolute *proportionate* errors

Figure 5.7 below details the minimum variance daily errors experienced for the three exotics, over World 1 and World 2 and without any scaling or proportion adjustments for option price. Note the higher peaks and lower deviation for cliquets compared to the other two exotics. This is largely due to the simple breakdown of a cliquet into two vanilla call options, which is more vanilla than the other two exotics. As expected, daily errors are fatter tailed in World 2 due to the increased volatility attached and are centred on zero. An interesting note is that Merton does not appear to perform too poorly when looking at the daily errors, though on closer observation most of the extreme tail values are from the Merton model for cliquet options.

Fig. 5.7: Minimum Variance Daily Errors



Chapter 6

Conclusion and Discussions

6.1 Conclusions

There are several considerations which contribute to a model's effectiveness, as experienced in this paper. Some points of flexibility are: the choice of approximation when calculating deltas and vegas, the method used to calculate the call option prices when calibrating (closed form vs characteristic functions); and the technique used to calculate the exotic price. Other considerations include: the hedging method chosen, the frequency of hedging and re-calibrating, the strikes used, the dynamics of the underlying 'real world' and the type of option hedged. Though the list above is long there are other factors which may lead to a model's demise, regardless of its true predictive and hedging power. In particular, the use of *fmincon* as an optimiser may lead to significant errors when hedging using the Merton model. A possible reason for the volatile calibration parameters is that the jumps in the Merton process cause problems with the very sensitive *fmincon*. Merton is the only asset pricing model with jumps that was investigated. It would be interesting to see if similar problems are encoutered with other jump processes, though this might fall more heavility into the scope of opitimisation, than hedging.

The hedge errors calculated in this paper have been proportional and absolute in nature. As seen in the histograms of errors in Figures 5.1-6, there is usually a spread of errors centred on zero. This means that the expected (non-absolute) error is close to zero, with profits in some simulations and losses in others. As the aim in this dissertation was to examine how close to zero each hedge could be, rather than to offset profit making with loss making hedges, the absolute error was used.

DOC's experience the largest change in errors between Worlds 1 and 2. This is largely due to the significant exotic feature that results in a zero payoff if the barrier is crossed. Since this is more likely to happen in World 2 than World 1, appropriate holdings are more difficult to determine in the former, which leads to larger errors.

The cliquet option can be deconstructed into a series of standard call options, resulting in features which are not as exotic as the other two options considered. This contributes to the lower hedge errors observed for cliquet options compared to lookbacks and DOC's.

When static hedging a DOC, Heston seems to outperform marginally in World 2, though in World 1 there is no clear winner. When statically hedging a cliquet, Black-Scholes performs best under World 1 while Merton performs best under World 2.

Regarding delta hedging, hedge errors are more robust to calibration difficulties with Merton still periodically outperforming the Heston and Black-Scholes model. Overall, in both World 1 and World 2 Heston outperforms when hedging a DOC. For lookbacks again there is no clear winner in World 1 but in World 2 Heston has an advantage overall. With cliquets, Black-Scholes tends to outperform slightly in World 2.

Comparing Minimum Variance hedging performance, we see that Black-Scholes and Heston have a similar performance throughout. It can be seen that Heston slightly outperforms in the hedging of lookbacks while Black-Scholes outperforms when hedging cliquets.

Addressing the aims in Chapter 1 we now confirm: Firstly, there is no overall winning asset pricing model in all circumstances. The outperformance varies between the type of exotic, strike and hedging method. What has been deduced, however, is a useful guideline for hedging exotic options and choosing which method to use. For example, when using minimum variance for a DOC, avoid using the Merton model under low volatility and non-stressed market conditions and when statically hedging a cliquet under stressed market conditions there is comfort in using Black-Scholes. The next issue then becomes identifying when the market is stressed and when it is not. Secondly, there is no hedging technique which performs best in all scenarios. With DOC's, static hedging outperforms. With cliquets, delta hedging and minimum variance hedging perform well (except for Merton). Finally, the danger of using simplified programming techniques is that there is a trade-off between computational efficiency and accuracy. This dissertation involved the calculation of a large amount of data over many loops, and involved multiple calibrations and simulations. Monte Carlo techniques were used, and sometimes nested Monte Carlo was required. In addition, better calibration techniques do exist and may lead to slightly different results. It is also necessary to understand the statistical difficulty in drawing confident conclusions by analysing the means and standard deviations of errors from 100 simulated paths. Also, as Monte Carlo techniques were used, it is possible that the observed differences in hedging errors are attributed to random variations in the Monte Carlo estimates.

More significant results would have been achieved if a more efficient programming language had been chosen, allowing far more sample paths to be run in the same amount of computing time. Compiled languages such as C++ are much faster and are more efficient if used properly.

6.2 Discussion of the Process and Avenues for Further Research

The structure of this dissertation provides a platform for multiple lines of further research. The 5-step process we use can easily be extended to other options with alternative strikes and terms to maturity, as well as other asset pricing models such as the Constant Elasticity of Variance (CEV) model. Asian options were not included in this dissertation as the payoff incorporates an inherent averaging of stock performance, which suppresses the potential for large hedging errors. However, it would be interesting to note how asset pricing models compare when hedging Asian options.

Further, alternative hedging strategies could be considered. In particular, the delta-vega hedging approach used by An and Suo (2009) could be explored, requiring little extra effort compared to what has been done above. An interesting angle would be to consider a few hedging strategies for an exotic option using past data actually observed in the market. This could be used to calculate the hedge error for a single observed path, where calibration is to actual call prices. This removes the need to simulate the real world, with the downside of not having a distribution of errors.

Perhaps most importantly, alternative calibration strategies could be explored which can improve accuracy, especially in the Merton case, without taking an inefficient amount of time as may be experienced when using a Genetic Algorithm.

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