Cosmological Dynamics Of Exponential Gravity

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Abstract

The objective of this thesis is to explore several hotly debated current issues in modern cosmology, with a focus on $f(R)$ gravity.

In chapter 1 we present a review of modern theoretical cosmology. We begin by introducing some fundamental cosmological concepts, followed by a discussion of the field equations of general relativity, which underlie both the behavior of global cosmological models and the isolated gravitating systems such as stars, black holes and galaxies. In particular we focus on the solutions for the Friedmann-Robertson-Walker Universe. Next we present a detailed discussion of the dark matter problem. Astrophysical observations indicate that these two components account for 25% of the total mass/energy of the observable Universe. We then present the big bang model, which represents the current standard model for the origin and the evolution of the Universe. In our discussion we introduce the inflationary scenario in some detail; specifically we present an example of quadratic inflaton to demonstrate how this scenario provides a solution to some of the problems with the standard model. Next we dissussed the dark energy model, which has been introduced to address the late-time acceleration problem. We then present the quintessence model, which has been proposed to address the coincidence and the magnitude problems. We conclude this chapter by a detailed discussion of the higher order theories of gravity with a particular we focus on $f(R)$-gravity, which is based on the idea of introducing corrections to the Einstein-Hilbert action that are negligible in the early Universe and only become effective at late times when the Ricci curvature R decreases. In our discussion we indicate how these corrections can be interpreted as an effective fluid of purely geometrical origin; we also discuss the phase space and stability of deSitter space in $f(R)$ gravity.

In chapter 2 we argue that, owing to the complexcity of the field equations, and the lack of an existence of an exact solutions, dynamical systems analysis is an extermely powerfull mathematical method in cosmology. It is very useful in understanding the qualitative behaviour of the cosmological dynamics and obtaining special exact solutions of the cosmological equations. This chapter contains an overview of this approach. We start by introducing some of the basic dynamical systems concepts and discuss the stability of 2-D linear dynamical systems. We then present the Hartman-Grobman Theorem, which completely solves the problem of determining the stability and qualitative behaviour in a neighbourhood of a hyperbolic critical point of a nonlinear system. Finally we present the center mainfold theorem indepth, which generalizes the Lyapunov theorem to the case when there exists a center mainfold $\mathbf{W}^c$ tangent to the center subspace $\mathbf{E}^c$ at a nonhyperbolic critical point.

In chapter 3 a report is presented, based on the original work completed in collaboration with Sante Carloni and Peter Dunsby which was recently submitted to the Classical and Quantum Gravity Journal. In this research paper we study a cosmological model based on the $\exp(-R/\Lambda)$ action. We apply the dynamical system approach to both the vacuum and matter cases, obtain my exact solutions of the field equations and investigating their stability in the finite and asymptotic regimes. These solutions

demonstrate in a clear manner how higher order gravity can provide a very natural cosmological setting for the early and the late-time accelerating Universe. In particular we find that for a large set of initial conditions a heteroclinic orbits with a non-zero cosmological constant is found to exist.

Finally in chapter 4 we conclude this work by a brief discussion of the possible constraints on the general $f(R)$ gravity Lagrangian and some open problems.

Throughout this thesis we will employ natural units ($\hbar = c = k_B = 8\pi G = 1$) and the signature $(+, -, -, -)$.

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Chapter 1

Introduction to theoretical cosmology

1.1 Introduction

Theoretical Cosmology is the scientific study which aims to explain the origin, composition, ultimate fate and evolution of the entire Universe as a single system. Cosmological theories are based on hypotheses about the Universe which make specific predictions for phenomena that can be tested with observations. The fundamental assumptions of modern cosmology are embodied in the cosmological principle and Einstein's theory of General Relativity. The cosmological principle is not a principle, but rather an assumption that follows from the observations of the Universe on large scales. It states that:

- The Universe is isotropic.
- The Universe is homogeneous.

Isotropy implies that there is no preferred direction in the Universe and, consequently, when the Universe is viewed from a particular point, it looks the same regardless of direction. Homogeneity means that at a given spacetime event the Universe appears the same everywhere. The cosmological principle is a generalization of the Copernican Principle, which states that the Earth is not the centre of the Universe. Homogeneity and isotropy lead to the invariance of the Hamiltonian of a free particle under translations and rotations. Thus the Cosmological Principle simplifies the study of the large scale structure of the Universe considerably. The strongest evidence for the cosmological principle is the observed isotropy of the Cosmic Microwave Background (CMB) [1].

The second fundamental assumption is that the General Theory of Relativity is the correct theory of gravitation. It replaced the Newton theory of gravity which was presented in the Principia in 1687. The fundamental idea in General Relativity is that gravity is a manifestation of the curvature of the spacetime, while in Newton's theory gravity acts directly as a force between bodies. The field equations of General Relativity contain all the information about the dynamics of the the Universe. Many of the predictions of General Relativity, such as the bending of starlight by gravity and a tiny shift in the orbit of the planet Mercury, have been quantitatively confirmed by experiment (for a review, see [2]).

Around 1930 Edwin Hubble observed a direct relationship between the distance between galaxies and the velocity with which they are receding; this relation can be written as,

$$v = H_0 D \,, \tag{1.1}$$

where v is the recessional velocity of the galaxy or other distant object, D is its apparent distance and H_0 is Hubble's constant, $H_{(t_0)} = 100\,h\;Km\,sec^{-1}Mpc^{-1}$ $(h = 0.72 \pm 0.07)$ [3]. Hubble's law has two possible explanations. Either we are at the centre of an explosion of galaxies which is inconsistent with the cosmological principle, or the Universe is uniformly expanding everywhere. This second interpretation of Hubble's observations represents the cornerstone of the Big Bang Theory [4], which is the most viable model in cosmology today.

1.2 Basic Cosmological Concepts

In this subsection we will introduce some of the basic concepts frequently employed in cosmology.

Spacetime metric: is the expression which describes how to compute the spacetime interval between two infinitesimally separated points (or events) in a given geometry, in terms of the coordinates of those points. The simplest spacetime metric, the metric of the flat spacetime, is given by

$$ds^2 = dt^2 - dr^2 \,, \tag{1.2}$$

where dr and dt represent the differences of the space and time coordinates of the two events respectively. The expansion of the Universe is owing to the change in the spacetime metric rather than objects moving in space. The line element of a homogeneous and isotropic Universe is given by Friedmann-Lemaitre-Robertson-Walker (FLRW) metric,

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right), \tag{1.3}$$

where r, θ and ϕ are comoving polar coordinates, $a(t)$ is the scale factor, which is a function of the cosmic time t.

Comoving coordinates: This is a system of coordinates fixed with respect to the overall Hubble flow of the Universe, so that a given galaxy's location in comoving coordinates does not change as the Universe expands. This can be pictured as a grid of lines spread across space, that stretches together with the expansion.

Cosmic Time: is the proper time of observers at rest with respect to the Hubble flow.

Scale factor: The large scale dynamic behaviour of the Universe can be described as a single function of time $a(t)$ called the scale factor. The scale factor describes how the spatial part of the Universe expands or contracts.

Comoving distance: The comoving distance d from an observer to a distant object (e.g. a galaxy) can be computed by the following formula:

$$d = \int_{t_0}^{t} \frac{dt'}{a(t')}, \tag{1.4}$$

where $a(t')$ is the scale factor, t_0 is the cosmic time of emission of the photons detected by the observer and t is the cosmic time "now". The comoving distance between two points remains constant during the evolution of the Universe, while the physical distance between two points does however change as the Universe expands.

Particle Horizon: is the maximum instantaneous distance at t travelled by light since $t = 0$. In terms of comoving distance the particle horizon d_H is equal to

$$d_H = a(t) \int_{t_0=0}^{t} \frac{dt'}{a(t')}. \tag{1.5}$$

The particle horizon coincides with the size of the Universe already seen at the time of observation t.

Redshift: The expansion of the Universe alters the wavelength of the light emitted by distant objects. The ratio of the observed wavelength to the emitted or "rest" wavelength defines the redshift z

$$1 + z = \frac{\lambda_0}{\lambda_e}, \tag{1.6}$$

where λ_0 and λ_e are the observed and emitted wavelength respectively. In terms of the scale of the Universe, the redshift observed now is given by the ratio of the scale of the Universe now to what it was when the light was emitted:

$$1 + z = \frac{a_0}{a(t)}. \tag{1.7}$$

1.3 The Field Equations of General Relativity

The central equations for physical cosmology are the Einstein field equations of General Relativity, which were first published in 1915. The action $S[g]$ that gives rise to Einstein field equations is given by the following integral

$$S[g] = \int [R - L_M] \sqrt{-g}\, d^4x, \tag{1.8}$$

where L_M is the matter Lagrangian, R the scalar curvature (the trace of the Ricci tensor R_{ij}), and g is the determinant of the metric tensor g_{ij}. This action is known as the Einstein-Hilbert action. The variation of $S[g]$, with respect to the metric yields the field equations of General Relativity

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij}, \tag{1.9}$$

where R_{ij} is the Ricci tensor and T_{ij} is the stress-energy tensor. The expression on the left side of the equation 1.9 describes the curvature of spacetime, while the expression on the right side represents the matter (energy) content of spacetime. The field equations 1.9 can then be interpreted as a set of equations dictating how the curvature of spacetime is related to the matter (energy) content of the Universe.

Before 1915, Einstein believed that the Universe was static, but the solutions of the field equations of General Relativity 1.9, suggest that the Universe should be expanding. Einstein therefore proposed a modification of his equations, to

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = -T_{ij}, \tag{1.10}$$

where Λ is a new free parameter, the cosmological constant. The role of the cosmological constant is to balance the attractive force of gravity and hence to allow for a static Universe. After the discovery of the Hubble redshift and the introduction of the expanding space paradigm, Einstein abandoned this concept. In 1998 measurements by two different groups of researchers of the apparent brightness of supernovae with redshifts near $z = 1$ showed that the expansion of the Universe is accelerating [5]. These results have generated a new interest in the old idea of the cosmological constant.

The geometric structure of the simplest Universe is homogeneous and isotropic. In General Relativity, a spatially homogeneous and isotropic Universe is described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. Throughout this thesis we will restrict ourselves to spacetimes which are characterized by such a metric.

1.3.1 The Friedmann-Lemaitre-Robertson-Walker metric

The FLRW metric 1.3 represents an exact solution of the Einstein field equations of General Relativity. It describes a homogeneous and isotropic model of an expanding or contracting Universe. Of course, this is only true on a large cosmological scales.
Given the matter content of the Universe, Einstein field equations allow us to determine $a(t)$. Geometrically, the parameter k which appears in FLRW metric describes the curvature of the spatial sections of spacetime (slices at constant cosmic time).

By solving the field equations 1.9 for the background metric 1.3, Alexander Friedmann in 1922 derived what is now termed the Friedmann equations and most cosmologists agree that the observable Universe is well approximated by such a model.

1.3.2 The Friedmann Model

A model of the Universe is a mathematical description of how the scale factor $a(t)$, and hence of how the Universe, evolves with time. In order to obtain an approximate model of how the Universe evolves, Alexander Friedmann considered a simplified distribution of energy and matter in the Universe. He assumed that the Universe is filled with a perfect fluid with a given energy density ρ and pressure density P. Calculation of the Einstein tensor using the FLRW metric gives

$$G^{00} = \frac{3}{a^2}\left(\dot{a}^2 + k\right), \tag{1.11}$$

$$G^{11} = G^{22} = G^{33} = -\frac{1}{a^2}\left(2a\ddot{a} + \dot{a}^2 + k\right). \tag{1.12}$$

The stress-energy tensor for a perfect fluid is given by

$$T^{ij} = \mathrm{diag}(\rho, P, P, P). \tag{1.13}$$

Given the Einstein tensor and an expression for the stress-energy tensor, Einstein equations 1.9 yield the two independent equations,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2}, \tag{1.14}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho - 3P)\,, \qquad (1.15)$$

where a dot denotes the derivative with respect to the cosmic time t. Equations 1.14 and 1.15 are termed the Friedmann and the Raychaudhuri equations, respectively. The Friedmann equation relates and constrains how the cosmic scale factor a evolves, while the Raychandhuri equation determines how the rate of the expansion of the Universe changes, i.e., whether it is slowing down or speeding up. Further, it is necessary to choose an equation of state, an equation that relates P with ρ. Generally, barotropic perfect fluids obey the equation of state.

$$P = \omega\rho\,, \qquad (1.16)$$

where ω is independent of time. By substituting 1.16 into 1.15, we obtain

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(1 + 3\omega)\,\rho\,. \qquad (1.17)$$

From equations 1.14 and 1.17 we can obtain the continuity equation

$$\dot{\rho} + 3H\,(1 + \omega)\,\rho = 0\,. \qquad (1.18)$$

Equation 1.18 is in fact the energy conservation law. The first term $\dot{\rho}$ tells us how fast density changes, while the second term is the loss of kinetic energy from the fluid, into gravitational potential energy. By integrating this equation we obtain

$$\rho \propto a^{-3(1+\omega)}\,. \qquad (1.19)$$

In cosmology, dust refers to a non-relativistic matter which possesses an equation of state parameter $\omega = 0$. For a dust-dominated Universe, equation 1.19 becomes

$$\rho_m \propto a^{-3}\,. \qquad (1.20)$$

This equation tells us that the density in a comoving volume decreases owing to Hubble expansion. For the radiation-dominated Universe where $\omega = \frac{1}{3}$, the energy density falls off as

$$\rho_r \propto a^{-4}\,. \qquad (1.21)$$

It is evident from equation 1.17, that the expansion rate is decelerating ($\ddot{a} < 0$) for dust and radiation dominated Universes. The radiation density decreases with a^4 rather than a^3 as in matter case. The extra a stems from the redshift effect on the wavelength of the radiation. The last case we need to consider is that of vacuum energy. For a Universe dominated by the cosmological constant $\omega = -1$, it follows that,

$$\rho_\Lambda = Const. \tag{1.22}$$

It is clear from equation 1.17 that when the Universe is dominated by a fluid with an equation of state ($\omega < -\frac{1}{3}$), $\ddot{a}$ will be positive and the expansion of the Universe will be accelerated. Such a fluid with negative pressure is referred to as dark energy. The recent astronomical observations strongly support the view that the Universe is now accelerating [5], which indicates that our Universe is currently dominated by some kind of dark energy.

1.3.3 Important cosmological Parameters

In this subsection we will introduce some of the important cosmological parameters that are frequently utilised to describe the expansion of the Universe.

Hubble's parameter: is defined as

$$H = \frac{\dot{a}(t)}{a(t)}, \tag{1.23}$$

where $a(t)$ is the scale factor. Hubble's parameter is a time-dependent function which measures the rate of expansion of the Universe. The Hubble constant $H_0 = H(t_0)$ is the Hubble parameter measured at the present time.

The Density parameter: In 1917, Einstein and de Sitter both developed cosmological models by assuming that the Universe is dominated by matter with zero curvature ($k = 0$). By substituting ($k = 0$) into Friedmann Equations and solving for ρ we obtain

$$\rho = \rho_{crit} = 3H^2. \tag{1.24}$$

The critical density ρ_{crit} is the amount of matter required for the Universe to be exactly flat ($k = 0$). The recent observations of the Degree Angular Scale Interferometer ($DASI$) suggest this type of geometry [6]. From 1.24 we notice that ρ_{crit} depends on the Hubble parameter, which means that ρ_{crit} changes as the Universe evolves.

The fraction $\Omega = \rho/\rho_{crit}$ is known as the density parameter. According to the recent ($DASI$) observations [6], the present value of Ω is $\Omega_0 = 1 \pm 0.04$.

In general, the total density Ω is the sum of the contributions from radiation, matter and vacuum energy. The Friedmann Equation 1.14 can now be written in the compact form

$$\Omega - 1 = \frac{k}{H^2 a^2}. \tag{1.25}$$

According to the equation 1.15, if $\ddot{a} < 0$ then the term $H^2 a^2 = \dot{a}^2$ will always be decreasing. Thus Ω will always evolve away from unity.

It is clear from equation 1.25 that the sign of k depends on the value of Ω

$$\begin{aligned}
\Omega &< 1 \text{ or } \rho < \rho_{crit} \leftrightarrow k = -1 \leftrightarrow \text{ open 3-spaces},\\
\Omega &= 1 \text{ or } \rho = \rho_{crit} \leftrightarrow k = 0 \leftrightarrow \text{ flat 3-spaces},\\
\Omega &> 1 \text{ or } \rho > \rho_{crit} \leftrightarrow k = +1 \leftrightarrow \text{ closed 3-spaces}.
\end{aligned}$$

By solving equation 1.25 for each case where one of the three terms dominates we obtain

$$\begin{aligned}
a(t) &\propto t^{\frac{1}{2}} \quad \text{radiation dominated},\\
a(t) &\propto t^{\frac{2}{3}} \quad \text{matter dominated},\\
a(t) &\propto e^{Ht} \quad \text{Cosmological constant dominated}.
\end{aligned}$$

In a Universe containing matter and radiation, matter falls off slower than radiation. At early times, radiation dominates the matter in Universe, and after some time from the Big Bang, there must be a time when $\rho_{matter} = \rho_{radiation}$. This is known as matter-radiation equality. Thereafter, matter will begin to dominate the expansion of the Universe.

Using equations 1.6, 1.20 and 1.21, we can write the ratio of ρ_m to ρ_r as

$$\frac{\rho_m}{\rho_r} = \frac{\rho_{m0}}{\rho_{r0}}(1+z)^{-1}. \tag{1.26}$$

The Deceleration parameter: The deceleration parameter q is another means of quantifying the rate of expansion of the Universe, which is defined (for $k = 0$) by the equation

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\rho + 3P}{2\rho}. \tag{1.27}$$

The observations of type Ia supernova [5] indicate that the Universe is accelerating i.e., $q_0 < 0$.

1.4 Dark Matter

Dark Matter is a hypothetical matter of unknown composition that does not emit or reflect enough electromagnetic radiation to be observed directly. The presence of this kind of matter can only be inferred from its gravitational effects on visible matter [11]. Therefore astronomers are faced with quite a challenge detecting it. Astrophysical observations indicate that dark matter accounts for 26% of the matter in the observable Universe [14][15].

One of the most important techniques in studying the motions of galaxies is the Doppler Shift, which results in either a redshift or blueshift. The spectral lines from an approaching astronomical light source result in a blueshift, while those of receding sources result in a redshift. We know that galaxies rotate because, when viewed edge-on, the light from one side of the galaxy is blue shifted and the light from the other side is redshifted. In order to determine the speed at which the galaxy is rotating we can use the formula [12]

$$v = \frac{(\lambda_o - \lambda_e)}{\lambda_e}, \tag{1.28}$$

where λ_o and λ_e are the observed and emitted wave lengths respectively. Knowing the rotation speed of the galaxy, we can estimate the mass of a galaxy. For example, let us consider a star of mass m near the edge of a spiral galaxy (distance R_g), moving in a circular orbit at speed v. The centripetal force F_c that acts on this mass is given by

$$F_c = \frac{mv^2}{R_g}. \tag{1.29}$$

From the Newtonian law of gravitation this force is equal to,

$$F_G = \frac{GmM}{R_g^2}, \tag{1.30}$$

where M is the mass inside the orbit. Units have been re-introduced. By equating equations 1.29 and 1.30 and solving for M, we obtain

$$M = \frac{R_g v^2}{G}. \tag{1.31}$$

Over half a century ago, when the astronomer Fritz Zwicky was studying the motions of distant galaxies, he estimated the total mass of a group of galaxies by measuring their brightness. When he computed the mass of the same cluster of galaxies by observing the rotation speed method, he came up with a number that was 400 times his original estimate [13].

According to Kepler's third law the speed v of individual stars is related to their distance R_g from the centre of the galaxy by the equation

$$v = \sqrt{\frac{M}{R_g}}. \tag{1.32}$$

If R_g lies outside the visible part of the galaxy, we expect that $v \propto 1/\sqrt{R_g}$. Instead, in most galaxies one finds that v becomes approximately constant out to the largest values of R_g where the rotation orbit can be measured. In some galaxies, the stars move at speeds that should rip the galaxy apart. This implies the existence of a dark halo, with mass $M \propto R_g$.

The nature of the dark matter remains an unresolved puzzle and the current claim is that most of the dark matter in the universe is of an exotic nature. Consequently, investigations of dark matter is not only vital to explain the gravitational motions of galaxies, but also important to test current theories of gravity.

Several categories of dark matter have been postulated:

- Hot dark matter.
- Cold dark matter.

Hot dark matter consists of particles that travel with relativistic velocities (the neutrinos are a prime example). On the other hand, cold dark matter is composed of objects sufficiently massive that they move at sub-relativistic velocities. The difference between cold dark matter and hot dark matter is significant in the formation of structure, because the high velocities of hot dark matter causes it to wipe out the structure on small scales.

1.5 The Big Bang Cosmology

We have previously mentioned Hubble's discovery of a direct relationship between the distance between galaxies and the velocity with which they are receding. If we consider this observational fact together with the cosmological principle we reach the conclusion that the Universe is expanding. Conversely, if this expansion has continued over the entire age of the Universe, then in the past, these distant, receding objects must once have been closer together. This idea gave rise to the Big Bang theory, the dominant model in cosmology today. In the next section we will discuss some of the observational evidence for this model.

1.6 Supporting Evidence for the Big Bang

1.6.1 The Cosmic Microwave Background (CMB)

The cosmic microwave background radiation (CMB) is a form of electromagnetic radiation that fills the entire Universe and radiates uniformly from all directions in the sky. It has a thermal 2.7 Kelvin black body spectrum which peaks in the microwave range at a frequency of 160.2 GHz (for a review, see [1]). Presently the Universe is dominated by dust, but because radiation dilutes more quickly with the expansion, this implies that at earlier times the Universe was dominated by radiation. Today the CMB fills the Universe with photons of temperature $T \sim 2.7$ K. In order to see how the CMB temperature varies with the expansion of the Universe let us consider a volume of photons. According to the first law of thermodynamics, the amount of heat dQ flowing into or out of the volume is given by

$$dQ = dE + P\,dV\,, \tag{1.33}$$

where dE is the resultant change in the internal energy inside the volume, and $P\,dV$ is the work done by/on the system. By assuming $dQ = 0$, equation 1.34 can be written as

$$\frac{dE}{dt} = -P(t)\frac{dV}{dt}\,. \tag{1.34}$$

By substituting $E = aT^4V$ and $P = \frac{1}{3}aT^4$, the equation 1.34 can be written as

$$4T^3\frac{dT}{dt}V + T^4\frac{dV}{dt} = -\frac{1}{3}T^4\frac{dV}{dt}\,. \tag{1.35}$$

Using $V \propto a(t)^3$, we obtain

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{3}\frac{1}{V}\frac{dV}{dt}\,, \tag{1.36}$$

which has the solution

$$T(t) \propto a(t)^{-1}\,. \tag{1.37}$$

It is clear that the CMB arises naturally if the Universe began in a hot and dense state. The discovery of the CMB radiation, secured the Big Bang as the best theory of the origin and evolution of the Universe.

1.6.2 The Big Bang Nucleosynthesis (BBN)

Nucleosynthesis is the process of creating new atomic nuclei from pre-existing nucleons (protons and neutrons). BBN occurred within the first three minutes of the Universe and is responsible for much of the abundance ratios of 1H (hydrogen), 2H (deuterium), 3He (helium-3), and 4He (helium-4), in the Universe. The BBN predicts mass abundances of approximately 75% of H-1, approximately 25% helium-4, and approximately 0.01% of deuterium. The observed abundances in the Universe are found to be consistent with these numbers. Thus BBN also provides strong evidence for the Big Bang theory.

1.6.3 The History of the Universe

The Big Bang model provides us with a fairly clear picture of how the Universe evolved. The key events in the history of the Universe are:

- 10^{-4} seconds: Formation of protons and neutrons;
- 1 second: Formation of light nuclei, via the nucleosynthesis process;
- 10^4 years: The Universe reaches the matter-radiation equality state;
- 10^5 years: Formation of the CMB owing to the decoupling of radiation from matter. This is coincident with the recombination epoch, when free electrons combine with the nuclei to form atoms, and
- 10^{10} years: The present Universe.

Theoretical support for the Big Bang theory is derived from the Friedmann model, which illustrates the consistency of the Big Bang theory with General Relativity. The hot Big Bang theory has been extremely successful in correlating the observable properties of our Universe.

There are however some difficulties with this simple picture and we summarize them below.

1.7 Problems with the Big Bang Theory

1.7.1 The Horizon Problem

Observations of the cosmic microwave background reveal that it is highly isotropic. Radiation on opposite sides of the observable Universe today appears uniform in temperature. If the cosmic microwave background is at such a uniform temperature, it should mean that the photons have been thermalized through repeated particle collisions.

If we consider the distance, a photon could have travelled since the beginning of the Big Bang at $t = 0$ to the time of recombination (the time when hydrogen and helium atoms begin to form) compared to the comoving radius at the present

$$\int_0^{t_{rec}} \frac{dt}{a(t)} << \int_{t_{rec}}^{t_0} \frac{dt}{a(t)},$$

we conclude that the horizon size established at the recombination time (when free electrons combine with the nuclei to form atoms) is much smaller than necessary to account for the isotropy observed in the CMB today. This is the so called horizon problem.

1.7.2 The Flatness Problem

Observations show that Ω is close to one [6], which means that it must have been much closer to unity in the past. For example, we require $\Omega - 1 < \mathcal{O}(10^{-16})$ at the epoch of the nucleosynthesis [8] and $\Omega - 1 < \mathcal{O}(10^{-64})$ at the Planck epoch [9]. The question that arises is how could the average density of the Universe be identical to the critical density so soon after the Big Bang? We know that the critical density ρ_{cri} changes as the Universe expands: thus for the Universe to be "fine-tuned" to this precision is highly improbable. This is called the Flatness Problem.

1.7.3 The Galaxy Formation Problem

One of the tenets of the Big Bang theory is that the beginning of the Universe was smooth and homogeneous. The question that arises in this instance is how could galactic structures have arisen from a smooth distribution of matter/energy in the early Universe? This is the so called Galaxy Formation Problem.

1.8 Inflationary Universe

In this section we will see how the introduction of an inflationary stage [10] in the early Universe offers an elegant solution to the above problems. Inflation is a general term for scenarios of the very early Universe, all of which share the common feature that the Universe experiences a finite period of accelerated expansion, while it is in a vacuum-like state containing only homogeneous classical fields. After inflation, the initial vacuum state decay into elementary particles.

According to equation 1.15, the accelerated expansion of the Universe can be achieved if the condition

$$\rho + 3P < 0 \tag{1.38}$$

is satisfied. For example, let us consider the Friedmann Equation 1.14. Since the contributions to the energy density are redshifted away as the Universe expands ($\rho_M \sim a^{-3}, \rho_R \sim a^{-4}$ and $\rho_K \sim a^{-2}$), the cosmological constant will eventually be the dominant form of energy. By ignoring all the contributions to the energy density other than the cosmological constant, the Friedmann equation 1.14 becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_\Lambda}{3}, \tag{1.39}$$

This differential equation has the solution

$$a(t) \sim e^{Ht}, \tag{1.40}$$

where $H = (\frac{\rho_\Lambda}{3})$. From the equation 1.25 it is evident that, as a and H increase by tremendous amounts during inflation, the right hand side of the equation 1.25 approaches zero. Thus, Ω is driven towards unity and the spatial slice of the Universe is made flat by inflation.

Inflation result in an overlap of the past histories of casual disadjoint regions. Thus these regions can interact and make the microwave background uniform [?]. Also inflation explains the origin of the large scale structure of the cosmos as resulting from quantum fluctuations in the microscopic inflationary region [18]. The theory states that the spontaneous fluctuations in the pre-inflationary epoch were greatly magnified by inflation. In the post-inflationary cosmos, these fluctuations produced regions just slightly denser than their surroundings. The differences in density were in turn amplified by gravity, which pulls matter into the denser regions. Thus inflation provides a mechanism to generate the seeds of density perturbations observed in the CMB anisotropies.

Currently no other theory can explain both why the Universe is so uniform overall, and yet contains exactly the kind of "ripples" represented by the distribution of galaxies through space and by the variations in the cosmic microwave background radiation.

The problem with the previous inflationary model is that there is no way to halt such rapid evolution once it has started, and eventually the Universe will be empty of matter and radiation. In 1980 Guth developed a new inflationary scenario, by introducing some kind of cosmological scalar field $\phi(t, r)$. Guth's intention was to limit the amount of time during which that this rapid expansion occurs. The mechanism of the Guth inflation corresponds to a phase transition in the early Universe. In the next section we present a brief review of this scenario.

1.8.1 Inflation and Scalar Fields

Scalar fields have long been used in many unified particle theories, for example, the Higgs field of the Standard Model, as well as the pion field mediating the strong nuclear interaction. Assuming the existence of a scalar field $\phi(t)$, with a potential energy density $V(\phi)$, we can define its Lagragian density L as

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{1.41}$$

The action is then

$$S = \int d^3x\, dt L = \int d^3x dt \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]. \tag{1.42}$$

The equation of motion for this theory is obtained by extremizing the action above with respect to the scalar field ϕ

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \tag{1.43}$$

where the prime represents differentiation with respect to ϕ. The term containing the Hubble constant serves as a kind of friction term resulting from the expansion. By using Noether's theorem one can define the stress-energy tensor

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}L. \tag{1.44}$$

Comparing 1.44 and 1.13 yields

$$\rho \;=\; T^{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.45}$$

$$P \;=\; \frac{1}{3}\sum_i T^{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.46}$$

In deriving these equations any gradient contributions are eliminated, because any inhomogeneity in the field will eventually be redshifted away by the expansion. The terms $\frac{1}{2}\dot{\phi}^2$ and $V(\phi)$ can be interpreted as the kinetic energy density, and the potential energy density of the inflation field respectively.

1.8.2 Slow Roll Approximation

The equations of inflation can be simplified by using the so called slow-roll approximation which assumes that the kinetic energy density of the inflation field $\frac{1}{2}\dot{\phi}$ is small compared to it's potential energy density $V(\phi)$, and that the inflation equation 1.43 is friction-dominated. According to these assumptions the acceleration term $\ddot{\phi}$ can be ignored compared to the force term $V'(\phi)$ and the friction term $3H\dot{\phi}$. By defining the slow roll parameters

$$\epsilon \equiv \frac{1}{16\pi}\left(\frac{V'}{V}\right)^2 , \qquad \eta = \frac{1}{8\pi}\frac{V''}{V} . \tag{1.47}$$

Then the slow roll approximation is valid when $\epsilon, |\eta| \ll 1$. ϵ and η basically measure the slope and the curvature of the scalar field potential $V(\phi)$. When ϵ and $|\eta|$ are small, the potential $V(\phi)$ is relatively flat during slow roll inflation. The inflationary phase ends when ϵ and $|\eta|$ grow to of order unity. Unfortunately $\epsilon, |\eta| \ll 1$ is a necessary but not sufficient to guarantee the slow-roll approximation. The parameters $\epsilon, |\eta|$ place restrictions on the form of the potential not the dynamical solutions. In general the validity of the slow-roll approximation equires an existence of asymptotic attractor solution. In literature there is aother form of the slow-roll approximation which places a conditions on the evolution of the Hubble parameter during the inflation[79].

1.8.3 The amount of inflation

An important quantity that describes the amount of inflation is the number of e-foldings, defined by

$$N_e(\phi_i \to \phi_f) \equiv ln\frac{a_f}{a_i} = \int_{t_i}^{t_f} H\, dt = \int_{\phi_i}^{\phi_f} \frac{H(\phi)}{\dot{\phi}}\, d\phi = -\int_{\phi_i}^{\phi_f} 3\frac{H^2}{V'}\, d\phi , \tag{1.48}$$

where i and f refer to the value of the quantities at the beginning and the end of the inflation respectively. In order to solve the flatness problem, the number of e-foldings should be $N_e > 70$ [62].

From the equations 1.45 and 1.46 we can obtain the following equation for the state of the inflation field

$$\omega = \frac{P}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} . \tag{1.49}$$

If the inflation field varies slowly $\dot{\phi}^2 \ll V(\phi)$ then the equation of state becomes

$$\omega = -1 . \tag{1.50}$$

Thus, in this case, the inflation field acts like a cosmological constant.

By utilising the slow roll approximation, the equations of inflation are reduced to

$$H^2 = \frac{1}{3}V , \tag{1.51}$$

$$3H\dot{\phi} = -V' . \tag{1.52}$$

Different inflationary scenarios correspond to the different choices of the potential $V(\phi)$, which are usually motivated by arguments stemming from particle physics. To clarify the idea of inflation, a simple example can be taken. Consider a polynomial inflation where the potential is given by $V(\phi) = \frac{1}{2}m^2\phi^2$, see fig [1.1]. Equations 1.51 read

$$H^2 \simeq \frac{m^2\phi^2}{6}, \qquad 3H\dot{\phi} + m^2\phi \simeq 0. \tag{1.53}$$

By combining these equations we obtain the solutions

$$\phi \simeq \phi_i - \frac{\sqrt{6}}{3} m t, \tag{1.54}$$

$$a \simeq a_i \exp\left[\frac{m}{\sqrt{6}}\left(\phi_i t - \frac{m}{\sqrt{6}} t^2\right)\right]. \tag{1.55}$$

where ϕ_i is an integration constant corresponding to the initial values of the inflation. The expansion rate slows down with the increase of the second term in the square bracket of equation 1.55. The total amount of inflation is given by

$$N \simeq 2\pi(\phi_i)^2 - \frac{1}{2}, \tag{1.56}$$

from which a constraint on ϕ_i can be placed.

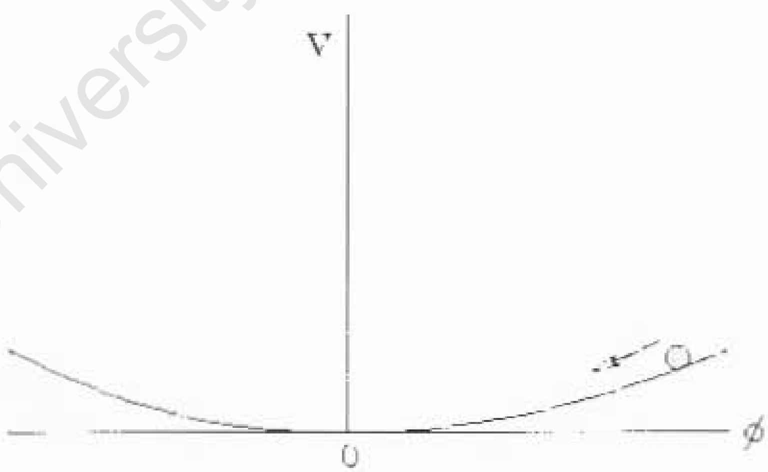

Figure 1.1: Toy models for a quadratic inflationary potential

1.9 Dark Energy

Measurements of the CMB indicate that the Universe is very close to having flat spatial sections [6]. The total amount of matter in the Universe (including baryons and dark matter), as measured by the CMB, accounts for only about 26% of the critical density [14][15]. This implies the existence of an additional form of energy to account for the remaining 74% [14][15]. Also, in an effort to measure how much the universal expansion has slowed over the last few billion years, astronomers recently began to observe very bright rare stars called supernova. In 1998, observations of type Ia supernova by the Supernova Cosmology Project at the Lawrence Berkeley National Laboratory and the High-z Supernova Search Team, surprisingly indicated that the universal expansion is speeding up, or accelerating [5]. These two observational facts indicate that the Universe may contain a new form of matter or energy that is, in effect, gravitationally repulsive. This is called dark energy. Although recently it has become quite clear how dark energy works, its nature remains an unsolved problem. The simplest explanation claims that the cosmological constant Λ leads to the so called ΛCD model [77].

As we mentioned previously, the concept of the cosmological constant was first introduced by Einstein so as to allow for a static Universe. Today, the cosmological constant is recognized as vacuum energy, an energy assigned to empty space. If the field equations are rewritten so that the cosmological constant appears on the right hand side of the equation, then the cosmological constant can be associated with a vacuum energy density. In the next subsection, we will discuss the consequences of such an interpretation of the cosmological constant.

1.9.1 Vacuum Energy

Consider a scalar field ϕ, with potential $V(\phi)$. The action is then

$$S = \int \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{1.57}$$

The corresponding energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}(g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi)\, g_{\mu\nu} - V(\phi)g_{\mu\nu}. \tag{1.58}$$

The vacuum energy can be obtained by setting $\partial_\mu\phi = 0$, and $V(\phi) = V(\phi_o)$, where $V(\phi_o)$ is the global minimum of $V(\phi)$. The vacuum energy momentum tensor is then

$$T^{(vac)}_{\mu\nu} = -\rho_{vac} g_{\mu\nu}, \tag{1.59}$$

where $\rho_{vac} = \rho_\Lambda = V(\phi_o) \equiv \Lambda$. Thus the pressure associated with the cosmological constant is then given by the relation

$$P_\Lambda = -\rho_\Lambda. \tag{1.60}$$

In this manner, the cosmological constant behaves gravitationally like matter and energy, except that it possesses negative pressure. If the energy density is positive, the associated negative pressure will drive an accelerated expansion of empty space. It is the fact that pressure also contributes to gravity that makes the inclusion of the cosmological constant interesting. The main attraction of the cosmological constant term is that it provides a remarkable agreement between theory and observations.

The physical interpretation of the cosmological constant as vacuum energy density is supported by the existence of the "zero point" energy predicted by quantum mechanics. A major outstanding problem stemming from associating the cosmological constant with quantum-mechanical vacuum energy appears when we make even a simple estimate of what this implies for its value. The following estimate is taken from [53]. Consider, for example, a simple harmonic oscillator of frequency ω; that is, a particle of mass m moving in a one dimensional potential well $V(x) = 1/2\omega^2 x^2$. We have chosen the potential such that it has a minimum $V(0) = 0$. The vacuum state has a zero-point energy,

$$E_0 = \omega \, . \tag{1.61}$$

A relativistic field may be thought of as a collection of harmonic oscillators of all possible frequencies. A simple example is provided by a scalar field ϕ of mass m. For this system, the vacuum energy is simply a sum of contributions,

$$E_0 = \sum_{j=1}^{N} \omega_j \, , \tag{1.62}$$

where the sum is over all possible modes of the field, i.e. over all wave vectors k. The sum can be calculated by placing the system in a box of volume L^3, and letting L go to infinity. If we impose periodic boundary conditions, forcing the wavelength (in, say, the ith direction) to be $\lambda_i = L/n_i$ for some integer n_i, then, since $k_i = 2p_i/\lambda_i$, there are $dk_i L/2\pi$ discrete values of k_i in the range $(k_i, k_i - dk_i)$. Therefore, the expression 1.62 becomes

$$E_0 = \frac{1}{2} L^3 \int \frac{d^3k}{(2\pi)^3} \omega_k \, . \tag{1.63}$$

The energy density $\rho_{\rm vac}$ is obtained by letting $L \to \infty$ while simultaneously dividing both sides by the volume L^3. In order to perform the integral, we must use $\omega_k^2 = k^2 + m^2$, and impose a cutoff at a maximum wave vector $k_{\rm max} >> m$. Then the integral gives,

$$\rho_{\rm vac} \equiv \lim_{x \to \infty} \frac{E_0}{L^3} = \frac{k^4_{max}}{16\pi^2} \, . \tag{1.64}$$

It is widely believed that the Planck energy $E^* \approx 10^{19} GeV \approx 10^{16} erg$ marks a point where conventional field theory breaks down owing to quantum gravitational effects. By choosing $k_{\max} = E^*$, we obtain

$$\rho_{\rm vac} \approx 10^{47}\, GeV^4 \approx 10^{92}\, g/cm^3 . \tag{1.65}$$

This is approximately 120 orders of magnitude larger than is allowed by observations. This problem is termed the magnitude problem. Such highly theoretical calculations of $\rho_{\rm vac}$ are a real limit to the plausibility of a non-zero cosmological constant. Even though theoretical calculations of the cosmological constant are not fully understood, the fact remains that the vacuum energy does exist. Since gravity couples all forms of energy, the cosmological constant remains as a physically plausible part of modern cosmology.

Although the cosmological constant is the simplest paradigm for cosmic acceleration, it is not favoured because of the magnitude of the problem. Furthermore, the effective equation of state should evolve with time [31], while ω of the cosmological constant is time-independent. As a response to this problem, a new model with dynamical vacuum energy, dubbed quintessence, has been proposed [76]. In the next section we will present a detailed discussion of this model.

1.10 Quintessence Q

Quintessence Q represents dynamical, spatially inhomogeneous, negative pressure cosmic scalar fields. The critical distinction between Q and Λ is that ω, the ratio of pressure P to the energy density ρ, is precisely -1 for Λ, whereas ω is $-1 < \omega < 0$ for quintessence; thus quintessence possesses a time-varying equation of state. An important motivation for considering the quintessence model is to address the coincidence problem.

The key idea of quintessence is to assume a scalar field $\phi(t)$ with a potential energy density $V(\phi)$. In order to derive the accelerated expansion, the scalar field should roll down its potential very slowly. The homogeneous component of that field has an equation of motion

$$\ddot{\phi} - 3H\dot{\phi} - \frac{\partial V(\phi)}{\partial \phi} = 0 , \tag{1.66}$$

where the dot denotes a derivative with respect to time and H is the Hubble parameter

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi}{3}\left(\rho_m - \rho_r + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right) . \tag{1.67}$$

It is clear from 1.66 that the expansion provides a friction that prevents the scalar field from quickly rolling to the minimum of its potential. The pressure and the energy density in the field are given by

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{1.68}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{1.69}$$

As for the inflation field, the first term, $\frac{1}{2}\dot{\phi}^2$ can be interpreted as the kinetic energy density and the second term, as the potential energy. While the equation of state parameter ω is given by

$$\omega_Q = \frac{KE - V}{KE + V}. \tag{1.70}$$

By multiplying 1.66 by $\dot{\phi}$, equation 1.66 can be written as

$$\left[\frac{1}{2}\dot{\phi}^2\right]^{\cdot} + 6H\left[\frac{1}{2}\dot{\phi}^2\right] = -\dot{V}. \tag{1.71}$$

By substituting for $\dot{\phi}$ from 1.68 and 1.69 we obtain the continuity equation for the quintessence field

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{1.72}$$

Using the general equation of state for a perfect fluid equation 1.72 reduced to

$$\dot{\rho} + 3H(1 + \omega_Q)\rho = 0. \tag{1.73}$$

where the dot denotes a derivative with respect to $\ln a$, then the energy density of the quintessence field can be reconstructed as

$$\rho(a) = \Omega_Q\, \rho_{crit} \exp\left(3\int [1 + \omega_Q(a)]\; dln\, a\right). \tag{1.74}$$

It is also possible to reconstruct the potential $V(a)$, kinetic energy $KE(a)$ and field evolution $Q(a)$.

$$V(a) = \frac{1}{2}\rho(a)\,[1 - \omega_Q(a)], \tag{1.75}$$

$$KE(a) = \frac{1}{2}\rho(a)\,[1 + \omega_Q(a)], \tag{1.76}$$

$$Q(a) = \int H^{-1}\sqrt{\rho(a)\,[1 + \omega_Q(a)]}\,dln\, a. \tag{1.77}$$

If the potential energy dominates $V >> K$, the scalar field will roll down its potential slowly enough; then the equation of state parameter will become $\omega_Q < 0$, producing a cosmic acceleration.

For the quintessence to address the coincidence and the magnitude problems, we must tune the parameters of the potential $V(\phi)$ and the initial conditions in the field ϕ. Another problem with quintessence is referred to as the cosmic coincidence problem, that is, the near equivalence of the matter and Λ contribution to the total energy density. In order to solve this problem, a quintessence model known as Tracker has been proposed.

1.10.1 Tracker Quintessence

Tracker is a particular class of quintessence models that contain an equation of motion with attractor solutions in which a very wide range of initial conditions rapidly converge to a common cosmic evolutionary track, so that the tracker solutions are insensitive to the initial conditions. TO study how the tracker solution is approched it is useful to rewrite the equation of motion equation 1.66 in the form,

$$\pm\frac{V'}{V} = 3\sqrt{\frac{8\pi(1+\omega_Q)}{3\Omega_Q}}\left[1+\frac{1}{6}\frac{d\ln x}{d\ln a}\right], \tag{1.78}$$

where the prime denotes the derivative with respect to ϕ, the $\pm$ sign depends on whether $V' > 0$ or $V' < 0$ respectively, and

$$x \equiv \frac{1/2\,\dot{\phi}^2}{V} = \frac{(1+\omega_Q)}{(1-\omega_t)}. \tag{1.79}$$

By taking the derivative of equation 1.78 with respect to ϕ we obtain

$$\Gamma \equiv \frac{V''V}{V'^2} = 1 - \frac{\omega_B - \omega_Q}{2(1+\omega_Q)} - \frac{1+\omega_B-2\omega_Q}{2(1+\omega_Q)}\frac{\dot{x}}{(6+\dot{x})} - \frac{2}{(1+\omega_Q)}\frac{\ddot{x}}{(6+\dot{x})^2}, \tag{1.80}$$

where $\dot{x} = d\ln x/d\ln a$ and ω_B is the equation of state parameter. Tracking occurs for any potential which satisfies $\omega < \omega_B$, $\Gamma > 1$ and $d(\Gamma-1)/dt << (\Gamma-1)H$, and the tracker equation of state ω_t is now given by the formula

$$\omega_t \simeq \frac{\omega_B - 2(\Gamma-1)}{1+2(\Gamma-1)}. \tag{1.81}$$

The tracker equation of state depends on what is happening in the rest of the Universe. If the Universe is radiation-dominated $\omega_B = 1/3$, the tracker behaves as if it were another radiation component with ω_t less than or equal to 1/3, and its energy density decreases in parallel with the dominant radiation component. When the Universe is matter-dominated $\omega_B = 0$, then $\omega_t < 0$ and the energy density in the tracker field begins to decrease less rapidly than the matter density. Eventually ρ_t surpasses the matter density and becomes the dominant component. At this point the Hubble damping of the field evolution becomes important, ϕ slows to a crawl and $\omega \to -1$ as $\Omega_t \to 1$ and the Universe is driven to an accelerating phase. The tracker only behaves as a negative pressure component after matter radiation equality, so that it can only overtake the matter density and induce cosmic acceleration after the matter has dominated the Universe for some period. This is an important example of an explanation of the fine-tuning and cosmic coincidence problems. The point is summarized in fig [1.2]. Although

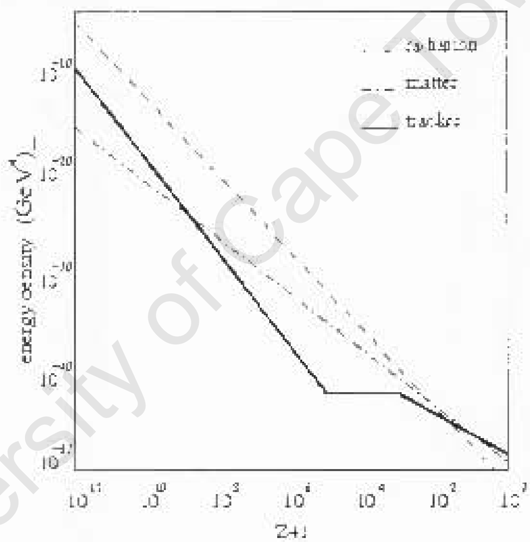

Figure 1.2: The energy density versus red shift for tracker field.

the success of quintessence in a scalar field explaining the cosmic acceleration and the coincidence problem is evident, it is still not clear where the scalar field arises and how to choose the self-interacting potential. A different means of solving the cosmic acceleration problem is called the extended theories of gravity [13 – 42]. In this frame work the standard theory of gravity GR is considered as a particular case of a more general theory. In the next section we present a detailed discussion of $f(R)$-gravity which is one of the most extensively studied extended theories of gravity.

1.11 f(R)-Gravity

Regarding the scales on which gravity should behave, essential classical cosmological observations indicate that General Relativity appears unable to offer a simple explanation of the phenomena observed. Currently serious attention is being paid to the classical formulation of gravity, in particular, the dynamics of galaxies and clusters of galaxies as well as the present accelerated expansion of the universe, the existence of dark matter and, most of all, that of dark energy.

Recently, cosmologists believe that there is more credibility in classical gravity than in general relativity; therefore they are investigating the possibilities of formulating a viable alternative theory of gravity, which could explain the low energy universe without recourse to new, unobserved particles. Various modifications have been proposed. Some examples of these alternatives are dark energy models of gravity, $f(R)$-gravity models of gravity [13 − 42], and braneworld models. Here we will restrict our discussion to the second classes of gravity models.

Modern theories such as String/M-theory indicate that a modification of the standard gravitational action 1.8 appears to be necessary. Furthermore, there is no prior reason to restrict the gravitational Lagrangian to the simple Einstein-Hilbert action when a more general formulation is allowed. Thus serious attention has been paid to the alternative formulations of gravity which are based on the Lagrangian to include higher order corrections with respect to the linear term in the scalar curvature (Einstein-Hilbert action), either in the form of higher order curvature invariants or non-minimally coupled scalar fields [35, 36, 37]. These theories possess a number of interesting features with regards to cosmological and astrophysical scales. In fact, they are known to admit natural inflation phases [33] and to explain the flattening of the galactic rotation curves [38]. Another very interesting feature of these models is that the higher order corrections to the Hilbert-Einstein action can be viewed as effectively a fluid which could mimic the properties of Dark Energy [34].

1.11.1 The Curvature Fluid

The idea of modified gravity consists of introducing corrections to the Einstein-Hilbert action that are negligible in the early universe and only become effective at late times when the Ricci curvature R decreases. Here we will only consider $f(R)$ gravity in the FRW universe. In this approach the curvature R in the Einstein-Hilbert action in vacuum is replaced by some generic function $f(R)$. In vacuum, the $f(R)$ gravity action takes the form

$$S[g] = \int [f(R(g))]\sqrt{-g}\, d^4x\,, \tag{1.82}$$

where $R(g) \equiv g^{\mu\nu}R_{\mu\nu}(g)$, g is the determinant of the metric tensor $g_{\mu\nu}$ and $R_{\mu\nu}(g)$ is the Ricci tensor defined in terms of a connection Γ as

$$R_{\mu\nu}(g) = -\partial_\mu \Gamma^\lambda_{\lambda\nu} + \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda}, \tag{1.83}$$

where

$$\Gamma^\lambda_{\alpha\beta} = \frac{g^{\lambda\rho}}{2}\left[\partial_\alpha g_{\rho\beta} - \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}\right]. \tag{1.84}$$

In the metric-$f(R)$ gravity varying the action 1.82 yields the field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(curv)}_{\mu\nu}, \tag{1.85}$$

where the prime denotes a derivative with respect to R, and the curvature stress-energy $T^{(curv)}_{\mu\nu}$ tensor is given by

$$T^{(curv)}_{\mu\nu} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\mu\nu}\left[f(R) - Rf'(R)\right] + f'(R)^{;\alpha\beta}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta})\right\}. \tag{1.86}$$

Written in this form, eqs (2.17) have a clear interpretation: the right hand side represents the sources that generate the spacetime curvature associated with its respective metrics. Though we initially started with a pure gravity theory in a vacuum, the higher order terms turned the vacuum gravity theory into Einstein's equations with additional curvature stress-energy tensor $T^{(curv)}_{\mu\nu}$ that acts as a matter source.

This new source can be interpreted as an effective fluid of purely geometrical origin. Depending on the scales, such a curvature fluid can play the role of dark energy and dark matter. When matter is considered, the action (1.82) takes the form

$$S[g] = \int [f(R(g)) - L_M]\sqrt{-g}\,d^4x, \tag{1.87}$$

where L_M is the standard matter Lagrangian density, varying (1.87) with respect to the metric, we obtain

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^{(curv)}_{\mu\nu} + \frac{1}{f'(R)}T^{(m)}_{\mu\nu}. \tag{1.88}$$

It is clear from (1.88) that the term $\frac{1}{f'(R)}$ couples matter non-minimally to geometry. Owing to the presence of the term $f'(R)^{;\mu\nu}$ equation (1.88) can be seen as a system of fourth order differential equations with respect to the metric. However, when $f(R) = R$, the curvature stress-energy tensor $T^{(curv)}_{\mu\nu}$ identically vanishes and equation (1.88) reduces to the standard second order Einstein field equations.

In the comoving coordinates (t, x, y, z) the field equations (1.88) reduce to

$$H^2 = \frac{1}{3f'}\left[\frac{Rf' - f}{2} - 3H\dot{R}f''\right] - \frac{k}{a^2}, \tag{1.89}$$

$$2\dot{H} + 3H^2 = -\frac{1}{f'}\left[f'''(\dot{R})^2 + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right], \tag{1.90}$$

where the dot denotes the differentiation with respect to t.

1.11.2 Curvature Fluid Equation of State

The field equations (1.89) and (1.90) result in the acceleration equation

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{1}{6}\left[\rho_{(tot)} + 3p_{(tot)}\right], \tag{1.91}$$

where

$$p_{(tot)} = p_{(R)} + p_{(m)} \qquad \rho_{(tot)} = \rho_{(R)} + \rho_{(m)}. \tag{1.92}$$

It is clear that the accelerated or decelerated behaviour depends on the sign of rhs. Using the curvature-stress-energy tensor, one can write a curvature pressure $p_{(R)}$ and a curvature energy density $\rho_{(R)}$ for $f(R)$ models as follows:

$$p_{(R)} = \frac{1}{f'(R)}\left\{2\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R) + \ddot{R}f''(R) + \dot{R}^2 f'''(R) - \frac{1}{2}\left[f(R) - Rf'(R)\right]\right\}, \tag{1.93}$$

and a curvature density

$$\rho_{(R)} = \frac{1}{f'(R)}\left\{\frac{1}{2}\left[f(R) - Rf'(R)\right] - 3\left(\frac{\dot{a}}{a}\right)\dot{R}f''(R)\right\}. \tag{1.94}$$

From eq.1.91, the accelerated behaviour is achieved if

$$\rho_{(R)} + 3p_{(R)} = \frac{3}{f'(R)}\left\{\dot{R}^2 f'''(R) + \left(\frac{\dot{a}}{a}\right)\dot{R}f''(R) + \ddot{R}f''(R) - \frac{1}{3}\left[f(R) - Rf'(R)\right]\right\} < 0. \tag{1.95}$$

Type Ia supernovae yield an equation of state $\omega \approx -1$ at present. For the model to mimic the de-Sitter equation of state with $\omega_R = -1$, it must satisfy

$$\frac{f'''}{f''} = \frac{\dot{R}H - \ddot{R}}{\dot{R}^2}. \tag{1.96}$$

1.12 The Phase Space of $f(R)$ Gravity

This section presents a review of the phase space of $f(R)$ cosmology [28]. The field equations 1.89 and 1.90 are fourth order in the scale factor $a(t)$. But when the curvature index $k = 0$, they can be converted to third order equations in the Hubble parameter $H \equiv \frac{\dot{a}}{a}$. In this discussion we will restrict ourselves to this case. By setting $k = 0$ the Hamiltonian constraint 1.89 can now be written as

$$\dot{R}(H, R) = \frac{Rf' - f - 6f'H^2}{6Hf''}. \tag{1.97}$$

It is clear that the dynamical variables $H, R, \dot{R}$ are not independent. Thus phase space of the field equation 1.89 and 1.90 can be reduced to two dimensions (H, R). The solutions of the field equations are confined to the surface defined by the Hamiltonian constraint equation 1.97 in the 3-D phase of space $(H, R, \dot{R})$.
In the phase space (H, R) the regions defined by

$$\left\{(H, R) : Rf'(R) - f(R) - 6H\dot{R}(H, R)f''(R) < 0\right\}, \tag{1.98}$$

and

$$\{(H, R) : f(R), f'(R), \text{ or } f''(R) \text{ are not defined}\}. \tag{1.99}$$

In these regions H^2 is found to be less than zero; therefore any orbit passing through these regions will not be considered as a physical solution.
By setting $\dot{H} = 0$ and $\dot{R} = 0$ the equilibrium points of the dynamical system 1.97 and 1.90 are found to be

$$(H, R) = \left(\pm\sqrt{\frac{f_0}{6f_0'}}, \frac{2f_0}{f_0'}\right), \tag{1.100}$$

where $f_0 \equiv f(R_0)$. If $f_0 \geq 0$ and $f_0' > 0$, two de Sitter equilibrium points always exist in the $\dot{R} = 0$ plane with $H_0^{(\pm)} = \pm\sqrt{\frac{f_0}{6f_0'}}$, the positive sign corresponding to an expanding solution while the negative sign represents a contracting solution.

1.12.1 The Stability of de Sitter Space

The stability of de Sitter space can be studied either with respect to the homogeneous perturbations, which depends only on time, or with respect to the more general inhomogeneous perturbations, which depends on both space and time. In [28] it is demonstrated that the stability conditions for these two cases coincide, because the inhomogeneities are found to be redshifted away. The stability condition for de Sitter space is found to be

$$\frac{(f_0')^2 - 2f_0f_0''}{f_0'f_0''} \geq 0. \tag{1.101}$$

This stability condition helps to rule out models from a theoretical point of view. As an example of the application of the stability condition see [29]. For $f(R)$ a theory, characterized by

$$f(R) = R - \frac{\mu^4}{R}. \tag{1.102}$$

The stability condition 1.101 reduces to

$$1 + \frac{6\mu^2}{R_0^2} - \frac{3\mu^8}{R_0^4} \leq 0. \tag{1.103}$$

The condition for the existence of de Sitter solutions is $R_0 = \sqrt{3}\,\mu^2$: therefore the stability condition 1.103 can never be satisfied. This situation can be compensated by a quadratic correction in the theory with

$$f(R) = R - \frac{\mu^4}{R} + \alpha R^2 . \tag{1.104}$$

The condition for the existence of de Sitter space is again $R_0 = \sqrt{3}\,\mu^2$, the stability condition 1.101 reduces to

$$\frac{1}{3\sqrt{3}\,\alpha\,\mu^2 - 1} \geq 0 , \tag{1.105}$$

and therefore the de Sitter space is stable only if

$$\alpha > \frac{1}{3\sqrt{3}\mu^2} , \tag{1.106}$$

otherwise it is unstable.

One of the main problems occurring in the study of higher order theories of gravity is that finding exact cosmological solutions is extremely difficult owing to the high degree of nonlinearity exhibited by these theories and the fact that the field equations arising from $f(R)$-action are typically of the fourth order. This problem can be partially addressed by employing a suitable choice of generalized coordinates. In this case, the field equations can be written as a system of first order autonomous differential equations together with a constraint equation [30]. In this way the methods of dynamical systems theory can be exploited [31] in order to both understand the qualitative behaviour of the cosmological dynamics and obtain special exact solutions of the cosmological equations. The general approach allowing one to analyze higher order gravity with dynamical systems techniques has been presented elsewhere [32].

Chapter 2

Dynamical Systems

2.1 Basic Concepts

This chapter provides an elementary introduction to the theory of dynamical systems. The evolution of physical systems is often given by a set of differential equations which can not always be solved analytically. The dydnamical system theory enable us to understand the space of solutions without solving the equations themselves. Instead of looking for a particular solution to a set of differential equations, dynamical systems theory looks at the qualitative properties of the set of all solutions to a system.

A dynamic system is a set of states that are normally described by a set of variables $x_1, ..., x_n$, each one depending on time t, along with a rule that describes how the present state is determined by the previous state and the initial conditions. This rule is given in terms of differential equations of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, ..., x_n), \\ \dot{x}_2 &= f_2(t, x_1, ..., x_n), \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, ..., x_n), \end{aligned} \tag{2.1}$$

where $\mathbf{x}_i \in \mathrm{R}$ and $f_i : \mathrm{R} \rightarrow \mathrm{R}$, $i = 1, ..., n$. By defining $\mathbf{x} = (x_1, ..., x_n)^T$ and $\mathbf{F}(t, \mathbf{x}) = (f_1, ...f_n)^T$, in the matrix form, the system 2.1 can be written as

$$\dot{\mathbf{x}} = \mathbf{F}(t, \mathbf{x}(t)). \tag{2.2}$$

One may use various methods to study the system 2.2; generally they are classified as (1) qualitative (2) analytical (3) numerical. This chapter focuses only on qualitative methods. We begin by defining some fundamental concepts closely related to the qualitative approach to dynamical systems.

Definition 1: Autonomous System

In the system 2.1, if none of the functions, f_j, depend on time, the system is termed autonomous. Autonomous systems can be written in the matrix form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}). \tag{2.3}$$

In the following sections we will focus only on autonomous systems.

Definition 3: Phase Space

The phase space is the set of all possible states $(x_1, ..., x_n)$ of the system; generally the phase Space is a subspace of $\mathbf{R}^n$ locally.

Definition 4: Trajectory

The set of points $(t, \mathbf{x}(t)); t \in \mathbf{R}$ which solve the system 2.3, and for which $\mathbf{x}(t_0) = x_0$, is called the trajectory or solution curve of the dynamical system passing through $\mathbf{x}_0$.

Definition 5: Phase Flow

The collection of all trajectories obtained by varying t_0 and $\mathbf{x}_0$ throughout all allowed values is called the flow of the dynamical system.

2.2 Linear Dynamical Systems

Theorem 1.2 *Let A be any $n \times n$ matrix. Then for a given $x_0 \in R^n$, the initial value problem*

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{2.4}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0, \tag{2.5}$$

where $\mathbf{x}_0$ is determined by the initial conditions.

The set of vectors $\mathbf{x} \in \mathbf{R}^n$ spans the phase space of the system 2.4, and the set of all solution curves $\mathbf{x}(t) \in \mathbf{R}^n$, defines the phase portrait of the system 2.4. The function

$$f(x) = e^{At}. \tag{2.6}$$

On the right-hand side of 2.5 defines a set of linear mappings $f : \mathbf{R}^n \to \mathbf{R}^n$. This set of mappings may be regarded as describing the motion of the points $\mathbf{x}_0 \in \mathbf{R}^n$, along trajectories of 2.4, where the point $\mathbf{x}_0 \in \mathbf{R}^n$, moves to the point $\mathbf{x}(t) \in \mathbf{R}^n$, given by 2.5 after time t. The set of all the mappings 2.6 is called the flow of the linear system 2.4.

Definition 7: Fixed/Critical Point

A point x_0 is called a fixed or critical point of the autonomous dynamic system 2.3 if

$$\dot{\mathbf{x}} = 0 \Longleftrightarrow A\mathbf{x}_0 = 0 \,, \tag{2.7}$$

which means that if a system is at a fixed point x_0 it will remain there forever.

2.2.1 Diagonalization

If the matrix A is diagonal then the linear system 2.4 is termed an uncoupled linear system. For uncoupled systems it is easy to compute the matrix e^{At} and hence obtain an explicit form for the solution 2.5. Since the matrix elements depend on the bases that are used to represent them, if A is not diagonal, we can look for the transformation that reduces the matrix elements of A to a diagonal form. This section presents the various cases for which this transformation exists.

Theorem 2.2: *If the eigenvalues $\lambda_1, ..., \lambda_n$ of an $n \times n$ matrix A are real and distinct, and if the linear transformation $T : R^n \to R^n$ is represented by the matrix A with respect to the standard basis $[e_1, ..., e_n]$, then, with respect to any basis of eigenvectors $[v_1, ..., v_n]$ of A, T is represented by*

$$\text{diag}[\lambda_1, ..., \lambda_n] = P^{-1}AP \,, \tag{2.8}$$

where the transformation matrix $P = [v_1 ... v_n]$.

With respect to the basis $[v_1, ..., v_n]$, the system 2.4 takes the form

$$P^{-1}\dot{\mathbf{x}} = P^{-1}APP^{-1}\mathbf{x}_0 \qquad \Longleftrightarrow \qquad \dot{\mathbf{y}} = \text{diag}[\lambda_1, ..., \lambda_n]\mathbf{y} \,, \tag{2.9}$$

where $\mathbf{y} = P^{-1}\mathbf{x}$. This uncoupled linear system has the solution

$$\mathbf{y}(t) = \text{diag}[e^{\lambda_1 t}, ..., e^{\lambda_n t}]\,\mathbf{y}(0) \,. \tag{2.10}$$

It follows that the linear system 2.4 has the solution

$$\mathbf{x}(t) = P^{-1}[e^{\lambda_1 t}, ..., e^{\lambda_n t}]P\,\mathbf{x}(0) \,. \tag{2.11}$$

Definition 8. An equilibrium point x_0 is called a hyperbolic equilibrium point of 2.4 if none of the eigenvalues of the matrix A contain a zero real part.

An important property of an equilibrium point is its stability. The following describes the basic cases:

- If all solutions that start close to an equilibrium converge to the equilibrium asymptotically as $t \to \infty$, the equilibrium is described as asymptotically stable.
- If all solutions in a sufficiently small neighbourhood of the equilibrium remain close to the equilibrium point, the equilibrium point is likewise regarded as stable. Note that asymptotic stability implies stability.
- If every neighbourhood of the equilibrium point contains solutions arbitrarily close to the equilibrium point that leaves the neighbourhood, we say the equilibrium point is unstable.

Definition 9: An equilibrium point x_0 of 2.4 is called a sink if all of the eigenvalues of the matrix A contain a negative real part, it is called a source if all of the eigenvalues of the matrix A have a positive real part and it is called a saddle if it is a hyperbolic point and A has at least one eigenvalue with a positive real part and at least one with a negative real part; it is called a center if whenever A has a pair of pure imaginary complex conjugate eigenvalues. The phase flows of these different cases are shown in fig [2.1].

Theorem 3.2: *If the $2n \times 2n$ real matrix A has $2n$ distinct complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding complex eigenvectors $w_j = u_j + iv_j$ and $\bar{w}_j = u_j - iv_j$, $j = 1, ..., n$, then $[u_1, v_1, ..., u_n, v_n]$ are the basis of R^n. The system 2.4 can be written in the diagonal form*

$$P^{-1}\dot{\mathbf{x}} = P^{-1}APP^{-1}\mathbf{x}_0 \quad \Longleftrightarrow \quad \dot{\mathbf{y}} = \mathrm{diag}[\lambda_1, ..., \lambda_n]\mathbf{y}, \tag{2.12}$$

where $P = [u_1 v_1 ... u_n v_n]$ and $P^{-1}AP$ is a $2n \times 2n$ with 2×2 block diagonal.

The solution of 2.12 is given by

$$\mathbf{y}(t) = \mathrm{diag}\, e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} \mathbf{y}_0.$$

It then follows that 2.4 has the solution

$$\mathbf{x}(t) = P\, \mathrm{diag}\, e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1}\mathbf{x}_0.$$

Definition 10: Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity $m \leq n$. Then for $k = 1, .., m$ any nonzero solution of $(A - \lambda I)^k v = 0$ is called a generalized vector of A.

Definition 11: An $n \times n$ matrix N is said to be nilpotent of order k if $N^{k-1} \neq 0$ and $N^k = 0$.

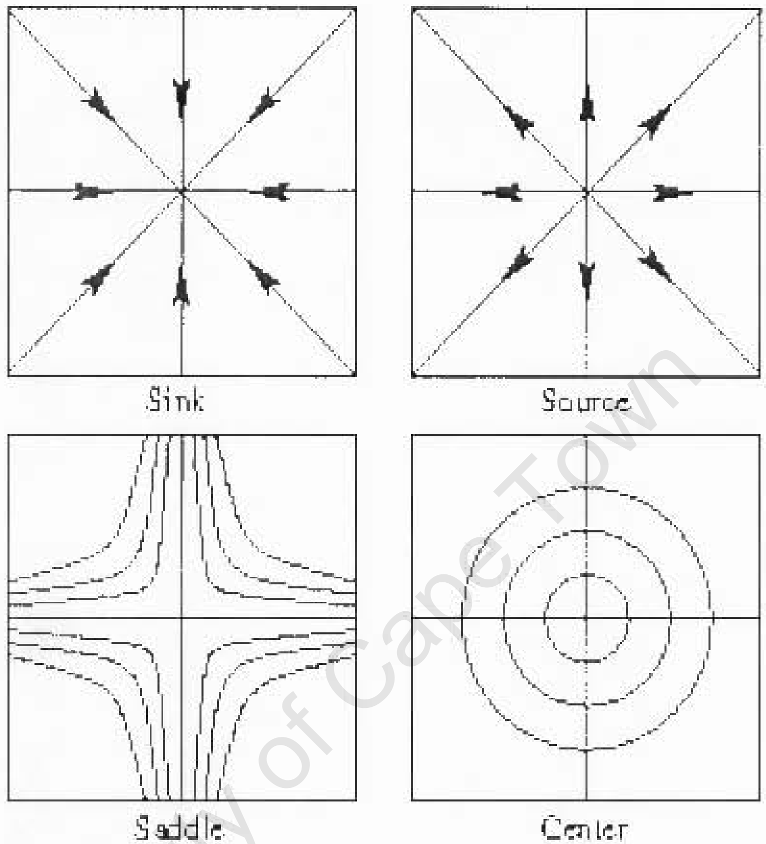

Figure 2.1: The phase portrait of a sink, source saddle and center respectively

Theorem 4.2: *Let A be a real $n \times n$ matrix with real eigenvalues $\lambda_1, ..., \lambda_n$, repeated according to their multiplicity. Also, a basis of generalized eigenvectors for R^n exists. If $[v_1, ..., v_n]$ forms any basis of generalized eigenvectors for R^n, the matrix P is invertable*

$$A = S + N, \tag{2.13}$$

where $N = A - S$ is nilpotent of order $k \leq n$.

The solution of 2.4 is given by

$$\mathbf{x}(t) = P \operatorname{diag}[e^{\lambda_j t}] \, P^{-1} \left[I + Nt + ... + \frac{N^{k-1}t^{k-1}}{(k-1)!} \right], \tag{2.14}$$

Finally let us consider the case of multiple complex eigenvalues. If the matrix A in 2.4 is $2n \times 2n$ with multiple complex eigenvalues, then the solution of 2.4 is given by

$$\mathbf{x}(t) = P \operatorname{diag} e^{a_j t} \begin{bmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix} P^{-1} \left[I + Nt + ... + \frac{N^k t^k}{(k)!} \right] \mathbf{x}_0 .$$

where $P = [v_1 u_1 ... v_n u_n]$ and $N = A - S$ is nilpotent of order $k \leq 2n$.

2.2.2 Invariant Subspaces

Theorem 5.2 *Let A be a real $n \times n$ matrix. Then R^n can be written as a direct sum of three subspaces denoted by E^s, E^u and E^c*

$$\mathbf{R}^n = \mathbf{E}^s + \mathbf{E}^u + \mathbf{E}^c. \tag{2.15}$$

where

- E^s *=span* $e_1, ..., e_s$ *, the stable subspace.*
- E^u *=span* $e_{s+1}, ..., e_{s+u}$ *, the unstable subspace.*
- E^c *=span* $e_{s+u+1}, ..., e_{s+u+c}$ *, the center subspace.*

$[e_1, ..., e_s]$ are the generalized eigenvectors of A corresponding to the eigenvalues of A having a negative real part, $[e_{s+1}, ..., e_{s+u}]$ are the generalized eigenvectors of A corresponding to the eigenvalues of A having a positive real part, and $[e_{s+u+1}, ..., e_{s+u+c}]$ are the generalized eigenvectors of A corresponding to the eigenvalues of A having a zero real part (see fig [2.2]). These three subspaces are invariant with respect to the flow

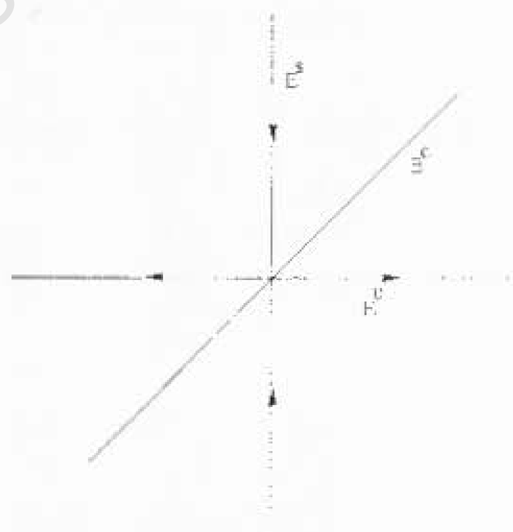

Figure 2.2: The stable, unstable and center subspaces.

e^{At} of 2.4, where the solutions of 2.4 with initial conditions entirely contained in either $\mathbf{E}^s$, $\mathbf{E}^u$, or $\mathbf{E}^c$ must forever remain in that particular subspace.

2.3 Nonlinear systems

In general it is not always possible to find an analytical solution for a given nonlinear system, and in such cases one is forced to use approximation methods. The following sections present some basic approximation techniques that enable one to deal with nonlinear systems.

2.3.1 Linearization

Let us consider the n dimensional nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) . \tag{2.16}$$

If $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ is an equilibrium point of 2.16, then by Taylor's theorem

$$\mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{0})\mathbf{x} + \frac{1}{2}D^2\mathbf{f}(\mathbf{0})(\mathbf{x},\mathbf{x}) + \dots , \tag{2.17}$$

it follows that the linear term $D\mathbf{f}(\mathbf{0})\mathbf{x}$ is a good first approximation to the nonlinear function $\mathbf{f}(\mathbf{x})$ near $\mathbf{x} = \mathbf{0}$. The nonlinear system 2.16 can now be approximated by the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} , \tag{2.18}$$

where

$$A = D\mathbf{f}(\mathbf{0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} ,$$

This is called the Jacobian matrix.

2.3.2 Hartman-Grobman Theorem

This theorem expresses one of the most important results in the local qualitative theory of the ordinary differential equations: according to it, near the hyperbolic point $\mathbf{x}_0$, the nonlinear system

$$\dot{\mathbf{x}} = f(\mathbf{x}) , \tag{2.19}$$

is topologically equivalent to the linear system

$$\dot{\mathbf{x}} = A\mathbf{x} , \tag{2.20}$$

which means that the two autonomous systems 2.19 and 2.20 have the same qualitative structure near the equilibrium point $\mathbf{x}_0$.

2.3.3 Lyapunov Theorem

If $A = Df(\mathbf{0})$ has no eigenvalues with zero real part; then there exists stable and unstable mainfolds of the nonlinear system, $\mathbf{W}^s$ and $\mathbf{W}^u$, which are tangential to $\mathbf{E}^s$ and $\mathbf{E}^u$ respectively at the fixed point (see fig [2.3]).

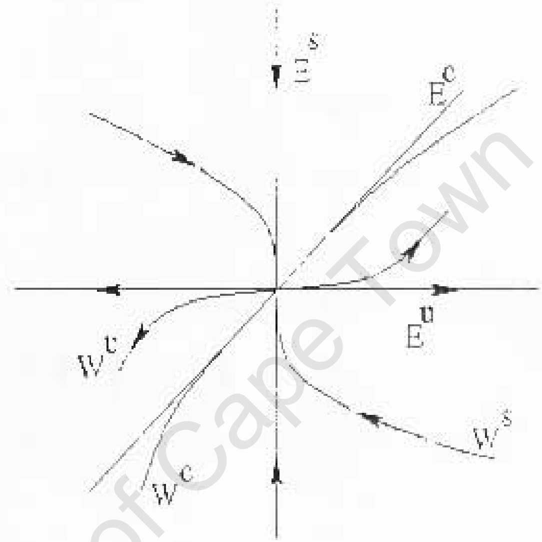

Figure 2.3: The stable, unstable and center manifolds.

All we know about these manifolds is the fact that they are tangential to the invariant subspaces in the neighbourhood of the fixed point.

2.4 The Center Manifold Theorem:

The previous section presented the Hartman-Grobman Theorem, which completely solves the problem of determining the stability and qualitative behaviour in the neighbourhood of a hyperbolic critical point of a nonlinear system. In this section we will discuss the problem of determining the stability and qualitative behaviour of the flow near a non-hyperbolic critical point of 2.19 using the centre manifold approach. The centre manifold theorem is a generalization of the Lyapunov theorem to the case when there exist a centre manifold $\mathbf{W}^c$ tangential to the centre subspace $\mathbf{E}^c$ at the fixed point. For simplicity we shall assume that the origin is a non-hyperbolic fixed point as regards this system (this assumption does not affect the generality of our treatment because it is always possible to change the coordinates to make the fixed point the origin of the new coordinate system). If $\mathbf{f} \in C^1(E)$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, then the system 2.19 can be written in the diagonal form

$$\dot{\mathbf{w}} = A\mathbf{w} + \mathbf{F}(\mathbf{w}), \tag{2.21}$$

where $A = \mathbf{Df}(\mathbf{0})$, $\mathbf{F}(\mathbf{w}) = \mathbf{w} - A\mathbf{w}$ and $\mathbf{F}(\mathbf{0}) = D\mathbf{F}(\mathbf{0}) = \mathbf{0}$. Now let us try to decouple the nonlinear system 2.21. We can find a linear transformation $\mathbf{T}$ which transforms 2.19 into the form

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \\ \dot{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix} + \begin{pmatrix} \mathbf{F}_s(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \mathbf{F}_u(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \mathbf{F}_c(\mathbf{x},\mathbf{y},\mathbf{z}) \end{pmatrix}, \tag{2.22}$$

where $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in R^s \times R^u \times R^c$, $s + u + c = n$, A_s is a diagonal $s \times s$ matrix having eigenvalues with a negative real part, A_u is a diagonal $u \times u$ matrix having eigenvalues with a positive real part, A_c is a diagonal $c \times c$ matrix having eigenvalues with a zero real part and $\mathbf{F}_s(\mathbf{x},\mathbf{y},\mathbf{z})$, $\mathbf{F}_u(\mathbf{x},\mathbf{y},\mathbf{z})$, and $\mathbf{F}_c(\mathbf{x},\mathbf{y},\mathbf{z})$ are the s, u and c components, respectively, of the vector $\mathbf{T}^{-1}F(\mathbf{T}(\mathbf{x},\mathbf{y},\mathbf{z}))$.

We did not decouple the stable and unstable states associated with the negative and positive eigenvalues respectively, from the states associated with the zero eigenvalue, because of the term $(\mathbf{F}_s(\mathbf{x},\mathbf{y},\mathbf{z}), \mathbf{F}_u(\mathbf{x},\mathbf{y},\mathbf{z}), \mathbf{F}_c(\mathbf{x},\mathbf{y},\mathbf{z}))^T$. In what follows, we will find two functions $\mathbf{h}^c_x$ relating $\mathbf{x}$ and $\mathbf{z}$ and $\mathbf{h}^c_y$ relating $\mathbf{y}$ and $\mathbf{z}$, i.e, $\mathbf{x} = \mathbf{h}^c_x(\mathbf{z})$ and $\mathbf{y} = \mathbf{h}^c_y(\mathbf{z})$. The centre manifold theorem guarantees the existence of such mapping.

2.4.1 The Local Center Manifold Theorem

Suppose 2.22 is $\mathbf{C}^r$, $r \geq 2$, then these exist as a $\mathbf{C}^r$ s-dimensional local stable manifold $\mathbf{W}^s_{loc}(\mathbf{0})$ tangent to the stable subspace $\mathbf{E}^s$ at the origin, a $\mathbf{C}^r$ u-dimensional local stable manifold $\mathbf{W}^{su}_{loc}(\mathbf{0})$ tangential to the unstable subspace $\mathbf{E}^u$ at the origin and $\mathbf{C}^r$ c-dimensional local centre manifold $\mathbf{W}^c_{loc}(\mathbf{0})$ tangential to the centre subspace $\mathbf{E}^c$ at the origin. In particular, we have

$$\begin{aligned}
\mathbf{W}^s_{loc}(\mathbf{0}) &= \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c | \mathbf{y} = \mathbf{h}^s_y(\mathbf{x}), \mathbf{z} = \mathbf{h}^s_z(\mathbf{x}); D\mathbf{h}^s_y = \mathbf{0}, D\mathbf{h}^s_z = \mathbf{0};\ |x| < \delta\right\}, \\
\mathbf{W}^u_{loc}(\mathbf{0}) &= \{(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c | \mathbf{x} = \mathbf{h}^u_x(\mathbf{y}), \mathbf{z} = \mathbf{h}^u_z(\mathbf{y}); D\mathbf{h}^u_x = \mathbf{0}, D\mathbf{h}^u_z = \mathbf{0};\ |y| < \delta\}, \\
\mathbf{W}^c_{loc}(\mathbf{0}) &= \left\{(\mathbf{x},\mathbf{y},\mathbf{z}) \in R^s \times R^u \times R^c | \mathbf{x} = \mathbf{h}^c_x(\mathbf{z}), \mathbf{y} = \mathbf{h}^c_y(\mathbf{z}); D\mathbf{h}^c_x = \mathbf{0}, D\mathbf{h}^c_y = \mathbf{0};\ |z| < \delta\right\},
\end{aligned}$$

where $\mathbf{h}^s_y(\mathbf{x}), \mathbf{h}^s_z(\mathbf{x}), \mathbf{h}^u_x(\mathbf{y}), \mathbf{h}^u_z(\mathbf{y}), \mathbf{h}^c_x(\mathbf{z})$ and $\mathbf{h}^c_y(\mathbf{z})$ are $\mathbf{C}^r$. In order to find the centre manifold we need to solve

$$\mathbf{Dh}^c_x(\mathbf{z})\left[A_c\mathbf{z} + F_c(\mathbf{h}^c_x(\mathbf{z}), \mathbf{h}^c_y(\mathbf{z}), \mathbf{z})\right] - A_s\mathbf{h}^c_x(\mathbf{z}) - F_s\left(\mathbf{h}^c_x(\mathbf{z}), \mathbf{h}^c_y(\mathbf{z}), \mathbf{z}\right) = 0, \tag{2.23}$$

$$\mathbf{Dh}_y^c(\mathbf{z})\left[A_c\mathbf{z} + F_c\left(\mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z}), \mathbf{z}\right)\right] - A_u\mathbf{h}_y^c(\mathbf{z}) - F_u(\mathbf{h}_x^c(\mathbf{z}), \mathbf{h}_y^c(\mathbf{z}), \mathbf{z}) = 0, \tag{2.24}$$

with the boundary conditions

$$\mathbf{h}_x^c(\mathbf{0}) = \mathbf{Dh}_x^c(\mathbf{0}) = \mathbf{0},$$

$$\mathbf{h}_y^c(\mathbf{0}) = \mathbf{Dh}_y^c(\mathbf{0}) = \mathbf{0}\,.$$

In general, these equations are difficult to solve analytically, so we resort to approximations. There are various approximations to the previous boundary value problem and we restrict ourselves to only the polynomial approximation method. In this method we approximate $\mathbf{h}_x^c(\mathbf{0})$ and $\mathbf{h}_y^c(\mathbf{0})$ as

$$\mathbf{h}_x^c(\mathbf{z}) = a_0 + a_1\mathbf{z} + a_2\mathbf{z}^2 + a_3\mathbf{z}^3 + \ldots, \tag{2.25}$$

$$\mathbf{h}_y^c(\mathbf{z}) = b_0 + b_1\mathbf{z} + b_2\mathbf{z}^2 + b_3\mathbf{z}^3 + \ldots. \tag{2.26}$$

form the boundary conditions $a_0 = b_0 = a_1 = b_1 = 0$. To find the other coefficient, substitute these polynomials into 2.23 and 2.24, and match the coefficients. Now the flow on the centre manifold $\mathbf{W}^c(\mathbf{0})$ in the neighbourhood of the origin is defined by the system

$$\dot{\mathbf{z}} = A_c\mathbf{z} + \mathbf{F}_c\left(\mathbf{z}, \mathbf{h}_x^c(\mathbf{z}), \mathbf{h}'_y(\mathbf{z})\right).$$

for all $\mathbf{z} \in \mathbf{R}^c$. Since the dimension of the centre manifold is typically less than n, this simplifies the problem of determining the qualitative behaviour of the system 2.19 near a non-hyperbolic critical point. In general, the flow on the centre manifold near the fixed point takes the form

$$\dot{\mathbf{z}} = a\,\mathbf{z}^r + \ldots.$$

If $r \geq 2$ and $a_r \neq 0$, then for r we have a saddle-node at the fixed point, for r odd and $a_r > 0$ we have an unstable node and for r odd and $a_r < 0$ we have a topological saddle.

Now let us consider the case when we have a double zero eigenvalue and one eigenvalue with a negative or a positive real part. Assume that we have the system

$$\begin{pmatrix} \dot{x} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_s & 0 & 0 \\ 0 & A_{c1} & 0 \\ 0 & 0 & A_{c2} \end{pmatrix}\begin{pmatrix} x \\ z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \mathbf{F}_s(x, z_1, z_2) \\ \mathbf{F}_{c1}(x, z_1, z_2) \\ \mathbf{F}_{c2}(x, z_1, z_2) \end{pmatrix}$$

where z_1 and z_2 are the states that correspond to the zero eigenvalues and x is the state that corresponds to the eigenvalue with a negative real part. In this case it follows from the local centre manifold theory, that a 2-dimensional invariant center manifold $\mathbf{W}^c_{local}(\mathbf{0})$ exists, defined by

$$\mathbf{W}^c_{local}(0) = \{(x, z_1, z_2) \in R^s \times R^c | \, x = \mathbf{h}(z_1, z_2) \text{ for} |z_1|, |z_2| < \delta\} ,$$

for some $\delta > 0$, where $\mathbf{h} \in C^r(N_\delta(\mathbf{0}))$, $\mathbf{h}(\mathbf{0})) = D\mathbf{h}(\mathbf{0}) = 0$. In order to find the centre manifold in this case, we need to solve

$$\begin{bmatrix} \frac{\partial \mathbf{h}(z_1,z_2)}{\partial z_1} & \frac{\partial \mathbf{h}(z_1,z_2)}{\partial z_2} \end{bmatrix} \begin{bmatrix} A_{c1} z_1 \\ A_{c2} z_2 \end{bmatrix} = A_s x + \mathbf{F}_s(x, \mathbf{h}(z_1, z_2)) ,$$

with the boundary conditions

$$\mathbf{h}(z_1, z_2) = \mathbf{D}\mathbf{h}(z_1, z_2) = 0 .$$

By considering these boundary conditions the function $\mathbf{h}(z_1, z_2)$ can be approximated by

$$\mathbf{h}(z_1, z_2) = a z_1^2 + b z_1 z_2 + d z_2^2 .$$

Now, the flow on the centre manifold $\mathbf{W}^c(\mathbf{0})$ in the neighbourhood of the origin is defined by the reduced system

$$\begin{aligned} \dot{z}_1 &= A_{c1} z_1 + \mathbf{F}_s(x, \mathbf{h}(z_1, z_2)) , \\ \dot{z}_2 &= A_{c1} z_2 + \mathbf{F}_s(x, \mathbf{h}(z_1, z_2)) . \end{aligned}$$

Example 2.1:
Consider the following nonlinear system

$$\dot{z} = z^2 x - z^5 , \qquad (2.27)$$

$$\dot{x} = -x + z^2 . \qquad (2.28)$$

In this case we have $A_c = [0]$, $A_s = [-1]$, $F_c = z^2 x - z^5$, and $F_s = z^2$. By substitution of the expansion

$$h^c_x(z) = a_2 z^2 + a_3 z^3 + \dots , \qquad Dh^c_x(z) = 2a_2 z + 3a_3 z + \dots ,$$

into the equation

$$Dh^c_x(z) \left[A_c z + F_c(h^c_x(z), h^c_y(z), z) \right] - A_s \, h^c_x(z) - F_s \left(h^c_x(z), h^c_y(z), z \right) = 0 ,$$

we obtain

$$(2az + 3bz^2 + \dots)(az^4 + bz^5 + \dots - z^5) + az^2 + bz^3 + \dots - z^2 = 0 .$$

Setting the coefficients of like powers of z equal to zero yields $a = 1, b = c = 0, \ldots$, it follows that

$$\mathbf{h}_x^c(z) = z^2 + 0(z^5)\,.$$

Substituting $x = h_x^c(z)$ into 2.26 yields

$$\dot{z} = z^4 + 0(z^5)\,.$$

This equation describes the flow of the original nonlinear system in the neighbourhood of the origin (see fig [2.4]). It is clear that the origin is a saddle-node. In the next section,

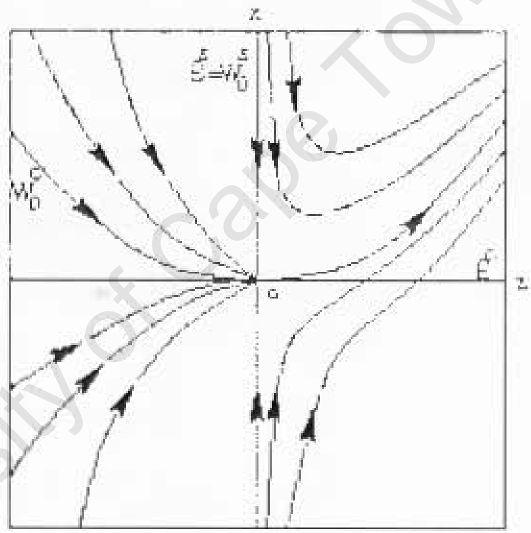

Figure 2.4: The phase portrait in the neighbourhood of the origin of the system in example 2.1.

we will investigate Friedmann-Lemaitre models with a cosmological constant using the dynamical systems approach.

2.4.2 The Einstein Static Universe

For the FRW metric background, the Einstein field equation 1.9 takes the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad (2.29)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P) + \frac{\Lambda}{3} \qquad (2.30)$$

Equation (2.29) can be written in a more compact form as

$$\dot{a}^2 + V(a) = -k\,,$$

where

$$V(a) = \frac{\rho_c}{3} a^{-(1-3\omega)} - \frac{\Lambda}{3} a^2 .$$

This equation is very useful in studying the dynamic stability of the Universe. For the Universe to be static $\ddot{a} = \dot{a} = 0$, this condition is satisfied when

$$\frac{dV(a)}{da} = 0 .$$

Thus the radius of the Einstein Static Universe is consequently given by

$$a_0^{3\gamma} = \frac{1}{2}(3\gamma - 2)\frac{\rho_c}{\Lambda} ,$$

where $\gamma = \omega + 1$. Fig [2.5] shows the graph of the V(a) a function of a. It is clear from

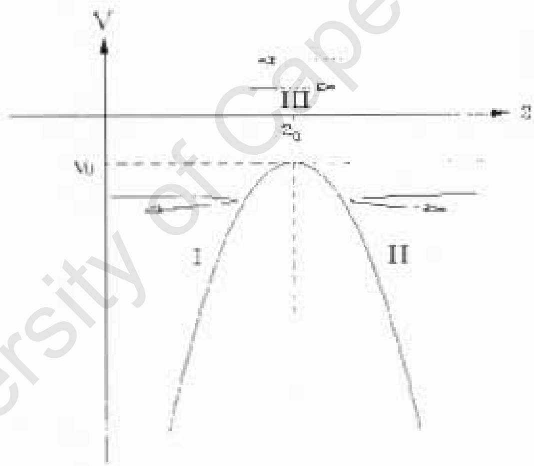

Figure 2.5: The graph of $V(a)$ a function of a.

fig [2.5] that the Einstein Static Universe, which take place at (v_0, a_0) is not stable, so that a small change in the energy density of the Universe will result in a fast collapse or infinite expansion. If the initial conditions are chosen in such a manner that the Universe starts in region I, then there will be three possible evolutions.

- $a(t)$ expands up to a maximum radius a_0 and then re-collapses ,
- $a(t)$ expands up to a maximum radius a_0 with $q < 0$ and then expands with $q > 0$,
- $a(t)$ expands to a_0 in an infinite time .

If the initial conditions are chosen in such a manner that the Universe starts in region II then,

- $a(t)$ contracts to a minimum radius a_0 and then expands.
- $a(t)$ contracts to a minimum radius a_0 with $q < 0$ and then contracts to zero with $q > 0$.
- $a(t)$ contracts to a_0 in an infinite time.

Fig [2.6] shows schematic diagrams of all these cases.

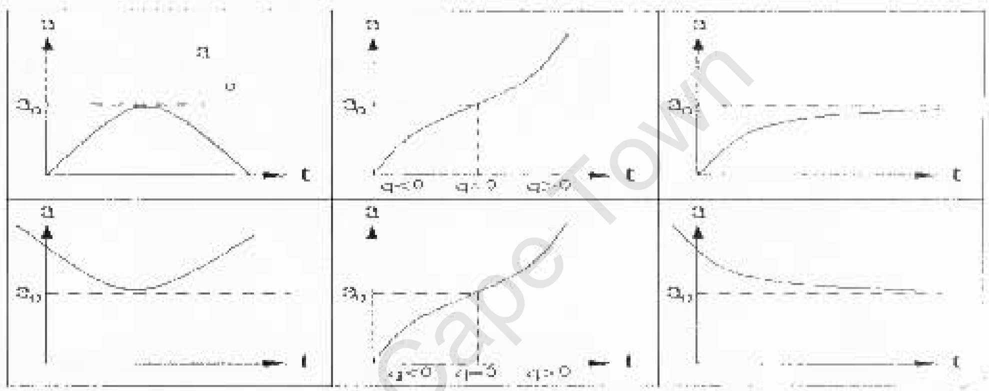

Figure 2.6: The evolution of the scale factor for Friedmann-Lemaitre models.

In region III, depending on the initial conditions, the Universe can either expand to infinity or contract to zero.

2.5 The stability analysis of Friedmann-Lemaitre models

Let us define the dimensionless parameters

$$K = \frac{k}{3H^2} \qquad \Omega = \frac{\rho}{3H^2} \Omega_\Lambda = \frac{\Lambda}{3H^2} . \tag{2.31}$$

In what follows, $H \neq 0$ is assumed. Using these variables, equation (2.29) can be written as a system of first order differential equations

$$\begin{aligned} K' &= ([3\gamma - 2]\Omega - 2\Omega_\Lambda)(\Omega + \Omega_\Lambda - 1) , \\ \Omega'_\Lambda &= 2(1 - \frac{3\gamma - 2}{2}\Omega - \Omega_\Lambda)\Omega_\Lambda , \\ \Omega' &= ([3\gamma - 2]\Omega - 2\Omega_\Lambda)(1 - K - \Omega_\Lambda) , \end{aligned} \tag{2.32}$$

where the prime represents the derivative with respect to the time variable $' = H^{-1} d/dt$. This system is completed by the Friedmann constraint

$$\Omega + \Omega_\Lambda + K = 1 .$$

By considering a weak energy condition and assuming $\Lambda \geq 0$, the dynamical variables proposed in (2.31) can be compactified to the range [0,1]. Using the Friedmann constraint the system (2.32) can be reduced to a two-dimensional system,

$$K' = ([3\gamma - 2]\Omega - 2\Omega_\Lambda)K\,,$$
$$\Omega'_\Lambda = 2(1 + \frac{3\gamma - 2}{2}\Omega - \Omega_\Lambda)\Omega_\Lambda\,.$$

Equilibrium point solutions, and the stability analysis of this system, are shown in Table [2.1]. In order to compactify the region $K > 0$ of the phase space we will begin by writing

Table 2.1: Coordinates of the fixed points, eigenvalues, solutions and the stability.

Fixed points (K, Ω_Λ)	Eigenvalues	Solution	Stability $0 < \gamma < 2/3$	$2/3 < \gamma < 2$
[0, 0]	$[3\gamma - 2, 3\gamma]$	F	saddle	source
[−1, 0]	$[-(3\gamma - 2), 2]$	M	source	saddle
[0, 1]	$[-2, -3\gamma]$	ds	sink	sink

(2.30) as

$$\rho = 3H^2 + \frac{3k}{a^2} - \Lambda\,.$$

Let us define the compact variables

$$Q = \frac{H}{D} \qquad \tilde{\Omega_\Lambda} = \frac{\Lambda}{3D^2}\,,$$

where $D = \sqrt{H^2 + k/a^2}$. Then by using the new time variable$' = D^{-1}d/dt$, the evolution equations for Q and $\tilde{\Omega_\Lambda}$ are given by

$$Q' = (1 - \frac{3\gamma}{2}[1 - \tilde{\Omega_\Lambda}])(1 - Q^2)\,,$$
$$\tilde{\Omega_\Lambda} = 3\gamma(1 - \tilde{\Omega_\Lambda})Q\tilde{\Omega_\Lambda}\,.$$

The equilibrium points, and the solutions of this system, are depicted in table [2.2] The compact space for all Friedmann models with $2/3 < \gamma < 2$ is illustrated in fig [2.7]. The left and right sides of the phase space correspond to expanding and contracting models respectively. In the $K < 0$ and $H > 0$ region orbits with $\Omega > 0$ and $\Lambda > 0$ evolve from F to ds. For $K > 0$ and $H > 0$ there are three possibilities:

- If Λ is sufficiently small, we obtain Friedmann-Lemaitre models,

Table 2.2: Coordinates of the fixed points, solutions and the stability.

Coordinates(Q, Ω_Λ)	Solution	Stability
$[1, 0]$	$+F$	Source
$[-1, 0]$	$-F$	Sink
$[1, 1]$	$+ds$	Sink
$[-1, 1]$	$-ds$	Source
$[0, \frac{3\gamma-2}{3\gamma}]$	E	Saddle

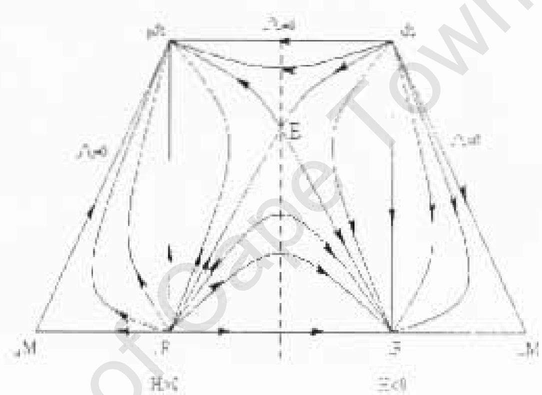

Figure 2.7: The full phase space of Friedmann-Lemaitre models.

- If Λ is large, we obtain Lemaitre models,
- For $\Lambda = \frac{3\gamma-2}{\gamma}\frac{k}{a_0^2}$, we obtain the Einstein static Universe.

There are also models which start with $H < 0$ and cross to $H > 0$. The models which passed asymptotic to the Einstein static Universe are called Eddington-Lemaitre models. The straight line $\Omega = 0$ from $-ds$ to $+ds$ correspond to a de-Sitter Universes.

Chapter 3

Cosmological Dynamics Of Exponential Gravity

3.1 Introduction

In this chapter a detailed investigation of the cosmological dynamics based on $\exp(-R/\Lambda)$ gravity will be presented. The dynamical system approach to an important class of theories with Lagrangian density will be applied

$$A = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} \left[e^{-\frac{R}{\Lambda}} + L_M \right] , \tag{3.1}$$

where Λ is the cosmological constant, R is the Ricci scalar and L_M is the Lagrangian of standard matter. This Lagrangian possesses some interesting features. Firstly, its treatment is equivalent to one involving a combination of powers of the Ricci scalar and, secondly, it reduces to

$$\exp\left(-\frac{R}{\Lambda}\right) = 1 - \frac{R}{\Lambda} - O[R^2] \tag{3.2}$$

in the small curvature limit, which is equivalent to the Hilbert-Einstein action.

3.2 Basic Equations

The general action for a fourth order theory of gravity in a homogeneous and isotropic spacetime is:

$$\mathcal{A} = \int d^4x \sqrt{-g} [f(R) - L_M] , \tag{3.3}$$

where $f(R)$ is a function of the Ricci scalar R. Varying this action with respect to the metric gives

$$G_{\mu\nu} = T^{TOT}_{\mu\nu} = \tilde{T}^{M}_{\mu\nu} + T^{R}_{\mu\nu} , \tag{3.4}$$

where $G_{\mu\nu}$ is the usual Einstein tensor, and

$$\tilde{T}^M_{\mu\nu} = \frac{1}{f'(R)} T^M_{\mu\nu}, \tag{3.5}$$

$$T^R_{\mu\nu} = \frac{1}{f'(R)} \left[\frac{1}{2} g_{\mu\nu} \left[f(R) - R f'(R) \right] + f'(R)^{;\alpha\beta} \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu} \right) \right]. \tag{3.6}$$

The prime denotes the derivative with respect to R. $T^M_{\mu\nu}$ is the energy-momentum tensor for standard matter, which is assumed to be a perfect fluid and $T^R_{\mu\nu}$ is the stress-energy tensor of an *effective fluid* (sometimes referred to as the curvature *fluid*), which represents the non-Einsteinian part of the gravitational interaction and constitutes an additional source term of purely geometrical origin [40]. By assuming $f(R) = \exp(-R/\Lambda)$ we obtain the field equations:

$$\begin{aligned} G_{\mu\nu} &= -\Lambda e^{R/\Lambda} \tilde{T}^M_{\mu\nu} + \\ &- e^{R/\Lambda} \left[\frac{1}{2} g_{\mu\nu} (\Lambda + R) + \frac{1}{\Lambda^2} (R^{;\alpha} R^{;\beta} + \Lambda R^{;\alpha\beta}) (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\beta} g_{\mu\nu}) \right]. \end{aligned} \tag{3.7}$$

In the case of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the above equations reduce to:

$$H^2 - \frac{k}{a^2} - \frac{H}{\Lambda}\dot{R} + \frac{R}{6} + \frac{\Lambda}{6} - \frac{\Lambda\rho}{3e^{-R/\Lambda}} = 0, \tag{3.8}$$

$$2\frac{\ddot{a}}{a} + \frac{R}{3} - \frac{H}{\Lambda}\dot{R} + \frac{1}{\Lambda^2}\dot{R}^2 - \frac{1}{\Lambda}\ddot{R} - \frac{\Lambda\rho}{3e^{-R/\Lambda}}(1 + 3w) + \frac{\Lambda}{3} = 0, \tag{3.9}$$

with,

$$R = -6\left(\frac{\ddot{a}}{a} - H^2 + \frac{k}{a^2}\right), \tag{3.10}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, a is the usual scale factor, k is the spatial curvature, ρ is the energy density of standard matter and w its barotropic factor. The Bianchi identities applied to the total stress-energy tensor $T^{TOT}_{\mu\nu}$ lead to the energy conservation equation for standard matter [31]:

$$\dot{\rho} - 3H\rho(1 + w) = 0 \tag{3.11}$$

3.3 The Vacuum Case

In the case of a vacuum, ($\rho = 0$) equation 3.8 can be written as a closed system of first order differential equations using the dimensionless variables:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad K = \frac{k}{a^2 H^2}. \tag{3.12}$$

Here the variables y and z are a measure of the expansion normalized Ricci curvature and the cosmological constant respectively. K is the spatial curvature parameter of the Friedmann model, while x is a measure of the time rate of the change of the Ricci curvature. The evolution equations for the variables 3.12 are given by

$$\begin{aligned}
\frac{dx}{dN} &= \varepsilon(2z + 2K - 2) + x\varepsilon(1 - x + y - K), \\
\frac{dy}{dN} &= x\varepsilon z + 2y\varepsilon(2 + y - K)\,, \\
\frac{dz}{dN} &= 2z\varepsilon(2 - y + K)\,, \\
\frac{dK}{dN} &= 2K\varepsilon(y + 1 - K)\,,
\end{aligned} \tag{3.13}$$

where the prime represents the derivatives with respect to the time variable $N = \ln|a|$ and $\varepsilon = |H|/H$. This system is completed with the Friedmann constraint,

$$1 - K + y + z - x = 0\,, \tag{3.14}$$

which defines a hyperplane in the total phase space of the system. Consequently, all solutions of the dynamical system will be located in a non-compact submanifold of the phase space associated with 3.13. The time derivative of 3.14 is nothing other than the Raychaudhuri equation.

3.3.1 Finite analysis

The dimensionality of the state space of the system 3.13 can be reduced by eliminating any one of the four variables using the constraint equation 3.14. If we choose to eliminate x[1], the dynamic equations become:

$$\begin{aligned}
\frac{dy}{dN} &= y\varepsilon(4 + 2K + 2y + z) + z\varepsilon(1 - K + z)\,, \\
\frac{dz}{dN} &= 2z\varepsilon(2 + K + y), \\
\frac{dK}{dN} &= 2K\varepsilon(1 + K - y)\,.
\end{aligned} \tag{3.15}$$

The system 3.15 admits two invariant submanifolds: $z = 0$ and $K = 0$. This means that the points contained in these submanifolds form an invariant set under the transformation

[1]Of course we could have chosen to eliminate any other variable of the system: our choice is motivated by the fact that the equation for x is by far the most complicated one to solve and that, with this choice, the number of invariant submanifolds is maximized. As we will see, this will assist in the investigation of the properties of the cosmology, particularly in the matter case.

defined by the system 3.15, that is, 3.15 sends points on these sets only to points of the same set. As a consequence, if we choose $z = 0$ ($K = 0$) as an initial condition for an orbit, this orbit will never leave the plane $z = 0$ ($K = 0$), while for orbits with an initial condition $z \neq 0$ ($K \neq 0$), the only way to smoothly approach $z = 0$ ($K = 0$) is to approach it asymptotically. This implies that no orbit crosses the $z = 0$ plane and consequently no global attractor can exist, because the phase space is divided into independent sectors which contain complete cosmological histories.

Setting $K' = 0$, $y' = 0$, $z' = 0$, we obtain four fixed points that we label with capital letters and the subscript v to indicate those which are found for the vacuum case (see Table 3.1).

We can obtain exact cosmological solutions at these points by using the Raychaudhuri equation,

$$\dot{H} = -(y - K + 2)H^2 \,. \tag{3.16}$$

In fact, at any fixed point with $y + K - 2 \neq 0$, the equation 3.16 reduces to

$$\dot{H} = -\frac{1}{\alpha}H^2 \,, \qquad \alpha = (y_* + K_* - 2)^{-1} \,, \tag{3.17}$$

where the subscript $*$" indicates that a quantity has been calculated at the fixed point. Equation 3.16 applies to both the matter and vacuum cases and describes a general power law evolution of the scale factor. In addition, integrating with respect to time we obtain

$$a = a_o(t - t_o)^{\alpha} \,, \quad \alpha \neq 0 \,. \tag{3.18}$$

This means that by finding the value of α at a given fixed point, we can obtain the solutions associated with it using equation 3.16 and this solution will be given by 3.18 for $\alpha \neq 0$.

In this way, points A_v and B_v are found to represent Milne and power-law evolutions respectively (see Table 3.1). However, by direct substitution into the cosmological equations it can be shown that these fixed points cannot be considered physical because, in order to satisfy 3.8, one needs to violate the weak energy condition $\rho \geq 0$. This does not constitute a problem because, as we will see below, these points are always unstable, which means that we can choose initial conditions as close to these points as we wish.

For the points C_v and D_v we have $\alpha = 0$ so that 3.16 reduces to $\dot{H} = 0$ and the scale factor is given by

$$a = a_o e^{\gamma(t - t_o)}. \tag{3.19}$$

The value of the constant γ can be obtained by direct substitution into equations 3.8. For both $\mathcal{C}_v$ and $\mathcal{D}_v$ we obtain

$$\gamma = \pm\sqrt{\frac{\Lambda}{6}}\,, \tag{3.20}$$

so that they represent an exponential evolution. The contracting or expanding nature of this solution depends on the direction of approach of the orbits with respect to the hypersurface $y + K + 2 = 0$. This hypersurface divides the phase space into two hypervolumes characterized by a contracting or expanding evolution. In particular, for $y < -K - 2$ the orbits describe a contracting universe, while for $y > -K - 2$ they represent an expanding one.

The stability of the hyperbolic fixed points $\mathcal{A}_v$, $\mathcal{B}_v$ and $\mathcal{D}_v$ is obtained by using the Hartman-Grobman theorem. The point $\mathcal{C}_v$, instead, is non-hyperbolic and must use the local centre manifold theorem in order to find its stability. A brief review of this theorem can be found in chapter 2 [41]. In our case, using the transformation

$$y = u_1 - 2u_2 - m\,,$$
$$z = 4m\,, \tag{3.21}$$
$$k = u_2$$

the system 3.15 can be written in the form

$$\dot{u}_1 = -4u_1\varepsilon + \varepsilon(12m^2 + 2mu_1 + 2u_1^2 - 4mu_2 - 2u_1)\,, \tag{3.22}$$

$$\dot{u}_2 = -2u_2\varepsilon + \varepsilon(-2mu_2 + 2u_1u_2 - 2u_2^2)\,, \tag{3.23}$$

$$\dot{m} = \varepsilon(-2m^2 + 2mu_1 - 2mu_2)\,, \tag{3.24}$$

where

$$F_{s1}(u_1, u_2, m) = \varepsilon(12m^2 + 2mu_1 + 2u_1^2 - 4mu_2 - 2u_1)\,, \tag{3.25}$$

$$F_{s2}(u_1, u_2, m) = \varepsilon(-2mu_2 + 2u_1u_2 - 2u_2^2)\,, \tag{3.26}$$

$$F_m(u_1, u_2, m) = \varepsilon(-2m^2 + 2mu_1 - 2mu_2)\,, \tag{3.27}$$

represent the nonlinear terms. By substituting the expansions

$$h1(m) = am^2 + bm^3 + O(m^4)\,, \tag{3.28}$$

$$h2(m) = cm^2 + dm^3 + O(m^4) \tag{3.29}$$

Table 3.1: Coordinates of the fixed points, eigenvalues, stability and solutions for $\exp(-\frac{R}{\Lambda})$-gravity in vacuum.

Point	Coordinates(y,z,K)	Eigenvalues	Stability	Solution
$\mathcal{A}_v$	$[0,0,0]$	$[2,4,4]$	repeller	$a = a_0(t-t_0)^{\frac{1}{2}}$
$\mathcal{B}_v$	$[0,0,-1]$	$[-2,2,2]$	Saddle	$a = a_0(t-t_0)$
$\mathcal{C}_v$	$[-2,0,0]$	$[-4,-2,0]$	Saddle-node	$a = a_0 e^{\gamma(t-t_0)}$
$\mathcal{D}_v$	$[-2,1,0]$	$[-\frac{(3+\sqrt{17})}{2},-2,\frac{(3+\sqrt{17})}{2}]$	Saddle	$a = a_0 e^{\gamma(t-t_0)}$

into equations 2.23 and 2.24 and then solving for the coefficients a, b, c and d, we obtain

$$h1(m) = 3m^2 + \frac{9}{2}m^3 + O(m^4), \qquad h2(m) = O(m^4)\,. \tag{3.30}$$

Substituting this result into equation 3.24 then yields

$$\dot{m} = -2m^2 + O(m^3) \tag{3.31}$$

on the centre manifold $W^c(\mathbf{0})$, around the point $\mathcal{C}_v$. This implies that the point $\mathcal{C}_v$ is a saddle-node, that is, it behaves like a saddle or an attractor depending on the direction from which the orbit approaches. The local phase portrait in the neighbourhood of $\mathcal{C}_v$ is depicted in figure 3.1.

If one considers the transformation (3.21) now, one realizes that $m \propto z$, so that $\mathcal{C}_v$ is an attractor for $z > 0$ and a saddle for $z < 0$. This is also clear from Figure 3.2 in which the invariant submanifold $K = 0$ is depicted.

Finally, it is useful to derive an expression for the deceleration parameter q in terms of the dynamical variables:

$$q = -\frac{\dot{H}}{H^2} - 1 = -(y + K + 1)\,. \tag{3.32}$$

This equation holds for both the vacuum and the matter case. Note that that $q > 0$ is realized only when $(y + K + 1) < 0$. This condition is satisfied only for the point $\mathcal{C}_v$ as is expected by looking at the solution associated with this fixed point (see Table [3.1]). In Figure 3.3 we give the location of the $q = 0$ plane relative to the fixed points $\mathcal{A}_v$, $\mathcal{C}_v$ and $\mathcal{B}_v$.

3.3.2 Asymptotic analysis

In this section we will determine the fixed points at infinity and study their stability. In order to simplify the asymptotic analysis we will compactify the phase space using the

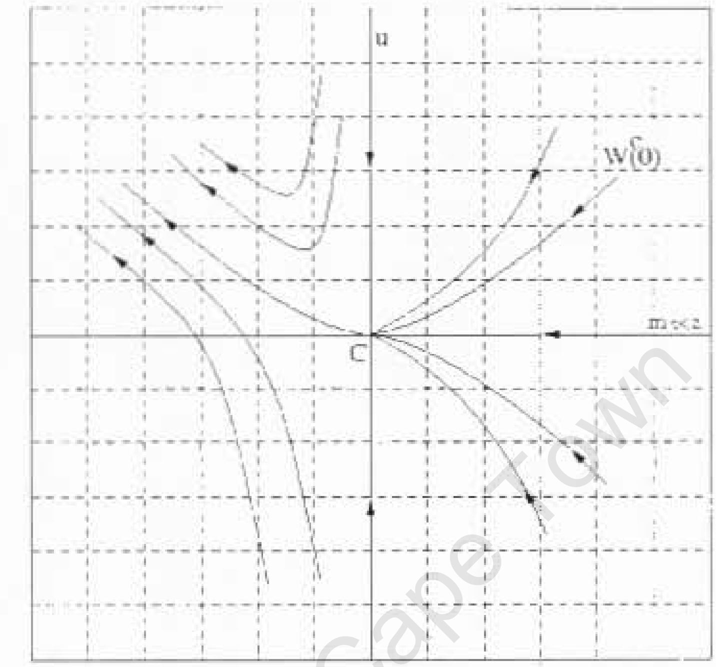

Figure 3.1: The phase portrait for the system 3.15 in the neighbourhood of the fixed point $\mathcal{C}$ for $\exp(-R/\Lambda)$-gravity in vacuum.

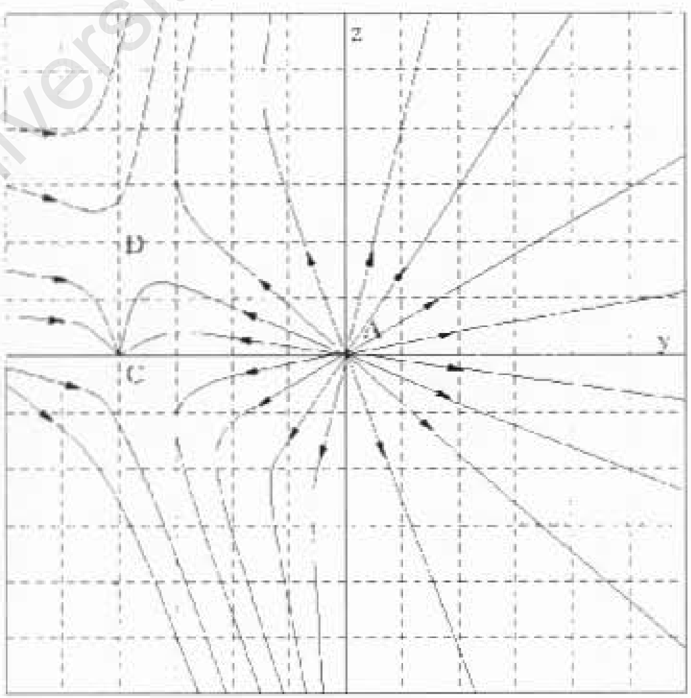

Figure 3.2: The invariant submanifold $K = 0$ for $\exp(-R/\Lambda)$-gravity in vacuum.

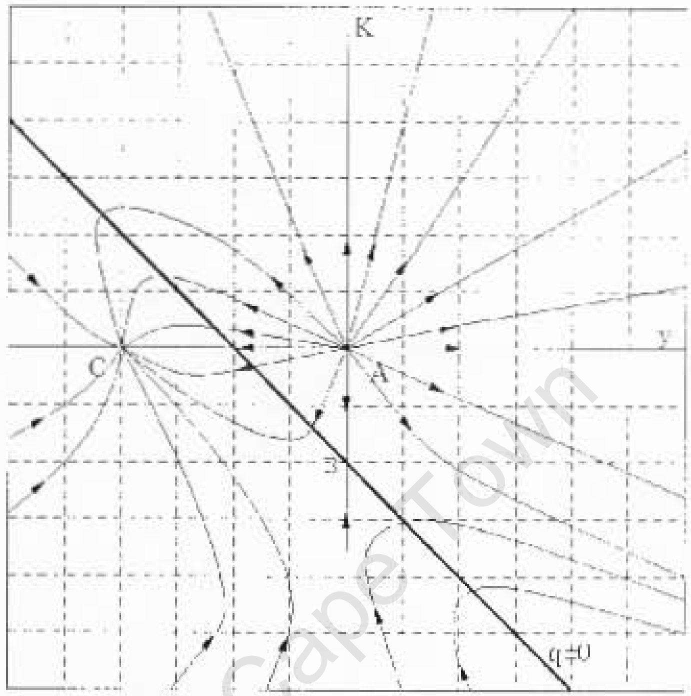

Figure 3.3: The invariant submanifold $z = 0$. We explicitly indicate the location of the $q = 0$ plane relative to the fixed points $\mathcal{A}_v$, $\mathcal{C}_v$ and $\mathcal{B}_v$ for $\exp(-R/\Lambda)$ gravity in vacuum.

Poincaré method. Transforming to polar coordinates (r, θ, ϕ):

$$z \to r\ \cos\theta,\ \ K \to r\ \sin\theta\ \cos\phi,\ \ y \to r\ \sin\theta\ \sin\phi$$

and substituting $r \to \frac{\mathcal{R}}{1-\mathcal{R}}$, the regime $r \to \infty$ corresponds to $\mathcal{R} \to 1$. Using this coordinate transformation and taking the limit $\mathcal{R} \to 1$, the system 3.15 can be written as

$$\begin{aligned} \mathcal{R}' \to \quad & \frac{1}{4}\Big\{8\cos\phi\sin^3\theta - \sin\phi\,(7\sin\theta + \sin 3\theta) \\ & + 8A\cos^2\theta\sin\theta + 4A\cos\theta\sin^2\theta\sin\phi\Big\}\,, \end{aligned} \tag{3.33}$$

$$\mathcal{R}\theta' \to \frac{\varepsilon\cos^2\theta\sin\phi\sin\theta}{\mathcal{R}-1}(A - \cot\theta)\,, \tag{3.34}$$

$$\mathcal{R}\phi' \to \frac{\varepsilon\cos\phi\cot\theta}{\mathcal{R}-1}(A + \cot\theta)\,, \tag{3.35}$$

where $A = \cos\phi + \sin\phi$. Since equation 3.33 does not depend on the coordinate $\mathcal{R}$, we can find the fixed points of the above system using equations 3.34 and 3.35 only. From equations (3.34) and (3.35) if

$$A + \cot\theta = 0\,, \tag{3.36}$$

then $\theta' = \phi' = 0$, which means that $\mathcal{O}_v^\infty = [-\text{arccot}\, A, \phi]$ represent a fixed subspace. The other fixed subspace is given by $\mathcal{I}_v^\infty = [\pi/2, \phi]$, and there are no single fixed points (see Table 3.2).

Let us now derive the solution for $\mathcal{I}_v^\infty$. In this subspace the first equation of the system 3.15 reduces to

$$\frac{dy}{dN} = 2\epsilon y^2(\cot\phi_0 + 1)\,, \quad \text{where} \quad \cot\phi_0 = K/y\,, \tag{3.37}$$

and equation 3.16 becomes

$$\dot{H} = -y(\cot\phi_0 - 1)H^2\,. \tag{3.38}$$

Integrating equation 3.37 we obtain

$$y = \frac{-1}{2\epsilon(1-\cot\phi_0)(N - N_\infty)}\,, \tag{3.39}$$

where N_∞ is an integration constant. Substituting y back into equation 3.38 and solving for N, we obtain

$$N - N_\infty = [\frac{1}{2\epsilon}(c_1 \pm c_o(t - t_o))]^2\,. \tag{3.40}$$

The same procedure can be employed to obtain solutions for $\mathcal{O}_v^\infty$ (see Table 3.2 for the result).

Using the 3.33 to take into account the radial behaviour of the orbits, the stability of $\mathcal{I}_v^\infty$ is:

$-\pi/4 < \phi < \pi/2$	Stable,
$\pi/2 < \phi < 3\pi/4$	Unstable,
$3\pi/4 < \phi < 3\pi/2$	Stable,
$3\pi/2 < \phi < 7\pi/4$	Unstable.

We obtain the stability of $\mathcal{O}_v^\infty$ in the same way from which it turns out that these points are never stable. For the values of ϕ for which $L_1(\phi)$ and $L_2(\phi)$ (L_1 and L_2 are functions of ϕ only they are too complicated to be recorded here (see Figure 3.4 for their plots)) are both positive, the points in $\mathcal{O}_v^\infty$ are repellers, while for the values of ϕ for which these functions have opposite signs, they are saddles.

In the next section we will observe how the introduction of matter modifies the picture we obtained in the vacuum case.

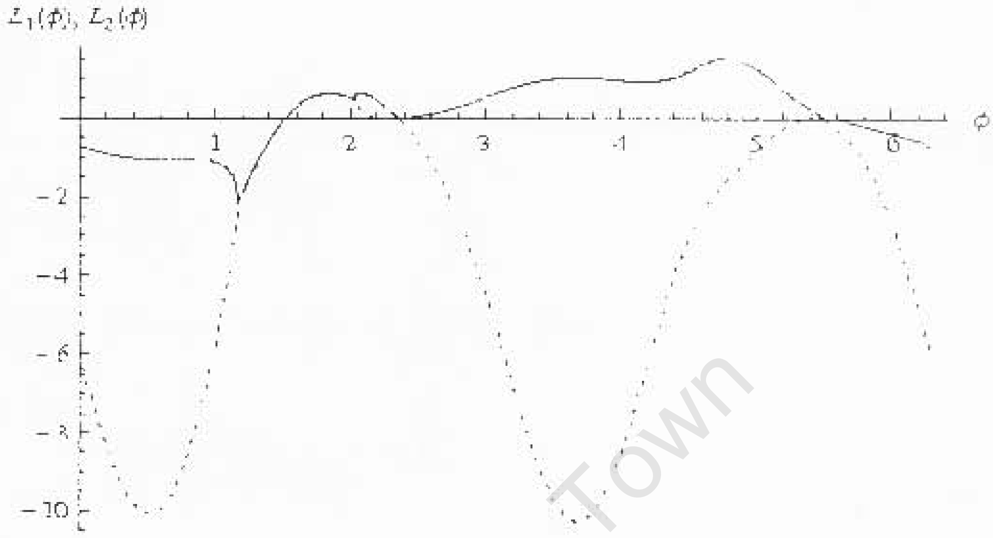

Figure 3.4: The graph of the eigenvalues of the fixed space as a function of ϕ. Here $L_1(\phi)$ is represented by a solid curve and $L_2(\phi)$ is the dashed one.

Table 3.2: Coordinates, eigenvalues and the stability of the fixed points in the asymptotic regime for $\exp(-\frac{R}{\Lambda})$ gravity in vacuum. L_1 and L_2 are functions of ϕ which are too complicated to be recorded here (see Figure 3.4 for their plots). $E = [2 + 2\cot\phi_0 + \cot\theta \operatorname{cosec}\phi_0 + A_0 \operatorname{cosec}\phi_0^2(1 - A_0^{-1}\cos\phi_0)]$ and A_0 is the value of A in ϕ_0

Point (θ, ϕ)	Eigenvalues	Solution
$\mathcal{I}_v^\infty$ $[\frac{\pi}{2}, \phi]$	$[0, -A\cos\phi]$	$(N - N_\infty) = [\frac{1}{2}(c_1 \pm c_0(t_1 - t_0))]^2$
$\mathcal{O}_v^\infty$ $[-\operatorname{arccot} A, \phi]$	$[L_1(\phi) < 0\ \forall\phi,\ L_2(\phi) > 0\ \forall\phi]$	$(N - N_\infty) = [\frac{E-1}{cE}(c_1 \pm c_0(t_1 - t_0))]^{\frac{E}{E-1}}$

3.4 The matter case

In this case we can utilise the same dynamical variables we used for the vacuum case together with one additional variable D, related to the matter energy density:

$$x = \frac{\dot{R}}{\Lambda H}, \quad y = \frac{R}{6H^2}, \quad z = \frac{\Lambda}{6H^2}, \quad K = \frac{k}{a^2H^2}, \quad D = \frac{\Lambda\rho}{3H^2e^{R/\Lambda}}. \tag{3.41}$$

The definition of the variables reveals that not all of the phase space corresponds to physical situations. This becomes clear if we divide D by z. We obtain

$$\frac{D}{z} = 2\rho\exp\left(-\frac{R}{\Lambda}\right), \tag{3.42}$$

which has the same sign as ρ. This means that the sectors in the phase space for which the sign of D is different from the sign of z contain orbits in which standard matter violates the weak energy condition $\rho > 0$, and have to be discarded as not physical. As we will observe, this implies restrictions on the cosmic evolution scenarios allowed in this model. Following the same procedure we used in the vacuum case, we obtain the autonomous system:

$$
\begin{aligned}
\frac{dx}{dN} &= \varepsilon(2+2z+2K) + x\,\varepsilon(1-x+y+K) - D\,\varepsilon(1-3w)\,, \\
\frac{dy}{dN} &= xz\,\varepsilon + 2y\,\varepsilon(2+y-K)\,, \\
\frac{dz}{dN} &= 2z\,\varepsilon(2+y+K)\,, \\
\frac{dK}{dN} &= 2K\,\varepsilon(y-1+K)\,, \\
\frac{dD}{dN} &= D\,\varepsilon(1-3w+2y+2K-x)\,,
\end{aligned}
\tag{3.43}
$$

together with the constraint equation

$$1 + K + x + y - z - D = 0\,, \tag{3.44}$$

where the prime again denotes the derivative with respect to the logarithmic time variable N.

3.4.1 Finite analysis

The system 3.43 can be further simplified using the constraint equation 3.44 to eliminate x. We obtain:

$$
\begin{aligned}
\frac{dy}{dN} &= y\varepsilon(4+2K+2y+z) + z\varepsilon(1+K+D+z)\,, \\
\frac{dK}{dN} &= 2K\varepsilon(1+K+y)\,, \\
\frac{dz}{dN} &= 2z\varepsilon(2+y+K)\,, \\
\frac{dD}{dN} &= D\varepsilon(2-3w+3K+D+3y+z)\,.
\end{aligned}
\tag{3.45}
$$

The structure of 3.45 reveals that in this case we have three invariant submanifolds: $K = 0$, $z = 0$ and $D = 0$; hence also in this case no global attractor can exist. Setting $K' = 0$, $y' = 0$, $z' = 0$ and $D' = 0$ we obtain seven fixed points (see Table 3.3).

As in the vacuum case, we can use the coordinates of these fixed points and equation 3.16 to find the behaviour of the scale factor at these points. In addition, the behaviour

Table 3.3: Coordinates of the fixed points, the eigenvalues, and solutions for $\exp(-R/\Lambda)$-gravity in the matter case.

Point	Coordinates(y,z,K,D)	Eigenvalues	Solution
$\mathcal{A}_m$	$[0,0,0,0]$	$[2-3w,2,4,4]$	$a = a_0(t-t_0)^{\frac{1}{2}}$
$\mathcal{B}_m$	$[0,0,-1,0]$	$[2,2,-2,-(1-3w)]$	$a = a_0(t-t_0)$
$\mathcal{C}_m$	$[0,0,0,3w-2]$	$[3w-2,2,4,4]$	$a = a_0(t-t_0)^{\frac{1}{2}}$
$\mathcal{D}_m$	$[0,0,-1,3w+1]$	$[2,2,-2,(1+3w)]$	$a = a_0(t-t_0)$
$\mathcal{E}_m$	$[-2,1,0,0]$	$[-\frac{\sqrt{17}+3}{2},\frac{\sqrt{17}-3}{2},-2,-3-3w]$	$a = a_0e^{\gamma(t-t_0)}$
$\mathcal{F}_m$	$[-2,0,0,0]$	$[-2,-4,-3w-4,0]$	$a = a_0e^{\gamma(t-t_0)}$
$\mathcal{G}_m$	$[-2,0,0,3w+4]$	$[3w+4,-2,-4,0]$	$a = a_0e^{\gamma(t-t_0)}$

of the energy density ρ can be obtained from equation 3.11, which at a fixed point reads

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\alpha}{t}\,, \tag{3.46}$$

where α is defined by 3.17. However, direct substitution in the cosmological equations reveals that all the fixed points correspond to vacuum states.

The exact solutions at the fixed points are summarized in Table 3.3. As in the vacuum case, we use the Hartman-Grobman theorem together with the centre manifold theorem to analyze the stability of all the fixed points. The results appear in Table 3.4.

3.4.2 Asymptotic analysis

We complete the analysis for the matter case by investigating the asymptotic behavior of the system 3.45. In order to achieve this we compactify the phase space by transforming it to 4-D polar coordinates. The transformation equations are

$$D \to r\,\cos\delta\,,\quad z \to r\,\sin\delta\,\cos\theta,\quad K \to r\,\sin\delta\,\sin\theta\,\cos\phi,$$

$$y \to r\,\sin\theta\,\sin\delta\,\sin\phi\,,$$

where $r \in [0,\infty]$, $\delta \in [0,\pi]$, $\theta \in [0,\pi]$, and $\phi \in [0,2\pi]$. We then transform the radial coordinate $r \to \frac{\mathcal{R}}{1-\mathcal{R}}$ and in the limit $\mathcal{R} \to 1$, the system 3.45 reduces to

$$\begin{aligned} \mathcal{R}' \to \cos^3\delta &+ \cos\delta\cos\theta\sin^2\delta\sin\theta\sin\phi - \cos^2\delta\sin\delta(\cos\theta \\ &+ 3A\sin\theta) + \sin^3\delta\sin\theta\Big[\cos\phi(2+B) \\ &+ \sin\phi(3\cos^2\theta + 2\sin^2\theta + B)\Big]\,, \end{aligned} \tag{3.47}$$

Table 3.4: Stability of the fixed points for $\exp(-R\Lambda)$-gravity in the matter case.

Point	$w=0$	$0<w<\frac{1}{3}$	$w=\frac{1}{3}$
$\mathcal{A}_m$	Repeller	Repeller	Repeller
$\mathcal{B}_m$	Saddle	Saddle	Saddle
$\mathcal{C}_m$	Saddle	Saddle	Saddle
$\mathcal{D}_m$	Saddle	Saddle	Saddle
$\mathcal{E}_m$	Saddle	Saddle	Saddle
$\mathcal{F}_m$	Saddle-node	Saddle-node	Saddle-node
$\mathcal{G}_m$	Saddle-node	Saddle-node	Saddle-node
Point	$\frac{1}{3}<w<\frac{2}{3}$	$w=\frac{2}{3}$	$\frac{2}{3}<w<1$
$\mathcal{A}_m$	Repeller	Saddle-node	Saddle
$\mathcal{B}_m$	Saddlee	Saddle	Saddle
$\mathcal{C}_m$	Saddle	Saddle-node	Repeller
$\mathcal{D}_m$	Saddle	Saddle	Saddle
$\mathcal{E}_m$	Saddle	Saddle	Saddle
$\mathcal{F}_m$	Saddle-node	Saddle-node	Saddle-node
$\mathcal{G}_m$	Saddle-node	Saddle-node	Saddle-node

$$\mathcal{R}\delta' \rightarrow \frac{\sin\delta\cos\delta}{8(\mathcal{R}-1)} \{8\cos\delta\,(B-1) - \sin\delta\,[\cos 3\theta + 8\cos\phi\sin\theta + 8\sin^3\theta\sin\phi + 7\cos\theta + 4\cos\theta\sin^2\theta\,(\cos 2\phi - \sin 2\phi)]\}\,, \tag{3.48}$$

$$\mathcal{R}\theta' \rightarrow \frac{\cos^2\theta}{2(\mathcal{R}-1)} \{2\cos\delta\sin\phi - \sin\delta\,[2\cos\theta\sin\phi + \sin\theta\,(1-\cos 2\phi - \sin 2\phi)]\}\,, \tag{3.49}$$

$$\mathcal{R}\phi' \rightarrow \frac{\cos\phi\,\{\cos\delta\cot\theta + \cos\theta\sin\delta\,[A+\cot\theta]\}}{\mathcal{R}-1}\,, \tag{3.50}$$

where $B = \cos\theta\sin\theta\sin\phi$. Notice that the first equation of the previous system does not depend on $\mathcal{R}$, which means that the fixed points of this system can be determined by the angular equations alone. As in the vacuum case, there are no isolated fixed points (see Table 3.5).

The solutions at the fixed points can be obtained by following the same procedure we used in the vacuum case.

Taking into account the radial behaviour of the orbits we can deduce the stability of the first two fixed subspaces. We have that $\mathcal{A}^{\infty}_m$, and $\mathcal{B}^{\infty}_m$ attractors for $0<\theta<3\pi/4$

and $0 < \theta < \pi/4$ respectively. The fixed subspaces $\mathcal{C}_m^\infty$ and $\mathcal{D}_m^\infty$ are unstable for all ϕ (see Figure 3.7). The stability analysis of the subspace $\mathcal{E}_m^\infty$ is complicated by

Table 3.5: Coordinates, eigenvalues, and the solutions for fixed points in the asymptotic regime for the $\exp(-R/\Lambda)$ gravity in matter case. Here f_1 and f_2 are functions of θ while g is a function of ϕ (see Figures 3.5 and 3.6). $S = [2 + \cot\theta + \cot^2\theta(1 + \cot\delta\sec\theta)]$ and $\tilde{S} = [-2 + \cot\theta - \cot^2\theta(1 + \cot\delta\sec\theta)]$

Point (δ, θ, ϕ)		Eigenvalues	Solution
$\mathcal{A}_m^\infty$	$[\operatorname{arccot}(-\sin\theta - \cos\theta), \theta, \frac{\pi}{2}]$	$[0, 0, f_1(\theta)]$	$(N - N_\infty) = [\frac{S}{\epsilon}^{-1}(c_1 \pm c_0(t - t_0))]$
$\mathcal{B}_m^\infty$	$[\operatorname{arccot}(\sin\theta - \cos\theta), \theta, \frac{3\pi}{2}]$	$[0, 0, f_2(\theta)]$	$(N - N_\infty) = [\frac{\tilde{S}-1}{\epsilon\tilde{S}}(c_1 \pm c_0(t - t_0))]$
$\mathcal{C}_m^\infty$	$[\operatorname{arccot}(-A), \frac{\pi}{2}, \phi]$	$[0, 0, g(\phi) > 0\ \forall\phi]$	$(N - N_\infty) = [\frac{1}{2\epsilon}(c_1 \pm c_0(t - t_0))]^2$
$\mathcal{D}_m^\infty$	$[\operatorname{arccot}(A), \frac{3\pi}{2}, \phi]$	$[0, 0, g(\phi) > 0\ \forall\phi]$	$(N - N_\infty) = [\frac{1}{2\epsilon}(c_1 \pm c_0(t - t_0))]^2$
$\mathcal{E}_m^\infty$	$[\operatorname{arccot}(-\cos\theta), \theta, \frac{3\pi}{4}]$	$[0, 0, 0]$	$a = ce^{\gamma(t-t_0)}$

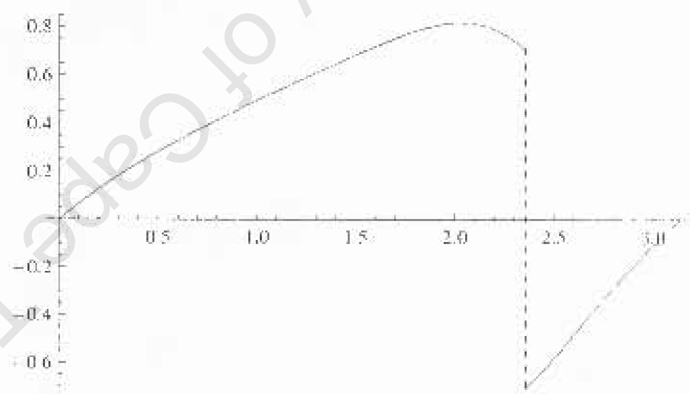

Figure 3.5: The graph of the function $f_1(\theta)$.

the fact that the eigenvalues are all zero. This means that the system for these points is not structurally stable or, in other words, their stability is determined mainly by the nonlinear terms. One can gain an idea of the stability of $\mathcal{E}_m^\infty$ by studying second order small perturbations around this fixed subspace. To the second order the evolution equations for the perturbations $(\delta, \theta, \phi, \mathcal{R})$ around $\mathcal{E}_m^\infty$ are

$$\begin{aligned}
\bar{\delta}' &= \varepsilon(a_1\bar{\delta} + a_2\bar{\delta}^2 + a_3\bar{\delta}\bar{\theta} + a_4\bar{\delta}\bar{\phi} - a_5\bar{\theta} + a_6\bar{\theta}^2 + a_7\bar{\phi}^2 - a_8\bar{\phi} + a_9\bar{\phi}\bar{\theta}),\\
\bar{\theta}' &= \varepsilon(b_1\bar{\theta} + b_2\bar{\theta}\bar{\phi} + b_3\bar{\delta} - b_4\bar{\delta}\bar{\phi} + b_5\bar{\delta}\bar{\theta} - b_6\bar{\phi} + b_7\bar{\theta}^2 + b_8\bar{\phi}^2), \qquad (3.51)\\
\bar{\phi}' &= \varepsilon(c_1\bar{\delta} + c_2\bar{\delta}\bar{\phi} + c_3\bar{\delta}\bar{\theta} + c_4\bar{\theta}\bar{\phi} + c_5\bar{\theta} + c_6\bar{\phi}^2 + c_7\bar{\phi} + c_8\bar{\theta}^2).
\end{aligned}$$

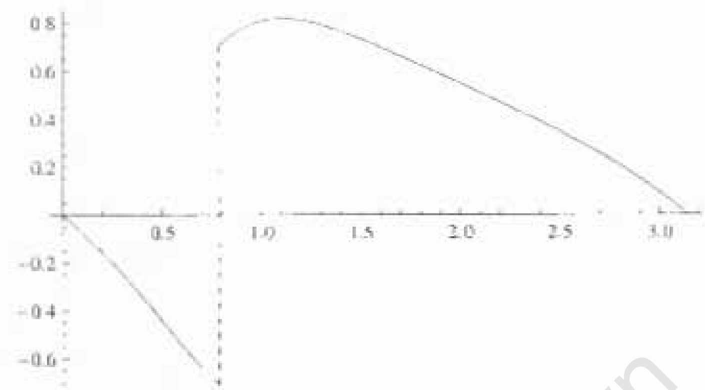

Figure 3.6: The graph of the function $f_2(\theta)$.

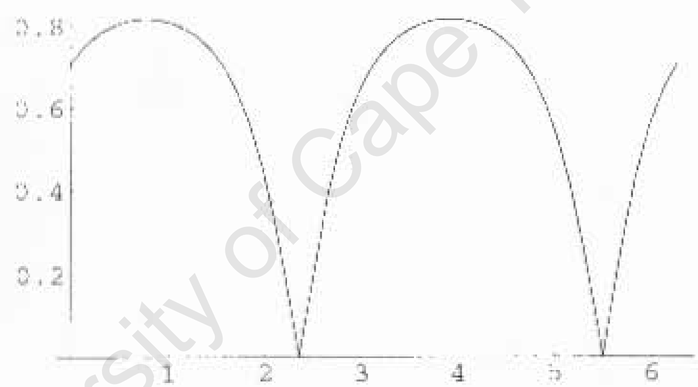

Figure 3.7: The graph of the function $g(\phi)$.

$$\mathcal{R}' = \varepsilon(d_1\bar{\delta}^2 + \theta d_3\delta + \bar{\phi}d_7\bar{\delta} + \bar{\theta}^2 d_5 + \bar{\phi}d_5 + \theta\phi d_8 + \bar{\phi}^2 d_9 + \bar{\delta}d_2 + \bar{\theta}d_4),$$

where

$$a_1 = -\frac{(3+\cos 2\theta_0)\sec\theta_0^2(-4+\sqrt{2}\sin 2\theta_0)}{8(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_2 = -(\frac{(3+\cos 2\theta_0)\sec\theta_0(-4+\sqrt{2}\sin 2\theta_0)\tan\theta_0^2}{8(1+\sec\theta_0^2)^{\frac{3}{2}}}),$$

$$a_3 = -\frac{\sqrt{2}\cos 2\theta_0 + (-2+\sec\theta_0^2)\tan\theta_0}{(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_4 = \frac{(4\cos 2\theta_0 - \sqrt{2}(\sin 2\theta_0 - 4\tan\theta_0))\tan\theta_0^2}{4(1-\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_5 = \frac{(2\sec\theta_0 - \sqrt{2}\sin\theta_0)\tan\theta_0}{2(1+\sec\theta_0^2)^{\frac{1}{2}}}.$$

$$a_6 = \frac{-5\sqrt{2}\sin\theta_0 + 2\sec\theta_0(1+\sqrt{2}\tan\theta_0)}{4(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_7 = \frac{\sin\theta_0\tan\theta_0}{(1+\sec\theta_0^2)^{\frac{3}{2}}}, \qquad a_8 = \frac{\sqrt{2}\sec\theta_0 - \sin\theta_0\tan\theta_0}{(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$a_9 = -\frac{\sqrt{2}\cos\theta_0 - 6\sin\theta_0 + \sec\theta_0(3\sqrt{2}2\tan\theta_0)}{2(1+\sec\theta_0^2)^{\frac{3}{2}}},$$

$$b_1 = \frac{\sin\theta_0\cos\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_2 = -\frac{2-6\cos 2\theta_0 + \sqrt{2}\sin 2\theta_0}{4(\sqrt{1+\sec\theta_0^2})},$$

$$b_3 = \frac{\cos\theta_0 + \cos\theta_0^3}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_4 = -\frac{\cos\theta_0(\sqrt{2}+\sqrt{2}\cos\theta_0^2 + \sin 2\theta_0)}{2(\sqrt{1+\sec\theta_0^2})},$$

$$b_5 = -\frac{(2+3\cos\theta_0^2)\sin\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_6 = \frac{\cos\theta_0\sin\theta_0}{\sqrt{1+\sec\theta_0^2}},$$

$$b_7 = \frac{3+5\cos 2\theta_0}{4\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad b_8 = -\frac{\cos\theta_0\sin\theta_0}{\sqrt{1+\sec\theta_0^2}},$$

$$c_1 = \frac{2\sec\theta_0 + \sin\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad c_2 = \frac{2\cos\theta_0 + \sqrt{2}(-2\csc\theta_0 + \sin\theta_0)}{2(\sqrt{1+\sec\theta_0^2})},$$

$$c_3 = \frac{\cos\theta_0 + 2\cot\theta_0\csc\theta_0 - \sec\theta_0}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})},$$

$$c_4 = -\frac{2(\frac{\cos\theta_0 - \sec\theta_0}{\sqrt{2}} - \sqrt{2}\cot\theta_0\csc\theta_0 - \sin\theta_0) - \sqrt{2} - 2\tan\theta_0}{\sqrt{1+\sec\theta_0^2)}},$$

$$c_5 = -\frac{1}{\sqrt{2}(\sqrt{1+\sec\theta_0^2})}, \qquad c_6 = -\frac{1}{\sqrt{1+\sec\theta_0^2}},$$

$$c_7 = \frac{1}{\sqrt{1+\sec\theta_0^2}}, \qquad c_8 = \frac{\cot\theta_0 + 2\tan\theta_0}{2\sqrt{2}\sqrt{1+\sec\theta_0^2}},$$

$$d_1 = -\frac{(\cos(2\theta_0) - 3)\sec^2(\theta_0)\left(\sqrt{2}\sin(2\theta_0) - 4\right)}{4\left(\sec^2(\theta_0) + 1\right)^{3/2}},$$

$$d_2 = \frac{(\cos(2\theta_0) - 3)\sec(\theta_0)\left(\sqrt{2}\tan(\theta_0) + 2\right)}{4\left(\sec^2(\theta_0) + 1\right)^{3/2}},$$

$$d_3 = -\frac{\sqrt{2}\sec^3(\theta_0) - 2\left(\sqrt{2} - 2\tan(\theta_0)\right)\sec(\theta_0) - 5\sqrt{2}\cos(\theta_0) + 2\sin(\theta_0)}{2\left(\sec^2(\theta_0) + 1\right)^{3/2}},$$

$$
\begin{aligned}
d_4 &= \frac{\tan(\theta_0)\left(\sqrt{2}\tan(\theta_0)+2\right)}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}, \\
d_5 &= \frac{\sqrt{2}\left(5-2\sec^2(\theta_0)\right)\tan(\theta_0)+2}{4\left(\sec^2(\theta_0)+1\right)^{3/2}}, \\
d_6 &= \frac{\sec^2(\theta_0)\left(3\sqrt{2}\cos(2\theta_0)+\sin(2\theta_0)+7\sqrt{2}\right)\tan(\theta_0)}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}, \\
d_7 &= -\frac{\sec(\theta_0)\left(7\sqrt{2}\cos(2\theta_0)+6\sin(2\theta_0)+9\sqrt{2}\right)\tan(\theta_0)}{4\left(\sec^2(\theta_0)+1\right)^{3/2}}, \\
d_8 &= \frac{\left(3\sqrt{2}-2\tan(\theta_0)\right)\sec^2(\theta_0)+6\tan(\theta_0)+7\sqrt{2}}{2\left(\sec^2(\theta_0)+1\right)^{3/2}}, \\
d_9 &= -\frac{\tan^2(\theta_0)}{\left(\sec^2(\theta_0)+1\right)^{3/2}}.
\end{aligned}
$$

The system above is very difficult to solve exactly, so we will limit ourselves to plotting the solutions for specific values of θ and to deduce the stability from them. Figure 3.8 shows the solutions of the perturbed system 3.51 for specific values of θ (i.e. $\theta = \pi/6, \pi/4, \pi/3, 3\pi/4$ and $5\pi/6$); it is clear from the graphs that all the solutions are unstable.

3.5 Discussion

We have applied the dynamical systems approach to the exponential gravity cosmological model, and found exact solutions together with their stability for both the vacuum and matter cases.

In the vacuum case, we identified four finite critical points $\mathcal{A}_v$, $\mathcal{B}_v$, $\mathcal{C}_v$ and $\mathcal{D}_v$, of which only two $\mathcal{C}_v$ and $\mathcal{D}_v$ are found to be physical. These last points were established to represent a solution whose nature depends on the parameter $\gamma(\Lambda)$; for $\Lambda > 0$ we can have either exponential expansion ($\gamma > 0$) or exponential contraction ($\gamma < 0$), while for $\Lambda < 0$ the solution oscillates.

From the stability point of view, the point $\mathcal{C}_v$, which resides in the invariant submanifold $z = 0$, is of particular interest because, since it is non-hyperbolic, it represents an attractor for $z > 0$ and a saddle for $z < 0$, while the other physical point $\mathcal{D}_v$ is found to be a saddle.

On the other hand, the solution connected with the non-physical points $\mathcal{A}_v$ and $\mathcal{B}_v$ is found to correspond to power law evolution.

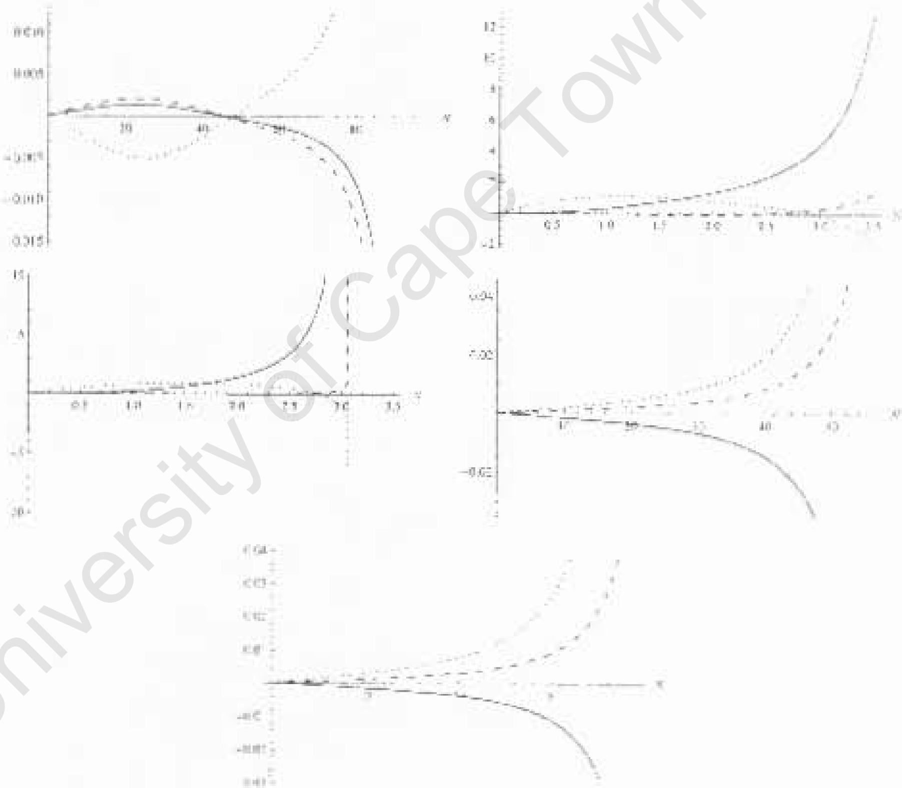

Figure 3.8: The solutions for $\bar{\delta}$, $\bar{\theta}$ and $\bar{\phi}$ of the system 3.51 for $\theta = \pi/56, \pi/4, \pi/3, 3\pi/4$ and $5\pi/6$ respectively. The solid, dashed and dotted lines represent $\bar{\delta}$, $\bar{\theta}$ and $\bar{\phi}$ respectively.

The invariant submanifold $z = 0$ divides the phase space into two regions, $z > 0$ and $z < 0$ which correspond to $\Lambda > 0$ and $\Lambda < 0$ respectively. The fact that no orbit can cross the plane $z = 0$ is then consistent with the fact that Λ must always have the same sign during a cosmic history.

In the vacuum case, we established that the region $z < 0$ does not contain any finite critical point. Thus, the only attractors in the region $z < 0$ are asymptotic, which means that all the models that begin their evolution in this region will re-collapse. However, in the plane $z = 0$, the physical point $\mathcal{C}_v$ represents a saddle and the non-physical points $\mathcal{A}_v$ and $\mathcal{B}_v$ are a repeller and saddle point respectively. This means that during the evolution towards the asymptotic attractors a transient de-Sitter or a power law phase(s) might be present, depending on the initial conditions.

From a physical point of view, the region $z > 0$ appears to be more interesting because in this region the point $\mathcal{C}_v$ represents a de-Sitter attractor which might be associated with a Dark Energy/inflation era. The same region also contains the point $\mathcal{D}_v$, which represents an unstable de Sitter phase (see Figure 3.1). This implies that the subset of the orbits which converge to $\mathcal{C}_v$ can also contain cosmic histories that present a second, unstable, de Sitter phase. In addition, orbits that evolve near the non-physical point $\mathcal{B}_v$ can also present an intermediate power law phase.

Finally, it is apparent from Figure 3 that the de Sitter phases $\mathcal{C}_v$ and $\mathcal{D}_v$ are separated from the past attractor $\mathcal{A}_v$ by the plane $q = 0$; therefore any model with initial conditions near the past attractor $\mathcal{A}_v$ and evolving toward the future de-Sitter attractor $\mathcal{C}_v$ will cross the plane $q = 0$, indicating a transition from an accelerating evolution to a decelerating one.

In the asymptotic regime, no isolated fixed point was found, but only two fixed subspaces: $\mathcal{I}_v^\infty$ and $\mathcal{O}_v^\infty$. The stability analysis reveals that the only asymptotic attractors are in $\mathcal{I}_v^\infty$. This means that all the models that evolve towards an asymptotic attractor are bound to re-collapse.

The introduction of matter into this model increases the dimensionality of the phase space, making it more difficult to visualize. By a direct substitution into the field equations we found that all the fixed points in the matter case do not represent physical solutions. This suggests that the exponential lagrangian does not present a Friedmann-like phase. On the other hand, since there is evidence that fourth-order lagrangian in form of polynomials can admint such phases one can conclude that somehow the presence of Friedmann-like phase is related to power-law terms in the gravitational lagrangian.

In the finite regime (see Table 3.3), we determined that the four points $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{F}_m$ and $\mathcal{E}_m$ are generalizations of the vacuum fixed points $\mathcal{A}_v$, $\mathcal{B}_v$, $\mathcal{C}_v$ and $\mathcal{D}_v$ respectively and present the same solutions.

The points $\mathcal{A}_m$ and $\mathcal{B}_m$ are found to represent Milne solutions while $\mathcal{C}_m$ and $\mathcal{D}_m$ represent a power law evolution. For points $\mathcal{E}_m$, $\mathcal{F}_m$ and $\mathcal{G}_m$ we established that $\dot{H} = 0$, which means that these points represent Einstein-de Sitter solutions.

In the asymptotic regime we identified three different classes of solutions (see Table 3.5). The first class contains the fixed points $\mathcal{A}_m^\infty$ and $\mathcal{B}_m^\infty$, and the solutions at these points depends on the value of $S(\delta,\theta)$ and $\bar{S}(\delta,\theta)$. The solutions at these points represent an expansion if $S, \bar{S} > 1$, a contraction if $S, \bar{S} < 1$. These fixed subspaces contain attractive parts (specifically $0 < \theta < 3\pi/4$ for $\mathcal{A}_m^\infty$ and $0 < \theta < \pi/4$ for $\mathcal{B}_m^\infty$).

The second class of solutions contains the two fixed points $\mathcal{C}_m^\infty$ and $\mathcal{D}_m^\infty$. These subspaces represent a recollapsing evolution, but are always unstable.

The third class contains a single fixed subspace $\mathcal{E}_m^\infty$ and the solution at this point depends on the parameter γ. By substituting this solution into the field equations 3.8, and by ignoring the subdominant terms we obtain

$$\gamma^2 = 0, \tag{3.52}$$

$$\gamma^2\left[-1 + \frac{144}{\Lambda}\left(\frac{k}{a^2}\right)^2 + 24\left(\frac{k}{a^2}\right)\right] - 3\left(\frac{k}{a^2}\right) = 0. \tag{3.53}$$

It is clear that the only consistent solution is $\gamma = 0$, when the spatial part of the spacetime is flat $k = 0$. The stability of the points in this subspace cannot be performed in general because of their non-hyperbolic character. We limited ourselves to investigate a few specific cases analyzing the evolution of the second order perturbations around a general point of $\mathcal{E}_m^\infty$. These points seem to be always unstable. Such a behaviour is interesting because it points towards the presence of bounce solutions for this cosmological model.

In conclusion, $\exp(-\frac{R}{\Lambda})$ gravity possesses a very rich structure that includes a series of diverse and interesting cosmological histories. Particularly important are the ones including multiple de Sitter phases because they could provide us with natural models describing the early and late time acceleration of the Universe. Unfortunately, as is clear from Figure 3.2, this scenario does not include a decelerated expansion phase between these two de Sitter phases. This implies that these cosmic histories will not, in general, admit a standard structure formation scenario (as it was claimed using a different argument in [64]). On the other hand, as recently found in [65], the evolution of scalar perturbations in fourth order gravity does not necessarily need the presence of a decelerated expansion phase. Only a detailed numerical analysis of these specific orbits (and on the perturbation evolution along them) will be able to clarify this matter.

Chapter 4

Conclusion

As already mentioned, cosmological observations are beginning to indicate that the standard theory of gravity (GR) is unable to provide a simple explanation of the gravitational dynamics of the low energy Universe. In an attempt to justify these issues, many modifications of the GR have been proposed. In general, these modifications can be classified into two main groups,

- Dark Energy Approach, and
- Alternative Gravity Approach .

The simplest explanation of dark energy is the cosmological constant, which presents two main problems: the problem of magnitude (the calculated value of the vacuum energy using QFT is 120 orders of magnitude more than that observed for the cosmological constant) and of timing (the so called coincidence problem: why does the cosmological constant begin to dominate just now?). On the other hand, authors who support the alternative gravity approach assume that the recent cosmological observations, in particular the dynamics of galaxies and clusters of galaxies as well as the present accelerated expansion of the universe, can be explained by adding correcting terms to the Hilbert-Einstein action (which have their origin in modern theories such as string & M-theory). Among the possible modifications of the gravitational action, those consisting of nonlinear functions of the scalar curvature (the so called $f(R)$ gravity) have recently been paid serious attention.

The key aim of $f(R)$ models is to discover an explicit form of the function $f(R)$ yielding solutions that are stable and compatible with the experimental data. Currently, there is no clear indication of what the gravity Lagrangian should be. Although the available data from the solar system satisfies the required precision, the fact that the gravitational dynamics of this system is governed only by the ordinary matter places a major limitation on its viability with regards to $f(R)$ cosmology. However, in general $f(R)$ models must satisfy the following conditions (see [25]).

Evolution of The Universe Constrains: The theory must reproduce the desired dynamics of the Universe including an inflationary era, followed by a radiation era (required and well-constrained by primordial baryogenesis and nucleosynthesis) and a matter era (required for the formation of structures) and, finally, by the present acceleration epoch (constrained by CMB experiments, supernova data, and large scale structure surveys). There is recent evidence [17] that the popular models $f(R) = R - \mu^{2(n+1)}/R^n$ with $\mu, n > 0$ do not allow a matter-dominated era and are therefore ruled out.

Post-Newtonian Constrains: The theory must have post-Newtonian limits compatible with the available Solar System tests. In [20] it has been shown that only $f(R)$ theories with $f'' < 0$ exhibit stable high-curvature limits and well -behaved cosmological solutions with a proper era of matter domination. In general, any $f(R)$ models with terms that become dominant at low cosmic curvatures are not viable at solar system scales. In [19] it is shown that the gravity Lagrangian at intermediate and low scalar curvature is bound by

$$-2\Lambda \leq f(R) \leq R - 2\Lambda + \frac{l^2R^2}{2}, \tag{4.1}$$

where l represents a length scale.

Stability Constrains: The theory must be stable at the classical and quantum level. In principle, there are several kinds of instabilities to consider [21]. Dolgov and Kawasaki [22] discovered an instability in the matter sector for the specific model $f(R) = R - \mu^4/R$, a result confirmed in [23], in which it is also shown that adding a term quadratic in R removes the instability. The Dolgow-Kawasaki instability has been generalized to arbitrary $f(R)$ theories by Faraoni, where it is shown in [24] that theories with $f'' > 0$ are not viable owing to the instability of Ricci scalar. Instabilities of de Sitter space in the gravity sector were found in [25], while stability with respect to black hole nucleation was studied in [26]. The consideration of physically different instabilities yields remarkably similar stability conditions. The theory should also be ghost-free: the presence of ghosts has been studied in [27]. In this communication we focus on the Dolgow-Kawasaki instability and generalize such instability with respect to arbitrary $f(R)$ theories.

Unfortunately the exponential model which has been discussed in the previous chapter doesn't fulfills any of the above constrains. We conclude with a brief discussion of some open problem of $f(R)$ gravity. The field equations of the action 1.82 can be derived using two different approaches. The first one is called the standard metric formalism in which the action 1.82 is varied with respect to the metric; that leads to a system of fourth-order equations. The second one is the Palatini formalism, in this formalism the metric and the connection are assumed to be independent fields; the action 1.82 is varied with respect to both the metric and the connection, that result in a second-order field equations. The standard metric and the Platini formalisms lead to the same field equations only if $f(R)$ is linear in R, while they lead to different dynamical equations for every other choice of $f(R)$. Thus the question of what is the right physical formalism is still an open one. There is clearly much interesting work still to do.

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