Shear-Free Perfect Fluid Theorems in General Relativity

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ABSTRACT

We present a detailed method for proving shear-free perfect fluid theorems in General Relativity. This method uses the $(1 + 3)$-covariant formalism to establish the consistency of the Einstein gravitational field equations under the barotropic shear-free perfect fluid condition. Using a Mathematica package *xTensor*, we were able to prove the following cases: the case where the pressure is constant, the acceleration vector is parallel to the vorticity, the components of a rescaled acceleration vector field orthogonal to the vorticity are *basic* and the case where the dot product of the rescaled acceleration vector field and the unit vorticity vector is *basic*, leading to the existence of a Killing vector along the vorticity.

Contents

This thesis is dedicated to all who supported me during my PhD work. Especially my supervisor and my family and friends.

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The contemplation of celestial things will make
a man both speak and think more sublimely and
magnificently when he descends to human affairs.

- Cicero

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Introduction

1.1 The Historical Perspective

In general relativity, the shear-free perfect fluid conjecture states that a shear-free velocity vector field of a barotropic perfect fluid with energy condition $\mu + p(\mu) \neq 0$ is either non-expanding or irrotational. The shear is the symmetric trace-free part of the spatial gradient of the velocity vector field, with μ and $p(\mu)$ being the energy density and pressure of the perfect fluid, respectively.

Let us begin by reviewing the historical background of the shear-free perfect fluid conjecture to gain some historical perspective. Gödel in references [1, 2], gave the first indication that the shear-free perfect fluid condition can have restrictive properties on the expansion and vorticity of a relativistic cosmological model. Schücking in reference [3] used a coordinate representation approach to prove the shear-free perfect fluid conjecture for spatially homogeneous spacetimes with zero pressure, the so-called dust solutions. Ellis in reference [4] used an orthonormal tetrad system and an associated coordinate system to prove the shear-free perfect fluid conjecture in general relativity for spatially inhomogeneous dust solutions. This proof generalised the Schücking result mentioned above. White and Collins in reference [5] showed that the Ellis result remains true even if the cosmological constant(Λ) is non-zero.

Banerji in reference [6] showed the validity of the shear-free perfect fluid conjecture for spatially homogeneous perfect fluid cosmological solutions with a γ-law equation of state of the form $p = (\gamma - 1)\mu$, where $\gamma \neq \frac{10}{9}$. Using a similar approach as Schücking, Banerji constructed this proof using a coordinate representation approach. Ellis and Treciokas, in reference [7], provided proof of the conjecture in the case of incoherent radiation, which has an equation of state of the form $p = \frac{1}{3}\mu$. They also used an orthonormal tetrad formalism and a specific coordinate system to establish the result.

Furthermore, in reference [8], Coley extended the Ellis -Treciokas result for incoherent radiation in the case of a non-vanishing cosmological constant. In Ellis and Treciokas, an outline of a proof, where the acceleration potential is given by $r = \int_{p_0}^{p} \frac{1}{p+\mu} dp$ and obeys the propagation equation $u^c \nabla_c r = \beta(r)$ was presented. In the case of spatially homogeneous spacetimes, this result suggests the validity of the conjecture for a general equation of state $p(\mu)$, see the following references [5, 6, 9]. The details of the proof outlined by Ellis and Treciokas [7] were provided by Lang and Collins in references [10, 11], where they verified the conjecture in the case where the expansion scalar Θ is a function of the energy density μ[1]. Sopuerta gave a fully covariant proof of the above result in reference [12]. The work done by Sopuerta and that of Senovilla et al. in reference [13] provided the necessary motivation for using covariant methods in the search for the proof of the shear-free perfect fluid conjecture.

Following the work done in references [5] and [13], which proved the conjecture where the acceleration vector field ($u^c\nabla_c u_a$) is parallel to the vorticity vector field (ω_a) and where the magnetic (H_{ab}) or the electric (E_{ab}) parts of the Weyl curvature tensor (${C^{ab}}_{cd}$) vanish - see references [11, 14, 15]. Collins in reference [16] conjectured that shear-free perfect fluid solutions which rotate and expand simultaneously do not exist in general except in exceptional conditions, and this is the subject of our investigation.

The journey to the full proof of the shear-free perfect fluid conjecture has taken many twists and turns, and a wide range of special cases exist where the conjecture has been proven in the literature. Specifically, the cases where $\frac{dp}{d\mu} = -\frac{1}{3}$ were shown in references [11, 15, 17] and where Θ is functionally dependent on the modulus of the vorticity ω, see Sopuerta [12]. The proofs for the Petrov type N and III spacetimes were provided in references [18--20]. Coley proved the case where a conformal Killing vector field parallel to the velocity vector field exists in reference [8].

The case where the divergence of the electric part of the Weyl tensor vanishes was shown in reference [21] and the case where the divergence of the magnetic part of the Weyl tensor is zero as shown in reference [22], the case of a solenoidal magnetic part of the Weyl curvature and a γ-law equation of state was presented in reference [23].

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[1] Where the conservation equation $u^c\nabla_c\mu = -(\mu+p)\Theta$ implies $u^c\nabla_c r = \beta(r)$.

Slobodeanu in reference [24] provided a proof of the conjecture for a γ-law equation of state with a vanishing cosmological constant, where the following cases were excluded $(\gamma - 1 = -\frac{1}{5}, -\frac{1}{6}, -\frac{1}{11}, -\frac{1}{21}, \frac{1}{15}, \frac{1}{4})$. Nzioki, Goswami, Dunsby and Ellis, in reference [25], proved the conjecture for the case where the Einstein field equations were linearised with respect to the FLRW background spacetime.

Carminati, in reference [26], tried to prove the conjecture in the case of a linear equation of state with a zero cosmological constant. Unfortunately, the proof was faulty because there was an inappropriate use of an equation-solving command in Maple. Van den Berg and Slobodeanu in reference [27] completed the proof that Carminati attempted in reference [26], proving the conjecture in the case of a linear equation of state, including the non-vanishing cosmological constant case. They reduced the problem to a Lemma, stating that if a rotating and expanding shear-free perfect fluid has a linear equation of state of the form $p = (\gamma - 1)\mu + constant$ and some defined *basic*[2] variables go to zero; then, a Killing vector field exists parallel to the vorticity vector field. They argued that this Lemma is valid even for a general barotropic equation of state, but the proof remains elusive.

In a recent paper, Sikhonde and Dunsby [28] presented a more compact and neat proof of the conjecture for constant pressure spacetimes using the $(1+3)$-covariant approach, followed by the case where the acceleration vector field is parallel to the vorticity. In both these cases, a non-vanishing cosmological constant was considered. These proofs were given by Senovilla et al. in reference [13] in their $(1+3)$-covariant form, without including the cosmological constant. The proof of the case where the ac-

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[2] The Lie derivative of a *basic* variable vanishes.

celeration vector field is orthogonal to the vorticity was provided by Van den Berg and Slobodeanu in reference [27] using the orthonormal tetrad approach. Consequently, this proof leads to the existence of a Killing vector field parallel to the vorticity. Recently, another attempted proof for a general equation of state came from Goswami and Ellis in reference [29], where a factorisation of the vorticity scalar Ω into a gradient-free part Ω_1 and Ω_2 which is constant in time was assumed. Unfortunately a counter-example of Ω was found, namely, $\frac{1}{2}\mu^{-\frac{1}{6}}$ by Van den Bergh and Carminati in reference [30], so clearly Ω does not admit this factorization for a general function μ.

Restricting the matter equation of state to that of a barotropic perfect fluid leads to a set of new constraint equations on the full Einstein field equations. For example, all non-rotating and expanding shear-free perfect fluids obeying a barotropic equation of state are known to exist; see Collins in reference [16]. On the contrary, not all rotating and non-expanding shear-free perfect fluids obeying a barotropic equation of state are known to exist; see Karimian in reference [31]. In particular, all stationary and rigidly rotating perfect fluids belong to this class. Additionally, a rotating and expanding shear-free perfect fluid (Bianchi IX) model obeying an equation of state of the form ($p = -\mu = constant$) exists. However, this matter equation of state is of the cosmological constant type and is outside of the scope of this thesis; see Obukhov et al. in reference [32].

The validity of the shear-free perfect fluid conjecture would be a striking feature of the full Einstein field equations. Contrary to Newtonian cosmology, where there exists rotating and expanding shear-free perfect fluids obeying a barotropic equation of state, see references [13, 33--35] for details. Whereas in the $f(R)$ theory of gravity, where R is the Ricci curvature scalar, there exist no analogue of the conjecture see counter-examples by Sofuoğlu et al. in reference [36], and in the case where the equations of $f(R)$ gravity were linearised about the FLRW background spacetime. For the details of this proof, see Abebe et al. in reference [37] where counter-examples were found. However, it would be interesting to consider extensions of General Relativity, for example, $f(R)$ gravity theories where $f(R) \approx R^{\delta}$ and δ is constant to investigate the stability of the GR results. Furthermore, it is interesting that the conjecture is a property of the Lorentzian signature since for the Riemannian signature, we can obtain a *Ricci-flat* metric where shear-free perfect fluid, rotating, and expanding solution exists, see Pantilie in reference [38].

1.2 The Consistency Check Procedure

In General Relativity, there are ten Einstein (gravitational) field equations (EFE)[3]. These equations are fully consistent provided one does not impose any geometric constraints on them and that the energy-momentum tensor (EMT) is conserved[4]. The EFE is a coupled system of quasi-linear second-order differential equations, four of these are the constraint equations, and the remaining six are the evolution equations. Owing to the conservation of the EMT, spacetime evolves in a particular way from the initial conditions that satisfy the four constraint equations.

Furthermore, since the solutions to the EFE are subject to diffeomorphism invariance, they can be expressed in various ways for the same geometrical framework. Therefore, one needs to answer the following question, which aspect of the solution is due to the choice of a coordinate system or the physical degrees of freedom? To successfully answer this question, one needs to choose an evolution scheme for the four constraint equations of the EFE; see MacCallum [39], Ellis and Van Elst [40].

Imposing physical or geometric constraints on the EFE breaks the complete symmetry of these equations. Hence the consistency of the EFE is compromised. Forcing these restrictions on the EFE produces new constraint (NC) equations [5], which may result from transforming a propagation equation into a constraint equation, for instance by setting the shear to zero.

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[3] $R_{ab} - \frac{1}{2}g_{ab}R = T_{ab} - \Lambda g_{ab}$.

[4] Meaning $\nabla_a T^{ab} = 0$, where T^{ab} is the EMT.

[5] Of the form $F(g_{ab}, T_{ab}) = 0$.

Therefore, to investigate the consistency of the EFE under the NC, we consider the time evolution and the spatial gradients of the NC subject to the time propagation and constraint equations. When these are identically satisfied, we have shown consistency. If not, one has to time propagate the remaining constraint equations repeatedly until we have either shown consistency or inconsistency, in which case the solution does not exist.

Sometimes it suffices to use the Extended Einstein Field Equations (EEFE). Here we relate to the metric tensor and matter fields the components of the Weyl curvature tensor[6] together with the Bianchi identities[7]. These equations determine the divergence of the Weyl curvature tensor, where the EMT is the source term, and they represent the field equations for the Weyl curvature tensor.

Subject to these new constraint equations, we can proceed to investigate the resulting consistency relations as described in the preceding paragraph. Furthermore, one requires the other evolution and constraint equations resulting from the electric and magnetic parts of the Weyl curvature tensor to close the system. In practice, this is an enormous amount of calculations, requiring significant computing power. As a result, consistency has been shown only under extraordinary simplifying assumptions.

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[6] C_{abcd} is the trace-free part of R_{abcd}

[7] $R_{ab[cd;e]} = 0$

1.3 The Different Formalisms For Establishing Consistency

Here, we will discuss the three main formalisms for checking the consistency of the EFE in light of the imposed constraints. These constraints can determine which one of these formalisms we will use in our investigation. Firstly, we have a co-ordinate representation formalism, the traditional framework. In this framework, one needs to be careful in keeping track of the coordinate degrees of freedom the geometry allows. Then we need to discern which effects are due to the choice of coordinates or are physical.

Secondly, we have the tetrad formalism, where one requires additional variables: the tetrad rotation coefficients. They only have physical meaning if we choose the tetrad vectors in a physically meaningful way. For instance, this can be done either for a null tetrad in the Newman-Penrose (NP) formalism or the analogous spinor formalism or an orthonormal tetrad (OT) formalism, see Ellis and MacCallum in references [4, 41] respectively. The NP formalism is appropriate for studying gravitational radiation and vacuum solutions. The OT formalism, on the other hand, is suitable for investigating the shear-free perfect fluid solutions, which are the subject of this thesis.

Lastly, a third formalism exists, the $(1 + 3)$ - covariant formalism. It is described in the following references; Ehlers et. al [42], Kristian and Sachs [43], Trümper [44], Hawking [45], and is summarized by Ellis and Van Elst in references [40, 46]. In the $(1+3)$-covariant formalism, decomposing all the kinematical and curvature variables of the theory into a one-time direction and three spacial directions proves beneficial.

For example, the Weyl curvature tensor is decomposed into its electric and magnetic parts, which are orthogonal to the chosen 4-velocity vector field.

We can then derive propagation and constraint equations for E_{ab} and H_{ab} resulting from the imposed geometric or physical constraints. Ellis and Van Elst, in reference [40], provided a comprehensive account of the full set of the $(1 + 3)$ - covariant equations of GR. These equations can be examined covariantly as far as possible, and then they can be developed appropriately in terms of either a specific coordinate basis or a tetrad frame. Furthermore, the $(1 + 3)$ - covariant formalism is especially appealing when viewed from a pedagogical standpoint, as discussed in Van Elst [47].

The rest of the thesis is organised in the following way. First, we present the next Chapter (2), which discusses the $(1+3)-$ covariant formalism, including the relevant covariant kinematic and dynamical variables together with their time propagation and constraint equations. Then Chapter (3) follows; this chapter discusses the shear-free perfect fluid kinematic and dynamical variables, including their corresponding time propagation and constraint equations.

In Chapter (4), we provide a covariant proof of the conjecture for the case of a constant pressure shear-free perfect fluid. Then we get to Chapter (5); this chapter discusses the general shear-free perfect fluid conjecture and the time propagation of the resulting new constraint equations. In Chapter (6), we present the proof of the general barotropic perfect fluid equation of state where the vorticity is parallel to the acceleration vector field.

In Chapter (7), we investigate the case for a general perfect fluid equation of state, where the vorticity is not parallel to the acceleration vector field. This reduces to the case where a Killing vector exists along the vorticity. We first show that the conjecture is true when the orthogonal components of a rescaled acceleration vector field lying in the space orthogonal to the vorticity are *basic*[8], where the rescaling function depends on the energy density explicitly. Then we show that one can draw the same conclusion even when the rescaling function implicitly depends on the energy density. In addition, we repeat the same procedure for the case where the dot product of the unit vorticity vector and the rescaled acceleration vector mentioned above is *basic*, ending the chapter with a discussion of the results.

In Chapter (8), we conclude our discussion by providing a summary of our previous proven theorems' results. A discussion about possible future work follows just before we put the Appendices, where we compile all the necessary propagation equations, the constraint equations and some identities required for the calculations. Then we provide an outline of a Mathematica code implementing xTensor, which we have used to do our calculations.

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[8] Meaning their spatially projected Lie derivatives vanish.

The circumstance that there is no objective, rational division of the four-dimensional continuum into three-dimensional space and a one-dimensional time continuum indicates that the laws of nature will assume a form which is logically most satisfactory when expressed as laws in the four-dimensional space-time continuum. Upon this depends the great advance in method which the theory of relativity owes to Minkowski.

- Albert Einstein (The Meaning of Relativity)

2

$(1+3)$ - Covariant Approach

2.1 THE BASIC COVARIANT VARIABLES

2.1.1 PRELIMINARIES

For simplicity, we have chosen to work with units where $\left[c = 1 = \frac{8\pi G}{c^2}\right]$[1]. In addition, we consider a space-time manifold $\mathcal{M}$ endowed with a metric $\mathbf{g}$. Furthermore, we define a co-moving future directed time-like four-velocity vector field $\mathbf{u}$ on $\mathcal{M}$, which

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[1] Where c and G are the constant speed of light and Newton's gravitational constant respectively.

is normalised in the following way $\mathbf{u} \cdot \mathbf{u} = -1$. The vector field $\mathbf{u}$ characterises a chosen reference congruence[2] on the manifold $\mathcal{M}$.

The constitution $(\mathcal{M}, \mathbf{g}, \mathbf{u})$, forms the basis for our relativistic cosmological model. The metric $\mathbf{g}$ is subject to the following second rank tensor equation of motion,

$$G_{ab} = T_{ab} - \Lambda g_{ab}. \tag{2.1}$$

These are the legendary Einstein Field Equations (EFE) presented in index form [40], where $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ is the Einstein tensor, T_{ab} is the energy-momentum tensor, Λ is the cosmological constant and g_{ab} represents the components of metric tensor $\mathbf{g}$. The curvature tensor R_{ab} and the scalar curvature R are the first and second metric contractions of the Riemann curvature tensor R_{acbd}. They are called the Ricci tensor and scalar, respectively.

2.1.2 THE $(1+3)$ - FORMALISM

During the 1950s and early 1960s, Raychaudhuri, Schücking, Ehlers, Sachs and Trümper developed a $(1+3)$ - foliation of $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ by means of fundamental observers, see Ehlers [48] for references. In 1971 Ellis [49] showed how to apply this formalism to the framework of relativistic cosmology. This formalism has proven very useful in addressing many problems in relativistic astrophysics and cosmology; see a few of such applications of the $(1+3)$-covariant formalism in the following references [28, 37, 50, 51] and references therein. This is why we use this framework to address the problems discussed in this thesis. Namely; the problem of time propagating the spatial constraints, calculating the time-space derivative commuta-

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[2] A co-moving observer geodetic curve.

tion relations of our $(1+3)$-decomposed kinematic and dynamical variables and their space-space derivative commutation relations, see references [28, 40, 47].

The four-velocity vector field $\mathbf{u}$ induces two relevant projection tensors: $U^a{}_b = -u^a u_b$ and $h^a{}_b = \delta^a{}_b - U^a{}_b$, where $\delta^a{}_b = g^a{}_c g^c{}_b$. The projection tensor $U^a{}_b$ projects tensors parallel to $\mathbf{u}$, whereas $h^a{}_b$ is a projection onto the spatial 3 dimensional hyper-surface orthogonal to $\mathbf{u}$. These projection tensors obey the following metric contraction rules,

$$U^a{}_b u^b = u^a \quad , U^a{}_c U^c{}_b = U^a{}_b, \quad U^a{}_a = 1 \tag{2.2a}$$

$$h^a{}_b u^b = 0 \quad , h^a{}_c h^c{}_b = h^a{}_b, \quad h^a{}_a = 3. \tag{2.2b}$$

The projection tensors mentioned above allow us to define a 3-volume element for the 3-hyper-surface orthogonal to $\mathbf{u}$ from the spacetime 4-volume element defined as ϵ_{abcd}. The 4-volume element has the following properties $\epsilon_{abcd} = \epsilon_{[abcd]}$ and $\epsilon_{0123} = \sqrt{|\text{det} g_{ab}|}$. Hence we can define the 3-volume element as,

$$\epsilon_{abc} = u^g \epsilon_{gdef} h^d{}_a h^e{}_b h^f{}_c. \tag{2.3}$$

Therefore, the 4-volume element can also be written as $\epsilon_{abcd} = 2\epsilon_{ab[c}u_{d]} - 2u_{[a}\epsilon_{b]cd}$. The 3-volume element satisfies the following useful and well-known metric contraction identities:

$$\epsilon^{abc}\epsilon_{def} = 3! h^{[a}{}_d h^b{}_e h^{c]}{}_f \quad , \epsilon^{abc}\epsilon_{cef} = 2! h^{[a}{}_e h^{b]}{}_f \quad , \epsilon^{abc}\epsilon_{bcf} = 2! h^a{}_f \quad , \epsilon^{abc}\epsilon_{abc} = 3!. \tag{2.4}$$

Owing to the four-velocity $\mathbf{u}$ and the projection tensor $h^a{}_b$, one can define two covariant derivatives, which are the covariant derivative along $\mathbf{u}$ (this will be denoted by a dot $(\cdot)$ as an operator) and the fully orthogonally (to $\mathbf{u}$) projected covariant derivative (denoted by $\tilde{\nabla}$ as an operator). Therefore, any tensor $T^{ab}{}_{cd}$ can be differentiated as follows,

$$\dot{T}^{ab}{}_{cd} = u^e \nabla_e T^{ab}{}_{cd}, \tag{2.5a}$$

$$\tilde{\nabla}_e T^{ab}{}_{cd} = h^a{}_f h^b{}_g h^p{}_c h^q{}_d h^r{}_e \nabla_r T^{fg}{}_{pq}. \tag{2.5b}$$

Furthermore, we make use of angle brackets on the indices of tensors to denote orthogonal projection of vectors and the fully orthogonally projected symmetric trace free parts (PSTF) of tensors in the following way, $T^{\langle ab\rangle} = \left[h^{(a}{}_f h^{b)}{}_g - \frac{1}{3}h^{ab}h_{fg}\right] T^{fg}$ and $v^{\langle a\rangle} = h^a{}_f v^f$. For convenience, we use the same brackets to denote orthogonal projection and the fully orthogonally PSTF parts of the Fermi derivatives of vectors and tensors as follows: $\dot{T}^{\langle ab\rangle} = \left[h^{(a}{}_f h^{b)}{}_g - \frac{1}{3}h^{ab}h_{fg}\right] \dot{T}^{fg}$ and $\dot{v}^{\langle a\rangle} = h^a{}_f \dot{v}^f$. From equation (2.5a) and (2.5b), it is evident that the following structural identities hold, these are $\tilde{\nabla}_a h_{bc} = 0$ and $\tilde{\nabla}_a \epsilon_{bcd} = 0$.

2.2 THE KINEMATICS

2.2.1 THE KINEMATIC QUANTITIES

The covariant derivative of $\mathbf{u}$ can be decomposed into its irreducible kinematic variables using the $(1+3)$-covariant formalism, see equation (2.6a) below. These quantities describe the acceleration vector field $(\dot{u}_a)$, the expansion scalar (Θ), the shear (σ_{ab}) and rotation $(\epsilon_{abc}\omega^c)$ tensors of the fluid respectively and are defined by equations (2.6b):

$$\nabla_a u_b = -u_a \dot{u}_b + \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \epsilon_{abc}\omega^c, \tag{2.6a}$$

$$\dot{u}^a := u^b \nabla_b u^a \quad , \quad \Theta := \nabla_a u^a \quad , \quad \sigma_{ab} := \tilde{\nabla}_{\langle a} u_{b\rangle} \quad , \quad \omega_a := \tfrac{1}{2}\epsilon_{abc}\tilde{\nabla}^b u^c. \tag{2.6b}$$

In equations (2.6) above, we have chosen to define the vorticity as $\omega_a = \frac{1}{2}\epsilon_{abc}\tilde{\nabla}^b u^c$ whereas in the literature it might be defined as $\omega_a = -\frac{1}{2}\epsilon_{abc}\tilde{\nabla}^b u^c$, one has freedom of choice in this regard as long as the chosen convention is maintained.

2.2.2 THE ENERGY MOMENTUM TENSOR

The matter in the universe can be described by a symmetric divergence free rank two tensor T_{ab}, the energy-momentum tensor (EMT). The EMT appears on the right-hand side of equation (2.1). This tensor can be decomposed into its fundamental matter variables, which are given by equation (2.7b) below,

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}, \tag{2.7a}$$

$$\mu := T_{ab}u^a u^b \quad , \quad p := \tfrac{1}{3}T_{ab}h^{ab} \quad , \quad q^a := -T_{cb}h^{ca}u^b \quad , \quad \pi_{ab} := T_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}, \tag{2.7b}$$

where in line (2.7b), μ is the energy density of matter, p is the pressure of the fluid, q^a is the energy flux density, and the anisotropic stress tensor is defined by π_{ab}.

2.2.3 THE RIEMANN CURVATURE TENSOR

The curvature of spacetime in General Relativity[3] is encoded in a four-dimensional rank four tensor, this tensor represents the extent to which commutativity of double

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[3] The assumption here is that ∇_a is torsion-free.

covariant differentiation of a vector field fails. This tensor is called the Riemann curvature tensor, and it can be represented in index notation as $R^a{}_{bcd}$. The following equation captures the role played by the Riemann tensor for any covariant four-vector field V_a,

$$2\nabla_{[a}\nabla_{b]}V^c = R_{ab}{}^c{}_d V^d. \tag{2.8}$$

Equation (2.8) above is called the Ricci identity for covariant vectors. As a result of the skew-symmetrised second covariant derivative on the left-hand side of equation (2.8), the Riemann tensor is antisymmetric over its first two indices. Therefore, it obeys the following tensorial identity $R^a{}_{[bcd]} = 0$. The covariant derivative of $R^a{}_{bcd}$ fulfils the following structural Bianchi identity $\nabla_{[a}R_{bc]e}{}^d = 0$, from which the Einstein tensor G_{ab} can be derived. The Riemann curvature tensor can be Ricci spit into its trace and trace-free parts as follows,

$$R^{ab}{}_{cd} = C^{ab}{}_{cd} + 2R^{[a}{}_{[c}g^{b]}{}_{d]} - \tfrac{1}{3}Rg^a{}_{[c}g^b{}_{d]}. \tag{2.9}$$

The fully trace-less part C_{abcd} on the right hand side of equation (2.9) is the Weyl curvature tensor, this tensor can itself be decomposed into two symmetric trace--less spatial rank-2 tensors, these are the electric $\left(E_{ab} := C_{cdef}h^c{}_a u^d h^e{}_b u^f\right)$ and the magnetic $\left(H_{ab} := -\tfrac{1}{2}\epsilon_{cdgh}C^{gh}{}_{ef}h^c{}_a u^d h^e{}_b u^f\right)$ parts of the Weyl tensor as follows,

$$C^{ab}{}_{cd} = 4E^{[a}{}_{[c}\left(h^{b]}{}_{d]} + u^{b]}u_{d]}\right) - 2\epsilon^{ab}{}_e H^e{}_{[c}u_{d]} - 2\epsilon_{cd}{}^e H_e{}^{[a}u^{b]}. \tag{2.10}$$

Equation (2.9) can be fully decomposed using equation (2.10), the EFE (2.1) and the EMT (2.7a) which gives us the Riemann curvature tensor in its $(1+3)$-covariant decomposition as follows,

$$\begin{aligned} R^{ab}{}_{cd} = {} & 4u^{[a}u_{[c}(E^{b]}{}_{d]} - \tfrac{1}{2}\pi^{b]}{}_{d]}) + 4h^{[a}{}_{[c}(E^{b]}{}_{d]} + \tfrac{1}{2}\pi^{b]}{}_{d]}) \\ & + \tfrac{2}{3}(\mu + 3p - 2\Lambda)u^{[a}u_{[c}h^{b]}{}_{d} + \tfrac{2}{3}(\mu + \Lambda)h^{[a}{}_{[c}h^{b]}{}_{d} \\ & + 2\epsilon^{abe}u_{[c}(H_{d]e} + \tfrac{1}{2}\epsilon_{d]ef}q^{f}) + 2\epsilon_{cde}u^{[a}(H^{b]e} + \tfrac{1}{2}\epsilon^{b]ef}q_{f}). \end{aligned} \tag{2.11}$$

This form of the Riemann curvature tensor will prove useful when performing calculations related to the propagation scheme of the constraint equations. For convenience, we shall also define the following useful scalars, namely; the vorticity modulus squared given by $\omega^2 = \omega_a\omega^a \geq 0$, the shear modulus squared given by $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} \geq 0$ and the modulus squared of the acceleration as $\dot{u}^2 = \dot{u}_a\dot{u}^a \geq 0$.

2.3 The Dynamics

2.3.1 The Ricci identities

The following propagation and constraint equations arise from the Ricci identity for the four-velocity **u**, i.e.,

$$2\nabla_{[a}\nabla_{b]}u^{c} = R_{ab}{}^{c}{}_{d}u^{d}, \tag{2.12}$$

by using the covariant derivative of **u** given by equation (2.6a) and the Riemann curvature tensor decomposition given by equation (2.11).

Time propagation equations:

Taking the trace of equation (2.12) with respect to the indices b and c, and then projecting the remaining free index parallel(to **u**), we obtain the Raychaudhuri equation

(2.13) shown below

$$\dot{\Theta} - \tilde{\nabla}_a \dot{u}^a = -\tfrac{1}{3}\Theta^2 + \dot{u}^2 - 2(\sigma^2 - \omega^2) - \tfrac{1}{2}(\mu + 3p) + \Lambda, \tag{2.13}$$

which describes gravitational attraction and illustrates the repulsive nature of a positive cosmological constant (Λ), a possible candidate for the dark energy density of spacetime. Then by extracting the orthogonally projected symmetric trace free (PSTF) part of equation (2.12), this gives us the shear propagation equation as follows

$$\dot{\sigma}^{\langle ab\rangle} - \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} = -\tfrac{2}{3}\Theta\sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \sigma^{\langle a}{}_c\sigma^{b\rangle c} - \omega^{\langle a}\omega^{b\rangle} - (E^{ab} - \tfrac{1}{2}\pi^{ab}), \tag{2.14}$$

this equation plays an important role in discussing shear-free perfect fluids since it gets converted into an extra constraint equation where the shear tensor is absent. The orthogonally projected antisymmetric part of the parallel(to **u**) projection of equation (2.12) gives us the vorticity propagation equation (2.15) below,

$$\dot{\omega}^{\langle a\rangle} - \tfrac{1}{2}\epsilon^{abc}\tilde{\nabla}_b\dot{u}_c = -\tfrac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b. \tag{2.15}$$

CONSTRAINT EQUATIONS:

Orthogonally projecting equation (2.12), then contracting indices b and c with the metric tensor, we obtain the following constraint equation;

$$0 = (C_1)^a := \tilde{\nabla}_b\sigma^{ab} - \tfrac{2}{3}\tilde{\nabla}^a\Theta + \epsilon^a{}_{bc}\left(\tilde{\nabla}^b\omega^c + 2\dot{u}^b\omega^c\right) + q^a, \tag{2.16}$$

and by contracting equation (2.12) with $\epsilon^{ab}{}_c$, we obtain the structural vorticity divergence equation (2.17) below,

$$0 = (C_2) := \tilde{\nabla}_a\omega^a - \dot{u}_a\omega^a. \tag{2.17}$$

Furthermore, contracting equation (2.12) with ϵ^{abe} and taking the PSTF part of the resulting equation, one arrives at the constraint equation for the magnetic part of the Weyl tensor H^{ab} shown in equation (2.18) below,

$$0 = (C_3)^{ab} := H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + \tilde{\nabla}^{\langle a}\omega^{b\rangle} - \epsilon^{cd\langle a}\tilde{\nabla}_c\sigma^{b\rangle}{}_d. \tag{2.18}$$

2.3.2 The (Contracted) second Bianchi identities

The second set of propagation and constraint equations result from the Bianchi identities expressed by (2.19) below,

$$\nabla_{[a}R_{bc]d}{}^{e} = 0. \tag{2.19}$$

This can be achieved by applying equations (2.6a) and (2.11) into equation (2.19), then performing the relevant metric contractions.

Time propagation equations:

Contracting indices a and e in equation (2.19) above and projecting parallel(to $\mathbf{u}$), one obtains two propagation equations. The first is the time propagation of the electric part of the Weyl tensor plus half the anisotropic stress tensor and is given by the following equation,

$$\begin{aligned}(\dot{E}^{\langle ab\rangle} + \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle}) - \epsilon^{cd\langle a}\tilde{\nabla}_c H^{b\rangle}{}_d = &-\Theta(E^{ab} + \tfrac{1}{6}\pi^{ab}) + \tfrac{1}{2}\tilde{\nabla}^{\langle a}q^{b\rangle} - \dot{u}^{\langle a}q^{b\rangle} \\ &- \tfrac{1}{2}(p+\mu)\sigma^{ab} + 3\sigma^{\langle a}{}_c(E^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}) \\ &+ \epsilon^{cd\langle a}\big[2\dot{u}_c H^{b\rangle}{}_d + \omega_c(E^{b\rangle}{}_d + \tfrac{1}{2}\pi^{b\rangle}{}_d)\big].\end{aligned} \tag{2.20}$$

The second is the time propagation of the magnetic part of the Weyl tensor given by equation (2.21) below,

$$\begin{aligned}\dot{H}^{\langle ab\rangle} + \epsilon^{cd\langle a}\tilde{\nabla}_c(E^{b\rangle}{}_d - \tfrac{1}{2}\pi^{b\rangle}{}_d) &= -\Theta H^{ab} + 3\sigma^{\langle a}{}_c H^{b\rangle c} + \tfrac{3}{2}\omega^{\langle a}q^{b\rangle} \\ &\quad - \epsilon^{cd\langle a}\big[2\dot{u}_c E^{b\rangle}{}_d - \tfrac{2}{2}\sigma^{b\rangle}{}_c q_d - \omega_c H^{b\rangle}{}_d\big]. \qquad (2.21)\end{aligned}$$

The two above equations (2.20) and (2.21) describe gravitational radiation and tidal effects, i.e their second time propagation form wave equations for E^{ab} and H^{ab} tensor fields respectively. Contracting the Bianchi identity (2.19) twice gives us the energy momentum conservation equation ($\nabla_a T^{ab} = 0$). Projecting $\nabla_a T^{ab} = 0$ along $\mathbf{u}$, we obtain the energy density conservation equation below,

$$\dot{\mu} + \tilde{\nabla}_a q^a = -\Theta\,(p + \mu) - 2\dot{u}_a q^a - \sigma_{ab}\pi^{ab}. \qquad (2.22)$$

Furthermore, if we project $\nabla_a T^{ab} = 0$ orthogonal to $\mathbf{u}$, we arrive at the momentum flux propagation equation shown below,

$$\begin{aligned}\dot{q}^{\langle a\rangle} + \tilde{\nabla}^a p + \tilde{\nabla}_b \pi^{ab} &= -\tfrac{4}{3}\Theta q^a - \sigma^a{}_b q^b - (p + \mu)\,\dot{u}^a \\ &\quad - \dot{u}_b \pi^{ab} - \epsilon^{abc}\omega_b q_c. \qquad (2.23)\end{aligned}$$

CONSTRAINT EQUATIONS:

Contracting equation (2.19) once and projecting the leftover free index orthogonal to $\mathbf{u}$, one obtains two constraint equations (2.24a) and (2.24b) for the divergence of E^{ab} (which is sourced by the spatial gradient of the energy density) and the divergence of

H^{ab} (which is sourced by the vorticity vector field) shown below respectively,

$$
\begin{aligned}
0 = (C_4)^a &:= \tilde{\nabla}_b(E^{ab} + \tfrac{1}{2}\pi^{ab}) - \tfrac{1}{3}\tilde{\nabla}^a\mu - \tfrac{1}{3}\Theta q^a - \tfrac{1}{2}\sigma^a{}_b q^b - 3H^a{}_b\omega^b \\
&\quad - \epsilon^{abc}\left[\sigma_{bd}H^d{}_c - \tfrac{3}{2}\omega_b q_c\right] && \text{(2.24a)} \\
0 = (C_5)^a &:= \tilde{\nabla}_b H^{ab} + (p+\mu)\omega^a + 3\omega^b(E^a{}_b - \tfrac{1}{6}\pi^a{}_b) + \epsilon^{abc}\big[\tfrac{1}{2}\tilde{\nabla}_b q_c \\
&\quad + \sigma_{bd}(E^d{}_c + \tfrac{1}{2}\pi^d{}_c)\big]. && \text{(2.24b)}
\end{aligned}
$$

2.4 TIME PROPAGATION OF THE CONSTRAINTS

The constraint equations $(C_1)^a$ to $(C_5)^a$ are close under time propagation and hence the EFE are fully consistent. This can be achieved by applying the commutation relations given by Van Elst in reference [47], where the modifications suggested by MacCallum in reference [39] have been taken into account. This gives us the following well-known propagation equations below,

$$
\begin{aligned}
(\dot{C}_1)^{\langle a\rangle} &= -\Theta(C_1)^a - \tfrac{3}{2}\sigma^a{}_b(C_1)^b + \tfrac{1}{2}\epsilon^{abc}\omega_b(C_1)_c - \tfrac{8}{3}\omega^a(C_2) \\
&\quad - \epsilon^{abc}\sigma_{bd}(C_3)_c{}^d - 3\omega_b(C_3)^{ab} - (C_4)^a, && \text{(2.25a)} \\
(\dot{C}_2) &= -\Theta(C_2), && \text{(2.25b)} \\
(\dot{C}_3)^{\langle ab\rangle} &= -\Theta(C_3)^{ab} + 3\sigma^{\langle a}{}_c(C_3)^{b\rangle c} + \epsilon^{cd\langle a}\omega_c(C_3)^{b\rangle}{}_d + \tfrac{3}{2}\omega^{\langle a}(C_1)^{b\rangle} \\
&\quad + \tfrac{1}{2}\epsilon^{cd\langle a}\sigma^{b\rangle}{}_c(C_1)_d, && \text{(2.25c)} \\
(\dot{C}_4)^{\langle a\rangle} - \tfrac{1}{2}\epsilon^{abc}\tilde{\nabla}_b(C_5)_c &= -\tfrac{4}{3}\Theta(C_4)^a + \tfrac{1}{2}\sigma^a{}_b(C_4)^b - \tfrac{1}{2}\epsilon^{abc}\omega_b(C_4)_c - \tfrac{1}{2}(\mu+p)(C_1)^a \\
&\quad - \tfrac{1}{2}\pi^a{}_b(C_1)^b + 2\epsilon^{abc}E_{bd}(C_3)_c{}^d + \tfrac{3}{2}\epsilon^{abc}\dot{u}_b(C_5)_c, && \text{(2.25d)} \\
(\dot{C}_5)^{\langle a\rangle} + \tfrac{1}{2}\epsilon^{abc}\tilde{\nabla}_b(C_4)_c &= -\tfrac{4}{3}\Theta(C_5)^a + \tfrac{1}{2}\sigma^a{}_b(C_5)^b - \tfrac{1}{2}\epsilon^{abc}\omega_b(C_5)_c - \tfrac{1}{2}\epsilon^{abc}q_b(C_1)_c \\
&\quad + \tfrac{3}{2}q^a(C_2) + 2\epsilon^{abc}H_{bd}(C_3)_c{}^d - \tfrac{3}{2}\epsilon^{abc}\dot{u}_b(C_4)_c. && \text{(2.25e)}
\end{aligned}
$$

The closure under time propagation of constraint equations $(C_1)^a$ to $(C_5)^a$ is not surprising since, at this stage, we have not broken any symmetry of the EFE, also none of the geometric and matter extra constraints have been imposed. It shall be seen later that when these additional constraints are set, the constraints $(C_1)^a$ to $(C_5)^a$ no longer fully close under time propagation. Being able to show that these equations hold is a good exercise before we can start to tackle the cases where the symmetry has been broken. In this regard, we have written a Mathematica code to reproduce these calculations, outlined in Appendices (10).

``I have become imbued with great respect for mathematics, the subtler parts of which I had in my simplemindedness regarded as pure luxury until now.''

- Albert Einstein (letter to A. Sommerfeld 1916)

3

The Shear-Free Perfect Fluid Equations

We are now ready to set up all the required equations to investigate the consistency of the EFE under shear-free perfect fluid conditions. Here, we have used a symbolic computer algebra package for handling tensors in Mathematica to obtain the results appearing in this chapter. This package is called xTensor, a sub-package of xAct, a platform for symbolic tensor calculations in Mathematica.

3.1 THE KINEMATICS

Restricting the propagation and constraint equations appearing in section (2.1) above to those of a perfect fluid with vanishing shear, we obtain the modified four gradient of **u** in equation (3.1a), the EMT in equation (3.2a) and the Riemann curvature tensor shown in equation (3.3a) below,

THE SHEAR-FREE COVARIANT DERIVATIVE OF **u**:

$$\nabla_a u_b = -u_a \dot{u}_b + \tfrac{1}{3}\Theta h_{ab} + \epsilon_{abc}\omega^c, \tag{3.1a}$$

THE ENERGY-MOMENTUM TENSOR FOR A PERFECT FLUID:

$$T_{ab} = \mu u_a u_b + p h_{ab}, \tag{3.2a}$$

THE RIEMANN CURVATURE TENSOR FOR A PERFECT FLUID:

$$\begin{aligned} R^{ab}{}_{cd} = {} & 4u^{[a}u_{[c}E^{b]}{}_{d]} + 4h^{[a}{}_{[c}E^{b]}{}_{d]} + 2\epsilon^{abe}u_{[c}H_{d]e} + 2\epsilon_{cde}u^{[a}H^{b]e} \\ & + \tfrac{2}{3}(\mu + 3p - 2\Lambda)u^{[a}u_{[c}h^{b]}{}_{d]} + \tfrac{2}{3}(\mu + \Lambda)h^{[a}{}_{[c}h^{b]}{}_{d]}, \end{aligned} \tag{3.3a}$$

$$E_{ab} := C_{cdef}h^c{}_a u^d h^e{}_b u^f \quad , \quad H_{ab} := -\tfrac{1}{2}\epsilon_{cdgh}C^{gh}{}_{ef}h^c{}_a u^d h^e{}_b u^f. \tag{3.3b}$$

We obtain the shear-free perfect fluid dynamical equation appearing in the following section (3.2), with an extra constraint equation given by equation (3.6).

3.2 The Dynamics

3.2.1 Ricci identities for a shear-free ($\sigma_{ab} = 0$) perfect fluid

The following equations are the time derivative equations for the expansion scalar Θ (3.4a) and the vorticity vector field (3.4b) respectively.

$$\dot{\Theta} = \tilde{\nabla}_a \dot{u}^a + \dot{u}^2 - \tfrac{1}{3}\Theta^2 + 2\omega^2 - \tfrac{1}{2}\big(\mu + 3p\big) + \Lambda, \tag{3.4a}$$

$$\dot{\omega}^{\langle a \rangle} = \tfrac{1}{2}\epsilon^{abc}\tilde{\nabla}_b \dot{u}_c - \tfrac{2}{3}\Theta\omega^a. \tag{3.4b}$$

Equation (3.5a), (3.5b) and (3.5c) below are the spatial constraint equations for the quantities appearing in equation (3.1a) and (3.3b) in the previous section.

$$0 = (C_1)^a := -\tfrac{2}{3}\tilde{\nabla}^a\Theta + \epsilon^a{}_{bc}\left(\tilde{\nabla}^b\omega^c + 2\dot{u}^b\omega^c\right), \tag{3.5a}$$

$$0 = (C_2) := \tilde{\nabla}_a\omega^a - \dot{u}_a\omega^a, \tag{3.5b}$$

$$0 = (C_3)^{ab} := H^{ab} + 2\dot{u}^{\langle a}\omega^{b\rangle} + \tilde{\nabla}^{\langle a}\omega^{b\rangle}. \tag{3.5c}$$

Setting the shear to zero in equation (2.14) gives us a new constraint equation, which needs to be checked for consistency. This is expressed by equation (3.6) below,

$$0 = (C_6)^{ab} := \tilde{\nabla}^{\langle a}\dot{u}^{b\rangle} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \omega^{\langle a}\omega^{b\rangle} - E^{ab}. \tag{3.6}$$

Equation (3.6) together with equation (3.5c) relate the Weyl curvature tensor electric and magnetic parts to the acceleration and vorticity of the perfect fluid in question. In our investigation, we shall exploit the symmetrical nature of these equations. This will become clear in the following chapters.

3.2.2 (Contracted) second Bianchi identities ($\sigma_{ab} = 0$)

The following are time derivative equations for the electric (3.7a) and magnetic (3.7b) parts of the Weyl curvature tensor, including the energy density conservation equation (3.7c) followed by the energy density time propagation equation (3.7d) for the shear-free perfect fluid,

$$\dot{E}^{\langle ab\rangle} - \epsilon^{cd\langle a}\tilde{\nabla}_c H^{b\rangle}{}_d = -\Theta E^{ab} + \epsilon^{cd\langle a}\left(2\dot{u}_c H^{b\rangle}{}_d + \omega_c E^{b\rangle}{}_d\right), \tag{3.7a}$$

$$\dot{H}^{\langle ab\rangle} + \epsilon^{cd\langle a}\tilde{\nabla}_c E^{b\rangle}{}_d = -\Theta H^{ab} - \epsilon^{cd\langle a}\left(2\dot{u}_c E^{b\rangle}{}_d - \omega_c H^{b\rangle}{}_d\right), \tag{3.7b}$$

$$\tilde{\nabla}^a p = -\left(p + \mu\right)\dot{u}^a, \tag{3.7c}$$

$$\dot{\mu} = -\Theta\left(p + \mu\right). \tag{3.7d}$$

The two equations below are the spatial constraint equations, which indicate that the gradient of the energy density μ source's the divergence of electric (E^{ab}) Weyl curvature tensor field (3.8a) and that the vorticity is the source of the magnetic (H^{ab}) Weyl curvature tensor field (3.8b),

$$0 = (C_4)^a := \tilde{\nabla}_b E^{ab} - \tfrac{1}{3}\tilde{\nabla}^a\mu - 3H^a{}_b\omega^b, \tag{3.8a}$$

$$0 = (C_5)^a := \tilde{\nabla}_b H^{ab} + (p + \mu)\omega^a + 3E^a{}_b\omega^b. \tag{3.8b}$$

Now that we are done setting up the shear-free perfect fluid equations, we are ready to time propagate our original constraint equation, including the new constraint (3.6), subject to the evolution equations given in (3.4) and (3.7).

3.3 Propagation of the shear-free constraints

Time propagating constraint $(C_1)^a$ in equation (3.5a) result in equation (3.9) below,

$$\begin{aligned}(\dot{C}_1)^{\langle a\rangle} + \tilde{\nabla}_c(C_6)^{ac} = &-\Theta(C_1)^a - \tfrac{1}{2}\epsilon^a{}_{cb}(C_1)^c\omega^b - \tfrac{8}{3}(C_2)\omega^a - 3(C_3)^a{}_c\omega^c \\ &- (C_4)^a - (C_6)^a{}_c\dot{u}^c, \end{aligned} \tag{3.9}$$

where relation (9.29), (9.31), (9.26b), (9.26e) and (3.4a) have been used. Equation (3.9) indicates that if constraint $(C_1)^a$ is zero initially, it remains zero at a later time if and only if the divergence of our new constraint $(C_6)^{ab}$ vanishes. The scalar constraint equation (C_2) closes under time propagation as shown by equation (3.10) below,

$$(\dot{C}_2) = -(C_2)\Theta. \tag{3.10}$$

In order to establish this relation (9.31), (9.26b) and (9.26e) have been used. The time propagation of $(C_3)^{ab}$ can be achieved by using relation (3.7b), (9.31), (9.26b) and (9.26e) to obtain the propagation equation (3.11) below,

$$\begin{aligned}(\dot{C}_3)^{\langle ab\rangle} - \epsilon^{cd\langle a}\tilde{\nabla}_c(C_6)^{b\rangle}{}_d = &-\Theta(C_3)^{ab} + \epsilon^{cd\langle a}\omega_c(C_3)^{b\rangle}{}_d + \epsilon^{cd\langle a}\dot{u}_c(C_6)^{b\rangle}{}_d \\ &+ \tfrac{3}{2}(C_1)^{\langle a}\omega^{b\rangle}. \end{aligned} \tag{3.11}$$

Equation (3.11) states that as the universe evolves, $(C_3)^{ab}$ will remain zero at a later time if and only if $\epsilon^{cd\langle a}\tilde{\nabla}_c(C_6)^{b\rangle}{}_d$ vanishes. Propagating constraint $(C_4)^a$ along $\mathbf{u}$ and using relation (9.38), (9.29), (3.7d), (3.7b) and (9.26e) for substitution, together with identities (9.40b) to (9.40f), one obtains the time propagation equation (3.12) below:

$$\begin{aligned}(\dot{C}_4)^{\langle a\rangle} - \tfrac{1}{2}\epsilon^a{}_{bc}\tilde{\nabla}^b(C_5)^c = {}& -2\epsilon^a{}_{cd}E_b{}^d(C_3)^{bc} + \epsilon^a{}_{cd}H_b{}^d(C_6)^{bc} - \tfrac{3}{2}\epsilon^a{}_{bc}(C_5)^b\dot{u}^c \\ & - \tfrac{4}{3}\Theta(C_4)^a - \tfrac{1}{2}\big(p+\mu\big)(C_1)^a + \tfrac{1}{2}\epsilon^a{}_{bc}(C_4)^b\omega^c. \qquad (3.12)\end{aligned}$$

This propagation equation indicates that $\epsilon^a{}_{bc}\tilde{\nabla}^b(C_5)^c$ must be vanish for $(C_4)^a$ to remain zero under time propagation. Since $(C_5)^a$ is one of our original constraints, then $(C_4)^a$ closes under time propagation. Time propagating constraint $(C_5)^a$ along **u**, and using relation (9.37), (3.7d), (3.7a) and (9.26e). Then applying the identities from (9.40g) to (9.40k), we obtain equation (3.13) below:

$$\begin{aligned}(\dot{C}_5)^{\langle a\rangle} + \tfrac{1}{2}\epsilon^a{}_{bc}\tilde{\nabla}^b(C_4)^c = {}& -\epsilon^a{}_{cd}E_b{}^d(C_6)^{bc} - 2\epsilon^a{}_{cd}H_b{}^d(C_3)^{bc} + \tfrac{3}{2}\epsilon^a{}_{bc}(C_4)^b\dot{u}^c \\ & - \tfrac{4}{3}\Theta(C_5)^a + \tfrac{1}{2}\epsilon^a{}_{bc}(C_5)^b\omega^c. \qquad (3.13)\end{aligned}$$

Complementary to $(\dot{C}_4)^{\langle a\rangle}$, equation (3.13) shows that $\epsilon^a{}_{bc}\tilde{\nabla}^b(C_4)^c$ must vanish in order for $(C_5)^a$ to remain zero under time propagation. Finally, time propagating our new constraint equation $(C_6)^{ab}$ which arises from setting the shear to zero, and applying the following relations for substitutions (9.32), (3.7a), (9.26e) and (9.26b) as a means to write the resulting expression in terms of the original constraints as much as possible, one arrives at the following time propagation equation,

$$\begin{aligned}(\dot{C}_6)^{\langle ab\rangle} = {}& -\Theta\left[(2\phi+\chi)\dot{u}^{\langle a}\dot{u}^{b\rangle} + (2-4\beta+\phi)\omega^{\langle a}\omega^{b\rangle} - \tfrac{1}{3}(2-3\phi)E^{ab} + (C_6)^{ab}\right] \\ & + \beta\tilde{\nabla}^{\langle a}\tilde{\nabla}^{b\rangle}\Theta + 2(\beta-\phi)\dot{u}^{\langle a}\tilde{\nabla}^{b\rangle}\Theta + \tfrac{1}{2}\tilde{\nabla}^{\langle a}(C_1)^{b\rangle} + (C_1)^{\langle a}\dot{u}^{b\rangle} \\ & - \epsilon^{cd\langle a}\tilde{\nabla}_c(C_3)^{b\rangle}{}_d - 2\epsilon^{cd\langle a}\dot{u}_c(C_3)^{b\rangle}{}_d + 2\epsilon^{cd\langle a}\omega_c(C_6)^{b\rangle}{}_d. \qquad (3.14)\end{aligned}$$

This result will play a central role in proving the shear-free perfect fluid theorems presented in this thesis. In equation (3.14) above, the terms on the right-hand side, which cannot be expressed in terms of our original constraints, form the next level constraint equation (3.15) below.

$$\begin{aligned} 0 = (C_7)^{ab} := -\Theta \left[(2\phi+\chi)\dot{u}^{\langle a}\dot{u}^{b\rangle} + (2-4\beta+\phi)\omega^{\langle a}\omega^{b\rangle} - \tfrac{1}{3}(2-3\phi)E^{ab}\right] \\ + \beta\tilde{\nabla}^{\langle a}\tilde{\nabla}^{b\rangle}\Theta + 2(\beta-\phi)\dot{u}^{\langle a}\tilde{\nabla}^{b\rangle}\Theta. \end{aligned} \tag{3.15}$$

Equation (3.15) above together with the spatial gradient of equation (3.6) must be time propagated in order to establish the consistency of the full EFE. Furthermore, equation (3.15) and (3.14) above have been simplified by defining $\left(p' = \frac{\partial p}{\partial \mu}\right)$ and $\left(\beta = \frac{1}{3} + p'\right)$ together with the following scalar variables;

$$\mathcal{E} := p + \mu, \tag{3.16a}$$

$$\phi := \frac{1}{3} + \mathcal{E}\frac{p''}{p'} - p', \tag{3.16b}$$

$$\chi := -\frac{5}{3} - \mathcal{E}^2\frac{p^{(3)}}{p'^2} + \phi\left(\frac{\phi}{p'} - \frac{5}{3p'} + 2\right) + \frac{4}{9p'} + p', \tag{3.16c}$$

ϕ is exactly the same as the variable $G(\mu)$ defined by White and Collins [52]. Another useful scalar to define is $\mathcal{J}$, which represents the divergence of the acceleration vector field,

$$\mathcal{J} := \dot{u}^2 + \tilde{\nabla}_b\dot{u}^b. \tag{3.17}$$

The scalar $\mathcal{J}$ vanishes in the dust case or in situations where the pressure is constant. This is the next chapter's subject, where we prove the shear-free perfect fluid conjecture for these exceptional cases.

The chief support of the theory is to be found less in that lent by observation hitherto than in its inherent logical consistency, in which it far transcends that of classical mechanics, and also in the fact that it solves the perplexing problem of gravitation and of the relativity of motion at one stroke in a manner highly satisfying to our reason.

-H. Weyl (Space,Time, Matters)

4

Constant Pressure Shear-Free Perfect Fluid Theorem

In his seminal paper, titled *The Dynamics of Pressure-Free Matter in General Relativity* [4] George F. R. Ellis showed that if a space contains shear-free dust[1], then either the vorticity or the expansion scalar vanishes. This proof was obtained by setting up an orthonormal tetrad and associated coordinate systems. In a recent paper, Senovilla

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[1] Pressure is zero.

et al., in reference [13], provided a covariant proof of the Ellis result mentioned above, extending it to the constant pressure case. The following is an improved version of the proof given by Senovilla et al. [13], which contains some added details that serve to illuminate the computational process followed to obtain the proof of Theorem (1) stated in the following section.

4.1 THE THEOREM

Theorem 1. *In general relativity, if the velocity vector field of a geodesic barotropic perfect fluid is shear-free with* $\mu + p \neq 0$*, then either the expansion or the rotation vanishes.*

The velocity vector field in our cosmological model described in Chapter (2) is the four-velocity unit vector field u^a, whose covariant derivative can be irreducibly decomposed in the following way,

$$\nabla_a u_b = \tfrac{1}{3}\Theta h_{ab} + \epsilon_{abc}\omega^c. \tag{4.1}$$

Where Θ and ω^c are the expansion scalar and the vorticity vector field, respectively. The condition described by Theorem (1) above can be encapsulated in the following statement below,

$$\mu + p \neq 0, \quad p = constant, \quad \dot{u}^a = 0 \;\text{ and }\; \sigma_{ab} = 0 \qquad \Longrightarrow \qquad \omega\theta = 0. \tag{4.2}$$

Since by equation (3.7c) the geodesic condition implies that $\tilde{\nabla}_a p = 0$. If $\dot{p} \neq 0$, we have that $\nabla_a p = -u_a \dot{p}$ and u_a is hyper-surface orthogonal hence $\omega = 0$ unless p is constant.

4.2 The Proof

Proof. For this proof, we shall focus our attention on the time propagation of constraint $(C_1)^a$. From equation (3.9) it is clear that the resulting constraint equation comes from the divergence of $(C_6)^{ab}$. To obtain this divergence, we substitute H^{ab} and E^{ab} from equation (3.5c) and (3.6) into equation (3.8a), which yields the following constraint equation,

$$(C_4)^a = -3(C_3)^a{}_b\omega^b + 3\omega_b\tilde{\nabla}^{\langle a}\omega^{b\rangle} - \tfrac{1}{3}\tilde{\nabla}^a\mu - \tilde{\nabla}_b(C_6)^{ab} - \tilde{\nabla}_b\big(\omega^{\langle a}\omega^{b\rangle}\big). \tag{4.3}$$

Upon rearranging the terms in equation (4.3), we can write down the divergence of $(C_6)^{ab}$ in the following way,

$$\tilde{\nabla}_b(C_6)^{ab} = -(C_4)^a - 3(C_3)^a{}_b\omega^b + 3\omega_b\tilde{\nabla}^{\langle a}\omega^{b\rangle} - \tfrac{1}{3}\tilde{\nabla}^a\mu - \tilde{\nabla}_b\big(\omega^{\langle a}\omega^{b\rangle}\big). \tag{4.4}$$

Expanding out the angle brackets, which indicate the PSTF part of tensor fields, equation (4.4) can be rewritten as

$$\tilde{\nabla}_b(C_6)^{ab} = -(C_4)^a - 3(C_3)^a{}_b\omega^b - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{13}{6}\omega\tilde{\nabla}^a\omega + \tfrac{1}{2}\omega^b\tilde{\nabla}_b\omega^a - 2\omega^a\tilde{\nabla}_b\omega^b. \tag{4.5}$$

Buy using the commutation relation (9.34) and constraint (C_2) from equation (3.5b) one then arrives at the following equation,

$$\begin{aligned}\tilde{\nabla}_b(C_6)^{ab} = -(C_4)^a - 2(C_2)\omega^a - 3(C_3)^a{}_b\omega^b + \tfrac{1}{2}\epsilon^a{}_{bc}(C_1)^b\omega^c \\ - \tfrac{1}{3}\tilde{\nabla}^a\mu + \tfrac{8}{3}\omega\tilde{\nabla}^a\omega - \tfrac{1}{3}\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\Theta.\end{aligned} \tag{4.6}$$

The last three non-zero terms on the right hand side of equation (4.6) form our next level constraint equation $(C_{\mathrm{v}7})^a$ as shown below,

$$0 = (C_{\mathrm{v}7})^a := \tilde{\nabla}^a \mu - 8\omega \tilde{\nabla}^a \omega + \epsilon^a{}_{cb} \omega^c \tilde{\nabla}^b \Theta. \tag{4.7}$$

Equation (4.7) is similar to equation (29) of Senovilla et. al [13], the only difference is the sign of the last term on the right-hand side. This depends on the definition of the orientation of the vorticity vector field ω^a. To establish consistency, we time propagate equation (4.7) and apply the following relations; (9.29), (9.31), (9.29), (9.26e), (9.26f) and (4.7) itself, in order to eliminate terms containing $\omega \tilde{\nabla}^a \omega$. This gives us the following time propagation equation;

$$\begin{aligned}(\dot{C}_{\mathrm{v}7})^a = &-\tfrac{5}{3}(C_{\mathrm{v}7})^a \Theta - \tfrac{1}{2}\epsilon^a{}_{cb}(C_{\mathrm{v}7})^c \omega^b - \left(p + \mu - \tfrac{29}{6}\omega^2\right) \tilde{\nabla}^a \Theta + \tfrac{1}{3}\Theta \tilde{\nabla}^a \mu \\ &+ \tfrac{1}{2}\omega^a \omega^c \tilde{\nabla}_c \Theta.\end{aligned} \tag{4.8}$$

In equation (4.8), we proceed in a similar fashion to equation (4.6); where we extract the non-zero last three terms on the right hand side of (4.8) to form the next level constraint $(C_{\mathrm{v}8})^a$ shown below;

$$0 = (C_{\mathrm{v}8})^a := \left(p + \mu - \tfrac{29}{6}\omega^2\right) \tilde{\nabla}^a \Theta - \tfrac{1}{3}\Theta \tilde{\nabla}^a \mu - \tfrac{1}{2}\omega^a \omega^c \tilde{\nabla}_c \Theta. \tag{4.9}$$

The above equation (4.9) resembles equation (31) in Senovilla et. al [13]. Furthermore, we take the time propagation of equation (4.9), and we use relations (9.29), (9.29), (9.26e), (9.26f), (4.7) and (3.7d). Subsequently, we multiply by 12Θ in order to apply equation (4.9) to get rid of terms involving $\Theta \tilde{\nabla}^a \mu$. This gives us the following constraint equation;

$$\begin{aligned}12\Theta(\dot{C}_{\mathrm{v}8})^a = {} & -6\Theta\mathcal{P}_1(C_{\mathrm{v}7})^a + 2\left(\mathcal{P}_2 - 14\Theta^2\right)(C_{\mathrm{v}8})^a - 3\Theta\big[4\epsilon^a{}_{cb}(C_{\mathrm{v}8})^c\omega^b \\ & - \omega^a(C_{\mathrm{v}7})^c\omega_c\big] - 2\big[\mathcal{P}_1(\mathcal{P}_2 - 4\Theta^2) - \tfrac{58}{3}\Theta^2\omega^2\big]\tilde{\nabla}^a\Theta \\ & + 6\Theta\mathcal{P}_1\epsilon^a{}_{cb}\omega^c\tilde{\nabla}^b\Theta + \mathcal{P}_2\omega^a\omega^c\tilde{\nabla}_c\Theta, \end{aligned} \tag{4.10}$$

where we have defined the following scalars;

$$\mathcal{P}_1 = p + \mu - \tfrac{29}{6}\omega^2, \tag{4.11}$$

$$\mathcal{P}_2 = -6p + 6\Lambda - 3\mathcal{E}_1 + 4\Theta^2 - \tfrac{5}{2}\omega^2, \tag{4.12}$$

for simplicity. Equation (4.10) above may seem like a long and complicated equation, but the non-zero terms on the right hand side form a linear combination of the following three vectors; $\tilde{\nabla}^a\Theta$, $\epsilon^a{}_{cb}\omega^c\tilde{\nabla}^b\Theta$ and ω^a, which form our next level constraint equations $(C_{\mathrm{v}9})^a$ below,

$$\begin{aligned}0 = (C_{\mathrm{v}9})^a := {} & -2\big[\mathcal{P}_1(\mathcal{P}_2 - 4\Theta^2) - \tfrac{58}{3}\Theta^2\omega^2\big]\tilde{\nabla}^a\Theta + 6\Theta\mathcal{P}_1\epsilon^a{}_{cb}\omega^c\tilde{\nabla}^b\Theta \\ & + \mathcal{P}_2\omega^a\omega^c\tilde{\nabla}_c\Theta. \end{aligned} \tag{4.13}$$

In equation (4.13) above, we observe that the second term which contains the vector $\epsilon^a{}_{cb}\omega^c\tilde{\nabla}^b\Theta$ can be removed by contracting with ω_a, since these two vectors are orthogonal to each other. Contracting equation (4.13) with ω_a gives us the following scalar constraint equation,

$$0 = (C_{\mathrm{s}9}) := \left[\tfrac{1}{4}\mathcal{P}_2\omega^2 - \mathcal{P}_1\left(\tfrac{1}{2}\mathcal{P}_2 + 2\Theta^2\right) + \tfrac{29}{3}\Theta^2\omega^2\right]\omega^a\tilde{\nabla}_a\Theta. \tag{4.14}$$

The constraint equation (4.14) presents us with two options, either $\omega^a\tilde{\nabla}_a\Theta = 0$ or the terms within the square brackets must be zero. Let us consider each of these two cases separately; (i) when the terms within the square parentheses in equation (4.14) vanish, we have the following new scalar constraint equation (C_{s10}) shown below;

$$0 = (C_{\mathrm{s10}}) := \tfrac{1}{4}\mathcal{P}_2\omega^2 - \mathcal{P}_1\left(\tfrac{1}{2}\mathcal{P}_2 + 2\Theta^2\right) + \tfrac{29}{3}\Theta^2\omega^2. \tag{4.15}$$

By time propagating equation (4.15) and making use of relations (9.26f), (3.7d), (3.4a) and (4.14), we obtain;

$$(\dot{C}_{\mathrm{s10}}) = -\tfrac{1}{24}\Theta\left[24(C_{\mathrm{s10}}) + 36\mathcal{P}_1\left(\mathcal{P}_1 - \omega^2\right) - \omega^2\left(64\mathcal{P}_2 - 512\Theta^2 - 9\omega^2\right)\right]. \tag{4.16}$$

From equation (4.16) above, we can extract the non-zero terms from the right-hand side to form the following constraint equation;

$$0 = (C_{\mathrm{s11}}) := -\tfrac{1}{24}\Theta\left[36\mathcal{P}_1\left(\mathcal{P}_1 - \omega^2\right) - \omega^2\left(64\mathcal{P}_2 - 512\Theta^2 - 9\omega^2\right)\right]. \tag{4.17}$$

For consistency, equation (4.17) is identically satisfied when $\Theta = 0$, in which case, we have the proof or when the terms in the square brackets vanish. Then from the latter, we form the following consistency constraint equation;

$$0 = (C_{\mathrm{s12}}) := -\tfrac{1}{24}\left[36\mathcal{P}_1\left(\mathcal{P}_1 - \omega^2\right) - \omega^2\left(64\mathcal{P}_2 - 512\Theta^2 - 9\omega^2\right)\right]. \tag{4.18}$$

Furthermore, time propagating (4.18) and using the following relation; (3.7d), (9.26f) and (4.18) itself, gives us;

$$(\dot{C}_{\mathrm{s12}}) = -2\Theta(C_{\mathrm{s12}}) + 8\Theta\omega^2\left(\mu + \tfrac{7}{3}p - \tfrac{40}{9}\omega^2 - \tfrac{4}{3}\Lambda\right). \tag{4.19}$$

In (4.19) above, we have substituted for $\mathcal{P}_1$ and $\mathcal{P}_2$ using equation (4.11) and (4.12), and then by extracting the non-zero part of equation (4.19) above we obtain the constraint equation similar to equation (48) in reference [13];

$$0 = (C_{\mathrm{s13}}) := \Theta\omega^2\big(\mu + \tfrac{7}{3}p - \tfrac{40}{9}\omega^2 - \tfrac{4}{3}\Lambda\big). \tag{4.20}$$

Equation (4.20) gives us two cases to consider; either $\Theta\omega^2 = 0$ and we are done, or the terms within the brackets must vanish, in which case we have the following new constraint equation;

$$0 = (C_{\mathrm{s14}}) := \mu + \tfrac{7}{3}p - \tfrac{40}{9}\omega^2 - \tfrac{4}{3}\Lambda. \tag{4.21}$$

Time propagating equation (4.21) and using equation (3.7d) and (9.26f), we obtain the following equation;

$$(\dot{C}_{\mathrm{s14}}) = -\Theta\big(p + \mu - \tfrac{160}{27}\omega^2\big). \tag{4.22}$$

Similarly, in equation (4.22) we require that the terms in the brackets on the right--hand side be zero; otherwise, the conjecture is true, giving us the new constraint equation shown below,

$$0 = (C_{\mathrm{s15}}) := p + \mu - \tfrac{160}{27}\omega^2. \tag{4.23}$$

The time evolution of equation (4.23), where relations (3.7d), (9.26f) and (4.23) itself have been used, gives us the following constraint equation;

$$0 = (\dot{C}_{\mathrm{s15}}) := -(C_{\mathrm{s15}})\Theta + \tfrac{160}{81}\Theta\omega^2. \tag{4.24}$$

The non-zero term on the right-hand side of equation (4.24) clearly shows that $\Theta\omega$ must vanish, and we have the proof. For completion, we also need to consider the other option given by equation (4.14). This is the case where the terms in the brackets do not vanish; then it follows that $\omega^a\tilde{\nabla}_a\Theta$ must be zero. From now on, equation (4.13) becomes the following constraint;

$$0 = (C_{\mathrm{v10}})^a := \Big[\tfrac{116}{3}\Theta^2\omega^2 + 6\left(p + \mu - \tfrac{29}{6}\omega^2\right)\left(3p + \mu - 2\Lambda - 4\omega^2\right)\Big]\tilde{\nabla}^a\Theta$$
$$+ 6\Theta\left(p + \mu - \tfrac{29}{6}\omega^2\right)\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\Theta. \quad (4.25)$$

The vectors $\tilde{\nabla}^a\Theta$ and $\epsilon^a{}_{cb}\omega^c\tilde{\nabla}^b\Theta$ in equation (4.25) above, are orthogonal, then by contracting equation (4.25) with $\epsilon_{acb}\omega^c\tilde{\nabla}^b\Theta$ and applying the condition $\omega^a\tilde{\nabla}_a\Theta = 0$, we obtain the useful constraint equation;

$$0 = (C_{\mathrm{s17}}) := \tfrac{3}{2}\Theta\omega^2\big(p + \mu - \tfrac{29}{6}\omega^2\big)\tilde{\nabla}_a\Theta\tilde{\nabla}^a\Theta. \quad (4.26)$$

Equation (4.26) presents us with three cases; either (1) $\Theta\omega = 0$, which proves the theorem or (2); the term within the brackets vanish, and this gives us our new constraint equation;

$$0 = (C_{\mathrm{s18}}) := p + \mu - \tfrac{29}{6}\omega^2, \quad (4.27)$$

or (3); $\tilde{\nabla}_a\Theta\tilde{\nabla}^a\Theta = 0$ which completes the proof. In case (2), we need to time propagate equation (4.27) again to reach the desired restriction on the expansion and vorticity. This gives us the following;

$$(\dot{C}_{\mathrm{s18}}) = -(C_{\mathrm{s18}})\Theta + \tfrac{29}{18}\Theta\omega^2. \quad (4.28)$$

As in equation (4.24) above, we have that $\Theta\omega = 0$ as our consistency condition, which completes the proof. Finally, in case (3) we can see that the norm of the spatial gradient vector of the expansion is zero, hence we have that $\tilde{\nabla}^a\Theta$ must vanish, then by equation (4.7) and (4.9) it follows that $\tilde{\nabla}^a\mu = \tilde{\nabla}^a\omega = 0$. Thus, according to relation (9.30), we have that $\dot{\omega} = 0$, which indicates that the vorticity is constant under time propagation and hence $\Theta\omega = 0$ by equation (9.26f). Alternatively, one can take the spatial divergence of constraint $(C_1)^a$ and by using relation (9.33), we obtain another useful constraint equation below;

$$0 = -\tfrac{3}{2}\tilde{\nabla}_a(C_1)^a := \Theta\omega^2 + \tilde{\nabla}_a\tilde{\nabla}^a\Theta. \tag{4.29}$$

It is clear from equation (4.29) above, that when $\tilde{\nabla}^a\Theta$ is zero, then $\Theta\omega$ must vanish and this completes the proof. □

We have proved the conjecture for the constant pressure case above, and we are now in a position to address the case for a general equation of state, which is a subject of the next chapter. Theorem (1) above is a source of inspiration for developing the covariant proofs, which are the subject of Chapter (6) and (7). The next chapter discusses the general shear-free perfect fluid conjecture indicating the necessary constraint equation to investigate for consistency of the EFE.

Quicquid excessit modum pendet instabilli loco.
(Whatever has exceeded its proper bounds is in a state of instability.)

-Seneca

5

The general shear free perfect fluid conjecture

5.1 THE CONJECTURE

The theorem discussed in the previous chapter is a special case of a more general shear-free perfect fluid conjecture which states that *In general relativity if the velocity vector field of a barotropic perfect fluid is shear-free, then either the expansion or* *the rotation vanishes.* Under the shear-free condition, the covariant derivative of the velocity vector field is given by the following equation;

$$\nabla_a u_b = -u_a \dot{u}_b + \tfrac{1}{3}\Theta h_{ab} + \epsilon_{abc}\omega^c, \tag{5.1}$$

and the statement of the shear-free perfect fluid conjecture can be stated symbolically in the following way;

$$\mu + p \neq 0, \quad p = p(\mu) \text{ and } \sigma_{ab} = 0 \qquad \Longrightarrow \qquad \omega\theta = 0. \tag{5.2}$$

5.2 Time Propagation of Constraint $(C_6)^{ab}$ and it's Spatial Gradients

By time propagating equation (3.7c) and applying relation (9.29), (3.7d) and (3.7c) itself, one obtains the propagation equation for the acceleration vector field as shown below;

$$\ddot{u}^{\langle a\rangle} = -\dot{u}^a\Theta\phi - \epsilon^a{}_{bc}\dot{u}^b\omega^c + p'\tilde{\nabla}^a\Theta. \tag{5.3}$$

The time evolution equation (5.3) is essential to our discussion in this thesis since the presence of the acceleration provides a point of departure from the constant pressure situation described by Theorem (1) in the preceding chapter. Equation (5.3) allows us to compute the time evolution of the scalar $\mathcal{J}$ given by equation (3.17), by using the time-space covariant derivative commutation relation (9.32) given in the Appendix by equation (9.32). This gives us the following time evolution equation;

$$\begin{aligned}\dot{\mathcal{J}} = &-\Theta\big(\mathcal{J}\big(\tfrac{1}{3} + \phi\big) - (1 - 2\phi - \chi)\dot{u}^2 + p'(1 - 9p')\omega^2\big) + (1 - 2\phi)\dot{u}^a\tilde{\nabla}_a\Theta \\ &- 3p'(C_1)^a\dot{u}_a - \tfrac{3}{2}p'\tilde{\nabla}_a(C_1)^a.\end{aligned} \tag{5.4}$$

Furthermore, equation (3.4b) can be simplified by substituting $\dot{u}^a$ from equation (3.7c) and applying equation (9.30), (3.7d) and (3.7c) itself. We then obtain the following evolution equation for the vorticity vector (5.5) and scalar (5.6) respectively,

$$\dot{\omega}^{\langle a\rangle} = \Theta\omega^a \left(p' - \tfrac{2}{3}\right), \tag{5.5}$$

$$\dot{\omega} = \Theta\omega \left(p' - \tfrac{2}{3}\right). \tag{5.6}$$

Using the time propagation of constraint $(C_6)^{ab}$ along $\mathbf{u}$, given by equation (3.14) and upon extracting the non-zero terms, then what remains become our new constraint $(C_7)^{ab}$, which is given by equation (5.7) below;

$$\begin{aligned} 0 = (C_7)^{ab} := &-\Theta\left[(2\phi+\chi)\dot{u}^{\langle a}\dot{u}^{b\rangle} + (2-4\beta+\phi)\omega^{\langle a}\omega^{b\rangle} - \tfrac{1}{3}(2-3\phi)E^{ab}\right] \\ &+ \beta\tilde{\nabla}^{\langle a}\tilde{\nabla}^{b\rangle}\Theta + 2(\beta-\phi)\dot{u}^{\langle a}\tilde{\nabla}^{b\rangle}\Theta. \end{aligned} \tag{5.7}$$

Equation (5.7) contains terms involving the expansion scalar's first and second spatial gradients Θ. These terms are problematic under time propagation because of the presence of the scalar $\mathcal{J}$ in equation (3.4a). Therefore, we would like to eliminate these terms if possible. In our pursuit to simplify equation (5.7), we are led to the realisation that we need to obtain other complementary constraint equations arising from taking the divergence and curl of constraint $(C_6)^{ab}$. We will then compare these equations to make progress towards the proof. For the rest of this chapter, the scalars (ψ) is given by the equation $(\psi = \frac{1}{3}(4 - 18\phi - 9\chi))$. Time propagating constraint $(C_1)^a$ given by equation (3.9) and applying the relations (9.35), (3.8a),(3.5c) and (3.6) itself, gives us the divergence of $(C_6)^{ab}$ as follows;

$$\begin{aligned}\tilde{\nabla}_b(C_6)^{ab} = {} & \tfrac{2}{9}\left(3\Lambda + 3\mathcal{J} - \Theta^2 + 3\mu + 6\omega^2\right)\dot{u}^a - \tfrac{1}{3}\Theta\left(2 - 9\phi\right)\epsilon^a{}_{bc}\dot{u}^b\omega^c \\ & + \tfrac{1}{3}\left(2\tilde{\nabla}^a\mathcal{J} - 2p'\Theta\tilde{\nabla}^a\Theta - \tilde{\nabla}^a\mu + 8\omega\tilde{\nabla}^a\omega\right) - (C_4)^a \\ & - \tfrac{1}{3}\left(1 - 9p'\right)\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\Theta - (C_6)^a{}_b\dot{u}^b - p'\Theta(C_1)^a \\ & - 2(C_2)\omega^a - 3(C_3)^a{}_b\omega^b + \tfrac{1}{2}\epsilon^a{}_{bc}(C_1)^b\omega^c. \end{aligned} \tag{5.8}$$

Equation (5.8) above contains the first spatial gradients of scalars, which are the expansion scalar (Θ), the scalar ($\mathcal{J}$), the energy density (μ) and the vorticity scalar (ω). Upon extracting the vanishing terms on the right-hand side of equation (5.8), we arrive at the following constraint equation;

$$\begin{aligned}0 = (C_8)^a := {} & \tfrac{2}{9}\left(3\Lambda + 3\mathcal{J} - \Theta^2 + 3\mu + 6\omega^2\right)\dot{u}^a - \tfrac{1}{3}\Theta\left(2 - 9\phi\right)\epsilon^a{}_{bc}\dot{u}^b\omega^c \\ & + \tfrac{1}{3}\left(2\tilde{\nabla}^a\mathcal{J} - 2p'\Theta\tilde{\nabla}^a\Theta - \tilde{\nabla}^a\mu + 8\omega\tilde{\nabla}^a\omega\right) \\ & - \tfrac{1}{3}\left(1 - 9p'\right)\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\Theta. \end{aligned} \tag{5.9}$$

To compare equation (5.7) and equation (5.9), it suffices to take the curl of constraint $(C_8)^a$ in order to generate second order spatial gradients of (Θ). Upon taking the curl both sides of equation (5.9) above, then taking into account the integrability conditions (9.30) and (9.36) for any scalar ($\mathcal{S}$) and the acceleration vector field ($\dot{u}^a$) respectively, we arrive at following constraint equation on the next page;

$$
\begin{aligned}
\epsilon^{a}{}_{bc}\tilde{\nabla}^{b}(C_8)^{c} = {} & \epsilon^{a}{}_{bc}(C_8)^{b}\dot{u}^{c} + \Big[(1-13p')(C_1)^{b}\dot{u}_{b} + \tfrac{1}{2}(1-13p')\tilde{\nabla}_{b}(C_1)^{b}\Big]\omega^{a} \\
& + \tfrac{1}{3}(C_2)\Big[(1-9p')\tilde{\nabla}^{a}\Theta - \Theta(2-9\phi)\dot{u}^{a}\Big] - \tfrac{1}{3}\Theta(2-9\phi)\omega^{b}\tilde{\nabla}_{b}\dot{u}^{a} \\
& + \tfrac{1}{3}\Big[2\Theta(1+3p')\mathcal{E} + \tfrac{1}{3}\Theta(2-39\phi)\mathcal{J} - \tfrac{1}{9}\Theta(16-162\phi-39\psi)\dot{u}_{b}\dot{u}^{b} \\
& - \tfrac{1}{3}\Theta\left(29+18p'-351p'^{2}\right)\omega^{2} + 2(5-9p'-13\phi)\dot{u}^{b}\tilde{\nabla}_{b}\Theta\Big]\omega^{a} \\
& - \Big[\Theta\psi\dot{u}^{b}\omega_{b} + \tfrac{1}{3}(2-9\phi)\omega^{b}\tilde{\nabla}_{b}\Theta\Big]\dot{u}^{a} + \tfrac{1}{3}\Theta(2-9\phi)\dot{u}^{b}\tilde{\nabla}_{b}\omega^{a} \\
& + \tfrac{2}{9}\Theta(1+3\phi)\epsilon^{a}{}_{bc}\dot{u}^{b}\tilde{\nabla}^{c}\Theta - \tfrac{1}{3}(1+9p'-9\phi)\dot{u}^{b}\omega_{b}\tilde{\nabla}^{a}\Theta \\
& + \tfrac{1}{3}(1-9p')\omega^{b}\tilde{\nabla}_{b}\tilde{\nabla}^{a}\Theta - \tfrac{1}{3}(1-9p')\tilde{\nabla}^{b}\Theta\tilde{\nabla}_{b}\omega^{a}. \qquad (5.10)
\end{aligned}
$$

The terms on the right-hand side of equation (5.10), which cannot be written in terms of our previously established constraints, give us the following;

$$
\begin{aligned}
0 = (C_9)^{a} := {} & \tfrac{1}{3}\Big[2\Theta(1+3p')\mathcal{E} + \tfrac{1}{3}\Theta(2-39\phi)\mathcal{J} - \tfrac{1}{9}\Theta(16-162\phi-39\psi)\dot{u}_{b}\dot{u}^{b} \\
& - \tfrac{1}{3}\Theta\left(29+18p'-351p'^{2}\right)\omega^{2} + 2(5-9p'-13\phi)\dot{u}^{b}\tilde{\nabla}_{b}\Theta\Big]\omega^{a} \\
& - \Big[\Theta\psi\dot{u}^{b}\omega_{b} + \tfrac{1}{3}(2-9\phi)\omega^{b}\tilde{\nabla}_{b}\Theta\Big]\dot{u}^{a} - \tfrac{1}{3}\Theta(2-9\phi)\omega^{b}\tilde{\nabla}_{b}\dot{u}^{a} \\
& + \tfrac{1}{3}\Theta(2-9\phi)\dot{u}^{b}\tilde{\nabla}_{b}\omega^{a} - \tfrac{1}{3}(1+9p'-9\phi)\dot{u}^{b}\omega_{b}\tilde{\nabla}^{a}\Theta \\
& + \tfrac{2}{9}\Theta(1+3\phi)\epsilon^{a}{}_{bc}\dot{u}^{b}\tilde{\nabla}^{c}\Theta - \tfrac{1}{3}(1-9p')\tilde{\nabla}^{b}\Theta\tilde{\nabla}_{b}\omega^{a} \\
& + \tfrac{1}{3}(1-9p')\omega^{b}\tilde{\nabla}_{b}\tilde{\nabla}^{a}\Theta. \qquad (5.11)
\end{aligned}
$$

The last term on the right-hand side of equation (5.11) contains the second-order spatial gradients of the expansion scalar (Θ), with one index projected along the vorticity direction. We also observe that upon taking the symmetric trace-free curl of constraint $(C_6)^{ab}$ in equation (3.6); the term that appears on the left-hand side of equation (3.11) results in the following constraint equation;

$$\begin{aligned}\epsilon^{\langle a}{}_{cd}\tilde{\nabla}^c(C_6)^{b\rangle d} = \epsilon^{\langle a}{}_{cd}\dot{u}^c(C_6)^{b\rangle d} - (C_1)^{(a}\omega^{b)} - 6\Theta p'(C_2)h^{ab} - 3\Theta\phi\dot{u}^{\langle a}\omega^{b\rangle}& \\ + p'\Theta\tilde{\nabla}^{(a}\omega^{b)} - (\tfrac{2}{3} - 3p')\omega^{(a}\tilde{\nabla}^{b)}\Theta + \epsilon^{\langle a}{}_{cd}\tilde{\nabla}^d E^{b\rangle c}& \\ - 2\epsilon^{(a}{}_{cd}\dot{u}^c\tilde{\nabla}^d\dot{u}^{b)} + \epsilon^{(a}{}_{cd}\omega^c\tilde{\nabla}^d\omega^{b)}& \\ + p'\big(\Theta\dot{u}^c\omega_c - 6\omega^c\tilde{\nabla}_c\Theta\big)h^{ab}.&\end{aligned} \tag{5.12}$$

Furthermore, as we have done previously, we can extract the non-vanishing terms from the right-hand side of equation (5.12) to form the constraint $(C_{10})^{ab}$ below;

$$\begin{aligned}0 = (C_{10})^{ab} := -3\Theta\phi\dot{u}^{\langle a}\omega^{b\rangle} + p'\Theta\tilde{\nabla}^{(a}\omega^{b)} - (\tfrac{2}{3} - 3p')\omega^{(a}\tilde{\nabla}^{b)}\Theta + \epsilon^{\langle a}{}_{cd}\tilde{\nabla}^d E^{b\rangle c}& \\ - 2\epsilon^{(a}{}_{cd}\dot{u}^c\tilde{\nabla}^d\dot{u}^{b)} + \epsilon^{(a}{}_{cd}\omega^c\tilde{\nabla}^d\omega^{b)} + p'\big(\Theta\dot{u}^c\omega_c - 6\omega^c\tilde{\nabla}_c\Theta\big)h^{ab}.&\end{aligned} \tag{5.13}$$

Incidentally, taking the divergence of $(C_{10})^{ab}$ in equation (5.13) above gives us double spatial gradients of the expansion scalar (Θ), which is what we are looking for. Here, we used the following key relations; (9.33), (9.35), (9.39) and the identity (9.40a) to obtain equation (5.14) shown in the next page. This equation may look long and complicated, but all it represents is a constraint equation involving the acceleration ($\dot{u}^a$), the expansion scalar (Θ), the vorticity (ω^a) combined with their spatial gradients. This represents a milestone in our journey towards the proof of the shear-free perfect fluid conjecture since these equations have never been seen before in their present covariant form. These appear in our paper; see Sikhonde and Dunsby [28].

$$
\begin{aligned}
\tilde{\nabla}_b(C_{10})^{ab} = &-\tfrac{1}{6}\big(2-9\phi\big)\Theta\dot{u}^b\tilde{\nabla}_b\omega^a - \tfrac{1}{2}\big(2-\phi\big)\Theta\omega^b\tilde{\nabla}_b\dot{u}^a \\
&+\tfrac{1}{6}\big(1-9p'\big)\tilde{\nabla}^b\Theta\tilde{\nabla}_b\omega^a - \tfrac{1}{6}\big(5+3p'\big)\omega^b\tilde{\nabla}_b\tilde{\nabla}^a\Theta \\
&-\tfrac{1}{2}\big(1+p'-\phi\big)\dot{u}^b\omega_b\tilde{\nabla}^a\Theta - \tfrac{1}{9}\big(1+3\phi\big)\Theta\epsilon^a{}_{bc}\dot{u}^b\tilde{\nabla}^c\Theta \\
&+\Big[\tfrac{1}{6}\big(2+9\phi\big)\Theta\mathcal{J} - \tfrac{1}{3}\big(1+3p'\big)\Theta\mathcal{E} - \tfrac{1}{2}\big(6\phi+\psi\big)\Theta\dot{u}_b\dot{u}^b \\
&+\tfrac{1}{6}\big(19-18p'-81p'^2\big)\Theta\omega^2 - \tfrac{1}{3}\big(5-9p'-9\phi\big)\dot{u}^b\tilde{\nabla}_b\Theta\Big]\omega^a \\
&-\Big[\tfrac{1}{18}\big(8-36\phi+3\psi\big)\dot{u}^b\Theta\omega_b + \tfrac{1}{6}\big(2+12p'-3\phi\big)\omega^b\tilde{\nabla}_b\Theta\Big]\dot{u}^a \\
&-\epsilon^a{}_{bc}\Big[\omega^b\tilde{\nabla}^c(C_2) + \tfrac{3}{2}(C_8)^b\dot{u}^c + \tfrac{1}{2}p'\Theta\tilde{\nabla}^c(C_1)^b + \tfrac{1}{2}\tilde{\nabla}^c(C_4)^b\Big] \\
&+\Big[2\epsilon^a{}_{cd}(C_3)_b{}^d - \epsilon_{bcd}(C_3)^{ad}\Big]\dot{u}^b\omega^c - 2\epsilon^a{}_{cd}(C_6)_b{}^d\Big[\dot{u}^b\dot{u}^c + \omega^b\omega^c\Big] \\
&-\tfrac{1}{4}\omega^b\Big[2\tilde{\nabla}^a(C_1)_b + 5\tilde{\nabla}_b(C_1)^a + 2\epsilon_{bcd}\tilde{\nabla}^d(C_3)^{ac} + 12\epsilon^a{}_{cd}\tilde{\nabla}^d(C_3)_b{}^c\Big] \\
&-(C_1)^b\Big[\tfrac{1}{4}\tilde{\nabla}_b\omega^a + \tfrac{1}{2}\epsilon^a{}_{bc}p'\tilde{\nabla}^c\Theta + \tfrac{1}{2}(5-9p')\dot{u}_b\omega^a + \tfrac{3}{2}(C_3)^a{}_b + \dot{u}^a\omega_b \\
&-\tfrac{1}{6}\epsilon^a{}_{bc}\big(1+3\phi+9p'\big)\dot{u}^c\Theta\Big] + \epsilon^a{}_{bd}\Big[-\tfrac{1}{2}\omega^b\tilde{\nabla}_c(C_3)^{cd} + \tfrac{3}{2}(C_3)_c{}^d\tilde{\nabla}^c\omega^b\Big] \\
&+\tfrac{1}{4}\Big[(C_1)^a\dot{u}^b\omega_b - 4p'\Theta\tilde{\nabla}^a(C_2) - 4(C_3)^a{}_b\tilde{\nabla}^b\Theta\Big] - \tfrac{1}{4}\big(2-9p'\big)\omega^a\tilde{\nabla}_b(C_1)^b \\
&+\tfrac{1}{2}(C_2)\Big[\big(3\phi-2p'\big)\dot{u}^a\Theta + \tfrac{1}{2}(C_1)^a + \big(1-3p'\big)\tilde{\nabla}^a\Theta\Big] \\
&+\big(1+p'\big)\Theta(C_6)^a{}_b\omega^b \qquad (5.14)
\end{aligned}
$$

In our usual fashion, we extract the non-vanishing terms from the right-hand side of equation (5.14) to obtain the following equation $(C_{11})^a$ below, which is the divergence of the curl of $(C_6)^{ab}$, where the seventh term on the right-hand side of constraint equation (5.15) below, is of interest since it contains the p''' term in (ψ).

$$\begin{aligned}0 = (C_{11})^a := {} & -\tfrac{1}{6}(2-9\phi)\Theta\dot{u}^b\tilde{\nabla}_b\omega^a - \tfrac{1}{2}(2-\phi)\Theta\omega^b\tilde{\nabla}_b\dot{u}^a \\ & + \tfrac{1}{6}(1-9p')\tilde{\nabla}^b\Theta\tilde{\nabla}_b\omega^a - \tfrac{1}{6}(5+3p')\omega^b\tilde{\nabla}_b\tilde{\nabla}^a\Theta \\ & - \tfrac{1}{2}(1+p'-\phi)\dot{u}^b\omega_b\tilde{\nabla}^a\Theta - \tfrac{1}{9}(1+3\phi)\Theta\epsilon^a{}_{bc}\dot{u}^b\tilde{\nabla}^c\Theta \\ & + \Big[\tfrac{1}{6}(2+9\phi)\Theta\mathcal{J} - \tfrac{1}{3}(1+3p')\Theta\mathcal{E} - \tfrac{1}{2}(6\phi+\psi)\Theta\dot{u}_b\dot{u}^b \\ & + \tfrac{1}{6}\big(19-18p'-81p'^2\big)\Theta\omega^2 - \tfrac{1}{3}(5-9p'-9\phi)\dot{u}^b\tilde{\nabla}_b\Theta\Big]\omega^a \\ & - \Big[\tfrac{1}{18}(8-36\phi+3\psi)\Theta\dot{u}^b\omega_b + \tfrac{1}{6}(2+12p'-3\phi)\omega^b\tilde{\nabla}_b\Theta\Big]\dot{u}^a. \end{aligned} \tag{5.15}$$

To compare equation (5.7) , (5.11) and (5.15), we need to contract constraint equation $(C_7)^{ab}$ in equation (5.7) with ω_b. This leads to the constraint equation (5.16) below, and the fifth term on the right-hand side is the term of interest.

$$\begin{aligned}(C_7)_b{}^a\omega^b = {} & \Big[\tfrac{1}{9}\big(2-9\phi+3\psi\big)\Theta\dot{u}^b\omega_b + \big(\tfrac{1}{3}+p'-\phi\big)\omega^b\tilde{\nabla}_b\Theta\Big]\dot{u}^a \\ & - \Big[\tfrac{1}{9}(2-3\phi)\Theta\mathcal{J} - \tfrac{1}{27}(4-3\psi)\Theta\dot{u}_b\dot{u}^b + \big(\tfrac{7}{9}-2p'+3p'^2\big)\Theta\omega^2 \\ & - \tfrac{2}{3}\phi\dot{u}^b\tilde{\nabla}_b\Theta\Big]\omega^a + \big(\tfrac{1}{3}+p'-\phi\big)\dot{u}^b\omega_b\tilde{\nabla}^a\Theta + \big(\tfrac{2}{3}-\phi\big)\Theta\omega^b\tilde{\nabla}_b\dot{u}^a \\ & + \tfrac{1}{3}(1+3p')\Big[\omega^b\tilde{\nabla}_b\tilde{\nabla}^a\Theta - (C_1)^b\dot{u}_b + \tfrac{1}{2}\tilde{\nabla}_b(C_1)^b\Big]\omega^a. \end{aligned} \tag{5.16}$$

Note that, at this point, we have what we need to eliminate the double spatial gradients of the expansion scalar from constraint $(C_7)^{ab}$ to retain terms only involving the spatial gradients of the acceleration vector ($\dot{u}^a$), the expansion (Θ), the vorticity (ω^a) and its spatial gradient. The terms on the right-hand side of equation (5.16) above, which cannot be written in terms of our original constraints, are extracted to form our new constraint equation $(C_{12})^a$ below,

$$\begin{aligned} 0 = (C_{12})^a := & \Big[\tfrac{1}{9}\big(2 - 9\phi + 3\psi\big)\Theta\dot{u}^b\omega_b + \big(\tfrac{1}{3} + p' - \phi\big)\omega^b\tilde{\nabla}_b\Theta\Big]\dot{u}^a \\ & - \Big[\tfrac{1}{9}(2 - 3\phi)\Theta\mathcal{J} - \tfrac{1}{27}(4 - 3\psi)\Theta\dot{u}_b\dot{u}^b + \big(\tfrac{7}{9} - 2p' + 3p'^2\big)\Theta\omega^2 \\ & - \tfrac{2}{3}\phi\dot{u}^b\tilde{\nabla}_b\Theta\Big]\omega^a + \big(\tfrac{1}{3} + p' - \phi\big)\dot{u}^b\omega_b\tilde{\nabla}^a\Theta + \big(\tfrac{2}{3} - \phi\big)\Theta\omega^b\tilde{\nabla}_b\dot{u}^a \\ & + \big(\tfrac{1}{3} + p'\big)\omega^b\tilde{\nabla}_b\tilde{\nabla}^a\Theta. \end{aligned} \tag{5.17}$$

Combining equation (5.17), (5.15) and (5.11), yields the following constraint equation (5.18) below. This is a much simpler constraint equation, since the double spatial gradients do not have coefficients involving the pressure derivative with respect to the energy density.

$$\begin{aligned} 0 = (C_{13})^a := & \tfrac{1}{2}\omega^a\big[\tfrac{1}{3}(2 + 15\phi)\Theta\mathcal{J} - (1 + 3p')\Theta\mathcal{E} + \tfrac{1}{9}(2 - 81\phi - 15\psi)\Theta\dot{u}^2 \\ & + \tfrac{1}{3}\big(25 - 18p' - 135p'^2\big)\Theta\omega^2 - (5 - 9p' - 10\phi)\dot{u}^b\tilde{\nabla}_b\Theta\big] \\ & - \tfrac{1}{4}\dot{u}^a\big[(2 - 9\phi)\Theta\dot{u}^b\omega_b + (1 + 9p')\omega^b\tilde{\nabla}_b\Theta\big] - \omega^b\tilde{\nabla}_b\tilde{\nabla}^a\Theta \\ & - \tfrac{1}{2}\dot{u}^b\omega_b\tilde{\nabla}^a\Theta - \Theta\omega^b\tilde{\nabla}_b\dot{u}^a - \tfrac{1}{4}(2 - 9\phi)\Theta\dot{u}^b\tilde{\nabla}_b\omega^a \\ & + \tfrac{1}{4}(1 - 9p')\tilde{\nabla}^b\Theta\tilde{\nabla}_b\omega^a - \tfrac{1}{6}(1 + 3\phi)\Theta\epsilon^a{}_{bc}\dot{u}^b\tilde{\nabla}^c\Theta. \end{aligned} \tag{5.18}$$

We have shown that whenever $\{\sigma_{ab} = 0, T_{ab} = \rho u_a u_b + p h_{ab}\}$ the following constraints must be satisfied; $(C_1)^a \to (C_5)^a$ and $(C_6)^{ab}$. The time propagation of these constraints is identically zero if $(C_7)^{ab}$ holds true, otherwise the next level consistency relations $(C_8)^a \to (C_{13})^a$ then follow. We still have to compute the time propagation of constraint $(C_{13})^a$ given by equation (5.18) to establish the consistency of the EFE under the shear-free perfect fluid condition.

``If only it weren't so damnably difficult to find

exact solutions!''

-Albert Einstein (letter to M. Born, c. 1936)

6

The case where the vorticity and acceleration vector fields are parallel

6.1 THE THEOREM

In this chapter, we impose a simplifying assumption to prove the conjecture in the case where the acceleration and vorticity vector fields are parallel; see the original proof by Collins and White in reference [52] given using the orthonormal tetrad formalism.

This scenario can be stated in the following theorem, provided in its covariant form by Senovilla et al. [13]. Here we included more details of the proof for completeness and clarity. This theorem states that,

Theorem 2. *If a rotating and expanding shear-free perfect fluid obeys a barotropic equation of state and* $\dot{u}^a = \psi\omega^a$*, then* $\Theta\omega = 0$*.*

6.2 THE PROOF

Proof. Let us consider the case where the acceleration vector field is proportional to the vorticity vector field with a proportionality space-time scalar field ψ given as follows;

$$\dot{u}^a = \psi\omega^a. \tag{6.1}$$

The symbol ψ in equation (6.1) is not the same as the one from the previous chapter, where the choice of ψ is so that we mimic the calculation done in reference [13], with the condition that $\psi \neq 0$. Henceforth, equation (9.26b) and (9.26e) can be rewritten as two evolution equations for the acceleration vector field in the following way;

$$\ddot{u}^{\langle a\rangle} = -\tfrac{3}{2}p'(C_1)^a - \left(\tfrac{1}{3} + \tfrac{\mathcal{E}p''}{p'} - p' - \tfrac{3p'^2}{\psi^2}\right)\Theta\dot{u}^a + \frac{3p'}{2\psi}\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\psi, \tag{6.2}$$

$$\ddot{u}^{\langle a\rangle} = \left[\tfrac{\dot{\psi}}{\psi} - \left(\tfrac{2}{3} - p'\right)\Theta\right]\dot{u}^a. \tag{6.3}$$

The comparison of equations (5.3) and (6.3) above yields the following information about the space-time scalar ψ shown below;

$$\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\psi = 0, \tag{6.4}$$

$$\dot{\psi} = \Big(\tfrac{1}{3} - \tfrac{\mathcal{E}p''}{p'} + \tfrac{3p'^2}{\psi^2}\Big)\Theta\psi. \tag{6.5}$$

The constraint equation (6.4) indicates that the spatial gradient of ψ and the vorticity vector field are parallel, and equation (6.5) is the evolution equation for ψ. By applying equation (6.1) and (6.4) to constraint $(C_1)^a$ given by equation (3.5a), we obtain the following simple-looking equation for the spatial gradient of the expansion scalar Θ, which is parallel to the vorticity vector field as well.

$$\tilde{\nabla}^a\Theta = -\tfrac{3}{2}(C_1)^a + \tfrac{3p'}{\psi}\Theta\omega^a. \tag{6.6}$$

By substituting for $\tilde{\nabla}^a\Theta$ which is given by equation (6.6) above, into the integrability identity (9.30), leads to the following evolution equation for the expansion scalar Θ,

$$\dot{\Theta} = \tfrac{3p'^2}{\psi^2}\Theta^2 + \tfrac{3}{4\omega^2}\epsilon_{abd}\omega^a\tilde{\nabla}^d(C_1)^b. \tag{6.7}$$

Upon taking the spatial divergence of constraint $(C_1)^a$ given by equation (3.5a), and using the following relations; (9.36), (9.34) and (9.33), we arrive at the constraint equation below;

$$0 = \tilde{\nabla}_c(C_1)^c := -\tfrac{2}{3}\tilde{\nabla}^2\Theta - \big(\tfrac{2}{3} - 6p'\big)\Theta\omega^2 - \tfrac{4}{3}\dot{u}^c\tilde{\nabla}_c\Theta - 2(C_1)^c\dot{u}_c. \tag{6.8}$$

When substituting for equation (6.6), (6.1) and (3.5b) into equation (6.8) above, we obtain the following constraint equation;

$$0 = -\tfrac{2p'}{\psi}\Theta(C_2) + \tfrac{3p'}{\psi}(C_1)^c\omega_c - 2\Big(\tfrac{1}{3} - \tfrac{\mathcal{E}p''}{p'} + \tfrac{3p'^2}{\psi^2}\Big)\Theta\omega^2 + \tfrac{2p'}{\psi^2}\Theta\omega^c\tilde{\nabla}_c\psi. \tag{6.9}$$

From equation (6.9) above, we can write down the following constraint equation (6.10) resulting from the terms on the right-hand side which cannot be expressed in terms of our original constraints,

$$0 = (C_{14}) := \Theta\Big[\tfrac{p'}{\psi^2}\omega^c\tilde{\nabla}_c\psi - \Big(\tfrac{1}{3} - \tfrac{\mathcal{E}p''}{p'} + \tfrac{3p'^2}{\psi^2}\Big)\omega^2\Big]. \tag{6.10}$$

From the constraint equation (6.10) above, it is clear that either $\Theta = 0$ and we are done with the proof, or the terms within the square parentheses must vanish. This will result in the following constraint equation;

$$0 = (C_{15}) := \tfrac{p'}{\psi^2}\omega^c\tilde{\nabla}_c\psi - \Big(\tfrac{1}{3} - \tfrac{\mathcal{E}p''}{p'} + \tfrac{3p'^2}{\psi^2}\Big)\omega^2. \tag{6.11}$$

Another expression for $\omega^c\tilde{\nabla}_c\psi$ can be found by substituting for $\dot{\Theta}$ from equation (6.7) into the Raychaudhuri equation (3.4a) and applying the condition (6.1) to obtain the following,

$$\begin{aligned} 0 = \Lambda - \tfrac{3}{2}\mathcal{E} + \mu - \Big(\tfrac{1}{3} + \tfrac{3p'^2}{\psi^2}\Big)\Theta^2 + 2(1+\psi^2)\omega^2 + \omega^a\tilde{\nabla}_a\psi + (C_2)\psi \\ - \tfrac{3}{4\omega^2}\epsilon_{abc}\omega^a\tilde{\nabla}^c(C_1)^b. \end{aligned} \tag{6.12}$$

From the terms on the right-hand side of equation (6.12) which cannot be expressed in terms of our original constraints, we can extract the constraint equation which involves the expression $\omega^c\tilde{\nabla}_c\psi$ shown below;

$$0 = (C_{16}) := \Lambda - \tfrac{3}{2}\mathcal{E} + \mu - \Big(\tfrac{1}{3} + \tfrac{3p'^2}{\psi^2}\Big)\Theta^2 + 2(1+\psi^2)\omega^2 + \omega^a\tilde{\nabla}_a\psi. \tag{6.13}$$

Comparing equation (6.11) and (6.13), we can eliminate $\omega^a \tilde{\nabla}_a \psi$. The resulting terms form the following constraint equation;

$$
\begin{aligned}
0 = (C_{17}) := &-\Big[3p'^2\big(2+3p'\big)\psi^2 - \Big(3\mathcal{E}p'' - p' - 6p'^2\Big)\psi^4\Big]\omega^2 \\
&+ p'^2\Big(9p'^2 + \psi^2\Big)\Theta^2 - \tfrac{3}{2}p'^2\psi^2\big(2\Lambda - 3\mathcal{E} + 2\mu\big). \qquad (6.14)
\end{aligned}
$$

Operating on equation (6.14) with the operator $\epsilon_{abc}\omega^b\tilde{\nabla}^c$ in order to eliminate $\epsilon_{abc}\omega^b\tilde{\nabla}^c\psi$ and $\tilde{\nabla}_a\Theta$ according to equation (6.4) and (6.6) respectively, yields the following constraint equation;

$$
\begin{aligned}
\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c(C_{17}) = 2\psi^2\Bigg[&\Big\{3p'^2\big(2+3p'\big) - \big(3\mathcal{E}p'' - p' - 6p'^2\big)\psi^2\Big\}\omega\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\omega \\
&- \tfrac{3}{2}p'^2\Big(1 + \tfrac{9p'^2}{\psi^2}\Big)\Theta\epsilon^a{}_{bc}(C_1)^b\omega^c\Bigg]. \qquad (6.15)
\end{aligned}
$$

We proceed similarly as before to extract the non-zero terms on the right-hand side of equation (6.15) to form the constraint equation shown below;

$$
0 = (C_{18})^a := \Big[3p'^2\big(2+3p'\big) - \big(3\mathcal{E}p'' - p' - 6p'^2\big)\psi^2\Big]\omega\epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\omega. \qquad (6.16)
$$

Equation (6.16) presents us with three options for consistency, either $\omega = 0$ and the proof is finished, or the terms in the square brackets must vanish (i), or the spatial gradient of the vorticity scalar is parallel to the vorticity vector field (ii). We shall discuss these cases separately as follows;

(i) Letting the terms within the square parentheses in equation (6.16) go to zero leads us to the constraint equation below;

$$
0 = (C_{19}) := 3p'^2\big(2+3p'\big) - \big(3\mathcal{E}p'' - p' - 6p'^2\big)\psi^2. \qquad (6.17)
$$

Taking equation (6.17) into account, equation (6.14) becomes the following constraint equation;

$$0 = (C_{20}) := \Lambda - \tfrac{3}{2}\mathcal{E} + \mu - \Big(\tfrac{1}{3} + \tfrac{3p'^2}{\psi^2}\Big)\Theta^2. \tag{6.18}$$

The time propagation of equation (6.18) results in the following instructive consistency constraint equation;

$$(\dot{C}_{20}) = \tfrac{1}{2}\big(1 + 3p'\big)\Theta\mathcal{E} - \tfrac{1}{2\omega^2}\Big(1 + \tfrac{9p'^2}{\psi^2}\Big)\Theta\epsilon_{abc}\omega^a\tilde{\nabla}^c(C_1)^b. \tag{6.19}$$

The first term on the right-hand side of equation (6.19) becomes our new consistency constraint equation as shown below;

$$0 = (C_{21}) := \big(1 + 3p'\big)\Theta\mathcal{E}, \tag{6.20}$$

where either $\Theta = 0$ and the proof is done, or $p' = -\frac{1}{3}$ which by equation (6.14) implies that $\psi^2 + 1 = 0$, which is not allowed since we are restricted to real number solutions. Therefore we must have $\Theta\omega = 0$ as our consistency condition, which completes the proof.

(ii) Or we have the following;

$$0 = (C_{22})^a := \epsilon^a{}_{bc}\omega^b\tilde{\nabla}^c\omega. \tag{6.21}$$

In this case, we need to go back and time propagate equation (6.14) which gives us the constraint equation below;

$$\begin{aligned}
(\dot{C}_{17}) = \Theta\Big\{ & \mathcal{E}\Big[9p'^3(4 + 3p') + \tfrac{9}{2}p'^2(1 + 9p')\psi^2\Big] - \big(\Lambda + \mu\big)\Big[6p'^3(4 + 3p') \\
& + 2p'^2(1 + 12p')\psi^2\Big] + \Big[4p'^3(2 + 3p' + 18p'^2) + \tfrac{18p'^5}{\psi^2}(4 + 3p') \\
& + \tfrac{2}{3}p'^2(1 + 12p')\psi^2\Big]\Theta^2 - \Big[27p'^4(2 + 3p') + 3p'^3(7 + 15p')\psi^2 \\
& + \tfrac{1}{3}p'(1 + 6p')(-2 + 9p')\psi^4\Big]\omega^2 + \tfrac{4}{3p'}\mathcal{Z}^2\omega^2 + 3\mathcal{E}^2p^{(3)}\psi^4\omega^2
\end{aligned}$$

$$
\begin{aligned}
&- \mathcal{Z}\Big[4p'\big(\Lambda - \tfrac{3}{2}\mathcal{E}p' + \mu\big) - 4p'\Big(\tfrac{1}{3} + \tfrac{3p'^2}{\psi^2}\Big)\Theta^2 - \Big(p'(8-3p') \\
&+ (2+3p')\psi^2\Big)\omega^2\Big] - \tfrac{3p'^2}{2\omega^2}\big(9p'^2 + \psi^2\big)\epsilon_{abc}\omega^a\tilde{\nabla}^c(C_1)^b\bigg\}.
\end{aligned}
\tag{6.22}
$$

In equation (6.22), we have defined $\mathcal{Z}$ as a linear combination (6.23) below since it repeatedly appears in calculations to follow.

$$
\mathcal{Z} = 3p'^2\big(2+3p'\big) - \big(3\mathcal{E}p'' - p' - 6p'^2\big)\psi^2. \tag{6.23}
$$

From the terms on the right-hand side of equation (6.22) which cannot be written in terms of our original constraint, including those we have constructed, we have that either $\Theta = 0$ and the proof is done or the non-vanishing terms within the curly brackets must vanish. This gives us the following constraint;

$$
\begin{aligned}
0 = (C_{23}) := {}& 9\mathcal{E}\Big[p'^3(4+3p') + \tfrac{1}{2}p'^2(1+9p')\psi^2\Big] - 2\big(\Lambda+\mu\big)\Big[3p'^3(4+3p') \\
&+ p'^2(1+12p')\psi^2\Big] - \mathcal{Z}\bigg\{4p'\Big[\big(\Lambda - \tfrac{3}{2}\mathcal{E}p' + \mu\big) - \Big(\tfrac{1}{3} + \tfrac{3p'^2}{\psi^2}\Big)\Theta^2\Big] \\
&- \Big[p'(8-3p') + (2+3p')\psi^2\Big]\omega^2\bigg\} + 2\Big[2p'^3(2+3p'+18p'^2) \\
&+ \tfrac{9p'^5}{\psi^2}(4+3p') + \tfrac{1}{3}p'^2(1+12p')\psi^2\Big]\Theta^2 - \Big[27p'^4(2+3p') \\
&+ 3p'^3(7+15p')\psi^2 + \tfrac{1}{3}p'(1+6p')(-2+9p')\psi^4\Big]\omega^2 \\
&+ \tfrac{4}{3p'}\mathcal{Z}^2\omega^2 + 3\mathcal{E}^2p^{(3)}\psi^4\omega^2.
\end{aligned}
\tag{6.24}
$$

Furthermore, when applying the operator $\omega^a\tilde{\nabla}_a$ in equation (6.14), and using equation (6.11), (6.1) and (6.6), we obtain the following constraint equation;

$$
\begin{aligned}
\omega^a\tilde{\nabla}_a(C_{17}) = \tfrac{\psi}{p'}\omega^2\bigg\{& 9p'^2\mathcal{E}\Big[p'(4+3p') + \tfrac{1}{2}(1+9p')\psi^2\Big] - 2p'^2\big(\Lambda+\mu\big)\Big[3p'(4 \\
&+ 3p') + (1+12p')\psi^2\Big] + 2p'^2\Big[2p'(2+3p'+18p'^2) + \tfrac{9p'^3}{\psi^2}(4
\end{aligned}
$$

$$\begin{aligned}
&+3p')+\tfrac{1}{3}(1+12p')\psi^2\Big]\Theta^2-p'\Big[27p'^3(2+3p')+3p'^2(7+\\
&15p')\psi^2+\tfrac{1}{3}(1+6p')(-2+9p')\psi^4\Big]\omega^2-\mathcal{Z}\Big[\tfrac{2p'\psi}{\omega}\omega^a\tilde{\nabla}_a\omega-\\
&4p'\Big(\tfrac{1}{3}+\tfrac{3p'^2}{\psi^2}\Big)\Theta^2+4p'\big(\Lambda-\tfrac{3}{2}\mathcal{E}p'+\mu\big)-\Big(p'(8-3p')+\\
&\tfrac{1}{3}(2+15p')\psi^2\Big)\omega^2\Big]-(C_{15})\psi^2\Big[6p'^2\big(2+3p'\big)+4\mathcal{Z}-\\
&\tfrac{p'^2}{\omega^2}\big(6\Lambda-9\mathcal{E}-2\Theta^2+6\mu\big)\Big]+\tfrac{3p'^3}{\omega^2}\Big(\tfrac{9p'^2}{\psi}+\psi\Big)\Theta(C_1)^a\omega_a\\
&+\tfrac{4}{3p'}\mathcal{Z}^2\omega^2+3\mathcal{E}^2p^{(3)}\psi^4\omega^2\Big\}.
\end{aligned}\tag{6.25}$$

From the constraint equation (6.25) above, one can see that either ω vanishes and we have the proof, or the non-zero terms within the curly brackets must vanish. This results in the constraint equation shown below;

$$\begin{aligned}
0=(C_{24}):=\tfrac{\psi}{p'}\omega^2\Big\{&9p'^2\mathcal{E}\Big[p'(4+3p')+\tfrac{1}{2}(1+9p')\psi^2\Big]-2p'^2\big(\Lambda+\mu\big)\Big[3p'(4\\
&+3p')+(1+12p')\psi^2\Big]+2p'^2\Big[2p'(2+3p'+18p'^2)+\tfrac{9p'^3}{\psi^2}(4\\
&+3p')+\tfrac{1}{3}(1+12p')\psi^2\Big]\Theta^2-p'\Big[27p'^3(2+3p')+3p'^2(7+\\
&15p')\psi^2+\tfrac{1}{3}(1+6p')(-2+9p')\psi^4\Big]\omega^2-\mathcal{Z}\Big[\tfrac{2p'\psi}{\omega}\omega^a\tilde{\nabla}_a\omega-\\
&4p'\Big(\tfrac{1}{3}+\tfrac{3p'^2}{\psi^2}\Big)\Theta^2+4p'\big(\Lambda-\tfrac{3}{2}\mathcal{E}p'+\mu\big)-\Big(p'(8-3p')+\\
&\tfrac{1}{3}(2+15p')\psi^2\Big)\omega^2\Big]+\tfrac{4}{3p'}\mathcal{Z}^2\omega^2+3\mathcal{E}^2p^{(3)}\psi^4\omega^2\Big\}.
\end{aligned}\tag{6.26}$$

Equations (6.24) and (6.26) when compared, yield a straightforward constraint equation shown below;

$$0=(C_{25}):=\mathcal{Z}\Big[\big(\tfrac{2}{3}-p'\big)\omega^3+\tfrac{p'}{\psi}\omega^a\tilde{\nabla}_a\omega\Big].\tag{6.27}$$

Since $\mathcal{Z}$ is equal to constraint (6.17) and thus $\mathcal{Z}$ cannot be zero, in this case, only the terms within the square parentheses in equation (6.27) must be zero; this leads us to the following constraint equation,

$$0 = (C_{26}) := \big(\tfrac{2}{3} - p'\big)\omega^3 + \tfrac{p'}{\psi}\omega^a\tilde{\nabla}_a\omega. \tag{6.28}$$

At this point, equation (6.21) and (6.28) become our new constraints which require our attention. As a result, the time propagating equation (6.21) above gives us the following constraint equation;

$$\begin{aligned}(\dot{C_{22}})^{\langle a\rangle} &= \big(2p' - \tfrac{5}{3}\big)\Theta(C_{22})^a + \tfrac{\psi}{p'}(C_{26})\omega^a + \big(\tfrac{3}{2}p' - 1\big)\omega\epsilon^a{}_{bc}(C_1)^b\omega^c \\ &\quad + \big(1 - \tfrac{2}{3p'}\big)\psi\omega^3\omega^a - \omega^2\tilde{\nabla}^a\omega. \end{aligned} \tag{6.29}$$

The terms on the right-hand side of equation (6.29) which cannot be written in terms of the other constraints, give us the new constraint equation below;

$$0 = (C_{27})^a := \omega^2\Big[\big(1 - \tfrac{2}{3p'}\big)\psi\omega\omega^a - \tilde{\nabla}^a\omega\Big]. \tag{6.30}$$

From equation (6.30), we can see that either the vorticity scalar ω vanishes and we are done or that the terms within the square brackets must vanish, giving us the following constraint equation;

$$0 = (C_{28})^a := \big(1 - \tfrac{2}{3p'}\big)\psi\omega\omega^a - \tilde{\nabla}^a\omega. \tag{6.31}$$

Equation (6.31) above shows that the spatial gradient of the vorticity is parallel to the vorticity vector field. The time propagation of the constraint equation (6.31) above vanishes as shown by the equation (6.32) below;

$$(\dot{C_{28}})^{\langle a\rangle} = \big(p' - 1\big)\Theta(C_{28})^a - \epsilon^a{}_{bc}(C_{28})^b\omega^c + \big(\tfrac{3}{2}p' - 1\big)\omega(C_1)^a. \tag{6.32}$$

Equation (6.31) is closed under time propagation and does not give us any new information. Similarly, equation (6.28) also closes under time-propagation and does not give us any new information either, as seen from equation (6.33) below;

$$(\dot{C}_{26}) = \Big(-2 + 2p' - \tfrac{3p'^2}{\psi^2}\Big)\Theta(C_{26}) + \tfrac{p'}{\psi}\Big(1 - \tfrac{3p'}{2}\Big)\omega(C_1)^a\omega_a. \tag{6.33}$$

At this point, it seems we cannot proceed further. However, we can consider one more integrability condition. This is the Ricci identities for the vorticity given by equation(9.33), together with all the relations we have obtained up to this point. This leads to the following constraint equation;

$$\begin{aligned} 0 = {} & (C_{28})_a(C_{28})^a - \tfrac{2}{p'}\left(1 + \tfrac{1}{3p'}\right)\psi^2\omega(C_{26}) + \tfrac{2}{3}\left(2 - \tfrac{1}{p'}\right)\psi\omega^2(C_2) \\ & - \frac{2}{3p'}\left(1 + \tfrac{1}{p'}\right)\psi^2\omega^2(C_{15}) - \left(\tfrac{2}{3}(C_{18}) + \mathcal{E} - \tfrac{2p'^2}{\psi^2}\Theta^2\right)\omega^2 \\ & - \left[\tfrac{2}{9p'^2}(C_{17}) - \left(1 + \tfrac{2}{3p'^2} - \tfrac{4}{3p'}\right)\psi^2\right]\omega^4 + (C_6)_{ab}\omega^a\omega^b \\ & - 2\left(1 - \tfrac{2}{3p'}\right)\psi\omega(C_{28})^a\omega_a - \omega^a\tilde{\nabla}_a(C_2) - \omega\tilde{\nabla}_a(C_{28})^a \\ & - \tilde{\nabla}_b\omega_a\tilde{\nabla}^b\omega^a. \end{aligned} \tag{6.34}$$

Extracting the non-zero terms from equation (6.34) above, we are left with the new constraint equation shown below;

$$0 = (C_{29}) := \Big(-\mathcal{E} + \tfrac{2p'^2}{\psi^2}\Theta^2\Big)\omega^2 + \Big(1 + \tfrac{2}{3p'^2} - \tfrac{4}{3p'}\Big)\psi^2\omega^4 - \tilde{\nabla}_b\omega_a\tilde{\nabla}^b\omega^a. \tag{6.35}$$

Equation (6.35) is equivalent to equation (79) in [13], if we defined a projection tensor Q_{ab}, which is orthogonal to both u^a and ω^a as follows;

$$Q_{ab} = h_{ab} - \hat{\omega}_a\hat{\omega}_b, \tag{6.36}$$

where we have defined $\hat{\omega}_a$ to be a unit vector in the direction of the vorticity, which has the property that $\dot{\omega}^{\langle a\rangle}=0$. Finally, the time propagation of equation (6.35) leads us to the following constraint equation below;

$$
\begin{aligned}
(\dot{C}_{29}) = {} & \mathcal{E}\big(\tfrac{1}{3}+p'\big)\Theta\omega^2 - 2\big(1-p'\big)\Theta(C_{29}) + \big(2-3p'\big)\omega(C_1)^a(C_{28})_a \\
& + \Big((C_2) - \big(3-\tfrac{4}{3p'}-3p'\big)\psi\omega^2\Big)(C_1)^a\omega_a - (C_1)^a\omega^b\tilde{\nabla}_b\omega_a \\
& - 2\Big(\tfrac{5p'}{\psi}+\psi\big(\tfrac{4}{3}+p'\big)+\tfrac{\mathcal{Z}}{3p'\psi}\Big)\Theta\omega(C_{28})^a\omega_a \\
& - \tfrac{2}{3}(C_2)\epsilon_{abc}\omega^a\tilde{\nabla}^c\omega^b + \tfrac{3p'^2}{\psi^2}\Theta\epsilon_{abc}\omega^a\tilde{\nabla}^c(C_1)^b \\
& - 2\Big(\tfrac{p'}{\psi}+\tfrac{1}{3}\psi\Big)\Theta\omega^2(C_2).
\end{aligned}
\tag{6.37}
$$

From equation (6.37) above, it is clear that the proof is done since we can extract our new constraint from the non-zero terms on the right-hand side, giving us the following constraint equation;

$$
0 = (C_{30}) := \mathcal{E}\big(\tfrac{1}{3}+p'\big)\Theta\omega^2. \tag{6.38}
$$

We have shown in case (i) above that the constraint equation (6.38) leads to the required condition for the theorem to be true; hence we have that $\Theta\omega^2=0$. This finishes the proof. The discussion of this section follows that of section (4) in reference [13]. But, we have provided a neat proof and more details than in [13]. The case where the acceleration and vorticity vector fields are orthogonal will be addressed in the next chapter. This leads to the situation where a Killing vector along the vorticity exists.

□

In my opinion, the theory presented here is the logically simplest relativistic field theory which is at all possible. But this does not mean that nature might not obey a more complex field theory.

-Albert Einstein (The Meaning of Relativity)

7

The case where the vorticity and acceleration vector fields are not parallel

In their work, see reference [27], Van den Bergh and Slobodeanu provided two general proofs for the following theorems; the first states that *if for a shear-free perfect fluid, obeying a barotropic equation of state,* $\left(\dot{U}_1 = \frac{\dot{u}_1}{p'\lambda}\right)$ *and* $\left(\dot{U}_2 = \frac{\dot{u}_2}{p'\lambda}\right)$ *are basic, then* $(\omega\Theta = 0)$, where the word *basic* means that $\left(\partial_0(\dot{U}_1) = 0\right)$ and $\left(\partial_0(\dot{U}_2) = 0\right)$. The second theorem states that *if for a rotating and expanding shear-free perfect fluid,* *obeying a barotropic equation of state,* $\left(\dot{U}_3 = \frac{\dot{u}_3}{p'\lambda}\right)$ *is basic, then a Killing vector along the vorticity exists.* These proofs were produced using the orthonormal tetrad formalism mentioned in Chapter (1), where $\left(\lambda = \exp\int \frac{d\mu}{3\mathcal{E}}\right)$ and the tetrad fixing convention is such that the vorticity is aligned with (∂_3), see reference [27] for details. Henceforth, in this chapter, we shall use the following general definition for a *basic* tensorial object;

Definition: *A spatially projected tensorial object* ζ *on a manifold* $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ *is called* ***basic*** *if its spatially projected Lie derivative along* $(\mathbf{u})$ *vanishes [24]*, with this definition, we proceed to provide covariant versions of the theorems mentioned above, extending these results to the case where the acceleration is rescaled by a general function of the energy density $f(\mu)$.

7.1 THE CASE WHERE THE COMPONENTS OF THE (RESCALED) ACCELERATION ORTHOGONAL TO THE VORTICITY ARE *basic*

If a barotropic equation of state governs a rotating and expanding shear-free perfect fluid with energy condition that $(p + \mu \neq 0)$, then there exists a unit vorticity vector field defined as follows;

$$\hat{\omega}_a = \frac{\omega_a}{\omega}. \tag{7.1}$$

The unit vorticity vector defined by equation (7.1) has the property that $(\dot{\hat{\omega}}_{\langle a \rangle} = 0)$, and it can be used to construct a projection tensor $(Q^a{}_b)$ which projects vectors onto the space orthogonal to the vorticity and is expressed in the following way;

$$Q^a{}_b = h^a{}_b - \hat{\omega}^a \hat{\omega}_b. \tag{7.2}$$

The projection tensor $(Q^a{}_b)$ has the following properties; $(Q^a{}_a = 2)$, $(Q^a{}_b Q^b{}_c = Q^a{}_c)$, $(\omega^a{}_b Q^b{}_c = \omega^a{}_c)$, $(\omega_c{}^a \omega^c{}_b = \omega^2 Q^a{}_b)$, $(\dot{Q}_{\langle ab\rangle} = 0)$ and $(Q^a{}_b)$ is naturally orthogonal to the unit vorticity vector field $(\hat{\omega}^a)$. The tensor (ω_{ab}) is the rotation tensor defined as $(\omega_{ab} = \epsilon_{abc}\omega^c)$. Let us use $(Q^a{}_b)$ to provide proof of Theorem (3) below, which is a covariant version of Theorem (1) given in reference [27].

Theorem 3. *If for a shear-free perfect fluid obeying a barotropic equation of state* $(\dot{U}_Q)^a$ *or* $(\dot{U}_\omega)^a$ *is basic, then* $\omega\Theta = 0$.

Proof. Let the rescaled acceleration vector field be $\dot{U}^a = \dot{u}^a\left(\frac{1}{p'\lambda^2}\right)$, notice that the rescaling factor differ by a factor of $(\frac{1}{\lambda})$ from the one used in reference [27]. This is because, here, we are dealing with projected (by $h^a{}_b$) Lie derivatives of vectors instead of Lie derivatives of scalars, as in the orthonormal tetrad formalism. However, this does not affect the results of the proof, as we shall see. In addition, this allows us to write the evolution equation for $(\dot{U}^a)$ in the following way;

$$\ddot{U}^{\langle a\rangle} = \frac{F^a}{\lambda^2} + \frac{1}{3}\Theta\dot{U}^a - \omega^a{}_c\dot{U}^c, \tag{7.3}$$

where equation (3.7d) and (5.3) have been used and the vector field (F^a) is defined as $(F^a = \Theta\dot{u}^a + \tilde{\nabla}^a\Theta)$. Equation (7.3) gives us the projected Lie derivative of $(\dot{U}^a)$ along the velocity vector field (u^a) shown below;

$$h^a{}_b\mathcal{L}_u\dot{U}^b = \frac{F^a}{\lambda^2}. \tag{7.4}$$

Let $(\dot{U}_Q)^a = Q^a{}_b\dot{U}^b$ and $(\dot{U}_\omega)^a = \frac{1}{\omega}\omega^a{}_b\dot{U}^b$ be the component vectors of the rescaled acceleration vector field lying on the plane orthogonal to the vorticity. Therefore, equation (7.4) together with $(\dot{Q}_{\langle ab\rangle} = 0)$ and $\left(\dot{\omega}_{\langle ab\rangle} = \left(p' - \frac{2}{3}\right)\omega_{ab}\right)$ allow us to calculate and write down the projected Lie derivative of $(\dot{U}_Q)^a$ and $(\dot{U}_\omega)^a$ as follows;

$$h^a{}_b\mathcal{L}_u(\dot{U}_Q)^b = \frac{F^bQ^a{}_b}{\lambda^2}, \tag{7.5}$$

$$h^a{}_b\mathcal{L}_u(\dot{U}_\omega)^b = \frac{F^b\omega^a{}_b}{\lambda^2\omega}. \tag{7.6}$$

The premise of this proof is that $(\dot{U}_Q)^a$ or $(\dot{U}_\omega)^a$ are *basic*, which implies that the left hand side of equation (7.5) and (7.6) above must vanish, giving us the following constraint equations;

$$F^bQ^a{}_b = 0 = F^b\omega^a{}_b. \tag{7.7}$$

The time propagation equation for the vector field (F^a) can be achieved by applying equation (5.3) and (5.9) resulting in the following evolution equation;

$$\begin{aligned}\dot{F}^{\langle a\rangle} = {} & \left(p'\dot{\Theta} - \left(\mathcal{E}p'' + \frac{1}{3}p'(3p'-2)\right)\Theta^2\right)\lambda^2\dot{U}^a - \frac{9}{2}\mathcal{E}p''\Theta\omega\lambda^2(\dot{U}_\omega)^a \\ & + (2p'-1)\Theta F^a. \end{aligned} \tag{7.8}$$

Aided by the propagation equation (7.8) above, we can propagate equation (7.7) on both sides and upon applying equation (7.7) itself to eliminate vanishing terms, we arrive at the following constraint equation;

$$\left(\dot{\Theta} - \left(\frac{\mathcal{E}p''}{p'} + \frac{1}{3}(3p'-2)\right)\Theta^2\right)(\dot{U}_Q)^a = \frac{9\mathcal{E}p''}{2p'}\Theta\omega(\dot{U}_\omega)^a. \tag{7.9}$$

Assuming that $\Theta\omega \neq 0$, constraint equation (7.9) above is impossible since $(\dot{U}_Q)^a$ and $(\dot{U}_\omega)^a$ are orthogonal non-zero vector fields by definition, except if their scalar coefficients vanish. It follows that $p'' = 0$, which is equivalent to equation (49) given in reference [27]. This condition leads to the following propagation equation for the expansion scalar (Θ);

$$\dot{\Theta} = \left(p' - \frac{2}{3}\right)\Theta^2. \tag{7.10}$$

By applying the Raychaudhuri equation (3.4b) into equation (7.10), we can show that equation (7.10) is equivalent to equation (50) in reference [27]. Furthermore, we note that equation (5.9) can be written in a much simpler form as follows;

$$\begin{aligned} 0 = \Theta\left(\left(\frac{2}{3} - p'\right)F^a + \frac{9\mathcal{E}p''}{2p'}\dot{u}^b\omega^a{}_b\right) + \tilde{\nabla}^a\dot{\Theta} \\ - \dot{u}^a\left(\left(\frac{2}{3} - p'\right)\Theta^2 - \dot{\Theta}\right). \end{aligned} \tag{7.11}$$

Upon applying equation (7.10) together with the condition that ($p'' = 0$) in equation (7.11), we obtain the following instructive constraint equation;

$$0 = \left(\frac{2}{3} - p'\right)\Theta F^a. \tag{7.12}$$

Equation (7.12) above, implies that either ($F^a = 0$) or ($\frac{2}{3} - p' = 0$). The vector field (F^a) can be written alternatively as $\left(e^{-\gamma}\tilde{\nabla}^a\mathfrak{F}\right)$, where $\left(\gamma = \int\frac{dp}{\mathcal{E}}\right)$ and ($\mathfrak{F} = e^{-\gamma}\Theta$). The evolution equation for ($\mathfrak{F}$) can be obtained by applying equation (7.10) giving us the following equation;

$$\dot{\mathfrak{F}} = \frac{2}{3}e^{\gamma}\left(3p' - 1\right)\mathfrak{F}^2. \tag{7.13}$$

In the case where ($F^a = 0$), we have that ($\tilde{\nabla}^a\mathfrak{F}$) vanishes, implying that the covariant derivative of ($\mathfrak{F}$) is given by the equation ($\nabla^a\mathfrak{F} = -u^a\dot{\mathfrak{F}}$). This shows that the velocity vector field (u^a) is hypersurface orthogonal, hence the vorticity (ω^a) vanishes unless ($\dot{\mathfrak{F}} = 0$), whence by equation (7.13) we have that either ($\mathfrak{F} = e^{-\gamma}\Theta = 0$) or $\left(p' = \frac{1}{3}\right)$, see reference [8] for the proof, in both of these cases, the shear-free perfect fluid conjecture is valid. Thus, if $(\omega\Theta \neq 0)$ in equation (7.12), we only have the case where $\left(\frac{2}{3} - p' = 0\right)$ to investigate. Under this condition, equation (7.10) together with the Raychaudhuri equation (3.4b) gives us the following constraint equation;

$$0 = \dot{\Theta} = -\frac{1}{3}\Theta^2 + \mathcal{J} + 2\omega^2 - \frac{1}{2}(\mu + 3p) + \Lambda, \tag{7.14}$$

this equation can be rewritten for convenience in the following way;

$$\frac{1}{3}\Theta^2 - 2\omega^2 - \Lambda = \mathcal{J} - \frac{1}{2}(\mu + 3p). \tag{7.15}$$

In equation (7.15) above, we have written the left-hand side such that its time propagation vanishes, since $(\dot{\Theta} = 0 = \dot{\omega})$. Furthermore, the evolution equation for $(\mathcal{J})$ which appears on the right-hand side of equation (7.15), originally given by equation (9.26d) can be simplified to give us the following evolution equation;

$$\dot{\mathcal{J}} = \frac{5}{3}\left(F^a\dot{u}_a + 2\Theta\omega^2\right). \tag{7.16}$$

Time propagating both sides of equation (7.15) and using equation (7.16) gives us the following constraint equation;

$$-2\Theta\omega^2 = F^a\dot{u}_a + \frac{9}{10}\Theta(p + \mu). \tag{7.17}$$

In order to time propagate equation (7.17) above, we need to rewrite equation (7.8) and (5.3) into their simplified form as follows;

$$\dot{F}^{\langle a\rangle} = \frac{1}{3}\Theta F^a, \tag{7.18}$$

$$\ddot{u}^{\langle a\rangle} = \frac{2}{3}F^a - \frac{1}{3}\Theta\dot{u}^a - \dot{u}^c\omega^a{}_c. \tag{7.19}$$

Time propagating equation (7.17), and using equations (7.18) and (7.19), we obtain the following constraint equation;

$$0 = F_a F^a - \frac{9}{4}\Theta^2(p+\mu). \tag{7.20}$$

Upon time propagating equation (7.20) above, and applying equation (7.18) and using equation (7.20) itself to eliminate the variable $(F^a F_a)$, we arrive at the following constraint equation;

$$0 = \Theta^2(p+\mu). \tag{7.21}$$

Therefore, (Θ) must vanish since $(p+\mu \neq 0)$ and we have the proof. $\square$

The above theorem (3) inspired the following theorem (4), where we generalise the acceleration vector field rescaling function to $f(\mu)$.

Theorem 4. *If for a shear-free perfect fluid obeying a barotropic equation of state* $(\mathfrak{V}_1)_a = Q_{ab} f \dot{u}^b$ *or* $(\mathfrak{V}_2)_a = \epsilon_{abc}\hat{\omega}^c f \dot{u}^b$ *are basic, then* $\omega\Theta = 0$.

Proof. It is useful to note that $(\mathfrak{V}_1)^a(\mathfrak{V}_2)_a = 0$. This observation will become useful later in the proof. The requirements that vectors $(\mathfrak{V}_1)_a$ and $(\mathfrak{V}_2)_a$ are *basic*, implies that the following equations must be satisfied;

$$h_a{}^b \mathcal{L}_u (\mathfrak{V}_1)_b = 0, \tag{7.22}$$

$$h_a{}^b \mathcal{L}_u (\mathfrak{V}_2)_b = 0. \tag{7.23}$$

Equation (7.22) and (7.23) above can be shown to be equivalent to the following constraint equations by using equation (5.3) given in Chapter (5);

$$Q_{ab}\tilde{\nabla}^b \mathcal{S} = 0, \tag{7.24}$$

$$\epsilon_{abc}\hat{\omega}^b \tilde{\nabla}^c \mathcal{S} = 0. \tag{7.25}$$

The scalar ($\mathcal{S}$) appearing in equation (7.24) and (7.25) above is defined by the following equation;

$$e^{\mathcal{S}} = e^{-\gamma}\left(p' f \Theta\right), \tag{7.26}$$

such that the time and space derivatives of ($\mathcal{S}$) are given as follows;

$$\dot{\mathcal{S}} = -\mathfrak{S}\Theta + \frac{\dot{\Theta}}{\Theta}, \tag{7.27}$$

$$\tilde{\nabla}^a \mathcal{S} = -\mathfrak{S}\frac{\dot{u}^a}{p'} + \frac{\tilde{\nabla}^a \Theta}{\Theta}. \tag{7.28}$$

In equation (7.27) and (7.28) above the scalar function ($\mathfrak{S}$) is given by equation (9.2) in Appendix (9), these equations can be used to obtain the time-space derivative commutator of ($\mathcal{S}$) shown below;

$$h^a{}_b\big(\tilde{\nabla}^b \mathcal{S}\big)^{\cdot} = (g_1)\dot{u}^a + (g_2)\omega(\mathfrak{V}_2)^a + (g_3)\tilde{\nabla}^a \mathcal{S}. \tag{7.29}$$

The scalar coefficients (g_1), (g_2) and (g_3) appearing in equation (7.29) above are given by equation (9.6), (9.7) and (9.8) in Appendix (9) respectively. To obtain equation (7.29) above, we take the time propagation of the constraint equation (3.5a), which can be written as follows;

$$\tilde{\nabla}^a \dot{\Theta} = \Theta\left[(g_3 + g_4\Theta)\dot{u}^a + (g_2)\omega(\mathfrak{V}_2)^a + (g_5)\Theta\tilde{\nabla}^a \mathcal{S}\right]. \tag{7.30}$$

In equation (7.30) the scalars (g_4) and (g_5) are given by equation (9.9) and (9.10) in Appendix (9) respectively. Furthermore, equation (7.24) and (7.25) above can be rotated into each other by the rotation tensor ($\omega^a{}_b$), hence we shall time propagate only one of them to check for consistency. Therefore, taking the time propagation of constraint equation (7.24) and applying equation (7.29) gives the following constraint equation;

$$0 = (g_1)(\mathfrak{V}_1)_a + (g_2) f \omega (\mathfrak{V}_2)_a. \tag{7.31}$$

In equation (7.31) above, we note that since $(\mathfrak{V}_1)_a$ is orthogonal to $(\mathfrak{V}_2)_a$ then consistency requires that the scalar coefficients (g_1 and g_2) must vanish otherwise $(\mathfrak{V}_1)_a$ and $(\mathfrak{V}_2)_a$ vanish, which implies that the acceleration vector field is parallel to the vorticity completing the proof by Theorem (2) discussed in the previous Chapter (6). Therefore, the ($g_1 = 0$ and $g_2 = 0$) requirement leads to the following new evolution equation for (Θ) and the energy density constraint equation respectively;

$$0 = \mathfrak{H}\Theta^2 + \mathfrak{S}\dot{\Theta}, \tag{7.32}$$

$$0 = (2 - 9\phi)p' - \mathfrak{S}(1 - 9p'). \tag{7.33}$$

In equation (7.32) above, the scalar function of the energy density ($\mathfrak{H}$) is given by equation (9.5) in Appendix (9) and the time propagation of equation (7.33) above, gives the following constraint equation;

$$0 = \mathfrak{I}(2 - 9\phi)p' + (1 - 9p')\left(9p'^2\chi + \mathfrak{H}(1 - 9p')\right), \tag{7.34}$$

where the scalar ($\mathfrak{I}$) is given by equation (9.11) in Appendix (9). Furthermore, we can substitute for ($\mathfrak{S}$) and ($\mathfrak{H}$) from equation (7.33) and (7.34) respectively into equation (7.32) giving us the following evolution equation for (Θ);

$$(2 - 9\phi)(1 - 9p')\dot{\Theta} = \big(\mathfrak{I}(2 - 9\phi) + 9p'(1 - 9p')\chi\big)\Theta^2. \tag{7.35}$$

Therefore, substituting for ($\dot{\Theta}$) and ($\mathfrak{S}$) from equations (7.35) and (7.33) into equation (7.30) we arrive at the following constraint equation;

$$0 = (\mathfrak{A})\dot{u}^a + \frac{1}{3}(\mathfrak{D})(2 - 9\phi)(1 - 9p')\tilde{\nabla}^a\mathcal{S}. \tag{7.36}$$

The scalars $(\mathfrak{A})$ and $(\mathfrak{D})$ appearing in equation (7.36) above are given by equation (9.19) and (9.17) in Appendix (9). Upon projecting with $({Q_a}^b)$ and applying equation (7.24), equation (7.36) simplifies to the following constraint equation;

$$0 = (\mathfrak{A})(\mathfrak{V}_1)^a. \tag{7.37}$$

Equation (7.37) above presents us with the following cases; the first case is where $(\mathfrak{V}_1)^a = 0$, and we have the proof according to Theorem (2) appearing in the previous Chapter (6). The second case is where $(\mathfrak{A} = 0)$, which by equation (9.19) in Appendix (9) gives us the following constraint equation;

$$0 = \mathfrak{D}^2 + (2 - 9\phi)\left(\frac{2}{3}\mathfrak{D}\mathfrak{K} + \frac{1}{36}\mathfrak{L}(2 - 9\phi) + 324\alpha p'^2(1 - 9p')^2\right), \tag{7.38}$$

where the scalars $(\mathfrak{K})$, $(\mathfrak{L})$ and (α) are given in Appendix (9) by equation (9.14), (9.15) and (9.20) respectively. Furthermore, when $(\mathfrak{A} = 0)$ equation (7.36) reduces to the following constraint equation;

$$0 = (\mathfrak{D})(2 - 9\phi)(1 - 9p')\tilde{\nabla}^a\mathcal{S}. \tag{7.39}$$

Equation (7.39) above gives us four cases to consider; we shall begin with the case where $(\mathfrak{D} = 0)$, whence by equation (9.17) in Appendix (9), we have the following equation;

$$0 = \mathfrak{J}(2 - 9\phi) + 54p'(1 - 9p')\chi. \tag{7.40}$$

Time propagating equation (7.40) above and applying the evolution equation for $(\mathfrak{J})$ given by equation (9.27) in Appendix (9), and upon using equation (7.40) itself in order to eliminate the variable (χ) we arrive at the following constraint equation;

$$0=\left(\frac{1}{36}\mathfrak{L}(2-9\phi)+324\alpha p'^2(1-9p')^2+6p'(1+3\phi)(2-9\phi)(1-9p')^2\right)\Theta. \quad (7.41)$$

The scalar function of the energy density ($\mathfrak{L}$) appearing in equation (7.41) above is given by equation (9.16) in Appendix (9). In addition, setting ($\mathfrak{D}=0$) in equation (7.38) gives us the following constraint equation;

$$0=(2-9\phi)\left(\frac{1}{36}\mathfrak{L}(2-9\phi)+324\alpha p'^2(1-9p')^2\right), \quad (7.42)$$

and since we are looking at the case where $(9\phi-2)(9p'-1)\neq 0$ in equation (7.39), equation (7.42) implies that

$$0=\frac{1}{36}\mathfrak{L}(2-9\phi)+324\alpha p'^2(1-9p')^2. \quad (7.43)$$

Equation (7.43) above enable us to write equation (7.41) as follows;

$$0=(1+3\phi)(2-9\phi)(1-9p')^2\Theta. \quad (7.44)$$

In equation (7.44) above, we have that $(1+3\phi)$ must vanish since $(9\phi-2)(9p'-1)\neq 0$ in this case, implying that ($0=\dot{\phi}=p'\chi\Theta$), where either ($\chi$) vanishes or the conjecture holds. Under these conditions equation (7.40) implies that ($\mathfrak{J}=0$), where by equation (9.13) in Appendix (9) together with the condition that ($1+3\phi=0$), we obtain the following constraint equation;

$$0=(2-3p')(1+3p'). \quad (7.45)$$

Equation (7.45) above, gives us that either ($2-3p'=0$) and then (Θ) vanishes by the previous Theorem (3) or that ($1+3p'=0$), implying that ($\Theta\omega=0$), see the following references [11, 15, 17, 53] for further details. The second case in equation (7.39) is where $(2-9\phi)=0$, which by equation (7.33) implies that ($\mathfrak{S}=0$), as a result equation (7.25) and (7.28) give us the following constraint equation;

$$0 = \omega^a{}_b \tilde{\nabla}^b \Theta. \tag{7.46}$$

Upon taking the spatial divergence of (7.46) above, one obtains the following constraint equation;

$$0 = \dot{\Theta}\omega^2 - \frac{1}{3}\tilde{\nabla}_a \Theta \tilde{\nabla}^a \Theta. \tag{7.47}$$

Time propagating equation (7.47) above and using it to eliminate the $\left(\tilde{\nabla}_a \Theta \tilde{\nabla}^a \Theta\right)$ terms, we arrive at the second order evolution equation for the expansion scalar (Θ) shown below,

$$0 = \ddot{\Theta} + \frac{2}{3}\Theta\dot{\Theta}. \tag{7.48}$$

Furthermore, equation (5.9) can be rewritten in terms of $(\dot{\Theta})$ in the following way;

$$0 = \tilde{\nabla}^a \dot{\Theta} + \dot{u}^a \dot{\Theta} + \left(\frac{2}{3} - p'\right) \Theta \tilde{\nabla}^a \Theta. \tag{7.49}$$

Time propagating equation (7.49) further and projecting with $(Q^c{}_a)$ leads to the following constraint equation;

$$0 = (\mathfrak{V}_1)^a \left(\ddot{\Theta} - \frac{1}{9}\Theta\dot{\Theta}\right) - \frac{1}{2}(\mathfrak{V}_2)^a \dot{\Theta}\omega + \frac{1}{2} f Q^a{}_c \tilde{\nabla}^c \ddot{\Theta} - \frac{1}{6} f \left(Q^a{}_c \Theta + 3\omega^a{}_c\right) \tilde{\nabla}^c \dot{\Theta}. \tag{7.50}$$

Applying equation (7.48) and (7.49) to substitute for $(\ddot{\Theta})$ and $(\tilde{\nabla}_a \dot{\Theta})$ into equation (7.50) above, we obtain the following instructive constraint equation;

$$0 = (\mathfrak{V}_1)^a \Theta \dot{\Theta}. \tag{7.51}$$

The only interesting part of equation (7.51), is the case where $(\dot{\Theta} = 0)$ since where $(\mathfrak{V}_1)^a \Theta = 0$, the conjecture is valid. Therefore, where $(\dot{\Theta} = 0)$ equation (7.47) gives us the following constraint equation;

$$0 = \tilde{\nabla}_a \Theta \tilde{\nabla}^a \Theta, \tag{7.52}$$

The condition (7.52) above implies that (Θ) is a spacetime constant. Consequently, constraint equation (3.5a) takes the following form;

$$0 = 2\epsilon^{a}{}_{bc}\dot{u}^{b}\omega^{c} + \epsilon^{a}{}_{bc}\tilde{\nabla}^{b}\omega^{c}. \tag{7.53}$$

Taking the spatial divergence of both sides of equation (7.53) above, we arrive at the following illuminating constraint equation;

$$0 = (1 - 9p')\Theta\omega^{2}. \tag{7.54}$$

In equation (7.54) above, we have that either ($\Theta\omega = 0$) and we have the proof or $(1 - 9p') = 0$, this case has been dealt with in reference [54], therefore ($\Theta\omega$) must vanish. The fourth and the last case in equation (7.39) is where ($\tilde{\nabla}_{a}\mathcal{S} = 0$), implying that ($\omega^{a} = 0$) unless ($\dot{\mathcal{S}} = 0$). This means ($\mathcal{S}$) is a spacetime constant. Therefore, equation (7.26) implies that $\left(\Theta = \frac{1}{p'f}e^{\gamma+\mathcal{S}}\right)$ indicating that (Θ) is a function of the energy density and the shear-free perfect fluid conjecture holds, see references [10--12] for further details. □

7.2 THE CASE WHERE THE COMPONENT OF THE (RESCALED) ACCELERATION PARALLEL TO THE VORTICITY IS *basic*

The component of the acceleration vector field ($\dot{u}^a$) which is parallel to the vorticity vector field (ω^a), is defined in the following way;

$$\left(\dot{u}^b\hat{\omega}_b\right)\hat{\omega}^a. \tag{7.55}$$

It has been shown that if the time evolution of $\left(\frac{\dot{u}_3}{\lambda p'}\right)$ vanishes, a Killing vector parallel to the vorticity exists, the proof can be found in reference [27] where ($\dot{u}_3$) represents the dot product ($\dot{u}^b\hat{\omega}_b$). This section aims to prove that we can draw the same conclusion for the case where the time evolution of $\left(f\dot{u}^b\hat{\omega}_b\right)$ vanishes, where the scalar function ($f = f(\mu)$) is purely a function of the energy density. Before we proceed, let us first prove that when the time evolution of $\left(\frac{\dot{u}^b\hat{\omega}_b}{\lambda p'}\right)$ vanishes, a Killing vector exists along the vorticity. The following theorem is a covariant version of Theorem (2) in references [27], produced using the orthonormal tetrad formalism.

Theorem 5. *If a rotating and expanding shear-free perfect fluid is obeying a barotropic equation of state and* $\left(\frac{\dot{u}^b\hat{\omega}_b}{\lambda p'}\right)$ *is basic, then a Killing vector parallel to the vorticity exists.*

Proof. The scalar field $\left(\frac{\dot{u}^b\hat{\omega}_b}{\lambda p'}\right)$ is called *basic* if its time evolution vanishes. Hence, by applying equation (5.3), this property leads us to the following constraint equation;

$$F^c\hat{\omega}_c = 0, \tag{7.56}$$

where the vector filed (F^a) appearing in equation (7.56) above is conveniently defined as ($F^a = \Theta\dot{u}^a + \tilde{\nabla}^a\Theta$). The time evolution equation for (F^a) can be obtained by applying equation (5.3) and (9.29), where the expansion scalar (Θ) is used instead of

($\mathcal{S}$), giving us the following time evolution equation;

$$\dot{F}^{\langle a\rangle} = (2p' - 1)\Theta F^a + \Big((1-\phi-2p')\Theta^2 + \dot{\Theta}\Big)\dot{u}^a + \frac{3}{2}\Big((3p'-1)F^c + (1-3\phi-3p')\Theta\dot{u}^c\Big)\omega^a{}_c. \tag{7.57}$$

The evolution equation (7.57), allows us to time propagate equation (7.56) on both sides, giving us the following constraint equation;

$$0 = \dot{u}^a\hat{\omega}_a\Big((1-\phi-2p')\Theta^2 + \dot{\Theta}\Big). \tag{7.58}$$

Therefore, in equation (7.58), we have that either ($\dot{u}^a\hat{\omega}_a = 0$) or the term enclosed by the brackets must vanish, resulting in the following constraint equation;

$$0 = (1-\phi-2p')\,\Theta^2 + \dot{\Theta}. \tag{7.59}$$

Equation (7.59) above is equivalent to equation (48) in references [27]. From the time propagation of equation (3.5c), one obtains the following equation;

$$\tilde{\nabla}_a\dot{\Theta} = \left(p' - \frac{2}{3}\right)\Theta F_a + \left(\left(\frac{2}{3} - p'\right)\Theta^2 - \dot{\Theta}\right)\dot{u}_a + \frac{1}{2}\Big((9p'-1)F^b + 3(1-3\phi-3p')\dot{u}^b\Theta\Big)\omega_{ab}. \tag{7.60}$$

Upon taking the spatial gradient of equation (7.59) and applying equation (7.60) above, we arrive at the following constraint equation;

$$0 = (1-3\phi-3p'+\chi)\Theta^2\dot{u}_a - \frac{1}{3}(4-6\phi-9p')\Theta F_a - \frac{1}{2}\Big((9p'-1)F^b + 3(1-3\phi-3p')\dot{u}^b\Theta\Big)\omega_{ab}. \tag{7.61}$$

Therefore, contracting equation (7.61) above with ($\hat{\omega}^a$) and taking equation (7.56) into account, we obtain the following constraint equation;

$$0 = (1 - 3\phi - 3p' + \chi)\Theta^2 \dot{u}^a \hat{\omega}_a. \tag{7.62}$$

Since we have chosen the case where ($\dot{u}^a \hat{\omega}_a \neq 0$) in equation (7.58), then the following constraint must be satisfied for the consistency of equation (7.62);

$$0 = 1 - 3\phi - 3p' + \chi. \tag{7.63}$$

Equation (7.63) above is equivalent to equation (51) appearing in reference [27], where the full expressions for (ϕ) and (χ) given by equations (3.16b) and (9.1) have been used. This allows us to simplify equation (7.61), giving us a new constraint equation shown below;

$$\begin{aligned} 0 = &\left(\frac{4}{3} - 2\phi - 3p'\right)\Theta F_a + \frac{1}{2}\left(9p' - 1\right)F^b\omega_{ab} \\ &- 3\left(1 - 3\phi - 3p'\right)\Theta \dot{u}^b \omega_{ab}. \end{aligned} \tag{7.64}$$

Substituting for (F_a) into equation (7.64) above, one obtains an equation which corresponds to equation (52) and (53) in reference [27]. Aided by equation (7.64) and (7.59), equation (7.57) simplifies and can be written as follows;

$$\dot{F}^{\langle a \rangle} = \left(2\phi + 5p' - \frac{7}{3}\right)\Theta F^a - \omega^a{}_b F^b. \tag{7.65}$$

Time propagating equation (7.64), and applying equation (7.65), (7.59) and (7.64) itself in order to eliminate the ($\dot{u}^b \omega_{ab}$) terms, we arrive at the following constraint equation;

$$0 = (h_1)\,\omega_{ab}F^b + (h_2)\,F_a, \tag{7.66}$$

where the scalar functions (h_1) and (h_1) appearing in equation (7.66) above are given by the following equations;

$$h_1 = \phi - \frac{2}{3}(1 - 6p'), \tag{7.67}$$

$$h_2 = \frac{2}{3}\left(\left(9{h_1}^2 + h_1(1 - 27p') + 5p'\right)\Theta + 18\dot{h}_1\right). \tag{7.68}$$

Therefore, equation (7.66) implies that either ($F_a = 0$) and then the vorticity vanishes by Theorem (3), and we are done or ($h_1 = 0$) and ($h_2 = \frac{10}{3}p'\Theta = 0$) implying that ($\Theta = 0$) and we have the proof. Finally, we need to return to the case where ($V = \dot{u}^a\hat{\omega}_a = 0$), which implies that ($Z = \hat{\omega}_a\tilde{\nabla}^a\Theta = 0$) since ($\hat{\omega}_a F^a = 0$) by equation (7.56). Hence, it can be shown that ($\dot{Z} = 0$) and ($\dot{V} = 0$) by employing equation (5.3) and (9.29). In addition, the spatial gradients of (Z) and (V) must vanish, giving us the following constraint equations respectively;

$$\omega^c\tilde{\nabla}_c\tilde{\nabla}_a\Theta = H_{ac}\tilde{\nabla}^c\Theta, \tag{7.69}$$

$$\omega^c\tilde{\nabla}_c\dot{u}_a = H_{ac}\dot{u}^c - \frac{1}{3}\mathcal{Y}\omega_a. \tag{7.70}$$

The scalar ($\mathcal{Y}$) appearing on the right hand side of equation (7.70) above is defined by the equation ($\epsilon_{abc}\dot{u}^b\tilde{\nabla}^c\Theta = \mathcal{Y}\omega_a$). Equation (7.69) and (7.70) allow us to rewrite constraint equation (5.17) and (5.11) appearing in Chapter (5) in the following way;

$$H^a{}_b\left((2 - 3\phi)\dot{u}^b\Theta + (1 + 3p')\tilde{\nabla}^b\Theta\right) = h_3\omega^a, \tag{7.71}$$

$$H^a{}_b\left((2 - 9\phi)\dot{u}^b\Theta + (9p' - 1)\tilde{\nabla}^b\Theta\right) = h_4\omega^a. \tag{7.72}$$

Equation (7.71) is associated with equation (59) and (60) appearing in reference [27], the extra constraint equation (7.72) is necessary to simplify our analysis from now on. In equation (7.71) and (7.72) above, the scalar functions (h_3) and (h_4) are defined as follows;

$$\begin{aligned}
h_3 =& \frac{1}{3}\Theta\left((2-3\phi)(\mathcal{J}+\mathcal{Y}) - 3\dot{u}^2(2\phi+\chi) + \left(7-18p'+27p'^2\right)\omega^2\right) \\
&- 2\phi\dot{u}^b\tilde{\nabla}_b\Theta, \qquad (7.73) \\
h_4 =& \frac{1}{2}\Theta\left(2\mathcal{E}(1+3p') - 8\mathcal{J} - (8-15\phi)\mathcal{Y} + 36\phi\dot{u}^2 - 8(5-9p')\omega^2\right) \\
&+ 6(1-3p')\dot{u}^b\tilde{\nabla}_b\Theta + \frac{13}{2}h_3. \qquad (7.74)
\end{aligned}$$

Furthermore, equation (7.71) and (7.72) can be simplified further, giving us the following two constraint equations;

$$H^a{}_b\dot{u}^b\Theta = h_5\ \omega^a, \qquad (7.75)$$

$$H^a{}_b\tilde{\nabla}^b\Theta = h_6\ \omega^a, \qquad (7.76)$$

where the scalar functions (h_5) and (h_6) are given by the following equations;

$$h_5 = -\frac{1}{12}\frac{p'}{\mathcal{E}p''}\left(h_4\left(1+3p'\right) + h_3\left(1-9p'\right)\right), \qquad (7.77)$$

$$h_6 = \frac{1}{4}(3h_3 - h_4) - h_5. \qquad (7.78)$$

It is useful to define scalars ($\mathfrak{B} = e^{-\gamma}(1-3p')\Theta$) and ($\mathfrak{G} = e^{-\gamma}(1-9p')\Theta$), such that the evolution equations for (H^{ab}) and ($\tilde{\nabla}_a\Theta$) are given in the following compact form;

$$\dot{H}^{\langle ab\rangle} = (p'-1)\Theta H^{ab} + 2\omega_c{}^{(a}H^{b)c} + e^{\gamma}\omega^{(a}\tilde{\nabla}^{b)}\mathfrak{B}, \qquad (7.79)$$

$$h^a{}_b(\tilde{\nabla}^b\Theta)^{\cdot} = (p'-1)\Theta\tilde{\nabla}^a\Theta + \omega_c{}^a\tilde{\nabla}^c\Theta + \frac{1}{2}e^{\gamma}\omega_c{}^a\tilde{\nabla}^c\mathfrak{G}. \qquad (7.80)$$

It is also useful to note that the scalar ($\mathfrak{G}$) can be used to rewrite equation (7.72) as follows;

$$H^a{}_b\tilde{\nabla}^b\mathfrak{G} = -e^{-\gamma}h_4\omega^a, \tag{7.81}$$

and equation (7.76) can be rotated by contracting both sides by the rotation tensor ($\omega^e{}_a$), giving us the following constraint equation;

$$0 = \omega^a{}_d H^d{}_b\tilde{\nabla}^b\Theta. \tag{7.82}$$

Time propagating equation (7.82) and applying equation (7.79), (7.80), (7.81) and equation (7.76) we obtain the following constraint equation;

$$0 = \left(H^{cb}Q_{cb}\right)\tilde{\nabla}^a\mathfrak{G}. \tag{7.83}$$

In equation (7.83) above, one can show that ($H^{bc}Q_{bc} = H^{bc}\hat{\omega}_b\hat{\omega}_c = \hat{\omega}^b\tilde{\nabla}_b\omega$). Furthermore, equation (7.83) indicate that either ($\tilde{\nabla}^a\mathfrak{G} = 0$) then the vorticity vanishes unless ($\mathfrak{G}$) is a constant, then (Θ) is purely a function of the energy density, and we have the proof (see references [10--12]) or that $\left(H^{cb}Q_{cb} = \hat{\omega}^b\tilde{\nabla}_b\omega = 0\right)$. Upon taking the spatial gradient of ($\hat{\omega}^b\tilde{\nabla}_b\omega = 0$) both sides, we are lead to the following equation;

$$\omega^b\tilde{\nabla}^a\tilde{\nabla}_b\omega = H^a{}_b\tilde{\nabla}^b\omega + \frac{1}{3}\mathcal{W}\omega^a. \tag{7.84}$$

The scalar function ($\mathcal{W}$) appearing on the right hand side of equation (7.84) above is defined by the equation ($\epsilon^a{}_{bc}\tilde{\nabla}^b\Theta\tilde{\nabla}^c\omega = \mathcal{W}\omega^a$) because both ($\tilde{\nabla}^a\Theta$) and ($\tilde{\nabla}^a\omega$) are orthogonal to the vorticity. Equation (7.84) above is essential in the following proof of the existence of a Killing vector (K^a) parallel to the vorticity, this Killing vector ($K^a = \hat{\omega}^a\mathcal{K}$) must obey the Killing equation ($\nabla^{(a}K^{b)} = 0$). This leads to the time evolution equation for ($\mathcal{K}$) (when we contract the Killing equation with $u_a\hat{\omega}_b$), the spatial gradient equation for ($\mathcal{K}$) (when we contract the Killing equation with $\hat{\omega}_b$ and project with $h^e{}_a$) and to the (H^{ab}) constraint equation (when the Killing equation is fully projected by ($h^b{}_a$) and we substitute for $K^a = \hat{\omega}^a \mathcal{K}$). This can be shown by taking constraint equation (3.5c) and writing it as an equation for the magnetic part of the Weyl tensor as follows;

$$H^{ab} = -\tilde{\nabla}^{\langle a}\omega^{b\rangle} - 2\dot{u}^{\langle a}\omega^{b\rangle}, \tag{7.85}$$

then by substituting for $\left(\omega^a = \frac{\omega}{\mathcal{K}} K^a\right)$, we obtain the following equation for (H^{ab}) in terms of (K^a) and scalar function (ζ);

$$H^{ab} = -\frac{\omega}{\mathcal{K}}\tilde{\nabla}^{\langle a}K^{b\rangle} + \omega^{\langle a}\tilde{\nabla}^{b\rangle}\zeta. \tag{7.86}$$

Therefore, since $\tilde{\nabla}^{\langle a}K^{b\rangle} = 0$ and $\omega^b\tilde{\nabla}_b\zeta = 0$, the Killing equations can be written as follows:

$$\dot{\mathcal{K}} = \frac{1}{3}\mathcal{K}\Theta, \tag{7.87}$$

$$\tilde{\nabla}^a\mathcal{K} = -\mathcal{K}\hat{\omega}^b\tilde{\nabla}_b\hat{\omega}^a, \tag{7.88}$$

$$H^{ab} = \omega^{(a}\tilde{\nabla}^{b)}\zeta. \tag{7.89}$$

Equation (7.89) satisfy our constraint equation (7.82) where equations (7.87) and (7.88) are equivalent to equation (61) which appears in reference [27], and in equation (7.89) the scalar (ζ) is given by ($\zeta = \ln\left(e^{2\gamma}\mathcal{K}\omega^{-1}\right)$), where its spatial gradient is given below;

$$\tilde{\nabla}^a\zeta = -2\dot{u}^a + \frac{1}{\mathcal{K}}\tilde{\nabla}^a\mathcal{K} - \frac{1}{\omega}\tilde{\nabla}^a\omega. \tag{7.90}$$

The vector field ($\hat{\omega}^b\tilde{\nabla}_b\hat{\omega}^a$) seen on the right hand side of equation (7.88) is orthogonal to the unit vorticity vector ($\hat{\omega}_a$), this is analogous to the acceleration vector ($\dot{u}_a$) being orthogonal to the four velocity vector field (u^a). To show that the integrability conditions given by equation (9.29) and (9.30) in Appendix A (9), for equations (7.87), (7.88) and (7.89) are satisfied, we need the following evolution equation for the gradient of the unit vorticity vector field;

$$h^a{}_e h^b{}_c (\tilde{\nabla}^e \hat{\omega}^c)^{\cdot} = -\frac{1}{3}\Theta\tilde{\nabla}^a\hat{\omega}^b - \omega^a{}_d\tilde{\nabla}^d\hat{\omega}^b - \omega^b{}_d\tilde{\nabla}^a\hat{\omega}^d - \frac{1}{3}\hat{\omega}^a F^b. \tag{7.91}$$

Time propagating both sides of equation (7.88) and applying the propagation equations (7.91) and (7.87) above, we obtain the following propagation equation for the gradient of the scalar function ($\mathcal{K}$);

$$h^a{}_c(\tilde{\nabla}^c\mathcal{K})^{\cdot} = \frac{1}{3}\mathcal{K}F^a - \omega^a{}_b\tilde{\nabla}^b\mathcal{K}. \tag{7.92}$$

Equation (7.92) above is equivalent to equation (9.29) appearing in Appendix A (9) when ($\mathcal{K}$) is substituted for ($\mathcal{S}$). Furthermore, equation (9.33), which is also in Appendix A (9) together with equation (7.84) and (7.89), enable us to write the curl of the acceleration-like vector field ($\hat{\omega}^b\tilde{\nabla}_b\hat{\omega}^a$) lying on the plane orthogonal to the vorticity as follows;

$$\epsilon^a{}_{bc}\tilde{\nabla}^c(\hat{\omega}^d\tilde{\nabla}_d\hat{\omega}^b) = 2\omega^a\frac{\dot{\mathcal{K}}}{\mathcal{K}}. \tag{7.93}$$

Therefore, taking the curl of both sides of equation (7.88) and applying equation (7.93) and (7.88) itself, we recover the usual integrability condition for the scalar function ($\mathcal{K}$) shown below;

$$\epsilon^a{}_{bc}\tilde{\nabla}^b\tilde{\nabla}^c\mathcal{K} = 2\omega^a\dot{\mathcal{K}}. \tag{7.94}$$

Finally, by contracting equation (7.89) with the vorticity vector, we obtain the following equation for the spatial gradient of the scalar (ζ);

$$\tilde{\nabla}^a\zeta = \frac{2}{\omega}H^a{}_b\hat{\omega}^b. \tag{7.95}$$

Time propagating equation (7.95) above, then applying equation (7.79) and (7.95) itself to substitute for terms containing $(H^a{}_b\hat{\omega}^b)$, we obtain the following equation;

$$h^a{}_c\big(\tilde{\nabla}^c\zeta\big)^{\cdot} = e^{\gamma}\tilde{\nabla}^a\mathfrak{B} - \frac{1}{3}\Theta\tilde{\nabla}^a\zeta - \omega^a{}_b\tilde{\nabla}^b\zeta, \tag{7.96}$$

where $\big(\mathfrak{B} = e^{-\gamma}(1-3p')\Theta = e^{-\gamma}\dot{\zeta}\big)$. Therefore, equation (7.96) is the usual time space derivative commutator for a scalar field given by equation (9.29) appearing in Appendix A (9). In addition, taking the curl of both sides of equation (7.95), it can be shown that the curl of the spatial gradient of (ζ) is given by the following;

$$\epsilon^e{}_{ab}\tilde{\nabla}^a\tilde{\nabla}^b\zeta = 2\omega^e\dot{\zeta}, \tag{7.97}$$

this implies the existence of a Killing vector field (K^a) parallel to the vorticity. However, it remains to show that if (K^a) exists, then either $(\Theta\omega = 0)$ or $(\Theta\omega \neq 0)$. In this case, we must investigate the consequences of equation (7.75) and (7.76). ☐

The following theorem (6) generalises the previous theorem (5) above, where the generalised acceleration vector field rescaling function is $f(\mu)$.

Theorem 6. *If a rotating and expanding shear-free perfect fluid obeys a barotropic equation of state* $p(\mu)$, *and* $\big(f\dot{u}^b\hat{\omega}_b\big)$ *is basic, where* $f = f(\mu)$. *Then either; a Killing vector parallel to the vorticity exists or* $\mathcal{F} = \mathcal{F}(\mu, p', p'', f') = 0$ *or* $\mathcal{S} = \mathcal{S}(\mu, p', p'', p^{(3)}, f', f'') = 0$.

Proof. We begin our discussion by noting that if $\big(f\dot{u}^b\hat{\omega}_b\big)$ is *basic*, then the following constraint equation must be satisfied;

$$0 = \mathcal{C}^b\hat{\omega}_b, \tag{7.98}$$

where equation (5.3) and $(\dot{f} = -f'\Theta\mathcal{E})$ have been used in the time propagation of $\big(f\dot{u}^b\hat{\omega}_b\big)$. The vector field $(\mathcal{C}^a)$ appearing in equation (7.98) above, is given by the following equation;

$$\mathcal{C}^a = \mathcal{F}\dot{U}^a\Theta - \tilde{\nabla}^a\Theta, \tag{7.99}$$

where $(\dot{U}^a)$ is a rescaled acceleration vector field defined as follows;

$$\dot{U}^a = \frac{1}{p'}\dot{u}^a. \tag{7.100}$$

The scalar function $(\mathcal{F})$ appearing in equation (7.99) above, is defined by the following equation;

$$\mathcal{F} = \phi + \mathcal{E}\frac{f'}{f}, \tag{7.101}$$

where $(f' = \partial_\mu f)$. Constraint equation (7.98) above implies that there exists a vector field $(\mathcal{C}^a)$ which is orthogonal to the unit vorticity vector field $(\hat{\omega}^a)$ and that its time evolution is also orthogonal to the vorticity $(0 = \dot{\mathcal{C}}^b\hat{\omega}_b)$. Therefore, we need the time propagation equation for $(\mathcal{C}^a)$, where equation (7.99) shows that we require the time evolution equations for the vectors $(\dot{U}^a)$ and $(\tilde{\nabla}^a\Theta)$. The following equations, respectively, give these;

$$\ddot{U}^{\langle a\rangle} = \left(\mathcal{F} + p' - \frac{1}{3}\right)\Theta\dot{U}^a - \omega^a{}_c\dot{U}^c - \mathcal{C}^a, \tag{7.102}$$

$$\begin{aligned} h^a{}_b(\tilde{\nabla}^b\Theta)^{\cdot} = {}& (1-p')\Theta\mathcal{C}^a + \frac{3}{2}(1-3p')\omega^a{}_c\mathcal{C}^c - \mathcal{F}(1-p')\Theta^2\dot{U}^a \\ & - \frac{1}{2}\big(3\mathcal{F}(1-3p') - (2-9\phi)p'\big)\Theta\omega^a{}_c\dot{U}^c. \end{aligned} \tag{7.103}$$

Therefore, the time evolution equation for $(\mathcal{C}^a)$ can be obtained by the use of equation (7.102) and (7.103), giving us the following;

$$\begin{aligned} \dot{\mathcal{C}}^{\langle a\rangle} = {}& -(\mathcal{F} + 1 - p')\Theta\mathcal{C}^a - \frac{3}{2}(1-3p')\omega^a{}_c\mathcal{C}^c + \left(\mathcal{I}\Theta^2 + \mathcal{F}\dot{\Theta}\right)\dot{U}^a \\ & + \frac{1}{2}\big(\mathcal{F}(1-9p') - (2-9\phi)p'\big)\Theta\omega^a{}_c\dot{U}^c. \end{aligned} \tag{7.104}$$

Time propagating constraint equation (7.98) and applying equation (7.104) yields the following constraint equation;

$$0 = \left(\mathcal{C}^c\hat{\omega}_c\right)^{\dot{}} = \dot{U}^c\hat{\omega}_c\left(\mathcal{I}\Theta^2 + \mathcal{F}\,\dot{\Theta}\right). \tag{7.105}$$

For consistency, equation (7.105) requires that either $\left(\dot{U}^c\hat{\omega}_c = 0\right)$ or $\left(\mathcal{I}\Theta^2 + \mathcal{F}\,\dot{\Theta} = 0\right)$. Following work done by Van den Bergh and Slobodeanu in reference [27], let us proceed by first considering the case where,

$$0 = \mathcal{I}\Theta^2 + \mathcal{F}\,\dot{\Theta}. \tag{7.106}$$

The scalar ($\mathcal{I}$) appearing in equation (7.106) above is given by the following equation;

$$\mathcal{I} = \frac{1}{3}\mathcal{F}(2 + 3\mathcal{F}) + \mathcal{G}, \tag{7.107}$$

where ($\mathcal{G}$) is given by the evolution equation $\dot{\mathcal{F}} = \mathcal{G}\Theta$ and can be written as follows;

$$\mathcal{G} = (\mathcal{F} - \phi)(\mathcal{F} - 1 - \phi - p') - \mathcal{E}^2\frac{f''}{f} + p'\chi. \tag{7.108}$$

Time propagating equation (3.5a) given in Chapter (1) and applying equation (7.99), we obtain the following equation for the spatial gradient of the time evolution of the expansion scalar (Θ);

$$\begin{aligned}\tilde{\nabla}^a\dot{\Theta} = {} & \frac{1}{9}(5 - \mathcal{M})\Theta\mathcal{C}^a - \frac{1}{2}\mathcal{M}\omega^a{}_b\mathcal{C}^b + \frac{1}{2}\mathcal{N}\Theta\omega^a{}_b\dot{U}^b \\ & + \frac{1}{9}\left(\mathcal{F}(\mathcal{M} - 5)\Theta^2 - (1 + \mathcal{M})\dot{\Theta}\right)\dot{U}^a.\end{aligned} \tag{7.109}$$

In equation (7.109) above, the scalar functions of the energy density ($\mathcal{M}$) and ($\mathcal{N}$)

are given as follows;

$$\mathcal{M} = -1 + 9p', \tag{7.110}$$

$$\mathcal{N} = (2 - 9\phi)p' + \mathcal{F}\mathcal{M}. \tag{7.111}$$

Acting on equation (7.106) with the operator $\left(\hat{\omega}_a \tilde{\nabla}^a\right)$ and applying equation (7.109) ,(7.99) and equation (7.106) itself, one obtains the following constraint equation;

$$0 = \mathcal{L}\dot{U}^a \hat{\omega}_a \Theta^2. \tag{7.112}$$

In equation (7.112) above, the scalar function of the energy density ($\mathcal{L}$) is given by the following equation;

$$\mathcal{L} = -\mathcal{F}\mathcal{O} + \frac{1}{108}\mathcal{F}^2(\mathcal{M} - 5)(3 + 6\mathcal{F} + \mathcal{M}) - \frac{1}{18}\mathcal{F}(2 + 27\mathcal{F} + 2\mathcal{M})\mathcal{K} + \frac{1}{4}\mathcal{K}^2, \tag{7.113}$$

where the scalar ($\mathcal{O}$) appearing in equation (7.113) is defined by the evolution equation ($\dot{\mathcal{I}} = \mathcal{O}\Theta$). The scalars ($\mathcal{K}$) and ($\mathcal{O}$) are given by the following equations respectively;

$$\mathcal{K} = \frac{1}{9}\mathcal{F}(\mathcal{M} - 5) + 2\mathcal{I}, \tag{7.114}$$

$$\mathcal{O} = \frac{2}{3}(1 + 3\mathcal{F})\mathcal{G} + \mathcal{H}. \tag{7.115}$$

In equation (7.115), the scalar ($\mathcal{H}$) is defined by the evolution equation ($\dot{\mathcal{G}} = \mathcal{H}\Theta$) and is shown in Appendix A (9) by equation (9.21). Furthermore, equation (7.112) implies that ($\mathcal{L}$) must vanish since we are dealing with the case where ($\dot{U}^a \hat{\omega}_a \neq 0$). Therefore, taking the spatial gradient of equation (7.106) and applying equation (7.109), (7.99) and equation (7.106) itself, we obtain the following constraint equation;

$$0 = \mathcal{F}\left(\mathcal{K}\Theta\mathcal{C}_a + \frac{1}{2}\mathcal{F}\left(\mathcal{M}\mathcal{C}^b - \mathcal{N}\Theta\dot{U}^b\right)\omega_{ab}\right). \tag{7.116}$$

Equation (7.116) above gives us two cases to consider, namely; the case where ($\mathcal{F} = 0$) and the case where the following constraint equation holds;

$$0 = \mathcal{K}\Theta\mathcal{C}_a + \frac{1}{2}\mathcal{F}\left(\mathcal{M}\mathcal{C}^b - \mathcal{N}\Theta\dot{U}^b\right)\omega_{ab}. \tag{7.117}$$

The time evolution equations for the scalars ($\mathcal{F}$), ($\mathcal{M}$), ($\mathcal{K}$) and ($\mathcal{N}$) which appear in equation (7.117) above are given by the following equations respectively;

$$\dot{\mathcal{F}} = \frac{1}{2}\left(\mathcal{K} - \frac{1}{9}\mathcal{F}(7 + 18\mathcal{F} + \mathcal{M})\right)\Theta, \tag{7.118}$$

$$\dot{\mathcal{M}} = \left(\mathcal{N} - \frac{1}{9}\mathcal{M}(1 + 9\mathcal{F} + \mathcal{M})\right)\Theta, \tag{7.119}$$

$$\dot{\mathcal{K}} = \left(\frac{\mathcal{K}^2}{2\mathcal{F}} - \frac{1}{6}\mathcal{K}(3 + 18\mathcal{F} + \mathcal{M}) + \frac{1}{9}\mathcal{F}\left(\mathcal{N} - \frac{1}{9}(5 + 5\mathcal{M} + 9\mathcal{F}\mathcal{M})\right)\right)\Theta, \tag{7.120}$$

$$\begin{aligned}\dot{\mathcal{N}} = &\left(-\mathcal{N}\left(\frac{1}{9}\mathcal{M} - \frac{\mathcal{F}(1-\mathcal{M})}{1+\mathcal{M}}\right) - \mathcal{F}\mathcal{M}\left(\frac{\mathcal{F}(2+\mathcal{M})}{1+\mathcal{M}} + \frac{1}{18}(9+\mathcal{M})\right)\right.\\ &\left. + \frac{\mathcal{N}^2}{1+\mathcal{M}} - \frac{1}{9}(1+\mathcal{M})^2\chi + \frac{1}{2}\mathcal{M}\mathcal{K}\right)\Theta.\end{aligned} \tag{7.121}$$

In addition, we can use equation (7.106), (7.117) and (7.110) to simplify the evolution equations for the vectors ($\dot{U}^a$) and ($\mathcal{C}_a$) which appear in equation (7.117), resulting in the following evolution equations;

$$\dot{U}^{\langle a\rangle} = \left(\mathcal{F} + \frac{1}{9}(\mathcal{M} - 2)\right)\dot{U}^a\Theta - \dot{U}^c\omega^a{}_c - \mathcal{C}^a, \tag{7.122}$$

$$\dot{\mathcal{C}}^{\langle a\rangle} = \left(\frac{1}{9}(\mathcal{M} - 8) - \mathcal{F} - \frac{\mathcal{K}}{\mathcal{F}}\right)\Theta\mathcal{C}^a - \mathcal{C}^c\omega^a{}_c. \tag{7.123}$$

Equipped with equation (7.118), (7.119), (7.120), (7.122) and (7.123), we can time propagate the constraint equation (7.117) and by applying equation (7.117) itself in order to eliminate terms containing only ($\dot{U}^c\omega^a{}_c$), we arrive at the following constraint equation;

$$0=\left(\frac{\dot{\mathcal{N}}}{\mathcal{N}}+\frac{2}{9}\left(\frac{\mathcal{F}}{18\mathcal{K}}(5+5\mathcal{M}+9\mathcal{F}\mathcal{M}-9\mathcal{N})+(1+18\mathcal{F}+\mathcal{M})+\frac{9}{2}\frac{\mathcal{K}}{\mathcal{F}}\right)\Theta\right)\mathcal{K}\Theta\mathcal{C}_a$$
$$+\left(\frac{\mathcal{F}\mathcal{M}}{2}\frac{\dot{\mathcal{N}}}{\mathcal{N}}+\left(\frac{1}{12}\mathcal{F}(3\mathcal{M}+18\mathcal{F}\mathcal{M}+\mathcal{M}^2-12\mathcal{N})+\frac{1}{4}\mathcal{M}\mathcal{K}\right)\Theta\right)\omega_{ab}\mathcal{C}^b. \tag{7.124}$$

Equation (7.124) above, is impossible unless either ($\mathcal{C}_a=0$) and then the vorticity vanishes because ($\mathcal{C}_a$) can be written as $\left(-\frac{e^{\gamma}}{\lambda p'f}\tilde{\nabla}_a\mathfrak{C}\right)$, where ($\mathfrak{C}=\frac{\lambda p'f}{e^{\gamma}}\Theta$). If ($\mathfrak{C}$) is a spacetime constant, then (Θ) is a function of the energy density, and the conjecture holds (see references [10--12]) or the terms enclosed by the brackets must vanish, giving us the following evolution equations;

$$\dot{\mathcal{N}}=\left(\frac{2}{\mathcal{M}}\mathcal{N}^2-\mathcal{N}\left(\frac{1}{6}(3+18\mathcal{F}+\mathcal{M})+\frac{\mathcal{K}}{2\mathcal{F}}\right)\right)\Theta, \tag{7.125}$$
$$\dot{\mathcal{N}}=\left(\frac{\mathcal{F}}{9\mathcal{K}}\mathcal{N}^2-\mathcal{N}\left(\frac{2}{9}(1+18\mathcal{F}+\mathcal{M})+\frac{1}{81}(5+5\mathcal{M}+9\mathcal{F}\mathcal{M})\frac{\mathcal{F}}{\mathcal{K}}+\frac{\mathcal{K}}{\mathcal{F}}\right)\right)\Theta, \tag{7.126}$$

Consequently, subtracting equation (7.125) from equation (7.126), we obtain the following constraint equation;

$$0=\mathcal{N}\left(\frac{1}{9}\left(\mathcal{F}\mathcal{M}+\frac{5}{9}(\mathcal{M}+1)-\mathcal{N}\right)\mathcal{F}+\left(\mathcal{F}+\frac{1}{18}(\mathcal{M}-5)+\frac{2\mathcal{N}}{\mathcal{M}}\right)\mathcal{K}+\frac{\mathcal{K}^2}{2\mathcal{F}}\right). \tag{7.127}$$

Equation (7.127) above gives us two cases to consider. The first case is where ($\mathcal{N}=0$), in which equation (7.117) can be written as follows;

$$0=\mathcal{F}\mathcal{M}\omega^2\mathcal{C}_a-2\mathcal{K}\Theta\mathcal{C}^b\omega_{ab}. \tag{7.128}$$

Equation (7.128) above implies that ($\mathcal{M} = -1 + 9p' = 0$), and the shear-free perfect fluid conjecture is true, see reference [27]. The second case of equation (7.127) is where the term within the square brackets vanishes, giving us the following constraint equation.

$$0 = \frac{1}{9}\left(\mathcal{F}\mathcal{M} + \frac{5}{9}(\mathcal{M}+1) - \mathcal{N}\right)\mathcal{F} + \left(\mathcal{F} + \frac{1}{18}(\mathcal{M}-5) + \frac{2\mathcal{N}}{\mathcal{M}}\right)\mathcal{K} + \frac{\mathcal{K}^2}{2\mathcal{F}}. \tag{7.129}$$

Equation (7.129) above can be time-propagated further until one establishes consistency. This can be achieved by applying the following new evolution equations for $\mathcal{F}$, $\mathcal{M}$, $\mathcal{K}$ and $\mathcal{N}$;

$$\dot{\mathcal{F}} = \frac{1}{2}\left(\mathcal{K} - \frac{1}{9}\mathcal{F}(7 + 18\mathcal{F} + \mathcal{M})\right)\Theta, \tag{7.130}$$

$$\dot{\mathcal{M}} = \left(\mathcal{N} - \frac{1}{9}\mathcal{M}(1 + 9\mathcal{F} + \mathcal{M})\right)\Theta, \tag{7.131}$$

$$\dot{\mathcal{K}} = \mathcal{K}\left(\frac{\mathcal{K}}{\mathcal{F}} - \left(\frac{1}{9}(7 + 18\mathcal{F} + \mathcal{M}) - \frac{2\mathcal{N}}{\mathcal{M}}\right)\right)\Theta, \tag{7.132}$$

$$\dot{\mathcal{N}} = \mathcal{N}\left(\frac{2\mathcal{N}}{\mathcal{M}} - \frac{1}{2}\left(\frac{1}{3}(3 + 18\mathcal{F} + \mathcal{M}) + \frac{\mathcal{K}}{\mathcal{F}}\right)\right)\Theta. \tag{7.133}$$

Time propagating equation (7.129) above, and applying evolution equations (7.130), (7.131), (7.132) and (7.133). By using equation (7.129) itself to eliminate the ($\mathcal{K}^2$) terms, we obtain the following constraint equation;

$$\begin{aligned}\mathcal{S} = &\left(\frac{3\mathcal{N}^2}{\mathcal{M}} - \frac{1}{6}(5 + 18\mathcal{F})\mathcal{N} - \frac{5}{108}\left(9 + 36\mathcal{F} + 10\mathcal{M} + 18\mathcal{F}\mathcal{M} + \mathcal{M}^2\right)\right)\mathcal{F} \\ &- \left(\frac{81\mathcal{N}^2}{\mathcal{M}^2} - \frac{3(5 - 2\mathcal{M})\mathcal{N}}{\mathcal{M}} - \frac{5}{4}(1 + 6\mathcal{F} + \mathcal{M})\right)\mathcal{K} = 0. \end{aligned} \tag{7.134}$$

As illustrated above, equation (7.134) can also be time-propagated as we have done for equation (7.129). Where at each stage of propagation, the scalars $\mathcal{K}$, $\mathcal{N}$ and $\mathcal{F}$ are eliminated in favour of $\mathcal{M}$. This process eventually leads to a vanishing polynomial of $\mathcal{M}$, which either has no solutions or has a linear equation of state solution for the pressure $p(\mu)$, where the case of a linear equation of state has been proven in reference [27]. However, we will have to show that this is true, the details of which will come as part of our future work. In an exceptional case, we can compare the coefficients of ($\mathcal{N}$) and ($\mathcal{N}^2$) on the right-hand side of equation (7.125) and (7.126), thereby obtaining the following equations for ($\mathcal{F}$) and ($\mathcal{K}$) respectively;

$$\mathcal{F} = -\frac{(40 + 30\mathcal{M} + 3\mathcal{M}^2)}{108\mathcal{M}}, \tag{7.135}$$

$$\mathcal{K} = -\frac{(40 + 30\mathcal{M} + 3\mathcal{M}^2)}{1944}. \tag{7.136}$$

Equation (7.135) and (7.136) above can be time propagated by applying equation (7.118), (7.119) and (7.120), then using equation (7.135) and (7.136) themselves to substitute for ($\mathcal{F}$) and ($\mathcal{K}$) leads us to the following two equations for ($\mathcal{N}$);

$$\mathcal{N} = \frac{(-10 + 3\mathcal{M})(-320 - 240\mathcal{M} + 9\mathcal{M}^3)}{108(-40 + 3\mathcal{M}^2)}, \tag{7.137}$$

$$\mathcal{N} = \frac{5(320 + 256\mathcal{M} + 36\mathcal{M}^2 + 12\mathcal{M}^3 + 3\mathcal{M}^4)}{216\mathcal{M}(5 + \mathcal{M})}. \tag{7.138}$$

The difference between equation (7.137) and (7.138) gives us the following constraint equation;

$$0 = (40 + 30\mathcal{M} + 3\mathcal{M}^2)(1600 + 880\mathcal{M} - 200\mathcal{M}^2 - 60\mathcal{M}^3 + 3\mathcal{M}^4), \tag{7.139}$$

this equation has linear equation of state solutions since ($\mathcal{M} = -1 + 9p'$). Thus, ($\Theta\omega = 0$) by Theorem (4) in reference [27]. However, when ($40 + 30\mathcal{M} + 3\mathcal{M}^2 = 0$), equation (7.135) implies that ($\mathcal{F}$) must vanish, which is a contradiction since we are looking at the case where ($\mathcal{F} \neq 0$). Let us turn back and look at the particular case of equation (7.116) where ($\mathcal{F} = 0$). In this case, equations (7.98) and (7.101) imply the following constraint equation;

$$0 = \hat{\omega}^b \tilde{\nabla}_b \Theta, \tag{7.140}$$

Taking the spatial gradient of both sides of equation (7.140) gives us the following constraint equation;

$$0 = \frac{\mathcal{U}\omega}{f} \tilde{\nabla}^a \Theta + \omega^b \tilde{\nabla}^a \tilde{\nabla}_b \Theta - H^a{}_b \tilde{\nabla}^b \Theta. \tag{7.141}$$

The time propagation of equation (7.141) above requires the evolution equation for ($H^a{}_b$) and ($\tilde{\nabla}^a \Theta$) which are given by equation (7.80) and (7.79) respectively, and the evolution equation for ($\omega^b \tilde{\nabla}^a \tilde{\nabla}_b \Theta$) can be obtained by applying evolution equation (9.28) for ($\tilde{\nabla}^a \tilde{\nabla}^b \Theta$) given in Appendix A (9). Time propagating equation (7.141) and applying equation (7.79), (7.80), (9.28), (5.3) and (5.5) one can show that equation (7.141) is closed under time propagation and is identically satisfied. Alternatively, taking the spatial gradient of ($\mathcal{U}$) and applying equation (3.5c) and (3.6) we obtain the following constraint equation;

$$\begin{aligned} \tilde{\nabla}^a \mathcal{U} = \frac{f}{3} \left(\mathcal{J} + 2\omega^2 \right) \hat{\omega}^a + \mathcal{U} \Bigg((\phi - 2p') \dot{U}^a - \frac{\tilde{\nabla}^a \omega}{\omega} \Bigg) \\ + f \Bigg(E^a{}_b \hat{\omega}^b - \frac{\dot{u}^b (3H^a{}_b - \epsilon^a{}_{bc} \tilde{\nabla}^c \Theta)}{3\omega} \Bigg). \end{aligned} \tag{7.142}$$

Given equations (7.141) and (7.142) above, we can rearrange and then simply equations (5.11) and (5.17), making the curvature terms containing ($H^a{}_b$) the subject on the left-hand side, we obtain the following constraint equations to be time propagated to establish consistency of the EFE in this case, where $(\mathcal{F} = 0)$;

$$\begin{aligned}\frac{\mathcal{E}p''}{p'}H^{a}{}_{b}f\tilde{\nabla}^{b}\Theta = \frac{1}{3}\mathcal{R}f\omega^{a} + \frac{1}{3}\Theta\Big((2-3\phi)(2-9\phi)\tilde{\nabla}^{a}\mathfrak{U} - \mathfrak{U}(f_1)\dot{U}^{a}\Big)\\ - \mathfrak{U}\big(2 - 5\phi - 9\phi^{2} - 3(2-3\phi)p'\big)\tilde{\nabla}^{a}\Theta\\ - \frac{1}{36}\Theta(2-3\phi)(1-12\phi)\epsilon^{a}{}_{bc}f\dot{u}^{b}\tilde{\nabla}^{c}\Theta.\end{aligned} \tag{7.143}$$

$$\begin{aligned}\frac{\mathcal{E}p''}{p'}H^{a}{}_{b}f\dot{u}^{b}\Theta = \frac{1}{3}\mathfrak{R}f\omega^{a} + \Theta\Big((2 - 5\phi - 10p' + 9\phi p')\tilde{\nabla}^{a}\mathfrak{U} - \mathfrak{U}(f_2)\dot{U}^{a}\Big)\\ + \mathfrak{U}(1 - 5\phi + 9\phi p' - 9p'^{2})\tilde{\nabla}^{a}\Theta\\ - \frac{1}{12}\Theta(1 - 5p' + 12\phi p')\epsilon^{a}{}_{bc}f\dot{u}^{b}\tilde{\nabla}^{c}\Theta.\end{aligned} \tag{7.144}$$

In equation (7.143) and (7.144) above, the scalar function $(\mathfrak{U} = \frac{1}{8}\mathcal{U}\omega)$, $(\mathcal{R})$ and $(\mathfrak{R})$ are given by equations (9.22) and (9.23) in Appendix A (9) respectively. Additionally, the scalars (f_1) and (f_2) which appear in the coefficients of $(\dot{U}^{a})$ on the left hand side of equations (7.143) and (7.144) are given in the Appendix A (9) by equations (9.24) and (9.25) respectively. At this juncture, one can investigate special cases of the energy density function (f), say $\big(f = \frac{1}{\lambda p'}\big)$ which is the case exhibited by Van den Bergh and Slobodeanu in reference [27]. It remains to show whether the $(\mathcal{F} = 0)$ consistency condition leads to the situation where there exists a Killing vector along the vorticity or not. This we shall leave as future work. In the other case of equation (7.105), we have that $(\dot{U}^{a}\hat{\omega}_{a} = 0)$ which leads to the existence of a Killing vector parallel to the vorticity vector field by Theorem (5). Whether the shear-free perfect fluid conjecture is true in this case is still an open question. □

7.3 DISCUSSION

This chapter aims to shed some light on the shear-free perfect fluid conjecture where the acceleration vector field is not parallel to the vorticity. We began our discussion with the case where some energy density function (f) rescales the acceleration vector field. Then we require that the dot product of this rescaled acceleration vector and the vorticity be *basic*, meaning that $(f\dot{u}^c\hat{\omega}_b)^{\cdot} = 0$. Incidentally, this leads to the case where ($\dot{u}^c\hat{\omega}_b = 0$) and the existence of a Killing vector along the vorticity; otherwise, the conjecture holds, and we have that ($\Theta\omega = 0$).

For completeness, we first showed that if the components of the rescaled acceleration vector field orthogonal to the vorticity are *basic*, then the conjecture holds. In Theorem (3), we prove that if the scaling function (f) is equal to $\left(\frac{1}{\lambda^2 p'}\right)$ the conjecture is true. Whereas in Theorem (4), we prove the conjecture in the case of a general function of the energy density $f = f(\mu)$. These theorems are important because we developed many techniques and gained some intuition for proving the subsequent theorems which are the subject of this chapter.

In Theorem (5), we proved that the conjecture is valid for the case where $\left(\frac{\dot{u}^c\hat{\omega}_b}{\lambda^2 p'}\right)^{\cdot} = 0$, otherwise ($\dot{u}^c\hat{\omega}_b = 0$) and a Killing vector parallel to the vorticity exists. However, the conjecture's validity is still an open question in this case. Finally, in Theorem (6), we proved that if $(f\dot{u}^c\hat{\omega}_b)^{\cdot} = 0$, then ($\dot{u}^c\hat{\omega}_b = 0$) implies the existence of a Killing vector along the vorticity by Theorem (5), on the other hand, the cases where ($\mathcal{F} = 0$) and where ($\mathcal{S} = 0$ given by equation (7.134)) remains elusive and has been left to the interest of the reader and future work.

8

Conclusions and Future Work

8.1 CONCLUSIONS

In Theorem (1) of Chapter (4) of this thesis, we used the $(1+3)$-covariant formalism to establish the consistency of the Einstein Field Equations under the shear-free perfect fluid condition, by providing a covariant proof for the case where the pressure is constant, and the cosmological constant is non-zero. This theorem incorporates the Ellis result for the dust case; see reference [4]. Senovilla et al. and Sopuerta in references [12] and [13] respectively, also used the $(1+3)$-covariant formalism to establish this result.

We then proceed to provide proof of the conjecture in the case where the acceleration and the vorticity vector fields are parallel, given in Theorem (2) of Chapter (6) of this thesis. This proof was first presented in its covariant form by Senovilla et al. in reference [13]. However, it lacked some details and did not include the case of a non-zero cosmological constant. This theorem is one of the milestones toward the proof of the shear-free perfect fluid conjecture. It provides confidence that the conjecture might be true in complete generality, even though this is still an exceptional case.

In Theorem (2), we saw that the conjecture is true when $(\dot{u}^a)$ is parallel to (ω^a). A natural alternative to this is to investigate what happens when the acceleration is not parallel to the vorticity. Following the method provided by Van den Bergh and Slobodeanu in reference [27], we began by showing that if the components of the rescaled acceleration vector field lying in the plane orthogonal to the vorticity are *basic*, then the conjecture holds. In Theorem (3), we proved that if a function of the energy density, namely, $\left(\frac{1}{\lambda^2 p'}\right)$ rescales the acceleration vector field the conjecture holds or a Killing vector exists along the vorticity. In contrast, in Theorem (4), we prove the conjecture where a general function of the energy density $f = f(\mu)$ rescales the acceleration vector field. In Theorem (5) and (6), we assumed that $\left(\frac{\dot{u}_a \omega^a}{\lambda^2 p'}\right)$ and $(f \dot{u}_a \omega^a)$ are *basic* scalars, respectively, these lead to the existence of a Killing vector along the vorticity, where we still need to check if the conjecture is true or not. In addition, the latter case (Theorem 6) leaves the question of the validity of the conjecture where $(\mathcal{F} = 0)$ or $(\mathcal{S} = 0$, given by equation (7.134)) open for investigation.

8.2 FUTURE WORK

We want to extend this work by looking at the case where a Killing vector exists along the vorticity, and the perfect fluid obeys a general barotropic equation of state. In addition, we would like to complete the proof of Theorem (6) in general and show whether ($\Theta\omega = 0$) or not in this case. The so-called *basic* variables may play an essential role in developing the proof of the shear-free perfect fluid conjecture in general or, at the very least, show that a shear-free perfect fluid obeying a barotropic equation of state can simultaneously expand and rotate.

Furthermore, we want to include modified gravity theories such as $f(R)$ gravity. Although counter-examples exist, see reference [36], the conjecture might still be true for some exceptional cases. As a start, we will begin by investigating the dust ($p = 0$) case first. Then we will examine whether the proofs provided here are stable under small perturbations of General Relativity, showing us if the shear-free perfect fluid conjecture is a property of GR or is general to relativistic theories of gravity.

There is some indication that another formalism might be necessary to get down to the core variables of the theory, and some aspects of the conjecture might be more apparent in the $(1+1+2)$-covariant formalism briefly outlined in reference [29]. This formalism provides a new arsenal for attempting the full proof of the shear-free perfect fluid conjecture in General Relativity and beyond.

9

Appendix A

DEFINATIONS

$$\mathcal{X} = -\frac{5}{3} + 2\phi - \frac{\mathcal{E}^2 p^{(3)}}{p'^2} + \frac{4}{9p'} - \frac{5\phi}{3p'} + \frac{\phi^2}{p'} + p', \tag{9.1}$$

$$\mathfrak{S} = -\frac{1}{3} + \phi + \frac{f'\mathcal{E}}{f} \tag{9.2}$$

$$\dot{\mathfrak{S}} = -\mathfrak{R}p'\Theta \tag{9.3}$$

$$\mathfrak{R} = -\frac{\mathfrak{S}^2}{p'} + \frac{\mathfrak{S}(1+6\phi+3p')}{3p'} + \frac{f''\mathcal{E}^2}{p'f} + \frac{(1-3\phi)(2+3\phi+3p')}{9p'} - \chi \tag{9.4}$$

$$\mathfrak{H} = \frac{1}{3}\mathfrak{S}(2+3\mathfrak{S}) - \mathfrak{R}p' \tag{9.5}$$

$$g_1 = -\frac{1}{p'}\left(\mathfrak{H}\Theta + \mathfrak{S}\frac{\dot{\Theta}}{\Theta}\right) \tag{9.6}$$

$$g_2 = \frac{1}{2f}\left(2 - 9\phi - (1-9p')\frac{\mathfrak{S}}{p'}\right) \tag{9.7}$$

$$g_3 = -(1+\mathfrak{S}-p')\Theta - \frac{\dot{\Theta}}{\Theta} \tag{9.8}$$

$$g_4 = 2\left(\frac{1-3p'}{1-9p'}\right)\left(3\phi + \frac{2g_2f}{3}\right) - \frac{(1+18p'-27p'^2)}{3(1-9p')} \tag{9.9}$$

$$g_5 = -\frac{1}{3}\left(2-3p'\right) \tag{9.10}$$

$$\mathfrak{I} = \phi\left(1+9p'\right) + 5p' - 1 \tag{9.11}$$

$$\mathfrak{J} = 6\mathfrak{I} + (2-3p')(1-9p') \tag{9.12}$$

$$\mathfrak{J} = -4 + 9p' + 27p'^2 + 6\phi(1 + 9p') \tag{9.13}$$

$$\mathfrak{K} = 6\mathfrak{J} + 7 - 54p' - 81p'^2 \tag{9.14}$$

$$\begin{aligned} \mathfrak{L} = {} & \mathfrak{K}^2 + 2\mathfrak{K}(11 + 72p' - 81p'^2) - 5112p' + 15390p'^2 - 46656p'^3 \\ & + 32805p'^4 + 1141 - 3744\phi + 2592\phi^2 \end{aligned} \tag{9.15}$$

$$\begin{aligned} \mathfrak{L} = {} & 3888\phi^2(1 + 6p' + 27p'^2) + 144\phi(-29 + 9p' + 324p'^2) \\ & + 12(88 - 630p' + 1431p'^2 - 2916p'^3 + 2187p'^4) \end{aligned} \tag{9.16}$$

$$\mathfrak{D} = -\mathfrak{J}(2 - 9\phi) - 54p'(1 - 9p')\chi \tag{9.17}$$

$$\mathfrak{D} = (2 - 9\phi)(6\phi(1 + 9p') + 9p'(1 + 3p') - 4) - 54p'(1 - 9p')\chi \tag{9.18}$$

$$\mathfrak{A} = -\frac{\mathfrak{D}^2}{36p'} - (2 - 9\phi)\left(\frac{\mathfrak{D}\mathfrak{K}}{54p'} + \frac{(2 - 9\phi)}{1296}\frac{\mathfrak{L}}{p'} + 9\alpha p'(1 - 9p')^2\right) \tag{9.19}$$

$$\begin{aligned} \alpha = {} & \frac{\mathcal{E}^3 p^{(4)}}{p'^3} - \frac{\phi^3}{p'^2} + \frac{\phi^2(4 - 3p')}{p'^2} - \frac{\phi(13 - 21p' + 9p'^2)}{3p'^2} \\ & + \frac{\chi(12\phi + 6p' - 13)}{3p'} + \frac{(1 - 3p')(28 - 24p' + 9p'^2)}{27p'^2} \end{aligned} \tag{9.20}$$

$$
\begin{aligned}
\mathcal{H} = \frac{f^{(3)}}{f}\mathcal{E}^3 &+ (3\mathcal{G} - \mathcal{F}^2)\mathcal{F} - 3(\mathcal{G} - \mathcal{F}^2)\big((1+\phi) + p'\big) \\
&- \mathcal{F}\left(2 + 6\phi + 3\phi^2 + \frac{1}{3}(13 + 15\phi)p' + p'^2 + 3p'\chi\right) \\
&+ \phi\left((1+\phi)(2+\phi) + \frac{1}{3}(13 + 6\phi)p' + p'^2\right) \\
&+ p'\left(\frac{2}{3}(5 + 3\phi) + 2p'\right)\chi + \alpha p'^2.
\end{aligned}
\tag{9.21}
$$

$$
\begin{aligned}
\mathcal{R} = \frac{1}{4}&\big(12 - 40\phi + 21\phi^2 - 36p' + 54\phi p'\big)\dot{u}^b\tilde{\nabla}_b\Theta \\
&+ \frac{1}{4}\Theta\Bigg(\mathcal{E}(2 - 3\phi)(1 + 3p') - \frac{1}{6}\mathcal{J}(2 - 3\phi)(2 + 21\phi) \\
&- \frac{1}{6}\omega^2(86 - 213\phi - 36p' + 270\phi p' - 594p'^2 + 567\phi p'^2) \\
&+ \frac{1}{2}\dot{u}^2(28\phi - 66\phi^2 - 22\chi + 21\phi\chi)\Bigg).
\end{aligned}
\tag{9.22}
$$

$$
\begin{aligned}
\mathfrak{R} = -\frac{3}{4}&(2 - 5\phi - 7\phi p' - 18p'^2)\dot{u}^b\tilde{\nabla}_b\Theta - \frac{3}{4}\Theta\Bigg(\frac{1}{6}\mathcal{J}(2 - 15\phi - 10p' - 21\phi p') \\
&+ \frac{1}{3}\mathcal{E}(1 + 3p')^2 - \frac{1}{6}\omega^2(5 + 89p' - 225p'^2 - 189p'^3) \\
&+ \frac{1}{2}\dot{u}^2(2\phi + 22\phi p' - 5\chi - 7p'\chi)\Bigg).
\end{aligned}
\tag{9.23}
$$

$$
f_1 = (2 - 9\phi)(2\phi - 3\phi^2 - 4p' + 18\phi p') - 3(2 + 9\phi)p'\chi. \tag{9.24}
$$

$$
f_2 = 2\phi - 5\phi^2 - 4p' + 4\phi p' + 9\phi^2 p' + 20p'^2 - 54\phi p'^2 + p'(5 - 9p')\chi. \tag{9.25}
$$

Propagation Equations

The following are some useful time propagation equations used in this thesis.

$$\begin{aligned}
u^c\nabla_c\dot{U}^a &= \lambda p' u^a\dot{U}_c\dot{U}^c + p'\dot{U}^a\Theta - \epsilon^a{}_{cb}\dot{U}^c\hat{\omega}^b\omega + Z^a\omega. && (9.26a)\\
\ddot{u}^{\langle a\rangle} &= -\dot{u}^a\Theta\phi - \epsilon^a{}_{bc}\dot{u}^b\omega^c + p'\tilde{\nabla}^a\Theta. && (9.26b)\\
h^a{}_b\mathcal{L}_u V^b &= \dot{V}^{\langle a\rangle} - \tfrac{1}{3}V^a\Theta + \epsilon^a{}_{bc}V^b\omega^c, && (9.26c)\\
\dot{\mathcal{J}} &= -\mathcal{J}\Theta(\tfrac{1}{3}+\phi) + \tfrac{1}{9}\dot{u}_a\dot{u}^a\Theta(5+3\psi) - p'(1-9p')\Theta\omega^2,\\
&\quad + (1-2\phi)\dot{u}^a\tilde{\nabla}_a\Theta - 3p'(C_1)^a\dot{u}_a - \tfrac{3}{2}p'\tilde{\nabla}_a(C_1)^a, && (9.26d)\\
\dot{\omega}^{\langle a\rangle} &= \Theta\omega^a\big(p' - \tfrac{2}{3}\big), && (9.26e)\\
\dot{\omega} &= \Theta\omega\big(p' - \tfrac{2}{3}\big). && (9.26f)
\end{aligned}$$

$$\begin{aligned}
\dot{\mathfrak{J}} = \Bigg(&\frac{6\phi^2(1+9p'+162p'^2)}{1-9p'} - \frac{\phi(16+126p'+567p'^2-3645p'^3)}{3(1-9p')}\\
&+ \frac{8+81p'-378p'^2-1701p'^3+4374p'^4}{9(1-9p')}\Bigg)\Theta && (9.27)
\end{aligned}$$

$$\begin{aligned}
h^a{}_e h^b{}_d\big(\tilde{\nabla}^e\tilde{\nabla}^d\Theta\big)^{\cdot} = &\left(p' - \frac{4}{3}\right)\Theta\tilde{\nabla}^a\tilde{\nabla}^b\Theta - \omega^a{}_c\left(\tilde{\nabla}^c\tilde{\nabla}^b\Theta + \dot{u}^b\tilde{\nabla}^c\Theta\right)\\
&+ \frac{3}{2}\omega^b{}_c\left((3p'-1)\tilde{\nabla}^a\tilde{\nabla}^c\Theta + \left(\frac{2}{3} - 3\phi\right)\Theta\tilde{\nabla}^a\dot{u}^c\right)\\
&- \frac{3}{2}\omega^b{}_c\left(3\phi\tilde{\nabla}^c\Theta + \dot{u}^c\left(\frac{2}{3} + 3\phi + 3\chi\right)\Theta - 2e^\gamma\tilde{\nabla}^c\mathfrak{H}\right)\dot{u}^a\\
&- \frac{8}{9}\Theta\dot{u}^a\tilde{\nabla}^b\Theta - \left(\frac{1}{3}\Theta\dot{u}^b - \left(1 - \frac{9}{2}\phi\right)\omega^b{}_c\dot{u}^c + \frac{11}{9}\tilde{\nabla}^b\Theta\right)\tilde{\nabla}^a\Theta\\
&- \frac{e^\gamma}{2}\left(\frac{5}{9}\tilde{\nabla}^b\Theta\tilde{\nabla}^a\mathfrak{G} - \epsilon^b{}_{cd}H^{ad}\tilde{\nabla}^c\mathfrak{G} - \left(\tilde{\nabla}_c\Theta\tilde{\nabla}^c\mathfrak{H}\right)h^{ab}\right)\\
&+ (\dot{u}^c\omega_c)\epsilon^{ab}{}_d\Big(\dot{u}^d\Theta - \frac{3}{2}e^\gamma\tilde{\nabla}^d\mathfrak{H}\Big) + \big(\dot{u}^c\tilde{\nabla}_c\Theta\big)\omega^{ab}. && (9.28)
\end{aligned}$$

9.0.1 Commutation relations

Time-space and space-space derivative commutator for any scalar S.

$$h^a{}_b\big[\tilde{\nabla}^b S\big]^{\cdot} = \tilde{\nabla}^a\dot{S} + \dot{u}^a\dot{S} - \tfrac{1}{3}\Theta\tilde{\nabla}^a S + \epsilon^a{}_{cb}\omega^c\tilde{\nabla}^b S, \tag{9.29}$$

$$\tilde{\nabla}^a\tilde{\nabla}^b S = \tilde{\nabla}^b\tilde{\nabla}^a S + 2\epsilon^{ab}{}_c\omega^c\dot{S} \tag{9.30}$$

Spatial gradients of 3-vectors: Time-space derivative commutators.

The following equation is the time-space derivative commutator for the vorticity vector field;

$$\begin{aligned} h^a{}_e h^b{}_d\big[\tilde{\nabla}^e\omega^d\big]^{\cdot} = {} & \tilde{\nabla}^a\dot{\omega}^{\langle b\rangle} + \dot{u}^a\dot{\omega}^{\langle b\rangle} - \tfrac{1}{3}\Theta\tilde{\nabla}^a\omega^b + \epsilon^a{}_{cd}\omega^c\tilde{\nabla}^d\omega^b + \epsilon^{ab}{}_d\omega^d\dot{u}^c\omega_c \\ & + \big[(C_3)^{ad} - H^{ad}\big]\epsilon^b{}_{cd}\omega^c + \omega^a\big[\tfrac{1}{2}(C_1)^b - \tfrac{1}{3}\Theta\dot{u}^b\big] \\ & - h^{ab}\big[\tfrac{1}{2}(C_1)^c - \tfrac{1}{3}\Theta\dot{u}^c\big]\omega_c. \end{aligned} \tag{9.31}$$

The following equation is the time-space derivative commutator for the acceleration vector field;

$$\begin{aligned} h^a{}_e h^b{}_d\big[\tilde{\nabla}^e\dot{u}^d\big]^{\cdot} = {} & \tilde{\nabla}^a\ddot{u}^{\langle b\rangle} - \tfrac{1}{3}\Theta\tilde{\nabla}^a\dot{u}^b + \epsilon^a{}_{cd}\big(\omega^c\tilde{\nabla}^d\dot{u}^b - \dot{u}^b\dot{u}^c\omega^d\big) + \epsilon^{ab}{}_d\omega^d\dot{u}_c\dot{u}^c \\ & + \big[(C_3)^{ad} - H^{ad}\big]\epsilon^b{}_{cd}\dot{u}^c + \dot{u}^a\big[\tfrac{1}{2}(C_1)^b + \ddot{u}^{\langle b\rangle} - \tfrac{1}{3}\Theta\dot{u}^b\big] \\ & - h^{ab}\big[\tfrac{1}{2}(C_1)^c - \tfrac{1}{3}\Theta\dot{u}^c\big]\dot{u}_c. \end{aligned} \tag{9.32}$$

Space-space derivative commutators:

The following equation is the space-space derivative commutator for the vorticity vector field;

$$\tilde{\nabla}^{[a}\tilde{\nabla}^{b]}\omega_c = E^{[a}{}_c\omega^{b]} + h^{[a}{}_cE^{b]}{}_d\omega^d + \tfrac{1}{3}(\Lambda - \tfrac{1}{3}\Theta^2 + \mu)h^{[a}{}_c\omega^{b]}$$
$$- \tfrac{1}{3}\Theta\epsilon^{[a}{}_{cd}\omega^{b]}\omega^d + \epsilon^{ab}{}_d\omega^d\dot{\omega}_{\langle c\rangle}. \tag{9.33}$$

The following equation is the anti-symmetric spatial derivative for the vorticity vector field;

$$\tilde{\nabla}_{[a}\omega_{b]} = \tfrac{1}{2}\epsilon_{abc}(C_1)^c - 2\dot{u}_{[a}\omega_{b]} + \tfrac{1}{3}\epsilon_{abc}\tilde{\nabla}^c\Theta. \tag{9.34}$$

The following equation is the space-space derivative commutator for the acceleration vector field;

$$\tilde{\nabla}^{[a}\tilde{\nabla}^{b]}\dot{u}_c = E^{[a}{}_c\dot{u}^{b]} + h^{[a}{}_cE^{b]}{}_d\dot{u}^d + \tfrac{1}{3}\big(\Lambda - \tfrac{1}{3}\Theta^2 + \mu - 3\omega^2\big)h^{[a}{}_c\dot{u}^{b]}$$
$$- \tfrac{1}{3}\Theta\epsilon^{[a}{}_{cd}\dot{u}^{b]}\omega^d - \tfrac{1}{3}\Theta h^{[a}{}_c\epsilon^{b]}{}_{de}\dot{u}^d\omega^e - \dot{u}^{[a}\omega^{b]}\omega_c$$
$$+ \dot{u}^d\omega_d h^{[a}{}_c\omega^{b]} + \epsilon^{ab}{}_d\omega^d\ddot{u}_{\langle c\rangle} \tag{9.35}$$

The following equation is the anti-symmetric spatial derivative for the acceleration vector field;

$$\tilde{\nabla}_{[a}\dot{u}_{b]} = \epsilon_{abc}\omega^c p'\Theta. \tag{9.36}$$

SPATIAL GRADIENTS OF 3-TENSORS: EVOLUTION OF SPATIAL DIVERGENCE TERMS

The following propagation equation is the time-space derivative commutator H_{ab};

$$h^a{}_b\big[\tilde{\nabla}_cH^{bc}\big]^{\cdot} = \tilde{\nabla}_b\dot{H}^{\langle ab\rangle} + \dot{u}_b\dot{H}^{\langle ab\rangle} - \tfrac{1}{3}\Theta\tilde{\nabla}_bH^{ab} - \epsilon_{bcd}\omega^b\tilde{\nabla}^dH^{ac}$$
$$+ \Theta\dot{u}_bH^{ab} + \epsilon_{bcd}H^{ad}\dot{u}^b\omega^c + \epsilon^a{}_{cd}H_b{}^d\dot{u}^b\omega^c$$

$$- \tfrac{3}{2} H^{a}{}_{b} (C_1)^{b} - \epsilon^{a}{}_{cd} (C_3)^{bc} H_{b}{}^{d}. \tag{9.37}$$

The following propagation equation is the time-space derivative commutator E_{ab};

$$\begin{aligned} h^{a}{}_{b} \big[\tilde{\nabla}_{c} E^{bc}\big]^{\cdot} = {} & \tilde{\nabla}_{b} \dot{E}^{\langle ab \rangle} + \dot{u}_{b} \dot{E}^{\langle ab \rangle} - \tfrac{1}{3} \Theta \tilde{\nabla}_{b} E^{ab} - \epsilon_{bcd} \omega^{b} \tilde{\nabla}^{d} E^{ac} - \epsilon^{a}{}_{cd} H_{b}{}^{d} E^{bc} \\ & + \Theta \dot{u}_{b} E^{ab} + \epsilon_{bcd} E^{ad} \dot{u}^{b} \omega^{c} + \epsilon^{a}{}_{cd} E_{b}{}^{d} \dot{u}^{b} \omega^{c} - \tfrac{3}{2} E^{a}{}_{b} (C_1)^{b} \\ & - \epsilon^{a}{}_{cd} (C_3)^{bc} E_{b}{}^{d}. \end{aligned} \tag{9.38}$$

Space-space derivative commutators: The following equation is the space-space derivative commutator for H_{ab};

$$\begin{aligned} \tilde{\nabla}^{[a} \tilde{\nabla}^{b]} E_{cd} = {} & -2E^{[a}{}_{(c} \omega^{b]} \omega_{d)} + 2h^{[a}{}_{(c} E^{b]e} E_{d)e} + \tfrac{2}{3} h^{[a}{}_{(c} E^{b]}{}_{d)} \big(\Lambda - \tfrac{1}{3}\Theta^{2} + \mu - 3\omega^{2}\big) \\ & + 2h^{[a}{}_{(c} \omega^{b]} E_{d)e} \omega^{e} - \tfrac{2}{3} \Theta \epsilon^{[a}{}_{(ce} \omega^{e} E^{b]}{}_{d)} + \tfrac{2}{3} \Theta h^{[a}{}_{(c} \epsilon^{b]}{}_{ej} \omega^{e} E_{d)}{}^{j} + \omega^{ab} \dot{E}_{\langle cd \rangle} \end{aligned} \tag{9.39}$$

IDENTITIES

The following algebraic and differential identities have been applied in the time propagation of constraint equations see reference [47, 50].

$$0 = \epsilon^{abc} \tilde{\nabla}_{\langle b} \omega_{d \rangle} \dot{u}_{c} \omega^{d} - \epsilon^{abc} \tilde{\nabla}_{\langle b} \omega_{d \rangle} \dot{u}^{d} \omega_{c} - \tilde{\nabla}^{\langle a} \omega^{b \rangle} \epsilon_{bcd} \dot{u}^{c} \omega^{d} \tag{9.40a}$$

$$0 = \epsilon^{a}{}_{bd} \dot{u}^{b} \tilde{\nabla}_{c} H^{cd} - \epsilon_{bcd} \dot{u}^{b} \tilde{\nabla}^{d} H^{ac} + \epsilon^{a}{}_{cd} \dot{u}^{b} \tilde{\nabla}^{d} H_{b}{}^{c} \tag{9.40b}$$

$$0 = \epsilon^{a}{}_{bd} \omega^{b} \tilde{\nabla}_{c} E^{cd} - \epsilon_{bcd} \omega^{b} \tilde{\nabla}^{d} E^{ac} + \epsilon^{a}{}_{cd} \omega^{b} \tilde{\nabla}^{d} E_{b}{}^{c} \tag{9.40c}$$

$$0 = -\epsilon_{bcd} H^{ad} \tilde{\nabla}^{c} \dot{u}^{b} + \epsilon^{a}{}_{cd} H_{b}{}^{d} \tilde{\nabla}^{c} \dot{u}^{b} - \epsilon^{a}{}_{bd} H_{c}{}^{d} \tilde{\nabla}^{c} \dot{u}^{b} \tag{9.40d}$$

$$0 = E_{c}{}^{d} \epsilon^{a}{}_{bd} \tilde{\nabla}^{c} \omega^{b} - E_{b}{}^{d} \epsilon^{a}{}_{cd} \tilde{\nabla}^{c} \omega^{b} + E^{ad} \epsilon_{bcd} \tilde{\nabla}^{c} \omega^{b} \tag{9.40e}$$

$$0 = -E_c{}^d \epsilon^a{}_{bd} \dot{u}^b \omega^c + E_b{}^d \epsilon^a{}_{cd} \dot{u}^b \omega^c - E^{ad} \epsilon_{bcd} \dot{u}^b \omega^c \tag{9.40f}$$

$$0 = \epsilon^a{}_{bd} \dot{u}^b \tilde{\nabla}_c E^{cd} - \epsilon_{bcd} \dot{u}^b \tilde{\nabla}^d E^{ac} + \epsilon^a{}_{cd} \dot{u}^b \tilde{\nabla}^d E_b{}^c \tag{9.40g}$$

$$0 = \epsilon^a{}_{bd} \omega^b \tilde{\nabla}_c H^{cd} - \epsilon_{bcd} \omega^b \tilde{\nabla}^d H^{ac} + \epsilon^a{}_{cd} \omega^b \tilde{\nabla}^d H_b{}^c \tag{9.40h}$$

$$0 = \epsilon_{bcd} H^{ad} \tilde{\nabla}^c \omega^b - \epsilon^a{}_{cd} H_b{}^d \tilde{\nabla}^c \omega^b + \epsilon^a{}_{bd} H_c{}^d \tilde{\nabla}^c \omega^b \tag{9.40i}$$

$$0 = -\epsilon_{bcd} H^{ad} \dot{u}^b \omega^c + \epsilon^a{}_{cd} H_b{}^d \dot{u}^b \omega^c - \epsilon^a{}_{bd} H_c{}^d \dot{u}^b \omega^c \tag{9.40j}$$

$$0 = -E_c{}^d \epsilon^a{}_{bd} \tilde{\nabla}^c \dot{u}^b + E_b{}^d \epsilon^a{}_{cd} \tilde{\nabla}^c \dot{u}^b - E^{ad} \epsilon_{bcd} \tilde{\nabla}^c \dot{u}^b \tag{9.40k}$$

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Appendix B

MATHEMATICA PSEUDOCODE FOR THE CASE WHERE THE PRESSURE IS CONSTANT.

The following describes the code used in this thesis to evaluate the space and time covariant derivative commutation relations. To perform these calculations, one needs to download and install a Mathematica package named **xTensor**, and all the necessary sub-packages, which can be found on the following xAct website:
“**http** : //**www.xact.es**/″; this package can be called inside a much broader package

called **xTras**. Additionally, one can download some smaller contributed packages, which can be found at "**http : //contrib.xact.es/**″.

We first load some required packages into Mathematica before we start; this can be done by typing the following in the Mathematica notebook command line.

```
<< xAct`xTras`            (*xAct is a set of packages for performing
symbolic computer algebra for tensors in the Wolfram Language*)
<< xAct`TexAct`         (*TexAct is a package for printing output in
Latex format for typesetting*)
<< xAct`TraceFree`   (*TraceFree is a package for defining traceless
tensors*)
```

After loading the above packages, we are now ready to set up our scenario $(\mathcal{M}, g_{ab}, u^a)$, followed by the definition of all necessary tensorial objects which are required for describing the General Theory of Relativity as follows,

```
dimension = 4;
DefManifold[M4, dimension]; (*Defining a 4 dimensional manifold M4*)
DefMetric[-1, gab[-a, -b]]; (*Defining the metric tensor on M4*)
DefTensor[ua[-a], {M4}] ; (*Defining the velocity 4vector on M4*)
DefMetric[1, hab[-a, -b]]; (*Defining the spatial induced metric*)
DefTensor[uadot[-c], {M4}, OrthogonalTo -> {ua[c]}, ProjectedWith ->
                {hab[c, -a]}]; (*Defining the acceleration vector*)
DefTensor[\Theta[], {M4}];  (*Defining the expansion scalar Theta*)
DefTensor[Eab[-c, -d], {M4}, Symmetric[{-c, -d}],
                TraceFree -> {{{-c, -d}, hab}},
```

```
OrthogonalTo -> {ua[c], ua[d]},
ProjectedWith -> {hab[c, -a], hab[d, -b]}]; (*Defining
the electric part of the Weyl tensor*)
```

The above rules of defining tensors are general; we can similarly define any tensor we require. To do any useful calculations in this framework, it suffices to define some tensorial identities in the form of automatic rules so that we don't worry about these identities from now onwards. These automated rules can help significantly simplify our calculations, and we can accumulate some of them as we go along. The Mathematica code and notebooks used to perform the calculations in this thesis can be found here: *https://www.dropbox.com/sh/ckzjqwudynlrjhd/AAAsw2DU22C3LkfDpitdPJ4fa?dl=0.*

References

[1] Gödel K. An example of a new type of cosmological solutions of Einstein's field equations of gravitation. Reviews of Modern Physics. 1949;21(3):447.

[2] Gödel K. Rotating universes Proc Int Cong Math (Camb, Mass). Ed LM Graves et al. 1952;1:175.

[3] Schücking E. Homogene scherungsfreie Weltmodelle in der relativistischen Kosmologie. Naturwissenschaften. 1957;44(19):507--507.

[4] Ellis GR. Dynamics of Pressure-Free Matter in General Relativity. Journal of Mathematical Physics. 1967;8(5):1171--1194.

[5] White A, Collins C. A class of shear-free perfect fluids in general relativity. I. Journal of mathematical physics. 1984;25(2):332--337.

[6] Banerji S. Homogeneous cosmological models without shear. Progress of Theoretical Physics. 1968;39(2):365--371.

[7] Treciokas R, Ellis G. Isotropic solutions of the Einstein-Boltzmann equations. Communications in Mathematical Physics. 1971;23(1):1--22.

[8] Coley AA. Fluid spacetimes admitting a conformal Killing vector parallel to the velocity vector. Classical and Quantum Gravity. 1991;8(5):955.

[9] King AR, Ellis GF. Tilted homogeneous cosmological models. Communications in Mathematical Physics. 1973;31(3):209--242.

[10] Lang J, Collins C. Observationally homogeneous shear-free perfect fluids. General relativity and gravitation. 1988;20(7):683--710.

[11] Lang JM. Contributions to the Study of General Relativistic Shear-Free Perfect Fluids-An Approach Involving Cartan's Equivalence Method, Differential Forms and Symbolic Computation. 1994;.

[12] Sopuerta CF. Covariant study of a conjecture on shear - free barotropic perfect fluids. Class Quant Grav. 1998;15:1043--1062.

[13] Senovilla JM, Sopuerta CF, Szekeres P. Theorems on shear-free perfect fluids with their Newtonian analogues. General Relativity and Gravitation. 1998;30(3):389--411.

[14] Collins C. Shear-free perfect fluids with zero magnetic Weyl tensor. Journal of mathematical physics. 1984;25(4):995--1000.

[15] Cyganowski S, Carminati J. Shear-free perfect fluids in general relativity: gravito-magnetic spacetimes. General Relativity and Gravitation. 2000;32(2):221--233.

[16] Collins C. Shear-free fluids in general relativity. Canadian journal of physics. 1986;64(2):191--199.

[17] Slobodeanu R. Harmonic morphisms and shear-free perfect fluids coupled with gravity. Annali di Matematica Pura ed Applicata (1923-). 2014;193(4):1139--1146.

[18] Carminati J. Shear-free perfect fluids in general relativity II. Aligned, Petrov type III space-times. Journal of mathematical physics. 1990;31(10):2434--2440.

[19] Carminati J, Cyganowski S. Shear-free perfect fluids in general relativity: III. Petrov type III spacetimes. Classical and Quantum Gravity. 1996;13(7):1805.

[20] Carminati J, Cyganowski S. Shear-free perfect fluids in general relativity: IV. Petrov type III spacetimes. Classical and Quantum Gravity. 1997;14(5):1167.

[21] Van den Bergh N, Carminati J, Karimian HR, Huf P. Shear-free perfect fluids with a solenoidal electric curvature. Classical and quantum gravity. 2012;29(10):105010.

[22] Carminati J, Karimian HR, Van Den Bergh N, Vu KT. Shear-free perfect fluids with a solenoidal magnetic curvature. Class Quant Grav. 2009;26:195002.

[23] Van den Bergh N, Carminati J, Karimian HR. Shear-free perfect fluids with solenoidal magnetic curvature and a γ-law equation of state. Classical and quantum gravity. 2007;24(14):3735.

[24] Slobodeanu R. Shear-free perfect fluids with linear equation of state. Classical and Quantum Gravity. 2014;31(12):125012.

[25] Nzioki AM, Goswami R, Dunsby PK, Ellis GF. Shear-free perturbations of Friedmann-Lemaître-Robertson-Walker universes. Physical Review D. 2011;84(12):124028.

[26] Carminati J. Shear-free perfect fluids with a\ gamma-law equation of state. General Relativity and Gravitation. 2015;47(4):1--10.

[27] Van den Bergh N, Slobodeanu R. Shear-free perfect fluids with a barotropic equation of state in general relativity: the present status. Classical and Quantum Gravity. 2016;33(8):085008.

[28] Sikhonde ME, Dunsby PK. Reviving the shear-free perfect fluid conjecture in general relativity. Classical and Quantum Gravity. 2017;34(24):245007.

[29] Goswami R, Ellis GFR. Shear free barotropic perfect fluids cannot rotate and expand simultaneously. 2021 09;Available from: https://arxiv.org/pdf/2109.07776.pdf.

[30] den Bergh NV, Carminati J. Comment on "Shear-free barotropic perfect fluids cannot rotate and expand simultaneously" by R. Goswami and G.F.R. Ellis. 2021 11;Available from: https://arxiv.org/pdf/2111.06303.pdf.

[31] Karimian H. Contributions to the study of shear-free and of purely radiative perfect fluids in general relativity. Ghent University; 2012.

[32] Obukhov YN, Chrobok T, Scherfner M. Shear-free rotating inflation. Physical Review D. 2002;66(4):043518.

[33] Ellis GF. Shear free solutions in general relativity theory. General Relativity and Gravitation. 2011;43(12):3253--3268.

[34] Heckmann O, Schücking E. Bemerkungen zur Newtonschen Kosmologie. I. Mit 3 Textabbildungen in 8 Einzeldarstellungen. Zeitschrift fur Astrophysik. 1955;38:95.

[35] Narlikar J, Hoyle F. Newtonian universes with shear and rotation. Monthly Notices of the Royal Astronomical Society. 1963;126(2):203--208.

[36] Sofuoğlu D, Mutuş H. Investigations of f (R)-gravity counterparts of the general relativistic shear-free conjecture by illustrative examples. General Relativity and Gravitation. 2014;46(12):1--25.

[37] Abebe A, Goswami R, Dunsby PK. Shear-free perturbations of f (R) gravity. Physical Review D. 2011;84(12):124027.

[38] Pantilie R. Harmonic morphisms with 1-dim fibres on 4-dim Einstein manifolds. Preprint, University of Leeds. 1999;.

[39] MacCallum MAH. Integrability in tetrad formalisms and conservation in cosmology. In: Rainer M, Schmidt HJ, editors. Current Topics in Mathematical Cosmology (Proceedings of the International Seminar). Singapore: World Scientific; 1998. p. 133--142.

[40] Ellis GF, Van Elst H. Cosmological models. In: Theoretical and Observational Cosmology. Springer; 1999. p. 1--116.

[41] Ellis GF, MacCallum MA. A class of homogeneous cosmological models. Communications in Mathematical Physics. 1969;12(2):108--141.

[42] Ehlers J, Jordan P, Sachs RK. Beiträge zur Theorie der reinen Gravitationsstrahlung. Verlag der Akademie der Wissenschaften und der Literatur in Mainz. 1961;.

[43] Kristian J, Sachs R. Observations in cosmology. The Astrophysical Journal. 1966;143:379.

[44] Trümper M. On a Special Class of Type-I Gravitational Fields. Journal of Mathematical Physics. 1965;6(4):584--589.

[45] Hawking SW. The occurrence of singularities in cosmology. III. Causality and singularities. In: Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. vol. 300. The Royal Society; 1967. p. 187--201.

[46] Ellis GF. Topology and cosmology. General Relativity and Gravitation. 1971;2(1):7--21.

[47] Van Elst H. Extensions and applications of 1+ 3 decomposition methods in general relativistic cosmological modelling. PhD thesis; 1996.

[48] Ehlers J. J. Ehlers, Abh. Math. Naturwiss. Kl., Akad. Wiss. Lit., Mainz 11, 793 (1961). Abh Math Naturwiss Kl, Akad Wiss Lit, Mainz. 1961;11:793.

[49] Ellis G. General Relativity and Cosmology, Proceedings of the XLVII Enrico Fermi Summer School. Academic, New York. 1971;.

[50] Maartens R. Linearization instability of gravity waves? Physical Review D. 1997;55(2):463.

[51] Maartens R, Bassett BA. Gravito-electromagnetism. Classical and Quantum Gravity. 1998;15(3):705.

[52] White AJ, Collins CB. A class of shear-free perfect fluids in general relativity. II. Journal of Mathematical Physics. 1984;25(2):332--337.

[53] den Bergh NV. The shear-free perfect fluid conjecture. Classical and Quantum Gravity. 1999 jan;16(1):117--129. Available from: https://doi.org/10.1088/0264-9381/16/1/009.

[54] Slobodeanu R. Shear-free perfect fluids with linear equation of state. Class Quant Grav. 2014;31:125012.