Periodic Solutions of Nonlinear Wave Equations

A Thesis presented to the University of Cape Town and prepared under the supervision of Professor R.I. Becker in fulfilment of the requirements of the Degree of Master of Science in Mathematics
by

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Chapter 1

Introduction

This thesis is devoted to the construction of periodic solutions for partial differential equations. The study of periodic solutions was introduced by H. Poincaré in the memoire he presented on the three body problem[1]. By looking at periodic and asymptotics solutions he discovered what today we call chaos, and showed that these special solutions were a powerful tool to resolve questions in the theory of ordinary differential equations.

Unlike the case of ordinary differential equations, there does not exist a theory of periodic solutions for nonlinear partial differential equations; until the sixties the construction of periodic solutions was only known in special cases. Rabinowitz was the first to prove a theorem in that direction. He considered the problem:

$$\begin{cases} \partial_{tt}u = \partial_{xx}u - f(u) \\ 0 \leq x \leq \pi \quad x, t \in \mathbb{R} \\ u(0,t) = u(\pi,t) = 0 \end{cases} \tag{1.0.1}$$

using variational methods he was only able to prove that (1.0.1) admits periodic solutions with rational frequency[2]. However, this was quite surprising since these restrictions never occur when dealing with ordinary differential equations. The technique that we present here was introduced by Craig and Wayne who constructed periodic solutions with frequency Ω in a Cantor set of non-zero measure (see [9]). So they proved that there are periodic

[1] One may consult the book [1] for more details about the memoire

[2] A survey of all the results using variational methods can be found in [6], but they all have the that restriction the frequency is rational

solutions with irrational period as well. They considered the problem:

$$\begin{cases} \partial_{tt}u = (\partial_{xx} - v(x))u - g(u,x) \\ 0 \leq x \leq \pi \quad x,t \in \mathbb{R} \\ u(0,t) = u(\pi,t) = 0 \\ \textbf{Dirichlet boundary condition.} \end{cases} \tag{1.0.2}$$

and proved that (1.0.2) has solutions, provided that it satisfies both a nonresonance and a twist condition. The first advantage over variational techniques in (1.0.2) is the fact that the linear operator $(\partial_{xx} - v(x))$ depends on x. This technique has also showed great flexibility if one wants to generalize these techniques to higher space dimension. In 1994 Bourgain was able to generalised the technique to higher dimensional space domain in [2], and in 1996 to construct almost-periodic solutions for one-dimensional problem (see [3]). Recently Li-Ta-tsien, Shi Yumming and Qin Tiehu considering a nonlinear wave equation (with a nonlinearity depending on u only), were able also to construct periodic solutions for equations in higher dimension (see [16]).

Another advantage of this technique is its applicability to the nonlinear Schrödinger equation. In the case of one space dimension Schrödinger equations, Craig and Wayne showed that the technique can contruct periodic solutions. Once again Bourgain extended these techniques to higher dimensional spaces problem for periodic solutions and to two spacial dimensions problems for quasi-periodic solutions in [3],[4],[5]. Grébert and Guillot also proved that the technique can be used for coupled nonlinear Schrödinger equations in [12] and [13].

Kuksin has also contributed to the field by extending the traditional KAM approach for ordinary differential equations to the construction of periodic and quasi-periodic solutions for one-dimesional wave and Schrödinger equations in [14] and with Pöschel in [15].

As in [9] we start with an ansatz $u = \sum_{j,k} \hat{u}(j,k)\psi_j(x)e^{ik\xi}$, where the $\psi_j(x)$ are the eigenfunctions of the Sturm-Liouville operator $(\partial_{xx} - v(x))$ with the boundary conditions $\psi_j(0) = \psi_j(\pi) = 0$ and eigenvalues denoted ω_j^2. The solution of (1.0.2) must satisfy:

$$F(u) = (\omega_j^2 - \Omega^2 k^2)\hat{u}(j,k) - W(u)(j,k) = 0, \tag{1.0.3}$$

where $W(u)$ is the nonlinear operator resulting from expanding the nonlinearity $g(u,x)$ in terms of the eigenfunctions. The values $\omega_j^2 - \Omega^2 k^2$ accumulate at zero, a phenomenom called the *small denominator problem in partial differential equations* [18], making it difficult to compute the inverse of $F'(u)$. One of the amazing features of this technique is the method of inverting the linearized operators[3] during Newton's method (in fact we use a variant of Newton's method called the Nash-Moser method) which originally comes from Quantum Mechanics and the work of Fröhlich and Spencer in [11]. The main difficulty in this technique is to prove that the sequence converges and to find suitable normed spaces to which $\hat{u}(j,k)$ must belong. Instead of using the normed spaces H_σ:

$$H_\sigma = \{\hat{u} \in l^2(\mathbb{N} \times \mathbb{Z}) \ ||u||_\sigma^2 = \sum_{j,k} |\hat{u}(j,k)|^2 e^{2\sigma(|j|+|k|)} < \infty\} \tag{1.0.4}$$

as in [9]; we will use the spaces $H^{m,s}$:

$$H^{m,s} = \{\hat{u} \in l^2(\mathbb{N} \times \mathbb{Z}) \ ||u||_{m,s}^2 = \alpha \sum_{l \in \mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |\hat{u}(l)|^2 e^{2m|l|} < \infty\} \tag{1.0.5}$$

which have the advantage of being a Banach Algebra and also simplifiy the relationship between the function u and its lattice representation $\hat{u}(j,k)$. Most of the properties of the spaces H_σ and $H^{m,s}$ depend on the fact that the eigenfunctions $\psi_j(x)$ decay very rapidly. We prove this property, which was enunciated without proof in [9] and this constitutes the whole of chapter 7. The change from H_σ to $H^{m,s}$ leads to various changes. We redefine the sequence ρ_n which measures the domain of analaticity of the approximate solution u_n at the n^{th} stage and consequently modify the proof of that Newton's method converges. The spaces $H^{m,s}$ were introduced in [10] in the case of the nonlinear Schröedinger equation; here we show that they are also suitable for the wave equation and the proof of the construction of the solution to (1.0.2) presented here is simpler than the one in [9].

We end the introduction with an outline of the thesis. In chapter 2, assuming some properties on the eigenfunctions (which we prove in chapter 7) we show some propeties of the spaces $H^{m,s}$ (many of theses properties were enunciated without proof in [10]). Then we introduce a new norm, $||S||_{m,s}$ slightly different from the operator norm used in [10]. This norm has the advantage over the one in [10] to be sub-multiplicative ($||ST||_{m,s} \leq ||S||_{m,s}||T||_{m,s}$). At the end of the chapter we prove a lemma on the inverse of self-adjoint operators and prove inequalities which will be used in the thesis. In chapter 3, using the Lyapounov-Schmidt decomposition we split (1.0.2) in an infinite dimensional and a finite dimensional problem. The

[3] The technique is used in chapter 4 to get a good estimate on the inverse of $F'(u)$

infinite dimensional problem is solved using the Nach-Moser method with the assumption that the derivative of $F(u)$ is invertible at each iterative step, and assuming a nonresonance condition on $g_1(x)$. To ensure the convergence of the Nash-Moser method with theses new normed spaces we have introduced a new sequence ρ_n, so the proof of the convergence is the Nash-Moser method differs from the one in [10]. The chapter 4 proves an estimate used in chapter 3 to ensure the convergence of the Nash-Moser scheme. In chapter 5 we find the parameters for which the derivative of $F'(u)$ is not invertible and the Twist condition is used to extract them from the interval to which the parameters belong. In chapter 6 we solve the finite dimensional problem and hence finish the construction of the solution of (1.0.2). The last chapter is the proof of an estimate of the rapid decay of the eigenfunctions. We choose the frequency Ω close to ω_1 the first frequency of the spectrum of $-\frac{d^2}{dx^2} + g_1(x)$. For $\Omega = \omega_1$ and $k = \pm 1$ we have a solution of the linear problem in (1.0.3) and thus we construct our solution as perturbation of the linear solution. As in [9] we get to the conclusion that:

Theorem 1.1 *Let $|p| < r_0$, there exists a Cantor set $\mathcal{M}$ of measure non-zero such that for $p \in \mathcal{M}$ there exists a periodic solution of (1.0.2) denoted $\widetilde{\phi(p)} + \widetilde{u}(p, \Omega(p))$ which satisfies the following estimate:*

$$|\widetilde{u}(p, \Omega(p))| < C|p|^2, \tag{1.0.6}$$

and

$$|\Omega(p) - \omega_1| < C|p|^2, \tag{1.0.7}$$

where $\widetilde{\phi(p)} = \frac{p}{2}\psi_1(x)e^{i\Omega t} + \frac{p}{2}\psi_1(x)e^{-i\Omega t}$

Chapter 2

Preliminaries

In this thesis we will construct time-periodic solutions of the Wave equation:

$$\begin{cases} \partial_{tt}u = (\partial^2_{xx} - v(x))u - g(u,x) \\ 0 \le x \le \pi \quad x, t \in \mathbb{R} \\ u(0,t) = u(\pi,t) = 0 \\ \textbf{Dirichlet boundary condition.} \end{cases} \tag{2.0.1}$$

Our assumptions on the v are:
v is analytic in the strip $|Im\ x| \le \overline{\sigma}$
v is real on the real axis
v is even
$v(0) = v(\pi) = 0$ and v is 2π-periodic.
We call L_v the Sturm-Liouville operator $-\frac{d^2}{dx^2} + v$ and we label its eigenfunctions(respectively eigenvalues) ψ_j (respectively ω_j^2, in the case of Dirichlet boundary condition there are only a finite possible number of negative eigenvalues), $j \in \mathbb{N}, j \ge 1$.

The preceding conditions ensure that the eigenfunctions of the Sturm-Liouville operator can be analytically extended to 2π odd analytic functions and admit an expansion of the form (see[9])

$$\psi_n(x) = \sum_{p \ge 1} \hat{\psi}_n(p) \sin px \ (\ \mathrm{x} \in \mathbb{C}.) \tag{2.0.2}$$

The inner product is defined as

$$(u,v) = \frac{1}{2\pi} \int_0^{2\pi} u(x)\overline{v(x)}dx$$

and we normalize our eigenfunctions such that

$$(\psi_j, \psi_j) = 1.$$

And if u and v are two sequences in $l^2(\mathbb{Z})$ we define the inner product as

$$(u, v)_{l^2(\mathbb{Z})} = \sum_{k \in Z} u(k)\overline{v(k)}.$$

The nonlinearity $g(u, x)$ is assumed to be polynomial of the form

$$g(u, x) = \sum_{p=3}^{k} a_p(x) u^p \tag{2.0.3}$$

where the $a_p(x)$ satisfy the condition

$$a_p(0) = a_p(\pi) = 0, \tag{2.0.4}$$

$g(x, u) = -g(-x, -u)$ and the nonlinearity g is of at least order 3 .
Rescaling the coordinate $t \longrightarrow \frac{\xi}{\Omega}$, the partial differential equation (2.0.1) becomes

$$\Omega^2 \partial^2_{\xi\xi} u = (\partial^2_{xx} - v(x))u - g(u, x)$$

and we search for solutions of period 2π in ξ. In this thesis we construct solution with frequency Ω close to ω_1, which we suppose positive (with little loss of generality). The frequency Ω is choosen very close to ω_1 so that $\omega_1/2 \leq \Omega_{min} \leq \Omega \leq \Omega_{max} \leq 2\omega_1$. We start by an ansatz and search for solutions of the form

$$u(x, \xi) = \sum_{j \in \mathbb{N}, k \in \mathbb{Z}} \hat{u}(j, k) \psi_j(x) \frac{e^{ik\xi}}{\sqrt{2\pi}} \tag{2.0.5}$$

and obtain formally

$$\sum_{j,k} \Omega^2 (i^2 k^2) \hat{u}(j, k) \psi_j(x) \frac{e^{ik\xi}}{\sqrt{2\pi}} = -L_v\Big(\sum_{j,k} \hat{u}(j, k) \psi_j(x) \frac{e^{ik\xi}}{\sqrt{2\pi}}\Big) - g(u, x)$$

$$\sum_{j,k} -\Omega^2 k^2 \psi_j(x) \frac{e^{ik\xi}}{\sqrt{2\pi}} = -\sum_{j,k} \hat{u}(j, k) \frac{e^{ik\xi}}{\sqrt{2\pi}} \omega_j^2 \psi_j(x) - g(u, x)$$

$$\sum_{j,k} \hat{u}(j, k)(\omega_j^2 - \Omega^2 k^2) \frac{e^{ik\xi}}{\sqrt{2\pi}} \psi_j(x) + g(u, x) = 0 \tag{2.0.6}$$

Our aim is to find the coefficients $\hat{u}(j,k)$ and then go back to get a solution in the x and ξ variables. The solution that we construct is an eigenfunction expansion so we will need a rapid decay of the eigenfunction to ensure the convergence of the series and so a solution in x and ξ variables. We will thefore define suitables spaces to which $\hat{u}(j,k)$ will belong.

2.1 Functions Spaces

The properties of the functions spaces depend heavily on some estimates on the eigenfunctions. However the proof of these estimates is very long so we have put it in chapter 7.
Let $H^{m,s}$ be:

$$H^{m,s} = \{\hat{u} \in l^2(\mathbb{N}\times\mathbb{Z})\ ||u||^2_{m,s} = \alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |\hat{u}(l)|^2 e^{2m|l|} < \infty\}. \tag{2.1.7}$$

Theorem 2.1 [10] *Let m be a positive integer, and a real number $s \geq 2$. We denote by $H^{m,s}$ the space of sequences*

$$H^{m,s} = \{\hat{u} \in l^2(\mathbb{N}\times\mathbb{Z})\ ||u||^2_{m,s} = \alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |\hat{u}(l)|^2 e^{2m|l|} < \infty\}. \tag{2.1.8}$$

There exists a positive real number α, such that $H^{m,s}$ is a Banach algebra. Furthermore if $m' \leq m$ then

$$||\hat{u}||_{m',s} \leq ||\hat{u}||_{m,s} \tag{2.1.9}$$

Proof:
Let $(u_n)_n$ be a Cauchy-sequence in $H^{m,s}$.

$$\forall \varepsilon, \exists N(\varepsilon) \text{ such that } p,q > N \implies ||u_p - u_q||_{m,s} < \varepsilon$$

which can be rewritten as

$$\sqrt{\alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |(u_p-u_q)(l)|^2 e^{2m|l|}} < \varepsilon$$

So

$$|(u_p-u_q)(l)| < \varepsilon \quad \forall l \in \mathbb{N}\times\mathbb{Z}.$$

For a given l, $(u_n(l))_n$ is a Cauchy-sequence in $\mathbb{C}$ which is complete and therefore $(u_n(l))_n$ is convergent. Let us denote $u(l)$ its limit $u(l) := \lim(u_n(l))_n$. We have to prove that $u \in H^{m,s}$. $(u_n)_n$ is a Cauchy-sequence in $H^{m,s}$ and therefore is bounded so

$$\alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |u_n(l)|^2 e^{2m|l|} < \infty,$$

and passing to the limit we deduce

$$\lim_n \alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |u_n(l)|^2 e^{2m|l|} < \infty$$

$$\alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s \lim_n |u_n(l)|^2 e^{2m|l|} < \infty$$

$$\alpha \sum_{l\in\mathbb{N}\times\mathbb{Z}} (1+|l|^2)^s |u(l)|^2 e^{2m|l|} < \infty$$

so $u \in H^{m,s}$. Let $u(x,\xi)$ be

$$u(x,\xi) = \sum_{j,k} \hat{u}(j,k)\psi_j(x)\frac{e^{ik\xi}}{\sqrt{2\pi}}$$

where

$$\hat{u}(j,k) = \int_0^\pi \int_0^{2\pi} u(x,\xi)\bar{\psi}_j(x)\frac{e^{-ik\xi}}{\sqrt{2\pi}}dx d\xi,$$

and

$$v(x,\xi) = \sum_{j',k'} \hat{v}(j',k')\psi_{j'}(x)\frac{e^{-ik'\xi}}{\sqrt{2\pi}}.$$

Let $k = (k_1,k_2) \in \mathbb{Z}^2$, $l = (l_1,l_2) \in \mathbb{Z}^2$ and $m = (m_1,m_2) \in \mathbb{Z}^2$

$$(uv)\hat{}(k) = \int_0^\pi \int_0^{2\pi} \sum_{l,m} \hat{u}(l_1,l_2)\psi_{l_1}(x)\hat{v}(m_1,m_2)\psi_{m_1}(x)\frac{e^{im_2\xi}}{\sqrt{2\pi}}\frac{e^{il_2\xi}}{\sqrt{2\pi}}\overline{\psi}_{k_1}(x)\frac{e^{-ik_2\xi}}{\sqrt{2\pi}}dx d\xi.$$

We denote here by $\hat{\psi}_k$ the coefficients of the eigenfunctions in the basis e^{ikx} and we have the expansion

$$\psi_k(x) = \sum_{r\in\mathbb{Z}} \hat{\psi}_k(r)e^{irx}.$$

We now prove that there exists α such that $||uv||_{m,s} \leq ||u||_{m,s}||v||_{m,s}$

$$(uv)\hat{}(k) = (2\pi)^{-3/2}\int_0^\pi \int_0^{2\pi} \sum_{l,m} \hat{u}(l_1,l_2)\hat{v}(m_1,m_2)\psi_{l_1}(x)\psi_{m_1}(x)\overline{\psi}_{k_1}(x)e^{i(l_2+m_2-k_2)\xi}dx d\xi.$$

We expand these eigenfunctions as Fourier series

$$\psi_{l_1}(x) = \sum_{q_1} \hat{\psi}_{l_1}(q_1) e^{iq_1 x}$$

$$\psi_{m_1}(x) = \sum_{\beta_1} \hat{\psi}_{m_1}(\beta_1) e^{i\beta_1 x}$$

$$\psi_{k_1}(x) = \sum_{r_1} \hat{\psi}_{k_1}(r_1) e^{ir_1 x}$$

$$\begin{aligned}
(uv)\hat{}(k) &= (2\pi)^{-3/2} \sum_{l,m} \hat{u}(l_1,l_2)\hat{v}(m_1,m_2) \int_0^{\pi}\int_0^{2\pi} \sum_{q_1,\beta_1,r_1} \hat{\psi}_{l_1}(q_1)\hat{\psi}_{m_1}(\beta_1)\overline{\hat{\psi}}_{k_1}(r_1) \\
&\quad e^{i(q_1+\beta_1-r_1)x} e^{i(l_2+m_2-k_2)\xi} dx d\xi \\
&= \sum_{l,m} \hat{u}(l_1,l_2)\hat{v}(m_1,m_2) \sum_{q_1,\beta_1,r_1} \hat{\psi}_{l_1}(q_1)\hat{\psi}_{m_1}(\beta_1)\overline{\hat{\psi}}_{k_1}(r_1) \\
&\quad \int_0^{\pi} e^{i(q_1+\beta_1-r_1)x} \int_0^{2\pi} e^{i(l_2+m_2-k_2)\xi} dx d\xi
\end{aligned}$$

$$\begin{cases} \int_0^{\pi} e^{i(q_1+\beta_1-r_1)x} dx = \pi \text{ if } \beta_1 = r_1 - q_1 \\ \int_0^{\pi} e^{i(q_1+\beta_1-r_1)x} = 0 \text{ otherwise} \end{cases} \tag{2.1.10}$$

and as well

$$\begin{cases} \int_0^{2\pi} e^{i(l_2+m_2-k_2)\xi} d\xi = 2\pi \text{ if } m_2 = k_2 - l_2 \\ \int_0^{2\pi} e^{i(l_2+m_2-k_2)\xi} d\xi = 0 \ \text{ otherwise.} \end{cases} \tag{2.1.11}$$

We can eliminate the summation over β_1 and m_2 to get

$$(uv)\hat{}(k) = \sum_{l_1,l_2,m_1} \sum_{q_1,r_1} \sqrt{2\pi}\hat{u}(l_1,l_2)\hat{v}(m_1,k_2-l_2)\hat{\psi}_{l_1}(q_1)\hat{\psi}_{m_1}(r_1-q_1)\overline{\hat{\psi}}_{k_1}(r_1) \tag{2.1.12}$$

$$|(uv)\hat{}(k)| = |\sum_{l_1,l_2,m_1} \sum_{q_1,r_1} \sqrt{2\pi}\hat{u}(l_1,l_2)\hat{v}(m_1,k_2-l_2)\hat{\psi}_{l_1}(q_1)\hat{\psi}_{m_1}(r_1-q_1)\overline{\hat{\psi}}_{k_1}(r_1)|,$$

applying now the estimate on the eigenfunctions (7.1.4) we have

$$\begin{aligned}
|(uv)\hat{}(k)| &\le C^3 \sum_{q_1,r_1,l_1,l_2,m_1} |\hat{u}(l_1,l_2)\hat{v}(m_1,k_2-l_2)| e^{-\sigma_*(||q_1-r_1|-|m_1||)} \\
&\quad e^{-\sigma_*(||l_1|-|q_1||)} e^{-\sigma_*(||k_1|-|r_1||)}.
\end{aligned}$$

We now obtain an estimate of $||uv||_{m,s}$. If denote $D := ||uv||^2_{m,s}$ we now obtain

$$\begin{aligned} D \quad \leq \quad & C^3\alpha\sum_k(1+|k|^2)^s e^{2m|k|} \sum_{l_1,l_2,m_1,q_1,r_1} |\hat{u}(l_1,l_2)\hat{v}(m_1,k_2-l_2)|e^{-\sigma_*(||q_1-r_1|-|m_1||)} \\ & e^{-\sigma_*(||l_1|-|q_1||)}e^{-\sigma_*(||k_1|-|r_1||)} \sum_{l'_1,l'_2,m'_1,q'_1,r'_1} |\hat{u}(l'_1,l'_2)\hat{v}(m'_1,k_2-l'_2)| \\ & e^{-\sigma_*(||q'_1-r'_1|-|m'_1||)}e^{-\sigma_*(||l'_1|-|q'_1||)}e^{-\sigma_*(||k_1|-|r'_1||)} \end{aligned} \tag{2.1.13}$$

$$|k_1| \leq |k_1| - |r_1| + |r_1|$$

$$|k_1| \leq |\ |k_1| - |r_1|\ | + |r_1|$$

$$|k_1| \leq |\ |k_1| - |r_1|\ | + |r_1 - q_1| + |q_1|$$

$$|k_1| \leq |\ |k_1| - |r_1|\ | + |r_1 - q_1| + ||q_1| - |l_1|| + |l_1|$$

$$|k_1| \leq |\ |k_1| - |r_1|\ | + |r_1 - q_1| - |m_1| + |m_1| + ||q_1| - |l_1|| + |l_1|$$

$$|k_1| + |k_2| \leq ||k_1| - |r_1|| + ||r_1 - q_1| - |m_1|| + |m_1| + ||q_1| - |l_1|| + |l_1| + |k_2 - l_2| + |l_2|$$

$$|k_1|+|k_2|-|l_1|-|l_2|-|m_1|-|k_2-l_2| \leq ||k_1|-|r_1||+||r_1-q_1|-|m_1||+|m_1|+||q_1|-|l_1||.$$

If we denote the left hand side by B and the right hand side by A, and let $\mu = \sigma_* - m$ the preceding inequality can be rewriten as $A \leq B$ which implies that

$$mA \leq mB$$

$$mA - mB - \mu B \leq -\mu B$$

$$mA - (m+\mu)B \leq -\mu B$$

$$mA - \sigma_* B \leq -\mu B,$$

inserting this bound into the exponential we obtain

$$e^{m|k|}.e^{-m|l|}.e^{-m|(m_1,k_2-l_2)|}.e^{-\sigma_*(||k_1|-|r_1||+||r_1-q_1|-|m_1||+|m_1|+||q_1|-|l_1||)}$$

$$\leq e^{-\mu||k_1|-|r_1||+||r_1-q_1|-|m_1||+|m_1|+||q_1|-|l_1||)}$$

which implies that

$$e^{m|k|}.e^{-\sigma_* B} \leq e^{-\mu B}.e^{m|l|}.e^{m|(m_1,k_2-l_2)|}.$$

The same inequality holds when l, m_1, q_1, r_1 are replaced by l', m'_1, q'_1, r'_1. Let

$$B' = |k_1| + |k_2| - |l'_1| - |l'_2| - |m'_1| - |k_2 - l'_2| \tag{2.1.14}$$

and

$$A' = ||k_1| - |r'_1|| + ||r'_1 - q'_1| - |m'_1|| + |m'_1| + ||q'_1| - |l'_1|| \tag{2.1.15}$$

we have

$$e^{m|k|}e^{-\sigma_* B'} \leq e^{-\mu B'}e^{m|l'|}e^{m|(m_1',k_2-l_2')|}.$$

Now inserting this into (2.1.13) and splitting the product

$$(1+|k|^2)^s e^{2m|k|} = (1+|k|^2)^{s/2}e^{m|k|}(1+|k|^2)^{s/2}e^{m|k|}$$

we have

$$\begin{aligned}
D \;\leq\;& \alpha C^6 \sum_k \sum_{l_1,l_2,m_1,q_1,r_1} (1+|l|^2)^{s/2}(1+|l|^2)^{-s/2}(1+|(m_1,k_2-l_2)|^2)^{s/2}(1+|(m_1,\\
& k_2-l_2)|^2)^{-s/2}|\hat{u}(l_1,l_2)||\hat{v}(m_1,k_2-l_2)|e^{m|l|}.e^{m|(m_1,k_2-l_2)|}e^{-\mu B}(1+|k|^2)^{s/2}\\
& \sum_{l_1',l_2',m_1',q_1',r_1'} (1+|l'|^2)^{s/2}(1+|l'|^2)^{-s/2}(1+|(m_1',k_2-l_2')|^2)^{s/2}(1+|(m_1',\\
& k_2-l_2')|^2)^{-s/2}|\hat{u}(l_1',l_2')||\hat{v}(m_1',k_2-l_2')|e^{m|l'|}.e^{m|(m_1',k_2-l_2')|}e^{-\mu B}(1+|k|^2)^{s/2}\\
\leq\;& \alpha C^6 \sum_k \sum_{l_1,l_2,m_1,q_1,r_1} [(1+|l|^2)^{s/2}|\hat{u}(l_1,l_2)|(1+|(m_1,k_2-l_2)|^2)^{s/2}e^{m|(m_1,k_2-l_2)|}\\
& |\hat{v}(m_1,k_2-l_2)|e^{-\mu/2B}][(1+|k|^2)^{s/2}(1+|l|^2)^{-s/2}(1+|(m_1,k_2-l_2)|^2)^{-s/2}\\
& e^{-\mu/2B}] \times \sum_{l_1',l_2',m_1',q_1',r_1'} [(1+|l'|^2)^{s/2}|\hat{u}(l_1',l_2')|(1+|(m_1',k_2-l_2')|^2)^{s/2}e^{m|(m_1',k_2-l_2')|}\\
& |\hat{v}(m_1',k_2-l_2')|e^{-\mu/2B}][(1+|k|^2)^{s/2}(1+|l'|^2)^{-s/2}(1+|(m_1',k_2-l_2')|^2)^{-s/2},
\end{aligned} \tag{2.1.16}$$

we can now apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}
D \;\leq\;& \alpha C^6 \sum_k [\sum_{l_1,l_2,m_1,q_1,r_1} (1+|l|^2)^s|\hat{u}(l_1,l_2)|^2(1+|(m_1,k_2-l_2)|^2)^s e^{2m|(m_1,k_2-l_2)|}\\
& |\hat{v}(m_1,k_2-l_2)|^2e^{-\mu B}]^{1/2}.[\sum_{l_1,l_2,m_1,q_1,r_1} (1+|k|^2)^s(1+|l|^2)^{-s}\\
& (1+|(m_1,k_2-l_2)|^2)^{-s}e^{-\mu B}]^{1/2} \times [\sum_{l_1',l_2',m_1',q_1',r_1'} (1+|l'|^2)^s|\hat{u}(l_1',l_2')|^2\\
& (1+|(m_1',k_2-l_2')|^2)^s e^{2m|(m_1',k_2-l_2')|}|\hat{v}(m_1',k_2-l_2')|^2e^{-\mu B}]^{1/2}.\\
& [\sum_{l_1',l_2',m_1',q_1',r_1'} (1+|k|^2)^s(1+|l'|^2)^{-s}(1+|(m_1',k_2-l_2')|^2)^{-s}e^{-\mu B}]^{1/2}
\end{aligned}$$

If we call

$$\beta := [\sum_{l_1,l_2,m_1,q_1,r_1} (1+|k|^2)^s(1+|l|^2)^{-s}(1+|(m_1,k_2-l_2)|^2)^{-s}e^{-\mu B}]^{1/2}$$

then $\beta \leq C(\mu, s)$. (The proof is provided in the next lemma.)

$$\begin{aligned}
D \leq\ & \alpha C^6 C(\mu)^2 \sum_k [\sum_{l_1,l_2,m_1,q_1,r_1} (1+|l|^2)^s |\hat{u}(l_1,l_2)|^2 (1+|(m_1,k_2-l_2)|^2)^s \\
& e^{2m|(m_1,k_2-l_2)|} |\hat{v}(m_1,k_2-l_2)|^2 e^{-\mu B}]^{1/2} \\
& \times [\sum_{l'_1,l'_2,m'_1,q'_1,r'_1} (1+|l'|^2)^s |\hat{u}(l'_1,l'_2)|^2 (1+|(m'_1,k_2-l'_2)|^2)^s \\
& e^{2m|(m'_1,k_2-l'_2)|} |\hat{v}(m'_1,k_2-l'_2)|^2 e^{-\mu B}]^{1/2}
\end{aligned}$$

Now applying the Cauchy inequality on the sum over k

$$\begin{aligned}
D \leq\ & \alpha C^6 C(\mu)^2 (\sum_{k_1,k_2,l_1,l_2,m_1,q_1,r_1} [(1+|l|^2)^s |\hat{u}(l_1,l_2)|^2 (1+|(m_1,k_2-l_2)|^2)^s \\
& e^{2m|(m_1,k_2-l_2)|} |\hat{v}(m_1,k_2-l_2)|^2 e^{-\mu B}])^{1/2} \\
& \times (\sum_{k_1,k_2,l'_1,l'_2,m'_1,q'_1,r'_1} [(1+|l'|^2)^s |\hat{u}(l'_1,l'_2)|^2 (1+|(m'_1,k_2-l'_2)|^2)^s \\
& e^{2m|(m'_1,k_2-l'_2)|} |\hat{v}(m'_1,k_2-l'_2)|^2 e^{-\mu B}])^{1/2} \\
\leq\ & \alpha C^6 C(\mu)^2 \sum_{k_1,k_2,l_1,l_2,m_1,q_1,r_1} (1+|l|^2)^s |\hat{u}(l_1,l_2)|^2 (1+|(m_1,k_2-l_2)|^2)^s \\
& e^{2m|(m_1,k_2-l_2)|} |\hat{v}(m_1,k_2-l_2)|^2 e^{-\mu B} \\
\leq\ & \frac{1}{\alpha} C^6 C(\mu)^2 \sum_{l_1,l_2} [\alpha (1+|l|^2)^s |\hat{u}(l_1,l_2)|^2 e^{m|(m_1,k_2-l_2)|} \\
& \times \sum_{k_2,m_1} \alpha (1+|(m_1,k_2-l_2)|^2)^s e^{m|(m_1,k_2-l_2)|} |\hat{v}(m_1,k_2-l_2)|^2 \sum_{r_1,q_1,k_1} e^{-\mu B}] \\
\leq\ & \frac{1}{\alpha} C^6 C(\mu)^2 ||u||^2_{m,s} ||v||^2_{m,s} \sum_{q_1,r_1} |e^{-\mu(||r_1-q_1|-|m_1||)} e^{-\mu(||q_1|-|l_1||)} \sum_{k_1} e^{-\mu(|k_1|-|r_1||)}
\end{aligned}$$

if we denote by $C_1(\mu)$ the sum $\sum_{k \in \mathbb{Z}} e^{-\mu|k|}$ then

$$\begin{aligned}
D \leq\ & \frac{1}{\alpha} C^6 C(\mu)^2 C_1(\mu) ||u||^2_{m,s} ||v||^2_{m,s} \sum_{q_1,r_1} |e^{-\mu(||r_1-q_1|-|m_1||)} e^{-\mu(||q_1|-|l_1||)} \\
\leq\ & \frac{1}{\alpha} C^6 C(\mu)^2 C_1(\mu) ||u||^2_{m,s} ||v||^2_{m,s} \sum_{q_1} e^{-\mu(||q_1|-|l_1||)} \sum_{r_1} e^{-\mu(||r_1-q_1|-|m_1||)} \\
\leq\ & \frac{1}{\alpha} C^6 C(\mu)^2 C_1^3(\mu) ||u||^2_{m,s} ||v||^2_{m,s}.
\end{aligned}$$

We can then choose $\alpha(\mu)$ greater then 1 such that we have the conclusion

$$||uv||^2_{m,s} \leq ||u||^2_{m,s} ||v||^2_{m,s}$$

and hence

$$||uv||_{m,s} \leq ||u||_{m,s}||v||_{m,s}.\square$$

The next lemma was enunciated in [10] whitout proof, here we provide a proof.

Lemma 2.1.1 *Let $s > 2$ and $\mu > 0$, then we have the following estimate:*

$$\sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [\frac{(1+|k|^2)e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}]^s \leq c(\mu, s) \tag{2.1.17}$$

Proof:

$$\begin{aligned}
\beta &= \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [\frac{1+|k|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}]^s \\
&\leq \sum_{l_1,l_2,q_1,m_1,r_1} [\frac{1+|k_1|^2+1+|k_2|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}]^s
\end{aligned}$$

Using a Clarkson-like inequality from [7]:

$$|a+b|^s \leq 2^{s-1}(|a|^s+|b|^s), \tag{2.1.18}$$

we now split the sum into A and B where

$$\begin{aligned}
A &= \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [\frac{2^{\frac{s-1}{s}}(1+|k_1|^2)e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}]^s \\
&\leq \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [\frac{2^{\frac{s-1}{s}}(|k_1|^2+|k_1|^2)e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}]^s \\
&\leq \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [\frac{2|k_1|^2e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}]^s \\
&\leq \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [\frac{2^2(|k_1-r_1|^2+|r_1|^2)e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}]^s.
\end{aligned}$$

Let E be

$$E = -\frac{\mu}{s}(||q_1 - r_1| - m_1| + ||l_1| - |q_1|| + ||k_1| - |r_1||)$$

$$\begin{aligned}
A &\le \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} 2^s[2\frac{|k_1 - r_1|^2 + |r_1|^2}{(1+|l|^2)(1 + (|m_1| + |k_2 - l_2|)^2))}e^E]^s \\
&\le \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [2^2\frac{|k_1 - r_1|^2}{(1+|l|^2)(1 + (|m_1| + |k_2 - l_2|)^2))}e^E]^s \\
&\quad + \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} 2^s[2\frac{|r_1|^2}{(1+|l|^2)(1 + (|m_1| + |k_2 - l_2|)^2))}e^E]^s \\
&\le \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} [2^2\frac{|k_1 - r_1|^2}{(1+|l|^2)(1 + (|m_1| + |k_2 - l_2|)^2))}e^E]^s \\
&\quad + \sum_{l_1, l_2, q_1, m_1, r_1} 2^{2s}[\frac{|r_1|^2}{(1+|l|^2)(1 + (|m_1| + |k_2 - l_2|)^2))}e^E]^s \\
&\le A_1 + A_2
\end{aligned}$$

$$A_1 = \sum_{\substack{l_1, l_2 \\ q_1, m_1, r_1}} 2^{2s}[\frac{|k_1 - r_1|^2}{(1+|l|^2)(1 + (|m_1| + |k_2 - l_2|)^2))}e^E]^s$$

and as

$$\frac{1}{1 + (|m_1| + |k_2 - l_2|)^2} \le \frac{1}{1 + |k_2 - l_2|^2}$$

we have

$$\begin{aligned}
A_1 &\le \sum_{l_1, l_2} \frac{1}{1+|l|^2}\frac{1}{1+|k_2 - l_2|^2} \sum_{q_1} e^{-\frac{\mu}{s}||q_1| - |l_1||} \sum_{r_1} |k_1 - r_1|^2 e^{-\frac{\mu}{s}||k_1| - |r_1||} \\
&\quad \sum_{m_1} e^{-\frac{\mu}{s}||q_1 - r_1| - m_1|} \\
&\le C(\mu) \sum_{l_1, l_2} \frac{1}{1+|l|^2}\frac{1}{1+|k_2 - l_2|^2} \\
&\le C(\mu)
\end{aligned}$$

$$\begin{aligned}
|r_1| &\leq |r_1-q_1|+|q_1| \\
&\leq ||r_1-q_1|+|q_1|| \\
&\leq ||r_1-q_1|-|m_1|+|m_1|+||q_1|-|l_1||+|l_1|| \\
&\leq |\ ||r_1-q_1|-|m_1||+|m_1|+||q_1|-|l_1||+|l_1|\ |
\end{aligned}$$

$$\begin{aligned}
|r_1|^2 &\leq |\ ||r_1-q_1|-|m_1||+|m_1|+||q_1|-|l_1||+|l_1|\ |^2 \\
&\leq 2[(||r_1-q_1|-|m_1||+|m_1|)^2+||q_1|-|l_1||+|l_1||^2] \\
&\leq 4[||r_1-q_1|-|m_1||^2+|m_1|^2+||q_1|-|l_1||+|l_1||^2]
\end{aligned}$$

$$\begin{aligned}
A_2 &= \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} 2^{2s}[\frac{|r_1|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s \\
&\leq \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} 2^{2s}[4\frac{||r_1-q_1|-|m_1||^2+|m_1|^2+||q_1|-|l_1||+|l_1||^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s.
\end{aligned}$$

As we have 4 terms in the sum we establish the inequality

$$\begin{aligned}
(a+b+c+d)^s &\leq 2^{s-1}((a+b)^s+(c+d)^s) \\
&\leq 2^{s-1}(2^{s-1}(a^s+b^s)+2^{s-1}(c^s+d^s)) \\
&\leq 2^{2(s-1)}(a^s+b^s+c^s+d^s) \qquad (2.1.19)
\end{aligned}$$

and we deduce

$$\begin{aligned}
A_2 \leq\ & 2^{2s}2^{2(s-1)} \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} \frac{||r_1-q_1|-|m_1||^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s \\
&+2^{2s}2^{2(s-1)} \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} \frac{||q_1-|l_1||^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s \\
&+\ 2^{2s}2^{2(s-1)} \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} \frac{|m_1|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s \\
&+ \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} \frac{|l_1|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s,
\end{aligned}$$

the first two terms can easily be bounded by a constant dependent on μ. We call the next two terms A_{22} and A_{23}

$$A_{22} = 2^{2s}2^{2(s-1)} \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} \frac{|m_1|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))} e^E]^s,$$

cancelling m_1 by the term in the denominator containing m_1

$$\begin{aligned}
A_{22} &\leq 2^{2s}2^{2(s-1)}\frac{1}{(1+|l|^2)^s}.e^{-\mu(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)} \\
&\leq \sum_{l_1,l_2} 2^{2s}2^{2(s-1)}\frac{1}{(1+|l|^2)^s}.\sum_{q_1} e^{-\mu||l_1|-|q_1|} \\
&\quad .\sum_{r_1} e^{-\mu||k_1|-|r_1||}.\sum_{m_1} e^{-\mu||q_1-r_1|-m_1|} \\
&\leq C(\mu)
\end{aligned}$$

A_{23} follows similarly by cancelling now by l_1.

$$\begin{aligned}
B &= \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} [\frac{2^{\frac{s-1}{s}}(1+|k_2|^2)}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^{-\frac{\mu}{s}(||q_1-r_1|-m_1|+||l_1|-|q_1||+||k_1|-|r_1||)}]^s \\
&\leq \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} 2^{s-1}[\frac{4(|k_2-l_2|^2+|l_2|^2)}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s \\
&\leq 2^{3s} \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} 2^s[\frac{4|k_2-l_2|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s \\
&\quad +2^{3s} \sum_{\substack{l_1,l_2\\ q_1,m_1,r_1}} 2^s[\frac{|l_2|^2}{(1+|l|^2)(1+(|m_1|+|k_2-l_2|)^2))}e^E]^s,
\end{aligned}$$

If we call these two terms B_1 and B_2 then we can see that B_1 is estimated as A_{22} and B_2 as A_{23} so

$$B \leq C(\mu).\square \tag{2.1.20}$$

Lemma 2.1.2 [10] *Let $f(u)$ be a polynomial in u of at least order 2 and of the form $f(u) = \sum_{p=2}^{k} a_k(x)u^p$, and suppose that there is a constant M such* *that $||\hat{u}||_{m,s} \leq M$. We have the following inequality*

$$||f(u)\hat{}||_{m,s} \leq C_f ||\hat{u}||^2_{m,s} \tag{2.1.21}$$

where $f(u)\hat{}$ denotes the coefficients of the eigenfunction expansion of $f(u)$.

Proof:

$$\begin{aligned}
||f(u)\hat{}||_{m,s} &= ||\sum_{p=2}^{k} a_p(x)u^p||_{m,s} \\
&= ||(u^2(\sum_{p=0}^{k} a_p(x)u^{p-2}))\hat{}||_{m,s} \\
&\leq ||(u^2)\hat{}||_{m,s}||[\sum_{p=0}^{k-2} a_p(x)u^{p-2})]\hat{}||_{m,s}) \\
&\leq ||\hat{u}||^2_{m,s}\cdot\sum_{p=0}^{k-2} ||[a_p(x)u^{p-2}]\hat{}||_{m,s}) \\
&\leq ||\hat{u}||^2_{m,s}\cdot\sum_{p=0}^{k-2} ||a_p(x)\hat{}||_{m,s}||(u^{p-2})\hat{}||_{m,s}) \\
&\leq ||\hat{u}||^2_{m,s}\cdot\sum_{p=0}^{k-2} ||a_p(x)\hat{}||_{m,s}||\hat{u}||^{p-2}_{m,s}) \\
&\leq ||\hat{u}||^2_{m,s}\sum_{p=0}^{k-2} ||a_p(x)\hat{}||_{m,s}M^{p-2} \\
&\leq ||\hat{u}||^2_{m,s}C_f
\end{aligned}$$

which implies that $||f(u)\hat{}||_{m,s} \leq ||\hat{u}||_{m,s}||^2 C_f$. □

As we have a norm on $H^{m,s}$ we now define a new operator norm as follows:

$$||S||^2_{m,s} = c_1 \max(\sup_{l}\sum_{l'}(1+|l-l'|^2)^s|S(l,l')|^2 e^{2m|l-l'|}),$$
$$\sup_{l'}\sum_{l}(1+|l-l'|^2)^s|S(l,l')|^2 e^{2m|l-l'|}). \tag{2.1.22}$$

We show that we can choose c_1 such that this norm is submultiplicative that is:

$$||ST||_{m,s} \leq ||S||_{m,s}||T||_{m,s}. \tag{2.1.23}$$

Lemma 2.1.3

$$||ST||_{m,s} \leq ||S||_{m,s}||T||_{m,s} \tag{2.1.24}$$

Proof:

$$||ST||_{m,s}^2 = c_1 \max(\sup_l \sum_{l'} (1+|l-l'|^2)^s |ST(l,l')|^2 e^{2m|l-l'|}),$$

$$\sup_{l'} \sum_l (1+|l-l'|^2)^s |ST(l,l')|^2 e^{2m|l-l'|})$$

Consider

$$X^2 = \sup_l \sum_{l'} (1+|l-l'|^2)^s |ST(l,l')|^2 e^{2m|l-l'|}$$

then

$$X^2 = \sup_l \sum_{l'} (1+|l-l'|^2)^s |\sum_k S(l,k)T(k,l')|^2 e^{2m|l-l'|}$$

$$X^2 = \sup_l \sum_{l'} (1+|l-l'|^2)^s |\sum_k S(l,k)T(k,l')\gamma_{l'k}^{s/2}(\frac{1}{\gamma_{l'k}})^{s/2}|^2 e^{2m|l-l'|} \qquad (2.1.25)$$

We now want to have the product of S^2 and T^2. We thus have to find a suitable $\gamma_{l'k}$ and break the sum in (2.1.25) two parts using Cauchy-Schwarz to get the product

$$(1+|l-k|^2)(1+|l'-k|^2)$$

so we choose

$$\gamma_{l'k}(1+|l-l'|^2) = (1+|l-k|^2)(1+|l'-k|^2)$$

we now have to prove that our $(\gamma_{l'k})^{-1}$ is summable

$$\gamma_{l'k} = (1+|l-k|^2)(1+|l'-k|^2)(1+|l-l'|^2)^{-1}$$

$$\frac{1}{\gamma_{l'k}} = \frac{(1+|l-l'|^2)}{(1+|l-k|^2)(1+|l'-k|^2)}$$

$$\frac{1}{\gamma_{l'k}} \le \frac{2(1+|l-k|^2)+2(1+|l'-k|^2)}{(1+|l-k|^2)(1+|l'-k|^2)}$$

$$\frac{1}{\gamma_{l'k}} \le 2(\frac{1}{(1+|l-k|^2)} + \frac{1}{(1+|l-k|^2)})$$

Let

$$\begin{aligned} c_{l'}^2 &= \sum_k (\frac{1}{\gamma_{l'k}})^s \\ &\le \sum_k 2^s (\frac{1}{(1+|l-k|^2)} + \frac{1}{(1+|l-k|^2)})^s \\ &\le 2^s 2^{s-1} \sum_k (\frac{1}{(1+|l-k|^2)})^s + (\frac{1}{(1+|l-k|^2)})^s \\ c_{l'}^2 &\le 2^{2s-1} 2\eta \end{aligned} \qquad (2.1.26)$$

where η is the sum of the double series $\sum_{k\in\mathbb{Z}^2}(\frac{1}{1+|k|^2})^s$ which converges if $s>1$.

We can now continue the proof.

$$X^2=(\sup_l\sum_{l'}(1+|l-l'|^2)^s|\sum_k S(l,k)T(k,l')\gamma_{l'k}^{s/2}(\frac{1}{\gamma_{l'k}})^{s/2}|^2e^{2m|l-l'|})$$

and applying Cauchy-Schwarz ot follows that

$$X^2\leq(\sup_l\sum_{l'}(1+|l-l'|^2)^s c_{l'}^2\sum_k|S(l,k)|^2|T(k,l')|^2\gamma_{l'k}^s e^{2m|l-l'|})$$

$$X^2\leq 4^s\eta(\sup_l\sum_{l'}\sum_k(1+|l-k|^2)(1+|l'-k|^2)|S(l,k)|^2|T(k,l')|^2e^{2m|l-k|}e^{2m|k-l'|}$$

$$X^2\leq 4^s\eta(\sup_l\sum_k(1+|l-k|^2)e^{2m|l-k|}|S(l,k)|^2\sum_{l'}|T(k,l')|^2(1+|l'-k|^2)e^{2m|k-l'|}$$

If we choose $c_1=4^s\eta$

$$c_1X^2\leq c_1(\sup_l\sum_k(1+|l-k|^2)e^{2m|l-k|}|S(l,k)|^2.\ c_1\sum_{l'}|T(k,l')|^2(1+|l'-k|^2)e^{2m|k-l'|}$$

$$||ST||_{m,s}^2\leq||S||_{m,s}^2||T||_{m,s}^2$$
$$||ST||_{m,s}\leq||S||_{m,s}||T||_{m,s}.\square \tag{2.1.27}$$

Definition 2.1 *Let $\hat{w},\hat{v}$ two elements of $H^{m,s}$ and their corresponding functions in the x and ξ variables be v and w. We define an operator on this space $\hat{W}$ as follows*

$$\hat{W}(\hat{v})=(wv)\hat{}$$

Where the hat denotes the coefficients of the eigenfunction expansion of the product of v and w.

The next lemma was enunciated in [10], we provide a proof

Lemma 2.1.4 *There exists a constant C_{op} such that:*

$$||\hat{W}(k,m)||_{m,s}\leq C_{op}||\hat{w}||_{m,s}. \tag{2.1.28}$$

Proof:

$$\hat{W}\hat{v}(k)=(wv)\hat{}(k)=\sum_m\hat{W}(k,m)\hat{v}(m),$$

We now show some properties of this operator:
If we consider $\hat{W}$ as a matrix operator then we can write

$$\hat{W}(k,m)=\sum_j\hat{W}(k,j)\delta_m(j)$$

where $\delta_m(j) = 0$ if $m \neq j$ and $\delta_j(j) = 1$. Recall now (2.1.12)

$$(wv)\hat{}(k) = \sqrt{2\pi} \sum_{l_1,q_1,r_1,j_1,l_2} \hat{w}(l_1,l_2)\hat{v}(j_1,k_2-l_2)\psi_{j_1}(q_1-r_1)\psi_{l_1}(q_1)\overline{\hat{\psi}_{k_1}(r_1)}.$$

If

$$\hat{v} = \hat{\delta}_m \text{ with } m = (m_1, m_2)$$

then

$$j_1 = m_1, \text{ and } k_2 - l_2 = m_2 \text{ so } l_2 = k_2 - m_2$$

$$\hat{W}(k,m) = \sum_{l_1,q_1,r_1} \hat{w}(l_1,k_2-m_2).1.\psi_{m_1}(q_1-r_1)\psi_{l_1}(q_1)\overline{\hat{\psi}_{k_1}(r_1)}$$

$$|\hat{W}(k,m)| \leq C^3 \sum_{l_1,q_1,r_1} |\hat{w}(l_1,k_2-m_2)| e^{-\sigma_*(||k_1|-|r_1||+||r_1-q_1|-|m_1||+||q_1|-|l_1||)}$$

$$\begin{aligned} ||\hat{W}(k,m)||^2_{m,s} \leq \; & c_1 C^3 \sup_k \sum_m (1+|k-m|^2)^s e^{2\sigma|k-m|} \sum_{l_1,q_1,r_1,l_1',q_1',r_1'} |\hat{w}(l_1,k_2-m_2)| \\ & |\hat{w}(l_1',k_2'-m_2)| e^{-\sigma_*(||k_1|-|r_1||+||r_1-q_1|-|m_1||+||q_1|-|l_1||)} \\ & e^{-\sigma_*(||k_1|-|r_1'||+||r_1'-q_1'|-|m_1||+||q_1'|-|l_1'||)} \end{aligned}$$

$$|r_1| \leq |r_1 - q_1| + |q_1| - |m_1| + |m_1|$$

$$|r_1| - |m_1| \leq |r_1 - q_1| - |m_1| + |q_1|$$

$$||r_1| - |m_1|| \leq ||r_1 - q_1| - |m_1|| + |q_1|$$

$$||r_1| - |m_1|| \leq ||r_1 - q_1| - |m_1|| + ||q_1| - |l_1|| + |l_1|$$

As the eigenfunctions are numbered only by positive integers k_1, m_1, l_1 the subscripts are all positive

$$\begin{aligned} |k_1 - m_1| &= ||k_1| - |m_1|| \\ &= ||k_1| - |r_1| + |r_1| - |m_1|| \\ |k_1 - m_1| &\leq ||k_1| - |r_1|| + ||r_1 - q_1| - |m_1|| + ||q_1| - |l_1|| + |l_1|. \end{aligned} \tag{2.1.29}$$

Let A be

$$\begin{aligned} A = \; & m(|k_1 - m_1| + |k_2 - m_2|) - m(|k_2 - m_2| + |l_1|) - \sigma_*(||k_1| - |r_1|| \\ & + ||r_1 - q_1| - |m_1|| + ||q_1| - |l_1||) \end{aligned} \tag{2.1.30}$$

$$A = m(|k_1 - m_1|) - m|l_1| - \sigma_*(||k_1| - |r_1|| + ||r_1 - q_1| - |m_1|| + ||q_1| - |l_1||)$$

using the inequality (2.1.29)

$$A \leq m(||k_1| - |r_1|| + ||r_1 - q_1| - |m_1|| + ||q_1| - |l_1|| + |l_1|)$$

$$-\sigma_*(||k_1|-|r_1||+||r_1-q_1|-|m_1||+||q_1|-|l_1||)$$
$$A \le -\mu(||k_1|-|r_1||+||r_1-q_1|-|m_1||+||q_1|-|l_1||)$$

where $\mu = \sigma_* - m$

$$\begin{aligned} ||\hat{W}(k,m)||^2_{m,s} \le\ & c_1 C^3 \sup_k \sum_m \sum_{l_1,q_1,r_1} (1+(|l_1|+|k_2-m_2|)^2)^s e^{m(|l_1|+|k_2-m_2|)} \\ & |\hat{w}(l_1,k_2-m_2)|e^{-m(|l_1|+|k_2-m_2|)}.e^{m|k-m|}(1+(|l_1|+|k_2-m_2|)^2)^{-s} \\ & \times \sum_{l_1',q_1',r_1'} (1+(|l_1'|+|k_2-m_2|)^2)^s e^{m(|l_1'|+|k_2-m_2|)}\hat{w}(l_1',k_2-m_2). \\ & e^{-m(|l_1'|+|k_2-m_2|)}.e^{m|k-m|}(1+(|l_1'|+|k_2-m_2|)^2)^{-s} \end{aligned} \tag{2.1.31}$$

$$B = ||k_1|-|r_1||+||r_1-q_1|-|m_1||+||q_1|-|l_1||$$

$$\begin{aligned} ||\hat{W}(k,m)||^2_{m,s} \le\ & c_1 C^3 \sup_k \sum_m \sum_{l_1,q_1,r_1} [(1+(|l_1|+|k_2-m_2|)^2)^{s/2} e^{m(|l_1|+|k_2-m_2|)}|\hat{w}(l_1,k_2-m_2)|]. \\ & [(1+|k-m|^2)^{s/2}e^{-\mu/2B}(1+(|l_1|+|k_2-m_2|)^2)^{-s/2}] \\ & \times \sum_{l_1',q_1',r_1'} [(1+(|l_1'|+|k_2-m_2'|)^2)^{s/2} e^{m(|l_1'|+|k_2-m_2'|)}|\hat{w}(l_1',k_2-m_2')|]. \\ & [(1+|k-m'|^2)^{s/2}e^{-\mu/2B}(1+(|l_1'|+|k_2-m_2'|)^2)^{-s/2}] \end{aligned}$$

Now applying the Cauchy-Schwarz inequality over the sum on l_1, q_1, r_1

$$\begin{aligned} ||\hat{W}(k,m)||^2_{m,s} \le\ & c_1 C^3 \sup_k \sum_m \sqrt{\sum_{l_1,q_1,r_1} (1+(|l_1|+|k_2-m_2|)^2)^s e^{2\sigma(|l_1|+|k_2-m_2|)}|\hat{w}(l_1,k_2-m_2)|^2} \\ & \sqrt{\sum_{l_1,q_1,r_1} (1+(|l_1|+|k_2-m_2|)^2)^{-s}(1+|k-m|^2)^s e^{-\mu B}} \\ & \times \sqrt{\sum_{l_1',q_1',r_1'} (1+(|l_1'|+|k_2-m_2'|)^2)^s e^{2\sigma(|l_1'|+|k_2-m_2|)}|\hat{w}(l_1',k_2-m_2)|^2} \\ & \sqrt{\sum_{l_1',q_1',r_1'} (1+(|l_1'|+|k_2-m_2|)^2)^{-s}(1+|k-m|^2)^s e^{-\mu B}} \\ =\ & c_1 C^3 \sup_k \sum_m \sum_{l_1,q_1,r_1} (1+(|l_1|+|k_2-m_2|)^2)^s e^{2\sigma(|l_1|+|k_2-m_2|)}|\hat{w}(l_1,k_2-m_2)|^2 \\ & \times \sum_{l_1,q_1,r_1} (1+(|l_1|+|k_2-m_2|)^2)^{-s}(1+|k-m|^2)^s e^{-\mu B} \\ \le\ & C \sum_{l_1,m_2} (1+(|l_1|+|k_2-m_2|)^2)^s e^{2\sigma(|l_1|+|k_2-m_2|)}|\hat{w}(l_1,k_2-m_2)|^2 \end{aligned}$$

$$\begin{aligned}
&\sum_{m_1,q_1,r_1}(1+(|l_1|+|k_2-m_2|)^2)^{-s}(1+|k-m|^2)^s e^{-\mu B}\\
\leq\ & C||\hat{w}||^2_{m,s}\sum_{m_1,q_1,r_1}\frac{(1+|k-m|^2)^s e^{-\mu B}}{(1+(|l_1|+|k_2-m_2|)^2)^s}\\
\leq\ & C||\hat{w}||^2_{m,s}\sum_{m_1,q_1,r_1}\frac{[(1+|k_1-m_1|)^2+(1+|k_2-m_2|)]^s e^{-\mu B}}{(1+(|l_1|+|k_2-m_2|)^2)^s}\\
\leq\ & C||\hat{w}||^2_{m,s}\sum_{m_1,q_1,r_1}\frac{[(1+|k_1-m_1|)^2]^s e^{-\mu B}}{(1+(|l_1|+|k_2-m_2|)^2)^s}\\
&+C||\hat{w}||^2_{m,s}\sum_{m_1,q_1,r_1}\frac{e^{-\mu B}}{(1+(|l_1|+|k_2-m_2|)^2)^s}.
\end{aligned}$$

We will denote these two sums A and B. As

$$|k_1-m_1|\leq||k_1|-|r_1||+||r_1-q_1|-|m_1||+||q_1|-|l_1||+|l_1|$$

we have

$$\begin{aligned}
A\ \leq\ & C||\hat{w}||^2_{m,s}\frac{(1+|l_1|)^2}{(1+(|l_1|+|k_2-m_2|)^2)}\sum_{q_1}||q_1|-|l_1||^2 e^{-\mu/s||q_1|-|l_1||}\\
&\sum_{m_1}||r_1-q_1|-|m_1||^2 e^{-\mu/s||r_1-q_1|-|m_1||}\sum_{r_1}||k_1|-|r_1||^2 e^{-\mu||k_1|-|r_1||}\\
\leq\ & c(\mu,s)||\hat{w}||^2_{m,s}
\end{aligned}$$

and similarly for B. Then

$$||\hat{W}(k,m)||^2_{m,s}\leq c(\mu,s)||\hat{w}||^2_{m,s}$$

hence

$$||\hat{W}(k,m)||_{m,s}\leq c||\hat{w}||_{m,s}.\square$$

The next lemma was enunciated in [10] without proof. Here we provide one.

Lemma 2.1.5 *Let S be an operator acting on $H^{m,s}$ and denote by $||S||^{op}$ the standard operator norm, we have the inequality*

$$||S||^{op}\leq||S||_{m,s} \tag{2.1.32}$$

Proof:

$$\begin{aligned}
||Su||^2_{m,s} &= c_1\sum_l(1+|l|^2)^s e^{2\sigma|l|}|(Su)(l)|^2\\
&= c_1\sum_l(1+|l|^2)^s e^{2\sigma|l|}|\sum_k S(l,k)u(k)|^2\\
&= c_1\sum_l(1+|l|^2)^s e^{2\sigma|l|}|\sum_k S(l,k)u(k)\gamma_{lk}^{s/2}\frac{1}{\gamma_{lk}^{s/2}}|^2
\end{aligned}$$

We decide to choose $\gamma_{lk} = \frac{(1+|l-k|^2)(1+|k|^2)}{1+|l|^2}$ and so applying Cauchy-Schwarz we get

$$||Su||^2_{m,s} \leq \sum_l (1+|l|^2)^s e^{2\sigma|l|} \sum_k |S(l,k)u(k)|^2 \gamma^s_{lk} \sum_k \frac{1}{\gamma^s_{lk}}$$

as we did before we can prove that $\sum_k \frac{1}{\gamma^s_{lk}} \leq 4^s \eta = c_1$ and so

$$\begin{aligned}
||Su||^2_{m,s} &\leq c_1 \sum_l (1+|l|^2)^s e^{2\sigma|l|} \sum_k |S(l,k)u(k)|^2 \gamma^s_{lk} c_1 \\
&\leq c_1^2 \sum_l \sum_k |S(l,k)u(k)|^2 (1+|l|^2)^s \gamma^s_{lk} e^{2\sigma|l|} \\
&\leq c_1^2 \sum_l \sum_k |S(l,k)u(k)|^2 \frac{(1+|l-k|^2)^s(1+|k|^2)^s}{(1+|l|^2)^s} (1+|l|^2)^s e^{2\sigma|l|} \\
&\leq c_1^2 \sum_k (1+|k|^2)^s |u(k)|^2 \sum_l |S(l,k)|^2 (1+|l-k|^2)^s e^{2\sigma|l-k|} e^{2\sigma|k|} \\
&\leq c_1 \sum_k (1+|k|^2)^s e^{2\sigma|k|} |u(k)|^2 \sum_l c_1 |S(l,k)|^2 (1+|l-k|^2)^s e^{2\sigma|l-k|} \\
||Su||^2_{m,s} &\leq ||u||^2_{m,s} ||S||^2_{m,s}
\end{aligned}$$

because we can always suppose $\alpha \geq c_1$ hence

$$||S||^{op} \leq ||S||_{m,s}. \square \tag{2.1.33}$$

Lemma 2.1.6 *Let $m_0, s, m > 0$ such that $m > m_0$ and $\hat{\psi}$ a vector in $H^{m,s}$. Then we have the following estimate:*

$$||S\psi||_{0,0} \leq (\frac{1}{1-e^{-m_0}}) ||S||_{m,s} ||\psi||_{0,0} \tag{2.1.34}$$

Proof:

$$\begin{aligned}
||S\psi||^2_{0,0} &= \alpha \sum_{l_1} |(S\psi)(l_1)|^2 \\
&= \alpha \sum_{l_1} | \sum_{l_2} S(l_1,l_2)\psi(l_2)|^2 \\
&= \alpha \sum_{l_1} | \sum_{l_2} (1+|l_1-l_2|^2)^{s/2} e^{m|l_1-l_2|} S(l_1,l_2)(1+|l_1-l_2|^2)^{-s/2} e^{-m|l_1-l_2|} \psi(l_2)|^2.
\end{aligned}$$

Applying now Cauchy-Schwarz to the sum over l_2 we get

$$||S\psi||^2_{0,0} \leq \alpha \sum_{l_1} \sum_{l_2} (1+|l_1-l_2|^2)^s e^{2m|l_1-l_2|} |S(l_1,l_2)|^2$$

$$
\begin{aligned}
& \sum_{l_2}(1+|l_1-l_2|^2)^{-s}e^{-2m|l_1-l_2|}|\psi(l_2)|^2 \\
\leq\ & \sum_{l_1}||S||^2_{m,s}\alpha\sum_{l_2}(1+|l_1-l_2|^2)^{-s}e^{-2m|l_1-l_2|}|\psi(l_2)|^2 \\
\leq\ & ||S||^2_{m,s}\sum_{l_2}\alpha|\psi(l_2)|^2\sum_{l_1}(1+|l_1-l_2|^2)^{-s}e^{-2m|l_1-l_2|} \\
\leq\ & ||S||^2_{m,s}\sum_{l_2}\alpha|\psi(l_2)|^2\sum_{l_1}e^{-2m|l_1-l_2|} \\
\leq\ & ||S||^2_{m,s}\sum_{l_2}|\alpha\psi(l_2)|^2\sum_{l_1}e^{-2m_0|l_1-l_2|} \\
\leq\ & ||S||^2_{m,s}||\psi||^2_{0,0}(\frac{1}{1-e^{-m_0}})^2
\end{aligned}
$$

and we conclude

$$||S\psi||_{0,0}\leq||S||_{m,s}||\psi||_{0,0}(\frac{1}{1-e^{-m_0}}).\square \tag{2.1.35}$$

Lemma 2.1.7 *Let X be a vector space of dimension n, and its canonical orthonormal basis $(e_i)_i$ $1\leq i\leq n$. Let A be a self-adjoint operator defined from X to X, and $(\lambda_i)_i$ its eigenvalues . We denote by $A(i,j)$ the matrix elements of the representation of the operator A in the basis $(e_i)_i$, and we define an inner product $\langle,\rangle$ and thus a norm.*
Suppose $|\lambda_i|>\delta$ $\forall i, 1\leq i\leq n$, then we have the relation

$$|A^{-1}(i,j)|\leq\frac{1}{\delta} \tag{2.1.36}$$

Proof:
As A is self-adjoint there exists a basis of orthogonal eigenvectors ψ_i and corresponding eigenvalues λ_i which we normalize $\langle\psi_i,\psi_i\rangle=1$. Let $x\in X$

$$x=\sum_{i=1}^{n}x_i\psi_i$$

$$Ax=\sum_{i=1}^{n}x_i\lambda_i\psi_i$$

$$\langle Ax,Ax\rangle=\sum_{i=1}^{n}|x_i\lambda_i|^2$$

$$||Ax||^2\leq\max_i|\lambda_i|^2\sum_{i=1}^{n}|x_i|^2$$

$$||Ax||^2 \leq \max_i |\lambda_i|^2 (x,x) \tag{2.1.37}$$

We have as well

$$||Ae_i||^2 = \sum_{j=1}^{n} |A(i,j)|^2 \text{ so}$$

$$||Ae_i|| \geq |A(i,j)| \tag{2.1.38}$$

and combining (2.1.37) and (2.1.38) we have

$$|A(i,j)| \leq ||Ae_i|| \leq \max_i |\lambda_i||e_i|^2 = \max_i |\lambda_i|. \tag{2.1.39}$$

Now A is self-adjoint and invertible, its inverse A^{-1} has eigenvalues $\frac{1}{\lambda_i}$, and using (2.1.39) we get

$$|A^{-1}(i,j)| \leq \max_i \frac{1}{|\lambda_i|}$$

and so

$$|A^{-1}(i,j)| \leq \frac{1}{\delta}.\Box$$

2.2 Some useful estimates

Lemma 2.2.1 *For $0 < \gamma < 1$ we have*

$$\sum_{l \in \mathbb{Z}^2} e^{-\gamma |l|} \leq \frac{64}{\gamma^2} \tag{2.2.40}$$

and

$$l^4 e^{-\gamma l} \leq \frac{4^4}{\gamma^4}. \tag{2.2.41}$$

Proof:

$$\text{Let } S := \sum_{l \in \mathbb{Z}^2} e^{-\gamma |l|} \tag{2.2.42}$$

$$\begin{aligned} S &= \sum_{l \in \mathbb{Z}^2} e^{-\gamma |l|} \\ &\leq 2 \sum_{l_1 \in \mathbb{N}} e^{-\gamma l_1} 2 \sum_{l_2 \in \mathbb{N}} e^{-\gamma l_2} \\ &\leq 4 (\frac{1}{1 - e^{-\gamma}})^2 \\ &\leq (\frac{64}{1 - e^{-\gamma}})^2. \end{aligned} \tag{2.2.43}$$

We now prove that for

$$0<\gamma<1,\ \frac{1}{1-e^{-\gamma}}\leq\frac{4}{\gamma}.$$

Let

$$g(\gamma)=4-4e^{-\gamma}-\gamma$$
$$g(0)=0$$
$$g'(\gamma)=4e^{-\gamma}-1$$
$$\text{as } \gamma\leq 1 \text{ then } e^{\gamma}\leq e$$
$$\text{and } e^{-\gamma}\geq\frac{1}{e}$$

so

$$g'(\gamma)=4e^{-\gamma}-1\geq 4\frac{1}{e}-1>0.$$

We can then conclude that $g(\gamma)\geq 0$ for $0\leq\gamma<1$ and

$$4-4e^{-\gamma}-\gamma\geq 0$$

so

$$4(1-e^{-\gamma})\geq\gamma$$

hence

$$\frac{1}{1-e^{-\gamma}}\leq\frac{4}{\gamma}$$

$$\begin{aligned} S &\leq 4.\frac{16}{\gamma^2} \\ &\leq \frac{64}{\gamma^2}. \end{aligned} \tag{2.2.44}$$

Let

$$f(l)=l^4e^{-\gamma l}\ (l\geq 0)$$
$$f'(l)=4l^3e^{-\gamma l}-\gamma l^4e^{-\gamma l}$$
$$f'(l)=l^3e^{-\gamma l}(4-\gamma l)$$
$$f'(l)=0 \text{ implies } l=\frac{2}{\gamma} \text{ which is a maximum because } \gamma>0.$$

We can deduce that

$$l^4e^{-\gamma l}\leq f(\frac{4}{\gamma})=\frac{4^4}{\gamma^4}e^{-4}$$
$$l^4e^{-\gamma l}\leq\frac{4^4}{\gamma^4}.\square \tag{2.2.45}$$

We denote the weights $(1+|y_1-y_2|^2)$ by $\langle y_1-y_2\rangle$. In the lattice $\mathbb{Z}\times\mathbb{N}$ we consider a path from y_1 to y_2 with Q steps. That is we number $\beta_0=y_1$, till $\beta_Q=y_2$. The object of this is to derive some properties of the weights. The next lemma was enunciated in [8] without proof. Here we provide one.

Lemma 2.2.2 *We have the properties:*

$$\langle y_1 - y_2\rangle^s \le c(Q,s)\sum_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle^s \tag{2.2.46}$$

where $\beta_0 = y_1$ and $\beta_Q = y_2$ *and*

$$\sum_{\beta: y_1, y_2\ Q steps} \frac{\langle y_1 - y_2\rangle^s}{\Pi_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle^s} \le c(Q,s) \tag{2.2.47}$$

Proof:
From [7] we have the relation

$$(a+b)^s \le 2^s(a^s + b^s) \text{ for } s > 0$$

$$\begin{aligned}
\langle y_1 - y_2\rangle &= 1 + |y_1 - y_2|^2 \\
&= 1 + |\sum_{q=1}^{Q}(\beta_q - \beta_{q-1})|^2 \\
&\le 1 + 2^{Q-1}\sum_{q=1}^{Q}|\beta_q - \beta_{q-1}|^2 \\
&\le 2^{Q-1}\sum_{q=1}^{Q}(1 + |\beta_q - \beta_{q-1}|^2) \\
&\le 2^{Q-1}\sum_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle
\end{aligned}$$

$$(\sum_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle)^s \le 2^{s(Q-1)}(\sum_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle)^s \tag{2.2.48}$$

$$\begin{aligned}
\langle y_1 - y_2\rangle^s &\le (2^{Q-1}\sum_{q=1}^{Q-1}\langle \beta_q - \beta_{q-1}\rangle)^s \\
&\le c(Q,s)\sum_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle^s.
\end{aligned}$$

and (2.2.46) is proved.
We here make the summmation over all the paths from y_1 to y_2 with Q steps . Let A be

$$A = \sum_{\beta: y_1, y_2\ Q steps} \frac{\langle y_1 - y_2\rangle^s}{\Pi_{q=1}^{Q}\langle \beta_q - \beta_{q-1}\rangle^s}. \tag{2.2.49}$$

Using (2.2.46) we deduce

$$\sum_{\beta:y_1,y_2\ Qsteps} \frac{\langle y_1-y_2\rangle^s}{\Pi_{q=1}^{Q}\langle \beta_q-\beta_{q-1}\rangle^s} \leq \sum_{\beta:y_1,y_2\ Qsteps} \frac{c(Q,s)\sum_{q=1}^{Q}\langle \beta_q-\beta_{q-1}\rangle^s}{\Pi_{q=1}^{Q}\langle \beta_q-\beta_{q-1}\rangle^s} \tag{2.2.50}$$

and

$$\begin{aligned}
A &\leq c(Q,s)\sum_{\beta:y_1,y_2\ Qsteps}\frac{1}{\langle\beta_1-\beta_2\rangle^s\langle\beta_2-\beta_3\rangle^s\ldots\langle y_2-\beta_{q-1}\rangle^s} \\
&\quad +\ldots+\frac{1}{\langle y_1-\beta_1\rangle^s\ldots\langle\beta_{q-2}-\beta_{q-1}\rangle^s} \\
&\leq c(Q,s)\sum_{\beta_2,\ldots,\beta_{q-1}}\frac{1}{\langle\beta_1-\beta_2\rangle^s\langle\beta_2-\beta_3\rangle^s\ldots\langle y_2-\beta_{q-1}\rangle^s} \\
&\quad +\ldots+\frac{1}{\langle y_1-\beta_1\rangle^s\ldots\langle\beta_{q-1}\rangle^s} \\
&\leq c(Q,s)\sum_{\beta_2,\ldots,\beta_{q-1}}\frac{1}{\langle\beta_2-\beta_3\rangle^s\ldots\langle y_2-\beta_{q-1}\rangle^s}\sum_{\beta_1}\frac{1}{\langle\beta_1-\beta_2\rangle^s} \\
&\quad +\ldots+\sum_{\beta_1,\ldots,\beta_{q-2}}\frac{1}{\langle y_1-\beta_1\rangle^s\ldots\langle\beta_{q-2}-\beta_{q-3}\rangle^s}\sum_{\beta_{q-1}}\frac{1}{\langle\beta_{q-2}-\beta_{q-1}\rangle^s} \\
&\leq c(Q,s)\eta^Q
\end{aligned}$$

where η is the sum of the series $\sum_{k\in\mathbb{Z}^2}\frac{1}{(1+|k|^2)^s}$. □

We now define a new diagonal operator $V(\Omega)$. Let $x=(j,k)$ and $y=(j',k')$

$$V(\Omega)(x,y)=(\omega_j^2-\Omega^2k^2)\delta(x,y) \tag{2.2.51}$$

We have the following property:

Lemma 2.2.3

$$||V(\Omega)w||_{m-\gamma,s}\leq\frac{c(\Omega)}{\gamma^2}||w||_{m,s} \tag{2.2.52}$$

Proof:

$$\begin{aligned}
||V(\Omega)w||^2_{m-\gamma,s} &= \alpha\sum_l(1+|l|^2)^s e^{2(m-\gamma)|l|}|V(\Omega)w(l)| \\
&= \alpha\sum_l(1+|l|^2)^s e^{2(m-\gamma)|l|}|\sum_k V(\Omega)(l,k)w(k)|^2
\end{aligned}$$

$$\begin{aligned} &= \alpha \sum_l (1+|l|^2)^s e^{2(m-\gamma)|l|} |V(\Omega)(l,l)w(l)|^2 \\ &= \alpha \sum_l (1+|l|^2)^s e^{2(m-\gamma)|l|} |(\omega_j^2 - k^2\Omega^2)w(l)|^2 \end{aligned}$$

From [9] we have the asymptotics $\omega_j^2 = j^2 + v^* + d(j)$. So for j large enough $\omega_j^2 \leq 2j^2$ and so there exists c such that $\omega_j^2 \leq (2+c)j^2$ for any $j \geq 1$.

$$\begin{aligned} ||V(\Omega)w||^2_{m-\gamma,s} &\leq \sum_l \alpha(1+|l|^2)^s e^{2(m-\gamma)|l|} [(2+c)(j^2+k^2\Omega^2)]^2 |w(l)|^2 \\ &\leq c(\Omega_{max}) \sum_l \alpha(1+|l|^2)^s e^{2(m-\gamma)|l|} |l|^4 |w(l)|^2 \\ &\leq c(\Omega_{max}) \sum_l \alpha(1+|l|^2)^s |w(l)|^2 e^{2\sigma|l|} |l|^4 . e^{-2\gamma|l|} \\ &\leq c(\Omega_{max}) \frac{4^4}{(2\gamma)^4} \sum_l \alpha(1+|l|^2)^s |w(l)|^2 e^{2\sigma|l|} \\ &\leq \frac{c(\Omega_{max})}{\gamma^4} ||w||^2_{m,s} \end{aligned}$$

and

$$||V(\Omega)w||_{m-\gamma,s} \leq \frac{c(\Omega_{max})}{\gamma^2} ||w||_{m,s}. \square \qquad (2.2.53)$$

Chapter 3

Construction of the solution

In this chapter we start the construction of the solution of (2.0.1). We use the technique outlined in [9] which we adapt to the spaces $H^{m,s}$ which are (as we proved in chapter 2) a Banach algebra. This property simplifies the proof of the construction of the solution. First we find an approximate solution u_0, then using the Nash-Moser method we construct a sequence u_n which converges to the solution of the infinite dimensional problem. Let L be a positive integer then we define $B(L)$ as

$$B(L) = \{(j,k) \in \mathbb{N} \times \mathbb{Z},\ |j| + |k| \leq L\}. \tag{3.0.1}$$

Here we use the sequence $L_p = 2^p L_0$. Let $N = \{(1,1),(1,-1)\}$; we denote by $B(\overline{L}) := B(L \setminus N)$

Definition 3.1 *A sequence $(\omega_j)_j$ is (d_0, L_0) nonresonant with ω_{j_0} if there exists $\tau \geq 1$ such that for all $(j,k) \in B(\overline{L_0})$ we have :*

$$|k^2\omega_{j_0}^2 - \omega_j^2| > d_0, \tag{3.0.2}$$

and

$$|k\omega_{j_0} - j| > \frac{d_0}{(|j| + |k|)^\tau} \ \textit{for} \ (j,k) \neq (0,0). \tag{3.0.3}$$

As in [9] we take $j_0 = 1$.

Definition 3.2 *A site (j,k) is said to be d_s-singular if $|k^2\Omega^2 - \omega_j^2| < d_s$.*

As we said in (2.0.5) and (2.0.6) we transform a continuous problem into a discrete one, which we solve. The systems of eigenfunctions $\psi_j(x)$ and $e^{ik\xi}$ are complete; and we can say that

$$[g(u)](j,k) = \int_0^{\pi}\int_0^{2\pi} g(u,x)\psi_j(x)\frac{e^{-ik\xi}}{\sqrt{2\pi}}dx\,d\xi. \tag{3.0.4}$$

We rewrite (2.0.6) as a vector equation to get

$$V(\Omega)u + W(u) = 0 \tag{3.0.5}$$

where $V(\Omega)$ is the operator defined in (2.2.51). W is the nonlinear operator defined by

$$[W(u)](j,k) = \int_0^{\pi}\int_0^{2\pi} g(u,x)\psi_j(x)\frac{e^{-ik\xi}}{\sqrt{2\pi}}dx\,d\xi. \tag{3.0.6}$$

We linearize the operator $F(u)$ about $u = 0$ to obtain the operator $H(u) = V(\Omega) + DW(u)$ which we call the Hamiltonian. We will have $H(0) = V(\Omega)$, which we will need to invert to apply the implicit function theorem. This leads us to find the kernel of $V(\Omega)$, that is the functions φ such that:

$$V(\Omega)\varphi = 0. \tag{3.0.7}$$

We choose our frequency Ω very close to ω_1 hence (3.0.7) implies that

$$(\omega_j^2 - \Omega^2k^2)\varphi(j,k) = 0 \tag{3.0.8}$$

which has two solutions $\Omega = \pm\frac{\omega_j}{k} = \pm\omega_1$ and φ has support in N so we split the space $l^2(\mathbb{N}\times\mathbb{Z})$ to get

$$l^2(\mathbb{N}\times\mathbb{Z}) = l^2(N) \oplus l^2((\mathbb{N}\times\mathbb{Z})\setminus N). \tag{3.0.9}$$

Definition 3.3 *Let $x = (j,k)$ we define the vector $\phi(p)$ as follows*

$$\phi(p)(x) = \frac{1}{2}(p_1 + ip_2)\delta_{(1,1)}(x) + \frac{1}{2}(p_1 - ip_2)\delta_{(1,-1)}(x). \tag{3.0.10}$$

$\phi(p)$ spans the subspace $l^2(N)$. Now followimg the Lyapounov procedure we split the problem (3.0.5) in a system of two equations:

$$\begin{cases} P(V(\Omega) + W(u)) = 0 \\ Q(V(\Omega) + W(u)) = 0. \end{cases} \tag{3.0.11}$$

P is the projection onto $B((\mathbb{N} \times \mathbb{Z}) \setminus N)$ while Q is the projection onto N. Having transform the problem into the form (3.0.11) we apply the Lyapounov-Schmidt procedure to solve it. It is solving the Q-equation that restricts the existence of solutions for $\Omega = \Omega(p)$ a smooth curve which Taylor expansion is given in chapter 6 in (6.0.16). For any $\hat{u} \in l^2(\mathbb{N} \times \mathbb{Z})$ we have the decomposition:

$$\hat{u}(x) = \phi(p)(x) + u(x) \tag{3.0.12}$$

where $u \in l^2((\mathbb{N} \times \mathbb{Z}) \setminus N)$ and $\phi(p) \in l^2(N)$. We denote by $*$ the lattice involution:

$$x = (j, k) \;\Rightarrow\; x^* = (j, -k)$$

Let $p = (p_1, p_2)$ and we will also denote by $p^* = (p_1, -p_2)$. Let $y = (1, 1)$ to write

$$\phi(p)(x) = \frac{1}{2}(p_1 + ip_2)\delta_y(x) + \frac{1}{2}(p_1 - ip_2)\delta_{y^*}(x). \tag{3.0.13}$$

The following covariant properties exist:

Lemma 3.0.4 [9]

$$\overline{\phi(p)(x)} = \phi(\overline{p^*})(x) = \phi(\overline{p})(x^*). \tag{3.0.14}$$

Proof:

$$\begin{aligned}
\overline{\phi(p)(x)} &= \frac{1}{2}(\overline{p_1} + \overline{ip_2})\delta_y(x) + \frac{1}{2}(\overline{p_1} - \overline{ip_2})\delta_{y^*}(x) \\
&= \frac{1}{2}(\overline{p_1} + i\overline{-p_2})\delta_y(x) + \frac{1}{2}(\overline{p_1} - i\overline{-p_2})\delta_{y^*}(x) \\
&= \phi(\overline{p_1}, \overline{-p_2})(x) && (3.0.15)\\
&= \phi(\overline{p}^*)(x) \\
&= \phi(\overline{p^*})(x) && (3.0.16)
\end{aligned}$$

We know that $x = y \Longleftrightarrow x^* = y^*$ and $x = y^* \Longleftrightarrow x^* = y$ so:

$$\begin{aligned}
\overline{\phi(p)(x)} &= \frac{1}{2}(\overline{p_1} - i\overline{p_2})\delta_{y^*}(x^*) + \frac{1}{2}(\overline{p_1} + i\overline{p_2})\delta_y(x^*) \\
&= \phi(\overline{p})(x^*). && (3.0.17)
\end{aligned}$$

Hence we have $\overline{\phi(p)(x)} = \phi(\overline{p^*})(x) = \phi(\overline{p})(x^*)$. □

Furthermore if p is real the property

$$\overline{\phi(p)(x)} = \phi(p)(x^*). \tag{3.0.18}$$

holds.
We denote by T_θ, the translation acting on sequences $u(j,k)$:

$$[T_\theta u](j,k) = e^{ik\theta} u(j,k). \tag{3.0.19}$$

For $p = (p_1, p_2)$ we define

$$T_\theta p = (\cos(\theta)p_1 - \sin(\theta)p_2, \sin(\theta)p_1 + \cos(\theta)p_2) \tag{3.0.20}$$

where the following property holds

Lemma 3.0.5 [9]

$$T_\theta \phi(p) = \phi(T_\theta p) \tag{3.0.21}$$

Proof:

$$\begin{aligned}
\phi(T_\theta p)(x) &= \phi((\cos(\theta)p_1 - \sin(\theta)p_2, \sin(\theta)p_1 + \cos(\theta)p_2))(x) \\
&= \frac{1}{2}[(\cos(\theta)p_1 - \sin(\theta)p_2) + i(\sin(\theta)p_1 + \cos(\theta)p_2)]\delta_y(x) \\
&\quad + \frac{1}{2}[\cos(\theta)p_1 - \sin(\theta)p_2 - i(\sin(\theta)p_1 + \cos(\theta)p_2)]\delta_{y^*}(x) \\
&= \frac{1}{2}[\cos(\theta)p_1 + i\sin(\theta)p_1 + i(\cos(\theta)p_2 + i\sin(\theta)p_2)]\delta_y(x) \\
&\quad + \frac{1}{2}[(\cos(\theta)p_1 - i\sin(\theta)p_1) - i(\cos(\theta)p_2 - i\sin(\theta)p_2)]\delta_{y^*}(x) \\
&= \frac{1}{2}[e^{i\theta}p_1 + e^{i\theta}ip_2]\delta_y(x) + \frac{1}{2}[e^{-i\theta}p_1 - e^{-i\theta}ip_2]\delta_{y^*}(x) \\
&= e^{i\theta}\frac{1}{2}(p_1 + ip_2)\delta_y(x) + e^{-i\theta}(p_1 - ip_2)\delta_{y^*}(x) \\
&= T_\theta\phi(p)(x). \square
\end{aligned}$$

Now if we denote by $\widetilde{\phi(p)}(x)$ a function associated with a vector $\phi(p)(x)$ then we have the relation:

Lemma 3.0.6

$$\overline{\widetilde{\phi(p)}} = \widetilde{\phi(\overline{p})} \tag{3.0.22}$$

Proof:

$$\widetilde{\phi(p)} = \frac{1}{2}(p_1 + ip_2)\psi_1(x)e^{i\xi} + \frac{1}{2}(p_1 - ip_2)\psi_1(x)e^{-i\xi} \tag{3.0.23}$$

$$\begin{aligned}
\overline{\widetilde{\phi(p)}} &= \overline{\frac{1}{2}(p_1 + ip_2)\psi_1(x)e^{i\xi}} + \overline{\frac{1}{2}(p_1 - ip_2)\psi_1(x)e^{-i\xi}} \\
&= \frac{1}{2}(\overline{p_1} - i\overline{p_2})\psi_1(x)e^{-i\xi} + \frac{1}{2}(\overline{p_1} + i\overline{p_2})\psi_1(x)e^{i\xi}
\end{aligned}$$

because the eigenfunctions are real on the real axis $\overline{\psi_1(x)} = \psi_1(x)$ whenever $x \in \mathbb{R}$ and

$$\begin{aligned}\overline{\widetilde{\phi(p)}} &= (\overline{p_1} + i\overline{p_2})\psi_1(x)e^{i\xi} + \frac{1}{2}(\overline{p_1} - i\overline{p_2})\psi_1(x)e^{-i\xi} \\ &= \widetilde{\phi(\overline{p})}.\square \end{aligned} \tag{3.0.24}$$

We have now a relation between the norm of $\phi(p)$ and the modulus of p:

Lemma 3.0.7

$$||\phi(p)||_{\overline{\sigma},s} \leq 2\sqrt{\alpha}5^{s/2}e^{2\overline{\sigma}}|p| \tag{3.0.25}$$

Proof:

$$\begin{aligned}||\phi(p)||^2_{\overline{\sigma},s} &= \alpha(1+2^2)^s|\frac{1}{2}(p_1+ip_2)|^2e^{2\overline{\sigma}(1+1)}\alpha(1+2^2)^s|\frac{1}{2}(p_1-ip_2)|^2e^{2\overline{\sigma}(1+1)} \\ &\leq \alpha 5^s e^{2\overline{\sigma}2}\frac{1}{2}(|p_1+ip_2|^2 + |p_1-ip_2|^2) \\ &\leq \alpha 5^s e^{4\overline{\sigma}}\frac{1}{2}[2(|p_1|^2+|p_2|^2) + 2(|p_1|^2+|p_2|^2)] \\ &\leq 2\alpha 5^s e^{4\overline{\sigma}}[(|p_1|^2+|p_2|^2) + (|p_1|^2+|p_2|^2)] \\ &\leq 4\alpha 5^s e^{4\overline{\sigma}}|p|^2\end{aligned}$$

so we have the relation

$$||\phi(p)||_{\overline{\sigma},s} \leq 2\sqrt{\alpha}5^{s/2}e^{2\overline{\sigma}}|p|. \tag{3.0.26}$$

We will denote by $K_0 := 2\sqrt{\alpha}5^{s/2}e^{\overline{\sigma}}$ and by $||p|| := ||\phi(p)||_{\overline{\sigma},s}$ For any $\sigma \leq \overline{\sigma}$ we also have

$$||\phi(p)||_{\sigma,s} \leq K_0|p|.\square$$

3.1 The approximate problem

The method of the construction of the solution uses Newton's method. We need to find a good the first approximate solution (called u_0) to ensure that iteration converges very rapidly.

3.1.1 Existence of the solution

We denote the projection onto $B(\overline{L_n})$ by P_n and by V_0(resp $W_0(u)$) the projection of V(resp $W_0(u)$) on $B(\overline{L_0})$. We want to solve

$$P(V(\Omega) + W(u)) = 0. \tag{3.1.27}$$

First solve the approximate problem $P_0F(\phi(p)+u)=0$. In this case we are resticted to a finite part of the lattice $\mathbb{N}\times\mathbb{Z}$ and the proof is an application of the implicit function theorem. We solve the approximate problem:

$$P_0(P(W(\hat{u})+V(\Omega)(\hat{u}))=0 \tag{3.1.28}$$

on a finite dimensional subspace $B(\overline{L_0})$. We can decompose

$$\hat{u}=\phi(p)+u,$$

where $\phi(p)$ is the projection of $\hat{u}$ on the sites $N=\{(1.1),(1,-1)\}$ and u has support in $(B(\overline{L_0})$, we then have :

$$P_0W(\phi(p)+u)+V(\Omega)(\phi(p)+u))=0 \tag{3.1.29}$$

$$P_0W(\phi(p)+u)+P_0V(\Omega)(\phi(p)+u)=0$$

$$P_0W(\phi(p)+u)+P_0V(\Omega)(\phi(p)+u)=0$$

$$P_0W(\phi(p)+u)+P_0V(\Omega)u=0.$$

this is because $P_0V(\Omega)(\phi(p))=0$. Let F_1 be

$$F_1((p,\Omega),u):=P_0W(\phi(p)+u)+P_0V(\Omega)(u)=0 \tag{3.1.30}$$

and differentiating (3.1.30) with respect to u we get

$$F_1'((p,\Omega),u)=V_0(\Omega)+P_0W_u'((p,\Omega),u).$$

We have

$$F_1(p=0,\Omega=\omega_1,u=0)=0.$$

We linearize the solution about $u=0$ and $(p=0,\Omega=\omega_1)$ to get

$$F_1'(p=0,\Omega=\omega_1,u=0)=V_0(\omega_1)$$

Now $(\omega_j)_j$ is (d_0,L_0)-nonresonant so that

$$(j,k)\in B(L_0\setminus N),\ |V(\omega_1)((j,k),(j,k))|\geq d_0$$

and $V_0(\omega_1)$ is invertible; we can apply the Implicit Function Theorem to obtain an approximate solution on $B(L_0\setminus N)$ which we call u_0. We want now to find a neighborhood in which the solution lies. For this we use the following lemma:

Lemma 3.1.1 *If* $|\Omega - \omega_1| \leq \frac{d_0}{L_0^3}$, *and* $|p| \leq \frac{d_0^2}{K_0 16^2 C_W^2}$ *we have*

$$||u_0||_{\sigma_0,s} \leq \frac{8C_W}{d_0}||p||^3. \tag{3.1.31}$$

Proof:
We have

$$P_0 W(\phi(p) + u) + V_0(\Omega)u = 0,$$

adding $-V_0(\omega_1)u$ to each side,

$$-V_0(\omega_1)u = P_0 W(\phi(p) + u) + V(\Omega)u - V_0(\omega_1)u$$

$$-V_0(\omega_1)u = P_0 W(\phi(p) + u) + V(\Omega)u - V_0(\omega_1)u$$

$$u = -V_0(\omega_1)^{-1}[P_0 W(\phi(p) + u) + V_0(\Omega)u - V_0(\omega_1)u]$$

so solutions of the approximate problem are fixed of the operator S defined by

$$S(u) = -V_0(\omega_1)^{-1}[P_0 W(\phi(p) + u) + V_0(\Omega)u - V_0(\omega_1)u]. \tag{3.1.32}$$

To get a neighborood in which the solution lies, we must find the conditions on p and Ω which will ensure that this operator is a contraction. Let

$$S = S_1 + S_2$$

where

$$S_1 = -V_0(\omega_1)^{-1}P_0 W(\phi(p) + u) \text{ and } S_2(u) = -V_0(\omega_1)^{-1}[V(\Omega)u - V_0(\omega_1)u]..$$

S_2 is a diagonal operator; let $l = (j, k)$

$$S_2(l, l) = \frac{1}{\omega_j^2 - \omega_1^2 k^2}(\omega_j^2 - \Omega^2 k^2 - (\omega_j^2 - \omega_1^2 k^2))$$

$$\begin{aligned}
|S_2[(j,k),(j,k)]| &\leq \frac{1}{d_0}|\Omega^2 - \omega_1^2|k^2 \\
&\leq \frac{k^2}{d_0}|\Omega + \omega_1|.|\Omega - \omega_1| \\
&\leq \frac{C(\Omega_{max})L_0^2}{d_0}|\Omega - \omega_1|,
\end{aligned}$$

$$\begin{aligned}
||S_2(\hat{u})||_{\sigma_0,s}^2 &= \alpha \sum_l (1 + |l|^2)^s e^{2\sigma_0|l|}.|(S_2\hat{u})(l)|^2 \\
&= \alpha \sum_l (1 + |l|^2)^s e^{2\sigma_0|l|}.|\sum_i S_2(l,i)\hat{u}(i)|^2
\end{aligned}$$

$$\begin{aligned}
&= \alpha\sum_l (1+|l|^2)^s e^{2\sigma_0|l|}.|S_2(l,l)\hat{u}(l)|^2 \\
&\leq (\frac{C(\Omega_{max})L_0^2}{d_0}|\Omega-\omega_1|)^2\alpha\sum_l(1+|l|^2)^s e^{2\sigma_0|l|}|\hat{u}(l)|^2 \\
&\leq (\frac{C(\Omega_{max})L_0^2}{d_0}|\Omega-\omega_1|)^2||\hat{u}||^2_{\sigma_0,s}
\end{aligned}$$

hence

$$||S_2(\hat{u})||_{\sigma_0,s} \leq \frac{C(\Omega_{max})L_0^2}{d_0}|\Omega-\omega_1|||\hat{u}||_{\sigma_0,s}.$$

$$S_1(\hat{u}) = -V_0(\omega_1)^{-1}P_0W(\hat{u}).$$

Here $-V_0(\omega_1)^{-1}$ is a diagonal operator which satisfies the property

$$|V_0(\omega_1)^{-1}((j,k),(j,k))| \leq \frac{1}{d_0}, \tag{3.1.33}$$

applied to the vector $P_0W(\hat{u})$, so as we proved for S_2 we have

$$||S_1(\hat{u})||_{\sigma_0,s} \leq \frac{1}{d_0}||P_0W(\hat{u})||_{\sigma_0,s}.$$

Using now the lemma 2.1.2 and the fact that our nonlinearity is of at least order 3 we deduce that

$$||S_1(\hat{u})||_{\sigma_0,s} \leq \frac{C_W}{d_0}||\hat{u}||^3_{\sigma_0,s}$$

and as $\hat{u} = \phi(p) + u_0$ we have

$$||S_1(\hat{u})||_{\sigma_0,s} \leq \frac{C_W}{d_0}||\phi(p)+u_0||^3_{\sigma_0,s}$$

$$||S_1(\hat{u})||_{\sigma_0,s} \leq \frac{C_W}{d_0}2^2(||\phi(p)||^3_{\sigma_0,s} + ||u_0||^3_{\sigma_0,s}).$$

We have the following restrictions $|p| \leq \frac{d_0^2}{K_0 16^2 C_W^2}$ so using lemma (3.0.7) we have $||p|| \leq \frac{d_0^2}{16^2 C_W^2}$. We have $|\Omega - \omega_1| \leq \frac{d_0}{L_0^3}$ we impose an additional one $||u_0||_{\sigma_0,s} \leq \frac{1}{2}$ to prove that

$$||Su_0||_{\sigma_0,s} \leq \frac{8C_W}{d_0}||p||^3 \text{ and } ||Su_0||_{\sigma_0,s} \leq \frac{1}{2}$$

which will imply the statement in the lemma

$$\begin{aligned}
||S_1u||_{\sigma_0,s} &\leq C_W\frac{1}{d_0}2^2(||p||^3 + ||u_0||^3_{\sigma_0,s}) \\
&\leq 4C_W\frac{1}{d_0}(||p||^3 + (\frac{8C_W}{d_0})^3||p||^9) \\
&\leq 4C_W\frac{1}{d_0}||p||^3(1+(\frac{8C_W}{d_0})^3||p||^6). \qquad (3.1.34)
\end{aligned}$$

Under the following restrictions on $||p|| \leq \frac{d_0^2}{16^2 C_W^2}$ which we introduce into (3.1.34) we get

$$\begin{aligned}
||S_1 u||_{\sigma_0,s} &\leq 4C_W \frac{1}{d_0}||p||^3(1+(\frac{8C_W}{d_0})^3(\frac{d_0^2}{16^2 C_W^2})^6) \\
&\leq 4C_W \frac{1}{d_0}||p||^3(1+\frac{8^3 C_W^3}{d_0^3}\frac{d_0^1 2}{8^2 4 C_W^1 2})) \\
&\leq 4C_W \frac{1}{d_0}||p||^3(1+\frac{d_0^9}{8^2 1 C_W^9}) \\
&\leq 4\frac{C_W}{d_0}||p||^3(1+\frac{d_0^9}{8^2 1 C_W^9}).
\end{aligned}$$

Now if we have d_0 small enough then

$$||S_1 u||_{\sigma_0,s} \leq \frac{3}{2}4\frac{C_W}{d_0}||p||^3_{\sigma_0,s}$$

And on the other term we had the following estimate

$$||S_2 u||_{\sigma_0,s} \leq \frac{C(\Omega_{max})L_0^2}{d_0}|\Omega-\omega_1|||\hat{u}||_{\sigma_0,s}.$$

If we add now the fact that $|\Omega-\omega_1| \leq \frac{d_0}{L_0^3}$ we can say that

$$||S_2 u||_{\sigma_0,s} \leq \frac{C(\Omega_{max})L_0^2}{d_0}\frac{d_0}{L_0^3}||\hat{u}||_{\sigma_0,s}$$

which implies that

$$||S_2 u||_{\sigma_0,s} \leq \frac{C(\Omega_{max})L_0^2}{d_0}\frac{d_0}{L_0^3}\frac{2.4C_W}{d_0}||p||^3.$$

So for L_0 large enough we will have $\frac{C(\Omega_{max})L_0^2}{d_0}\frac{2d_0}{L_0^3} = \frac{C(\Omega_{max})}{L_0} \leq \frac{1}{2}$ and hence

$$||S\hat{u}||_{\sigma_0,s} \leq (\frac{3}{2}+\frac{1}{2})\frac{4C_W}{d_0}||p||^3$$

and

$$||S\hat{u}||_{\sigma_0,s} \leq \frac{8C_W}{d_0}||p||^3$$

and as $||p|| \leq \frac{d_0^2}{16^2 C_W^2}$ we have also

$$||S\hat{u}||_{\sigma_0,s} \leq (\frac{3}{2}+\frac{1}{2})\frac{8C_W}{d_0}(\frac{d_0^2}{16^2 C_W^2})^3$$

$$||S\hat{u}||_{\sigma_0,s} \leq (\frac{3}{2}+\frac{1}{2})\frac{8C_W}{d_0}\frac{d_0^6}{16^6C_W^6}$$

$$||S\hat{u}||_{\sigma_0,s} \leq \frac{d_0^5}{16^5C_W^5}$$

and for d_0 small enough we have

$$||S\hat{u}||_{\sigma_0,s} \leq \frac{1}{2}$$

which proves that the solution will lie in this neighborhood of the origin. We show now that S is a contraction to prove that the fixed point lies in the ball Y of centre 0 and raduis $\frac{8C_W}{d_0}||p||^3$. Let

$$B(u_2-u_1) := -V_0(\omega_1)^{-1}(V(\Omega)(u_2-u_1)-V_0(\omega_1))(u_2-u_1)$$

$$\begin{aligned}
||B(u_2-u_1)||_{\sigma_0,s} &= ||-V_0(\omega_1)^{-1}[V(\Omega)-V_0(\omega_1)](u_2-u_1)||_{\sigma_0,s} \\
&\leq \sup_{l\in L_0\setminus N}|(-V_0(\omega_1)^{-1}[V(\Omega)-V_0(\omega_1)])(l,l)|||u_2-u_1||_{\sigma_0,s} \\
&\leq \sup_{l\in L_0\setminus N}|-V_0(\omega_1)^{-1}(l,l)\sup_{l\in L_0\setminus N}[V(\Omega)-V_0(\omega_1)](l,l)|||u_2-u_1||_{\sigma_0,s} \\
&\leq \frac{1}{d_0}|\frac{L_0^2}{d_0}C(\Omega_{max})(\Omega-\omega_1)|.||u_2-u_1||_{\sigma_0,s} \\
&\leq \frac{1}{d_0}|L_0^2C(\Omega_{max})|\frac{d_0}{L_0^3}||u_2-u_1||_{\sigma_0,s} \\
&\leq \frac{C(\Omega_{max})}{L_0}||u_2-u_1||_{\sigma_0,s}.
\end{aligned}$$

So for L_0 large enough

$$||B(u_2-u_1)||_{\sigma_0,s} < \frac{1}{2}||u_2-u_1||_{\sigma_0,s}.$$

Let

$$A(u_2-u_1) := V_0^{-1}(\omega_1)(W_0(\phi(p)+u_2)-W_0(\phi(p)+u_1)).$$

To apply the fixed point theorem we need to show that S is a contraction from Y to Y

$$\begin{aligned}
||A(u_2-u_1)||_{\sigma_0,s} &= ||V_0^{-1}(\omega_1)(W_0(\phi(p)+u_2)-W_0(\phi(p)+u_1)||_{\sigma_0,s} \\
&\leq \sup_{l\in L_0\setminus N}|-V_0(\omega_1)^{-1}(l,l)|||W_0(\phi(p)+u_2)-W_0(\phi(p)+u_1)||_{\sigma_0,s} \\
&\leq \frac{1}{d_0}||\int_0^1 DW(\phi(p)+u_2+\tau(u_2-u_1))(u_2-u_1)d\tau||_{\sigma_0,s} \\
&\leq \frac{1}{d_0}\sup_{0\leq\tau\leq 1}||D_uW(\phi(p)+u_1+\tau(u_2-u_1)(u_2-u_1)||_{\sigma_0,s}
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{d_0} \sup_{0\leq\tau\leq 1} \|D_u W(\phi(p) + u_1 + \tau(u_2 - u_1))\|_{\sigma_0,s} \|u_1 - v_2\|_{\sigma_0,s} \\ &\leq \frac{C_{op}}{d_0} C_{DW} (\|\phi(p)\| + \|u_1\| + \tau\|u_2 - u_1\|)^2 \|u_1 - u_2\|_{\sigma_0,s}. \end{aligned}$$

The C_{op} comes from the fact that we have replaced the operator DW by the polynomial DW as in lemma 2.1.4,

$$\|A(u_2 - u_1)\|_{\sigma_0,s} \leq \frac{C_{op}}{d_0} C_{DW} (\|p\| + \frac{8C_W}{d_0}\|p\|^3 + 2.\frac{8C_W}{d_0}\|p\|^3)^2 \|u_1 - v_2\|_{\sigma_0,s}.$$

We have assumed that the nonlinearity is a polynomial of order 3 at least and the first derivative is of at least order 2 then fact that $\|p\| \leq \frac{d_0^2}{16^2 C_W^2}$ implies

$$\begin{aligned} \|A(u_2 - u_1)\|_{\sigma_0,s} &\leq \frac{C_{op}}{d_0} C_{DW} \|p\|^2 (1 + \frac{24C_W}{d_0}\|p\|^2)^2 \|u_1 - u_2\|_{\sigma_0,s} \\ &\leq \frac{C_{op}}{d_0} C_{DW} (\frac{d_0^2}{16^2 C_W^2})^2 (1 + \frac{24C_W}{d_0}(\frac{d_0^2}{16^2 C_W^2})^2)^2 \|u_1 - u_2\|_{\sigma_0,s} \\ &\leq \frac{C_{op} d_0^3}{16^4 C_W^3} (1 + \frac{24 d_0^3}{16^4 C_W^3}) \|u_1 - u_2\|_{\sigma_0,s}. \end{aligned}$$

So for d_0 small enough we have

$$\|A(u_2 - u_1)\|_{\sigma_0,s} < \frac{1}{2}\|u_1 - u_2\|_{\sigma_0,s}$$

$$\|S(u_2) - S(u_1)\| \leq \|A(u_2 - u_1)\|_{\sigma_0,s} + \|B(u_2 - u_1)\|_{\sigma_0,s} < \|u_1 - u_2\|_{\sigma_0,s}$$

hence S is a contraction. □

Let $r_0 = \min(\frac{1}{2}\frac{d_0}{L_0^3}, \frac{1}{2}\frac{d_0^2}{K_0 16^2 C_W^2})$, we now define $\mathcal{N}_0$ as

$$\mathcal{N}_0 = \{(p, \Omega) \in \mathbb{R}^2 \times \mathbb{R} \text{ such that } |p| < r_0 \text{ and } |\Omega - \omega_1| < r_0\} \tag{3.1.35}$$

and the ρ_0 complex neighborhood of $\mathcal{N}_0$, $(\mathcal{N}_0, \rho_0)$ by:

$$(\mathcal{N}_0, \rho_0) = \{(p_1, \Omega_1) \in \mathbb{C}^2 \times \mathbb{C} \text{ where } (p_1, \Omega_1) \in \mathcal{N}_0 \text{ and } \sqrt{|p - p_1|^2 + |\Omega - \Omega_1|^2} \leq \rho_0\} \tag{3.1.36}$$

We will for convenience take $\rho_0 = r_0$ and we have the following corollary:

Corollary 1 *For $(p, \Omega) \in (\mathcal{N}_0, \rho_0)$ we have the following estimate*

$$\|u_0\|_{\sigma_0,s} \leq 8\frac{C_W}{d_0}\|p\|^3 \tag{3.1.37}$$

Proof:
As long as $(p,\Omega) \in (\mathcal{N}_0, \rho_0)$ we have

$$|p| \leq r_0 + \rho_0 \leq \frac{d_0}{L_0^3} \tag{3.1.38}$$

and as well

$$|\Omega - \omega_1| \leq r_0 + \rho_0 \leq \frac{d_0^2}{K_0 16^2 C_W^2} \tag{3.1.39}$$

so we can use the lemma 3.1.1 to deduce that

$$||u_0||_{\sigma_0,s} \leq 8\frac{C_W}{d_0}||p||^3.\square \tag{3.1.40}$$

We also define rewrite $r_0 := L_0^{-2-\eta}$. For any η greater than 2 we can take L_0 large enough such that $r_0 := L_0^{-2-\eta}$. We now show some covariant properties of the solution u_0.

3.1.2 Properties of u_0

Lemma 3.1.2 [9] *The solution u_0 of (3.1.28) satisfies the following covariant properties:*

$$\begin{cases} W(T_\theta u_0) = T_\theta W(u_0) \\ \overline{V(\Omega)u_0(x)} = V(\overline{\Omega})\overline{u_0(x)} \\ \overline{W(u_0)} = W(\overline{u_0(x)}) \\ \widetilde{\overline{u_0}} = \overline{\widetilde{u_0}} \end{cases} \tag{3.1.41}$$

$$u_0(\overline{p}, \overline{\Omega}, x) = \overline{u_0(p,\Omega)(x^*)} \tag{3.1.42}$$

$$u_0(\overline{p^*}, \overline{\Omega}, x) = \overline{u_0(p,\Omega,x)} \tag{3.1.43}$$

and furthermore

$$V(\Omega)T_\theta = T_\theta V(\Omega). \tag{3.1.44}$$

Proof:
The relation in (3.1.44) comes from the fact that $V(\Omega)$ and T_θ are diagonal. Let $u(j,k)$ be the lattice representation of a function $\widetilde{u}(x,\xi)$. Then $e^{ik\theta}u(j,k)$ is the lattice representation of $\widetilde{u}(x,\xi+\theta)$:

$$\widetilde{u}(x,\xi) = \sum_{j,k} u(j,k)e^{ik\xi}\psi_j(x)$$

$$\widetilde{u}(x,\xi+\theta) = \sum_{j,k} u(j,k)e^{ik(\xi+\theta)}\psi_j(x)$$

$$\widetilde{u}(x,\xi+\theta)=\sum_{j,k}e^{ik\theta}u(j,k)e^{ik\xi}\psi_j(x)$$

$$W(u)(j,k)=\int_0^\pi\int_0^{2\pi}\overline{\psi_j(x)}e^{-ik\xi}g(\widetilde{u}(x,\xi))dxd\xi$$

$$\begin{aligned}T_\theta W(u)(j,k) &= e^{ik\theta}\int_0^\pi\int_0^{2\pi}\overline{\psi_j(x)}e^{-ikx}g(\widetilde{u}(x,\xi))dxd\xi\\ &= \int_0^\pi\int_0^{2\pi}\overline{\psi_j(x)}e^{-ik(\xi-\theta)}g(\widetilde{u}(x,\xi))dxd\xi.\end{aligned}$$

We now make the following change of variables $\eta=\xi-\theta$ and we get

$$\begin{aligned}T_\theta W(u)(j,k) &= \int_0^{2\pi}\int_{-\theta}^{2\pi-\theta}\overline{\psi_j(x)}e^{-ik\eta}g(\widetilde{u}(x,\eta+\theta))dxd\eta\\ &= \int_0^{2\pi}\int_{-\theta}^{2\pi-\theta}\overline{\psi_j(x)}e^{-ik\eta}g(\widetilde{T_\theta u}(x,\eta))dxd\eta\end{aligned}$$

and as the integrand is of period 2π in η we have

$$\begin{aligned}&= \int_0^{2\pi}\int_0^{2\pi}\overline{\psi_j(x)}e^{-ik\eta}g(\widetilde{T_\theta u}(x,\eta))dxd\eta\\ &= W(T_\theta u).\end{aligned} \tag{3.1.45}$$

Now going back to equation (3.1.29), u_0 satisfies

$$P_0W(\phi(p)+u_0(p,\Omega))+V(\Omega)(\phi(p)+u_0(p,\Omega))=0 \tag{3.1.46}$$

$$T_\theta P_0W(\phi(p)+u_0(p,\Omega))+V(\Omega)(\phi(p)+u_0(p,\Omega))=0 \tag{3.1.47}$$

$$P_0W(T_\theta\phi(p)+T_\theta u_0(p,\Omega))+T_\theta V(\Omega)(\phi(p)+u_0(p,\Omega))=0. \tag{3.1.48}$$

As $V(\Omega)$ is diagonal it commutes with T_θ so

$$P_0W(T_\theta\phi(p)+T_\theta u_0(p,\Omega))+V(\Omega)(T_\theta\phi(p)+T_\theta u_0(p,\Omega))=0 \tag{3.1.49}$$

and as $T_\theta\phi(p)=\phi(T_\theta p)$

$$P_0W(\phi(T_\theta p)+T_\theta u_0(p,\Omega))+V(\Omega)(\phi(T_\theta p)+T_\theta u_0(p,\Omega))=0. \tag{3.1.50}$$

The Implicit Function Theorem provides the uniqueness of the solution so we can conclude that $T_\theta u_0(p,\Omega)=u_0(T_\theta p,\Omega)$. Restarting again by 3.1.29 u_0 satisfies

$$\int_0^\pi\int_0^{2\pi}\psi_j(x)e^{-ik\xi}g(\widetilde{u_0}+\widetilde{\phi(p)})(x,\xi)dxd\xi+V(\Omega)u_0(j,k)=0 \tag{3.1.51}$$

$$\overline{\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{-ik\xi} g(\widetilde{u_0} + \widetilde{\phi(p)})(x,\xi) dx d\xi + V(\Omega) u_0(j,k)} = 0 \tag{3.1.52}$$

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} \overline{g(\widetilde{u_0} + \widetilde{\phi(p)})(x,\xi)} dx d\xi + \overline{V(\Omega) u_0(j,k)} = 0. \tag{3.1.53}$$

Now we evaluate $\overline{\widetilde{u}(p,\Omega)}$

$$\begin{aligned}
\overline{\widetilde{u}(p,\Omega)} &= \overline{\sum_{j,k} u(p,\Omega)(j,k)\psi_j(x)e^{ik\xi}} \\
&= \sum_{j,k} \overline{u}(p,\Omega)(j,k)\psi_j(x)e^{-ik\xi} \\
&= \sum_{j,k} \overline{u}(p,\Omega)(j,-k)\psi_j(x)e^{ik\xi}.
\end{aligned}$$

So $\overline{\widetilde{u}(p,\Omega)}(j,k) = \widetilde{\overline{u}}(p,\Omega)(j,-k)$ and $\overline{\widetilde{u}(p,\Omega,x)} = \overline{\widetilde{u(p,\Omega,x^*)}}$

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} \overline{g(\widetilde{u_0}(p,\Omega)(x) + \widetilde{\phi(p)}(x))}(x,\xi) dx d\xi + \overline{V(\Omega) u_0(p,\Omega)(x)(j,k)} = 0 \tag{3.1.54}$$

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} g(\overline{\widetilde{u_0(p,\Omega)}(x^*)} + \phi\widetilde{(\overline{p})(x)})(x,\xi) dx d\xi + V(\overline{\Omega}) \overline{u_0(p,\Omega)(x)(j,k)} = 0 \tag{3.1.55}$$

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} g(\overline{\widetilde{u_0(p,\Omega)}(x^*)} + \phi\widetilde{(\overline{p})(x)})(x,\xi) dx d\xi + V(\overline{\Omega}) \overline{u_0(p,\Omega)(x^*)}(j,-k) = 0. \tag{3.1.56}$$

By the uniqueness of the Implicit Function Theorem we know that the only function v that satisfies

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{-ik\xi} g(\widetilde{v} + \phi\widetilde{(\overline{p})(x)})(x,\xi) dx d\xi + V(\overline{\Omega}) v(j,k) = 0 \tag{3.1.57}$$

or

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} g(\widetilde{v} + \phi\widetilde{(\overline{p})(x)})(x,\xi) dx d\xi + V(\overline{\Omega}) v(j,-k) = 0 \tag{3.1.58}$$

is $u_0(\overline{p},\overline{\Omega},x)$ so we conclude that $u_0(\overline{p},\overline{\Omega},x) = \overline{u_0(p,\Omega)(x^*)}$ and we can as well deduce another property of u_0 from (3.1.56):

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} g(\overline{\widetilde{u_0(p,\Omega)}(x^*)} + \phi\widetilde{(\overline{p})(x)})(x,\xi) dx d\xi + V(\overline{\Omega}) \overline{u_0(p,\Omega)(x^*)}(j,-k) = 0$$

so if we change $x \to x^*$ then

$$\int_0^\pi \int_0^{2\pi} \psi_j(x) e^{ik\xi} g(\overline{\widetilde{u_0(p,\Omega)}(x)} + \phi\widetilde{(\overline{p})(x^*)})(x,\xi) dx d\xi + V(\overline{\Omega}) \overline{u_0(p,\Omega)(x)}(j,-k) = 0$$

$$\int_0^{\pi}\int_0^{2\pi}\psi_j(x)e^{ik\xi}g(\widetilde{\overline{u_0(p,\Omega)(x)}}+\phi(\widetilde{\overline{p^*}})(x))(x,\xi)dxd\xi+V(\overline{\Omega})\overline{u_0(p,\Omega)(x)}(j,-k)=0$$

and as before we deduce that $u_0(\overline{p^*},\overline{\Omega},x)=\overline{u_0(p,\Omega,x)}$

$$\tilde{u}=\sum_{j,k}u(j,k)\psi_j(x)e^{ik\xi}$$

$$u(j,k)=\int_0^{\pi}\int_0^{2\pi}\overline{\psi_j(x)}e^{-ik\xi}u(x,\xi)dxd\xi$$

$$\overline{u(j,k)}=\int_0^{\pi}\int_0^{2\pi}\overline{\overline{\psi_j(x)}}e^{ik\xi}\overline{u(x,\xi)}dxd\xi=(\overline{u})(j,-k)$$

so

$$\begin{aligned}\overline{\tilde{u}} &= \sum_{j,k}\overline{u(j,k)}\psi_j(x)e^{-ik\xi}\\ &= \sum_{j,k}\overline{u}(j,-k)\psi_j(x)e^{-ik\xi}\\ &= \tilde{\overline{u}}\ \square.\end{aligned}$$

3.2 The Nash-Moser method

Our aim is to solve the problem

$$F(\hat{u})=P(W(\hat{u})+V(\Omega)(\hat{u}))=0$$

Having solved the approximate problem

$$P_0(P(W(\hat{u})+V(\Omega)(\hat{u}))=0 \tag{3.2.59}$$

on a finite dimensional subspace $B(\overline{L_0})$, we now construct a sequence u_n which will converge to u the solution of $F(u)=0$.
Taylor's formula states that:

$$F(\phi(p)+u_n+\delta u)=F(\phi(p)+u_n)+F'(\phi(p)+u_n)\delta u+\int_0^1 F''(\phi(p)+u_n+\tau\delta u)\delta u.\delta u d\tau \tag{3.2.60}$$

we write

$$u_n+\delta u=u_{n+1} \tag{3.2.61}$$

and we can then rewrite

$$F(\phi(p)+u_{n+1}) = F(\phi(p)+u_n)+F'(\phi(p)+u_n)\delta u+\int_0^1 F''(\phi(p)+u_n+\tau\delta u)\delta u\delta u d\tau$$

our goal is how to prove that:

$$\lim_{n\to\infty} F(\phi(p)+u_{n+1}) = 0.$$

We denote the projection onto $B(L_n \setminus N)$ by P_n and by P the projection onto $l^2(\mathbb{N}\times\mathbb{Z}\setminus N)$. L_0 will be chosen very large to ensure that our algorithm to construct the solution converges. We know that $F'(u) = V(\Omega) + DW(u)$ which we have to invert at each iterative step of Newton's method to get a better approximate solution. As we just proved in the lemma 2.2.3 $V(\Omega)$ is a mapping from $H^{\sigma-\gamma,s}$ to $H^{\sigma,s}$ so its inverse (whenever it exists) will be from $H^{\sigma,s}$ to $H^{\sigma-\gamma,s}$. We will have to choose the sequence γ_n carefully so that $\sigma - \sum_n \gamma_n > 0$. Before we start Newton's method we already expect some loss of decay at each stage of the iteration. We now start by a simple lemma:

Lemma 3.2.1 *Let $\gamma > 0$ and denote by Π_n the projection on L_n then*

$$||(I-\Pi_n)u||_{\sigma,s} \leq e^{-\gamma L_n}||u||_{\sigma+\gamma,s} \tag{3.2.62}$$

Proof:
Since $\Pi_n((I-\Pi_n)u) = 0$ then

$$\begin{aligned}
||(I-\Pi_n)u||^2_{\sigma,s} &= \sum_{|l|\geq L_n}(1+|l|^2)^s|u(l)|^2e^{2\sigma} \\
&= \sum_{|l|\geq L_n}(1+|l|^2)^s|u(l)|^2e^{2(\sigma+\gamma-\gamma)|l|} \\
&= \sum_{|l|\geq L_n}(1+|l|^2)^s|u(l)|^2e^{2(\sigma+\gamma)|l|}e^{-2\gamma|l|} \\
&\leq e^{-2\gamma L_n}\sum_{|l|\geq L_n}(1+|l|^2)^s|u(l)|^2e^{2(\sigma+\gamma)|l|} \\
&\leq e^{-2\gamma L_n}||u||^2_{\sigma+\gamma,s}
\end{aligned} \tag{3.2.63}$$

and

$$||(I-\Pi_n)u||_{\sigma,s} \leq e^{-\gamma L_n}||u||_{\sigma+\gamma,s}.\square \tag{3.2.64}$$

We now prove an estimate on $F(\phi(p)+u_0)$.

Theorem 3.1 *Let u_0 be the solution of the approximate problem (3.1.28). Then there exists ε_0 such that we have the following estimate*

$$||F(\phi(p)+u_0)||_{\sigma_0-2\gamma_0,s} \leq ||p||^2\varepsilon_0 \tag{3.2.65}$$

Proof:
Since

$$P_0F(\phi(p)+u_0)=0$$

then

$$\begin{aligned} ||F(\phi(p)+u_0)||_{\sigma_0-2\gamma_0,s} &= ||(1-P_{L_0})F(\phi(p)+u_0)||_{\sigma_0-2\gamma_0,s} \\ &= ||(1-P_{L_0})(V(\Omega)(u_0+\phi(p))+W(\phi(p)+u_0)||_{\sigma_0-\gamma_0,s} \end{aligned}$$

$V(\Omega)$ is diagonal so

$$(1-P_{L_0})(V(\Omega)(u_0+\phi(p)) = (V(\Omega)(1-P_{L_0})(u_0+\phi(p))=0 \tag{3.2.66}$$

$$\begin{aligned} ||F(\phi(p)+u_0)||_{\sigma_0-2\gamma_0,s} &= ||(1-P_{L_0})W(\phi(p)+u_0)||_{\sigma_0-\gamma_0,s} \\ &\leq e^{-\gamma_0L_0}||W(\phi(p)+u_0)||_{\sigma_0-\gamma_0,s} \\ &\leq e^{-\gamma_0L_0}C_W||\phi(p)+u_0||^3_{\sigma_0-\gamma_0,s}) \\ &\leq e^{-\gamma_0L_0}C_W(||p||+8C_W\frac{||p||^3}{d_0})^3 \\ &\leq e^{-\gamma_0L_0}C_W||p||^3(1+8C_W\frac{||p||^2}{d_0})^3 \end{aligned} \tag{3.2.67}$$

For L_0 large, r_0 becomes very small, and as $||p|| \leq r_0$ we can deduce

$$\begin{aligned} ||F(\phi(p)+u_0)||_{\sigma_0-2\gamma_0,s} &\leq ||p||^2e^{-\gamma_0L_0} \\ &\leq ||p||^2\varepsilon_0. \end{aligned} \tag{3.2.68}$$

where $\varepsilon_0 = e^{-c_0L_0}$ and $c_0 \leq \gamma_0$. □

At each iterative step we want to get an estimate of the norm of $F(\phi(p)+u_{n+1})$ which is of the form :

$$||F(\phi(p)+u_{n+1})||_{\sigma_n,s} \leq \varepsilon_{n+1}||p||, \tag{3.2.69}$$

where σ_n is a sequence with a strictly positive lower bound ($\sigma_n > \frac{\sigma_0}{2} > 0\ \forall p$) and ε_n is a sequence converging towards zero as n tends to infinity. We will proceed in two steps :

First we split the operator F into two parts

$$P_{n+1}F(\phi(p)+u_{n+1}) \text{ and } (I-P_{n+1})F(\phi(p)+u_{n+1})$$

and we want to get the estimates

$$||P_{n+1}F(\phi(p)+u_{n+1})||_{\sigma_{n+1}} \leq \frac{1}{2}\varepsilon_{n+1}||p||$$

$$||(I-P_{n+1})F(\phi(p)+u_{n+1})||_{\sigma_{n+1}} \leq \frac{1}{2}\varepsilon_{n+1}||p||$$

where the sequence u_n, converges so that its limit u satisfies:

$$F(\phi(p)+u) = 0. \tag{3.2.70}$$

$$\begin{aligned} P_{n+1}F(\phi(p)+u_{n+1}) &= P_{n+1}(F(\phi(p)+u_n) + F'(\phi(p)+u_n)\delta u) \\ &\quad + P_{n+1}\left(\int_0^1 F''(\phi(p)+v_n+\tau\delta u)\delta u.\delta u d\tau\right) \end{aligned} \tag{3.2.71}$$

We now decompose $P_{n+1}F(\phi(p)+u_{n+1})$ in two parts:

$$P_{n+1}F(\phi(p)+u_{n+1}) = A_{n+1} + B_{n+1}$$

where

$$\begin{cases} A_{n+1} = P_{n+1}(F(\phi(p)+u_n) + F'(\phi(p)+u_n)\delta u) \\ B_{n+1} = P_{n+1}\int_0^1 F''(\phi(p)+u_n+\tau\delta u)\delta u\delta u d\tau. \end{cases} \tag{3.2.72}$$

The Nash-Moser method consists of finding δu such that

$$\begin{cases} A_{n+1} = 0, \\ \text{and } \textit{support of } \delta u \subset \overline{L_{n+1}} \text{ with } P_{n+1}\delta u = \delta u. \end{cases}$$

Now

$$A_{n+1} = P_{n+1}F(\phi(p)+u_n) + P_{n+1}F'(\phi(p)+u_n)\delta u \tag{3.2.73}$$

and also

$$A_{n+1} = P_{n+1}F(\phi(p)+u_n) + P_{n+1}F'(\phi(p)+u_n)P_{n+1}\delta u$$

because $P_{n+1}\delta u = \delta u$. We take

$$\delta u = -(P_{n+1}F'(\phi(p)+u_n)P_{n+1})^{-1}P_{n+1}(F(\phi(p)+u_n)). \tag{3.2.74}$$

and we need the restriction of the Hamiltonian H to $\overline{L_{n+1}}$:

$$H_{n+1} = P_{n+1}F'(\phi(p) + u_n)P_{n+1} \tag{3.2.75}$$

to be invertible. (3.2.74) can be rewritten as

$$\delta u = -H_{n+1}^{-1}P_{n+1}F(\phi(p) + u_n) \tag{3.2.76}$$

and substituting (3.2.76) into (3.2.73) we have

$$\begin{aligned} A_{n+1} &= P_{n+1}(F(\phi(p) + u_n)) - (P_{n+1}(F'(u_n)P_{n+1})(P_{n+1}(F'(\phi(p) + u_n)P_{n+1})^{-1} \\ &\quad .P_{n+1}(F(\phi(p) + u_n)) \end{aligned} \tag{3.2.77}$$

so

$$A_{n+1} = P_{n+1}(F(\phi(p) + u_n)) - P_{n+1}(F(\phi(p) + u_n)) = 0.$$

We denote by G_{n+1} the inverse H_{n+1}. We check now that at each iterative step all the conditions are fulfilled to invert the Hamiltonian and we introduce now the following sequences

$$\begin{cases} \varepsilon_n = {\varepsilon_0}^{k^n} \\ \delta_n = L_n^{-\alpha} = 2^{-n\alpha}L_0^{-\alpha} \\ \gamma_n = \frac{\sigma_0}{32(1+n^2)} \\ \sigma_{n+1} = \sigma_n - 5\gamma_n \end{cases}$$

which are used during the iteration. where M_n is defined by

$$M_n := ||u_n + \phi(p)||_{\sigma_n}. \tag{3.2.78}$$

Taking the norm of (3.2.76)

$$||\delta u_{n+1}||_{\sigma_n - 2\gamma_n} \leq ||G_{n+1}(u_n)||_{\sigma_n - 2\gamma_n}||P_{n+1}F(u_n)||_{\sigma_n - 2\gamma_n} \tag{3.2.79}$$

Since $||F(u_n + \phi(p))||_{\sigma_n} \leq \varepsilon_n||p||$ we want to prove that at the $n + 1^{th}$ stage we still have

$$||F(u_{n+1} + \phi(p))||_{\sigma_{n+1}} \leq \varepsilon_{n+1}||p||. \tag{3.2.80}$$

The estimate on $||G_{n+1}(u_n + \phi(p))||_{\sigma_n - 2\gamma_n}$,

$$||G_{n+1}(u_n + \phi(p))||_{\sigma_n - 2\gamma_n} \leq \frac{||p||^2}{d_0\delta_{n+1}\gamma_n^2} \tag{3.2.81}$$

that we use in the next equation is the object of the whole of chapter 3. We use it here and provide the proof later.

$$||\delta u_{n+1}||_{\sigma_n-2\gamma_n} \leq \frac{||p||^2}{d_0\delta_{n+1}\gamma_n}||P_{n+1}F(u_n+\phi(p))||_{\sigma_n}$$

as $||P_{n+1}F(u_n+\phi(p))|| \leq ||F(u_n+\phi(p))||$ so

$$||\delta u_{n+1}||_{\sigma_n-2\gamma_n} \leq \frac{||p||^2}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n||p||. \tag{3.2.82}$$

We have

$$\sigma_n - 2\gamma_n \geq \sigma_{n+1}$$
$$M_{n+1} = ||u_{n+1}+\phi(p)||_{\sigma_{n+1}}$$
$$M_{n+1} \leq ||u_n+\phi(p)||_{\sigma_{n+1}} + ||\delta u_{n+1}||_{\sigma_{n+1}}$$
$$M_{n+1} \leq M_n + ||\delta u_{n+1}||_{\sigma_{n+1}}. \tag{3.2.83}$$

We derive another estimate of M_{n+1}

$$\begin{aligned} M_{n+1} &\leq M_n + \frac{||p||^2}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n||p|| \\ &\leq M_0 + \sum_{r=0}^{n}\frac{||p||^2}{d_0\delta_{r+1}\gamma_n^2}\varepsilon_n||p|| \\ &\leq ||p|| + 8\frac{C_W}{d_0}||p||^3 + N||p||^3 \end{aligned} \tag{3.2.84}$$

where

$$N = \sum_{n=0}^{\infty}\frac{\varepsilon_n}{d_0\delta_{r+1}\gamma_n^2}. \tag{3.2.85}$$

and we define the number C_3 by

$$C_3 = 8\frac{C_W}{d_0}K_0^3 + N \tag{3.2.86}$$

and we choose L_0 very large such that

$$C_3 r_0 \leq \frac{1}{2}. \tag{3.2.87}$$

We now have to prove an estimate on $||u_{n+1}||_{\sigma_{n+1}-2\gamma_{n+1}}$,

$$\text{where } u_{n+1} = u_0 + \sum_{r=0}^{n}\delta u_{r+1}.$$

We note that the sequence $\sigma_n - 2\gamma_n$ is decreasing :

$$\sigma_{n+1} - 2\gamma_{n+1} = \sigma_n - 5\gamma_n - 2\gamma_{n+1}$$

$$\sigma_{n+1} - 2\gamma_{n+1} = \sigma_n - 2\gamma_n - 3\gamma_n - 2\gamma_{n+1}$$

$$\sigma_{n+1} - 2\gamma_{n+1} \leq \sigma_n - 2\gamma_n. \tag{3.2.88}$$

$$\begin{aligned} ||u_{n+1}||_{\sigma_{n+1}-2\gamma_{n+1}} &\leq ||u_0||_{\sigma_0-2\gamma_0} + \sum_{r=0}^{n} ||\delta u_{r+1}||_{\sigma_{r+1}-2\gamma_{r+1}} \\ &\leq ||u_0||_{\sigma_0-2\gamma_0} + \sum_{r=0}^{n} \frac{||p||^2}{d_0\delta_{r+1}(2\gamma_{r+1})^2}\varepsilon_n||p||, \end{aligned}$$

and so

$$\begin{aligned} ||u_{n+1}||_{\sigma_{n+1}-2\gamma_{n+1}} &\leq ||u_0||_{\sigma_0-2\gamma_0} + \frac{||p||^3}{4d_0}\sum_{r=0}^{n} \frac{{\varepsilon_0}^{k^r}}{\delta_{r+1}(\gamma_{r+1}^2)} \\ &\leq ||u_0||_{\sigma_0-2\gamma_0} + \frac{32||p||^3}{4d_0\sigma_0}\sum_{r=0}^{n} e^{k^r \ln \varepsilon_0} \\ &\qquad e^{-r\alpha \ln 2}.e^{\alpha \ln L_0}.e^{2\ln(1+r^2)}. \end{aligned} \tag{3.2.89}$$

Since L_0 large, $1 < k$, and $k^r \ln \varepsilon_0 = -c_0 L_0 k^r$, $e^{k^r \ln \varepsilon_0}$ dominates the other terms in the sum. Hence

$$||u_{n+1}||_{\sigma_{n+1}-2\gamma_{n+1}} \leq ||p||^2. \tag{3.2.90}$$

And similarly as for L_0 large enough the sum in (3.2.89) is convergent and the condition of equation (3.2.87) is as well satisfied. We continue from (3.2.84) to get

$$M_n \leq 2||p|| \; \forall n. \tag{3.2.91}$$

Let

$$R_n = ||P_{n+1}F(\phi(p) + u_{n+1})||_{\sigma_n-2\gamma_n} \tag{3.2.92}$$

$$\begin{aligned} R_n &= ||P_{n+1}(\int_0^1 F''(\phi(p) + u_n + \tau\delta u_{n+1})\delta u_{n+1}.\delta u_{n+1} d\tau)||_{\sigma_n-2\gamma_n} \\ &\leq \sup_{\tau\in[0,1]} ||F''(\phi(p) + u_n + \tau\delta u_{n+1})\delta u_{n+1}.\delta u_{n+1}||_{\sigma_n-2\gamma_n} \\ &\leq \sup_{\tau\in[0,1]} ||F''(\phi(p) + u_n + \tau\delta u_{n+1})||_{\sigma_n-2\gamma_n}.||\delta u_{n+1}||_{\sigma_n-2\gamma_n}.||\delta u_{n+1}||_{\sigma_n-2\gamma_n} \\ &\leq \sup_{\tau\in[0,1]} C_{F''}||\phi(p) + u_n + \tau\delta u_{n+1}||_{\sigma_n-2\gamma_n}.||\delta u_{n+1}||^2_{\sigma_n-2\gamma_n}, \end{aligned}$$

where $C_{F''}$ is the constant corresponding to the polynomial F''. Therefore

$$R_n \leq C(||p|| + ||u_n||_{\sigma_n-2\gamma_n} + ||\delta u_{n+1}||_{\sigma_n-2\gamma_n})||\delta u_{n+1}||^2_{\sigma_n-2\gamma_n} \tag{3.2.93}$$

We have the relation

$$||G_{n+1}(\phi(p) + u_{n+1})||_{\sigma_n-2\gamma_n} \leq \frac{||p||^2}{d_0\delta_{n+1}\gamma_n}.$$

This implies that

$$||\delta u_{n+1}||_{\sigma_n-2\gamma_n} \leq \frac{\varepsilon_n||p||^3}{d_0\delta_{n+1}\gamma_n}. \tag{3.2.94}$$

Recalling (3.2.93) we deduce

$$\begin{aligned} ||P_{n+1}F(\phi(p) + u_{n+1})||_{\sigma_n-2\gamma_n} &\leq C(||p|| + ||p||^2 + \frac{\varepsilon_n||p||^3}{d_0\delta_{n+1}(\gamma_n)})(\frac{\varepsilon_n||p||^3}{d_0\delta_{n+1}(\gamma_n)})^2 \\ &\leq C(||p|| + ||p||^2 + C||p||)(\frac{\varepsilon_n||p||^3}{d_0\delta_{n+1}(\gamma_n)})^2. \end{aligned}$$

As we want

$$||P_{n+1}F(\phi(p) + u_{n+1})|| \leq \frac{1}{2}\varepsilon_{n+1}||p||$$

the following condition is needed

$$M||p||(\frac{\varepsilon_n||p||^3}{d_0\delta_{n+1}(\gamma_n)})^2 \leq \frac{1}{2}\varepsilon_{n+1}||p||$$

$$M||p||^7\frac{\varepsilon_n^2}{d_0^2\delta_{n+1}^2\gamma_n^2} \leq \frac{1}{2}{\varepsilon_0}^{k^{n+1}}||p||$$

$$M||p||^6\varepsilon_0^{2k^n}L_0^{2\alpha}2^{(n+1)2\alpha}.\frac{32(1+n^2)^2}{\sigma_0^2} \leq \frac{1}{2}\varepsilon_0^{k^{n+1}}$$

$$M||p||^6\varepsilon_0^{(2-k)k^n}2^{2(n+1)\alpha-1}L_0^{2\alpha}.\frac{32^2}{\sigma_0^2}.e^{2\ln(1+n^2)} \leq 1$$

$$\frac{32^2}{\sigma_0^2}ML_0^{-12-6\eta}e^{(2-k)k^n\ln\varepsilon_0}.e^{2(n+1)\alpha-1)\ln 2\ +2\alpha\ln L_0} \leq 1$$

$$\frac{32^2}{\sigma_0^2}ML_0^{-12-6\eta}e^{-c_0L_0(2-k)k^n}e^{2(n+1)\alpha-1)\ln 2\ +2\alpha\ln L_0} \leq 1. \tag{3.2.95}$$

For n large as $1 < k < 2$ we have $(2-k)k^n \gg 2(n+1)\alpha - 1$ and (3.2.95) is satisfied. For n small the linear growth of L_0 in $-c_0L_0(2-k)k^n$ will dominate

$2\alpha \ln L_0$ so choosing L_0 very large (3.2.95) holds any n.
We now prove that

$$||(I-P_{n+1})F(\phi(p)+u_{n+1})||_{\sigma_{n+1}} \leq \frac{1}{2}\varepsilon_{n+1}||p||.$$

Applying now the formula on the residuum, we now want an estimate of the form

$$Ce^{-L_{n+1}\gamma_n}||p|| \leq \frac{1}{2}\varepsilon_{n+1}||p||. \tag{3.2.96}$$

Let $I_{n+1} = ||(I-P_{n+1})F(\phi(p)+u_{n+1})||_{\sigma_{n+1}}$

$$\begin{aligned}
I_{n+1} &= ||(I-P_{n+1})[V(\Omega)u_{n+1}+W(\phi(p)+u_{n+1})]||_{\sigma_{n+1}} \\
&= ||(I-P_{n+1})V(\Omega)u_{n+1}+(I-P_{n+1})W(\phi(p)+u_{n+1})||_{\sigma_{n+1}} \\
&= ||(I-P_{n+1})W(\phi(p)+u_{n+1})||_{\sigma_{n+1}} \\
&\leq ||(I-P_{n+1})W(\phi(p)+u_{n+1})||_{\sigma_n-3\gamma_n} \\
&\leq e^{-L_{n+1}\gamma_n}||W(\phi(p)+u_{n+1})||_{\sigma_n-3\gamma_n+\gamma_n} \\
&\leq e^{-L_{n+1}\gamma_n}||(W(\phi(p)+u_{n+1})||_{\sigma_n-2\gamma_n} \\
&\leq e^{-L_{n+1}\gamma_n}C_W||\phi(p)+u_{n+1}||^3_{\sigma_n-2\gamma_n}) \\
&\leq Ce^{-L_{n+1}\gamma_n}C_W||\phi(p)+u_n+\delta u_{n+1}||^3_{\sigma_n-2\gamma_n} \\
&\leq Ce^{-L_{n+1}\gamma_n}C_W(||p||^3+||u_n||^3_{\sigma_n-2\gamma_n}+||\delta u_{n+1}||^3_{\sigma_n-2\gamma_n}) \\
&\leq Ce^{-L_{n+1}\gamma_n}C_W(||p||^3+||p||^6+||\delta u_{n+1}||^3_{\sigma_n-2\gamma_n}) \\
&\leq Ce^{-L_{n+1}\gamma_n}C_W(||p||^3+||p||^6+||G(\phi(p)+u_n)||^3_{\sigma_n-2\gamma_n} \\
&\quad \times||P_{n+1}F(\phi(p)+u_n)||^3_{\sigma_n-2\gamma_n} \\
&\leq Ce^{-L_{n+1}\gamma_n}C_W(||p||^3+||p||^6+\frac{||p||^6}{d_0\delta_n 2\gamma_n}\varepsilon_n||p||^3) \\
&\leq Ce^{-L_{n+1}\gamma_n}||p||^3
\end{aligned} \tag{3.2.97}$$

and we now go back to (3.2.96) and require

$$Ce^{-L_{n+1}\gamma_n}||p|| \leq \frac{1}{2}\varepsilon_{n+1}||p||. \tag{3.2.98}$$

We need:

$$C.e^{\frac{-\sigma_0}{32(1+n^2)}.2^{n+1}L_0}.e^{-k^{n+1}.\ln\varepsilon_0} \leq 1.$$

As $2 \geq k$, 2^{n+1} grows quicker than k^{n+1} and by choosing $c_0 \leq \frac{\sigma_0}{32}$ and L_0 large we have

$$\frac{\sigma_0}{32}L_0\frac{2^{n+1}}{(1+n^2)} \gg -\ln\varepsilon_0.k^{n+1} = c_0L_0k^{n+1}$$

and (3.2.96) is satisfied.
We will denote δu_{n+1} by v_n so that we can write

$$u_{n+1} = u_n + v_n. \tag{3.2.99}$$

In Newton's method we find the "correction" v_n such that

$$P_{n+1}(F(\phi(p)+u_n(p,\Omega,x)+DF(\phi(p)+u_n(p,\Omega,x))v_n(p,\Omega,x)) = 0 \tag{3.2.100}$$

Lemma 3.2.2 [10] *The spectrum of $DF(\phi(T_\theta p)+u(T_\theta p,\Omega))$ is independent of θ.*

Proof:

$$T_\theta F(\phi(p)+u(p,\Omega)) = F(T_\theta(\phi(p)+u(p.\Omega))) \tag{3.2.101}$$

We differentiate (3.2.101) with respect to $\phi(p)+u(p,\Omega)$ and apply the chain rule to the right hand side to get

$$T_\theta DF(\phi(p)+u(p,\Omega)) = DF(T_\theta(\phi(p)+u(p,\Omega)))T_\theta \tag{3.2.102}$$

$$T_\theta DF(\phi(p)+u(p,\Omega))T_\theta^{-1} = DF(\phi(T_\theta p)+u(T_\theta p,\Omega)).$$

$DF(\phi(p)+u(p,\Omega))$ and $DF(\phi(T_\theta p)+u(T_\theta p,\Omega))$ are unitarily equivalent. The spectrum of an operator is independent of the basis in which it is written so $DF(\phi(p)+u(p,\Omega))$ and $F(\phi(T_\theta p)+u(T_\theta p,\Omega))$ have the same spectrum. □

Lemma 3.2.3 [10] *The correction $v_n(p,\Omega)$ satisfies the property*

$$T_\theta v_n(p,\Omega) = v_n(T_\theta p,\Omega) \tag{3.2.103}$$

Proof:

$$P_{n+1}(F(\phi(p)+u_n(p,\Omega,x)+DF(\phi(p)+u_n(p,\Omega,x))v_n(p,\Omega,x)) = 0 \tag{3.2.104}$$

We start by multiplying (3.2.104) by T_θ to get

$$T_\theta P_{n+1}(F(\phi(p)+u_n(p,\Omega,x)+DF(\phi(p)+u_n(p,\Omega,x))v_n(p,\Omega,x)) = 0. \tag{3.2.105}$$

Since T_θ and P_{n+1} commute

$$P_{n+1}(T_\theta F(\phi(p)+u_n(p,\Omega,x)+T_\theta DF(\phi(p)+u_n(p,\Omega.x))v_n(p,\Omega,x)) = 0$$

So we can use (3.2.102) to deduce that

$$P_{n+1}(F(\phi(T_\theta p)+u_n(T_\theta p,\Omega,x)+DF(\phi(T_\theta p)+u_n(T_\theta p,\Omega,x))T_\theta v_n(p,\Omega,x)) = 0$$

and hence $T_\theta v_n(p,\Omega,x) = v_n(T_\theta p,\Omega,x)$. □

Lemma 3.2.4 *The "correction" v_n satisfies the following covariant property:*
if $p \in \mathbb{R} \times \mathbb{R}$ and $\Omega \in \mathbb{R}$ we have $\widetilde{v_n} \in \mathbb{R}$

Proof:

$$P_{n+1}(F(\phi(p) + u_n(p,\Omega,x) + DF(\phi(p) + u_n(p,\Omega,x))v_n(p,\Omega,x)) = 0 \quad (3.2.106)$$

so

$$P_{n+1}V(\Omega)(u_n + \phi(p) + v_n) + P_{n+1}W(u_n + \phi(p) + v_n)$$
$$+P_{n+1}V(\Omega)v_n + P_{n+1}DW(u_n + \phi(p) + v_n)v_n = 0.$$

At the site $(j,k) \in B(\overline{L_{n+1}})$ we have

$$(\omega_j^2 - \Omega^2 k^2)[u_n(j,k) + v_n(j,k)] + \int_0^{2\pi}\int_0^{\pi} \overline{\psi_j(x)} e^{-ik\xi} g(\widetilde{u_n} + \widetilde{v_n} + \widetilde{\phi(p)}) dx d\xi$$

$$+(\omega_j^2 - \Omega^2 k^2)v_n(j,k) + \int_0^{2\pi}\int_0^{\pi} \overline{\psi_j(x)} e^{-ik\xi} g'(\widetilde{u_n} + \widetilde{v_n} + \widetilde{\phi(p)})\widetilde{v_n} dx d\xi = 0 \quad (3.2.107)$$

and taking the conjugate of (3.2.107) we get

$$\overline{(\omega_j^2 - \Omega^2 k^2)[u_n(j,k) + v_n(j,k)]} + \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\psi_j(x)} e^{-ik\xi} g(\widetilde{u_n} + \widetilde{v_n} + \widetilde{\phi(p)})} dx d\xi$$

$$+\overline{(\omega_j^2 - \Omega^2 k^2)v_n(j,k)} + \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\psi_j(x)} e^{-ik\xi} g'(\widetilde{u_n} + \widetilde{v_n} + \widetilde{\phi(p)})\widetilde{v_n}} dx d\xi = 0. \quad (3.2.108)$$

$$(\omega_j^2 - \overline{\Omega}^2 k^2)(\overline{u_n(j,k)} + \overline{v_n(j,k)}) + \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\psi_j(x)}} e^{ik\xi} g(\overline{\widetilde{u_n}} + \overline{\widetilde{v_n}} + \overline{\widetilde{\phi(p)}}) dx d\xi$$

$$(\omega_j^2 - \overline{\Omega}^2 k^2)\overline{v_n(j,k)} + \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\psi_j(x)}} e^{ik\xi} g'(\overline{\widetilde{u_n}} + \overline{\widetilde{v_n}} + \overline{\widetilde{\phi(p)}})\overline{\widetilde{v_n}} dx d\xi = 0. \quad (3.2.109)$$

We now make the restriction $\Omega \in \mathbb{R}$ and $p \in \mathbb{R}$ so we can use (3.1.42),

$$\overline{u_n(j,k)} = u_n(j,-k),\ \widetilde{u_n}, \widetilde{\phi(p)} \in \mathbb{R}, \quad (3.2.110)$$

and introduce these relations into (3.2.109) to get

$$(\omega_j^2 - \Omega^2 k^2)(u_n(j,-k) + \overline{v_n(j,k)}) + \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\psi_j(x)}} e^{ik\xi} g(\widetilde{u_n} + \widetilde{v_n} + \widetilde{\phi(p)}) dx d\xi$$

$$(\omega_j^2 - \Omega^2 k^2)\overline{v_n(j,k)} + \int_0^{2\pi}\int_0^{\pi} \overline{\overline{\psi_j(x)}} e^{ik\xi} g'(\widetilde{u_n} + \overline{\widetilde{v_n}} + \widetilde{\phi(p)})\overline{\widetilde{v_n}} dx d\xi = 0. \quad (3.2.111)$$

Since v_n is unique we deduce that

$$v_n(j,-k) = \overline{v_n(j,k)}. \tag{3.2.112}$$

Now consider $x \in [0,\pi]$

$$\begin{aligned} \widetilde{v_n}(x,\xi) &= \sum_{(j,k)\in\mathbb{N}\times\mathbb{Z}} v_n(j,k)\psi_j(x)e^{ik\xi} \\ &= \sum_{j\in\mathbb{N}} \psi_j(x) \sum_{k\in\mathbb{Z}} (v_n(j,k)e^{ik\xi} + v_n(j,-k)e^{-ik\xi}) \\ &= \sum_{j\in\mathbb{N}} \psi_j(x) \sum_{k\in\mathbb{Z}} (v_n(j,k)e^{ik\xi} + \overline{v_n(j,k)e^{ik\xi}}) \end{aligned} \tag{3.2.113}$$

and since $v_n(j,-k) = \overline{v_n(j,k)}$ and $\psi_j(x)$ is real when x is real we have $\widetilde{v_n} \in \mathbb{R}$ for real x. □

Lemma 3.2.5 *For $(p_0,\Omega_0) \in (\mathcal{N}_n, \frac{\rho_n}{2})$ we have the following estimates on v_n*

$$||\frac{\partial v_n}{\partial p}(p,\Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{4}{\pi\rho_{n+1}}\frac{r_0^3}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n \tag{3.2.114}$$

$$||\frac{\partial v_n}{\partial \Omega}(p,\Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{4}{\pi\rho_{n+1}}\frac{r_0^3}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n \tag{3.2.115}$$

and therefore there exist constants M_1', M_{1p}' and $M_{1\Omega}'$ such that

$$\sum_{n=0}^{\infty} ||v_n(p,\Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq M_{1p}'r_0^2 \leq r_0^2 \tag{3.2.116}$$

$$\sum_{n=0}^{\infty} ||\frac{\partial v_n}{\partial p}(p,\Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq M_{1p}'r_0 \leq \frac{1}{2}r_0 \tag{3.2.117}$$

$$\sum_{n=0}^{\infty} ||\frac{\partial v_n}{\partial \Omega}(p,\Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq M_{1\Omega}'r_0 \leq \frac{1}{2}r_0 \tag{3.2.118}$$

Proof:
Let $(p_1,\Omega_1) \in \mathcal{N}_n$, denote R by $R := \rho_n$. Take $(p_0,\Omega_0) \in (\mathcal{N}_n, \frac{\rho_n}{2})$, C_1 (resp C_2) to be the circle of center p_1(resp Ω_1) and radius R(resp R). The analytic dependence

of v_n with respect to the parameters helps us to derive the estimates on $v_n(p,\Omega)$ using the Cauchy estimates.
Recalling (3.2.82) we have

$$||v_n||_{\sigma_n-2\gamma_n} \leq \frac{||p||^2}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n||p|| \tag{3.2.119}$$

when $(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})\in(\mathcal{N}_n,\rho_n)$, and when p is on the circle $\mathcal{C}_1$

$$|p-p_1|=R=\rho_n \tag{3.2.120}$$

so

$$|p-p_0|=|p-p_1+p_1-p_0|\geq|p-p_1|-|p_1-p_0|\geq R-\frac{R}{2}=\frac{R}{2} \tag{3.2.121}$$

and also when $\Omega\in\mathcal{C}_2$

$$|\Omega-\Omega_0|\geq\frac{R}{2}. \tag{3.2.122}$$

Using the Cauchy integral formula we have :

$$\frac{\partial v}{\partial p}(p_0,\Omega_0)=\frac{1}{-4\pi^2}\oint_{C_1}\oint_{C_2}\frac{v(p,\Omega)}{(p-p_0)^2(\Omega-\Omega_0)}dpd\Omega \tag{3.2.123}$$

and we deduce the formula

$$\frac{\partial v}{\partial p}(p_0,\Omega_0)=\frac{-R^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})}{(p_1+Re^{i\phi}-p_0)^2(\Omega_1+Re^{i\psi}-\Omega_0)}d\phi d\psi \tag{3.2.124}$$

$$\begin{aligned}
|\frac{\partial v}{\partial p}(p_0,\Omega_0)(j,k)| &= |\frac{-R^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)}{(p_1+Re^{i\phi}-p_0)^2(\Omega_1+Re^{i\psi}-\Omega_0)}d\phi d\psi| \\
&= \frac{R^2}{4\pi^2}|\int_0^{2\pi}\int_0^{2\pi}\frac{v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)}{(p_1+Re^{i\phi}-p_0)^2(\Omega_1+Re^{i\psi}-\Omega_0)}d\phi d\psi| \\
&\leq \frac{R^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{|v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)|}{|p_1+Re^{i\phi}-p_0|^2|\Omega_1+Re^{i\psi}-\Omega_0|}d\phi d\psi.
\end{aligned}$$

When p is on the circle $\mathcal{C}_1$, $|p-p_0|\geq\frac{R}{2}$ and when $\Omega\in\mathcal{C}_2$, $|\Omega-\Omega_0|\geq\frac{R}{2}$. So we have:

$$\begin{aligned}
|\frac{\partial v}{\partial p}(p_0,\Omega_0)(j,k)| &\leq \frac{R^2}{4\pi^2}(\frac{2}{R})^3\int_0^{2\pi}\int_0^{2\pi}|v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)|d\phi d\psi \\
&\leq \frac{2}{R\pi^2}\int_0^{2\pi}\int_0^{2\pi}|v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)|d\phi d\psi.
\end{aligned}$$

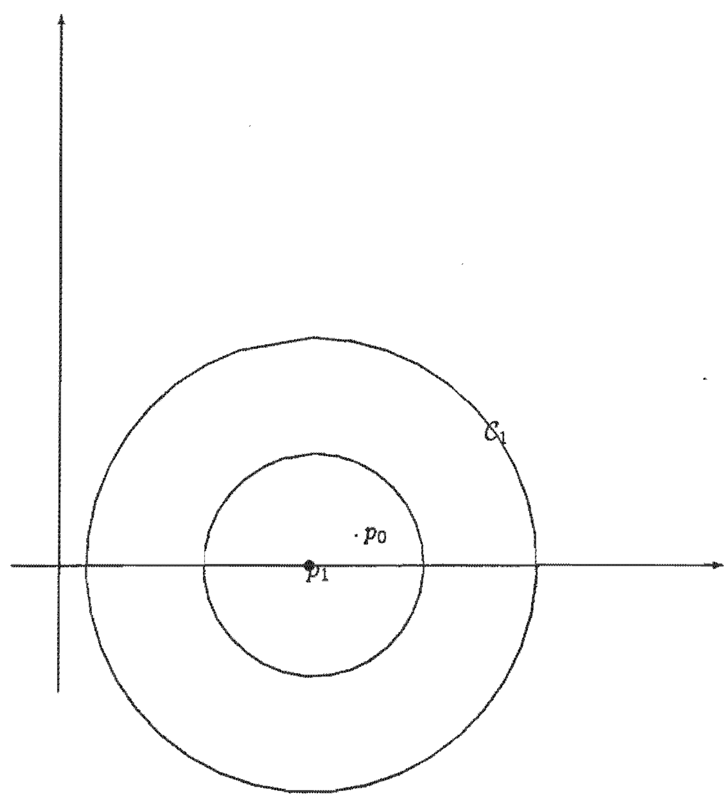

Figure 3.1: The circle C_1 of center p_1 and radius R

Applying Cauchy-Schwarz we get

$$
\begin{aligned}
|\frac{\partial v}{\partial p}(p_0,\Omega_0)(j,k)|^2 &\leq (\frac{2}{R\pi^2})^2(2\pi)^2\int_0^{2\pi}\int_0^{2\pi}|v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)|^2 d\phi d\psi \\
&\leq \frac{16}{\pi^2R^2}\int_0^{2\pi}\int_0^{2\pi}|v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)|^2 d\phi d\psi \quad (3.2.125)
\end{aligned}
$$

We are now able to get an estimate on the norm of $\frac{\partial v}{\partial p}(p_0,\Omega_0)$:

$$
||\frac{\partial v}{\partial p}(p_0,\Omega_0)||^2_{m,s} = \sum_l (1+|l|^2)^s e^{2m|l|}|\frac{\partial v}{\partial p}(p_0,\Omega_0)(j,k)|^2
$$

and using the estimate (3.2.125)

$$
\begin{aligned}
||\frac{\partial v}{\partial p}(p_0,\Omega_0)||^2_{m,s} &\leq \sum_l (1+|l|^2)^s e^{2m|l|}\int_0^{2\pi}\int_0^{2\pi}|v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})(j,k)|^2 d\phi d\psi \\
&\leq \int_0^{2\pi}\int_0^{2\pi}\frac{16}{\pi^2R^2}||v(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})||^2_{m,s}d\phi d\psi \qquad (3.2.126)
\end{aligned}
$$

As in (3.2.126) we can write

$$
||\frac{\partial v_n}{\partial p}(p,\Omega)||^2_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{16}{\pi^2R^2}\int_0^{2\pi}\int_0^{2\pi}||v_n(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})||^2_{\sigma_{n+1}-2\gamma_{n+1},s}d\phi d\psi \qquad (3.2.127)
$$

and using (3.2.82) we get

$$
||\frac{\partial v_n}{\partial p}(p,\Omega)||^2_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{16}{\pi^2R^2}\frac{r_0^6}{d_0^2\delta_{n+1}^2\gamma_n^4}\varepsilon_n^2. \qquad (3.2.128)
$$

We now introduce the value of $R=\rho_{n+1}$ and get

$$
||\frac{\partial v_n}{\partial p}(p,\Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{4}{\pi\rho_{n+1}}\frac{r_0^3}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n. \qquad (3.2.129)
$$

We define $\rho_{n+1}=\rho_0\frac{\delta_{n+1}}{32.2L^2_{n+1}l^2_{n+1}}$:

$$
\begin{aligned}
\rho_{n+1} &= \rho_0\frac{L_{n+1}^{-\alpha}}{64L^2_{n+1}L^{2\beta}_{n+1}} \\
&= \rho_0\frac{L_{n+1}^{-\alpha-2\beta-2}}{64} \\
&= \frac{\rho_0}{64}(2^{n+1}L_0)^{-\alpha-2\beta-2} \\
&= \frac{\rho_0}{64}L_0^{-\alpha-2\beta-2}2^{(n+1)(-\alpha-2\beta-2)} \\
&= \frac{\rho_0}{64}L_0^{-\alpha-2\beta-2}e^{(n+1)(-\alpha-2\beta-2)\ln 2}.
\end{aligned}
$$

Let us define

$$S_m := \sum_{n=0}^{m} ||\frac{\partial v_n}{\partial p}(p_0, \Omega_0)||_{\sigma_{n+1}-2\gamma_{n+1},s} \leq \frac{4}{\pi}\frac{32r_0^3}{4d_0\sigma_0}\sum_{n=0}^{m} e^{k^n \ln\varepsilon_0} e^{-n\alpha\ln 2} e^{\alpha\ln L_0} e^{2\ln(1+n^2)} \rho_{n+1}^{-1}. \tag{3.2.130}$$

Once again as $k^n \ln\varepsilon_0 = -c_0 L_0 k^n$ and $k > 1$ so $e^{k^n \ln\varepsilon_0}$ dominates the other terms in the sum and we will prove that

$$S := \lim_m S_m \leq \frac{1}{2} r_0. \tag{3.2.131}$$

$$\begin{aligned}
S_m &\leq \sum_{n=0}^{m} \frac{4r_0^3 \varepsilon_n}{\rho_{n+1} d_0 \delta_{n+1} \gamma_{n+1}^2} \\
&\leq \sum_{n=0}^{m} \frac{64r_0^3}{\rho_0} L_0^{\alpha+2\beta+2} e^{(n+1)(\alpha+2\beta+2)\ln 2} \frac{32}{4d_0\sigma_0} e^{k^n \ln\varepsilon_0} . e^{-n\alpha\ln 2} . e^{\alpha\ln L_0} . e^{2\ln(1+n^2)} \\
&\leq \frac{64r_0^3}{\rho_0}\frac{32}{4d_0\sigma_0}\sum_{n=0}^{m} L_0^{\alpha+2\beta+2} e^{-c_0 L_0 k^n} . e^{-n\alpha\ln 2} . e^{\alpha\ln L_0} . e^{2\ln(1+n^2)} e^{(n+1)(\alpha+2\beta+2)\ln 2} \\
&\leq \frac{64r_0^3}{\rho_0}\frac{32}{4d_0\sigma_0} T_m
\end{aligned} \tag{3.2.132}$$

where $T_m = \sum_{n=0}^{m} L_0^{\alpha+2\beta+2} e^{-c_0 L_0 k^n} . e^{-n\alpha\ln 2} . e^{\alpha\ln L_0} . e^{2\ln(1+n^2)} e^{(n+1)(\alpha+2\beta+2)\ln 2}$. T_m will converge to a number, say T. Hence we have the estimate:

$$\sum_{n=0}^{m} ||\frac{\partial v_n}{\partial p}(p, \Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq T\frac{512r_0^3}{\rho_0 d_0 \sigma_0}. \tag{3.2.133}$$

Now $r_0 = \rho_0$ and for L_0 very large we have

$$\sum_{n=0}^{m} ||\frac{\partial v_n}{\partial p}(p, \Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{1}{2} r_0. \tag{3.2.134}$$

Similarly we derive estimate for the derivative with respect to Ω, and

$$||\frac{\partial v_n}{\partial \Omega}(p, \Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{4}{\rho_{n+1}}\frac{r_0^3}{d_0\delta_{n+1}\gamma_n^2}\varepsilon_n, \tag{3.2.135}$$

hence

$$\sum_{n=0}^{m} ||\frac{\partial v_n}{\partial \Omega}(p, \Omega)||_{\sigma_{n+1}-2\gamma_{n+1}} \leq \frac{1}{2} r_0. \tag{3.2.136}$$

□

Lemma 3.2.6 *u_0 satisfies the following estimate*

$$||\partial_\Omega u_0(p_0,\Omega)||_{\sigma_0,s} \leq C\frac{r_0^4}{d_0} \tag{3.2.137}$$

Proof:
Here $R = \rho_0$ and $(p,\Omega) \in (\mathcal{N}_0, \rho_0)$. From (3.1.37) we have

$$||u(p_0,\Omega_0)||_{m,s} \leq 8\frac{C_W}{d_0}r_0^3 \tag{3.2.138}$$

so as in (3.2.126) we get

$$||\partial_\Omega u_0(p_0,\Omega_0)||^2_{\sigma_0,s} \leq \frac{16}{\pi^2R^2}\int_0^{2\pi}\int_0^{2\pi}||u(p_1+Re^{i\phi},\Omega_1+Re^{i\psi})||^2_{\sigma_0,s}d\phi d\psi \tag{3.2.139}$$

implying

$$||\partial_\Omega u_0(p_0,\Omega)||^2_{\sigma_0,s} \leq \frac{16}{\pi^2\rho_0^2}2\pi^2 8\frac{C_W}{d_0^2}r_0^6 \tag{3.2.140}$$

and $\rho_0 = r_0$ so

$$||\partial_\Omega u_0(p_0,\Omega)||_{\sigma_0,s} \leq C\frac{r_0^4}{d_0}.\square \tag{3.2.141}$$

Chapter 4

The inversion of the Hamiltonian

We have constructed the solution of the P-equation provided that we have a good estimate on G_{n+1} (equation (3.2.81)) and in this chapter we provide the proof of this result.
The Hamiltonian H can be written in the form $H = V(\Omega) + DW(u)$. We make the distinction between the sites l at which $|V(\Omega)(l,l)| \geq d_s$, (called nonsingular sites) and the one at which they are small. Intuitively we can see that, when u is very small, $H(p,\Omega)$ is just a perturbation of $V(\Omega)$ and therefore on nonsingular regions the operator $H(p,\Omega)$ will be easy to invert. In this chapter we start by inverting the operators on nonsingular regions and afterwards we proceed to the singular ones. Let us denote by $e_i(p,\Omega)$ the eigenvalues of $H(p,\Omega)$. In this chapter we invert $H(p,\Omega)$ for values of (p,Ω) such that $e_i(p,\Omega)$ satisfies $|e_i(p,\Omega)| \geq \delta_n$. In the next lemma we prove that $DW(u)$ is self-adjoint.

Lemma 4.0.7 [9] *$DW(u)$ is a self-adjoint operator*

Proof:
Let

$$\tilde{h} = h\psi_{j'}(x)e^{ik'\xi} \tag{4.0.1}$$

and $\tilde{u}$ the function associated with the lattice vector $u(j,k)$. We have:

$$\lim_{h\to 0} |\frac{g(\tilde{u}+\tilde{h}) - Dg(\tilde{u})\tilde{h})}{\tilde{h}}| = 0. \tag{4.0.2}$$

We denote by $W(u)$ the vector

$$W(u)(j,k) = \int_0^{2\pi}\int_0^{\pi} \overline{\psi_j(x)} e^{-ik\xi} g(\tilde{u}) dx d\xi \tag{4.0.3}$$

and $D_u W(u)(j,k)(j',k')$ is the partial derivative of $W(u)(j,k)$ with refer to the component (j',k'),

$$D_u W(u)(j,k)(j',k') = \int_0^{2\pi}\int_0^{\pi} \overline{\psi_j(x)} e^{-ik\xi} \psi_{j'}(x) e^{ik'\xi} Dg(\tilde{u}) dx d\xi. \tag{4.0.4}$$

Indeed let l be defined by:

$$l = \lim_{h\to 0} \frac{W(u+h)(j,k) - W(u)(j,k) - D_u W(u)h(j,k)}{h}. \tag{4.0.5}$$

We have

$$\begin{aligned}
l &= \lim_{h\to 0} \frac{W(u+h)(j,k) - W(u)(j,k) - D_u W(u)h(j,k)}{h} \\
&= \int_0^{2\pi}\int_0^{\pi} e^{-ik\xi}\overline{\psi_j(x)} \frac{g(\tilde{u}+\tilde{h}) - g(u) - Dg(u)\psi_{j'}(x)e^{ik'\xi}h}{h} dx d\xi \\
&= \int_0^{2\pi}\int_0^{\pi} e^{-ik\xi}\overline{\psi_j(x)} \lim_{h\to 0} \frac{g(\tilde{u}+\tilde{h}) - g(\tilde{u}) - Dg(u)\psi_{j'}(x)e^{ik'\xi}h}{h} dx d\xi \\
&= \int_0^{2\pi}\int_0^{\pi} e^{-ik\xi}\overline{\psi_j(x)} \lim_{h\to 0} \psi_{j'}(x)e^{ik'\xi} \frac{g(\tilde{u}+\tilde{h}) - g(\tilde{u}) - Dg(u)\tilde{h}}{h\psi_{j'}(x)e^{ik'\xi}} dx d\xi \\
&= \int_0^{2\pi}\int_0^{\pi} e^{-ik\xi}\overline{\psi_j(x)} \lim_{\tilde{h}\to 0} \psi_{j'}(x)e^{ik'\xi} \frac{g(\tilde{u}+\tilde{h}) - g(\tilde{u}) - Dg(u)\tilde{h}}{\tilde{h}} dx d\xi \\
&= \int_0^{2\pi}\int_0^{\pi} e^{-ik\xi}\overline{\psi_j(x)}\psi_{j'}(x)e^{ik'\xi} \lim_{\tilde{h}\to 0} \frac{g(\tilde{u}+\tilde{h}) - g(\tilde{u}) - Dg(u)\tilde{h}}{\tilde{h}} dx d\xi. \\
&= 0.
\end{aligned} \tag{4.0.6}$$

Now we prove that the operator $DW(u)$ is self adjoint.

$$\begin{aligned}
\overline{DW(u)(j,k)(j',k')} &= \overline{\int_0^{2\pi}\int_0^{\pi} \overline{\psi_j(x)} e^{-ik\xi}\psi_{j'}(x)e^{ik'\xi} Dg(\tilde{u}) dx d\xi} \\
&= \int_0^{2\pi}\int_0^{\pi} \psi_j(x) e^{ik\xi}\overline{\psi_{j'}(x)} e^{-ik'\xi} \overline{Dg(\tilde{u})} dx d\xi
\end{aligned}$$

and as $\tilde{u}$ is real on the real axis we get

$$\begin{aligned}
\overline{DW(u)(j,k)(j',k')} &= \int_0^{2\pi}\int_0^{\pi} \psi_j(x) e^{ik\xi}\overline{\psi_{j'}(x)} e^{-ik'\xi} Dg(\tilde{u}) dx d\xi \\
&= DW(u)(j',k')(j,k). \square
\end{aligned} \tag{4.0.7}$$

4.1 Inversion of the local Hamiltonian on singular regions

4.1.1 The nonsingular case

The estimates of G_A where A is a nonsingular region are easily computed using Neumann series:

Lemma 4.1.1 [8] *Let $A \subset \mathbb{N} \times \mathbb{Z}$ be a nonsingular region, denote by G (resp G_A) the inverse of H (resp H_A) then for r_0 small enough we have the following estimate:*

$$||G_A||_{\sigma_n,s} \leq \frac{4}{d_0} \tag{4.1.8}$$

Proof:

$$H_A = V_A(\Omega) + W'_A(\phi(p) + u_n)$$

$$\begin{aligned} G_A &= (V_A(\Omega) + W'_A(\phi(p) + u_n))^{-1} \\ &= [(I + W'_A(\phi(p) + u_n)V_A^{-1})V_A]^{-1} \\ &= V_A^{-1}(I + W'_A(\phi(p) + u_n)V_A^{-1}) \\ &= V_A^{-1}\sum_p[-W'_A(\phi(p) + u_n)V_A^{-1}]^p. \end{aligned} \tag{4.1.9}$$

As DW is a polynomial of at least order 2 we get

$$||DW_A(\phi(p) + u_n))||_{\sigma_n,s} \leq C_{DW}||\phi(p) + u_n||^2_{\sigma_n,s}.$$

A is nonsingular, we have $|V(\Omega)(j,k)(j,k)| > d_s = \frac{d_0}{2}$ for $(j,k) \in A$, hence

$$||V_A^{-1}(\Omega)||_{\sigma_n,s} \leq \sqrt{c_1}\frac{2}{d_0}.$$

From (4.1.9) we deduce the inequality:

$$||G_A||_{\sigma_n,s} \leq ||V_A^{-1}||_{\sigma_n,s}||\sum_p[-DW_A(\phi(p) + u_n)V_A^{-1}]^p||_{\sigma_n,s}. \tag{4.1.10}$$

We must now prove that the serie (4.1.10) converges for r_0 small enough. We require that

$$||-DW_A(\phi(p) + u_n)||_{\sigma_n,s} \leq \frac{1}{4c_1}\frac{d_0}{2}$$

however it is known that

$$||-DW_A(\phi(p) + u_n)||_{\sigma_n,s} \leq C_{DW,c_1}||(\phi(p) + u_n)||^2_{\sigma_n,s}$$

So and use (3.2.91) to require the condition:

$$C_{DW,\varsigma_1}||(\phi(p)+u_n)||^2_{\sigma_n,s} \leq \frac{C_{DW,\alpha}}{4}||p|| \leq \frac{d_0}{4}.$$

This will be satisfied for r_0 small enough (which means L_0 large), then

$$||-DW_A(\phi(p)+u_n)V_A^{-1}||_{\sigma_n,s} \leq \frac{1}{2},$$

our series converges and the following bound follows:

$$||G_A||_{\sigma_n,s} \leq \frac{2}{d_0}\frac{1}{1-\frac{1}{2}}$$

$$||G_A||_{\sigma_n,s} \leq \frac{4}{d_0}.\square$$

4.1.2 The singular case

Let S be a region containing only singular sites. We define the l_n-neighborhood of S, C_{l_n}:

$$C_{l_n} = C(S) = \{x = (j,k) \in \mathbb{N} \times \mathbb{Z} \ dist(x,S) < l_n\}.$$

In the case of case of Dirichlet boundary condition ($u(0,t) = u(\pi,t) = 0$) the singular sites consist of isolated points but for other boundary problems (for example periodic boundary condition $u(0,t) = u(\pi,t)$) they may consist of bounded regions of the lattice. In this thesis we assume only that the singular regions are bounded so the proof provided here is applicable to a large variety of boundary problem.
Here the technique that we use relies on the fact the the sites $l = (j,k)$ at which $|V(\Omega)(l,l)|$ is small are very far from each other making, the Neumann series used to invert it convergent.
This chapter is about getting an upper bound on $||G_{n+1}||_{\sigma_n-2\gamma_n}$ so we start by getting an estimate on the matrix elements $|G_{C_{l_n}}(l_1,l_2)|$:

Lemma 4.1.2 [8]

$$|G_{C_{l_n}}(l_1,l_2)| \leq \frac{C}{\delta_n}e^{-\sigma_n|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2} \qquad (4.1.11)$$

Proof:
As in the Poschel lemma from [17] we will use a decomposition of the local Hamiltonian in block diagonal and off-diagonal parts :

$$H_{C_{l_n}} = H_D \oplus H_S + \Gamma_{S,D} \tag{4.1.12}$$

$$\text{where } C_{l_n} = S \cup D,\ \ D \text{ is the nonsingular region around } S \text{ in } C_{l_n}. \tag{4.1.13}$$

We use the following resolvent identities

$$G_{C_{l_n}} = (G_S \oplus G_D) - (G_S \oplus G_D)\Gamma_{S,D} G_{C_{l_n}},$$

or

$$G_{C_{l_n}} = (G_S \oplus G_D) - G_{C_{l_n}}\Gamma_{S,D}(G_S \oplus G_D). \tag{4.1.14}$$

(The proof of these identities is provided in chapter 6 in (7.1.30).) These formulas are combined to get :

$$G_{C_{l_n}} = (G_S \oplus G_D) - (G_S \oplus G_D)\Gamma_{S,D}(G_S \oplus G_D) + (G_S \oplus G_D)\Gamma_{S,D} G_{C_{l_n}}(G_S \oplus G_D) \tag{4.1.15}$$

We will consider the 4 cases that can occur when l_1 and l_2 are in C_{l_n}

$$\begin{cases} \text{Case 1. } l_1, l_2 \in S \\ \text{Case 2. } l_1 \in S\ , l_2 \notin S \\ \text{Case 3. } l_1 \notin S\ , l_2 \in S \text{ this case is similar to case 2} \\ \text{Case 4. } l_1, l_2 \notin S. \end{cases}$$

Starting with Case 1:
Let e_i an eigenvalue of $H_{C_{l_n}}$. We have the property that $|e_i(p,\Omega)| \geq \delta_n$ for $(p,\Omega) \in \mathcal{N}_{n+1}$, (see the lemma 5.0.5 and equations (5.0.5) and (5.0.25)), furthermore $H_{C_{l_n}}$ is self-adjoint so using the lemma 2.1.7 we have the inequality

$$|G_{C_{l_n}}(l_1, l_2)| \leq \frac{1}{\delta_n}. \tag{4.1.16}$$

Let C_0 be the constant

$$C_0 = \sup_{l_1, l_2 \in S} (e^{\sigma_n |l_1 - l_2|}(1 + |l_1 - l_2|^2)^{s/2})$$

The property

$$1 \leq C_0(e^{-\sigma_n |l_1 - l_2|}(1 + |l_1 - l_2|^2)^{-s/2})\ \forall\, l_1, l_2 \in S \tag{4.1.17}$$

then implies that

$$|G_{C_{l_n}}(l_1, l_2)| \leq \frac{C_0}{\delta_n} e^{-\sigma_n |l_1 - l_2|}(1 + |l_1 - l_2|^2)^{-s/2}.$$

Case 2:
l_1 and l_2 are not in the same blocks

$$(G_S \oplus G_D)(l_1, l_2) = G_S(l_1, l_2) + G_D(l_1, l_2) = 0,$$

applying (4.1.14) the relation

$$|G_{C_{l_n}}(l_1, l_2)| = |\sum_{\beta_1 \in S, \beta_2 \in D} G_{C_{l_n}}(l_1, \beta_1)\Gamma_{S,D}(\beta_1, \beta_2)G_D(\beta_2, l_2)|. \tag{4.1.18}$$

follows. Now using in addition

$$H_{C_{l_n}} = H_D \oplus H_S + \Gamma_{S,D}$$

we get an estimate on the norm of $\Gamma_{S,D}$. $\Gamma_{S,D}$ is given by:

$$\begin{aligned} \Gamma_{S,D} &= H_{C_{l_n}} - H_D \oplus H_S \\ &= (PV(\Omega) + DPW(\phi(p) + u))\,|_{C_{l_n}} - (PV(\Omega) + DPW(\phi(p) + u))\,|_D \\ &\quad -(PV(\Omega) + DPW(\phi(p) + u))\,|_S\,. \end{aligned}$$

Using the fact that $V(\Omega)$ is diagonal we deduce

$$\begin{aligned} \Gamma_{S,D} &= DPW(\phi(p) + u))\,|_{C_{l_n}} - DPW(\phi(p) + u)\,|_D \\ &\quad DPW(\phi(p) + u)\,|_S, \end{aligned}$$

and thus

$$\begin{aligned} ||\Gamma_{S,D}||_{\sigma_n,s} &\leq 3||DPW(\phi(p) + u)||_{\sigma_n,s} \\ &\leq 3C_{W,DW}||\phi(p) + u||^2_{\sigma_n,s} \\ &\leq c^2 C_{W,DW}||p||^2 \\ &\leq 2||p||^2. \end{aligned} \tag{4.1.19}$$

D is a nonsingular domain so using (4.1.8) we have

$$||G_D||_{\sigma_n,s} \leq \frac{4}{d_0}$$

and the following estimates are deduced:

$$G_D(x, y) \leq \frac{4}{d_0}(1 + |x - y|^2)^{-s/2} e^{-\sigma_n|x-y|}$$

$$\Gamma_{S,D}(x, y) \leq C||p||^2(1 + |x - y|^2)^{-s/2} e^{-\sigma_n|x-y|}.$$

Now introducing (4.1.17) into (4.1.18),

$$\begin{aligned} |G_{C_{l_n}}(l_1, l_2)| \leq &\sum_{\beta_1 \in S, \beta_2 \in D} \frac{C_0}{\delta_n} e^{-\sigma_n|l_1-\beta_1|}(1 + |l_1 - \beta_1|^2)^{-s/2}||p||^2 \\ &(1 + |\beta_1 - \beta_2|^2)^{-s/2} e^{-\sigma_n|\beta_1-\beta_2|} \frac{4}{d_0}(1 + |\beta_2 - l_2|^2)^{-s/2} e^{-\sigma_n|\beta_2-l_2|} \end{aligned}$$

$$
\begin{aligned}
|G_{C_{l_n}}(l_1,l_2)| &\leq \frac{C_0}{\delta_n}||p||^2\frac{4}{d_0}\sum_{\beta_1\in S,\beta_2\in D} e^{-\sigma_n|l_1-\beta_1|}(1+|l_1-\beta_1|^2)^{-s/2} \\
&\quad (1+|\beta_1-\beta_2|^2)^{-s/2}e^{-\sigma_n|\beta_1-\beta_2|}(1+|\beta_2-l_2|^2)^{-s/2}e^{-\sigma_n|\beta_2-l_2|} \\
&\leq \frac{C_0}{\delta_n}||p||^2\frac{4}{d_0}\sum_{\beta_1\in S,\beta_2\in D} e^{-\sigma_n|l_1-l_2|} \\
&\quad (1+|l_1-\beta_1|^2)^{-s/2}(1+|\beta_1-\beta_2|^2)^{-s/2}(1+|\beta_2-l_2|^2)^{-s/2} \\
&\leq \frac{C_0}{\delta_n}||p||^2\frac{4}{d_0}e^{-\sigma_n|l_1-l_2|}\sum_{\beta_1\in S,\beta_2\in D} \\
&\quad (1+|l_1-\beta_1|^2)^{s/2}(1+|\beta_1-\beta_2|^2)^{-s/2}(1+|\beta_2-l_2|^2)^{-s/2},
\end{aligned}
$$

using the lemma 2.2.2 finally find

$$
|G_{C_{l_n}}(l_1,l_2)| \leq \frac{C_0}{\delta_n}||p||^2\frac{4}{d_0}e^{-\sigma_n|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2}.
$$

Case 3:The third case is similar to the second one.
Case 4:We now consider the fourth case, we use the formula on the weights and (4.1.14).

$$
\begin{aligned}
G_{C_{l_n}} &= (G_S\oplus G_D)-(G_S\oplus G_D)\Gamma_{S,D}(G_S\oplus G_D) \\
&\quad +(G_S\oplus G_D)\Gamma_{S,D}G_{C_{l_n}}(G_S\oplus G_D)\Gamma_{S,D}).
\end{aligned}
$$

As l_1 and l_2 are in the same block D we have

$$
(G_S\oplus G_D)\Gamma_{S,D}(G_S\oplus G_D)(l_1,l_2)=\sum_{\beta_1,\beta_2\in D} G_D(l_1,\beta_1)\Gamma_{S,D}(\beta_1,\beta_2)G_D(\beta_1,l_2)
$$

but $\Gamma_{S,D}(\beta_1,\beta_2)=0$ because β_1 and β_2 are in the same block. This means that $(G_S\oplus G_D)\Gamma_{S,D}(G_S\oplus G_D)(l_1,l_2)=0$. The estimate on $(G_S\oplus G_D)(l_1,l_2)$ can be found by using (4.1.8) as D is a nonsingular region.
We now establish the estimate on the third term. Let

$$
x(l_1,l_2)=(G_S\oplus G_D)\Gamma_{S,D}G_{C_{l_n}}(G_S\oplus G_D)\Gamma_{S,D})(l_1,l_2), \tag{4.1.20}
$$

$$
\begin{aligned}
x(l_1,l_2) &= \sum_{\beta_1,\beta_4\in D;\beta_3,\beta_2\in S} G_D(l_1,\beta_1)\Gamma_{S,D}(\beta_1,\beta_2)G_C(\beta_2,\beta_3)\Gamma_{S;D}(\beta_3,\beta_4) \\
&\quad \times G_D(\beta_4,l_2)
\end{aligned} \tag{4.1.21}
$$

$$
\begin{aligned}
|x(l_1,l_2)| &\leq \sum_{\beta_1,\beta_4\in D;\beta_3,\beta_2\in S} \frac{4}{d_0}e^{-\sigma_n|l_1-\beta_1|}(1+|l_1-\beta_1|^2)^{-s/2}||p||^2 \\
&\quad (1+|\beta_1-\beta_2|^2)^{-s/2}e^{-\sigma_n|\beta_1-\beta_2|}\frac{C_0}{\delta_n}(1+|\beta_2-\beta_3|^2)^{-s/2}e^{-\sigma_n|\beta_2-\beta_3|}
\end{aligned}
$$

$$
\begin{aligned}
& ||p||^2(1+|\beta_2-\beta_3|^2)^{-s/2}e^{-\sigma_n|\beta_2-\beta_3|}\frac{4}{d_0}e^{-\sigma_n|\beta_3-l_2|} \\
& (1+|\beta_3-l_2|^2)^{-s/2} \\
\leq \; & \frac{C}{\delta_n}e^{-\sigma_n|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2}
\end{aligned}
$$

hence we have shown that:

$$|G_{C_{l_n}}(l_1,l_2)| \leq \frac{C}{\delta_n}e^{-\sigma_n|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2}.\square \qquad (4.1.22)$$

4.2 The general case

Lemma 4.2.1 [1] *If l_1 and l_2 are two distincts singular sites in $L_n \setminus L_{n-1}$ then $|l_1-l_2| \geq l_n$.*

We have proved an estimate on $G_{C_{l_n}}(l_1,l_2)$ for $l_1,l_2 \in C(S)$, we must extend this estimate for any l_1,l_2 in $L_n \setminus L_{n-1}$. Let us denote by S_j, any singular region. Here $A=(B_n \setminus N)\setminus(\cup_j S_j)$ is a nonsingular region surrounding the S_j and $A'=(B_n\setminus N)\setminus(\cup_j C(S_j))$ a region surrounding the l_n-neighborhood of singular region. We write the region $\overline{L_n}$ in two ways $: A \cup (\cup_j S_j)$ and $A' \cup (\cup_j C(S_j))$.
Now we prove an estimate on $|G(l_1,l_2)|$ for any l_1,l_2 in $L_n \setminus L_{n-1}$ and we deduce an estimate on $||G_{n+1}||_{\sigma_n-2\gamma_n,s}$:

Lemma 4.2.2 [10] *We have the following estimate of the G_{n+1}:*

$$||G_{n+1}||_{\sigma_n-2\gamma_n,s} \leq \frac{r_0^2}{d_0^2}\frac{C}{\delta_n(2\gamma_n)^2}. \qquad (4.2.23)$$

Proof:
H_n the local Hamiltonian can be written as:

$$H_n = H_A \oplus H_S + \Gamma_{AS}$$

$$\text{or } H_n = H_{A'} \oplus H_S + \Gamma_{A'S}$$

where $S=\cup_j S_j$. Now as in (7.1.29), we have

$$G_{n+1} = G_A \oplus \cup_j G_{S_j} - (G_A \oplus \cup_j G_{S_j})\Gamma_{AS}G_{n+1}. \qquad (4.2.24)$$

[1] This lemma is proved in theorem 4.1

and

$$G_{n+1} = G_{A'} \oplus \cup_j G_{C(S_j)} - (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S} G_{n+1}. \tag{4.2.25}$$

Now we put (4.2.24) into (4.2.25) to get:

$$\begin{aligned} G_{n+1} &= G_{A'} \oplus \cup_j G_{C(S_j)} - (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S}[(G_A \oplus \cup_j G_{S_j}) \\ &\quad -(G_A \oplus \cup_j G_{S_j})\Gamma_{AS} G_{n+1}] \\ &= G_{A'} \oplus \cup_j G_{C(S_j)} - (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S} G_A \oplus \cup_j G_{S_j} \\ &\quad +(G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S} G_A \oplus \cup_j G_{S_j} \Gamma_{AS} G_{n+1}. \end{aligned} \tag{4.2.26}$$

In (4.2.26) we have a formula of the form:

$$G_{n+1} = \alpha + \beta G_{n+1} \tag{4.2.27}$$

where

$$\alpha = G_{A'} \oplus \cup_j G_{C(S_j)} - (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S} G_A \oplus \cup_j G_{S_j}, \tag{4.2.28}$$

and

$$\beta = (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S} G_A \oplus \cup_j G_{S_j} \Gamma_{AS}; \tag{4.2.29}$$

so we have

$$(I - \beta)G = \alpha \tag{4.2.30}$$

and formally

$$\begin{aligned} G &= (I-\beta)^{-1}\alpha \\ &= \sum_{m=0}^{\infty} \beta^m \alpha. \end{aligned} \tag{4.2.31}$$

Therefore

$$\begin{aligned} G_{n+1} &= G_{A'} \oplus \cup_j G_{C(S_j)} + (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S}(G_A \oplus \cup_j G_{S_j}) \\ &\quad \sum_{m=1}^{\infty}((G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S}(G_A \oplus \cup_j G_{S_j})\Gamma_{AS}))^m (G_{A'} \oplus \cup_j G_{C(S_j)} + \\ &\quad G_{A'} \oplus \cup_j G_{C(S_j)} \Gamma_{A'S} G_A \oplus \cup_j G_{S_j}). \end{aligned}$$

Let define the matrices r, q and s as follows:

$$\begin{cases} s = G_{A'} \oplus \cup_j G_{C(S_j)} \\ r = (G_{A'} \oplus \cup_j G_{C(S_j)} \Gamma_{A'S}(G_A \oplus \cup_j G_{S_j}) \\ q = (G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S}(G_A \oplus \cup_j G_{S_j})\Gamma_{AS}) \end{cases}$$

$$q(l_1, l_2) := ((G_{A'} \oplus \cup_j G_{C(S_j)})\Gamma_{A'S}(G_A \oplus \cup_j G_{S_j})\Gamma_{AS})(l_1, l_2). \tag{4.2.32}$$

We show that this geometric series converges; that is $||q(l_1,l_2)||_{\sigma_n,s} < 1$,

$$\begin{aligned} q(l_1,l_2) &= \sum_{\beta_1,\beta_2,\beta_3} (G_{A'} \oplus \cup_j G_{C(S_j)})(l_1,\beta_1)\Gamma_{A'S}(\beta_1,\beta_2) \\ & (G_A \oplus \cup_j G(S_j))(\beta_2,\beta_3)\Gamma_{AS}(\beta_3,l_2). \end{aligned} \tag{4.2.33}$$

Here we have to consider 4 cases:

$$\begin{cases} l_1 \in C(S) \ or \ l_1 \in A' \\ l_2 \in S \ or \ l_2 \in A \end{cases}$$

We start by $l_1 \in C(S), l_2 \in A$;
$G_{C(S)}$ has only terms in $C(S)$ so $\beta_1 \in C(S)$. Then $\beta_1 \in A'$ and $\Gamma_{A'C(S)}$ has only interactions terms between A' and S so $\beta_2 \in A'$. Now starting from the left, Γ_{AS} has only interactions terms between A and S, as $l_2 \in A$ we have $\beta_3 \in A$ and hence $\beta_2 \in S$ which contradicts the fact that β_2 should be in A' therefore the case $l_1 \in C(S), l_2 \in A$ is impossible.
$l_1 \in C(S), l_2 \in S$;
$\beta_1 \in C(S) \Longrightarrow \beta_2 \in A'$ and Γ_{AS} has only interactions terms so $l_2 \in S \Longrightarrow \beta_3 \in A$. $\beta_2 \in A$ because of G_A and recalling the fact that $A \subset A'$ it is clear that the length of the path taken from l_1 to l_2 is longer than l_n because $\beta_2 \in A'$ and $l_2 \in S$.

$$\begin{aligned} |q(l_1,l_2)| &= |\sum_{\beta_1,\beta_2,\beta_3} G_{C(S)}(l_1,\beta_1)\Gamma_{A'C(S)}(\beta_1,\beta_2)G_A(\beta_2,\beta_3)\Gamma_{AS}(\beta_3,l_2)| \\ &\leq \sum_{\beta_1,\beta_2,\beta_3} \frac{1}{\delta_n} e^{-\sigma_n|l_1-\beta_1|}(1+|l_1-\beta_1|^2)^{-s/2}||p||^2 e^{-\sigma_n|\beta_1-\beta_2|} \\ & (1+|\beta_1-\beta_2|^2)^{-s/2}\frac{C}{d_0}e^{-\sigma_n|\beta_2-\beta_3|}(1+|\beta_2-\beta_3|^2)^{-s/2}||p||^2 \\ & e^{-\sigma_n|\beta_3-l_2|}(1+|\beta_3-l_2|^2)^{-s/2}. \end{aligned}$$

Let

$$\beta = |l_1-\beta_1|+|\beta_1-\beta_2|+|\beta_2-\beta_3|+|\beta_3-l_2|. \tag{4.2.34}$$

Here we have $Q=4$ steps while $\beta \geq l_n$ and $\beta \geq |l_1-l_2|$ then

$$\begin{aligned} e^{-\sigma_n\beta} &= e^{(-\sigma_n+\gamma_n-\gamma_n)\beta} \\ &\leq e^{-(\sigma_n-\gamma_n)\beta}e^{-\gamma_n\beta} \\ &\leq e^{-\gamma_n l_n}e^{-(\sigma_n-\gamma_n)|l_1-l_2|}. \end{aligned} \tag{4.2.35}$$

Using (4.2.35) we deduce

$$\begin{aligned} |q(l_1,l_2)| &\leq \frac{||p||^4}{d_0}\frac{1}{\delta_n}e^{-\gamma_n l_n}e^{-(\sigma_n-\gamma_n)|l_1-l_2|}\sum_{\beta_1,\beta_2,\beta_3}(1+|l_1-\beta_1|^2)^{-s/2} \\ & (1+|\beta_1-\beta_2|^2)^{-s/2}(1+|\beta_2-\beta_3|^2)^{-s/2}(1+|\beta_3-l_2|^2)^{-s/2} \end{aligned}$$

and recalling now lemma 2.2.2, we get

$$|q(l_1,l_2)| \leq \frac{||p||^4}{d_0}\frac{C}{\delta_n}e^{-\gamma_n l_n}e^{-(\sigma_n-\gamma_n)|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2}. \tag{4.2.36}$$

Now consider the case: $l_1 \in A'$ and $l_2 \in S$
$l_1 \in A'$ and $G_{A'}$ has only terms in A', so $\beta_1 \in A'$ and $\beta_2 \in C(S)$. Now $\beta_3 \in A$ as $l_2 \in S$ and Γ_{AS} has only interactions terms between A and S. The length of the path (which we will call β) is greater than l_n because $l_1 \in A'$ and $l_2 \in S$:

$$\begin{aligned}
|q(l_1,l_2)| &= |\sum_{\beta_1,\beta_2,\beta_3} G_{A'}(l_1,\beta_1)\Gamma_{A'C(S)}(\beta_1,\beta_2)G_A(\beta_2,\beta_3)\Gamma_{AS}(\beta_3,l_2)| \\
&\leq \sum_{\beta_1,\beta_2,\beta_3} \frac{1}{d_0}e^{-\sigma_n|l_1-\beta_1|}(1+|l_1-\beta_1|^2)^{-s/2}\frac{||p||^2}{d_0}e^{-\sigma_n|\beta_1-\beta_2|} \\
&\quad (1+|\beta_1-\beta_2|^2)^{-s/2}\frac{1}{d_0}e^{-\sigma_n|\beta_2-\beta_3|}(1+|\beta_2-\beta_3|^2)^{-s/2}\frac{||p||^2}{d_0} \\
&\quad e^{-\sigma_n|\beta_2-\beta_3|}(1+|\beta_2-\beta_3|^2)^{-s/2}
\end{aligned}$$

and using lemma 2.2.2 we get

$$|q(l_1,l_2)| \leq C||p||^4 e^{-\gamma_n l_n}e^{-(\sigma_n-\gamma_n)|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2}. \tag{4.2.37}$$

We consider now : $l_1 \in A'$ and $l_2 \in A$.
This leads to $\beta_1 \in A'$ and $\beta_2 \in C(S), \beta_3 \in S$. The path $\beta \geq l_n$ because $l_1 \in A'$ and $\beta_3 \in S$

$$\begin{aligned}
|q(l_1,l_2)| &= |\sum_{\beta_1,\beta_2,\beta_3} G_{A'}(l_1,\beta_1)\Gamma_{A'C(S)}(\beta_1,\beta_2)G_S(\beta_2,\beta_3)\Gamma_{AS}(\beta_3,l_2)| \\
&\leq \sum_{\beta_1,\beta_2,\beta_3} \frac{4}{d_0}e^{-\sigma_n|l_1-\beta_1|}(1+|l_1-\beta_1|^2)^{-s/2}||p||^2 e^{-\sigma_n|\beta_1-\beta_2|}\cdot \\
&\quad (1+|\beta_1-\beta_2|^2)^{-s/2}C_0 e^{-\sigma_n|\beta_2-\beta_3|}(1+|\beta_2-\beta_3|^2)^{-s/2}\frac{||p||^2}{d_0} \\
&\quad e^{-\sigma_n|\beta_3-l_2|}(1+|\beta_1-l_2|^2)^{-s/2} \\
&\leq \frac{||p||^4}{d_0^2}e^{-\gamma_n l_n}e^{-(\sigma_n-\gamma_n)|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2},
\end{aligned} \tag{4.2.38}$$

combining (4.2.36), (4.2.37) and (4.2.38) we deduce:

$$|q(l_1,l_2)| \leq \frac{||p||^4}{d_0}\frac{C}{\delta_n}e^{-\gamma_n l_n}e^{-(\sigma_n-\gamma_n)|l_1-l_2|}(1+|l_1-l_2|^2)^{-s/2}, \tag{4.2.39}$$

for any l_1, l_2. We now use the estimates on the matrix elements of G_{n+1} to get an estimate on the its norm; so we start by getting an estimate on the norm of q:

If

$$||q||^2_{\sigma_n-2\gamma_n,s} = \sup_{l_2}\sum_{l_1}(1+|l_1-l_2|^2)^s e^{2(\sigma_n-2\gamma_n)|l_1-l_2|}|q(l_1,l_2)|^2$$

then, using (4.2.38) we continue and deduce

$$\begin{aligned}
||q||^2_{\sigma_n-2\gamma_n,s} &\leq \sup_{l_2}\sum_{l_1}(1+|l_1-l_2|^2)^s e^{2(\sigma_n-2\gamma_n)|l_1-l_2|}\frac{1}{\delta_n^2}(\frac{||p||^4}{d_0^2})^2 e^{-2\gamma_n l_n} e^{-2(\sigma_n-\gamma_n)|l_1-l_2|} \\
&\quad (1+|l_1-l_2|^2)^{-s} \\
&\leq C\frac{1}{\delta_n^2}(\frac{||p||^2}{d_0^2})^2 e^{-2\gamma_n l_n}\sup_{l_2}\sum_{l_1} e^{-2\gamma_n|l_1-l_2|} \\
&\leq \frac{1}{\delta_n^2}(\frac{||p||^4}{d_0^2})^2 e^{-2\gamma_n l_n}\sup_{l_2}\frac{80q}{(2\gamma_n)^2} \\
&\leq \frac{1}{\delta_n^2}C(\frac{||p||^4}{d_0^2})^2 e^{-2\gamma_n l_n}\frac{1}{(2\gamma_n)^2}. \qquad (4.2.40)
\end{aligned}$$

Hence

$$||q||_{\sigma_n-2\gamma_n,s} \leq C\frac{1}{\delta_n}\frac{||p||^4}{d_0^2}e^{-\gamma_n l_n}\frac{1}{\gamma_n} \qquad (4.2.41)$$

and if

$$||q||^2_{\sigma_n-2\gamma_n,s} = \sup_{l_1}\sum_{l_2}(1+|l_1-l_2|^2)^s e^{2(\sigma_n-2\gamma_n)|l_1-l_2|}|q(l_1,l_2)|^2$$

the estimate (4.2.41) follows in the same fashion. To ensure the convergence of the series (4.2.41) we require that

$$\frac{1}{\delta_n}\frac{||p||^4}{d_0^2}e^{-\gamma_n l_n}\frac{1}{\gamma_n^2} < \frac{1}{2}, \qquad (4.2.42)$$

as $||p|| \leq Cr_0$ we impose the condition

$$\frac{1}{\delta_n}\frac{r_0^2}{d_0^2}e^{-\gamma_n l_n}\frac{1}{\gamma_n^2} < \frac{1}{2}. \qquad (4.2.43)$$

Replacing each sequence by its definition we get

$$2^{\alpha n}L_0^{\alpha}\frac{L_0^{-4-2\eta}}{d_0^2}e^{-\frac{\sigma_0}{32(1+n^2)}2^{-n\beta}L_0^{-\beta}}(\frac{32(1+n^2)}{\sigma_0})^2 < \frac{1}{2}$$

$$e^{n\alpha\ln 2+\alpha\ln L_0}.e^{(-4-2\eta)\ln L_0}e^{-\frac{\sigma_0}{32(1+n^2)}2^{-n\beta}L_0^{-\beta}}.e^{2\ln\frac{32(1+n^2)}{\sigma_0}} < \frac{1}{2}$$

$$e^{n\alpha\ln 2+\alpha\ln L_0}e^{(-4-2\eta)\ln L_0}.e^{-\frac{\sigma_0}{32(1+n^2)}2^{-n\beta}L_0^{-\beta}}.e^{2\ln\frac{32(1+n^2)}{\sigma_0}}.e^{\ln 2} < 1.$$

As

$$(-4-2\eta)\ln L_0+n\alpha\ln 2+\alpha\ln L_0-\frac{\sigma_0}{32(1+n^2)}2^{-n\beta}L_0^{-\beta}+2\ln\frac{32(1+n^2)}{\sigma_0}+\ln 2\longrightarrow-\infty$$

when $n\longrightarrow\infty$ the inequality (4.2.43) will be satisfied for n large enough. For the first steps of iteration we can choose L_0 very large so that the condition (4.2.43) is satisfied.

$$\begin{aligned} G_{n+1} &= G_{A'}\oplus\cup_j G_{C(S_j)}+(G_{A'}\oplus\cup_j G_{C(S_j)}\Gamma_{A'S}(G_A\oplus\cup_j G_{S_j}) \\ &\sum_{m=1}^{\infty}((G_{A'}\oplus\cup_j G_{C(S_j)})\Gamma_{A'S}(G_A\oplus\cup_j G_{S_j})\Gamma_{AS}))^m(G_{A'}\oplus\cup_j G_{C(S_j)}+ \\ &G_{A'}\oplus\cup_j G_{C(S_j)}\Gamma_{A'S}G_A\oplus\cup_j G_{S_j}) \end{aligned}$$

We now estimate $|s(l_1,l_2)|$:

$$s=G_{A'}\oplus\cup_j G_{C(S_j)}; \tag{4.2.44}$$

if $l_1,l_2\in A'$ then

$$|s(l_1,l_2)|\le\frac{4}{d_0}(1+|l_1-l_2|^2)^{-s/2}e^{-\sigma_n|l_1-l_2|} \tag{4.2.45}$$

because A' is a nonsingular region. And if $l_1,l_2\in C(S))$ then

$$|s(l_1,l_2)|\le\frac{C}{\delta_n}(1+|l_1-l_2|^2)^{-s/2}e^{-\sigma_n|l_1-l_2|}. \tag{4.2.46}$$

If

$$||s||^2_{\sigma_n-2,s}=\sup_{l_2}\sum_{l_1}e^{2(\sigma_n-2\gamma_n)|l_1-l_2|}|s(l_1,l_2)|^2$$

then

$$\begin{aligned} ||s||^2_{\sigma_n-2\gamma_n,s} &= \sup_{l_2}\sum_{l_1}e^{2(\sigma_n-2\gamma_n)|l_1-l_2|}|s(l_1,l_2)|^2 \\ &\le \sup_{l_2}\sum_{l_1}\frac{C}{\delta_n}(1+|l_1-l_2|^2)^{-s}e^{-2m|l_1-l_2|}(1+|l_1-l_2|^2)^s e^{2(\sigma_n-2)\gamma_n|l_1-l_2|} \\ &\le \frac{C}{\delta_n}\sup_{l_2}\sum_{l_1}e^{-2\gamma_n|l_1-l_2|}, \end{aligned} \tag{4.2.47}$$

$$||s||_{\sigma_n-2\gamma_n,s}\le\frac{C}{\delta_n}\frac{C}{(2\gamma_n)^2}. \tag{4.2.48}$$

And if

$$||s||^2_{\sigma_n-2,s}=\sup_{l_1}\sum_{l_2}e^{2(\sigma_n-2\gamma_n)|l_1-l_2|}|s(l_1,l_2)|^2 \tag{4.2.49}$$

the estimate on s follows in the same fashion. Now consider r,

$$r(l_1,l_2) = \sum_{\beta_1,\beta_2} G_{A'}\Gamma_{A'C(S)}(\beta_1,\beta_2)G_A(\beta_2,l_2). \tag{4.2.50}$$

We will consider the four possibles cases:

$$\begin{cases} 1.\ l_1 \in A'\ l_2 \in A \\ 2.\ l_1 \in A'\ l_2 \in S \\ 3.\ l_1 \in C(S)\ l_2 \in A \\ 4.\ l_1 \in C(S)\ l_2 \in S. \end{cases}$$

We start by $1:\ l_1 \in A'$ and $l_2 \in A$

$$\begin{aligned} |r(l_1,l_2)| &= |\sum_{\beta_1,\beta_2} G_{A'}\Gamma_{A'C(S)}(\beta_1,\beta_2)G_A(\beta_2,l_2)| \\ &\leq \sum_{\beta_1,\beta_2}\frac{4}{d_0}(1+|l_1-\beta_1|^2)^{-s/2}e^{-\sigma_n|l_1-\beta_1|}C_3||p||^2(1+|\beta_1-\beta_2|^2)^{-s/2} \\ &\quad e^{-\sigma_n|\beta_1-\beta_2|}\frac{4}{d_0}(1+|\beta_2-l_2|^2)^{-s/2}e^{-\sigma_n|\beta_2-l_2|} \\ &\leq C\frac{r_0^2}{d_0^2}(1+|l_1-l_2|^2)^{-s/2}e^{-\sigma_n|l_1-l_2|}. \end{aligned}$$

$2:\ l_1 \in A'$ and $l_2 \in S$

$$\begin{aligned} |r(l_1,l_2)| &= |\sum_{\beta_1,\beta_2} G_{C(S)}\Gamma_{A'C(S)}(\beta_1,\beta_2)G_A(\beta_2,l_2)| \\ &\leq \sum_{\beta_1,\beta_2}\frac{C_0}{d_n}(1+|l_1-\beta_1|^2)^{-s/2}e^{-\sigma_n|l_1-\beta_1|}\frac{C_3||p||^2(1+|\beta_1-\beta_2|^2)^{-s/2}}{d_0} \\ &\quad e^{-\sigma_n|\beta_1-\beta_2|}\frac{4}{d_0}(1+|\beta_2-l_2|^2)^{-s/2}e^{-\sigma_n|\beta_2-l_2|} \\ &\leq C\frac{r_0^2}{d_0^2}\frac{1}{d_n}(1+|l_1-l_2|^2)^{-s/2}e^{-\sigma_n|l_1-l_2|}. \end{aligned}$$

$3:\ l_1 \in C(S)$ and $l_2 \in A$:

$$\begin{aligned} |r(l_1,l_2)| &= \sum_{\beta_1,\beta_2} G_{A'}(l_1,\beta_1)\Gamma_{A'C(S)}(\beta_1,\beta_2)G_S(\beta_2,l_2) \\ &\leq \sum_{\beta_1,\beta_2}\frac{4}{d_0}(1+|l_1-\beta_1|^2)^{-s/2}e^{-\sigma_n|l_1-\beta_1|}C_3||p||^2(1+|\beta_1-\beta_2|^2)^{-s/2}e^{-\sigma_n|\beta_1-\beta_2|} \\ &\quad C_0(1+|\beta_2-l_2|^2)^{-s/2}e^{-\sigma_n|\beta_2-l_2|} \\ &\leq C\frac{r_0^2}{d_0^2}(1+|l_1-l_2|^2)^{-s/2}e^{-\sigma_n|l_1-l_2|} \end{aligned} \tag{4.2.51}$$

4. $l_1 \in C(S)$ and $l_2 \in S$:
$l_1 \in C(S)$ so $\beta_1 \in C(S)$ and $\beta_2 \in A'$. However beginning from the left in (4.2.50) we have $\beta_2 \in S$ which contradicts $\beta_2 \in A'$ so

$$r(l_1, l_2) = 0.$$

As we did in (4.2.48), we can conclude that for any l_1 and l_2

$$|r(l_1, l_2)| \leq \frac{C}{\delta_n}\frac{r_0^2}{d_0^2}(1 + |l_1 - l_2|^2)^{-s/2}e^{-\sigma_n|l_1 - l_2|}$$

$$||r||_{\sigma_n - 2\gamma_n, s} \leq \frac{r_0^2}{d_0^2}\frac{C}{\delta_n(2\gamma_n)^2}.$$

Using (4.2.41), (4.2.48) and (4.2.51) we are now able to get an estimate on G_{n+1}

$$||G_{n+1}||_{\sigma_n - 2\gamma_n, s} \leq ||s||_{\sigma_n - 2\gamma_n, s} + ||r||_{\sigma_n - 2\gamma_n, s} + ||r||_{\sigma_n - 2\gamma_n, s}\sum_{m=1}^{\infty}||q||^m_{\sigma_n - 2\gamma_n, s}$$

$$||G_{n+1}||_{\sigma_n - 2\gamma_n, s} \leq ||s||_{\sigma_n - 2\gamma_n, s} + ||r||_{\sigma_n - 2\gamma_n, s} + ||r||_{\sigma_n - 2\gamma_n, s}\frac{1}{1 - \frac{1}{2}}$$

$$||G_{n+1}||_{\sigma_n - 2\gamma_n, s} \leq \frac{r_0^2}{d_0^2}\frac{C}{\delta_n(2\gamma_n)^2}. \square \tag{4.2.52}$$

We have assumed that the sites $l_1, l_2 \in L_{n+1} \setminus L_n$ such that $|V(\Omega)(l, l)| \leq d_s$ were such that $|l_1 - l_2| \geq l_n$ to ensure the convergence of the Neumann series. Now we provide the proof:

Theorem 4.1 [9] *The Singular sites are well separated if L_0 is large enough.*
Let $x = (j, k)$ and $x' = (j', k')$ two singular sites in $B_{n+1} \setminus B_n$ then

$$|x - x'| \geq 2l_{n+1} \quad (l_{n+1} = L^{\beta}_{n+1}) \tag{4.2.53}$$

The point of the lemma is to show that the sites where $|V(\Omega)(j, k)(j, k)| < d_s$ are very far from each other. This property will help us to get an estimate of the inverse of the local Hamiltonians.
Proof:
We proceed in two steps. We first prove that singular sites are well separated for $n \leq N_0$ a constant that we define later in the proof.
Assume that $|x - x'| \leq 2l_{n+1}$. We will prove that if L_0 is large enough and β small enough we have a contradiction.
Let $x = (j, k)$ singular in $B_{n+1} \setminus B_n$, (so $|x| \geq L_0$) and we have

$$|\omega_j^2 - k^2\Omega^2| \leq d_s$$

$$-d_s \leq -\omega_j^2 + k^2\Omega^2 \leq d_s$$
$$-d_s + \omega_j^2 \leq k^2\Omega^2 \leq d_s + \omega_j^2$$
$$-d_s + j^2 + v^* + d(j) \leq k^2\Omega^2 \leq d_s + j^2 + v^* + d(j). \tag{4.2.54}$$

We have here used the asymptotics of ω_j^2 given in [9]. The sequence $d(j)$ satisfies $\sum_j |d(j)|^2 < \infty$ hence $|d(j)|$ is bounded and there exists an M such that for $j \geq M$ we have

$$\frac{1}{2}j^2 \leq -d_s + j^2 + v^* + d(j) \leq k^2\Omega^2 \leq d_s + k^2 + v^* + d(j) \leq 2j^2 \tag{4.2.55}$$

and thus

$$\frac{1}{2}j^2 \leq k^2\Omega^2 \leq 2j^2. \tag{4.2.56}$$

We have also

$$k^2\Omega^2 \leq d_s + j^2 + v^* + d(j)$$

and so

$$k^2\Omega_{min}^2 \leq k^2\Omega^2 \leq d_s + j^2 + v^* + d(j) \tag{4.2.57}$$
$$k^2 \leq \frac{1}{\Omega_{min}^2}(d_s + j^2 + v^* + d(j)) \leq \frac{2j^2}{\Omega_{min}^2} \tag{4.2.58}$$

so that whenever x is singular $|k| \leq c(\Omega_{min}, M)$. If we choose now

$$L_0 \geq M + c(\Omega_{min}, M) \geq |j| + |k|,$$

we are sure that if x is singular in $B_{n+1} \setminus B_n$ then $|k| \geq M$. If this was not the case then we would have had $|x| \leq L_0$ which contradicts the fact that $x \in B_{n+1} \setminus B_n$. Recalling now (4.2.58) we have

$$L_n \leq |j| + |k| \leq |k| \leq |j| + \sqrt{\frac{2}{\Omega_{min}}}|j|$$

$$L_n \leq (1 + \sqrt{\frac{2}{\Omega_{min}}})|j|$$
$$L_n \leq c(\Omega_{min})|j|.$$

And we conclude that $|j| \geq M_1 L_n$ whenever x is singular where $M_1 = \frac{1}{c(\Omega_{min})}$. Similarly using (4.2.56) we can see that $|j| \leq \sqrt{2}\Omega_{max}|k|$, hence $L_n \leq c(\Omega_{max})|k|$ so there exists M_2 such that whenever x is singular

$$|k| \geq M_2 L_n. \tag{4.2.59}$$

For the Dirichlet problem we have

$$|\omega_j - j| < \frac{C_v}{j} \quad (j \geq 1) \text{ see [9]}, \tag{4.2.60}$$

x is in $B_{n+1} \setminus B_n$ singular then $|\Omega^2 k^2 - \omega_j^2| < d_s$

$$|j\Omega - \omega_k| \quad \leq \quad \frac{d_s}{|j\Omega + \omega_k|} \tag{4.2.61}$$

and

$$\frac{1}{2}j^2 \leq \omega_j^2 \leq 2j^2$$

$$\frac{1}{\sqrt{2}}|j| \leq |\omega_j| \leq \sqrt{2}|j|$$

so when $k, k' \geq 0$ we have

$$\begin{aligned} |k\Omega + \omega_j| = k\Omega + \omega_j &\geq k\Omega_{min} + \frac{1}{\sqrt{2}}|j| \\ &\geq \inf(\Omega_{min}, \frac{1}{\sqrt{2}})|k| + \inf(\Omega_{min}, \frac{1}{\sqrt{2}})|j| \\ &\geq \inf(\Omega_{min}, \frac{1}{\sqrt{2}})(|j| + |k|) \\ &\geq c(\Omega_{min}, \frac{1}{\sqrt{2}})(|j| + |k|). \end{aligned} \tag{4.2.62}$$

$$\begin{aligned} |k\Omega - \omega_j| &\leq \frac{d_s}{|k\Omega + \omega_j|} \\ &\leq \frac{d_s}{c(\Omega_{min}, \frac{1}{\sqrt{2}})(|j| + |k|)} \\ &\leq \frac{d_s}{c(\Omega_{min}, \frac{1}{\sqrt{2}})L_n}, \end{aligned} \tag{4.2.63}$$

$$\begin{aligned} |(k - k')\Omega - (j - j')| &= |(k\Omega - \omega_j) - (k'\Omega - \omega_{j'}) + (\omega_j - \omega_{j'}) - (j - j')| \\ &\leq |(k\Omega - \omega_j) - (k'\Omega - \omega_{j'})| + |(\omega_j - j) - (\omega_{j'} - j')| \\ &\leq 2.\frac{C(\Omega_{min})}{L_n} + 2\frac{C_v}{L_n} \text{ using (4.2.63) and (4.2.60)} \\ &\leq \frac{C}{L_n}. \end{aligned}$$

Now we consider the case $k, k' \leq 0$

$$|k^2\Omega^2 - \omega_j^2| < d_s$$

$$|k\Omega - \omega_j|.|k\Omega + \omega_j| < d_s$$

$$|-k\Omega + \omega_j|.|k\Omega + \omega_j| < d_s$$

$$|k\Omega + \omega_j| < \frac{d_s}{|-k\Omega + \omega_j|}$$

and $-k$ is positive so

$$|k\Omega + \omega_j| < \frac{d_s}{c(\Omega_{min}, \frac{1}{\sqrt{2}})L_n}. \tag{4.2.64}$$

We now evaluate

$$\begin{aligned} |(k-k')\Omega + (j-j')| &= |(k\Omega + \omega_j) - (k'\Omega + \omega_{j'}) + (-\omega_j + \omega_{j'}) + (j-j')| \\ &\leq |(k\Omega + \omega_j) - (k'\Omega + \omega_{j'})| + |(j - \omega_j) + (\omega'_j - j')| \\ &\leq 2.\frac{C(\Omega_{min})}{L_n} + 2\frac{C_v}{L_n} \text{ using (4.2.60) and (4.2.64)} \\ &\leq \frac{C}{L_n} \end{aligned}$$

$$|(k-k')\Omega - (j-j')| \geq |(k-k')\omega_1 - (j-j')| - |(j-j')(\Omega - \omega_1)|$$

If k and k' are of different sign they are necesserely separated;
this can be shown as for L_0 large enough, recalling (4.2.56) we have

$$\frac{1}{2}j^2 \leq k^2\Omega^2 \leq 2j^2. \tag{4.2.65}$$

hence

$$j^2 \leq 2\Omega^2 k^2$$
$$|j| \leq \sqrt{2}\Omega_{max}|k|$$

and as

$$L_n \leq |j| + |k| \leq (1 + \sqrt{2}\Omega_{max})|k|$$
$$|k| \geq \frac{1}{(1+\sqrt{2}\Omega_{max})}L_n. \tag{4.2.66}$$

Now we want to get the estimate

$$\frac{1}{C}L_n \geq 2l_{n+1} = 2(2L_n)^\beta \tag{4.2.67}$$

where $C = 1 + \sqrt{2}\Omega_{max}$, which is equivalent to

$$L_n \geq C2 2^\beta L_n^\beta$$
$$L_n^{1-\beta} \geq C2^{\beta+1}.$$

We will take $\beta < \frac{1}{2}$ so

$$L_n^{1-\beta} \geq \sqrt{L_n} \geq C2^{1+\frac{1}{2}} \geq C.2^{\beta+1}$$

and

$$\sqrt{L_n} \geq \sqrt{L_0} \geq C.2^2$$
$$\sqrt{L_0} \geq C.2^2$$

and thus

$$|k-k'| = |k| + |k'| \geq 2\frac{L_n}{C} \geq 4l_{n+1}. \tag{4.2.68}$$

This condition is fulfilled for L_0 large enough and so

$$L_n^{1-\beta} \geq \sqrt{L_n} \geq \sqrt{L_0} \geq C.2^2 \geq C2^{1+\frac{1}{2}} \geq C.2^{\beta+1}$$

and 4.2.67 is satisfied.
We decide to choose $N_0 = [\frac{\ln L_0}{\beta}]$ where [ ] denotes the integer part of a real number :

$$\begin{aligned}
2l_{N_0+1} &= 2(2^{N_0+1}L_0)^\beta \\
&\leq 22^\beta(2^{\frac{\ln L_0}{\beta}}L_0)^\beta \\
&\leq 2^{\beta+1}e^{\ln L_0 \ln 2}L_0^\beta \\
&\leq 2^{\beta+1}L_0^{\beta+\ln 2} \\
&\leq L_0
\end{aligned} \tag{4.2.69}$$

for β small and $\beta + \ln 2 < 1$ and $2l_{n+1} \leq L_0$.
We have assumed the following non-resonant condition

$$|j\omega_1 - k| \geq \frac{d_0}{(|j|+|k|)^{\tau_0}} \text{ for } (j,k) \neq (0,0) \text{ and } |j|+|k| \leq L_0 \tag{4.2.70}$$

so as long as $2l_{n+1} \leq L_0$ we have

$$|j-j'| + |k-k'| \leq 2l_{n+1} \leq L_0$$

$$\begin{aligned}
|(k-k')\Omega - (j-j')| &\geq |(k-k')\omega_1 - (k-k')| - |(j-j')(\Omega-\omega_1)| \\
&\geq \frac{d_0}{(|j-j'|+|k-k'|)^{\tau_0}} - |(k-k')(\Omega-\omega_1)| \\
&\geq \frac{d_0}{(2l_{n+1})^{\tau_0}} - |(k-k')(\Omega-\omega_1)| \ (\tau_0 > 0).
\end{aligned} \tag{4.2.71}$$

As $|k-k'| \leq 2l_{n+1}$

$$-|k-k'| \geq -2l_{n+1}$$
$$-|k-k'||(\Omega-\omega_1)| \geq -2l_{n+1}|(\Omega-\omega_1)|$$

and

$$|(k-k')\Omega - (k-k')| \geq \frac{d_0}{(2l_{n+1})^{\tau_0}} - 2l_{n+1}|(\Omega-\omega_1)|.$$

We want $2l_{n+1}|(\Omega-\omega_1)| \leq \frac{1}{2}\frac{d_0}{(2l_{n+1})^{\tau_0}}$ i.e

$$|(\Omega-\omega_1)| \leq \frac{1}{4l_{n+1}}\frac{d_s}{(2l_{n+1})^{\tau_0}}$$

as $2l_{n+1} \leq L_0$, $\frac{1}{L_0} \leq \frac{1}{2l_{n+1}}$. We will require

$$|\Omega-\omega_1| \leq \frac{1}{2L_0}\frac{d_s}{L_0^{\tau_0}} \tag{4.2.72}$$

which is the case for $\tau_0 = 1$ and $d_s = \frac{d_0}{2}$; and we have the estimate

$$|(k-k')\Omega-(j-j')| \geq \frac{1}{2}\frac{d_s}{(2l_{n+1})^{\tau_0}} \tag{4.2.73}$$

We then have the following inequality

$$\frac{1}{2}\frac{d_s}{(2l_{n+1})^{\tau_0}} \leq |(k-k')\Omega-(j-j')| \leq \frac{C}{L_n} \tag{4.2.74}$$

$$\begin{aligned}
\frac{1}{2}\frac{d_s}{(2l_{n+1})^{\tau_0}} &\leq \frac{C}{L_n} \\
\frac{1}{2}\frac{d_s}{2^{\tau_0}L_{n+1}^{\beta\tau_0}} &\leq \frac{C}{L_n} \\
\frac{1}{2}2^{-\tau_0}L_{n+1}^{-\beta\tau_0}L_n &\leq \frac{C}{d_s} \\
\frac{1}{2}2^{-\tau_0}L_{n+1}^{-\beta\tau_0}\frac{1}{2}L_{n+1} &\leq \frac{C}{d_s} \\
2^{-\tau_0-2}L_{n+1}^{1-\beta\tau_0} &\leq \frac{C}{d_s} \\
L_{n+1}^{1-\beta\tau_0} &\leq 2^{\tau_0+2}\frac{C}{d_s}
\end{aligned} \tag{4.2.75}$$

this leads us to the condition:

$$d_0 \leq 2^{\tau_0+2}\frac{C}{L_{n+1}^{1-\beta\tau_0}} \tag{4.2.76}$$

and we choose $d_0 = \frac{1}{\sqrt{L_0}} = L_0^{-0.5}$ and for β small, $1-\beta\tau_0 \geq 0.8$ and L_0 large enough

$$L_0^{-0.5} > \frac{2^{\tau_0+2}}{L_0^{-0.8}} \geq \frac{C}{L_{n+1}^{0.5}} \geq 2^{\tau_0+2}\frac{C}{L_{n+1}^{1-\beta\tau}} \tag{4.2.77}$$

and we have a contradiction. Hence The singular sites are well separated for $n \leq N_0$. For larger values of n we will require some restrictions on Ω. We denote by $I_1 = [\omega_1 - r_0, \omega_1 + r_0]$, we have the lemma:

Lemma 4.2.3 *Let I be*

$$I = \{\Omega \in I_1, |k\Omega - j| < \frac{d}{(|j|+|k|)^\tau}, \text{ for } l_n \le |j|+|k| \le 4l_{n+1}\} \quad (4.2.78)$$

then

$$Meas\ I \le 16 l_{n+1}^2 \frac{2d}{l_n^\tau} \quad (4.2.79)$$

Proof:
Let (j,k) be a site such that $l_n \le |j|+|k| \le 4l_{n+1}$ and Ω_1 and Ω_2 two reals numbers in I, and such that
$|k\Omega_1 - j| < \frac{d}{(|j|+|k|)^\tau}$ and $|k\Omega_2 - j| < \frac{d}{(|j|+|k|)^\tau}$. We then have

$$-\frac{d}{(|j|+|k|)^\tau} \le j - \Omega_1 k \le \frac{d}{(|j|+|k|)^\tau}$$

and

$$-\frac{d}{(|j|+|k|)^\tau} \le \Omega_2 k - j \le \frac{d}{(|j|+|k|)^\tau}.$$

So adding the two inequalities we get

$$-2\frac{d}{(|j|+|k|)^\tau} \le (\Omega_2 - \Omega_1)k \le 2\frac{d}{(|j|+|k|)^\tau}$$

$$|(\Omega_2 - \Omega_1)k| \le 2\frac{d}{(|j|+|k|)^\tau}$$

$$|\Omega_2 - \Omega_1| \le \frac{2d}{|k|(|j|+|k|)^\tau}.$$

We can suppose that $|k| \ge 1$ because:
if $k=0$, $|0\Omega - j| < \frac{d}{(|0|+|j|)^\tau}$ is impossible for $d<1$ (this condition is required for $(j,k) \ne (0,0)$), and as $l_n \le |j|+|k|$ we have

$$|\Omega_2 - \Omega_1| \le \frac{2d}{l_n^\tau}. \quad (4.2.80)$$

There are at most $16l_{n+1}^2$ sites that satisfy $l_n \le |j|+|k| \le 4l_{n+1}$ so

$$Meas\ I \le 16 l_{n+1}^2 \frac{2d}{l_n^\tau}. \quad (4.2.81)$$

We define the set $\mathcal{N}_{n+1}^{(1)}$ by

$$\mathcal{N}_{n+1}^{(1)} = \mathcal{N}_n^{(1)} \setminus \{\Omega \in \mathcal{N}_0\ |k\Omega - j| < d(|j|+|k|)^{-\tau};\ l_n \le |j|+|k| \le 4l_{n+1}\} \quad (4.2.82)$$

For $\Omega \in \mathcal{N}_{n+1}^{(1)}$ we have

$$|k\Omega - j| \geq \frac{d}{(|j| + |k|)^{\tau}}.$$

We can now continue the proof of the separation of sites for values such that $n \geq N_0 + 1$.
If $l_{N_0+1} \leq |j| + |k| \leq 4l_{n+1}$ then from the left expression of (4.2.71) we get:

$$|(k-k')\Omega - (j-j')| \geq \frac{d}{(|k-k'| + |j-j'|)^{\tau}} \quad \text{when } l_{N_0+1} \leq |k-k'| + |j-j'| \leq 4l_{n+1}.$$

We now derive an inequality like (4.2.75) where τ replaces τ_0 and we choose again $d = d_s = \frac{d_0}{2}$ but $\tau > 0$ is now very large with the condition $1 - \beta\tau > 0.8$ we have again a contradiction:

$$\frac{d}{2l_{N_0+1}^{\tau}} \leq |(k-k')\Omega - (j-j')| \leq \frac{C}{L_n} \tag{4.2.83}$$

but

$$2l_{N_0+1} \leq L_0 \tag{4.2.84}$$

so

$$\frac{d}{L_0^{\tau}} \leq \frac{1}{(2l_{N_0+1})^{\tau}} \leq \frac{C}{2^n L_0} \tag{4.2.85}$$

then as

$$N_0 = [\frac{\ln L_0}{\beta}] \tag{4.2.86}$$

we have

$$\frac{1}{2^{N_0+1}} \leq \frac{1}{L_0^{\ln 2/\beta}} \tag{4.2.87}$$

hence

$$\frac{d}{L_0^{\tau}} \leq \frac{C}{L_0^{\ln 2/\beta} L_0} \tag{4.2.88}$$

and by choosing again $d = d_s = \frac{1}{2}L_0^{-0.5}$ and β very small so that

$$\tau - \ln 2/\beta - 1 \leq -1 \tag{4.2.89}$$

we have a contradiction.
If

$$0 < |k-k'| + |j-j'| \leq l_{N_0+1} \leq l_{N_0+1} \leq L_0 \tag{4.2.90}$$

then going back to (4.2.72) we get

$$|\Omega(k-k') - (j-j')| \geq \frac{d_s}{(2l_{N_0+1})^{\tau_0}} \tag{4.2.91}$$

hence

$$\frac{1}{2}\frac{d_s}{(L_0)^{\tau_0}} \leq \frac{1}{2}\frac{d_s}{(2l_{N_0+1})^{\tau_0}} \leq |(k-k')\Omega - (j-j')| \leq \frac{C}{L_n} \leq \frac{C}{2^{N_0+1}L_0} \tag{4.2.92}$$

and

$$\frac{1}{2}\frac{d_s}{(L_0)^{\tau_0}} \leq \frac{C}{2^{N_0+1}L_0} \tag{4.2.93}$$

thus

$$d_0 \leq \frac{CL_0^{\tau_0}}{2^{N_0+1}L_0} \tag{4.2.94}$$

$$d_0 \leq \frac{CL_0^{\tau_0}}{2^{\ln L_0/\beta}L_0} \tag{4.2.95}$$

and since $\tau_0 = 1$ we have

$$d_0 \leq \frac{C}{L_0^{\ln 2/\beta}} \tag{4.2.96}$$

so for $d_0 = L_0^{-0.5}$, $\beta < 0.5$ and L_0 large we have a contradiction; hence we conclude that:

$$|x - x'| \leq 2l_{n+1}. \ \square \tag{4.2.97}$$

Chapter 5

The extraction of parameters

To invert the locals Hamiltonians $H_{C(S)}$ at each iterative stage we needed to extract from $\mathcal{N}_0$ the values of parametetrs (p, Ω) such that $|e_i(p, \Omega)| \leq \delta_n$. In this chapter we prove that we have not excised all the values of $(p, \Omega) \in \mathcal{N}_0$. Most of the lemmas we enunciate here were originated in the paper [9] here we prove that they still hold using the norm spaces $H^{m,s}$.

The excised set

Let us denote by $e_i(p, \Omega)$ the eigenvalues of $H_{C(S)}$. We have the following lemma:

Lemma 5.0.4 [9] *For $(p, \Omega) \in \mathcal{N}_0$ each eigenvalue $e_i(p, \Omega)$ is strictly monotone decreasing in $Re\Omega$.*

Proof:
We denote by $\langle , \rangle$ the $l^2(\mathbb{N} \times \mathbb{Z})$ inner product. For S a singular region $S \subset B_{n+1} \backslash B_n$ every singular site $x = (j, k) \in S$ satisfies $|k| > 2M_3 L_n$ with $M_3 = \frac{1}{2} M_1$, M_1 is the constant introduced in (4.2.59). The radius of $C(S)$ is $l_{n+1} \ll L_n$ then we have $|k| > M_3 L_n$ also for $x \in C(S)$. The Hamiltonian $H_{C(S)}$ is monotone in ReΩ:
Let Ω be real, then

$$H_{C(S)}(p, \Omega; u_n) = V(\Omega) + D_u W(\phi(p) + u_n) \mid_{C(S)} \tag{5.0.1}$$

so

$$\partial_\Omega H_{C(S)}(p, \Omega; u_n) = \partial_\Omega V(\Omega) \mid_{C(S)} + D_u^2 W(\phi(p) + u_n)(\partial_\Omega u_n) \mid_{C(S)} \tag{5.0.2}$$

and for every

$$x = (j, k) \in C(S), \text{ so } |k| > M_3 L_n$$

$$\partial_\Omega V(\Omega) \mid_{C(S)} (x,y) = -2\Omega k^2 \delta(x,y) \leq -2\Omega_{min} M_3^2 L_n^2 \delta(x,y) \quad (\Omega > 0). \quad (5.0.3)$$

The second term in (5.0.2) has a norm bounded by

$$\|D_u^2 W(\phi(p)+u_n)(\partial_\Omega u_n, w)\|_{\sigma_0/2,s} \leq c\|\partial_\Omega u_n\|_{\sigma_0/2,s}\|w\|_{\sigma_0/2,s} \quad (5.0.4)$$

$\partial_\Omega u_n$ is bounded using the Cauchy estimates whereas u_n is bounded in norm by r_0. Let denote by ψ_i, the normalized eigenfunctions ($\langle \psi_i, \psi_i\rangle = 1$) of $H_{C(S)}$. $H_{C(S)}$ is self-adjoint so its eigenvalues $e_i(p,\Omega)$ are reals and satisfy the relation

$$e_i(p,\Omega) = \langle \psi_i, H_{C(S)}\psi_i\rangle. \quad (5.0.5)$$

We now show that $\partial_\Omega e_i(p,\Omega) \leq 0$

$$\begin{aligned}
\partial_\Omega e_i(p,\Omega) &= \langle \psi_i, \partial_\Omega [H_{C(S)}\psi_i]\rangle + \langle \partial_\Omega \psi_i, H_{C(S)}\psi_i\rangle \\
&= \langle \psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle + \langle \psi_i, H_{C(S)}\partial_\Omega \psi_i\rangle + \langle \partial_\Omega \psi_i, H_{C(S)}\psi_i\rangle \\
&= \langle \psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle + \langle H_{C(S)}\psi_i, \partial_\Omega \psi_i\rangle + \langle \partial_\Omega \psi_i, e_i\psi_i\rangle \\
&= \langle \psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle + e_i\langle \psi_i, \partial_\Omega \psi_i\rangle + e_i\langle \partial_\Omega \psi_i, \psi_i\rangle \\
&= \langle \psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle + e_i(\langle \psi_i, \partial_\Omega \psi_i\rangle + \langle \partial_\Omega \psi_i, \psi_i\rangle).
\end{aligned}$$

$\langle \psi_i, \psi_i\rangle = 1$ and we differentiate with respect to Ω so

$$\langle \partial_\Omega \psi_i, \psi_i\rangle + \langle \psi_i, \partial_\Omega \psi_i\rangle = 0,$$

and then

$$\partial_\Omega e_i(p,\Omega) = \langle \psi_i, \partial_\Omega H_{C(S)}\psi_i\rangle. \quad (5.0.6)$$

$$\partial_\Omega e_i(p,\Omega) = \langle \psi_i, (\partial_\Omega V(\Omega) + D_u^2 W(\phi(p)+u_n)(\partial_\Omega u_n)\psi_i\rangle$$

and now using (5.0.2)

$$\partial_\Omega e_i(p,\Omega) = \langle \psi_i, \partial_\Omega V(\Omega)\psi_i\rangle + \langle \psi_i, D_u^2 W(\phi(p)+u_n)(\partial_\Omega u_n)\psi_i\rangle$$

and using (5.0.3)

$$\begin{aligned}
\partial_\Omega e_i(p,\Omega) &\leq -2\Omega_{min} M_3^2 L_n^2 \langle \psi_i, \psi_i\rangle + \\
&\quad \langle \psi_i, D_u^2 W(\phi(p)+u_n)(\partial_\Omega u_n)\psi_i\rangle \quad (5.0.7)\\
&\leq -2\Omega_{min} M_3^2 L_n^2 + \langle \psi_i, D_u^2 W(\phi(p)+u_n)(\partial_\Omega u_n)\psi_i\rangle \\
&\leq -2\Omega_{min} M_3^2 L_n^2 + \|\psi_i\|_{0,0}\|D_u^2 W(\phi(p)+u_n)\partial_\Omega u_n \psi_i\|_{0,0} \\
&\leq -2\Omega_{min} M_3^2 L_n^2 + \|D_u^2 W(\phi(p)+u_n)\partial_\Omega u_n \psi_i\|_{0,0}
\end{aligned}$$

Now using the lemma 2.1.6

$$\begin{aligned}
\partial_\Omega e_i(p,\Omega) &\leq -2\Omega_{min} M_3^2 L_n^2 + \frac{C_{op}C_{D^2W}}{1-e^{\sigma_0/2}}\|D_u^2 W(\phi(p)+u_n)\partial_\Omega u_n\|_{\sigma_0/2,s}\|\psi_i\|_{0,0} \\
&\leq -2\Omega_{min} M_3^2 L_n^2 + \frac{C_{op}C_{D^2W}}{1-e^{\sigma_0/2}}\|W(\phi(p)+u_n)\|_{\sigma_0/2,s}\|\partial_\Omega u_n\|_{\sigma_0/2,s}.
\end{aligned}$$

We know that

$$||u_n(p,\Omega)||_{\sigma_0/2,s} \leq ||p||^2 \tag{5.0.8}$$

and we now define the following numbers

$$M' = \sup_{(p,\Omega)\in\mathcal{N}_n} ||u_n(p,\Omega)||_{\sigma_0/2} \tag{5.0.9}$$

$$M'_p = \sup_{(p,\Omega)\in\mathcal{N}_n} ||\partial_p u_n(p,\Omega)||_{\sigma_0/2} \tag{5.0.10}$$

$$M'_\Omega = \sup_{(p,\Omega)\in\mathcal{N}_n} ||\partial_\Omega u_n(p,\Omega)||_{\sigma_0/2} \tag{5.0.11}$$

using lemma 3.2.5 and 3.2.6 we can say that for r_0 small enough $M', M'_p, M'_\Omega \leq 2$ hence

$$\begin{aligned} -\partial_\Omega e_i(p,\Omega) &\geq 2\Omega_{min}M_3^2L_n^2 - \frac{C_{op}C_{D^2W}}{1-e^{\sigma_0/2}}||D_u^2W(\phi(p)+u_n)||_{\sigma_0/2,s}||\partial_\Omega u_n||_{\sigma_0/2,s} \\ &\geq 2\Omega_{min}M_3^2L_n^2 - C(D_u^2W,\sigma_0,s)MM'_\Omega||p|| \\ &\geq \Omega_{min}M_3^2L_n^2 \text{ for } r_0 \text{ small enough.} \quad \square \end{aligned} \tag{5.0.12}$$

We define the sets Z_i by

$$Z_i = \{(p,\Omega)\in\mathcal{N}_0,\ \ e_i(p,\Omega)=0\} \tag{5.0.13}$$

We have proved that for fixed p the $e_i(p,\Omega)$ are monotone decreasing in Ω so for each p there is at most one Ω; hence Z_i defines a surface $(p,\Omega_i(p))$. Furthermore since lemma 3.2.2 the spectrum of H is invariant under T_θ hence the sets Z_i are also invariant under T_θ.
It is important to locate the values of (p,Ω) for which $|e_i(p,\Omega)|$ is small and then excise them at each iterative stage so that the remaining parameters satisfy $|e_i(p,\Omega)| \geq \delta_n$. We now have the lemma:

Lemma 5.0.5 [9] *Let $e_i(p,\Omega_1)$ be an eigenvalue of $H_{C(S)}$ such that $|e_i(p,\Omega_1)| < 2\delta_{n+1}$ then there exists Ω_2 with*

$$|\Omega_2-\Omega_1| \leq C\frac{\delta_{n+1}}{L_n^2} \tag{5.0.14}$$

such that $e_i(p,\Omega_2)=0$.

The point of this lemma is to show that small eigenvalues of $H_{C(S)}$ lie in small neighborhoods of the set Z_i:
Proof:
By the Fatou lemma we have

$$e_i(p,\Omega_1) - e_i(p,\Omega) = \int_{\Omega_1}^{\Omega} -\partial_\Omega e_i(p,\omega)d\omega. \tag{5.0.15}$$

We take now a small positive eigenvalue $e_i(p,\Omega_1)$

$$0 \le e_i(p,\Omega_1) \le 2\delta_{n+1} \tag{5.0.16}$$

then by substracting $e_i(p,\Omega)$ from (5.0.16) we obtain

$$2\delta_{n+1} - e_i(p,\Omega) \quad \ge \quad e_i(p,\Omega_1) - e_i(p.\Omega)$$

using now Fatou's Lemma we obtain

$$\begin{aligned} 2\delta_{n+1} - e_i(p,\Omega) &\ge \int_{\Omega_1}^{\Omega} -\partial_\Omega e_i(p,\omega)d\omega \\ &\ge (\Omega-\Omega_1)\Omega_{min}M_3^2L_n^2 \text{ using (5.0.12).} \end{aligned}$$

Then

$$2\delta_{n+1} - e_i(p,\Omega) \ge \Omega_{min}M_3^2L_n^2(\Omega-\Omega_1) \tag{5.0.17}$$

and

$$e_i(p,\Omega) \le 2\delta_{n+1} - \Omega_{min}M_3^2L_n^2(\Omega-\Omega_1) \tag{5.0.18}$$

This shows that when Ω increases, the second term in the right hand side of (5.0.18) becomes very large and thus the right hand side of 5.0.18 vanishes for a value, say Ω_*. This implies that $e_i(p,\Omega)$ will vanish for a value Ω_2 close to Ω_1.

$$0 = 2\delta_{n+1} - \Omega_{min}M_3^2L_n^2(\Omega_*-\Omega_1) \tag{5.0.19}$$

and $\Omega_1 \le \Omega_2 \le \Omega_*$ because for p fixed $e_i(p,\Omega)$ is decreasing in Ω, and from (5.0.19) we have

$$\Omega_* - \Omega_1 = \frac{2\delta_{n+1}}{\Omega_{min}M_3^2L_n^2} \tag{5.0.20}$$

then

$$|\Omega_2-\Omega_1| = \Omega_2-\Omega_1 \le \Omega_*-\Omega_1 = \frac{2\delta_{n+1}}{\Omega_{min}.M_3^2L_n^2}. \tag{5.0.21}$$

If now $e_i(p,\Omega_1)$ is a negative small eigenvalue

$$-2\delta_{n+1} \le e_i(p,\Omega_1) \le 0$$

then $2\delta_{n+1} \ge -e_i(p,\Omega_1) \ge 0$.

$$\begin{aligned} 2\delta_{n+1} + e_i(p,\Omega) &\ge e_i(p,\Omega) - e_i(p,\Omega_1) \\ &\ge \int_{\Omega}^{\Omega_1} -\partial_\Omega e_i(p,\Omega)d\omega \\ &\ge (\Omega_1-\Omega)\Omega_{min}M_3^2L_n^2 \end{aligned}$$

$$e_i(p,\Omega) \ge (\Omega_1-\Omega)\Omega_{min}M_3^2L_n^2 - 2\delta_{n+1}$$

and similarly, for some value $\Omega_2 \leq \Omega_1$, $e_i(p,\Omega)$ will vanish. Once again we have

$$|\Omega_2 - \Omega_1| \leq \frac{2\delta_{n+1}}{\Omega_{min} M_3^2 L_n^2} \square \tag{5.0.22}$$

and 5.0.14 is deduced. We can conclude that the values of (p,Ω) which provide small eigenvalues lie in a neighborhood of width $C\delta_{n+1}/L_n^2$ of the surface Z_i.

The number of singular regions in $\overline{B_{n+1}} \setminus \overline{B_n}$ is bounded by L_{n+1}^2. For each singular region we excise a neighborood of width $C\delta_{n+1}/L_n^2$ of all zero sets Z_i. There are at most 2 eigenvalues for S, at most $4l_{n+1}^2$ for $C_{l_{n+1}}$, because the radius of C(S) is l_{n+1} and each of these neighborhood is invariant under T_θ. Thus the total measure excised of $\mathcal{N}_0$ is bounded by

$$L_{n+1}^2 l_{n+1}^2 (\delta_{n+1}/L_n^2)\pi r_0^2. \tag{5.0.23}$$

$\mathcal{N}_{n+1}$ is then inductively defined by the parameters that remain after extracting the singular and almost singular eigenvalues from the local Hamiltonian at the $n+1^{th}$ stage and the values of Ω so that our singular sites are separated.

As $\delta_n = L_n^{-\alpha}$ the excised region is inversely proportional to α and a choice of a large value of α will allow the sum of all the excised regions to be less than r_0 (Increasing the value of α means increasing of tolerance to the small eigenvalues.)

The subset of $\mathcal{N}_n$ obtained by excising the neighborhood of Z_i is called $\mathcal{N}_{n+1}^{(2)}$. For $(p,\Omega) \in \mathcal{N}_{n+1}^{(2)}$ we have:

$$|e_i(p,\Omega)| \geq 2\delta_{n+1}. \tag{5.0.24}$$

We define $\mathcal{N}_{n+1}$ by:

$$\mathcal{N}_{n+1} = \mathcal{N}_{n+1}^{(1)} \cap \mathcal{N}_{n+1}^{(2)}. \tag{5.0.25}$$

We want to extend the estimate in (5.0.24) to the ρ_{n+1} complex neighborhood of $\mathcal{N}_{n+1}$, $(\mathcal{N}_{n+1}, \rho_{n+1})$. We have the following lemma:

Lemma 5.0.6 [9] *For $(p,\Omega) \in (\mathcal{N}_{n+1}, \rho_{n+1})$ we have $|e_i(p,\Omega)| \geq \delta_{n+1}$*

Proof:

Let $(p,\Omega) \in (\mathcal{N}_{n+1}, \rho_{n+1})$, then there exists $(p_1,\Omega_1) \in \mathcal{N}_{n+1}$ such that

$$\sqrt{|p-p_1|^2 + |\Omega - \Omega_1|^2} \leq \rho_{n+1}. \tag{5.0.26}$$

We show that when $(p,\Omega) \in (\mathcal{N}_{n+1}, \rho_{n+1})$ then

$$||H_{C(S)}(p,\Omega,u_n) - H_{C(S)}(p_1,\Omega_1,u_n)||_0 \leq \frac{\delta_{n+1}}{8l_{n+1}^2} \tag{5.0.27}$$

and thus for $|\xi| < \delta_{n+1}$, $H_{C(S)}(p,\Omega,u_n) - \xi I$ is invertible using the Neumann series.

$$||H_{C(S)}(p,\Omega,u_n) - H_{C(S)}(p_1,\Omega_1,u_n)||_0 \leq ||V(\Omega)_{C(S)} - V(\Omega_1)_{C(S)}||_0$$
$$+||D_u W(\phi(p)+u_n) - D_u W(\phi(p_1)+u_n)||_0$$

$$\begin{aligned}
||V(\Omega)_{C(S)} - V(\Omega_1)_{C(S)}||_0 &= \sup_{l_1} \sum_{l_2} |V(\Omega)_{C(S)} - V(\Omega_1)_{C(S)}(l_1,l_2)| \\
&= \sup_{l_1} |V(\Omega)_{C(S)} - V(\Omega_1)_{C(S)}(l_1,l_1)| \\
&= \sup_{l_1} |(\Omega^2 k^2 - \omega_j^2) - (\Omega_1^2 k^2 - \omega_j^2)| \;\; (l_1 = (j,k)) \\
&\leq |(\Omega - \Omega_1^2)k^2| \\
&\leq |\Omega - \Omega_1^2| k^2 \\
&\leq |\Omega - \Omega_1|.|\Omega + \Omega_1| L_{n+1}^2 \\
&\leq \rho_{n+1} C(\Omega_{max}) L_{n+1}^2 .
\end{aligned} \tag{5.0.28}$$

Taylor formula states

$$f(x+h,y+k) - f(x,y) = \int_0^1 f'(x+\tau h, y+\tau k)(h,k)d\tau \tag{5.0.29}$$

Let $\delta := D_u W(\phi(p)+u_n) - D_u W(\phi(p_1)+u_n)$

$$\begin{aligned}
\delta &= \int_0^1 D^2 W(u_n(p_1+\tau(p-p_1)), \Omega_1 + \tau(\Omega-\Omega_1))\partial_p u_n(p_1+\tau(p-p_1)), \Omega_1 \\
&\quad +\tau(\Omega-\Omega_1))(p-p_1) + D^2 W(u_n((p_1+\tau(p-p_1)), \Omega_1+\tau(\Omega-\Omega_1)) \\
&\quad \partial_\Omega u_n((p_1+\tau(p-p_1)), \Omega_1+\tau(\Omega-\Omega_1))(\Omega-\Omega_1)d\tau.
\end{aligned}$$

Taking the norm of δ we get

$$\begin{aligned}
||\delta||_{0,s} &\leq \sup_{\tau\in[0,1]} ||D^2 W(u_n(p_1+\tau(p-p_1)), \Omega_1+\tau(\Omega-\Omega_1))\partial_p u_n(p_1+\tau(p-p_1)), \Omega_1 \\
&\quad +\tau(\Omega-\Omega_1))(p-p_1)||_{0,s} + ||D^2 W(u_n((p_1+\tau(p-p_1)), \Omega_1+\tau(\Omega-\Omega_1)) \\
&\quad \partial_\Omega u_n((p_1+\tau(p-p_1)), \Omega_1+\tau(\Omega-\Omega_1))(\Omega-\Omega_1)||_{0,s} \\
&\leq \sup_{\tau\in[0,1]} ||D^2 W(u_n(p_1+\tau(p-p_1)), \Omega_1+\tau(\Omega-\Omega_1))||_{0,s} ||\partial_p u_n(p_1+\tau(p-p_1)), \\
&\quad \Omega_1+\tau(\Omega-\Omega_1))(p-p_1)||_{0,s}
\end{aligned}$$

$$
\begin{aligned}
&+||D^2W(u_n((p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}\\
&\times||\partial_\Omega u_n((p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))(\Omega-\Omega_1)||_{0,s}\\
\leq\ & \sup_{\tau\in[0,1]}||D^2W(u_n(p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}||\partial_p u_n(p_1+\tau(p-p_1),\\
&\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}|p-p_1|+||D^2W(u_n((p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}\\
&\times||\partial_\Omega u_n((p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}|\Omega-\Omega_1|\\
\leq\ & \sup_{\tau\in[0,1]} C_{D^2W}||u_n(p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1)||_{0,s}||\partial_p u_n(p_1+\tau(p-p_1),\\
&\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}\rho_{n+1}+C_{D^2W}||u_n((p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}\\
&\times||\partial_\Omega u_n(p_1+\tau(p-p_1),\Omega_1+\tau(\Omega-\Omega_1))||_{0,s}\rho_{n+1}
\end{aligned}
$$

and as $p_1+\tau(p-p_1)\in(\mathcal{N}_{n+1},\rho_{n+1})$ we call M_4 the supremum of the sequence u_n when $(p,\Omega)\in\mathcal{N}_{n+1}$ and M_2 the supremum of u_n $(p,\Omega)\in(\mathcal{N}_{n+1},\rho_{n+1})$.

$$
\begin{aligned}
||\delta||_{0,s} &\leq \sup_{\tau\in[0,1]} C_{D^2W}M_4M_2\rho_{n+1}+\sup_{\tau\in[0,1]} C_{D^2W}M_4M_2\rho_{n+1}\\
&\leq 2C_{D^2W}M_4M_2\rho_{n+1}.
\end{aligned}
$$

So we can now say that

$$||H_{C(S)}(p,\Omega,u_n)-H_{C(S)}(p_1,\Omega_1,u_n)||_0 \ \leq\ \rho_{n+1}C(\Omega_{max})L_{n+1}^2+2C_{D^2W}M_4M_2\rho_{n+1}.$$

We want this quantity to be less than $\frac{1}{2}\frac{\delta_{n+1}}{8l_{n+1}^2}$ so we require that

$$\rho_{n+1}C(\Omega_{max})L_{n+1}^2\leq\frac{\delta_{n+1}}{32l_{n+1}^2}$$

and

$$2C_{D^2W}M_4M_2\rho_{n+1}\leq\frac{\delta_{n+1}}{32l_{n+1}^2}$$

and we choose L_0 large so that $M_4,M_2\leq 1$. Then with $\rho_{n+1}=\rho_0\frac{\delta_{n+1}}{32.2L_{n+1}^2l_{n+1}^2}$ we get the result we want:

$$||H_{C(S)}(p,\Omega,u_n)-H_{C(S)}(p_1,\Omega_1,u_n)||_0\leq\frac{1}{2}\frac{\delta_{n+1}}{8l_{n+1}^2}. \qquad (5.0.30)$$

We want to extend the bound on the eigenfunctions to the complex plane so we want to prove that $(H_{C(S)}(p,\Omega,u_n)-\xi I)^{-1}$ exists whenever $(p,\Omega)\in(\mathcal{N}_{n+1},\rho_{n+1})$. In the next lines we will denote by $H:=H_{C(S)}(p,\Omega,u_n)$ and $H_1:=H_{C(S)}(p_1,\Omega_1,u_n)$.

$$
\begin{aligned}
(H-\xi I)^{-1} &= (H-H_1+H_1-\xi I)^{-1}\\
&= [(H_1-\xi I)((H_1-\xi I)^{-1}(H-H_1)+I)]^{-1}\\
&= [(H_1-\xi I)^{-1}(H-H_1)+I]^{-1}(H_1-\xi I)^{-1}
\end{aligned}
$$

and formally we have

$$(H-\xi I)^{-1} = \sum_{p\geq 1}[(-1)(H-\xi I)^{-1}(H-H_1)]^p.(H_1-\xi I)^{-1}.$$

We will use the inequality 5.0.30 to get the estimate

$$\begin{aligned} ||(H-\xi I)^{-1}_{C(S)}||_0 &\leq ||\sum_{p\geq 1}[(-1)(H-\xi I)^{-1}_{C(S)}(H-H_1)_{C(S)}]^p||_0.||(H_1-\xi I)^{-1}_{C(S)}||_0 \\ &\leq \sum_{p\geq 1}|||[(-1)(H-\xi I)^{-1}(H-H_1)]||_0^p||(H_1-\xi I)^{-1}_{C(S)}||_0 \end{aligned}$$

Using the lemma (2.1.7) and (5.0.24) we have the inequality

$$|(H_1-\xi I)^{-1}_{C(S)}(i,j)| \leq \frac{2}{\delta_{n+1}} \tag{5.0.31}$$

and as the dimension of $C(S)$ is $4l_{n+1}^2$ we have

$$||(H_1-\xi I)^{-1}_{C(S)}||_0 \leq \frac{8l_{n+1}^2}{\delta_{n+1}}. \tag{5.0.32}$$

We are now able to get an estimate of $||[(-1)(H-\xi I)^{-1}(H-H_1)]||_0$

$$\begin{aligned} ||(-1)(H-\xi I)^{-1}(H-H_1)||_0 &\leq ||(H-\xi I)^{-1}||_0||H-H_1||_0 \\ &\leq \frac{8l_{n+1}^2}{\delta_{n+1}}\frac{1}{2}\frac{\delta_{n+1}}{8l_{n+1}^2} \\ &\leq \frac{1}{2} \end{aligned} \tag{5.0.33}$$

and the Neumann series converges. Thus

$$(H-\xi I)^{-1}_{C(S)} \text{ exists and for } |\xi| \leq 2\delta_{n+1}. \tag{5.0.34}$$

Hence any eigenvalue of H, satisfies

$$|e_i(p,\Omega)| \geq 2\delta_{n+1} \tag{5.0.35}$$

as long as $(p,\Omega) \in (\mathcal{N}_{n+1}, \rho_{n+1})$. □

We have estimated the measure of the region $\mathcal{N}_0$ that has been excised in (5.0.23) to ensure that the eigenvalues of the Hamiltonians are not too singular ($|e_i(p,\Omega)| \geq \delta_n$). We have to solve the Q-equation as well; that is we still have to prove that the intersection of the curve C, $(p,\Omega(p))$ of the Q-equation results in a non-zero measure set. The Taylor expansion of the curve C is computed in chapter 6 in (6.0.16) and is

$$\Omega(p) = \omega_1 + Kp^2 + o(p^2). \tag{5.0.36}$$

We have

$$H_{C(S)}(p,\Omega;u_n) = V(\Omega) + D_u W(\phi(p)+u_n)\mid_{C(S)}. \qquad (5.0.37)$$

For $p = 0$ we have $u_n = 0$ so $H_{C(S)}(p,\Omega;0)\mid_{C(S)} = V(\Omega)\mid_{C(S)}$ hence the eigenvalues e_i are

$$e_i(p,\Omega) = \omega_l^2 - \Omega^2 m^2 \quad \text{for} \quad (l,m) \in C(S) \qquad (5.0.38)$$

so if $(p,\Omega_i(p)) \in Z_i$ we have

$$0 = \Omega^2 m^2 - \omega_l^2 \Leftrightarrow \Omega = \frac{\omega_l}{m} \qquad (5.0.39)$$

hence

$$\Omega_i(0) = \frac{\omega_l}{m} \text{ for some } (l,m) \in C(S). \qquad (5.0.40)$$

We now describe the geometry of the set Z_i which have to be excluded.

Lemma 5.0.7 [9] *There is a constant C_1 for every zero set Z_i, there exists $(l,m) \in C_{l_{n+1}}$ such that Z_i lies between the cones*

$$\frac{\omega_l}{m} \pm \frac{C_1}{L_n^2}|p| \qquad (5.0.41)$$

Proof:
$e_i(0,\omega_l/m) = 0$ for some $x = (l,m) \in C(S)$. We start by proving that the eigenvalue $e_i(p,\Omega)$ is monotone decreasing along the line:

$$(p(s),\Omega^+(s)) = (sp_0, w_l/m + sC/L_n^2) \in \mathcal{N}_0. \qquad (5.0.42)$$

We have

$$\begin{aligned} \frac{d}{ds}H_{C(S)}(p(s),\Omega(s);u_n) &= \frac{d\Omega}{ds}\partial_\Omega H_{C(S)} + \frac{dp}{ds}\partial_p H_{C(S)} \\ &= \frac{C_1}{L_n^2}\partial_\Omega H_{C(S)} + p_0\partial_p H_{C(S)}. \end{aligned} \qquad (5.0.43)$$

The p derivative is bounded by

$$\begin{aligned} \|p_0\partial_p H_{C(S)}\|_{\sigma_0/2,s} &\leq \|D_u^2 W(\phi(p)+u_n)(\partial_p(u_n+\phi))\|_{\sigma_0/2,s}|p_0| \\ &\leq C_{DW}|p_0|\|\partial_p(u_n+\phi))\|_{\sigma_0/2,s} \\ &\leq C_{DW}r_0. \end{aligned} \qquad (5.0.44)$$

We show that we can choose C_1 such that the monotonicity of $V(\Omega)$ dominates the other operators in $\frac{d}{ds}H_{C(S)}$. As in (5.0.6) we have

$$\frac{d}{ds}e_i(p,\Omega) = \langle \psi_i, \frac{d}{ds}H_{C(S)}\psi_i\rangle \qquad (5.0.45)$$

and recalling (5.0.43)

$$\begin{aligned}
\frac{d}{ds}e_i(p(s),\Omega(s)) &= \langle\psi_i,(\frac{C_1}{L_n^2}\partial_\Omega H_{C(S)}+p_0\partial_p H_{C(S)})\psi_i\rangle \\
&= \langle\psi_i,(\frac{C_1}{L_n^2}\partial_\Omega V(\Omega)_{C(S)}+D_u^2W(\phi(p)+u_n)(\partial_\Omega u_n)_{C(S)} \\
&\quad +p_0\partial_p H_{C(S)})\psi_i\rangle \\
&= \langle\psi_i,\frac{C_1}{L_n^2}\partial_\Omega V(\Omega)_{C(S)}\psi_i\rangle+\langle\psi_i,D_u^2W(\phi(p)+u_n)(\partial_\Omega u_n)_{C(S)} \\
&\quad +p_0\partial_p H_{C(S)})\psi_i\rangle. \qquad (5.0.46)
\end{aligned}$$

Using now (5.0.4) and (5.0.44) we deduce that the second term is very small. Furthermore recalling (5.0.3) we can see that taking C_1 large the variations of $\frac{C_1}{L_n^2}\partial_\Omega V(\Omega)_{C(S)}$ will dominate the variation of $p\partial_p H_{C(S)}$ and hence

$$\begin{aligned}
\frac{d}{ds}e_i(p(s),\Omega(s)) &\leq -2\Omega_{min}L_n^2\frac{C_1}{L_n^2}\langle\psi_i,\psi_i\rangle+C_Wr_0 \\
&\leq -2\Omega_{min}C_1+C_Wr_0\leq 0. \qquad (5.0.47)
\end{aligned}$$

Thus the eigenvalues are decreasing along the lines $(sp_0, w_l/m+sC/L_n^2)$.
Now to show that Z_i lies to the left of the cone $\frac{\omega_l}{m}+\frac{C_1}{L_n^2}|p|$ consider:

$$\begin{aligned}
e_i(p(s),\Omega^+(s)) &= e_i(p(s),\Omega^+(s))-e_i(0,\omega_l/m) \\
&= e_i(p(s),\Omega^+(s))-e_i(p(0),\Omega(0)) \\
&\leq \int_0^s\frac{d}{dt}e_i(p(t),\Omega(t)dt \\
&\leq 0
\end{aligned}$$

$e_i(p(s),\Omega^+(s))\leq 0=e_i(p(s),\Omega_i(p(s)))$ where $(p(s),\Omega_i(p(s)))\in Z_i$ but for fixed p $e_i(p,\Omega)$ is decreasing in Ω so $\Omega_i(p(s))\leq\Omega^+(s)$. Hence the set Z_i lies to the right of the cone $w_l/m-sC_1/L_n^2$. The operator $H_{C(S)}$ is monotone increasing along the curve

$$(p(s),\Omega^-(s))=(sp_0,w_l/m-sC_1/L_n^2)\in\mathcal{N}_0.$$

In the same way we have

$$\begin{aligned}
\frac{d}{ds}H_{C(S)}(p(s),\Omega^-(s);u_n) &= \frac{d\Omega}{ds}\partial_\Omega H_{C(S)}+\frac{dp}{ds}\partial_p H_{C(S)} \\
&= -\frac{C_1}{L_n^2}\partial_\Omega H_{C(S)}+p_0\partial_p H_{C(S)}
\end{aligned}$$

so as in (5.0.46) and (5.0.47) we have $\frac{d}{ds}e_i(p(s),\Omega^-(s))\geq 0$.

$$\begin{aligned}
e_i(p(s),\Omega^-(s)) &= e_i(p(s),\Omega^-(s))-e_i(0,\omega_l/m) \\
&= e_i(p(s),\Omega^-(s))-e_i(p(0),\Omega^-(0))
\end{aligned}$$

and as e_i is increasing along the line $(p(s), \Omega^-(s))$

$$\begin{aligned} e_i(p(s), \Omega^-(s)) &\geq \int_0^s \frac{d}{dt} e_i(p(t), \Omega(t) dt \\ &\geq 0 \end{aligned}$$

$e_i(p(s), \Omega^-(s)) \geq 0 = e_i(p(s), \Omega)$ where $(p(s), \Omega) \in Z_i$, but for fixed p, $e_i(p, \Omega)$ is decreasing in Ω so $\Omega^- \leq \Omega$. We can then conclude that Z_i lies between the cones

$$\frac{\omega_l}{m} \pm \frac{C_1}{L_n^2}|p|. \tag{5.0.48}$$

The sets Z_i are restricted by paraboloids when they intersect $\mathcal{N}_{n+1}$, we have the lemma:

Lemma 5.0.8 [9] *Let (p_1, Ω_1) and (p_2, Ω_2) lying in $Z_i \cap \mathcal{N}_{n+1}$, then we have the following property*

$$|\Omega_1 - \Omega_2| < \frac{C_1}{L_n^2}||p_2|^2 - |p_1|^2|. \tag{5.0.49}$$

Proof:
Let $(p_1, \Omega_1) \in Z_i \cap \mathcal{N}_{n+1}$ and consider a path:

$$(p(s), \Omega_\pm(s)) = (p_1 + sp_1/|p_1|, \Omega_1 \pm C_1(|p(s)|^2 - |p_1|^2)/2L_n^2) \tag{5.0.50}$$

$$\begin{aligned} \frac{d}{ds} e_i &= \partial_\Omega e_i \frac{d\Omega_\pm}{ds} + \partial_p e_i \frac{dp}{ds} \\ &= \pm \langle \psi_i, \partial_\Omega H_C \psi_i \rangle \frac{C_1}{L_n^2} 2 \langle dp(s)/ds, p(s) \rangle + \\ &\quad \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \\ &= \pm \langle \psi_i, \partial_\Omega H_C \psi_i \rangle \frac{C_1}{L_n^2} 2 \langle p_1/|p_1|, p_1 + sp_1/|p_1| \rangle \\ &\quad + \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \\ &= \pm \langle \psi_i, \partial_\Omega H_C \psi_i \rangle \frac{C_1}{L_n^2} (|p_1|^2/|p_1| + s) + \\ &\quad \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \\ &= \pm \langle \psi_i, \partial_\Omega H_C \psi_i \rangle \frac{C_1}{L_n^2} (|p_1| + s) + \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \\ &= \pm \langle \psi_i, \partial_\Omega H_C \psi_i \rangle \frac{C_1}{L_n^2} |p(s)| + \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1|, \end{aligned}$$

we then take the line $(p(s), \Omega_+(s)) = (p_1 + sp_1/|p_1|, \Omega_1 + C_1(|p(s)|^2 - |p_1|^2)/2L_n^2)$

$$\begin{aligned} \frac{d}{ds} e_{i_+} &= \langle \psi_i, \partial_\Omega H_C \psi_i \rangle \frac{C_1}{L_n^2} |p(s)| + \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \\ &= \partial_\Omega e_i \frac{C_1}{L_n^2} |p(s)| + \langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1|. \end{aligned} \tag{5.0.51}$$

We now start by getting an estimate of the second term of (5.0.51):

$$|\langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \,| = |\langle \psi_i, \partial_p H_C \psi_i \rangle|$$

$$\begin{aligned} |\langle \psi_i, \partial_p H_C \psi_i \rangle| &\leq \langle \psi_i, D_u^2 W(\phi(p) + u_n)(\partial_p(u_n + \phi(p))\psi_i \rangle \\ &\leq ||\psi_i||_{0,0} ||D_u^2 W(\phi(p) + u_n)\partial_p(u_n + \phi(p))\psi_i||_{0,0} \\ &\leq ||D_u^2 W(\phi(p) + u_n)\partial_p(u_n + \phi(p))\psi_i||_{0,0} \\ &\leq \frac{C_{op} C_{D^2W}}{1 - e^{\sigma_0/2}} ||W(\phi(p) + u_n)\partial_p(\phi(p) + u_n)||_{\sigma_0/2,s} ||\psi_i||_{0,0} \\ &\leq \frac{C_{op} C_{D^2W}}{1 - e^{\sigma_0/2}} ||\phi(p) + u_n||_{\sigma_0/2,s} ||\partial_p(\phi(p) + u_n)||_{\sigma_0/2,s} \end{aligned}$$

hence

$$|\langle \psi_i, \partial_p H_C \psi_i \rangle p_1/|p_1| \,| \leq C_2 |p(s)|. \tag{5.0.52}$$

Recalling (5.0.12) we deduce that

$$\begin{aligned} \frac{d}{ds} e_{i_+} &\leq -\Omega_{min} M_3 L_n^2 (C_1/L_n^2)|p(s)| + C_2|p(s)| \\ &\leq (-\Omega_{min} M_3 C_1 + C_2)|p(s)|. \end{aligned} \tag{5.0.53}$$

If the constant C_1 is chosen sufficiently large i.e $M_3 \Omega_{min} C_1 > C_2$ then $de_i/ds \leq 0$ along the curve $(p(s), \Omega_+(s))$ and

$$e_i(p(s), \Omega_+(s)) - e_i(p_1, \Omega_1) \leq \int_0^s \frac{d}{dt} e_i(p(t), \Omega(t) dt \leq 0$$

so

$$e_i(p(s), \Omega_+(s)) \leq 0 = e_i(p(s), \Omega_i(p(s)) \tag{5.0.54}$$

hence $\Omega_+(s) \geq \Omega_i(p(s))$ and Z_i remains to the left of $(p(s), \Omega_+(s))$. Similarly Z_i lies as well to the right of the curve $(p(s), \Omega_-(s))$.
Let (p_1, Ω_1) and $(p_2, \Omega_2) \in Z_i \cap \mathcal{N}_{n+1}$ with $|p_1| \leq |p_2|$. We can assume that p_1 and p_2 are parallel since Z_i is invariant under T_θ so on the curve $(p(s), \Omega_\pm(s))$ there exists t such that $p(t) = p_2$. If $(p(t), \Omega_\pm(t)) \in \mathcal{N}_{n+1}$ then

$$\Omega_-(t) \leq \Omega_2 \leq \Omega_+(t) \tag{5.0.55}$$

from which we can deduce that

$$\begin{aligned} \Omega_-(t) &\leq \Omega_2 \\ \Omega_1 - (C_1/2L_n^2)(|p_2|^2 - |p_1|^2) &\leq \Omega_2 \ (p(t) = p_2) \\ \Omega_1 - \Omega_2 &\leq C_1/2L_n^2(|p_2|^2 - |p_1|^2). \end{aligned} \tag{5.0.56}$$

From (5.0.55) we have:

$$\Omega_2 \leq \Omega_+(t) \tag{5.0.57}$$

then

$$\begin{aligned} \Omega_2 &\leq C_1(|p_2|^2 - |p_1|^2)/2L_n^2 \\ \Omega_2 - \Omega_1 &\leq C_1(|p_2|^2 - |p_1|^2)/2L_n^2 \end{aligned}$$

hence

$$|\Omega_1 - \Omega_2| \leq C_1/L_n^2 ||p_2|^2 - |p_1|^2|. \quad \Box \tag{5.0.58}$$

Let two points $(p_1, \Omega(p_1))$ and $(p, \Omega(p))$ on the curve C that may have been excised to ensure that the Hamiltonians have eigenvalues such that $|e_i(p, \Omega)| \geq \delta_n$. Since the lemma 5.0.5 we have the inequalities:

$$|\Omega_i(p_1) - \Omega(p_1)| \leq C\delta_{n+1}/L_n^2 \tag{5.0.59}$$

and

$$|\Omega_i(p) - \Omega(p)| \leq C\delta_{n+1}/L_n^2. \tag{5.0.60}$$

As $(p_1, \Omega(p_1))$ and $(p, \Omega(p))$ are on the curve C we can use the Taylor expansion (6.0.16) to deduce that

$$|\Omega(p) - \Omega(p_1)| \geq \frac{K}{2}||p|^2 - |p_1|^2|. \tag{5.0.61}$$

Since lemma 5.0.8, we have also the relation

$$|\Omega_i(p_1) - \Omega_i(p)| \leq \frac{C_1}{L_n^2}||p|^2 - |p_1|^2| \tag{5.0.62}$$

on Z_i so we will get an estimate of $||p|^2 - |p_1|^2|$ and deduce an upper bound of the measure of the set of points on the curve C excised.

$$\begin{aligned} |\Omega(p) - \Omega(p_1)| &= |\Omega(p) - \Omega_i(p) + \Omega_i(p) - \Omega_i(p_1) + \Omega_i(p_1) - \Omega(p_1)| \\ &\leq C\delta_{n+1}/L_n^2 + \frac{C_1}{L_n^2}||p|^2 - |p_1|^2| + C\delta_{n+1}/L_n^2. \end{aligned} \tag{5.0.63}$$

Now for L_0 large enough we have

$$\frac{C_1}{L_n^2} \leq \frac{|K|}{4} \tag{5.0.64}$$

so we have

$$|\Omega(p) - \Omega(p_1)| \quad \leq \quad C\delta_{n+1}/L_n^2 + \frac{|K|}{4}\left||p|^2 - |p_1|^2\right| + C\delta_{n+1}/L_n^2$$

and using now equation (5.0.61) we have

$$\frac{|K|}{2}\left||p|^2 - |p_1|^2\right| \leq C\delta_{n+1}/L_n^2 + \frac{|K|}{4}\left||p|^2 - |p_1|^2\right| + C\delta_{n+1}/L_n^2 \tag{5.0.65}$$

hence

$$\frac{|K|}{4}\left||p|^2 - |p_1|^2\right| \leq 2C\delta_{n+1}/L_n^2. \tag{5.0.66}$$

We have the estimates

$$\left||p|^2 - |p_1|^2\right| \leq \frac{8}{|K|}C\delta_{n+1}/L_n^2$$

$$-\frac{8}{|K|}C\delta_{n+1}/L_n^2 \leq |p|^2 - |p_1|^2 \leq \frac{8}{|K|}C\delta_{n+1}/L_n^2$$

$$-\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2 < |p|^2 < \frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2. \tag{5.0.67}$$

If

$$-\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2 > 0$$

then we can say that

$$\sqrt{-\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2} < |p| < \sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2}.$$

The length of this interval is

$$\begin{aligned} l &= \sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2} - \sqrt{-\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2} \\ &= \frac{\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2 - (-\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2)}{\sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2} + \sqrt{-\frac{8}{|K|}C\delta_{n+1}/L_n^2 + |p_1|^2}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\frac{8}{|K|}C\delta_{n+1}/L_n^2}{\sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2+|p_1|^2}+\sqrt{-\frac{8}{|K|}C\delta_{n+1}/L_n^2+|p_1|^2}} \\
&\leq \frac{2\frac{8}{|K|}C\delta_{n+1}/L_n^2}{\sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2}} \\
&\leq 2\sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2}. \qquad (5.0.68)
\end{aligned}
$$

We sum now for all singular sites and all possible eigenvalue e_i and get an estimate of measure of the excised set which we denote by $Meas$

$$Meas \leq L_{n+1}^2 l_n^2 2\sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2}. \qquad (5.0.69)$$

And if

$$-\frac{8}{|K|}C\delta_{n+1}/L_n^2+|p_1|^2<0 \qquad (5.0.70)$$

then from (5.0.67) we have

$$|p|^2 \leq \frac{8}{|K|}C\delta_{n+1}/L_n^2+|p_1|^2$$

and now using (5.0.70) we deduce

$$|p|^2 \leq 2\frac{8}{|K|}C\delta_{n+1}/L_n^2 \qquad (5.0.71)$$

hence

$$|p| \leq \sqrt{2\frac{8}{|K|}C\delta_{n+1}/L_n^2}. \qquad (5.0.72)$$

So the measure of the excised set satisfies also (5.0.69). We have to prove that at the end of the iteration there is a non-zero set of parameters for which we have solution .
Let $V_n := L_{n+1}^2 l_n^2 2\sqrt{\frac{8}{|K|}C\delta_{n+1}/L_n^2}$

$$
\begin{aligned}
V_n &= (2^{n+1}L_0)^2 4(2^nL_0)^{2\beta}\sqrt{C(2^nL_0)^{-\alpha}8(2^nL_0)^{-2}4L_0^{\nu}} \\
&= 4C2^{2(n+1)}L_0^2 4 2^{2n\beta}L_0^{2\beta}2^{-n\alpha/2}L_0^{-\alpha/2}2^{-n}L_0^{-1}4L_0^{\nu/2} \\
&= 64CL_0^{2\beta+2+\nu/2-\alpha/2}2^{2n+2n\beta-\alpha/2-n} \\
&= 64CL_0^{2\beta+2+\nu/2-\alpha/2}2^{n(1+\beta-\alpha/2)} \\
&\leq L_0^{2\beta+1+2+\nu/2-\alpha/2}2^{n(1+\beta-\alpha/2)}. \qquad (5.0.73)
\end{aligned}
$$

We first require that

$$1+\beta-\alpha/2 \leq -1 \tag{5.0.74}$$

and as we want an the measure of the excised set to be less than r_0 impose the condition:

$$\sum_n V_n < \frac{1}{10} r_0 \tag{5.0.75}$$

and we will require that

$$2\beta+2+\nu/2-\alpha/2+1+1 < -(2+\eta). \tag{5.0.76}$$

That will be enough to ensure that $\sum_n V_n < \frac{1}{10} r_0$. We now prove that the measure of the set excised to ensure the separation of sites has a measure less than r_0. Let (j,k) a site such that $l_n \leq |j|+|k| \leq 4l_{n+1}$, and two frequencies $\Omega(p_1)$ and $\Omega(p_2)$ eventually excised so that we have the separation of sites. These frequencies satisfy:

$$|k\Omega - j| \leq \frac{d}{(|j|+|k|)^\tau}. \tag{5.0.77}$$

But we have proved in (4.2.80) that this implies

$$|\Omega(p_1)-\Omega(p_2)| \leq \frac{2d}{l_n^\tau}. \tag{5.0.78}$$

On C we have also the estimate:

$$||p_1|^2-|p_2|^2| \leq \frac{2}{|K|}|\Omega(p_1)-\Omega(p_2)| \tag{5.0.79}$$

then

$$||p_1|^2-|p_2|^2| \leq \frac{2}{|K|}\frac{2d}{l_n^\tau} \tag{5.0.80}$$

so as before we deduce that the length l of the segment has the estimate:

$$l \leq 4\sqrt{\frac{2}{|K|}\frac{2d}{l_n^\tau}}. \tag{5.0.81}$$

Now to get an estimate of the measure of the excised set. There exist at most $16l_{n+1}^2$ sites (j,k) such that $l_n \leq |j|+|k| \leq 4l_{n+1}$. Then the measure of the excised set is bounded by

$$L = 128 l_{n+1}^2 \sqrt{\frac{1}{|K|}\frac{d}{l_n^\tau}}. \tag{5.0.82}$$

We want the serie of general term $128 l_{n+1}^2 \sqrt{\frac{1}{|K|}\frac{d}{l_n^\tau}}$ to be less than $\frac{1}{10} r_0 = \frac{1}{10} L_0^{-2-\eta}$ so we will prove that for τ large enough and L_0 the series:

$$s_{N_0} = \sum_{n \geq N_0+1} 128 l_{n+1}^2 \sqrt{\frac{1}{|K|}\frac{d}{l_n^\tau}} \leq \frac{1}{10} L_0^{-2-\eta}. \tag{5.0.83}$$

This is equivalent to the condition

$$s'_{N_0} = 10 \sum_{n \geq N_0+1} 128 l_{n+1}^2 \sqrt{\frac{1}{|K|}\frac{d}{l_n^\tau}} \leq L_0^{-2-\eta}. \tag{5.0.84}$$

$$\begin{aligned} s'_{N_0} &= \sum_{n \geq N_0+1} \sqrt{12802^{2\beta}} \sqrt{d} L_0^{\nu/2} \sqrt{(L_0 2^n)^{4\beta - \tau\beta}} \\ &= \sqrt{1280} \sqrt{d} L_0^{\nu/2} 2^{2\beta} \sqrt{(L_0 2^{N_0+1})^{4\beta-\tau\beta}} \\ &\quad \times \sum_{n=0} \sqrt{(L_0 2^n)^{4\beta-\tau\beta}}. \end{aligned} \tag{5.0.85}$$

Now

$$N_0 = [\frac{\ln L_0}{\beta}] \tag{5.0.86}$$

so $N_0 + 1 \geq \frac{\ln L_0}{\beta}$ and as $4\beta - \tau\beta < 0$ for τ large we deduce that

$$\begin{aligned} s'_{N_0} &\leq \sqrt{2560} \sqrt{d} L_0^{\nu/2} \sqrt{(L_0 2^{\frac{\ln L_0}{\beta}})^{4\beta-\tau\beta}} \sum_{n=0} \sqrt{(L_0 2^n)^{4\beta-\tau\beta}} \\ &\leq \sqrt{2560} \sqrt{d} L_0^{\nu/2} L_0^{1/2} \sqrt{e^{\ln 2 \; 2 \ln L_0 - \ln 2 \ln L_0 \tau}} \sum_{n=0} \sqrt{(L_0 2^n)^{4\beta-\tau\beta}} \\ &\leq \sqrt{2560} \sqrt{d} L_0^{\nu/2} L_0^{1/2} \sqrt{L_0^{2\ln 2} L_0^{-\tau \ln 2}} \sqrt{L_0^{4\beta-\tau\beta}} \sum_{n=0} \sqrt{2^{n(\beta-\tau 3)}} \\ &\leq L_0 L_0^{-0.5} L_0^{\nu/2+1/2+1/2} L_0^{-\tau \ln 2}, \end{aligned} \tag{5.0.87}$$

for L_0 large enough.
We want

$$L_0^{\nu/2+3/2} L_0^{-\tau \ln 2} \leq L_0^{-2-\eta} \tag{5.0.88}$$

so we choose

$$\tau \geq \frac{\nu/2 + 4 + \eta}{\ln 2}. \square \tag{5.0.89}$$

Chapter 6

The Q-equation

The P-equation has been solved, and after having extracted from $\mathcal{N}_0$ the values of parameters providing the small eigenvalues we remain with a Cantor set. The Lyapounov-Schmidt scheme requires that we now solve the Q-equation. Let $u(p,\Omega)$ the solution of the P-equation. We have at this stage p and Ω independent and now a simple application of the Morse-Lemma will restrict the existence of solution for a curve $\Omega = \Omega(p)$. We will prove that there are solutions for $p = (p_1, 0), p_1 \in \mathbb{R}$. We denote by

$$G(p,\Omega) = Q(W(\phi(p) + u(p,\Omega)) + V(\Omega)\phi(p)). \tag{6.0.1}$$

Using lemma 3.1.2 we have the property

$$T_\theta G(p,\Omega) = G(T_\theta p, \Omega)$$

so the zero set of G is T_θ invariant. We also have the property

$$G(\overline{p},\overline{\Omega})(1,-1) = \overline{G(p,\Omega)(1,1)} \tag{6.0.2}$$

so we can just study the zero set of $G(p,\Omega)(1,1)$. At $p=0$ we have $u_0 = 0$ and hence $u = 0$. This means that we have a trivial branch of solution $p = 0$, $\Omega \in [\omega_1 - r_0, \omega_1 + r_0]$. We now find another family of nontrivial solutions.

$$\begin{aligned} G(p,\Omega)(1,1) &= W(\phi(p) + u(p,\Omega))(1,1) + (\omega_1^2 - \Omega^2 k^2)\frac{1}{2}p_1 \\ \frac{\partial G}{\partial p}(p,\Omega)(1,1) &= DW(\phi(p) + u(p,\Omega))(\partial_p\phi(p) + \partial_p u(p,\Omega))(1,1) - 2\Omega\frac{\partial\Omega}{\partial p}\frac{1}{2}p_1 \\ &\quad + \frac{\omega_1^2 - \Omega^2}{2}. \end{aligned}$$

At $p = 0, \Omega = \omega_1$ we have $u_0 = 0$ and hence $u = 0$.

$$\begin{aligned}
\frac{\partial G}{\partial p}(0,\omega_1)(1,1) &= DW(0)(\partial_p\phi(0) + \partial_p u(0,\omega_1))(1,1) - 2\omega_1\frac{\partial\Omega}{\partial p}\frac{1}{2}.0 \\
&\quad + \frac{\omega_1^2 - \omega_1^2}{2} \\
&= 0 \qquad (6.0.3)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial G}{\partial \Omega}(p,\Omega)(1,1) &= DW(\phi(p) + u(p,\Omega))(\partial_\Omega\phi(p) + \partial_\Omega u(p,\Omega_1))(1,1) - 2\Omega\frac{1}{2}p_1 \\
&= DW(\phi(p) + u(p,\Omega))(\partial_\Omega u(p,\Omega))(1,1) - 2\Omega\frac{1}{2}p_1 \qquad (6.0.4)
\end{aligned}$$

and at $p = 0, \Omega = \omega_1$, $u = 0$ hence again

$$\frac{\partial G}{\partial \Omega}(0,\omega_1)(1,1) = 0. \qquad (6.0.5)$$

$$\begin{aligned}
\frac{\partial^2 G}{\partial \Omega^2}(p,\Omega)(1,1) &= \frac{\partial}{\partial\Omega}DW(\phi(p) + u(p,\Omega))(\partial_\Omega u(p,\Omega))(1,1) - 2\Omega\frac{1}{2}p_1 \\
&= D^2W(\phi(p) + u(p,\Omega))(\partial_\Omega\phi(p) + \partial_\Omega u(p,\Omega))(\partial_\Omega u(p,\omega_1))(1,1) \\
&\quad + DW(\phi(p) + u(p,\Omega))(\partial_\Omega\Omega u(p,\Omega))(1,1) - p_1.
\end{aligned}$$

At $p = 0, \Omega = \omega_1$, $u = 0$ so we have

$$\frac{\partial^2 G}{\partial^2 \Omega}(0,\omega_1)(1,1) = 0. \qquad (6.0.6)$$

$$\begin{aligned}
T_\pi G(p,\Omega)(1,1) &= G(T_\pi p,\Omega)(1,1) \\
&= W(\phi(-p) + u(-p,\Omega)) + V(\Omega)\phi(-p))
\end{aligned}$$

and $G(p,\Omega)$ is odd in p thus $\frac{\partial^2 G}{\partial \Omega^2}(0,\omega_1)(1,1) = 0$ and $\frac{\partial^2 G}{\partial p^2}(0,\omega_1)(1,1) = 0$.

$$\begin{aligned}
\frac{\partial^2 G}{\partial p \partial \Omega}(p,\Omega) &= D^2W(\phi(p) + u(p,\Omega))(\partial_p\phi(p) + \partial_p u(p,\Omega))(\partial_\Omega u(p,\Omega))(1,1) \\
&\quad + DW(\phi(p) + u(p,\Omega))(\partial_p\partial_\Omega u(p,\Omega))(1,1) - \Omega. \qquad (6.0.7)
\end{aligned}$$

So at $\Omega = \omega_1$ and $p = 0$ we have

$$\frac{\partial^2 G}{\partial p \partial \Omega}(0,\omega_1) = -\omega_1 \neq 0 \qquad (6.0.8)$$

Hence the Hessian is invertible and we can apply the Morse Lemma and deduce that there exists a curve $\Omega(p)$ in a neighborhood of the origin on which $G(p,\Omega(p)) = 0$. This means that there exists ϵ such that for

$$|p| \leq \epsilon, \; G(p,\Omega(p)) = 0. \tag{6.0.9}$$

$r_0 \leq \epsilon$ for L_0 large enough.
We denote by $\Omega_p := \frac{\partial \Omega}{\partial p}$. Let us now compute the derivatives of $\Omega(p)$.
At $p = 0, \Omega = \omega_1$ we have $u = 0$ so $\Omega(0) = \omega_1$.

$$\partial_p G(p,\Omega(p)) = DW(\phi(p) + u(p,\Omega))(\partial_p\phi(p) + \partial_p u(p,\Omega)) - 2\Omega\Omega_p \frac{1}{2} p + \frac{\omega_1 - \Omega^2}{2}$$

$$\begin{aligned}\partial_{pp}G(p,\Omega(p))(1,1) &= D^2W(\phi(p) + u(p,\Omega))(\partial_p\phi(p) + \partial_p u(p,\Omega))(\partial_p\phi(p) \\ &\quad +\partial_p u(p,\Omega)) + DW(\phi(p) + u(p,\Omega))(\partial_{pp}\phi(p) \\ &\quad +\partial_{pp}u(p,\Omega))(1,1) - \Omega_p^2 p - \Omega\Omega_{pp}p - \Omega_p\Omega \\ &\quad -\Omega\Omega_p \end{aligned} \tag{6.0.10}$$

$$\partial_{pp}G(0,\Omega(0))(1,1) = -2\omega_1\Omega_p(0,\omega_1). \tag{6.0.11}$$

For $|p| \leq \epsilon$ we have $G(p,\Omega(p))(1,1) = 0$ hence all the derivatives of G with refer to p also vanish and we deduce that

$$-2\omega_1\Omega_p(0,\omega_1) = 0$$

thus

$$\Omega_p(0,\omega_1) = 0. \tag{6.0.12}$$

We now evaluate the second derivative of $\Omega(p)$. Let denote by
$G_{p^3} = \partial_{ppp}G(p,\Omega(p))(1,1)$

$$\begin{aligned} G_{p^3} &= D^3W(\phi(p) + u(p,\Omega))(\partial_p\phi(p) + \partial_p u(p,\Omega))^3 \\ &\quad +2D^2W((\phi(p) + u(p,\Omega))(\partial_{pp}\phi(p)\phi(p) + \partial_{pp}u(p,\Omega)) \\ &\quad +\partial_{pp}u(p,\Omega)) + (-\Omega_{pp}\Omega_p p - \Omega_p(\Omega_p p)_p - \Omega_p(\Omega_{pp}p + \Omega_p) - \Omega(\Omega_{ppp}p \\ &\quad +\Omega_{pp} + \Omega_{pp}) - \Omega_p\Omega_p + D^2W(\phi(p) + u(p,\Omega))(\partial_p\phi(p) + \partial_p u(p,\Omega))(\partial_{pp}\phi(p) \\ &\quad +\partial_{pp}u(p,\Omega)) + DW(\phi(p) + u(p,\Omega))(\partial_{ppp}\phi(p) + \partial_{ppp}u(p,\Omega)) \end{aligned} \tag{6.0.13}$$

so we have

$$\partial_{ppp}G(0,\omega_1))(1,1) = D^3W(0)(\partial_p\phi(0) + \partial_p u(0,\omega_1))^3 - \omega_1\Omega_{pp}(0) \tag{6.0.14}$$

and hence

$$\Omega_{pp}(0,\omega_1) = \frac{1}{\omega_1} D^3W(0)(\partial_p\phi(0))^3(1,1) \tag{6.0.15}$$

We now require the condition $K = D^3W(\phi(0))(\partial_p\phi(0))^3(1,1) \neq 0$ which we call the Twist condition to ensure that in the process of excising the parameters we end with a non-zero mesure set. For any $\nu > 0$ we can find L_0 large enough such that $|K| \geq L_0^{-\nu}$. The curve $\Omega(p)$ has the following representation

$$\Omega(p) = \omega_1 + Kp^2 + o(p^2). \tag{6.0.16}$$

To conclude we now have an estimate between the function $\tilde{u}(x,\xi)$ and its lattice representation.

Lemma 6.0.9 *For any $\sigma > 0$, there exists $C(\sigma,s)$ such that we have the relation*

$$\sup_{x,\xi\in\mathbb{R}} |\tilde{u}(x,\xi)| \leq C(\sigma,s)||u||_{\sigma,s} \tag{6.0.17}$$

Proof:
Let $x,\xi \in \mathbb{R}$ we have

$$\begin{aligned}
|\tilde{u}(x,\xi)| &= |\sum_{j,k} u(j,k)\psi_j(x)e^{ik\xi}| \\
&\leq \sum_{j,k} |u(j,k)| \sum_m \psi_j(m)e^{imx}||e^{ik\xi}| \\
&\leq \sum_{j,k} |u(j,k)| \sum_m Ce^{-\sigma_*|j-|m||}|e^{imx}| \\
&\leq \sum_{j,k} |u(j,k)| \sum_m Ce^{-\sigma_*|j-|m||} \\
&\leq C\sum_{j,k} |u(j,k)| \\
&\leq C\sum_{j,k} [|u(j,k)|(1+(|l|+|k|)^2)^{s/2}e^{\sigma(|l|+|k|)}][(1+(|l|+|k|)^2)^{-s/2}e^{-\sigma(|l|+|k|)}]
\end{aligned}$$

and now applying Cauchy-Schwarz we obtain

$$\begin{aligned}
|\tilde{u}(x,\xi)| &\leq C\sqrt{\sum_{j,k}|u(j,k)|^2(1+(|l|+|k|)^2)^s e^{2\sigma(|l|+|k|)}} \\
&\quad \sqrt{\sum_{j,k}(1+(|l|+|k|)^2)^{-s}e^{-2\sigma(|l|+|k|)}} \\
&\leq C||u||_{\sigma,s}. \quad \square
\end{aligned} \tag{6.0.18}$$

We started our proof of the convergence of the solution by writing a decomposition of $l^2(\mathbb{N}\times\mathbb{Z})$ into:

$$l^2(\mathbb{N}\times\mathbb{Z}) = l^2(N)\oplus l^2((\mathbb{N}\times\mathbb{Z})\setminus N). \tag{6.0.19}$$

And as in (3.0.12) any function $\hat{u}$ can be decompose as:

$$\hat{u}(x) = \phi(p)(x) + u(x). \quad (x\in\mathbb{N}\times\mathbb{Z}) \tag{6.0.20}$$

The restriction of the solution to $l^2((\mathbb{N}\times\mathbb{Z})\setminus N)$ was called $u(p,\Omega)$ so the solution of (1.0.2) is $\phi(p)+u(p,\Omega(p))$. We enunciate the final theorem of the thesis. After having done all the excisions for the separation of sites and for the singular eigenvalues of the Hamiltonians at each iterative step, the parameters which remain will provide the solutions of (1.0.2):

Theorem 6.1 [9] *Let $|p|<r_0$, there exists a Cantor set $\mathcal{M}$ of measure non-zero such that for $p\in\mathcal{M}$ there exists a periodic solution of (1.0.2) denoted $\widetilde{\phi(p)}+\widetilde{u}(p,\Omega(p))$ which satisfies the following estimate:*

$$|\widetilde{u}(p,\Omega(p))| < C|p|^2. \tag{6.0.21}$$

and

$$|\Omega(p)-\omega_1| < C|p|^2, \tag{6.0.22}$$

where $\widetilde{\phi(p)} = \frac{p}{2}\psi_1(x)e^{i\Omega t} + \frac{p}{2}\psi_1(x)e^{-i\Omega t}$

Proof:
Now we have $\sigma_n \geq \frac{\sigma_0}{2}$ $\forall n$ and recalling (3.2.90) we have

$$||u_n||_{\sigma_0/2,s} \leq ||p||^2. \tag{6.0.23}$$

Now using the lemma 6.0.9 we have

$$\sup_{x,\xi\in\mathbb{R}} |\widetilde{u}(x,\xi)| \leq C(\frac{\sigma_0}{2},s)||u||_{\frac{\sigma_0}{2},s}. \tag{6.0.24}$$

Hence we can conclude that

$$\sup_{x,\xi\in\mathbb{R}} |\widetilde{u}(x,\xi)| \leq C||p||^2, \tag{6.0.25}$$

and using the lemma 3.0.7 we can conclude that

$$\sup_{x,\xi\in\mathbb{R}} |\widetilde{u}(x,\xi)| \leq C|p|^2. \tag{6.0.26}$$

The relation (6.0.22) is deduced by using (6.0.16) $\square$.

Chapter 7

Estimates on the eigenfunctions

This chapter is devoted to the proof of a technical estimate of the exponentional decay of the eigenfunctions.
The following theorem was enunciated without proof in [9], and we provide a proof:

Lemma 7.0.10 *Let ψ_n be a normalized eigenfunction of the Sturm-Liouville problem. ψ_n admits an expansion of the form*

$$\psi_n(x) = \sum_{p\geq 1} \hat{\psi}_n(p) \sin px$$

which converges uniformly for $x \in [0, \pi]$ and its derivative has the expansion

$$\psi'_n(x) = \sum_{p\geq 1} p\hat{\psi}_n(p) \cos px \tag{7.0.1}$$

which also converges uniformly for $x \in [0, \pi]$

Proof: Let ψ_n is an eigenfuction of the Sturm-Liouville problem with Dirichlet boundary condition, ψ_n satisfies $\psi_n(0) = \psi_n(\pi) = 0$ and

$$\psi_n(x) = \sum_{p\geq 1} \hat{\psi}_n(p) \sin px$$

where $\hat{\psi}_n \in l^2(\mathbb{Z})$. Writing the sine function in the exponential form, we can rewrite this as

$$\psi_n(x) = \sum_{p\geq 1} \hat{\psi}_n(p) \frac{1}{2i}(e^{ipx} - e^{-ipx})$$

$$\psi_n(x) = \sum_{p\geq 1} \hat{\psi}_n(p)\frac{1}{2i}e^{ipx} - \hat{\psi}_n(p)\frac{1}{2i}e^{-ipx}$$
$$\psi_n(x) = \sum_{m\in\mathbb{Z}} \chi_n(m)e^{imx}$$

where

$$\chi_n(m) = \frac{1}{2i}\hat{\psi}_n(m) \text{ if } m = p \geq 1,\ \chi_n(0) = 0$$
$$\chi_n(m) = -\chi_n(-m) \text{ if } m \leq -1 \tag{7.0.2}$$
$$\chi_n(m) = \frac{1}{2\pi}\int_0^{2\pi} \psi_n(x)e^{-imx}dx.$$

We consider only the case $m \geq 0$ since for $m \leq -1$ we can use (7.0.2) to get the estimate for nonpositive m . This integral over the rectangular contour $0 \longrightarrow -i\overline{\sigma} \longrightarrow 2\pi - i\overline{\sigma} \longrightarrow 2\pi \longrightarrow 0$ is zero because the integrand is analytic.
Denote by $\mathcal{C}_1$ the straight line $0 \longrightarrow -i\overline{\sigma}$, by $\mathcal{C}_2$ the straight line $-i\overline{\sigma} \longrightarrow 2\pi - i\overline{\sigma}$, by $\mathcal{C}_3$ the straight line $2\pi - i\overline{\sigma} \longrightarrow 2\pi$. Then we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-imx}\psi_n(x)dx = \frac{1}{2\pi}\Big(\int_{\mathcal{C}_1} e^{-imx}\psi_n(x)dx + \int_{\mathcal{C}_2} e^{-imx}\psi_n(x)dx + \int_{\mathcal{C}_3} e^{-imx}\psi_n(x)dx\Big). \tag{7.0.3}$$

Using the fact that ψ_n is of period 2π and the fact that the first and the third integral are oriented in opposite directions we have

$$\chi_n(m) = \frac{1}{2\pi}\int_{\mathcal{C}_2} e^{-imx}\psi_n(x)dx.$$

We parametrize $\mathcal{C}_2$ by $x = x_1 - i\overline{\sigma},\ 0 \leq x_1 \leq 2\pi$.

$$\chi_n(m) = \frac{1}{2\pi}\int_0^{2\pi} e^{-imx-m\overline{\sigma}}\psi_n(x - i\overline{\sigma})dx$$
$$|\chi_n(m)| = \frac{1}{2\pi}\Big|\int_0^{2\pi} e^{-imx-m\overline{\sigma}}\psi_n(x - i\overline{\sigma})dx\Big|$$
$$|\chi_n(m)| \leq \frac{1}{2\pi}\int_0^{2\pi} e^{-m\overline{\sigma}}|\psi_n(x - i\overline{\sigma})|dx.$$

As ψ_n is analytic it is bounded on the line $\mathcal{C}_2$ and so

$$|\chi_n(m)| \leq C(\psi_n)e^{-m\overline{\sigma}}$$

and also

$$|\hat{\psi}_n(p)| \leq C(\psi_n)e^{-p\overline{\sigma}}$$

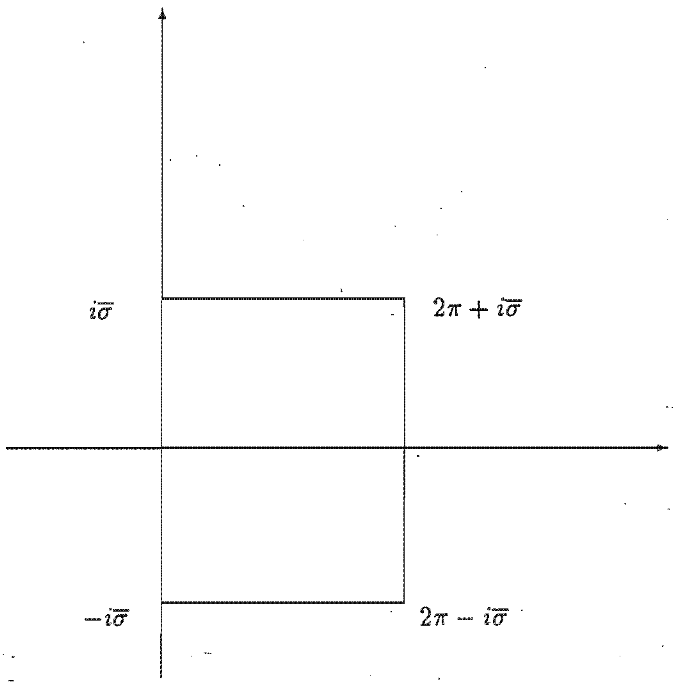

Figure 7.1: the rectangular contour

$$\sum_{p\geq 0} e^{-p\overline{\sigma}} = \frac{1}{1-e^{-\overline{\sigma}}}.$$

By Weierstrass M-test the serie $\sum_{p\geq 1} \hat{\psi}_n(p)\sin px$ converges uniformly on the real line. The expansions of the first and second derivative are established similarly. □

7.1 Properties of the eigenfunctions

Theorem 7.1 *The Fourier coefficients of the eigenfunctions of the Dirichlet problem decay exponentially. More precisely they admit an expansion of the form* $\psi_n(x) = \sum_{m\in\mathbb{Z}} \chi_n(m)e^{imx}$ *and for any* $\overline{\sigma}_*$, σ , *such that* $\sigma_* < \sigma < \overline{\sigma}$ *there is a constant* $C_1(v)$ *independent of* m *such that*

$$|\chi_n(m)| \leq C_1 e^{-\sigma_*|n-|m||}. \tag{7.1.4}$$

Proof:
Let ψ_n an eigenfunction of the Sturm-Liouville problem with Dirichlet boundary conditions:

$$\psi_n(x) = \sum_{p\geq 1} \hat{\psi}_n(p)\sin px$$

where $\hat{\psi}_n \in l^2(\mathbb{N})$.

$$(-\frac{d^2}{dx^2} + v(x))\psi_n(x) = \omega_n^2\psi_n(x)$$

$$(-\frac{d^2}{dx^2} + v(x))\sum_{p\geq 1}\hat{\psi}_n(p)\sin px = \omega_n^2\sum_{p\geq 1}\hat{\psi}_n(p)\sin px.$$

As the first and the second derivative have expansion that converges uniformly we can differentiate and get

$$\sum_{p\geq 1} p^2\hat{\psi}_n(p)\sin px + v(x)\sum_{p\geq 1}\hat{\psi}_n(p)\sin px = \omega_n^2\sum_{p\geq 1}\hat{\psi}_n(p)\sin px.$$

If we now use the identity $\sin px = \frac{1}{2i}(e^{ipx} - e^{-ipx})$ as in the preceding lemma we get a sum over $\mathbb{Z}$

$$\sum_{p\geq 1} p^2\hat{\psi}_n(p)\frac{(e^{ipx}-e^{-ipx})}{2i} + v(x)\sum_{p\geq 1}\hat{\psi}_n(p)\frac{(e^{ipx}-e^{-ipx})}{2i} = \omega_n^2\sum_{p\geq 1}\hat{\psi}_n(p)\frac{(e^{ipx}-e^{-ipx})}{2i}$$

$$\psi_n(x) = \sum_{p\geq 1}\hat{\psi}_n(p)\frac{1}{2i}(e^{ipx} - e^{-ipx})$$

$$\psi_n(x) = \sum_{p\geq 1} \hat{\psi}_n(p)\frac{1}{2i}e^{ipx} - \hat{\psi}_n(p)\frac{1}{2i}e^{-ipx}$$

$$\psi_n(x) = \sum_{m\in\mathbb{Z}} \chi_n(m)e^{imx}$$

where $\chi_n(m)$ is as before. Then

$$\sum_m (m^2\chi_n(m) + v(x)\chi_n(m))e^{imx} = \omega_n^2 \sum_m \chi_n(m)e^{imx}$$

Let denote by $e_k = e^{ikx}$ and take the inner product of this equation with e^{ikx} we get

$$\sum_j (v(x)e_j, e_k)\chi_n(j) + (k^2 - \omega_n^2)\chi_n(k) = 0.$$

This is an operator equation

$$(V_n + D)\chi_n = 0 \tag{7.1.5}$$

where V_n is a diagonal operator from $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$

$$V_n(k,k) = k^2 - \omega_n^2 \text{ and } V_n(m,k) = 0 \text{ if } m \neq k \tag{7.1.6}$$

and $D : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$

$$D(j,k) = \int_0^{2\pi} e^{ix(j-k)}v(x)dx$$

$$|D(j,k)| \leq C(v)e^{-\overline{\sigma}|j-k|}. \tag{7.1.7}$$

Assume: $j - k \geq 0$.

The integral over the rectangular contour $0 \longrightarrow i\overline{\sigma} \longrightarrow 2\pi + i\overline{\sigma} \longrightarrow 2\pi \longrightarrow 0$ is zero because the integrand is analytic. Denote by $\mathcal{C}_1$ the straight line $0 \longrightarrow i\overline{\sigma}, \mathcal{C}_2$ the straight line $i\overline{\sigma} \longrightarrow 2\pi + i\overline{\sigma}, \mathcal{C}_3$ the straight line $2\pi + i\overline{\sigma} \longrightarrow 2\pi$

$$\int_0^{2\pi} e^{ix(j-k)}v(x)dx = \int_{\mathcal{C}_1} e^{ix(j-k)}v(x)dx + \int_{\mathcal{C}_2} e^{ix(j-k)}v(x)dx + \int_{\mathcal{C}_3} e^{ix(j-k)}v(x)dx \tag{7.1.8}$$

As v is 2π-periodic and since the first and the third integral are oriented in opposite direction we have

$$\int_0^{2\pi} e^{ix(j-k)}v(x)dx = \int_{\mathcal{C}_2} e^{ix(j-k)}v(x)dx$$

$$\int_0^{2\pi} e^{ix(j-k)}v(x)dx = \int_{\mathcal{C}_2} e^{ix(j-k)}v(x)dx = \int_0^{2\pi} v(x_1 + i\overline{\sigma})e^{ix_1(j-k)}e^{i^2\overline{\sigma}(j-k)}dx_1$$

where $x = x_1 + i\overline{\sigma}$ and $0 \leq x_1 \leq 2\pi$

$$|\int_0^{2\pi} e^{ix(j-k)} v(x)dx| \leq \int_0^{2\pi} |v(x_1 + i\overline{\sigma})||e^{ix_1(j-k)} e^{i^2\overline{\sigma}(j-k)}|dx_1$$

$$|\int_0^{2\pi} e^{ix(j-k)} v(x)dx| \leq \int_0^{2\pi} |v(x_1 + i\overline{\sigma})||e^{-\overline{\sigma}(j-k)}|dx_1$$

since v being analytic is bounded on that line segment

$$|\int_0^{2\pi} e^{ix(j-k)} v(x)dx| \leq \int_0^{2\pi} C(v)|e^{-\overline{\sigma}(j-k)}|dx_1$$

$$|\int_0^{2\pi} e^{ix(j-k)} v(x)dx| \leq C(v)e^{-\overline{\sigma}(j-k)}.$$

Assume now : $j - k \leq 0$.
We must use the rectangle contour $0 \longrightarrow -i\overline{\sigma} \longrightarrow 2\pi - i\overline{\sigma} \longrightarrow 2\pi \longrightarrow 0$ and we get the estimate

$$\int_0^{2\pi} e^{ix(j-k)} v(x)dx \leq C(v)e^{-\overline{\sigma}(k-j)} = C(v)e^{-\overline{\sigma}|k-j|}.$$

The equation (7.1.5) can be multiplied by $\frac{1}{n}$

$$\frac{1}{n}(V_n + D)\chi_n = 0. \tag{7.1.9}$$

This means that χ_n is a zero-eigenvalue eigenfunction of $\frac{1}{n}(V_n + D)$, our $\chi_n \in l^2(\mathbb{Z})$ and our eigenfunctions ψ_n are normalized, so that

$$\begin{aligned}
1 = (\psi_n, \psi_n) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_k \chi_n(k)e^{ikx}\overline{\sum_m \chi_n(m)e^{imx}}dx \\
&= \frac{1}{2\pi}\sum_k\sum_m \chi_n(k)\overline{\chi_n(m)}\int_0^{2\pi} e^{i(k-m)x}dx \\
&= \frac{1}{2\pi}\sum_k \chi_n(k)\overline{\chi_n(k)}2\pi \\
&= \sum_k |\chi_n(k)|^2 \\
&= |\chi_n|^2_{l^2(\mathbb{Z})}
\end{aligned}$$

and so

$$|\chi_n|_{l^2(\mathbb{Z})} = 1.\square \tag{7.1.10}$$

Let us define the operator H_n by

$$H_n = \frac{1}{n}(V_n + D).$$

We now define P the projection of the operator H_n on the small eigenvalues. We prove in the next lemma that when ξ is in an annulus A which we define later. $(H_n - \xi I)^{-1}$ is self-adjoint and compact and so P can be written as

$$P = \frac{1}{2\pi i}\int_C \frac{d\xi}{H_n - \xi} \tag{7.1.11}$$

where C is a circle surrounding the small eigenvalues of H_n .
The operator H_n is a perturbation of $\frac{1}{n}V_n$, we must study its properties.

Lemma 7.1.1 *$\frac{1}{n}V_n$ is a diagonal operator which has a gap in his spectrum, that is, there is a constant $c_2 > 0$ so that:
for $|k| \neq n$ $|\frac{1}{n}V_n(k,k)| \geq c_2$ and when $|k| = n$, $|\frac{1}{n}V_n(k,k)| \leq \frac{c_2}{8}$.*

Proof:
Assume $|k| < n$:
ω_j^2 satisfies the following formula, (see[9])

$$\omega_j^2 = j^2 + v^* + d(j) \tag{7.1.12}$$

where $v^* = \int_0^\pi v(x)dx$ and $d(j)$ are the terms of a sequence in $l^2(\mathbb{N})$

$$\begin{aligned}
|\frac{1}{n}V_n(k,k)| &= \frac{1}{n}|k^2 - \omega_n^2| \\
&= \frac{1}{n}|k^2 - (n^2 + v^* + d(n))| \\
&= \frac{1}{n}|k^2 - n^2 - (v^* + d(n))| \\
&\geq \frac{1}{n}|k^2 - n^2| - \frac{1}{n}|(v^* + d(n))| \\
&\geq \frac{1}{n}(|k^2 - n^2|) - .005 \quad \text{for n large enough} \\
&\geq \frac{1}{n}(n^2 - k^2) - .005 \\
&\geq \frac{1}{n}(n^2 - (n-1)^2) - .005 \\
&\geq \frac{1}{n}(2n - 1) - .005
\end{aligned}$$

Thus $\frac{1}{n}V_n(k,k)$ is greater than a given positive constant smaller than 2, say c_2 (for n large enough).
Assume now $|k| > n$

$$k = n + p \text{ or } k = -n - p \text{ and } k^2 = n^2 + 2np + p^2 \text{ where p} > 0.$$

$$\begin{aligned}
|\frac{1}{n}V_n(k,k)| &= \frac{1}{n}|k^2 - \omega_n^2| \\
&= \frac{1}{n}|k^2 - (n^2 + v^* + d(n))| \\
&= \frac{1}{n}|n^2 + 2np + p^2 - n^2 - v^* - d(n)| \\
&= |n + 2p + \frac{p^2}{n} - \frac{v^*}{n} - \frac{d(n)}{n}| \\
&\geq |n + 2p + \frac{p^2}{n}| - .005
\end{aligned}$$

for n large enough, n independent of p). Hence $|\frac{1}{n}V_n(k,k)| \geq c_2$.
Assume $k = n$:

$$\begin{aligned}
|\frac{1}{n}V_n(k,k)| &= \frac{1}{n}|n - n\sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}||n + n\sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}| \\
&= n|1 - \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}||1 + \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}|.
\end{aligned}$$

Let us first prove that the product of the two terms of the preceding product converges to zero.
Let

$$\begin{aligned}
A &= n|1 - \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}| \\
&= n|\frac{1 - (1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2})}{1 + \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}}| \\
&= n|\frac{\frac{v^*}{n^2} + \frac{d(n)}{n^2}}{1 + \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}}| \\
&\leq |\frac{\frac{v^*}{n} + \frac{d(n)}{n}}{1 + \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}}|
\end{aligned}$$

which tends to zero.
Let B the second term in the product

$$B = |1 + \sqrt{1 + \frac{v^*}{n^2} + \frac{d(n)}{n^2}}|$$

Then $B \leq 3$ for large n. So $\frac{1}{n}V_n(k,k) \leq \frac{c_2}{8}$ for n large enough. The case $k = -n$ follows similarly.
Therefore we can conclude that there is a a gap in the spectrum of the diagonal operator $\frac{1}{n}V_n$ between $\frac{c_2}{8}$ and c_2. □
We now define an operator norm $||T||_{\overline{\sigma}}$

$$||T||_{\overline{\sigma}} = \max(\sup_j \sum_k |T(j,k)|e^{\overline{\sigma}|j-k|}, \sup_k \sum_j |T(j,k)|e^{\overline{\sigma}|j-k|}).$$

Lemma 7.1.2 *Let S, D two linear operators on $l^2(\mathbb{Z})$. We have the following properties :*

$$||DS||_{\sigma'} \leq ||D||_{\sigma'}||S||_{\sigma'} \tag{7.1.13}$$

and for $\sigma' > 0$,

$$||S||_{op} \leq c(\sigma')||S||_{\sigma'}$$

where $||S||_{op}$ denotes the standard operator norm.

Proof:
Let $u \in l^2(\mathbb{Z})$

$$\begin{aligned}
||Su||^2_{l^2(\mathbb{Z})} &= \sum_k |Su(k)|^2 \\
&= \sum_k |\sum_l S(k,l)u(l)|^2 \\
&= \sum_k |\sum_l S(k,l)e^{\sigma/2'|k-l|}e^{-\sigma/2'|k-l|}u(l)|^2 \\
||Su||^2_{l^2(\mathbb{Z})} &\leq \sum_k \sum_l |S(k,l)|^2 e^{\sigma'|k-l|} \sum_l e^{-\sigma'|k-l|}|u(l)|^2. \qquad (7.1.14)
\end{aligned}$$

We now consider the inequality

$$\sum_k |S(k,l)|e^{\sigma'|k-l|} \leq ||S||_{\sigma'}.$$

Hence we have

$$|S(k,l)e^{\sigma'|k-l|}| \leq ||S||_{\sigma'}$$
$$|S(k,l)|^2 e^{2\sigma'|k-l|} \leq ||S||^2_{\sigma'}$$
$$\Rightarrow |S(k,l)|^2 e^{\sigma'|k-l|} \leq ||S||^2_{\sigma'} e^{-\sigma'|k-l|}$$

and introducing this estimate into (7.1.14) we get

$$||Su||^2_{l^2(\mathbb{Z})} \leq \sum_k \sum_l e^{-\sigma'|k-l|}|u(l)|^2 \sum_l ||S||^2_{\sigma'} e^{-\sigma'|k-l|}$$

$$
\begin{aligned}
&\le ||S||^2_{\sigma'} \sum_k \sum_l e^{-\sigma'|k-l|}|u(l)|^2 \sum_l e^{-\sigma'|k-l|} \\
&\le ||S||^2_{\sigma'} \sum_k \sum_l |u(l)|^2 e^{-\sigma'|k-l|} c(\sigma') \\
&\le ||S||^2_{\sigma'} c(\sigma') \sum_k \sum_l e^{-\sigma'|k-l|}|u(l)|^2 \\
&\le ||S||^2_{\sigma'} c(\sigma') \sum_l |u(l)|^2 \sum_k e^{-\sigma'|k-l|} \\
&\le ||S||^2_{\sigma'} c(\sigma') \sum_l |u(l)|^2 c(\sigma').
\end{aligned}
$$

Hence

$$||Su||^2_{l^2(\mathbb{Z})} \le ||S||^2_{\sigma'} c^2(\sigma') ||u||^2_{l^2(\mathbb{Z})}$$

$||Su|| \le c(\sigma')||S||_{\sigma'}||u||_{l^2(\mathbb{Z})}$ and so $||S||_{op} \le c(\sigma')||S||_{\sigma'}$.
For the case

$$||DS||_{\overline{\sigma}} = \sup_j \sum_k |DS(j,k)| e^{\overline{\sigma}|j-k|}$$

we have

$$
\begin{aligned}
||DS||_{\overline{\sigma}} &= \sup_j \sum_k |\sum_l D(j,l)S(l,k)e^{\overline{\sigma}|j-k|}| \\
&\le \sup_j \sum_k \sum_l |D(j,l)||S(l,k)|e^{\overline{\sigma}|j-l|}e^{\overline{\sigma}|l-k|} \\
&\le \sup_j \sum_l |D(j,l)|e^{\overline{\sigma}|j-l|} \sum_k |S(l,k)|e^{\overline{\sigma}|l-k|} \\
&\le \sup_j \sum_l |D(j,l)|e^{\overline{\sigma}|j-l|} ||S||_{\overline{\sigma}} \\
&\le ||S||_{\overline{\sigma}} ||D||_{\overline{\sigma}}
\end{aligned}
$$

and if $||DS||_{\overline{\sigma}} = \sup_k \sum_j |DS(j,k)|e^{\overline{\sigma}|j-k|}$ the result follows similarly, so

$$||DS||_{\overline{\sigma}} \le ||S||_{\overline{\sigma}} ||D||_{\overline{\sigma}} \square \tag{7.1.15}$$

Lemma 7.1.3 *There is an annulus A around the origin such that for $\xi \in A$ the operator $(H_n - \xi)^{-1}$ is compact.*

Proof:
Let A be the annulus between the circles of radius $\frac{3c_2}{8}$ and $\frac{3c_2}{4}$ and ξ in A we have: $|\xi - \lambda| \ge \frac{c_2}{4}$ for any λ eigenvalue of $\frac{1}{n}\mathrm{V_n}$ so

$$\frac{1}{|\xi - \lambda|} \le \frac{4}{c_2} \tag{7.1.16}$$

We choose the circle C to be a circle centered at the origin of raduis $\frac{c_2}{2}$, this circle is a subset of the annulus. We have the following diagram and let $V_{n,\xi} := \frac{1}{n}V_n - \xi$. Then $V_{n,\xi}^{-1} := (\frac{1}{n}V_n - \xi)^{-1}$ exists and hence

$$\frac{1}{n}(k^2 - \omega_n^2) - \xi \neq 0.$$

$$V_{n,\xi}^{-1}(k,k) = \frac{1}{\frac{1}{n}(k^2 - \omega_n^2) - \xi}$$

$$V_{n,\xi}^{-1}(k,k) = n\frac{1}{k^2 - \omega_n^2 - n\xi}$$

For given ξ and n, and k large enough we have

$$|k^2 - \omega_n^2 - n\xi| \geq |k^2 - |\omega_n^2 - n\xi||$$

$$|k^2 - \omega_n^2 - n\xi| \geq \frac{1}{2}k^2$$

$$\sum_k \frac{n}{|k^2 - \omega_n^2 - n\xi|} = \alpha(n,\xi) < \infty \tag{7.1.17}$$

which leads us to the conclusion that $V_{n,\xi}^{-1}$ is nuclear and so bounded and compact. Using the relation (7.1.16), V_n and the fact that $\frac{1}{n}V_n$ is diagonal we get

$$||V_{n,\xi}^{-1}||_\sigma = ||V_{n,\xi}^{-1}||_0 \leq \frac{4}{c_2} \tag{7.1.18}$$

$$\begin{aligned}
(H_n - \xi I)^{-1} &= (V_{n,\xi} + \frac{D}{n})^{-1} \\
&= [(I + \frac{D}{n}V_{n,\xi}^{-1})V_{n,\xi}]^{-1} \\
&= V_{n,\xi}^{-1}(I + \frac{D}{n}V_{n,\xi}^{-1})^{-1}
\end{aligned}$$

and formally

$$(H_n - \xi I)^{-1} = V_{n,\xi}^{-1}(\sum_p (-\frac{D}{n}V_{n,\xi}^{-1})^p).$$

We now have a decomposition of $(H_n - \xi I)^{-1}$ in two operators. $V_{n,\xi}^{-1}$ is a compact operator so to prove that $(H_n - \xi I)^{-1}$ is compact we prove that $(I + \frac{D}{n}V_{n,\xi}^{-1})^{-1}$ is bounded. By the hypothesis of the lemma $\sigma < \overline{\sigma}$ so there exists $\gamma > 0$ such that

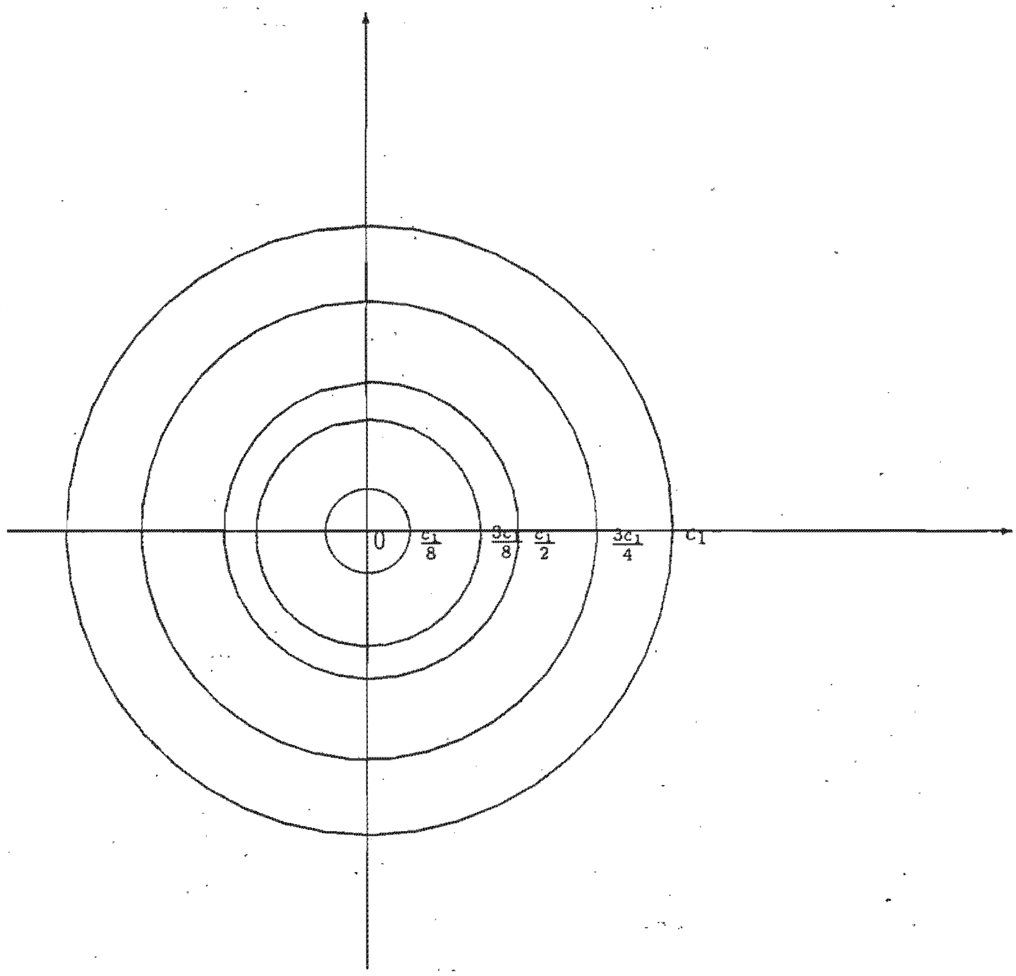

Figure 7.2: circles

$\sigma \leq \bar{\sigma} - \gamma$.
If

$$\begin{aligned}
\|\frac{D}{n}\|_{\bar{\sigma}-\gamma} &= \sup_j \sum_k \frac{1}{n}|D(j,k)|e^{(\bar{\sigma}-\gamma)|j-k|} \\
&\leq \sup_j \frac{1}{n}\sum_k C(v)e^{-\bar{\sigma}|j-k|}e^{\bar{\sigma}-\gamma|j-k|} \\
&\leq \frac{1}{n}\sum_k C(v)e^{-\gamma|j-k|} \\
&\leq \frac{1}{n}C(v)c(\gamma) \qquad (7.1.19)
\end{aligned}$$

and if

$$\|\frac{D}{n}\|_{\bar{\sigma}-\gamma} = \sup_k \sum_j \frac{1}{n}|D(j,k)|e^{(\bar{\sigma}-\gamma)|j-k|} \qquad (7.1.20)$$

then the estimate (7.1.19) follows similarly, and as before we conclude that

$$\|\frac{D}{n}\|_{\bar{\sigma}-\gamma} \leq \frac{1}{n}C(v)c(\gamma). \qquad (7.1.21)$$

We rename the preceding constant $C_1 := C(v)c(\gamma)$. Then

$$\begin{aligned}
\|\frac{D}{n}V_{n,\xi}^{-1}\|_{\bar{\sigma}-\gamma} &\leq \|\frac{D}{n}\|_{\bar{\sigma}-\gamma}\|V_{n,\xi}^{-1}\|_{\bar{\sigma}-\gamma} \\
&\leq \frac{C_1}{n}\frac{4}{c_2} \\
&\leq \frac{1}{2} \text{ for n large enough.} \qquad (7.1.22)
\end{aligned}$$

Thus we can conclude that $(H_n - \xi I)^{-1}$ is compact and

$$\begin{aligned}
\|(H_n - \xi I)^{-1}\|_{\bar{\sigma}-\gamma} &\leq 2\|V_{n,\xi}^{-1}\|_{\bar{\sigma}-\gamma} \\
&\leq 2\frac{4}{c_2} \\
\|(H_n - \xi I)^{-1}\|_{\bar{\sigma}-\gamma} &\leq \frac{8}{c_2} \qquad (7.1.23)
\end{aligned}$$

and also

$$\|(H_n - \xi I)^{-1}\|_{\sigma} \leq \frac{8}{c_2}. \Box \qquad (7.1.24)$$

Lemma 7.1.4 *For ξ_0 real, $\xi_0 \in A$ the operator $(H_n - \xi_0 I)^{-1}$ is also self-adjoint. Therefore for any $\xi \in A, (H_n - \xi)^{-1}$ has a countable basis of eigenfunctions. Moreover $H_n, H_n - \xi, (H_n - \xi)^{-1}$ have the same eigenfunctions.*

Proof:
$H_n = \frac{1}{n}(V_n + D)$. We start by proving that D is self-adjoint:
$D(j,k) = \int_0^{2\pi} v(x)e^{ix(j-k)}dx$ so

$$\overline{D(j,k)} = \int_0^{2\pi} v(x)\overline{e^{ix(j-k)}}dx = \int_0^{2\pi} v(x)e^{ix(k-j)}dx = D(k,j) \ .$$

because v is real on the real axis. $V_n(k,k) = \omega_n - k^2$ so $V_n(k,k)$ is real and thus H_n is self-adjoint.
$(H_n - \xi_0 I)^{-1}$ is then a self-adjoint compact operator and has a basis of eigenfunctions $\{\phi_p\}$. Let μ_p be the eigenvalue corresponding to ϕ_p.

$$(H_n - \xi_0)^{-1}\phi_p = \mu_p\phi_p \text{ and } \mu_p \neq 0 \text{ because } (H_n - \xi_0)^{-1} \text{ is invertible}$$

$$\Leftrightarrow \ (H_n - \xi_0)(H_n - \xi_0)^{-1}\phi_p = (H_n - \xi_0)\mu_p\phi_p$$

$$\Leftrightarrow \ \phi_p = H_n\mu_p\phi_p - \xi_0\mu_p\phi_p$$

$$\Leftrightarrow \ \mu_p H_n\phi_p = (1 + \xi_0\mu_p)^{-1}\phi_p$$

$$\Leftrightarrow \ H_n\phi_p = \frac{(1 + \xi_0\mu_p)}{\mu_p}\phi_p$$

$$\Leftrightarrow \ (H_n - \xi)\phi_p = \frac{(1 + (\xi_0 - \xi)\mu_p)}{\mu_p}\phi_p$$

so H_n, $H_n - \xi I$, $(H_n - \xi I)^{-1}$ have the same eigenfunctions. □

Lemma 7.1.5 *Let $\mathcal{C}$ be the circle of radius $\frac{c_2}{2}$. The operator $P = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{H_n - \xi}d\xi$ is the projection on the eigenvalues which modulus is less than $\frac{c_2}{2}$.*

Proof:
Let $(\chi_p)_p$ a basis of eigenfunctions of H_n and their eigenvalues λ_p.

$$H_n\chi_p = \lambda_p\chi_p \text{ and}$$

$$(H_n - \xi)^{-1}\chi_p = \frac{1}{\lambda_p - \xi}\chi_p \ .$$

$$P\chi_p = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{d\xi}{H_n - \xi}\chi_p$$

$$P\chi_p(k) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\chi_p(k)d\xi}{\lambda_p - \xi}.$$

We have $|\lambda_p| \leq \frac{3c_2}{8} \leq \frac{c_2}{2}$ or $|\lambda_p| \geq \frac{3c_2}{4}$ and so

$$P\chi_p(k) = \chi_p(k) \text{ for } |\lambda_p| \leq \frac{3c_2}{8} \leq \frac{c_2}{2} \tag{7.1.25}$$

and for

$$|\lambda_p| \geq \frac{3c_2}{4} \geq \frac{c_2}{2} \ P\chi_p = 0 \qquad (7.1.26)$$

Let χ_p an eigenvector of associated with the eigenvalue λ_p of H_n such that $|\lambda_p| \leq \frac{c_2}{8}$ and $||\chi_n||_{l^2(\mathbb{Z})} = 1$. We augment this vector by eigenvectors $(\chi_{p_k})_k$ to form a basis of M the eigenspace spanned by the eigenvectors corresponding to the smalls eigenvalues. Let $x \in l^2(\mathbb{Z})$ we have a decomposition

$$x = \sum_p x_p \chi_p$$

$$Px = P \sum_p x_p \chi_p$$

and using (7.1.25) and (7.1.26) we get

$$Px = \sum_k x_{p_k} \chi_{p_k}$$

so P is the projection on the small eigenvalues. □

We introduce now a relation between the coefficients of the eigenfunctions and the projection operator. It was first introduced in [9] for the case of periodic boundary condition.

Lemma 7.1.6 *Let ψ_n be a normalized eigenfunction of the Sturm-Liouville problem. We have the following inequality:*

$$|\chi_n(j)| \leq \sqrt{|P(j,j)|} \qquad (7.1.27)$$

This inequality will be very useful in proving (7.1.4) because it shows that if we can compute the matrix elements of the projection operator we can have an estimate for $|\chi_n(j)|$.

Proof:

Let us denote by e^j an element of the canonical base of $l^2(\mathbb{Z})$,

$$Pe^j \in M$$

$$\begin{aligned}
(e^j, Pe^j)_{l^2(\mathbb{Z})} &= (\sum_p (e_j, \chi_p)\chi_p, \sum_k (e^j, \chi_{p_k})\chi_{p_k}) \\
&= \sum_k (e^j, \chi_{p_k})\overline{(e^j, \chi_{p_k})}(\chi_{p_k}, \chi_{p_k}) \\
&= \sum_k |(e^j, \chi_{p_k})||\chi_{p_k}|^2
\end{aligned}$$

We know that χ_n is a zero-eigenvalue eigenfunction of H_n (7.1.5) and (7.1.10) and $|\chi_n|_{l^2(\mathbb{Z})} = 1$ so

$$(e^j, Pe^j)_{l^2(\mathbb{Z})} \geq |(e^j, \chi_n)|^2 . |\chi_n|^2 = |(e^j, \chi_n)|^2.$$

At the site j

$$\begin{aligned} |\chi_n(j)| &= |(e^j, \chi_n)_{l^2(\mathbb{Z})}| \\ &\leq \sqrt{|(e^j, Pe^j)_{l^2(\mathbb{Z})}|}. \end{aligned}$$

We compute the k^{th} components of the vector $P(e^j)$.

$$Pe^j(k) = \sum_l P(k,l)e^j(l) = P(k,j)$$

Then

$$\begin{aligned} (e^j, Pe^j)_{l^2(\mathbb{Z})} &= (e^j, \sum_{k\in\mathbb{Z}} Pe^j(k)e^k)_{l^2(\mathbb{Z})} \\ &= (e^j, \sum_{k\in\mathbb{Z}} P(k,j)e^k)_{l^2(\mathbb{Z})} \\ &= (e^j, P(j,j)e^j)_{l^2(\mathbb{Z})} \\ &= \overline{P(j,j)}. \text{ (The basis } e^j \text{ is orthonormal.)} \end{aligned}$$

Then $|\overline{P(j,j)}| = |(e^j, Pe^j)|$ so

$$|\chi_n(j)| \leq \sqrt{|P(j,j)|} \tag{7.1.28}$$

and this proves the lemma. □

We know that $P = \frac{1}{2\pi i}\int_C \frac{1}{H_n - \xi} d\xi$ and we had established that $||(H_n - \xi I)^{-1}||_0 \leq \frac{8}{c_2}$ in (7.1.23). But we need a stronger estimate, an estimate which proves the exponentional decay of the coefficients of the eigenfunctions. We denote by $G(\xi) = (H_n - \xi I)^{-1}$. Let $S = \{n, -n\}$ and $NS = \mathbb{Z} \setminus S$, we split the Hilbert space $l^2(\mathbb{Z})$ into a direct sum of two subspaces

$$l^2(\mathbb{Z}) = l^2(S) \oplus l^2(NS).$$

Where $l^2(S)$ (resp $l^2(NS)$) is the set of square sommables sequences $u(k)$ with $k \in S$(resp $k \in NS$) only. To get an estimate on $P(k,k)$ we need an estimate on $G(\xi)(k,k)$. In the case $k = \pm n$ the estimate (7.1.4) is easy to prove

$$|\chi_n(\pm n)| = |\frac{1}{2\pi}\int_0^{2\pi} \psi_n(x)e^{\pm inx}dx|$$

$$|\chi_n(\pm n)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\psi_n(x)|.|e^{\pm inx}|dx$$

$$|\chi_n(\pm n)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\psi_n(x)|dx$$

$$|\chi_n(\pm n)| \leq \frac{1}{2\pi}\sqrt{\int_0^{2\pi} |\psi_n(x)|^2 dx}\sqrt{\int_0^{2\pi} dx}$$

$$|\chi_n(\pm n)| \leq \frac{1}{2\pi}\sqrt{\int_0^{2\pi} dx}$$

$$|\chi_n(\pm n)| \leq \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}e^{-\sigma_\circ||n|-n|} \ (n \geq 1)$$

so in this case the inequality of the theorem is satisfied. We now want to get an estimate on $P(k,k)$ when $k \in NS$. We denote by G_{NS} the restrictions of G to the sites in NS, i.e

$$G_{NS}(j,k) = G(j,k) \text{ if } (j,k) \in NS \times NS \text{ and } 0 \text{ otherwise.}$$

We prove that the operator $G(\xi)$ satisfies

$$G(\xi) = G_S(\xi) \oplus G_{NS}(\xi) - (G_S(\xi) \oplus G_{NS}(\xi))\Gamma G(\xi) \tag{7.1.29}$$

where

$$\Gamma = H_n - H_{NS} \oplus H_S.$$

Let $R = G.(H_n - \xi I)$ we have to prove that $R = I$

$$\begin{aligned} R &= [G_S(\xi) \oplus G_{NS}(\xi) - (G_S(\xi) \oplus G_{NS}(\xi))\Gamma G(\xi)].(H_n - \xi I) \\ &= [G_S(\xi) \oplus G_{NS}(\xi)][(H_n - \xi I)_S + (H_n - \xi I)_{NS} + \Gamma)] \\ &\quad -[(G_S(\xi) \oplus G_{NS}(\xi))\Gamma]I \\ &= I_S + I_{NS} + [G_S(\xi) \oplus G_{NS}(\xi)]\Gamma - [(G_S(\xi) \oplus G_{NS}(\xi))\Gamma]I \\ &= I. \end{aligned}$$

We just proved that

$$G(\xi) = G_S(\xi) \oplus G_{NS}(\xi) - (G_S(\xi) \oplus G_{NS}(\xi))\Gamma G(\xi) \tag{7.1.30}$$

so

$$(I + (G_S(\xi) \oplus G_{NS}(\xi))\Gamma)G(\xi) = G_S(\xi) \oplus G_{NS}(\xi) \tag{7.1.31}$$

$$G(\xi) = (I + (G_S(\xi) \oplus G_{NS}(\xi))\Gamma)^{-1}(G_S(\xi) \oplus G_{NS}(\xi)) \tag{7.1.32}$$

and formally we get

$$G(\xi) = \sum_{l=0}^{\infty}(-1)^l[G_S(\xi) \oplus G_{NS}(\xi))\Gamma]^l(G_S(\xi) \oplus G_{NS}(\xi)) \tag{7.1.33}$$

$$G(\xi) = G_S(\xi) \oplus G_{NS}(\xi) + \sum_{l=1}^{\infty}(-1)^l (G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi)). \quad (7.1.34)$$

To prove the convergence of the series we need an estimate on the term $(G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l(G_S(\xi) \oplus G_{NS}(\xi))$ so we enunciate the following lemma on the first term of the product.

Lemma 7.1.7 *For $p \in NS$, $m \in S$, $l \geq 0$, there exits a constant $c'(N_0)$ such that for $n \geq N_0$*

$$|[(G_S(\xi) \oplus G_{NS}(\xi))\Gamma]^l(m,p)| \leq (\frac{c'}{n})^l \quad (7.1.35)$$

Proof:
If $l = 0$ the result is trivial because the matrix is the identity. We start by proving now the case $l = 1$. We have the following operators

$$H_0 = \frac{1}{n} V_n \mid_{NS}$$

$$H_S = \frac{1}{n}(V_n + D) \mid_S$$

$$H_{NS} = \frac{1}{n}(V_n + D) \mid_{NS}$$

which we call Hamiltonians and their inverses,

$$G_0(\xi) = (H_0 - \xi I) \mid_{NS}^{-1}$$

$$G_S(\xi) = (H_0 - \xi I) \mid_S^{-1}$$

$$G_{NS}(\xi) = (H_{NS} - \xi I) \mid_{NS}^{-1}$$

$$G(\xi) = (H_n - \xi I)^{-1}.$$

In (7.1.23) we have established the result

$$\|(H_n - \xi I)^{-1}\|_0 \leq \frac{8}{c_2}$$

which implies that

$$|G(\xi)(j,k)| \leq \frac{8}{c_2}. \quad (7.1.36)$$

$$\begin{aligned} G(\xi) &= G_S(\xi) \oplus G_{NS}(\xi) - (G_S(\xi) \oplus G_{NS}(\xi))\Gamma(G_S(\xi) \oplus G_{NS}(\xi)) \\ &+ \sum_{l=2}^{\infty}(-1)^l (G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi)). \quad (7.1.37) \end{aligned}$$

Assume now $k \in NS$:

$$G(\xi)(k,k) = G_{NS}(\xi)(k,k) - (G_S(\xi) \oplus G_{NS}(\xi))\Gamma(G_S(\xi) \oplus G_{NS}(\xi))(k,k)$$
$$+\sum_{l=2}^{\infty}(-1)^l(G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l(G_S(\xi) \oplus G_{NS}(\xi)(k,k).$$

We must first get an estimate on Γ and on $(G_S(\xi) \oplus G_{NS}(\xi))\Gamma$

$$\Gamma = H_n - H_{NS} \oplus H_S.$$

Since $V(\Omega)$ is diagonal $V(\Omega) = V(\Omega)_S \oplus V(\Omega)_{NS}$ and

$$\Gamma = \frac{D}{n} - \frac{D}{n}|_{NS} \oplus \frac{D}{n}|_S$$

so we can deduce that

$$\Gamma(q,m) = \frac{D}{n}(q,m) - \frac{D}{n}|_{NS}(q,m) \oplus \frac{D}{n}|_S(q,m)$$

and using and estimate like (7.1.21) we have

$$||\Gamma||_{\overline{\sigma}-\gamma/2} \le \frac{C(v)C(\overline{\sigma}-(\overline{\sigma}-\gamma/2))}{n}$$

$$||\Gamma||_{\overline{\sigma}-\gamma/2} \le \frac{C(v)C(\gamma/2)}{n}$$

$$|\Gamma(q,m)| \le \frac{C_1'}{n}e^{-(\overline{\sigma}-\gamma/2)|q-m|}.$$

where $C_1' = C(v)C(\gamma/2)$. Let $p \in NS$ and $m \in S$:

$$(G_S(\xi) \oplus G_{NS}(\xi))\Gamma(p,m) = \sum_q (G_S(\xi) \oplus G_{NS}(\xi))(p,q)\Gamma(q,m)$$

$p \in NS \Rightarrow q \in NS \Rightarrow m \in S$ (Γ has only interaction terms.) Using now the estimate on the resolvant established in (7.1.23) we have

$$|G_S(\xi) \oplus G_{NS}(\xi))(p,q)| \le \frac{8}{c_2}e^{-(\overline{\sigma}-\gamma)|p-q|}$$

$$\begin{aligned}(G_S(\xi) \oplus G_{NS}(\xi))\Gamma(p,m) &= \sum_{q\in NS}(G_S(\xi) \oplus G_{NS}(\xi))(p,q)\Gamma(q,m)\\ &\le \frac{16}{C_1}e^{-(\overline{\sigma}-\gamma)|p-q|}\frac{C_1'}{n}e^{-(\overline{\sigma}-\gamma/2)|q-m|}\\ &\le \frac{c'}{n}\sum_q e^{-(\overline{\sigma}-\gamma)|p-q|}e^{-(\overline{\sigma}-\gamma)|q-m|}e^{-\gamma/2|q-m|}\end{aligned}$$

$$
\begin{aligned}
& \leq \frac{c'}{n}\sum_q e^{-(\overline{\sigma}-\gamma)(|p-q|+|q-m|)}e^{-\gamma/2|q-m|} \\
& \leq \frac{c'}{n}\sum_q e^{-(\overline{\sigma}-\gamma)|p-m|}e^{-\gamma/2|q-m|} \\
& \leq \frac{c'}{n}e^{-(\overline{\sigma}-\gamma)|p-m|}\sum_q e^{-\gamma/2|q-m|} \\
& \leq \frac{c'}{n}e^{-(\overline{\sigma}-\gamma)|p-m|}\sum_q e^{-\gamma/2|q|} \\
(G_S(\xi)\oplus G_{NS}(\xi))\Gamma(p,m) & \leq \frac{c'}{n}e^{-(\overline{\sigma}-\gamma)|p-m|}2\frac{1}{1-e^{-\gamma/2}}. \qquad (7.1.38)
\end{aligned}
$$

If we take now the particular case $\gamma = \overline{\sigma}/2$ we have as well

$$(G_S(\xi)\oplus G_{NS}(\xi))\Gamma(p,m) \leq \frac{c'}{n}e^{-(\overline{\sigma}/2)|p-m|}2\frac{1}{1-e^{\overline{\sigma}/4}} \leq \frac{c'}{n}\frac{1}{1-e^{\overline{\sigma}/4}}.$$

The c' in the lemma is in fact $c'\frac{1}{1-e^{\overline{\sigma}/4}}$. We consider now the case $l \geq 2$:

$$||((G_S(\xi)\oplus G_{NS}(\xi))\Gamma)^l||_{\overline{\sigma}-2\gamma} \leq ||(G_S(\xi)\oplus G_{NS}(\xi))\Gamma||^l_{\overline{\sigma}-2\gamma}.$$

If

$$||(G_S(\xi)\oplus G_{NS}(\xi))\Gamma||_{\overline{\sigma}-2\gamma} = (\sup_y\sum_x e^{(\overline{\sigma}-2\gamma)|x-y|}|((G_S(\xi)\oplus G_{NS}(\xi))\Gamma)(x,y)|)$$

then using (7.1.38) we get

$$
\begin{aligned}
||((G_S(\xi)\oplus G_{NS}(\xi))\Gamma)^l||_{\overline{\sigma}-2\gamma} & \leq (\sup_y\sum_x e^{(\overline{\sigma}-2\gamma)|x-y|}\frac{c}{n}e^{-(\overline{\sigma}-\gamma)|x-y|})^l \\
& \leq (\sup_y\sum_x e^{(\overline{\sigma}-2\gamma)|x-y|}e^{-(\overline{\sigma}-2\gamma)|x-y|}\frac{c}{n}e^{-\gamma|x-y|})^l \\
& \leq (\frac{c}{n}\frac{1}{\gamma^2})^l
\end{aligned}
$$

We have used here the inequality $\sum_l e^{-\gamma|l|} \leq \frac{64}{\gamma^2}$ which we prove latter in this chapter. For n large enough we will have $\frac{c}{n}\frac{1}{\gamma^2} < 1$ and we expect that the Neumann sum will converge. This proves the lemma. □

We are now in a position to prove the theorem 7.1. If

$$
\begin{aligned}
||((G_S(\xi)\oplus G_{NS}(\xi))\Gamma)||_{\overline{\sigma}-2\gamma} = & (\sup_x\sum_y e^{(\overline{\sigma}-2\gamma)|x-y|} \\
& |((G_S(\xi)\oplus G_{NS}(\xi))\Gamma)(x,y)|).
\end{aligned}
$$

then the estimate follows similarly. We now consider the product

$$B(l) = [(G_S(\xi) \oplus G_{NS}(\xi))\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi))](k,k)$$

which can be rewritten as a sum

$$= \sum_{m,p,r,q,j} G_{NS}(k,j,\xi)\Gamma(j,m)(G_S(\xi) \oplus G_{NS}(\xi))\Gamma)^{l-2}(m,p)$$

$$G_S(p,r;\xi)\Gamma(r,q)G_{NS}(q,k;\xi).$$

We have $k \in NS \implies j \in NS \implies m \in S$ and beginning from the right $k \in NS \implies q \in NS \implies r \in S \implies p \in S$ so

$$= \sum_{m,p,r \in S;q,j \in NS} G_{NS}(k,j,\xi)\Gamma(j,m)(G_S(\xi) \oplus G_{NS}(\xi))\Gamma)^{l-2}(m,p)$$

$$G_S(p,r;\xi)\Gamma(r,q)G_{NS}(q,k;\xi).$$

In the next lines we show that the integral over the contour C of the two first terms in the Neumann series (i.e for $l = 0$ and $l = 1$)in (7.1.33) vanish whenever $k \in NS$. We start by proving that $\frac{1}{2\pi i}\int_C G_{NS}(\xi)(k,k)d\xi = 0$: We established in lemma 1.1.1 that the eigenvalues of $\frac{1}{n}V_n \mid_{NS}$ are greater then c_2 (the restrictions on NS correspond to the case $|k| \neq n$). Let $|\xi| \leq \frac{3c_2}{4}$, we have again the relation $|\xi - \lambda| < \frac{4}{c_2}$ for any λ eigenvalue of $\frac{1}{n}V_n \mid_{NS}$ thus like in (7.1.18)

$$||V_{n,\xi}^{-1} \mid_{NS} ||_{\overline{\sigma}-\gamma} \leq \frac{4}{c_2}$$

and

$$\begin{aligned}
H_{NS}(\xi) &= (\frac{1}{n}V_n + \frac{D}{n} - \xi I) \mid_{NS} \\
&= (\frac{1}{n}V_n - \xi I + \frac{D}{n}) \mid_{NS} \\
&= (V_{n,\xi} \mid_{NS} + \frac{D}{n} \mid_{NS}) \\
&= (I - \frac{D}{n} \mid_{NS} V_{n,\xi} \mid_{NS}^{-1}) V_{n,\xi} \mid_{NS}
\end{aligned}$$

and formally we get

$$H_{NS}(\xi)^{-1} = V_{n,\xi} \mid_{NS}^{-1} \sum_{l=0}^{\infty} (\frac{D}{n} \mid_{NS} \frac{D}{n} \mid_{NS})^l.$$

For n large enough as in (7.1.22) we have once again

$$||\frac{D}{n}V_{n,\xi} \mid_{NS}^{-1} ||_{\overline{\sigma}-\gamma} \leq ||\frac{D}{n} \mid_{NS} ||_{\overline{\sigma}-\gamma} ||V_{n,\xi} \mid_{NS}^{-1} ||_{\overline{\sigma}-\gamma} \leq \frac{C_1}{n}\frac{4}{c_2} < \frac{1}{2}.$$

This shows that this operator will be invertible whenever $|\xi| \leq \frac{3c_2}{4}$ and

$$\|(H_n - \xi I)^{-1}\|_\sigma \leq \frac{8}{c_2}. \tag{7.1.39}$$

The circle $\mathcal{C}$ has radius $\frac{c_2}{2} < \frac{3c_2}{4}$ so the operator $\int_{\mathcal{C}} G_{NS}(\xi)d\xi$ which defines the projection on the eigenvalues of H_{NS} which modulus is less than the raduis of $\mathcal{C}$ that is $\frac{c_2}{2}$. But as we just proved that if $|\xi| \leq \frac{3c_2}{4}$ the operator $H_{NS}(\xi)$ is invertible there are not any eigenvalues of modulus less than $\frac{c_2}{2}$ therefore

$$\int_{\mathcal{C}} G_{NS}(\xi)(k,k)d\xi = 0$$

Furthermore as $k \in NS$ and Γ has only interactions terms between S and NS we also have

$$(G_S(\xi) \oplus G_{NS}(\xi))\Gamma(G_S(\xi) \oplus G_{NS}(\xi))(k,k) = 0$$

hence the second term in the infinite sum in (7.1.33) vanish and so the infinite sum starts with $l = 2$. We can use now (7.1.36) to get the estimate on G_S since the circle $\mathcal{C}$ lies in the annulus A and for G_{NS} we use (7.1.39)

$$B(l) \leq \sum_{m,p,r\in S;q,j\in NS} \frac{16}{C_1}e^{-(\overline{\sigma}-\gamma)|j-k|}\frac{C_1}{n}e^{-\overline{\sigma}|j-m|}\left(\frac{c}{n}\frac{1}{\gamma^2}\right)^{l-2}\frac{16}{C_1}$$
$$\frac{C_1}{n}e^{-\overline{\sigma}|r-q|}\frac{16}{C_1}e^{-(\overline{\sigma}-\gamma)|k-q|}$$
$$\leq \left(\frac{c}{n}\frac{1}{\gamma^2}\right)^{l-2}\sum_{m,p,r\in S}\sum_j \frac{16}{C_1}\frac{C_1}{n}\frac{16}{C_1}e^{-(\overline{\sigma}-\gamma)|j-k|}e^{-\overline{\sigma}|j-m|}\frac{C_1}{n}\sum_q e^{-\overline{\sigma}|r-q|}\frac{16}{C_1}e^{-(\overline{\sigma}-\gamma)|k-q|}.$$

Let E be defined by

$$\begin{aligned}
E &= -(\overline{\sigma}-\gamma)|k-j| - \overline{\sigma}|j-m| \\
&= -(\overline{\sigma}-\gamma)(|k-j|+|j-m|) - \gamma|j-m| \\
&\leq -(\overline{\sigma}-\gamma)|k-j+j-m| - \gamma|j-m| \\
&\leq -(\overline{\sigma}-\gamma)|k-m| - \gamma|j-m| \\
&\leq -(\overline{\sigma}-\gamma)||k|-|m|| - \gamma|j-m|
\end{aligned}$$

as m,p,r can only take the values $\pm n$ we have $|m| = |r| = |p| = n$ $(n \geq 1)$ and so

$$E \leq -(\overline{\sigma}-\gamma)||k|-n|| - \gamma|j-m|$$

$$\begin{aligned}
B(l) &\leq \left(\frac{c}{n}\frac{1}{\gamma^2}\right)^{l-2}\frac{16}{C_1}\sum_{m,p,r\in S}\frac{C_1}{n}\frac{16}{C_1}\frac{C_1}{n}e^{-(\overline{\sigma}-\gamma)||k|-n|}e^{-\overline{\sigma}|r-q|}\frac{16}{C_1}e^{-(\overline{\sigma}-\gamma)|k-q|} \\
&\leq C\left(\frac{c}{n}\frac{1}{\gamma^2}\right)^{l-2}e^{-2(\overline{\sigma}-\gamma)||k|-n|}
\end{aligned}$$

The same technique is applied to get an estimate for the sum over q.
$P(k,k) = \frac{1}{2\pi i}\int_C G(\xi)(k,k)d\xi$ we will start the expansion at $l = 2$.

$$G(\xi) = G_S(\xi) \oplus G_{NS}(\xi) + \sum_{l=1}^{\infty}(-1)^l (G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi))$$

As $k \in NS$ $G_S(\xi) \oplus G_{NS}(\xi)(k,k) = G_{NS}(\xi)(k,k)$ which implies that

$$G(\xi)(k,k) = G_{NS}(\xi)(k,k) + \sum_{l=1}^{\infty}(-1)^l (G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi))(k,k)$$

$$|P(k,k)| = |\frac{1}{2\pi i}\int_C G_{NS}(\xi)(k,k)d\xi$$

$$+\frac{1}{2\pi i}\int_C \sum_{l=1}^{\infty}(-1)^l (G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi))(k,k)d\xi|$$

$$|(e^k, Pe^k)| = |\frac{1}{2\pi i}\int_C \sum_{l=2}^{\infty}(G_S(\xi) \oplus G_{NS}(\xi)\Gamma)^l (G_S(\xi) \oplus G_{NS}(\xi))(k,k)d\xi|.$$

$$|(e^k, Pe^k)| = |\frac{1}{2\pi i}\int_C \sum_{l=2}^{\infty} B(l)d\xi|$$

$$|(e^k, Pe^k)| \leq \frac{1}{2\pi}2\pi\frac{c_2}{2} \sup_{|\xi|\leq\frac{c_2}{2}} \sum_{l=2}^{\infty}|B(l)|$$

$$|(e^k, Pe^k)| \leq \frac{c_2}{2}\sum_{l=2}^{\infty} C(\frac{c}{n}\frac{1}{\gamma^2})^{l-2} e^{-2(\overline{\sigma}-\gamma)||k|-n|}.$$

For n large enough ($n \geq N_0$) $\frac{c}{n}\frac{1}{\gamma^2} < \frac{1}{2}$ and we deduce

$$|(e^k, Pe^k)| \leq e^{-2(\overline{\sigma}-\gamma)||k|-n|}C$$

$$|\chi_n(k)| \leq \sqrt{|(e^k, Pe^k)|}$$

and hence

$$|\chi_n(k)| \leq \sqrt{e^{-2(\overline{\sigma}-\gamma)||k|-n|}C}$$

$$|\chi_n(k)| \leq Ce^{-(\overline{\sigma}-\gamma)||k|-n|}$$

and we have the estimate of the theorem since for any $\sigma_* < \sigma$ there existed γ such that $\sigma \leq \overline{\sigma} - \gamma$ and hence $\sigma_* < \overline{\sigma} - \gamma$. So we have prove that the estimate (7.1.4)

holds for large enough n.
For $n \leq N_0$ it is very simple to check that we have (7.1.4)

$$\psi_n(m) = \int_0^{2\pi} e^{-imx}\psi_n(x)dx \tag{7.1.40}$$

and using contour integration as before we get obtain

$$\begin{aligned} |\psi_n(m)| &\leq C(\psi_n)e^{-\overline{\sigma}|m|} \\ &\leq e^{\overline{\sigma}n}C(\psi_n)e^{-\overline{\sigma}n-\overline{\sigma}|m|}. \end{aligned}$$

We have as well

$$|n-|m|| \leq n+|m| \Rightarrow -\overline{\sigma}(n+|m|) \leq -\overline{\sigma}|n-|m||$$

and thus

$$\begin{aligned} |\psi_n(m)| &\leq \sup_{n\leq N_0}(e^{\overline{\sigma}n}C(\psi_n)).e^{-\overline{\sigma}n-\overline{\sigma}|m|} \\ &\leq \sup_{n\leq N_0}(e^{\overline{\sigma}n}C(\psi_n))e^{-\overline{\sigma}|n-|m||} \\ |\psi_n(m)| &\leq Ce^{-\overline{\sigma}|n-|m||}.\square \end{aligned}$$

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