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*Variable Modeling of Fuzzy Phenomena
with Industrial Applications*



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Declaration

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Abstract

The thesis title is “Variable modeling of fuzzy phenomena with industrial application”.

Fuzzy uncertainty is an intrinsic part of real world surrounding us and it often coexists with random uncertainty. The fundamental difficulty in fuzzy phenomena modeling lies in the data information being available in the fuzzy set forms. The fuzzy credibility measure theory founded by Liu (2004) paves the way towards scalar variable modeling of the fuzzy phenomena and thus opens the chance of industrial applications in convenient and efficient approaches.

In this thesis the axiomatic foundation is heavily reviewed. Then we develop the scalar fuzzy variable concept and its credibility measure based characterizations. We use the sample entropy composition to search for a optimal data-assimilating membership function under the Maximum Entropy Principle and thus close the controversy about the subjectively membership function in fuzzy set theory literature. We also investigate the scalar variable modeling idea for random fuzzy phenomena. We define the random fuzzy concept and its average chance measure based characterizations. We explore the Maximum Average Chance Principle and seek the optimal data-assimilating average chance distribution for the random sum variable under investigation. Finally, we demonstrate how the scalar variable idea can be applied in industries.

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Chapter 1. Introduction

1.1 Fuzzy Uncertainty and Variable Modeling of Fuzzy phenomena

Fuzzy uncertainty problem appeared in many industrial engineering problems. Here in this thesis, we mainly examine fuzzy problems concerned with engineering reliability. In reliability analysis and maintenance planning, fuzzy problems address reduced system structural clarity, reduce information on the underlying mechanism of the interaction between sub-systems and reduce overall information of the system as a whole. The methodology to solve the fuzzy reliability problems via the complex system should be developed in terms of the basic concepts of fuzzy mathematics.

Based on Liu's non-classical credibility measure theory, we develop the scalar fuzzy variable concept associated with random fuzzy concept, to resolve the dilemmas for fuzzy mathematics in practical modeling, and to establish variable modeling for application within the industrial field such as in reliability failure times.

The idea to use random fuzzy failure time models for facilitating repairable system modeling arises from our basic understanding on fuzzy phenomenon, described by fuzzy set in earlier fuzzy mathematics. A possibility measure based theoretical framework using fuzzy variables under a fuzzy credibility measure theory allows for the fuzzy parameter to specify the system failure model. Logically speaking, randomness and fuzziness are two different types of uncertainty. Randomness is logically a breakdown of the law of causality because of the absence of identified conditions under which the event occurrence is inevitable. Randomness is traditionally a well-received formalization of uncertainty in terms of the usage of probability calculus by science and engineering. Fuzziness is logically the breakdown of the law of excluding the middle, but is less well known and is often ignored by the communities of engineering and management, particularly, reliability engineering. Evolution or emergence appears in all aspects around us, whether in natural world, social phenomena, or engineering practices. Any system includes many factors, many strata, and many intermediaries. These include interconnections and interactions, and within a system, strata must have intermediate links. Therefore holding on on the intermediate strata of the system structure is a necessary step to understanding underlying system dynamics in their entirety. A fuzzy membership function is an appropriate mathematical mechanism reflecting the evolving system state from one stratum to another.

1.2 Aims and Objectives

The overall objective of this thesis is:

- We develop the Liu (Liu, 2006)'s non-classical credibility measure theory, which belongs to a scalar fuzzy variable concept. We define the random fuzzy concept and further discuss an the entropy principle and average chance measure. A maximum entropy principle and maximum average chance principle follow and finally lead to an explanation of a data-assimilating membership function and its average chance distribution.

The aims of this thesis are:

- To create a random fuzzy variable concept structure frame. This includes concepts such as fuzzy variable and its membership function, conditional credibility measure, credibility-hazard function, credibility copula. (chapter 2.4, 3.1, 3.2, 3.3)
- To propose a criterion for data-assimilating parameter to specify the average chance distribution of a random fuzzy lifetime – Maximum Average Chance Principle. (chapter 5)
- To explore a two-parameter bathtub hazard family and the MLE procedure. (chapter 6)

1.3 Overview of Thesis

In Chapter 2, we compile or convert relevant concepts and results for (\vee, \cdot) -credibility measure theory based on Liu's foundational work (Liu, 2004; 2006). In Chapter 3, we propose a fuzzy hazard function and a credibility copula using a fuzzy credibility measure. We further propose a data-assimilating parameter approach under a maximum entropy principle in Chapter 4.

In Chapter 5, we present a definition of random fuzzy a variable from the preparation of reliability engineers and managers, and explore the average chance distribution. We accept and develop the Maximum Average chance principle and explore an accelerating life model in terms of two-parameter bathtub hazard family under random fuzzy environment in Chapter 6.

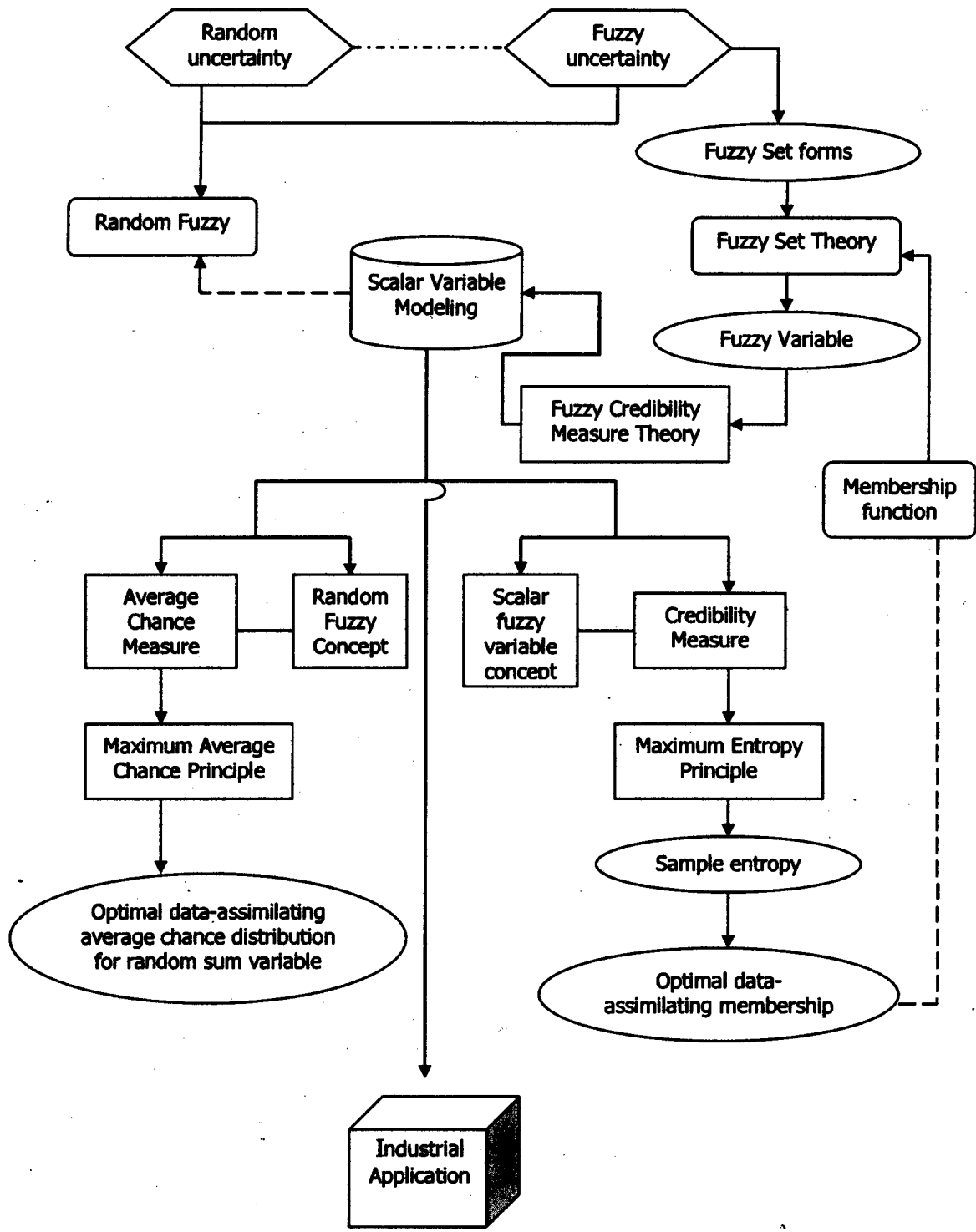


Figure 1.3.1 Overview of the Thesis

Chapter 2. Review of Fuzzy Credibility Measure Theory

Fuzzy mathematics initiated by Zadeh (1965) allowed a foundation dealing with vague phenomena in maintenance modeling. However, the fuzzy mathematical foundation initiated by Zadeh (1965, 1978) is membership function and possibility measure based, and used widely. The possibility measure, which was originally expected to play the role of probability measure in probability theory, but it did not, because possibility measure does not possess a self-duality property to parallel self-duality in probability theory. In standard probability theory, random variable and the distribution function play important roles for converting set-based arguments into variable-based arguments, which permit and empower applications. Kaufmann (1975) first proposed the concept of fuzzy variable with the intention of creating the counterpart to probability theory. However, Kaufmann's fuzzy variable is another name for fuzzy subset and the mathematical operations are difficult to handle. Therefore, it is necessary to develop a suitable and rigorous mathematical foundation for variable modeling of fuzzy phenomena because the fuzzy mathematical foundation established by Zadeh (1978) is not sufficient for variable modeling.

To resolve the three dilemmas, Liu (2004, 2006) proposed an axiomatic foundation for modeling fuzzy phenomena, fuzzy credibility measure theory. The fuzzy credibility measure possesses self-duality property and is able to play a role that parallel probability theory. The fuzzy variable concept and its (credibility) distribution are developed. However, the conditional credibility measure based on Liu's standard credibility theory gives rise to a fundamental difficulty (Liu, 2004), and he proposed a non-classical credibility theory for dealing with the conditioning related modeling (Liu, 2006).

Credibility theory is a branch of mathematics that studies the behaviour of fuzzy phenomena. Zadeh (1965) defined a fuzzy set in terms of membership function which is a natural extension of indicator function of a Cantor set. Later Zadeh (1978) proposed the concept of possibility measure which was intended as a counterpart of that in probability theory. However, possibility measure does not possess self-dual property that is absolutely critical both in theoretical developments and applications. Liu and Liu (2002) proposed the concept of credibility measure with self-duality and Liu (2004) established an axiomatic foundation of credibility theory for fuzzy mathematics.

As Liu later stated, fuzzy credibility measure theory has two versions, the classical credibility measure, and a non-classical credibility theory. It is noted that classical credibility theory is a (\vee, \wedge) -Axiomatic system within which it is inevitable to lose information. An (\vee, \cdot) -Axiomatic system, or equivalently, (\vee, \times) -Axiomatic system offers the possibility of the defining a conditional credibility measure, which is the critical concept for further theoretical developments, such as conditional credibility distribution, hazard function, fuzzy process etc. The requirements of reliability modeling and analysis suggest conditional measures, so we prefer the non-classical theory and thus introduce it systematically as a review in this section.

To achieve a self containment, we compile or convert relevant concepts and results for credibility measure theory based on Liu's foundational work (2004 & 2006) so that the (\vee, \cdot) -fuzzy credibility (non-classical credibility theory) measure is the unique starting point.

2.1 Axiomatic Foundation for Fuzzy Credibility Measure

Let Θ be a nonempty set, and 2^Θ be the power set of Θ . Each element, let us say, $A \subset \Theta$, $A \in 2^\Theta$, is called an event. A number denoted as $\text{Cr}\{A\}$, $0 \leq \text{Cr}\{A\} \leq 1$, is assigned to event $A \in 2^\Theta$, which indicates the credibility that event $A \in 2^\Theta$ occurs. A set function $\text{Cr}\{A\}$ satisfies following axioms Liu (2004, 2006).

Axiom 1: $\text{Cr}\{\Theta\} = 1$

Axiom 2: $\text{Cr}\{\cdot\}$ is non-decreasing, i.e., $\text{Cr}\{A\} \leq \text{Cr}\{B\}$ whenever $A \subset B$.

Axiom 3: $\text{Cr}\{\cdot\}$ is self-dual, i.e., $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ for any $A \in 2^\Theta$.

Axiom 4: $\text{Cr}\{\cup_i A_i\} \wedge 0.5 = \sup_i [\text{Cr}\{A_i\}]$ for any $\{A_i\}$ with $\text{Cr}\{A_i\} \leq 0.5$.

Axiom 5: Let set functions $\text{Cr}_k\{\cdot\}: 2^{\Theta_k} \rightarrow [0,1]$ satisfy Axioms 1-4, and $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_p$, then:

$$\text{Cr}\{\theta_1, \theta_2, \dots, \theta_p\} = \begin{cases} \frac{1}{2} \prod_{k=1}^p (2\text{Cr}_k\{\theta_k\} \wedge 1) & \text{if } \min_{1 \leq k \leq p} \{\text{Cr}_k\{\theta_k\}\} < 0.5 \\ \min_{1 \leq k \leq p} \{\text{Cr}_k\{\theta_k\}\} & \text{if } \min_{1 \leq k \leq p} \{\text{Cr}_k\{\theta_k\}\} \geq 0.5 \end{cases} \quad (2.1)$$

for each $\{\theta_1, \theta_2, \dots, \theta_p\} \in 2^\Theta$. In this case, we write $\text{Cr} = \text{Cr}_1 \times \text{Cr}_2 \times \dots \times \text{Cr}_p$.

Definition 2.1 (Liu, 2006) A set function $\text{Cr}: 2^\Theta \rightarrow [0,1]$ which satisfies **Axioms 1-4** is called a (\vee, \cdot) -fuzzy credibility measure (or non-classical fuzzy credibility measure). The triple $(\Theta, 2^\Theta, \text{Cr})$ is called the (\vee, \cdot) -fuzzy credibility measure space.

Remark 2.2 Fuzzy credibility measure in general does not possess additivity property analogical to the probability measure. In other words, it is a non-additive measure but it is sub-additive and null-additive.

2.2 Fuzzy Variable

Kaufmann (1975) first proposed the concept of fuzzy variable with the intention that is the counterpart of random variable. However, it remains as a fuzzy set formation and it does not behave as the random variable in probability theory. Liu's axioms (2004) for fuzzy credibility measure successfully convert a fuzzy set variable into scalar fuzzy variable form, which is similar to the scalar random variable.

Definition 2.3 (Liu, 2004) A fuzzy variable is defined as a real-valued mapping from a credibility space $(\Theta, \mathfrak{P}(\Theta), \text{Cr})$ to the set of real numbers, i.e., a fuzzy variable $\xi: \Theta \rightarrow \mathbb{R}$ such that for any $B \in \mathfrak{P}(\mathbb{R})$, $\{\theta \mid \xi(\theta) \in B, B \in \mathfrak{P}(\mathbb{R})\} \in \mathfrak{P}(\Theta)$. In other words, fuzzy variable is a Cr -measurable real-valued mapping from $(\Theta, \mathfrak{P}(\Theta))$ to $(\mathbb{R}, \mathfrak{P}(\mathbb{R}))$.

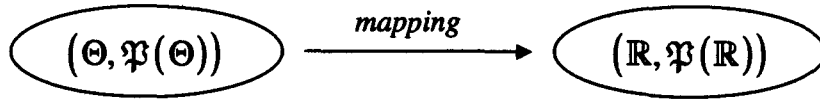


Figure 2.2.1 Set mapping illustration

Definition 2.4 (Liu, 2004) The (induced) membership function of a fuzzy variable ξ on $(\Theta, 2^\Theta, \text{Cr})$ is:

$$\mu(x) = (2\text{Cr}\{\xi = x\}) \wedge 1, \quad x \in \mathbb{R} \quad (2.2)$$

Conversely, for given membership function $\mu(\cdot)$ the fuzzy credibility measure is determined by the credibility inversion theorem.

Theorem 2.5 (Liu, 2004) Let ξ be a fuzzy variable with a membership function $\mu(\cdot)$. Then for $\forall B \subset \mathbb{R}$,

$$\text{Cr}\{\xi \in B\} = \frac{1}{2} \left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x) \right) \quad (2.3)$$

For example, if the set B is degenerate at a point $x \in \mathbb{R}$, then:

$$\text{Cr}\{\xi = x\} = \frac{1}{2} \left(\mu(x) + 1 - \sup_{y \neq x} \mu(y) \right) \quad (2.4)$$

Definition 2.6 (Liu, 2004) The credibility distribution $\Phi: \mathbb{R} \rightarrow [0, 1]$ of a fuzzy variable ξ on $(\Theta, 2^\Theta, \text{Cr})$ is:

$$\Phi(x) = \text{Cr}\{\theta \in \Theta \mid \xi(\theta) \leq x\} \quad (2.5)$$

The credibility distribution $\Phi(x)$ of a fuzzy variable ξ is the accumulated fuzzy credibility grade that the fuzzy variable ξ takes a value less than or equal to a real-number $x \in \mathbb{R}$. Generally speaking, the credibility distribution Φ is neither left-continuous nor right-continuous in contrast to probability distribution function in probability theory.

Definition 2.10 Let $(\xi_1, \xi_2, \dots, \xi_n)$ be a fuzzy vector defined on $(\Theta, \mathfrak{P}(\Theta), Cr)$. Then the joint credibility distribution $\Phi: [-\infty, +\infty]^n \rightarrow [0, 1]$ is defined by:

$$\Phi(x_1, x_2, \dots, x_n) = Cr(\theta \in \Theta: \xi_1(\theta) \leq x_1, \xi_2(\theta) \leq x_2, \dots, \xi_n(\theta) \leq x_n) \quad (2.11)$$

And the joint credibility density function $\phi: [-\infty, +\infty]^n \rightarrow [0, +\infty]$ of fuzzy vector $(\xi_1, \xi_2, \dots, \xi_n)$ is the function such that:

$$\Phi(x_1, x_2, \dots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \dots \int_{-\infty}^{x_n} \phi(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \quad (2.12)$$

holds for all $(x_1, x_2, \dots, x_n) \in [-\infty, +\infty]^n$, where $\Phi(x_1, x_2, \dots, x_n)$ is the joint credibility distribution for fuzzy vector $(\xi_1, \xi_2, \dots, \xi_n)$.

Definition 2.11 (Liu, 2004) Let ξ be a fuzzy variable on $(\Theta, 2^\Theta, Cr)$ and $\zeta: \mathbb{R} \rightarrow \mathbb{R}$, a real-valued function. Then the expected value of

$$E[\zeta(\xi)] = \int_0^{+\infty} Cr\{\zeta(r) \geq r\} dr - \int_{-\infty}^0 Cr\{\zeta(r) \leq r\} dr \quad (2.13)$$

Remark 2.12 In random variable theory, the expectation of function of a real-valued random variable, denoted by $\zeta(\xi)$, is the Lebesgue-Stieltjes integral of $\zeta(\xi)$ with respect to its probability distribution function, $F_{\zeta(\xi)}(\cdot)$, as long as the integral exists. However, in general, for fuzzy variable case,

$$E[\zeta(\xi)] = \int_{-\infty}^{+\infty} \zeta(r) d\Phi_{\zeta(\xi)}(r) \quad (2.14)$$

where $\Phi_{\zeta(\xi)}(\cdot)$ is the credibility distribution for the real-valued function of fuzzy variable ξ .

For a given fuzzy credibility distribution, $\Phi: \mathbb{R} \rightarrow [0, 1]$, the membership function of fuzzy variable ξ is given by

$$\mu(x) = \begin{cases} 2\Phi(x) & \text{if } \Phi(x) < 0.5 \\ 1 & \text{if } \lim_{y \uparrow x} \Phi(y) < 0.5 \leq \Phi(x) \\ 2 - 2\Phi(x) & \text{if } 0.5 \leq \lim_{y \uparrow x} \Phi(y) \end{cases} \quad (2.15)$$

2.3 Non-classical Fuzzy Arithmetic

Fuzzy arithmetic plays an important role when multiple fuzzy variables are involved. On the non-classical credibility measure theory platform, operations do not in general result in the same forms as these in classical credibility measure theory.

Theorem 2.13 (Liu, 2004) ((\vee, \cdot) -Product Credibility Measure Theorem) Let Θ_k be nonempty set and $C\bar{\tau}_k$ is a credibility measure on power set 2^{Θ_k} , $k = 1, 2, \dots, p$ respectively. Let $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_p$, then $C\bar{\tau} = C\bar{\tau}_1 \times C\bar{\tau}_2 \times \dots \times C\bar{\tau}_p$ defined by **Axiom 5** has unique extension to a credibility measure on power set 2^Θ . The triple $(\Theta, 2^\Theta, C\bar{\tau})$ is called the (\vee, \cdot) -credibility measure space.

Definition 2.14 (Liu, 2004) Let Θ be nonempty set and $C\bar{\tau}_k$ be a credibility measure on power set 2^{Θ} , let function $\varsigma: \mathbb{R}^n \rightarrow \mathbb{R}$ and let fuzzy variables $\xi_1, \xi_2, \dots, \xi_n$ be defined on the common fuzzy credibility space $(\Theta, 2^\Theta, C\bar{\tau})$. Then $\xi = \varsigma(\xi_1, \xi_2, \dots, \xi_n)$ is a fuzzy variable on $(\Theta, 2^\Theta, C\bar{\tau})$ and defined as

$$\xi(\theta) = \varsigma(\xi_1(\theta), \xi_2(\theta), \dots, \xi_n(\theta)), \forall \theta \in \Theta \quad (2.16)$$

Definition 2.15 (Liu, 2004) Let function $\varsigma: \mathbb{R}^n \rightarrow \mathbb{R}$, let ξ_i be defined on fuzzy credibility space $(\Theta, 2^\Theta, C\bar{\tau})$, $i = 1, 2, \dots, n$, respectively, and let $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_p$. Then $\xi = \varsigma(\xi_1, \xi_2, \dots, \xi_n)$ is a fuzzy variable on $(\Theta, 2^\Theta, C\bar{\tau})$ and defined as

$$\xi(\theta_1, \theta_2, \dots, \theta_n) = \varsigma(\xi_1(\theta_1), \xi_2(\theta_2), \dots, \xi_n(\theta_n)) \quad (2.17)$$

for any $(\theta_1, \theta_2, \dots, \theta_n) \in \Theta$.

Definition 2.16 (Liu, 2004) The fuzzy variables $\xi_1, \xi_2, \dots, \xi_m$ are independent if and only if

$$2\text{Cr}\left(\bigcap_{i=1}^m \{\xi_i \in B_i\}\right) \wedge 1 = \prod_{i=1}^m (\text{Cr}\{\xi_i \in B_i\}) \wedge 1 \quad (2.18)$$

for any sets $B_1, B_2, \dots, B_m \subset \mathbb{R}$.

Theorem 2.17 (Liu, 2004) ((\vee, \cdot) -Extension Principle) Assume that $\xi_1, \xi_2, \dots, \xi_n$ are independent fuzzy variables with membership functions $\mu_1, \mu_2, \dots, \mu_n$ respectively. Let ς be a mapping $\varsigma: \mathbb{R}^n \rightarrow \mathbb{R}$. Then fuzzy variable $\xi = \varsigma(\xi_1, \xi_2, \dots, \xi_n)$ has a membership function

$$\mu(x) = \sup_{x=\varsigma(x_1, x_2, \dots, x_n)} \left[\prod_{i=1}^n \mu_i(x_i) \right], x \in \mathbb{R} \quad (2.19)$$

Theorem 2.18 (Liu, 2004) ((\vee, \cdot) -Credibility Inversion Theorem) Assume that $\xi_1, \xi_2, \dots, \xi_n$ are independent fuzzy variables with membership functions $\mu_1, \mu_2, \dots, \mu_n$ respectively. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^q$. Then for any set $B \subset \mathbb{R}^q$, the credibility measure $\text{Cr}\{g(\xi_1, \xi_2, \dots, \xi_n) \in B\}$ is:

$$\frac{1}{2} \left(\sup_{s(\xi_1, \xi_2, \dots, \xi_n) \in B} \left[\prod_{i=1}^n \mu_i(x_i) \right] + 1 - \sup_{s(\xi_1, \xi_2, \dots, \xi_n) \in B^c} \left[\prod_{i=1}^n \mu_i(x_i) \right] \right), x \in \mathbb{R} \quad (2.20)$$

2.4 Conditional Credibility Measure

Now, we are ready to discuss the conditional credibility measure. In other words, we now consider the credibility of an event A after it has been learned that some other event B has occurred. This new credibility of A is called the conditional credibility of the event A given B , denoted by $\text{Cr}(A|B)$.

Definition 2.19 (Liu, 2006) Let $(\Theta, 2^\Theta, \text{C}\bar{\text{r}})$ be a credibility measure space and $A, B \in 2^\Theta$ be arbitrary. Then the conditional credibility measure of A given B , denoted by $\text{C}\bar{\text{r}}(A|B)$, is

$$\text{C}\bar{\text{r}}\{A|B\} = \frac{1}{2} \left(\frac{(2\text{C}\bar{\text{r}}\{A \cap B\}) \wedge 1}{(2\text{C}\bar{\text{r}}\{B\}) \wedge 1} + 1 - \frac{(2\text{C}\bar{\text{r}}\{A^c \cap B\}) \wedge 1}{(2\text{C}\bar{\text{r}}\{B\}) \wedge 1} \right) \quad (2.21)$$

provided $\text{C}\bar{\text{r}}\{B\} > 0$.

Based on the conditional credibility measure, the conditional credibility distribution, conditional credibility density function, conditional expected value for conditional fuzzy variable, etc. can be defined.

Definition 2.20 Let $(\Theta, 2^\Theta, \text{C}\bar{\text{r}})$ be a credibility measure space and select arbitrary $B \in 2^\Theta$. Then the conditional fuzzy credibility distribution, $\Phi: \mathbb{R} \times 2^\Theta \rightarrow [0, 1]$, of a fuzzy variable ξ given a fuzzy event B is defined by

$$\Phi(x|B) = \text{C}\bar{\text{r}}\{\xi(\theta) \leq x|B\} \quad (2.22)$$

provided $\text{C}\bar{\text{r}}\{B\} > 0$.

Definition 2.21 Let $(\Theta, 2^\Theta, \text{C}\bar{\text{r}})$ be a credibility measure space and select arbitrary $B \in 2^\Theta$. Then the conditional fuzzy credibility density distribution, $\phi: \mathbb{R} \times 2^\Theta \rightarrow [0, +\infty)$, of a fuzzy variable ξ given a fuzzy event B is a function such that

$$\Phi(x|B) = \int_{-\infty}^x \phi(s|B) ds \quad (2.23)$$

holds for all $x \in [-\infty, +\infty]$, where $\Phi(\cdot|B)$ is the conditional fuzzy credibility distribution of a fuzzy variable ξ given a fuzzy event B .

Definition 2.22 Let $(\Theta, 2^\Theta, \text{C}\bar{\text{r}})$ be a credibility measure space and select arbitrary $B \in 2^\Theta$. Then the conditional expectation of fuzzy variable ξ given a fuzzy event B is defined by

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r | B\} dr - \int_{-\infty}^0 Cr\{\xi \leq r | B\} dr \tag{2.24}$$

provided that at least of one the two integrals exists.

Let (ξ, η) be a pair of fuzzy variables with joint membership function $\mu(x, y)$. If $\sup \mu(r, y) > 0$ for some y , it follows from conditional credibility measure definition:

$$\mu(x | \eta = y) = \frac{\mu(x, y)}{\sup \mu(r, y)} \tag{2.25}$$

Table 2.4.1 Comparison of random uncertainty and fuzzy uncertainty

	<i>Probability Theory</i>	<i>Credibility Theory</i>
<i>Set</i>	Ω	Θ
<i>Set Class</i>	σ -algebra \mathcal{A} of Ω	Power set $\mathcal{P} = 2^\Theta$
<i>Membership</i>	$\vartheta_A : \Omega \rightarrow \{0,1\}$	$\mu(x) : \Theta \rightarrow [0,1]$
<i>Axioms</i>	Axiom1: $Pr\{\Omega\} = 1$ Axiom2: $Pr\{A\} \geq 0$, for $A \in \mathcal{A}$ Axiom3: $Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} Pr\{A_i\}$ Countable Additivity	Axiom1: $Cr\{\Theta\} = 1$ Axiom2: $Cr\{A\} \leq Cr\{B\}$ if $A \subset B$ Axiom3: $Cr\{A\} + Cr\{A^c\} = 1$ Axiom4: $Cr\left\{\bigcup_i A_i\right\} = \sup Cr\{A_i\}$ for $\forall \{A_i\}$ with $\sup Cr\{A_i\} < 0.5$
<i>Mapping</i>	$Pr : \mathcal{A} \rightarrow [0,1]$	$Cr : 2^\Theta \rightarrow [0,1]$
<i>Measure</i>	Pr is called a probability measure if it satisfies the three Axioms	Cr is called a credibility measure if it satisfies the four Axioms
<i>Space</i>	$(\Omega, \mathcal{A}, Pr)$ Probability space	$(\Theta, 2^\Theta, Cr)$ Credibility space
<i>Variable</i>	Random variable $X : (\Omega, \mathcal{A}, Pr) \rightarrow \mathbb{R}$ $\{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{A}$	Fuzzy variable $\xi : (\Theta, 2^\Theta, Cr) \rightarrow \mathbb{R}$
<i>Distribution</i>	$F_X(x) = Pr\{X(\omega) \leq x\}$	$\Phi_\xi(u) = Cr\{\xi(\theta) \leq u\}$

Table 2.4.1 contrasts set-theoretical foundation for random uncertainty and fuzzy uncertainty respectively.

Chapter 3. Fuzzy Lifetime Variable and Its Credibility Measure based Characterizations

The reliability of a system is ideally a quality index that the system completes the specified function within specified time in mutually harmonious manner under the specified conditions (including hard conditions: say, operating environments, input materials and soft conditions: say, system management and operator team, and computerized monitoring atomization subsystems etc.). Vagueness is intrinsic and inherent to the system and its operating environment and inevitably engaging fuzzy mathematics.

As previously stated, the non-classical credibility theory, i.e., (\vee, \cdot) -credibility measure theory facilitates the necessary structure for fuzzy variable reliability modeling of system lifetimes. In this Chapter, based on the conditional distribution of fuzzy variable under Liu's non-classical credibility measure theory (i.e., (\vee, \cdot) -credibility measure theory), we propose the hazard concept associated with fuzzy lifetime and its related theoretical framework, which is expected to be the counterpart of that under probability theory. We explore the basic property of uniform-distributed fuzzy variable and explore the joint uniform distribution for the fuzzy bivariate variables. Then, we propose the concept of a fuzzy copula on the ground of (\vee, \cdot) -credibility measure theory, named the credibility-copula, for the full characterization of the relationship among fuzzy variables. Finally, we explore a decomposition of the credibility-copula function into product copula and an adjusted dependence function.

3.1 Credibility Fuzzy Reliability

System lifetimes in fuzzy reliability are fuzzy variables, similar to random variables. To emphasize the variable modeling idea in fuzzy reliability (in contrast to these fuzzy subset treatments), we examine the fuzzy lifetime (variable), its distribution function, survival function and hazard function because these concepts constitute the foundation of the (credibility) fuzzy reliability theory.

3.1.1 Basic Concepts of Credibility Fuzzy Reliability

Definition 3.1 Let T be a fuzzy lifetime of a system, we denote $F_T(t) = \text{C}\bar{r}\{\theta \in \Theta | T(\theta) \leq t\}$ as the cumulative distribution for the fuzzy lifetime T and we denote $S_T(t) = \text{C}\bar{r}\{\theta \in \Theta | T(\theta) \geq t\} = 1 - F_T(t)$ as the survival function for the fuzzy life time T .

We first consider the credibility of an event A after it has been learned that some other event B has occurred. This new credibility of A is called the conditional credibility measure of the event A given B , denoted by $\text{C}\bar{r}(A|B)$.

Definition 3.2 Let $(\Theta, 2^\Theta, \text{C}\bar{r})$ be a credibility measure space and let $\forall A, B \in 2^\Theta$. Then the conditional credibility measure of A given event B , denoted as $\text{C}\bar{r}(A|B)$:

$$\text{C}\bar{r}(A|B) = \frac{1}{2} \left(\frac{(2\text{C}\bar{r}(A \cap B)) \wedge 1}{(2\text{C}\bar{r}(B)) \wedge 1} + 1 - \frac{(2\text{C}\bar{r}(A^c \cap B)) \wedge 1}{(2\text{C}\bar{r}(B)) \wedge 1} \right) \quad (3.1)$$

provided $\text{C}\bar{r}(B) > 0$.

Based on the conditional credibility measure, similar to the developments in Chapter 2, the notion conditional credibility distribution, conditional credibility density function, conditional expected value for fuzzy variable, etc. can be defined.

Let event $A = \{\theta : t \leq T(\theta) < t + \Delta t\}$ and event $B = \{\theta : T(\theta) \geq t\}$, therefore:

$$F_{TB}(\Delta t | t) = \text{C}\bar{r}(A|B) = \text{C}\bar{r}(\{\theta : t \leq T(\theta) < t + \Delta t\} | \{\theta : T(\theta) \geq t\}) \quad (3.2)$$

By that:

$$\{\theta : t \leq T(\theta) < t + \Delta t\}^c = \{\theta : T(\theta) < t\} \cup \{\theta : T(\theta) \geq t + \Delta t\} \quad (3.3)$$

and also for $\forall s > 0$, $\{\theta : T(\theta) \geq t + \Delta t\} \cap \{\theta : T(\theta) < t\} = \emptyset$, we have,

$$\begin{aligned} & \{\theta : t \leq T(\theta) < t + \Delta t\}^c \cap \{\theta : T(\theta) \geq t\} \\ &= (\{\theta : T(\theta) \geq t + \Delta t\} \cup \{\theta : T(\theta) < t\}) \cap \{\theta : T(\theta) \geq t\} \\ &= \{\{\theta : T(\theta) \geq t + \Delta t\} \cap \{\theta : T(\theta) \geq t\}\} \cup \{\{\theta : T(\theta) < t\} \cap \{\theta : T(\theta) \geq t\}\} \\ &= (\{\theta : T(\theta) \geq t + \Delta t\} \cap \{\theta : T(\theta) \geq t\}) \cup \emptyset \\ &= \{\theta : T(\theta) \geq t + \Delta t\} \end{aligned} \quad (3.4)$$

From Eq. (3.1), the conditional credibility measure of event A given event B , we note the set operation results in Eq. (3.4): $\{\theta: t \leq T(\theta) < t + \Delta t\}^c \cap \{\theta: T(\theta) > t\} = \{\theta: T(\theta) \geq t + \Delta t\}$ and $\{\theta: t \leq T(\theta) < t + \Delta t\} \cap \{\theta: T(\theta) \geq t\} = \{\theta: t \leq T(\theta) < t + \Delta t\}$ to obtain:

$$\begin{aligned}
& 2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\} | \{\theta: T(\theta) \geq t\}) \\
&= \frac{(2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\} \cap \{\theta: T(\theta) \geq t\})) \wedge 1}{(2\overline{CF}(\{\theta: T(\theta) \geq t\})) \wedge 1} \\
&+ 1 - \frac{(2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\}^c \cap \{\theta: T(\theta) \geq t\})) \wedge 1}{(2\overline{CF}(\{\theta: T(\theta) \geq t\})) \wedge 1} \quad (3.5) \\
&= \frac{(2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\})) \wedge 1}{(2\overline{CF}(\{\theta: T(\theta) \geq t\})) \wedge 1} + 1 - \frac{(2\overline{CF}(\{\theta: T(\theta) \geq t + \Delta t\})) \wedge 1}{(2\overline{CF}(\{\theta: T(\theta) \geq t\})) \wedge 1}
\end{aligned}$$

Then

$$\begin{aligned}
& 2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\} | \{\theta: T(\theta) > t\}) \\
&= \frac{(2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\})) \wedge 1}{(2\overline{CF}(\{\theta: T(\theta) \geq t\})) \wedge 1} + 1 - \frac{(2\overline{CF}(\{\theta: T(\theta) \geq t + \Delta t\})) \wedge 1}{(2\overline{CF}(\{\theta: T(\theta) \geq t\})) \wedge 1} \quad (3.6) \\
&= \frac{(2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\})) \wedge 1}{(2[1 - F_T(t)]) \wedge 1} + 1 - \frac{(2[1 - F_T(t + \Delta t)]) \wedge 1}{(2[1 - F_T(t)]) \wedge 1}
\end{aligned}$$

In other words,

$$F_T(\Delta t | t) = \frac{1}{2} \left[\frac{(2\overline{CF}(\{\theta: t \leq T(\theta) < t + \Delta t\})) \wedge 1}{(2[1 - F_T(t)]) \wedge 1} + 1 - \frac{(2[1 - F_T(t + \Delta t)]) \wedge 1}{(2[1 - F_T(t)]) \wedge 1} \right] \quad (3.7)$$

Definition 3.3 The credibility hazard function, denoted by $h_T(t)$, is defined as:

$$h_T(t) = \lim_{\Delta t \rightarrow 0} \frac{F(\Delta t | t)}{\Delta t} \quad (3.8)$$

where f is the credibility density function for the fuzzy life time $T(\theta)$. Therefore, in general, the credibility hazard function is:

$$h_T(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{2\Delta t} \left[\frac{(2C\bar{F}(\{\theta: t \leq T(\theta) < t + \Delta t\})) \wedge 1}{(2[1 - F_T(t)]) \wedge 1} + 1 - \frac{(2[1 - F_T(t + \Delta t)]) \wedge 1}{(2[1 - F_T(t)]) \wedge 1} \right] \quad (3.9)$$

Proposition 3.4 The credibility hazard function is

$$h_T(t) = \frac{1}{2} \frac{[2f_T(t)] \wedge 1}{(2[1 - F_T(t)]) \wedge 1} = \frac{1}{2} \frac{[2f_T(t)] \wedge 1}{(2S_T(t)) \wedge 1} \quad (3.10)$$

Proof: When Δt tends to zero, i.e., Δt is very small and if $C\bar{F}(\{\theta: t \leq T(\theta) < t + \Delta t\}) = \int_t^{t+\Delta t} f_T(t) dt$,

then: $C\bar{F}(\{\theta: t \leq T(\theta) < t + \Delta t\}) \approx f_T(t)\Delta t$. Also, as $\Delta t \rightarrow 0$, $F_T(t + \Delta t) \rightarrow F_T(t)$, thus:

$$h_T(t) = \lim_{\Delta t \rightarrow 0} \frac{F(\Delta t|t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{1}{2\Delta t} \left[\frac{[2f_T(t)\Delta t] \wedge 1}{(2[1 - F_T(t)]) \wedge 1} + 1 - \frac{(2[1 - F_T(t + \Delta t)]) \wedge 1}{(2[1 - F_T(t)]) \wedge 1} \right] = \frac{1}{2} \frac{[2f_T(t)] \wedge 1}{(2[1 - F_T(t)]) \wedge 1}$$

We observe that the function form in general is dissimilar to its counterpart in probabilistic reliability theory.

3.1.2 Fuzzy Lifetime with Exponential Credibility Distribution

The simplest lifetime distribution family in probabilistic reliability theory is the exponential distribution family. We examine the counterpart in credibility fuzzy reliability. Let the lifetime T have a credibility distribution

$$F_T(t) = C\bar{F}(T(\theta) \leq t) = \begin{cases} 1 - \exp\left(-\frac{t}{\beta}\right) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3.11)$$

Thus the membership function is:

$$\mu_T(t) = \begin{cases} 2 \left[1 - \exp\left(-\frac{t}{\beta}\right) \right] & \text{if } 0 \leq t < \beta \ln 2 \\ 2 \exp\left(-\frac{t}{\beta}\right) & \text{if } t \geq \beta \ln 2 \\ 0 & \text{otherwise} \end{cases} \quad (3.12)$$

In the next section, we explore the expression of the credibility hazard function, we establish that the credibility hazard function $h_r(t)$ does not always take a constant value analogous to the constant hazard of random lifetime with exponential probability distribution.

3.2 Credibility-Hazard Function

In probabilistic reliability theory, the hazard function plays an important role. Lawless (1982) stated that hazard function represents an aspect of distribution that has direct physical meaning and that information about the nature the hazard function is helping in selecting a model. We propose a fuzzy hazard function concept associated with fuzzy variables based upon fuzzy credibility measure, and then denote the fuzzy hazard function as credibility hazard function. Furthermore, we explore credibility-hazard maintenance models parallel to those in probabilistic maintenance theory.

3.2.1 Credibility Distribution for Fuzzy Failure Time

In probabilistic reliability engineering context, time is assumed to be a random but observable quantity. Parallel to a probabilistic context, system failure time may be vague in nature and therefore a fuzzy variable. For example, a specified time for a production task need not be a deterministic quantity simply because there are various covariates that are deterministic.

Definition 3.5 A continuous fuzzy failure time, denoted by T , is a mapping from a credibility space $(\Theta, 2^\Theta, Cr)$ to the space of Borel sets $([0, +\infty), \mathcal{B}[0, +\infty))$. The credibility distribution, denoted by $F : [0, +\infty) \rightarrow [0, 1]$, of the fuzzy failure time T , is defined as:

$$F(t) = Cr\{T(\theta) \leq t\}, \quad \forall t \in [0, +\infty) \quad (3.13)$$

As defined in probabilistic case, the survival function, or the reliability function, denoted by $\bar{F}(t)$ or $R(t)$, is the credibility measure of event $\{T > t\}$. In other words, it is the credibility

measure that the individual item does not fail before it reaches an age t . $\bar{F}(t)$ is often called the reliability which is related to the failure distribution $F(t)$ and often denoted by $R(t)$:

$$R(t; \pi) = \bar{F}(t; \pi) = \text{C}\bar{r}(\theta \in 2^\theta : \bar{T}(\theta) > t) = 1 - \text{C}\bar{r}(\theta \in 2^\theta : \bar{T}(\theta) \leq t) = 1 - F(t; \pi). \text{ i.e.,}$$

$$R(t) = \text{C}\bar{r}\{T(\theta) > t\}, \quad \forall t \in [0, +\infty) \quad (3.14)$$

In the later development, the conditional credibility measure that the item fails in the interval $[t, t+s]$ given that it has not failed by age t . Here, let event $A = \{\theta : t \leq T(\theta) \leq t+s\}$ and event $B = \{\theta : T(\theta) > t\}$. Therefore:

$$F(s|t) = \text{C}\bar{r}(A|B) = \text{C}\bar{r}(\{\theta : t < T(\theta) \leq t+s\} | \{\theta : \bar{T}(\theta) > t\}) \quad (3.15)$$

By that:

$$\begin{aligned} \{\theta : t < \bar{T}(\theta) \leq t+s\}^c &= \{\theta : T(\theta) > t+s\} \cup \{\theta : T(\theta) \leq t\} \\ &= \{\theta : T(\theta) \leq t+s\}^c \cup \{\theta : T(\theta) > t\}^c \end{aligned} \quad (3.16)$$

and also $\{\theta : T(\theta) > t+s\} \cap \{\theta : T(\theta) \leq t\} = \emptyset$, we obtain,

$$\begin{aligned} \{\theta : t < \bar{T}(\theta) \leq t+s\}^c \cap \{\theta : \bar{T}(\theta) > t\} &= (\{\theta : \bar{T}(\theta) > t+s\} \cup \{\theta : \bar{T}(\theta) \leq t\}) \cap \{\theta : \bar{T}(\theta) > t\} \\ &= \{\theta : \bar{T}(\theta) > t+s\} \cup \emptyset \\ &= \{\theta : \bar{T}(\theta) > t+s\} \end{aligned} \quad (3.17)$$

According to **Definition 2.19**, for arbitrary $B \in 2^\theta$, the conditional credibility measure of event A given event B is

$$\begin{aligned} &\text{C}\bar{r}\{\{t \leq T(\theta) < t+s\} | \{T(\theta) \geq t\}\} \\ &= \frac{1}{2} \frac{(2\text{C}\bar{r}\{\{t \leq T(\theta) < t+s\} \cap \{T(\theta) \geq t\}\}) \wedge 1}{(2\text{C}\bar{r}\{T(\theta) \geq t\}) \wedge 1} + \frac{1}{2} \left[1 - \frac{(2\text{C}\bar{r}\{\{t \leq T(\theta) < t+s\}^c \cap \{T(\theta) \geq t\}\}) \wedge 1}{(2\text{C}\bar{r}\{T(\theta) \geq t\}) \wedge 1} \right] \end{aligned} \quad (3.18)$$

Then

$$\text{Cr}\left(\left\{\theta: t < \tilde{T}(\theta) \leq t+s\right\} \mid \left\{\theta: \tilde{T}(\theta) > t\right\}\right) = \frac{\text{Cr}\left(\left\{\theta: T(\theta) \leq t+s\right\}\right) - \text{Cr}\left(\left\{\theta: T(\theta) \leq t\right\}\right)}{\text{Cr}\left(\left\{\theta: \tilde{T}(\theta) > t\right\}\right)} \quad (3.19)$$

In other words,

$$F(s|t) = \frac{F(t+s) - F(t)}{1 - F(t)} \quad (3.20)$$

3.2.2 Credibility Hazard Concept

If we purely follow the route to defining the probabilistic hazard function, we have

$$h(t) = \lim_{s \rightarrow 0} \frac{\text{Cr}\left(\left\{t \leq T(\theta) < t+s\right\} \mid \left\{T(\theta) \geq t\right\}\right)}{s} \quad (3.21)$$

However, we may end at a messy complicated expression. Therefore, we propose an operational definition for credibility hazard function.

Definition 3.6 Let $T: (\Theta, 2^\Theta, \text{Cr}) \rightarrow [0, +\infty)$ be a fuzzy failure time with credibility reliability function $R(\cdot) = 1 - F(\cdot)$ and credibility density function $f(\cdot)$ respectively, then function

$$h(t) = \frac{f(t)}{R(t)}, \quad \forall t \in [0, +\infty) \quad (3.22)$$

is called as the credibility hazard function for fuzzy failure time T .

Remark 3.7 The hazard function derived by starting with Eq. (3.21) may not end at the same expression for a credibility hazard function derived from Eq. (3.22). However, we believe that **Definition 3.8** is a logical choice for defining a credibility hazard function even there may exist inconsistency between the two different starting points.

The operational definition of credibility hazard function takes the same form as its counterpart in probabilistic reliability theory. This structure brings us great convenience so that many basic relations in probabilistic reliability theory can be borrowed and used in the (v, \cdot) -credibility measure based reliability theory.

For example the credibility reliability function can be expressed as by credibility hazard $h(\cdot)$:

$$R(t) = \exp\left(-\int_0^t h(u) du\right) \quad (3.23)$$

3.2.3 Models of Credibility hazard function

The credibility hazard function is a new development and there is no existing model available in the literature. Therefore, we propose three credibility hazard function models.

- (1) Parallel to probabilistic reliability theory, a very typical example of hazard function for fuzzy failure time is constant, i.e., $h(t; \lambda) = \lambda > 0$. Then the corresponding credibility reliability function is $R(t, \lambda) = e^{-\lambda t}$, $t \geq 0$, and the credibility failure time distribution is $F(t, \lambda) = 1 - e^{-\lambda t}$, $t \geq 0$. Then the membership function for the fuzzy age $T(\theta)$ is:

$$\mu_T(t) = \begin{cases} 2(1 - e^{-\lambda t}) & \text{if } t < \ln 2/\lambda \\ 1 & \text{if } t = \ln 2/\lambda \\ 2e^{-\lambda t} & \text{if } t > \ln 2/\lambda \end{cases} \quad (3.24)$$

- (2) Another typical hazard function used in probabilistic reliability theory is the Weibull hazard function. Let us assume that $h(t; (\alpha, \beta)) = (\beta/\alpha)(t/\alpha)^{\beta-1}$, $\alpha > 0, \beta > 0$ is the hazard function for a fuzzy failure time. If $\beta = 1$, it is a hazard function taking a constant value. The credibility reliability function $R(t; (\alpha, \beta)) = \exp(-(t/\alpha)^\beta)$, $t \geq 0$, and the failure time credibility distribution is $F(t; (\alpha, \beta)) = 1 - \exp(-(t/\alpha)^\beta)$, then the membership function for the fuzzy lifetime $T(\theta)$ is

$$\mu_T(t) = \begin{cases} 2(1 - e^{-(t/\alpha)^\beta}) & \text{if } t < \alpha \sqrt[\beta]{\ln 2} \\ 1 & \text{if } t = \alpha \sqrt[\beta]{\ln 2} \\ 2e^{-(t/\alpha)^\beta} & \text{if } t > \alpha \sqrt[\beta]{\ln 2} \end{cases} \quad (3.25)$$

- (3) The third hazard function used in probabilistic reliability theory is the bathtub hazard function. Let us assume that a fuzzy failure time has a hazard function of form

$$h(t;(\alpha,\beta)) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left(-\left(\frac{t}{\alpha}\right)^\beta\right) \quad \alpha > 0, \beta > 0 \tag{3.26}$$

Then the credibility reliability function for the fuzzy failure time is

$$R(t,(\alpha,\beta)) = \exp\left(-\exp\left(-\left(\frac{t}{\alpha}\right)^\beta\right)\right) \quad \alpha > 0, \beta > 0 \tag{3.27}$$

and the failure time credibility distribution is

$$F(t,(\alpha,\beta)) = 1 - \exp\left(-\exp\left(-\left(\frac{t}{\alpha}\right)^\beta\right)\right) \quad \alpha > 0, \beta > 0 \tag{3.28}$$

Accordingly, we can find the a membership function taking the form

$$\mu_T(t) = \begin{cases} 2\left(1 - \exp\left(-e^{(t/\alpha)^\beta}\right)\right) & \text{if } t < \alpha\sqrt[\beta]{\ln(\ln 2)} \\ 1 & \text{if } t = \alpha\sqrt[\beta]{\ln(\ln 2)} \\ 2\exp\left(-e^{(t/\alpha)^\beta}\right) & \text{if } t > \alpha\sqrt[\beta]{\ln(\ln 2)} \end{cases} \tag{3.29}$$

Table 3.2.1 Three proposed credibility hazard function models

Name	Credibility density function & hazard function	
Exponential	hazard	λ
	reliability	$\exp(-\lambda t)$
Weibull	hazard	$(\beta/\alpha)(t/\alpha)^{\beta-1}$
	reliability	$\exp\left(-\left(t/\alpha\right)^\beta\right)(\beta/\alpha)(t/\alpha)^{\beta-1}$
Bathtub	hazard	$(\beta/\alpha)(t/\alpha)^{\beta-1} \exp\left(-\left(t/\alpha\right)^\beta\right)$
	reliability	$\exp\left(-\exp\left(-\left(t/\alpha\right)^\beta\right)\right)$

3.2.4 Proportional Hazards Model

The Cox's proportional hazards formulation (1972) can be used here too,

$$h(t; \Pi) = h_0(t; \Pi) \varphi(y; \omega) \quad (3.30)$$

where $\varphi: \mathbf{R}^\delta \rightarrow \mathbf{R}^+$. $\delta = \dim(y)$, the dimensionality of covariate vector y .

A typical form of φ is

$$\varphi(y; \omega) = \exp\left(\sum_{j=0}^{\delta} y_j \omega_j\right) \quad (3.31)$$

where $y_0 = 1$.

3.3 Credibility-Copula

In probability theory, copula is the concept for describing the dependence structure between two random variables and has obtained more and more attention in financial risk analysis. Parallel to that in probability theory, we defined a copula concept for credibility measure based fuzzy variables and investigate its the applicability in fuzzy reliability analysis.

We explore the basic property of a uniformly distributed fuzzy variable and then the credibility distribution transformation based on the (ν, \cdot) -credibility measure.

3.3.1 Uniform Distributed Fuzzy Variable

Uniform distributions and the distribution transformation play very important roles in probabilistic context. We explore whether similar developments exist within credibility measure theory. We say a fuzzy variable ν to be (standard) uniformly distributed if its credibility density takes the form:

$$\phi_\nu(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (3.32)$$

The credibility distribution of fuzzy standard uniform variable ν is:

$$\Phi_v(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (3.33)$$

And thus the induced membership function for the fuzzy of fuzzy standard uniform variable v is:

$$\mu_v(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leq x < 0.5 \\ 2-2x & 0.5 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \quad (3.34)$$

Conversely, the isosceles triangle membership located at (0,0), (0.5,1), and (1,0) characterizes a fuzzy standard uniform-distributed variable.

3.3.2 Credibility Distribution Transformation

Let ξ be a fuzzy variable with credibility distribution function $\Phi_\xi(\cdot)$. We are interested in the distribution for fuzzy variable $\Phi_\xi(\xi)$. Recall that the membership function for fuzzy variable ξ is,

$$\mu_\xi(\xi) = \begin{cases} 2\Phi_\xi(\xi) & \text{if } \Phi_\xi(\xi) < 0.5 \\ 1 & \text{if } \lim_{y \uparrow \xi} \Phi_\xi(y) < 0.5 \leq \Phi_\xi(\xi) \\ 2-2\Phi_\xi(\xi) & \text{if } 0.5 \leq \lim_{y \uparrow \xi} \Phi_\xi(y) \end{cases} \quad (3.35)$$

Then for any given value $x = \Phi_\xi(\xi)$, $\xi = \Phi^{-1}(x)$, thus we obtain the membership function for $X = \Phi_\xi(\xi)$,

$$\mu_{\Phi_\xi(\xi)}(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leq x < 0.5 \\ 2-2x & 0.5 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \quad (3.36)$$

In other words, the transformation of a fuzzy variable by its credibility distribution results in a fuzzy standard uniform distributed variable.

3.3.3 Definition of Bivariate Credibility-Copula

Similar to probabilistic copula theory, copula is a dependence index for a pair of fuzzy variables.

Let X and Y be two continuous fuzzy variables with credibility distributions Φ_X and

Φ_Y respectively and $H_{XY}(\cdot, \cdot)$ be their joint credibility distribution. Let $I = [0, 1]$.

Definition 3.9 A bivariate credibility-copula is a function $C: I \times I \rightarrow I$ such that:

1. $C(0, x) = C(x, 0) = 0$ and $C(1, x) = C(x, 1) = x$ for all $x \in I$;
2. $C(\cdot, \cdot)$ is 2-increasing: for $\forall a, b, c, d \in I$, $a < b$, and $c < d$,

$$\nu_c([a, b] \times [c, d]) = C(b, d) - C(a, d) - C(b, c) + C(a, c) \geq 0 \quad (3.37)$$

The function ν_c is called the C -volume of the rectangle $[a, b] \times [c, d]$.

In other words, a credibility-copula is the restriction to the unit square $I \times I = [0, 1] \times [0, 1]$ of a bivariate credibility distribution function whose margins are standard uniform. More formally, a copula C induces a credibility measure on $I \times I$ in terms of ν_c , i.e.,

$$\nu_c([0, u] \times [0, v]) = C(u, v) \quad (3.38)$$

Now let us state the credibility theoretical version of Sklar's theorem (Sklar, 1959)

Theorem 3.10 Let H be a bivariate credibility distribution function with marginal credibility distributions Φ_X and Φ_Y respectively. Then there exists a credibility-copula C such that

$H(x, y) = C(\Phi_X(x), \Phi_Y(y))$. Conversely, for any credibility distribution functions Φ_X and

Φ_Y and any credibility-copula C , the function H defined above is a two-dimensional credibility distribution with margins Φ_X and Φ_Y respectively. Furthermore, if Φ_X and Φ_Y are continuous, the credibility-copula is unique.

Based on the credibility theoretical version of Sklar's theorem, given a two-dimensional credibility distribution $H(x, y)$ and marginal credibility distributions Φ_X and Φ_Y , the credibility-copula is,

$$C(u, v) = H(\Phi_X^{-1}(u), \Phi_Y^{-1}(v)) \quad (3.39)$$

where the inverse of the credibility distribution denoted by $F^{-1}(\cdot)$ is defined as:

$$F^{-1}(u) = \sup\{x \mid F(x) \leq u\} \quad (3.40)$$

By noting the credibility distribution transformation arguments in Section 3.2.2, we observe that $U = \Phi_x(X)$ and $V = \Phi_y(Y)$ are fuzzy standard uniform-distributed variables on $[0,1]$, the credibility-copula is the joint uniform credibility distribution of two uniform-distributed fuzzy variable on $[0,1] \times [0,1]$.

Accordingly, we have

Definition 3.11 Let $H(x_1, x_2)$ be the joint credibility distribution of the fuzzy variable pair (X_1, X_2) and the continuous marginal credibility distributions be $F_1(x_1)$ and $F_2(x_2)$ respectively. Then for any $u_1, u_2 \in [0,1] \times [0,1]$, $C(u_1, u_2) = H(F_1^{-1}(u_1), F_2^{-1}(u_2))$ where the inverse of credibility distribution $F(\cdot)$ is $F^{-1}(u) = \inf\{x \in \mathbf{R} : F(x) \geq u, \forall u \in [0,1]\}$.

To investigate the joint survival function of credibility fuzzy lifetimes (X_1, X_2) , $\bar{H}(x_1, x_2)$, and their marginal credibility survival distributions are defined by $\bar{F}_1(x_1) = 1 - F_1(x_1)$ and $\bar{F}_2(x_2) = 1 - F_2(x_2)$ respectively, we note

$$\begin{aligned} \bar{H}(x_1, x_2) &= 1 - F_1(x_1) - F_2(x_2) + H(x_1, x_2) \\ &= \bar{F}_1(x_1) + \bar{F}_2(x_2) - 1 + C(F_1(x_1), F_2(x_2)) \\ &= \bar{F}_1(x_1) + \bar{F}_2(x_2) - 1 + C(1 - \bar{F}_1(x_1), 1 - \bar{F}_2(x_2)) \end{aligned} \quad (3.41)$$

Definition 3.12 Let $C(u_1, u_2)$ be the credibility copula for credibility fuzzy variable pair (X_1, X_2) with continuous marginal credibility survival distributions $\bar{F}_1(x_1)$ and $\bar{F}_2(x_2)$ respectively. Then

$$\hat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2) \quad (3.42)$$

is called the survival copula (reliability copula) and the joint credibility survival function is

$$\bar{H}(x_1, x_2) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2)) \quad (3.43)$$

Therefore, credibility-copula fully describes the functional relation between two fuzzy variables.

3.3.4 Properties of Bivariate Credibility-Copulas

If $C(u, v)$, in nature a joint credibility uniform distribution, possesses a joint credibility density, denoted by $\partial C(u, v)/\partial u \partial v$, we say $C(u, v)$ is absolutely continuous. Given a two-dimensional credibility distribution $H(x, y)$ and marginal credibility distributions Φ_x and Φ_y , then it can be shown that:

$$\max\{\Phi_x(x) + \Phi_y(y) - 1, 0\} \leq H(x, y) \leq \min\{\Phi_x(x), \Phi_y(y)\} \quad (3.44)$$

Or equivalently,

$$\varpi(u, v) = \max\{u + v - 1, 0\} \leq C(u, v) \leq \min\{u, v\} = \omega(u, v) \quad (3.45)$$

It can also be shown that $\varpi(u, v)$, $\omega(u, v)$, and $\chi(u, v) = uv$ are credibility-copulas. We also note that for continuous fuzzy variables X and Y , (i) if and only if X and Y is an increasing function of the other, then $C(u, v) = \omega(u, v)$; (ii) if and only if X and Y is a decreasing function of the other, then $C(u, v) = \varpi(u, v)$; and (iii) if and only if X and Y are independent, then $C(u, v) = \chi(u, v) = uv$ (which is referred to as *product copula*).

3.3.5 Archimedean Family of Bivariate Credibility-Copulas

Parallel to probabilistic copula theory, the Archimedean credibility-copula is a function $C: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{I}$ defined by the generator ϕ such that:

$$C(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)) \quad (3.46)$$

where $\phi^{[-1]}$ is the pseudo-inverse of ϕ :

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & t \in [0, \phi(0)] \\ 0 & t > \phi(0) \end{cases} \quad (3.47)$$

The generator $\phi: \mathbf{I} \rightarrow [0, \infty]$, is a strictly monotone decreasing and continuous convex function. An important property of Archimedean copula family is associativity: i.e., $C(C(u, v), w) = C(u, C(v, w))$. A simple example of Archimedean copula is the generator of the form $\ln((1 - \theta(1 - t))/t)$ and the

generating copula is $C(u, v) = uv / (1 - \theta(1-u)(1-v))$. But note that a one-parameter bivariate copula is not necessarily an Archimedean copula. A typical example is $C(u, v) = uv + \alpha uv(1-u)(1-v)$, $|\alpha| \leq 1$.

3.3.6 A Decomposition of Bivariate Copulas

Recently, Dos Anjos (2005) pointed out that in probabilistic context there exists a local dependence measure, which provides explicit and precise information of the underlying dependence structure and helps to reformulate a bivariate distribution and its associated copula. Now we explore parallel developments based upon the credibility theory.

Definition 3.13 (Expected value and variance) Let X be a fuzzy variable with credibility distribution Φ_x . If

$$\lim_{x \rightarrow -\infty} \Phi_x(x) = 0, \quad \lim_{x \rightarrow \infty} \Phi_x(x) = 1 \quad (3.48)$$

and the Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} x d\Phi_x(x)$ is finite, then we define $\int_{-\infty}^{\infty} x d\Phi_x(x)$ as the expected value of fuzzy variable X , denoted as $E[X]$. Furthermore, we define $E[(X - E[X])^2]$ as the variance of fuzzy variable, denoted by $V[X]$.

Definition 3.14 (Covariance) Let two fuzzy variables X and Y have a bivariate credibility distribution $H(x, y)$ and the two marginal credibility distributions Φ_x and Φ_y , with finite expected values $E[X]$ and $E[Y]$ respectively. Then $E[(X - E[X])(Y - E[Y])]$ and $E[(X - E[X])(Y - E[Y])] / (\sqrt{V[X]}\sqrt{V[Y]})$ called the covariance and correlation of fuzzy variables X and Y respectively.

Remark 3.15 Under specific conditions, it can be shown that

$$E[(X - E[X])(Y - E[Y])] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - E[X])(y - E[Y]) dH(x, y) \quad (3.49)$$

Definition 3.16 Let two fuzzy variables X and Y have a bivariate credibility distribution $H(x, y)$ and the two marginal credibility distributions Φ_x and Φ_y . The underlying Spearman function ρ_H is

the correlation coefficient between $\vartheta\{\Phi_x(X) \leq \Phi_x(x)\}$ and $\vartheta\{\Phi_y(Y) \leq \Phi_y(y)\}$ for each $(x, y) \in [-\infty, \infty] \times [-\infty, \infty]$

$$\rho_H(x, y) = \frac{H(x, y) - \Phi_x(x)\Phi_y(y)}{\sqrt{\Phi_x(x)\Phi_y(y)(1-\Phi_x(x))(1-\Phi_y(y))}} \quad (3.50)$$

Similarly, for copulas

$$\rho_C(u, v) = \frac{C(u, v) - uv}{\sqrt{uv(1-u)(1-v)}} \quad (3.51)$$

Theorem 3.17 Let two fuzzy variables X and Y have a bivariate credibility distribution $H(x, y)$ and the two marginal credibility distributions Φ_x and Φ_y have associated copula $C(u, v)$, then

$$C(u, v) = uv + \rho_C(u, v)\sqrt{uv(1-u)(1-v)} \quad (3.52)$$

and

$$H(x, y) = \Phi_x(x)\Phi_y(y) + \rho_H(x, y)\sqrt{\Phi_x(x)\Phi_y(y)(1-\Phi_x(x))(1-\Phi_y(y))} \quad (3.53)$$

Furthermore,

$$\rho_H(\Phi_x(x), \Phi_y(y)) = \rho_C(u, v) \quad (3.54)$$

for all $(x, y) \in [-\infty, \infty] \times [-\infty, \infty]$ such that $u = \Phi_x(x)$ and $v = \Phi_y(y)$.

Remark 3.18 The product copula, $\chi(u, v) = uv$, represents non-dependence or independence. The value of product copula $\chi(u, v)$, measures the degree of dependence. The alternative representation of copula Eq. (3.52) reveals a fundamental fact that the degree of dependence between two fuzzy variables is the sum of the product copula factor $\chi(u, v) = uv$ and an adjusted local dependence factor, $D(u, v)$. In this sense, copula is a quantity composed of two factors: independence measure and dependence measure. Therefore, the distance between copula, $C(u, v)$ and the product copula, $\chi(u, v)$, $D(u, v) = C(u, v) - \chi(u, v)$, measures the true degree of dependence, i.e., the degree apart from independence, while ρ_C in Eq. (3.51) is the standardized true degree of dependence in a local sense, i.e., at about given point $(u, v) \in I \times I$. The decomposition of a copula offers a refined

understanding of the measure of dependence. For example, let $C(u, v) = uv / (1 - \theta(1-u)(1-v))$, then $D(u, v) = uv(1 - 1 / (1 - \theta(1-u)(1-v)))$ measures the pure bivariate dependence. Another example is $C(u, v) = uv + \theta uv(1-u)(1-v)$, $|\theta| \leq 1$, for which $D(u, v) = uv\theta(1-u)(1-v)$ and $\rho_c(u, v) = \theta \sqrt{uv(1-u)(1-v)}$.

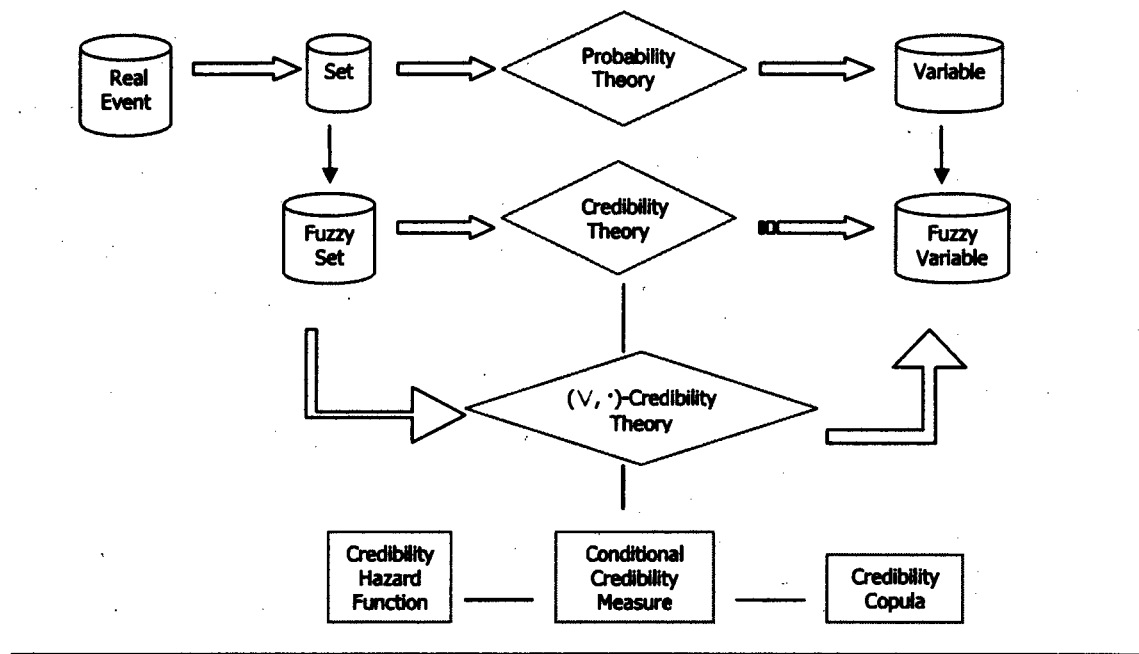


Figure 3.3.1 Relational Structure of Concepts

Figure 3.3.1 illustrates the concepts of this chapter and their derivation, and relationships between concepts.

Chapter 4. Maximum Entropy Principle and the Optimal Data-assimilating Membership Function

As we stated in Chapter 1, since Zadeh (1965) proposed Fuzzy mathematics, the attack on the subjectivity of the fuzzy methodology has not ended. In this Chapter, we propose a data-assimilating parameter approach to resolve this controversy. We further investigate the data-assimilating fuzzy credibility distribution based on a maximum entropy principle. Finally an illustrative example represents the maximum entropy estimation through a truncated hyperbolic tangent membership function.

4.1 Maximum Entropy Principle

A fundamental and thorny issue in fuzzy mathematical developments is the determination of the functional form of the fuzzy membership function and associated parameters. In contrast to its probabilistic counterpart, the parameter estimation is very difficult to conduct.

4.1.1 Fuzzy Entropy

Entropy is a measure of uncertainty. The entropy of De Luca and Termini (1972) characterizes uncertainty resulting primarily from the linguistic vagueness rather than resulting from information deficiency, and vanishes when the fuzzy variable takes all the values with membership degree 1. However, we hope that the degree of uncertainty is 0 when the fuzzy variable degenerates to a crisp number, and is maximum when the fuzzy variable has an equally possible form, i.e., all values have the same possibility. In order to meet such a requirement, Li and Liu (2007) proposed a new definition based on credibility measure.

Definition 4.1 (Fuzzy Entropy) Let ξ be a continuous fuzzy variable defined on a credibility space $(\Theta, 2^\Theta, Cr)$, then the fuzzy entropy, $H[\xi]$, is defined by

$$H[\xi] = \int_{-\infty}^{\infty} S(\text{Cr}(\{\theta : \xi(\theta) = u\})) du \quad (4.1)$$

where

$$S(t) = -t \ln t - (1-t) \ln(1-t) \quad (4.2)$$

For convenience, we define function $S(\cdot)$ as the entropy density

4.1.2 Maximum Entropy Principle

There are possibly infinitely many members in the family of membership functions. The maximum entropy principle provides a guideline for selecting of parameter(s) that maximizes the value of entropy and satisfies the given constraints, namely specifying a membership function of given function form.

In mathematical language,

$$\begin{aligned} & \max_{\Pi} H[\xi] \\ & \text{s.t.} \\ & \zeta(\Pi) = \zeta_0 \end{aligned} \quad (4.3)$$

where Π is the parameter space and the parameter constraint is $\zeta(\Pi) = \zeta_0$.

4.1.3 Entropy Estimation

Note that

$$\text{Cr}(\{\theta : \xi(\theta) = x\}) = \frac{1}{2} \left(\mu(x) + 1 - \sup_{y < x} \mu(y) \right) \quad (4.4)$$

For example, if the credibility-hazard function is assumed to have the form of

$$h(t; (\alpha, \beta)) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha} \right)^{\beta-1}, \quad \alpha > 0, \beta > 0, \text{ then membership function } \mu(x) = 2 \left(1 - e^{-(x/\alpha)^\beta} \right) \text{ is monotone}$$

increasing for $x \in [0, \alpha \sqrt[\beta]{\ln 4})$ and therefore, for any given $x_0 \in [0, \alpha \sqrt[\beta]{\ln 4})$, $\sup_{y < x_0} \mu(y) = 2 \left(1 - e^{-(x_0/\alpha)^\beta} \right)$.

While, for $\forall x_0 \geq \alpha^{\beta/\ln 4}$, membership $\mu(x) = 2e^{-(x/\alpha)^\beta}$ is monotone decreasing from 1, and thus

$$\sup_{y < x_0} \mu(y) = 2e^{-(y/\alpha)^\beta} = 1$$

$$\sup_{y < x} \mu(y) = \begin{cases} 2(1 - e^{-(x/\alpha)^\beta}) & x < \alpha^{\beta/\ln 4} \\ 1 & x \geq \alpha^{\beta/\ln 4} \end{cases} \quad (4.5)$$

Accordingly,

$$\text{Cr}(\{\theta : \xi(\theta) = x\}) = \begin{cases} \frac{1}{2} & x < \alpha^{\beta/\ln 4} \\ 1 - e^{-(x/\alpha)^\beta} & x \geq \alpha^{\beta/\ln 4} \end{cases} \quad (4.6)$$

which implies a function

$$S(\text{Cr}(\{\theta : \xi(\theta) = x\})) = \begin{cases} \ln 2 & x < \alpha^{\beta/\ln 4} \\ -(1 - e^{-(x/\alpha)^\beta}) \ln(1 - e^{-(x/\alpha)^\beta}) + (x/\alpha)^\beta e^{-(x/\alpha)^\beta} & x \geq \alpha^{\beta/\ln 4} \end{cases} \quad (4.7)$$

The entropy function for the fuzzy life time \tilde{T} , is given by

$$\begin{aligned} H[\tilde{T}] &= \int_0^\infty S(\text{Cr}(\{\theta : \tilde{T}(\theta) = t\})) dt \\ &= \int_0^{\alpha^{\beta/\ln 2}} (\ln 2) dx + \int_{\alpha^{\beta/\ln 2}}^\infty \left[-(1 - e^{-(x/\alpha)^\beta}) \ln(1 - e^{-(x/\alpha)^\beta}) + (x/\alpha)^\beta e^{-(x/\alpha)^\beta} \right] dx \\ &= \alpha(\ln 2)^{1+\frac{1}{\beta}} + \int_{\alpha^{\beta/\ln 2}}^\infty \left[-(1 - e^{-(x/\alpha)^\beta}) \ln(1 - e^{-(x/\alpha)^\beta}) + (x/\alpha)^\beta e^{-(x/\alpha)^\beta} \right] dx \end{aligned} \quad (4.8)$$

The maximization of entropy function is reached by searching (α, β) , for which the solution satisfies the nonlinear equation system,

$$\begin{cases} \frac{\partial H[\tilde{T}]}{\partial \alpha} = 0 \\ \frac{\partial H[\tilde{T}]}{\partial \beta} = 0 \end{cases} \quad (4.9)$$

with appropriate constraints from system performance data. This structure implies a constrained solution search for (α, β) .

However, we note that the setting of constraints extracted from read data is difficult for most of the engineering circumstances. Therefore, we suggest an empirical object function for parameter searching since the optimal value of the data-dependent object function has to reflect the constraints specified by system performance data implicitly. The data constrained object function is the average of entropies evaluated at $\{x_1, x_2, \dots, x_N\}$, i.e.,

$$J(\pi | x_1, x_2, \dots, x_N) = \frac{1}{N} \sum_{i=1}^N H[\xi = x_i; \pi] \quad (4.10)$$

Notice that as $N \rightarrow \infty$, $J \rightarrow H[\xi]$ asymptotically with parameter π which constrained by data structure.

Therefore, in engineering circumstances, we work with the problem of maximizing object function $J(\pi | x_1, x_2, \dots, x_N)$ which can be achieved by solving nonlinear equation system

$$\begin{cases} \frac{\partial J(\pi | x_1, x_2, \dots, x_N)}{\partial \pi_1} = 0 \\ \frac{\partial J(\pi | x_1, x_2, \dots, x_N)}{\partial \pi_2} = 0 \\ \vdots \\ \frac{\partial J(\pi | x_1, x_2, \dots, x_N)}{\partial \pi_q} = 0 \end{cases} \quad (4.11)$$

We can use Newton Raphson method to search solution $(\pi_1, \pi_2, \dots, \pi_q)$.

4.1.4 An Industrial Example

We explore an estimation problem for the parameters associated with credibility-hazard function by proposing the sample mean of entropy as object function in terms of maximum entropy principle. To use cement roller data as an illustrative example, Love and Guo (1990) analyzed Cement Roller data collected from a Canadian firm.

Table 4.1.1 Lifetime, failure status, covariates (D,B,W) and repair cost

<i>Lifetime</i>	<i>Failure</i>	<i>D</i>	<i>B</i>	<i>W</i>	<i>Cost of Repair</i>
54	pm	12	10	800	93
133	failure	13	16	1200	142

147	pm	15	12	1000	300
72	failure	12	15	1100	237
105	failure	13	16	1200	0
115	pm	11	13	900	525
141	pm	16	13	1000	493
59	failure	8	16	1100	427
107	pm	9	11	800	48
59	pm	8	10	900	1115
36	failure	11	13	1000	356
210	pm	8	10	800	382
45	failure	10	19	1300	37
69	pm	12	14	1100	128
55	failure	13	18	1200	37
74	pm	15	12	800	93
124	failure	12	17	1100	735
147	failure	13	16	1100	1983
171	pm	11	13	900	350
40	failure	13	16	1100	9
77	failure	14	17	1100	1262
98	failure	12	15	1100	142
108	failure	12	15	1100	167
110	pm	16	14	1100	457
85	failure	8	19	1300	166
100	failure	12	15	1000	144
115	failure	13	16	1200	24
217	pm	9	11	900	474
25	failure	15	18	1200	0
50	failure	11	13	1100	738
55	pm	8	10	800	119

In Table 1, in the failure (status) column, pm is referred to as planned maintenance and failure is an unexpected outage event.

For the illustrative purposes, we use credibility-hazard function with form of

$h(t;(\alpha, \beta)) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1}$, $\alpha > 0, \beta > 0$. The data set includes $N = 31$ points. A critical issue for using

the object function $J(\alpha, \beta | x_1, x_2, \dots, x_n)$ is that the expression of $\text{Cr}(\{\theta: \bar{T}(\theta) = t; \alpha, \beta\})$ is dependent upon parameters, α and β .

As a matter of fact, for the given credibility-hazard function, its membership function has membership grade 1 at $t = \alpha^{\beta/\ln 2}$ (which is the turning point of membership function induced by credibility-hazard $h(t; (\alpha, \beta)) = (\beta/\alpha)(t/\alpha)^{\beta-1}$). The determination of the turning point definitely deserves further research. However, at the present, we just search for α and β by solving the following nonlinear equation system:

$$\begin{cases} \frac{\partial J(\alpha, \beta; \{t_1, t_2, \dots, t_{31}\})}{\partial \alpha} = 0 \\ \frac{\partial J(\alpha, \beta; \{t_1, t_2, \dots, t_{31}\})}{\partial \beta} = 0 \end{cases} \quad (4.12)$$

where

$$J(\alpha, \beta; \{t_1, t_2, \dots, t_{31}\}) = \frac{1}{31} \sum_{i=1}^{31} H[\tilde{T} = t_i; \alpha, \beta] \quad (4.13)$$

From this example, it is obvious that the turning point of the parameter-dependent credibility-hazard-induced membership function is worthy of future investigation.

4.2 Sample Entropy and Maximum Entropy Data-assimilation

What we aim at is not obtaining parameters from the theoretical entropy expression rather determining the parameters based on observations of the fuzzy variable Z . In other words, we need to develop a criterion to obtain a data-assimilating membership function. Therefore, we suggest an empirical object function for parameter estimation since the optimal value of the data-dependent object function has to reflect the constraints specified by observational data implicitly. The data assimilated object function is the average of sample entropy densities evaluated at $\{z_1, z_2, \dots, z_n\}$ respectively, i.e.,

$$J = \frac{1}{n} \sum_{i=1}^n S(\text{Cr}\{Z(\theta) = z_i; \Pi\}) \quad (4.14)$$

Definition 4.2 (Fuzzy sample entropy) For a given fuzzy variable Z defined on a credibility space $(\Theta, 2^\Theta, \text{Cr})$, if a sample of Z is taken and denoted as $\{z_1, z_2, \dots, z_n\}$, then we call the sample function J defined by Eq. (4.14) a sample entropy.

What we expect is that the average of sample entropy densities should tend to the corresponding theoretical entropy as the sample size tends to infinity. In mathematical language,

$$J \rightarrow H[Z; \Pi] \quad (4.15)$$

as the sample size $n \rightarrow +\infty$.

Theorem 4.3 A necessary condition for the sample entropy J defined by Eq. (4.14) to tend to its theoretical entropy (**Definition 4.1**), as that $H[Z; \Pi]$ is the membership of fuzzy variable Z defined on a finite interval.

Denote the sample entropy as J , which satisfies the necessary condition stated in **Theorem 4.3**. Let $\Pi = (\pi_1, \pi_2, \dots, \pi_p)^T$, then the maximum entropy estimate $\hat{\Pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_p)^T$, satisfy the following equation system

$$\begin{cases} \frac{\partial J}{\partial \pi_1} = 0 \\ \frac{\partial J}{\partial \pi_2} = 0 \\ \vdots \\ \frac{\partial J}{\partial \pi_p} = 0 \end{cases} \quad (4.16)$$

It is possible that there is a set of constraints required for a unique solution.

4.3 Fuzzy Failure Time with a Sample Membership Function

Although we propose three credibility hazard function models we should be fully aware that all the three hazards are defined on $[0, +\infty)$, which has an infinite length. The three credibility hazard

models may not be guaranteed to permit maximum entropy estimation. However, even we work within a finite-domain fuzzy failure time model, for example, with a trapezoidal membership function

$$\mu(t) = \begin{cases} \frac{t-a}{b-a} & \text{if } a < t \leq b \\ 1 & \text{if } b < t \leq c \\ \frac{d-t}{d-c} & \text{if } c < t \leq d \\ 0 & \text{otherwise} \end{cases} \quad (4.17)$$

it is still difficult to specify the sample entropy function because the unknown parameters (a, b, c, d) become the threshold values during the parameter estimating.

4.3.1 Sample Entropy for Fuzzy Failure Time

As an illustrative example, we investigate a fuzzy failure time with a truncated hyperbolic tangent membership function proposed by Guo et al (2007),

$$\mu(t) = \begin{cases} \frac{1}{2} \left(\tanh\left(\frac{t-\delta}{\eta}\right) + 1 \right) & t \in [0, L] \\ 0 & t < 0 \text{ or } t > L \end{cases} \quad (4.18)$$

Then we have

$$\text{Cr}\{\{\theta: T(\theta) = t\}\} = \begin{cases} \frac{1}{4} \left(\tanh\left(\frac{t-\delta}{\eta}\right) + 1 \right) & t \in [0, L] \\ 0 & t < L, \text{ or } t > L \end{cases} \quad (4.19)$$

For a given observation set $\{t_1, t_2, \dots, t_n\}$ from a fuzzy failure time T , in terms of Eq. (2.4), for each observation t_i , $i = 1, 2, \dots, n$, the sample entropy density, $S(\text{Cr}\{\{\theta: T(\theta) = t_i\}\})$, $i = 1, 2, \dots, n$, has to be evaluated.

Then sample entropy is

$$J(\delta, \eta) = \frac{1}{n} \sum_{i=1}^n S(\text{Cr}\{T(\theta) = t_i\}) \quad (4.20)$$

where

$$\begin{aligned} S(\text{Cr}\{T(\theta) = t_i\}) = & -\text{Cr}\{T(\theta) = t_i\} \ln \text{Cr}\{T(\theta) = t_i\} \\ & - (1 - \text{Cr}\{T(\theta) = t_i\}) \ln(1 - \text{Cr}\{T(\theta) = t_i\}) \end{aligned} \quad (4.21)$$

4.3.2 Entropy Estimation for Fuzzy Failure Time

In order to obtain the data-assimilating parameter pair (δ, η) , we need to solve the following equation system

$$\begin{cases} \frac{\partial J(\delta, \eta)}{\partial \delta} = 0 \\ \frac{\partial J(\delta, \eta)}{\partial \eta} = 0 \end{cases} \quad (4.22)$$

where

$$J(\delta, \eta) = \frac{1}{n} \sum_{i=1}^n S\left(\frac{1}{4} \left(\tanh\left(\frac{t - \delta}{\eta}\right) + 1 \right)\right) \quad (4.23)$$

Note that the truncated hyperbolic tangent membership function is a two parameter family and there is no threshold point involved in the definition of the sample entropy and parameter estimation.

4.4 Data-assimilation of 1-dimensional Credibility Distribution

Finding an estimated credibility distribution based on observed data from a fuzzy variable is a highly critical task in practice because a serious researcher has to defend an important principle-objectivity. In other words, we should establish the credibility distribution in terms of data information collected objectively from the fuzzy variable itself. We explore a non-parametric approach-kernel estimation to propose the data-assimilation under the maximum entropy principle.

4.4.1 Kernel Estimation in Probability Theory

Definition 4.4 A kernel is a function $K(x) = c\kappa(\|x\|^2)$ mapping from \mathbb{R}^d to $[0, \infty)$, where $\kappa(\cdot)$ is a piecewise nonnegative monotone decreasing function such that $\int_0^\infty k(r)dr < \infty$ and c is a constant.

Two common kernels used in statistical theory are the Gaussian kernel, $(1/\sqrt{2\pi})e^{-t^2/2}$, and the Epanechnikov kernel, $(3/4)(1-\|x\|^2)$. For finite support, both kernel reduce to $K(x/h) = 0$, if $\|x\| > h$, where parameter $h > 0$ is called the bandwidth for the kernel function K . For the bivariate case, the Epanechnikov product kernel takes the form

$$K(x, y; h) = \begin{cases} \frac{9}{16} \left(1 - \left(\frac{x}{h}\right)^2\right) \left(1 - \left(\frac{y}{h}\right)^2\right) & \text{if } |x| < h, |y| < h \\ 0 & \text{otherwise} \end{cases} \quad (4.24)$$

and the Epanechnikov radial kernel takes the form

$$K(x, y; h) = \frac{3}{4} \left(1 - \frac{x^2 + y^2}{h^2}\right) \quad (4.25)$$

For data assimilation purposes, for a one-dimensional data sample $\{x_1, x_2, \dots, x_n\}$ the credibility kernel density takes a form

$$f_h(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x_i - x}{h}\right) \quad (4.26)$$

While for the two-dimensional case, data sample $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$, the credibility kernel density takes the form

$$f_h(x, y) = \frac{1}{nh} \sum_{i=1}^n K(x, y; h) \quad (4.27)$$

It is obvious that in order to have the highest degree of data assimilation an optimal bandwidth $h > 0$ has to be selected. Different from inference in the probabilistic sense, the inference based

on credibility theory must develop its own criterion. One of the criteria is the maximum entropy principle.

4.4.2 Maximum Entropy Kernel Estimation of 1-Dimensional Credibility Distribution

Similar to the development of Maximum Entropy in Section 4.1, we now argue for a start from a credibility kernel density to a credibility distribution, then to an induced membership function and finally a fuzzy entropy.

For a one-dimensional case, given the sample $\{x_1, x_2, \dots, x_n\}$, the credibility kernel density is

$f_h(x) = (1/nh) \sum_{i=1}^n K((x_i - x)/h)$, and the credibility distribution takes the form

$$\hat{\Phi}_x(x) = \frac{1}{nh} \sum_{i=1}^n \int_{-\infty}^x K\left(\frac{x_i - u}{h}\right) du \quad (4.28)$$

Thus the membership function can be determined by

$$\hat{\mu}_x(x) = \begin{cases} \frac{2}{nh} \sum_{i=1}^n \int_{-\infty}^x K\left(\frac{x_i - u}{h}\right) du & \text{if } \hat{\Phi}_x(x) < 0.5 \\ 1 & \text{if } \lim_{y \uparrow x} \hat{\Phi}_x(y) < 0.5 \leq \hat{\Phi}_x(x) \\ 2 - \frac{2}{nh} \sum_{i=1}^n \int_{-\infty}^x K\left(\frac{x_i - u}{h}\right) du & \text{if } 0.5 \leq \lim_{y \uparrow x} \hat{\Phi}_x(y) \end{cases} \quad (4.29)$$

Note that $\text{Cr}(\{\theta: \xi(\theta) = x\}) = \frac{1}{2} \left(\mu(x) + 1 - \sup_{y < x} \mu(y) \right)$, and that the fuzzy entropy is bandwidth-dependent

$$H[\xi; h] = \int_{-\infty}^{\infty} S_h(\text{Cr}(\{\theta: \xi(\theta) = u\})) du \quad (4.30)$$

However, we note that the setting of suitable constraints requires extracting evidences from data and is difficult for most engineering circumstances. Therefore, we suggest an empirical object function for parameter to estimate since the optimal value of the data-dependent object function

has to reflect the constraints specified by system performance data implicitly. The data constrained object function is the average of entropies evaluated at $\{x_1, x_2, \dots, x_n\}$, i.e.,

$$J(h | x_1, x_2, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n H[\xi = x_i; h] \quad (4.31)$$

Note that as $n \rightarrow \infty$, $J_h \rightarrow H[\xi; h]$ asymptotically with the estimate parameter constrained by data structure. Finally, the seeking an optimal bandwidth value is simply the problem

$$\max_{h>0} J(h | x_1, x_2, \dots, x_n) \quad (4.32)$$

or alternatively, finding the solution of the equation, denoted by \hat{h} , of

$$\frac{dJ(h | x_1, x_2, \dots, x_n)}{dh} = 0 \quad (4.33)$$

From the optimal bandwidth, \hat{h} , the maximum entropy kernel credibility distribution based on sample data $\{x_1, x_2, \dots, x_n\}$ is obtained, denoted by, $\hat{\Phi}_{x,h}(x)$. Accordingly, the values of the kernel credibility distribution at the points $\{x_1, x_2, \dots, x_n\}$ are denoted by $\{\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n\}$, where $\hat{u}_i = \hat{\Phi}_{x,h}(x_i)$, $i = 1, 2, \dots, n$. Similarly, the kernel credibility distribution $\hat{\Phi}_{y,h}(y)$ can be obtained and values of the kernel credibility distribution at the points $\{y_1, y_2, \dots, y_n\}$ are denoted by $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n\}$, where $\hat{v}_i = \hat{\Phi}_{y,h}(y_i)$, $i = 1, 2, \dots, n$.

4.5 Data-assimilation of Bivariate Credibility Copula

A bivariate credibility-copula is in essential a bivariate joint uniform distribution. However, the copula representation has an advantage within its simple forms, particularly, the one-parameter copula family. This feature may make the data assimilation process easier than that of the kernel estimation of the credibility distribution directly.

In terms of the arguments of Section 4.4, for a bivariate sample data $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$, the entropy kernel estimated bivariate uniform values are $\{(\hat{u}_1, \hat{v}_1), (\hat{u}_2, \hat{v}_2), \dots, (\hat{u}_n, \hat{v}_n)\}$.

Recall from Section 3.3.3 that a copula, $C(u, v)$, in nature a bivariate joint credibility uniform distribution, possesses a joint credibility density, denoted by $\partial^2 C(u, v)/\partial u \partial v$. We say $C(u, v)$ is absolutely continuous. Given a two-dimensional credibility distribution $H(x, y)$ and marginal credibility distributions Φ_x and Φ_y , then its associated copula is $C(u, v)$. Conversely, given a copula $C(u, v)$ and marginal credibility distributions Φ_x and Φ_y , then the bivariate joint distribution $H(x, y)$ can be found.

The joint bivariate membership function is defined by

$$\mu(x, y) = (2\text{Cr}((X, Y) = (x, y))) \wedge 1 \quad (4.34)$$

In terms of the equality: $H(x, y) = C(\Phi_x(x), \Phi_y(y))$, we have

$$\mu(x, y) = \begin{cases} C(\Phi_x(x), \Phi_y(y)) & \text{if } C(\Phi_x(x), \Phi_y(y)) < 0.5 \\ 1 & \text{if } \lim_{(s,t) \uparrow (x,y)} C(\Phi_x(s), \Phi_y(t)) < 0.5 \leq C(\Phi_x(x), \Phi_y(y)) \\ 2 - C(\Phi_x(x), \Phi_y(y)) & \text{if } 0.5 \leq \lim_{(s,t) \uparrow (x,y)} C(\Phi_x(s), \Phi_y(t)) \end{cases} \quad (4.35)$$

$$\text{Also, } \text{Cr}(\{\theta : (X, Y)(\theta) = (x, y)\}) = \frac{1}{2} \left(\mu(x, y) + 1 - \sup_{(s,t) \uparrow (x,y)} \mu(s, t) \right)$$

Accordingly, the entropy takes the form

$$H[(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\text{Cr}(\{\theta : (X, Y)(\theta) = (s, t)\})) ds dt \quad (4.36)$$

where $S(t) = -t \ln t - (1-t) \ln(1-t)$

Recall that the kernel estimated bivariate joint uniform distribution has estimated sample data

$\{(\hat{u}_1, \hat{v}_1), (\hat{u}_2, \hat{v}_2), \dots, (\hat{u}_n, \hat{v}_n)\}$, thus $\{C(\hat{u}_1, \hat{v}_1), C(\hat{u}_2, \hat{v}_2), \dots, C(\hat{u}_n, \hat{v}_n)\}$ are the observed copula values.

Accordingly, the observed entropy can be calculated in the form

$$J(\alpha | C(\hat{u}_i, \hat{v}_i), i = 1, 2, \dots, n) = \frac{1}{n} \sum_{i=1}^n H[(X, Y) = (x_i, y_i); \alpha] \quad (4.37)$$

Solving equation $dJ/d\alpha = 0$ gives the maximum entropy estimation of the copula $C(u, v; \alpha)$.

4.6 Simulation of Fuzzy Failure Times

In probabilistic reliability modeling, particularly in the system maintenance optimal planning models, simulation plays more and more important roles because of current powers of computation.

For fuzzy credibility measure based reliability modeling and analysis, the credibility measure grade is the critical quantity to be simulated. Liu (2004) gives a detailed algorithm for performing the simulation.

Let function $\psi: \mathbb{R} \rightarrow \mathbb{R}$, and ξ be a fuzzy failure time on the credibility space $(\Theta, 2^\Theta, \text{Cr})$. Let

$$L = \text{Cr}\{\psi(\xi) \leq 0\} \quad (4.38)$$

Simulation Algorithm:

Step 1. Random generate θ_k from Θ such that

$$\text{Cr}\{\theta_k\} \geq \frac{\varepsilon}{2} \quad (4.39)$$

Define ν_k such that $\nu_k = (2\text{Cr}\{\theta_k\}) \wedge 1$

Then generate $\xi(\theta_k)$, $k = 1, 2, \dots, N$, respectively for given sufficiently small $\varepsilon > 0$.

Step 2. Return L in terms of the following estimation formula

$$L = \frac{1}{2} \max_{1 \leq k \leq N} \{v_k \mid \psi(\xi(\theta_k)) \leq 0\} + \frac{1}{2} \min_{1 \leq k \leq N} \{1 - v_k \mid \psi(\xi(\theta_k)) > 0\} \quad (4.40)$$

So far, we have established a framework for fuzzy failure time modeling and analysis.

Chapter 5. Random Fuzzy Variable and its Average Chance distribution

It is a common knowledge that dealing with real-valued numbers is much easier than dealing with subsets or events. In standard probability theory, random variable and the distribution function play important roles for converting set-based arguments into variable-based arguments, which result in great conveniences in applications. As a mathematical operation, we prefer to dealing with real-valued number a . We call such a modeling idea as *variable modeling*. If we can have a variable oriented approach for every uncertainty case, then the modeling efforts are greatly simplified.

However, in system lifetime data, random uncertainty and fuzzy uncertainty often coexist intrinsically. Thus we face the complexities generated by mathematical treatments for uncertain phenomena at subset or event level except that in probability theory. Besides random uncertainty theory or probability theory, the mathematical treatment of other types of uncertain phenomena, say, fuzzy events, random interval-events, or random fuzzy events etc, did not exhibit a consciousness of uncertainty variable modeling until the general uncertainty theoretical framework proposed by Liu (2004).

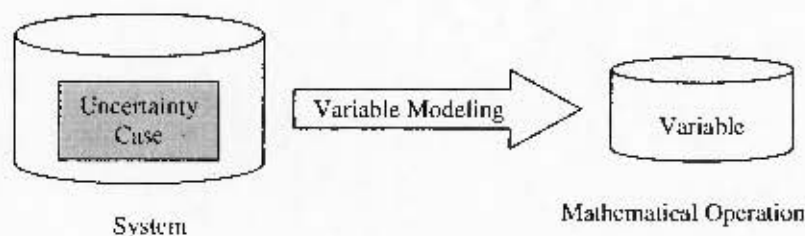


Figure 5.1.1 Direction converting illustration

In this Chapter, we review Liu's Chance distribution theory for exploring a *variable modeling* idea for random fuzzy events and accordingly propose a data-assimilating membership function. We explore the random fuzzy modeling of system lifetimes and propose a two-tier approach for seeking a data-assimilating membership function and statistically estimated probability distribution for a full data-assimilating chance distribution of system lifetimes.

5.1 A Review of (\vee, \cdot) -Chance Measure Theory

(\vee, \cdot) -Chance measure theory is built on axiomatic credibility measure theory as well as probability theory. Here we initially discuss the chance measure theory because of its role in variable modeling of random fuzzy system lifetimes.

Definition 5.1 A random fuzzy variable is a mapping from the credibility space $(\Theta, 2^\Theta, Cr)$ to a set of random variables.

Example 5.2 Let ξ follow a Gaussian distribution of mean ϖ and variance σ^2 , with ϖ being a fuzzy variable with membership function:

$$\mu_\varpi(u) = \begin{cases} 1 - e^{-\beta u} & u \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.1)$$

then ξ is a random fuzzy variable. This example hints an important fact about the relation between random fuzzy variable (event) and fuzzy variable. Liu (2004) stated as a theorem:

Theorem 5.3 Let ξ be a random fuzzy variable. If the expectation $E_p[\xi(\theta_0)]$ exists for any given $\theta_0 \in \Theta$, then $E_p[\xi(\cdot)]$ is a fuzzy variable.

It is obvious that in Example 5.3., $E_p[\xi] = \varpi$, which is the fuzzy variable with membership function defined by Eq. (2.3). This theorem may pave a way toward a data-assimilating fuzzy membership for a set of observations from a random fuzzy variable, denoted as $\{x_1, x_2, \dots, x_n\}$.

Definition 5.4 Let ξ be a random fuzzy variable and B a Borel set of real numbers. Then the chance measure of random fuzzy event $\{\xi \in B\}$ is a function mapping from $(0,1]$ to $[0,1]$,

$$Ch\{\xi \in B\}(\alpha) = \sup_{Cr\{A\} \geq \alpha} \inf_{\theta \in A} Pr\{\theta : \xi(\theta) \in B\} \quad (5.2)$$

Example 5.5 (Liu, 2004) Let ζ be a random fuzzy variable with “uniformly distributed random variable” values:

$$\zeta = \begin{cases} U(1,3) & \text{with membership degree 1} \\ U(3,5) & \text{with membership degree 0.8} \end{cases} \quad (5.3)$$

From the definition, α is the *lower bound* of the credibility measure of event $\{\theta: \zeta(\theta) \in B\}$. Then for event $\{\theta: 1 \leq \zeta(\theta) \leq 4\}$, we can look at sub-events: $\{\theta: 1 \leq \zeta(\theta) \leq 2\}$, $\{\theta: 2 \leq \zeta(\theta) \leq 3\}$, $\{\theta: 3 \leq \zeta(\theta) \leq 4\}$, ..., etc and establish that:

$$\text{Cr}\{\theta: 1 \leq \zeta(\theta) \leq 4\} = \frac{1}{2} \left(\sup_{x \in [1,4]} \mu(x) + 1 - \sup_{x \in [1,4]^c} \mu(x) \right) = \frac{1+1-0.8}{2} = 0.6 \quad (5.4)$$

gives the lower bound value, $\alpha = 0.6$. In other words, i.e., the lower bound α is set as 0.6. As to $\text{Pr}\{\theta: 1 \leq \zeta(\theta) \leq 4\}$, it could be either determined in chance measure for random fuzzy event $\{\zeta \in B\}$ and can be evaluated by noting that for $B = [1,4]$, if $\alpha \leq 0.6$, we have the event probability, $\text{Pr}\{\theta: 1 \leq \zeta(\theta) \leq 4\} \geq 1$, if $\alpha > 0.6$, only those sub-events possible, say, $\{\theta: 3 \leq \zeta(\theta) \leq 4\}$ gives the lower bound of the probability, 0.5. Thus:

$$\text{Ch}\{1 \leq \zeta \leq 4\}(\alpha) = \begin{cases} 1 & \text{if } \alpha \leq 0.6 \\ 0.5 & \text{if } \alpha > 0.6 \end{cases} \quad (5.5)$$

Similarly, we can argue that:

$$\text{Ch}\{2 \leq \zeta \leq 4\}(\alpha) = 0.5, \text{ for } \forall \alpha \in (0,1] \quad (5.6)$$

and

$$\text{Ch}\{2 \leq \zeta \leq 5\}(\alpha) = \begin{cases} 1 & \text{if } \alpha \leq 0.4 \\ 0.5 & \text{if } \alpha > 0.4 \end{cases} \quad (5.7)$$

From example 5.5, we can now understand chance measure of an event $\{\zeta \in B\}$ intuitively as a mapping from the set of credibility measures of all sub-events $\{\zeta \in B, : B, C \subset B\}$ into interval $[0,1]$.

The challenge to be faced here is to seek the appropriate chance measure function form.

Theorem 5.6 (Liu, 2004) Let ξ be a random fuzzy variable. Then for each $\alpha \in (0,1]$,

- (i) $\text{Ch}\{\xi \in \emptyset\}(\alpha) = 0$;
- (ii) $\text{Ch}\{\xi \in \mathbb{R}\}(\alpha) = 1$;
- (iii) $0 \leq \text{Ch}\{\xi \in \mathbb{R}\}(\alpha) \leq 1$ for every Borel set $B \subset \mathbb{R}$;
- (iv) $\text{Ch}\{\xi \in B\}(\alpha)$ is an increasing function on Borel set $B \subset \mathbb{R}$.

For any Borel set $B \subset \mathbb{R}$, $\text{Ch}\{\xi \in B\}(\alpha)$ is a decreasing function of $\alpha \in (0,1]$ and

$$\begin{aligned}\lim_{\alpha \downarrow 0} \text{Ch}\{\xi \in B\}(\alpha) &= \sup_{\theta \in \Theta^+} \text{Pr}\{\xi(\theta) \in B\} \\ \text{Ch}\{\xi \in B\}(1) &= \inf_{\theta \in \Theta^+} \text{Pr}\{\xi(\theta) \in B\}\end{aligned}\quad (5.8)$$

where Θ^+ is the kernel of credibility space $(\Theta, 2^\Theta, \text{Cr})$.

Definition 5.7 (Zhu and Liu, 2004) The chance distribution of a random fuzzy variable ξ , denoted by $\Phi: \mathbb{R} \times (0, 1] \rightarrow [0, 1]$, is defined by

$$\Phi(x, \alpha) = \text{Ch}\{\xi \leq x\}(\alpha) \quad (5.9)$$

Theorem 5.8 (Zhu and Liu, 2004) A function $\Phi: \mathbb{R} \times (0, 1] \rightarrow [0, 1]$ is a chance distribution of random fuzzy variable ξ if and only if $\Phi(x, \alpha)$ is

- (i) a decreasing and left-continuous function of α for each fixed x ;
- (ii) an increasing function of x for each α , and

$$\lim_{x \rightarrow -\infty} \Phi(x, \alpha) = 0, \text{ if } \alpha > 0.5 \quad (5.10)$$

$$\lim_{x \rightarrow +\infty} \Phi(x, \alpha) = 1, \text{ if } \alpha < 0.5 \quad (5.11)$$

$$\lim_{y \downarrow x} \Phi(y, \alpha) = \Phi(x, \alpha), \text{ if } \alpha > 0.5 \quad (5.12)$$

Definition 5.9 (Zhu and Liu, 2004) The chance density $\phi(x; \alpha): \mathbb{R} \times (0, 1] \rightarrow [0, +\infty)$ of a random fuzzy variable ξ is a function such that:

$$\begin{aligned}\Phi(x; \alpha) &= \int_{-\infty}^x \phi(y; \alpha) dy, \forall x \in \mathbb{R}, \alpha \in (0, 1] \\ \int_{-\infty}^{+\infty} \phi(y; \alpha) dy &= 1, \forall \alpha \in (0, 1]\end{aligned}\quad (5.13)$$

where $\Phi(\cdot, \cdot)$ is the chance distribution of random fuzzy variable ξ .

Definition 5.10 (Liu and Liu, 2003) The expected value of a random fuzzy variable ξ is:

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \leq r\} dr \quad (5.14)$$

provided at least one of the two integrals are finite. Furthermore, let $e = E[\xi]$, the variance of a random fuzzy variable ξ is defined as $V[\xi] = E[(\xi - e)^2]$.

We should emphasize again that the expectation and variance of a random fuzzy variable ξ are numbers.

5.2 Theory of Random Interval and Empirical Chance Distribution

5.2.1 Theory of Random Interval and Fuzzy Set

In this section, we review a random interval theory developed by Heilpern (1990). This review leads to a maximum entropy based kernel-estimation approach for obtaining data-assimilating chance distribution for random fuzzy system lifetimes, as examples of random fuzzy variables.

Let \mathbb{R} be the set of all real numbers. A random interval of the universal space of closed intervals, denoted by \mathbb{I}_X , $X \subset \mathbb{R}$, is a measurable mapping from the probability space $(\Omega, \mathfrak{A}, P)$ to the power set of \mathbb{I}_X , $2^{\mathbb{I}_X}$. Let W be a random interval taking value on \mathbb{I}_X , i.e.,

$$W(\omega) = [W'(\omega), W''(\omega)], \omega \in \Omega, W'(\omega) \leq W''(\omega) \quad (5.15)$$

A typical treatment is to convert every closed interval $[W', W''] \in \mathbb{I}_X$ into the point in the two-dimensional space:

$$\mathbb{R}_Z^2 = \{(w', w''), w', w'' \in \mathbb{R}, w' \leq w''\} \subset \mathbb{R}^2 \quad (5.16)$$

and thus each of the random interval W is treated as a (bivariate) random variable taking values on \mathbb{R}_Z^2 . Let $p(x, y)$ be the density function, assuming W to be continuous random variable, such that:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) dy dx = 1 \quad (5.17)$$

Let $\tilde{\varphi}(W)$ be the fuzzy set generated by the random interval W , then the membership function is expressed by:

$$\mu_{\tilde{A}(W)}(z) = \int_{-\infty}^z \int_x^{\infty} p(x, y) dy dx \quad (5.18)$$

In general, we assume fuzzy set \tilde{A} of normal form, i.e.,

$$\mu_{\tilde{A}}(z) = \begin{cases} l(z) & \text{if } a \leq z < c \\ 1 & \text{if } c \leq z \leq d \\ r(z) & \text{if } d < z \leq b \\ 0 & \text{otherwise} \end{cases} \quad (5.19)$$

where $l(\cdot)$ is monotone-increasing from zero to one and $r(\cdot)$ is monotone-decreasing function. Furthermore, Heilpern (1990) pointed out that the joint distribution function with the largest entropy has the form:

$$F(x, y) = \begin{cases} l(x) & \text{for } y = b \\ 1 - r(y) & \text{for } x = c \\ l(x)(1 - r(y)) & \text{for } (x, y) \in \text{Int } D \\ 0 & \text{otherwise} \end{cases} \quad (5.20)$$

The link between random interval and fuzzy set seems strange but is mathematically inevitable and intrinsic. We note that the probability of a random interval W is a set mapping from a power set to $P_W : 2^U \rightarrow [0, 1]$, i.e., defined on $(U, 2^U, P_W)$, while the credibility measure for fuzzy subsets is also defined on credibility space $(U, 2^U, Cr)$, i.e., credibility measure is a set function $Cr : 2^U \rightarrow [0, 1]$. This parallel structure is the reason why the link between probability measure and credibility measure exists, and a measure called probability-credibility consistence measure has been defined for the degree of link.

5.2.2 Empirical Chance Distribution via Kernel Estimation

Based on the arguments in previous sections, as long as a set of interval observations of system lifetimes can be collected, denoted as $\{(x_i^l, x_i^u), i = 1, 2, \dots, n\}$, we can estimate the joint density of interval variable (X^l, X^u) , denoted by $p_X(x^l, x^u)$. A typical nonparametric approach is kernel density estimation – kernel estimation under a maximum entropy principle. Note that fuzzy entropy is well-defined in the fuzzy mathematical literature.

According to the entropy described by De Luca and Termini (1972), the sample entropy is defined by:

$$J[\xi] = \frac{1}{n} \sum_{i=1}^n S(\text{Cr}(\{\theta: \xi(\theta) = x_i\})) \quad (5.21)$$

The kernel related parameters are thus chosen by maximizing the sample entropy $J[\xi]$, and then the data-assimilating membership function of the generating fuzzy set $\tilde{\varphi}(X', X'')$ is established:

$$\mu_{\tilde{\varphi}(X', X'')}(z) = \int_{-\infty}^z \int_{x''}^{\infty} p_K(x', x'') dx'' dx' \quad (5.22)$$

Also the data-assimilating credibility measure can be evaluated in terms of:

$$\text{Cr}(\theta: \xi(\theta) = z) = \frac{1}{2} \left(\mu_{\tilde{\varphi}(X', X'')}(z) + 1 - \sup_{y=z} \left[\mu_{\tilde{\varphi}(X', X'')}(z) \right] \right) \quad (5.23)$$

As to the probability measure of event $P[\xi(\theta) \in B]$, we have:

$$P[\xi(\theta) \in B] = \iint_B p_K(x', x'') dx'' dx' \quad (5.24)$$

Thus, we obtain the data-assimilating chance measure:

$$\text{Ch}\{\xi \in B\}(\alpha) = \sup_{\text{Cr}\{A\} \geq \alpha} \inf_{\theta \in A} \iint_B p_K(x', x'') dx'' dx' \quad (5.25)$$

where the credibility measure $\text{Cr}(\cdot)$ is given by Eq. (5.21).

5.2.3 An Industrial Example

Here we re-use the example that appeared in Section 4.1.4 (Chapter 4). It is the first order one-variable grey differential equation model for creating the interval-valued records for each of the failure (or PM) observation from the Cement Roller data (Love and Guo, 1991). The details were discussed on the paper by Guo and Love (2005).

Table 5.2.1 Lifetime, failure status, grey interval for lifetime

<i>Waiting time</i>	<i>Functioning time</i>	<i>Failure mode</i>	<i>Lower bound</i>	<i>Upper bound</i>
54	54	pm	54.012	54.413
187	133	failure	132.767	134.513
334	147	pm	147.066	148.013
406	72	failure	71.922	73.086
511	105	failure	104.546	106.883
626	115	pm	114.568	116.853
767	141	pm	140.922	142.251
826	59	failure	59.256	59.359
933	107	pm	106.546	108.886
992	59	pm	59.256	59.359
1028	36	failure	34.432	37.101
1238	210	pm	206.587	213.004
1283	45	failure	44.288	45.732
1352	69	pm	68.990	69.940
1407	55	failure	55.084	55.380
1481	74	pm	73.880	75.176
1605	124	failure	123.641	125.728
1752	147	failure	147.066	148.013
1923	171	pm	170.717	171.876
1963	40	failure	38.829	40.931
2040	77	failure	76.822	78.301
2138	98	failure	97.569	99.832
2246	108	failure	107.546	109.886
2356	110	pm	109.549	111.883
2441	85	failure	84.693	86.572
2541	100	failure	99.559	101.852
2656	115	failure	114.568	116.853
2873	217	pm	212.908	220.494
2898	25	failure	22.199	26.619
2948	50	failure	49.706	50.549
3003	55	pm	55.084	55.380

In Table 5.2.1, the lower bound is denoted as x_i^l and the upper bound is denoted as x_i^u for the grey interval of the i^{th} system lifetime X_i . The grey intervals $\{[x_i^l, x_i^u], i = 1, 2, \dots, 31\}$ are regarded as the realizations for the random fuzzy system lifetimes, which are used to perform kernel estimation and obtain the kernel density $p_K(x^l, x^u)$.

5.3 Random Fuzzy Variable and Average Chance Measure

In this Section, we propose a constructive definition for a random fuzzy variable which similar to the definition of a stochastic process. Then, we explore the average chance measure and average chance distribution framework proposed by Liu and Liu (2002).

5.3.1 Random Fuzzy Variable

Liu (2004) defined that *a random fuzzy variable as a mapping from the credibility space $(\Theta, 2^\Theta, Cr)$ to a set of random variables*. For reliability engineers and managers armed with introductory probability and statistics, this definition is difficult to understand. For a more intuitive understanding, we would like to present a definition similar to that of stochastic process in probability theory and expect readers who are familiar with the basic concept of stochastic processes can understand our alternative definition.

Definition 5.11 A random fuzzy variable, denoted as $\xi = \{X_{\beta(\theta)}, \theta \in \Theta\}$, is a collection of random variables X_{β} defined on the common probability space $(\Omega, \mathcal{A}, Pr)$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, Cr)$.

Similar to the interpretation of a stochastic process $X = \{X_t, t \in \mathbf{R}^+\}$, a random fuzzy variable is also a bivariate mapping from $(\Omega \times \Theta, \mathcal{A} \times 2^\Theta)$ to the space $(\mathbf{R}, \mathcal{B})$, which is a set of random variables. As to the index set, in stochastic process theory, the index set used is referred to as *time* typically, which is a positive (scalar variable), while in the random fuzzy variable theory, the “index” is a fuzzy variable β . Using an uncertainty parameter as index is not new in this random fuzzy variable definition. In stochastic process theory we already know that the stochastic process $X = \{X_{\tau(\omega)}, \omega \in \Omega\}$ uses stopping time $\tau(\omega)$, $\omega \in \Omega$, which is an (uncertain) random variable as its index.

5.3.2 Average Chance Measure and Average Chance Distribution for a Random Fuzzy variable

As we stated in previous section, Liu and Liu (2002) described a random fuzzy variable in terms of a chance measure concept. However, we note the potential mathematical complexity associated with the chance measure formulation. Therefore, it is necessary to explore a convenient way to deal with the chance measure specification. Recall that in probability theory, the distribution of a random variable ξ on probability space $(\Omega, \mathfrak{A}, \Pr)$, $F_\xi(\cdot)$ links to the probability measure of event $\forall \{\omega: \xi(\omega) \leq x\} \in \mathfrak{A}$

$$F_\xi(x) = \Pr\{\omega: \xi(\omega) \leq x\} \quad (5.26)$$

In random fuzzy variable theory, we may say that that average chance measure plays an equivalent role similar to probability measure, denoted as \Pr , in probability theory.

Definition 5.12 (Liu and Liu, 2002) Let ξ be a random fuzzy variable, then the average chance measure, denoted as $\text{ch}\{\cdot\}$, of a random fuzzy event $\{\xi \leq x\}$, is

$$\text{ch}\{\xi \leq x\} = \int_0^1 \text{Cr}\{\theta \in \Theta | \Pr\{\xi(\theta) \leq x\} \geq \alpha\} d\alpha \quad (5.27)$$

Then function $\Phi(\cdot)$ is called as average chance distribution if and only if

$$\Phi(x) = \text{ch}\{\xi \leq x\} \quad (5.28)$$

Now, we are required to establish a theoretical framework in terms of average chance measure concepts. Once the average chance measure for the basic event form $\{\xi \leq x\}$ is given, then the average chance measure for any event A should be established in terms of the basic event $\{\xi \leq x\}$.

In this way, we may define average chance measure for an arbitrary event A . The triple space $(\Omega \times \Theta, \mathfrak{A}, P \times 2^\circ, \text{ch})$ is called an average chance measure space.

Proposition 5.13 Let $\text{ch}(\cdot)$ be an average chance measure on a product measure space

$(\Omega \times \Theta, \mathfrak{A}, P \times 2^\circ)$. Then

- (i) $\text{ch}\{\emptyset\} = 0$ and $\text{ch}\{\Theta\} = 1$;
- (ii) (Normality) $\forall A \in 2^\circ$, $0 \leq \text{ch}\{A\} \leq 1$;
- (iii) (Self-Duality) For $\forall A \in 2^\circ$, then $\text{ch}\{A^c\} = 1 - \text{ch}\{A\}$

(iv) (Weak monotone increasing) For $\forall A \subset B, A, B \in 2^\Theta$, $\text{ch}\{A\} \leq \text{ch}\{B\}$;

(v) (Semi-Continuity) For $\forall A_n, A \in 2^\Theta$, $n = 1, 2, \dots$, if $A_n \rightarrow A$, then

$$\lim_{A_n \rightarrow A} \text{ch}\{A_n\} = \text{ch}\{A\} \quad (5.29)$$

if and only if one of the following conditions holds:

(a) $\text{Cr}\{A_n\} \leq 0.5$ and $A_n \uparrow A$,

(b) $\lim_{n \rightarrow \infty} \text{Cr}\{A_n\} < 0.5$ and $A_n \uparrow A$,

(c) $\text{Cr}\{A_n\} \geq 0.5$ and $A_n \downarrow A$,

(d) $\lim_{n \rightarrow \infty} \text{Cr}\{A_n\} > 0.5$ and $A_n \downarrow A$.

(vi) (Sub-Additivity) For $\forall A \subset B, A, B \in 2^\Theta$,

$$\text{ch}\{A \cup B\} \leq \text{ch}\{A\} + \text{ch}\{B\} \quad (5.30)$$

Proposition 5.14 Let $\Phi_\xi(\cdot)$ be average chance distribution of fuzzy random variable ξ on the chance measure space $(\Omega \times \Theta, \mathfrak{A} \times 2^\Theta, \text{ch})$. Then

(i) $\Phi_\xi(-\infty) = 0$ and $\Phi_\xi(+\infty) = 1$;

(ii) for $\forall x \in \mathbb{R} = (-\infty, +\infty)$, $0 \leq \Phi_\xi(x) \leq 1$;

(iii) a nonnegative real-valued function $\phi_\xi(\cdot)$ is called average chance density for a random fuzzy variable ξ if for $\phi_\xi(x) \geq 0, x \in \mathbb{R}$ and

$$\Phi_\xi(x) = \int_{-\infty}^x \phi_\xi(u) du \quad (5.31)$$

5.3.3 Construction of a Random Fuzzy Variable

Liu (2004) mentioned an exponentially distributed random fuzzy variable ξ has a density function

$$\phi(x) = \begin{cases} \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.32)$$

if the value of β is assumed to be a fuzzy variable, then ξ is a random fuzzy variable.

Accordingly, let fuzzy parameter β be defined by a membership function, $\mu_\beta(\cdot)$, and the

probability distribution is defined by Eq. (5.26), then the random fuzzy variable ξ is said to be exponentially distributed. This example leads to a constructive definition for specifying or a random fuzzy variable or equivalently average chance distribution.

Definition 5.15 Let $\{F(x;\beta(\theta)), \theta \in \Theta\}$ be a family of probability distributions on the probability space $(\Omega, \mathfrak{A}, \text{Pr})$ with a common fuzzy parameter β on the credibility measure space $(\Theta, 2^\Theta, \text{Cr})$, which induces a membership function, $\mu_\beta(\cdot)$, then the average distribution derived from $(F(x,\beta), \mu_\beta)$ defines a random fuzzy variable ξ .

5.4 Random Fuzzy Lifetime Theory

Different from the statistical lifetime modeling and analysis, where the random lifetimes are concerned, also different from the fuzzy lifetime modeling and analysis, where the fuzzy lifetimes are concerned, random fuzzy lifetime modeling analysis provides a general theoretical foundation.

Liu (2004) defined random fuzzy variable in a very formal way. However, it might be difficult for the reliability engineers. Therefore, we give an intuitive and constructive definition.

Definition 5.16 A random fuzzy lifetime, denoted as $\xi = \{X_{\beta(\theta)}, \theta \in \Theta\}$, is a collection of positive real-valued random variables $X_{\beta} > 0$ defined on the common probability space $(\Omega, \mathfrak{A}, \text{Pr})$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, \text{Cr})$.

A random fuzzy lifetime, denoted by ξ , is a special case of random fuzzy variable. In other words, random fuzzy lifetime is a bivariate mapping from $(\Omega \times \Theta, \mathfrak{A} \times 2^\Theta)$ to the space $(\mathbb{R}^+, \mathfrak{B}(\mathbb{R}^+))$.

5.4.1 Basic Construction of Continuous Random Fuzzy Lifetimes

In statistical lifetime modeling and analysis, the probability distribution contains the full information on system lifetime and there are many related concepts linking to the physical structure of a system. Particularly, the hazard function reveals an aspect of lifetime distribution.

Theorem 5.17 Let ξ be a continuous random fuzzy lifetime having probability distribution function $F(t;\beta(\theta))$, where the fuzzy parameter β is defined on the credibility measure space $(\Theta, 2^\Theta, Cr)$ with membership $\mu_\beta(\cdot)$. Then function $\Pi(\cdot)$ can uniquely define the random fuzzy lifetime ξ if the operator or function Λ satisfies $F(t;\beta) = \Lambda(\Pi(t;\beta))$.

The proof of the theorem is a straight application of **Definition 5.11** and **Definition 5.15**.

Table 5.4.1 lists four commonly used operator or functions.

Table 5.4.1 Examples of operators or functions

Name	Form of $\Pi(t;\beta)$	$\Lambda(\cdot)$
Survival function	$\bar{F}(t;\beta) = 1 - F(t;\beta)$	$F(t;\beta) = 1 - \bar{F}(t;\beta)$
Density function	$f(t;\beta) = dF(t;\beta)/dt$	$F(t;\beta) = \int_0^t f(u;\beta) du$
Hazard function	$h(t;\beta) = f(t)/[1 - F(t;\beta)]$	$F(t;\beta) = 1 - \exp\left(-\int_0^t h(u;\beta) du\right)$
Moment generating function	$m(\theta;\beta) = \int_0^{+\infty} e^{\theta t} dF(t;\beta)$	$F(t;\beta) = \int_0^t \left(\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} m(s;\beta) e^{su} ds \right) du$

5.4.2 Continuous Random Fuzzy Lifetime Model

In statistical lifetime modeling and analysis, the elementary lifetime models are exponential, Weibull, Log-normal, gamma, bathtub, etc.. These are essential for the construction of random fuzzy lifetimes. Table 5.4.2 lists these models.

Table 5.4.2 Commonly used distributional lifetime models

Name	Probability density & hazard function	
Exponential	density	$\beta \exp(-\beta t)$
	hazard	β
Weibull	density	$(\beta/\eta)(t/\eta)^{\beta-1} \exp(-(t/\eta)^\beta)$
	hazard	$(\beta/\eta)(t/\eta)^{\beta-1}$
Extreme - value	density	$(1/u) \exp((t-b)/u) \exp(-\exp((t-b)/u))$
	hazard	$(1/u) \exp((t-b)/u)$
Log-Normal	density	$(1/(\sqrt{2\pi}\sigma t)) \exp(-(\ln t - \mu)^2 / 2\sigma^2)$
	hazard	$((1/(\sqrt{2\pi}\sigma t)) \exp(-(\ln t - \mu)^2 / 2\sigma^2)) / (1 - \Phi((\ln t - \mu)/\sigma))$

<i>Gamma</i>	density	$(\lambda(\lambda t)^{\beta-1} / \Gamma(\beta)) e^{-\lambda t}$
	hazard	$(\lambda(\lambda t)^{\beta-1} / \Gamma(\beta)) e^{-\lambda t} / (1 - I(\beta, \lambda t))$
<i>Bathtub</i>	density	$(\beta/\eta)(t/\eta)^{\beta-1} \exp((t/\eta)^{\beta-1}) \exp(-\exp((t/\eta)^{\beta-1}))$
	hazard	$(\beta/\eta)(t/\eta)^{\beta-1} \exp((t/\eta)^{\beta-1})$

In Table 5.4.2, $I(\beta, \lambda t)$ denotes the incomplete gamma function of the first-type and $\Phi(\cdot)$ represents the cumulative distribution of a standard normal variable.

5.4.3 Accelerated Life Testing Models

Accelerated life testing is an important methodology in new product design and warranty policy decision making. The basic assumption is that a change in stress factors only alters the scale, only the shape, of the failure time distribution.

Definition 5.18 (Accelerated random fuzzy life model) Let $F_0(t; \beta)$ be the baseline failure time distribution function for a random fuzzy lifetime ξ having a fuzzy parameter β defined on the credibility measure space $(\Theta, 2^\Theta, Cr)$ with membership $\mu_\beta(\cdot)$, then the accelerated random fuzzy life model specifies the probability distribution for the random fuzzy failure time under time-independent stress variable z as

$$F(t; \beta, z) = F_0(t\varsigma(z); \beta) \tag{5.33}$$

where satisfies function of stress variable $\varsigma: \mathbb{R} \rightarrow \mathbb{R}^+$.

The average chance distribution with stress variable z is therefore

$$\Phi(t; z) = \int_0^1 Cr\{\theta: F_0(t\varsigma(z); \beta(\theta)) \geq \alpha\} d\alpha \tag{5.34}$$

where stress variable z may be assumed to be either fuzzy or deterministic.

The function, $\varsigma: \mathbb{R} \rightarrow \mathbb{R}^+$, is usually defined in terms of the relationship between the parameter β of lifetime distribution and stress variable(s). Well-known accelerated life models include power rule model

$$\beta = \frac{\lambda}{z^c}, \quad c > 0 \tag{5.35}$$

$$\beta = \lambda \exp\left(\frac{\delta}{T}\right), \quad T > 0 \quad (5.36)$$

Combined power rule and Arrhenius reaction rate model

$$\beta = \lambda z^{-c} \exp\left(\frac{\delta}{T}\right), \quad T > 0, c > 0 \quad (5.37)$$

Jurkov's model

$$\beta = \lambda \exp\left(\frac{\delta - Cz}{T}\right), \quad T > 0 \quad (5.38)$$

Generalized Eyring model

$$\beta = \lambda T \exp\left(\frac{\delta}{T}\right) \exp\left(Cz + \frac{dz}{T}\right), \quad T > 0 \quad (5.39)$$

and others.

5.4.4 Proportional hazard models

Besides accelerated testing model, another covariate model which plays a very important role in lifetime analysis is Cox's (1972) proportional hazards (abbreviated as PH) model

$$h(t; \beta, \gamma) = h_0(t; \beta) \varsigma(\gamma^T y) \quad (5.40)$$

where $h_0(t; \beta)$ is the baseline hazard function with a fuzzy parameter β defined on the credibility measure space $(\Theta, \mathcal{C}\bar{r})$ with membership $\mu_\beta(\cdot)$, while $\varsigma: \mathbf{R} \rightarrow \mathbf{R}^+$ with

$$\gamma^T y = \gamma_0 + \gamma_1 y_1 + \dots + \gamma_p y_p \quad (5.41)$$

where $y = (1, y_1, \dots, y_p)^T$ is covariate vector and $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_p)^T$ is covariate effect parameter vector.

A typically function of $\varsigma: \mathbf{R} \rightarrow \mathbf{R}^+$ used is the exponential function $\varsigma(x) = \exp(x)$. If covariate y is not time-dependent, it is easy to show that the accumulated hazard

$$H(t; \beta, \gamma) = H_0(t; \beta) e^{\gamma^T y} \quad (5.42)$$

And therefore the average chance distribution with covariate y is

$$\Phi(t, y) = \int_0^1 \mathcal{C}\bar{r}\left\{(\theta_1, \theta_2): H_0(t; \beta(\theta_1)) e^{\gamma^T y(\theta_2)} \geq -\ln(1-\alpha)\right\} d\alpha \quad (5.43)$$

where covariate y is assumed to be fuzzy but parameter γ is assumed to be deterministic, other options can be formulated.

It is necessary to note here that the two types of covariate models are not only powerful in product reliability design and analysis but also useful in repairable system maintenance optimal planning and analysis. In the probabilistic reliability literature, researchers have many useful developments. Therefore, in random fuzzy repairable system analysis it will be necessary to incorporate those development.

5.5 Exponentially Distributed Random Fuzzy Lifetimes

From the section 5.3.2, it is easy to see that exponential random fuzzy failure times are probably the simplest model to handle.

Let us use exponentially distributed random fuzzy lifetime which has probability density

$$f(t; \beta) = \begin{cases} 0 & t \leq 0 \\ \beta e^{-\beta t} & t > 0 \end{cases} \quad (5.44)$$

with a trapezoidal membership function

$$\mu_{\beta}(x) = \begin{cases} \frac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \leq c \\ \frac{d-x}{d-c} & c < x \leq d \\ 0 & \text{otherwise} \end{cases} \quad (5.45)$$

as an example of random fuzzy lifetime modeling and analysis.

Note that the factor in Eq. (5.27)

$$\Pr\{\xi(\theta) \leq t\} = 1 - e^{-\beta(\theta)t} \quad (5.46)$$

Therefore event $\{\theta : \Pr\{\xi(\theta) \leq t\} \geq \alpha\}$ is a fuzzy event and is equivalent to the fuzzy event $\{\theta : \beta(\theta) \geq -\ln(1-\alpha)/t\}$. As a critical step toward the derivation of the average chance distribution,

it is necessary to calculate the credibility measure for fuzzy event $\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$, i.e., obtain the expression for

$$\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\} \quad (5.47)$$

Recall that for the trapezoidal fuzzy variable (parameter), β , the credibility measure takes the form

$$\text{Cr}\{\theta: \beta(\theta) \leq x\} = \begin{cases} 0 & x \leq a \\ \frac{x-a}{2(b-a)} & a < x \leq b \\ \frac{1}{2} & b < x \leq c \\ \frac{x+d-2c}{2(d-c)} & c < x \leq d \\ 1 & x > d \end{cases} \quad (5.48)$$

Therefore the credibility measure of the complement event $\{\theta: \beta(\theta) \geq x\}$ is

$$\text{Cr}\{\theta: \beta(\theta) \geq x\} = \begin{cases} 1 & x \leq a \\ \frac{2b-x-a}{2(b-a)} & a < x \leq b \\ \frac{1}{2} & b < x \leq c \\ \frac{d-x}{2(d-c)} & c < x \leq d \\ 0 & x > d \end{cases} \quad (5.49)$$

Accordingly, the range of integration for α can be determined as shown in Table 5.5.1. Recall that the expression of $x = -\ln(1-\alpha)/t$ appears in Eq. (5.40) and Eq. (5.41), which facilitates the link between intermediate variable α and average chance measure.

Table 5.5.1 Range analysis for α

Range for x	α and credibility measure expression	
$-\infty < x \leq a$	Range for α	$0 \leq \alpha \leq 1 - e^{-a}$
	$\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$	1
$a < x \leq b$	Range for α	$1 - e^{-a} < \alpha \leq 1 - e^{-b}$
	$\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$	$1 - (x-a)/(2(b-a))$
$b < x \leq c$	Range for α	$1 - e^{-b} < \alpha \leq 1 - e^{-c}$
	$\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$	0.5
$c < x \leq d$	Range for α	$1 - e^{-c} < \alpha \leq 1 - e^{-d}$
	$\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$	$(d-x)/(2(d-c))$
$d < x < +\infty$	Range for α	$1 - e^{-d} < \alpha \leq 1$
	$\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$	0

The average chance distribution for the exponentially distributed random fuzzy lifetime is then derived by splitting the integration into five terms according to the range of α and the corresponding mathematical expression for the credibility measure $\text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\}$, which is detailed in Table 5.5.1. Then the exponential random fuzzy lifetime has an average chance distribution function:

$$\begin{aligned} \Phi_{\xi}(t) &= \int_0^1 \text{Cr}\{\theta: \beta(\theta) \geq -\ln(1-\alpha)/t\} d\alpha \\ &= 1 + \frac{e^{-bt} - e^{-at}}{2(b-a)t} + \frac{e^{-dt} - e^{-ct}}{2(d-c)t} \end{aligned} \tag{5.50}$$

The average chance density for the exponentially distributed random fuzzy lifetime is then the derivative with respect to t

$$\phi_{\xi}(t) = \frac{e^{-at} - e^{-bt}}{2(b-a)t^2} + \frac{be^{-bt} - ae^{-at}}{2(b-a)t} + \frac{e^{-ct} - e^{-dt}}{2(d-c)t^2} + \frac{ce^{-ct} - de^{-dt}}{2(d-c)t} \tag{5.51}$$

Recall how we defined the survival function in Section 3.2.1(Chapter 3), we now define a reliability function or survival function for a random fuzzy lifetime and accordingly name it as the average chance reliability function, which is specified accordingly as

$$R_{\xi}(t) = 1 - \Phi_{\xi}(t) \quad (5.52)$$

Then, for exponential random fuzzy lifetime, its average chance reliability function is

$$R_{\xi}(t) = \frac{e^{-at} - e^{-bt}}{2(b-a)t} + \frac{e^{-ct} - e^{-dt}}{2(d-c)t} \quad (5.53)$$

We present all the technical details in step-by-step manner to exhibit the insight of the average chance distribution of an exponential random fuzzy failure time. The forms of exponential random fuzzy failure time with triangular membership function in this subsection are merely special cases of the trapezoidal membership function.

In standard statistical lifetime modeling and analysis reliability function reveals the system functioning behavior. The average chance reliability function should play similar roles in random fuzzy lifetime modeling and analysis. In order to obtain an intuitive perceptions of the average chance reliability function, let us assume that the trapezoidal membership function defined by (0.1, 0.15, 0.25, 0.30), i.e., the parameters for specifying the membership function are $a = 0.1$, $b = 0.15$, $c = 0.25$, $d = 0.30$. For comparison purposes, we define an exponentially distributed random lifetime with fixed valued parameter, 0.20, which is obtained by

$$m_{\beta} = E(\beta) = 0.20 \quad (5.54)$$

Then the reliability function for the exponentially distributed random lifetime with parameter $m_{\beta} = 0.20$ is

$$R(t; 0.20) = \exp(-0.2t) \quad (5.55)$$

The corresponding average chance reliability function, $R_{\xi}(t; \beta)$:

$$R_{\xi}(t; \beta) = \frac{10(e^{-at} - e^{-bt})}{t} + \frac{10(e^{-ct} - e^{-dt})}{t} \quad (5.56)$$

Figure 1 gives a comparison between $R_{\xi}(t; \beta)$ and $R(t; 0.20)$.

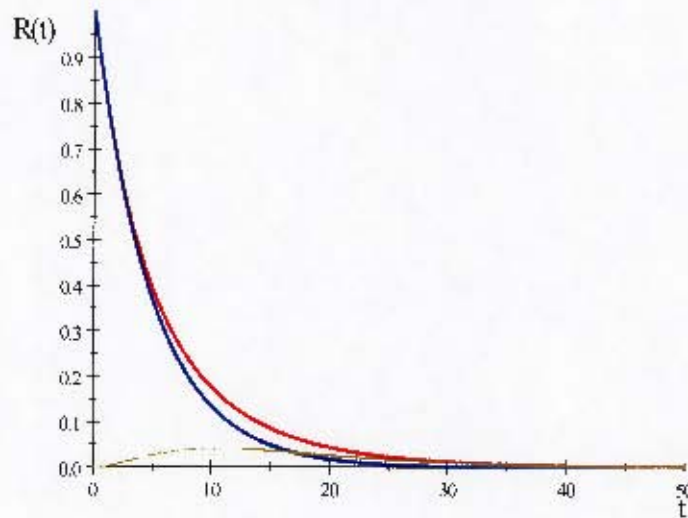


Figure 5.5.1 Exponential random fuzzy Lifetime average chance reliability (Red), corresponding exponential lifetime reliability (Blue), and the difference function (Sienna)

Intuitively, we can see that given two systems: first an exponentially distributed random fuzzy system with trapezoidal membership function $b = (0.10, 0.15, 0.25, 0.30)$ and second an exponentially distributed random system with parameter $m_b = 0.20$, the first one enjoys a higher reliability than that of the second one. A rigorous mathematical proof should be pursued before stating this impression as a general statement.

However, the purpose of developing random fuzzy lifetime analysis theory is a serious effort to formulate a foundation for analyzing reliability data collected from system performance. In other words, we need to estimate the parameters (a, b, c, d) . The estimation, or more accurately, a data-assimilated parameter value, which is implemented in terms of the Maximum Average Chance Principle discussed in Section 5, by finding the derivatives of the average chance density $f_x(t)$ with respect to (a, b, c, d) and setting them equal to zero, i.e.,

$$\frac{\partial}{\partial a} \frac{\int \ln f_x(t_i)}{\int q_i} = 0, i = 1, 2, 3, 4 \tag{5.57}$$

where $(\theta_1, \theta_2, \theta_3, \theta_4) = (a, b, c, d)$. Note in the expression of $\phi_\xi(t_i)$ parameter (a, b) and (c, d) are not tied each other, therefore, we have two separately equation systems for parameter pairs (a, b) and (c, d) respectively:

$$\left\{ \begin{array}{l} \sum_{i=1}^n \frac{(1+at_i - abt_i^2 + a^2t_i^2)e^{-at_i} - (1+bt_i)e^{-bt_i}}{(1+at_i)e^{-at_i} - (1+bt_i)e^{-bt_i}} = 0 \\ \sum_{i=1}^n \frac{(1+abt_i^2 - b^2t_i^2)e^{-at_i} - (1+bt_i - abt_i^2)e^{-bt_i}}{(1+at_i)e^{-at_i} - (1+bt_i)e^{-bt_i}} = 0 \end{array} \right. \quad (5.58)$$

and

$$\left\{ \begin{array}{l} \sum_{i=1}^n \frac{(1+ct_i - cdt_i^2 + c^2t_i^2)e^{-ct_i} - (1+dt_i)e^{-dt_i}}{(1+ct_i)e^{-ct_i} - (1+dt_i)e^{-dt_i}} = 0 \\ \sum_{i=1}^n \frac{(1+ct_i)e^{-ct_i} - (1+dt_i - cdt_i^2 + d^2t_i^2)e^{-dt_i}}{(1+ct_i)e^{-ct_i} - (1+dt_i)e^{-dt_i}} = 0 \end{array} \right. \quad (5.59)$$

We have therefore illustrated some models developments for random fuzzy lifetimes.

Chapter 6. Maximum Average Chance Principle and Data-Assimilation

It is often assumed that the model error structure follows multivariate normal distribution and thus the likelihood function can be obtained in linear model theory. For random fuzzy variable theory, an important distribution is the average chance distribution by which the average chance function can be defined, for theoretical details, recall what we stated in Section 2.1 (Chapter 2).

In this Chapter, we propose a data-assimilating algorithm for determining the unknown parameters of the average chance distribution. We use data-assimilation for deferring it from statistical estimation because statistical estimation and data-assimilation are both determining unknown parameters in terms of sampling data, however, statistical estimation is performed according the hypothesizing (random) population probability distribution while the data-assimilation is performed according to the chance distribution, particularly, the average chance distribution, which is not population probability distribution at all.

6.1 Maximum Average Chance Principle

Let us assume that n lifetimes, t_1, t_2, \dots, t_n are observed independently from the same (continuous) average chance distribution $\Phi(t, \theta)$, where θ is the parameter (either scalar or vector) to specify the average chance distribution, i.e., the membership function μ_β . Similar to the probabilistic developments, the M -estimates, $\hat{\theta}_n$ to the parameter are the solution to the equation

$$\sum_{i=1}^n \psi(t_i, \hat{\theta}_n) = 0 \quad (6.1)$$

Let

$$\sum_{i=1}^n \psi(t_i, \hat{\theta}_n) = 0 \quad (6.2)$$

where $\phi(t, \theta)$ is the average chance density for the n lifetimes, t_1, t_2, \dots, t_n . It is obvious that Equation (22) looks like the log-likelihood function in maximum likelihood estimation theory, but it is not. It is the log-average chance density for the random fuzzy lifetime sampled.

For n *i.i.d.* lifetimes, t_1, t_2, \dots, t_n , the joint average chance density is

$$\varphi(t_1, t_2, \dots, t_n, \theta) = \prod_{i=1}^n \phi(t_i, \theta) \quad (6.3)$$

Now we are ready to state the maximum average chance principle for seeking data-assimilating parameter.

Definition 6.1 (Maximum Average Chance Principle) For given random fuzzy lifetime sample t_1, t_2, \dots, t_n , the best data-assimilating parameter is the sample function $\theta(t_1, t_2, \dots, t_n)$ to maximize the joint average chance density $\varphi(t_1, t_2, \dots, t_n, \theta)$. Accordingly, the sample function $\theta(t_1, t_2, \dots, t_n)$ maximizing $\varphi(t_1, t_2, \dots, t_n, \theta)$ is called a maximum average chance estimates.

It is obvious that maximizing the joint average chance density $\varphi(t_1, t_2, \dots, t_n, \theta)$ is equivalent to maximizing the log-average chance function:

$$\ln(\varphi(t_1, t_2, \dots, t_n, \theta)) = \sum_{i=1}^n \ln \phi(t_i, \theta) \quad (6.4)$$

Find the first-derivatives with respect to parameter θ_k , $k = 1, 2, \dots, p$ and set them equal to zero, we obtain a nonlinear equation system

$$\sum_{i=1}^n \frac{\partial \ln \phi(t_i, \hat{\theta}_k)}{\partial \theta_k} = 0, \quad k = 1, 2, \dots, p \quad (6.5)$$

The solution to equation system in Eq.(6.5) is a maximum average chance estimate for $\theta = (\theta_1, \dots, \theta_p)$.

6.2 Regression Modeling under Maximum Average Chance Principle

In statistics, the common used principle is maximum likelihood estimation, where the estimated parameter(s) maximize the likelihood function. Parallel to maximum likelihood estimation, we defined average chance function according to hypothesizing random fuzzy population average chance distribution and then seeking the data-assimilating parameter(s) for maximize the average chance function, which may be regarded as a counterpart of likelihood function.

6.2.1 Normal Random Fuzzy Variable with Triangular fuzzy parameter

As for primary preparation, before we represent the regression modeling, we now state the normal random fuzzy variable with Triangular fuzzy parameter.

Let ϕ and Φ be the density and cdf for standard normal random variable respectively. Then, there is a normal random fuzzy variable with a triangular fuzzy mean of parameters (a, b, c) . In this case, the average chance distribution for mean is

$$\begin{aligned} \Psi(x) = & \frac{x-a}{2(b-a)} \left(\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(\frac{x-b}{\sigma}\right) \right) + \frac{x+c-2b}{2(c-b)} \left(\Phi\left(\frac{x-b}{\sigma}\right) - \Phi\left(\frac{x-c}{\sigma}\right) \right) \\ & + \Phi\left(\frac{x-c}{\sigma}\right) - \frac{\sigma}{2(b-a)} \int_{\frac{x-b}{\sigma}}^{\frac{x-a}{\sigma}} u \phi(u) du - \frac{\sigma}{2(c-b)} \int_{\frac{x-c}{\sigma}}^{\frac{x-b}{\sigma}} u \phi(u) du \end{aligned} \quad (6.6)$$

The average chance density is

$$\begin{aligned} \psi(x) = & \frac{1}{2(b-a)} \left(\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(\frac{x-b}{\sigma}\right) \right) + \frac{x-a}{2(b-a)\sigma} \left(\phi\left(\frac{x-a}{\sigma}\right) - \phi\left(\frac{x-b}{\sigma}\right) \right) \\ & + \frac{1}{2(c-b)} \left(\Phi\left(\frac{x-b}{\sigma}\right) - \Phi\left(\frac{x-c}{\sigma}\right) \right) + \frac{x+c-2b}{2(c-b)\sigma} \left(\phi\left(\frac{x-b}{\sigma}\right) - \phi\left(\frac{x-c}{\sigma}\right) \right) + \frac{1}{\sigma} \phi\left(\frac{x-c}{\sigma}\right) \\ & - \frac{1}{2(b-a)} \left(\frac{x-a}{\sigma} \phi\left(\frac{x-a}{\sigma}\right) - \frac{x-b}{\sigma} \phi\left(\frac{x-b}{\sigma}\right) \right) - \frac{1}{2(c-b)} \left(\frac{x-b}{\sigma} \phi\left(\frac{x-b}{\sigma}\right) - \frac{x-c}{\sigma} \phi\left(\frac{x-c}{\sigma}\right) \right) \end{aligned} \quad (6.7)$$

which is utilizing the differentiation formula of an integral:

$$\frac{d}{dt} \left(\int_{a(t)}^{b(t)} f(x,t) dx \right) = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx + f(b(t),t) \frac{db(t)}{dt} - f(a(t),t) \frac{da(t)}{dt} \quad (6.8)$$

As usual, sampling distributions are critical for the construction of hypothesis testing. We will address the sampling average chance distributions.

The sample mean, denoted as $\bar{\xi} = \sum_{i=1}^n x_i / n$, is a normal random fuzzy variable and the average chance distribution of the sampling mean can be obtained by substituting σ by σ/\sqrt{n} . The reason is obvious because

$$\begin{aligned} \Psi(x) = & \frac{x-a}{2(b-a)} \left(\Phi \left(\frac{x-a}{\sigma/\sqrt{n}} \right) - \Phi \left(\frac{x-b}{\sigma/\sqrt{n}} \right) \right) + \frac{x+c-2b}{2(c-b)} \left(\Phi \left(\frac{x-b}{\sigma/\sqrt{n}} \right) - \Phi \left(\frac{x-c}{\sigma/\sqrt{n}} \right) \right) \\ & + \Phi \left(\frac{x-c}{\sigma/\sqrt{n}} \right) - \frac{\sigma/\sqrt{n}}{2(b-a)} \int_{\frac{x-b}{\sigma/\sqrt{n}}}^{\frac{x-a}{\sigma/\sqrt{n}}} u\phi(u)du - \frac{\sigma/\sqrt{n}}{2(c-b)} \int_{\frac{x-c}{\sigma/\sqrt{n}}}^{\frac{x-b}{\sigma/\sqrt{n}}} u\phi(u)du \end{aligned} \quad (6.9)$$

6.2.2 The Theoretical Foundation for Regression Modeling

Definition 6.2 (Average chance function) Let $\{x_1, x_2, \dots, x_n\}$ be a simple random sample drawing from a give population hypothesizing probability distribution $F(x, \theta)$, where parameter-vector $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_\gamma)^T$, $\gamma \geq 1$ and parameter component θ_1 is a fuzzy variable with credibility distribution $\Lambda_{\theta_0}(y)$ defined by parameter-vector $\underline{\rho}$ and accordingly the average chance distribution is derivable and denoted as $\Psi(x)$. Then the joint average chance densities, denoted as

$$C(\underline{\rho}^T, \theta_2, \dots, \theta_\gamma | \{x_1, x_2, \dots, x_n\}) = \prod_{i=1}^n \psi(x_i | \underline{\rho}^T, \theta_2, \dots, \theta_\gamma) \quad (6.10)$$

is called the (average) chance function. Similar to log-likelihood function definition, function

$$\begin{aligned} l_c(\underline{\rho}^T, \theta_2, \dots, \theta_\gamma | \{x_1, x_2, \dots, x_n\}) &= \ln C(\underline{\rho}^T, \theta_2, \dots, \theta_\gamma | \{x_1, x_2, \dots, x_n\}) \\ &= \sum_{i=1}^n \ln \psi(x_i | \underline{\rho}^T, \theta_2, \dots, \theta_\gamma) \end{aligned} \quad (6.11)$$

is called the log-chance function given the simple random sample $\{x_1, x_2, \dots, x_n\}$.

Therefore, here for a given simple random sample $\{x_1, x_2, \dots, x_n\}$, the **Maximum Average Chance Principle** is the optimal data-assimilating parameter-vector $(\underline{\rho}^T, \theta_2, \dots, \theta_r)^T$, which is maximizing the average chance function or equivalently, maximizes the log-chance function.

Later in the next Section, we will show you an industrial application, which is an example as the illustration of the searching an maximum average chance estimation of normal random fuzzy with triangular credibility distributed fuzzy mean with parameter (a, b, c) and fixed variance parameter σ^2 .

To a full data-assimilating parameter estimation of the coupled regression model specified by the univariate DEMR model (R.Guo, D.Guo, 2007) recall that

$$y_i - \underline{x}_i \underline{\alpha} \sim N(\bar{\epsilon}, \sigma^2) \quad (6.12)$$

where

$$\underline{x}_i = (1, \hat{x}(i), \Delta x(i), \dots, \Delta^{p-1} x(i)) \quad (6.13)$$

and $y_i = \Delta^r x(i)$, therefore the contribution of i^{th} sample to the average chance function is

$$\begin{aligned} \psi(x_i) = & \frac{1}{2(b-a)} \left(\Phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - a}{\sigma} \right) - \Phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \right) \right) \\ & + \frac{(y_i - \underline{x}_i \underline{\alpha}) - a}{2(b-a)\sigma} \left(\phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - a}{\sigma} \right) - \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \right) \right) \\ & + \frac{1}{2(c-b)} \left(\Phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \right) - \Phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - c}{\sigma} \right) \right) \\ & + \frac{(y_i - \underline{x}_i \underline{\alpha}) + c - 2b}{2(c-b)\sigma} \left(\phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \right) - \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - c}{\sigma} \right) \right) + \frac{1}{\sigma} \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - c}{\sigma} \right) \\ & - \frac{1}{2(b-a)} \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - a}{\sigma} \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - a}{\sigma} \right) - \frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \right) \right) \\ & - \frac{1}{2(c-b)} \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - b}{\sigma} \right) - \frac{(y_i - \underline{x}_i \underline{\alpha}) - c}{\sigma} \phi \left(\frac{(y_i - \underline{x}_i \underline{\alpha}) - c}{\sigma} \right) \right) \end{aligned} \quad (6.14)$$

the parameter to be estimated are $\underline{\alpha} = (\alpha_0, \alpha_2, \dots, \alpha_{p-1})$ for specifying the trending in the coupled regression DEMR model defined in Eq. (5), for the general setting, let (a, b, c) specify fuzzy mean \tilde{e} and σ^2 for the variance, then the full log-chance function is

$$l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\}) = \sum_{i=1}^n \ln(\psi(x_i | \underline{\alpha}, (a, b, c), \sigma)) \quad (6.15)$$

Then searching the unknown parameters, the optimization problem may be converted into the problem of solving $(p+4)$ nonlinear equation system as follows

$$\left\{ \begin{array}{l} \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial \alpha_0} = 0 \\ \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial \alpha_1} = 0 \\ \vdots \\ \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial \alpha_{p-1}} = 0 \\ \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial a} = 0 \\ \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial b} = 0 \\ \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial c} = 0 \\ \frac{\partial l_c(\underline{\alpha}, (a, b, c), \sigma | \{x_1, x_2, \dots, x_n\})}{\partial \sigma} = 0 \end{array} \right. \quad (6.16)$$

It is obvious that searching the appropriate optimal data-assimilating parameter will be a difficult task and deserving a separate paper to deal with it.

6.3 A Random Fuzzy Accelerating Life Modeling via Bathtub Hazard

Today, the fast-changing industrial environments pose an information shortage problem during new product design and testing stage and thus the product controllable parameters can only be specified as fuzzy ones by combining similar product parameter specifications, new product parameter planned specifications and product designing team's experiences.

In this Section, we explore accelerating life model in terms of two-parameter bathtub hazard family under random fuzzy environment. A foundational framework for the small sample reliability modeling and analysis using two-parameter bathtub hazard family under the theories of fuzzy credibility measure and small sample asymptotic are discussed. An illustrative example of accelerating test is given by using TDDDB Data of Al/20W Hf-doped TaO_x/Si.

6.3.1 Accelerating Life Modeling

AL model, in Ushakov and Harrison's definition, is

$$F_T(t; \underline{z}) = F(g(\underline{z})t) \quad (6.17)$$

where $\underline{z} = (z_1, z_2, \dots, z_p)^T$ is covariate vector and $g(\cdot)$ is a positive real-valued function. The intrinsic and inherent character of accelerating life model is the covariate impact on system lifetime behaves as a scale (function) for lifetime. The covariate function in aggregation in AL model plays a role to stretch or condense the functional life time of the item under test. Different from AL model, proportional hazards model (abbreviated as PH model) proposed by Cox (1972), The covariate function in aggregation in PH model plays a role to stretch or condense the functional hazard of the item under test.

$$h(t; \underline{z}) = h_0(t)g(\underline{z}) \quad (6.18)$$

AL model is a very critical approach in new product design stage because the product reliability information can be obtained in a very short time period by applying higher stress level.

In today's accelerating life test model, reliability analysts often debate the engagements of bathtub hazard function, particularly, in electronic industries. However, most of the analyses were

still Weibull-extreme value distribution related. Table 6.1.1 collects the bathtub hazard family, since Smith and Bain.

Table 6.3.1 Bathtub hazard families

No	$h(t)$	Authors
1	$\theta(t)^{\theta-1} \exp(-(t)^\theta)$	Smith and Bain (1975) (Product model)
2	$\theta\pi(\pi t)^{\theta-1} \exp(-(\pi t)^\theta)$	Smith and Bain (1975) (Product model)
3	$\frac{\beta}{t+\gamma} + \delta t$	Hjorth (1980) (Additive model)
4	$\alpha + \beta t + \gamma t^2$	Blische and Murthy (2000)
5	$\exp(\alpha + \beta t + \gamma t^2)$	Blische and Murthy (2000)
6	$(\pi t)^{\theta-1} + (\pi t)^{\theta+1}, \theta \in (0,1)$	(Additive model)
7	$\theta + (1-\theta)(t+\theta)^{\theta-1} + \theta t^\theta$	Guo, Kuo, and Zhao (2007) (Additive model)
8	$\theta + (1-\theta)(\pi t + \theta)^{\theta-1} + \theta(\pi t)^\theta$	Guo, Kuo, and Zhao (2007) (Additive Model)

6.3.2 One-parameter Bathtub Hazard Families

Role of One-Parameter Hazard

Model efficiency is not only reflected by the goodness-of-fit but also reflected the number of the parameters, particularly in case of small sample. Less number of parameters with reasonable data-assimilation is the goal to pursue. In these aspects, the two one-parameter bathtub hazard deserve attention, Guo, Kuo, and Zhao.

$$h(t; \theta) = \theta + (1-\theta)(t+\theta)^{\theta-1} + \theta t^\theta \quad (6.19)$$

and Smith and Bain

$$h(t; \theta) = \theta t^{\theta-1} e^{-t^\theta} \quad (6.20)$$

As seen in later section, both Eq. (6.19) and Eq. (6.20) are special cases of the two-parameter bathtub hazard family with the setting the scale parameter $\pi = 1$. However, Guo, Kuo, and Zhao's bathtub is an additive model, while Smith and Bain's bathtub is a product model in nature.

Critical Indices of Bathtub Hazard

Working with bathtub hazard family, it is inevitable to ask for the indices for evaluating the hazard or criteria for the choice of an appropriate hazard family.

Definition 6.3 The value of $\min_{t \geq 0} h(t)$ is called the bottom-rate of the bathtub, the value t_b , such that $h(t_b) = \min_{t \geq 0} (h(t))$ is called as the bottom point of the bathtub.

Definition 6.4 In the interval $(0, t_d]$, hazard function demonstrates sharp-decreasing pattern, the system state when $t \in (0, 1]$ is called the infant mortality (i.e., the so-called burn-in) stage. t_d is called the down turning point of the bathtub.

Definition 6.5 The upper turning point of a bathtub is denoted as t_u , from which the system starts its wear-out state with a monotone increasing hazard rate.

Definition 6.6 The interval $[t_d, t_u]$ is called the bottom-interval, where the system is governed by memory-less failure law, and $l_h = t_u - t_d$ is called as the bottom-length of a bathtub hazard.

Definition 6.7 When $t \in [t_d, t_u]$, the hazard function changes slowly and demonstrates a (relatively) flat trend. In other words, for $\forall t \in [t_0, t_1]$ and the hazard rate $\left| h(t) - \min_{t \geq 0} h(t) \right| \leq \epsilon$, for a pre-set $\epsilon > 0$. We call $r_f(h) = (h(t_d) + h(t_u))/2$ as the flat rate, which is regarded as the system being governed by a "constant" rate. The system is called in a pseudo memory-less state because it can be treated as exponentially distributed with flat rate as its parameter.

Table 6.3.2 Indices specifying bathtub hazard

<i>Term</i>	<i>Notation</i>	<i>Functional role</i>
<i>Bottom rate</i>	$h(t_b)$	The lowest value of failure rate

<i>Down turning point</i>	t_d	The end of infant mortality (the so-called burn-in) state or the start of (pseudo) random failure state
<i>Upper turning point</i>	t_u	The end of (pseudo) memory-less state or the start of the wear-out state with increasing failure rate
<i>Bottom-Interval</i>	$[t_d, t_u]$	The interval where the system shows a (pseudo) memory-less state
<i>Bottom-length</i>	l_h	The duration of (pseudo) constant failure rate state
<i>Flat rate</i>	$r_f(h)$	The average failure rate of pseudo memoryless state

The critical indices defined will help us to have a non-probabilistic analysis on bathtub hazard. The reason why we perform an exploration the one-parameter bathtub hazard rate function in a non-probabilistic manner is due to a fundamental fact that the hazard function represents an aspect of a distribution that has physical meaning and that information about the nature of the hazard function is helpful in selecting a model.

A Non-Probabilistic Examination on Additive Model

For the one-parameter additive bathtub hazard model. We will have an over view in terms of the three parts: Table 6.3.3 , three regression models, and Figure 6.3.1, six bathtub curves under six grids of θ from 0 to 0.3.

Table 6.3.3 Bathtub bottom-interval and θ

θ	<i>Bottom-interval</i> $[t_d, t_u]$	<i>Bottom-length</i> l_b	<i>Flat rate</i> $r_f(h)$
0.05	[20,500]	>500	≈ 0.125
0.10	[20.0,130.0]	110	≈ 0.285
0.15	[20.0,60.0]	40	≈ 0.45
0.20	[10.0,40.0]	30	≈ 0.643
0.25	[6.0,25.0]	19	≈ 0.838
0.30	[4,0,12,0]	8	≈ 1.13

The flat rate demonstrates a very strong linear relation with the parameter θ , a straight line is fitted for reflecting the linear functional.

$$r_h(\theta) = -0.1092 + 3.9297 \theta \quad (R^2 = 0.9878) \quad (6.21)$$

(0.0425) (0.2185)

The length also demonstrates an functional relation with parameter value of θ , an exponential trend is obtained as

$$l_b(\theta) = 681.8968 e^{-15.2618 \theta} \quad (R^2 = 0.9301) \quad (6.22)$$

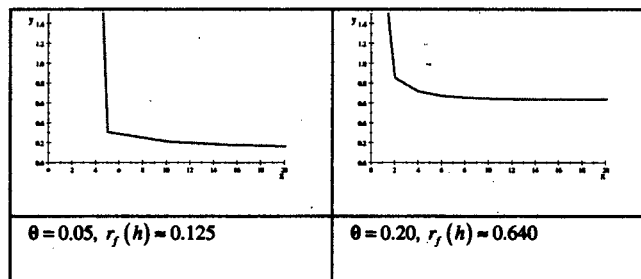
(1.50302) (2.09261)

The flat rate and the bottom-length information may help the design of the product since it is possible for given bottom-length to decide the parameter θ and thus determine the bathtub hazard $h(t; \theta)$ in terms of Eq. (6.19) and Eq. (6.20) initially.

$$\theta = 0.40987 - 0.06094 \ln(l_b) \quad (R^2 = 0.9301) \quad (6.23)$$

(0.034126) (0.0083557)

Since the information flat rate $r_f(h)$ and the bottom-length l_b are critical in designing the structure of a system under development, a drawing of bathtub curve will definitely help to understand the properties of the bathtub hazard family, particularly, the relevant indices: t_d, t_u, l_b , and $r_f(h)$ etc. Note here that for comparison purpose, we set the units of X-axis (time) and Y-Axis (hazard rate) the same. Therefore, the drawing of the curves are all focused on (or limited) time axis is ranged from 0 to 20 with the intention to see the infant mortality pattern behavior, the down-turning point, t_d , and the starting (part) of the memoryless failure hazard rate changing trend, the value of flat rate $r_f(h)$, and even the wear-out state in the last curve with $\theta = 0.30$.



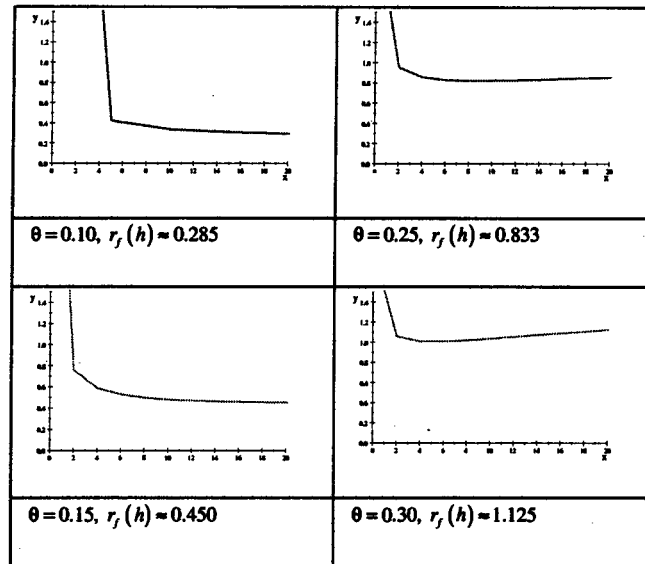


Figure 6.3.1 Patterns of the Additive Bathtub Curves

We found out that as θ evolves from 0.0 to 0.3, the length of the “bottom of the bathtub” (denoted as bottom-interval, $[t_L, t_U]$, such that $h(t) \approx \text{constant}$ for $t \in [t_L, t_U]$) decreases. Furthermore, it looks that the down-turning point in the additive model case is relatively easy to be recognized. Such a feature may offer some conveniences for early design of the product and later data analysis.

An Exploratory Look at Product Bathtub Model

For the product type one-parameter bathtub hazard, we perform similar analyses.

For the one-parameter product bathtub hazard model. We will have an over view also in terms of the three parts: Table IV, three regression models, and Figure II, six product bathtub curves under six grids of θ from 0 to 0.3. However, the details are not all the same as these in additive bathtub case.

Table 6.3.4 Bathtub down-turning point, bottom-rate, and θ

θ	Down-turning point t_d	Bottom-rate $h(t_b)$	Flat rate
0.05	3.759E+25	4.96E-17	0.01464
0.10	3.487E+09	2.09E-06	0.03086
0.15	1.052E+05	2.33E-03	0.05635

0.20	1.024E+03	4.27E-02	0.05040
0.25	8.100E+01	1.86E-01	0.20492
0.30	1.685E+01	4.28E-01	0.44863

Note here we do not list the down-turning points in table as we do in Table 6.3.3 is because the chosen of them are slightly intuitive by eyes.

The bottom-rate demonstrates a weak linear relation with the parameter θ , a straight line is fitted for reflecting the linear functional.

$$h_b(\theta) = -0.1641 + 1.5659 \theta \quad (R^2 = 0.7274) \quad (6.24)$$

(0.0933)
(0.4793)

The flat-rate and θ also demonstrates a weak linear relation as

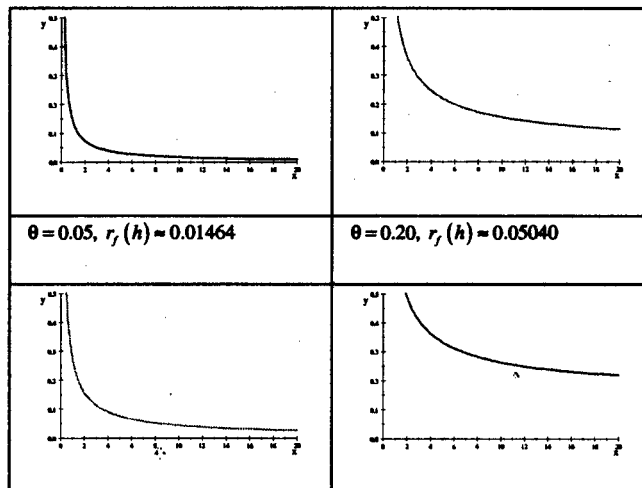
$$r_f(h) = -0.13432 + 1.53497 \theta \quad (R^2 = 0.7263) \quad (6.24)$$

(0.09174)
(0.47113)

The bottom-point and θ demonstrates some functional relation. We perform log-log-transformation and successfully fit a linear relation.

$$\ln(\ln(t_b(\theta))) = 4.39737 - 11.7297 \theta \quad (R^2 = 0.9736) \quad (6.25)$$

(0.18817)
(0.96634)



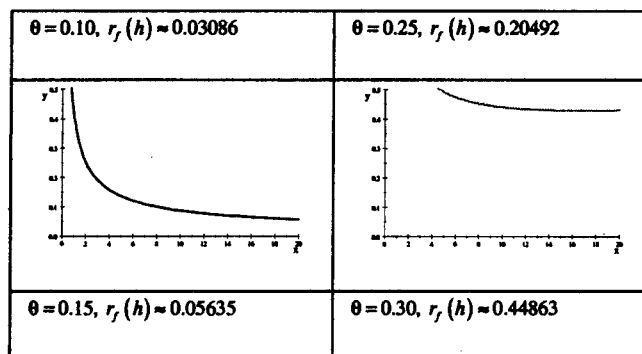


Figure 6.3.2 Patterns of the Product Bathtub Curves

Comparing to the additive bathtub hazard curves, product bathtub hazard curves demonstrate much lower bottom-point, t_b , and accordingly lower flat-rate. However, the turning points of the bathtub are difficult to identify, which may cause inconveniences in design and analysis.

Finally, from the above subsection C and D, these non-probabilistic exploratory analyses will help us to be confident on the capacity of an adequate flexibility for data-assimilation.

We expect these intuitive explorations will provide useful information either in product specification design or in maximum likelihood estimation (initial value settings of parameter). Even, before the estimation starts, we may obtain these estimates of the indices, say, $r_f(h)$, t_d , and θ respectively.

Properties of Additive Bathtub Hazard

The survival function of the one-parameter bathtub hazard family is

$$S(t; \theta) = \exp\left((1-\theta)\theta^{\theta-1} - \left(\theta t + \frac{1-\theta}{\theta}(t+\theta)^\theta + \frac{\theta}{\theta+1}t^{\theta+1}\right)\right) \tag{6.26}$$

Accordingly, the distribution function is

$$F(t; \theta) = 1 - \exp\left((1-\theta)\theta^{\theta-1} - \left(\theta t + \frac{1-\theta}{\theta}(t+\theta)^\theta + \frac{\theta}{\theta+1}t^{\theta+1}\right)\right) \tag{6.27}$$

The moment generating function of the one-parameter bathtub family is difficult to find, however, we can calculate the sample moment generating function as

$$\hat{m}(\varphi) = \frac{1}{n} \sum_{i=1}^n e^{\varphi x_i} \quad (6.28)$$

And thus we can find the sample cumulant generating function

$$\hat{k}(\varphi) = \ln(\hat{m}(\varphi)) = \ln\left(\frac{1}{n} \sum_{i=1}^n e^{\varphi x_i}\right) \quad (6.29)$$

Then an empirical small sample asymptotic analysis may be engaged with the help of sample cumulant generating function:

Finally, we have to point out that the proposal of the one-parameter hazard model does not have theoretical justification but just an engineering intuition and an empirical data analysis motivation - less parameter and good flexibility in model fitting.

6.3.3 A Two-parameter Additive Bathtub Family

As Table 6.3.1 shows the two-parameter bathtub hazard proposed by Guo, Kuo, and Zhao is more complicated in its functional form than that of Smith and Bain. We expect such complexity in formation will help us to gain more flexibility in explaining the behavior of hazard function.

The Density Function of a Two-parameter Bathtub

The two-parameter bathtub hazard takes the form

$$h(r; \theta, \pi) = \theta + (1 - \theta)(\pi r + \theta)^{\theta-1} + \theta(\pi r)^{\theta} \quad (6.30)$$

The survival distribution function is

$$S(r; \theta, \pi) = e^{-\frac{(1-\theta)\theta^{\theta-1}}{\pi}} \exp\left(-\left(\theta r + \frac{1-\theta}{\pi\theta}(\pi r + \theta)^{\theta} + \frac{\theta}{(1+\theta)}(\pi r)^{\theta+1}\right)\right) \quad (6.31)$$

And the density function is

$$\begin{aligned}
f(t; \theta, \pi) &= h(t; \theta, \pi) S(t; \theta, \pi) \\
&= \left(\theta + (1 - \theta)(\pi t + \theta)^{\theta - 1} + \theta(\pi t)^{\theta} \right) \times \\
&\quad e^{-\frac{(1 - \theta)\theta^{\theta - 1}}{\pi}} \exp \left(- \left(\theta t + \frac{1 - \theta}{\pi \theta} (\pi t + \theta)^{\theta} + \frac{\theta}{(1 + \theta)} (\pi t)^{\theta + 1} \right) \right)
\end{aligned} \tag{6.32}$$

Log-Likelihood Function for θ

Note that the two-parameter bathtub hazard is rooted in the one-parameter one with the scale factor, π , added in. The value of π comes from the data collected for a given system or product. Therefore, the estimation is still one-parameter bathtub hazard problem with π being fixed at given level. For later convenience, we define a series of intermediate function of θ and t .

$$A(t, \theta) = (\pi t + \theta)^{\theta} \tag{6.33}$$

$$B(t, \theta) = (\pi t)^{\theta + 1} \tag{6.34}$$

$$C(\theta) = \frac{1}{\pi} (1 - \theta) \theta^{\theta - 1} \tag{6.35}$$

and

$$H(t, \theta) = \theta + \frac{1}{\pi t + \theta} A(t, \theta) + \frac{1}{\pi t} B(t, \theta) \tag{6.36}$$

respectively.

Then, the density of the two-parameter bathtub hazard is

$$f(t; \theta, \pi) = e^{C(\theta)H(t, \theta)} \exp \left(- \left(\theta t + \frac{1}{\pi \theta} A(t, \theta) + \frac{1}{(1 - \theta)\pi} B(t, \theta) \right) \right) \tag{6.37}$$

Let the observation of the data take the form

$$K = \{t_1, t_2, \dots, t_n; \pi_0\} \tag{6.38}$$

Then the likelihood function is

$$L(\theta | K) = \prod_{i=1}^n f(t_i; \theta, \pi_0) \quad (6.39)$$

The log-likelihood function is

$$\begin{aligned} l(\theta | K) &= \sum_{i=1}^n \ln(f(t_i; \theta, \pi_0)) \\ &= nC(\theta) + \sum_{i=1}^n \ln H(t_i, \theta) - \theta \sum_{i=1}^n t_i - \frac{1}{\pi_0 \theta} \sum_{i=1}^n A(t_i, \theta) - \frac{1}{(1+\theta)\pi_0} \sum_{i=1}^n B(t_i, \theta) \end{aligned} \quad (6.40)$$

Maximum Likelihood Estimate for θ

As a standard maximum likelihood procedure, it is necessary to derive the first-order derivative of the log-likelihood function with respect to parameter θ and set it to zero.

$$\frac{d}{d\theta} l(\theta | K) = 0 \quad (6.41)$$

Now, we need to derive the expression of $dl(\theta | K)/d\theta$. We notice it as above. i.e.,

$$\begin{aligned} \frac{d}{d\theta} l(\theta | K) &= nC'(\theta) + \sum_{i=1}^n \frac{H'(t_i, \theta)}{H(t_i, \theta)} - \sum_{i=1}^n t_i + \frac{1}{\pi_0 \theta^2} \sum_{i=1}^n A(t_i, \theta) \\ &+ \frac{1}{(1+\theta)^2 \pi_0} \sum_{i=1}^n B(t_i, \theta) - \frac{1}{\pi_0 \theta} \sum_{i=1}^n A'(t_i, \theta) - \frac{1}{(1+\theta)\pi_0} \sum_{i=1}^n B'(t_i, \theta) \end{aligned} \quad (6.42)$$

where

$$A'(t, \theta) = (\pi t + \theta)^\theta (1 + \ln(\pi t + \theta)) \quad (6.43)$$

$$B'(t, \theta) = (\pi t)^{\theta+1} \ln(\pi t) \quad (6.44)$$

$$C'(\theta) = \frac{1}{\pi} (1-\theta)\theta^{\theta-1} (\ln \theta - 1) \quad (6.45)$$

and

$$H'(t, \theta) = 1 - \frac{1}{(\pi t + \theta)^2} A(t, \theta) + \frac{1}{(\pi t + \theta)} A'(t, \theta) + \frac{1}{\pi t} B'(t, \theta) \quad (6.46)$$

respectively.

Asymptotic Inference on θ

The asymptotic inference is involved the second-order derivative of $l(\theta|K)$ with respect to θ , i.e., $d^2l(\theta|K)/d\theta^2$.

$$\begin{aligned} \frac{d^2}{d\theta^2}l(\theta|K) &= nC'(\theta) + \sum_{i=1}^n \frac{H^*(t_i, \theta) - (H'(t_i, \theta))^2}{H(t_i, \theta)} \\ &\quad - \frac{2}{\pi_0 \theta^3} \sum_{i=1}^n A(t_i, \theta) - \frac{2}{(1+\theta)^3 \pi_0} \sum_{i=1}^n B(t_i, \theta) \\ &\quad + \frac{2}{\pi_0 \theta^2} \sum_{i=1}^n A'(t_i, \theta) + \frac{2}{(1+\theta)^2 \pi_0} \sum_{i=1}^n B'(t_i, \theta) \\ &\quad - \frac{1}{\pi_0 \theta} \sum_{i=1}^n A''(t_i, \theta) - \frac{1}{(1+\theta) \pi_0} \sum_{i=1}^n B''(t_i, \theta) \end{aligned} \quad (6.47)$$

where

$$A^*(t, \theta) = (\pi t + \theta)^0 \left((1 + \ln(\pi t + \theta))^2 + \frac{1}{\pi t + \theta} \right) \quad (6.48)$$

$$B^*(t, \theta) = (\pi t)^{\theta+1} (\ln(\pi t))^2 \quad (6.49)$$

$$C'(\theta) = \frac{1}{\pi} (1-\theta) \theta^{\theta-1} \left((\ln \theta - 1)^2 + \frac{1}{\theta} \right) \quad (6.50)$$

and

$$H^*(t, \theta) = \frac{2}{(\pi t + \theta)^3} A(t, \theta) - \frac{2}{(\pi t + \theta)^2} A'(t, \theta) + \frac{1}{(\pi t + \theta)} A''(t, \theta) + \frac{1}{\pi t} B^*(t, \theta) \quad (6.51)$$

respectively.

Let us denote the maximum like estimate as $\hat{\theta}$ and also denoted the variance as

$$V = \frac{1}{\frac{d^2l(\theta|K)}{d\theta^2}} \quad (6.52)$$

Then quantity $(\hat{\theta} - \theta)/\sqrt{V}$ is asymptotically distributed as a standard normal random variable. The Hypothesis and confidence interval for θ can be constructed based on such fact.

6.3.4 A Two-parameter Product Bathtub Family

The two-parameter product bathtub hazard is proposed by Smith and Bain

$$h(t; \theta, \pi) = \theta \pi (\pi t)^{\theta-1} e^{(\pi t)^\theta}, \quad \theta \in (0, 1), \quad \pi > 0 \quad (6.53)$$

Note that the Survival function is

$$s(t; \theta, \pi) = e \exp\left(-e^{(\pi t)^\theta}\right), \quad \theta \in (0, 1), \quad \pi > 0 \quad (6.54)$$

And the density function is

$$h(t; \theta, \pi) = \theta \pi (\pi t)^{\theta-1} e^{(\pi t)^\theta} \exp\left(-e^{(\pi t)^\theta}\right) \quad (6.54)$$

For given observation $K = \{t_1, t_2, \dots, t_n; \pi_0\}$, the likelihood function is

$$L(\theta | K) = \prod_{i=1}^n \left(\theta \pi (\pi t_i)^{\theta-1} e^{(\pi t_i)^\theta} \exp\left(-e^{(\pi t_i)^\theta}\right) \right) \quad (6.55)$$

and the log-likelihood function is

$$l(\theta | K) = n \ln(e\pi\theta) + (\theta-1) \sum_{i=1}^n \ln(\pi t_i) + \sum_{i=1}^n Q(t_i, \theta) - \sum_{i=1}^n e^{Q(t_i, \theta)} \quad (6.56)$$

Now we are ready to derive the first-order derivative of $l(\theta | K)$ with respect to θ

$$\frac{dl(\theta | K)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln(\pi t_i) + \sum_{i=1}^n Q(t_i, \theta) \ln(\pi t_i) - \sum_{i=1}^n e^{Q(t_i, \theta)} Q(t_i, \theta) \ln(\pi t_i) \quad (6.57)$$

Set $dl(\theta | K)/d\theta = 0$, we obtain a nonlinear equation:

$$\frac{n}{\theta} + \sum_{i=1}^n \ln(\pi t_i) + \sum_{i=1}^n Q(t_i, \theta) \ln(\pi t_i) - \sum_{i=1}^n e^{Q(t_i, \theta)} Q(t_i, \theta) \ln(\pi t_i) = 0 \quad (6.58)$$

The solution of the Eq. (6.58) is called the maximum likelihood estimate, denoted as $\hat{\theta}$.

The second-order derivative of $l(\theta | K)$ with respect to θ

$$\frac{d^2l(\theta|K)}{d\theta^2} = -\frac{n}{\theta^2} + \sum_{i=1}^n Q(t_i, \theta) (\ln(\pi t_i))^2 - \sum_{i=1}^n e^{Q(t_i, \theta)} Q(t_i, \theta) (\ln(\pi t_i))^2 (Q(t_i, \theta) + 1) \quad (6.59)$$

Let

$$V = -\left[\frac{d^2l(\theta|K)}{d\theta^2} \right]_{\theta=\hat{\theta}}^{-1} \quad (6.60)$$

which is the asymptotic variance for the MLE.

An example of maximum likelihood estimation is shown in Section 6.4.1.

6.3.5 Fuzzy Variable in AL modeling

Product design specification is one thing and the product in use is another thing because product must be tuned up to a particular application environment. Furthermore, in the product design stage, there is often a shortage of the information on the product under designing but there is very rich knowledge on its former sequence of similar products. Therefore, using a (scalar) fuzzy (variable) parameter to replace a fixed one would be logical and lead to realistic analysis for further decision making of the product design and production. If the scale parameter of the two-parameter bathtub hazard family is a triangular fuzzy variable with membership function.

$$\mu_{\pi}(y) = \begin{cases} \frac{y-a_{\pi}}{b_{\pi}-a_{\pi}} & a_{\pi} \leq y < b_{\pi} \\ \frac{c_{\pi}-y}{c_{\pi}-b_{\pi}} & b_{\pi} \leq y < c_{\pi} \\ 0 & \text{otherwise} \end{cases} \quad (6.61)$$

Then according to Liu, this will result a random fuzzy variable, which could be described by average chance distribution. For Liu's credibility measure theory and random fuzzy variable theory in general, we have a brief Appendix for interested readers, although the materials are quite abstract. For reliability engineers, it is not necessary to battle with Appendix, rather, to follow the arguments in main text.

The average chance measure, which is the counterpart of the probability measure, of event $\{\xi \leq t\}$ is

$$\text{ch}\{\xi \leq t\} = \int_0^1 \text{Cr}\{\pi : \Pr\{\xi(\pi, \omega) \leq t\} \geq \alpha\} d\alpha \tag{6.62}$$

The critical step is to derive the expression of $\{\pi : \Pr\{\xi(\pi, \omega) \leq t\} \geq \alpha\}$:

Note that event $\{\pi : \Pr\{\xi(\pi, \omega) \leq t\} \geq \alpha\}$ is equivalent to event $\{\pi : \pi \geq \sqrt[t]{\ln(-\ln((1-\alpha)/e))}/t\}$ since

$$\begin{aligned} & \{\pi : \Pr\{\xi(\pi, \omega) \leq t\} \geq \alpha\} \\ & \Leftrightarrow \\ & \{\pi : 1 - S(t; \pi) \geq \alpha\} \\ & \Leftrightarrow \\ & \{\pi : e \exp(-e^{(\pi t)^{\theta}}) \leq 1 - \alpha\} \\ & \Leftrightarrow \\ & \left\{ \pi : (\pi t)^{\theta} \geq \ln\left(-\ln\left(\frac{1-\alpha}{e}\right)\right) \right\} \\ & \Leftrightarrow \\ & \left\{ \pi : \pi \geq \frac{1}{t} \sqrt[\theta]{\ln\left(-\ln\left(\frac{1-\alpha}{e}\right)\right)} \right\} \end{aligned} \tag{6.63}$$

Further we notice that from Eq. (6.61), the credibility measure of event $\text{Cr}\{\pi \geq y\}$ is

$$\text{Cr}\{\pi \geq y\} = \begin{cases} 1 & \text{if } y \leq a_x \\ \frac{2b_x - a_x - y}{2(b_x - a_x)} & \text{if } a_x \leq y < b_x \\ \frac{c_x - y}{2(c_x - b_x)} & \text{if } b_x \leq y < a_x \\ 0 & \text{if } y \geq c_x \end{cases} \tag{6.64}$$

Table 6.3.5 Integration with respect to $\alpha \in [0,1]$

$g(\alpha): \frac{1}{t} \sqrt[\theta]{\ln\left(-\ln\left(\frac{1-\alpha}{e}\right)\right)}$	Range for α	The integrand
		$\{\pi : \Pr\{\xi(\pi, \omega) \leq t\} \geq \alpha\}$

$0 < g(\alpha) \leq a_x$	$\alpha_{a_x} < \alpha \leq 1.0$	1
$a_x < g(\alpha) \leq b_x$	$\alpha_{b_x} < \alpha \leq \alpha_{a_x}$	$\frac{2b_x - a_x - g(\alpha)}{2(b_x - a_x)}$
$b_x < g(\alpha) \leq c_x$	$\alpha_{c_x} < \alpha \leq \alpha_{b_x}$	$\frac{c_x - g(\alpha)}{2(c_x - b_x)}$
$c_x < g(\alpha) \leq +\infty$	$0 < \alpha \leq \alpha_{c_x}$	0

where

$$\alpha_x = 1 - e \exp\left(-e^{(x_x)^0}\right) \quad (6.65)$$

with

$$x = a_x, b_x, \text{ or } c_x \quad (6.66)$$

respectively. Then, we will obtain the average chance distribution

$$\begin{aligned} \Psi(t) &= \text{ch}\{\xi \leq t\} \\ &= \int_{\alpha_{a_x}}^1 1 \times d\alpha + \int_{\alpha_{b_x}}^{\alpha_{a_x}} \frac{2b_x - a_x - g(\alpha)}{2(b_x - a_x)} d\alpha + \int_{\alpha_{c_x}}^{\alpha_{b_x}} \frac{c_x - g(\alpha)}{2(c_x - b_x)} d\alpha + \int_0^{\alpha_{c_x}} 0 \times d\alpha \end{aligned} \quad (6.67)$$

Now, in terms of change of variable, we finally reach

$$\begin{aligned} \Psi(t) &= \text{ch}\{\xi \leq t\} \\ &= e \exp\left(-e^{(a_x)^0}\right) + \frac{(2b_x - a_x)e}{2(b_x - a_x)} \left(\exp\left(-e^{(b_x)^0}\right) - \exp\left(-e^{(a_x)^0}\right) \right) \\ &\quad + \frac{c_x e}{2(c_x - b_x)} \left(\exp\left(-e^{(c_x)^0}\right) - \exp\left(-e^{(b_x)^0}\right) \right) \\ &\quad + \frac{1}{2(b_x - a_x)} \left(\exp\left(-e^{(a_x)^0}\right) - \exp\left(-e^{(b_x)^0}\right) \right) \\ &\quad + \frac{1}{2(c_x - b_x)} \left(\exp\left(-e^{(b_x)^0}\right) - \exp\left(-e^{(c_x)^0}\right) \right) \end{aligned} \quad (6.68)$$

6.4 Random Fuzzy Survival Analysis with Smith-Bain Hazard Function

Life-time survival analysis is critical in reliability engineering. An abbreviation of accelerating life model is a one of the commonly used approaches in new product design stage because the product reliability information can be obtained in a very short time period by applying higher stress level.

Definitely, if the stress level is unknown, we can search parameter pair (θ, π) by solving the following nonlinear equation system

$$\begin{cases} \frac{n}{\theta} + \sum_{i=1}^n \ln(\pi t_i) + \sum_{i=1}^n Q(t_i, \theta) \ln(\pi t_i) - \sum_{i=1}^n e^{Q(t_i, \theta)} Q(t_i, \theta) \ln(\pi t_i) = 0 \\ \frac{n}{\pi} + (\theta - 1) \sum_{i=1}^n \frac{t_i}{\pi} + \sum_{i=1}^n Q(t_i, \theta) (\pi t_i)^{\theta-1} t_i - \sum_{i=1}^n e^{Q(t_i, \theta)} Q(t_i, \theta) (\pi t_i)^{\theta-1} t_i = 0 \end{cases} \quad (6.69)$$

6.4.1 Maximum likelihood estimation under random fuzzy lifetime context

In this section, we look at the life-time differently, i.e., we treat the lifetime as random fuzzy observations, which may include censoring lifetime observations. Here we can “borrow” the likelihood function construction idea from standard statistical lifetime data analysis to form a “random fuzzy likelihood” function. For the bathtub (Smith Bain, 1976) case, we still have the two parameter pair (θ, π) . However, now we assume the “stress” (or scale) parameter π be a fuzzy variable having a trapezoidal membership function with parameter (a, b, c, d)

$$\mu_{\pi}(u) = \begin{cases} \frac{u-a}{b-a} & a \leq u < b \\ 1 & b \leq u < c \\ \frac{d-u}{d-c} & c \leq u < d \\ 0 & \text{otherwise} \end{cases} \quad (6.70)$$

Then for Smith and Bain [9] bathtub hazard, the average chance distribution is

$$\begin{aligned}
\Psi(t; \theta, a, b, c, d) &= \text{ch}\{T \leq t\} \\
&= e \exp\left(-\exp\left((at)^\beta\right)\right) + \frac{e}{2} \left(\exp\left(-\exp\left((ct)^\beta\right)\right) - \exp\left(-\exp\left((bt)^\beta\right)\right) \right) \\
&\quad + \frac{(2b-a)e}{2(b-a)} \left(\exp\left(-\exp\left((bt)^\beta\right)\right) - \exp\left(-\exp\left((at)^\beta\right)\right) \right) \\
&\quad - \frac{e}{2(b-a)} \left(b \exp\left(-\exp\left((bt)^\beta\right)\right) - a \exp\left(-\exp\left((at)^\beta\right)\right) \right) + \frac{e}{2(b-a)t} \int_a^{bt} \left(\exp\left(-\exp\left(y^\beta\right)\right) \right) dy \\
&\quad + \frac{de}{2(d-c)} \left(\exp\left(-\exp\left((dt)^\beta\right)\right) - \exp\left(-\exp\left((ct)^\beta\right)\right) \right) \\
&\quad - \frac{e}{2(d-c)} \left(d \exp\left(-\exp\left((dt)^\beta\right)\right) - c \exp\left(-\exp\left((ct)^\beta\right)\right) \right) + \frac{e}{2(d-c)t} \int_c^{dt} \left(\exp\left(-\exp\left(y^\beta\right)\right) \right) dy
\end{aligned} \tag{6.71}$$

where

$$\begin{aligned}
A(t; \theta, a) &= \exp\left(-\exp\left((at)^\beta\right)\right), \\
B(t; \theta, b) &= \exp\left(-\exp\left((bt)^\beta\right)\right), \\
C(t; \theta, c) &= \exp\left(-\exp\left((ct)^\beta\right)\right), \\
D(t; \theta, d) &= \exp\left(-\exp\left((dt)^\beta\right)\right)
\end{aligned} \tag{6.72}$$

Then the average chance distribution can be simplified as in terms of re-expressing Eq. (6.71) via Eq. (6.72)

$$\Psi(t; \theta, a, b, c, d) = \frac{e}{2(b-a)t} \int_a^{bt} \left(\exp\left(-\exp\left(y^\beta\right)\right) \right) dy + \frac{e}{2(d-c)t} \int_c^{dt} \left(\exp\left(-\exp\left(y^\beta\right)\right) \right) dy \tag{6.73}$$

Therefore, the average density is

$$\begin{aligned}
\psi(t; \theta, a, b, c, d) &= -\frac{e}{2(b-a)t^2} \int_a^{bt} \left(\exp\left(-\exp\left(y^\beta\right)\right) \right) dy - \frac{e}{2(d-c)t^2} \int_c^{dt} \left(\exp\left(-\exp\left(y^\beta\right)\right) \right) dy \\
&\quad + \frac{e}{2(b-a)t} \left(b \exp\left(-\exp\left((bt)^\beta\right)\right) - a \exp\left(-\exp\left((at)^\beta\right)\right) \right) \\
&\quad + \frac{e}{2(d-c)t} \left(d \exp\left(-\exp\left((dt)^\beta\right)\right) - c \exp\left(-\exp\left((ct)^\beta\right)\right) \right)
\end{aligned} \tag{6.74}$$

Then the “likelihood function” given the random fuzzy lifetime observations $K^* = \{t_1, t_2, \dots, t_n\}$ takes the form

$$L(\theta, a, b, c, d | K^*) = \prod_{i=1}^n \psi(t_i; \theta, a, b, c, d) \quad (6.75)$$

which leads to a “log-likelihood function”

$$l(\theta, a, b, c, d | K^*) = \sum_{i=1}^n \ln \psi(t_i; \theta, a, b, c, d) \quad (6.76)$$

See Appendix A for the first-order partial derivatives and the non-linear equation system.

6.4.2 Maximum average chance estimation

Although we did obtain a general M -function system for the parameter $(\theta, a, b, c, d)^T$, there is still some fundamental issue links to the “likelihood” function of random fuzzy lifetimes.

Liu (2007) states a Maximum Uncertainty Principle: “For any event, if there are multiple reasonable values that a measure may take, then the value as close as 0.5 as possible is assigned to the event”.

Then a maximum average chance principle can similarly stated although the average chance measure does not belong to the uncertainty measure (Liu, 2004) family. Let event sequence $\{A_1, A_2, \dots, A_n\}$, then we expect

$$J = \sum_{i=1}^n (\text{ch} \{A_i\} - 0.5)^2 \quad (6.77)$$

is minimized.

For given random fuzzy life times $K^* = \{t_1, t_2, \dots, t_n\}$, we are expecting the parameter $(\theta, a, b, c, d)^T$ should minimizes the maximum average chance uncertainty criterion

$$J(\theta, a, b, c, d | K^*) = \sum_{i=1}^n (\Psi(t_i; \theta, a, b, c, d) - 0.5)^2 \quad (6.78)$$

Then an average chance M -estimator for $(\theta, a, b, c, d)^T$, denoted as $(\hat{\theta}, \hat{a}, \hat{b}, \hat{c}, \hat{d})^T$ should satisfy the following equation system

$$\begin{cases} \sum_{i=1}^n (\Psi(t_i; \theta, a, b, c, d) - 0.5) \frac{\partial}{\partial \theta} \Psi(t_i; \theta, a, b, c, d) = 0 \\ \sum_{i=1}^n (\Psi(t_i; \theta, a, b, c, d) - 0.5) \frac{\partial}{\partial a} \Psi(t_i; \theta, a, b, c, d) = 0 \\ \sum_{i=1}^n (\Psi(t_i; \theta, a, b, c, d) - 0.5) \frac{\partial}{\partial b} \Psi(t_i; \theta, a, b, c, d) = 0 \\ \sum_{i=1}^n (\Psi(t_i; \theta, a, b, c, d) - 0.5) \frac{\partial}{\partial c} \Psi(t_i; \theta, a, b, c, d) = 0 \\ \sum_{i=1}^n (\Psi(t_i; \theta, a, b, c, d) - 0.5) \frac{\partial}{\partial d} \Psi(t_i; \theta, a, b, c, d) = 0 \end{cases} \quad (6.79)$$

where the partial derivatives of $\Psi(\cdot)$ with respect to parameters are

$$\frac{\partial}{\partial \theta} \Psi(t) = \frac{e}{2(a-b)t} \cdot \int_a^{bt} [y^\theta \cdot e^{y^\theta} \cdot e^{-e^{y^\theta}} \cdot \ln(y)] dy + \frac{e}{2(c-d)t} \cdot \int_c^{dt} [y^\theta \cdot e^{y^\theta} \cdot e^{-e^{y^\theta}} \cdot \ln(y)] dy \quad (6.80)$$

$$\frac{\partial}{\partial a} \Psi(t) = \frac{e}{2(b-a)^2 t} \cdot \int_a^{bt} (e^{-e^{y^\theta}}) dy - \frac{e}{2(b-a)} \cdot e^{-e^{(bt)^\theta}} \quad (6.81)$$

$$\frac{\partial}{\partial b} \Psi(t) = -\frac{e}{2(b-a)^2 t} \cdot \int_a^{bt} (e^{-e^{y^\theta}}) dy + \frac{e}{2(b-a)} \cdot e^{-e^{(bt)^\theta}} \quad (6.82)$$

$$\frac{\partial}{\partial c} \Psi(t) = \frac{e}{2(d-c)^2 t} \cdot \int_c^{dt} (e^{-e^{y^\theta}}) dy - \frac{e}{2(d-c)} \cdot e^{-e^{(dt)^\theta}} \quad (6.83)$$

$$\frac{\partial}{\partial d} \Psi(t) = -\frac{e}{2(d-c)^2 t} \cdot \int_c^{dt} (e^{-e^{y^\theta}}) dy + \frac{e}{2(d-c)} \cdot e^{-e^{(dt)^\theta}} \quad (6.84)$$

6.5 An Industrial Example of Random Fuzzy Analysis

MOS capacitor, with the structure of Al/high- k /Si, is a high k dielectric film between aluminum gate electrode and silicon substrate.

These capacitors are fabricated and tested. The high- k dielectric under test is *20W Hf-doped TaO_x film*. See Figure 6.3.1 for the schematic layout of the sample and Table 6.3.1 for information of the sample capacitors.

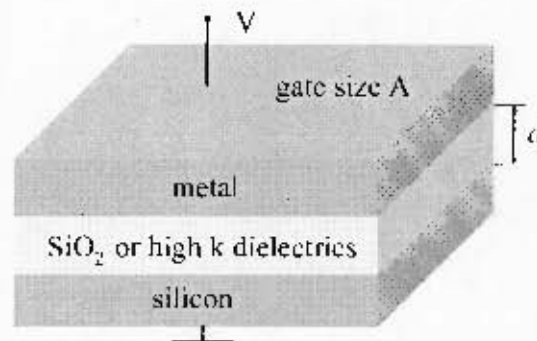


Figure 6.3.3 Schematic structure of a MOS capacitor

Table 6.3.6 High- k dielectric films under test

High- k Dielectric THIN FILM	Thickness (10^4 cm^2)	Gate Size (nm)	EOT* (nm)	K (at 1MHz)
20W HF-DOPE TAO _x	62-63	3.83	2.19	11.3

* Equivalent Oxide Thickness

Constant voltage stress tests are performed on the capacitors to collect time-depended dielectric breakdown (TDDB) data. The jump of leakage current at the moment of breakdown can be clearly identified in the tests. Each is tested individually and breaks down independently. The ordered breakdown times in seconds at the *accelerated stress* of 7.7 MV/cm are recorded as: 9 18 20 25 29 66 124 127 175 221 249 341 362 552 630 760 782 794 906 932 968 1378 1386 1664 1728 2229 2249 2338 4058 4986 6312 6400 6847 8474 ($n = 34$)

6.5.1 Data Computation of Fuzzy Variable in AL modeling

The data computation is based on the bathtub hazard function (Smith Bain, 1976)

$h(t; \eta, \beta) = \eta\beta(\eta t)^{\beta-1} \exp\left(-(\eta t)^\beta\right)$. The data are time-dependent dielectric breakdown (TDDB) data (Kuo, W, 2006)

To search for a value of the random fuzzy parameter, we divide the arithmetic into two parts.

First part we assume that η is a constant as for the condition. We tried value 7.7 and 1, respectively.

For log-likelihood function of the above hazard, take first order derivative with respect to β :

$$\frac{\partial}{\partial \beta} l(\eta, \beta) = \frac{n}{\beta} + \sum_{i=1}^n \ln(\eta t_i) + \sum_{i=1}^n (\eta t_i)^\beta \ln(\eta t_i) - \sum_{i=1}^n \exp\left(-(\eta t_i)^\beta\right) (\eta t_i)^\beta \ln(\eta t_i) \quad (\text{B1})$$

where $n = 34$ in this case.

We here use do-loop method to search the estimate of β . For $\eta = 7.7$, we search until

$\beta = 0.0372206478$, so that $\frac{\partial}{\partial \beta} l(\eta, \beta) = 2.57084\text{E-}07$; For $\eta = 1$, we search until $\beta = 0.048108959$,

so that $\frac{\partial}{\partial \beta} l(\eta, \beta) = 8.95259\text{E-}07$; Here the figure below illustrates spreadsheet the computing process.

	A	B	C	D	E	F	G	H	
1	r_i	β	η	$\frac{n}{\beta}$	$\sum_{i=1}^n \ln(\eta_i)$	$\sum_{i=1}^n (\eta_i)^\beta \ln(\eta_i)$	$\sum_{i=1}^n \exp((\eta_i)^\beta) (\eta_i)^\beta \ln(\eta_i)$	$\frac{\partial}{\partial \beta} l(\eta, \beta) \rightarrow 0$	
2	9	0.0100000000	7.7	3400	207.7144572	309.7121183	923.1054975	30711.376578	
3	10	0.1000000000		340	207.7144572	690.0922191	8890.778755	-7576.972079	
4	20	0.1000000000		1700	283.7144572	338.1862692	1110.171990	1205.728741	
5	25	0.1310000000		1133.333333	283.7144572	369.3796919	1365.010125	421.4093165	
6	29	0.1400000000		650	283.7144572	413.5411875	1690.727079	-103.4529943	
7	35	0.1310000000		1106.774194	283.7144572	372.6588455	1393.710065	509.4300403	
8	12	0.0322000000		1042.5	283.7144572	375.367967	1423.109191	290.991032	
9	127	0.0300000000		894.7368421	283.7144572	395.4714666	1618.177364	43.2045983	
10	175	0.0371000000		916.4420485	283.7144572	393.3236751	1666.772307	6.707388461	
11	221	0.0372000000		913.9784940	277.7144572	395.6721496	1500.217507	1.147919316	
12	238	0.0371000000		911.5201501	277.7144572	394.0209434	1513.472174	-4.4076523534	
13	241	0.0372100000		913.7320675	277.7144572	393.7070146	1510.662404	0.691735247	
14	252	0.0372200000		913.4871724	273.7144572	393.7418829	1500.907113	0.0359995	
15	552	0.0372300000		913.2420091	283.7144572	391.7767542	1501.252119	-0.619698067	
16	630	0.0372210000		913.4620301	283.7144572	390.7453658	1500.942229	-0.019671972	
17	760	0.0372207000		913.4271327	277.7144572	390.7440337	1500.931874	-0.01291117	
18	792	0.0372206478		7.7	913.4714735	283.7144572	391.7441417	1500.930072	2.57084107
19	794								
20	906	0.0100000000	1	3400	214.312366	229.6248214	671.004719	3171.703016	
21	902	0.1000000000		340	214.312366	430.161047	3446.137619	-2700.661926	
22	968	0.1100000000		1700	214.312366	245.9786010	775.8098105	1384.362717	
23	1378	0.1100000000		650	214.312366	302.717900	1060.098342	-3.56747139	
26		0.048108959			706.7330716	214.312366	298.7561881	1119.786225	8.95259007
27									

Figure 6.4.1 Computing process when η is constant

Then, take the second order of derivative with respect to β :

$$\frac{\partial^2}{\partial \beta^2} l(\eta, \beta) = -\frac{n}{\beta^2} + \sum_{i=1}^n (\eta_i)^\beta (\ln(\eta_i))^2 - \sum_{i=1}^n \exp((\eta_i)^\beta) (\ln(\eta_i))^2 ((\eta_i)^{2\beta} + (\eta_i)^\beta) \quad (B2)$$

We obtain the result of variance V, $v = \left(\frac{\partial^2}{\partial \beta^2} l(\eta, \beta) \right)^{-1}$.

Table 6.4.1 Comparison of result when η is constant

	β	$\frac{\partial}{\partial \beta} l(\eta, \beta)$	V
$\eta = 7.7$	0.0372206478	0.000000257084	0.00001799
$\eta = 1$	0.048108959	0.000000895259	0.00002966

Apparently, the result of variance under $\eta = 7.7$ is more close to target zero than it is under $\eta = 1$.

Second part Now consider η as a fuzzy parameter following a trapezoidal membership function, defined by parameters a, b, c, and d:

$$\mu_{\eta}(z) = \begin{cases} (z-a)/(b-a) & a \leq z < b \\ 1 & a \leq z < b \\ (d-z)/(d-c) & a \leq z < b \\ 0 & \text{otherwise} \end{cases} \quad (B3)$$

The equations system that need to be solved are listed in Appendix A.

We still use a do-loop method to search for the parameter value, but since the number of estimated parameters now become not one but five, the do-loop directions become more complicated. Through the whole computing process, we not only obtain parameter values, but also investigate the association and correlation in between parameters. The VBA code is provided in Appendix B. See detail in the spreadsheet file which is in the attached CD.

β	d	b	c	f	$ a $	$ b $	$\frac{\partial \beta}{\partial a} \rightarrow 0$	$\frac{\partial \beta}{\partial b} \rightarrow 0$	$\frac{\partial \beta}{\partial c} \rightarrow 0$	$\frac{\partial \beta}{\partial d} \rightarrow 0$	$\frac{\partial \beta}{\partial f} \rightarrow 0$
0.022000000	0.3	2978.548	2978.548	5956.996	2978.3	154.799169861	0.200000000	0.000000000	-0.000145816	0.000804397	
0.022500000	0.3	2978.548	2978.548	5956.996	2978.3	31.061160668	-0.202752115	0.001025857	-0.000752601	0.001666745	
0.022700000	0.3	2978.548	2978.548	5956.996	2978.3	1.800367859	0.002816273	0.001011425	-0.000756644	0.000662203	
0.022710000	0.3	2978.548	2978.548	5956.996	2978.3	0.109776953	-0.003914636	0.001010221	-0.000159812	0.000651499	
0.022712000	0.3	2978.548	2978.548	5956.996	2978.3	0.047580420	-0.003915237	0.001011578	-0.000156722	0.000651346	
0.022712100	0.3	2978.548	2978.548	5956.996	2978.3	0.033384110	-0.003915267	0.001011571	-0.000156723	0.000651339	
0.022712126	0.3	2978.548	2978.548	5956.996	2978.3	0.000078438	-0.003915336	0.001011576	-0.000156726	0.000651329	
0.022712326	0.3	2978.548	2978.548	5956.996	2978.3	0.000035477	-0.003915395	0.001011556	-0.000156726	0.000651329	
0.022712326	0.3	2978.548	2978.548	5956.996	2978.3	-0.00008127	-0.003915395	0.001011556	-0.000156726	0.000651329	
0.022712326	0.3	2978.548	2978.548	5956.996	2978.3	-3236.6462584	-0.003915371	0.001011578	-0.000156726	0.000651329	
0.022712326	0.001	2978.548	2978.548	5956.996	2978.3	7.586797917	0.001011572	0.001011572	-0.000156723	0.000651303	
0.022712326	7.493	2986.047	2986.047	5944.196	2978.3	-2.145967366	-0.001568295	0.001008286	0.000156620	0.000651643	
0.022712326	778	3748.348	3748.348	6725.136	2978.3	-96.51682204	-0.001405281	0.000904449	-0.000133015	0.000657072	
0.022712326	17000	79878.35	79878.35	82356.7	2978.3	1044.77085267	0.000101341	0.000763656	-0.000029955	0.000759726	
0.022712326	7700000	7712978	7702978	7706957	2978.3	0.1726030075106	-0.000001150	0.000569901	-0.000002162	0.000650957	
0.010000000	7700000	7712978	7702978	7706957	2978.3	1804.862930240	0.000000004	0.000627335	-0.000003106	0.002627291	
0.014400000	7700000	7702978	7702978	7706957	2978.3	0.732411757	-0.000000044	0.001446865	0.000000057	0.001446822	
0.014452000	7700000	7702978	7702978	7706957	2978.3	0.007371519	-0.000000044	0.001446514	-0.000000057	0.001447169	
0.014452020	7700000	7702978	7702978	7706957	2978.3	0.000444564	-0.000000044	0.001446510	-0.000000057	0.001447166	
0.0144520212	7700000	7702978	7702978	7706957	2978.3	0.000000065	-0.000000044	0.001446510	-0.000000057	0.001446969	
0.0144520713	7700000	7702978	7702978	7706957	2978.3	0.000000077	-0.000000044	0.001446510	-0.000000057	0.001446466	
0.011000000	7700000	7702978	7702978	7706957	2978.3	-19.328380347	-0.000000045	0.001447083	-0.000000057	0.001446044	

Figure 6.4.2 Spreadsheet output for computation of average chance distribution for AL modeling

6.5.2 Data Computation of Fuzzy Variable for Random Fuzzy Survival Analysis

The computation process follows up the same procedure of the AL modeling, while the modeling formulas changed. See detail of the computation process in the attached CD.

Figure 6.4.3 shows the output result after computation. The results reflect the highly precision estimation value.

AG13	AL	AM	AN	AO	AP		AR	AS	AT	AU	AV
	a	b	c	d		$ b - c - d $	$\frac{\partial \mu_a}{\partial a} \rightarrow 0$	$\frac{\partial \mu_a}{\partial a} \rightarrow 0$	$\frac{\partial \mu_a}{\partial b} \rightarrow 0$	$\frac{\partial \mu_a}{\partial c} \rightarrow 0$	
1											
2	0.0372206	7.5	7.7	7.7	7.7	7.2-0-7.2	34.1702736000	0.0050160916	0.0096126066	0.0039452637	0.003088578
3	0.0372206	7.04205	7.05705	7.06205	7.06705	6.051-4-7.4	34.4136917779	0.0111426879	0.0111416269	0.0026629138	0.0030197346
4	0.0372206	6.04205	6.05205	6.05705	6.06205	5.037-0-7.4	34.6737540553	0.0070461522	0.0070462079	0.0050081549	0.0031091816
5	0.0372206	5.04205	5.05205	5.05705	5.06205	4.023-0-7.4	34.9338163327	0.0029496810	0.0029497367	0.0030080452	0.0031031196
6											
7	0.0372206	6.999	7.009	7.015	7.020	1.5-0.001-1.5	34.751909046	0.0050951769	0.0050212315	0.0048613667	0.0045613662
8	0.0372206	6.2	7.5	7.9	9.1	1.2-0.4-1.2	34.7517538113	0.0060821199	0.0051025246	0.0047923125	0.0045403067
9											
10	0.0372206	7.0591	7.1021	7.1004	9.5004	2.051-0.0011-2.4	34.7556451170	0.0056628797	0.0062026228	0.0051858117	0.0045661514
11	0.0372206	7.6385	7.6395	7.6382	7.7022	1.014-0.0005-2.022	34.7561131095	0.0052041705	0.0052039205	0.0052039469	0.0047022462
12	0.0372206	7.63346	7.6395	7.6398	7.70075	0.0001-0.0005-0.001	34.7561445090	0.0051905099	0.0052300670	0.0052001908	0.0052066255
13											
14	0.0372206	7.699	7.7	7.7	7.701	0.001-0-0.001	34.7612033385	0.0052037958	0.0052035456	0.0052090456	0.0052027902
15	0.0372206	7.699	7.7	7.7	7.701	0.001-0-0.001	34.7612033385	0.0052037958	0.0052035456	0.0052090456	0.0052027902
16											
17	0.05	7.699	7.7	7.7	7.701	0.001-0-0.001	25.9624800341	0.0060410300	0.0060407964	0.0060407154	0.0060399795
18	0.1	7.699	7.7	7.7	7.701	0.001-0-0.001	2.1576232651	0.0011693104	0.0013032223	0.0013030260	0.001303269206
19	0.105	7.699	7.7	7.7	7.701	0.001-0-0.001	1.7121172682	0.0011256306	0.0011234147	0.0011230408	0.00112301542
20	0.11	7.699	7.7	7.7	7.701	0.001-0-0.001	1.3904903003	0.0010517501	0.000962917	0.0009611070	0.0009610603
21	0.15	7.699	7.7	7.7	7.701	0.001-0-0.001	0.1067884791	0.000192291	0.0001912042	0.0001911963	0.0001911322
22	0.18	7.699	7.7	7.7	7.701	0.001-0-0.001	0.0245539574	0.0002330745	0.0000143887	0.0000333671	0.0000333613
23	0.2	7.699	7.7	7.7	7.701	0.001-0-0.001	0.0047533798	0.000072427	0.0000072411	0.0000072378	0.0000072311
24	0.25	7.699	7.7	7.7	7.701	0.001-0-0.001	0.0000049639	0.0000000095	0.0000000096	0.0000000095	0.0000000095
25	0.3	7.699	7.7	7.7	7.701	0.001-0-0.001	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
26											
27											

Figure 6.4.3 Spreadsheet output for computation of average chance distribution for random fuzzy survival analysis

Chapter 7. Conclusions

In this thesis, we first reviewed non-classical credibility measure theory, and extract basis discussion, developed the Liu (2006) Credibility measure theory, and then a scalar fuzzy variable.

We defined the random fuzzy concept and discussed the Entropy Principle and Average Chance Measure to specify a theoretical frame for modeling random fuzzy lifetimes and the average chance distributions. We propose a criterion for a data-assimilating parameter specifying the average chance distribution of a random fuzzy lifetime, using a Maximum Average Chance Principle. We use exponentially distributed random fuzzy lifetime with a trapezoidal membership function as an example to illustrate the model developments within random fuzzy lifetimes.

At last, we propose a two-parameter bathtub hazard family and developed the MLE procedure. An illustrative example of accelerating test is given by using TDDB Data of Al/20W Hf-doped TaO_x/Si. We further use Maximum Uncertainty Principle for developing a random fuzzy M -estimation of parameter for Smith and Bain bathtub hazard distribution and analyzed the data TDDB Data of Al/20W Hf-doped TaO_x/Si. The analysis evidently showed that the vagueness has very little impacts.

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Appendix A: Computation Equations for Maximum Likelihood Analysis

For carrying out the computation, we did the mathematic derivation start from Smith and Bain's

Bathtub Functions which takes the form $h(t; \eta, \beta) = \eta \beta (\eta t)^{\beta-1} \exp(-(\eta t)^\beta)$

So the target Equations System which need fuzzy parameter β , a , b , c , d to satisfy is

$$\begin{cases} \frac{1}{\psi(t_1)} \cdot \frac{\partial}{\partial \beta} [\psi(t_1)] + \frac{1}{\psi(t_2)} \cdot \frac{\partial}{\partial \beta} [\psi(t_2)] + \dots + \frac{1}{\psi(t_i)} \cdot \frac{\partial}{\partial \beta} [\psi(t_i)] = 0 \\ \frac{1}{\psi(t_1)} \cdot \frac{\partial}{\partial a} [\psi(t_1)] + \frac{1}{\psi(t_2)} \cdot \frac{\partial}{\partial a} [\psi(t_2)] + \dots + \frac{1}{\psi(t_i)} \cdot \frac{\partial}{\partial a} [\psi(t_i)] = 0 \\ \frac{1}{\psi(t_1)} \cdot \frac{\partial}{\partial b} [\psi(t_1)] + \frac{1}{\psi(t_2)} \cdot \frac{\partial}{\partial b} [\psi(t_2)] + \dots + \frac{1}{\psi(t_i)} \cdot \frac{\partial}{\partial b} [\psi(t_i)] = 0 \\ \frac{1}{\psi(t_1)} \cdot \frac{\partial}{\partial c} [\psi(t_1)] + \frac{1}{\psi(t_2)} \cdot \frac{\partial}{\partial c} [\psi(t_2)] + \dots + \frac{1}{\psi(t_i)} \cdot \frac{\partial}{\partial c} [\psi(t_i)] = 0 \\ \frac{1}{\psi(t_1)} \cdot \frac{\partial}{\partial d} [\psi(t_1)] + \frac{1}{\psi(t_2)} \cdot \frac{\partial}{\partial d} [\psi(t_2)] + \dots + \frac{1}{\psi(t_i)} \cdot \frac{\partial}{\partial d} [\psi(t_i)] = 0 \end{cases} \quad (A1)$$

where

$$\begin{aligned} \psi(t) = & \frac{be}{2(b-a)t} e^{-e^{(bt)^\beta}} - \frac{ae}{2(b-a)t} e^{-e^{(at)^\beta}} + \frac{de}{2(d-c)t} e^{-e^{(dt)^\beta}} - \frac{ce}{2(d-c)t} e^{-e^{(ct)^\beta}} \\ & - \frac{e}{2(b-a)t^2} \int_{at}^{bt} \left(e^{-e^{(y)^\beta}} \right) dy - \frac{e}{2(d-c)t^2} \int_{ct}^{dt} \left(e^{-e^{(y)^\beta}} \right) dy \end{aligned} \quad (A2)$$

$$\begin{aligned} \frac{\partial}{\partial \beta} \psi(t_i) = & -\frac{be}{2(b-a)t_i} (bt_i)^\beta e^{(bt_i)^\beta} e^{-e^{(bt_i)^\beta}} \ln(bt_i) + \frac{ae}{2(b-a)t_i} (at_i)^\beta e^{(at_i)^\beta} e^{-e^{(at_i)^\beta}} \ln(at_i) \\ & - \frac{de}{2(d-c)t_i} (dt_i)^\beta e^{(dt_i)^\beta} e^{-e^{(dt_i)^\beta}} \ln(dt_i) + \frac{ce}{2(d-c)t_i} (ct_i)^\beta e^{(ct_i)^\beta} e^{-e^{(ct_i)^\beta}} \ln(ct_i) \\ & + \frac{e}{2(b-a)t_i^2} \int_{at_i}^{bt_i} \left(y^\beta e^{y^\beta} e^{-e^{y^\beta}} \ln(y) \right) dy + \frac{e}{2(d-c)t_i^2} \int_{ct_i}^{dt_i} \left(y^\beta e^{y^\beta} e^{-e^{y^\beta}} \ln(y) \right) dy \end{aligned} \quad (A3)$$

$$\begin{aligned} \frac{\partial}{\partial a} \psi(t_i) = & \frac{be}{2(b-a)^2 t_i} e^{-e^{(a_i)^\beta}} - \frac{be}{2(b-a)^2 t_i} e^{-e^{(b_i)^\beta}} + \frac{ae\beta}{2(b-a)} (at)^{\beta-1} e^{(a_i)^\beta} e^{-e^{(a_i)^\beta}} \\ & + \frac{e}{2(b-a)t_i} e^{-e^{(a_i)^\beta}} - \frac{e}{2(b-a)^2 t_i^2} \int_{a_i}^{bt_i} (e^{-e^{y^\beta}}) dy \end{aligned} \quad (A4)$$

$$\begin{aligned} \frac{\partial}{\partial b} \psi(t_i) = & \frac{ae}{2(b-a)^2 t_i} e^{-e^{(a_i)^\beta}} - \frac{ae}{2(b-a)^2 t_i} e^{-e^{(b_i)^\beta}} - \frac{be\beta}{2(b-a)} (bt)^{\beta-1} e^{(b_i)^\beta} e^{-e^{(b_i)^\beta}} \\ & - \frac{e}{2(b-a)t_i} e^{-e^{(b_i)^\beta}} - \frac{e}{2(b-a)^2 t_i^2} \int_{a_i}^{bt_i} (e^{-e^{y^\beta}}) dy \end{aligned} \quad (A5)$$

$$\begin{aligned} \frac{\partial}{\partial c} \psi(t_i) = & \frac{de}{2(d-c)^2 t_i} e^{-e^{(c_i)^\beta}} - \frac{de}{2(d-c)^2 t_i} e^{-e^{(d_i)^\beta}} + \frac{ce\beta}{2(d-c)} (ct)^{\beta-1} e^{(c_i)^\beta} e^{-e^{(c_i)^\beta}} \\ & + \frac{e}{2(d-c)t_i} e^{-e^{(c_i)^\beta}} - \frac{e}{2(d-c)^2 t_i^2} \int_{c_i}^{dt_i} (e^{-e^{y^\beta}}) dy \end{aligned} \quad (A6)$$

$$\begin{aligned} \frac{\partial}{\partial d} \psi(t_i) = & \frac{ce}{2(d-c)^2 t_i} e^{-e^{(c_i)^\beta}} - \frac{ce}{2(d-c)^2 t_i} e^{-e^{(d_i)^\beta}} - \frac{de\beta}{2(d-c)} (dt)^{\beta-1} e^{(d_i)^\beta} e^{-e^{(d_i)^\beta}} \\ & - \frac{e}{2(d-c)t_i} e^{-e^{(d_i)^\beta}} - \frac{e}{2(d-c)^2 t_i^2} \int_{c_i}^{dt_i} (e^{-e^{y^\beta}}) dy \end{aligned} \quad (A7)$$

Appendix B: Computation Programme for Maximum Likelihood Analysis

1. Below is the Program code written in VBA, for computation to search the parameter β in terms of the constant parameter η in the two parameters Bathtub Hazard function.

```
Sub Column_E()
```

```
Dim i As Integer
```

```
Dim ETA, sum As Double
```

```
ETA = ActiveSheet.Cells(1, 3).Value
```

```
sum = 0
```

```
For i = 2 To 35
```

```
    sum = sum + Log(ActiveSheet.Cells(i, 1).Value * ETA)
```

```
Next
```

```
ActiveSheet.Cells(2, 5).Value = sum
```

```
End Sub
```

```
Sub Column_F()
```

```
Dim i As Integer
```

```
Dim ETA, beta, sum As Double
```

```
ETA = ActiveSheet.Cells(1, 3).Value
```

```
beta = ActiveSheet.Cells(18, 2).Value
```

```
sum = 0
```

```
For i = 2 To 35
```

```
    sum = sum + (ActiveSheet.Cells(i, 1).Value * ETA) ^ beta * Log(ActiveSheet.Cells(i, 1).Value * ETA)
```

```
Next
```

```
ActiveSheet.Cells(18, 6).Value = sum
```

```
End Sub
```

```
Sub Column_G()
```

Dim i As Integer

Dim ETA, beta, sum As Double

ETA = ActiveSheet.Cells(1, 3).Value

beta = ActiveSheet.Cells(18, 2).Value

sum = 0

For i = 2 To 35

sum = sum + Exp((ActiveSheet.Cells(i, 1).Value * ETA) ^ beta) * ((ActiveSheet.Cells(i, 1).Value * ETA) ^ beta) * Log(ActiveSheet.Cells(i, 1).Value * ETA)

Next

ActiveSheet.Cells(18, 7).Value = sum

End Sub

Sub Column_K()

Dim i As Integer

Dim ETA, beta, sum As Double

ETA = ActiveSheet.Cells(1, 3).Value

beta = ActiveSheet.Cells(18, 2).Value

sum = 0

For i = 2 To 35

sum = sum + (ActiveSheet.Cells(i, 1).Value * ETA) ^ beta * (Log(ActiveSheet.Cells(i, 1).Value * ETA)) ^ 2

Next

ActiveSheet.Cells(18, 11).Value = sum

End Sub

Sub Column_L()

Dim i As Integer

Dim ETA, beta, sum As Double

ETA = ActiveSheet.Cells(1, 3).Value

beta = ActiveSheet.Cells(18, 2).Value

```

sum = 0
For i = 2 To 35

    sum = sum + Exp((ActiveSheet.Cells(i, 1).Value * ETA) ^ beta) * ((ActiveSheet.Cells(i,
1).Value * ETA) ^ beta) * (Log(ActiveSheet.Cells(i, 1).Value * ETA)) ^ 2 * ((ActiveSheet.Cells(i,
1).Value * ETA) ^ beta + 1)

Next
ActiveSheet.Cells(18, 12).Value = sum

End Sub

```

2. Below is the Program code written in VBA, for estimating the fuzzy parameters β , a , b , c , d , under the Average Chance Distribution.

```

Sub Colum_AR()

Dim i As Integer
Dim beta, a, b, c, d, Psi, PsiBeta, sum As Double

beta = ActiveSheet.Cells(2, 38).Value
a = ActiveSheet.Cells(46, 39).Value
b = ActiveSheet.Cells(46, 40).Value
c = ActiveSheet.Cells(46, 41).Value
d = ActiveSheet.Cells(46, 42).Value

sum = 0
For i = 2 To 35

    Psi = b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    - a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    + d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    - c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    - Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 49).Value)
    - Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)

```

```

PsiBeta = -b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (b * ActiveSheet.Cells(i,
1).Value) ^ beta * Exp((b * ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta)) * Log(b * ActiveSheet.Cells(i, 1).Value) _
+ a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (a * ActiveSheet.Cells(i,
1).Value) ^ beta * Exp((a * ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta)) * Log(a * ActiveSheet.Cells(i, 1).Value) _
- d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (d * ActiveSheet.Cells(i,
1).Value) ^ beta * Exp((d * ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta)) * Log(d * ActiveSheet.Cells(i, 1).Value) _
+ c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (c * ActiveSheet.Cells(i,
1).Value) ^ beta * Exp((c * ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta)) * Log(c * ActiveSheet.Cells(i, 1).Value) _
+ Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
51).Value) _
+ Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
52).Value)

```

```
sum = sum + (1 / Psi) * PsiBeta
```

```
Next
```

```
ActiveSheet.Cells(48, 44).Value = sum
```

```
End Sub
```

```
Sub Colum_AS()
```

```
Dim i As Integer
```

```
Dim beta, a, b, c, d, Psi, PsiA, sum As Double
```

```
beta = ActiveSheet.Cells(2, 38).Value
```

```
a = ActiveSheet.Cells(46, 39).Value
```

```
b = ActiveSheet.Cells(46, 40).Value
```

```
c = ActiveSheet.Cells(46, 41).Value
```

```
d = ActiveSheet.Cells(46, 42).Value
```

```
sum = 0
```

```
For i = 2 To 35
```

```

Psi = b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
+ d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _

```

```

- Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 49).Value)
-
- Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)

PsiA = b * Exp(1) / (2 * (b - a) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- b * Exp(1) / (2 * (b - a) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
+ a * Exp(1) * beta / (2 * (b - a)) * (a * ActiveSheet.Cells(i, 1).Value) ^ (beta - 1) * Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((a * ActiveSheet.Cells(i, 1).Value) ^ beta)) _
+ Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((a * ActiveSheet.Cells(i,
1).Value) ^ beta)) _
- Exp(1) / (2 * (b - a) ^ 2 * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
49).Value)

sum = sum + (1 / Psi) * PsiA

Next
ActiveSheet.Cells(46, 45).Value = sum

End Sub

Sub Colum_AT()

Dim i As Integer
Dim beta, a, b, c, d, Psi, PsiB, sum As Double

beta = ActiveSheet.Cells(2, 38).Value
a = ActiveSheet.Cells(46, 39).Value
b = ActiveSheet.Cells(46, 40).Value
c = ActiveSheet.Cells(46, 41).Value
d = ActiveSheet.Cells(46, 42).Value

sum = 0
For i = 2 To 35

Psi = b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
+ d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 49).Value)
-
- Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)

```

```

PsiB = a * Exp(1) / (2 * (b - a) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- a * Exp(1) / (2 * (b - a) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- b * Exp(1) * beta / (2 * (b - a)) * (b * ActiveSheet.Cells(i, 1).Value) ^ (beta - 1) * Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((b * ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((b * ActiveSheet.Cells(i,
1).Value) ^ beta)) _
- Exp(1) / (2 * (b - a) ^ 2 * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
49).Value)

```

```
sum = sum + (1 / Psi) * PsiB
```

```
Next
```

```
ActiveSheet.Cells(46, 46).Value = sum
```

```
End Sub
```

```
Sub Colum_AU()
```

```
Dim i As Integer
```

```
Dim beta, a, b, c, d, Psi, PsiC, sum As Double
```

```
beta = ActiveSheet.Cells(2, 38).Value
```

```
a = ActiveSheet.Cells(46, 39).Value
```

```
b = ActiveSheet.Cells(46, 40).Value
```

```
c = ActiveSheet.Cells(46, 41).Value
```

```
d = ActiveSheet.Cells(46, 42).Value
```

```
sum = 0
```

```
For i = 2 To 35
```

```

Psi = b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
+ d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
- Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 49).Value)
- Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)

```

```

PsiC = d * Exp(1) / (2 * (d - c) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _

```

```

    - d * Exp(1) / (2 * (d - c) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
    + c * Exp(1) * beta / (2 * (d - c)) * (c * ActiveSheet.Cells(i, 1).Value) ^ (beta - 1) * Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((c * ActiveSheet.Cells(i, 1).Value) ^ beta)) _
    + Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((c * ActiveSheet.Cells(i,
1).Value) ^ beta)) _
    - Exp(1) / (2 * (d - c) ^ 2 * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
50).Value)

```

```
sum = sum + (1 / Psi) * PsiC
```

```
Next
```

```
ActiveSheet.Cells(46, 47).Value = sum
```

```
End Sub
```

```
Sub Colum_AV()
```

```
Dim i As Integer
```

```
Dim beta, a, b, c, d, Psi, PsiD, sum As Double
```

```
beta = ActiveSheet.Cells(2, 38).Value
```

```
a = ActiveSheet.Cells(46, 39).Value
```

```
b = ActiveSheet.Cells(46, 40).Value
```

```
c = ActiveSheet.Cells(46, 41).Value
```

```
d = ActiveSheet.Cells(46, 42).Value
```

```
sum = 0
```

```
For i = 2 To 35
```

```

    Psi = b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    - a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    + d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    - c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
    - Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 49).Value)
    -
    - Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)

```

```

    PsiD = c * Exp(1) / (2 * (d - c) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _

```

```

    - c * Exp(1) / (2 * (d - c) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _

```

```

    - d * Exp(1) * beta / (2 * (d - c)) * (d * ActiveSheet.Cells(i, 1).Value) ^ (beta - 1) * Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((d * ActiveSheet.Cells(i, 1).Value) ^ beta)) _
    - Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((d * ActiveSheet.Cells(i,
1).Value) ^ beta)) _
    - Exp(1) / (2 * (d - c) ^ 2 * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
50).Value)

```

```
sum = sum + (1 / Psi) * PsiD
```

Next

```
ActiveSheet.Cells(46, 48).Value = sum
```

End Sub

This is a program work for 5 results together by one time

```
Sub Colum_AR_AS_AT_AU_AV()
```

```
Dim i As Integer
```

```
Dim beta, a, b, c, d, Psi, PsiBeta, PsiA, PsiB, PsiC, PsiD, sum, sum1, sum2, sum3, sum4, sum5 As
Double
```

```
beta = ActiveSheet.Cells(70, 38).Value
```

```
a = ActiveSheet.Cells(70, 39).Value
```

```
b = ActiveSheet.Cells(70, 40).Value
```

```
c = ActiveSheet.Cells(70, 41).Value
```

```
d = ActiveSheet.Cells(70, 42).Value
```

```
sum1 = 0
```

```
sum2 = 0
```

```
sum3 = 0
```

```
sum4 = 0
```

```
sum5 = 0
```

```
For i = 2 To 35
```

```
    Psi = b * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((b *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
```

```
    - a * Exp(1) / (2 * (b - a) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((a *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
```

```
    + d * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
```

```
    - c * Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * (Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta))) _
```

```
    - Exp(1) / (2 * (b - a) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 49).Value)
```

```
    - Exp(1) / (2 * (d - c) * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)
```



```
- Exp(1) / (2 * (d - c) ^ 2 * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i, 50).Value)
```

```
PsiD = c * Exp(1) / (2 * (d - c) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((c *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- c * Exp(1) / (2 * (d - c) ^ 2 * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- d * Exp(1) * beta / (2 * (d - c)) * (d * ActiveSheet.Cells(i, 1).Value) ^ (beta - 1) * Exp((d *
ActiveSheet.Cells(i, 1).Value) ^ beta) * Exp(-Exp((d * ActiveSheet.Cells(i, 1).Value) ^ beta)) _
- Exp(1) / (2 * (d - c) * ActiveSheet.Cells(i, 1).Value) * Exp(-Exp((d * ActiveSheet.Cells(i,
1).Value) ^ beta)) _
- Exp(1) / (2 * (d - c) ^ 2 * (ActiveSheet.Cells(i, 1).Value) ^ 2) * (ActiveSheet.Cells(i,
50).Value)
```

```
sum1 = sum1 + (1 / Psi) * PsiBeta
sum2 = sum2 + (1 / Psi) * PsiA
sum3 = sum3 + (1 / Psi) * PsiB
sum4 = sum4 + (1 / Psi) * PsiC
sum5 = sum5 + (1 / Psi) * PsiD
```

Next

```
ActiveSheet.Cells(70, 44).Value = sum1
ActiveSheet.Cells(70, 45).Value = sum2
ActiveSheet.Cells(70, 46).Value = sum3
ActiveSheet.Cells(70, 47).Value = sum4
ActiveSheet.Cells(70, 48).Value = sum5
```

End Sub

3. Here below is the programme design for computing the integral terms which are contained in the Average Chance function's equations system

```
Public Function integral_ba()
```

```
Dim i, j, k As Integer
Dim beta, a, b, h, XO, sum As Double
Dim rng As Range
```

```
beta = ActiveSheet.Cells(2, 2).Value
a = ActiveSheet.Cells(2, 3).Value
b = ActiveSheet.Cells(2, 4).Value
```

```
h = 1
```

```
'Set tiny distance for cutting the trapezoid piecewise
```

```

v = Worksheets("sheet2").Range("A2:A35").Value
Set rng = ActiveSheet.Range("A2:A35")

With rng
For i = 1 To .Columns.Count
For j = 1 To .Rows.Count

XO = a * v(j, i) 'Set lowerlimit boundray

sum = 0
For k = 1 To ((b - a) * v(j, i) / h - 1)

    sum = sum + ((XO + k * h) ^ beta) * Exp((XO + k * h) ^ beta) * Exp(-Exp((XO + k * h) ^ beta))
    * Log(XO + k * h)

Next
ActiveSheet.Cells(j + 1, i + 12).Value = h * sum + 0.5 * h * (XO ^ beta * Exp(XO ^ beta) * Exp(-
Exp(XO ^ beta)) * Log(XO) + _
(XO + (b - a) * v(j, i)) ^ beta * Exp((XO + (b - a) * v(j, i)) ^ beta) * Exp(-Exp((XO + (b - a) * v(j,
i)) ^ beta)) * Log(XO + (b - a) * v(j, i)))

Next j
Next i
End With

End Function

Public Function integral_dc()

Dim i, j, k As Integer
Dim beta, c, d, h, XO, sum As Double
Dim rng As Range

beta = ActiveSheet.Cells(2, 2).Value
c = ActiveSheet.Cells(2, 5).Value
d = ActiveSheet.Cells(2, 6).Value

h = 1 'Set tiny distance for cutting the trapezoid piecewise

v = Worksheets("sheet2").Range("A2:A35").Value
Set rng = ActiveSheet.Range("A2:A35")

With rng
For i = 1 To .Columns.Count
For j = 1 To .Rows.Count

XO = c * v(j, i) 'Set lowerlimit boundray

```

```

sum = 0
For k = 1 To ((d - c) * v(j, i) / h - 1)

    sum = sum + ((XO + k * h) ^ beta) * Exp((XO + k * h) ^ beta) * Exp(-Exp((XO + k * h) ^ beta))
    * Log(XO + k * h)

Next
ActiveSheet.Cells(j + 1, i + 19).Value = h * sum + 0.5 * h * (XO ^ beta * Exp(XO ^ beta) * Exp(-
Exp(XO ^ beta)) * Log(XO) +
(XO + (d - c) * v(j, i)) ^ beta * Exp((XO + (d - c) * v(j, i)) ^ beta) * Exp(-Exp((XO + (d - c) * v(j,
i)) ^ beta)) * Log(XO + (d - c) * v(j, i)))

Next j
Next i
End With

End Function

```

'Design for computing the first integral term, which are done by MATLAB.
 'Here just dispaly it and make sence of double check the accuracy of the result of MATLAB.

```

Public Function integral_try_ba()

Dim i, j, k As Integer
Dim beta, a, b, h, XO, sum As Double
Dim rng As Range

beta = ActiveSheet.Cells(2, 2).Value
a = ActiveSheet.Cells(2, 3).Value
b = ActiveSheet.Cells(2, 4).Value

h = 0.1

v = Worksheets("sheet2").Range("A2:A35").Value
Set rng = ActiveSheet.Range("A2:A35")

With rng
For i = 1 To .Columns.Count
For j = 1 To .Rows.Count

XO = a * v(j, i) 'Set lowerlimit boundray

sum = 0
For k = 1 To ((b - a) * v(j, i) / h - 1)

    sum = sum + Exp(-Exp((XO + k * h) ^ beta))

```

```

Next
ActiveSheet.Cells(j + 1, i + 9).Value = h * sum + 0.5 * h * (Exp(-Exp(XO ^ beta)) + Exp(-
Exp((XO + (b - a) * v(j, i)) ^ beta)))

Next j
Next i
End With

End Function

Public Function integral_try_dc()

Dim i, j, k As Integer
Dim beta, a, b, h, XO, sum As Double
Dim rng As Range

beta = ActiveSheet.Cells(2, 2).Value
c = ActiveSheet.Cells(2, 5).Value
d = ActiveSheet.Cells(2, 6).Value

h = 16

v = Worksheets("sheet2").Range("A2:A35").Value
Set rng = ActiveSheet.Range("A2:A35")

With rng
For i = 1 To .Columns.Count
For j = 1 To .Rows.Count

XO = a * v(j, i) 'Set lowerlimit boundray

sum = 0
For k = 1 To ((d - c) * v(j, i) / h - 1)

sum = sum + Exp(-Exp((XO + k * h) ^ beta))

Next
ActiveSheet.Cells(j + 1, i + 12).Value = h * sum + 0.5 * h * (Exp(-Exp(XO ^ beta)) + Exp(-
Exp((XO + (d - c) * v(j, i)) ^ beta)))

Next j
Next i
End With

End Function

```

```
Public Function integral_bam()
```

```
Dim i, j, k As Integer
Dim beta, a, b, m, XO, sum As Double
Dim rng As Range
```

```
beta = ActiveSheet.Cells(2, 2).Value
a = ActiveSheet.Cells(2, 3).Value
b = ActiveSheet.Cells(2, 4).Value
```

```
m = 35
'Set tiny distance for cutting the trapezoid piecewise
```

```
v = Worksheets("sheet2").Range("A2:A35").Value
Set rng = ActiveSheet.Range("A2:A35")
```

```
With rng
For i = 1 To .Columns.Count
For j = 1 To .Rows.Count
```

```
XO = a * v(j, i) 'Set lowerlimit boundray
```

```
sum = 0
For k = 1 To ((b - a) * v(j, i) / ActiveSheet.Cells(m, 21).Value - 1)
```

```
sum = sum + ((XO + k * ActiveSheet.Cells(m, 21).Value) ^ beta) * Exp((XO + k *
ActiveSheet.Cells(m, 21).Value) ^ beta) * Exp(-Exp((XO + k * ActiveSheet.Cells(m, 21).Value) ^
beta)) * Log(XO + k * ActiveSheet.Cells(m, 21).Value)
```

```
Next
ActiveSheet.Cells(j + 1, i + 12).Value = ActiveSheet.Cells(m, 21).Value * sum + 0.5 *
ActiveSheet.Cells(m, 21).Value * (XO ^ beta * Exp(XO ^ beta) * Exp(-Exp(XO ^ beta)) *
Log(XO) +
(XO + (b - a) * v(j, i)) ^ beta * Exp((XO + (b - a) * v(j, i)) ^ beta) * Exp(-Exp((XO + (b - a) * v(j,
i)) ^ beta)) * Log(XO + (b - a) * v(j, i)))
```

```
Next j
Next i
End With
```

```
End Function
```

```
Public Function integral_dcm()
```

```
Dim i, j, k As Integer
Dim beta, c, d, m, XO, sum As Double
Dim rng As Range
```

```

beta = ActiveSheet.Cells(2, 2).Value
c = ActiveSheet.Cells(2, 5).Value
d = ActiveSheet.Cells(2, 6).Value

```

```

m = 34 'Set tiny distance for cutting the trapezoid piecewise

```

```

v = Worksheets("sheet2").Range("A2:A35").Value
Set rng = ActiveSheet.Range("A2:A35")

```

```

With rng
For i = 1 To .Columns.Count
For j = 1 To .Rows.Count

```

```

XO = c * v(j, i) 'Set lowerlimit boundray

```

```

sum = 0
For k = 1 To ((d - c) * v(j, i) / ActiveSheet.Cells(m, 21).Value - 1)

```

```

    sum = sum + ((XO + k * ActiveSheet.Cells(m, 21).Value) ^ beta) * Exp((XO + k *
ActiveSheet.Cells(m, 21).Value) ^ beta) * Exp(-Exp((XO + k * ActiveSheet.Cells(m, 21).Value) ^
beta)) * Log(XO + k * ActiveSheet.Cells(m, 21).Value)

```

```

Next
ActiveSheet.Cells(j + 1, i + 19).Value = ActiveSheet.Cells(m, 21).Value * sum + 0.5 *
ActiveSheet.Cells(m, 21).Value * (XO ^ beta * Exp(XO ^ beta) * Exp(-Exp(XO ^ beta)) *
Log(XO) + _
(XO + (d - c) * v(j, i)) ^ beta * Exp((XO + (d - c) * v(j, i)) ^ beta) * Exp(-Exp((XO + (d - c) * v(j,
i)) ^ beta)) * Log(XO + (d - c) * v(j, i)))

```

```

Next j
Next i
End With

```

```

End Function

```

Appendix C: Programme for Matlab command and Excel linking Matlab

In the computation, some part of the module, I use Matlab to get result as for its high efficiency. (But as for pursuit high precision of the target result data, I also use VBA to write code to help for computation)

Here below represents the programme for matlab command and Excel links Matlab.

function y = myfun(x)
y = exp(-exp(x.^0.0227)); '0.0227 is the experimental parameter that is input manually.

lowerlimit = [

2.7

5.4

6

7.5

8.7

19.8

37.2

38.1

52.5

66.3

74.7

102.3

108.6

165.6

189

228

234.6

238.2

271.8

279.6

290.4

413.4

415.8

499.2

518.4

668.7

674.7

701.4

1217.4

1495.8

```
1893.6
1920
2054.1
2542.2
];
```

```
upperlimit = [
26807.83193
53615.66385
59572.95984
74466.1998
86380.79176
196590.7675
369352.351
378288.295
521263.3986
658281.2062
741683.35
1015718.965
1078270.573
1644213.692
1876548.235
2263772.474
2329302.73
2365046.506
2698655.081
2776099.928
2883331.256
4104576.933
4128406.117
4956470.259
5147103.73
6639406.374
6698979.334
6964079.005
12087353.55
14851538.89
18801226.12
19063347.15
20394802.8
25241063.08
];
Ans = upperlimit;
```

```
for i = 1:34
```

```
    Ans(i) = quad(@myfun,lowerlimit(i),upperlimit(i))
end
```