UNIVERSITY OF CAPE TOWN

DEPARTMENT OF MATHEMATICS

Finitely Generated Function Algebras

by

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CONTENTS

INTRODUCTION

The theory of function algebras has been an active field of research over the past two decades and its coming of age has been heralded by the appearance within the last twelve months of three textbooks devoted entirely to them, namely the books by Browder [6], Leibowitz [4] and Gamelin [9]. One of the attractive features of the theory of function algebras is that it draws on diverse specialities like the theory of Banach algebras, harmonic analysis and the theory of analytic functions of several complex variables.

The last mentioned theory has led to some of the most powerful results in the theory of function algebras. Not surprisingly, many of these results, for example Rossi's local maximum modulus principle theorem 2.24, were first proved for finitely generated and then extended to arbitrary function algebras. This observation, together with the fact that there has been no systematic study of finitely generated function algebras, led to the writing of this thesis. We have made use of some of the results of the theory of analytic functions of several complex variables, though we have not specifically used the methods thereof. What we have looked for is ways in which the functions of finitely generated function algebras behave like analytic functions and then tried to see if

arbitrarily generated function algebras behave in a similar way.

This angle of approach is justified in Chapter I. Firstly it is shown how every finitely generated function algebra is metrically and algebraically equivalent to P(K), the uniform closure of the algebra of polynomials on a certain subset K, which depends on the generators, of $\mathbb{C}^n$, where n is the number of generators. Moreover the maximal ideal space of such a function algebra is homeomorphic to K. What is surprising is that P(K) is equal to the uniform closure of the algebra of all functions holomorphic in a neighbourhood of K, a deep result from the theory of analytic functions of several complex variables. That the maximal ideal space of a finitely generated function algebra is homeomorphic to a subset of $\mathbb{C}^n$ led us to the conclusion that every finitely generated function algebra must necessarily be defined on a metrizable space, a fact which is made use of in Chapter II. In 1.20 an interesting example of a finitely generated function algebra is given. The remainder of Chapter I is devoted to a discussion of Rickart's recent paper [8] on holomorphic functions of an infinite number of complex variables and use is made of his generalized Oka polynomial approximation theorem to extend the equivalence of finitely generated function algebras and algebras of holomorphic functions to the arbitrarily generated case.

In Chapter II "boundary" phenomena of function algebras are discussed, with the emphasis on boundaries of finitely generated function algebras. There are certain subsets, called boundaries, of the maximal ideal space of a function algebra on which every member achieves its maximum modulus. This is another "analytic" characteristic of function algebras. We have also included an exposition of Rossi's local maximum modulus principle, emphasizing the role of finitely generated function algebras.

Chapter III is an attempt to explore the connection between analytic function theory and finitely generated function algebras from another direction. We first establish under what conditions adjoining a continuous function to a function algebra leaves its maximal ideal space (and its Shilov boundary) unaltered. We then apply these results to obtain a continuous algebraic epimorphism from certain finitely generated function algebras onto the uniform closure of the space of continuous functions on a compact subset of the plane which are analytic in the interior of this subset.

Chapter IV was inspired by Royden [16], whose treatment of algebras as direct limits of their finitely generated subalgebras led us to investigate what properties of function algebras are inherited from their finitely generated sub-function algebras. Unfortunately we had to restrict ourselves to function algebras containing a 1:1 function, but with this

restriction managed to obtain results relating direct systems of finitely generated sub-function algebras and the inverse systems of their spectra to the corresponding objects in analytic function theory. We also found that analyticity (in the function algebraic sense of the word) is preserved under the taking of direct limits and that function algebras containing a 1:1 function which are defined on a connected space imitate the behaviour of analytic functions in that any member of such a function algebra which vanishes on an open subset, vanishes identically.

To keep the length of the thesis within reasonable bounds we had to omit the proofs of some results. In every case where this is done a reference is given to a textbook in which the result in question and its proof can be found. The reader is referred to the bibliographical notes for a detailed account of to whom the results contained in the thesis can be attributed.

In conclusion I wish to express my deep indebtedness to my supervisor, Dr. W. Kotzé, for his many valuable suggestions and constant encouragement.

CHAPTER I.

FINITELY GENERATED FUNCTION ALGEBRAS AND ANALYTIC FUNCTIONS.

In general we shall be concerned with certain subalgebras of $C(X)$, the space of all continuous complex-valued functions on a compact Hausdorff space X. If we define addition, scalar multiplication and multiplication as these operations applied pointwise to the members of $C(X)$, then the latter is an algebra.

Putting

$$\|f\| = \sup \{|f(x)| : x \in X\}$$

for each f in $C(X)$, we furthermore obtain that $C(X)$ is a Banach algebra.

A subset A of $C(X)$ is called a function algebra if A is a subalgebra of $C(X)$ satisfying:

1. If x, y are in X and $x \neq y$, then there is an f in A such that $f(x) \neq f(y)$ i.e. A separates the points of X.
2. The constant function 1, and hence every constant function, belongs to A.
3. A is closed with respect to the uniform norm; thus A itself is a Banach algebra.

Definition 1.1: Let B be any Banach algebra. A map

$m : B \to \mathbb{C}$ satisfying:

1. m is a non-zero linear functional.
2. $m(xy) = m(x)m(y)$ for all $x, y \in B$

is called a complex homomorphism of B. We denote the set of all complex homomorphisms of B by Spec B.

The following proposition shows that every complex homomorphism of a Banach algebra with an identity is bounded and has norm equal to one. (Note that we can assume that the identity has norm equal to one).

Proposition 1.2:

If B is a Banach algebra with identity e of norm one, then $m(e) = 1$ and $\|m\| = 1$ for all m in Spec B.

Proof: Let m belong to Spec B. Since $m \neq o$, there is an x in B such that $m(x) \neq o$. Now $m(x) = m(xe) = m(x)m(e)$. Therefore $m(e) = 1$ and $\|m\| \geq 1$. For any $y \in B$ we have $|m(y)|^n = |[m(y)^n]| = |m(y^n)| \leq \|m\| \|y^n\| \leq \|m\| \|y\|^n$. Therefore $|m(y)| \leq \|m\|^{1/n} \|y\|$, which implies that $|m(y)| \leq \|y\|$. Hence $\|m\| \leq 1$ and so $\|m\| = 1$.

Note 1.3: It is well-known from the General Theory of Banach algebras that if B is a commutative Banach algebra with an identity (as is every function algebra) then there is a 1:1 correspondence between Spec B and the set of all maximal

ideals of B, usually denoted M_B. This correspondence is obtained by mapping each complex homomorphism onto its kernel, which is a maximal ideal. See e.g. [2].

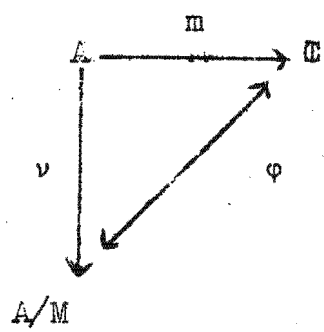
Theorem 1.4:

Let A be a function algebra. Define for each f in A and each m in Spec A

$$\hat{f}(m) = m(f).$$

Then there exists a unique Hausdorff topology on Spec A such that Spec A is compact and each $\hat{f}$ is continuous on Spec A.

Proof: The theorem is just a restatement of the well-known theorem in the General Theory of Banach algebras dealing with the Gelfand Representation of a commutative Banach algebra with an identity. In that setting, for fixed m in Spec A and f in A, we have the following commutative diagram yielding the Gelfand Representation.



where M is the kernel of m, ν is the canonical map from A onto the quotient A/M and φ is the isomorphism existing by virtue of the Gelfand-Mazur theorem. Since we have identified m with M, $\hat{f}(m) = \hat{f}(M) = m(f)$, and the result follows. Explicitly, the topology for Spec A is the weakest one which makes each $\hat{f}$ continuous.

In view of the previous theorem, we shall refer to $\hat{f}$ as the Gelfand transform of f.

Theorem 1.5:

There is a map $i:X \to \text{Spec } A$ which embeds X homeomorphically as a closed subset of Spec A. For each $f \in A$, $f \circ i^{-1}$ coincides with $\hat{f}$ on the image of X under i. Moreover, for each f in A and each m in Spec A

$$|\hat{f}(m)| \leq \sup\{|f(x)| : x \in X\} = \sup\{|\hat{f}(i(x))| : x \in X\}$$

and so $\|\hat{f}\| = \|f\|$.

Proof: For each x in X, define m_x in Spec A by $m_x(f) = f(x)$. The required map is then given by $i(x) = m_x$. Suppose $x \neq y$. Then there is a function g in A such that $g(x) \neq g(y)$. Hence $m_x(g) \neq m_y(g)$ and so i is 1:1. To show that i is continuous as a map from X onto i(X) (provided with the relative topology induced by Spec A), consider the open neighbourhood of m_{x_o} in i(X) defined by

$$U(m_{x_o};\ \varepsilon\ ;\ f_1,\ldots,f_r)$$
$$= \{m_x \in i(X):\ |\hat{f}_i(m_x) - \hat{f}_i(m_{x_o})| < \varepsilon \text{ for } i = 1,\ldots,r\}$$
$$= \{m_x \in i(X):\ |m_x(f_i) - m_{x_o}(f_i)| < \varepsilon \text{ for } i = 1,\ldots,r\}$$
$$= \{m_x \in i(X):\ |f_i(x) - f_i(x_o)| < \varepsilon \text{ for } i = 1,\ldots,r\}$$

Since each f_i is continuous there is a neighbourhood V_i of x_o for each $i = 1,\ldots,r$ such that

$$x \in V_i \Rightarrow |f_i(x) - f_i(x_o)| < \varepsilon .$$

Clearly $V = \bigcap_1^r V_i$ is a neighbourhood of x_o such that

$$x \in V \Rightarrow i(x) \in U(m_{x_o};\ \varepsilon\ ;\ f_1,\ldots,f_r).$$

The subspace $i(X)$ of Spec A is Hausdorff. Thus i is a 1:1 map from a compact space onto a Hausdorff space, and hence a homeomorphism. Being the continuous image of the compact space X, $i(X)$ is itself compact and is thus a closed subset of Spec A.

For $i(x) \in$ Spec A we have

$$\hat{f}(i(x)) = \hat{f}(m_x) = m_x(f) = f(x) = f(i^{-1}(i(x)))$$

and hence $f \circ i^{-1}$ coincides with $\hat{f}$ on $i(X)$.

For f in A and m in Spec A we have that

$$|\hat{f}(m)| = |m(f)| \leq \|m\|\ \|f\| = \|f\| , \text{ by 1.2.}$$

But $\|f\| = \sup\{|f(x)| : x \in X\} = \sup\{|\hat{f}(m_x)| : x \in X\}$

Hence $\hat{f}$ achieves its maximum modulus on $i(X)$ and $\|\hat{f}\| = \|f\|$.

If we let $\hat{A} = \{\hat{f} : f \in A\}$, then it follows from the two previous theorems that A is isometrically isomorphic to $\hat{A}$.

Definition 1.6:

For n a positive integer, $\mathbb{C}^n$ denotes the cartesian product of n copies of the complex plane $\mathbb{C}$.

i.e. $z \in \mathbb{C}^n$ iff $z = (z_1,\ldots,z_n)$ where $z_i \in \mathbb{C}$ for $i = 1,\ldots,n$.

We provide $\mathbb{C}^n$ with the metric $|z| = \max \{|z_i| : i = 1,\ldots,n\}$.

It can be shown that this metric is equivalent to the Euclidean metric.

A polynomial on $\mathbb{C}^n$ is simply a polynomial in n complex variables. Such a polynomial will be a finite sum of the form $\Sigma a_{r_1\ldots r_n} z_1^{r_1}\ldots z_n^{r_n}$, where the $a_{r_1\ldots r_n}$ are complex constants.

If K is a compact subset of $\mathbb{C}^n$ we denote by $P_o(K)$ the algebra of polynomials on $\mathbb{C}^n$ restricted to K. The uniform closure of $P_o(K)$ will be denoted by $P(K)$.

Definition 1.7: Let K be a compact subset of $\mathbb{C}^n$. The set hull $(K) = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K$ for each polynomial $p\}$, where $\|p\|_K = \sup \{|p(z)| : z \in K\}$, is called the polynomially convex hull of K.

We say that K is polynomially convex if hull (K) = K.

Proposition 1.8:

If K is a compact subset of $\mathbb{C}^n$ then so is hull (K), and hull (K) is the largest set containing K to which every element of

P(K) can be extended with preservation of the norm.

Proof: Since hull (K) is an intersection of closed sets it is a closed subset of the space $\mathbb{C}^n$. It is also bounded, for if z_i denotes the i-th coordinate function on $\mathbb{C}^n$ and $a_i = \sup\{|z_i| : z \in K\}$, then any point $w = (w_1,\ldots,w_n)$ which is such that $|w_i| > a_i$ for some i cannot lie in hull (K). Hence hull (K) is a closed and bounded subset of $\mathbb{C}^n$, and since the metric we have defined on $\mathbb{C}^n$ is equivalent to the Euclidean metric, it follows that hull (K) is compact.

Suppose that $f \in P(K)$. Then there is a sequence $\{p_n\}$ of polynomials converging uniformly on K to f. By the definition of hull (K), $\{p_n\}$ also converges uniformly on hull (K). Thus we can define an extension $\hat{f}$ of f to hull (K) by setting $\hat{f}(z) = \lim_{n\to\infty} p_n(z)$. It is a routine matter to check that $\hat{f}$ is well-defined.

Hence hull (K) is the largest set containing K to which each member of P(K) can be continuously extended with preservation of the norm. We shall soon see that hull (K) = Spec P(K) and that $\hat{f}$ as we have just defined it corresponds with the Gelfand transform of f for f in P(K).

Definition 1.9: Let A be a function algebra. If there is a subset $B \subset A$ such that A is the uniform closure of the set of all polynomials in finitely many members of B then we say

that A is generated by B. If $B = \{f_1,\ldots,f_n\}$ is a finite subset of A then A is said to be finitely generated. If $\{f_1,\ldots,f_n\}$ generate A we write $A = [f_1,\ldots,f_n]$.

Theorem 1.10:

Let $A = [f_1,\ldots,f_n]$. Then there is a compact polynomially convex subset $\mathbb{K}$ of $\mathbb{C}^n$ such that Spec A is homeomorphic to K and A is isometrically isomorphic to P(K).

Proof: Define the map $\mathbb{F}$: Spec A $\to \mathbb{C}^n$ by $F(m) = (\hat{f}_1(m),\ldots,\hat{f}_n(m))$ for each m in Spec A. F thus defined is continuous since the composition of F with the respective projections from $\mathbb{C}^n$ into $\mathbb{C}$ simply yields the Gelfand transform of each of the generators of A, which are all continuous. To show that F is 1:1 let m and m' in Spec A be such that $m \neq m'$. Then there exists an f in A such that $m(f) \neq m'(f)$. Since f is a uniform limit of polynomials in the f_i, $i = 1,\ldots n$, for some j, $1 \leq j \leq n$, we must have that $m(f_j) \neq m'(f_j)$. Hence $f_j(m) \neq f_j(m')$ and so $F(m) \neq F(m')$.

We denote the image of Spec A under F i.e. the set $\prod_1^n R(\hat{f}_i)$, where $R(\hat{f}_i)$ is the range of $\hat{f}_i$, by K. Since $\mathbb{C}^n$ is Hausdorff, so is K and thus F is a continuous 1:1 map from a compact space onto a Hausdorff space. Hence F is a homeomorphism.

To show that K is polynomially convex it is sufficient

to show that hull (K) $\subset$ K. Suppose that z $\in$ hull (K). The map $m_z : p(f_1,\ldots,f_n) \to p(z)$ which sends elements of A expressible as polynomials in the f_i onto $\mathbb{C}$ is a complex homomorphism on such elements. It is also bounded since $|p(z)| \leq \|p\|_{\text{hull (K)}} = \|p(\hat{f}_1,\ldots,\hat{f}_n)\|_{\text{Spec A}} = \|p(f_1,\ldots f_n)\|_X$, and so can be extended by continuity to the whole of A to become an element m $\in$ Spec A.

Hence $p(z) = m(p(f_1,\ldots,f_n)) = p(m(f_1),\ldots,m(f_n))$
$= p(\hat{f}_1(m),\ldots,f_n(m)) = p(z')$

for some z' in K. Therefore z = z' and so z is in K.

Clearly there is an isomorphism between the set of all uniform limits of polynomials in the $\hat{f}_i$ and the set of all uniform limits of polynomials in n complex variables on K. This isomorphism is norm-preserving, since the norm on P(K) is the uniform norm. Since A is isometrically isomorphic to $\hat{A}$, it follows that A is isometrically isomorphic to P(K).

Proposition 1.11:

If K is a compact subset of $\mathbb{C}^n$, then Spec P(K) is homeomorphic to hull (K).

Proof: Suppose z $\in$ hull (K). Then the map $m_z : P_o(K) \to \mathbb{C}$ defined by $m_z(p) = p(z)$ is a complex homomorphism on $P_o(K)$. Also

$\|m_z\| = \sup\{|p(z)|/\|p\|_K : p \in P_o(K)\} \leq 1$. In fact

$\| m_z \| = 1$ since 1 is in $P_o(K)$.

Conversely, if m_z is a complex homomorphism on $P_o(K)$ such that $\| m_z \| = 1$, then $z \in$ hull (K).

Hence $z \in$ hull (K) iff $\| m_z \| = 1$. But $\| m_z \| = 1$ iff m_z can be extended by continuity to P(K). Thus $z \in$ hull (K) iff there exists a complex homomorphism m_z in Spec P(K) such that $m_z(p) = p(z)$ for every polynomial p on $\mathbb{C}^n$.

The map F : Spec P(K) → hull (K) defined by

$$F(m_z) = (m_z(z_1),\ldots,m_z(z_n)) = (z_1,\ldots,z_n)$$

is clearly a homeomorphism from Spec P(K) onto hull (K).

Corollary 1.12: If K is a compact polynomially convex subset of $\mathbb{C}^n$, then Spec P(K) is homeomorphic to K.

Proof: Immediate from the previous proposition.

Thus we have shown that a finitely generated function algebra $A = [f_1,\ldots,f_n]$ is isometrically isomorphic to P(K) where $K = \prod_1^n \overline{R(\hat{f}_i)}$ and that Spec A is homeomorphic to K which can be identified with Spec P(K).

Definition 1.13:

Let $D \subset \mathbb{C}^n$ be open. A complex-valued function f defined on D is said to be holomorphic in D if for each point $w \in D$ there is an open neighbourhood U of w contained in D, such that f has a power series expansion

$$f(z) = \sum_{r_1 \ldots r_n = 0}^{\infty} a_{r_1 \ldots r_n} (z_1 - w_1)^{r_1} \ldots (z_n - w_n)^{r_n}$$

which converges for all $z \in U$

Definition 1.14: Let K be a compact polynomially convex subset of $\mathbb{C}^n$. We denote by A(K) the uniform closure of the algebra of all functions holomorphic in a neighbourhood of K.

Theorem 1.15: (Oka's polynomial approximation theorem). If K is a compact polynomially convex subset of $\mathbb{C}^n$, then $A(K) = P(K)$.

Proof: See e.g. [3]. p.56.

Corollary 1.16: A finitely generated function algebra $A = [f_1, \ldots, f_n]$ is isometrically isomorphic to A(K) where $K = \prod_1^n R(\hat{f}_i)$ and Spec A is homeomorphic to K.

Proof: The assertion follows from 1.10 and 1.15.

The fact that Spec A is homeomorphic to a subset of $\mathbb{C}^n$ when A is finitely generated forces us to the conclusion that Spec A, and hence X, is metrizable. This can be proved independently, as we now show.

Theorem 1.17: (Urysohn's metrization theorem).

A T_3 topological space whose topology has a countable base is metrizable.

Proof: See e.g. [5] p.125

Proposition 1.18:

If A is a function algebra on the compact Hausdorff space X, then the topology on X is the weakest topology for which every function in A is continuous.

Proof: Let $\mathfrak{I}$ be the given topology on X and $\mathfrak{W}$ the weak A topology on X. Then the identity map $(X,\mathfrak{I}) \to (X,\mathfrak{W})$ is continuous by definition of the weak topology. Since A separates the points of X, $(X,\mathfrak{W})$ must be Hausdorff. Hence the identity map is a continuous map from the compact space $(X,\mathfrak{I})$ onto the Hausdorff space $(X,\mathfrak{W})$, and thus a homeomorphism.

Theorem 1.19: If A is a finitely generated function algebra on a compact Hausdorff space X, then X is metrizable.

Proof: Let $A = [f_1,\ldots,f_n]$.

Put B =

$$\{\sum_{j=1}^{k} a_j f_1^{m_{1j}}\ldots f_n^{m_{nj}} : a_j \in \mathbb{C} \text{ and } a_j \text{ has rational real and imaginary parts}\}.$$

Clearly, $\overline{B} = A$ and so A is separable since B is countable. By 1.17 it is sufficient for us to show that the topology on X is countably based. We show in fact that the set

$\{f^{-1}(U) : f \in B$ and U is an open disc in the complex plane

with rational radius and centre} forms a countable base for the topology on X.

Let $\mathcal{B}$ be the family of all open discs in the complex plane with rational radius and with centre having rational real and imaginary parts. $\mathcal{B}$ forms a countable base for the topology of the plane. Since for any family of subsets $\mathcal{F}$,
$f^{-1}(\cup \{F : F \in \mathcal{F}\}) = \cup \{f^{-1}(F) : F \in \mathcal{F}\}$,
it is clear that the set $\{f^{-1}(U): f \in A, U \in \mathcal{B}\}$ forms a base for the topology of X, as the latter topology consists in all possible pre-images of open sets in the plane. We make the even stronger claim that the countable set
$\{f^{-1}(U): f \in B, U \in \mathcal{B}\}$ forms a base for the topology on X, in which case the theorem is proved.

Let $f \in A$ and $U \in \mathcal{B}$. Since $\overline{B} = A$, there is a sequence $\{f_n\} \subset B$ such that f_n converges to f uniformly on X.
Now $x \in f^{-1}(U)$ implies $f(x) \in U$. For n large enough,
$f_n(x) \in U$ i.e. $x \in f_n^{-1}(U)$.
Hence $x \in \bigcup_n f_n^{-1}(U)$, and so $f^{-1}(U) \subset \bigcup_n f_n^{-1}(U)$.

In view of the preceding theorem, we ask whether it is possible to produce a metric for Spec A when A is a finitely generated function algebra. We can do so, using the homeomorphism

$$F : \text{Spec } A \to \prod_{i=1}^{n} R(\hat{f}_i) = K \subset \mathbb{C}^n.$$

Two points m and m' in Spec A correspond to the points $(\hat{f}_1(m),\ldots,\hat{f}_n(m))$ and $(\hat{f}_1(m'),\ldots,\hat{f}(m'))$ in K respectively. The distance between the latter two points is given by $\max\{|\hat{f}_i(m) - \hat{f}_i(m')|:i = 1,\ldots,n\}$.

We can define the metric d on Spec A by setting

$d(m,m') = \max\{\hat{f}_i(m) - \hat{f}_i(m'):i = 1,\ldots,n\}$.

This metric will be consistent with the topology on Spec A, since Spec A and K are homeomorphic. Since X is homeomorphically embedded in Spec A, d restricted to X is given by

$d(x,y) = \max\{|f_i(x) - f_i(y)|:i = 1,\ldots n\}$.

Example of a finitely generated function algebra 1.20.

Definition: Let X be a compact subset of $\mathbb{C}^n$. We denote by R(X) the uniform closure of the set $R_o(X)$ of all rational functions on X i.e. the uniform closure on X of the set of all quotients p/q where p and q are polynomials and q does not vanish on X.

The rationally convex hull of X is the set

R - hull $(X) = \{z \in \mathbb{C}^n:|g(z)| \leq \|g\|_X$ for all g rational on X$\}$.

$P(X) = [z_1,\ldots,z_n]$, the function algebra generated by the n coordinate functions. Clearly $P(X) \subset R(X)$. It turns out that R(X) is also finitely generated.

Lemma: If A is a Banach algebra with an identity e, then $\|x-e\| < 1$ implies that x has an inverse.

Proof: See [7] p.177.

Theorem:

Let X be a compact set in $\mathbb{C}^n$. Then there is a function f in R(X) such that $R(X) = [z_1,\ldots,z_n,f]$.

Proof: Let Q_1 be the family of all rational functions of the form $1/p$ where p is a polynomial vanishing nowhere on X. Then $z_1,\ldots,z_n$ and Q_1 generate R(X). Let P_1 be the family of all polynomials p on $\mathbb{C}^n$ such that p vanishes nowhere on X and all the coefficients in p have rational real and imaginary parts. P_1 is countable. Enumerate its members as $p_1,p_2,\ldots$. Since X is compact, the family $Q = \{1/p : p \in P_1\}$ is dense in Q_1. So R(X) is generated by the countable set $Q \cup \{z_1,\ldots,z_n\}$.

Choose a sequence $\{a_i\}$ of positive numbers as follows: Let $a_1 = 1$ and

$$a_{k+1} = 1/2^{k+2} \| 1/p^{k+1} \|^{-1} \min\{a_1/\|p_1\|,\ldots,a_k/\|p_k\|\}$$

for $k = 1,2,\ldots$.

Then $\sum_{i=k+1}^{\infty} a_i \| 1/p_i \| \leq a_k/\|p_k\| \sum_{i=k+1}^{\infty} 1/2^{i+1} \|1/p_i\|^{-1} \| 1/p_i \|$

$$< a_k \|p_k\|$$

Put $f = \sum_{i=1}^{\infty} a_i/p_i$. Since $\sum_{i=2}^{\infty} a_i \| 1/p_i \| < 1/\| p_1 \| < \infty$ and $a_i/p_i \in R(X)$ for each i, it follows that $f \in R(X)$.

We prove that in fact $R(X) = [z_1,\ldots,z_n,f]$.

To do this it is sufficient to prove that

$1/p_i \in [z_1,\ldots,z_n,f] = A$ for each $i = 1, 2, \ldots$.

Since $\| p_1 f - 1 \| = \| \sum_{i=1}^{\infty} p_1 a_i/p_i - 1 \| = \| \sum_{i=2}^{\infty} p_1 a_i/p_i \| \leq \sum_{i=2}^{\infty} p_1 \| a_i/p_i \| < 1$

it follows from the previous lemma that $p_1 f$ is invertible in A.

But then $p_1^{-1} = f(p_1 f)^{-1} \in A$. Now assume that

$1/p_1,\ldots,1/p_k \in A$. Then

$$\| p_{k+1} a_{k+1} (f - \sum_{i=1}^{k} a_i/p_i) - 1 \| = \| p_{k+1} a_{k+1} \sum_{i=k+1}^{\infty} a_i/p_i - 1 \|$$

$$= \| p_{k+1} a_{k+1} \sum_{i=k+2}^{\infty} a_i/p_i \| < 1$$

and so $p_{k+1} a_{k+1}^{-1} (f - \sum_{i=1}^{k} a_i/p_i)$ is an invertible element of A.

Put $g_k = a_{k+1}^{-1} (f - \sum_{i=1}^{k} a_i/p_i)$. Then $g_k \in A$ by hypothesis.

Therefore $p_{k+1}^{-1} = g_k (p_{k+1} g_k)^{-1} \in A$.

By induction, every $1/p_i$ is in A, so the theorem is proved.

Theorem:

Let X be a compact subset of $\mathbb{C}^n$. Then

(a) $z \in R$ - hull (X) iff $p(z) \in p(X)$ for every polynomial p on $\mathbb{C}^n$.

(b) Spec R(X) = R - hull (X).

Proof: (a) If there is a polynomial p such that $p(z) \notin p(X)$

then $p - p(z)$ is a polynomial with no zeros on X, so the rational function $(p - p(z))^{-1}$ is finite on X. Since this function has value ∞ at z, $z \notin$ R-hull (X). Conversely, if $z \notin$ R-hull (X), then there are polynomials p and q such that $|q| > o$ on X and $|p(z)/q(z)| > \|p/q\|_X$. Multiplying by a suitable constant we can assume that $p(z)/q(z) = 1$. Then $r = p - q$ is a polynomial, and $r(z) = o \notin r(X)$.

(b) If $z \in$ R-hull (X), then $g \to g(z)$ is a continuous algebra homomorphism from $R_o(X)$ onto $\mathbb{C}$, so it extends to a complex homomorphism m_z of R(X).

Conversely, let $m \in$ Spec R(X). Then the restriction of m to P(X) is a complex homomorphism of P(X), so there is a point $z \in$ hull (X) such that $m(p) = p(z)$ for every polynomial p. Given $g \in R_o(X)$, $g = p/q$ where p and q are polynomials and q vanishes nowhere on X. Since q is an invertible element of R(X), $q(z) \neq o$(otherwise q would belong to the maximal ideal $M = \{g \in R(X): g(z) = o\}$). Thus $p = qg$ and so $p(z) = q(z)m(g)$ and $m(g) = p(z)/q(z)$.

Note that condition (a) has the following equivalent form: $z \in R(X)$ iff every polynomial which vanishes at z also vanishes somewhere on X. So we see that $z \in$ R-hull (X), and m agrees with m_z on a dense subalgebra. Hence $m = m_z$.

Corollary: Let X be a compact subset of $\mathbb{C}^n$. Then

$R(X) = [z_1,\ldots,z_n,f]$ is isometrically isomorphic to $P(K)$, where

$K = R\text{-hull}(X) \times \hat{f}(R\text{-hull}(X)) \subset \mathbb{C}^{n+1}$.

We have shown that a finitely generated function algebra $A = [f_1,\ldots,f_n]$ is isometrically isomorphic to the uniform closure of the algebra of all holomorphic functions about the compact polynomially convex set $F(\text{Spec } A) \subset \mathbb{C}^n$, where $F(m) = (\hat{f}_1(m),\ldots,\hat{f}_n(m))$. The question arises whether one can construct a similar equivalence for any function algebra A. It is in fact possible to do this, but first we have to define the concept of a holomorphic function of an infinite number of complex variables, a notion due to Rickart [8].

Holomorphic functions of an infinite number of complex variables.

Definition 1.21:

$\mathbb{C}^{\Omega} = \prod_{\omega\in\Omega} \mathbb{C}^{\omega}$, Ω any indexing set, provided with the usual product topology. Thus if $w = \{w_\omega : \omega\in\Omega\}$ is in $\mathbb{C}^{\Omega}$ then a basic system of neighbourhoods for w consists in sets of the form

$\{z \in \mathbb{C}^{\Omega} : |z_\omega - w_\omega| < \varepsilon \text{ for } \omega\in\pi\}$ where $\varepsilon>0$ and π is a finite subset of Ω. Hereafter π will always denote a finite subset of Ω.

For each finite subset $\pi \subset \Omega$, we define the

projection $p_\pi : \mathbb{C}^\Omega \to \mathbb{C}^\pi$ by

$$p_\pi\{z_\omega : \omega \in \Omega\} = \{z_\omega : \omega \in \pi\}.$$

A set X contained in $\mathbb{C}^\Omega$ is said to be determined by π, or simply finitely determined, if $X = p_\pi^{-1}(p_\pi(X))$, which means that $z \in \mathbb{C}^\Omega$ being in X depends only on the values of z_ω for $\omega \in \pi$.

Suppose that f is a function with domain D in $\mathbb{C}^\Omega$. We say that f is determined by π or is finitely determined if D is determined by the finite subset π of Ω and $f(z) = f(w)$ for all z, w in D such that $p_\pi(z) = p_\pi(w)$.

If f is determined by π we can unambiguously define the projection f_π of f into $\mathbb{C}^\pi$ by setting $f_\pi(p_\pi(z)) = f(z)$ for all z in D.

Definition 1.22: By a polynomial on $\mathbb{C}^\Omega$ we mean an ordinary polynomial in a finite number of the complex variables z_ω, $\omega \in \Omega$. Every polynomial on $\mathbb{C}^\Omega$ is thus finitely determined and continuous.

Let X be a compact subset of $\mathbb{C}^\Omega$. The set

$$\text{hull}(K) = \{z \in \mathbb{C}^\Omega : |p(z)| \leq \|p\|_K \text{ for all polynomials } p \text{ on } \mathbb{C}^\Omega\}$$

is called the polynomially convex hull of K in $\mathbb{C}^\Omega$.
K is said to be polynomially convex if K = hull (K).

As in the finite case we let $P_o(K)$ be the algebra of all polynomials on K and P(K) the uniform closure of $P_o(K)$.

Now let K be any compact polynomially convex set in $\mathbb{C}^{\Omega}$ and let π be a finite subset of Ω. We denote by K_π the set hull $(p_\pi K) \subset \mathbb{C}^{\pi}$.

(Note that in general $p_\pi K$ is not polynomially convex.)

Lemma 1.23: ([8])

(1) If $\pi \subset \pi'$ then $p_{\pi'}^{-1}(K_{\pi'}) \subset p_\pi^{-1}(K_\pi)$

(2) For arbitrary π_o, $K = \bigcap_{\pi \supset \pi_o} p_\pi^{-1}(K_\pi)$

(3) Suppose $K \subset G \subset \mathbb{C}^{\Omega}$, G open. Then for arbitrary π_o there exists $\pi \supset \pi_o$ such that $K_\pi \subset p_\pi G$.

Proof: (1) For $\pi \subset \pi'$ we have

$$p_{\pi'} K \subset p_\pi K \times \mathbb{C}^{\pi'-\pi} \subset K_\pi \times \mathbb{C}^{\pi'-\pi}.$$

Since $K_\pi \times \mathbb{C}^{\pi'-\pi}$ is clearly polynomially convex in $\mathbb{C}^{\pi'}$ we have that $K_{\pi'} \subset K_\pi \times \mathbb{C}^{\pi'-\pi}$. Hence

$$p_{\pi'}^{-1}(K_{\pi'}) \subset p_{\pi'}^{-1}(K_\pi \times \mathbb{C}^{\pi'-\pi}) = p_\pi^{-1}(K_\pi)$$

(2) since $K \subset p_\pi^{-1}(p_\pi K) \subset p_\pi^{-1}(K_\pi)$ for all finite π, $K \subset \bigcap_{\pi \supset \pi_o} p_\pi^{-1}(K_\pi)$. To obtain the reverse inclusion, suppose that z is in $\mathbb{C}^{\Omega} - K$. Since K is polynomially convex there is a polynomial P on $\mathbb{C}^{\Omega}$ such that $|P(z)| > \|P\|$. Let π be any finite subset of Ω which contains π_o along with all $\omega \in \Omega$ such that P involves the variables z_ω. Then P is determined by π and hence P_π, the projection of P into $\mathbb{C}^{\pi}$, satisfies

$$|P_\pi(p_\pi(z))| = |P(z)| > \|P\|_K = \|P_\pi\|_{p_\pi K}.$$

Since P_π is a polynomial on C^π, it follows that $p_\pi(z) \notin \text{hull}\,(p_\pi K) = K_\pi$, that is, $z \notin p_\pi^{-1}(K_\pi)$ and hence $z \notin \bigcap_{\pi \supset \pi_o} p_\pi^{-1}(K_\pi)$.

(3) Let $L = \{z \in \mathbb{C}^\Omega : |z_\omega| \leq \sup_{\omega \in \Omega} |w_\omega|,\ w \in K\}$

L is a compact polydisc containing K. Since $K \subset L \cap G$, $K \cap (L - G) = \emptyset$. But by (2), $K = \bigcap_{\pi \supset \pi_o} p_\pi^{-1}(K_\pi)$.

Hence $(L - G) \cap [\bigcap_{\pi \supset \pi_o} p_\pi^{-1}(K_\pi)] = \emptyset$.

By the compactness of K there are $\pi_1, \ldots, \pi_k$ with $\pi_i \supset \pi_o$, $i = 1, \ldots k$, such that

$L \cap p_{\pi_1}^{-1}(K_{\pi_1}) \cap \ldots \cap p_{\pi_k}^{-1}(K_{\pi_k}) \subset G$.

Let $\pi = \pi_1 \cup \ldots \cup \pi_k$. Then by (1) we have that

$$p_\pi^{-1}(K_\pi) \subset p_{\pi_i}^{-1}(K_{\pi_i}).$$

Hence $p_\pi^{-1}(K_\pi) \subset \bigcap_{i=1}^{k} p_{\pi_i}(K_{\pi_i})$ and so

$L \cap p_\pi^{-1}(K_\pi) \subset G$. This implies that

$p_\pi L \cap K_\pi \subset p_\pi G$. The set $p_\pi L$, being a polydisc in $\mathbb{C}^\pi$, is polynomially convex. Since $p_\pi K \subset p_\pi L$ it follows that $K_\pi \subset p_\pi L$, and hence that $K_\pi \subset p_\pi G$.

Definition 1.24: A complex-valued function h is said to be holomorphic in $\mathbb{C}^\Omega$ if it is defined in an open subset of $\mathbb{C}^\Omega$ and is locally a uniform limit of polynomials. Clearly, if Ω is finite, then a holomorphic function in $\mathbb{C}^\Omega$ is an ordinary holomorphic function of several complex variables.

The following lemma enables us to reduce questions concerning holomorphic functions in $\mathbb{C}^{\Omega}$ to the finite dimensional case.

Lemma 1.25: ([8])
Let h be a function which is holomorphic in some neighbourhood G of a compact set K in $\mathbb{C}^{\Omega}$. Then there exists a neighbourhood G_o of K contained in G on which h is finitely determined.

Proof: For each w in K we can find a basic neighbourhood U of w contained in G, such that h restricted to U is bounded and a uniform limit of polynomials. Since K is compact, it is covered by a finite number of such neighbourhoods, say $U_1, \ldots U_k$. Let $G_o = U_1 \cup \ldots \cup U_k$. Then

$$K \subset G_o \subset G.$$

To show that h is finitely determined on G_o we note first that each U_i, being a basic neighbourhood, is finitely determined. Suppose

$$U_i = \{z \in \mathbb{C}^{\Omega} : |w_\omega - z_\omega| < \varepsilon ; \omega \in \pi_i\}.$$

Then G_o is determined by $\pi = \bigcup_{i=1}^{k} \pi_i$ and is hence also finitely determined.

Now the function h is bounded on U_i for each i, so the polynomials which approximate h on U_i are also bounded. But since $U_i = p_\pi^{-1}(p_\pi U_i)$, a polynomial which is bounded on U_i obviously cannot depend on any of the variables z_ω for $\omega \notin \pi$. It follows from this that the restriction of h to U_i is determined by π.

Finally, suppose z and w are any two points in G_o such that $p_\pi w = p_\pi z$. For some i, $w \in U_i$. Then $z \in p_\pi^{-1}(p_\pi U_i) = U_i$. But we have just shown that h restricted to U_i is determined by π, hence $h(z) = h(w)$. Thus on G_o, h is finitely determined.

Theorem 1.26: ([8]) (Generalized Oka Polynomial Approximation Theorem).

If f is a function holomorphic in a neighbourhood of the compact polynomially convex set K in $\mathbb{C}^\Omega$, then f is a uniform limit on K of polynomials.

Proof: By the above lemma we may assume that there is an open set G containing K in which f is holomorphic and determined by some finite subset π_o of Ω. By 1.23 (3) there exists a finite set π containing π_o such that $K_\pi \subset p_\pi G_o$. Since f is finitely determined on G_o, we can define the projection f_π of f into $\mathbb{C}^\pi$ by $f_\pi(p_\pi(z)) = f(z)$, in which case f_π is a holomorphic function in the ordinary sense on the open set $p_\pi G_o$ which contains the compact polynomially convex set K_π. Thus we can apply Oka's Polynomial Approximation Theorem for finite dimensions to obtain f_π as a uniform limit on K_π of polynomials in the variables z_ω, $\omega \in \pi$. Hence f is a uniform limit on $p_\pi^{-1}K_\pi$ of polynomials on $\mathbb{C}^\Omega$. But $K \subset p_\pi^{-1}K_\pi$, and so f is a uniform limit also on K of polynomials on $\mathbb{C}^\Omega$.

The preceding results obtained by Rickart on holomorphic functions of an infinite number of complex variables enable us to generalize certain of our earlier results to function algebras which are not necessarily finitely generated.

Proposition 1.27: Let K be a compact polynomially convex subset of $\mathbb{C}^{\Omega}$. Then Spec P(K) is homeomorphic to hull (K).

Proof: Suppose $z \in$ hull (K). Then the map

$m_z : P_o(K) \to \mathbb{C}$ defined by $m_z(p) = p(z)$ is a complex homomorphism on $P_o(K)$. (Note that p involves only finitely many of the components of z). Also

$\| m_z \| = \sup \{ |p(z)| / \| p \|_K : p \in P_o(K) \} \leq 1$. In fact $\| m_z \| = 1$, since 1 is in $P_o(K)$.

Conversely, if m_z is a complex homomorphism on $P_o(K)$ such that $\| m_z \| = 1$, then $z \in$ hull (K).

Hence $z \in$ hull (K) if and only if $\| m_z \| = 1$.

But $\| m_z \| = 1$ if and only if there exists a complex homomorphism m_z in Spec P(K) such that $m_z(p) = p(z)$ for every polynomial p on $\mathbb{C}^{\Omega}$.

The map F: Spec P(K) $\to$ hull (K) defined by

$F(m_z) = \{ m_z(z_\omega) : \omega \in \Omega \} = \{ z_\omega : \omega \in \Omega \}$ is clearly a homeomorphism from Spec P(K) onto hull (K).

Corollary 1.28: If K is a compact polynomially convex subset of $\mathbb{C}^{\Omega}$ then Spec P(K) is homeomorphic to K.

Theorem 1.29: Let A be a function algebra and suppose that $B \subset A$ is a set of generators for A. Then there is a compact polynomially convex subset $K \subset \mathbb{C}^B$ such that Spec A is homeomorphic to K and A is isometrically isomorphicto P(K) (and hence to the uniform closure of the algebra of functions holomorphic in a neighbourhood of K). Moreover, Spec P(K) is homeomorphic to K.

Proof: Define the map $F:\text{Spec } A \to \mathbb{C}^B$ by

$F(m) = \{\hat{f}(m): f \in B\}$. As before, Spec A is homeomorphic to the set $K = F(\text{Spec } A) = \prod_{f \in B} R(\hat{f}) \subset \mathbb{C}^B$.

Hence K is compact. A is clearly isometrically isomorphic to P(K) (and hence to the uniform closure of the algebra of functions holomorphic in a neighbourhood of K, by the Generalized Oka Polynomial Approximation Theorem).

To show that K is polynomially convex it is sufficient to show that hull (K) $\subset$ K. Suppose that

$z \in$ hull (K). Then the map

$m_z: p(f_1,\ldots,f_k) \to p(z)$ is a complex homomorphism of $P_o(K)$, where $\{f_1,\ldots,f_k\} \subset B$.

It is also bounded since

$|p(z)| \leq \|p\|_{\text{hull}(K)} = \|p(\hat{f}_1,\ldots,\hat{f}_k)\|_{\text{Spec } A} = \|p(f_1,\ldots,f_k)\|_X$

and so can be extended by continuity to the whole of A to become an element $m \in$ Spec A.

Hence $p(z) = m(p(f_1,\ldots,f_k)) = p(m(f_1),\ldots,m(f_k))$

$= p(\hat{f}_1(m),\ldots\hat{f}_k(m)) = p(z')$ for some $z' \in K$.

Hence $z = z'$ and so $z \in K$.

Every function algebra has, of course, at least one set of generators, namely itself, so that 1.29 is always true for $B = A$.

CHAPTER II.

BOUNDARIES OF FINITELY GENERATED FUNCTION ALGEBRAS.

We saw earlier (Chapter I, 1.5) that for any function algebra A, if f is a member of A then $\hat{f}$ achieves its maximum modulus on X regarded as a subset of Spec A. In this sense $\hat{f}$ behaves like an analytic function, if we consider X as a "boundary" of Spec A. The question arises, are there any other subsets of Spec A which behave like "boundaries", and if so, is there a unique minimal one amongst them?

Definition 2.1: Let A be a function algebra on X. A boundary of Spec A for A is a subset E of Spec A such that for each f in A there is a point m in E such that

$$|\hat{f}(m)| = \|\hat{f}\| .$$

Clearly, according to this definition, X is a boundary of Spec A for A.

Theorem 2.2: For every function algebra A there is a unique minimal closed boundary of Spec A.

Proof: See e.g. [9], p. 9.

Definition 2.3: The unique minimal closed boundary of Spec A which exists by virtue of 2.2 is called the Shilov boundary of A, and is denoted by bA.

Definition 2.4: A point m_o in Spec A is said to be a peak point for A if there is an f in A such that

$$\hat{f}(m_o) = 1 \text{ and } |\hat{f}(m)| < 1 \text{ for all } m \neq m_o.$$

Note that in general peak points for A need not exist, even if A = C(X). However, in the case that X is compact metrizable, Bishop proved the following.

Theorem 2.5: If A is a function algebra on a compact metrizable space X then the set of peak points for A is dense in bA. Furthermore, the set of peak points for A is the unique minimal (not necessarily closed) boundary for A.

Proof: See [10].

We denote the set of peak points of a function algebra A by M(A). If A is defined on a metrizable space we shall refer to M(A) as the minimal boundary for A.

Proposition 2.6: Let A be a function algebra on X. Then

$$M(A) \subset bA \subset X \subset \text{Spec } A.$$

Corollary 2.7: If A is a finitely generated function algebra

on X, then the Shilov boundary of A is the closure in the weak topology of the minimal boundary for A.

Proof: By 1.19, X is metrizable. Hence by 2.5, M(A) is dense in bA and is the unique minimal boundary for A.

Thus when dealing with the Shilov boundary of a finitely generated function algebra we need only essentially consider the peak points for A.

In the same paper [10], Bishop also proved the following characterization of the minimal boundary.

Theorem 2.8: Let A be a function algebra on the compact metrizable space X, and let d be a metric on X. For each positive integer n and each x in X let

$D_n(x) = \{y \in X : d(x,y) \geq 1/n\}$. Let

$U_n = \{x \in X : \exists$ an f in A with $\|f\| \leq 1$, $|f(x)| > 3/4$ and $|f(y)| < 1/4$ for all y in $D_n(x)\}$.

Then U_n is open and $\cap_n U_n = M(A)$.

Proof: For each f in A let

$\sigma_n(f) = \{x \in X : |f(x)| > 3/4$ and $|f(y)| < 1/4$ for all y in $D_n(x)\}$.

The set $\sigma_n(f)$ is open, for if $x_o \in \sigma_n(f)$, we can find a neighbourhood of x_o on which $|f| > 3/4$ and which is contained in

$X - D_n(x_o)$ i.e. each point of such a neighbourhood also belongs to $\sigma_n(f)$.

Now $U_n = \cup\{\sigma_n(f): f \in A \text{ and } \|f\| \leq 1\}$, and is therefore open.

Let $x \in M(A)$ i.e. x is a peak point for some function f in A. We can assume that $\|f\| = 1$ and hence that $|f(x) = 1$. For each n, $D_n(x)$ is a closed and therefore compact set. But $|f(y)| < 1$ for $y \in D_n(x)$ and thus for some positive integer p_n,

$$|f(y)|^{p_n} < 1/4 \text{ for } y \in D_n(x).$$

Thus $x \in \sigma_n(f^{p_n})$ and so $x \in U_n$. Since this is true for each n, it follows that $x \in \bigcap_n U_n$. Therefore $M(A) \subset \bigcap_n U_n$.

To prove the reverse inclusion, let x be a fixed element of $\cap U_n$. We construct by induction a sequence $\{g_n\} \subset A$ satisfying the following conditions.

(1) $\|g_{n+1} - g_n\| \leq 2^{-n+1}$

(2) $\|g_n\| \leq 3(1 - 2^{-n-1})$

(3) $g_n(x) = 3(1 - 2^{-n})$

(4) $|g_{n+1}(y) - g_n(y)| < 2^{-n-1}$ for $y \in D_n(x)$

We start by constructing g_1. Since $x \in U_1$, there is a function f in A such that $\|f\| \leq 1$ and $x \in \sigma_1(f)$. Let

$$g_1 = 3/2\,[f(x)]^{-1} f.$$

Since $|f(x)| > 3/4$, it follows that

$$\|g_1\| = 3/2.4/3\ \|f\| \leq 3/2.4/3 = 2 < 3(1 - 2^{-2})$$

so that g_1 satisfies (2). Also

$$g_1(x) = 3/2\,[f(x)]^{-1} f(x) = 3/2 = 3(1 - 2^{-1}).$$

Hence $g_1(x)$ satisfies all the relevant conditions.

Assume now that $g_1,\ldots,g_k$ have been chosen to satisfy (1) - (4). Since $g_k(x) = 3(1 - 2^{-k})$ and g_k is continuous, there exists an integer $j > k$ such that

$$|g_k(y)| < 3(1 - 2^{-k}) + 2^{-k-2} \text{ for } d(x,y) < 1/j$$

i.e. for $y \in X - D_j(x)$. Since $x \in U_j$, there is an f in A such that $\|f\| \leq 1$ and $x \in \sigma_j(f)$. Let

$$h = 3.2^{-k-1}[f(x)]^{-1}f.$$ Then $h(x) = 3.2^{-k-1}$.

Since $\|f\| \leq 1$ and $|f(x)| > 3/4$, $\|h\| = 3.2^{-k-1}.4/3 = 2^{-k+1}$.

Since also $|f(y)| < 1/4$ for $y \in D_j(x)$ it follows that

$|h(y)| < 3.2^{-k-1}.4/3.1/4 = 2^{-k-1}$ for $y \in D_j(x)$.

Let $g_{k+1} = g_k + h$.

Then $\|g_{k+1} - g_k\| = \|g_k + h - g_k\| = \|h\| \leq 2^{-k+1}$ ... (1)

Now $j > k \Rightarrow D_k(x) \subset D_j(x)$. Hence

$|g_{k+1}(y) - g_k(y)| = |h(y)| < 2^{-k-1}$ for $y \in D_k(x)$ ... (4)

Also, $g_{k+1}(x) = g_k(x) + h(x) = 3(1 - 2^{-k}) + 3.2^{-k-1}$

$$= 3 - 3(2^{-k} - 2^{-k-1})$$

$$= 3(1 - 2^{-k-1}) \quad \ldots (3)$$

If $y \in D_j(x)$, then

$$|g_{k+1}(y)| = |g_k(y) + h(y)| \leq |g_k(y)| + |h(y)|$$

$$< \|g_k\| + 2^{-k-1} \leq 3(1 - 2^{-k-1}) + 2^{-k-1}$$

$$= 3 - 2^{-k} < 3(1 - 2^{-k-2})$$

since $2^{-k} > 3.2^{-k-2}$.

If $y \in X - D_j(x)$, then

$$\begin{aligned} |g_{k+1}(y)| &\leq |g_k(y)| + |h(y)| \\ &< 3(1 - 2^{-k}) + 2^{-k-2} + \|h\| \\ &\leq 3(1 - 2^{-k}) + 2^{-k-2} + 2^{-k+1} \\ &= 3(1 - 2^{-k-2}). \end{aligned}$$

Hence $\|g_{k+1}\| \leq 3(1 - 2^{-k-2})$ ... (2)

Thus g_{k+1} satisfies (1) - (4). We have thus constructed the sequence $\{g_n\}$ by induction.

By condition (1), $\{g_n\}$ converges uniformly on X to a function g in A.

By (2), $\|g\| \leq 3$.

By (3), $g_n(x) = 3$.

If $y \in D_n(x)$, then

$$\begin{aligned} |g(y)| &\leq \|g_n\| + \sum_{k=n}^{\infty} |g_{k+1}(y) - g_k(y)| \\ &< 3(1 - 2^{-n-1}) + \sum_{k=n}^{\infty} 2^{-k-1}, \text{ by (4)} \\ &< 3 \end{aligned}$$

Hence x is a peak point for g and so $x \in M(A)$, as was to be proved.

Recalling that if A is the finitely generated function algebra $[f_1,\ldots,f_n]$, then X must be metrizable and the metric on X can be taken to be

$d(x,y) = \max\{|f_i(x) - f_i(y)| : i = 1,\ldots,n\}$, we can rewrite Bishop's characterization of the minimal boundary as follows.

Corollary 2.9: Let $A = [f_1, \ldots f_n]$. For each positive integer n and each $x \in X$ let

$D_n(x) = \{y \in X : \max_i |f_i(x) - f_i(y)| \geq 1/n\}$. Let

$U_n = \{x \in X : \exists$ f in A with $\|f\| \leq 1$, $|f(x)| > 3/4$ and $|f(y)| < 1/4$ for all $y \in D_n(x)\}$.

Then U_n is open and $\cap_n U_n = M(A)$.

We can obtain a very similar result for arbitrary function algebras by applying some of the techniques developed by Rickart in his paper [8], which we discussed in Chapter I.

Theorem 2.10: Let A be a function algebra on a (not necessarily metrizable) compact Hausdorff space X.

Then $m \in$ Spec A is a peak point for A if there exists some finite subset $\pi \subset A$ such that

$m \in F^{-1}[\cap_n p_\pi^{-1}(U_n)]$, where

F:Spec $A \to K = \prod R(\hat{f})$ and

$U_n = \{z \in p_\pi K : \exists$ a function f in $A(p_\pi K)$ such that $\|f\| \leq 1$, $|f(z)| > 3/4$, $|f(w)| < 1/4$ for $w \in D_n(z)\}$, where

$D_n(z) = \{w \in p_\pi K : |z - w| \geq 1/n\}$

Proof: Suppose that $m \in$ Spec A is a peak point for A. We showed in Chapter I that A is isometrically isomorphic to A(K). Therefore $F(m) \in K$ is a peak point for some function g which is holomorphic in a neighbourhood G of K. By 1.25

there is an open set G_o containing K and contained in G on which g is determined by some finite subset $\pi \subset A$. Now g_π, the projection of g into $\mathbb{C}^\pi$, is a holomorphic function of a finite number of complex variables on the open set $p_\pi G_o$. Since $g_\pi(p_\pi(z)) = g(z)$ for $z \in G_o$, it follows that $p_\pi(F(m))$ is a peak point for $A(p_\pi K)$, which is a function algebra on the compact metric space $p_\pi K$. Hence by Bishop's characterization,

$p_\pi(F(m)) \in \bigcap_n U_n$. But

$F(m) \in p_\pi^{-1}(\bigcap_n U_n) = \bigcap p_\pi^{-1}(U_n)$, and hence

$m \in F^{-1}[\bigcap_n p_\pi^{-1}(U_n)]$

Conversely, suppose that for some finite subset $\pi \subset A$, $p_\pi(F(m)) \in \bigcap_n U_n$. Then by Bishop's characterization

$p_\pi(F(m))$ is a peak point for $A(p_\pi K)$. Hence there exists an open set $G^\pi \supset p_\pi K$ and a function g_π holomorphic in G^π such that $p_\pi(F(m))$ is a peak point for g_π. The function $g \in A(K)$ defined by

$g(z) = g_\pi(p_\pi(z))$ for $z \in p_\pi^{-1}(G^\pi)$ is holomorphic in the open set $p_\pi^{-1}(G^\pi)$ containing $p_\pi^{-1}(p_\pi K) \supset K$, and clearly F(m) is a peak point for A(K). Hence m is a peak point for A.

The Local Maximum Modulus Principle.

H. Rossi in [11] proved his now famous local maximum modulus principle for function algebras. G. Stolzenberg in [12] proved the same result for finitely generated function algebras.

We shall be taking Stolzenberg's path to Rossi's result.

A result which we shall need, and which often crops up in other contexts, is the following theorem due to Shilov.

Theorem 2.11: (Shilov's idempotent theorem) ([13])
Let A be a finitely generated function algebra on X. If Spec A = S ∪ T where S and T are nonempty disjoint closed subsets, then there is an f in A such that
$\hat{f} = 0$ on S and $\hat{f} = 1$ on T.

Proof: A is isometrically isomorphic to P(K), as was shown in Chapter I, 1.10, and Spec A is homeomorphic to K. Let S_1 and T_1 be the subsets of K corresponding to the S and T under the homeomorphism F. Thus S_1 and T_1 are nonempty disjoint closed subsets of K whose union is K. Let W_1 and W_2 be disjoint neighbourhoods of S_1 and T_1 in $\mathbb{C}^n$. Then $W = W_1 \cup W_2$ is a neighbourhood of K and the function g defined as 0 in W_1 and as 1 in W_2 is holomorphic in W. By Oka's polynomial approximation theorem, the restriction of g to K lies in P(K). The function in A which corresponds to g is then the desired element f.

Theorem 2.12:
Let A be any function algebra on X. If Spec A = S ∪ T where S and T are nonempty disjoint closed subsets, then there is an

f in A such that $\hat{f} = 0$ on S and $\hat{f} = 1$ on T.

Proof: Using 1.29 and 1.26 we can repeat the proof of 2.11.

An interesting consequence of Shilov's theorem is the following

Corollary 2.13: Let A be a function algebra on X. If X is connected, then so is Spec A. In particular, if X is a connected compact set in $\mathbb{C}^n$, then hull(X) is also connected.

Proof: If Spec A is not connected, let S and T be nonempty disjoint closed subsets of Spec A such that Spec $A = S \cup T$. By the preceding theorem, we can find an f in A such that $\hat{f} = 0$ on S and $\hat{f} = 1$ on T. Thus f restricted to X takes only the values 0 or 1. Since X is connected, either $f \equiv 0$ or $f \equiv 1$. Neither possibility is consistent with the properties of $\hat{f}$. Hence Spec A is connected.

Definition 2.14: ([14])

Let D be an open set in $\mathbb{C}^n$ and $z \in D$. Consider functions which are each defined in some neighbourhood of z in D. Call two such functions equivalent if they coincide on a neighbourhood of z. This defines an equivalence relation, and the equivalence class of a function f at z, denoted by $[f]_z$, is called the germ of f at z.

If f is a holomorphic function, then $[f]_z$ amounts to a convergent power series.

Germs at z form a ring with the obvious definition of addition and multiplication.

The ring $\mathcal{O}_z$ of germs of holomorphic functions is a commutative integral domain with identity and unique factorization.

Topologize the space of germs $\mathcal{O} = \bigcup_{z \in D} \mathcal{O}_z$ by defining the following basis for the open sets:

let $k \in \mathcal{O}$, then $k \in \mathcal{O}_{z_o}$ and so $k = [f]_{z_o}$, where f is defined in an ε-neighbourhood N_{z_o} of z_o in D. At each $z \in N_{z_o}$ take that class in $\mathcal{O}_z$ containing the direct analytic continuation of f i.e. take $[f]_z$. Then define $\bigcup_{z \in N_{z_o}} [f]_z$ to be an open set and the collection of such sets to be the basis of open sets.

A <u>holomorphic function</u> f in D amounts to a continuous mapping $f : D \to \mathcal{O}$ which assigns to each point in D a holomorphic germ over that point.

Now form the quotient field M_z of $\mathcal{O}_z$ for each z in D.

Topologize $M = \bigcup_{z \in D} M_z$ as follows: let $\lambda \in M$, then $\lambda \in M_{z_o}$ and

$\lambda = [\alpha/\beta]_{z_o}$ and is represented by $[f_1]_{z_o}/[f_2]_{z_o}$, where

f_1 and f_2 are holomorphic functions at z_o, and because of unique factorization we may take f_1 and f_2 to be coprime at z_o. Let N_{z_o} be a neighbourhood of z_o in which f_1 and f_2 are defined and are still coprime. At each $w \in N_{z_o}$ take that class in M_w represented by $[f_1]_w/[f_2]_w$. The union over N_{z_o} of these

classes we define as an open set and the collection of all such sets we take as the basis for the topology.

The elements of M_z are called the <u>germs of meromorphic functions over z</u>.

A <u>meromorphic function in D</u> is a continuous mapping which assigns to each point of D a meromorphic germ over that point. At a point $z_o \in D$, a meromorphic function g is defined by the quotient of two functions f_1 and f_2 coprime and holomorphic at z_o.

(a) If $f_2(z_o) \neq o$ i.e. f_2 is a unit, then f_1/f_2 is holomorphic there and hence g is holomorphic at z_o and therefore in a neighbourhood of z_o. The point z_o is called a <u>regular point</u> of g.

(b) If $f_2(z_o) = o$ and $f_1(z_o) \neq o$, then g is said to have a <u>pole at z_o</u>.

(c) If $f_2(z_o) = o$ and $f_1(z_o) = o$, then z_o is called a point of indeterminacy of g.

<u>Definition 2.15</u>:

A <u>polynomial polyhedron</u> in $\mathbb{C}^n$ is a compact set K of the form

$$K = \{z \in U : |f_j(z)| \leq k_j,\ j = 1,\ldots,r\},$$

where U is an open subset of $\mathbb{C}^n$, the f_j are polynomials on U and the k_j are non-negative constants.

By an <u>open polynomial polyhedron</u> we mean the interior of a polynomial polyhedron.

Lemma 2.16: Let X be a compact subset of $\mathbb{C}^n$. Then hull(X) can be represented as the intersection of a descending chain of open polynomial polyhedra O_i, with $O_i \supset \overline{O}_{i+1}$.

Proof: Let p be any polynomial on $\mathbb{C}^n$. Let

$$O_i = \{z \in \mathbb{C}^n : |p(z)| < \|p\|_{\text{hull}(X)} + 1/i\}$$

Theorem 2.17: (Solution of the Cousin 1 problem on an open polynomial polyhedron)

Let O be an open polynomial polyhedron. Then every Cousin 1 problem on O admits a solution i.e. if $\{O_i\}$ is an open cover of O and we assign meromorphic functions g_i on O_i such that $g_i - g_j$ is holomorphic on $U_i \cap U_j$ for all i, j, then there exists a function g meromorphic on O such that for each i, $g - g_i$ is holomorphic on O_i.

Proof: See [15]

Definition 2.18:

A point m_o in Spec A is a local peak point for the function algebra A iff there is an open neighbourhood U of m_o in Spec A and a function f in A such that

$$\hat{f}(m_o) = 1 \text{ and } |\hat{f}| < 1 \text{ on } U - \{m_o\}.$$

Theorem 2.19: ([12])

Let X be a compact subset of $\mathbb{C}^n$. Then every local peak point

for P(X) is actually a peak point for P(X).

Proof: Let m_o be a local peak point for P(X). Then there is a neighbourhood U of m_o in hull(X) and an f in P(X) such that $1 = \hat{f}(m_o) > |\hat{f}(m)|$ for all m in $U - \{m_o\}$. By shrinking U, we can assume that in fact

$$1 = \hat{f}(m_o) > \sup\{|\hat{f}(m)| : m \in \overline{U} - \{m_o\}\}.$$

Let $F : \text{hull}(X) \to \mathbb{C}^{n+1}$ be defined by

$$F(z_1,\ldots,z_n) = (z_1,\ldots z_n, f(z_1,\ldots,z_n)).$$

From the fact that hull(X) is the maximal ideal space of P(X), it follows easily that $F(\text{hull}(X)) = \text{hull}(F(X))$. Therefore we can assume that $\text{hull}(X) \subset \mathbb{C}^{n+1}$ and that $f = z_{n+1}$, the (n+1)-st coordinate function. By Bishop's characterization of the minimal boundary, it is sufficient to show that for each neighbourhood V of m_o in hull(X) there is a g in P(X) such that $|\hat{g}| \leq 1$ on hull (X), $|\hat{g}| < 1/4$ on $\text{hull}(X) - V$ and $|\hat{g}(m_o)| > 3/4$. By Oka's Polynomial Approximation Theorem, it is sufficient to find such a g which is holomorphic in a neighbourhood of hull(X). Choose a compact subset C of $\mathbb{C}^n$ such that

$C \cap \text{hull}(X) = \overline{U}$, $m_o \in \text{int } C \cap \text{hull}(X) \subset U$, and $\hat{f}$ is holomorphic in a neighbourhood of C. Then f is holomorphic and non-constant on the component of int C = U_o containing m_o and so $f(U_o)$ contains a neighbourhood of $1 = \hat{f}(m_o)$. Hence for a sufficiently small $\varepsilon > 0$, $f(U_o)$ contains the

interval $[1, 1 + \varepsilon]$. Let T be an open polynomial polyhedron containing $[1, 1 + \varepsilon]$.

On $T \times U_o$ define

$$H_t = \{z \in U_o : z_{n+1} - t = o\}.$$

Express hull(X) as the intersection of a descending chain of relatively compact open sets O_i i.e. open polynomial polyhedra such that $O_i \supset \overline{O}_{i+1}$ (this is possible in view of 2.16). We claim that if i is large enough, then every $H_t \cap O_i$ is closed in O_i. Suppose not. Then for each i there exists a t_i in $[1, 1 + \varepsilon]$ and a point s_i which belongs to the closure of $H_{t_i} \cap O_i$ in O_i, but not to H_{t_i}. Then s_i belongs to the compact set C, but not to its interior U_o, because H_{t_i} is closed in U_o. Also, by continuity, $f(s_i) = t_i$. By passing to subsequences we can arrange that t_i converges to t^* in $[1, 1 + \varepsilon]$ and s_i converges to s^* on the boundary of C, since $[1, 1 + \varepsilon]$ and bdry C are compact. Then

$$f(s^*) = t^* \geq 1. \quad \text{Also,}$$

$$s^* \in \bigcap_i O_i = \text{hull}(X).$$

Therefore $s^* \in \text{hull}(X) \cap \text{bdry } C = \text{bdry } U$ and $f(s^*) \geq 1$, which contradicts the fact that $\hat{f}(m_o) > \sup\{|\hat{f}(m)| : m \in \overline{U} - \{m_o\}\}$.

Hence for i large enough, every $H_t \cap O_i$ is closed in O_i. Choose $i = i_o$ that large and let $O = O_{i_o}$.

Let $W = \{(t, z) \in T \times U_o : f(z) - t = 0\}$. Then W is closed in $T \times O$. Cover $T \times O$ by the open sets $T \times U_o$

and $(T \times 0) - W$ and assign the meromorphic functions $1/z_{n+1} - t$ on $T \times U_o$ and o on $(T \times 0) - W$. Since $T \times 0$ is again an open polynomial polyhedron we can solve the Cousin 1 problem on it to obtain a meromorphic function G on all of $T \times 0$ such that G is holomorphic off W and on $T \times U_o$, $G = 1/z_{n+1} - t + E$, where E is holomorphic on $T \times U_o$. Choose a neighbourhood V of m_o in hull(X) such that $\overline{V} \subset U_o \cap \text{hull}(X)$. Since the pole set of G intersects $[1, 1 + \varepsilon] \times \text{hull}(X)$ only in $\{(1, m_o)\}$, $|G|$ is bounded on $[1, 1 + \varepsilon] \times (\text{hull}(X) - V)$, and $G(t, z)$ is holomorphic about hull(X) for each (fixed) $t > 1$. Also $\sup|G(t, z)|$ on hull(X) tends to infinity as t converges to 1 from above. Hence there is a δ, $o < \delta < \varepsilon$, such that for each $t \in (1, 1 + \delta]$, $|G(t, z)|$ attains its maximum over hull(X) only on V. Also there is a constant $c > o$ such that $|E| < c$ on $[1, \delta] \times \overline{V}$. Consider the ratio

$R_t = |G(t, m_o)|/\sup\{|G(t, z)| : z \in \text{hull}(X)\}$ for each $t \in (1, 1 + \delta]$. Then $R_t \leq 1$, and

$$R_t = |(1 - t)^{-1} + E(t, m_o)|/\sup_{\overline{V}}|(z_{n+1} - t)^{-1} + E(t, z)|$$

$$\geq [(t - 1) - c]/[\sup_{\overline{V}}|(z_{n+1} - t^{-1})| + c]$$

However on $\overline{V}$, $|z_{n+1}| \leq 1$ and $z_{n+1} = 1$ precisely at m_o. Therefore, for $t \in (1, 1 + \delta]$, $\sup_{\overline{V}}|z_{n+1} - t|^{-1} = (t - 1)^{-1}$.

Also, for $t > 1$, close enough to 1, we have $(t-1)^{-1} > c$. So $1 \geq Rt \geq [(t-1)^{-1} - c]/[(t-1)^{-1} + c] \to 1$ as $t \to 1$ from above. Hence for $t \in (1, 1+\delta]$ sufficiently close to 1, if we define

$$g(z) = G(t,z)/\sup\{|G(t,z)| : z \in \text{hull}(X)\} \text{ on } O$$

then g will be holomorphic on O and we will have $|\hat{g}| \leq 1$ on hull (X), $|\hat{g}| < 1/4$ on hull(X) - V and $|g(m_o)| > 3/4$.

Remark: Stolzenberg proved his result for P(X), X a compact subset of $\mathbb{C}^n$. His result clearly implies the local peak point theorem for finitely generated function algebras i.e. if A is a finitely generated function algebra then every local peak point for A is also a peak point for A. We can extend Stolzenberg's result to the following, again using Rickart's basic lemma 1.25.

Theorem 2.20: Let A be any function algebra. If m_o is a local peak point for A, then m_o is in fact a peak point for A.

Proof: By 1.29, A is isometrically isomorphic to the function algebra A(K), K a compact polynomially convex subset of $\mathbb{C}^A$. Let w be a local peak point for A(K). Then there exists a neighbourhood U of w and a member f of A(K) such that $f(w) = 1$ and $|f| < 1$ on $U - \{w\}$. Now f is holomorphic in a neighbourhood G of K and so by 1.25 there is an open set G

containing K and contained in G on which f is determined by some finite subset $\pi_o \subset A$. By 1.23(3) there is a finite set $\pi \supset \pi_o$ such that $K_\pi \subset p_\pi G_o$. Now f_π, the projection of f into $\mathbb{C}^\pi$, is a holomorphic function of a finite number of complex variables on the open set $p_\pi G_o$. Since $f_\pi (p_\pi(z)) = f(z)$ for $z \in G_o$ it follows that $p_\pi(w) \in p_\pi K \subset K_\pi \subset p_\pi G_o$ is a local peak point for $A(K_\pi) = P(K_\pi)$. Now K_π is a compact subset of $\mathbb{C}^\pi$ and so by 2.19 it follows that $p_\pi(w)$ is in fact a peak point for $P(K_\pi)$ i.e. $\exists$ a function g on K_π such that $g((p_\pi(w)) = 1$ and $|g| < 1$ on K_π. Define the function h on $p_\pi^{-1}(K_\pi)$ by $h(z) = g(p_\pi(z))$ then h is finitely determined and $h_\pi = g$. Since $p_\pi^{-1}(K_\pi)$ contains K, it follows that $h(w) = 1$ and $|h| < 1$ on K, and hence that w is a peak point for A(K).

Definition 2.21:

A compact set $K \subset \text{Spec } A$ is called a local peak set for the function algebra A if there is an open set U in Spec A containing K and an element $f \in A$ such that $\hat{f}|K = 1$ and $|\hat{f}| < 1$ on $U - K$; K is a peak set for A if U can be taken to be Spec A.

The next theorem generalizes 2.20 to local peak sets.

Theorem 2.22:

Let A be any function algebra. If K is a local peak set for A

then K is in fact a peak set for A.

Proof: Choose an open set U containing K and an element f in A such that $\hat{f} = 1$ on K and $|\hat{f}| < 1$ on U - K.

Let $H = \{m \in \text{Spec } A : \hat{f}(m) = 1\}$. By hypothesis $K = H \cap U = H \cap \overline{U}$ and so K is open and closed in H. Suppose m is not a member of H. Then the function $g = 1 - f$, which is in A, is such that $\hat{g}(m) \neq 0$ but $\hat{g} = 0$ on H. So H is the set of common zeros in Spec A of its kernel $I = \{g \in A : \hat{g} = 0 \text{ on } H\}$. Hence H is the maximal ideal space of the Banach algebra $B = A/I$, and the Gelfand map identifies $\hat{B}$ with $\hat{A}$ restricted to H.

Now K is an open-closed subset of H = Spec B. Therefore by Shilov's theorem on idempotents, $\exists\, x \in B$ such that $\hat{x} = 0$ on K and $\hat{x} = 1$ on H - K. Thus K is the zero-set of its kernel $J = \{h \in A : \hat{h} = 0 \text{ on } K\}$. (Consider any $m \in \text{Spec } A - K$. If $m \notin H$, then $\exists g \in A$ such that $\hat{g}(m) \neq 0$ but $\hat{g} = 0$ on H, which contains K. If $m \in H$, then $m \in H - K$ and so $\hat{x}(m) = 1$, $\hat{x} = 0$ on K In either case m is not in the zero-set of J.)

Let $A_K = \{g \in A : \hat{g} \text{ is constant on } K\}$

$= J + \mathbb{C} = \{g + \alpha : g \in J\}$.

Then J is a closed ideal of A and A_K is the closed subalgebra of A which is just J with an identity adjoined. So Spec A_K is the one-point compactification of Spec J. Since J is a closed ideal of A and K = zero-set (J), Spec J = Spec A - K. Hence Spec A_K is Spec A with K identified to a point (If $m \in \text{Spec } A_K$ and $m(J) = \{0\}$, then $m(g + \alpha) = \alpha$, so there is

only one such homomorphism. If $m(J) \neq \{o\}$, extend m uniquely to a complex homomorphism $\tilde{m}$ of A by setting

$\tilde{m}(h) = m(gh)/m(g)$, where g is any member of J such that $m(g) \neq o$. Thus we have Spec A_K equal to the one-point compactification of Spec J). Since $\hat{f} = 1$ on K, $f \in A_K$. If m_o is the point $\{K\}$ in Spec A_K, then $\hat{f}(m_o) = 1$. Also, if $m \in$ Spec $A_K \cap U$ and $|\hat{f}(m)| = 1$, then $\hat{f}(m) = 1$ and so $m \in H \cap U = K$; hence $m = m_o$. Thus on Spec A_K, $\hat{f}$ peaks within U exactly at m_o i.e. m_o is a local peak point for A_K. Therefore by 2.20, m_o is a peak point for A_K. Thus $\exists\ h \in A$ such that $\hat{h} = 1$ on K and $|\hat{h}| < 1$ on Spec A - K. Hence K is a peak set for A.

Theorem 2.23:

(Local Maximum Modulus Principle (Version 1))

Let U be an open set in Spec A. Suppose that for some $f \in A$ (1) $|\hat{f}| \leq 1$ on U, (2) $|\hat{f}| < 1$ on bdry U and (3) $\hat{f}(m_o) = 1$ for some $m_o \in U$. Then there is an element $g \in A$ such that

(a) $|\hat{g}| \leq 1$ on Spec A;

(b) $\{m \in \text{Spec } A : |\hat{g}(m)| = 1\} = \{m : \hat{g}(m) = 1\}$

$= \{m : \hat{f}(m) = 1\} \cap U$

Proof: Replace f by $\frac{1}{2}(1 + f)$. Then if $m \in U$, $|\hat{f}(m)| = 1$ iff $\hat{f}(m) = 1$. Let $K = \{m \in U : \hat{f}(m) = 1\}$. By hypothesis (3), K is nonempty and by (2), $K = \{m \in \overline{U} : \hat{f}(m) = 1\}$, so

K is compact. Since $\hat{f} = 1$ on K and $|\hat{f}| < 1$ on U - K, K is a local peak set for A. By 2.22, K is a peak set for A, so there is some $g \in A$ such that $\hat{g} = 1$ on K and $|\hat{g}| < 1$ on Spec A - K. Since $\hat{g} = 1$ precisely on K, (a) and (b) hold.

Theorem 2.24:

(Local Maximum Modulus Principle (Version 11))

Let U be an open set in Spec A disjoint from the Shilov boundary of A. Then

(a) bdry U is nonempty.

(b) $\sup_U |\hat{f}| = \sup_{\text{bdry } U} |\hat{f}|$ for every $f \in A$.

If we let A_U denote the closure of $\hat{A}|U$ in $C(\overline{U})$ then $bA_U \subset$ bdry U.

Proof: U is not closed in Spec A, for if U were open and closed then Shilov's theorem would imply that there is an f in A such that $\hat{f} = 1$ on U, $\hat{f} = 0$ on Spec A - U, which contains bA. This is impossible. Hence bdry U is nonempty.

Assume that $\sup_U |\hat{f}| \neq \sup_{\text{bdry } U} |\hat{f}| = \max_{\text{bdry } U} |\hat{f}|$ for some f in A. If $\max_{\text{bdry } U} |\hat{f}| > a = \sup_U |\hat{f}|$, then $|\hat{f}(m)| > a$ for some $m \in$ bdry U. By continuity, $|\hat{f}| > a$ on a neighbourhood of m and this neighbourhood must contain a point of U, so we have a contradiction. If $\max_{\text{bdry } U} |\hat{f}| < a$, we can multiply by a scalar to get $|\hat{f}| < 1 = \sup_U |\hat{f}|$ on bdry U. Since $\hat{f}$ is continuous and U has compact closure, we have

$\sup_U |\hat{f}| = |\hat{f}(m_o)| = 1$ for some $m_o \in \bar{U}$. Replace f by $e^{-i\theta}f$, where $\hat{f}(m_o) = e^{i\theta}$. Then f satisfies the hypothesis of Version 1, so there is a g in A with $|\hat{g}| \leq 1$ on Spec A and $\{m : |\hat{g}(m)| = 1\} \subset U$. Since U is disjoint from bA and $\hat{g}$ attains its maximum modulus only within U, we have a contradiction once more. Hence $\sup_U |\hat{f}| = \sup_{\text{bdry } U} |\hat{f}|$.

CHAPTER III.

EXTENSIONS OF FINITELY GENERATED FUNCTION ALGEBRAS.

Let A be a function algebra on Spec A. If $f \in C(\text{Spec}A)$, $[A,f]$ will denote the function algebra generated by A and f. By 2.6 it follows that $b[A,f] \subset \text{Spec } A \subset \text{Spec}[A,f]$. It is natural to ask under what conditions $\text{Spec } A = \text{Spec}[A,f]$ and $bA = b[A,f]$.

We firstly prove some theorems describing conditions on f which ensure that $\text{Spec } A = \text{Spec}[A,f]$ and $bA = b[A,f]$. Many of these results depend on the local maximum modulus principle, in particular those results in which f behaves in a sense like a holomorphic function.

Having done this, we attempt to apply this theory of extensions of function algebras to the finitely generated case. Again we find a close connection with the theory of analytic functions.

Definition 3.1: Let A be a function algebra. We say that $f \in C(\text{Spec } A)$ is A-holomorphic at a point $m \in \text{Spec } A$ if there is a neighbourhood U of m such that f can be approximated uniformly on U by functions in A. A function f is A-holomorphic on a subset S of Spec A if f is A-holomorphic at every point of S.

Lemma 3.2: Let A be a function algebra, U an open subset of Spec A and $f \in C(\text{Spec } A)$ a function which is A-holomorphic on U. Then

$$b[A,f] \subset bA \cup (\text{Spec } A - U).$$

Proof: Suppose $m \in U - bA$. There is an open set V containing m such that V and bA are disjoint, while f can be approximated uniformly on V by functions in A. Hence the closure of $[A,f]$ in $C(\overline{V})$ coincides with the closure $(A|V)^{-}$ of A in $C(\overline{V})$.

By the local maximum modulus principle 2.24, the Shilov boundary of $(A|V)^{-}$ is contained in bdry V. Thus $b[A,f]$ cannot meet V. Hence $b[A,f]$ is disjoint from $U - bA$.

Definition 3.3: Let A be a function algebra on X. The A-convex hull of a closed subset E of X is the set

$$A\text{-hull}(E) = \{m \in \text{Spec } A : |f(m)| \leq \|f\|_E \text{ for all } f \in A\}.$$

E is said to be A-convex iff $E = A\text{-hull}(E)$. Equivalently we can define A-hull (E) as the set of all complex homomorphisms which extend continuously to $(A|E)^{-}$. Since every complex homomorphism of $(A|E)^{-}$ yields a member of Spec A, we have the following

Theorem 3.4: Let A be a function algebra on X, and E a closed subset of X. Then $A\text{-hull}(E) = \text{Spec } (A|E)^{-}$.

Theorem 3.5: (Glicksberg's Lemma)

Let A be a function algebra on X, and let U be a nonempty open subset of X. Every function $f \in A$ which vanishes on U also vanishes on Spec A - A-hull(X - U). If $X = bA$, then $V = \text{Spec } A - \text{A-hull}(X - U)$ is an open subset of Spec A and $V \cap U$ is dense in U.

Proof: See [9] p.39.

Theorem 3.6: Let A be a function algebra. Let $f \in C(\text{Spec } A)$ be such that f is A-holomorphic on Spec A - $f^{-1}(o)$.

Then Spec[A,f] = Spec A and b[A,f] = bA.

Proof: Let $B = [A,f]$. By 3.2, $bB \subset bA \cup f^{-1}(o)$.

Suppose $bB \neq bA$. Then $bB - bA$ is a nonempty relatively open subset of bB on which f vanishes. By Glicksberg's lemma, f vanishes on an open subset U of Spec A which meets $bB - bA$. But then $A|U = B|U$ and therefore

$$U \cap (bB - bA) \subset b(B|U)^{-} = b(A|U)^{-} \subset \text{bdry } U \cup bA.$$

This is a contradiction and hence $bB = bA$.

Let r be the projection of Spec B onto Spec A i.e. if $m \in \text{Spec } B$ then r(m) is the restriction of m to A. We must show that every fiber $r^{-1}(m')$ consists of exactly one point i.e. that every $m' \in \text{Spec } A$ extends uniquely to B.

Let $g = f \circ r \in C(\text{Spec } B)$. Then g is constant on every fiber $r^{-1}(m')$. Now g is B-holomorphic on Spec B - $g^{-1}(o)$.

By the first part of the theorem, $b[B,f] = bB = bA$. However, g coincides with f on Spec A and, in particular, on bA. Therefore $g = f$, and f is constant on every fiber $r^{-1}(m')$. It follows that $r^{-1}(m')$ consists of no more than one point. Hence r is in fact a homeomorphism and Spec B = Spec A.

Corollary 3.7: Let A be a function algebra. Let B be a function algebra on Spec A such that $A \subset B$ and such that every $g \in B$ is A-holomorphic on Spec A. Then

Spec B = Spec A and $bB = bA$.

Proof: By 3.2 every function in B attains its maximum modulus on bA. Therefore $bB = bA$. By 3.6, every $m \in$ Spec A extends uniquely to each $g \in B$. Hence Spec A = Spec B.

Theorem 3.8:

Let A be a function algebra. Suppose $f \in C(\text{Spec } A)$ satisfies a relation of the form

$$f^n + g_{n-1}f^{n-1} + \cdots + g_1 f + g_o = o,$$

where $g_o, \ldots, g_{n-1} \in A$. Then $\text{Spec}[A,f] = \text{Spec } A$ and

$b[A,f] = bA$.

Proof: We can assume, by induction, that the theorem is true for all function algebras and all continuous roots of monic polynomials of degree $n - 1$. Consider the formal derivative

$$h = nf^{n-1} + (n-1)g_{n-1}f^{n-2} + \cdots + g_1.$$

In a neighbourhood of a point m at which $h(m) \neq o$, the function f can be expressed as a convergent power series in the coefficients $g_o, \ldots, g_{n-1}$. Hence f is A-holomorphic off the set $h^{-1}(o)$. It follows that h is A-holomorphic off the set $h^{-1}(o)$. By 3.6,

$$b[A,h] = bA \text{ and } \mathrm{Spec}[A,h] = \mathrm{Spec}\ A.$$

Now f satisfies a monic polynomial of degree n - 1 with coefficients in [A,h], namely

$$f^{n-1} + (n-1)/n.g_{n-1}f^{n-2} + \cdots + (g_1 - h)/n = o.$$

By the induction assumption, $\mathrm{Spec}[A,h,f] = \mathrm{Spec}\ A$ and $b[A,h,f] = bA$. But $[A,h,f] = [A,f]$ and hence $\mathrm{Spec}[A,f] = \mathrm{Spec}\ A$ and $b[A,f] = bA$.

Corollary 3.9: Let A be a function algebra, and let $f \in C(\mathrm{Spec}\ A)$.

Let S_j, $1 \leq j \leq n$, be subsets of Spec A such that $f|S_j \in A|S_j$, $1 \leq j \leq n$, while $\bigcup_{j=1}^{n} S_j = \mathrm{Spec}\ A$. Then $\mathrm{Spec}[A,f] = \mathrm{Spec}\ A$ and $b[A,f] = bA$.

Proof: If $g_j \in A$ coincides with f on S_j, then $\prod_{j=1}^{n}(f - g_j) = o$. The result follows on application of 3.8.

Theorem 3.10: (Baire Category Theorem)

Let X be a compact topological space. If $\{E_j\}_{j=1}^{\infty}$ is any countable closed covering of X then for at least one m,

int E_m is nonempty.

Proof: See [17], p. 250.

Note: The above theorem due to R. Baire is in fact true for a wider class of spaces called Baire spaces.

Theorem 3.11: Let A be a function algebra on X and let $\{E_j\}_{j=1}^{\infty}$ be a countable closed cover of Spec A. If $f \in C(\text{Spec } A)$ satisfies $f|E_j \in A|E_j$, $1 \leq j < \infty$, then Spec [A,f] = Spec A and b[A,f] = bA.

Proof: We show first that b[A,f] = bA. Suppose not. Then by the Baire category theorem 3.10, there is an index m and a nonempty relatively open subset U of b[A,f] - bA such that $U \subset E_m$. Choose $f_m \in A$ such that $f = f_m$ on E_m. By Glicksberg's lemma 3.5, $f - f_m$ vanishes on an open subset V of Spec A Such that $V \cap U$ is dense in U. By the local maximum modulus principle 2.24, b[A,f] cannot meet V - bA. This contradicts the fact that V-bA contains the nonempty subset $U \cap V$ of b[A,f]. Hence b[A,f] = bA.

To show that Spec[A,f] = Spec A, we employ an argument due to Quigley (unpublished). Let r : Spec[A,f] → Spec A be the natural projection which restricts the members of Spec[A,f] to A. If $g \in A$, then $\hat{g} = g \circ r$. Now $\hat{f} \circ r \in [A,f]$ on each set of the closed cover $\{r^{-1}(E_j)\}_{j=1}^{\infty}$ of Spec [A,f]. Applying what we have already proved to the algebras

$[A,\hat{f}]$ and $[A,\hat{f},f \circ r]$, considered as function algebras on Spec $[A,f]$, we find that $b[A,\hat{f},f \circ r] = b[A,\hat{f}] = bA$. Since $\hat{f} - f \circ r$ vanishes on $b[A,\hat{f},f \circ r]$, $\hat{f} - f \circ r$ must vanish on Spec $[A,\hat{f},f \circ r] \supset$ Spec$[A,f]$. Consequently, $\hat{f} = f \circ r$, and $\hat{f}$ is constant on each fiber $r^{-1}(m)$. It follows that r is 1:1, and hence that Spec$[A,f]$ = Spec A.

Theorem 3.12: Let A be a function algebra on Spec A and let $f \in C(\text{Spec } A)$. Let $B = [A,f]$. For $z \in R(f)$, let $T_z = \{m \in \text{Spec } A : f(m) = z\}$. Then

Spec B = Spec A $\Rightarrow T_z$ is A-convex for each $z \in R(f)$.

Proof: Put M_z = A-hull (T_z) (taken in Spec A). Let $K \subset$ Spec A x $\mathbb{C}$ be defined by

$K = \{(m,f(m)) : m \in \text{Spec } A\}$. Then K is an imbedding of Spec A in Spec A x $\mathbb{C}$. Let

$H = \{(m,w) \in \text{Spec } A \times \mathbb{C} : m \in M_w\}$.

H can be imbedded in Spec B. Let r denote the natural projection of Spec B onto Spec A i.e.

$m = r(m')$ iff $g(m) = \hat{g}(m')$ for all $g \in A$. (Since $A \subset B$, the functions in A extend continuously to functions on Spec B). Define a mapping s : Spec B $\to$ Spec A x $\mathbb{C}$ by $s(m)=(r(m),f(m))$. We show that s is an imbedding of Spec B into Spec A x $\mathbb{C}$ such that $H \subset s(\text{Spec } B)$.

(i) s is 1:1. If $s(m_1) = s(m_2)$, then $r(m_1) = r(m_2)$, so that $g(m_1) = g(r(m_1)) = g(r(m_2)) = g(m_2)$ for all $g \in A$. Also, $f(m_1) = f(m_2)$. Hence $h(m_1) = h(m_2)$ for all $h \in B$ and so $m_1 = m_2$.

(ii) $H \subset s(\text{Spec } B)$. For $(m,w) \in H$ we define a complex homomorphism $m'(m,w)$ on B as follows. Let B' be the dense subalgebra of B consisting of all polynomials in f with coefficients from A. Define $m'(m,w)$ on B' by

$m'(m,w)\ (\Sigma g_i f^i) = \Sigma g_i(m) w^i$, $g_i \in A$.

Then $m'(m,w)$ is clearly a complex homomorphism on B'. To extend it to B it is sufficient to show that $m'(m,w)$ is bounded on B'. Now

$|m'(m,w)\ (\Sigma g_i f^i)| = |\Sigma g_i(m) w^i| \leq ||\Sigma g_i w^i||_{T_w} = ||\Sigma g_i f^i||_{T_w}$

since $\Sigma g_i w^i \in A$ and $(m, w) \in H$ i.e. $m \in M_w$. Thus $m'(m,w)$ extends to a complex homomorphism on B. Also, for $g \in A$, $m'(m,w)\ (g) = g(m)$, which implies that $r(m'(m,w)) = m$; and $m'(m,w)\ (f) = w$. Therefore $s(m'(m,w)) = (m,w)$ and $H \subset s(\text{Spec } B)$.

(iii) $s(\text{Spec } A) = K$. For $m \in \text{Spec } A \subset \text{Spec } B$, $r(m) = m$. Hence $s(m) = (m, f(m)) \in K$.

If we now identify Spec A and Spec B with their imbeddings under s in Spec A x $\mathbb{C}$ we have

$$\text{Spec } A \subset H \subset \text{Spec } B.$$

Lemma 3.13: Let A, f and B be as in the previous theorem. If f(Spec B) properly contains f(Spec A), then each point $w \in f(\text{Spec } B) - f(\text{Spec } A)$ lies in a bounded component of $\mathbb{C} - f(\text{Spec } A)$. Moreover, f(Spec B) is the union of f(Spec A) with those bounded components of $\mathbb{C} - f(\text{Spec } A)$ which meet f(Spec B) (i.e. f(Spec A) can only be enlarged to f(Spec B) by completely filling in some holes).

Proof: Let $m \in \text{Spec } B$ be such that f(m) is not in f(Spec A). If f(m) lies in the unbounded component of $\mathbb{C} - f(\text{Spec } A)$, then there is a polynomial p in one complex variable such that $|p(f(m))| > \|p\|_{f(\text{Spec } A)}$. Since $g = p(f) \in B$, we have $|g(m)| > \|g\|_{\text{Spec}(A)}$. This contradicts the fact that B as a function algebra on Spec A has its Shilov boundary contained in Spec A.

Alternately, suppose that f(m) lies in some bounded component C of $\mathbb{C} - f(\text{Spec } A)$. If C is not contained in f(Spec B), there is a point w on the boundary of f(Spec B) and contained in $C - f(\text{Spec } A)$. Let $w = f(m_o)$.

Choose $a \in C - f(\text{Spec } B)$ such that

$$|a - w| < \inf\{|w - f(m')| : m' \in \text{Spec } A\}.$$

Then $g(z) = 1/(z - a)$ is analytic on a neighbourhood of f(Spec B), so that $g(f) \in B$. But

$|g(f(m_o))| > \|g(f)\|_{\text{Spec } A}$, which, as above, is impossible.

Theorem 3.14: (Lavrentiev)

Let K be a compact subset of the complex plane.

The conditions

(1) int K is empty

(2) the complement of K is connected

are necessary and sufficient for $P(K) = C(K)$

Proof: See [4] p. 104.

Theorem 3.15: Let H be as in 3.12 i.e.

$H = \{ (m,w) \in \text{Spec } A \times \mathbb{C} : m \in M_w \}$. If f(Spec A) is a compact subset of the plane with connected complement and empty interior, then Spec B = H.

Proof: By 3.13, f(Spec B) = f(Spec A). As in 3.12 consider the imbedding $s : \text{Spec } B \to \text{Spec } A \times \mathbb{C}$ given by $s(m) = (r(m), f(m))$, where r is the natural projection of Spec B onto Spec A. Then

$$s(\text{Spec } A) \subset H \subset s(\text{Spec } B) \subset \text{Spec } A \times f(\text{Spec } A)$$

and we wish to show that $H = s(\text{Spec } B)$. If $s(m) \in s(\text{Spec } B) - H$, from the definition of H there exists a function g in A such that

$1 = |g(r(m))| > \|g\|_{M_{f(m)}}$. Since s(m) is not in s(Spec A), we can choose a neighbourhood V of f(m) in $\mathbb{C}$ such that

$$(m',z') \in s(\text{Spec})A \cap [\text{Spec } A \times (V \cap f(\text{Spec } A)] \Rightarrow$$

$$|g(m')| < |g(r(m))|.$$

Also, since f(Spec A) is a compact subset of the plane with connected complement and empty interior, by Lavrentiev's theorem 3.14, every continuous function on f(Spec A) can be uniformly approximated on f(Spec A) by polynomials. In particular, if h is a function which peaks at f(m) i.e. $h(f(m))=1$, $|h(z)| < 1$ for $z \in f(\text{Spec } A)$, $z \neq f(m)$, then h can be uniformly approximated by polynomials. But $f(\text{Spec } A) = f(\text{Spec } B) \Rightarrow h(f) \in B$. Choose an integer N such that $\| h^N \|_{f(\text{Spec } A)} - V < 1/\| g \|$. Let $h' = h^N(f)g$. Then $h' \in B$, $|h'(m)| = 1$ and $\| h' \|_{\text{Spec } A} < 1$, a contradiction as in 3.13.

Corollary 3.16: Let A be a function algebra on Spec A. Let $H = \{(m,w) \in \text{Spec } A \times \mathbb{C} : m \in M_w\}$. If f(Spec A) is a compact subset of the plane with connected complement and empty interior, and in addition $T_w = M_w$ for each $w \in f(\text{Spec } A)$ i.e. the set $f^{-1}(w)$ is A-convex for each $w \in f(\text{Spec } A)$, then Spec A = H and hence Spec A = Spec B.

Proof: We had previously that Spec A $\subset$ H if we identify Spec A with $K = \{(m,f(m)) : m \in \text{Spec } A\}$. But

$$H = \cup \{M_w \times \{w\} : w \in f(\text{Spec } A)\}$$
$$= \cup \{T_w \times \{w\} : w \in f(\text{Spec } A)\}$$
$$= K.$$

Hence Spec A = H. By 3.15, H = Spec B and so Spec A = Spec B.

Remark: There is an interesting interpretation of the condition in 3.16 that $T_w = f^{-1}(w)$ be A-convex for each $w \in f(\text{Spec } A)$. It has been shown earlier (see 3.4) that if E is any closed subset of Spec A, then $\text{Spec}(A|E)^- = \text{A-hull}(E)$.

Since f is constant on each T_w,

$([A,f]|T_w)^- = (A|T_w)^-$. Hence

$\text{Spec}([A,f]|T_w)^- = \text{Spec}(A|T_w)^- = \text{A-hull}(T_w) = T_w$

for each $w \in f(\text{Spec } A)$. Thus $\text{Spec}[A,f] = \text{Spec } A$ if this is true for the restrictions of [A,f] and A to each T_w.

Theorem 3.12 stated that

$\text{Spec } A = \text{Spec}[A,f] \Rightarrow T_w$ is A-convex for all $w \in f(\text{Spec } A)$. What we have just proved in 3.16 is a partial converse to this i.e. if T_w is A-convex for all $w \in f(\text{Spec } A)$ and in addition $f(\text{Spec } A)$ is a compact subset of the plane with connected complement and empty interior, then $\text{Spec } A = \text{Spec}[A,f]$.

Gamelin and Wilken in [18] have produced the following counter-example to show that T_w is A-convex for each $w \in \text{Spec } A$ does not necessarily imply that $\text{Spec}[A,f] = \text{Spec } A$:

let $f(z) = z^2\bar{z}$, $z \in \Delta$, where Δ is the unit disc. The function f is a homeomorphism of Δ and $\text{Spec}[P(\Delta),f]$ is strictly larger than $\Delta = \text{Spec } P(\Delta)$. Notice that $f(\text{Spec } P(\Delta))$ does not have empty interior.

Remark: We can extend 3.12, 3.13, 3.15 and 3.16 to the more

general situation in which a finite number of functions are adjoined to A. Then the statements of these results have to be formulated in terms of intersections of the level sets of the functions. The statements are the same except that instead of adjoining $f \in C(\text{Spec } A)$ to A, we adjoin $\{f_1,\ldots,f_n\} \subset C(\text{Spec } A)$ and replace

T_w by $T(w_1,\ldots,w_n) = \{m \in \text{Spec } A : f_i(m) = w_i,\ i = 1,\ldots,n\}$,

M_w by $M(w_1,\ldots,w_n) = A\text{-hull}(T(w_1,\ldots,w_n))$ and

H by $H' = \{(m,w_1,\ldots,w_n) \in \text{Spec } A \times \mathbb{C}^n : m \in M(w_1,\ldots,w_n)\}$.

We now come to the application of the foregoing to finitely generated function algebras, but first we need to state a couple of theorems.

Definition 3.17: For K a compact subset of the complex plane, $A_o(K)$ will denote the set of all $f \in C(K)$ which are analytic in the interior of K.

Theorem 3.18: (Mergelyan).
Let K be a compact subset of the plane. If K has connected complement, then $P(K) = A_o(K)$.
Proof: See [9] p.48.

Theorem 3.19: A compact subset K of the plane is polynomially convex iff its complement is connected.

Proof: See [9] p.68.

Now let A = [g] be a singly generated function algebra on Spec A. Then A is isometrically isomorphic to P(K), where $K = R(\hat{g}) \subset \mathbb{C}$. But K is compact and polynomially convex and hence by Mergelyan's theorem 3.18 and the characterization of polynomial convexity in the plane 3.19, A is in fact isometrically isomorphic to $A_o(K)$. Suppose we adjoin to A a function $f \in C(\text{Spec } A)$ which is such that Spec A = Spec[A,f]. Notice that [A,f] is isometrically isomorphic to P(K'), where $K' = R(\hat{g}) \times R(\hat{f})$. Since K is homeomorphic to Spec A and K' is homeomorphic to Spec [A,f], it follows that K is homeomorphic to K'. In particular, the homeomorphism $\varphi : K \to K'$ is given by

$\varphi(z) = (z,w)$.

Theorem 3.20: Let A = [g] be a singly generated function algebra on Spec A and let $f \in C(\text{Spec } A)$ be such that Spec A = Spec[A,f]. Then there exists a continuous algebraic epimorphism $\Psi : [A,f] \to A_o(K)$.

Proof: Define the map $\Phi : P_o(K') \to P_o(K)$ by

$\Phi\,(p(z,w)) = p(\varphi^{-1}(z,w))$, where $p \in P_o(K')$

i.e. $\Phi\left(\sum_{k=0}^{m} \sum_{i+j=k} a_{ij} z^i w^j\right) = \sum_{k=0}^{m} \left(\sum_{i+j=k} a_{ij}\right) z^k$

thus defined is clearly onto, but not necessarily 1:1.

Φ is linear: let $p_1 = \Sigma\Sigma a_{ij} z^i w^j$, $p_2 = \Sigma\Sigma b_{ij} z^i w^j$ (we can assume for the sake of simplicity that z and w occur to the same powers). Then

$$\begin{aligned}\Phi(p_1 + p_2) &= \Phi(\Sigma\Sigma a_{ij} z^i w^j + \Sigma\Sigma b_{ij} z^i w^j) \\ &= \Phi(\Sigma\Sigma(a_{ij} + b_{ij}) z^i w^j) \\ &= \sum_k (\Sigma a_{ij} + b_{ij}) z^k \\ &= \sum_k (\Sigma a_{ij}) z^k + \sum_k (\Sigma b_{ij}) z^k \\ &= \Phi(p_1) + \Phi(p_2)\end{aligned}$$

Similarly $\Phi(\alpha p) = \alpha\Phi(p)$ for $p \in P_o(K')$, $\alpha \in \mathbb{C}$, and $\Phi(p_1 p_2) = \Phi(p_1)\Phi(p_2)$.

Φ is continuous : let $\varepsilon > o$ be given and let $p_1 = \Sigma\Sigma a_{ij} z^i w^j$. Choose $p_2 = \Sigma\Sigma b_{ij} z^i w^j$ such that

$\sum_{i+j=k} |a_{ij} - b_{ij}| = \sum_{i+j=k} |c_{ij}| < \varepsilon / 2^k |z|^k$. Then

$|\Phi(p_1) - \Phi(p_2)| = |\Sigma\Sigma c_{ij} z^k| \leq \Sigma\Sigma |c_{ij}| \, |z|^k < \varepsilon$.

Thus far we have shown that Φ is a continuous algebraic epimorphism from $P_o(K')$ onto $P_o(K)$. It can be extended by continuity to $\tilde{\Phi} : P(K') \to P(K)$ as follows: if $\tilde{p} \in P(K')$, define $\tilde{\Phi}(\tilde{p})$ to be the uniform limit of $\Phi(p_n)$, where $p_n \to \tilde{p}$ uniformly. We now show that $\tilde{\Phi}$ thus defined has the same properties as Φ.

Let $f \in P(K)$. We want to show that $f \in \tilde{\Phi}(P(K'))$ i.e. that $\tilde{\Phi}$ is onto. Now $\forall\, \varepsilon > o \; \exists f_1 \in P_o(K)$ such that $\| f - f_1 \| < \varepsilon$. Since Φ is onto, $\exists\, g_1 \in P_o(K')$ such that

$\Phi(g_1) = f_1$. Hence $\|f - \Phi(g_1)\| < \varepsilon$. Therefore

$f \in \overline{\Phi(P_o(K'))} = \tilde{\Phi}(\overline{P_o(K')}) = \tilde{\Phi}(P(K'))$.

$\tilde{\Phi}$ has the same algebraic properties as Φ. Let $f_1, f_2 \in P(K')$ and $\alpha, \beta \in \mathbb{C}$. We must show that $\tilde{\Phi}(\alpha f_1 + \beta f_2) = \alpha\tilde{\Phi}(f_1) + \beta\tilde{\Phi}(f_2)$, or equivalently that for any $\varepsilon > 0$ we can get

$$\|\tilde{\Phi}(\alpha f_1 + \beta f_2) - \alpha\tilde{\Phi}(f_1) - \beta\tilde{\Phi}(f_2)\| < \varepsilon.$$

Now

$$\begin{aligned}
&\|\tilde{\Phi}(\alpha f_1 + \beta f_2) - \alpha\tilde{\Phi}(f_1) - \beta\tilde{\Phi}(f_2)\| \\
&= \|\tilde{\Phi}(\alpha f_1 + \beta f_2) - \tilde{\Phi}(\alpha f_1' + \beta f_2') + \tilde{\Phi}(\alpha f_1' + \beta f_2') - \alpha\tilde{\Phi}(f_1) - \beta\tilde{\Phi}(f_2)\| \\
&= \|\tilde{\Phi}(\alpha f_1 + \beta f_2) - \tilde{\Phi}(\alpha f_1' + \beta f_2') + \Phi(\alpha f_1' + \beta f_2') - \alpha\tilde{\Phi}(f_1) - \beta\tilde{\Phi}(f_2)\| \\
&= \|\tilde{\Phi}(\alpha f_1 + \beta f_2) - \tilde{\Phi}(\alpha f_1' + \beta f_2') + \alpha(\tilde{\Phi}(f_1') - \tilde{\Phi}(f_1)) + \beta(\tilde{\Phi}(f_2') - \tilde{\Phi}(f_2))\| \\
&\leq \|\tilde{\Phi}(\alpha f_1 + \beta f_2) - \tilde{\Phi}(\alpha f_1' + \beta f_2')\| + |\alpha|\,\|\tilde{\Phi}(f_1') - \tilde{\Phi}(f_1)\| + |\beta|\,\|\tilde{\Phi}(f_2') - \tilde{\Phi}(f_2)\| \\
&< \varepsilon' + |\alpha|\varepsilon'' + |\beta|\varepsilon''' < \varepsilon
\end{aligned}$$

for f_1' and $f_2' \in P_o(K')$ chosen suitably close to f_1 and f_2 respectively.

Similarly, $\tilde{\Phi}(f_1 f_2) = \tilde{\Phi}(f_1)\,\tilde{\Phi}(f_2)$.

Thus we have a continuous algebraic epimorphism

$$\tilde{\Phi} : P(K') \to P(K) = A_o(K).$$

But $[A, f]$ is isometrically isomorphic to $P(K')$ and hence there is a continous algebraic epimorphism

$$\Psi : [A, f] \to A_o(K).$$

We can strengthen 3.20 if, in addition $b[A,f] = bA$.
To do so, we need a couple of further results.

Theorem 3.21: Let K be a compact subset of the plane. Then the Shilov boundary of P(K) is the topological boundary of hull(K).

Proof: See [4] p.85.

Proposition 3.22: Let $A = [g]$ be a singly generated function algebra. Then $bA = \text{Spec } A \Rightarrow A = C(X)$.

Proof: We prove the equivalent assertion that

$P(K) = C(K)$, where $K = R(g)$.

Now $bA = \text{Spec } A \Rightarrow \text{bdry } K = K$, by 3.21

$\Rightarrow \overline{K} - \text{int } K = K$

$\Rightarrow$ int K is empty, since $K = \overline{K}$

$\Rightarrow$ K is a compact subset of the plane

with empty interior and connected complement, by 3.19.

Corollary 3.23: Let $A = [g]$ be a singly generated function algebra on Spec A such that $bA = \text{Spec } A$. Let $f \in C(\text{Spec } A)$ be such that $\text{Spec}[A,f] = \text{Spec } A$ and $bA = b[A,f]$. Then there exists a continuous algebraic epimorphism

$$\Psi : [A,f] \to C(K).$$

Proof: The result follows immediately from 3.20 and 3.22.

Remarks: (i) The function $f \in C(\text{Spec } A)$ in 3.20 could be such that it satisfies the conditions of any one of 3.6 - 3.9, 3.11 and 3.16. The function $f \in C(\text{Spec } A)$ in 3.23 could be such that it satisfies the conditions of any one of 3.6 - 3.9 and 3.11.

(ii) Theorem 3.20 can be extended to the case where a finite number of functions, say $\{f_1,\ldots,f_n\}$, are adjoined to A, provided that Spec $[A,f_1,\ldots,f_n]$ = Spec A. e.g. if $\{f_1,\ldots,f_n\}$ are A-holomorphic on Spec A (by corollary 3.7) or if $\{f_1,\ldots,f_n\}$ satisfy the conditions of corollary 3.16 (see remark preceding definition 3.17).

(iii) The definition of an A-holomorphic function (see Definition 3.1) is identical to that of a function which is locally approximable by the functions of a function algebra A. Moreover, functions locally approximable by the functions of A are also locally in A. Thus all results flowing from theorem 3.6 are true for functions locally approximable by functions in A and functions locally in A. For a discussion of the adjunction of such functions to a function algebra see the M.Sc. thesis of J.P.G. Ewer, "Approximation Properties of Function Algebras" [35], done at the University of Cape Town.

CHAPTER IV.

FUNCTION ALGEBRAS AS DIRECT LIMITS OF THEIR FINITELY GENERATED SUB-FUNCTION ALGEBRAS.

In the case that a function algebra A contains a 1 : 1 function, we can express it as the direct limit of its finitely generated sub-function algebras. The value of this procedure will be seen later on when we shall show how information about such a function algebra can be obtained from the characteristics of its finitely generated sub-function algebras and vice versa.

Definition 4.1: A directed set D is a set with a partial order relation $\leqslant$ such that for $\alpha, \beta \in D$ there is a $\gamma \in D$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$.

A direct system of algebras and algebra homomorphisms $\{A^\alpha, h_\alpha^\beta\}$ consists in a collection $\{A^\alpha\}$ of algebras indexed by a directed set D, and a collection of algebra homomorphisms $h_\alpha^\beta : A^\alpha \to A^\beta$ for every $\alpha, \beta \in D$ with $\alpha \leqslant \beta$ such that

(a) h_α^α = identity of A^α

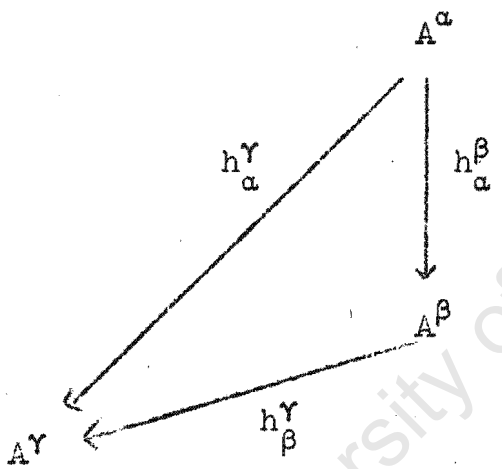
(b) $h_\alpha^\gamma = h_\beta^\gamma \circ h_\alpha^\beta : A^\alpha \to A^\beta$ for $\alpha \leqslant \beta \leqslant \gamma$ in D.

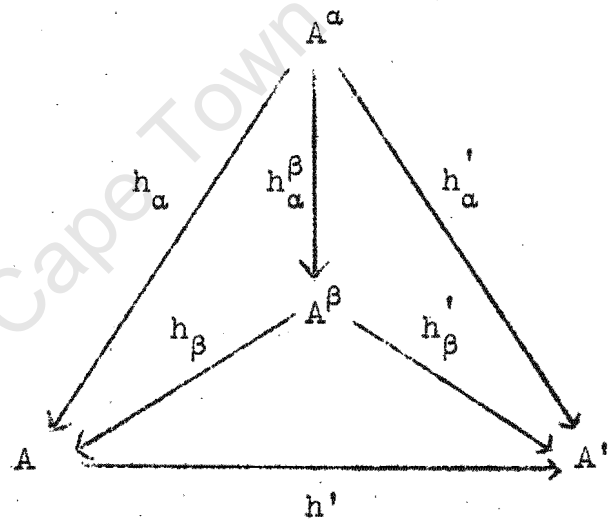
A direct limit of a direct system of algebras and algebra homomorphisms $\{A^\alpha, h_\alpha^\beta\}$ is an algebra A together with a family of algebra homomorphisms $h_\alpha : A^\alpha \to A$ (α in D) satisfying the

conditions

(1) $h_\beta \circ h_\alpha^\beta = h_\alpha$ if $\alpha \leqslant \beta$ in D, and

(2) for any algebra A' and any collection of algebra homomorphisms $h'_\alpha : A^\alpha \to A'$ such that $h'_\beta \circ h_\alpha^\beta = h'_\alpha$ if $\alpha \leqslant \beta$ in D, there exists a unique isomorphism $h' : A \to A'$ such that $h' \circ h_\alpha = h'_\alpha$ for all $\alpha \in D$. The situation is summed up in the following diagrams in which all the triangles commute.





A is also denoted by lim.dir. A^α.

The direct limit of a direct system $\{A^\alpha, h_\alpha^\beta\}$ has a canonical form, which can be constructed as follows.

Let $\sum_{\alpha \in D} A^\alpha$ be the direct sum of the A^α i.e. $\sum_{\alpha \in D} A^\alpha$ consists of all $f \in \prod_{\alpha \in D} A^\alpha$ such that only a finite number of coordinates of f are not zero. For each $\alpha \in D$ there is an embedding

$i_\alpha : A^\alpha \to \Sigma A^\alpha$; given f^α in A^α, $i_\alpha(f^\alpha)$ is the element of ΣA^α with α^{-th} coordinate f^α and all others zero.

Let S be the submodule of ΣA^α generated by the ranges

$(i_\beta \circ h^\beta_\alpha - i_\alpha)(A^\alpha)$ for all $(\alpha, \beta) \in D \times D$ with $\alpha \leqslant \beta$.

Then $\Sigma A^\alpha/S$ is a direct limit for $\{A^\alpha, h^\beta_\alpha\}$ where $h_\alpha : A^\alpha \to \Sigma A^\alpha/S$ is the composite of the canonical map from ΣA^α onto $\Sigma A^\alpha/S$ and i_α. (see [4]. p.211.)

Definition 4.2: An inverse system of topological spaces and continuous maps $\{X_\alpha, r^\beta_\alpha\}$ consists in a collection $\{X_\alpha\}$ of topological spaces indexed by a directed set D, and a collection of continuous functions $r^\beta_\alpha : X_\beta \to X_\alpha$ for $\alpha \leqslant \beta$ in D such that

(a) r^α_α = identity of X_α

(b) $r^\gamma_\alpha = r^\beta_\alpha \circ r^\gamma_\beta : X_\gamma \to X_\alpha$ for $\alpha \leqslant \beta \leqslant \gamma$ in D.

An inverse limit of an inverse system of topological spaces and continuous maps is a topological space X together with a family of continuous maps $r_\alpha : X \to X_\alpha$, $\alpha \in D$, satisfying the following conditions

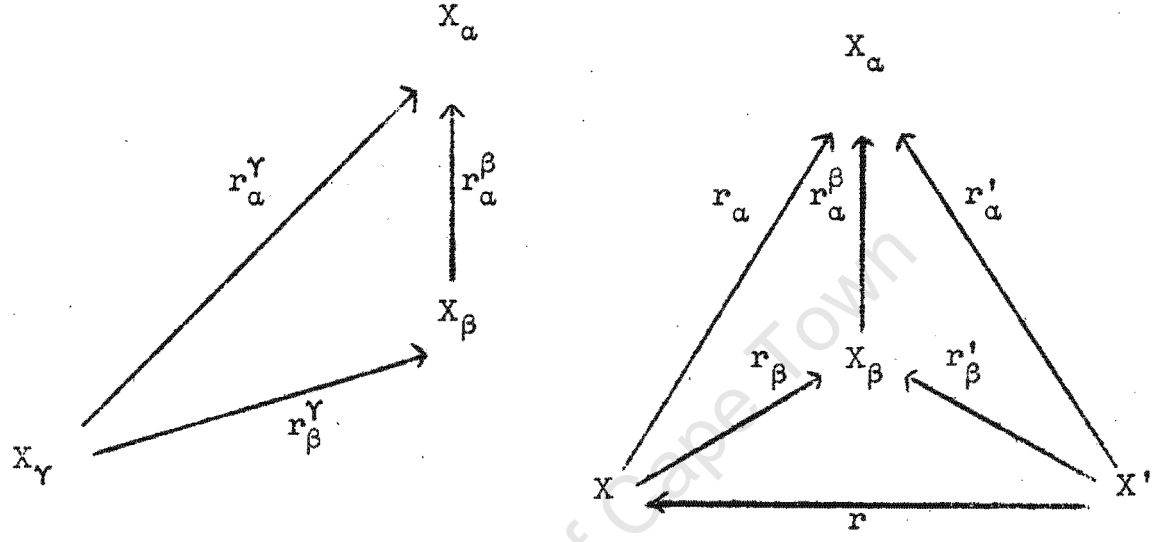
(1) $r^\beta_\alpha \circ r_\beta = r_\alpha$

(2) for any topological space X' and any family of continuous maps $r'_\alpha : X' \to X$ such that

$r^\beta_\alpha \circ r'_\beta = r'_\alpha$ for $(\alpha, \beta) \in D \times D$ with $\alpha \leqslant \beta$,

there exists a unique continuous map $r : X' \to X$ such that

$r_\alpha \circ r = r'_\alpha$ for all $\alpha \in D$. We can sum up by saying that all triangles in the following diagrams commute.



The inverse limit is also denoted by lim.inv. $\{X_\alpha, r_\alpha^\beta\}$.

We can construct an inverse limit of an inverse system of topological spaces and continuous maps as follows:

Define a <u>thread</u> to be an element x of the topological product $\prod_{\alpha \in D} X_\alpha$ such that $r_\alpha^\beta(x_\beta) = x_\alpha$ wherever $\alpha \leqslant \beta$.

Let X be the set of all threads provided with the relative topology induced by $\prod X_\alpha$, and let $r_\alpha(x) = x_\alpha$ for $x \in X$. i.e. r_α is the restriction to X of the projection from $\prod X_\alpha$ onto X_α. Then $\{X, r_\alpha\}$ is an inverse limit of the inverse system $\{X_\alpha, r_\alpha^\beta\}$. For if $\{X', r'_\alpha\}$ is such that $r'_\alpha = r_\alpha^\beta \circ r'_\beta$ for all $(\alpha,\beta) \in D \times D$ such that $\alpha \leqslant \beta$, then each element x' of X' defines a thread $(r'_\alpha(x'))\ \alpha \in D$. The map $r : X' \to X$

which sends $x' \in X'$ onto its corresponding thread is easily verified to be a commuting homeomorphism and is unique by definition. Henceforth we shall refer to $\{X, r_\alpha\}$ as just defined as the inverse limit of the inverse system.

Proposition 4.3:

Let $\{X_\alpha, r_\alpha^\beta\}$ be an inverse system of compact Hausdorff spaces and continuous maps. Then the inverse limit of the system is also compact.

Proof: See e.g. [4]. p.210.

Now suppose we are given a function algebra A containing a 1:1 function g. Let D be the family of finite subsets of A which include g as a member i.e. $\alpha \in D$ iff $\alpha = \{g, f_1, \ldots, f_k\}$ where the f_i and g belong to A. If we direct D by inclusion, then D is clearly a directed set. Put $A^\alpha = [g, f_1, \ldots, f_k]$. Clearly, $A = \bigcup_{\alpha \in D} A^\alpha$. Let $i_\alpha^\beta : A^\alpha \to A^\beta$ be the inclusion map for $\alpha \leq \beta$. Then $\{A^\alpha, i_\alpha^\beta\}$ forms a direct system of algebras and algebra homomorphisms, having as direct limit $A = \bigcup_{\alpha \in D} A^\alpha$, with the $i_\alpha : A^\alpha \to A$ taken as the inclusion map. For suppose $\{A', i'_\alpha\}$ is such that $i'_\beta \circ i_\alpha^\beta = i'_\alpha$ for all $(\alpha, \beta) \in (D \times D)$ with $\alpha \leq \beta$. We can define an algebra homomorphism $i : A \to A'$ as follows: if $f \in A$ then there is an $\alpha \in D$ such that $f \in A^\alpha$. Let $i(f) = i'_\alpha(i_\alpha^{-1}(f))$, which is well defined since i_α is the

inclusion map. It is easy to show that i has the required properties.

Note that if we were to define D simply as the family of all finite subsets of A we would have no guarantee that the subalgebras generated by these finite subsets would in fact be function algebras, since they may fail to separate points, even though A does so. Purely as an algebra, however, a function algebra is the direct limit of its finitely generated subalgebras. Indeed, if A did not contain a 1:1 function we could not even use our somewhat artificial tactic to ensure that A is the direct limit of certain of its finitely generated sub-function algebras. That A contains a 1:1 function is a sufficient condition for A to be so expressed. We do not know if it is necessary. Henceforth, whenever we discuss function algebras as direct limits we shall tacitly assume that A contains a 1:1 function.

Now let A be a function algebra containg a 1:1 function g and let D be the directed set described above.

Define r_α^β : Spec A^β → Spec A^α for $\alpha \leq \beta$ in D as follows:

if $m \in$ Spec A^β, then $r_\alpha^\beta(m) = m$ restricted to A^α.

Then $\{\text{Spec } A^\alpha, r_\alpha^\beta\}$ is an inverse system of compact Hausdorff spaces and continuous maps. Let

r'_α : Spec A → Spec A^α be the map sending each element of Spec A onto its restriction to A^α.

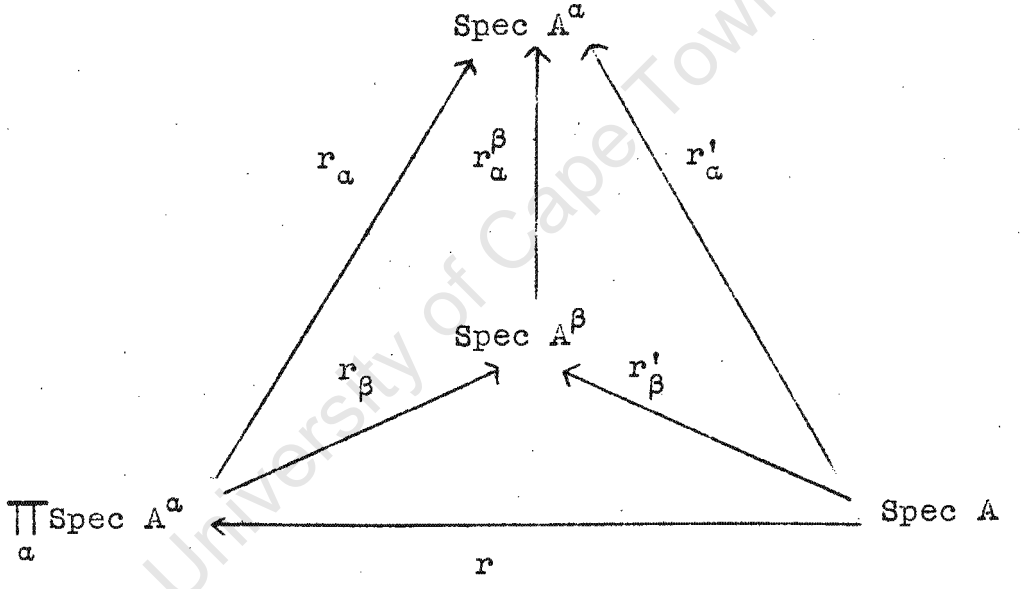
Theorem 4.4: (Royden, [16])

$\{$ Spec A, $r'_\alpha\}$ is an inverse limit of the inverse system

$\{$ Spec A^α, $r^\beta_\alpha\}$.

Proof: Clearly, lim.inv. $\{\text{Spec } A^\alpha, r^\beta_\alpha\} = \prod_{\alpha \in D} \text{Spec } A^\alpha$, since the r^β_α are simply restriction maps, and

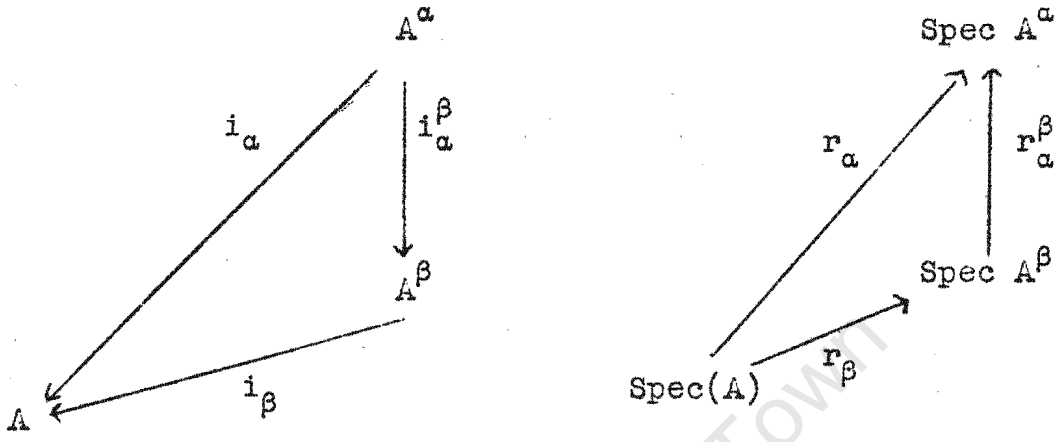
$r_\alpha : \prod_\alpha \text{Spec } A^\alpha \to \text{Spec } A^\alpha$ is simply the α^{-th} projection map.



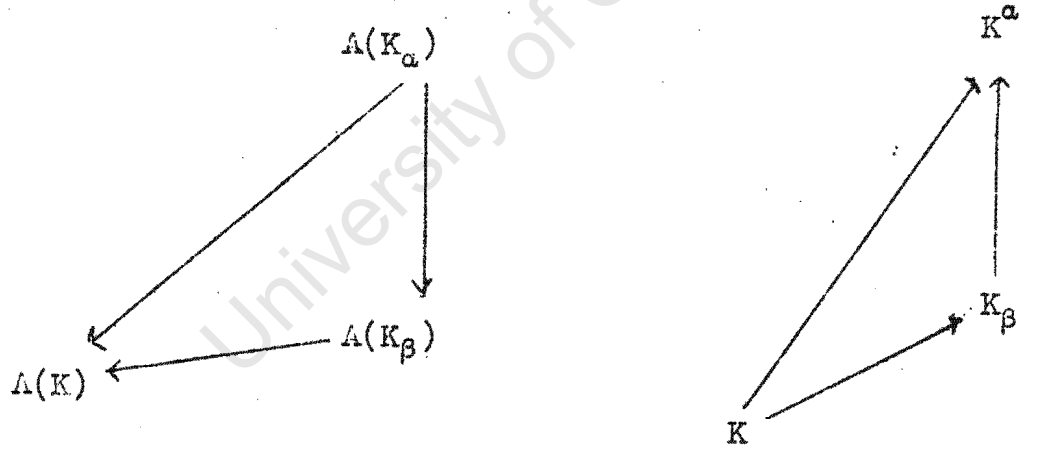
Define $r : \text{Spec } A \to \prod_\alpha \text{Spec } A^\alpha$ by letting the α^{-th} coordinate of r(m) be the restriction of m to A^α, for each m in Spec A. This map is easily seen to be a homeomorphism which makes the above diagram commutative. Since the inverse limit of an inverse system of topological spaces and continuous maps is unique up to commuting homeomorphism, we can consider

$\{$Spec A, $r'_\alpha\}$ as the inverse limit of $\{$Spec A^α, $r^\beta_\alpha\}$.

We now show how we can replace the scheme



by the equivalent scheme



Where $K_\alpha = F(\text{Spec } A^\alpha)$, $K = F(\text{Spec } A)$

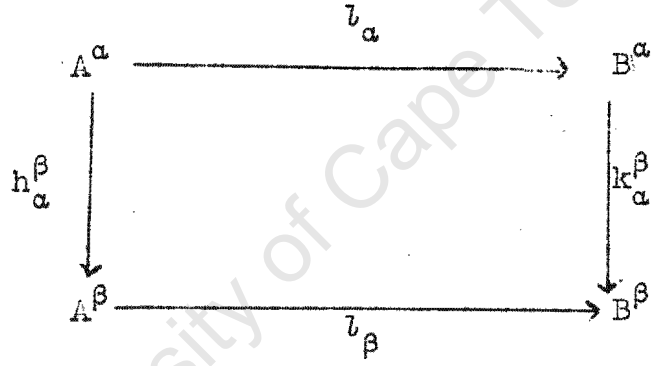
$A(K_\alpha)$ = uniform closure of the algebra of functions holomorphic in a neighbourhood of K_α and

A(K) = uniform closure of the algebra of functions holomorphic in a neighbourhood of K.

Definition 4.5:

Let $\{A^\alpha, h_\alpha^\beta\}$ and $\{B^\alpha, k_\alpha^\beta\}$ be two direct systems of function algebras and algebra homomorphisms with the same directed set D. An isomorphism $\{l_\alpha\}$ from $\{A^\alpha, h_\alpha^\beta\}$ onto $\{B^\alpha, k_\alpha^\beta\}$ is a collection of isomorphisms

$l_\alpha : A^\alpha \to B^\alpha$ such that for $(\alpha,\beta) \in (D \times D)$ with $\alpha \leq \beta$ the following diagram is commutative



Proposition 4.6: Let $\{A^\alpha, h_\alpha^\beta\}$ be a direct system of algebras and algebra homomorphisms with direct limit $\{A, h_\alpha\}$. Suppose that $f^\alpha \in A^\alpha$ and $f^\beta \in A^\beta$. Then $h_\alpha(f^\alpha) = h_\beta(f^\beta)$ iff $h_\alpha^\gamma(f^\alpha) = h_\beta^\gamma(f^\beta)$ for some γ in D such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

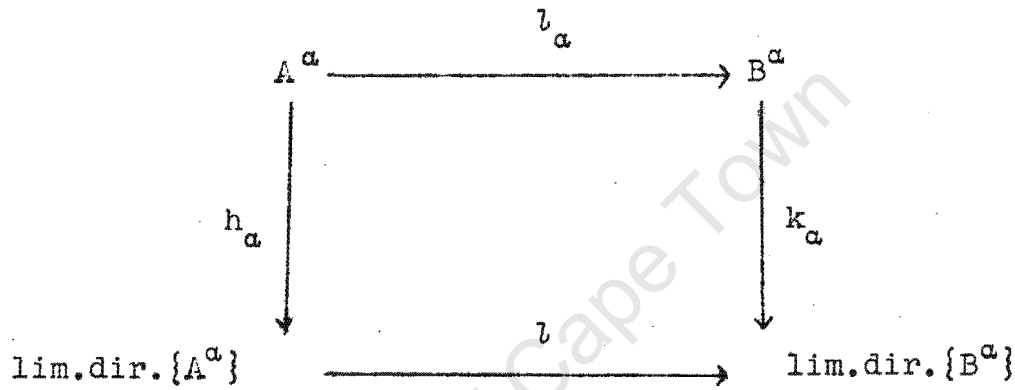
Proof: See [4]. p.212.

Proposition 4.7:

Let $\{A^\alpha, h_\alpha^\beta\}$ and $\{B^\alpha, k_\alpha^\beta\}$ be two direct systems of algebras

and algebra homomorphisms with the same directed set D such that the h_α^β and the k_α^β are monomorphisms.

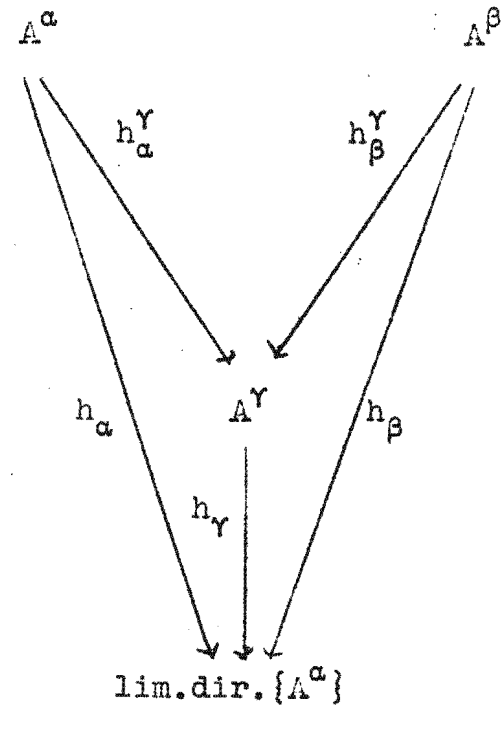
Then any isomorphism $\{l_\alpha\} : \{A^\alpha, h_\alpha^\beta\} \to \{B^\alpha, k_\alpha^\beta\}$ induces an isomorphism l : lim. dir. $\{A^\alpha, h_\alpha^\beta\} \to$ lim. dir. $\{B^\alpha, k_\alpha^\beta\}$ which makes the following diagram commutative for all $\alpha \in D$.

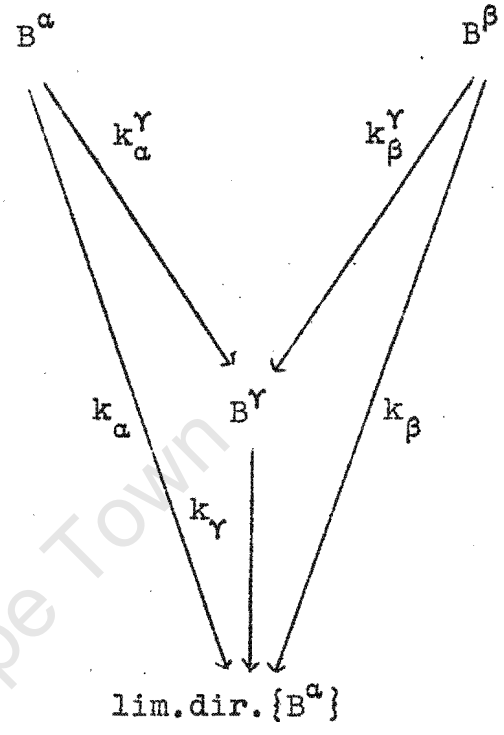


Proof: Clearly, the h_α and the k_α are monomorphisms.

Let $a \in$ lim. dir. $\{A^\alpha, h_\alpha^\beta\}$. Then for some α, $h_\alpha^{-1}(a) \in A^\alpha$.

Define $l(a) = (k_\alpha \circ l_\alpha \circ h_\alpha^{-1})\,(a)$. Such a definition clearly makes the diagram of the statement commutative. We must show, however, that l is well-defined i.e. that $l(a)$ is independent of the choice of α.





Suppose $h_\alpha^{-1}(a) \in A^\alpha$ and $h_\beta^{-1}(a) \in A^\beta$. We must show that

$$(k_\alpha \circ l_\alpha \circ h_\alpha^{-1})(a) = (k_\beta \circ l_\beta \circ h_\beta^{-1})(a).$$

By 4.6, there is a $\gamma \in D$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$ such that

$$(h_\alpha^\gamma \circ h_\alpha^{-1})(a) = (h_\beta^\gamma \circ h_\beta^{-1})(a).$$

Therefore $(l_\gamma \circ h_\alpha^\gamma \circ h_\alpha^{-1})(a) = (l_\gamma \circ h_\beta^\gamma \circ h_\beta^{-1})(a)$

and so $(k_\alpha^\gamma \circ l_\alpha \circ h_\alpha^{-1})(a) = (k_\beta^\gamma \circ l_\beta \circ h_\beta^{-1})(a)$.

Hence $(l_\alpha \circ h_\alpha^{-1})(a) \in B^\alpha$ and $(l_\beta \circ h_\beta^{-1})(a) \in B^\beta$

have a common successor in B^γ, and therefore

$$(k_\alpha \circ l_\alpha \circ h_\alpha^{-1})(a) = (k_\beta \circ l_\beta \circ h_\beta^{-1})(a),$$

which was to be shown.

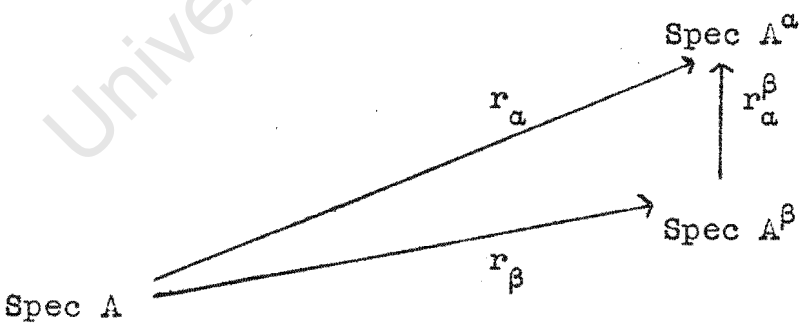
The map l is thus well-defined and is clearly an isomorphism.

Corollary 4.8: Let A be a function algebra containing a 1:1 function g and let $\{A^{\alpha}, i_{\alpha}^{\beta}\}$ be the direct system of subfunction algebras described earlier. Then $\{P(K_{\alpha}), i_{\alpha}^{\beta}\}$ is also a direct system (i_{α}^{β} denotes inclusion in both cases) and A is isometrically isomorphic to lim. dir. $\{P(K_{\alpha})\}$.

Proof: The collection $A^{\alpha} \to P(K_{\alpha})$ is an isomorphism, in fact an isometric isomorphism. The result then follows from the previous proposition.

If we set $K = \prod_{f \in A} R(\hat{f})$ then lim. dir. $\{P(K_{\alpha}), i_{\alpha}^{\beta}\}$ is isometrically isomorphic to P(K), so that 4.8 agrees with our general result that any function algebra A is isometrically isomorphic to P(K).

We now turn to the dual situation of the inverse system



Definition 4.9: Let $\{X_\alpha, r_\alpha^\beta\}$ and $\{Y_\alpha, s_\alpha^\beta\}$ be two systems of compact Hausdorff spaces and continuous maps with the same directed set D. A homeomorphism $\{g_\alpha\}$ from $\{X_\alpha, r_\alpha^\beta\}$ onto $\{Y_\alpha, s_\alpha^\beta\}$ is a collection of homeomorphisms $g_\alpha : X_\alpha \to Y_\alpha$ such that the following diagram is commutative for all $(\alpha, \beta) \in D \times D$ with $\alpha \leq \beta$.

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\quad g_\alpha \quad} & Y_\alpha \\ {\scriptstyle r_\alpha^\beta}\uparrow & & \uparrow{\scriptstyle s_\alpha^\beta} \\ X_\beta & \xrightarrow[\quad g_\beta \quad]{} & Y_\beta \end{array}$$

The proofs on direct systems are easily dualized to yield the following assertions, whose proofs we omit.

Proposition 4.10: Let $\{X_\alpha, r_\alpha^\beta\}$ and $\{Y_\alpha, s_\alpha^\beta\}$ be two inverse systems of compact Hausdorff spaces such that the r_α^β and the s_α^β are homeomorphisms. Then any homeomorphism $\{g_\alpha\} : \{X_\alpha, r_\alpha^\beta\} \to \{Y_\alpha, s_\alpha^\beta\}$ induces a homeomorphism g : lim. inv. $\{X_\alpha\}$ → lim. inv. $\{Y_\alpha\}$ which makes the following diagram commutative for all $\alpha \in D$.

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\quad g_\alpha \quad} & Y_\alpha \\ {\scriptstyle r_\alpha}\uparrow & & \uparrow{\scriptstyle s_\alpha} \\ \text{lim.inv.}\{X_\alpha\} & \xrightarrow[\quad g \quad]{} & \text{lim.inv.}\{Y_\alpha\} \end{array}$$

Theorem 4.11: Let $K_\alpha = F(\text{Spec } A^\alpha)$ and $r_\alpha^\beta : K_\beta \to K_\alpha$ be the projection onto K_α for $\alpha \leq \beta$. Then $\{K_\alpha, r_\alpha^\beta\}$ is an inverse system of compact Hausdorff spaces and continuous maps, and lim. inv. $\{K_\alpha, r_\alpha^\beta\}$ is homeomorphic to Spec A.

Topologies on the spectra of a direct system.

We started off with a function algebra A containing a 1:1 function g on a compact Hausdorff space X, and then formed the direct system of finitely generated sub-function algebras containing g, $\{A^\alpha, i_\alpha^\beta\}$. Since A^α is defined on X, Spec A^α contains X for each $\alpha \in D$. However, we know that Spec A^α has the weak topology induced on it by $\hat{A}^\alpha$, and so the relative topology on X changes as α changes. Since the topology on Spec A, which is homeomorphic to lim. inv. $\{\text{Spec } A^\alpha\}$, is the weak topology induced by $\hat{A}$, which is the direct limit of the system $\{A^\alpha, i_\alpha^\beta\}$, and the latter equals $\bigcup_{\alpha \in D} \hat{A}^\alpha$, it follows that the open sets of Spec A determine the topology of X in such a way that the topology on X is just the union of all the topologies induced on X by the Spec A^α, $\alpha \in D$.

Definition 4.12: Let A be a function algebra on a compact Hausdorff space X.

A is antisymmetric iff f is in A and f is real-valued imply that f is constant.

A is pervasive iff $(A|F)^- = C(F)$ for every proper closed subset F of X.

A is analytic iff the only element of A which vanishes on a nonvoid open set in X is the zero function.

A is essential iff A contains no non-zero ideal of C(X) (this is equivalent to the assertion that there is no proper closed subset F of X such that every continuous function vanishing on F belongs to A).

A is a maximal subalgebra of C(X) iff $A \neq C(X)$ and A is contained properly in no proper closed subalgebra of C(X).

We have the following theorem relating the above properties of a function algebra.

Theorem 4.13:

For a maximal subalgebra A of C(X), the following are equivalent.

(1) A is pervasive

(2) A is analytic

(3) A is an integral domain

(4) A is essential.

Proof: See [4] p.175.

Theorem 4.14: Let A be a function algebra on X containing

a 1:1 function g. Then A is analytic iff each of its sub-function algebras of the form $A^{\alpha} = [g, f_1, \ldots, f_k]$ is analytic.

Proof: Suppose that A is analytic and that $f \in A^{\alpha}$ vanishes on the open set U, which is open in the weak topology on X induced by A^{α}. By the above discussion on the topologies on the spectra of the direct system $\{A^{\alpha}, i_{\alpha}^{\beta}\}$ it follows that U is also open with respect to the topology on X induced by A. Hence f vanishes on X.

Conversely, suppose that A^{α} is analytic for each $\alpha \in D$. Let $f \in A$ vanish on the set U open in the weak A-topology. Then for some $\alpha \in D$, U is open in the weak A^{α}-topology on X and hence f vanishes identically on X.

Corollary 4.15:

If A is a maximal subalgebra of C(X) containing a 1:1 function g, then the following are equivalent.

(1) A is pervasive

(2) A is analytic

(3) A is an integral domain

(4) A is essential

(5) A^{α} is analytic for each $\alpha \in D$.

Proof: The assertion follows from 4.13 and 4.14.

Theorem 4.16: (Identity theorem)

If f(z) and g(z) are holomorphic functions in a connected open

set D contained in $\mathbb{C}^n$ and if $f(z) = g(z)$ for all points z in a nonempty subset U contained in D, then $f(z) = g(z)$ for all points z in D.

Proof: See [3] p.6.

Corollary 4.17: Let D be an open connected subset of $\mathbb{C}^n$, and let f(z) be holomorphic in D. If f vanishes on an open subset U of D then f vanishes identically on D.

Theorem 4.18: Let X be any topological space. The union of any family of connected subsets having at least one point in common is also connected.

Proof: See [17] p.108.

Lemma 4.19: Let $A = [f_1,\ldots,f_n]$ be a finitely generated function algebra on a connected compact Hausdorff space X. Then K = F(Spec A) is contained in a connected neighbourhood of $\mathbb{C}^n$.

Proof: By 2.13 we saw that K itself is a connected subset of $\mathbb{C}^n$. We can cover the topological boundary of K with overlapping connected neighbourhoods whose union, say L, will be connected by 4.18. By the same theorem $K \cup L$ will be a connected neighbourhood of K.

Theorem 4.20: If A is a finitely generated function algebra on a connected compact Hausdorff space X, then A is analytic.

Proof: Suppose that $A = [f_1,\ldots,f_n]$. Then A is isometrically isomorphic to A(K), the uniform closure of all functions holomorphic in a neighbourhood of $K = \prod_{i=1}^{n} \hat{R}(f_i)$. By the previous lemma K is contained in a connected open set L. Suppose f in A vanishes on the open set U in X. To f in A there corresponds a holomorphic function h in A(K) which vanishes on the corresponding set $U' = F(U)$ in K. By choosing a suitable V open in $\mathbb{C}^n$ and intersecting it with K we can assume that U' is also open in L and that by continuity, h vanishes on U'. Hence by the Identity theorem, h vanishes on L and hence on K. Thus f vanishes on X.

The previous theorem can be generalized to function algebras containing a 1:1 function.

Theorem 4.21: Let A be a function algebra containing a 1:1 function g on the compact Hausdorff space X. If X is also connected, then A is analytic.

Proof: A is the direct limit of its finitely generated subfunction algebras $A^\alpha = [g,f_1,\ldots,f_k]$, and Spec A is the inverse limit of the Spec A^α. We claim that for every α, Spec A^α is connected. Suppose not. Then for some α there exists an

open set $U_\alpha \subset \text{Spec } A^\alpha$ such that $\text{Spec } A^\alpha = U_\alpha \cup (\text{Spec } A^\alpha - U_\alpha)$. By Shilov's theorem on idempotents, there exists an $f_\alpha \in A^\alpha$ such that $\hat{f}_\alpha = 0$ on U_α and $\hat{f}_\alpha = 1$ on $\text{Spec } A^\alpha - U_\alpha$. But $\hat{f}_\alpha$ also belongs to $\hat{A}$ and U_α is also a member of the topology on Spec A. This would imply that Spec A is disconnected, a contradiction to the fact that by 2.13 Spec A must be connected. By 4.20 we have that A^α is analytic for each α and by 4.14 it follows that A itself is analytic.

BIBLIOGRAPHICAL NOTES

Chapter I: Gelfand in [19] and [20] originated the study of Banach algebras. Proposition 1.2 and theorems 1.4 and 1.5 are ultimately due to him, though we have followed Wermer's presentation of them in [1]. The early work on the correspondence between function algebras and algebras of polynomials in 1.8 - 1.12 was done by Shilov. See [13] and [1]. Polynomial and rational convexity were first thoroughly investigated by Stolzenberg in [12]. Oka's polynomial approximation theorem 1.15 is actually due to Shilov [13]. Proposition 1.18 is a bit of folklore - the proof we have given is to be found in [6] - and we have used it to prove theorem 1.19. The example of a finitely generated function algebra discussed in 1.20 is to be found in [4], though it appears to be based on a similar example in a more abstract setting due to Rossi [21].

Rickart in [8] recently defined the concept of a holomorphic function of an infinite number of complex variables and generalized many finite-dimensional results to the infinite-dimensional case. Lemmas 1.23 and 1.25 are due to him and he used them to prove his generalized Oka polynomial approximation theorem 1.26. We have used Rickart's results to obtain in proposition 1.27, corollary 1.28 and theorem 1.29 extensions

of earlier results on finitely generated function algebras to the arbitrarily generated case.

Chapter II: Theorem 2.2 was proved by Shilov in [22]. Peak points were first investigated by Bishop [10] and theorem 2.5 was the outcome. In the same paper Bishop gave an elegant constructive proof of his characterization of the minimal boundary, theorem 2.8, which we have rewritten in a different form for finitely generated function algebras in corollary 2.9. Using Bishop's characterization of the minimal boundary and Rickart's lemma 1.25 we have obtained theorem 2.10. Shilov's idempotent theorem 2.11 is another result from [13].

The solution of the Cousin 1 problem on an open polynomial polyhedron (theorem 2.17) is due to Oka [23] who based his work on the earlier efforts of Cousin [24]. The local peak point theorem 2.19 was first proved by Rossi in [11], though we have given Stolzenberg's proof of the same result for the finitely generated case in [12]. Theorem 2.20 is an extension to arbitrary function algebras of theorem 2.19 and was also first proved by Rossi in [11]. However, the proof we have given is based on Rickart's lemma 1.25. Theorems 2.22, 2.23 and 2.24 are the climax of Rossi's seminal paper, [11].We have presented them as in [4].

Chapter III: Lemma 3.2 is one of the principle applications of the local maximum modulus principle, which is often invoked in the investigation of extensions of function algebras. Many authors have studied such extensions. Glicksberg in [25] provided another very useful tool, theorem 3.5. The statements about the Shilov boundaries in theorems 3.6 and 3.8 are also due to him. See [26]. The statement about the maximal ideal space in theorem 3.6 is due to Rickart [27], though the proof given here is an unpublished argument of Quigley. The statement about the maximal ideal space in theorem 3.8 appears to have been discovered first by Björk [28]. Corollary 3.9 is due to Stolzenberg [29]. We have followed Gamelin [9] in our presentation of the above-mentioned results. In a recent paper [18] Gamelin and Wilken extended Stolzenberg's corollary 3.9 to obtain theorem 3.11.

Wilken in [30] investigated under what conditions the maximal ideal space of an extended function algebra is strictly larger than the maximal ideal space of the original algebra and produced theorem 3.12, lemma 3.13 and theorem 3.15. Theorem 3.14 was proved by Lavrentiev [31]. Corollary 3.16 is a partial converse that we have obtained to theorem 3.12.

Theorem 3.18 is an approximation theorem due to Mergelyan [32] and theorem 3.19 was essentially proved by Runge [33]. We have used all the foregoing results of this

chapter to obtain theorem 3.20, much of which was suggested by our supervisor Dr. W. Kotzé. Proposition 3.22 appears as problem 5.4 no. 5 in [4] and we have used it to obtain corollary 3.23.

Chapter IV. Except for proposition 4.3, theorem 4.4, proposition 4.6, theorem 4.13, theorem 4.16, corollary 4.17 and theorem 4.18, the results in this chapter are apparently new. Theorem 4.4 was essentially proved by Royden in [16] and theorem 4.13 is to be found in the paper by Hoffman and Singer [34].

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