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Pricing Options in a Fuzzy Environment



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Department of Statistical Sciences, University of Cape Town, South Africa.

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ABSTRACT

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It is well known that psychological impacts of market participants can affect the prices of both stocks and financial derivatives. These impacts have in the past been difficult to model as traditional methods within probability theory are notoriously hard to quantify events which are not clearly defined. Fuzzy logic on the other hand is suited to model events which are seen as vague.

Although Fuzzy Logic is not new, it is however only since 2004 that an axiomatic theory has been created that has all the desirable effects of Fuzzy Logic. This theory, named Credibility theory was proposed by Dr. Liu. Within this thesis we aim to utilize credibility theory to model the psychological impacts of market participants on European options. Specifically this is done by modifying the approach that was originally taken by Black and Scholes. The new model, which is known as the fuzzy drift parameter model, begins by replacing the deterministic drift within Brownian motion with a fuzzy parameter. This fuzzy parameter models the psychological impacts of market participants. Naturally as we are dealing in Chance theory ¹ the risk neutral dynamics change from that of Black and Scholes and thus so does the price of European call options.

The fuzzy drift parameter model is seen as desirable as it displays leptokurtic, volatility skews and has the ability to model psychological impacts of market participants. In addition to developing a theoretical framework for pricing European options

¹A system which displays both random and fuzzy elements resides within Chance theory.

a method to calibrate this model to the market will be developed as well. Specifically a method to estimate the parameters of the Average Chance Measure will be proposed and implemented.

In this paper we conclude that the fuzzy drift parameter model performs adequately in predicting at-the-money European call options

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set of Θ . A set function is defined as a credibility measure, $\text{Cr}\{\Theta\} : 2^\Theta \rightarrow [0, 1]$, if it satisfies the following four axioms

Axiom 1. (Normality) $\text{Cr}\{\Theta\} = 1$.

Axiom 2. (Monotonicity) $\text{Cr}\{A\} \leq \text{Cr}\{B\}$ whenever $A \subset B$.

Axiom 3. (Self-Duality) $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ for any event $A \in 2^\Theta$.

Axiom 4. (Maximality) $\text{Cr}\{\cup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any events $\{A_i\}$ with $\sup_i \text{Cr}\{A_i\} < 0.5$.

Remark 2 The credibility measure is defined on the largest σ -algebra - the power set of Θ .

Remark 3 The triple $(\Theta, 2^\Theta, \text{Cr})$ is called the credibility measure space.

Often one defines a credibility measure by assigning a credibility to each singleton of a power set, Θ , however is this credibility measure fully and uniquely determined? The next theorem answers this question.^[13]

Theorem 4 (Credibility Extension Theorem) Suppose that Θ is a nonempty set. If Cr is a credibility measure, then we have^[13]

$$\begin{aligned} \sup_{\theta \in \Theta} \text{Cr}\{\theta\} &\geq 0.5 \\ \text{Cr}\{\theta^*\} + \sup_{\theta \neq \theta^*} \text{Cr}\{\theta\} &= 1 \quad \text{if } \text{Cr}\{\theta^*\} \geq 0.5 \end{aligned}$$

Definition 5 (Fuzzy Variable) A fuzzy variable, ξ , is defined to be a measurable function from a credibility space $(\Theta, 2^\Theta, \text{Cr})$ to the set of real numbers.

As in probability theory, in credibility theory we can define the notion of an expectation, a cumulative distribution, independence and variance.^[13]

Definition 6 (*Credibility Distribution*) The credibility distribution $\Phi : \mathbb{R} \rightarrow [0, 1]$ of a fuzzy variable ξ is defined by

$$\Phi(x) = Cr\{\theta \in \Theta | \xi(\theta) \leq x\}$$

Definition 7 (*Credibility Expectation*) Let ξ be a fuzzy variable, and $f : \mathbb{R} \rightarrow \mathbb{R}$ a function. Then the expected value of $f(\xi)$ is

$$\mathbb{E}[f(\xi)] = \int_0^{\infty} Cr\{f(\xi) \geq r\}dr - \int_{-\infty}^0 Cr\{f(\xi) \leq r\}dr$$

provided that one of the integrals is finite.

Definition 8 (*Credibility Independence*) The fuzzy variables $\xi_1, \xi_2, \dots, \xi_m$ are said to be independent if

$$Cr\left\{\bigcap_{i=1}^m \{\xi_i \in B_i\}\right\} = \min_{1 \leq i \leq m} Cr\{\xi_i \in B_i\}.$$

Definition 9 (*Credibility Variance*) Let ξ be a fuzzy variable with finite expected value e . Then the variance of ξ is defined by $\mathbb{V}[\xi] = \mathbb{E}[(\xi - e)^2]$.

In Liu's credibility measure theory the credibility measure is the starting point and the membership function is derived from it. The membership function has desirable properties in that it has intuitive features¹ and provides a link between the fuzzy mathematics of Zadeh and those of Liu's credibility measure theory.

Definition 10 (*Membership Function*) Let ξ be a fuzzy variable defined on the credibility space $(\Theta, 2^\Theta, Cr)$. Then its membership function is derived from the credibility measure by

$$\mu(x) = (2 Cr\{\xi = x\}) \wedge 1, \quad x \in \mathbb{R}.$$

¹The membership function can be seen as a replacement of the indicator function. Thus it can be seen as a measure of the extent to which an element is within a set.

Theorem 11 (*Sufficient and Necessary Condition for Membership Function*) A function $\mu : \mathbb{R} \rightarrow [0, 1]$ is a membership function if and only if $\sup \mu(x) = 1$.

From the above, we see that every credibility measure characterizes a membership function. An important theorem in credibility theory states that every membership function can also be used for defining a credibility measure. A consequence of this theorem is that we can fully describe a fuzzy variable by its membership function. However, the roles of credibility measure and membership function are not the same. The credibility measure gives a measure of uncertainty whereas the membership function gives an indication of the extent an outcome falls within a certain event.

Theorem 12 (*Credibility Inversion Theorem*): Let ξ be a fuzzy variable with a membership function μ . Then for any set B of real numbers, we have

$$\text{Cr}\{\xi \in B\} = \frac{1}{2} \left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x) \right).$$

In this thesis three fuzzy variables will be used, namely the triangular, trapezoidal and normal fuzzy variables. The membership functions for these variables are presented below.

Definition 13 (*Triangular Fuzzy Variable*) A triangular fuzzy variable is fully determined by the triplet (a, b, c) of real numbers with $a < b < c$ and membership given by

$$\mu(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ \frac{x-c}{b-c}, & \text{if } b \leq x \leq c \\ 0, & \text{otherwise} \end{cases}$$

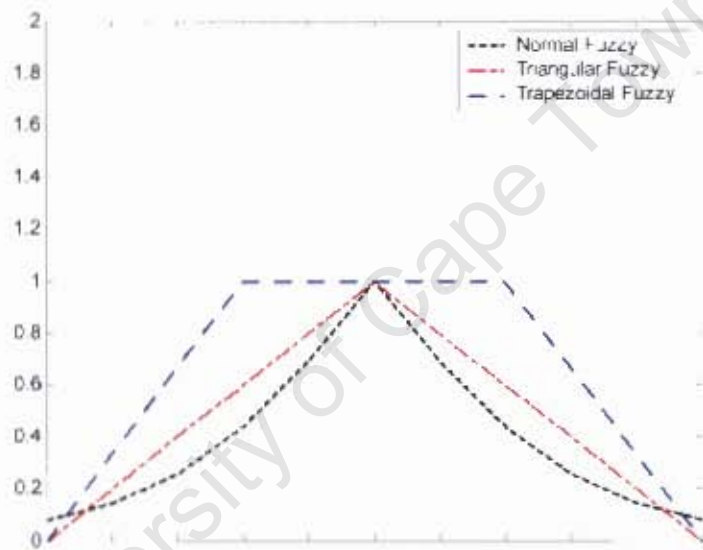


Figure 2.1: Membership functions of a Normal, Triangular and Trapezoidal fuzzy variable all centred around the same point.

Definition 14 (*Trapezoidal Fuzzy Variable*) A trapezoidal fuzzy variable is fully determined by the quadruplet (a, b, c, d) of real numbers with $a < b < c < d$ and membership given by

$$\mu(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c}, & \text{if } c \leq x \leq d \\ 0, & \text{otherwise} \end{cases}$$

Definition 15 (*Normal Fuzzy Variable*) A normal fuzzy variable is fully determined by the double (e, σ_n) of crisp² numbers with $\sigma_n > 0$ and membership given by

$$\mu(x) = 2 \left(1 + \exp \left(\frac{\pi |x - e|}{\sqrt{6} \sigma_n} \right) \right)^{-1}$$

Within this thesis, we will want to compare fuzzy variables. This task can be done by comparing the variance and entropy of various fuzzy variables. Previously we have defined the notion of variance within credibility theory. Entropy within credibility theory is defined as

Definition 16 (*Credibility Entropy*) Let ξ be a continuous fuzzy variable. Then its entropy is defined by

$$H[\xi] = \int_{-\infty}^{\infty} S(Cr\{\xi = x\}) dx$$

where

$$\begin{aligned} S(t) &= -t \ln t - (1-t) \ln(1-t) && \text{if } 0 < t < 1 \\ &= 0 && \text{if } t = 0 \text{ or } t = 1^3 \end{aligned}$$

²A real number is known as crisp

is defined as

$$\{\xi \in B\} = \{(\theta, \omega) \in \Theta \times \Omega \mid \xi(\theta, \omega) \in B\}$$

Remark 19 *Chance theory generalises the notion of a random variable and a fuzzy variable. That is both a random variable and a fuzzy variable can be viewed as a hybrid variable.*

The typical examples of hybrid variables are fuzzy random variables and random fuzzy variables. Guo^{[10][20]} defines a random fuzzy variable as a measurable mapping from the credibility space $(\Theta, 2^\Theta, \text{Cr})$ to a set of random variables. Note that a random fuzzy variable takes real numbers as its values and thus behaves similarly to a random process

Definition 20 *(Guo et al) A random fuzzy variable, denoted as $\xi = \{X_{\beta(\theta)}, \theta \in \Theta\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, \mathcal{A}, \text{Pr})$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, \text{Cr})$.*

Similar to the interpretation of a stochastic process $X = \{X_t, t \in \mathbb{R}^+\}$, a random fuzzy variable is a bivariate mapping from $(\Theta \times \Omega, 2^\Theta \times \sigma)$ to the space $(\mathbb{R}, B(\mathbb{R}))$, where $B(\mathbb{R})$ denotes the Borel- σ algebra on the real numbers. There are many ways for generating a random fuzzy variable. For a simplest example, if η is normally distributed with zero mean and variance σ^2 , i.e. $\eta \sim N(0, \sigma^2)$ and ξ is a fuzzy variable, then $\eta + \xi \sim N(\xi, \sigma^2)$.^[20]

The product measure $\text{Cr} \times \text{Pr}$, which is also known as the chance measure, may take different forms and need only satisfy the axioms of probability and credibility

theory on random and fuzzy variables respectively. The form of our product measure will alter our results and thus a careful choice must be made. Currently there are two logical choices for this measure, the chance measure proposed by Liu and the average chance measure used in this thesis.

Liu's chance measure is based on the logic of looking for the "most probable" or "likely event". Let $\xi(\theta, \varpi)$ be the hybrid variable within which we are working, and $\xi(\theta, \varpi) \in B$, be the event that we are looking to measure. Here θ is a fuzzy variable and ϖ is a random variable. For each fuzzy outcome, $\theta = \theta_0$, there is a credibility measure associated with it. This value of the fuzzy variable, implies a probability, $\Pr\{\xi(\theta_0, \varpi) \in B\}$. The quantity $\Pr\{\xi(\theta_0, \varpi) \in B\}$ relates to the form of our hybrid variable and to our event however it does not take into account how "unlikely" or "improbable" the fuzzy outcome is. A simple modification would be to look at the quantity $\text{Cr}\{\theta = \theta_0\} \wedge \Pr\{\xi(\theta_0, \varpi) \in B\}$.⁴ This quantity takes both the form of the hybrid variable and outcome into account as well as how "unlikely" the fuzzy event is. Liu defines his chance measure as the supremum of this quantity, or in simple terms "the most likely".

Definition 21 (*Chance Measure - Liu*) Let $(\Theta \times \Omega, 2^\Theta \times \mathcal{A}, \text{Cr} \times \text{Pr})$ be a chance space.

⁴This modification is also inspired by the necessity to have the chance measure satisfy the axioms of credibility and probability theory.

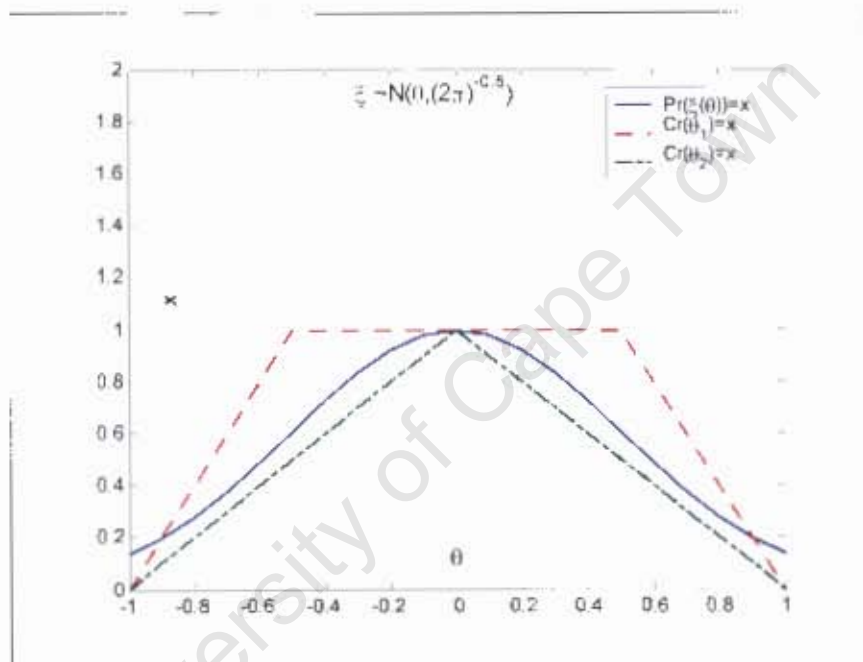


Figure 2.2: Illustration of Liu chance measure not distinguishing between two hybrid variables, that have different input fuzzy variables, for the event $Ch\{\xi(\theta)\} = 0$. Where the hybrid variable, ξ , is distributed normally with mean θ and variance $\frac{1}{\sqrt{2\pi}}$.

Then the chance of an event Γ occurring is defined as

$$\text{Ch}\{\Gamma\} = \begin{cases} \sup_{\theta \in \Theta} (\text{Cr}\{\theta\} \wedge \text{Pr}\{\Gamma(\theta)\}), \\ \text{if } \sup_{\theta \in \Theta} (\text{Cr}\{\theta\} \wedge \text{Pr}\{\Gamma(\theta)\}) < 0.5 \ ; \\ 1 - \sup_{\theta \in \Theta} (\text{Cr}\{\theta\} \wedge \text{Pr}\{\Gamma^c(\theta)\}) \\ \text{otherwise} \end{cases}$$

Liu's chance measure has the disadvantage of being computationally difficult.

It is not fully intuitive as the chance measure is determined by one point of the fuzzy variable, θ_0 . It thus disregards the shape of the membership function of θ . Figure (2.2) illustrates this point. An approach that would take the shape of the membership function into account would be to take the expectation of $\text{Pr}\{\xi(\theta, \varpi) \in B\}$. The quantity, $\text{Pr}\{\xi(\theta, \varpi) \in B\}$, depends on the fuzzy variable θ and thus the expectation must be interpreted in a credibilistic sense.

Definition 22 (Average Chance Measure) *Let $(\Theta \times \Omega, 2^\Theta \times \mathcal{A}, \text{Cr} \times \text{Pr})$ be a chance space and let $\xi(\theta, \varpi)$ be a hybrid variable, where θ is an absolutely continuous fuzzy variable and ϖ a random variable. Then the chance of an event $\xi(\theta, \varpi) \in B$ occurring is defined as*

$$\begin{aligned} \text{Ch}\{\xi(\theta, \varpi) \in B\} &= E_{\text{Cr}}[\text{Pr}\{\xi(\theta, \varpi) \in B\}] \\ &= \int_0^1 \text{Cr}\{\theta \in \Theta \mid \text{Pr}\{\xi(\theta, \varpi) \in B\} > \alpha\} d\alpha \end{aligned} \quad (2.1)$$

Remark 23 *The average chance measure is not in general sub-additive.*

Remark 24 *The average chance measure is only defined on fuzzy variables that have continuous membership functions. We thus cannot define an average chance measure*

for a hybrid variable that utilises an equipossible fuzzy variable.⁵

Remark 25 *The average chance measure can only be defined on a hybrid variable that depends on one continuous fuzzy variable and one random variable. Liu's chance measure however can be applied to a hybrid variable that depends on multiple fuzzy variables.*

The average chance measure has the added advantage that it is often possible to calculate closed forms for the chance of events happening. In this thesis, the hybrid variable that will be used is normally distributed with a fuzzy mean. The average chance measure for this hybrid variable under the relevant events has a closed form. In chance theory we define the notion of an expectation as

Definition 26 (*Chance Expectation*) *Let ξ be a hybrid variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{\infty} \text{Ch}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Ch}\{\xi \leq r\} dr$$

provided that one of the integrals is finite.

According to Liu, independence in chance theory is defined as^[13]

Definition 27 (*Chance Independence - Liu*) *The hybrid variables $\xi_1, \xi_2, \dots, \xi_n$ are said*

⁵The membership function for the equipossible variable is given by

$$\mu(x) = \begin{cases} 1, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

to be chance independent if

$$\mathbb{E} \left[\sum_{i=1}^n f_i(\xi_i) \right] = \sum_{i=1}^n \mathbb{E} [f_i(\xi_i)]$$

Remark 28 *This definition has the desirable properties that a random and fuzzy variable are chance independent and that two independent fuzzy variables are chance independent. However it also has the consequence that all random variables are chance independent.*

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CHAPTER 3.

STANDARD OPTION PRICING THEORY

3.1 Background

Options are derivative instruments¹ that grant the holder, the right but not the obligation to enter into some transaction in the future on a specified underlying security. Typically a transaction will involve the purchase or sale of the underlying security at a specified price (known as the strike) at or up to a specified time in the future (known as the maturity). An option granting the right to purchase a specified security is known as a call option whereas an option granting the right to sell a specified security is known as a put option. Originally the underlying security of options were stocks. However now they include stocks, indices, foreign currencies, commodities and even prevailing weather conditions. There is great flexibility in the specification of an option and the only limit on whether a specific type of option can be traded is whether a willing counter party to the trade can be found.

There are two basic styles of options, European and American. European options can only be exercised at maturity whereas American options can be exercised any time prior to the maturity date. In this thesis, we will be concerned with European call and put options. In 1973 Black and Scholes published^[2] their model on option pricing. This paper gave a method of pricing European options which has subsequently become the industry benchmark. This chapter will give the necessary background and derivation

¹Hull^[1] defines a derivative as " ... a financial instrument whose value depends on the value of other, more basic underlying variables.

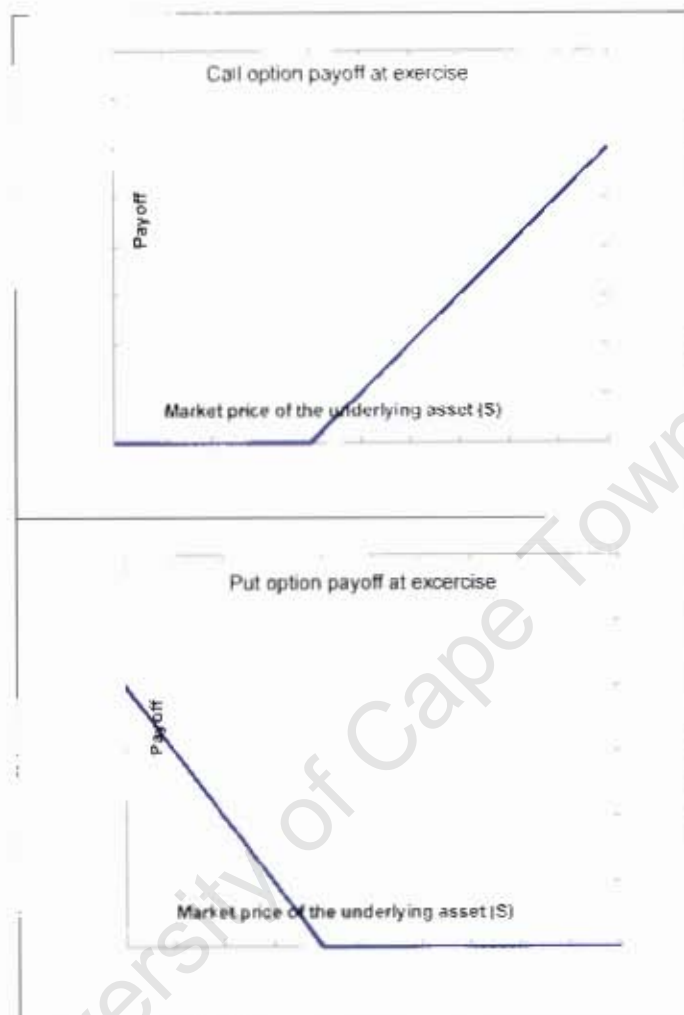


Table 3.1: Payoff functions at maturity to the holder of a European call and put option.

of the Black-Scholes formulae for pricing options and the subsequent method of risk-neutral valuation.

3.2 The Black-Scholes Model

The Black-Scholes model makes several ideal assumptions about the market.^{[2][11]}

1. The market is frictionless: there are no transaction costs or taxes and information is distributed freely throughout the market.

2. Short selling is permitted and there is no penalty or cost for doing so.
3. There are no dividends paid during the life of the option.
4. There are no arbitrage opportunities.
5. Assets are perfectly divisible.
6. The market trades continuously with respect to time. Thus there are no discrete jumps or "crashes" in the market.
7. The risk-free interest rate remains constant over the life of the option and it is possible to lend and borrow at this rate.
8. The underlying asset follows a geometric Brownian motion², with pre-deterministic drift μ and constant volatility σ .

$$\frac{dS}{S} = \mu dt + \sigma dW_t \quad (3.1)$$

The assumption that the underlying asset follows geometric Brownian motion is consistent with the efficient market hypothesis³. It also implies that the continuous return of the asset is normally distributed and thus that the drift and volatility of the underlying asset are probability independent of the current price. The above assumptions ensure that the market is "ideal", in that there are no complicating factors.

²See Appendix A: Theorems from Stochastic Finance. for a precise definition of Brownian motion.

³The weak form of the efficient market hypothesis states that stock prices reflect all information known to the market. A consequence of this assumption is that all changes in price are random. This assumption is equivalent mathematically to saying that stock prices are a Markov processes.

$$\mathbb{E}[S_T | S_t \forall t \leq k] = \mathbb{E}[S_T | S_k]$$

Geometric Brownian Motion is a Markov Process.

There are two distinct approaches to the pricing of options in financial mathematics. One approach is to construct a PDE based on arbitrage arguments and the other is to use the probabilistic argument of risk-neutral valuation. For the sake of completeness both views will be presented.

3.3 Black-Scholes PDE

A major problem in dealing with option pricing is that the asset dynamics (3.1) are uncertain. The dW_t term introduces an uncertainty into our model, which is problematic as each person perceives the risk from uncertainty differently. For instance one person may be willing to take on far more risk from uncertainty than another. Black and Scholes noted that if we could negate the uncertainty introduced from the dW_t term then it would be possible to obtain an option price that everyone could agree on.

We follow the original derivation of the Black-Scholes PDE by Black and Scholes.^[2] Suppose we have a portfolio Π that is short one derivative, V , and long Δ shares, S . Then the value of the portfolio is

$$\Pi(S, t) = -V(S, t) + \Delta S(t)$$

The instantaneous change in the portfolio is

$$d\Pi = -dV + \Delta dS$$

¹According to the Ito's lemma for differentiating stochastic processes, we should have

$$d\Pi = -dV + \Delta dS + S(d\Delta) + (d\Delta)(dS)$$

We can get around this if we assume that our portfolio is self financing and thus that $S(d\Delta) + (d\Delta)(dS) = 0$.

By Ito's lemma we have that the instantaneous change in the derivative is

$$dV = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t$$

and thus the instantaneous change of our portfolio is

$$d\Pi = - \left(\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial S} - \Delta \right) \mu S + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - \left(\frac{\partial V}{\partial S} - \Delta \right) \sigma S dW_t$$

Notice that if we let $\Delta = \frac{\partial V}{\partial S}$, then the risky part of our portfolio is eliminated.

This strategy however means that we must constantly be holding $\frac{\partial V}{\partial S}$ number of shares.

In order to calculate $\frac{\partial V}{\partial S}$ we require that μ and σ are pre-deterministic. As $\frac{\partial V}{\partial S}$ is continuously changing and thus the number of shares we are holding is continuously changing, we also require that the market is frictionless. By letting $\Delta = \frac{\partial V}{\partial S}$ we obtain

$$d\Pi = - \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

Our portfolio is now riskless and, by the no arbitrage assumption, must earn the risk-free rate.

$$\begin{aligned} d\Pi &= - \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \\ &= r\Pi dt \\ &= r \left(-V + \frac{\partial V}{\partial S} S \right) dt \end{aligned}$$

Rearranging we obtain the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (3.2)$$

The Black-Scholes PDE is a linear, 2nd order, parabolic diffusion equation. In our derivations we have not specified what type of derivative V is. In the case of V being a European call option, we have the final pay-off condition

$$V(S, T) = \max\{S - K, 0\} = (S - K)^+$$

where K is the strike price. Similarly in the case that V is a European put option, we have the final pay-off condition

$$V(S, T) = \max\{K - S, 0\} = (K - S)^+$$

where T represents the time at maturity. Solving for V_0 under the relevant final pay-off condition will give the price of the desired option.

3.4 Risk-Neutral Valuation

In this section a brief introduction to Risk-Neutral valuation is presented. For further reference and proofs regarding Risk-Neutral valuation see Bingham and Kiesel.^[1] The relevant theorems needed for this section are stated in Appendix A.

Risk-Neutral valuation approaches the problem of option pricing from a probabilistic view. The dynamics of (3.1) are specified under the "objective" or real world measure \mathbb{P} . Pricing options under these dynamics are problematic as one has to consider the risk preference of each market participant. However using standard results from probability theory, it is possible to change the probability measure under which the assets are specified. Specifically it is possible to change the real world measure \mathbb{P} to

an equivalent martingale measure (EMM) \mathbb{Q} where the option being priced is equal to the discounted expected payoff. The rationale for making this transformation is that the drift of the asset dynamics will change to the risk-free rate r . Under these new dynamics we do not need to consider the risk preference of each market participant as all assets have the same expected return.

In order for a derivative to equal the discounted expected payoff under \mathbb{Q} we require a number of conditions. Firstly we require that the discounted asset prices are martingales ⁵ under \mathbb{Q} . We also require that the market is complete. ⁶ The Martingale Representation Theorem implies that an EMM exists and by the fact that the market is complete we know that the EMM is unique.^[1] Under these conditions, the principle of Risk-Neutral valuation is applicable. That is we have

$$V_0 = \mathbb{E}_{\mathbb{Q}} [e^{-rT} V_T] \quad (3.3)$$

where V is the derivative we are trying to price and \mathbb{Q} is our unique EMM. We can change the dynamics of (3.1) from the real world measure \mathbb{P} , to the EMM measure \mathbb{Q} by an application of the Girsanov theorem. We begin with the dynamics of (3.1)

$$\frac{dS}{S} = \mu dt + \sigma dW_t$$

⁵A martingale is a process whose expected value at a future date, conditional on the current information, is its current value.

$$e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [S_T | F_t] = S_t$$

⁶A market is complete if every contingent claim is attainable. A contingent claim is a security whose value is a random variable. In our case, the contingent claim is the option being priced.

Applying the Girsanov theorem to these dynamics we obtain the \mathbb{Q} dynamics

$$\begin{aligned}\frac{dS}{S} &= \mu dt + \sigma \left(d\widetilde{W}_t - \lambda dt \right) \\ \frac{dS}{S} &= (\mu - \sigma\lambda)dt + \sigma d\widetilde{W}_t\end{aligned}$$

We require the discounted assets of S under \mathbb{Q} to be a martingale. Thus by letting $\lambda = \frac{\mu-r}{\sigma}$ ⁷ we obtain

$$\frac{dS}{S} = rdt + \sigma d\widetilde{W}_t \quad (3.4)$$

These dynamics are called risk-neutral as the drift is equal to the instantaneous risk-free rate r and thus investors have no additional compensation for taking on risk.

Calculating an European Call Option :

In order to calculate a European option, we first need to know the distributional properties of S under our risk-neutral dynamics. By Ito's lemma we obtain

$$\ln S_t \sim N \left(\ln S_0 + \left(r - \frac{1}{2}\sigma^2 \right) t + \sigma \widetilde{W}_t \right)$$

Thus S is distributed log-normally. By the principle of risk-neutral valuation and the fact that \mathbb{Q} is a martingale measure

$$\begin{aligned}C_0 &= e^{-rT} \mathbb{E}_{\mathbb{Q}} [\max\{S - K, 0\}] \\ &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\infty} (e^{\ln S_T} - K)^+ \exp \left(\frac{-(\ln S_T - (\ln S_0 + (r - \frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T} \right) d \ln S_T \\ &= \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}} \int_{\ln k}^{\infty} (e^u - K)^+ \exp \left(\frac{-(u - (\ln S_0 + (r - \frac{1}{2}\sigma^2)T))^2}{2\sigma^2 T} \right) du\end{aligned}$$

⁷ $\lambda = \frac{\mu-r}{\sigma}$ is known as the "market price of risk".

which can be simplified to

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2) \quad (3.5)$$

where $\Phi(\cdot)$ represents the standard normal cumulative density function and

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad (3.6)$$

$$d_2 = d_1 - \sigma\sqrt{T} \quad (3.7)$$

Equation (3.5) to (3.7) form the Black-Scholes formula for the price of a European call option on an asset that pays no dividends over the life of the option. In a similar fashion to the above development or by the put-call parity condition ⁸ we can obtain the Black-Scholes price for a European put option

$$P_0 = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1) \quad (3.8)$$

⁸The put-call parity condition states that

$$C - P = S - PV(K)$$

where C represents the price of a call, P represents the price of a put, S is the current price of the underlying asset and $PV(K)$ is the present value of the strike.

CHAPTER 4.

FUZZY DRIFT PARAMETER MODEL

4.1 Introduction

The Black-Scholes model represented a great breakthrough in option pricing by eliminating an individual's risk preference from the pricing formula. It does however have several limitations. If asset prices were to follow the dynamics of geometric Brownian motion (3.1) then continuous returns should be normally distributed. However numerous empirical studies^{[4][7]} have found that the market distribution has a leptokurtic feature. In other words the return distribution has a higher peak, is skewed to the left and has two fatter tails than those of the normal distribution. Another empirical inconsistency arises from the implied volatility of market options. The Black-Scholes formula has a one-to-one relationship with the volatility of the underlying asset. Thus given the price of an European option, we can calculate the implied volatility of this option. If the Black-Scholes formula is correct then the implied volatility should be constant. However empirical studies^[4] have found that the volatility curve represents a "skew", in other words it is a convex curve of the strike price.

A number of researchers have suggested alternative models based on the Black-Scholes setting that would correct for these empirical phenomenon. Several methods have been to argue that the volatility is not constant. These methods include the GARCH models,^[11] constant elasticity of variance model (CEV)^[11] and the stochastic volatility models.^[1] An assumption that is often overlooked in the literature is that of the drift being pre-deterministic. In simple terms this assumption translates to being

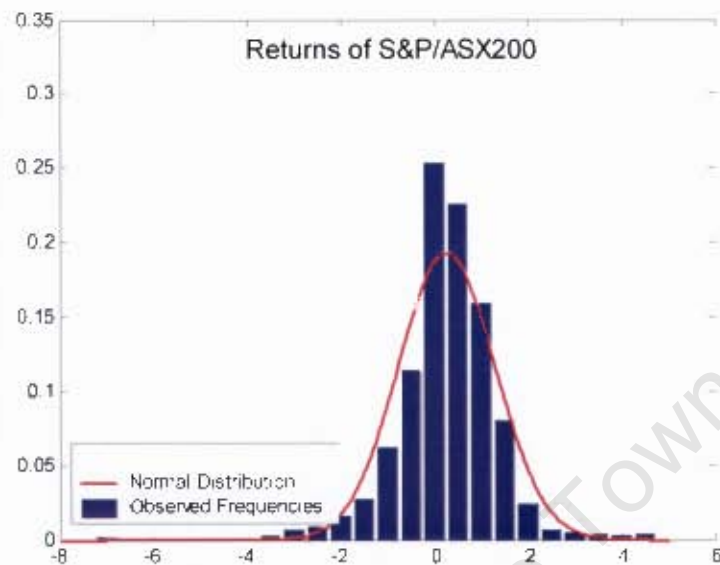


Figure 4.1: *Returns of the S&P/ASX200 illustrating the departure from normality.*

able to calculate at any instant what the drift currently is. The efficient market hypothesis stipulates that past performances are no guarantee for future performance. Thus arguing that the historical drift should be the same as the current drift would be folly. One could argue that the current drift could be calculated by current information based on economic, financial and political situations. This argument however assumes that each person will interpret subjective information in the same way. Studies by Wärneryd^[23] have shown the intuitive result that this is not the case. Market participants are prone to being influenced by their emotions. The most common emotion being that of social herding.^[23] In a bull market, market participants are prone to over-reacting to good news and under-reacting to bad. The opposite is true in a bear market.

Fuzzy variables are suited to describe variables where each event is not clearly

distinguishable. At any given moment, it is not clear what value the drift is, as this value depends on the interpretation of the person measuring it. A simple method to try and model the psychological impact of market participants on the assets dynamics is to replace the constant pre-deterministic drift μ with a fuzzy variable η .

$$\frac{dS}{S} = \eta dt + \sigma dWt \quad (4.1)$$

The above model will be named the fuzzy drift parameter model as the drift has been replaced by a fuzzy variable η .

In this chapter, European call options will be priced within the framework of the fuzzy drift parameter model and compared to those derived in the Black-Scholes framework.

4.2 Derivation of Risk-Neutral Dynamics

The dynamics of (4.1) has solution^[13]

$$S(T) = S(0) \exp \left(\left(\eta - \frac{\sigma^2}{2} \right) T + \sigma W(T) \right). \quad (4.2)$$

$S(T)$ is thus a random fuzzy variable and we will be working within the context of chance theory. We recall the definition of a chance space to be the product space of a credibility space and a probability space. $(\Theta, 2^\Theta, Cr) \times (\Omega, \mathcal{A}, Pr)$. As credibility space is defined on the power set, no filtration can be defined on chance space. As such there is no martingale theory and thus no risk neutral pricing theory. In the event that η is a crisp number, our chance space is isomorphic to our probability space and thus our

results should be identical to those of Black and Scholes. In order to generalise the results of Black and Scholes we will naively define the price of a contingent claim as ¹

Definition 29 (*Risk Neutral Valuation under chance theoretical platform*) Suppose X is an attainable contingent claim, and that \mathbb{Q} is a martingale measure. Then

$$X_0 = \mathbb{E}_{\mathbb{Q}} [e^{-rT} X_T] \quad (4.3)$$

Remark 30 By martingale measure, it is meant $S_0 = \mathbb{E}_{\mathbb{Q}} [e^{-rT} S_T]$

In order to apply the definition of risk neutral valuation within chance theory, we need to change the real world dynamics of (4.1) to something that is risk-neutral. Within chance theory no theorem exists that will allow us to define change measures. ² In order to get around this limitation, we will assume that we can apply the Girsanov theorem to our dynamics (4.1) and apply a heuristic derivation. Recall the definition of the average chance measure

$$\text{Ch}\{\xi(\eta, \varpi) \in B\} = \int_0^1 Cr\{\eta_1 \in \eta \mid \text{Pr}\xi(\eta_1, \varpi) \in B\} > d\alpha$$

In calculating the quantity $\text{Pr}\xi(\eta_1, \varpi) \in B$, the fuzzy variable η_1 is constant. As η_1 is constant when calculating $\text{Pr}\xi(\eta_1, \varpi) \in B$, we can apply the Girsanov theorem for each $\eta_1 \in \mathbb{R}$. For a specific $\eta_1 \in \mathbb{R}$ the dynamics of (4.1) are

$$\frac{dS}{S} = \eta_1 dt + \sigma dWt \quad (4.4)$$

¹An alternative development by the Liu school has suggested that we price an option as the discounted expected pay off under the real world dynamics (4.1). This approach however is inconsistent with the Black Scholes formula when our fuzzy variable η reduces to a crisp number.

²Specifically there is no Radon-Nikodym theorem.

where all our parameters are constant. We change the Brownian motion by application of the Girsanov theorem to

$$\widetilde{W}_t = W_t - \frac{\eta_2 - r}{\sigma} dt$$

where all parameters are constant and r represents the risk free rate. Substituting this form into (4.4), we obtain

$$\frac{dS}{S} = r + (\eta_1 - \eta_2)dt + \sigma dW_t$$

Usually here one would choose $\eta_1 = \eta_2$ to obtain something that is risk neutral within the framework of probability theory. In the original Black-Scholes setting, this choice is possible as the drift is assumed to be pre-deterministic. We have made the assumption that the true drift is not pre-deterministic, constant and lies somewhere within a pre-described range with credibility derived from its membership function. Each market participant assigns some value to the drift which need not be the same as the true drift. Within our model the membership function of the true drift is assumed to be known to all participants. Thus it is intuitive that the market participants' assigned drift be a fuzzy variable with the same membership function as that of the true drift. In the above η_1 represents the value of the drift that a market participant chooses and η_2 represents the value of the true drift. These substitutions can be made for all values of $\eta_1, \eta_2 \in \mathbb{R}$. As such it is claimed that the "risk neutral" dynamics in the fuzzy drift parameter model is

Name	Parameters of membership for η	Parameters of membership for $\Delta\eta$
Triangle	(a, b, c)	$(-(c - a), 0, (c - a))$
Trapezoidal	(a, b, c, d)	$(-(d - a), -(c - b), (c - b), (d - a))$
Normal	(e, σ_n)	$(0, 2\sigma_n)$

Table 4..1: Membership functions for symmetrical fuzzy variables

$$\frac{dS}{S} = (r + \Delta\eta)dt + \sigma dW_t \quad (4.5)$$

Here $\Delta\eta = \eta_1 - \eta_2$ and η_1, η_2 have the same membership function. Under these dynamics the underlying asset S is a martingale as the fuzzy variable $\Delta\eta$ is independent of the other variables and the expected value of $\Delta\eta$ is 0³. The heuristic derivation above assumes that we can apply the Girsanov theorem for each η_1 and that the membership function of the market participant is that of the true drift. An alternative development of option pricing is to assume that the psychological impact of market participants affects the risk neutral dynamics of probability theory directly. In this case the dynamics of (4.5) are assumed. Table 4.1 illustrates the various membership functions of $\Delta\eta$. For proof see Appendix B: Fuzzy Variables.

4.3 Pricing an European Call Option

The payoff of a call option at maturity, T , is defined to be $(S(T) - K)^+$. From the principle of risk neutral valuation in chance theory, we have that the price of a European call option is

³The expectation of $\Delta\eta$ is 0 as the fuzzy variable $\Delta\eta$ is symmetric. For the formula of expectations for fuzzy variables. see Liu.^[13]

$$\begin{aligned}
C_0 &= e^{-rT} \mathbb{E}_{\mathbb{Q}} [(S(T) - K)^+] \\
&= e^{-rT} \int_0^{\infty} \text{Ch} \left\{ S(0) \exp \left[\left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T + \sigma W(T) \right] - K \geq \alpha \right\} d\alpha \\
&= e^{-rT} S(0) \int_{\frac{K}{S(0)}}^{\infty} \text{Ch} \left\{ \exp \left[\left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T + \sigma W(T) \right] \geq u \right\} du \\
&= e^{-rT} S(0) \int_{\frac{K}{S(0)}}^{\infty} \text{Ch} \{ \xi \geq u \} du
\end{aligned}$$

where we have used the definition of expectation under chance theory and

$$\xi = \exp \left[\left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T + \sigma W(T) \right]$$

For our purposes , a call price is modified to

$$C_0 = \min \left\{ e^{-rT} S(0) \int_{\frac{K}{S(0)}}^{\infty} \text{Ch} \{ \xi \geq u \} du, S(0) \right\} \quad (4.6)$$

The justification for the above modification is that a call option cannot be priced above the initial stock price. Thus if our expectation turns out to be more than this, we merely take the initial stock price as our call price. The average chance measure for the event $\text{ch} \{ \xi \geq u \}$ is

$$1 - \text{ch} \{ \xi \leq u \} = 1 - \int_0^1 \text{Cr} \{ \Pr \{ \xi \leq u \} \geq \alpha \} d\alpha \quad (4.7)$$

In the above we first evaluate the probability, $\Pr \{ \xi \leq u \}$.

$$\begin{aligned}
\Pr \{ \xi \leq u \} &= \Pr \{ \ln \eta \leq \ln u \} \\
&= \Pr \left\{ \left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T + \sigma W(T) \leq \ln u \right\} \\
&= \Phi \left(\frac{\ln u - \left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right)
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable. Next we concentrate on obtaining the expression for $\text{Cr}\{\cdot\}$ within the integral. To do this we must solve the inequality, $\Pr \{ \xi \leq u \} \geq \alpha$, with respect to fuzzy variable $\Delta\eta$

$$\begin{aligned}
&\left\{ \Delta\eta : \Phi \left(\frac{\ln u - \left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right) \geq \alpha \right\} \\
&\Leftrightarrow \left\{ \Delta\eta : \frac{\ln u - \left((r + \Delta\eta) - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \geq \Phi^{-1}(\alpha) \right\} \\
&\Leftrightarrow \left\{ \Delta\eta : \Delta\eta \leq \frac{\ln u}{T} - r + \frac{\sigma^2}{2} - \frac{\sigma}{\sigma \sqrt{T}} \Phi^{-1}(\alpha) \right\}
\end{aligned}$$

Thus we have that

$$\Psi(u) = \text{ch} \{ \xi \leq u \} = \int_0^1 \text{Cr} \left\{ \Delta\eta : \Delta\eta \leq \frac{\ln u}{T} - r + \frac{\sigma^2}{2} - \frac{\sigma}{\sqrt{T}} \Phi^{-1}(\alpha) \right\} d\alpha \quad (4.8)$$

which is the average chance distribution for a log normal random variable with a known and fuzzy mean. By making the substitution of $x = \frac{\ln u}{T} - r + \frac{\sigma^2}{2}$ and $\sigma_* = \frac{\sigma}{\sqrt{T}}$ the above is transformed to

$$\text{ch} \{ \xi \leq u \} = \int_0^1 \text{Cr} \{ \Delta\eta : \Delta\eta \leq x - \sigma_* \Phi^{-1}(\alpha) \} d\alpha$$

Type of Fuzzy Variable	Average chance distribution of $\varsigma \sim N(\eta, \sigma^2)$.
Triangular with parameters (a, b, c)	$\frac{x-a}{2(b-a)} \left(\Phi \left(\frac{x-a}{\sigma} \right) - \Phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{x+c-2b}{2(c-b)} \left(\Phi \left(\frac{x-b}{\sigma} \right) - \Phi \left(\frac{x-c}{\sigma} \right) \right)$ $+ \Phi \left(\frac{x-c}{\sigma} \right) + \frac{\sigma}{2(b-a)} \left(\phi \left(\frac{x-a}{\sigma} \right) - \phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{\sigma}{2(c-b)} \left(\phi \left(\frac{x-b}{\sigma} \right) - \phi \left(\frac{x-c}{\sigma} \right) \right)$
Trapezoidal with parameters (a, b, c, d)	$\frac{x-a}{2(b-a)} \left(\Phi \left(\frac{x-a}{\sigma} \right) - \Phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{x+d-2c}{2(d-c)} \left(\Phi \left(\frac{x-c}{\sigma} \right) - \Phi \left(\frac{x-d}{\sigma} \right) \right)$ $+ \frac{1}{2} \left(\Phi \left(\frac{x-b}{\sigma} \right) - \Phi \left(\frac{x-c}{\sigma} \right) \right) + \Phi \left(\frac{x-d}{\sigma} \right)$ $+ \frac{\sigma}{2(b-a)} \left(\phi \left(\frac{x-a}{\sigma} \right) - \phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{\sigma}{2(d-c)} \left(\phi \left(\frac{x-c}{\sigma} \right) - \phi \left(\frac{x-d}{\sigma} \right) \right)$
Normal with parameters (e, σ_n)	$\Phi \left(\frac{x-e}{\sigma} \right) - \int_{-\infty}^{\frac{x-e}{\sigma}} \frac{\phi(s)ds}{1+\exp\left(\frac{-\pi(\sigma S-x+e)}{\sqrt{6}\sigma_n}\right)} + \int_{\frac{x-e}{\sigma}}^{\infty} \frac{\phi(s)ds}{1+\exp\left(\frac{\pi(\sigma S-x+e)}{\sqrt{6}\sigma_n}\right)}$

Table 4..2: (Formulas) Average chance distribution for a normal random variable with fuzzy mean and known variance.

which is the average chance distribution for a normal random variable with fuzzy mean, $\Delta\eta$, and known variance σ_n^2 . In this case, the average chance distribution can be calculated. Tables 4.2 and 4.3 illustrate the various average chance distributions.⁴ By the relevant average chance distributions given in Table (4.2), it is possible to calculate the event in (4.7) and thus to numerically calculate the price of a call option price as defined in (4.6).

Numerical integrations were done in MATLAB using the *quad* command which utilises the adaptive Simpson quadrature technique. When calculating the price of an option as defined in (4.6), one formally has to integrate from $\frac{K}{S(0)}$ to ∞ . This task poses a problem as MATLAB can only do finite integrations. The event as defined in (4.7) decreases rapidly however and in this thesis we approximate the price of an option by

⁴The calculations for the average chance distributions can be found in Appendix C: Average Chance Measures.

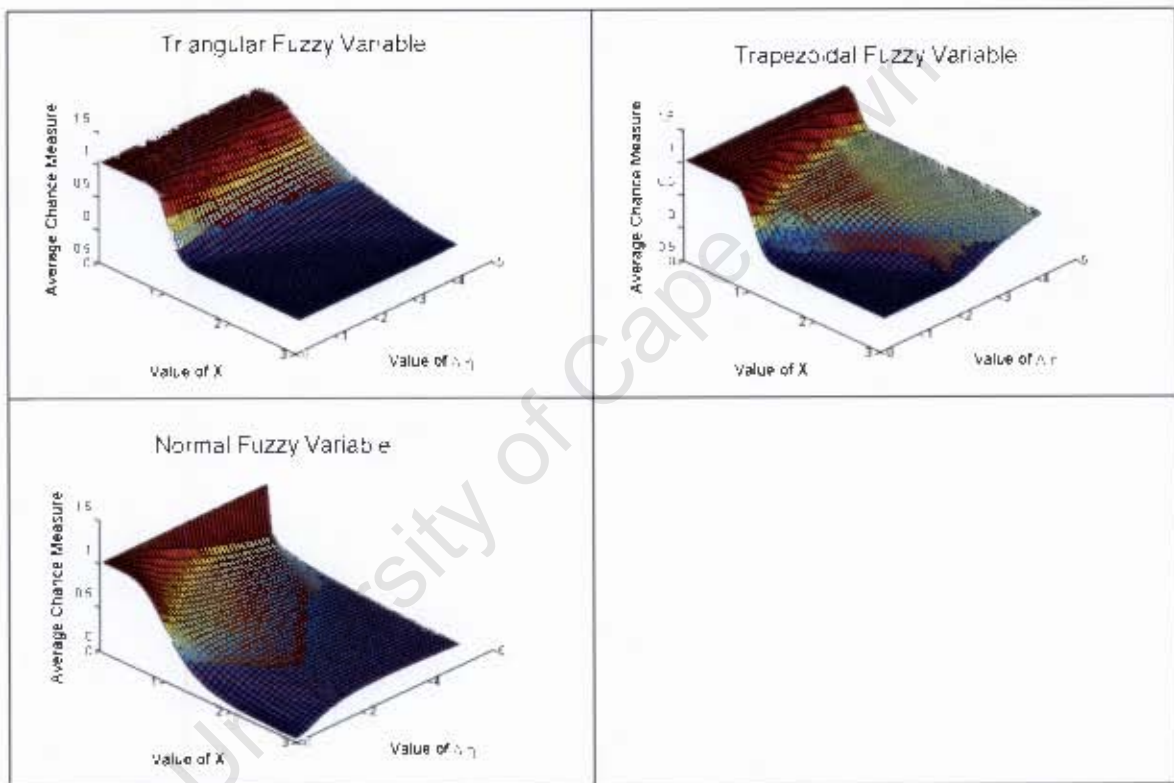


Table 4.3: (Plots) Average chance distribution for a normal random variable with a symmetrical fuzzy mean and known variance.

integrating from $\frac{K}{S(0)}$ to 10.

4.4 Various comparisons with respect to call option pricing

In the previous section a method to numerically calculate the price of a European call option was outlined. In this section we aim to explore what effect the choice of different fuzzy variables will have on the pricing a European call option. As the fuzzy drift parameter model has been designed to generalise the Black-Scholes model, we will have the Black Scholes model as a control in all comparisons. Three different methods of comparison are used. In the first method, the length of the support of the membership functions of our fuzzy variables are set to be equal.⁵ In the second method we set the entropy of each fuzzy variable to be equal and in the third method the variances of all the fuzzy variables are set equal.

In each method we use the special case where $S_0 = 30, K = 20, r = 0.08, T = 0.25$ and $\sigma = 0.25$. No abnormal deviations were obtained from changing these parameters.

Comparisons with Equal Support Intervals :

In this method of comparison, the support intervals where the fuzzy variables are non-zero are set to be equal. It is noted that the normal fuzzy variable is defined on an infinite interval. We artificially compared this fuzzy variable to the others by interpreting its standard deviation as its interval. As the intervals approached zero, all of our answers approached the Black Scholes derivation. This feature is consistent with

⁵The support of a membership function is defined as $\text{supp}(\tilde{\Delta\eta}) = \{u | \mu_{\tilde{\Delta\eta}}(u) > 0\}$

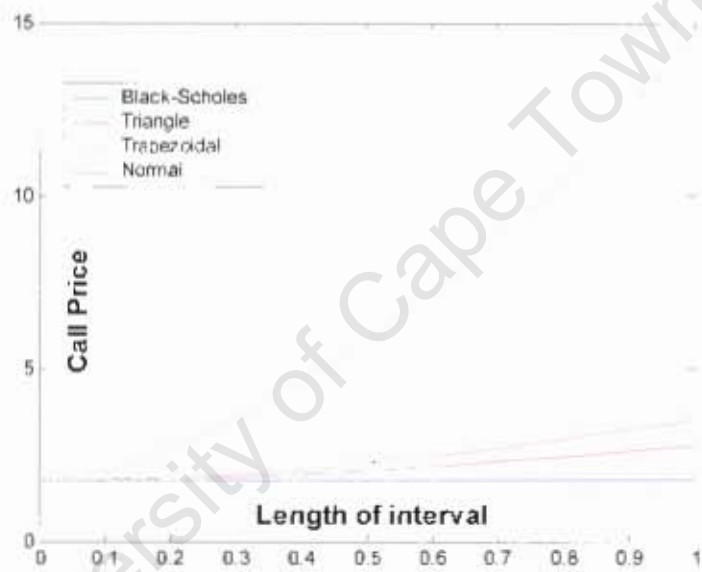


Figure 4.2: Comparison of European call prices with respect to the interval of the fuzzy variable being non-zero.

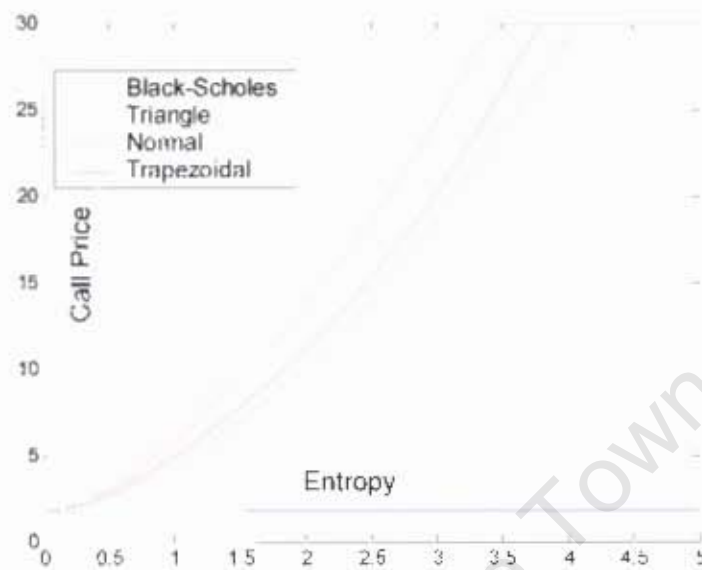


Figure 4.3: Comparison of European call prices with respect to the entropy.

the thinking that the fuzzy drift parameter model should generalise the framework of the Black-Scholes. The "closer" the fuzzy parameter got to a crisp number, the closer the resulting European options got to each other. The rankings for the finite interval fuzzy variables are intuitive, in that if a variable is symmetric trapezoidal we have less information about its whereabouts than if it were symmetric triangle. With less information on what the market drift is, we naturally expect our call prices to rise as this may be interpreted as taking on a new form of risk, specifically the risk of measuring the drift incorrectly.

Comparisons with Equal Entropies :

Under the interval comparison we had a problem of comparing a finite interval fuzzy variable with an infinite one. To counter this problem we compare each variable by its

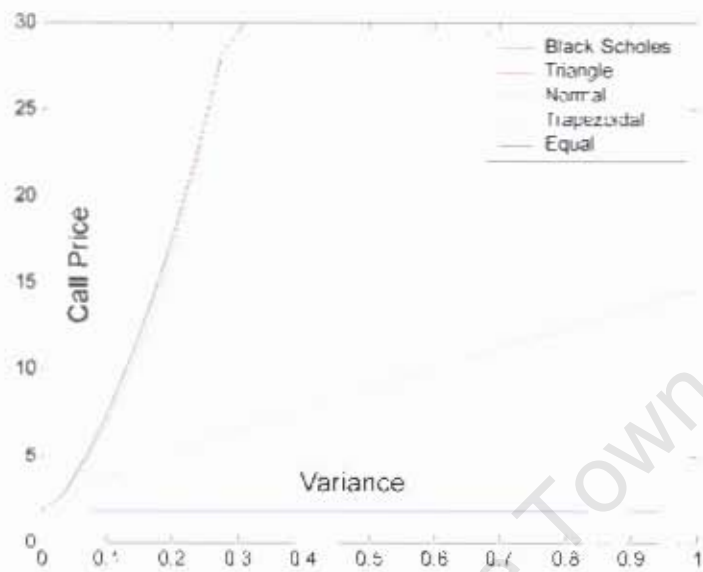


Figure 4.4: Comparison of European call prices with respect to the variance.

entropy. Entropy, a measure of disorder, can loosely be associated with the uncertainty of each variable. For our finite interval variables we have the same rankings as before. The normal fuzzy variable now however is at the lowest ranking. The relevant formulas for entropy can be found in table 4.4.

Comparisons with Equal Variances :

The finite interval variables all displayed nearly the same behavior. This suggests that there might be some asymptotic behavior with respect to its variance. Interestingly our normal fuzzy variable displays completely different behavior and a much slower growth with respect to its variance. The relevant calculations for variance can be found in table (4.4).

Type of Fuzzy Variable	Entropy	Variance
Symmetrical triangular with parameters (a, b, c)	$\frac{c-a}{2}$	$\frac{(c-a)^2}{24}$
Symmetrical trapezoidal with parameters (a, b, c, d)	$\frac{d-a}{2}$	$\frac{(d-a)^2+(d-a)(c-b)+(c-b)^2}{24}$
Normal with parameters (e, σ_n)	$\frac{\sqrt{6}\pi\sigma_n}{3}$	σ_n^2

Table 4.4: Entropy and variance of relevant fuzzy variables

4.5 Volatility Skew

Empirical studies have shown that the market often exhibits a volatility "skew" in that the volatility implied by the Black-Scholes model increases as the strike price decreases.^{[4][7]} ⁶A common justification for this behavior is that, as a stock decreases so does the fear that the stock will decrease even further, which translates into a higher implied volatility. Black first noted this phenomenon in 1973 and labelled it the "leverage effect".^[2]

The left column of Table 4.5 illustrates that the fuzzy drift parameter model asymptotically demonstrates the volatility skew for all fuzzy variables. ⁷ Note that as the fuzzy variable drifts further away from a crisp number, the volatility skew becomes more pronounced. This pattern is intuitive if we think that a fuzzy variable can be associated with the risk of not measuring the drift correctly. As the fuzzy variable

⁶Markets often exhibit the "volatility smile" as well.

⁷These implied volatilities are under the special case $S_0 = 30, K = 20, r = 0.08, T = 0.25$ and $\sigma = 0.25$

drifts further away from a crisp number, the risk associated with not measuring the drift correctly increases.

While asymptotically the fuzzy drift parameter model demonstrates the volatility skew however locally there is great variation. This is illustrated for various specific cases in the right column of Table 4.5. In the triangular case we have the special case where $(c - a) = 0.548$. Here there is a slight volatility smile between the strike prices of 10 and 30. For the trapezoidal case we have the special case where $(d - a) = 0.6650$ and $(c - b) = 0.1664$. The implied volatility decreases rapidly until the strike price of 25. In the range of strike prices between 25 and 30 the implied volatility levels off and for strike prices greater than 30 the implied volatility decreases again. In the normal case we have the special case where $\sigma_n = 0.098$. Here there is a slight volatility smile between the strike prices of 20 and 80.

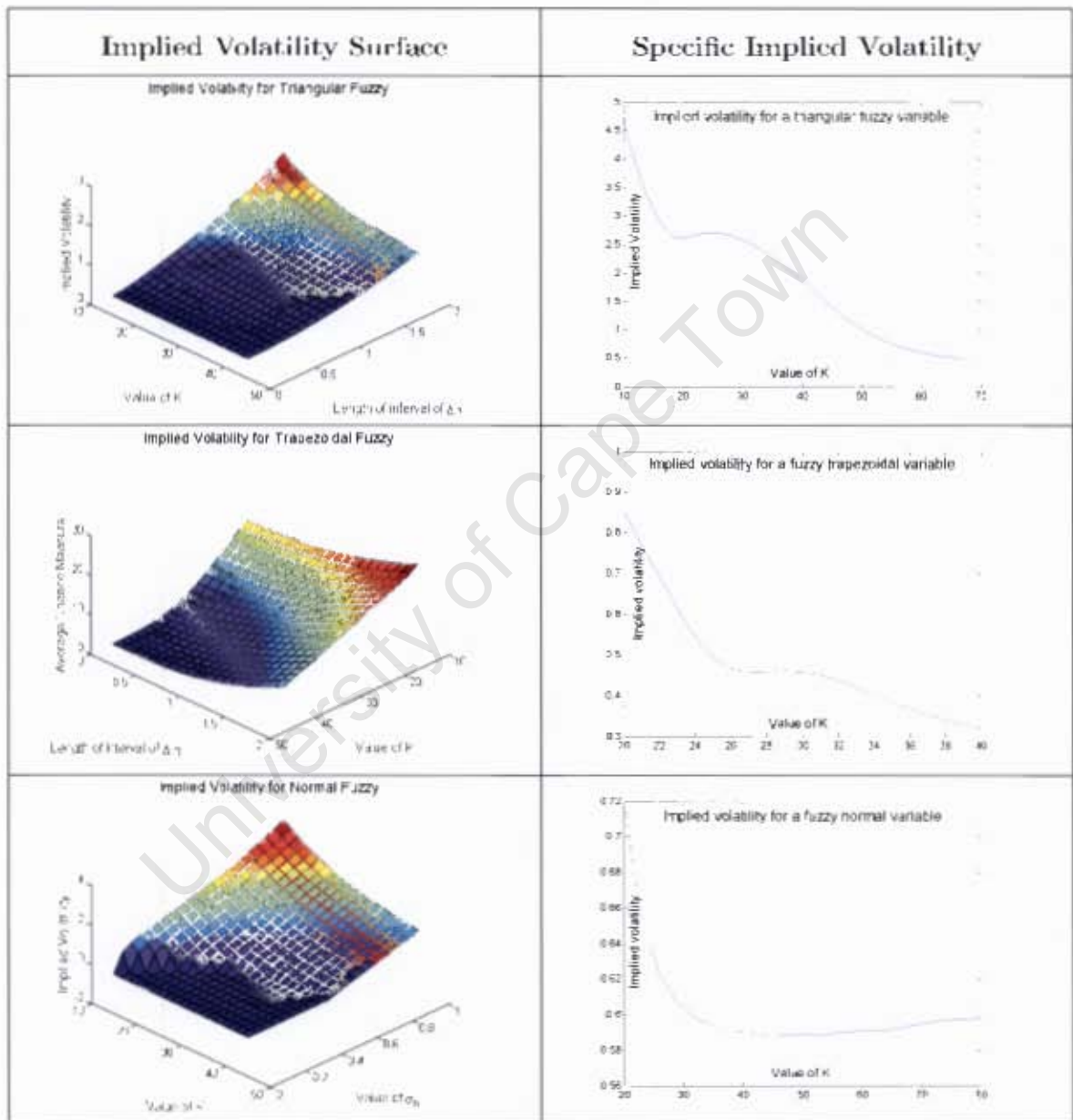


Table 4.5: (Plots) Volatility skews for the fuzzy drift parameter model

CHAPTER 5.

PARAMETER ESTIMATION

5.1 Introduction

In the previous chapter the fuzzy drift parameter model was developed. However no mention was made on how to estimate the parameters of the model or how to calibrate the model to the market. At present no universally accepted method has been developed for estimating parameters within chance and credibility theory. Some of the more common methods of estimating parameters will be discussed in this chapter and then a modified method of parameter estimation will be suggested and developed.

5.2 Background

When we aim to calibrate the fuzzy drift parameter model to market data we are given a series of data points, these points being the stock prices, S_{t_i} , and the respective dates t_i . Thus when estimating the parameters of the fuzzy drift parameter model we have the list of data points

$$\begin{array}{ll} t_1 & S_{t_1} \\ t_2 & S_{t_2} \\ \vdots & \vdots \\ t_n & S_{t_n} \end{array}$$

where we make the simplifying assumption that $\Delta t = t_i - t_{i-1} = t_{i-1} - t_{i-2} = 1$ for all i . From these quantities one can define the continuous rate of return as $\varepsilon_i = \ln \frac{S_{t_i}}{S_{t_{i-1}}}$.¹

¹Note that we now have $n - 1$ data points.

A naive way of calculating the deterministic drift, μ , under geometric Brownian motion is to let

$$\mu = \frac{1}{n-1} \sum_{i=1}^{n-1} \varepsilon_i = \bar{\varepsilon} \quad (5.1)$$

This assumption says that deterministic drift is equal to the historical return. By subtracting the historical return from our continuous returns we obtain a set of returns that are centred around 0. Under geometric Brownian motion these centred continuous returns, which are often referred to as the "errors", are distributed $N(0, \sigma^2 \Delta t)$ and should be perfectly symmetrical around 0. In the fuzzy drift parameter model however these returns are distributed $N((\eta - \bar{\varepsilon})\Delta t, \sigma^2 \Delta t)$ ² and need not be perfectly symmetrical. This possible asymmetry is seen as one of the desirable properties of the fuzzy drift parameter model as it can display skewed leptokurtic distributions.

The debated question now is how do we choose the optimal parameters for the fuzzy variable η and the normal variance σ^2 . One would like to choose the set of parameters that make the errors, which we will assume to be independent, "most likely". A mathematical interpretation of this thought is to find the parameters that maximise

$$\text{ch} \left\{ \bigcap_{i=1}^{n-1} \{\xi(\theta) = \hat{\varepsilon}_i\} \right\} \quad (5.2)$$

where $\hat{\varepsilon}_i = \varepsilon_i - \bar{\varepsilon}$. Work by Guo has suggested that we temporarily view chance independence in (5.2) in the probabilistic sense and thus have that the optimal parameters are found by maximising^{[8]3}

²The result follows as the errors are a random fuzzy process in the fuzzy drift parameter model.

³Equivalently, we use the probabilistic maximum likelihood estimator.

$$J = \max_{\theta \in \Theta} \prod_{i=1}^{n-1} \text{ch} \{ \xi(\theta) = \hat{\varepsilon}_i \} \quad (5.3)$$

The method above is known as the maximum average chance principle. An alternative development for estimating parameters by Guo^[9] is to utilise the maximum uncertainty principle which is defined as

Definition 31 (*Maximum Uncertainty Principle*) *For any event, if there are multiple reasonable values that a measure may take, then the value as close to 0.5 as possible is assigned to the event.*

The event that we will be examining is the distribution of errors which is defined as $\Psi(x) = \text{ch} \{ \xi \leq x \}$. It is intuitive that we would like our errors to be as close to 0 as possible. An error of 0 would result in a distribution of 0.5 in the case that our fuzzy variable is symmetrical around 0. Mathematically we can express the optimal parameters as being one that minimises the object function

$$J = \sum_{i=1}^{n-1} (\Psi(\hat{\varepsilon}_i) - 0.5)^2 \quad (5.4)$$

An advantage that parameter estimation by the maximum uncertainty principle has over parameter estimation by the maximum average chance principle is that it follows the definition of independence as given by chance theory. A disadvantage is that it can only be applied to find the optimal parameters for symmetrical ⁴fuzzy variables.

⁴By symmetrical it is meant that the membership function is even. This property implies that the parameters for the triangular fuzzy variable are $(-h, 0, h)$ those for the trapezoidal fuzzy variable are $(-h_2, -h_1, h_1, h_2)$ and those for the normal fuzzy variable are $(0, \sigma_n)$.

5.3 Parameter Estimation by the Maximum Average Chance Principle

The maximum average chance principle states that the best parameters are those that satisfy

$$\max_{\theta \in \Theta} \prod_{i=1}^{n-1} \text{ch} \{ \xi(\theta) = \hat{\varepsilon}_i \} \quad (5.5)$$

where ξ is the hybrid variable we are working with and is distributed $N((\eta - \bar{\varepsilon})\Delta t, \sigma^2 \Delta t)$.

Earlier work by Guo suggested that we interpret the measure $\text{ch} \{ \xi(\theta) = x \}$ as $\frac{d}{dx} \Psi(x)$.^[8]

Although this interpretation is true in probability theory it is not necessarily true in chance theory. A more consistent view of $\text{ch} \{ \xi(\theta) = x \}$ with our established chance theory is to apply the definition of the average chance measure to it. By assumption we have that our continuous returns minus our historical returns are distributed $N((\eta - \bar{\varepsilon})\Delta t, \sigma^2 \Delta t)$. Let us label the hybrid variable that has this distribution as ξ . The average chance measure for the event $\text{ch} \{ \xi = x \}$ is

$$\begin{aligned} \text{ch} \{ \xi = x \} &= \int_0^1 \text{Cr} \{ \text{Pr} \{ \xi = x \} \geq \alpha \} d\alpha \\ &= \int_0^1 \text{Cr} \left\{ \eta_1 : \frac{1}{\sigma} \phi \left(\frac{x - \eta_1}{\sigma} \right) \geq \alpha \right\} d\alpha \end{aligned} \quad (5.6)$$

where $\eta_1 = \eta - \bar{\varepsilon}$ and $\phi(\cdot)$ is the standard normal density function. In order to proceed one would want to rearrange the inequality above to have η_1 as its subject. This calculation however is problematic as $\phi(\cdot)$ is not a one to one function and thus its inverse is not unique. We do however know that if $\phi(x) = a$ then $\phi(-x) = a$ as well. Let us define $\phi_+^{-1}(\alpha)$ to be the positive inverse and $\phi_-^{-1}(\alpha)$ to be the negative. Due to

the normal density function not being one to one, we have that the calculation of our inequality will be different if x is positive or negative. In the case that x is positive we have that

$$\begin{aligned}
& \text{Cr} \left\{ \frac{1}{\sigma} \phi \left(\frac{x - \eta}{\sigma} \right) \geq \alpha \right\} \\
&= \text{Cr} \left\{ \frac{x - \eta_1}{\sigma} \leq \phi_+^{-1}(\sigma\alpha) \right\} \\
&= \text{Cr} \{ \eta_1 \geq x - \sigma\phi_+^{-1}(\sigma\alpha) \}
\end{aligned} \tag{5.7}$$

⁵ Note that we have the implied restriction that $\frac{x - \eta_1}{\sigma} \geq 0$ as $\phi(\cdot)$ is not one to one. In the case that x is negative

$$\begin{aligned}
& \text{Cr} \left\{ \frac{1}{\sigma} \phi \left(\frac{x - \eta_1}{\sigma} \right) \geq \alpha \right\} \\
&= \text{Cr} \left\{ \frac{x - \eta_1}{\sigma} \geq \phi_-^{-1}(\sigma\alpha) \right\} \\
&= \text{Cr} \{ \eta_1 \leq x - \sigma\phi_-^{-1}(\sigma\alpha) \}
\end{aligned} \tag{5.8}$$

where we have the implied restriction that $\frac{x - \eta_1}{\sigma} \leq 0$. Thus calculating the average

⁵Formally we have that

$$\begin{aligned}
& \text{Cr} \left\{ \frac{1}{\sigma} \phi \left(\frac{x - \eta}{\sigma} \right) \geq \alpha \right\} \\
&= \text{Cr} \left\{ \left\{ \frac{x - \eta}{\sigma} \leq \phi_+^{-1}(\sigma\alpha) \right\} \cup \left\{ \frac{x - \eta}{\sigma} \geq \phi_-^{-1}(\sigma\alpha) \right\} \right\}
\end{aligned}$$

Where each interval is justified by the need to have η as close to 0 as possible as this would result in the highest credibility rating. Each interval is credibility independent and by the definition of independence in credibility theory we have

$$= \max \left\{ \text{Cr} \left\{ \frac{x - \eta}{\sigma} \leq \phi_+^{-1}(\sigma\alpha) \right\}, \text{Cr} \left\{ \frac{x - \eta}{\sigma} \geq \phi_-^{-1}(\sigma\alpha) \right\} \right\}$$

In the case that x is positive we have that $\text{Cr} \left\{ \frac{x - \eta}{\sigma} \leq \phi_+^{-1}(\sigma\alpha) \right\} \geq \text{Cr} \left\{ \frac{x - \eta}{\sigma} \geq \phi_-^{-1}(\sigma\alpha) \right\}$ whereas in the case that x is negative we have that $\text{Cr} \left\{ \frac{x - \eta}{\sigma} \geq \phi_-^{-1}(\sigma\alpha) \right\} \geq \text{Cr} \left\{ \frac{x - \eta}{\sigma} \leq \phi_+^{-1}(\sigma\alpha) \right\}$.

Type of Fuzzy Variable	Average chance density of $\varsigma \sim N(\eta_1, \sigma^2)$.
Triangular with parameters $(a, 0, c)$	$\frac{1}{2a} (\Phi(\frac{F_1}{\sigma}) - \Phi(\frac{x}{\sigma})) + \frac{1}{2c} (\Phi(\frac{x}{\sigma}) - \Phi(\frac{x-c}{\sigma})) + \frac{a-x+F_1}{2\sigma a} \phi(\frac{F_1}{\sigma})$ if $x < 0$ and $F_1 = \min\{x-a, 0\}$
	$\frac{1}{2a} (\Phi(\frac{x-a}{\sigma}) - \Phi(\frac{x}{\sigma})) + \frac{1}{2c} (\Phi(\frac{x}{\sigma}) - \Phi(\frac{F_1}{\sigma})) + \frac{c-x+F_1}{2\sigma c} \phi(\frac{F_1}{\sigma})$ if $x \geq 0$ and $F_1 = \max\{x-c, 0\}$
Trapezoidal with parameters (a, b, c, d) where $a < b < 0 < c < d$	$\frac{1}{2(d-c)} (\Phi(\frac{x-c}{\sigma}) - \Phi(\frac{x-d}{\sigma})) + \frac{1}{2(b-a)} (\Phi(\frac{F_2}{\sigma}) - \Phi(\frac{F_1}{\sigma}))$ $+ \frac{1}{\sigma} \phi(\frac{F_1}{\sigma}) (\frac{1}{2} - \frac{x-a-F_1}{2(b-a)}) + \frac{1}{\sigma} \phi(\frac{F_2}{\sigma}) (\frac{x-a-F_2}{2(d-c)})$ if $x < 0$ and $F_1 = \min\{x-b, 0\}$ and $F_2 = \min\{x-a, 0\}$
	$\frac{1}{2(b-a)} (\Phi(\frac{x-a}{\sigma}) - \Phi(\frac{x-b}{\sigma})) + \frac{1}{2(d-c)} (\Phi(\frac{F_1}{\sigma}) - \Phi(\frac{F_2}{\sigma}))$ $+ \frac{1}{\sigma} \phi(\frac{F_1}{\sigma}) (\frac{1}{2} - \frac{d-x+F_1}{2(d-c)}) + \frac{1}{\sigma} \phi(\frac{F_2}{\sigma}) (\frac{d-x+F_2}{2(d-c)})$ if $x \geq 0$ and $F_1 = \max\{x-c, 0\}$ and $F_2 = \max\{x-d, 0\}$
Normal with parameters (e, σ_n)	$\frac{1}{\sigma} \phi(\frac{x}{\sigma}) + \frac{1}{\sigma} \int_{-\infty}^{\frac{x}{\sigma}} \frac{s\phi(s)ds}{1+\exp(\frac{x-\sigma s}{\sqrt{6}\sigma_n})} - \frac{1}{\sigma} \int_{\frac{x}{\sigma}}^0 \frac{s\phi(s)ds}{1+\exp(\frac{\sigma s-x}{\sqrt{6}\sigma_n})}$ if $x < 0$
	$\frac{1}{\sigma} \phi(\frac{x}{\sigma}) + \frac{1}{\sigma} \int_0^{\frac{x}{\sigma}} \frac{s\phi(s)ds}{1+\exp(\frac{x-\sigma s}{\sqrt{6}\sigma_n})} - \frac{1}{\sigma} \int_{\frac{x}{\sigma}}^{\infty} \frac{s\phi(s)ds}{1+\exp(\frac{\sigma s-x}{\sqrt{6}\sigma_n})}$ if $x \geq 0$

Table 5.1: (Formulas) Average chance density for a normal random variable with fuzzy mean and known variance.

chance density depends on the sign of x as well as the membership function of the fuzzy variable. In this thesis three fuzzy variables for η_1 are used. Namely the triangular, trapezoidal and normal. Table (5.1) illustrates the various average chance densities that are used in this thesis. The calculations for these measures can be found in Appendix C: Average Chance Measures.

From Table 5.1 and our object function in (5.5) it is possible to calculate the best parameters according to the maximum average chance principle. However there

is a problem. In probability theory we have that the area underneath the probability density distribution is one, whereas in chance theory we do not necessarily have this property. This difference poses a problem as by having $\sigma \rightarrow 0$ and allowing the absolute value of our other parameters $\rightarrow \infty$ we have that the average chance density $\rightarrow \infty$. In order to counter this contradiction and make our method of estimation closer to that found in probability theory, on which this method is based on, we modify our definition of $\text{ch}\{\xi(\theta) = x\}$ to $\frac{1}{K} \text{ch}\{\xi(\theta) = x\}$, where K is equal to the area underneath the average chance density graph. This modification ensures that the area underneath our new average chance density is one. Thus we have that the ideal parameters are chosen by the parameters that satisfy the following objective function

$$\max_{\theta \in \theta} \prod_{i=1}^{n-1} \frac{1}{K} \text{ch}\{\xi(\theta) = \hat{\varepsilon}_i\}$$

where $K = \int_{-\infty}^{\infty} \text{ch}\{\xi(\theta) = x\} dx$.

5.4 Parameter Estimation by the Maximum Uncertainty Principle

The maximum uncertainty principle implies that the best parameters are those that satisfy

$$\min_{\theta \in \theta} \sum_{i=1}^{n-1} (\text{ch}\{\xi \leq \hat{\varepsilon}_i(\theta)\} - 0.5)^2 \quad (5.9)$$

where ξ is a symmetrical fuzzy variable. Table (5.2) lists the average chance distributions for symmetrical fuzzy variables. In all cases we see that the average chance distribution $\rightarrow 0.5$ as $\sigma \rightarrow \infty$. In the triangular case the first and third term $\rightarrow 0$ as

Type of Fuzzy Variable	Average chance distribution of $\varsigma \sim N(\eta, \sigma^2)$.
Symmetrical Triangular with parameters $(-h, 0, h)$	$\frac{x+h}{2h} (\Phi(\frac{x+h}{\sigma}) - \Phi(\frac{x-h}{\sigma})) + \Phi(\frac{x-h}{\sigma})$ $+ \frac{\sigma}{2h} (\phi(\frac{x+h}{\sigma}) - \phi(\frac{x-h}{\sigma}))$
Trapezoidal with parameters $(-h_2, -h_1, h_1, h_2)$	$\frac{x+h_2}{2(h_2-h_1)} (\Phi(\frac{x+h_2}{\sigma}) - \Phi(\frac{x+h_1}{\sigma})) + \frac{x+h_2-2h_1}{2(h_2-h_1)} (\Phi(\frac{x-h_1}{\sigma}) - \Phi(\frac{x-h_2}{\sigma}))$ $+ \frac{1}{2} (\Phi(\frac{x+h_1}{\sigma}) - \Phi(\frac{x-h_1}{\sigma})) + \Phi(\frac{x-d}{\sigma})$ $+ \frac{\sigma}{2(h_2-h_1)} (\phi(\frac{x+h_2}{\sigma}) - \phi(\frac{x+h_1}{\sigma})) + \frac{\sigma}{2(h_2-h_1)} (\phi(\frac{x-h_1}{\sigma}) - \phi(\frac{x-h_2}{\sigma}))$
Normal with parameters $(0, \sigma_n)$	$\Phi(\frac{x-e}{\sigma})$

Table 5..2: (Formulas) Average chance distribution for a normal random variable with fuzzy mean and known variance.

$\sigma \rightarrow \infty$.⁶ As such all we are left with is the second term, $\Phi(\frac{x-h}{\sigma})$, which $\rightarrow \frac{1}{2}$ as $\sigma \rightarrow \infty$. In the trapezoidal case all the terms except the fourth term $\rightarrow 0$ as $\sigma \rightarrow \infty$. The fourth term, $\Phi(\frac{x-d}{\sigma})$, $\rightarrow \frac{1}{2}$ as $\sigma \rightarrow \infty$. In the normal case we only have one term, $\Phi(\frac{x-e}{\sigma})$, which $\rightarrow \frac{1}{2}$ as $\sigma \rightarrow \infty$.

Parameter estimation by the maximum average chance principle has the possibility of giving unreasonable parameters. As such, it is necessary to place restrictions on the problem. A possible restriction is to set the variance of ξ to equal the observed variance of historical returns. An alternative restriction is to directly set σ^2 equal to the observed variance of historical returns. In this thesis parameter estimation will be performed by the maximum average chance principle only. This choice is made due

⁶For the triangular and trapezoidal fuzzy case we are required to calculate a limit of the form $\lim_{\sigma \rightarrow \infty} \sigma(\phi(\frac{a}{\sigma}) - \phi(\frac{b}{\sigma}))$ which is equal to 0. This can be easily proven by taking the Taylor expansion of $\phi(\frac{a}{\sigma}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}}$ and $\phi(\frac{b}{\sigma}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2\sigma^2}}$.

to the difficulty of choosing meaningful restrictions for parameter estimation by the maximum uncertainty principle.

5.5 Particle Swarm Optimisation

Introduction The modified version of the maximum average chance principle states that the best parameters are those that satisfy

$$\max_{\theta \in \theta} \prod_{i=1}^{n-1} \frac{1}{K} \text{ch} \{ \xi(\theta) = \hat{\varepsilon}_i \} \quad (5.10)$$

where $K = \int_{-\infty}^{\infty} \text{ch} \{ \xi(\theta) = x \} dx$ and $\text{ch} \{ \xi(\theta) = \hat{\varepsilon}_i \}$ is given by Table (5.1) for the various membership functions. Presently no analytical solution is known and thus a numerical method must be implemented. A current commonly used method for numerical optimisation within Computer Science is that of particle swarm optimisation. Particle swarm optimisation works similarly to standard Monte Carlo methods in that it is an iterative method and each step has a random component to it. It differs however in that many simulations are run simultaneously (each simulation being known as a particle) and that each iterative step is influenced by the best solution the current particle has found and by the best solution the particle within its group has found. The simplest group is that which comprises all the particles. In this case, the particles are said to be fully informed and will be the method used for optimisation within this thesis. Two good references for particle swarm optimisation can be found in Carlisle and Mendes.^{[3][17]}

Method

The standard method of particle swarm optimisation will be used in this thesis. A reference for this method can be found in Mendes.^[17] In this method, each particle is assigned a "velocity" and its new position is calculated by adding the velocity to its position.

$$\vec{V}_{t+1} = \alpha \vec{V}_t + U [0, \varphi_1] \times (\vec{P}_i - \vec{X}_t) + U [0, \varphi_2] \times (\vec{P}_g - \vec{X}_t) \quad (5.11)$$

$$\vec{X}_{t+1} = \vec{X}_t + \vec{V}_{t+1} \quad (5.12)$$

Here the first equation relates to the particle's velocity and the second to its position. The terms $(\vec{P}_i - \vec{X}_t)$ and $(\vec{P}_g - \vec{X}_t)$ point the velocity vector in the direction of the particle's local best and global best solution respectively. The constant term α can be viewed as a "momentum" effect. It determines how much the new velocity vector is determined by its previous one. The two terms $U [0, \varphi_1]$, $U [0, \varphi_2]$ are two uniformly random numbers between 0 and φ_1 and 0 and φ_2 respectively. These terms together with the "momentum" effect add a random component to particle swarm optimisation that in practice help to ensure the optimal solution found is not a local optimal solution.

In the optimisation problems we have defined there are a few restrictions that need to be considered. For instance σ always has to be positive or in the case of the triangular fuzzy variable we have the restriction $a < b < c$. In the event that a parameter breaks a bound, the simple solution of making the parameter equal to the bound and adding (or subtracting for some specific bounds) 0.0001 is implemented.

Particle Swarm Optimisation was performed in VBA in Excel with the parameters of $\alpha = 0.9$, $\varphi_1 = \varphi_2 = 2$. The initial conditions were implemented by placing

equally spaced particles in an "effective" parameter space. By "effective" parameter space it is meant that initial bounds were chosen for parameters that were determined as "reasonable". For instance the initial bounds for the triangular fuzzy variable were set at $a = -1$ and $c = 1$. Particle swarm optimisation was implemented with thirty particles and 200 iterations.

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CHAPTER 6.

PREDICTION RESULTS

6.1 Introduction

Within this chapter we aim to test how well the fuzzy drift parameter model performs in predicting European call option prices. In order to assess the performance, we first calibrate the fuzzy drift parameter model to market share data. Once the calibration is complete, theoretical prices for European call options can be calculated and compared to observed market prices.

6.2 Data

The fuzzy drift parameter model will be calibrated to weekly data (obtained from McGregor) of the S&P/ASX 200 index from the 14th of August 2006 until the 14th of July 2008 which results in 100 data points. Weekly returns were chosen as they are seen as a desirable mix between the erraticness of daily returns and the relative stability of monthly returns. Each weekly return will be weighted equally.

The S&P/ASX200 index is a market-cap-weighted index that tracks the progress of Australia's top 200 companies. This index has the desirable properties that it is highly traded and consists of stocks that represent the entire stock market. In addition the options that are traded on this index are numerous, European style and highly traded. As options on the S&P/ASX200 are highly traded, liquidity risk is minimised. We thus make the simplifying assumption that liquidity risk is negligible.

Once the fuzzy drift parameter model has been calibrated to weekly data we

are in a position to calculate theoretical European call options and compare them to observed market prices (obtained from the Australian Stock Exchange). As the model has been calibrated for the 14th of July 2008, it is natural that all the options should be priced and compared on this date. Thirty different options are priced and compared with varying maturities and strike prices. Fifteen options expire on the 21st of August 2008 (34 days till maturity), nine options expire on the 18th of September 2008 (62 days till maturity) and finally eight options expire on the 18th of December 2008 (153 days till maturity). All the call options were checked to determine whether they had been regularly traded in the last day (more than 30 trades) and whether they satisfied the arbitrage lower bound

$$S_t - Ke^{-r(T-t)} \leq C(S, t)$$

where S_t is the current stock price, K is the strike and r is the risk-free rate. The home page of the Australian stock exchange recommends that the risk-free rate is 7.25 % which was the current Australian overnight rate. However the organisation for economic cooperation and development (OECD)^[18] suggests that one should use the relevant bank accepted bill rate as the risk-free rate. Table (6.1) shows the various bank accepted bill rates found in Australia on the 14th of July 2008. The risk free rate for the options expiring on the 21st of August 2008 (34 days till maturity) was found by linearly interpolating the 30 and 90 day bank accepted bill rates. The risk free rate for the options expiring on the 18th of September 2008 (62 days till maturity) and the 18th of December 2008 (153 days till maturity) were found in a similar way by linearly interpolating the two nearest bank accepted bill rates.

Maturity	Yield
30 Days	7.54%
90 Days	7.75%
180 Days	7.9%

Table 6..1: Bank accepted bill rates

6.3 Results of Calibration

Parameter estimation for the fuzzy drift parameter model was estimated by the maximum average chance principle. This principle states that the best parameters are those that satisfy

$$\max_{\theta \in \Theta} \prod_{i=1}^{n-1} \frac{1}{K} \text{ch} \{ \xi(\theta) = \hat{\varepsilon}_i \} \quad (6.1)$$

where θ is our parameter vector, $K = \int_{-\infty}^{\infty} \text{ch} \{ \xi(\theta) = x \} dx$ and $\hat{\varepsilon}_i$ are our adjusted returns. The optimal parameters were found by the numerical method of particle swarm optimisation. Thirty particles were used with 200 iterations. The optimal parameters for weekly returns are illustrated in Table (6.2). It is the industry norm to have parameters quoted in yearly returns. In Australia, each year consists of 262 trading days and each week consists of 5. Thus we have one week is equal to $\frac{5}{262}$ trading days. In order to convert our parameters from weekly to yearly, we multiply each parameter by $\sqrt{\frac{262}{5}}$. This ensures that the variance of the hybrid variable under weekly parameters is equal to that under yearly parameters.¹

¹The dynamics of the fuzzy drift parameter model are given by

$$\frac{dS}{S} = \eta dT + \sigma dW_t$$

Type of Fuzzy Variable	Optimal Parameters			
Triangular fuzzy mean	$a = -0.6079\%$	$b = 0\%$	$c = 0.6090\%$	
with parameters $(a, 0, c)$	$\sigma = 1.9859\%$			
Trapezoidal fuzzy mean	$a = -1.7271\%$	$b = -0.001\%$	$c = 0.5777\%$	$d = 0.5787\%$
with parameters (a, b, c, d)	$\sigma = 1.8144\%$			
Normal fuzzy mean	$e = 0\%$	$\sigma_n = 0.3847\%$		
with parameters $(0, \sigma_n)$	$\sigma = 0.7776\%$			

Table 6..2: Optimal parameters for weekly calibration

6.4 Results of Option Pricing

Now that the parameters for the fuzzy drift parameter model are calibrated we are in a position to price European call options and compare them to market data. The risk-neutral dynamics of the fuzzy drift parameter model are giving by

$$\frac{dS}{S} = (r + \Delta\eta)dt + \sigma dW_t$$

It is argued that the fuzzy variable represent a general uncertainty about what the drift is. The main justification being that to measure the drift requires interpreting subjective information which depends on the psychology of the practitioner. The simplest assumption would be that this uncertainty does not change with the maturity

As such the continuous returns are distributed $N(\eta T, \sigma^2 T)$. This hybrid variable can be broken up into the addition of two parts. The first being purely fuzzy ηT , and the second being purely random and distributed $N(0, \sigma^2 T)$. These two variables are independent and thus the variance of the hybrid variable is equal to the addition of the variances of the fuzzy and probabilistic variables.^[13]

If one had to change T to aT , then in order to keep the variance the same for the fuzzy part one has to multiply the parameter vector θ by $\frac{1}{\sqrt{a}}$. (See Table 4.4 for fuzzy variances). Similarly for the probabilistic part, one has to multiply σ by $\frac{1}{\sqrt{a}}$ to keep the same variance.

of the option. Mathematically this view is equivalent to stating that the fuzzy variable $\Delta\eta T$ should be the same for all maturities. This consequence is easily modelled by dividing the parameters of $\Delta\eta$ by T before pricing the option.

Two methods were used for checking how the fuzzy drift parameter model performs in predicting call options as given by the market. In the first method we examine the relative error of the call prices. Conventionally this criterion is defined as

$$E_{Conventional} = \frac{|C_{Model} - C_{Market}|}{C_{Model}}$$

Naturally if the relative error is small, the model is seen as giving a good approximation to the market. In our workings we would like to see whether the fuzzy drift parameter model over or underestimates the market data. Thus we modify the relative error of call prices to

$$E_{Modified} = \frac{C_{Model} - C_{Market}}{\frac{C_{Model} + C_{Market}}{2}}$$

The denominator has been modified so it wouldn't (apart from the sign) make a difference if the numerator was $C_{Market} - C_{Model}$. The second method of checking the predictive powers of the fuzzy drift parameter model is to compare the implied volatilities of the observed market data and that of the fuzzy drift parameter model. This method may not be appropriate however, as the concept of volatility in the fuzzy drift parameter model does not have the same meaning as that of the conventional Black-Scholes model.

On the 14th of July 2008, the underlying price of the S&P/ASX 200 index was 4840.4. Table (6.3) illustrates the various modified relative errors of the fuzzy drift

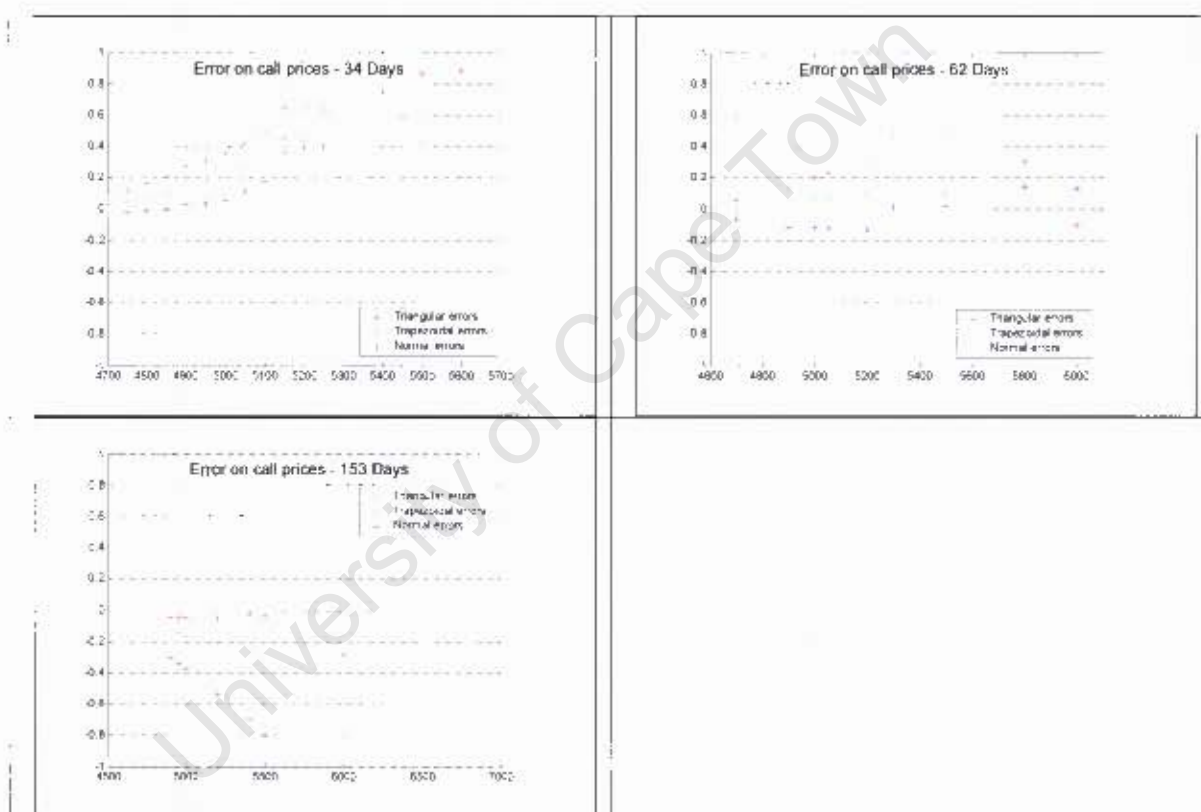


Table 6.3: Error of call prices at different maturities

parameter model as compared to the market. For all options it was found that the fuzzy drift parameter model fared the best for options relatively at-the-money. For options with a short maturity (34 days), the normal fuzzy variable fared the best. Between the strike prices of 4700 and 5100, the fuzzy drift parameter model resulted in modified errors below 10% while between the strike prices of 4750 and 5000, the modified errors were below 5%.

For options with a medium maturity (62 days), the trapezoidal fuzzy variable fared the best. Between the strike prices of 4700 and 5800, the modified relative error were below 10% while between the strike prices of 4700 and 5200, the modified errors were below 5%.

Finally for options with a long maturity (153 days), the triangular fuzzy variable fared the best. Between the strike prices of 4900 and 6000, the modified relative errors were below 10% and between the strike prices of 4900 and 5500, the modified relative errors were below 5%.

Table 6.4 illustrates the various implied volatilities of the fuzzy drift parameter model as well as that of the market's. Examining the modified relative errors, one would conclude that the normal fuzzy variable fares the best for options with a short maturity. However from table 6.4, we can see that the normal fuzzy variable with the parameters we have chosen has a completely the wrong shape for implied volatilities. This theme runs through for all maturities and all fuzzy variables. It is unclear whether this shape is a problem as volatility is interpreted differently in the Black-Scholes model from that of the fuzzy drift parameter model. (Within the fuzzy drift parameter model, we have two sources of uncertainty, these being the fuzzy parameter as well as the variance. In the

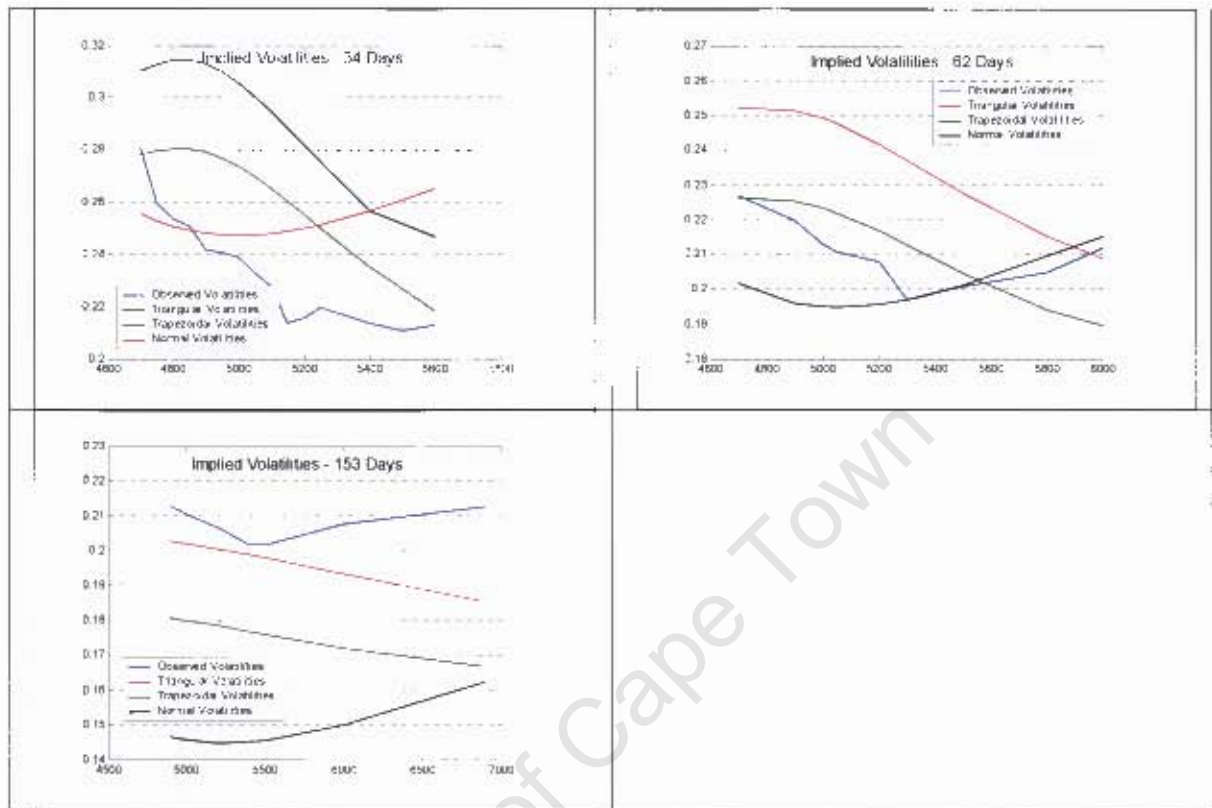


Table 6.4: Implied Volatilities at different maturities

Black-Scholes model however we have only one, namely the variance.) The difference in the implied volatilities suggests an alternative method to fitting parameters to any given data. That is we fit the parameters so that the implied volatility curves are as "close" to each other as possible.

CHAPTER 7.

CONCLUSIONS

The purpose of this thesis was to implement credibility measure theory within financial markets. Implementing fuzzy logic and in particular credibility theory to financial markets is a relatively new concept to the literature. In this thesis we attempted to model psychological impacts of market practitioners by replacing the drift with a fuzzy variable.

The argument for this approach is that uncertainties that are caused by market practitioners are introduced into the drift of the stock. These uncertainties are vague in nature and thus a fuzzy variable is best suited in describing them. A semi-closed form for European options was developed and then the fuzzy drift parameter model was tested to observed market data.

The primary method of testing the fuzzy drift parameter model was to examine the relative errors it produced when compared to actual market data. Three different types of fuzzy variables were used within this thesis, the triangular, the trapezoidal and the normal.

- The normal fuzzy variable was found to be best suited to describe at-the-money options with a short (34 days) maturity.
- The trapezoidal fuzzy variable was found to be best suited to describe at-the-money options with a medium (62 days) maturity.
- The triangle fuzzy variable was found to be best suited to describe at-the-money options with a long (153 days) maturity.

All three types of fuzzy variables were found to adequately describe at-the-money options.

Future Research It appears that this thesis was the first time credibility theory was implemented to model market uncertainties by altering the drift of a stock. Naturally there is a lot of scope for further research. Possible areas that were noted include

- Within parameter estimation historical returns were used to estimate parameters for the fuzzy variables. A method that ranks recent observations as more important could provide to be fruitful.
- An alternative to the current parameter estimation would be to estimate the parameters using some other proxy. The risk-neutral dynamics of the fuzzy drift parameter model are

$$\frac{dS}{S} = (r + \Delta\eta)dt + \sigma dW_t$$

Thus we have that the fuzzy parameter, $\Delta\eta$, alters the risk-free interest rate. One could thus estimate the parameters of the fuzzy term $\Delta\eta$ from past risk-free interest rates.

- In the fuzzy drift parameter model we argued that the drift was a fuzzy term. One could equally argue that the variance, σ^2 , is a fuzzy term. That is further research could be examining the fuzzy variance parameter model.

- Within this thesis we have assumed that the parameters of the fuzzy model are constant over time. Future work could be to relax this assumption.

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APPENDIX A.

THEOREMS FROM STOCHASTIC FINANCE

The relevant theorems and concepts from Stochastic Finance are briefly presented in this appendix. For further reference and proofs see Bingham and Kiesel.^[1]

Definition 32 (*Brownian Motion*)

A Brownian Motion with drift, μ , and variance, σ^2 , is a stochastic process $(X_t : t \geq 0)$ such that

1. (*continuous sample Paths*) : X_t is continuous almost surely.
2. (*Independent Increments*) : Given $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$, the random variables

$$X_{t_k} - X_{t_{k-1}}, \quad k = 1, \dots, n$$

are mutually independent.

3. (*Normally Distributed Increments*) : If $0 \leq s \leq t$, then

$$B_t - B_s \sim N(\mu(t - s), \sigma^2(t - s))$$

where $X \sim N(\mu, \sigma^2)$, means that the random variable X is normally distributed with mean μ and standard deviation σ .

Remark 33 A standard Brownian Motion (Wiener Process) is a Brownian Motion where $\mu = 0$ and $\sigma = 1$.

Theorem 34 (*Risk Neutral Valuation*) Suppose that X is an attainable contingent claim, and that \mathbb{Q} is an EMM (Equivalent Martingale Measure) for numéraire N . Then

$$X_t = N_t \mathbb{E}_{\mathbb{Q}} \left[\frac{X_T}{N_T} \middle| \mathcal{F}_t \right]$$

Theorem 35 (*Change of Numeraires*) Suppose that $\alpha_1(t), \alpha_2(t)$ are numeraires, and that $\mathbb{Q}_1, \mathbb{Q}_2$ are their associated EMM's. Then for any random variable X we have

$$\alpha_1(t) \mathbb{E}_{\mathbb{Q}_1} \left[\frac{X}{\alpha_1(T)} \middle| \mathcal{F}_t \right] = \alpha_2(t) \mathbb{E}_{\mathbb{Q}_2} \left[\frac{X}{\alpha_2(T)} \middle| \mathcal{F}_t \right]$$

Theorem 36 (*Girsanov Theorem for Brownian Motion*) Suppose a process Y has \mathbb{P} -dynamics

$$dY_t = \mu_t dt + \sigma_t dW_t \quad (t \leq T)$$

where W is a standard \mathbb{P} -Brownian motion, $\mu_t(w) \in \mathbb{R}, \sigma_t(w) \in \mathbb{R}$. Let $\lambda_t(w) \in \mathbb{R}$ be predictable. Define a measure \mathbb{Q} on \mathcal{F}_T by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \varepsilon(\lambda \cdot W)_T$$

Assume that Novikov's condition holds

$$\mathbb{E} \left[e^{\frac{1}{2} \int_0^T \|\lambda_s\|^2 ds} \right] < \infty$$

Then

1. \mathbb{Q} is a probability measure on \mathcal{F}_T
2. $\widetilde{W}_t = W_t - \int_0^t \lambda_s ds$ is a \mathbb{Q} -Brownian motion.
3. The \mathbb{Q} -dynamics of Y are given by

$$dY_t = (\mu_t + \sigma_t \lambda_t) dt + \sigma_t d\widetilde{W}_t$$

APPENDIX B.

SYMMETRIC FUZZY VARIABLES

The "risk-neutral" dynamics of the fuzzy drift parameter model are given by

$$\frac{dS}{S} = (r + \Delta\eta) dt + \sigma dW_t$$

where $\Delta\eta = \eta_1 - \eta_2$ and η_1, η_2 have the same membership function. In order to make sense of this equation, one needs to fully understand the fuzzy variable $\Delta\eta$, that is, what membership function does $\Delta\eta$ have. In order to calculate this function we utilise Zadeh's extension principle.^[13]

Theorem 37 *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent fuzzy variables with membership functions $\mu_1, \mu_2, \dots, \mu_n$ respectively, and $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a function. Then the membership function μ of $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ is derived from membership functions $\mu_1, \mu_2, \dots, \mu_n$ by*

$$\mu(x) = \sup_{x=f(x_1, x_2, \dots, x_n)} \min_{1 \leq i \leq n} \mu_i(x_i)$$

for any $x \in \mathbb{R}$. Here we set $\mu(x) = 0$ if there are no real numbers x_1, x_2, \dots, x_n such that $x = f(x_1, x_2, \dots, x_n)$.

Thus our problem of finding the membership function of $\eta_1 - \eta_2$, is reduced to finding

$$\mu_{\Delta\eta}(x) = \sup_{x=\eta_1-\eta_2} \min_{1 \leq i \leq 2} \mu_i(x_i) \tag{2.01}$$

The above depends on the membership functions of η_i . In this thesis, we use three different fuzzy variables. Namely, the triangular, trapezoidal and normal fuzzy

variables. We make the observation that for all three fuzzy variables (2.01) is maximised when $\mu_1(x_1) = \mu_2(x_2)$

2.1 Symmetrical Triangular Fuzzy Variable

In utilising Zadeh's extension principle to calculate the membership function of $\Delta\eta$, we are faced with two cases. The first being that $\Delta\eta \geq 0$ and thus that $x_1 \geq x_2$. The second being $\Delta\eta < 0$ and thus $x_1 < x_2$.

Case 1: $\Delta\eta \geq 0$ If $\Delta\eta \geq 0$, then $x_1 \geq x_2$. We have the following two equations

$$\begin{aligned}\Delta\eta &= x_1 - x_2 \\ \frac{x_2 - a}{b - a} &= \frac{x_1 - c}{b - c}\end{aligned}$$

The first equation, follows by definition of $\Delta\eta$. The second equation follows from our observation that the two membership functions are equal. Solving the second equation for x_2 we obtain

$$x_2 = \frac{(b - a)x_1}{b - c} - \frac{c(b - a)}{b - c} + a$$

substituting this value in our first equation and solving for x_1 we obtain

$$x_1 = \frac{(\Delta\eta + a)(b - c) - c(b - a)}{a - c}$$

Now substitute our value of x_1 into its known membership function.

$$\begin{aligned}
& \frac{x_1 - c}{b - c} \\
= & \frac{\Delta\eta(b - c) - b(c - a) - c(a - c)}{(a - c)(b - c)} \\
= & \frac{\Delta\eta - (c - a)}{-(c - a)}
\end{aligned}$$

As such we have found the membership function for $\Delta\eta \geq 0$.

Case 2. $\Delta\eta < 0$ If $\Delta\eta \leq 0$, then $x_2 > x_1$. In a similar fashion to the previous case, we have the following two equations

$$\begin{aligned}
\Delta\eta &= x_1 - x_2 \\
\frac{x_1 - a}{b - a} &= \frac{x_2 - c}{b - c}
\end{aligned}$$

Solving the second equation for x_1 we obtain

$$x_1 = \frac{(b - a)x_2}{b - c} - \frac{c(b - a)}{(b - c)} + a - x_2$$

Substituting this value in our first equation and solving for x_2 we obtain

$$x_2 = \frac{\Delta\eta(b - c) + b(c - a)}{(c - a)}$$

Substituting our value of x_2 into its known membership function, we obtain the membership function of $\Delta\eta < 0$

$$\begin{aligned}
& \frac{x_2 - c}{b - c} \\
= & \frac{\Delta\eta(b - c) + b(c - a) - c(c - a)}{(b - c)(c - a)} \\
= & \frac{\Delta\eta + (c - a)}{(c - a)}
\end{aligned}$$

Putting the membership functions of $\Delta\eta \geq 0$ and $\Delta\eta < 0$ together we arrive at the membership function of $\Delta\eta$.

$$\mu(\Delta\eta) = \begin{cases} \frac{\Delta\eta + (c - a)}{(c - a)}, & \text{if } \Delta\beta < 0 \\ \frac{\Delta\eta - (c - a)}{-(c - a)}, & \text{if } \Delta\beta \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (2.12)$$

That is, $\Delta\eta$ is a triangular fuzzy variable with parameters $(-(c - a), 0, (c - a))$.

2.2 Symmetrical Trapezoidal Fuzzy Variable

In calculating the symmetrical trapezoidal fuzzy variable we have four cases.

Case 1: $|\Delta\eta| \geq (d - a)$ In the first case we have that $\mu(\Delta\eta) = 0$. This is equality easily verified by examining the membership function of the trapezoidal fuzzy variable and Zadeh's extension principle.

Case 2: $(c - b) \leq \Delta\eta \leq (d - a)$ In the case of $(c - b) \leq \Delta\eta \leq (d - a)$, we have $c \leq x_1 \leq d$ and $a \leq x_2 \leq b$ and thus that

$$\begin{aligned}\Delta\eta &= x_1 - x_2 \\ \frac{x_1 - d}{c - d} &= \frac{x_2 - a}{b - a}\end{aligned}$$

where the first equation follows from the definition of $\Delta\eta$ and the second from the membership function of the trapezoidal fuzzy variable. Solving for x_2 we obtain

$$x_2 = \frac{(\Delta\eta - d)(b - a) + a(c - d)}{(c - b) - (b + d)}$$

substituting this value into our membership function for x_2 we obtain

$$\frac{\Delta\eta - (d - a)}{(c - b) - (d - a)}$$

which is the membership function of $\Delta\eta$ in the case that $(c - b) \leq \Delta\eta \leq (d - a)$.

Case 3: $-(d - a) \leq \Delta\eta \leq -(c - b)$ In the case of $-(d - a) \leq \Delta\eta \leq -(c - b)$, we have

$a \leq x_1 \leq b$ and $c \leq x_2 \leq d$ and thus that

$$\begin{aligned}\Delta\eta &= x_1 - x_2 \\ \frac{x_2 - d}{c - d} &= \frac{x_1 - a}{b - a}\end{aligned}$$

where the first equation follows from the definition of $\Delta\eta$ and the second from the membership function of the trapezoidal fuzzy variable. Solving for x_1 we obtain

$$\eta_1 = \frac{(\Delta\eta - d)(b - a) + a(c - d)}{-(c - b) - (-(d - a))}$$

substituting this value into our membership function for x_1 we obtain

$$\frac{\Delta\eta - (-(d-a))}{(c-b) - (d-a)}$$

which is the membership function of $\Delta\eta$ in the case that $-(d-a) \leq \Delta\eta \leq -(c-b)$.

Case 4: $|\Delta\eta| \leq (c-b)$ In the last case we have that $\mu(\Delta\eta) = \frac{1}{2}$. This equation is easily verified by examining the membership function of the trapezoidal fuzzy variable and Zadeh's extension principle.

Putting all the cases together we obtain the membership function of $\Delta\eta$.

$$\mu_{\Delta\eta} = \begin{cases} \frac{\Delta\eta - (-(d-a))}{(c-b) - (d-a)} & \text{if } (c-b) \leq \Delta\eta \leq (d-a) \\ \frac{1}{2} & \text{if } |\Delta\eta| \leq (c-b) \\ \frac{\Delta\eta - (d-a)}{(c-b) - (d-a)} & \text{if } -(d-a) \leq \Delta\eta \leq -(c-b) \\ 0 & \text{otherwise} \end{cases} \quad (2.23)$$

That is, $\Delta\eta$ is a trapezoidal fuzzy variable with parameters $(-(d-a), -(c-b), (c-b), (d-a))$.

2.3 Symmetrical Normal Fuzzy

In utilising Zadeh's extension principle to calculate the membership function of $\Delta\eta$, we are faced with two cases. The first being that $\Delta\eta \geq 0$ and thus that $x_1 \geq x_2$. The second being $\Delta\eta < 0$ and thus $x_1 < x_2$.

Case 1: $\Delta\eta \geq 0$ If $\Delta\eta \geq 0$, then $x_1 \geq x_2$. Thus we have the following two equations

$$\begin{aligned}\Delta\eta &= x_1 - x_2 \\ 2(1 + \exp(\frac{\pi(x_1 - e)}{\sqrt{6}\sigma})) &= 2(1 + \exp(\frac{\pi(x_2 - e)}{\sqrt{6}\sigma}))\end{aligned}$$

The first equation, follows by definition of $\Delta\eta$. The second equation follows from our observation that the two membership functions are equal and symmetrical about e . Solving for x_1 we obtain

$$x_1 = e - \frac{\Delta\eta}{2}$$

substituting this value into it's respective membership function we obtain the membership function for $\mu(\Delta\eta)$ where $\Delta\eta \geq 0$

$$\mu(\Delta\eta) = 2(1 + \exp(\frac{\pi\Delta\eta}{2\sqrt{6}\sigma}))$$

Case 2: $\Delta\eta < 0$ If $\Delta\eta \geq 0$, then $x_2 \geq x_1$. Thus we have the following two equations

$$\begin{aligned}\Delta\eta &= x_1 - x_2 \\ 2(1 + \exp(\frac{\pi(x_1 - e)}{\sqrt{6}\sigma})) &= 2(1 + \exp(\frac{\pi(x_2 - e)}{\sqrt{6}\sigma}))\end{aligned}$$

The first equation, follows by definition of $\Delta\eta$. The second equation follows from our observation that the two membership functions are equal and symmetrical about e . Solving for x_1 we obtain

$$x_1 = e - \frac{\Delta\eta}{2}$$

substituting this value into it's respective membership function we obtain the membership function for $\mu(\Delta\eta)$ where $\Delta\eta < 0$

$$\mu(\Delta\eta) = 2(1 + \exp(\frac{-\pi\Delta\eta}{2\sqrt{6}\sigma}))$$

Putting both cases together we obtain the membership function of $\Delta\eta$

$$\mu(\Delta\eta) = \begin{cases} \mu(\Delta\eta) = 2(1 + \exp(\frac{-\pi\Delta\eta}{2\sqrt{6}\sigma})) & \text{if } \Delta\eta < 0 \\ \mu(\Delta\eta) = 2(1 + \exp(\frac{\pi\Delta\eta}{2\sqrt{6}\sigma})) & \text{if } \Delta\eta \geq 0 \end{cases}$$

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APPENDIX C.

AVERAGE CHANCE MEASURES

Suppose ξ is a hybrid variable that is distributed normally with fuzzy mean, η , and known variance σ^2 . That is $\xi \sim N(\eta, \sigma^2)$. Within this thesis, we are interested in two events. These being $\text{ch}\{\xi \leq x\}$ and $\text{ch}\{\xi = x\}$ where $\text{ch}\{\cdot\}$ represents the average chance measure as given by (2.1). The first event, $\text{ch}\{\xi \leq x\}$, is known as the average chance distribution for a random normal variable with fuzzy mean and the second as the average chance density for a random normal variable with fuzzy mean. In this chapter, the relevant distributions and densities will be calculated.

3.1 Average Chance Distribution for a Random Normal variable with fuzzy mean

The average chance distribution is required when calculating the theoretical price of an European option. Suppose ξ is distributed normally with fuzzy mean, η , and known variance σ^2 . That is $\xi \sim N(\eta, \sigma^2)$. The average chance measure for this event is defined as

$$\begin{aligned}\text{ch}\{\xi \leq x\} &= \int_0^1 \text{Cr}\{\text{Pr}\{\xi \leq x\} \geq \alpha\} d\alpha \\ &= \int_0^1 \text{Cr}\left\{\Phi\left(\frac{x-\eta}{\sigma}\right) \geq \alpha\right\} d\alpha\end{aligned}$$

where $\Phi(\cdot)$ is the normal cumulative distribution function. Rearranging the inner bracket, we obtain

$$\begin{aligned}\text{ch}\{\xi \leq x\} &= \int_0^1 \text{Cr}\left\{\frac{x-\eta}{\sigma} \geq \Phi^{-1}(\alpha)\right\} d\alpha \\ &= \int_0^1 \text{Cr}\{\eta \leq x - \sigma\Phi^{-1}(\alpha)\} d\alpha\end{aligned}\tag{3.11}$$

The average chance distribution now depends on the fuzzy variable η . In this thesis three fuzzy variables are used, namely the Triangular, Trapezoidal and Normal.

Case 1: Triangular Fuzzy Variable In this case, we assume that η is a triangular fuzzy variable with parameters (a, b, c) . Recall that a triangular fuzzy variable has membership function

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{x-c}{b-c} & \text{if } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

and credibility distribution

$$\text{Cr}\{\eta \leq x\} = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)} & \text{if } b \leq x \leq c \\ 1 & \text{if } x > c \end{cases}$$

Substituting this credibility distribution in (3.11) we obtain

$$\text{Cr}\{\eta \leq x - \sigma\Phi^{-1}(\alpha)\} = \begin{cases} 0 & \text{if } \Phi\left(\frac{x-a}{\sigma}\right) < \alpha \leq 1 \\ \frac{x - \sigma\Phi^{-1}(\alpha) - a}{2(b-a)} & \text{if } \Phi\left(\frac{x-b}{\sigma}\right) \leq \alpha \leq \Phi\left(\frac{x-a}{\sigma}\right) \\ \frac{x - \sigma\Phi^{-1}(\alpha) + c - 2b}{2(c-b)} & \text{if } \Phi\left(\frac{x-c}{\sigma}\right) < \alpha \leq \Phi\left(\frac{x-b}{\sigma}\right) \\ 1 & \text{if } 0 \leq \alpha \leq \Phi\left(\frac{x-c}{\sigma}\right) \end{cases}$$

which results in the integral

$$\int_{\Phi\left(\frac{x-b}{\sigma}\right)}^{\Phi\left(\frac{x-a}{\sigma}\right)} \frac{x - \sigma\Phi^{-1}(\alpha) - a}{2(b-a)} d\alpha + \int_{\Phi\left(\frac{x-c}{\sigma}\right)}^{\Phi\left(\frac{x-b}{\sigma}\right)} \frac{x - \sigma\Phi^{-1}(\alpha) + c - 2b}{2(c-b)} d\alpha + \int_0^{\Phi\left(\frac{x-c}{\sigma}\right)} 1 d\alpha$$

Calculating this integral, we obtain ¹

$$\begin{aligned} \text{ch}\{\xi \leq x\} &= \frac{x-a}{2(b-a)} \left(\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(\frac{x-b}{\sigma}\right) \right) + \frac{x+c-2b}{2(c-b)} \left(\Phi\left(\frac{x-b}{\sigma}\right) - \Phi\left(\frac{x-c}{\sigma}\right) \right) \\ &+ \Phi\left(\frac{x-c}{\sigma}\right) + \frac{\sigma}{2(b-a)} \left(\phi\left(\frac{x-a}{\sigma}\right) - \phi\left(\frac{x-b}{\sigma}\right) \right) + \frac{\sigma}{2(c-b)} \left(\phi\left(\frac{x-b}{\sigma}\right) - \phi\left(\frac{x-c}{\sigma}\right) \right) \end{aligned} \quad (3.12)$$

Case 2: Trapezoidal Fuzzy Variable In this case, we assume that η is a trapezoidal fuzzy variable with parameters (a, b, c, d) . Recall that a trapezoidal fuzzy variable has membership function

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d} & \text{if } c \leq x \leq d \\ 0 & \text{otherwise} \end{cases}$$

and credibility distribution

$$\Lambda(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{1}{2} & \text{if } b \leq x \leq c \\ \frac{x+d-2c}{2(d-c)} & \text{if } c \leq x \leq d \\ 1 & \text{if } x > d \end{cases}$$

Substituting this credibility distribution in (3.11) we obtain

¹In order to calculate the integral $\int \Phi^{-1}(\alpha) d\alpha$, one has to make the substitution $\Phi(s) = \alpha$. This substitution results in the integral $\int s\phi(s)ds$ which is equal to $-\phi(s)$. Note that the bounds change after the first substitution.

$$\text{Cr} \{ \eta \leq x - \sigma \Phi^{-1}(\alpha) \} = \begin{cases} 0 & \text{if } \Phi \left(\frac{x-a}{\sigma} \right) < \alpha \leq 1 \\ \frac{x - \sigma \Phi^{-1}(\alpha) - a}{2(b-a)} & \text{if } \Phi \left(\frac{x-b}{\sigma} \right) \leq \alpha \leq \Phi \left(\frac{x-a}{\sigma} \right) \\ \frac{1}{2} & \text{if } \Phi \left(\frac{x-c}{\sigma} \right) < \alpha \leq \Phi \left(\frac{x-b}{\sigma} \right) \\ \frac{x - \sigma \Phi^{-1}(\alpha) + d - 2c}{2(d-c)} & \text{if } \Phi \left(\frac{x-d}{\sigma} \right) < \alpha \leq \Phi \left(\frac{x-c}{\sigma} \right) \\ 1 & \text{if } 0 \leq \alpha \leq \Phi \left(\frac{x-d}{\sigma} \right) \end{cases}$$

which results in the integral

$$\int_{\Phi \left(\frac{x-b}{\sigma} \right)}^{\Phi \left(\frac{x-a}{\sigma} \right)} \frac{x - \sigma \Phi^{-1}(\alpha) - a}{2(b-a)} d\alpha + \int_{\Phi \left(\frac{x-c}{\sigma} \right)}^{\Phi \left(\frac{x-b}{\sigma} \right)} \frac{1}{2} d\alpha + \int_{\Phi \left(\frac{x-d}{\sigma} \right)}^{\Phi \left(\frac{x-c}{\sigma} \right)} \frac{x - \sigma \Phi^{-1}(\alpha) + d - 2c}{2(d-c)} d\alpha + \int_0^{\Phi \left(\frac{x-d}{\sigma} \right)} 1 d\alpha$$

Calculating this integral, we obtain

$$\begin{aligned} \text{ch} \{ \xi \leq x \} &= \frac{x-a}{2(b-a)} \left(\Phi \left(\frac{x-a}{\sigma} \right) - \Phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{x+d-2c}{2(d-c)} \left(\Phi \left(\frac{x-c}{\sigma} \right) - \Phi \left(\frac{x-d}{\sigma} \right) \right) \\ &+ \frac{1}{2} \left(\Phi \left(\frac{x-b}{\sigma} \right) - \Phi \left(\frac{x-c}{\sigma} \right) \right) + \Phi \left(\frac{x-d}{\sigma} \right) \\ &+ \frac{\sigma}{2(b-a)} \left(\phi \left(\frac{x-a}{\sigma} \right) - \phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{\sigma}{2(d-c)} \left(\phi \left(\frac{x-c}{\sigma} \right) - \phi \left(\frac{x-d}{\sigma} \right) \right) \end{aligned} \quad (3.13)$$

Case 3: Normal Fuzzy Variable In this case, we assume that η is a normal fuzzy variable with parameters (e, σ_n) . Recall that a normal fuzzy variable has membership function

$$\mu(x) = \frac{2}{\left(1 + \exp \left(\frac{\pi |x-e|}{\sqrt{6}\sigma_n} \right) \right)}$$

and credibility distribution

$$\Lambda(x) = \begin{cases} \frac{1}{\left(1 + \exp \left(\frac{-\pi(x-e)}{\sqrt{6}\sigma_n} \right) \right)}, & \text{if } x < e \\ 1 - \frac{1}{\left(1 + \exp \left(\frac{\pi(x-e)}{\sqrt{6}\sigma_n} \right) \right)}, & \text{if } x \geq e \end{cases}$$

Substituting this credibility distribution in (3.11) we obtain

$$\text{Cr} \{ \eta \leq x - \sigma \Phi^{-1}(\alpha) \} = \begin{cases} \frac{1}{\left(1 + \exp\left(\frac{-\pi(x - \sigma \Phi^{-1}(\alpha) - e)}{\sqrt{6}\sigma_n}\right)\right)} & \text{if } \Phi\left(\frac{x-e}{\sigma}\right) < \alpha \leq 1 \\ 1 - \frac{1}{\left(1 + \exp\left(\frac{\pi(x - \sigma \Phi^{-1}(\alpha) - e)}{\sqrt{6}\sigma_n}\right)\right)} & \text{if } 0 \leq \alpha \leq \Phi\left(\frac{x-e}{\sigma}\right) \end{cases}$$

which results in the integral

$$\int_{\Phi\left(\frac{x-e}{\sigma}\right)}^1 \frac{1}{\left(1 + \exp\left(\frac{-\pi(x - \sigma \Phi^{-1}(\alpha) - e)}{\sqrt{6}\sigma_n}\right)\right)} d\alpha + \int_0^{\Phi\left(\frac{x-e}{\sigma}\right)} 1 - \frac{1}{\left(1 + \exp\left(\frac{\pi(x - \sigma \Phi^{-1}(\alpha) - e)}{\sqrt{6}\sigma_n}\right)\right)} d\alpha$$

Calculating this integral, we obtain

$$\Phi\left(\frac{x-e}{\sigma}\right) - \int_{-\infty}^{\frac{x-e}{\sigma}} \frac{\phi(s) ds}{1 + \exp\left(\frac{-\pi(\sigma S - x + e)}{\sqrt{6}\sigma_n}\right)} + \int_{\frac{x-e}{\sigma}}^{\infty} \frac{\phi(s) ds}{1 + \exp\left(\frac{\pi(\sigma S - x + e)}{\sqrt{6}\sigma_n}\right)} \quad (3.14)$$

3.2 Average Chance Density for a Random Normal variable with fuzzy mean

The average chance density is required when estimating parameters for the fuzzy drift parameter model. Suppose ξ is distributed normally with fuzzy mean, η and known variance σ^2 . We impose the added restriction that η is centred around 0. Thus we have that $\xi \sim N(\eta, \sigma^2)$. The average chance measure for the event $\text{ch} \{ \xi = x \}$ is defined as

$$\begin{aligned} \text{ch} \{ \xi = x \} &= \int_0^1 \text{Cr} \{ \text{Pr} \{ \xi = x \} \geq \alpha \} d\alpha \\ &= \int_0^1 \text{Cr} \left\{ \frac{1}{\sigma} \phi\left(\frac{x - \eta}{\sigma}\right) \geq \alpha \right\} d\alpha \end{aligned} \quad (3.25)$$

where $\phi(\cdot)$ is the normal density function. To proceed one would need to rearrange the inequality $\frac{1}{\sigma} \phi\left(\frac{x - \eta}{\sigma}\right) \geq \alpha$ to have η as its subject. However there is a problem in that the normal density function is not one to one and thus there is no unique inverse. We

do however know that if $\phi(x) = \alpha$ then $\phi(-x) = \alpha$ as well. Let us define $\phi_+^{-1}(\alpha)$ to be the positive inverse and $\phi_-^{-1}(\alpha)$ to be the negative. Due to the normal density function not being one to one, we have that the calculation of our inequality will be different if x is positive or negative. In the case that x is positive we have that

$$\begin{aligned}
& \text{Cr} \left\{ \frac{1}{\sigma} \phi \left(\frac{x - \eta}{\sigma} \right) \geq \alpha \right\} \\
&= \text{Cr} \left\{ \frac{x - \eta}{\sigma} \leq \phi_+^{-1}(\sigma\alpha) \right\} \\
&= \text{Cr} \left\{ \eta \geq x - \sigma\phi_+^{-1}(\sigma\alpha) \right\}
\end{aligned} \tag{3.26}$$

². Note that we have the implied restriction that $\frac{x-\eta}{\sigma} \geq 0$ as $\phi(\cdot)$ is not one to one. In the case that x is negative

$$\begin{aligned}
& \text{Cr} \left\{ \frac{1}{\sigma} \phi \left(\frac{x - \eta}{\sigma} \right) \geq \alpha \right\} \\
&= \text{Cr} \left\{ \frac{x - \eta}{\sigma} \geq \phi_-^{-1}(\sigma\alpha) \right\} \\
&= \text{Cr} \left\{ \eta \leq x - \sigma\phi_-^{-1}(\sigma\alpha) \right\}
\end{aligned} \tag{3.27}$$

where we have the implied restriction that $\frac{x-\eta}{\sigma} \leq 0$. Thus calculating the average chance density depends on the sign of x as well as the membership function of the fuzzy variable. In this thesis three fuzzy variables are used. These variables are the triangular, trapezoidal and normal. The average chance densities

Case 1 : Triangular Fuzzy Variable In this case, we assume that η is a triangular fuzzy variable with parameters (a, b, c) . Here we have that $a < 0 = b < c$, where the signs

²This result follows by the definition of independence within credibility theory.

of each parameter are required by the necessity of having the fuzzy variable centred around 0. Recall that a triangular fuzzy variable has membership function

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{x-c}{b-c} & \text{if } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}$$

with events

$$\text{Cr} \{ \eta \leq x \} = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)} & \text{if } b \leq x \leq c \\ 1 & \text{if } x > c \end{cases} \quad (3.28)$$

and

$$\text{Cr} \{ \eta \geq x \} = \begin{cases} 1 & \text{if } x < a \\ \frac{-a-x}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{c-x}{2(c-b)} & \text{if } b \leq x \leq c \\ 0 & \text{if } x > c \end{cases} \quad (3.29)$$

Subcase 1: $x \geq 0$ By (3.26), the average chance measure for this event is

$$\int_0^1 \text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \} d\alpha \quad (3.210)$$

By (3.26) and (3.28) the event $\text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \}$ is defined as

$$\text{Cr} \left\{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \right\} = \begin{cases} 1 & \text{if } 0 \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right) \\ \frac{(x - \sigma \phi_+^{-1}(\sigma \alpha)) + a}{2a} & \text{if } \frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right) \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) \\ \frac{c - (x - \sigma \phi_+^{-1}(\sigma \alpha))}{2c} & \text{if } \frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right) \\ 0 & \text{if } \alpha \geq \frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right) \end{cases}$$

where $F_1 = \max \{x - c, 0\}$ and is motivated by the restriction $\frac{x-\eta}{\sigma} \geq 0$. We have also used the fact that $b = 0$ in the above. This credibility function results in the integral

$$\int_0^{\frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right)} 1 d\alpha + \int_{\frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right)}^{\frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right)} \frac{(x - \sigma \phi_+^{-1}(\sigma \alpha)) + a}{2a} d\alpha + \int_{\frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right)}^{\frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right)} \frac{c - (x - \sigma \phi_+^{-1}(\sigma \alpha))}{2c} d\alpha$$

Calculating the integral, we obtain ³

$$\frac{1}{2a} \left(\Phi \left(\frac{x-a}{\sigma} \right) - \Phi \left(\frac{x}{\sigma} \right) \right) + \frac{1}{2c} \left(\Phi \left(\frac{x}{\sigma} \right) - \Phi \left(\frac{F_1}{\sigma} \right) \right) + \frac{c-x+F_1}{2\sigma c} \phi \left(\frac{F_1}{\sigma} \right)$$

Subcase 2: $x < 0$ By (3.27) , the average chance measure for this event is

$$\int_0^1 \text{Cr} \left\{ \eta \leq x - \sigma \phi_-^{-1}(\sigma \alpha) \right\} d\alpha \quad (3.211)$$

By (3.27) and (3.29) the event $\text{Cr} \left\{ \eta \leq x - \sigma \phi_-^{-1}(\sigma \alpha) \right\}$ is defined as

³In order to calculate the integral two substitutions are required. These substitutions are $\sigma \alpha = \alpha_*$ and $\phi(\alpha_*) = s$. The positive inverse is used here by the bounds of the problem.

$$\text{Cr} \{ \eta \leq x - \sigma \phi_{-}^{-1}(\sigma \alpha) \} = \begin{cases} 1 & \text{if } 0 \leq \alpha < \phi \left(\frac{x-c}{\sigma} \right) \\ \frac{x - \sigma \phi_{-}^{-1}(\sigma \alpha) + c - 2b}{2c} & \text{if } \phi \left(\frac{x-c}{\sigma} \right) \leq \alpha \leq \phi \left(\frac{x}{\sigma} \right) \\ \frac{-x + \sigma \phi_{-}^{-1}(\sigma \alpha) + a}{2a} & \text{if } \phi \left(\frac{x}{\sigma} \right) \leq \alpha \leq \phi \left(\frac{F_1}{\sigma} \right) \\ 0 & \text{if } \alpha > \phi \left(\frac{F_1}{\sigma} \right) \end{cases}$$

where $F_1 = \min \{ x - a, 0 \}$ and is motivated by the restriction $\frac{x-\eta}{\sigma} \leq 0$. We have also used the fact that $b = 0$ in the above. The above results in the integral

$$\int_0^{\phi \left(\frac{x-c}{\sigma} \right)} 1 d\alpha + \int_{\phi \left(\frac{x-c}{\sigma} \right)}^{\phi \left(\frac{x}{\sigma} \right)} \frac{x - \sigma \phi_{-}^{-1}(\sigma \alpha) + c - 2b}{2c} d\alpha + \int_{\phi \left(\frac{x}{\sigma} \right)}^{\phi \left(\frac{F_1}{\sigma} \right)} \frac{-x + \sigma \phi_{-}^{-1}(\sigma \alpha) + a}{2a} d\alpha$$

Calculating the integral, we obtain

$$\frac{1}{2a} \left(\Phi \left(\frac{F_1}{\sigma} \right) - \Phi \left(\frac{x}{\sigma} \right) \right) + \frac{1}{2c} \left(\Phi \left(\frac{x}{\sigma} \right) - \Phi \left(\frac{x-c}{\sigma} \right) \right) + \frac{a-x+F_1}{2\sigma a} \phi \left(\frac{F_1}{\sigma} \right)$$

Putting subcase 1 and subcase 2 together, we obtain

$$\text{ch} \{ \xi = x \} = \begin{cases} \frac{1}{2a} \left(\Phi \left(\frac{F_1}{\sigma} \right) - \Phi \left(\frac{x}{\sigma} \right) \right) + \frac{1}{2c} \left(\Phi \left(\frac{x}{\sigma} \right) - \Phi \left(\frac{x-c}{\sigma} \right) \right) + \frac{a-x+F_1}{2\sigma a} \phi \left(\frac{F_1}{\sigma} \right), \\ \text{if } x < 0 \text{ and } F_1 = \min \{ x - a, 0 \} \\ \frac{1}{2a} \left(\Phi \left(\frac{x-a}{\sigma} \right) - \Phi \left(\frac{x}{\sigma} \right) \right) + \frac{1}{2c} \left(\Phi \left(\frac{x}{\sigma} \right) - \Phi \left(\frac{F_1}{\sigma} \right) \right) + \frac{c-x+F_1}{2\sigma c} \phi \left(\frac{F_1}{\sigma} \right), \\ \text{if } x \geq 0 \text{ and } F_1 = \max \{ x - c, 0 \} \end{cases} \quad (3.212)$$

Case 2: Trapezoidal Fuzzy Variable In this case, we assume that η is a trapezoidal fuzzy variable with parameters (a, b, c, d) . Here we have that $a < b < 0 < c < d$, where

the signs of each parameter are required by the necessity of having the fuzzy variable centred around 0. Recall that a trapezoidal fuzzy variable has membership function

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d} & \text{if } c \leq x \leq d \\ 0 & \text{otherwise} \end{cases}$$

with events

$$\text{Cr}\{\eta \leq x\} = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{1}{2} & \text{if } b \leq x \leq c \\ \frac{x+d-2c}{2(d-c)} & \text{if } c \leq x \leq d \\ 1 & \text{if } x > d \end{cases} \quad (3.213)$$

and

$$\text{Cr}\{\eta \geq x\} = \begin{cases} 1 & \text{if } x < a \\ \frac{2b-x-a}{2(b-a)} & \text{if } a \leq x \leq b \\ \frac{1}{2} & \text{if } b \leq x \leq c \\ \frac{d-x}{2(d-c)} & \text{if } c \leq x \leq d \\ 0 & \text{if } x > d \end{cases} \quad (3.214)$$

Subcase 1: $x \geq 0$ By (3.26) , the average chance measure for this event is

$$\int_0^1 \text{Cr}\{\eta \geq x - \sigma\phi_+^{-1}(\sigma\alpha)\} d\alpha \quad (3.215)$$

By (3.26) and (3.214) the event $\text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \}$ is defined as

$$\text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \} = \begin{cases} 1 & \text{if } 0 \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right) \\ \frac{2b - (x - \sigma \phi_+^{-1}(\sigma \alpha)) - a}{2(b-a)} & \text{if } \frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right) \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{x-b}{\sigma} \right) \\ \frac{1}{2} & \text{if } \frac{1}{\sigma} \phi \left(\frac{x-b}{\sigma} \right) \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right) \\ \frac{d - (x - \sigma \phi_+^{-1}(\sigma \alpha))}{2(d-c)} & \text{if } \frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right) \leq \alpha < \frac{1}{\sigma} \phi \left(\frac{F_2}{\sigma} \right) \\ 0 & \text{if } \alpha \geq \frac{1}{\sigma} \phi \left(\frac{F_2}{\sigma} \right) \end{cases}$$

where $F_1 = \max \{ x - c, 0 \}$ and $F_2 = \max \{ x - d, 0 \}$. F_1 and F_2 are motivated by the restriction $\frac{x-\eta}{\sigma} \geq 0$. The credibility function results in the integral

$$\begin{aligned} & \int_0^{\frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right)} 1 d\alpha + \int_{\frac{1}{\sigma} \phi \left(\frac{x-a}{\sigma} \right)}^{\frac{1}{\sigma} \phi \left(\frac{x-b}{\sigma} \right)} \frac{2b - (x - \sigma \phi_+^{-1}(\sigma \alpha)) - a}{2(b-a)} d\alpha \\ & + \int_{\frac{1}{\sigma} \phi \left(\frac{x-b}{\sigma} \right)}^{\frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right)} \frac{1}{2} d\alpha + \int_{\frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right)}^{\frac{1}{\sigma} \phi \left(\frac{F_2}{\sigma} \right)} \frac{d - (x - \sigma \phi_+^{-1}(\sigma \alpha))}{2(d-c)} d\alpha \end{aligned}$$

Calculating this integral, we obtain

$$\begin{aligned} & \frac{1}{2(b-a)} \left(\Phi \left(\frac{x-a}{\sigma} \right) - \Phi \left(\frac{x-b}{\sigma} \right) \right) + \frac{1}{2(d-c)} \left(\Phi \left(\frac{F_1}{\sigma} \right) - \Phi \left(\frac{F_2}{\sigma} \right) \right) \\ & + \frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right) \left(\frac{1}{2} - \frac{d-x+F_1}{2(d-c)} \right) + \frac{1}{\sigma} \phi \left(\frac{F_2}{\sigma} \right) \left(\frac{d-x+F_2}{2(d-c)} \right) \end{aligned}$$

Subcase 2: $x < 0$ By (3.27), the average chance measure for this event is

$$\int_0^1 \text{Cr} \{ \eta \leq x - \sigma \phi_-^{-1}(\sigma \alpha) \} d\alpha \quad (3.216)$$

By (3.27) and (3.215) the event $\text{Cr} \{ \eta \leq x - \sigma \phi_-^{-1}(\sigma \alpha) \}$ is defined as

$$\text{Cr} \left\{ \eta \leq x - \sigma \phi_{-}^{-1}(\sigma \alpha) \right\} = \begin{cases} 1 & \text{if } 0 \leq \alpha < \phi \left(\frac{x-d}{\sigma} \right) \\ \frac{x - \sigma \phi_{-}^{-1}(\sigma \alpha) + d - 2c}{2(d-c)} & \text{if } \phi \left(\frac{x-d}{\sigma} \right) \leq \alpha \leq \phi \left(\frac{x-c}{\sigma} \right) \\ \frac{1}{2} & \text{if } \phi \left(\frac{x-c}{\sigma} \right) \leq \alpha \leq \phi \left(\frac{F_1}{\sigma} \right) \\ \frac{x - \sigma \phi_{-}^{-1}(\sigma \alpha) - a}{2(b-a)} & \text{if } \phi \left(\frac{F_1}{\sigma} \right) \leq \alpha \leq \phi \left(\frac{F_2}{\sigma} \right) \\ 0 & \text{if } \alpha > \phi \left(\frac{F_2}{\sigma} \right) \end{cases}$$

where $F_1 = \min \{x - b, 0\}$ and $F_2 = \min \{x - a, 0\}$. F_1 and F_2 are motivated by the restriction $\frac{x-\eta}{\sigma} \leq 0$. The credibility function results in the integral

$$\begin{aligned} & \int_0^1 1 d\alpha + \int_{\frac{1}{\sigma}\phi\left(\frac{x-d}{\sigma}\right)}^{\frac{1}{\sigma}\phi\left(\frac{x-c}{\sigma}\right)} \frac{x - \sigma \phi_{-}^{-1}(\sigma \alpha) + d - 2c}{2(d-c)} d\alpha \\ & + \int_{\frac{1}{\sigma}\phi\left(\frac{x-c}{\sigma}\right)}^{\frac{1}{\sigma}\phi\left(\frac{F_1}{\sigma}\right)} \frac{1}{2} d\alpha + \int_{\frac{1}{\sigma}\phi\left(\frac{F_1}{\sigma}\right)}^{\frac{1}{\sigma}\phi\left(\frac{F_2}{\sigma}\right)} \frac{x - \sigma \phi_{-}^{-1}(\sigma \alpha) - a}{2(b-a)} d\alpha \end{aligned}$$

Calculating this integral, we obtain

$$\begin{aligned} & \frac{1}{2(d-c)} \left(\Phi \left(\frac{x-c}{\sigma} \right) - \Phi \left(\frac{x-d}{\sigma} \right) \right) + \frac{1}{2(b-a)} \left(\Phi \left(\frac{F_2}{\sigma} \right) - \Phi \left(\frac{F_1}{\sigma} \right) \right) \\ & + \frac{1}{\sigma} \phi \left(\frac{F_1}{\sigma} \right) \left(\frac{1}{2} - \frac{x-a-F_1}{2(b-a)} \right) + \frac{1}{\sigma} \phi \left(\frac{F_2}{\sigma} \right) \left(\frac{x-a-F_2}{2(d-c)} \right) \end{aligned}$$

Putting subcase 1 and subcase 2 together, we obtain

$$\text{ch}\{\xi = x\} = \begin{cases} \frac{1}{2(d-c)} \left(\Phi\left(\frac{x-c}{\sigma}\right) - \Phi\left(\frac{x-d}{\sigma}\right) \right) + \frac{1}{2(b-a)} \left(\Phi\left(\frac{F_2}{\sigma}\right) - \Phi\left(\frac{F_1}{\sigma}\right) \right) \\ + \frac{1}{\sigma} \phi\left(\frac{F_1}{\sigma}\right) \left(\frac{1}{2} - \frac{x-a-F_1}{2(b-a)} \right) + \frac{1}{\sigma} \phi\left(\frac{F_2}{\sigma}\right) \left(\frac{x-a-F_2}{2(d-c)} \right) \\ \text{if } x < 0 \text{ and } F_1 = \min\{x-b, 0\} \text{ and } F_2 = \min\{x-a, 0\} \\ \frac{1}{2(b-a)} \left(\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(\frac{x-b}{\sigma}\right) \right) + \frac{1}{2(d-c)} \left(\Phi\left(\frac{F_1}{\sigma}\right) - \Phi\left(\frac{F_2}{\sigma}\right) \right) \\ + \frac{1}{\sigma} \phi\left(\frac{F_1}{\sigma}\right) \left(\frac{1}{2} - \frac{d-x+F_1}{2(d-c)} \right) + \frac{1}{\sigma} \phi\left(\frac{F_2}{\sigma}\right) \left(\frac{d-x+F_2}{2(d-c)} \right) \\ \text{if } x \geq 0 \text{ and } F_1 = \max\{x-c, 0\} \text{ and } F_2 = \max\{x-d, 0\} \end{cases} \quad (3.217)$$

Case 3: Normal Fuzzy Variable In this case, we assume that η is a normal fuzzy variable with parameters $(0, \sigma_n)$. The need for the first parameter being equal to 0 is a consequence of the fact that we assume that the fuzzy variable is centred around 0. Recall that a normal fuzzy variable has membership function

$$\mu(x) = \frac{2}{\left(1 + \exp\left(\frac{\pi|x-e|}{\sqrt{6}\sigma_n}\right)\right)}$$

with events

$$\text{Cr}\{\eta \leq x\} = \begin{cases} \frac{1}{\left(1 + \exp\left(\frac{-\pi x}{\sqrt{6}\sigma_n}\right)\right)}, \text{ if } x < 0 \\ 1 - \frac{1}{\left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\sigma_n}\right)\right)}, \text{ if } x \geq 0 \end{cases} \quad (3.218)$$

and

$$\text{Cr}\{\eta \geq x\} = \begin{cases} 1 - \frac{1}{\left(1 + \exp\left(\frac{-\pi x}{\sqrt{6}\sigma_n}\right)\right)}, \text{ if } x < 0 \\ \frac{1}{\left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\sigma_n}\right)\right)}, \text{ if } x \geq 0 \end{cases} \quad (3.219)$$

Subcase 1: $x \geq 0$ By (3.26), the average chance measure for this event is

$$\int_0^1 \text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \} d\alpha \quad (3.220)$$

By (3.26) and (3.218) the event $\text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \}$ is defined as

$$\text{Cr} \{ \eta \geq x - \sigma \phi_+^{-1}(\sigma \alpha) \} = \begin{cases} 1 - \frac{1}{\left(1 + \exp\left(\frac{-\pi(x - \sigma \phi_+^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n}\right)\right)}, & \text{if } 0 \leq \alpha \leq \frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right) \\ \frac{1}{\left(1 + \exp\left(\frac{\pi(x - \sigma \phi_+^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n}\right)\right)}, & \text{if } \frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right) \leq \alpha \leq \frac{1}{\sigma} \phi(0) \end{cases}$$

Here we have used the fact that the maximum value the right hand side of (3.220) can have is $\frac{1}{\sigma} \phi(0)$, due to $\phi(\cdot)$ being one to one. The credibility function results in the integral

$$\int_0^{\frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right)} 1 - \frac{1}{\left(1 + \exp\left(\frac{-\pi(x - \sigma \phi_+^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n}\right)\right)} d\alpha + \int_{\frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right)}^{\frac{1}{\sigma} \phi(0)} \frac{1}{\left(1 + \exp\left(\frac{\pi(x - \sigma \phi_+^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n}\right)\right)} d\alpha$$

which can be calculated to be

$$\frac{1}{\sigma} \phi\left(\frac{x}{\sigma}\right) + \frac{1}{\sigma} \int_0^{\frac{x}{\sigma}} \frac{s \phi(s) ds}{1 + \exp\left(\frac{x - \sigma s}{\sqrt{6}\sigma_n}\right)} - \frac{1}{\sigma} \int_{\frac{x}{\sigma}}^{\infty} \frac{s \phi(s) ds}{1 + \exp\left(\frac{\sigma s - x}{\sqrt{6}\sigma_n}\right)}$$

Subcase 2: $x < 0$ By (3.27), the average chance measure for this event is

$$\int_0^1 \text{Cr} \{ \eta \leq x - \sigma \phi_-^{-1}(\sigma \alpha) \} d\alpha \quad (3.221)$$

By (3.27) and (3.219) the event $\text{Cr} \{ \eta \leq x - \sigma \phi_-^{-1}(\sigma \alpha) \}$ is defined as

$$\text{Cr} \{ \eta \leq x - \sigma \phi^{-1}(\sigma \alpha) \} = \begin{cases} 1 - \frac{1}{\left(1 + \exp \left(\frac{-\pi(x - \sigma \phi^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n} \right) \right)}, & \text{if } 0 \leq \alpha \leq \frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) \\ \frac{1}{\left(1 + \exp \left(\frac{\pi(x - \sigma \phi^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n} \right) \right)}, & \text{if } \frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) \leq \alpha \leq \frac{1}{\sigma} \phi(0) \end{cases}$$

Here we have used the fact that the maximum value the right hand side of (3.221) can have is $\frac{1}{\sigma} \phi(0)$, due to $\phi(\cdot)$ being one to one. The credibility function results in the integral

$$\int_0^{\frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right)} 1 - \frac{1}{\left(1 + \exp \left(\frac{-\pi(x - \sigma \phi^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n} \right) \right)} d\alpha + \int_{\frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right)}^{\frac{1}{\sigma} \phi(0)} \frac{1}{\left(1 + \exp \left(\frac{\pi(x - \sigma \phi^{-1}(\sigma \alpha))}{\sqrt{6}\sigma_n} \right) \right)} d\alpha$$

which can be calculated to be

$$\frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) + \frac{1}{\sigma} \int_{-\infty}^{\frac{x}{\sigma}} \frac{s \phi(s) ds}{1 + \exp \left(\frac{x - \sigma s}{\sqrt{6}\sigma_n} \right)} - \frac{1}{\sigma} \int_{\frac{x}{\sigma}}^0 \frac{s \phi(s) ds}{1 + \exp \left(\frac{\sigma s - x}{\sqrt{6}\sigma_n} \right)}$$

Putting subcase 1 and subcase 2 together we obtain

$$\text{ch} \{ \xi = x \} = \begin{cases} \frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) + \frac{1}{\sigma} \int_{-\infty}^{\frac{x}{\sigma}} \frac{s \phi(s) ds}{1 + \exp \left(\frac{x - \sigma s}{\sqrt{6}\sigma_n} \right)} - \frac{1}{\sigma} \int_{\frac{x}{\sigma}}^0 \frac{s \phi(s) ds}{1 + \exp \left(\frac{\sigma s - x}{\sqrt{6}\sigma_n} \right)} & \text{if } x < 0 \\ \frac{1}{\sigma} \phi \left(\frac{x}{\sigma} \right) + \frac{1}{\sigma} \int_0^{\frac{x}{\sigma}} \frac{s \phi(s) ds}{1 + \exp \left(\frac{x - \sigma s}{\sqrt{6}\sigma_n} \right)} - \frac{1}{\sigma} \int_{\frac{x}{\sigma}}^{\infty} \frac{s \phi(s) ds}{1 + \exp \left(\frac{\sigma s - x}{\sqrt{6}\sigma_n} \right)} & \text{if } x \geq 0 \end{cases} \quad (3.222)$$

APPENDIX D.

MATLAB AND EXCEL CODE

4.1 Excel Code

This thesis implements the particle swarm optimisation method to find the optimal parameters for the fuzzy drift parameter model. The code is very similar for the triangular, trapezoidal and normal case. In order to keep this appendix as short as possible, the code for only the triangular case will be presented. The code used for this thesis can be found in the Excel file entitled "ParameterEstimation.xls" on the CD provided with the thesis.

Within Module - Credibility Measures

```
Public Function calculateKTriangle(a As Double, c As Double, sigma As Double)
```

```
As Double
```

```
Dim i As Integer, tempAnswer As Double
```

```
i = 200
```

```
tempAnswer = credibilityTriangle(-12, a, c, sigma)
```

```
For i = 1 To i - 1
```

```
tempAnswer = tempAnswer + 2 * credibilityTriangle(-12 + (i / 200) * 24,
```

```
a, c, sigma)
```

```
Next i
```

```
tempAnswer = tempAnswer + credibilityTriangle(12, a, c, sigma)
```

```
tempAnswer = tempAnswer * 12 / i
```

```
calculateKTriangle = tempAnswer
```

```
End Function
```

```
Public Function calculateProductFunctionTriangle(a As Double, b As Double,  
sigma As Double)
```

```
Dim i As Integer, tempAnswer As Double, temp As Double, k As Double
```

```
noOfObservations = Sheets("Calibration").Cells(3, 4).Value
```

```
If sigma = 0 Then sigma = 0.01
```

```
tempAnswer = 1
```

```
k = calculateKTriangle(a, b, sigma)
```

```
For i = 1 To noOfObservations
```

```
temp = Sheets("Data").Cells(4 + i, 6).Value
```

```
tempAnswer = tempAnswer * credibilityTriangle(Sheets("Data").Cells(4  
+ i, 7).Value, a, b, sigma) * 10
```

```
tempAnswer = tempAnswer / k
```

```
Next i
```

```
calculateProductFunctionTriangle = tempAnswer * 10
```

```
End Function
```

```
Public Function credibilityTriangle(x As Double, a As Double, b As Double,  
sigma As Double) As Double
```

```
Dim F As Double, tempAnswer As Double, temp As Double
```

```

If x >= 0 Then
    F = WorksheetFunction.Max(0, x - b)
    tempAnswer = (1 / sigma) * WorksheetFunction.NormDist(F / sigma, 0,
1, False) * (b - x + F) / (2 * b)
    tempAnswer = tempAnswer + (1 / (2 * a)) * (WorksheetFunction.NormSDist((x
+ a) / sigma) - WorksheetFunction.NormSDist(x
/ sigma))
    tempAnswer = tempAnswer + (1 / (2 * b)) * (WorksheetFunction.NormSDist(x
/ sigma) - WorksheetFunction.NormSDist(
F / sigma))
Else
    F = WorksheetFunction.Min(0, x + a)
    tempAnswer = (1 / sigma) * WorksheetFunction.NormDist(F / sigma, 0,
1, False) * (a + x - F) / (2 * a)
    tempAnswer = tempAnswer + (1 / (2 * b)) * (WorksheetFunction.NormSDist(x
/ sigma) - WorksheetFunction.NormSDist(
(x - b) / sigma))
    tempAnswer = tempAnswer + (1 / (2 * a)) * (WorksheetFunction.NormSDist(F
/ sigma) - WorksheetFunction.NormSDist(
x / sigma))
End If
credibilityTriangle = tempAnswer
End Function

```

Within Module - Particles

```
Public Sub performParticleSwarmTriangle()
```

```
Dim temp As Double, iterations As Integer, i As Integer, j As Integer,
```

```
k As Double
```

```
Call SetUpParticlesTriangle
```

```
iterations = 2
```

```
For i = 1 To iterations
```

```
    For j = 1 To NoOfParticles
```

```
        Set particles(j) = CalculateNewPosition(particles(j))
```

```
        Set particles(j) = calculateNewVelocity(particles(j))
```

```
    Next j
```

```
Next i
```

```
Sheets("Calibration").Range("A8:E30").Clear
```

```
Sheets("Calibration").Cells(15, 1).Value = "The optimal parameters for your  
search were found to be"
```

```
Sheets("Calibration").Cells(17, 1).Value = "a was found to be equal to "
```

```
& Str(globalBestParticle.aBest)
```

```
Sheets("Calibration").Cells(18, 1).Value = "b was found to be equal to "
```

```
& Str(globalBestParticle.bBest)
```

```
Sheets("Calibration").Cells(19, 1).Value = "sigma by definition was"
```

```
& Str(globalBestParticle.sigmaBest)
```

```
Sheets("Calibration").Cells(20, 1).Value = "The product function was "
```

```

& Str(globalBestParticle.BestMEstimation)

Sheets("SummaryStats").Cells(3, 1).Value = "The optimal parameters for your
search were found to be"

Sheets("SummaryStats").Cells(5, 1).Value = "a was found to be equal to "
Sheets("SummaryStats").Cells(5, 4).Value = globalBestParticle.aBest
Sheets("SummaryStats").Cells(6, 1).Value = "b was found to be equal to "
Sheets("SummaryStats").Cells(6, 4).Value = globalBestParticle.bBest
Sheets("SummaryStats").Cells(7, 1).Value = "sigma by definition was"
Sheets("SummaryStats").Cells(7, 4).Value = globalBestParticle.sigmaBest
Sheets("SummaryStats").Cells(8, 1).Value = "The product function was "
Sheets("SummaryStats").Cells(8, 4).Value = globalBestParticle.BestMEstimation

If Sheets("SummaryStats").Cells(8, 7).Value < globalBestParticle.BestMEstimation
Then
    Sheets("SummaryStats").Cells(5, 7).Value = globalBestParticle.aBest
    Sheets("SummaryStats").Cells(6, 7).Value = globalBestParticle.bBest
    Sheets("SummaryStats").Cells(7, 7).Value = globalBestParticle.sigmaBest
    Sheets("SummaryStats").Cells(8, 7).Value = globalBestParticle.BestMEstimation
End If

k = calculateKTriangle(globalBestParticle.aBest, globalBestParticle.bBest, glob-
alBestParticle.sigmaBest)

For i = 1 To 61

    Sheets("GraphValues").Cells(37 + i, 3).Value = credibilityTriangle(-6 +

```

```

    (i - 1) * 0.2, globalBestParticle.aBest, globalBestParticle.bBest, globalBest-
Particle.sigmaBest) / k

    Sheets("GraphValues").Cells(37 + i, 2).Value = WorksheetFunction.NormDist
(-6 + (i - 1) * 0.2, 0, observedStd, False)

    Sheets("GraphValues").Cells(37 + i, 1).Value = -6 + (i - 1) * 0.2

Next i

Sheets("GraphValues").Cells(33, 3).Value = observedStd

End Sub

Private Function calculateNewVelocity(Position As TriangleParticle) As
TriangleParticle

Dim myRandomNumber As Double, myRandomNumber2 As Double

Set calculateNewVelocity = Position

myRandomNumber = Rnd() * phi
myRandomNumber2 = Rnd() * phi

If calculateNewVelocity.BestMEstimation > globalBestParticle.BestMEstimation

Then

    Set globalBestParticle = Position

End If

calculateNewVelocity.aVelocity = Dampening * (calculateNewVelocity
.aVelocity + myRandomNumber * ((calculateNewVelocity.aBest -
calculateNewVelocity.a)) + myRandomNumber2 * ((globalBestParticle.

```

aBest - calculateNewVelocity.a)))

calculateNewVelocity.bVelocity = Dampening * (calculateNewVelocity
.bVelocity + myRandomNumber * ((calculateNewVelocity.bBest -
calculateNewVelocity.b)) + myRandomNumber2 * ((globalBestParticle.
bBest - calculateNewVelocity.b)))

calculateNewVelocity.sigmaVelocity = Dampening * (calculateNewVelocity
.sigmaVelocity + myRandomNumber * ((calculateNewVelocity.sigmaBest
- calculateNewVelocity.sigma)) + myRandomNumber2 * ((globalBestParticle
.sigmaBest - calculateNewVelocity.sigma)))

End Function

Private Function CalculateNewPosition(Position As TriangleParticle) As
TriangleParticle

Dim temp As Double

Set CalculateNewPosition = Position

CalculateNewPosition.a = Position.a + Position.aVelocity

CalculateNewPosition.b = Position.b + Position.bVelocity

CalculateNewPosition.sigma = Position.sigma + Position.sigmaVelocity

If CalculateNewPosition.a > CalculateNewPosition.b Then

 CalculateNewPosition.b = CalculateNewPosition.a + 0.01

End If

If CalculateNewPosition.a < 0 Then

```

        CalculateNewPosition.a = 0.001
    End If
    If CalculateNewPosition.b < 0 Then
        CalculateNewPosition.b = 0.001
    End If
    If CalculateNewPosition.b > 10 Then
        CalculateNewPosition.b = 10
    End If
    temp = calculateProductFunctionTriangle(CalculateNewPosition.a,
    CalculateNewPosition.b, CalculateNewPosition.sigma)
    If temp > CalculateNewPosition.BestMEstimation Then
        CalculateNewPosition.aBest = CalculateNewPosition.a
        CalculateNewPosition.bBest = CalculateNewPosition.b
        CalculateNewPosition.sigmaBest = CalculateNewPosition.sigma
        CalculateNewPosition.BestMEstimation = temp
    End If
End Function

Public Sub SetUpParticlesTriangle()
    Dim i As Integer, magnitude As Double, direction As Double
    Dim temp As Double, tempBest As Double, counter As Integer

    noOfObservations = Sheets("Calibration").Cells(3, 4).Value
    observedStd = Sheets("Data").Cells(4 + noOfObservations, 6).Value

```

```

NoOfParticles = 30

Dampening = 0.8

phi = 2

ReDim particles(1 To NoOfParticles)

tempBest = 0

particles(1).aBest = Sheets("SummaryStats").Cells(5, 7).Value
particles(1).bBest = Sheets("SummaryStats").Cells(6, 7).Value
particles(1).sigmaBest = Sheets("SummaryStats").Cells(7, 7).Value
particles(1).BestMEstimation = Sheets("SummaryStats").Cells(8, 7).
Value

particles(1).a = particles(1).aBest
particles(1).b = particles(1).bBest
particles(1).sigma = particles(1).sigmaBest

For i = 2 To NoOfParticles
    particles(i).a = Rnd() * 8
    direction = Rnd()

    If direction > 0.5 Then
        direction = 1
    Else
        direction = -1
    End If

    magnitude = Rnd() * 2

```

```

particles(i).aVelocity = magnitude * direction
particles(i).b = Rnd() * 2
direction = Rnd()

    If direction > 0.5 Then
        direction = 1
    Else
        direction = -1
    End If

magnitude = Rnd() * 8
particles(i).bVelocity = magnitude * direction
particles(i).sigma = Sheets("Data").Cells(4 + noOfObservations, 6)
.Value * 3 * Rnd

direction = Rnd()

    If direction > 0.5 Then
        direction = 1
    Else
        direction = -1
    End If

magnitude = Rnd() * 2
particles(i).sigmaVelocity = magnitude * direction
particles(i).BestMEstimation = calculateProductFunctionTriangle
(particles(i).a, particles(i).b, particles(i).sigma)

particles(i).aBest = particles(i).a

```

```

particles(i).bBest = particles(i).b

particles(i).sigmaBest = particles(i).sigma

    If particles(i).BestMEstimation > temp Then

        temp = particles(i).BestMEstimation

        Set globalBestParticle = particles(i)

    End If

Next i

Set globalBestParticle = particles(1)

End Sub

```

4.2 MATLAB Code

MATLAB was used extensively throughout this thesis to simulate the fuzzy drift parameter model. The code for the triangular, trapezoidal and normal case are very similar. In order to keep this thesis as short as possible, only the code for the triangular case shall be presented. The MATLAB code can be found in the CD in the "MATLAB Files - Average Chance Measure" folder.

Code used to generate plots of the Average Chance Distribution :

Provided is the code used to generate the plots of the Average Chance Distribution within Table 4.3. Only the code for the triangular case will be provided as the code for the other variables is very similar.

TestTriangleAverageChance.m

```
function outPut = TestTriangleAverageChance
```

```

K = 20;

sigma = 0.25;

Y0 = 30;

r = 0.08;

T = 0.25;

noOfEntries = 20;

noOfXEntries = noOfEntries;

Answers = ones(3,noOfEntries);

b = 0.08;

tempValue = 0.08;

startX = 0.000000001;

deltaX = (3-startX)/noOfXEntries;

delta = 4/(2*noOfEntries);

tempB = [2*delta:2*delta : 4+2*delta];

tempX = [startX : deltaX : 3];

Answers = chanceMeasureGreaterMatrix (tempX , tempB , T , sigma,r)';

xlabel('Value of X');

ylabel('Value of \Delta \beta');

zlabel('Average Chance Measure');

title('Triangular Fuzzy Variable');

VIEW(45,45)

surface(tempX, tempB , Answers);

```

ChanceMeasureGreaterMatrix.m

```
function cMeasureG = chanceMeasureGreaterMatrix (x , deltaB , T , sigma,r)

sizeOfX = size(x)

sizeOfDeltaB = size(deltaB)

cMeasureG = ones(sizeOfX(2),sizeOfDeltaB(2));

for j = 1 : sizeOfDeltaB(2)

temp= chanceMeasureGreater(x , deltaB(j) , T , sigma,r);

cMeasureG(:,j) = temp';

end
```

ChanceMeasureGreater.m

```
function cMeasureG = chanceMeasureGreater (x , deltaB , T , sigma,r)

% Chance measure greater than

% a , b , c are what our triangle fuzzy variable will take

sizeOfX = size(x);

n = 100 ;% 1000 steps

%delta = (c - a) / n;

for j = 1 :sizeOfX(2)

yStar = log(x(j))/T -r +0.5*(sigma^2);

yA = ((T^0.5)/sigma)*(yStar-(deltaB));

yB = ((T^0.5)/sigma)*(yStar);

yC = ((T^0.5)/sigma)*(yStar+(deltaB));

temp = (yStar/(2*(deltaB)))*(-norms(yA)+norms(yC))+0.5*(norms(yA)+norms(yC));
```

```

temp = temp - (sigma/(2*(deltaB)*(T^0.5)))*quad(@normalSpecial,yA,yC);
cMeasureG(1,j) = 1-temp;

end

```

Code used to graph the comparison of fuzzy variables :

Provided is the code used to generate the figures of the comparison of fuzzy variables within the fuzzy drift parameter model and used within Figures 4.2 to 4.4. Only the code for the triangular case will be provided as the code for the other variables is very similar.

TestAll.m

```

function outPut = TestAll

K = 30;

sigma = 0.25;

Y0 = 30;

r = 0.08;

T = 0.25;

noOfEntries = 30;

BSArray = ones(2,noOfEntries);

temp = BlackScholes(K, sigma,Y0,r,T);

BSArray = BSArray*temp;

Answers = ones(5,noOfEntries);

delta = 1/(noOfEntries-1);

b = 0.08;

```

```

Answers(1,1) = 0;

Answers(2,1) = BSAArray(1,1);

BSAArray(1,1) = 0;

tempValue = 0.08;

%Do Normal Fuzzy

normalSigma = [delta : delta : (noOfEntries-1)*delta];

temp = [Answers(2,1) ,NormalFuzzy(normalSigma , T , sigma,r,K,Y0)];

Answers(3,:) = temp;

%Do Normal Triangular

a=[-delta/2 : -delta/2 : -(noOfEntries-1)*delta/2 ];

b=ones(1,noOfEntries-1);

b=b*0;

c=[delta/2 : delta/2: (noOfEntries-1)*delta/2 ];

temp = [Answers(2,1) ,TriangleFuzzy(a , b , c , T , sigma,r,K,Y0)];

Answers(2,:) = temp;

%Do Normal Trapezoidal

a=[-2*delta/4 + tempValue : -2*delta/4 : -(noOfEntries-1)*2*delta/4 + temp-
Value ];

b=[-delta/4 + tempValue : -delta/4 : -(noOfEntries-1)*delta/4 + tempValue ];

c=[delta/4 + tempValue : delta/4 : (noOfEntries-1)*delta/4 + tempValue ];

d=[2*delta/4 + tempValue : 2*delta/4 : (noOfEntries-1)*2*delta/4 + temp-
Value ];

temp = [Answers(2,1) ,TrapezoidalFuzzy(a , b , c , d , T , sigma,r,K,Y0)];

```

```

Answers(4,:) = temp;

%Do Trapezoidal Equipossible

a=[-delta/2 : -delta/2 : -(noOfEntries-1)*delta/2];
c=[delta/2: delta/2: (noOfEntries-1)*delta/2];

temp = [Answers(2,1) ,equalfuzzy(a , c , T , sigma,r,K,Y0)];

Answers(5,:) = temp;

Answers(1 , :)= [0: delta: ((noOfEntries-1) *delta)];
BSArray(1 , :)= [0: delta: ((noOfEntries-1) *delta)];

Answers;

BSArray;

hold off

plot(BSArray(1,:),BSArray(2:,:),'b')

hold on

plot(Answers(1,:),Answers(2,:),'r') %Triangle
plot(Answers(1,:),Answers(3,:),'g') %Normal
plot(Answers(1,:),Answers(4,:),'m') %Trapezoidal

xlabel('Value of \Delta \eta');

ylabel('Call Price');

title('Call prices for different fuzzy drifts')

legend('Black Scholes','Triangle', 'Normal', 'Trapezoidal')

```

TriangleFuzzy.m

```
function Calculate = TriangleFuzzy( a , b , c , T , sigma,r,K,Y0)
```

```

SizeA = size(a);
sizeT = size(T);
Calculate = ones(sizeT(2),SizeA(2));
whos
for j = 1: sizeT(2)
    for i = 1 : SizeA(2)
        if a(1,i) == c(1,i)
            Calculate(j,i) = BlackScholes(K,sigma,Y0,r,T(1,j));
        else
            temp =quadr(@chanceMeasureGreaterTriangle , K/Y0 ,10, [],[],
,a(1,i) , b(1,i) ,c(1,i) , T(1,j) , sigma,r) ;
            Calculate(j,i) = temp;
            Calculate(j,i) = Calculate(j,i) * Y0* exp(-r * T(1,j));
            if Calculate(1,i) > Y0
                Calculate(1,i) = Y0
            end
        end
    end
end
end
Calculate;

```

ChanceMeasureGreaterTriangle.m

```
function cMeasureG = chanceMeasureGreaterTriangle (x , a , b , c , T , sigma)
```

```

% Chance measure greater than

% a , b, c are what our triangle fuzzy variable will take

sizeOfX = size(x);

n = 100 ;% 1000 steps

delta = (c - a) / n;

f

or j = 1 :sizeOfX(2)

    yA = ((T^0.5)/sigma)*(yStar-(c-a));

    yB = ((T^0.5)/sigma)*(yStar);

    yC = ((T^0.5)/sigma)*(yStar+(c-a));

    temp = -(yStar/(2*(c-a)))*(norms(yA)-norms(yC))+0.5*(norms(yA)+norms(yC));

    temp = temp - (sigma/(2*(c-a)*(T^0.5)))*quad(@normalSpecial,yA,yC);

    cMeasureG(1,j) = 1-temp;

end

```

NormalSpecial.m

```

function temp = normalSpecial(x)

temp = x.*(1 / (sqrt(2 * pi)) * exp(- (x.^2) / 2));

```

Code used to generate the implied volatilities :

Provided is the code used to generate the implied volatilities of different fuzzy variables within the fuzzy drift parameter model and used within Table 4.5. Only the code for the triangular case will be provided as the code for the other variables is very similar. Note that the commented code was used to generate the surface and the non commented

code was used to generate the 2D graph.

TestTriangleFuzzy1.m

```
function outPut = TestTriangleFuzzy

K = 30;

sigma = 0.2;

Y0 = 30;

r = 0.0725;

noOfEntries = 2;

noOfTEntries = 1

%BSArray = ones(2,noOfEntries);

%temp = BlackScholes(K, sigma,Y0,r,T);

%BSArray = BSArray*temp;

%Answers = ones(3,noOfEntries);

T=0.25;

delta = 1/noOfEntries;

b = 0;

sigma = sigma;

a = -0.0607951/(T);

c = 0.060988/(T);

b = 0;

K=[10:5:70];

deltaBeta = [0 : 2*delta : (noOfEntries-1)*2*delta];
```

```

%for i = 2:noOfEntries

% c = tempValue + (i-1)*delta;

% a = tempValue - (i-1) *delta;

temp =TriangleFuzzy(a , b , c , T , sigma,r,K,Y0)

% whos

% Answers(2,:) = temp;

%Answers(1,i) = (i-1)*delta;

% Answers(1 , :)= [0: 2*delta: 2*((noOfEntries-1) *delta)];

%BSArray(1 , i)= (i-1)*delta;

%BSArray(1 , :)= [0: 2*delta: 2*((noOfEntries-1) *delta)];

%end

plot(K, temp)

%surface(deltaBeta, T , temp)

%xlabel('Value of \Delta \beta');

%ylabel('Value of T');

%zLabel('Average Chance Measure');

title('Triangular Fuzzy Variable');

%VIEW(45,45)

```

TriangleFuzzy.m

```

function Calculate = TriangleFuzzy( a , b , c , T , sigma,r,K,Y0)

SizeA = size(a);

sizeT = size(T);

```

```

sizeK = size(K);

%Im going to essentially overload this procedure

%If T is a vector, then the matrix outputted is with respect to T, else it is
outputted with respect to K

if sizeT(2) > 1
    Calculate = ones(sizeT(2),SizeA(2));
    for j = 1: sizeT(2)
        for i = 1 : SizeA(2)
            if a(1,i) == c(1,i)
                Calculate(j,i) = BlackScholes(K,sigma,Y0,r,T(1,j));
                %bits for implied vol
                %Calculate(j,i) = impliedVolatility(Calculate(j,i) ,K,Y0,r,T(1,j));
            else
                temp =quadr(@chanceMeasureGreaterTriangle , K/Y0
,10, [], [], a(1,i) , b(1,i) ,c(1,i) , T(1,j) , sigma,r) ;
                Calculate(j,i) = temp;
                Calculate(j,i) = min(Calculate(j,i) * Y0* exp(-r * T(1,j)),
Y0);

                %bits for implied vol
                %Calculate(j,i) = impliedVolatility(Calculate(j,i) ,K,Y0,r,T(1,j));
            end
        end
    end
end

```

```

        end

    else

        Calculate = ones(sizeK(2),SizeA(2));

        %whos

        for j = 1: sizeK(2)

            for i = 1 : SizeA(2)

                if a(1,i) == c(1,i)

                    Calculate(j,i) = BlackScholes(K(1,j),sigma,Y0,r,T);

                    %bits for implied vol

                    Calculate(j,i) = impliedVolatility(Calculate(j,i) ,K(1,j),Y0,r,T);

                else

                    temp =quadl(@chanceMeasureGreaterTriangle , K(1,j)/Y0
,3, [], [], a(1,i) , b(1,i) ,c(1,i) , T , sigma,r) ;

                    Calculate(j,i) = temp;

                    Calculate(j,i) = min(Calculate(j,i) * Y0* exp(-r * T) ,

Y0);

                    %bits for implied vol

                    Calculate(j,i) = impliedVolatility(Calculate(j,i) ,K(1,j),Y0,r,T);

                end

            end

        end

    end

end

```

end

ImpliedVolatility.m

```
function temp = impliedVolatility(BS,K,Y0,r,T)
```

```
%BS is Black scholes price
```

```
%K is strike price
```

```
%Y0 is initial price
```

```
%r is risk free rate
```

```
%T is time
```

```
%we assume that volatility lies between 0 and 2
```

```
temp = fzero(@BSForImplied,0.3,[],BS,K,Y0,r,T);
```

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