MANIFOLD JOINS AND JUMP CONDITIONS

IN

GENERAL RELATIVITY

Thesis

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CONTENTS

INTRODUCTION

This thesis has as its aim the analysis of a possible manifold structure on V, a join of two individual manifolds V^+ and V^-, and analysing the physics across the join, as implied by Einstein's theory of General Relativity.

There are several reasons why one might want to study such a situation. Firstly, the joining of manifolds is useful in the study of shock waves, be they of gravitational or other origin - we will be able to characterise the propagation of energy in the join. Secondly piecing together manifolds is a potentially fruitful way of obtaining e xact solutions of Einstein's equations which do not exhibit any symmetries in the large, and are yet sufficiently homogeneous (in some sense) to enable one to model the apparant Universe - the prototype of this is the Swiss-Cheese model, used to study light transmission in an inhomogeneous Universe. Thirdly, discontinuities in the fundamental quantities in Relativity are of prime importance in the study of singularities, and in particular, it is of prime importance to single out the contributions of the differential geometry and metric structure of the Universe to the existence and nature of such singularities. Closely linked to these problems is the problem of linking the small scale structure of the Universe (which is manifestly complicated and potentially full of singularities) and the large scale structure, which seems so well modelled by assumptions of homogeneity and isotropy. In this regard, the techniques of Regge (1961), originally proposed to provide approximate solutions to the Einstein equations, assume a new theoretical importance, for the delta-type singularities in the curvature he used, in a smoothing process, to represent the (assumed) continuous curvature of space, could in themselves play a distinguished role representing the small scale structure of the Universe. Furthermore, the matching together of blocks of space-time with sharp edges and corners may enable to develop a manifold like structure in which the tangent spaces of some points had a surfeit or defficiency of vectors, so that the differential geometry of the resulting spacetime forced discontinuities and singularities in the metric structure of the Universe. Although this may be aphysical, it may be a reasonable way of seeking further understanding of the Universe.

This thesis tackles only the matching problem when a finite number of manifolds with smooth boundaries are joined together - that is, it develops only explicit results in this case. However, the theory developed in Chapters one and three is hopefully a basis for the study of the wider problem.

Chapter one provides the tools to prove the existence and uniqueness of differentiable structures on a matching of two manifolds across a common boundary. The theory of r-objects developed there is in fact the theory of r-jets, (see Dodson (1980)), something not known when the theory was developed; the work of section 1.3, and the notion of a product family and its relation to a chart is new.

Chapter two examines the uniqueness of the structure for V, given that the union of the metrics on the joining spaces is to be C^r with respect to the structure for V, again being new material.

Chapter three develops a theory of distributions on manifolds, in a constructive fashion and without the need of any kind of metric on the space, or of comparison with real distributions copied onto the manifold (see e.g. Choquet-Bruhat et al,

CHAPTER ONE : MATCHING MANIFOLDS WITH BOUNDARY

This chapter is concerned with some of the problems in Differential Geometry which arise when two manifolds are joined together along their boundaries, and a differentiable structure sought for their union.

The first task is the setting up of tangent like spaces at each point of a manifold, which contain, broadly speaking, differential operators of more than just the first order - this provides machinery by means of which differentiability conditions of more than just the first order can be established across the joined boundaries of the manifolds, and concentrates at a point much of the information supplied near that point by a co-ordinate chart.

The actual joining problem is then faced. Theorem 1.1 establishes ~~an~~ existence and uniqueness, up to a diffeomorphism, of a differentiable structure for the joined manifold, together with a characterization of possible structures on the joined manifold. Then the joining of the newly defined tangent spaces of points of the boundaries is investigated and a more complete uniqueness result (Theorem 1.2) is the result.

If at times the mechanisms seem too considerable for the tasks, then this is because they were designed also for the later study of manifold like structures, whose boundaries are not smooth, such as a cube with its faces and edges included, in the hope of consolidating the theoretical basis of Regge Calculus, which might later prove useful in discussions of the relation between small and large scale views of the Universe.

1 Real maps and notational conventions

Throughout the thesis, Roman indices range over {1,2,...,n} and Greek indices over {1,2,...,(n-1)} except the underlined Greek indices will denote n- or m-tuples of non-negative integers. Thus $(\mu^1,\mu^2,\ldots,\mu^n)$ and by $|\underline{\mu}|$ we will mean the sum $\mu^1+\mu^2+\ldots+\mu^n$ and by $\underline{\mu}!$ the product $\mu^1!\mu^2!\ldots\mu^n!$. Also $\underline{\mu}\leq\underline{\nu}$ implies that $\mu^i\leq\nu^i$ for all i. In general, $\underline{\mu},\underline{\nu}$ will range over all possible values such that $1\leq|\underline{\mu}|\leq s$ for some s although occasionally $\underline{\mu}=\underline{0}$ will be included. If $\underline{x}$ is a real n-tuple or an n-tuple of maps to R, then $\underline{x}^{\underline{\mu}}$ will indicate $(x^1)^{\mu^1}\ldots(x^n)^{\mu^n}$

$|\underline{x}|=|x^1|+|x^2|+\ldots|x^n|$. $\binom{\underline{\mu}}{\underline{\nu}}=\binom{\mu^1}{\nu^1}\binom{\mu^2}{\nu^2}\cdots\binom{\mu^n}{\nu^n}$, provided $\underline{\nu}\leq\underline{\mu}$.

$\delta^{\underline{\mu}}_{\underline{\nu}} = \delta^{\mu^1}_{\nu^1}\delta^{\mu^2}_{\nu^2}\cdots\delta^{\mu^n}_{\nu^n}$.

For $\underline{x}\,\epsilon U$, an open set in R^n, let $C^k(U)$ denote the set of maps $f:R^n\to R$ such that f and its derivatives $\frac{\partial}{\partial\underline{x}^{\underline{\mu}}}f = \frac{\partial^{|\underline{\mu}|}}{(\partial x^1)^{\mu^1}\ldots(\partial x^n)^{\mu^n}}f$ for $1\leq|\underline{\mu}|\leq k$ are continuous on U. If $h:R^n\to R^m$ maps $\underline{x}$ to $(y^1(\underline{x}),\ldots,y^m(\underline{x}))$, then $h\epsilon C^k(U,R^n)$, i.e. h is C^k on U, provided each y^i is in $C^k(U)$; in the sequel, we will frequently desire a definition of a C^k map whose domain is not open in R^n. Munkres(1966) defines $C^k(U)$ to be all $f:U\to R$ such that for each $\underline{x}$ in U, there exists an open neighbourhood N_x of $\underline{x}$ and an extension f_x of f to N_x such that $f_x\,\epsilon C^k(N_x)$ and $f_x|_{N_x\cap U}=f$, that is, f is locally extendible to a C^k map. He shows that this is equivalent to f being extendible to a C^k map on a neighbourhood of U.

Let U be open in R^n, and let $\overline{U}$ be the closure of U; let f be C^k on $\overline{U}$ and suppose that f_1 and f_2 are extensions of f to a neighbourhood of $\overline{U}$ in R^n, both C^k. On U, $\frac{\partial}{\partial\underline{x}^{\underline{\mu}}}f_1 = \frac{\partial}{\partial\underline{x}^{\underline{\mu}}}f_2$; it follows that this is also true on $\overline{U}$ (by continuity), so that $\frac{\partial}{\partial\underline{x}^{\underline{\mu}}}f$ is uniquely defined on $\overline{U}$. Generalizing further, if $U\subset R^n$ is such that for every $\underline{x}$ in U, there is an R^n open set $O\subset U$ with $\underline{x}$ in $\overline{O}$, then $\frac{\partial}{\partial\underline{x}^{\underline{\mu}}}f$ are uniquely defined on U; such a set will be called a D-set.

It is clear from the definition that products, sums and compositions of C^k maps are C^k maps, and their derivatives obey the chain rule.

1.2 Manifolds with boundary

We will very slightly broaden the definition of a manifold with boundary, and then extend the notions of tangent spaces at points of a manifold to provide machinery to examine differentiability across a join of two manifolds at their boundaries.

V is an n-dimensional manifold (with smooth boundary) if

a) V is a Hausdorff paracompact topological space (equivalently, V is Hausdorff and either every connected component is a countable union of compact sets, or has a countable base for its open sets - Narashimhan (1968) p53)

b) for all p in V, there is an open neighbourhood of p, U say, and a homeomorphism h of U with R^n such that

$$h: U \to R^n \quad \text{(type I)}$$

$$\text{or} \quad h: U \to R^{n-1} \times [0,\infty) \quad \text{(type II)}$$

$$\text{or} \quad h: U \to R^{n-1} \times (-\infty,0] \quad \text{(type III)}$$

Such a pair is called a p-chart (or just a chart) of V.

As usual, a collection of such charts, which collectively cover V, is an atlas for V; a C^k atlas is one such that for charts (U,h) and (U',h'), the composition $h' \circ h^{-1}$ is C^k on $h(U \cap U')$. A complete C^k atlas, or a C^k structure for V is an atlas to which no chart can be added without disrupting the differentiability requirement. Given a C^k atlas A for V, there exists an unique complete C^k structure $\hat{A}$ for V such that $A \subset \hat{A}$. (existence is a consequence of Zorn's lemma, with partial ordering by inclusion; uniqueness follows trivially from the differentiability and completeness requirements). The set of points mapped by homeomorphisms of types II and III to a subset of $R^{n-1} \times \{0\}$ is called the (manifold) boundary of V, denoted by ∂V. It inherits a natural C^k structure from V and is thus an (n-1) dimensional manifold.

A C^r map $(0 \leq r \leq k)$ $\phi: V \rightarrow R^m$ is a continuous map such that, for each (U,h) such that $U \cap D$ (D=domain of ϕ) is not empty, $\phi \circ h^{-1}$ is in $C^k(h(U \cap D), R^m)$. If V,W are both C^k manifolds and $f: V \rightarrow W$ is continuous, then f is a C^k map provided $\varphi \circ f$ is C^k whenever φ is C^k on W with Domain(φ)$\cap$Range(f) not empty.

For charts (U,h) and (U',h') if $U \cap U' \neq \phi$, $h(U \cap U')$ is a D set, and the Jacobean of $h' \circ h^{-1}$ is well defined. If it is possible to select an atlas for V such that the Jacobeans $(h' \circ h^{-1})$ are positive for every intersecting pair of charts, V is said to be orientable, and the atlas defines an orientation for V. It is always possible to require that such an atlas consist of type I and either type II or type III charts.

A C^k submanifold, A say, of a C^k manifold V is a subset of V, endowed with a C^k structure, such that any chart (U,h) in A is a restriction to A of a chart in V, where for the moment R^m is considered in its canonical imbedding into R^n. V induces a manifold structure onto each of its open sets, and a submanifold results.

1.2.1 Germs and r-Object spaces

Following Narashimhan (1968) p 55, we say for $p \in V$ and $f,g : U_f, U_g \to R$ with $p \in U_f \cap U_g$ that f is equivalent to g at p if f and g agree on an open neighbourhood of p. We write $f \sim g$. $\sim$ is clearly an equivalence relation, and divides the set of functions C^s in a neighbourhood of p into a set of equivalence classes or germs, to be denoted by $C_{p,s}(V)$ (or just $C_{p,s}$). A germ containing the function g will be denoted by [g]. The value v([g]) of a germ [g] at p is well defined to be g(p) for a representative g. A C^s function near p is stationary of order r (with $1 \leq r \leq s$) at p if there exists a p-chart (U,h) such that at h(p), $f \circ h^{-1}$ has zero derivatives of orders 1 to r. (This is invariant of the choice of the chart (U,h), by several applications of the chain rule.) A germ at p is stationary of order r if it contains a function stationary of order r at p. (This is invariant of the choice of representative). The set of C^s r stationary germs at p will be denoted by $S_{p,s,r}(V)$.

Addition of germs is well defined by addition of representatives and an abelian group results. Multiplication by a $\lambda \in R$ is defined by $\lambda[g] = [\lambda g]$, and a linear vector space results. Multiplication of germs is also well defined by multiplication of representatives, and an algebra results. Each $S_{p,s,r}(V)$ is a normal subgroup of $C_{p,s}(V)$ under addition, since clearly the addition of r stationary germs yields an r stationary germ so that $S_{p,s,r}(V)$ is a subgroup of $C_{p,s}(V)$, and since $C_{p,s}(V)$ with addition is abelian. Also $S_{p,s,r}(V)$ is a subspace of $C_{p,s}(V)$ (as a vector space). It follows that the factor group $T^*_{p,r}(V) = C_{p,r}/S_{p,r,r}$ is well defined. Following the notation of algebra (see e.g. Fraleigh (1982)), we will denote the elements of $T^*_{p,r}(V)$ by $[g] + S_{p,r,r}(V)$, indicating the factor containg the germ [g]. These elements will be called covariant r-objects (or r-geometric objects) at p. Denote by $T_{p,r}(V)$ the dual (algebraic) of $T^*_{p,r}(V)$. Its elements will be called contravariant r-objects at p. The canonical map ϕ: $C_{p,r} \to T^*_{p,r}$ by $\phi[g] = [g] + S_{p,r,r}$ enables one to regard a element of $T_{p,r}(V)$, X say, as a linear map on $C_{p,r}$ by equating X and X o ϕ . $T_{p,r}$ may therefore also be considered as the set of linear maps of $C_{p,r}$ to R which are zero on $S_{p,r,r}$; in this way, $T_{p,r}$ may also be considered as a subset of $C^*_{p,r}$. Now clearly $C_{p,r+1}$ is a subspace of $C_{p,r}$ and $S_{p,r+1,r+1}$ is a subspace of $S_{p,r,r}$. It follows that the restriction of an X in $T_{p,r}$ to $T_{p,r+1}$ is well defined. We will denote this restriction operation by I.

The two ways of defining $T_{p,r}$ will be used interchangably in the sequel. Since (as will be shown below) $T_{p,r}(V)$ is finite dimensional, the usual identification of $T^*_{p,r}(V)^{**}$ with $T^*_{p,r}(V)$ is possible so that $T_{p,r}(V)^*$ is essentially the same as $T^*_{p,r}(V)$.

1.2.2 Bases for $T_{p,r}(V)$ and $T^*_{p,r}(V)$

If f is C^r at p and (U,h) is a p chart with U $\subset$ Domain f such that $h(p)=\underline{0}$ and $h(q)=\underline{x}(q) \in R^n$, then h(U) is a D set as defined earlier so that $\frac{\partial}{\partial \underline{x}^{\underline{\mu}}}(f \circ h^{-1})$ is defined on h(U), uniquely. Set g:U $\to$R by

$$g(h^{-1}(\underline{x})) = f(h^{-1}(\underline{x})) - f(p) - \sum_{1 \leq |\underline{\mu}| \leq r} \frac{1}{\underline{\mu}!}(\underline{x} \circ h^{-1})^{\underline{\mu}} \left(\frac{\partial}{\partial \underline{x}^{\underline{\mu}}}(f \circ h^{-1})\right)\Big|_{\underline{0}}$$

Clearly, g is C^r at p and [g] is stationary of order r at p. Hence, for $X \in T_{p,r}(V)$,

$$X[g] = 0 = X[f] - \sum_{1\le|\underline{\mu}|\le r} X[(\underline{x})^{\underline{\mu}}] \,/\, \underline{\mu}! \cdot \frac{\partial}{\partial \underline{x}^{\underline{\mu}}} (f \circ h^{-1}) \Big|_{\underline{0}} \qquad \ldots(1.1)$$

whence we may write

$$X = \sum_{1\le|\underline{\mu}|\le r} X^{\underline{\mu}} \left(\frac{\partial}{\partial \underline{x}^{\underline{\mu}}} \right)_p \quad \text{with} \quad X^{\underline{\mu}} = \frac{1}{\underline{\mu}!} X[(\underline{x})^{\underline{\mu}}] \qquad \ldots(1.2)$$

where

$$\left(\frac{\partial}{\partial \underline{x}^{\underline{\mu}}} \right)_p [g] = \frac{\partial}{\partial \underline{x}^{\underline{\mu}}} (g \circ h^{-1}) \Big|_{\underline{0}} \qquad \ldots(1.3)$$

It immediately follows that $\{ (\frac{\partial}{\partial \underline{x}^{\underline{\mu}}})_p \mid 1\le |\underline{\mu}| \le r \}$ spans $T_{p,r}(V)$. Further, since $(\frac{\partial}{\partial \underline{x}^{\underline{\mu}}})_p [(\underline{x})^{\underline{\nu}}] = \delta^{\underline{\nu}}_{\underline{\mu}}$, this set is independent and hence a basis. The dual of this basis is $\{[\underline{x}^{\underline{\nu}}] + S_{p,r,r}\}$ which is a basis for $T^*_{p,r}(V)$. The individual elements of this basis will be denoted by $(d_{,r}\underline{x}^{\underline{\nu}})_p / \underline{\nu}!$.
(This basis may be used to justify the view that $T^*_{p,r}(V)$ is the set of classes formed by regarding two C^r functions as equivalent if their partial derivatives agree to order r in a neighbourhood of p).

If we introduce another p chart (U',h') say with $h'(q)=\underline{y}(q)$ for $q \in U'$, then

$$\left(\frac{\partial}{\partial \underline{y}^{\underline{\mu}}}\right)_p = \left\{ \left(\frac{\partial}{\partial \underline{y}^{\underline{\mu}}}\right)_p \left((d_{,r}\, \underline{x}^{\underline{\nu}})_p \,/\, \underline{\nu}! \right) \right\} \left(\frac{\partial}{\partial \underline{x}^{\underline{\nu}}}\right)_p$$

(the summation convention is operative); the first factor may be expressed as a sum of products of derivatives of $\underline{x}$ with respect to $\underline{y}$ of order at most $|\underline{\mu}|$; it follows that $X = X^{\underline{\mu}} X^{\underline{\nu}}_{\underline{\mu}} (\frac{\partial}{\partial \underline{x}^{\underline{\nu}}})_p$, where $X^{\underline{\nu}}_{\underline{\mu}}$ is the first factor.
This constitutes a 'change of basis' formula.

1.2.3 The spaces $T_p(V)$ and $T^*_p(V)$

Narashimhan (1968) performs the above analysis only for the case r=1; if we set $T_{p,1} = T_p$ and $T^*_{p,1} = T^*_p$ then we regain the usual tangent and cotangent spaces. We will use the ordinary notation for these spaces whenever desirable.

1.2.4 Object bundles and object fields

Define the contravariant r-object bundle by $T_{,r}(V) = \underset{p\varepsilon V}{U}\ T_{p,r}(V)$ and similarly define $T^*_{,r}(V)$. Let $\pi: T_{,r} \to V$ so that $\pi(X_p) = p$ (for X_p in $T_{p,r}(V)$). A C^{k-r} structure for $T_{,r}(V)$ (together with a topology) is uniquely defined by the following:

> let (U,h) be a p chart in V and let $\hat{h} : \pi^{-1}(U) \to U \times R^{N_r}$ (where N_r is the dimension of $T_{p,r}(V)$) and let π_1, π_2 be the projections to either factor. Suppose that $\pi_1 \circ \hat{h} = \pi$ and $\pi_2 \circ \hat{h}\,|_{T_p(V)}$ is an isomorphism of $T_{p,r}(V)$ with R^{N_r} (both regarded as vector spaces). Set $H = (h \circ \pi\ ,\ \pi_2 \circ \hat{h})$ and $\underline{U} = \pi^{-1}(U)$, and require $(\underline{U},H)$ to be a chart of $T_{,r}(V)$.

(The proof that this does indeed yield a structure on $T_{,r}(V)$ is a simple extension of the case r=1 given in Narashimhan).

Similarly, a C^{k-r} structure is defined on $T^*_{,r}$.

We will denote $T_{U,r}(V) = \pi^{-1}(U)$; a particular example of a $T_{,r}(V)$ chart is obtained by letting $\left(\frac{\partial}{\partial x^{\underline{\mu}}} \right)_p$ be a co-ordinate derived basis of $T_{p,r}(V)$ and for X_p in $T_{p,r}$ with $X_p = X^{\underline{\mu}} \left(\frac{\partial}{\partial x^{\underline{\mu}}}\right)_p$, setting $\hat{h}(X_p) = (p, X^{\underline{\mu}})$, and $H(X_p) = (h(p), X^{\underline{\mu}})$. ...(1.4)

A map assigning an X_p in $T_{p,r}(V)$ to each p in A, a subset of V, will be called a contravariant r-object field, or r-field on A. A covariant r-field is similarly defined. Being maps between differentiable manifolds, one may ask if they are differentiable. The space of all contravariant r-fields in V which are C^{k-r} will be denoted by $X_{,r}(V)$. Defining addition of X,Y in $X_{,r}$ in the obvious fashion, and multiplication by a C^{k-r} function by $fX : p \to f(p)X(p)$ as usual, results in $X_{,r}(V)$ being a module over the algebra of C^{k-r} maps of V to R. (Strictly speaking, addition and multiplication of object fields are only defined when the appropriate domains co-incide).

$X^*_{,r}$ is similarly defined.

Clearly $\frac{\partial}{\partial x^{\underline{\mu}}} : p \to \left(\frac{\partial}{\partial x^{\underline{\mu}}} \right)_p$ and $d_{,r}\, x^{\underline{\nu}} : p \to (d_{,r}\, x^{\underline{\nu}})_p$ are in $X_{,r}$ and $X^*_{,r}$ respectively; thus any X in $X_{,r}(V)$ may locally be written as $X^{\underline{\mu}} \frac{\partial}{\partial x^{\underline{\mu}}}$, where necessarily and sufficiently the $X^{\underline{\mu}}$ are C^{k-r} functions on V. (Summation is ofcourse implied over $1 \leq |\underline{\mu}| \leq r$). Similarly ω in $X^*_{,r}(V)$ implies $\omega = \omega_{\underline{\nu}} \left(d_{,r}\, x^{\underline{\nu}} / \underline{\nu}! \right)$. Because of these relations, $\frac{\partial}{\partial x^{\underline{\mu}}}$ and $d_{,r}\, x^{\underline{\nu}} / \underline{\nu}!$ will be called local co-ordinate bases of $X_{,r}(V)$ and $X^*_{,r}(V)$.

If f is a C^s ($s \leq k$) map on U to R, then for each $r \leq s$, f defines a germ $[f]_p$ in $C_{p,r}$ for p in U and hence a member of $T^*_{p,r}(V)$, which we will denote by $(d_{,r}f)_p$. At p, $(d_{,r}f)_p$ may be written out in terms of $(d_{,r}x^{\underline{\nu}} / \underline{\nu}!)_p$ by using the dual basis $\left(\frac{\partial}{\partial x^{\underline{\mu}}} \right)_p$ thus :

$$(d_{,r}\, f)_p = \left(\frac{\partial}{\partial x^{\underline{\mu}}} \right)_p (d_{,r}f)_p \, (d_{,r}x^{\underline{\mu}} / \underline{\mu}!)_p$$

Now

$$\left(\frac{\partial}{\partial x^{\underline{\mu}}} \right)_p (d_{,r}f)_p = \left(\frac{\partial}{\partial x^{\underline{\mu}}} \right)_p ([f] + S_{p,r,r}) = \frac{\partial}{\partial x^{\underline{\mu}}} \left(f \circ h^{-1} \right) \Big|_{h(p)}$$

by definition; the covariant r-object field $d_{,r}f : p \to (d_{,r}f)_p$ is then C^{s-r} on the domain of f. In particular, if f is C^k, $d_{,r}f$ is in $X^*_{,r}(V)$. In any case if X is in $X_{,r}(V)$, $X(d_{,r}f)$ is a C^{s-r} map.

So now let X be in $X_{,r}$ (V) and Y in $X_{,s}(V)$, both defined in a neighbourhood of p, and let f be a C^{s+r} real map near p. Then $Y(d_{,s}f)$ is C^{r} near p so that $d_{,r}(Y(d_{,s}f))$ is continuous near p and so is $X(d_{,r}(Y(d_{,s}f)))$. Thus for [f] in $C_{p,r+s}$ define

$$(XY)_p\,[f] = X_p\,(d_{,r}\,(Y\,(d_{,s}\,f)))\qquad \text{for any } f \text{ in } [f]$$

and the map $XY : p \to XY_p$ is in $X_{,r+s}$ for calculating :

$$(XY)_p\,[f] = X^{\underline{\mu}}(p)\left(\frac{\partial}{\partial \underline{x}^{\underline{\mu}}}\left((Y^{\underline{\nu}} \circ h^{-1})\,\frac{\partial}{\partial \underline{x}^{\underline{\nu}}}\,(f \circ h^{-1})\right)\right)\Big|_{h(p)}$$

(sum $1 \le |\underline{\nu}| \le s$ and $1 \le |\underline{\mu}| \le r$) in a p chart (U,h) with X and Y appropriately expressed in terms of their components. Using Leibniz formula (John 1978) :

$$(XY)_p[f] = X^{\underline{\mu}}(p)\sum_{\underline{\mu}' \le \underline{\mu}} \binom{\underline{\mu}}{\underline{\mu}'}\,\frac{\partial}{\partial \underline{x}^{\underline{\mu}'}}(Y^{\underline{\nu}} \circ h^{-1})\,\frac{\partial}{\partial \underline{x}^{\underline{\mu}+\underline{\nu}-\underline{\mu}'}}\,(f \circ h^{-1})\Big|_{h(p)}$$

and XY is $C^{k-(s+r)}$ (check the terms!)

Thus a product mapping of $X_{,r}(V) \times X_{,s}(V) \to X_{,r+s}(V)$ is defined.

Recalling the canonical restriction I of $T_{p,r}(V)$ to $T_{p,r+1}(V)$, we may associate with X in $X_{,r}$ the field IX in $X_{,r+1}$ where $IX : p \to I(X_p)$ so that we may regard $X_{,r} \subset X_{,r+1}$. Noting that $\frac{\partial}{\partial \underline{x}^{\underline{\mu}}} = \frac{\partial}{\partial \underline{x}^{\underline{\mu}-\underline{\nu}}}\,\frac{\partial}{\partial \underline{x}^{\underline{\nu}}}$ and in general that $\frac{\partial}{\partial \underline{x}^{\underline{\mu}}}\,\frac{\partial}{\partial \underline{x}^{\underline{\nu}}} = \frac{\partial}{\partial \underline{x}^{\underline{\mu}+\underline{\nu}}}$, so that the co-ordinate basis of $X_{,r}$ may be constructed as a series of products of elements of the co-ordinate basis in $X_{,1}$, we may regard $X_{,r}$ as an r-fold product of $X_{,1}$.

The commutator of two fields X in $X_{,r}$ and Y in $X_{,s}$ with $r+s\leq k$ is defined by $[X,Y] = XY-YX$ and is an element of $X_{,r+s-1}$, as is easily verified by computation. The Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ is satisfied, by virtue of the anti-symmetry of the bracket and a Lie algebra is defined on $X_{,1}(V) = X(V)$, as usual.

1.2.5 The submanifolds $T_{A,r}$ and $T^*_{A,r}$

Recalling that $T_{A,r}(V) = \bigcup_{p\in A} T_{p,r}(V)$, if A is a submanifold (differentiability C^k) of V then obviously $T_{A,r}(V)$ is a submanifold of $T_{,r}(V)$. If $X:A \to T_{A,r}$ then X is a C^s covariant field on A if it is a C^s map. $X_{A,r}(V)$ is the set of all C^{k-r} r-fields on A in V. $T^*_{A,r}$ and $X^*_{A,r}$ are similarly defined. An example, of importance in this thesis, is the case when $A = \partial V$ - one is then considering r-fields defined only on ∂V, whose image is an r-object in $T_{,r}(V)$, for example the normal and null fields of Chapter Two.

1.2.6 The extended spaces $\tilde{T}_{p,r}$

By way of theoretical interest, the spaces $\tilde{T}_{p,r}(V)$ are introduced here. The value map v on $C_{p,r}(V)$ is of course a linear functional on $C_{p,r}$, and hence in $C^*_{p,r}$. Recalling that $T_{p,r}$ may be considered a subspace of $C^*_{p,r}$, and noting that v is independent of every vector in $T_{p,r}$, we may form the subspace $T_{p,r} \oplus \langle v\rangle = \tilde{T}_{p,r}$ (where $\langle v\rangle$ denotes the span of v and $\oplus$ is the direct sum of subspaces). If we denote v by $\frac{\partial}{\partial \underline{x}^{\underline{o}}}$, then the set $\{ \frac{\partial}{\partial \underline{x}^{\underline{\mu}}} \mid 0 \leq |\underline{\mu}| \leq r \}$ is a basis for $\tilde{T}_{p,r}$. Since the dual of a direct sum is the direct sum of the duals, $\tilde{T}^*_{p,r} = T^*_{p,r} \oplus \langle v^*\rangle$ where the dual of v is v^*, which will be denoted by $(d_{,r}\, \underline{x}^{\underline{o}})_p$.

Defining $\tilde{T}_{,r}(V)$ in the obvious way leads to a C^{k-r} manifold, and $\tilde{X}_{,r}(V)$ is then well defined (as the space of C^{k-r} maps of V to $\tilde{T}_{,r}$). Simply define

$\tilde{T}_{p,0}(V) = \langle v \rangle$ and obtain $\tilde{T}^*_{p,0}$, $\tilde{T}_{,0}$ and $\tilde{X}_{,0}$.

Again, given a C^s function $f:U \to R$, U a neighbourhood of p, [f] is in $C_{p,s}$ and $(d_{,r}f)_p \overset{def}{=} f(p)(d_{,r}\underline{x}^o)_p + (d_{,r}f)_p$ for $r \leq s$.

A product $\tilde{X}_{,r} \times \tilde{X}_{,s} \to \tilde{X}_{,r+s}$ is obviously defined. $\frac{\partial}{\partial \underline{x}}o$ may be regarded as the identity of this product, since for X in $\tilde{X}_{,s}$:

$$\begin{aligned}(\tfrac{\partial}{\partial \underline{x}}o\ X)_p\ [f] &= (\tfrac{\partial}{\partial \underline{x}}o)_p\ (d_{,0}\ (X\ d_{,r}f)) \\ &= (\tfrac{\partial}{\partial \underline{x}}o)_p\ [\ X\ d_{,r}f\ ] \\ &= (X\ d_{,r}f)\ (p) = X_p\ d_{,r}f = X_p\ [f] \quad \text{for all } [f]\end{aligned}$$

We have not yet defined C^∞ manifolds - they are defined in the usual way. On such manifolds, an infinite sequence of the spaces $\tilde{T}_{p,r}$ (r=0,1, ...) $\tilde{T}_{,r}$ and $\tilde{X}_{,r}$ are defined. $\tilde{\tilde{T}}_{,r}$ is a C^∞ manifold itself and $\tilde{X}_{,r}$ is a module over C^∞ functions. We may then consider formal sums of finitely many terms $X_0 \dot{+} X_1 \dot{+} \ldots . \dot{+} X_m$ where X_i is in $\tilde{X}_{,i}$. Such formal sums may be added termwise and multiplied, using the object product in a formally distributive fashion. The result will be a graded algebra with unity.

1.2.7 Differentiable maps between manifolds

Let $f:V \to W$ be a C^k map, where both V and W are C^k differentiable manifolds, of dimensions n and m respectively. For X in $T_{p,r}(V)$ and [g] in $C_{f(p),r}(W)$, define f_*X by $f_*X[g]=X[g \circ f]$ (We assume f defined on a domain D with the property that for p in D, there exists an open set O in V with p in $\overline{O}$, the closure of O, such that $O \subset D$; it then follows that if (U,h) is a p chart in V, $h(U \cap D)$ is a D set in R^n and the partial derivatives of $g \circ f \circ h^{-1}$ are well defined at p, so that $d_{,r}(g \circ f)$ and hence $X[g \circ f]$ is well defined.)

f_*X is easily shown to be linear, and therefore in $C^*_{f(p),r}(W)$. Repeated applications of the chain rule show that since X is zero on $S_{p,r,r}(V)$, f_*X is zero on $S_{f(p),r,r}(W)$. It therefore follows that $f_*(X)$ is in $T_{f(p),r}(W)$. f_* is therefore a map from $T_{p,r}(V)$ to $T_{f(p),r}(W)$. It is linear -

$$f_*(\alpha X + \beta Y)[g] = (\alpha X + \beta Y)[g \circ f] = \alpha(X[g \circ f]) + \beta(Y[g \circ f])$$

$$= (\alpha f_* X + \beta f_* Y)[g] \quad (\text{ here, } \alpha, \beta \text{ are in R and } X,Y \text{ in } T_{p,r}(V) \text{ }).$$

The rank of f_* is therefore well defined, for each r. The rank of f at p will be defined to be the rank of f_* at p regarded as a map on $T_{p,1}(V)$. We say that f is a C^k immersion if f is C^k and rank f_*= dimension V at each p in the domain of f. If, additionally, f is a homeomorphism onto its image, it is an imbedding. If yet further, f is a V,W homeomorphism, it is a diffeomorphism (equivalently, f is a diffeomorphism if both f and f^{-1} are imbeddings). By corollary I to the inverse function theorem (Appendix I), an immersion is locally a diffeomorphism, i.e. for each p in domain f, there exists a neighbourhood N_p in domain f with $f_{|N_p}$ a diffeomorphism of N_p with $f(N_p)$. A consequence of corollaries II and III (Appendix I, again) is that the inclusion map i of a subset A of a manifold V, endowed with its own differentiable structure, is an imbedding iff A is a submanifold of V. (These results are extensions of similar results in the non-boundary case - see e.g. Kahn (1980) p 21).

Now recall that $f_*(T_{p,r}(V))$ is a subspace of $T_{f(p),r}(W)$, since $f_*$is a linear map, assuming only that f is C^k; then on the dual of this subspace, $(f_* T_{p,r}(V))^*$, which may be considered a subset of $T^*_{f(p),r}(W)$ (regard $T_{f(p),r}(W)$ as a direct sum of $f_*(T_{p,r}(V))$ and its algebraic complement in $T_{f(p),r}(W)$ and $T^*_{f(p),r}(W)$ is the direct sum of the duals of these spaces), a map f^* is defined by $X(f^*\omega) = (f_*X)\omega$ for ω in $(f_*(T_{p,r}(V))^*$ and X in $T_{p,r}(V)$. If f_* is of full rank on $T_{p,r}(V)$, then $(f_*T_{p,r}(V))^*$ is isomorphic to $T^*_{p,r}(V)$, under f^*; in this case, define f_* on T^* (V) to be simply $(f^*)^{-1}$; regarding again $(f_*(T_{p,r}(V)))^*$ as a subspace of $T^*_{f(p),r}(W)$, f_*: $T^*_{p,r}(V) \rightarrow T^*_{f(p),r}(W)$ as a linear map of full rank.

If f is a diffeomorphism, then for [g] in $C_{p,r}(V)$, define $f_*[g] = [g \circ f^{-1}]$. This is a good definition, since $g_1 \sim g_2$ at p iff $g_1 \circ f^{-1} \sim g_2 \circ f^{-1}$ at f(p). It leads also to a map $f'([g] + S_{p,r,r}) \rightarrow (f_*[g] + S_{f(p),r,r})$ since, again repeatedly applying the chain rule, $[g \circ f^{-1}]$ is r stationary iff [g] is r stationary. This map immediately accords with the previous definition of f_* on $T^*_{p,r}(V)$ (since $f_*X(f'[g]) = f_*X[g \circ f^{-1}] = X[g]$, for all X in $T_{p,r}(V)$, and [g] in $C_{p,r}(V)$). Also, in this case, $T_{p,r}(V)$ is isomorphic to $T_{f(p),r}(W)$, by f_*(as is seen by noting that $(f^{-1})_* = (f_*)^{-1}$ which is therefore defined on all of $T_{f(p),r}(W)$).

If f is a C^k map, then f_* is ofcourse also a map from $T_{,r}(V)$ into $T_{,r}(W)$ by $X_q \rightarrow f_*(X_q)$. If f is a C^k diffeomorphism, immersion, imbedding, then this f_* will be a C^k diffeomorphism, immersion, imbedding as can be seen thus :

let (U,h) be a p chart of V, and (U̲,H) the derived chart (eqn 1.4) Since an immersion is locally an imbedding, when may restrict the the domain of f, and if necessary, U, until an imbedding is obtained. It then follows that f(U) is a submanifold of W (cf corrolaries II and III of the inverse function theorem) so that $(f(U), h \circ f^{-1})$ (with f^{-1} defined only on f(U), f being a homeomorphism on U) is the restriction of some chart (U',h') in W. In terms of the charts (U̲,H) and (U̲',H') (derived from (U',h')) f_* is simply an imbedding of R^N into R^M, where N=dim $T_{,r}(V)$ and M=dim $T_{,r}(W)$, by means of the inclusion map of R^N into R^M (the fact that f is locally an imbedding implies that $N \leq M$); it immediately follows that f_* is of rank N (now ofcourse f_* is being regarded as a map between manifolds - the rank implied here is the rank of $(f_*)_*$ on $T_{p,r}(T_{p,r}(V))$.

If f is C^k, f_* is C^{k-r}, by the same argument (using again the charts $(\underline{U},H)$ and $(\underline{U}',H)$); if f is a homeomorphism onto its range, so is f_* and if f is a homeomorphism, f_* must be.

Let now f be an imbedding, defined on all of V (f being C^k); then f_* is an imbedding (C^{k-r}) of $T_{,r}(V)$ to $T_{,r}(W)$, and f_* is a diffeomorphism of $T_{,r}(V)$ with $T_{f(V),r}(W)$. If X is in $X_{,r}(V)$, defined on A, then necessarily, $f_*X : f_* \circ X \circ f^{-1}$ is a C^{k-r} map, hence in $X_{f(V),r}(f(V))$. $f_* : X_{,r}(V)$ to $X_{f(v),r}(W)$. It is a product homomorphism with respect to these two spaces as is seen by :

$$\begin{aligned} (f_*(XY))_{f(p)}[g] &\overset{def}{=} (XY)_p[g \circ f] \\ &\overset{def}{=} X_p(d_{,r} (Y d_{,s}(g \circ f))) \\ &= X_p d_{,r} (f_*Y (d_{,s} g) \circ f) \\ &= f_*X (d_{,r} (f_*Y (d_{,s}g)) \circ f) (p) \\ &= (f_*X f_*Y)_{f(p)}[g] \end{aligned}$$

for [g] in $C_{f(p),r+s}(W)$.

If f is a diffeomorphism, then from the above, $f : X_{,r}(V)$ to $X_{,r}(W)$ and f_* is a product homomorphism with respect to these spaces and their associated products.

1.2.8 Product families

The above may be extended to the spaces $\tilde{T}_{,r}$, $\tilde{X}_{,r}$ etc; in particular, $f_*((\frac{\partial}{\partial \underline{x}}\underline{o})_p)[g] = g(f(p))$. The family $\{\frac{\partial}{\partial \underline{x}}\underline{\mu} \mid 0 \leq |\underline{\mu}| \leq r\}$ for $r \leq k$ satisfies the conditions $\frac{\partial}{\partial \underline{x}}\underline{\mu} = \frac{\partial}{\partial \underline{x}}\underline{\mu_1} \frac{\partial}{\partial \underline{x}}\underline{\mu_2}$ when $\underline{\mu_1} + \underline{\mu_2} = \underline{\mu}$. We use this as a model for a product family :

if $\underline{n}$ is an m-tuple of non-negative integers, and $A = \{X_{\underline{n}} \mid X_{\underline{n}} \in X_{,|\underline{n}|}(V)$, for $0 \leq |\underline{n}| \leq r\}$ for $r \leq k$ satisfying $X_{\underline{n}} = X_{\underline{n}_1} X_{\underline{n}_2}$ whenever $\underline{n}_1 + \underline{n}_2 = \underline{n}$, then the family A is called a product family; note that since $\underline{0} + \underline{n} = \underline{n}$, $X_{\underline{n}} = X_{\underline{0}} X_{\underline{n}}$, whence necessarily, $X_{\underline{0}} = \frac{\partial}{\partial \underline{x}^{\underline{o}}}$; a knowledge of the set $\{ X_{\underline{n}} \mid |\underline{n}| = 1 \}$ therefore determines A (ofcourse, the $X_{\underline{n}}$ must be known in a neighbourhood for the product to be defined, and thus we will assume the $X_{\underline{n}}$ defined on some open set in V).

Suppose that $k \geq 2$. Let $\underline{n}_1$, $\underline{n}_2$ be such that $|\underline{n}_1| = |\underline{n}_2| = 1$; then let $\underline{n} = \underline{n}_1 + \underline{n}_2$ and $X_{\underline{n}} = X_{\underline{n}_1} X_{\underline{n}_2} = X_{\underline{n}_2} X_{\underline{n}_1}$, whence the Lie bracket (commutator)

$$[X_{\underline{n}_1}, X_{\underline{n}_2}] = X_{\underline{n}_1} X_{\underline{n}_2} - X_{\underline{n}_2} X_{\underline{n}_1} = 0 \qquad \ldots(1.5)$$

Noting that both $X_{\underline{n}_1}$ and $X_{\underline{n}_2}$ are vector fields, of differentiabilty C^{k-1}, both generate C^k integral curves in a neighbourhood of any point of their domain at which they are not zero (Choquet-Bruhat et al (1977), p 141).

Suppose now that $m \leq n$ and that the collection $\{ X_{\underline{n}} \mid |\underline{n}| = 1 \}$ is an independent set of non zero vectors in $T_p(V)$ for some p in the domain A of $X_{\underline{n}}$. By continuity, this independence and non zero property extends to a neighbourhood of p in V. For ease of notation, let $X_{\underline{n}} = X_I$ if $n^J = \delta^J_I$ (I,J,L = 1,2,...,m). If $h: U \to R^n$ is a diffeomorphism of a neighbourhood of p, U, with h(U), with $h(p) = \underline{0}$, such as say a co-ordinate chart at p, let $h(q)=\underline{x}(q)$ for q near p and denote by $\frac{\partial}{\partial x^i}$ (i=1,...,n) the 'co-ordinate' like basis fields given by

$$\left(\frac{\partial}{\partial x^i}\right)_q [g] = \frac{\partial}{\partial x^i} (g \circ h^{-1}) \Big|_{h(q)}$$

(we will assume that the image h(U) is a D set so that the right hand side is well defined on h(U)).

Write :

$$X_I = X^i{}_I \frac{\partial}{\partial x^i} \qquad \text{(summation over } i = 1,\ldots,n)$$

The $X^i{}_I$ are C^{k-1} functions of position on $A \cap U$ - without loss of generality, we may assume $X^1{}_1 > 0$ at and near p. The integral curves of X_1 therefore, are such that to each q near p in $h^{-1}(\{x^1=0\})$ there is one and only one curve γ_q say, which intesects $h^{-1}(\{x^1=0\})$ in q, its only point of intesection with this hyperplane in V. Provided y^1 is sufficiently small, we may define functions X^i of y^j by

$$X^i(y^j) = x^i(\gamma_q(y^1)), \quad q=h^{-1}(0,y^2,\ldots,y^n) \qquad \ldots(1.6)$$

(provided also that $(0,y^2,\ldots,y^n)$ is in h(U)).
X^i give the 'co-ordinates' (i.e. the image under h) of a point a parameter distance y^1 along the X_1 integral curve through $q = h^{-1}(0,y^2,\ldots,y^n)$.

At p, we may compute the $\frac{\partial X^i}{\partial y^j}$ thus :

if $i,j \geq 2$, $y^1=0$, then $X^i(y^j) = y^i$ and $\frac{\partial X^i}{\partial y^j} = \delta^i_j$;

if $i=1$, $j \geq 2$, $y^1=0$, then $X^1(y^j) = 0$ and $\frac{\partial X^i}{\partial y^j} = 0$;

if $y^2,\ldots,y^n$ are held constant, and y^1 varied, the image under h of γ_p is traced, so $\frac{\partial X^i}{\partial y^1} = X^i{}_1$ ...(1.7)

It follows that $\det \{ \frac{\partial X^i}{\partial y^j} \} = X^1{}_1 > 0$.

We now have two possibilities; either $p \in \partial V$ or not. If not, then by the inverse function theorem (appendix I), since the X^i are C^k functions of the y^j (Choquet-Bruhat et al (1977) p 141), the mapping implied by (1.6) is a C^k diffeomorphism in a neighbourhood of $\underline{0}$ (with a neighbourhood of $\underline{0}$, since (1.6) maps $\underline{0}$ to $\underline{0}$) (appendix I).

If p $\varepsilon\partial V$, we may (since the functions in (1.6) are C^k as observed - the Choquet-Bruhat proof will extend to the boundary manifold case) extend (1.6) to an open neighbourhood of $\underline{0}$, apply the reasoning of the previous paragraph, and then restrict (1.6) again, obtaining a C^k diffeomorphism, still of a neighbourhood of $\underline{0}$ in the R^n induced topology on h(U) with its image, which, by virtue of this diffeomorphism being a restriction of an R^n homeomorphism, is still a D set.

In both cases, the result is a C^k diffeomorphism h' of a neighbourhood U' of p in V, with its image in R^n, such that if $h'(q) = \underline{y}(q)$, for q near p, the integral curves of X_1 are given by

$$(h')^{-1}\ (t, y^2, \ldots, y^n) \qquad \text{with} \qquad X_1 = \frac{\partial}{\partial y^1} \qquad \ldots(1.8)$$

$(y^2, \ldots, y^n$ held constant, t varying).

The process may now be repeated, expressing $X_2 = Y^i{}_2 \frac{\partial}{\partial y^i}$, and, by the non zero and independence assumption, $Y^2{}_2 > 0$ near p. A diffeomorphism (C^k still) results - call it h'' defined on the neighbourhood U'' of p in V. Supposing that $h''(q)=\underline{w}(q)$, the integral curves of X_2 are given by

$$(h'')^{-1}\ (w^1, t, w^3, \ldots w^n) \qquad \text{with} \qquad X_2 = \frac{\partial}{\partial w^2}$$

$(w^1, w^3, \ldots, w^n$ held constant, t varying).

Now $X_1 = \frac{\partial w^k}{\partial y^1} \frac{\partial}{\partial w^k}$ (by (1.8)) so that by (1.5) :

$0 = [X_1, X_2]$ implies that $\frac{\partial}{\partial w^2} \frac{\partial w^k}{\partial y^1} = 0$

which implies that $\frac{\partial w^k}{\partial y^1}(w^1, w^2, w^3, \ldots w^n) = \frac{\partial w^k}{\partial y^1}(w^1, 0, w^2, \ldots, w^n)$

which is δ^k_1 (by (1.7) and $X_1 = \frac{\partial}{\partial w^1}$

so that, the X_1 integral curves are given by 'varying the first component' i.e.

$$(h'')^{-1}\ (t, w^2, \ldots w^n) \qquad (w^2, \ldots, w^n \text{ constant, t varying})$$

Repeating the process enables one to find a C^k diffeomorphism into R^n, $\tilde{h}$ say, mapping a neighbourhood $\tilde{U}$ of p in V, to $\tilde{h}(\tilde{U})$, with say $\tilde{h}(q)=\underline{z}(q)$ and

$$X_I = \frac{\partial}{\partial z^I} \qquad (I = 1,\ldots,m)$$

Finally, noting that $\tilde{h}$ is a product homomorphism,

$$X_{\underline{\eta}} = \frac{\partial}{\partial \underline{z}^{\underline{\eta}}} \qquad \text{where } \frac{\partial}{\partial \underline{z}^{\underline{\eta}}} = \frac{\partial^{\eta^1}}{\partial (z^1)^{\eta^1}} \cdots \frac{\partial^{\eta^m}}{\partial (z^m)^{\eta^m}}.$$

The result may be broadened slightly and stated as a lemma :

LEMMA 1.1

Let $\{X_{\underline{\eta}}\ ,\ 0 \leq |\underline{\eta}| \leq r\}$ be a product family defined in a neighbourhood of p in V. Suppose that at p, the set of vectors $\{X_{\underline{\eta}} \mid |\underline{\eta}| = 1\}$ is a set of non zero independent vectors. Then there is a neighbourhood U of p and a C^r diffeomorphism h of U with $h(U) \subset R^n$, given a C^k structure for V ($r \leq k$) such that :

i) h(U) is a D set

ii) if $h(q) = \underline{z}(q)$, q in U, then $X_{\underline{\eta}} = \frac{\partial}{\partial \underline{z}^{\underline{\eta}}}$

iii) the integral curves of $X_{\underline{\eta}}$, with $\eta^I = \delta^I_J$ (I,J = 1,...,m) are given by $h^{-1}(z^1,z^2,\ldots,z^n)$ with z^J alone varying.

($\frac{\partial}{\partial \underline{z}^{\underline{\eta}}}$ is defined by $\frac{\partial}{\partial \underline{z}^{\underline{\eta}}}[g] = \frac{\partial^{|\underline{\eta}|}}{\partial (z^1)^{\eta^1} \cdots \partial (z^m)^{\eta^m}} (g \circ h^{-1}) \big|_{h(q)}$ which is well defined by (i))

COROLLARY 1

If m = n and p is not in ∂V, then the pair (U,h) of the lemma is a p chart.

COROLLARY 2

if $m = n$ and p is in ∂V with each $X_{\underline{\eta}}$ such that

$\eta^i = \delta^i_j \Rightarrow (X_{\underline{\eta}})_q$ is in $T_q(\partial V)$ q in $\partial V \cap U$

$\eta^m = 1,\ \eta^\alpha = 0 \Rightarrow (X_{\underline{\eta}})_q$ is in $T_q(V) \setminus T_q(\partial V)$ q in ∂V U

then (U,h) may be chosen to be a p chart.

The correspondence between product families and charts is of fundamental importance in the problem of matching manifolds.

1.3 Matching Manifolds

Let $V^\pm$ be C^k n-dimensional manifolds with boundary, denoted $\partial V^\pm$ respectively, and let $f: \partial V^+ \to \partial V^-$ be a C^k diffeomorphism, relative to the induced structures on $\partial V^\pm$. Let U_f denote the union operation (between subsets, maps etc.) in which points x in ∂V^+ and $f(x)$ in ∂V^- are identified.

Let $V = V^+ \cup_f V^-$. This section seeks to characterize C^k structures for V such that the inclusion maps of $V^\pm$ to themselves in V are C^k imbeddings. (We will work only with structures that have this property).

Firstly, notice that since f is a C^k diffeomorphism, the C^k structure induced on $\Sigma = V^+ \cap_f V^-$ in V by $V^\pm$ is necessarily the same. (To see this, notice that f may be regarded as the identity on Σ).

Secondly, it is a consequence of a Theorem of Munkres (1966) p 56, that there exist product meighbourhoods $N^\pm$ of $\partial V^\pm$ in $V^\pm$ respectively and C^k diffeomorphisms

$P^+ : N^+ \to V^+ \times [0,1)$ $\quad P^+(x) = (x,0)$ if x is in V^+

$P^- : N^- \to V^- \times (-1,0]$ $\quad P^-(y) = (y,0)$ if y is in V^-

Now $P = P^{+} U_f P^{-}$ maps $N = N^{+} U_f N^{-}$ to Σx (-1,-1). The map is clearly a bijection, we may require it to be a homeomorphism (thus topologising N) and a C^k diffeomorphism (thus giving a C^k structure to N). Since $P^{\pm}$ are diffeomorphisms, these requirements are consistent with the requirements that the inclusion maps ($I_f^{\pm}$) mapping $V^{\pm}$ to themselves in V be imbeddings, and together with this requirement, they specify a topology and a C^k structure for V. The topology is Haussdorff and paracompact (since that of Σ x (-1,-1) is) and V is a C^k manifold.

A chart of V is either of the form (U,h) with $U \cap \Sigma = \phi$ and (U,h) a V^{+} or V^{-} chart, or obtainable by a C^k diffeomorphism from a chart constructed thus :

Let (U_Σ, h_Σ) be a Σ chart. If $U = P^{-1}(U_\Sigma x(-1,-1))$ and q is in U, with $P(q) = (\tilde{q}, r)$, define $h(q) = (h_\Sigma(\tilde{q}), r)$. (U,h) is a V chart.

So a C^k structure exists for V. To what extent is this unique? We will answer this question progressively, first showing that the technique used above to construct a structure for V characterises all structures for V, then showing that all C^k structures for V are C^k diffeomorphic, and finally investigating the matching of the object spaces at p with those at f(p), for p in ∂V^{+}, which will lay the groundwork for selecting a specific C^k structure for V on metric and physical grounds.

Lemma 1 of appendix I yields the following :

If F is any structure for $V = V^{+} U_f V^{-}$ such that V are C^k imbedded in V, by the inclusion maps $I_f^{\pm}$, then there is a neighbourhood N of Σ in V and a C^k diffeomorphism

$P : N \rightarrow \Sigma$ x (-1,-1) such that

i) x in $\Sigma \rightarrow P(x) = (x,0)$

ii) x in $V^{+}\backslash\Sigma \rightarrow P(x) = (x,r)$ and $r > 0$

iii) x in $V^{-}\backslash\Sigma \rightarrow P(x) = (x,r)$ and $r < 0$

Define $N^{\pm} = N \cap V$; let $P^{\pm} = P_{|N^{\pm}}$. Then the pairs $(P^{\pm}, N^{\pm})$ may be used as above to construct a structure F' for V, by requiring as before that $P = P^{+} \cup_f P^{-}$ be a C^k diffeomorphism. But then, PP^{-1} is the identity map on V and is a C^k diffeomorphism of (V,F) with (V,F') so that F = F'. Thus every C^k structure for V may be constructed in the manner outlined. This enables us to prove:

THEOREM 1.1

If $V^{\pm}$ are C^k n-dimensional manifolds with boundary and $f:\partial V^{+} \rightarrow \partial V^{-}$ is a C^k diffeomorphism on ∂V^{+} onto ∂V^{-}, then there exists up to a C^k diffeomorphism, an unique C^k structure for V such that the inclusion maps $I_f^{\pm}: V \rightarrow V : p \mapsto p$ are C^k imbeddings of $V^{\pm}$ into V and $\partial V = \phi$.

Proof:

The existence of a structure has already been proven. It remains to prove that it is unique up to a diffeomorphism (C^k). Suppose therefore that F,F' are two structures for V; by the characterization remark, we may assume these constructed by means of maps P,P' of N with Σ x (-1,-1) (the domains of P and P' may, without loss of generality, be assumed the same, as will follow by sufficiently restricting the domains of P_1, P_1' used to construct them in lemma 1 of appendix I).

Noting that Σ is an unbounded manifold, let $g = P'P^{-1}$; this maps a neighbourhood W of $\Sigma \times \{0\}$ in $\Sigma \times R$ into $\Sigma \times R$, is the identity on $\Sigma \times [0,\infty) \cap W = W^{+}$ &

$\Sigma \times (-\infty,0] \cap W = W^-$. By lemma 2 of appendix I, there exists a map $\tilde{g}$ which is a homeomorphism of W with $P'P^{-1}(W)$, is $P'P^{-1}$ in a neighbourhood of the complement of W in V, a C^k diffeomorphism on W and the identity in a neighbourhood of $\Sigma \times \{\underline{0}\}$.

Let $h:(V,F) \to (V,F')$ be defined by $h(x) = ((P')^{-1} \circ \tilde{g} \circ P)(x)$ on the neighbourhood $P^{-1}(W)$ of Σ in (V,F); in a neighbourhood of the complement of $P^{-1}(W)$, h is the identity and may be extended to the whole of V by defining it to be the identity on $V \backslash P^{-1}(W)$; the result is a C^k diffeomorphism.

In general, h will not be the identity on V; if it is, then necessarily F and F' are the same. This lack of uniqueness, roughly, arises because f specifies (via f_*) only a matching of the object spaces $T_{p,r}(\partial V^+)$ and $T_{f(p),r}(\partial V^-)$, not of the full spaces $T_{p,r}(V^+)$ and $T_{f(p),r}(V^-)$ - i.e., no unique concept of differentiability across Σ in V is specified by f. Until such a concept is introduced (which is the task of Theorem 1.2), V may be viewed as consisting of two blocks of jelly, which, although fixed together on a common face, have no unique natural position relative to one another.

Let us introduce, given a structure F for V, a mapping implied by this structure between the full spaces $T_{p,r}(V^+)$ and $T_{f(p),r}(V^-)$ for p in ∂V^+. Define

$$\tilde{f}_* = (((I_f^-)_*)^{-1} \circ (I_f^+)_*) \qquad \ldots(1.9)$$

$\tilde{f}_*$ is defined for each p in V ; it is a map of full rank (n) on $T_{p,1}(V^+)$, since $I_f^{\pm}$ are imbeddings. It establishes the join of geometric object spaces across Σ in V. When restricted to $T_{p,r}(\partial V^+)$, $\tilde{f}_*$ clearly is f_* *(Condition 1)* (here ofcourse $1 \leq r \leq k$). As usual (see e.g. Choquet-Bruhat et al (1977) p 140) $\tilde{f}_*$ may be extended to a linear map of the tensor space $T_p\binom{r}{s}(V^+)$ to $T_{f(p)}\binom{r}{s}(V^-)$ by the requirement that it be a homomorphism with respect to the tensor product, and that it commutes with contractions.

Now I_f^+ being an imbedding, $(I_f^+)_*$ is a C^{k-r} imbedding of $T_{,r}(V^+)$ into $T_{,r}(V)$. In particular, when restricted to $T_{\partial V^+,r}(V^+)$, it is a C^{k-r} diffeomorphism with its image $T_{\Sigma,r}(V)$. Likewise, $(I_f^-)_*$ is a C^{k-r} diffeomorphism of $T_{\partial V^+,r}(V^+)$ with $T_{\Sigma,r}(V)$, so that $\tilde{f}_*$ is a C^{k-r} diffeomorphism of $T_{\partial V^+,r}(V^+)$ with $T_{\partial V^-,r}(V^-)$. *(CONDITION 2)*. If $X \in X_{\partial V^+,r}(V)$, let $\tilde{f}_* X$ be defined by (for q in ∂V^-) :

$$\tilde{f}_* X : q \to \tilde{f}_*(X_{f^{-1}(q)});$$

then $\tilde{f}_* X$ is in $X_{\partial V^-,r}(V)$.

Suppose for p in ∂V^+, that $\{X_{\eta}^+\}$ (η an m-tuple, $0 \le |\eta| \le r \le k$, say) is a product family defined on U, a neighbourhood of p in V^+; suppose that $\{X_{\eta} \mid |\eta| = 1\}$ is a set of independent non zero vectors at p, and let $h : U \to R^n$ be a diffeomorphism of U with h(U) in R^n of the form given by lemma 1.1 (restricting U if necessary). Then $I_f^+ \circ h^{-1}$ is an imbedding of h(U) into V and, as in the proof of corrollary II of the inverse function theorem (Appendix I), it follows that there is a chart of V $(\hat{U},\hat{h})$ say, with $I_f^+(U) \subset \hat{U}$, $\hat{h}_{|U} = h \circ (I_f^+)^{-1}$. The product family generated by the first m co-ordinate object fields of this chart is an extension of $(I_f^+)_*(X_{\eta}^+{}_{|U})$ to $\hat{U}$; necessarily, $\hat{U} \cap I_f^-(V^-) \neq \emptyset$, and $\hat{U} \cap I_f^-(V^-)$ provides a chart for V^-, and hence, a product family $\{X_{\eta}^-\}$ such that

$$X_{\eta}^-|_{\partial V^- \cap \hat{U}} = \tilde{f}_* \left(X_{\eta}^+|_{\partial V^+ \cap \hat{U}} \right) .$$

Performing this calculation at each p in ∂V^+ for which the X_{η}^+ are defined, and satisfy the nonzero and independence requirements given above clearly validates the following *(CONDITION 3)* :

If $\{X_{\eta}^+\}$ is a product family defined on $\mathcal{D}^+ \subset V^+$ such that $\{X_{\eta}^+ \mid |\eta|=1\}$ is independent and non-zero on $\mathcal{D}^+ \cap \partial V^+$ then there is a family $\{X_{\eta}^-\}$ such that $X_{\eta}^-|_{f(\mathcal{D}^+ \cap \partial V^+)} = f_* (X_{\eta}^+ | \mathcal{D}^+ \cap V^+)$.

Notice that condition 3 implies condition 2 - simply apply condition 3 to a co-ordinate induced product family on V^+ - the product family induced on V^- clearly specifies a chart in V^-, and then, since f is a C^k diffeomorphism and $\tilde{f}_*$ is linear, condition 2 is obtained.

Lastly, we obtain that <u>$\tilde{f}_*$ is orientation preserving</u> *(CONDITION 4)* : if p is in ∂V^+, (U^+,h^+), (U^-,h^-) are p and f(p) charts of types II and III respectively (cf section 1.2) i.e.

$$h^+ : U \to R^{n-1} \times [0,\infty)$$

$$h^- : U \to R^{n-1} \times (-\infty,0] \ ,$$

then, by the imbedding corollary of Appendix I, there exists a chart (U,h) in V with $U \cap V^+ = U^+$ and $h_{|U^+} = h^+$ which provides a chart $(U \cap V^-, h_{|U \cap N^-})$ for V^-, since V^- is imbedded in V. It is clear that for q in $U \cap U^-$ the jacobean of $h_{|U\cap V^-} \circ (h^-)^{-1}$ is positive - i.e. given

$$f_*\left(\frac{\partial}{\partial x^{\underline{\mu}}}\right)^+ = X^{\underline{\nu}}_{\underline{\mu}} \left(\frac{\partial}{\partial x^{\underline{\nu}}}\right)^- \quad (|\underline{\mu}| = |\underline{\nu}| = 1, \text{ sum over } |\nu| = 1)$$

then det X^{ν}_{μ} is positive; if as we may, we assume that (U,h) is of the type constructed in the proof of theorem 1.1, then this requirement is obvious. Broadly speaking, this is the requirement that a vector pointing towards the boundary in V^- should be the image of a vector in ∂V^+ which points away from the boundary in V^+.

Conditions 1 to 4 characterise $\tilde{f}_*$- like maps between $T_{\partial V^+, \tau}(V^+)$ and $T_{\partial V^-, \tau}(V^-)$, as is shown by the following theorem, which also implies that if two C^k structures for $V^+ \cup_f V^-$ induce the same $\tilde{f}_*$ then the structures are identical.

THEOREM 1.2

Let $V^{\pm}$ be C^k n-dimensional manifolds with boundaries $\partial V^{\pm}$. Suppose that $f:\partial V^+ \to \partial V^-$ is a C^k diffeomorphism. Let g be given as an isomorphism of $T_{p,r}(V^+)$ with $T_{f(p),r}(V^-)$ for each r, $1 \leq r \leq k$, and each p in ∂V^+, and as a map from $C_{p,k}(\partial V^+)$ to $C_{f(p),k}(\partial V^-)$ which satisfies :

a) g agrees with f_* on $T_{p,k}(\partial V^+)$ and $C_{p,k}(\partial V^+)$

b) if $\{X^+_{\underline{\eta}}\}$ is a product family in V, defined on $\mathcal{D}^+$(open in V^+) with $\mathcal{D}^+ \cap V^+ \neq \phi$, then there exists a product family $\{X^-_{\underline{\eta}}\}$ in $\mathcal{D}^-$ (open in V^-) with $(\mathcal{D}^- \cap \partial V^-) = f(\mathcal{D}^+ \cap \partial V^+)$ such that $g(X^+_{\underline{\eta}}|_{\mathcal{D}^+\cap\partial V^+}) = X^-_{\eta}|_{\mathcal{D}^-\cap\partial V^-}$.

c) if $(U^{\pm},h^{\pm})$ are type II and III charts in $V^{\pm}$ respectively, with $f(U^+\cap\partial V^+) \subset U^-\cap\partial V^-$, and $(\frac{\partial}{\partial x^i})^+$ and $(\frac{\partial}{\partial x^i})^-$ are the associated basis vector fields, then $g\,(\frac{\partial}{\partial x^i})^+ = X^j_i(\frac{\partial}{\partial x^j})^-$ implies det $\{X^j_i\}$ is positive,

then there exists an unique C^k structure for V, F say, whose $\tilde{f}_*$ is g.

Proof

Let (U^+,h^+) be a p chart in V^+, p in ∂V^+. On U^+ this induces the product family $(\frac{\partial}{\partial \underline{x}}\underline{\mu})$ (if $h^+(q) = \underline{x}(q)$).
Let $Y_{\underline{\mu}}$ be the product family in V^- whose domain is U^- and whose restriction to $U^-\cap\partial V^-$ is $g(\frac{\partial}{\partial \underline{x}}\underline{\mu}|_{U^+\cap\partial V^+})$. By the lemma 1.1, there exists a chart (U^-,h^-) say with $h^-(q) = \underline{y}(q)$, such that $Y_{\underline{\mu}} = (\frac{\partial}{\partial \underline{y}}\underline{\mu})$ ...(1.10)

Since f is a diffeomorphism and g agrees with f_* on $T_{p,k}(\partial V^+)$, there is no loss of generality in assuming that $h^-(f(q)) = h^+(q)$, for all q in $U^+ \cap \partial V^+$ - this because both $\partial V^\pm$ are imbedded in their respective $V^\pm$ and $g = f_*$ on $T_{\partial V^+, r}(V^+)$.

It follows by (c) that if $h^+ : U \to R^{n-1} \times [0,\infty)$, then $h^- : U \to R^{n-1} \times (-\infty,0]$ (noting that $X^\alpha_\beta = \delta^\alpha_\beta$ here).

Thus $h = h^+ \cup_f h^-$ is well defined on $U = U^+ \cup_f U^-$, and is a bijection of U with h(U) (recalling that it is a bijection on $U^\pm$ and using the previous paragraph).

Requiring that h be a homeomorphism and also, and consistently, requiring that the inclusion maps $I^\pm_f$ of $V^+ \setminus \partial V^+$ and $V^- \setminus \partial V^-$ be homeomorphisms specifies a topology for V, which is clearly hausdorff and locally finite, hence paracompact (since these are the properties of the topologies of $V^\pm$).

(U,h) is therefore a chart for V. Let (U',h') be another such chart, with $U \cap U' \neq \emptyset$; let (U'^+, h'^+) and (U'^-, h'^-) be the associated charts in $V^\pm$. Set

$$h'^+(q) = \underline{x}'(q) \text{ and } h'^-(q) = \underline{y}'(q).$$

Then, using the change of basis formula of section 1.2.2, for q in $\partial V^+ \cap U^+$:

$$\begin{aligned}(X^{\underline{\nu}}_{\underline{\mu}})^-_{f(q)} \left(\frac{\partial}{\partial \underline{y}}{}^{\underline{\nu}}\right)_{f(q)} &= \left(\frac{\partial}{\partial \underline{y}'}, \underline{\mu}\right)_{f(q)} \\ &= g\left(\left(\frac{\partial}{\partial \underline{x}'}, \underline{\mu}\right)_q\right) \\ &= g\left((X^{\underline{\nu}}_{\underline{\mu}})^+_q \left(\frac{\partial}{\partial \underline{x}}{}^{\underline{\nu}}\right)_q\right) \\ &= (X^{\underline{\nu}}_{\underline{\mu}})^+_q \left(\frac{\partial}{\partial \underline{y}}{}^{\underline{\nu}}\right)_{f(q)}\end{aligned}$$

whence $(X^{\underline{\nu}}_{\underline{\mu}})^-_{f(q)} = (X^{\underline{\nu}}_{\underline{\mu}})^+_q \qquad 1 \leq |\underline{\mu}|, |\underline{\nu}| \leq k_1.$

In particular, for $|\underline{\nu}| = 1$, $(X^{\underline{\nu}}_{\underline{\mu}})^-_{f(q)} = \left(\frac{\partial}{\partial \underline{y}'}, \underline{\mu}\right)_q [(\underline{y} - \underline{y}(q))^{\underline{\nu}}]$ and $(X^{\underline{\nu}}_{\underline{\mu}})^+_q = \left(\frac{\partial}{\partial \underline{x}'}, \underline{\mu}\right)_q [(\underline{x} - \underline{x}(q))^{\underline{\nu}}]$ whence

$$\frac{\partial}{\partial y'_{\mu}}(y^i) = \frac{\partial}{\partial x'_{\mu}}(x^i) \qquad (i = 1,\dots n, \quad 1 \leq |\mu| \leq k)$$

and $h \circ (h')^{-1}$ is C^k. ...(1.11)

It follows that by considering all such charts, together (and consistently) with the requirement that the imbedding inclusions $I_f^{\pm}$ be C^k imbeddings a C^k atlas and hence a C^k structure for V is defined, and $g = \tilde{f}_*$ (by 1.10). (Denote this structure by F).

The uniqueness of F follows by the argument (1.11) applied to the case when F' is another such structure for V, using the $\tilde{f}_*$ implied by F' for g with (U,h) in F and (U',h') in F'; $h \circ (h')^{-1}$ C^k then implies that F = F'.

COROLLARY

If F and F' are C^k structures for $V = V^+ \cup_f V^-$ such that $I_f^{\pm}$ are C^r imbeddings, which induce the maps $\tilde{f}_*$ and $\tilde{f}'_*$ respectively, then

a) $\tilde{f}_* = \tilde{f}'_*$ on $T_{\partial V^+, s}(V^+)$ for $0 \leq s \leq r$ $(r \leq k)$,

b) F and F' may be completed to the same C^r structure for V,

are equivalent.

COMMENT

We may replace condition (b) by the following condition :

b') for each p in V, there exist product families $\{X_{\eta}^{\pm} \mid 0 \leq |\eta| \leq k\}$ defined on $V^{\pm}$neighbourhoods $D^{\pm}$ of p and f(p) respectively, such that :

i) $\tilde{f}_*\left(X^+_{\underline{\eta}}|_{D^+\cap\partial V^+}\right) = X^-_{\underline{\eta}}|_{D^-\cap\partial V^-}$

ii) $\{(X^{\pm}_{\underline{\eta}})_p \mid |\underline{\eta}| = 1\}$ is an independent set, with $(X^{\pm}_{\underline{\eta}})_q$ in $T_q(\partial V^{\pm})$ for q in $D^{\pm}\cap\partial V^{\pm}$, $|\underline{\eta}| = 1$, $\eta^n = 0$, and $(X^{\pm}_{\underline{\eta}})_q$ in $T_q(V^{\pm}) \setminus T_q(\partial V^{\pm})$, for $\eta^\alpha = 0$ and $\eta^n = 1$.

It is obvious that the proof will proceed as before, except that the chart (U^+, h^+) will be specified by $X^+_{\underline{\eta}}$ (using corollary 2 of lemma 1.1 - the lemma on product families). The conclusion reached is the same.

CHAPTER TWO : JOINING METRIC AND CONNECTION STRUCTURE

In this chapter, the problem of matching metric structures defined on two manifolds $V^{\pm}$ which are to be joined across a common boundary is investigated.

After showing how unique fields, the normal and null fields, may be defined on the boundaries $\partial V^{\pm}$, and constructing charts based on these fields in neighbourhoods of $\partial V^{\pm}$, a study of the mechanics of matching the tensor spaces on $\partial V^{\pm}$ is continued from Chapter one, and compatibility conditions stated under which the union metric $g = g^{+} \cup_{f} g^{-}$ of the metrics $g^{\pm}$ on V is a C^{r} metric with respect to a C^{k} structure F for $V = V^{+} \cup_{f} V^{-}$.

We show (theorems 2.1 and 2.2) that there can be at most one C^{r+1} structure for V for which g is a C^{r} metric, and prove that a structure exists on which g is continuous, provided only that f is an isometry (theorem 2.3). This last theorem proves the existence of a unique set of 'admissible co-ordinates', much in the sense of Lichnerowicz (1955), and clarifies their meaning.

The above is applicable irrespective of the nature of $\partial V^{\pm}$ (i.e. null, spacelike or timelike or a combination), but for completeness and comparison with previous results, the concept of extrinsic curvature in the non-null case is discussed, and we show that the continuity of g and the extrinsic curvature tensor, is equivalent to the metric g being C^{1} on V, (given C^{1} at least on V), a result we will need to compare our results with those of Israel (1966) and others.

Finally, the theory is applied to the case of a purely impulsive plane gravitational wave, and mention is made of non-uniqueness problems arising in the definition of the boundary matching diffeomorphism f, caused by isometries acting within $\partial V^{\pm}$.

2.1 Metrics on $V^{\pm}$

As before, we shall suppose that $V^{\pm}$ are C^k manifolds with boundary, and $f : \partial V^+ \to \partial V^-$ is a C^k diffeomorphism on ∂V^+ onto ∂V^-; further, we shall suppose that $V^{\pm}$ are endowed with C^{k-1} locally Lorenzian metrics (signature +2); we will require that $k \geq 2$ in this chapter, and higher requirements will be needed later, when the Einstein equations are employed in $V^{\pm}$. As mentioned in Chapter one, the summation convention is active; lower case Romans range over $1,\ldots,n$, lower case Greeks over $1,\ldots,(n-1)$ and for the present, upper case Romans over $1,\ldots(n-2)$. Furthermore throughout the rest of this thesis, for convenience, sub or superscripts + or - on a kernel letter will indicate an object, set etc. in V^+ or V^-, whilst a kernel letter alone will indicate an object etc. in $V = V^+ \cup_f V^-$. For example, (U^+,h^+) will be a chart in a structure for V^+. In any chart of type II or III, $\partial V^{\pm}$ are locally given by

$$(h^{\pm})^{-1} \{ \underline{x} \mid x^n_{\pm} = 0, \ \underline{x} \text{ in } h^{\pm}(U^{\pm}) \} .$$

When a metric g is defined on a manifold V, the inner product g_p at p induces a map of $T_p(V)$ to $T^*_p(V)$ by $v_p \to v^g{}_p$ where $v^g{}_p(w_p) = g(v_p,w_p)$ for v_p,w_p both in $T_p(V)$. If v is in $X_{,1}(V)$, then v^g is in $X^*_{,1}(V)$ provided g is C^{k-1} at least. This is of course the map implied by the 'lowering' of indices using g, for if $v'_p = v^i_p (\frac{\partial}{\partial x^i})_p$, then $v^g{}_p = v^i_p \, g_{ij}(p) \, (dx^j)_p$. The map is an isomorphism and its inverse is a map from $T^*_p(V) \to T_p(V)$ by $\omega_p \to \omega^{g^{-1}}_p$; we have that $v^{gg^{-1}} = v$ and $\omega^{g^{-1}g} = \omega$.

Recall (cf appendix I, lemma 1) the existence of C^k functions on V^+ and on V^- such that

$$q \text{ in } V^+\backslash\partial V^+ \quad \text{implies} \quad r_+(q) > 0$$

$$q \text{ in } \partial V^+ \quad \text{implies} \quad r_+(q) = 0$$

$$q \text{ in } V^-\backslash\partial V^- \quad \text{implies} \quad r_-(q) < 0$$

$$q \text{ in } \partial V^- \quad \text{implies} \quad r_-(q) = 0$$

and such that rank $r_\pm = 1$ on $\partial V^\pm$, so that for p in $\partial V^\pm$ v_p in $T_p(V^\pm)\backslash T_p(\partial V^\pm)$,

$$v_p^\pm[r_\pm] \neq 0 \qquad \ldots(2.1)$$

We define the normal form on $\partial V^\pm$ by $dr_\pm$ and the normal vectors on $\partial V^\pm$ by

$$n^\pm = (dr_\pm)^{(g)^{\pm})^{-1}} \qquad \ldots(2.2)$$

By (2.1), $dr_\pm \neq 0$ and $n^\pm \neq 0$.

(Notice that $r = r_+ \cup_f r_-$ is well defined on $V = V^+ \cup_f V^-$, and it is possible to choose $r_\pm$ such that for a given C^k structure F for V, r is C^k on V, and the normal form $dr = (dr_+) \cup_f (dr_-)$ is well defined on Σ).

$\partial V^\pm$ will be termed of nature timelike, null or spacelike at p according as

$$e(n_p^\pm) = g\,(n_p^\pm, n_p^\pm) > 0 \quad (n_p \text{ is spacelike})$$
$$= 0 \quad (n_p \text{ is null})$$
$$< 0 \quad (n_p \text{ is timelike})$$

Without loss of generality, we may assume that $e(n_p^\pm) = 1, 0, -1$ by suitably rescaling r ; the fields $n^\pm$ are C^{k-1} on $\partial V^\pm$ since g is C^{k-1} on $V^\pm$.

If $n_p^\pm$ is null, i.e. if $e(n_p^\pm) = 0$, then $n_p^\pm$ is in $T_p(\partial V^\pm)$ for p in $\partial V^\pm$ (to see this, notice that $(dr_+)(n^+) = (dr_+)^{(g^+)^{-1}(g^+)}(n^+) = n^{+g^+}(n^+) = g^+(n^+, n^+) = 0$ which implies n^+ in $T_p(\partial V^+)$, for example), otherwise, by a similar argument, $n^{\pm}_p$ is in $T_p(V^{\pm})\setminus T_p(\partial V^{\pm})$. In this case, there exist neighbourhoods $N^{\pm}$ of $\partial V^{\pm}$ to which the field may be extended by $\nabla^+_{n^+} n^+ = 0$ or $\nabla^-_{n^-} n^- = 0$ ...(2.3)

(where ∇^+ and ∇^- denote the metric symmetric connections on V^+ and V^-).

Dealing for a moment only with n^+,

$$\nabla^+_n \, e(n^+) = 2\, g^+((\nabla^+_n \, n^+, n^+) = 0$$

(by the metric nature of g^+), so that $e(n^+)$ is preserved in the extension of n^+. If $(h_{\partial V^+}, U_{\partial V^+})$ is a ∂V^+ chart near p in ∂V^+, with co-ordinate induced fields $\frac{\partial}{\partial x_+^{\alpha}}$, we may extend the $\frac{\partial}{\partial x_+^{\alpha}}$ off of ∂V^+ to a neighbourhood of p by requiring

$$[n^+, \frac{\partial}{\partial x_+^{\alpha}}] = 0 \qquad \ldots(2.4)$$

(here, α is an n-tuple, with $\alpha^n=0$)

Noting that by the Jacobi identity

$$0 = [n^+, [\frac{\partial}{\partial x_+^{\alpha}}, \frac{\partial}{\partial x_+^{\beta}}]] + [\frac{\partial}{\partial x_+^{\alpha}}, [\frac{\partial}{\partial x_+^{\beta}}, n^+]] + [\frac{\partial}{\partial x_+^{\beta}}, [n^+, \frac{\partial}{\partial x_+^{\alpha}}]]$$

we obtain

$$Ł_{n^+} [\frac{\partial}{\partial x_+^{\alpha}}, \frac{\partial}{\partial x_+^{\beta}}] = [n^+, [\frac{\partial}{\partial x_+^{\alpha}}, \frac{\partial}{\partial x_+^{\beta}}]] = 0$$

where $Ł_{n+}$ denotes the Lie derivative with respect to n^+).

It follows (cf Choquet-Bruhat et al (1977) p.148) that $[\frac{\partial}{\partial x_+^{\alpha}}, \frac{\partial}{\partial x_+^{\beta}}]$ is invariant under the one parameter group of transformations generated by n^+ and hence, since zero on ∂V^+, zero where defined, i.e.

$$[\frac{\partial}{\partial x_+^{\alpha}}, \frac{\partial}{\partial x_+^{\beta}}] = 0 \qquad \ldots(2.5)$$

By this fact and (2.4), the family of fields generated by taking at most (k-1) products of the fields $\frac{\partial}{\partial x_+^{\alpha}}$, n^+ is a product family near p, satisfying the criteria of corollary 2 of lemma 1.1, and we obtain what we shall call the normal chart (U^+, h^+) near p.

This whole argument is equally applicable to n^-.

To interpret the normal charts is simple. Again, working for the moment only with n^+ : given q in V^+ near p (which is in ∂V^+), there is an unique integral curve γ_q of n^+ through q; the notion of the parameter distance r_q (for instance arclength) from q to ∂V^+ (signed +ve - signed -ve when n^- is used) is well defined; let γ_q intersect ∂V^+ in q' and set $h^+(q)=(h_{\partial V^+}(q'),r_q)$ which agrees with the previous definition of h^+ by virtue of (2.4).

Notice that the union $n = n^+ \cup_f n^-$ may not even be defined. We can seek conditions under which such an n is well defined and C^r on $V = V^+ \cup_f V^-$; this will solve the uniqueness problem of this chapter only when $\partial V^{\pm}$ are not null in a neighbourhood of p and f(p) respectively. In order to include the null case, we introduce the null fields $1^{\pm}$.

2.2 The Null Fields $1^{\pm}$.

We seek the definition of vector fields in the vicinity of $\partial V^{\pm}$, which can be uniquely defined irrespective of the nature of $\partial V^{\pm}$ using $g^{\pm}$. Let us work again only with ∂V^+ in V^+ : let p be in ∂V^+. Since g^+ is of signature +2, and C^{k-1}, we may find on ∂V^+, near p, (n-2) C^{k-1} vector fields e^+_A each of which is space-like, and which together form an orthonormal (independent) set, with each $(e^+_A)_q$ in $T_q(\partial V^+)$, for q in ∂V^+, near q. Since $T_q(\partial V^+)$ is (n-1) dimensional, we may find a C^{k-1} vector field k^+, defined on ∂V^+ near p, which is independent of the e^+_A such that $\{(e^+_A)_q, k^+\}$ is a basis of $T_q(\partial V^+)$. Define a vector field 1^+ by

a) $\overset{+}{g}(1^+_q, 1^+_q) = 0$ ...(2.6)

$\overset{+}{g}(1^+_q, (e^+_A)_q) = 0$ ...(2.7)

$\overset{+}{g}(1^+_q, k^+_q) = -1$ ...(2.8)

b) $1^+_q[r_+] > 0$, where r_+ is the map of $V^+ \to R$ described above.

An unique 1^+_q exists* which satisfies these conditions. To see this, complete $\{e^+_A\}$ into an orthonormal basis set of vector fields near p in ∂V^+, e^+_j say, with, again all the e^+_j C^{k-1} on ∂V^+ (g^+ being C^{k-1}); without loss of generality we will assume e^+_{n-1} spacelike and e^+_n timelike. Then any vector satisfing (2.7) is a linear combination of e^+_{n-1} and e^+_n . Tentatively set

$$1^+_q = \alpha(e^+_{n-1})_q + \beta(e^+_n)_q \qquad (\alpha, \beta \text{ in } R).$$

(2.6) implies

$$\alpha^2 = \beta^2 \qquad \ldots(2.9)$$

Now since k^+_q is independent of $(e^+_A)_q$, we may write

$$k^+_q = a(q)\ (e^+_{n-1})_q + b(q)\ (e^+_n)_q \qquad ..(2.10)$$

for C^{k-1} functions a,b on ∂V^+.

(2.10) and (2.8) imply

$$a(q)\alpha + \beta b(q) = -1 \qquad ..(2.11)$$

Since not both a(q) and b(q) are zero, (2.9) and (2.11) determine either two solutions for 1^+_q, one of which is selected using (b), or just one solution, which may be made consistent with (b) by an appropriate choice of the signs of $(e^+_{n-1})_q$ [r] and $(e^+_n)_q$ [r] and of k^+ itself. It is clear that 1^+ is C^{k-1} on ∂V^+, and $\{(e^+_A)_p, k^+_p, 1^+_p\}$ forms a basis for $T_p(V^+)$ (The independence of k^+_p and 1^+_p is easily seen).

The argument used to produce the normal chart (U^+, h^+) in the previous section may now be applied to create the null chart (U'^+, h'^+), based on 1^+.

This argument may be applied in V^- to determine an 1^- and charts (U'^-, h'^-).

These null charts are defined independently of any structure we may have established for $V = V^+ \cup_f V^-$, and independently of the nature of $\partial V^\pm$.

* l^+_q is unique when k^+_q is given

2.3 The maps $\tilde{f}_*$

Recall that

$$\tilde{f}_* \overset{def}{=} ((I_f^-)^{-1})_* \circ (I_f^+)_*$$

(equation (1.9) was defined as a map on $T_p(V^+)$ (for p in ∂V^+), and, recalling the theory presented in paragraph 2 of section 1.2.7, as a map on $T_p^*(V^+)$. We may extend $\tilde{f}_*$ to the tensor space $T_p\binom{s}{t}(V^+)$ (s,t in N), by requiring that it be a product homomorphism (w.r.t the tensor product), and to $C_{p,r}(\partial V^+)$ by equating it with f_*. Notice that on $T_p(\partial V^+)$ and on $T_p^*(\partial V^+)$ and hence on $T_p\binom{s}{t}(\partial V^+)$, $\tilde{f}_* = f_*$. By way of an example, if on ∂V^+, g^+ is given by

$$g_{ij}^+ \; dx_+^i \otimes dx_+^j,$$

then on V^-,

$$\tilde{f}_*(g^+)_q = (\tilde{f}_* \, g_{ij}^+)_q \, ((\tilde{f}_* \, dx_+^i)_q \otimes (\tilde{f}_* \, dx_+^j)_q)$$
$$= g_{ij}^+((f)^{-1}(q)) \, (\tilde{f}_* \, dx_+^i)_q \otimes (\tilde{f}_* \, dx_+^j)_q$$

2.4 Compatibility Conditions and Uniqueness

2.4.1 The Metric Compatibility Condition (m.c.c.)

A C^1 structure F for $V = V^+ \cup_f V^-$, is said to satisfy the metric compatibility condition if

$$\tilde{f}_*(g^+|_{V^+}) = g^-|_{V^-} \qquad ..(2.12)$$

The metric compatibility condition is a necessary and sufficient condition for the union metric $g = g^+ \cup_f g^-$ to be defined and continuous (C^0) on V.

It is clear that if the metric compatibility condition is satisfied, then

a) ∂V^+, ∂V^- and Σ are all of the same nature at every (corresponding) point (i.e. null, spacelike or timelike)

b) if ∂V^+, ∂V^- are not null, then the normal field defined by

$$n = n^+ \cup_f n^- \qquad ..(2.13)$$

is well defined (since $f_*(n^+)_{V^+} = n^-_{V^-}$ by the remark following equation (2.2)) and continuous near Σ.

c) irrespective of the nature of $\partial V^+, \partial V^-$ and Σ, the fields $1^\pm$ obey

$$f_* (1^+|_{V^+}) = 1^-|_{V^-} \qquad ..(2.14)$$

and are continuous on V.

It is clear that the equation (2.14) is independent of the structure F, and depends only on the fact that F obeys the m.c.c., and the fact that f is a diffeomorphism. Thus the m.c.c. provides a means of mapping vectors in $T_{\partial V^+}(V^+) \setminus T(\partial V^+) \longrightarrow T_{\partial V^-}(V^-) \setminus T(V^-)$. It follows that any two C^1 structures for V which satisfy the m.c.c. must induce $\tilde{f}_*$ which agree on $T_{\partial V^+}(V^+)$. Using the corollary of theorem 1.2 we have :

THEOREM 2.1

Let V be C^k manifolds and $g^\pm$ locally Lorentzian metrics (signature ± 2) which are continuous on V ; then, if a C^1 structure F exists for $V = V^+ \cup_f V^-$, which satisfies the metric compatibility condition, it is unique.

This theorem is independent of the nature of $\partial V^\pm$ and Σ , as remarked. The union of null charts (U'^+, h'^+) and (U'^-, h'^-) such that

$$U'^- \cap V^- = f(U'^+ \cap V^+) \qquad ..(2.15)$$

and

$$h'^{-}|_{V^-} \circ f = h'^{+}|_{V^+} \qquad ..(2.15')$$

given by

(U',h') with $U' = U'^{+} \cup_f U'^{-}$ and $h' = h'^{+} \cup_f h'^{-}$

is a chart in the unique C^1 structure for V, if it exists, termed (obviously) a null chart on V.

Likewise, the union of two normal charts satisfying conditions similar to (2.15) and (2.15'), will be called a normal chart, and will be a chart in any structure (C^1) for V which satisfies the m.c.c. (defined only where Σ is not null).

These charts will be important in practical applications.

2.4.2 The Connection Compatibility Condition (c.c.c.)

Consider now $(V^{\pm}, g^{\pm})$ as before and let u be a continuous and v a C^1 vector field on $V = V^+ \cup_f V^-$ under a structure F for V which satisfies the m.c.c.; if in addition, $\tilde{f}_* ((\Gamma^+(u,v))_p) = (\Gamma^-(u,v))_{f(p)}$ for all p in V^+, then F satisfies the connection compatibility condition (where Γ^+ and Γ^- are connections, metric and symmetric, induced on V^+ and V^- by g^+ and g^- resp.)

Under this condition, the joint metric connection

$$\nabla = \nabla^+ \cup_f \nabla^-$$

is continuous (i.e. relative to F its components Γ^i_{jk} are continuous) and defined by the union metric

$$g = g^+ \cup_f g^-.$$

It is clearly seen that if g is C^1 on V, under F, then F satisfies the connection compatibility condition, and conversely, since

$$\frac{\partial}{\partial x^a}\left[g\left(\frac{\partial}{\partial x^b};\frac{\partial}{\partial x^c}\right)\right] = \nabla_a\, g\left(\frac{\partial}{\partial x^b},\frac{\partial}{\partial x^c}\right)$$

$$= g\left(\nabla_a\frac{\partial}{\partial x^\beta},\frac{\partial}{\partial x^c}\right) + g\left(\frac{\partial}{\partial x^b},\nabla_a\frac{\partial}{\partial x^c}\right)$$

and hence C^0. (Notice that $\nabla_a = \nabla_{\frac{\partial}{\partial x^a}}$).

It is clearly necessary and sufficient for F to satisfy the c.c.c. that

$$\tilde{f}_*\left(\nabla^+_a \frac{\partial}{\partial x^b}\right) = \nabla^-_a \frac{\partial}{\partial x^b}$$

2.4.3 The Connection Compatibility Condition of order (r) (c.c.c.(r))

For ease of discussion, if ∇ is a metric symmetric connection on a Riemannian space, and $v_1,\ldots v_s,u$ are vector fields of appropriate differentiability, define

$$\nabla^{(s)}(v_1,v_2,\ldots,v_s,u) = \nabla_{v_1}\nabla_{v_2}\ \cdots\ \nabla_{v_s}\ u$$

where the metric is assumed of differentiability C^r and $s \leq r$.

Consider now $(V^\pm,g^\pm)$ and F as before, and let the <u>connection compatibility condition of order (r)</u> (c.c.c.(r)) be satisfied by F if F satisfies the m.c.c. (which will be refered to as the connection compatibility condition of order (0)) and if

$$\tilde{f}_*\left((\nabla^{+(s)}(v_1,\ldots,v_s,u)_p\right) = (\nabla^{-(s)}(v_1,\ldots,v_s,u))_{f(p)}$$

for all p in V^+, and for all $s \leq r$.

The c.c.c.(r) is satisfied iff the joint metric g is C^r w.r.t. the structure F. (The only if part is obvious, and the 'if' part follows by repeated applications of the technique of the previous section).

Immediately, for $1 \leq r \leq (k-1)$, recalling that $V^{\pm}$ are C^k, the joint null field $l = l^+ \cup_f l^-$ is a C^r field with respect to any structure $F\ (C^{r+1})$ satisfying the c.c.c.(r). Using the same chart (U_Σ, h_Σ) to construct null charts (U'^+, h'^+) and (U'^-, h'^-) (cf sections 2.2 and 2.3) and using the null co-ordinate vector fields defined by these charts to generate a pair of product families, together with the comment following theorem 1.2, yields (together with the result already stated for $r = 0$, theorem 2.1):

THEOREM 2.2

Let $V^{\pm}$ be C^k manifolds with boundary, and let $f : \partial V^+ \to \partial V^-$ be a C^k diffeomorphism as in theorem 1.2 . If $g^{\pm}$ are Lorentzian metrics (signature +2) of differentiability C^r on V with $0 \leq r \leq k-1$, then there is at most one C^{r+1} structure for $V = V^+ \cup_f V^-$ which satisfies the connection compatibility condition of order (r), equivalently, in respect of which the union metric $g = g^+ \cup_f g^-$ is C^r.

In other words, the imposition of a C^r metric on the manifold V uniquely fixes the C^{r+1} structure on V.

2.5 Existence of a structure satisfying the Metric Compatibility Condition

Our next result is of considerable theoretical and practical importance. We show that a necessary and sufficient condition for there to be a C^1 structure F for V which satisfies the metric compatibility condition is that f be an isometry between V^+ and V^- :

THEOREM 2.3

Let $(V^{\pm}, g^{\pm})$, $f : \partial V^{+} \to \partial V^{-}$ be as before. Then there exists a C^1 structure for $V = V^{+} \cup_f V^{-}$ obeying the metric compatibility condition iff f is an isometry w.r.t. the induced metric on ∂V^{+} and ∂V^{-}.

Proof :

The necessity in the theorem is trivial, since $\hat{f}_* = f_*$ on $T_p(\partial V^{+})$, for p in ∂V^{+}.

Otherwise, suppose that f is an isometry; we may require that

$$\tilde{f}_* \; e^{+}_{A} = e^{-}_{A} \qquad \text{(cf section 2.2)} \qquad ..(2.16)$$

and

$$\tilde{f}_* \; k^{+} = k^{-} \qquad ..(2.16')$$

It is necessary and sufficient for the satisfaction of the m.c.c. that $f_* \, 1^{+} = 1^{-}$; accordingly, define $g : T_p(V^{+}) \to T_{f(p)}(V^{-})$ by :

$$g(e^{+}_{A})_p = (e^{-}_{A})_{f(p)}$$

$$g(k^{+})_p = (k^{-})_{f(p)}$$

$$g(1^{+})_p = (1^{-})_{f(p)}$$

and on $C_p(\partial V^{+})$ to $C_p(\partial V^{-})$ by $g = f_*$

The result then follows by the comment to theorem 1.2 (recalling that 1^{+} and 1^{-} satisfy $1^{+}[r] > 0$ and $1^{-}[r] < 0$ so that condition (c) of theorem 1.2 is satisfied, using product charts of $\partial V^{\pm}$ constructed using r (cf section 1.3)).

This theorem is the main theorem on Riemannian geometry in this thesis. Its practical application will lie at the root of the work done in matching manifolds of the final chapters. When attempting to match two manifolds, across their common boundary, it assures us that we need check only that the induced metrics on this boundary agree, and then an unique C^1 structure, implying the matching of vectors across this boundary, is obtained. This will enable the matching of co-ordinates across the boundary, and will single out 'admissible' co-ordinates, much in the sense of Lichnerowicz (1955), on which calculations should be based. Notice that our treatment ensures the applicability of this result irrespective of the nature of the boundary.

2.6 The Extrinsic Curvature Condition.

It is useful, for comparison with earlier work (cf especially Israel (1966)) to show that in the case Σ not null, extrinsic curvature tensors may be introduced on $\partial V^{\pm}$, and their matching under $\hat{f}_*$ made equivalent to the c.c.c.

Given n^+ we may define a tensor field on V^+ in V^+ by :

$$K^+ = dx_+^{\alpha} \otimes \overset{+}{\nabla}_{\alpha} n^+ \qquad ..(2.17)$$

This definition requires n^+ to be defined only on ∂V^+; an alternative definition, requirng a C^{k-1} extension of n^+ to a neighbourhood of V^+ is :

$$K^+ = dx^c \otimes \overset{+}{\nabla}_c n^+ \qquad ..(2.17')$$

but this definition reduces to the previous one on V^+ irrespective of the extension used (cf Hawking and Ellis (1973) p46). We will use (2.17') together with the extension of n^+ via $\overset{+}{\nabla}_{n^+} n^+ = 0$ given earlier.

Then we have :

$$\overset{+}{\nabla}_c n^+ = K_c{}^d \frac{\partial}{\partial x_+^d} \overset{*}{=} \overset{+}{\Gamma}{}^d{}_{c4} \frac{\partial}{\partial x_+^d} \qquad ..(2.18)$$

where in the last term, we have assumed that a normal chart was being used.

Also, if ${}^{(n-1)}\nabla^{+}$ is the intrinsic metric connection on ∂V^{+} induced by g^{+}, we have :

$$ {}^{(n-1)}\nabla_{\alpha}^{+} \frac{\partial}{\partial x^{\beta}} = K^{+}_{\alpha\beta}\, n\, e(n) \quad + \quad \nabla_{\alpha}^{+} \frac{\partial}{\partial x^{\beta}} \qquad ..(2.19) $$

(the Gauss-Weingarten equations - cf Israel (1966))

(assuming that the $\frac{\partial}{\partial x^{\alpha}}$ are co-ordinate basis fields in ∂V^{+}).

K^{-} is similarly defined.

The extrinsic curvature condition (e.c.c.) is satisfied by any structure F for $V = V^{+} \cup_f V^{-}$ which in addition to satisfying the metric compatibility condition also satisfies :

$$ \tilde{f}_{*} K^{+}{}_{|V^{+}} \doteq K^{-}{}_{|V^{-}} $$

Since $\nabla^{+}_{n^{+}} n^{+} = 0 = \nabla^{-}_{n^{-}} n^{-}$, it follows that if Σ is not null, then the existence of a structure satisfying the e.c.c. implies the existence of a (unique) structure satisfying the c.c.c. Formally :

LEMMA 2.1

Let $V^{\pm}$, $g^{\pm}$ be as in theorem 2.3; suppose that a structure F exists for $V = V^{+} \cup_f V^{-}$ which satisfies the Extrinsic Curvature Condition. Then there exists an unique C^1 structure for V satisfying the Connection Compatibility Condition.

2.7 Practical application of the compatibility conditions - The purely impulsive gravitational wave

By way of an example of the notions of this and the previous chapter, we include the following example of a purely impulsive gravitational wave - cf Penrose (1972).

Let V^- be a flat (Minkowski) spacetime with metric given by

$$-2\, d\underset{-}{U}\, d\underset{-}{V} + (d\underset{-}{X})^2 + (d\underset{-}{Y})^2 = (d\underset{-}{s})^2 \quad \text{for} \quad \underset{-}{V} \leq 0$$

and V^+ also a flat spacetime, with metric

$$-2\, d\underset{+}{U}\, d\underset{+}{V} + (1+\underset{+}{V})^2(d\underset{+}{X})^2 + (1-\underset{+}{V})^2(d\underset{+}{Y})^2 = (d\underset{+}{s})^2 \quad \text{for} \quad \underset{+}{V} \geq 0$$

$\partial V^- \equiv \underset{-}{V} = 0$ and $\partial V^+ \equiv \underset{+}{V} = 0.$

Set $f : \partial V^+ \to \partial V^-$ by $(\underset{-}{U},\underset{-}{X},\underset{-}{Y}) = (\underset{+}{U},\underset{+}{X},\underset{+}{Y}) = f(\underset{+}{U},\underset{+}{X},\underset{+}{Y})$.

The induced metrics on V are :

$$\partial V^+ : \quad (d\underset{+}{X})^2 + (d\underset{+}{Y})^2 = (d\underset{+}{s})^2$$

$$\partial V^- : \quad (d\underset{-}{X})^2 + (d\underset{-}{Y})^2 = (d\underset{-}{s})^2$$

We next solve for $l^{\pm}$ as sugested above. Let

$$e_1^{\pm} = \frac{\partial}{\partial X_{\pm}} \qquad e_2 = \frac{\partial}{\partial Y_{\pm}} \qquad k = \frac{\partial}{\partial U_{\pm}}$$

Let $e_3^{\pm} = \frac{\partial}{\partial U_{\pm}} - \frac{\partial}{\partial V_{\pm}}$ (then $g^{\pm}(e_3,e_3) = 1$)

$e_4^{\pm} = \frac{\partial}{\partial U_{\pm}} - \frac{\partial}{\partial V_{\pm}}$ (then $g^{\pm}(e_4,e_4) = -1$)

Supposing that $l^{\pm} = \alpha e_3^{\pm} + \beta e_4^{\pm}$, and noting $k^{\pm} = 1/2\,(e_3^{\pm} + e_4^{\pm})$ obtain from (2.9)

$$\alpha_{\pm}^2 = \beta_{\pm}^2 \quad \text{and}$$

$$(1/2)\alpha^{\pm} - (1/2)\beta^{\pm} = -1 \qquad \text{(from (2.11))}$$

whence only $\alpha = -\beta = -1$

and $l^{\pm} = 2 \frac{\partial}{\partial v_{\pm}}$.

Specifying $\tilde{f}_*$ by requiring that $\tilde{f}_*(\frac{\partial}{\partial v_+})_p = (\frac{\partial}{\partial v_-})_{f(p)}$ for all p in V^+ specifies an unique C^1 structure for $V = V^+ \cup_f V^-$. This problem is examined further in a later chapter, where further individual matching problems will be examined.

2.8 Isometries within $\partial V^{\pm}$

All the results given so far depend on a given diffeomorphism between ∂V^+ and ∂V^-. Uniqueness under a given condition is uniqueness only for a given f. In physical cases, we shall require that at least the metric compatibility condition is satisfied, so that any physically acceptable choice of f must be an isometry. f is then either non-existent (i.e. the m.c.c. cannot be satisfied) or unique up to modification by isometries acting within ∂V^+ and ∂V^-. In general, we cannot hope to say more than this. It is possible that distinct choices of f may be made which lead to physically distinct $V(f) = V^+ \cup_f V^-$. It is also possible that requirements higher than the m.c.c. will lift the ambiguity.

In practice, however, it may be reasonable to expect that an isometry acting in V^+ or V^- is a restriction of a symmetry (in matter distribution for instance) operating some distance 'into' V^+ or V^-; specifically, it may be reasonable to suppose that there exist neighbourhoods N^+ and N^- of ∂V^+ and ∂V^- in which for instance either the fields l^+ and l^- or (in non-null cases) n^+ and n^-, or perhaps some other field whose restriction to ∂V^+ or ∂V^- lies out of the tangent space of $\partial V^{\pm}$ are Killing vectors. Then, if we work in null or normal charts, and if w^+, say is a Killing vector field in ∂V^+, we have (using $Ł_{w^+}(g^+) = 0$, $Ł_{l^+}(g^+) = 0$ and $[l^+, w^+] = Ł_{l^+}(w^+) = 0$, cf eqntn (2.4)) that $Ł_{l^+}(L_{w^+}(g)) = 0$ so that $Ł_{w^+}(g) = 0$ in a neighbourhood of ∂V^+, given the extension of w^+ off ∂V^+ using 1^+ (cf section 2.1). The result is that, although in global terms, V(f) may perhaps be defined in distinct ways, depending on f, there will be a neighbourhood of Σ in $V^+ \cup_f V^-$ which will look physically the same in every possible $V^+ \cup_f V^-$.

CHAPTER THREE : DISTRIBUTIONS ON MANIFOLDS

This chapter has two goals : to develop tools of functional analysis needed to deal with field and conservation equations across the join Σ of the manifolds V^+ and V^- of chapters one and two, and to apply these tools to derive functional equivalents of the Riemann and related tensors and the Bianchi identities for use on V. (The physical discussion of these results is the subject of chapter four).

The chapter recalls the notions of integration on manifolds we will need, and then discusses in detail the construction of spaces of distributions on manifolds. This latter theory is carried well beyond the requirements of the thesis itself - the reason is the hope, at a later stage, to broaden the scope of the thesis, as remarked in the introduction to chapter one.

It proves impossible to consider a distributional form of the Riemann tensor, for the covariant definition of the Riemann tensor fails, when the metric is not C^2, to extend from a multilinear functional on $D\binom{0}{1}(V) \times (D\binom{1}{0}(V))^3$ to a continuous functional on $D\binom{3}{1}(V)$. It is possible, however, to consider a functional analogue of the Riemann tensor - a combination of $\binom{1}{0}$ tensor distributions - and thereby to suggest a Ricci functional and a functional form of the Bianchi identities. Using these, the concept of a 'conservative' manifold may be introduced - this is a manifold in which (roughly speaking) matter-energy cannot be lost or gained.

3.1 Preliminaries

In this and the following sections, V will be assumed to be a C^k manifold, with boundary and with $k \geq 1$. Where necessary, we will use the fact that a C^k structure for V contains at least one C^∞ structure for V (which is in general not uniquely determined). g will be a Riemannian (pseudo) metric locally Lorentzian, signature +2, which will usually be required to be at least continuous on V.

If f is a function, tensor etc., define

$$\text{support of } f = \text{supp } f = \text{Cl}\{p \mid p \in V \text{ and } f(p) \neq 0\} ,$$

where Cl denotes topological closure; f is of compact support if supp f is a compact set in V.

Define the exterior product $\wedge$ of an r (exterior differential) form α with an s form β by

$$(\alpha \wedge \beta)(v_1, v_2, \ldots, v_{r+s}) = \frac{1}{r!s!} \sum_{\pi} \text{sign}(\pi) \; \alpha(v_{\pi(1)}, \ldots v_{\pi(r)}) \cdot \beta(v_{\pi(r+1)}, \ldots v_{\pi(r+s)}) \qquad \ldots(3.1)$$

where $v_1, v_2, \ldots, v_{r+s}$ are vector (fields), and the sum ranges over all permutations π of the integers $1, 2, \ldots, r+s$. The constant factor in this definition varies from text to text, which will mean differing factors in Stokes' and Gauss' theorems given below.

The exterior derivative is defined as usual by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta \qquad \ldots(3.2)$$

where $d(d\alpha) = 0$, $df = \frac{\partial f}{\partial x^i} dx^i$ for C^1 real map f on V etc.

If A is open in V, then A is itself a C^k manifold with boundary, with the structure induced by V; in particular, A is Haussdorff and paracompact.

A Hausdorff paracompact space is a countable union of compact sets (cf section 1.2) and therefore locally compact (every point in the space has a compact neighbourhood), and, being Hausdorff, has the property that every compact set is closed (cf Choquet-Bruhat et al (1977) p 15).

The class of compact sets in V will be denoted by K(V).

If $\{(U_i,h_i)\}$ is any locally finite atlas for V, there exists a locally finite partition of unity $\{(V_i,\psi_i)\}$ subordinate to the atlas, such that each $V_i = \text{supp}\,\psi_i$ is compact. (Cf Sternberg (1964), p56; this is a further consequence of the paracompact nature of V).

3.2 Integration on Manifolds.

A Borel σ-algebra for a topological space is the σ-algebra generated by the topology, i.e. the open sets or equivalently, the closed sets, of the space. A manifold may be equipped with a σ-algebra; notice that since a compact subset of a manifold is closed, it is measurable. If B is Borel measurable in V, and $B \subset U$ for some chart (U,h), then necessarily (and sufficiently), since h is an homeomorphism, h(B) is Borel measurable in R^n.

Let ω be an n-form with compact support in U, for a chart (U,h) of V; with respect to this chart, write $\omega = \omega_{1\,2\,\ldots\ldots\,n}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$; suppose that

$$\int_U \omega \overset{\text{def}}{=} \int_{h(U)} \omega_{1\,2\,\ldots\ldots\,n}\, d\mu$$

exists, where $d\mu$ denotes Lebesgue measure on R^n. Then ω is locally integrable.

Suppose next that $\{(U_i, h_i)\}$ is a locally finite orientated atlas for V, and $\{(V_i, \psi_i)\}$ a subordinate locally finite partition of unity (with the V_i compact). Then for a general n-form ω, ω is integrable if its integral

$$\int \omega = \sum_i \int_{U_i} \omega\, \psi_i \qquad \ldots(3.3)$$

exists and if the individual $\omega\psi_i$ are integrable. Since, using ordinary measure theory these latter are integrable iff $\int_{h(U_j)} |\psi_j \omega|\, d\mu$ exists, the form defined by

$$|\omega| \overset{\text{def}}{=} \sum_i |\omega_{1\,2\,\ldots\ldots\,n}|\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

is also integrable.

It is true that the definition of $\int \omega$ is independent of the choices of the atlas and partition of unity (cf Choquet-Bruhat (1977) p 204).

If A is a subset of V, χ_A is the characteristic function of A.
ω is said to be locally integrable if $\int_K \omega = \int \chi_K \omega$ exists for all K in K(V).

Integrals defined here have the usual properties of additivity etc.

Remaining in an orientated atlas, suppose that the well defined form given locally by $\eta = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n$ where $g = \det \{g_{ij}\}$ is locally integrable; since $\sqrt{|g|} > 0$ on V, $\eta(A) \overset{\text{det}}{=} \int_A \eta$ for measurable A is positive or zero and η may be regarded as a measure on V . Formally η satisfies the following :

(i) $\eta(\emptyset) = 0$

(ii) $\eta(A) \geq 0$ (for measurable A)

(iii) $\eta(\overset{\infty}{\underset{1}{\cup}} E_n) = \overset{\infty}{\underset{1}{\Sigma}} (E_n)$ (for a disjoint sequence of measurable E_i)

Further, η is σ-finite : V is a countable union of compact sets, and by assumption, $\eta(K)<\infty$ for K in K(V). η will be called metric or volume measure.

The usual results concerning integration using a measure now apply; f, a real function on V is integrable if $\int f\eta$ exists, in which instance, $\int |f|\eta$ exists also. If T is an $\binom{r}{s}$ tensor field, on V, we say that T is locally integrable if $\int_U |T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}|\eta$ exists for every chart U C K for a given K in K(V), and for all components $T^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$.

3.2.1 Orientation of boundaries

Suppose an orientated atlas of V is given consisting only of type I and type III charts; this induces an atlas on ∂V and, clearly, an orientation on ∂V. If n = dimension V is odd, this is the orientation we choose for ∂V; if n is even, we take the opposite orientation - that is, we replace each chart in our originally induced atlas for ∂V by a chart in which the sign of the first, or any one of the first (n-1) components is changed. (The dependence on the parity of n arises because locally, ∂V is given by $x^n = 0$, in this work, whereas normally (cf Choquet-Bruhat p 205 ff) it is given by $x^1 = 0$).
The orientation thus chosen for ∂V is called the orientation induced upon ∂V by the orientated atlas for V, or just the orientation induced on ∂V by the orientation on V.

Calculation reveals that

$$v\cdot\eta = \sqrt{|g|}\sum_{i=1}^{n} v^i(-1)^{(i-1)}(dx^1\wedge\ldots\wedge\widehat{dx^i}\wedge\ldots\wedge dx^n)$$

where the caret ^ denotes an omitted term. It follows that

$$d(v\cdot\eta) = \sqrt{|g|}\ v^i{}_{;i}\ dx^1\wedge\ldots\wedge dx^n$$

$$= \nabla\cdot v\ \ \eta$$

whence Gauss' theorem

$$\int_A \nabla\cdot v\eta = \int_{\partial A}\sqrt{|g|}\sum_{i=1}^{n} v^i(dx^1\wedge\ldots\wedge\widehat{dx^i}\wedge\ldots\wedge dx^n) \qquad \ldots(3.6)$$

Supposing that $\partial A \subset U$, for some co-ordinate chart U, in the (orientated) atlas chosen for A, the L.H.S. of (3.6) becomes :

$$\int_{\partial A}\sqrt{|g|}\ \ v^n(-1)^{n-1}\,dx^1\wedge\ldots\wedge dx^{n-1}$$

$$=\int_{h(A)}\sqrt{|g|}\ \ v^n\ \ dx^1dx^2\ldots dx^{n-1} \qquad \ldots(3.7)$$

using (3.3').

3.2.3 Integration on a joined manifold

Finally, turning to the case of manifold joins, given $V^{\pm}$, $g^{\pm}$ as before, suppose $V^{\pm}$ to be orientable with $\eta^{\pm}$ well defined volume measures. If $V^{\pm}$ are equipped with orientated atlases consisting of type I and type III charts only and thereby induce opposite orientations on $\Sigma = \partial V^+ \cup_f \partial V^-$, then $V = V^+ \cup_f V^-$ is orientable in any structure imbedding $V^{\pm}$ in V.

If F is an orientable structure for V which satisfies the metric compatibility condition, η^+ defined on V^+ using an orientated atlas of types I and II charts, and η^- defined on V^- using an orientated atlas of types I and III charts, $\eta = \eta^+ \cup_f \eta^-$ is well defined on V.

The Borel σ-algebra on V is clearly the union of the Borel σ-algebras on V^+ and V^- (recall that $V^\pm$ are closed in V, which implies that any closed set in V is the union of a closed set in V^+ and a closed set in V^-).

If the union metric $g = g^+ \cup_f g^-$ on V is not C^1 on V (i.e. if the connection compatibility conditions of order 1 and higher are not satisfied), then there is no guarantee that Stokes' and Gauss' theorems will hold for an arbitrary submanifold A of V, but they will hold on $V^\pm \cap A = A^\pm$, given that the $g^\pm$ are at least C^1.

3.3 Distributions on a manifold.

3.3.1 The Spaces $D^m(V)$, $D^m(^r_s)(V)$

Given a C^k structure for a manifold V (with $k \geq 1$), denote by $C^m_o(V)$ the set of all real C^m maps on V, with $0 \leq m \leq k$, which have compact support in V, and by $C^m_{oK}(V)$ the set of φ in $C^m_o(V)$ with support contained in K, for K in K(V). Likewise, denote by $C^m_o(^r_s)(V)$ the space of all (^r_s) tensor fields, of differentiability C^m, on V, with compact support in V, and by $C^m_{oK}(^r_s)(V)$ the space of all T in $C^m_o(^r_s)(V)$ with support in K, K in K(V). All these spaces are vector fields over R, and modules over the rings of C^m functions on V.

Begin with a compact set $K \subset U$ for some chart (U,h) of a C^m atlas for V; a norm for $C^m_{oK}(V)$ is given by

$$\|\varphi\|_{K,m} = \sup_{\substack{p \text{ in } K \\ 0 \leq |\mu| \leq m}} \left| \left(\frac{\partial}{\partial x^\mu}\right)_p [\varphi]_p \right| \quad \text{if } K \neq \emptyset \qquad \ldots(3.8)$$

$$= 0 \quad \text{if } K = \emptyset$$

(this exists by the continuity of $(\frac{\partial}{\partial x^{\mu}})[\phi]$, for every ϕ in $C^m(V)$).

Now let $\{(U_i,h_i)\}$ be a locally finite C^m atlas for V, and let $\{(K_j)\}$ be a subordinate locally finite covering of V by compact sets; assign one U_i containing K_j to K_j for each K_j. By lemma 1 of appendix II, if K is an arbitrary compact subset of V, there are at most a finite number of K_j such that $K \cap K_j \neq \emptyset$. Accordingly, define a norm on $C^m_{oK}(V)$ by

$$\|\varphi\|_{K,m} = \sup_j \|\varphi\|_{K\cap K_j, m} \qquad \ldots(3.9)$$

This is the supremum of a finite set, and exists for all φ in $C^m(V)$.

Notice that $\|\varphi\|_{K,0} = \sup_{p \text{ in } K} |\varphi(p)|$.

Similarly, for T in $C^m(^r_s)$ (V), and compact $K \subset U$, (U,h) a chart in a C^m atlas for V, set

$$\|T\|_{K,m(^r_s)} = \sup \|T^{i_1\ldots i_r}{}_{j_1\ldots j_s}\|_{K,m} \qquad ..(3.10)$$

where the supremum is taken over all components of T in the chart (U,h). For a given, locally finite atlas $\{(U_i,h_i)\}$ with subordinate compact covering $\{(K_j)\}$, K in K(V)

$$\|T\|_{K,m(^r_s)} = \sup_j \|T\|_{K\cap K_j,m(^r_s)} \qquad ..(3.11)$$

which define norms on $C^m_{oK}(^r_s)(V)$.

The norms (3.9) and (3.11) define topologies on the linear spaces $C^m_{oK}(V)$ and $C^m_{oK}(^r_s)(V)$ respectively - the topological normed linear spaces which result will be denoted by $D^m_K(V)$ and $D^m_K(^r_s)(V)$ respectively (cf the case $V = R^n$ in Choquet-Bruhat p 347).

We have

LEMMA 3.1

The topologies on $D^m_K(V)$ and $D^m_K(^r_s)(V)$ are independent of the choices of atlas and of subordinate compact cover used in their definition.

Proof

We prove only the assertion for $D^m_K(V)$. Let $\{(U'_i,h'_i)\}$ and $\{(K'_j)\}$ be a further locally finite C^m atlas and subordinate covering by compact sets. Denote by $\| \ \|'_{K,m}$ the norms they define on $C^m_K(V)$.

First of all, suppose that $K_i \cap K'_j \neq \emptyset$, with $K_i \subset U$ and $K_j \subset U'$, where (U,h) is in $\{(U_i,h_i)\}$ and (U',h') is in $\{(U'_i,h'_i)\}$. Then for φ in $C^m_K(V)$, and for $h(q)=\underline{x}(q)$, $h'(q)=\underline{y}(q)$, K in K(V),

$$\|\varphi\|_{K\cap K_i\cap K'_j,\, m} = \sup_{\substack{p \text{ in } K\cap K_i\cap K'_j \\ 0\le|\mu|\le m}} \left|\left(\frac{\partial}{\partial x^{\mu}}\right)_p [\varphi]_p\right|$$

$$= \sup_{\substack{p\in K\cap K_i\cap K'_j \\ 0\le|\mu|\le m}} \left| X^{\underline{\nu}}_{\underline{\mu}}(p)\left(\frac{\partial}{\partial y^{\underline{\nu}}}\right)_p [\varphi]_p \right| \quad \left(\text{sum } 0\le|\underline{\nu}|\le|\underline{\mu}|\right)$$

$$\le \sup_{\substack{p\in K\cap K_i\cap K'_j \\ 0\le|\underline{\mu}|\le m \\ 0\le|\underline{\nu}|\le|\underline{\mu}|}} \left|X^{\underline{\nu}}_{\underline{\mu}}(p)\right| \cdot \sup_{\substack{p\in K\cap K'_j\cap K_i \\ 0\le\underline{\nu}\le m}} \left|\left(\frac{\partial}{\partial y^{\underline{\nu}}}\right)_p [\varphi]_p\right| \cdot \sum_{0}^{|\underline{\mu}|=m} 1$$

$$= C_{i,j} \ \|\varphi\|'_{K\cap K_i\cap K'_j,\, m}$$

where $X^{\underline{\nu}}_{\underline{\mu}}$ is a C^m function on $U_i\cap U'_j$ (see the change of basis formula, section 1.2.2), so that the $C_{i,j}$ are well defined.

Now it is easily seen that $\|\varphi\|_{K,m} = \sup_{i,j} \|\varphi\|_{K\cap K_i\cap K'_j}$, a supremum of a finite set; setting $C(K) = \sup_{i,j} C_{i,j}$, the above implies :

$$\|\varphi\|_{K,m} \leq C(K) \|\varphi\|'_{K,m} .$$

By symmetry, then, $\| \ \|_{K,m}$ and $\| \ \|'_{K,m}$ are equivalent norms, and hence define the same topology on $C^m_{oK}(V)$

The proof for $D^m_K(^r_s)(V)$ is similar.

LEMMA 3.2

$C^m_{oK}(V)$ and $C^m_{oK}(^r_s)(V)$, with specific norms $\| \ \|_{K,m}$ and $\| \ \|_{K,m}(^r_s)$ are Banach spaces. Therefore, $D^m_K(V)$ and $D^m_K(^r_s)(V)$ are Frechet spaces.

Proof

We will not show that $\| \ \|_{K,m}$ is a norm - this is obvious. We show the completeness of $D^m_K(V)$ - the result for $D^m_K(^r_s)(V)$ follows similarly.

(a) suppose that $K \subset U$ for some chart (U,h) of V. Then the completeness of $D^m_K(V)$ follows by the usual results of convergence of uniformly convergent sequences of real functions.

(b) for general K in $K(V)$, let $\{\varphi_i\}$ be a Cauchy sequence in $D^m_K(V)$, and let $\{(U_i,h_i)\}$ be a locally finite C^m atlas for V with a subordinate C^m partition of unity $\{(K_j,\psi_j)\}$ (K_j = supp ψ_j in $K(V)$). Then, in terms of the norms associated with these families

$$\|\psi_j(\varphi_k - \varphi_l)\|_{K_j \cap K,m}$$
$$= \sup_{\substack{p \in K_j \cap K \\ 0 \leq |\mu| \leq m}} |(\partial_{x^\mu})_p [\psi_j(\varphi_k - \varphi_l)]_p|$$

$$\leq C\|\psi_j\|_{K\cap K_j,m} \|\phi_k - \phi_l\|_{K\cap K_j,m}$$

by Leibniz rule, where C dependes only on m. Thus $\{\psi_j \phi_k\}$ is a Cauchy sequence, for each j, converging in $D^m_{K\cap K_j}(V)$, by (a), to Φ_j, say. Set $\phi = \sum_j \Phi_j$; at most a finite number of the Φ_j are not the zero function, and it follows that $\{\phi_k\}$ converges to ϕ which is in $D^m_K(V)$.

The notion of a "Locally Frechet" or LF space is frequently used in the theory of distributions. See for instance Treves (1967) pp 126-135. We now show that the criteria for $C^m_o(V)$ to be given a topology resulting in the LF space $D^m(V)$ are met.

Since V is paracompact, it is a countable union of compact sets, as stated before, and there therefore exists an increasing sequence of compact sets K_j which collectively cover V - we assume that V is connected, or has countably many components. Clearly $D^m_{K_j}(V)$ is an increasing sequence of subspaces of $C^m_o(V)$, with, firstly, the topology induced on $C^m_{K_j}(V)$ by $D^m_{K_{j+1}}(V)$ being the same as that of $D^m_{K_j}(V)$ (to see this, note that the restriction to $C^m_{K_j}(V)$ of any norm on $D^m_{K_{j+1}}(V)$ is equivalent to any norm on $D^m_{K_j}(V)$) and secondly, each $D^m_{K_j}(V)$ is a Frechet space. A topology on $C^m_o(V)$, the strict inductive limit topology of the $D^m_{K_j}(V)$ is defined by :

let O be a convex subset of $C^m_o(V)$, containing its zero. (O is convex iff for x,y in O, ax + (1-a)y is in O for all a, $0 \leq a \leq 1$). O is a neighbourhood of zero in the strict inductive limit topology for $C^m_o(V)$ (i.e., O contains an open set containing zero) iff $O \cap D^m_{K_j}(V)$ is a neighbourhood of zero in $D^m_{K_j}(V)$. (A set is open in this topology for $C^m_o(V)$ iff it contains a neighbourhood of all its points).

The resulting topological space will be denoted by $D^m(V)$.

LEMMA 3.3

$D^m(V)$ has the following properties :

i) A convex subset O of $D^m(V)$ is a neighbourhood of zero in $D^m(V)$ iff for all K in K(V), $O \cap D^m_K(V)$ is a neighbourhood of zero in $D^m_K(V)$,

ii) The topology of $D^m(V)$ is independent of the choi ce of the sequence $\{K_i\}$ used in its definition,

iii) The topology induced on $C^m_{oK}(V)$ by $D^m(V)$ is that of $D^m_K(V)$,

iv) A linear functional on $D^m(V)$ is continuous iff its restriction to $D^m_K(V)$ for K in K(V) is continuous on $D^m_K(V)$, and

v) $D^m(V)$ is a locally convex complete space (although not metrizable).

Proof

i) given K in K(V), there is a K_j from the sequence used in the definition such that $K \subset K_j$; as has been said, the topology induced on C^m_{oK} by $D^m_{K_j}(V)$ is that of $D^m_K(V)$; the result is immediate.

ii) follows from (i)

iii) standard result on LF spaces - uses (i) and the result in Treves, p127.

iv) as (iii) - see Treves p 128

v) as (iii) - see Treves p 129

The spaces $D^m\binom{r}{s}(V)$ are similarly defined, with properties akin to those of $D^m(V)$.

3.3.2 The Spaces D(V) and $D(^r_s)(V)$

Next, suppose that F is a C^∞ structure on V (and recall that every C^k structure for a manifold V, with $k \geq 1$, contains at least one C^∞ structure (Munkres 1966) so that what follows is defined, though perhaps not uniquely, for every differentiable manifold). For K in K(V), the family of norms $\| \|_{K,m}$ for m = 0,1,2,... is a family of seminorms on $C_{oK}(V)$, is countable and separated (for terminology, see Choquet-Bruhat, p 343 ff) and hence defines a metrizable topology on $C_{oK}(V)$, resulting in the topological space $D_K(V)$. That $D_K(V)$ is a Frechet space (i.e. complete) follows as in lemma 3.2, from the Frechet nature of $D_K(V)$ for $K \subset U$, (U,h) a chart in F.

The same construction used to produce $D^m(V)$, as an LF space, will produce a topology on $C_o(V)$, yielding the space D(V). $D(^r_s)(V)$ is produced in a similar fashion, and the results of lemma 3.3 hold with $m = \infty$.

3.3.3 Distributions : the spaces D'(V) and $D'(^r_s)(V)$

Given a specific C^∞ structure F for a manifold V, construct D(V) and $D(^r_s)(V)$. A distribution on V is a continuous real functional on D(V). An (^r_s) tensor distribution V is a continuous real functional on $D(^s_r)(V)$. Distributions are elements of the dual space D'(V), whilst an (^r_s) tensor distribution is an element of the dual space $D'(^s_r)(V)$ (note the reversal of r and s).

Examples

Assume V orientated, let ω be a locally integrable form on V and let A be any measurable subset of a compact subset of V; then $\chi_A \omega$ is integrable and defines a distribution on V by

$$\omega_A(\varphi) = \int_A \varphi\omega \qquad ..(3.12)$$

(to show that ω_A is a distribution, notice that if φ is in $D_K(V)$, for any K in K(V), then

$$\begin{aligned} |\omega_A(\varphi)| &= \left|\int_A \varphi\omega\right| \\ &\leq \sup_{p \text{ in } A} |\varphi(p)| \int_A |\omega| \\ &\leq C \cdot \|\varphi\|_{K,0} \end{aligned}$$

so that ω_A is continuous*- we use the result of Treves (p 64) that a linear functional f on a linear space L with a topology defined by a family of semi-norms, is continuous if $|f(\varphi)| \leq C \cdot \|\varphi\|$ for some continuous semi-norm $\| \ \|$ on L)

Similarly, if V is endowed with a Riemannian metric g, whose metric form η is locally integrable, we define, for a locally integrable real map f on V,

$$f_A(\varphi) = \int_A f\varphi\,\eta \qquad ..(3.13)$$

Finally, is T is a locally integrable $\binom{r}{s}$ tensor field, an $\binom{r}{s}$ tensor distribution is defined by

$$T_A(S) = \int_A T(S)\,\eta \qquad ..(3.14)$$

for S in $D\binom{s}{r}(V)$.

Distributions expressible in one of the forms illustrated above will be termed <u>regular</u>. $T_A(S)$ is extended as an integral to all locally integrable $\binom{s}{r}$ fields S.

* strictly, we have proven that $\omega_{A|D_K(V)}$ is continuous for every K in K(V) which, by lemma 3.3(iv), with $m = \infty$, implies ω_A continuous on D(V)

3.3.4 The order of a distribution.

A distribution f is of order q if a least possible non-negative integer q exists such that for every K in K(V),

$$|f(\varphi)| \leq C(K)\|\varphi\|_{K,q} \qquad \text{for all } \varphi \text{ in } D_K(V) \qquad ..(3.15)$$

where C(K) is a constant, depending only on K.

The order of a distribution may be undefined - by virtue of part (iv) of lemma 3.3, adapted to D(V), and the comment on proving the continuity of a linear form on $D_K(V)$ of the previous section, (3.15) holds for each K in K(V) for some q(K), the order of the distribution f then being the supremum of the set

$$\{q(K) \mid K \text{ in } K(V)\}$$

if it exists.

The order of a tensor distribution is similarly defined.

Note that regular distributions are of order 0.

A fundamental lemma is the following

LEMMA 3.4

i) A distribution is of order q iff it can be extended to a continuous linear functional on $D^q(V)$

ii) Similarly, a tensor distribution T in $D'\binom{r}{s}(V)$ is of order q iff it can be extended to an element of $D^{q\prime}\binom{r}{s}(V)$.

iii) The extensions of (i) and (ii), if they exist, are unique.

The proof of this ~~theorem~~ lemma is given in appendix II.

This lemma will enable us to consider only C^r structures on V when considering qth order distributions and tensor distributions on V with $q \leq r$. It then follows, by Theorem 2.2 that the space of distributions of order $q \leq r$ on the joint manifold $V^+ \cup_f V^-$ is uniquely defined if the structure on $V^+ \cup_f V^-$ satisfies the connection compatibility condition order r-1, even if that structure contains several distinct C^∞ structures. In practice, this means that the spaces of distributions which concern us are uniquely defined.

3.3.5 Piecing distributions together.

Since often it will be convenient to work "in the small" i.e. on open neighbourhoods in a manifold V, and then to piece the result together into a global one, we give the following lemma :

LEMMA 3.5

Let V be endowed with a C^∞ structure F, and let U_i be an open covering of V. Let T_i be in $D(U_i)$ for each i such that for every φ in $D(U_i \cap U_j)$, $T_i(\varphi) = T_j(\varphi)$. Then there is an unique distribution T on V such that $T_{|D(U_i)} = T_i$.

Proof

This is proven in the same way as Choquet-Bruhat et al prove the lemma for $V = R^n$ (see p 360) - the proof rests on the fact that V has a locally finite partition of unity.

3.3.6 Products of real functions and distributions

If f is a distribution on V and ψ is in $C^{\infty}(V)$, define a distribution on V by

$$(\psi f)\ (\varphi) = f\ (\psi\varphi)$$

If f is a distribution of order q and ψ is in $C^q(V)$, then $f\psi$ is still a distribution (using Lemma 3.4), of order q (by Liebniz rule).

In this way, $D'(V)$ and $D^{m'}(V)$, also $D'\binom{r}{s}(V)$ and $D^{m'}\binom{r}{s}(V)$ may be viewed as modules over the spaces $C^{\infty}(V)$ and $C^m(V)$.

3.3.7 Covariant differentiation of distributions

Let A be a submanifold of V, with boundary ∂A; assume that A is orientated using an atlas of types I and III charts, inducing also an orientation of ∂A. Assume that the metric g is C^1 on A, so the Stokes' and Gauss' theorems are applicable on A.

Let V be a C^1 vector field on A, and set T_A to be the regular distribution defined on $D\binom{s}{r}(A)$ by the $\binom{r}{s}$ C^1 tensor field T on A. For S in $D\binom{s}{r}$, we have :

$$\begin{aligned}
&\int_{\partial A} T(S)\ v\cdot\eta \\
&= \int_A \nabla\cdot\ (T(S)v)\ \eta \\
&= \int_A dx^i(\ \nabla_i(T(S)v))\ \eta \\
&= \int_A dx^i((\nabla_i\ (T(S))\ )\ v\ ) + T(S)\ \nabla\cdot\ v\ \ \eta \\
&= \int_A \nabla_v\ (T(S))\ +\ T(S)\ \nabla\cdot v\ \eta \\
&= \int_A (\nabla_v(T))(S)\ +\ T(\nabla_v(S))\ +\ T(S)\nabla\cdot v\ \eta
\end{aligned}$$

whence the regular distribution defined on $D\binom{r}{s}(V)$ by $\nabla_v T$ is given by

$$(\nabla_v\ T_A)\ (S)\ =\ \int_A (\nabla_v T)\ (S)\ \eta$$

$$= \int_{\partial A} T(S) \; v\cdot\eta \;-\; \int_A T(\nabla_v S) + T(S)\; \nabla\cdot v \;\; \eta$$

..(3.16)

We have, for K in K(V) and S in $D_K(A)$

$$|(\nabla_v T_A)(S)| \le \int_{\partial A \cap K} T(S) \;\; v\cdot\eta \;+\; |T_A(\nabla_v S)| \;+\; |\nabla\cdot v \; T_A(S)| \qquad ..(3.17)$$

$$\le \;\; \| T \|_{K,0} \| S \|_{K,0}\binom{s}{r} \cdot \int_{\partial A \cap K} v\cdot\eta$$

$$+ \;\; C_1 \| S \|_{K,1}\binom{s}{r} \;+\; C_2 \| S \|_{K,0}\binom{s}{r}$$

where C_1 and C_2 are constants depending only on K

(take $C_1 = (\eta(A)) \cdot \| T \|_{K,0}\binom{r}{s} \; \| v \|_{K,0}\binom{1}{0}$

and $C_2 = \| T \|_{K,0}\binom{r}{s} \| v \|_{K,0}\binom{1}{0} \cdot C_3$,

where C_3 depends only on the first derivatives of g in the norm-defining charts intersecting K)

It is therefore true that for any locally integrable $\binom{r}{s}$ tensor field T on A, the distribution defined by (3.16), i.e.

$$\nabla_v \, T_A \,(S) \;=\; \left(\int_{\partial A} T(S) \;\; v\cdot\eta \;\right) \;-\; T_A(\nabla_v S + (\nabla\cdot v)S) \qquad ..(3.18)$$

is a first order distribution on D(A) - uniquely extendible to an element of $D^{1\prime}\binom{r}{s}(A)$. Indeed, if $\partial A = \emptyset$, (3.18) defines $(\nabla_v T_A)$ for all distributions T_A on D(A) and the argument at (3.17) will establish both that $(\nabla_v T_A)$ is a distribution (i.e. continuous) and that, if T is of order q, $\nabla_v T$ is of order at most (q+1), this last, given that the metric is at least C^{q+1} and v is at least C^q.

In the event that g fails to be C^1 on a set of measure zero on A, but with $\frac{\partial}{\partial x^i} g_{jk}$ locally integrable in every co-ordinate system of A, (3.16) remains a good definition of $\nabla_v \, T_A$, provided that T is a regular distribution (for $\int_A T(S)\eta = \int_{A\setminus M} T(S)\eta$ where $\eta(M) = 0$).

The next step is to calculate a double covariant derivative of a distribution; suppose to begin with, that T,g,u,v are all C^2 on A (u,v, being vector fields on A) and notice from (3.16) that

$$\begin{aligned} \nabla_v(\nabla_u T_A)(S) &= \int_{\partial A} (\nabla_u T)(S)\, v\cdot\eta - \nabla_u(T_A)\,(\nabla_v S + (\nabla\cdot v)(S)) \\ &= \int_{\partial A} (\nabla_u T)(S)\, v\cdot\eta - \int_{\partial A} T(\nabla_v S + (\nabla\cdot v)S)\, u\cdot\eta \\ &\quad + T_A(\nabla_u(\nabla_v S + (\nabla\cdot v)S + \nabla\cdot u(\nabla_v S + (\nabla\cdot v)S)) \end{aligned} \qquad ..(3.19)$$

As before (viz 3.17) we have that for any tensor T such that $\nabla_u T, \frac{\partial^2}{\partial x^k \partial x^l} g_{ij}$ are locally integrable on A and T is locally integrable on A, $\nabla_v(\nabla_u T_A)$ is a second order distribution, uniquely extendible to an element of $D^{2\prime}(^r_s)(A)$ Indeed, again for $\partial A = \emptyset$ (3.19) defines $\nabla_v(\nabla_u T_A)$ for all distributions T_A, and the result is a distribution of at most two orders more than T_A, if T_A is of finite order - in this case, sufficient differentiability requirements on g,u,v are that they be C^{q+2} for T_A of order q.

Also, as previously, if g fails to be C^2 on a set of measure zero in A, but with $\frac{\partial^2}{\partial x^i \partial x^j} g_{kl}$ still locally integrable in any chart of A, (3.19) remains a good definition for $\nabla_v \nabla_u T_A$, if T_A is regular.

3.3.8 Generalization of the domain A, and D(A) etc.

We have assumed, in 3.3.7 and 3.2 in the covariant differentiation of distributions and the statements of Stokes' and Gauss' theorems that the domain A was a submanifold of V, with boundary ∂A.

The above results will generalize to the case in which A is a domain of integration and ∂A its boundary (cf Choquet-Bruhat et al (1977) p 207). In particular, we will be using for A either a compact submanifold of V, or, in the case in which

$V = V^+ \cup_f V^-$ is a joined manifold, sets of the form $A^\pm = A \cap V^\pm$, with A a compact submanifold of V. Clearly $A^\pm$ will be compact on V , and although $A^\pm$ will not necessarily be manifolds in the true sense of the word, the notions applied previously in the definition of the spaces D(V) etc will still hold good if we replace V by A, and talk of D(A), which will be the space of all C^∞ real functions on A, of compact support in A, which will mean, since A is compact, that the support of some elements of D(A) will be A itself, with these maps non-zero on ∂A.

3.4 Distributions on Manifold joins

Let $V^\pm$ be C^k manifolds with boundaries $\partial V^\pm$, $f : \partial V^+ \to \partial V^-$ a C^k diffeomorphism on ∂V^+ onto ∂V^- and $g^\pm$ C^k metrics (signature +2) on V ; $k \geq 1$. These C^k structures contain (though perhaps not uniquely) C^∞ structures for V (cf Munkres (1966, p57)) and hence, the spaces $D'(V^\pm)$ and $D'\binom{r}{s}(V^\pm)$ may be defined. In particular, by lemma 3.4, $D^{k'}(V^\pm)$ and $D^{k'}\binom{r}{s}(V^\pm)$ are uniquely defined. Letting $V = V^+ \cup_f V^-$, V carries a C^k structure (again perhaps not uniquely definable) such that $V^\pm$ are imbedded in V, and D'(V) and $D'\binom{r}{s}(V)$ can be defined.

Given any tensor field S, let $S^\pm = ((I_f^\pm)_*)^{-1} (S_{|(I_f^\pm(V))})$ on V .

If K is compact in V, then $K^\pm = K \cap V^\pm$ are closed subspaces of K (regarded as a compact Hausdorff space), by virtue of the fact that $V^\pm$ are closed in V; thus $K^\pm$ are compact in $V^\pm$ and hence in V . (theorem, cf Choquet-Bruhat p 15). It follows that if S is in $D\binom{s}{r}(V)$, $S^\pm$ are in $D\binom{s}{r}(V^\pm)$.

Now define $g = g^+ \cup_f g^-$ on $V \backslash \Sigma$ and $\eta = \eta^+ \cup_f \eta^-$ on $V \backslash \Sigma$. Setting $\eta(\Sigma) = 0$,

η defines a measure on V, volume measure, even if g cannot be continuously extended to V. Accordingly, if T is an η-locally integrable tensor field, defined by $T = T^+ \cup_f T^-$ where $T^\pm$ are η-locally integrable tensors on $V^\pm$, then T defines a regular distribution by

$$\begin{aligned} T_A(S) &\overset{\text{def}}{=} \int_A T(S)\,\eta \\ &= \int_{A^+} T^+(S^+)\,\eta^+ + \int_{A^-} T^-(S^-)\,\eta^- \\ &= T_{A^+}(S) + T_{A^-}(S) \qquad (\text{ S is in } D\binom{s}{r}(V)\ ) \end{aligned} \qquad ..(3.20)$$

T_A is of order 0, and hence, (by Lemma 3.4) defined independently of the structure on V.

Now let $\nabla = \nabla^+ \cup_f \nabla^-$ on $V\backslash\Sigma$. If v is a C^1 vector field on V, we may, via (3.16) define $\nabla_v T_A$ by

$$\begin{aligned} (\nabla_v T_A)(S) &= -\,T_A(\nabla_v S + (\nabla\cdot v)\,S) + \int_{\partial A} T(S)\ v\cdot\eta \\ &= -\,T_{A^+}(\nabla^+_{v^+} S^+ + (\nabla^+\cdot v^+)\,S^+) - T_{A^-}(\nabla^-_{v^-} S^- + (\nabla^-\cdot v^-)\,S^-) \\ &\quad + \int_{\partial A^+\backslash\partial V^+} T^+(S^+)\ v^+\cdot\eta^+ + \int_{\partial A^-\backslash\partial V^-} T^-(S^-)\ v^-\cdot\eta^- \end{aligned} \qquad ..(3.21)$$

For arbitrary T, we must expect $(\nabla_v T_A)$ to be of order one. $\nabla_v T_A$ depends on the structure chosen for V (since $v^\pm$ do), but only on C^1 structure. If the structure for V satisfies the metric compatibility condition, then its C^1 structure is uniquely defined (Theorem 2.1), and $\nabla_v T_A$ is uniquely defined. For this reason and for the reason that in a sense the metric compatibility condition is a physical requirement (cf chapter 4), we will henceforth work only with structures for V which satisfy it.

It is a consequence of (3.21) and (3.16) that

$$\begin{aligned} (\nabla_v T_A)\ (S) &= (\nabla^+_{v^+} T^+_{A^+})(S^+) + (\nabla^-_{v^-} T^-_{A^-})(S^-) \\ &\quad - \int_{\partial A^+\cap\partial V^+} T^+_{A^+}(S^+)\ v^+\cdot\eta^+ - \int_{\partial A^-\cap\partial V^-} T^-_{A^-}(S^-)\ v^-\cdot\eta^- \end{aligned} \qquad ..(3.22)$$

Except where needed for clarity, we will omit the superscripts + and - on T,S etc. Since the metric compatibility conditions is assumed to hold, and since we will assume that V is orientable, $\eta = \eta^+ U_f \eta^-$ is well defined.

Similarly, we can define $(\nabla_v \nabla_u \; T_A)$ for v,u, C^2 on V by

$$\begin{aligned}\nabla_v \nabla_u \; T_A (S) &= T_A(\nabla_u(\nabla_v S + (\nabla\cdot v)S) + \nabla\cdot u(\nabla_v S + (\nabla\cdot v)S)) \\ &- \int_{\partial A^+\setminus\partial V^+} \{ T^+(\nabla_v^+ S + (\nabla^+\cdot v)S)\} u\cdot\eta - (\nabla_u^+ T)(S)\, v\cdot\eta \\ &- \int_{\partial A^-\setminus\partial V^-} \{ T^-(\nabla_v^- S + (\nabla^-\cdot v)S)\} u\cdot\eta - (\nabla_u^- T)(S)\, v\cdot\eta\end{aligned}$$

obtaining, provided ∇_u T are locally integrable on A :

$$\begin{aligned}\nabla_v \nabla_u \; T_A (S) = \; &\nabla_v^+ \nabla_u^+ T_{A^+}(S) + \nabla_v^- \nabla_u^- T_{A^-}(S) \\ &+ \int_{\partial A^+\cap\partial V^+} \{ T^+(\nabla_v^+ S + (\nabla^+\cdot v)S)\} u\cdot\eta - (\nabla_u^+ T^+)(S)\, v\cdot\eta \\ &+ \int_{\partial A^-\cap\partial V^-} \{ T^-(\nabla_v^- S + (\nabla^-\cdot v)S)\} u\cdot\eta - (\nabla_u^- T^-)(S)\, v\cdot\eta \qquad ..(3.23)\end{aligned}$$

For arbitrary T,$\nabla_v \nabla_u \; T_A$ is of order two; if the connection compatibility condition of order 1 is satisfied by the structure for V, then the C^2 structure of V is uniquely defined (via theorem 2.2) and $\nabla_v \nabla_u \; T_A$ is defineable quite uniquely. We will not, however, in general require the satisfaction of the c.c.c., so that $\nabla_v \nabla_u \; T_A$ will be ambiguously defined in general.

If we specialize T somewhat, then (3.22) and (3.23) simplify somewhat because some of the terms in the boundary integrals will cancel, using continuities of T across Σ.

3.5 The Riemann and related functionals on $V^+ U_f V^-$

We assume that a C^k structure exists for $V = V^+ U_f V^-$, which satisfies the metric compatibility condition, so that $g = g^+ U_f g^-$ is continuous on V; g are assumed C^k with $k \geq 2$. It is unfortunately not possible to construct a distributional form of the Riemann tensor, unless the connection compatibility condition, order 2, is satisfied, but we will define a multilinear real functional whose arguments are vector and one-form fields, based on a combination of distributions, which will serve the purpose of the Riemann tensor on V and enable us to write functional forms of the Einstein equations and the Bianchi identities.

Let ω be in $D(^0_1)(A)$ and let x,u,v be in $D(^1_0)(A)$. Define :

$$\mathrm{Riemann}_A(\omega,x,u,v) = \nabla_u(\nabla_v x)_A(\omega) - \nabla_v(\nabla_u x)_A(\omega) - (\nabla_{[u,v]}x)_A(\omega) \qquad ..(3.24)$$

where $(\nabla_v x)_A(\omega) = \int_{A^+} \omega(\nabla_v^+ x)\eta + \int_{A^-} \omega(\nabla_v^- x)\eta$ etc.

The form of Riemann is fairly obviously based on the usual definition of the Riemann tensor. By (3.21)

$$\begin{aligned}\mathrm{Riemann}_A(\omega,x,u,v) = & -(\nabla_v x)_A(\nabla_u\omega + (\nabla\cdot u)\omega) + (\nabla_u x)_A(\nabla_v\omega + (\nabla\cdot v)\omega) - (\nabla_{[u,v]}x)_A(\omega)\\ & + \int_{\partial A^-\cap\partial V^-} \omega(\nabla_u^- x)\,v\cdot\eta - \omega(\nabla_v^- x)\,u\cdot\eta\\ & + \int_{\partial A^+\cap\partial V^+} \omega(\nabla_u^+ x)\,v\cdot\eta - \omega(\nabla_v^+ x)\,u\cdot\eta\end{aligned} \qquad ..(3.25)$$

where

$$(\nabla_v x)_A(\nabla_u\omega + (\nabla\cdot u)\omega) = \int_A (\nabla_u\omega + (\nabla\cdot u)\omega)(\nabla_v x)\,\eta$$

etc.

This definition is based on an extension of $(\nabla_v X)_A$ to the space of locally integrable one form fields by means of an integral representation. This is not necessarily the only extension we could use. It is a natural choice, and if the connection compatibility condition is satisfied, the only choice (to see this, note that if the c.c.c. is satisfied, $(\nabla_u\omega + \nabla\cdot u\,\omega)$ is in $D^0(^0_1)(A)$, and $(\nabla_v x)_A$ being of order zero is uniquely extendible to $D^0(^0_1)(A)$ by lemma 3.4). This uniqueness of choice (when the c.c.c. is satisfied) is the reason also for basing Riemann on $(\nabla_v x)_A$ and not on $\nabla_v(x)_A$, for in the latter case, an expansion akin to (3.25) would yield terms such as $(x)_A(\nabla_v(\nabla_u\omega + (\nabla\cdot u)\omega) + \bar{\nabla}\cdot v(\nabla_u\omega + (\nabla\cdot u)\omega))$ whose uniqueness is in question until the connection compatibility condition (order 2) is satisfied, which is generally too stringent a requirement.

It follows from (3.25) and (3.22) that

$$\begin{aligned} \text{Riemann}_A(\omega,x,u,v) &= \text{Riemann}_{A^+}(\omega,x,u,v) + \text{Riemann}_{A^-}(\omega,x,u,v) \\ &+ \int_{\partial A^-\cap\partial V^-} \omega(\nabla^-_u x)\, v\cdot\eta - \omega(\nabla^-_v x)\, u\cdot\eta \\ &+ \int_{\partial A^+\cap\partial V^+} \omega(\nabla^+_u x)\, v\cdot\eta - \omega(\nabla^+_v x)\, u\cdot\eta \end{aligned} \quad ..(3.26)$$

where now, since $g^\pm$ are C^2, Riemann_A are in fact given by $\binom{1}{3}$ regular tensor distributions, based on the Riemann tensor fields $R^\pm$ in V ;

$\text{Riemann}_{A^+}(,x,u,v) = R^+_{A^+}(\otimes x \otimes u \otimes v)$.

Based on (3.26) we can extend Riemann_A to all fields ω,x,u,v which are C^1 on $V\setminus\Sigma$ and continuous on V.

Taking the integral representation of (3.26), for u,v, in $D\binom{1}{0}(A)$, we may define, covariantly :

$$\begin{aligned} \text{Ricci}_A(u,v) &= \text{Ricci}_{A^+}(u,v) + \text{Ricci}_{A^-}(u,v) \\ &+ \int_{\partial A^-\cap\partial V^-} (dx^a\,\nabla^-_a u)\, v\cdot\eta - (dx^a)(\nabla^-_v u)\, \tfrac{\partial}{\partial x^a}\cdot\eta \\ &+ \int_{\partial A^+\cap\partial V^+} (dx^a\,\nabla^+_a u)\, v\cdot\eta - (dx^a)(\nabla^+_v u)\, \tfrac{\partial}{\partial x^a}\cdot\eta \\ &= \text{Ricci}_{A^+}(u,v) + \text{Ricci}_{A^-}(u,v) \\ &+ \int_{\partial A^-\cap\partial V^-} ((\nabla^-\cdot u)v - \nabla^-_v u)\cdot\eta \\ &+ \int_{\partial A^+\cap\partial V^+} ((\nabla^+\cdot u)v - \nabla^+_v u)\cdot\eta \end{aligned} \quad ..(3.27)$$

where Ricci_{A^+} and Ricci_{A^-} are given in terms of the regular distributions based on the Ricci tensors in V^+ and V^-. (3.27) will be the basis for the geometric part of the functional Einstein equations across Σ in V.

Letting ω, χ be in $D\binom{0}{1}(A)$, u,v in $D\binom{1}{0}(A)$, and setting $x = \chi^{g^{-1}}$

(the metric conjugate of $\mathcal{X}$, cf chapter 2), we define

$$\mathrm{Riemann}_A(\omega,\mathcal{X},u,v) = \mathrm{Riemann}_A(\omega,x,u,v) \qquad ..(3.28)$$

the right hand side being well defined by the comment after equation (3.26).

3.6 Bianchi identities - the functional form

For w a C^1 vector field on V, we define

$$(\nabla_W \mathrm{Riemann})_A(\omega,x,u,v) = \nabla_W(\nabla_u(\nabla_v x)_A)(\omega) - \nabla_W(\nabla_v(\nabla_u x)_A)(\omega) - \nabla_W(\nabla_{[u,v]}x)_A(\omega) \qquad ..(3.29)$$

Expanding these, using (3.23) we obtain :

$$\begin{aligned}
&(\nabla_W(\mathrm{Riemann}))_A(\omega,x,u,v) \\
&= (\nabla^+_{w^+}\mathrm{Riemann})_{A^+}(\omega,x,u,v) + (\nabla^-_{w^-}\mathrm{Riemann})_{A^-}(\omega,x,u,v) \\
&+ \int_{\partial A^+\cap\partial V^+} \{(\nabla^+_W\omega + (\nabla^+\cdot w)\omega)\nabla^+_v x\}\, u\cdot\eta - \omega(\nabla^+_u(\nabla^+_v x))\, w\cdot\eta \\
&+ \int_{\partial A^-\cap\partial V^-} \{(\nabla^-_W\omega + (\nabla^-\cdot w)\omega)\nabla^-_v x\}\, u\cdot\eta - \omega(\nabla^-_u(\nabla^-_v x))\, w\cdot\eta \\
&- \int_{\partial A^+\cap\partial V^+} \{(\nabla^+_W\omega + (\nabla^+\cdot w)\omega)\nabla^+_u x\}\, v\cdot\eta - \omega(\nabla^+_v(\nabla^+_u x))\, w\cdot\eta + \int_{\partial A^+\cap\partial V^+}(\nabla^+_{[u,v]}x)(\omega)\, w\cdot\eta \\
&- \int_{\partial A^-\cap\partial V^-} \{(\nabla^-_W\omega + (\nabla^-\cdot w)\omega)\nabla^-_u x\}\, v\cdot\eta - \omega(\nabla^-_v(\nabla^-_u x))\, w\cdot\eta + \int_{\partial A^-\cap\partial V^-}(\nabla^-_{[u,v]}x)(\omega)\, w\cdot\eta \\
&= \nabla^+_W(R^+_{A^+})(\omega\otimes x\otimes u\otimes v) + \nabla^-_W(R^-_{A^-})(\omega\otimes x\otimes u\otimes v) \\
&- \int_{\partial A^+\cap\partial V^+} R^+(\omega,x,u,v)\, w\cdot\eta - \int_{\partial A^-\cap\partial V^-} R^-(\omega,x,u,v)\, w\cdot\eta \\
&+ \int_{\partial A^+\cap\partial V^+} \{(\nabla^+_v x)(\nabla^+_W\omega + (\nabla^+\cdot w)\omega)\}\, u\cdot\eta - \{\nabla^+_u x\, (\nabla^+_W\omega + (\nabla\cdot w)\omega)\}\, v\cdot\eta \\
&+ \int_{\partial A^-\cap\partial V^+} \{(\nabla^-_v x)(\nabla^-_W\omega + (\nabla^-\cdot w)\omega)\}\, u\cdot\eta - \{\nabla^-_u x\, (\nabla^-_W\omega + (\nabla^-\cdot w)\omega)\}\, v\cdot\eta
\end{aligned} \qquad ..(3.30)$$

where as before $R^{\pm}$ denote the Riemann tensors in $V^{\pm}$ and, assuming $k \geq 3$,

$(\nabla_W R^{\pm}_A)$ are regular distributions.

Employing the Bianchi identities in $V^{\pm}$, obtain the functional Bianchi identity on V as :

$$\sum_{c(w,u,v)} (\nabla_w \text{Riemann})_A(\omega,x,u,v)$$
$$= \sum_{c(w,u,v)} \Big\{ -\int_{\partial A^+\cap\partial V^+} R^+(\omega,x,u,v)\, w\cdot\eta \;-\; \int_{\partial A^-\cap\partial V^-} R^-(\omega,x,u,v)\, w\cdot\eta$$
$$+ \int_{\partial A^+\cap\partial V^+} (\nabla^+_w\omega + (\nabla^+\cdot w)\omega)\nabla^+_v x \;\; u\cdot\eta - (\nabla^+_w\omega + \nabla^+\cdot w\,\omega)\nabla^+_u x \;\; v\cdot\eta$$
$$+ \int_{\partial A^-\cap\partial V^-} (\nabla^-_w\omega + (\nabla^-\cdot w)\omega)\nabla^-_v x \;\; u\cdot\eta - (\nabla^-_w\omega + \nabla^-\cdot w\,\omega)\nabla^-_u x \;\; v\cdot\eta \Big\} \qquad ..(3.31)$$

where $\sum_{c(w,u,v)}$ denotes the sum over the cyclic permutation of (w,u,v).

If the structure on V satisfies the connection compatibility condition, the full functional Bianchi identities are

$$\sum_c (\nabla_w \text{Riemann})_A(\omega,x,u,v) = \sum_{c(w,u,v)} \Big\{ \int_{\partial A^+\cap\partial V^+} R^+(\omega,x,u,v)\, w\cdot\eta + \int_{\partial A^-\cap\partial V^-} R^-(\omega,x,u,v)\, w\cdot\eta \Big\}$$
$$= \sum_{c(w,u,v)} \Big\{ \int_{A\cap\Sigma} (R^- - R^+)(\omega,x,u,v)\, w\cdot\eta \Big\} \qquad ..(3.32)$$

(since the remaining integrands under the integrals over $\partial A^+\cap\partial V^+$ and $\partial A^-\cap\partial V^-$ are now equal, whilst the integrals induce an opposite sense on $A\cap\Sigma$ so that they cancel)

Irrespective of whether the c.c.c. is satisfied, we will say the V is a conservative manifold if

$$\sum_{c(w,u,v)} (\nabla_w \text{Riemann})_A(\omega,x,u,v) = 0 \qquad ..(3.33)$$

for all vector fields x,u,v and form fields ω which are C^1 on $V\backslash\Sigma$ and continuous on V.

If as before, χ is a one form field, also C^1 on $V\backslash\Sigma$ and continuous on V, we may derive (3.31) and (3.32) with χ in place of x. (Cf (3.28))

3.7 Contracted Bianchi identities - functional form.

We now imagine A contained in some co-ordinate chart.

3.7.1 First contration

Consider :

$$\sum_{c(w,\frac{\partial}{\partial x^a},v)} (\nabla_w \text{ Riemann})_A (dx^a, x, \tfrac{\partial}{\partial x^a}, v)$$

$$= (\nabla_w \text{ Riemann})_A (dx^a, x, \tfrac{\partial}{\partial x^a}, v) + (\nabla_a \text{ Riemann})_A (dx^a, X, v, w) + (\nabla_v \text{ Riemann})_A (dx^a, X, w, \tfrac{\partial}{\partial x^a})$$

$$= \sum_{c(w,\frac{\partial}{\partial x^a},v)} -\int_{\partial A^+\cap\partial V^+} R^+(dx^a, X, \tfrac{\partial}{\partial x^a}, v) w\cdot\eta - \int_{\partial A^-\cap\partial V^-} R^-(dx^a, x, \tfrac{\partial}{\partial x^a}, v)\, w\cdot\eta$$
$$+ \int_{\partial A^+\cap\partial V^+} \{(\nabla_w^+ dx^a + (\nabla^+\cdot w)\, dx^a)\, \nabla_v^+ X\} \tfrac{\partial}{\partial x^a}\cdot\eta - \{(\nabla_w^+ dx^a + (\nabla^+\cdot w)\, dx^a)\, \nabla_a^+ X\}\, v\cdot\eta$$
$$+ \int_{\partial A^-\cap\partial V^-} \{(\nabla_w^- dx^a + (\nabla^-\cdot w)\, dx^a)\, \nabla_v^- X\} \tfrac{\partial}{\partial x^a}\cdot\eta - \{(\nabla_w^- dx^a + (\nabla^-\cdot w)\, dx^a)\, \nabla_a^- X\}\, v\cdot\eta$$

..(3.34)

The first term of (3.34) yields :

$$- \int_{\partial A^+\cap\partial V^+} R^+(x,v)\, w\cdot\eta - \int_{\partial A^-\cap\partial V^-} R^-(x,v)\, w\cdot\eta$$
$$+ \int_{\partial A^+\cap\partial V^+} \{(\nabla_w^+ dx^a + (\nabla^+\cdot w)\, dx^a)\, \nabla_v^+ x\} \tfrac{\partial}{\partial x^a}\cdot\eta - \{(\nabla_w^+ dx^a + (\nabla^+\cdot w)\, dx^a)\, \nabla_a^+ x\}\, v\cdot\eta$$
$$+ \int_{\partial A^-\cap\partial V^-} \{(\nabla_w^- dx^a + (\nabla^-\cdot w)\, dx^a)\, \nabla_v^- x\} \tfrac{\partial}{\partial x^a}\cdot\eta - \{(\nabla_w^- dx^a + (\nabla^-\cdot w)\, dx^a)\, \nabla_a^- x\}\, v\cdot\eta \qquad .(3.34a)$$

The third term of (3.34) yields :

$$+ \int_{\partial A^+\cap\partial V^+} R^+(x,w)\, v\cdot\eta + \int_{\partial A^-\cap\partial V^-} R^-(x,w)\, v\cdot\eta$$
$$- \int_{\partial A^+\cap\partial V^+} \{(\nabla_v^+ dx^a + (\nabla^+\cdot v)\, dx^a)\, \nabla_w^+ x\} \tfrac{\partial}{\partial x^a}\cdot\eta - \{(\nabla_v^+ dx^a + (\nabla\cdot v)\, dx^a)\, \nabla_a^+ x\}\, w\cdot\eta$$
$$- \int_{\partial A^-\cap\partial V^-} \{(\nabla_v^- dx^a + (\nabla^-\cdot v)\, dx^a)\, \nabla_w^- x\} \tfrac{\partial}{\partial x^a}\cdot\eta - \{(\nabla_v^- dx^a + (\nabla^-\cdot v)\, dx^a)\, \nabla_a^- x\}\, w\cdot\eta \qquad .(3.34b)$$

The second term of (3.34) yields :

$$- \int_{\partial A^+\cap\partial V^+} R^+(dx^a, X, v, w) \tfrac{\partial}{\partial x^a}\cdot\eta - \int_{\partial A^-} R^-(dx^a, X, v, w) \tfrac{\partial}{\partial x^a}\cdot\eta$$
$$+ \int_{\partial A^+\cap\partial V^+} \{(\nabla_a^+ dx^a + (\nabla^+\cdot \tfrac{\partial}{\partial x^a})\, dx^a)\, \nabla_w^+ X\}\, v\cdot\eta - \{(\nabla_a^+ dx^a + (\nabla^+\cdot \tfrac{\partial}{\partial x^a})\, dx^a)\, \nabla_v^+ X\}\, w\cdot\eta$$
$$+ \int_{\partial A^-\cap\partial V^-} \{(\nabla_a^- dx^a + (\nabla^-\cdot \tfrac{\partial}{\partial x^a})\, dx^a)\, \nabla_w^- X\}\, v\cdot\eta - \{(\nabla_a^- dx^a + (\nabla^-\cdot \tfrac{\partial}{\partial x^a})\, dx^a)\, \nabla_v^- X\}\, w\cdot\eta \qquad .(3.34c)$$

3.7.2 Second Contraction

In the above, we may regard $x = \chi g^{-1}$ for a C^1 one-form χ on V, and in particular, regard $x = (dx^b)g^{-1}$. The second contraction is then on $\chi = dx^b$ and $w = \frac{\partial}{\partial x^b}$ Perform this on (3.34a), (3.34b), (3.34c) :

(3.34a) yields :

$$
\begin{aligned}
& - \int_{\partial A^+ \cap \partial V^+} R^+(dx^b, v) \frac{\partial}{\partial x^b} \cdot \eta \;-\; \int_{\partial A^- \cap \partial V^-} R^-(dx^b, v) \frac{\partial}{\partial x^b} \cdot \eta \\
& + \int_{\partial A^+ \cap \partial V^+} \{ (\nabla_b^+ dx^a + (\nabla^+ \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla_v^+ (dx^b) g^{-1} \} \frac{\partial}{\partial x^a} \cdot \eta \\
& \qquad - \{ (\nabla_b^+ dx^a + (\nabla^+ \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla_a^+ (dx^b) g^{-1} \} \, v \cdot \eta \\
& + \int_{\partial A^- \cap \partial V^-} \{ (\nabla_b^- dx^a + (\nabla^- \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla_v^- (dx^b) g^{-1} \} \frac{\partial}{\partial x^a} \cdot \eta \\
& \qquad - \{ (\nabla_b^- dx^a + (\nabla^- \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla_a^- (dx^b) g^{-1} \} \, v \cdot \eta
\end{aligned} \qquad (3.35a)
$$

(3.34b) yields :

$$
\begin{aligned}
& + \int_{\partial A^+ \cap \partial V^+} R^+(dx^b, \tfrac{\partial}{\partial x^b}) \, v \cdot \eta \;+\; \int_{\partial A^- \cap \partial V^-} R^+(dx^b, \tfrac{\partial}{\partial x^b}) \, v \cdot \eta \\
& - \int_{\partial A^+ \cap \partial V^+} \{ (\nabla_v^+ dx^a + (\nabla^+ \cdot v) dx^a)(\nabla_b^+ (dx^b) g^{-1}) \} \frac{\partial}{\partial x^a} \cdot \eta \\
& \qquad - \{ (\nabla_v^+ dx^a + (\nabla^+ \cdot v) dx^a)(\nabla_a^+ (dx^b) g^{-1}) \} \frac{\partial}{\partial x^b} \cdot \eta \\
& - \int_{\partial A^- \cap \partial V^-} \{ (\nabla_v^- dx^a + (\nabla^- \cdot v) dx^a)(\nabla_b^- (dx^b) g^{-1}) \} \frac{\partial}{\partial x^a} \cdot \eta \\
& \qquad - \{ (\nabla_v^- dx^a + (\nabla^- \cdot v) dx^a)(\nabla_a^- (dx^b) g^{-1}) \} \frac{\partial}{\partial x^b} \cdot \eta
\end{aligned} \qquad (3.35b)
$$

(3.34c) yields :

$$
\begin{aligned}
& - \int_{\partial A^+ \cap \partial V^+} R^+(dx^a, dx^b, v, \tfrac{\partial}{\partial x^b}) \frac{\partial}{\partial x^a} \cdot \eta \;-\; \int_{\partial A^- \cap \partial V^-} R^-(dx^a, dx^b, v \tfrac{\partial}{\partial x^b}) \frac{\partial}{\partial x^a} \cdot \eta \\
& + \int_{\partial A^+ \cap \partial V^+} \{ (\nabla_a^+ dx^a + (\nabla^+ \cdot \tfrac{\partial}{\partial x^a}) dx^a)(\nabla_b^+ (dx^b) g^{-1}) \} \, v \cdot \eta \\
& \qquad - \{ (\nabla_a^+ dx^a + (\nabla^+ \cdot \tfrac{\partial}{\partial x^a}) dx^a)(\nabla_v^+ (dx^b) g^{-1}) \} \frac{\partial}{\partial x^b} \cdot \eta \\
& + \int_{\partial A^- \cap \partial V^-} \{ (\nabla_a^- dx^a + (\nabla^- \cdot \tfrac{\partial}{\partial x^a}) dx^a)(\nabla_b^- (dx^b) g^{-1}) \} \, v \cdot \eta \\
& \qquad - \{ (\nabla_a^- dx^a + (\nabla^- \cdot \tfrac{\partial}{\partial x^a}) dx^a)(\nabla_v^- (dx^b) g^{-1}) \} \frac{\partial}{\partial x^b} \cdot \eta
\end{aligned} \qquad (3.35c)
$$

Consolidating the first pair of integrals in (3.35a), (3.35b) and (3.35c) into one expression yields in index notation (and with a few changes in the dummy summation indecii) :

$$\int_{\partial A^+ \cap \partial V^+} \{ - R^{+b}{}_c + \delta^b_c R^+ - R^{+ba}{}_{ca} \} v^c \left(\tfrac{\partial}{\partial x^b} \cdot \eta \right)$$
$$+ \int_{\partial A^- \cap \partial V^-} \{ - R^{-b}{}_c + \delta^b_c R^- - R^{-ba}{}_{ca} \} v^c \left(\tfrac{\partial}{\partial x^b} \cdot \eta \right)$$
$$= -2 \left(\int_{\partial A^+ \cap \partial V^+} G^+ (dx^b, v) \tfrac{\partial}{\partial x^b} \cdot \eta + \int_{\partial A^- \cap \partial V^-} G^- (dx^b, v) \tfrac{\partial}{\partial x^b} \cdot \eta \right) \qquad ..(3.36)$$

which expression will be used to express the basic conservation of ordinary material in the joined Universe V (see chapter 4).

Furthermore, it is easily seen in (3.35c) that

$$\nabla^{\pm}_a \, dx^a + (\nabla^{\pm} \cdot \tfrac{\partial}{\partial x} a) \, dx^a$$
$$= -\Gamma^{\pm a}_{ac} \, dx^c + \Gamma^{\pm c}_{ca} \, dx^a = 0$$

so that the last two surface integrals of (3.35c) vanish.

We are left with :

$$\sum_{c(\frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^a}, v)} (\nabla_b \text{ Riemann})_A \left(dx^a, dx^b, \tfrac{\partial}{\partial x^a}, v \right)$$
$$= -2 \left\{ \int_{\partial A^+ \cap \partial V^+} G^+ (dx^b, v) \tfrac{\partial}{\partial x^b} \cdot \eta + \int_{\partial A^- \cap \partial V^-} G^- (dx^b, v) \tfrac{\partial}{\partial x^b} \cdot \eta \right\}$$
$$+ \int_{\partial A^+ \cap \partial V^+} \{ (\nabla^+_b dx^a + (\nabla^+ \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla^+_v (dx^b) g^{-1} \} \tfrac{\partial}{\partial x^a} \cdot \eta$$
$$- \{ (\nabla^+_b dx^a + (\nabla^+ \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla^+_a (dx^b) g^{-1} \} v \cdot \eta$$
$$+ \int_{\partial A^- \cap \partial V^-} \{ (\nabla^-_b dx^a + (\nabla^- \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla^-_v (dx^b) g^{-1} \} \tfrac{\partial}{\partial x^a} \cdot \eta$$
$$- \{ (\nabla^-_b dx^a + (\nabla^- \cdot \tfrac{\partial}{\partial x^b}) dx^a) \nabla^-_a (dx^b) g^{-1} \} v \cdot \eta$$
$$- \int_{\partial A^+ \cap \partial V^+} \{ (\nabla^+_v dx^a + (\nabla^+ \cdot v) dx^a) \nabla^+_b (dx^b) g^{-1} \} \tfrac{\partial}{\partial x^a} \cdot \eta$$
$$- \{ (\nabla^+_v dx^a + (\nabla^+ \cdot v) dx^a) \nabla^+_a (dx^b) g^{-1} \} \tfrac{\partial}{\partial x^b} \cdot \eta$$
$$- \int_{\partial A^- \cap \partial V^-} \{ (\nabla^-_v dx^a + (\nabla^- \cdot v) dx^a) \nabla^-_b (dx^b) g^{-1} \} \tfrac{\partial}{\partial x^a} \cdot \eta$$
$$- \{ (\nabla^-_v dx^a + (\nabla^- \cdot v) dx^a) \nabla^-_a (dx^b) g^{-1} \} \tfrac{\partial}{\partial x^b} \cdot \eta \qquad ..(3.37)$$

which, if V is conservative, must be set to zero, which will result in the equations of matter-energy conservation in V and the equations of motion of the material (surface layer) 'trapped' in Σ. The next chapter will discuss these concepts.

CHAPTER FOUR :

GENERAL RELATIVITY ON THE JOINED MANIFOLD.

In this chapter, we consider the definition of a functional form of the Einstein Equations, and work out, both in the null and non-null cases the implications of its satisfaction. We also work out the consequences of a functional form of the Bianchi identities in a conservative manifold.

We show that a physically reasonable manifold join must satisfy the metric compatibility conditions, and introduce the concept of a surface energy momentum density on the join of two manifolds. We show that the absence of such a surface layer implies, in the case Σ non-null, the existence of a C^1 structure for V satisfying the connection compatibility condition and note that when Σ is spacelike, the dominant energy condition implies that there can be no surface layer, whilst in the case Σ null, we show that it is possible (even with no surface layer present) that there be no C^1 structure for V satisfying the connection compatibility condition. This corresponds to the case of an impulsive, purely gravitational, wave.

In the case when the c.c.c.(1) holds, irrespective of whether Σ is null or non-null, we use the functional form of the Bianchi identities in a conservative manifold to deduce the continuity of energy momentum flow across Σ (and hence, via an equation of state, in the spacelike case, the continuity of an energy momentum tensor describing fluid flow across Σ).

When a surface layer is present (in the non-null case) an equation for the interchange of energy-momentum between the ordinary fluid in V and the surface layer is presented, and analysed.

The chapter ends with a table outlining the possibilities just presented.

Throughout this chapter, $V^{\pm}$ are C^k 4-dimensional manifolds, $g^{\pm}$ are C^{k-1} metrics (signature +2) on V , $f : \partial V^+ \to \partial V^-$, $k \geq 4$, $V = V^+ \cup_f V^-$ and $\Sigma = \partial V^+ \cup_f \partial V^-$.

4.1 Physics across Σ.

We outline in this section the physical assumptions we shall make to do physics across Σ in V. Material in V will be supposed to be found in two ways ; (a) as 'ordinary' material, found in $V^{\pm}$ before the join was conceived, and described via a C^1 tensor field T^+ in V^+ and a field T^- in V^-, and (b) as 'surface layer' material whose history in spacetime is given by Σ (such as, for instance, the energy carried in a shock front whose space-time history is idealized to be Σ). We will allow the interaction of these two kinds of material, but we will (in the tradition of physics) allow ourselves the luxury of test particles, which may pass through Σ without interacting with the surface layer material in Σ, and make certain assumptions about these particles as they cross Σ or travel in Σ .

	PHYSICS		MATHEMATICS
P1	nearby particles remain nearby	M1	The map f is a homeomorphism and if a particle enters Σ at x^+ say, it exists at $x^- = f(x^+)$

P2	The four momentum of a test particle which free falls across Σ is continuous	M2	the tangent to the world line of a particle is continuous across Σ: at least a C^1 structure is available on V.
P3	The energy of a particle free falling across Σ is conserved.	M3	if p is future pointing, then $g^+(p,p) = g^-(p,p)$

We also need to make large scale assumptions about the physics in $V^{\pm}$ viewed in V :

P4	The physics in $V^{\pm}$ viewed once they are joined to form V is the same as the physics in $V^{\pm}$ when they are viewed separately	M4	$V^{\pm}$ are imbedded in V, and, in particular, they induce the same C^k structure on Σ i.e. f is a diffeomorphism.
P5	The physics of $V^{\pm}$ are governed by the usual laws of General Relativity (with zero cosmological term)	M5	$V^{\pm}$ are at least C^4 manifolds, $g^{\pm}$ are at least C^3; the Einstein equations are satisfied, and the Bianchi identities hold in $V^{\pm}$.

Finally, we need some assumptions about the large scale behaviour of physics in V :

P6	The Einstein equations hold in functional form in V	M6	The exact functional form is given in section 4.2

P7 Matter energy is neither created nor destroyed in V.	M7 The manifold V is conservative.

We will also assume that the manifold V is orientable (an assumption sometimes made in other studies of integral formulations of G-R see e.g. Marsden and Fischer (1979))

P3 implies that the metric compatibility condition holds in the given C^1 structure for V: Given any future pointing timelike vector p, P3 implies that

$$g^+(p,p) = g^-(p,p).$$

Take two such vectors, p_1 and p_2 and their sum p_1+p_2 is also necessarily future pointing and timelike. It follows that

$$g^+(p_1,p_1) + 2\, g^+(p_1,p_2) + g^+(p_2,p_2)$$
$$= g^+(p_1+p_2,p_1+p_2)$$
$$= g^-(p_1+p_2,p_1+p_2)$$
$$= g^-(p_1,p_1) + 2\, g^-(p_1,p_2) + g^-(p_2,p_2)$$

whence

$$g^+(p_1,p_2) = g^-(p_1,p_2) \qquad \ldots(4.1)$$

and the m.c.c. holds (since a basis for $T_p(V)$ can always be found consisting of future pointing timelike vectors.)

The consequences of the above assumptions which are of interest to us are :

1. The theory of joining of manifolds given in chapters one and two is applicable to V, and in particular, by theorem 2.1 and 2.3, there exists an unique C^1 structure for V which satisfies the m.c.c..The normal and null charts of chapter two are charts in this structure.

2. V, being a C^1 manifold has a C^r structure for all $r \geq 1$, which is compatible with the existing C^1 structure (Munkres (1966) p 57); if this structure satifies the connection compatibility condition of order r-1 it is unique in doing so, and must contain the normal and null charts of chapter two. It follows that if the connection components expressed in a normal or null chart for V are discontinuous, then no structure for V can satisfy the connection compatibility condition - effectively, the connection is not continuous across the manifold join.

3. The spaces of zero and first order distributions on V are uniquely defined.

4. f is a diffeomorphism and an isometry. Whatever metric structure is induced on ∂V^+ by V^+ and hence on Σ, agrees with the structure induced on Σ by ∂V^-.

5. The joint metric (volume) measure $\eta = \eta^+ \cup_f \eta^-$ is well defined.

In the sequel, A is a subset of V of the kind described in section 3.3.8.

4.2 Functional Einstein Equation in V.

Given $T^{\pm}$ in $V^{\pm}$ (energy momentum tensors) let $T^{*\pm}(u,v) = T^{\pm}(u,v) - \frac{1}{2}\, g(u,v)\, T^{\pm}$ so that (in suitable units)

$$R^{\pm}(u,v) = 8\pi\, T^{*\pm}(u,v) \qquad \ldots(4.2)$$

where $R^{\pm}$ are the Ricci tensors in $V^{\pm}$, u,v arbitrary C^1 vector fields on V - these are the Einstein equations. We may replace them by the functional forms

$$R_{A^{\pm}}(u,v) = 8\pi\, T^*_{A^{\pm}}(u,v) \qquad \ldots(4.3)$$

(where $R^{\pm}_{A^{\pm}}$ are defined as if they where regular distributions, and similarly for $T^{\pm}_{A^{\pm}}$). By equation (3.27) we have (for u,v in D(A))

$$\begin{aligned} \mathrm{Ricci}_A(u,v) &= \mathrm{Ricci}_{A^+}(u,v) + \mathrm{Ricci}_{A^-}(u,v) \\ &\quad + \int_{\partial A^-\cap\partial V^-} ((\nabla^-\cdot u)v - \nabla^-_v u)\cdot\eta \\ &\quad + \int_{\partial A^+\cap\partial V^+} ((\nabla^+\cdot u)v - \nabla^+_v u)\cdot\eta \\ &= 8\pi\, T^*_{A^+}(u,v) + 8\pi\, T^*_{A^-}(u,v) \\ &\quad + \int_{\partial A^-\cap\partial V^-} \{(\nabla^-\cdot u)v - \nabla^-_v u\}\cdot\eta + \int_{\partial A^+\cap\partial V^+} \{(\nabla^+\cdot u)v - \nabla^+_v u\}\cdot\eta \\ &\overset{\mathrm{def}}{=} 8\pi\, T^*_A(u,v) \end{aligned} \qquad \ldots(4.4)$$

We view the multilinear functional T^*_A as defined by

$$T^*_{A^+}(u,v) + T^*_{A^-}(u,v) + I_{A\cap\Sigma}(u,v) \qquad \ldots(4.5)$$

where $8\pi\, I_{A\cap\Sigma}(u,v) = \int_{A\cap\Sigma} [\{(\nabla^-\cdot u)v - \nabla^-_v u\} - \{(\nabla^+\cdot u)v - \nabla^+_v u\}]\cdot\eta$ ...(4.6)

and where $A\cap\Sigma$ is regarded as having the orientation of $\partial A^-\cap\partial V^-$ (since we will usually regard matter as flowing from V^- to V^+, with V^+ usually to the future of V^-, if possible).

We calculate $I_{A\cap\Sigma}(u,v)$ thus (assuming that A is contained in some chart of V and using equation (3.7)) :

$$\begin{aligned} I_{A\cap\Sigma}(u,v) &= \int_{h^-(A\cap\Sigma)} \sqrt{|g|}\; dx^n \left((\nabla^-\cdot u)v - \nabla^-_v u - (\nabla^+\cdot u)v + \nabla^+_v u \right) dx^1 dx^2 \cdots dx^{n-1} \\ &= \int_{h^-(A\cap\Sigma)} (v^n u^a_{,a} + u^b \Gamma^{-a}_{ab} - v^a u^n_{,a} - u^b \Gamma^{-n}_{ab} v^a - v^n u^a_{,a} - u^b \Gamma^{+a}_{ab} + v^a u^n_{,a} + \\ &\qquad u^b \Gamma^{+n}_{ab} v^a)\sqrt{|g|}\, dx^1 \ldots dx^{n-1} \qquad (n \equiv 4) \\ &= \int_{h^-(A\cap\Sigma)} u^a v^b (\delta^4_b(\Gamma^{-c}_{ca} - \Gamma^{+c}_{ca}) - (\Gamma^{-4}_{ab} - \Gamma^{+4}_{ab}))\sqrt{|g|}\, dx^1 \cdot dx^3 \end{aligned}$$

We will set

$$8\pi\, S^*_{ab} = \delta^4_b(\Gamma^{-c}_{ca} - \Gamma^{+c}_{ca}) - (\Gamma^{-4}_{ab} - \Gamma^{+4}_{ab}) \qquad \ldots(4.7)$$

to describe the energy momentum of a surface layer of material whose history in space time is given by Σ. Then $I_{A\cap\Sigma}(u,v) = \int_{A\cap\Sigma} S^*(u,v) \;\; \frac{\partial}{\partial x^4}\cdot\eta$ (the use of the metric compatibility condition and the C^1 nature of u,v is needed to derive (4.7)). We then 'postulate' the functional form of the Einstein equations :

$\mathrm{Ricci}_A(u,v) \;=\; 8\pi\, T^*_A(u,v)$ must hold for all compact submanifolds A of V contained in charts in V and for all C^1 vector fields u,v in V (given that V satisfies the metric compatibility condition) ...(4.8)

For an heuristic justification of the term 'surface layer' see Israel (1966). Our terminology and results are the same as those of Israel when Σ is non-null, but our formalism extends to the case Σ null.

4.2.1 Absence of a Surface Layer

In this case, Σ is termed a hypersurface of higher order by Israel (1966). By (4.7) we must have :

$$\delta^4_b\,(\Gamma^{-c}_{\;\;ca} - \Gamma^{+c}_{\;\;ca}) - (\Gamma^{-4}_{\;\;ab} - \Gamma^{+4}_{\;\;ab}) \;=\; 0 \qquad \text{...(4.9)}$$

It is convenient to consider this equation separately for the cases Σ non-null and null.

Σ non-null

In this instance, we may work in a normal chart across V, when $\frac{\partial}{\partial x^4} = n$. By the Gauss-Weingarten equation (2.19) we have that

$$\Gamma^{-4}_{\alpha\beta} - \Gamma^{+4}_{\alpha\beta} \;=\; e(n)\,(K^{+}_{\alpha\beta} - K^{-}_{\alpha\beta}) \qquad \text{..(4.10)}$$

and the implication of (4.9) is that the structure for V satisfies the extrinsic curvature condition, and may, via lemma 2.1, be assumed to satisfy the connection compatibility condition, when (4.9) is identically satisfied. It follows that the Riemann and Weyl tensors are at worst discontinuous across Σ.

An example of this kind of match (which is the usually desired non-null match) is the Swiss-Cheese Universe (Lake(1980), and Kantowski (1969)), which is a special case of the matching of Szekeres solutions, given in Chapter 5.

Work in this case appears in

Edelen (1962) : satisfaction of the m.c.c. is considered in a special case

Edelen and Thomas (1962 and 1963) : the pioneering papers explicitly requiring a C^1 metric on V

Israel (1966)

Σ null

Let us work in a null chart across Σ in V (cf chapter 2 for terminology). If the connection compatibility is ever to be satisfied for V, it must be satisfied in an atlas containing the null charts for V, as observed, and so there can be no loss of generality caused by spurious co-ordinate discontinuities if we work in these co-ordinates.

Noting that the hypersurface Σ is null, we may assume it to be generated by a family of null geodesics whose tangent field we will take to be k. (in fact, $k = (dx^4)^{g^{-1}} = n$ in chapter 2). Suppose that near p in Σ (at which we will evaluate (4.9)), $k = \frac{\partial}{\partial x^1}$ and that co-ordinates in a chart in Σ are given by x^1, x^2, x^3. Regard $l = \frac{\partial}{\partial x^4}$. It is no loss of generality to assume that $\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ are spacelike (in fact they must be !) and to assume at p a metric of the form

$$(g_{ab})_p = (g^{ab})_p = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Near p, in Σ, the g_{4a} are constant (by the construction of a null neighbourhood)

Near p, define the hypersurface $\Sigma^{\perp}$ by holding $x^1 = x^1(p)$ constant and varying x^2,

x^3 and x^4. $\Sigma^{\perp}$ is clearly null, and intersects Σ in the 2-surface obtained by varying x^2, x^3 and setting $x^1=x^1(p)$, $x^4=x^4(p)$. l is the null field generated by $\Sigma^{\perp}$. (Notice that by the m.c.c., $\Sigma^{\perp}$ is a C^1 submanifold of V).

In these co-ordinates, several simplifications in the computation of the components of the connection occur. The results of these are shown in table 4.1, (which shows also the results obtained when there is a surface layer in Σ). In particular

$$\Gamma^{\pm 3}_{24} = \Gamma^{\pm 2}_{34}, \quad \Gamma^{\pm 1}_{14} = \Gamma^{\pm 4}_{44} = \Gamma^{\pm d}_{44} = \Gamma^{\pm d}_{11} = 0 \qquad ..(4.11)$$

Following Penrose(1972) p 108 (also Penrose(1968) , p 166), we define $m = \frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}$ and $\bar{m} = \frac{\partial}{\partial x^2} - i\frac{\partial}{\partial x^3}$ -(k,m,$\bar{m}$,l) is then a null tetrad.

Set

$$\rho^{\pm} = m^a \bar{m}^b \overset{\pm}{\nabla}_b l_a = \Gamma^{\pm 2}_{24} + \Gamma^{\pm 3}_{34} + i\,(\Gamma^{\pm 3}_{24} - \Gamma^{\pm 2}_{34}) \qquad ..(4.12)$$

$$\sigma^{\pm} = m^a m^b \overset{\pm}{\nabla}_b l_a = \Gamma^{\pm 2}_{24} - \Gamma^{\pm 3}_{34} + i\,(\Gamma^{\pm 3}_{24} + \Gamma^{\pm 2}_{34}) \qquad ..(4.13)$$

ρ is the complex expression of the convergence (Reρ) and the rotation (Imρ) of the congruence of null geodesics comprising $\Sigma^{\perp}$ (which we may view as a test wavefront of light passed through Σ) - by construction, the rotation is zero. σ is the complex description of the shearing of a small pencil of light rays (tangent field l) passing through Σ. The results obtained by combining (4.12), (4.13), (4.11), the satisfaction of the m.c.c. and (4.9) (the absence of a surface layer) are shown in column 3 of table 4.1

In particular, notice that Re ρ = 0 by (4.9) with a=b=4, using $\Gamma^{\pm 4}_{44}=0$ and $\Gamma^{\pm 1}_{14}=0$ (4.11).

The important conclusion is that in the null case, the absence of a surface layer does not neccesarily imply the satisfaction of the connection compatibility condition. In the non-null case, the behaviour of the Weyl tensor across Σ was immaterial to the physics implied by the functional Einstein equations. In the null case the Weyl tensor intrudes via its control of the shear (see Penrose (1972) p 109); a jump in the shear implies a delta type distribution in the Weyl tensor - precisely, surface terms in a functional form of the Weyl tensor which do not cancel. In this situation (which might later be investigated using the functional definition of the Riemann tensor, or better, spinor methods), Σ represents the history of a purely impulsive gravitational wave.

Apart from the work of Penrose, Papapetrou (preprint) has dealt with the case when Σ is null and the c.c.c. is satisfied (using the Newman-Penrose formalism) whilst Lichnerowicz (1971) has shown as we have that only when Σ is null can the connection compatibility condition fail to be satisfied, when there is no surface layer present.

Chapter 5 presents the purely gravitational impulsive plane wave as an example of this theory.

TABLE 4.1 — CONNECTION DISCONTINUITIES IN NULL CO-ORDINATES — Σ NULL

1 Connection coefficient	2 Remark	3 No surface layer	4 Surface layer
1) Γ^{4}_{44}	0 by construct	continuous	continuous
2) Γ^{α}_{44}	ditto	ditto	ditto
3) $\Gamma^{4}_{\alpha 4}$		continuous (by 4.9)	jump determined by $S^*_{4\alpha}$ by 4.7
4) $\Gamma^{4}_{\alpha\beta}$	= 0	continuous (by 4.9)	jump determined by $S^*_{\alpha\beta}$ by 4.7
5i) Γ^{B}_{A4}		jump determined by jump in shear. Rotation and jump in convergence zero.	jump determined by jump in σ and $\mathrm{Re}\,\rho$ Rotation = 0 by construct. 8π Jump ρ = S^*_{44}
5ii) Γ^{A}_{14}	$= -\Gamma^{4}_{4A}$	continuous by (3)	jump determined by $S^*_{4\alpha}$
5iii) Γ^{1}_{A4}	$= 0 \left(= -\frac{1}{2}\frac{\partial g_{44}}{\partial x^A}\right)$	continuous	continuous
5iv) Γ^{1}_{14}	$= 0 \left(= -\frac{1}{2}\frac{\partial g_{44}}{\partial x^1}\right)$	ditto	ditto
6) Γ^{1}_{11}, Γ^{A}_{11}	= 0 by $\nabla_k k = 0$	ditto	ditto
7) $\Gamma^{1}_{1\alpha}$		continuous by (3) and (4.9)	determined by $S^*_{\alpha 4}$ and $S^*_{4\alpha}$ via (3),(4.9)
8) Γ^{1}_{AB}	$= \frac{1}{2}(\Gamma^{B}_{4A} + \Gamma^{A}_{4B})$	determined via (5) by shear jump	determined via (5) from shear jump, S^*_{44}
9) Γ^{A}_{B1} Γ^{A}_{BC}	$\Gamma^{A}_{\alpha\beta}$	continuous by m.c.c.	continuous by m.c.c.

In the above lower case romans run over 1,2,3,4; lower greek over 1,2,3 and upper romans over 2,3.

4.2.2 Surface layer present

Let us suppose that $S^*_{ab} \neq 0$, i.e. there is material present in Σ. Again, we split consideration to cases in which Σ is locally either null or non-null.

Σ non-null

We continue to use a normal co-ordinate system as in the previous section. By (4.7), S^*_{ab} is given by :

$$S^*_{ab} = \{\delta^4_b(\Gamma^{-c}_{ca} - \Gamma^{+c}_{ca}) - (\Gamma^{-4}_{ab} - \Gamma^{+4}_{ab})\}/8\pi \quad ..(4.14)$$

Using (4.10)

$$8\pi S^*_{\alpha\beta} = e(n)\,(K^-_{\alpha\beta} - K^+_{\alpha\beta}) \quad ..(4.15)$$

Recalling that g(n,n) = e(n) is constant where defined, we have that

$$0 = \nabla^\pm_\alpha g(n,n) = 2\, g(\nabla^\pm_\alpha n, n)$$

whence $dx^4(\nabla^\pm_\alpha n) = 0$ and $\Gamma^{4}_{a4} = 0$ which implies that

$$S^*_{4\alpha} = 0 \quad ..(4.16)$$

Notice next that since the metric compatibility condition is satisfied by V,

$\Gamma^{+\alpha}_{\beta\gamma} = \Gamma^{-\alpha}_{\beta\gamma} = {}^3\Gamma^\alpha_{\beta\gamma}$ where ${}^3\Gamma^\alpha_{\beta\gamma}$ are the components of the metric connection induced in Σ by g . Thus

$$\begin{aligned} 8\pi S^*_{\alpha 4} &= (\Gamma^{-c}_{c\alpha} - \Gamma^{+c}_{c\alpha}) \\ &= (\Gamma^{-4}_{4\alpha} - \Gamma^{+4}_{4\alpha}) + ({}^3\Gamma^\gamma_{\gamma\alpha} - {}^3\Gamma^\gamma_{\gamma\alpha}) \\ &= 0 \end{aligned} \quad ..(4.17)$$

Finally,

$$\begin{aligned} 8\pi S^*_{44} &= (\Gamma^{-c}_{c4} - \Gamma^{+c}_{c4}) \\ &= (\Gamma^{-\gamma}_{\gamma 4} - \Gamma^{+\gamma}_{\gamma 4}) \qquad (\text{since } \Gamma^{+4}_{44} = \Gamma^{-4}_{44} = 0) \\ &= (K^{-\gamma}_{\gamma} - K^{+\gamma}_{\gamma}) \end{aligned} \quad ..(4.18)$$

Noting the form of $(T^*)^{\pm}$ in (4.2), we set

$$S_{ab} = S^*_{ab} - \frac{1}{2} g_{ab} S^{*d}_{d} \quad ..(4.19)$$

In normal co-ordinates, $g_{4a} = 0$ if $a \neq 4$ and $g_{44} = e(n)$ so that the metric may be bordered on thus :

$$\{g_{ab}\} = \begin{Bmatrix} & & & 0 \\ & (g_{\alpha\beta}) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & e(n) \end{Bmatrix}$$

where $g_{\alpha\beta}$ is the induced metric on Σ. It follows, by (4.15) and (4.18) that

$$S^{*a}_{a} = 2 \frac{e(n)}{8\pi} (K^{-\gamma}_{\gamma} - K^{+\gamma}_{\gamma}) \quad ..(4.20)$$

whence, by (4.19)

$$\begin{aligned} S_{44} &= 0 \\ S_{4\alpha} &= S_{\alpha 4} = 0 \\ S_{\alpha\beta} &= \frac{e(n)}{8\pi} (K^{-}_{\alpha\beta} - K^{+}_{\alpha\beta} - g_{\alpha\beta} (K^{-\gamma}_{\gamma} - K^{+\gamma}_{\gamma})) \end{aligned} \quad ..(4.21)$$

(a formulation entirely agreeing with Israel (1966)).

An immediate consequence, if S_{ab} is to model realistic material, that is material satisfying the dominant energy condition (Hawking and Ellis (1973) p 41), is that, if $\frac{\partial}{\partial x^4} = n$ is timelike, that is, Σ is spacelike, then $S_{ab} = 0$ (in an orthonormal frame, with $\frac{\partial}{\partial x^4}$ timelike, S_{44} must dominate in magnitude every other component of the energy-momentum tensor). It is physically reasonable, therefore, to rule out the occurence of a surface energy on Σ when Σ is spacelike - it then follows that V satisfies the connection compatibility condition of necessity.

In the next section, we will develop a conservation equation for S, along with an equation of motion.

Σ null

In column 4 of table 4.1 (p87) we have indicated how the jumps in the connection in the null case may be related to the components of S^*_{ab} in a null chart. These results follow directly from equations (4.7) and (4.11).

By way of a specific example, consider the case of a radiation fluid flowing in Σ - we assume that the fluid momentum field is null, and that the flow lines of the fluid (in 4-space) are null. It then follows, since Σ is null that the fluid flow lines are the null generating geodesics of Σ and the momentum field is, up to a scalar multiplier function, k. Accordingly set

$$S = \mu \, k \otimes k$$

whence

$$S_{44} = S^*_{44} = \mu$$

$$S_{4\alpha} = S^*_{4\alpha} = S_{\alpha 4} = S^*_{\alpha 4} = S_{\alpha\beta} = S^*_{\alpha\beta} = 0$$

It follows, by equation (4.12) that $\rho^- - \rho^+ = S^*_{44}/8\pi = \mu/8\pi$ and the presence of the radiation fluid introduces a jump in the convergence of a small bundle of (test) light rays passing through Σ in the $l = l^+ \cup_f l^-$ direction, the situation otherwise remaining unchanged from the case of no surface layer.

4.3 Functional Bianchi identities - conservation of matter-energy

The final structure we impose on V is to require that V be conservative.

CASE Σ NON-NULL

The theory of distributional derivatives used in chapter three may again be applied to the integrals of equations (3.35a) to (3.35c). Since the subset $A \subset V$ of that section is a compact submanifold of V, we may assume that $\sigma^{\pm} = \partial(\partial A^{\pm} \cap \partial V^{\pm}) \neq \emptyset$. Using a normal chart for V, consider an integral of the form

$$I = \int_{\partial A^{+} \cap \partial V^{+}} \phi \; v \cdot \eta$$

By equation (3.7) this is zero unless $v^4 \neq 0$, when we have

$$I = \int_{\partial A^{+} \cap \partial V^{+}} v^4 \varphi \sqrt{|g|} \, dx^1 \wedge \ldots \wedge dx^{n-1}$$

Noting that $g_{4\alpha} = 0$, $g_{44} = e(n)$, and letting ^3g be the metric induced on Σ by g, we have that $\sqrt{|g|} = \sqrt{|^3g|}$ whence

$$\sqrt{|g|} \; dx^1 \wedge \ldots \wedge dx^{n-1} = \eta_\Sigma$$

the metric (volume) measure induced on Σ by g. By Gauss' theorem, we then have :

$$\int_{\sigma^{+} = \partial(\partial A^{+} \cap \partial V^{+})} T(S) \; v \cdot \eta_\Sigma$$

$$= \int_{\partial A^{+} \cap \partial V^{+}} {}^3\nabla \cdot \left(T(S) \; v_{/\!/} \right) \eta_\Sigma$$

$$= \int_{\partial A^{+} \cap \partial V^{+}} {}^3\nabla_{v_{/\!/}} \left(T(S) \right) + T(S) \; {}^3\nabla \cdot v_{/\!/} \;\; \eta_\Sigma \qquad \ldots(4.23)$$

where

$$v_{/\!/} = v^\alpha \frac{\partial}{\partial x^\alpha}$$

in a normal chart, v a vector field on ∂V^+ in V^+ and $^3\nabla$ the metric connection induced on Σ. (This arises from equation 3.6 with n replaced by (n-1)).

Noting that

$${}^3\nabla_{v_{/\!/}} (T(S)) = v_{/\!/}[T(S)] = {}^4\nabla_{v_{/\!/}} (T(S)) = ({}^4\nabla_{v_{/\!/}} T)(S) + T({}^4\nabla_{v_{/\!/}} S)$$

and, for $v_{/\!/} = v$,

$${}^3\nabla \cdot v = {}^4\nabla \cdot v$$

we may write

$$\int_{\partial A^+ \cap \partial V^+} T(\nabla_v(S)) + \nabla \cdot v \; (T(S)) \quad \eta_\Sigma$$

$$= \int_{\sigma^+} T(S) \quad v \cdot \eta_\Sigma \quad - \int_{\partial A^+ \cap \partial V^+} (\nabla_v T)(S) \; \eta_\Sigma \qquad ..(4.24)$$

We use this equation in the second pair of surface integrals in (3.35a) to (3.35c); we shall not derive the full consequences of the contracted Bianchi identities in a conservative manifold, but only those obtained by setting v equal to $\frac{\partial}{\partial x}{}^\gamma$. Working with '+' terms only (and omitting + as superscript) from (3.35a) :

$$I_1 = \int_{\partial V^+ \cap \partial A^+} \{ (\nabla_b \, dx^a + (\nabla \cdot \tfrac{\partial}{\partial x}{}^b) \, dx^a) \, (\nabla_v \, (dx^b)^{g^{-1}}) \} \, \tfrac{\partial}{\partial x}{}^a \cdot \eta$$
$$- \{ \nabla_b \, dx^a + (\nabla \cdot \tfrac{\partial}{\partial x}{}^b) \, dx^a) \, (\nabla_a \, (dx^b)^{g^{-1}}) \} \; v \cdot \eta$$

$$= \int_{\partial V^+ \cap \partial A^+} (\nabla_4 \, dx^4 + (\nabla \cdot \tfrac{\partial}{\partial x}{}^4) \, dx^4) \, (\nabla_\gamma \, (dx^4)^{g^{-1}})$$
$$+ (\nabla_\beta \, dx^4 + (\nabla \cdot \tfrac{\partial}{\partial x}{}^\beta) \, dx^4) \, (\nabla_\gamma \, (dx^\beta)^{g^{-1}}) \qquad \eta_\Sigma$$

Notice that since $\Gamma^{+4}_{4a} = 0$ (see previous section) $\nabla^+_4 \, dx^4 = 0$ and since ∇^+ is metric

$$((\nabla \cdot \tfrac{\partial}{\partial x}{}^4) \quad dx^4) \; (\nabla_\gamma \; (dx^4)^{g^{-1}})$$
$$= ((\nabla \cdot \tfrac{\partial}{\partial x}{}^4) \quad dx^4) \; (\nabla_\gamma \; dx^4)^{g^{-1}} \quad = \quad (\nabla \cdot \tfrac{\partial}{\partial x}{}^4) \, (\nabla_\gamma \, dx^4)(n) = 0$$

The first term of the integrand is therefore zero, and we may employ (4.24) to obtain

$$I_1 = \int_{\sigma^+} (\nabla_\gamma \, (dx^\beta)^{g^{-1}}) \, (dx^4) \quad \tfrac{\partial}{\partial x}{}^\beta \cdot \eta_\Sigma$$
$$- \int_{\partial A^+ \cap \partial V^+} (dx^4) (\nabla_\beta \nabla_\gamma \, (dx^\beta)^{g^{-1}}) \; \eta_\Sigma \qquad ..(4.25a)$$

Next obtain from (3.35b) :

$$I_2 = -\int_{\partial V^+ \cap \partial A^+} \{ (\nabla_V\, dx^a + (\nabla \cdot V)\, dx^a)\, (\nabla_b (dx^b) g^{-1}) \} \frac{\partial}{\partial x^a} \cdot \eta$$
$$- \{ (\nabla_V\, dx^a + (\nabla \cdot V)\, dx^a)\, (\nabla_a (dx^b) g^{-1}) \} \frac{\partial}{\partial x^b} \cdot \eta$$

$$= -\int_{\partial V^+ \cap \partial A^+} (\nabla_\gamma\, dx^4 \quad (\nabla \cdot \tfrac{\partial}{\partial x}\gamma)\, dx^4)(\nabla_\beta (dx^\beta) g^{-1})$$
$$- (\nabla_\gamma\, dx^\alpha + dx^\alpha (\nabla \cdot \tfrac{\partial}{\partial x}\gamma))\, (\nabla_\alpha \quad n \quad) \quad \eta_\Sigma$$

(where we have used $\nabla_n\, n = 0$ and $(dx^4) g^{-1} = n$) whence :

$$I_2 = -\int_{\sigma^+} (dx^4 \,(\nabla_\beta (dx^\beta) g^{-1}) - dx^\alpha\, \nabla_\alpha n \,) \frac{\partial}{\partial x}\gamma \cdot \eta_\Sigma$$

$$+ \int_{\partial V^+ \cap \partial A^+} dx^4\, (\nabla_\gamma \nabla_\beta\, (dx^\beta) g^{-1}) - dx^\alpha (\nabla_\gamma \nabla_\alpha n) \quad \eta_\Sigma \qquad ..(4.25b)$$

Finally from (3.35c)

$$I_3 = \int_{\partial V^+ \cap \partial A^+} ((\nabla_a\, dx^a) + (\nabla \cdot \tfrac{\partial}{\partial x^a})\, dx^a)(\nabla_b\, (dx^b) g^{-1})\; V \cdot \eta$$
$$- (\nabla_a\, dx^a + (\nabla \cdot \tfrac{\partial}{\partial x^a})\, dx^a)(\nabla_V (dx^b) g^{-1}) \quad \frac{\partial}{\partial x^b} \cdot \eta$$

$$= -\int_{\partial V^+ \cap \partial A^+} (\nabla_4\, dx^4 + (\nabla \cdot \tfrac{\partial}{\partial x^4})\, dx^4)(\nabla_\gamma\, n)$$
$$+ (\nabla_\alpha\, dx^\alpha + (\nabla \cdot \tfrac{\partial}{\partial x^\alpha})\, dx^\alpha)(\nabla_\gamma\, n) \qquad \eta_\Sigma$$

$$= -\int_{\sigma^+} dx^\alpha (\nabla_\gamma n) \quad \frac{\partial}{\partial x^\alpha} \cdot \eta_\Sigma$$
$$+ \int_{\partial V^+ \cap \partial A^+} dx^\alpha (\nabla_\alpha \nabla_\gamma\, n) \quad \eta_\Sigma \qquad ..(4.25c)$$

It is straightforward to show, in components that

$$(\nabla_\alpha \nabla_\gamma\, dx^\beta) = -K_{\gamma,\alpha}^{\ \beta}\, dx^4 + K_\gamma^{\ \beta}\, \Gamma^4_{\alpha c}\, dx^c$$
$$- \Gamma^{\ \beta}_{\gamma\lambda,\alpha}\, dx^\lambda + \Gamma^{\ \beta}_{\gamma\lambda}\, \Gamma^{\ \lambda}_{\alpha\epsilon}\, dx^\epsilon$$
$$+ \Gamma^{\ \beta}_{\gamma\lambda}\, K_\alpha^{\ \lambda}\, dx^4$$

whence, using $\Gamma^4_{a4} = 0$:

$$dx^4 (\nabla_\alpha \nabla_\gamma dx^\beta) g^{-1} = (\nabla_\alpha \nabla_\gamma dx^\beta) n$$
$$= -K_{\gamma}{}^{\beta}{}_{,\alpha} + \Gamma^{\beta}_{\gamma\lambda} K_\alpha{}^\lambda = -K_\gamma{}^\beta{}_{,\alpha} + {}^3\Gamma^\beta_{\gamma\lambda} K_\alpha{}^\lambda \qquad ..(4.26)$$

Substituting this in I_1 to I_3 yields the sum of integrals over $\partial V^+ \cap \partial A^+$ to be

$$- \int_{\partial V^+ \cap \partial A^+} (K_\beta{}^\beta{}_{,\gamma} - K_\gamma{}^\beta{}_{,\beta} + K_\beta{}^\lambda\, {}^3\Gamma^\beta_{\gamma\lambda} - K_\gamma{}^\lambda\, {}^3\Gamma^\beta_{\beta\lambda} + K_\alpha{}^\alpha{}_{,\gamma} + K_\alpha{}^\beta\, {}^3\Gamma^\alpha_{\gamma\beta} - K_\gamma{}^\alpha{}_{,\alpha} - K_\gamma{}^\mu\, {}^3\Gamma^\alpha_{\alpha\mu}) \quad \eta_\Sigma$$

$$= 2 \int_{\partial V^+ \cap \partial A^+} K_\gamma{}^\beta{}_{|\beta} - K_\beta{}^\beta{}_{|\gamma} \quad \eta_\Sigma \qquad ..(4.27)$$

where we have omitted the superscript + and the vertical bar represents covariant differentiation with respect to ${}^3g$ in Σ .

The surface integrals over σ^+ sum to

$$-2 \int_{\sigma^+} (K_\gamma{}^\beta - \delta^\beta_\gamma K_\mu{}^\mu) \frac{\partial}{\partial x^\beta} \cdot \eta_\Sigma \qquad ..(4.28)$$

Writing these integrals also with '-' we obtain, as a consequence of the contracted Bianchi identities, via (3.35a), (3.35b), (3.35c) and (3.36), and the requirement that V be conservative :

$$\int_{\partial A^+\cap\partial V^+} G^+(dx^4, \tfrac{\partial}{\partial x}\gamma)\,\eta_\Sigma \;+\; \int_{\partial A^-\cap\partial V^-} G^-(dx^4, \tfrac{\partial}{\partial x}\gamma)\,\eta_\Sigma$$

$$= \int_{\partial V^+\cap\partial A^+} K^{+}_{\gamma}{}^{\beta}{}_{|\beta} - K^{+}_{\beta}{}^{\beta}{}_{|\gamma}\quad \eta_\Sigma$$

$$+ \int_{\partial V^-\cap\partial A^-} K^{-}_{\gamma}{}^{\beta}{}_{|\beta} - K^{-}_{\beta}{}^{\beta}{}_{|\gamma}\quad \eta_\Sigma$$

$$- \int_{\sigma^+} (K^{+}_{\gamma}{}^{\beta} - \delta_\gamma^\beta\, K^{+}_{\mu}{}^{\mu})\, \tfrac{\partial}{\partial x^\beta}\cdot\eta_\Sigma$$

$$- \int_{\sigma^-} (K^{-}_{\gamma}{}^{\beta} - \delta_\gamma^\beta\, K^{-}_{\mu}{}^{\mu})\, \tfrac{\partial}{\partial x^\beta}\cdot\eta_\Sigma \qquad ..(4.29)$$

which, using (4.21) and inducing an orientation on Σ by the orientation on ∂V^- and using the Einstein equations in V^+ and V^- gives:

$$\int_{A\cap\Sigma} (T^- - T^+)(dx^4, \tfrac{\partial}{\partial x}\gamma)\,\eta_\Sigma \;+\; \int_{\partial(A\cap\Sigma)} S_\gamma{}^\beta\, \tfrac{\partial}{\partial x^\beta}\cdot\eta_\Sigma$$

$$= \int_{A\cap\Sigma} S_\gamma{}^\beta{}_{|\beta}\;\eta_\Sigma \qquad ..(4.30)$$

which is, surely, a most beautiful integral conservation equation (and the reason for the term 'a conservative manifold').

The left hand side of (4.30) may be regarded as a measure of the surface layer material created or destroyed in $A\cap\Sigma$, comprising a measure of the flow of surface layer material into and out of A $(\int_{\partial(A\cap\Sigma)} S_\gamma{}^\beta \tfrac{\partial}{\partial x^\beta}\cdot\eta_\Sigma)$ and a measure of the conversion of surface to ordinary material $(\int_{A\cap\Sigma}(T^- - T^+)(dx^4,\tfrac{\partial}{\partial x}\gamma)\,\eta_\Sigma)$. We may reasonably require these two terms together to be zero (to prevent the creation or destruction of material) and thence deduce that

$$0 = \int_{A\cap\Sigma} S_\gamma{}^\beta{}_{|\beta}\;\eta_\Sigma$$

for all A, whence, since $S_\gamma{}^\beta{}_{|\beta}$ is at least continuous,

$$S_\gamma{}^\beta{}_{|\beta} = 0 \qquad ..(4.31)$$

which generalizes a similar result of Israel (1966), and which is to be expected of ordinary material. (see e.g. Hawking and Ellis (1973) p 61).

If there is no surface layer in Σ, then, as remarked, we may assume V to satisfy the connection compatibility condition, and it follows by the Bianchi identities in the form (3.37) that

$$0 = \int_{A\cap\Sigma} G^+(dx^4, v) - G^-(dx^4, v) \quad \eta_\Sigma$$

for all A, whence, by the continuity of G^+, G^-, we must have on Σ that

$$T^+(dx^4, v) = T^-(dx^4, v) \qquad (4.32)$$

(using the Einstein equation in V^+ and V^-). This is the mathematical statement of the physical fact that the normal flow of energy momentum across Σ is conserved (again verifying a result of Israel).

If Σ is spacelike, then, as observed, the connection compatibility condition must be satisfied. Suppose that T^+ and T^- represent perfect fluids :

$$T^\pm_{4a} = -p^\pm \delta_{4a} + (\rho^\pm + p^\pm)\, u^\pm_a\, u^\pm_4 \qquad ..(4.33)$$

Equating $T^\pm_{4a} = T^{\pm 4}{}_a$ together with the definiteness condition

$$1 = u_4^+ = u_4^- \qquad ..(4.34)$$

(which we may impose, since u must be future pointing timelike) and an equation of state

$$p^\pm = P(\rho^\pm) \qquad ..(4.35)$$

which may be assumed to hold on both sides of Σ, we obtain, given ρ^-, p^-, u^-

six equations in the six unknowns ρ^+, p^+, u^+.

By (4.34) and (4.33) with a = 4 we have

$$\rho^+ = \rho^-$$

whence, by (4.35)

$$p^+ = p^-$$

and it follows that $u^+ = u^-$ (by (4.33)) so that

$$T^+ = T^-$$

This process is invalid when Σ is timelike, for then we cannot assume $u^4 \neq 0$. If we can still make this assumption, however then the deduction $T^+ = T^-$ holds from which we deduce that a discontinuity in a fluid energy-momentum tensor must occur on a timelike hypersurface*, co-moving with the fluid. In this case $u^4 = 0$, and it is an immediate consequence of (4.33) that the pressure p is continuous across Σ. A discontinuity in the pressure of a relativistic perfect fluid can only occur on a timelike hypersurface carrying a surface energy.

<u>CASE Σ NULL</u>

We will not handle this case in its full generality, although such a consideration should be possible from equation (3.35a) etc. If the connection compatibility condition is satisfied by the structure on V (i.e. there is no surface layer on Σ and we are not dealing with an impulsive gravitational wave), then again by (3.37)

---

* this result needs the continuity result of the case Σ null, to rule out the possibility of a null hypersurface of discontinuity in the fluid energy momentum

we must have that the flow of energy momentum across Σ in the 1 direction is conserved. (4.32) in null co-ordinates implies

$$T^-_{1b} = T^+_{1b} \qquad ..(4.36)$$

Assuming that at the point of evaluation, g has the form given on page 84, we obtain, for fluid flow of ordinary matter across Σ (with flow fields u^+, u^-, both future timelike)

$$T^{\pm}_{1b} = p^{\pm} g_{1b} + (\rho^{\pm} + p^{\pm})\, u^{\pm}_1 u^{\pm}_b \qquad ..(4.37)$$

Append to (4.36) the definiteness requirement (again possible, since u^+, u^- are to be timelike)

$$u^+_1 = u^-_1 = 1 \qquad ..(4.38)$$

and the equation of state (4.35); obtain from (4.36), with b = 1

$$p^+ + \rho^+ = p^- + \rho^- \qquad ..(4.39)$$

Given p^- and ρ^-, (4.39) and (4.35) imply

$$P(\rho^+) + \rho^+ = \text{constant} \qquad ..(4.40)$$

Now $dP/d\rho$ is the adiabatic speed of sound in the medium described by (4.35), and we may assume that $|dP/d\rho| < 1$ i.e., the speed of sound is less than the speed of light (cf Hawking and Ellis, p 91), a reasonable assumption for ordinary matter. Then the left hand side of (4.40) has positive derivative with respect to ρ and so is a monotonically increasing function of ρ . (4.40) can therefore have at most one solution, and $\rho^+ = \rho^-$. Hence $p^+ = p^-$. Then, from $T^+_{1b} = T^-_{1b}$ with b = 2,3,4 we easily deduce $u^+ = u^-$ and thence, $T^+ = T^-$, and the flow of ordinary material across a null hypersurface is necessarily continuous.

4.4 SUMMARY OF RESULTS

For ease, we present a summary of the results implied by imposing the functional Einstein Equations onto V, and then requiring that V be conservative. See table 4.2 overleaf.

TABLE 4.2 SUMMARY OF PHYSICS ON AND ACROSS Σ

Case : Σ spacelike

no physically reasonable surface layer is possible

the connection compatibility condition is satisfied (so an unique C^2 structure exists for V and the joint metric is C^1)

the Weyl tensor can at most be jump discontinuous (probably not)

if $V^{\pm}$ contain reasonable physical fluids (perfect) then the joint energy-momentum tensor $T = T^{+} U_f T^{-}$ is continuous

Case : Σ null

surface layer absent		surface layer present
c.c.c satisfied	c.c.c not satisfied	c.c.c not satisfied
$G^4{}_b$ continuous ordinary fluid T cont. at worst a jump in the Weyl tensor, representing a wave carrying no energy	delta type discontinuity in Weyl tensor impulsive, purely gravitational waves	delta type disc. possible in Weyl tensor, though not necessary. A delta type disc. in Ricci and Einstein tensors representing impulsive wave.

TABLE 4.2 CONTINUED

Case : Σ timelike		
surface layer absent		surface layer present
c.c.c. satisfied	c.c.c. not satisfied	c.c.c. not satisfied
$G^4_{\ b}$ continuous a jump in a fluid T possible only if Σ is co-moving with the fluid Weyl at worst jump disc.	not possible	a delta discontinuity in Ricci and Einstein tensors - a fluid shock wave carrying energy is propagating for instance.

In this chapter, we examine the matching of two pairs of manifolds, on which energy-momentum fields satisfying the Einstein equations have been specified.

The first, the matching of two Szekeres solutions, affords examples of the matchings of spaces across a timelike hypersurface, and includes as special examples, the matching of a Schwarzchild and a Robertson Walker solution (the classical Swiss-Cheese match of Einstein and Strauss (1945) used by Kantowski(1969) and others to model light propogation in a non homogeneous Universe), and a match of a Schwarzchild to a Minkowski universe (when a surface layer is present) which will give a measure of discontinuities implied by treating the solar system as a Schwarzchild solution.

The second is the case of an impulsive, purely gravitational plane wave - an example of the case, on a null hypersurface of discontinuity, when no surface energy is present and yet there is a discontinuity in the connection across the hypersurface.

5.1 Matching Szekeres solutions.

Following Bonnor (1976), we outline the form of the Szekeres solutions. We are interested in solutions of the form

$$ds^2 = e^{\lambda} dr^2 + e^{\omega} (dy^2 + dz^2) - dt^2$$

where λ, ω are functions of r,y,z,t. The field equations for dust impose the following requirements :

$$e^{\omega/2} = \varphi(r,t) [P(r,y,z)]^{-1}$$

$$e^{\lambda/2} = P(r,y,z) [W(r)]^{-1} \frac{\partial}{\partial r} (e^{\omega/2})$$

where

$$P = a(r)(y^2 + z^2) + 2f(r)y + 2\,g(r)z + c(r)$$

$$ac - f^2 - g^2 = 1/4$$

$$\left(\frac{\partial}{\partial t}\phi\right)^2 = W^2 - 1 + S(r)/\phi$$

there being presumed an initial condition

$$\phi(0,r) = H(r)$$

for the last equation.

The co-ordinates used are comoving co-ordinates (w.r.t. dust fluid)

The dust density is given by

$$8\pi\rho = (PS_r - 3SP_r)\,\phi^{-2}\,(P\phi_r - \phi P_r)^{-1}$$

(where a subscript refers to differentiation with respect to r)

Given arbitrary a,c,f,W,S and H, the above equations specify an exact solution of Einsteins equations for dust.

It is possible (cf Bonnor, section 3) to consider the two-surfaces $r = r_1$ = constant, $t = t_1$ = constant, and then, by a transformation of the form $y' = ey + k_1$, $z' = ez + k_2$, for a particular hypersurface, we may obtain an induced metric

$$d\sigma^2 = \phi_*^2\,[\,dy'^2 + dz'^2\,]\,[\,1/2\,(\,1 + y'^2 + z'^2)\,]^{-2}$$

given

$$\phi_* = \phi(r_1,t_1).$$

Setting

$$y' = \cot 1/2\theta \quad \cos\Phi \qquad , \quad z' = \cot 1/2\theta \quad \sin\Phi$$

we can further obtain

$$d\sigma^2 = \phi_*^2\,[\,d\theta^2 + \sin^2\theta \;\; d\Phi^2\,]$$

so that the 2-surfaces are spheres of radius ϕ_*.

In the particular case of complete spherical symmetry of solutions (i.e. when

$a_r/a = f_r/f = g_r/g = c_r/c$ (cf Bonnor p 194 or Szekeres (1975), p 59), this transformation may be made to hold irrespective of r.

Bonnor has shown that two Szekeres solutions for dust may be matched under certain conditions. We prove this also, using our framework.

Let r_0 positive, be given; let V^+ be a Szekeres spacetime with domain $r_+ \geq r_0$, specified by the functions $a_+, f_+, g_+, c_+, S_+, W_+, H_+$ and V^- a similar space-time, specified by $a_-, f_-, g_-, c_-, S_-, W_-, H_-$ and r_- less than or equal to r_0. $\partial V^\pm$ is then the 3-surface specified by $r_\pm = r_0$ in $V^\pm$. By means of the transformation discussed above, we may assume that the metric inherited by $\partial V^\pm$ from $V^\pm$ has the form :

$$ds_\pm^{\ 2} = \varphi_\pm^2(r_0, t_\pm)(\, d\theta_\pm^2 + \sin^2\theta_\pm \, d\Phi_\pm^2\,) - dt^2$$

Identifying the points of $\partial V^\pm$ by identifying $t_+ = t_-$ $\theta_+ = \theta_-$ $\Phi_+ = \Phi_-$, specifies the map $f : \partial V^+ \rightarrow \partial V^-$ of previous chapters. (This matching is equivalent to the requirement that $y_+ = y_-$, $z_+ = z_-$, $t_+ = t_-$ - the matching by identification of co-moving co-ordinates ensures that in the case of no surface layer, the functional Bianchi identities of the last chapter will be satisfied, and in any case $T^{+4}{}_a = T^{-4}{}_a$ across Σ).

<u>STEP ONE</u>

the m.c.c. is satified only if $\varphi_+(r_0, t) = \varphi_-(r_0, t)$ for all t, which we therefore require.

<u>STEP TWO</u>

calculate the unit normals n^+ and n^- on $\partial V^\pm$.

n^+ : the full metric is

$$ds_+^2 = e^\lambda \, dr_+^2 + e^\omega (\, dy_+^2 + dz_+^2\,) - dt_+^2$$

and the unit normal is $e^{-\lambda/2}\ \partial/\partial r = n^{+}$

STEP 3

calculate the differences (in appropriate normal co-ordinates)

$\Gamma^{+4}_{\alpha\beta} - \Gamma^{-4}_{\alpha\beta}$

Now

$$\Gamma^{+4}_{\alpha\beta} = -1/2 \frac{\partial g^{+}_{\alpha\beta}}{\partial x^4}$$

in normal co-ordinates. Denoting normal co-ordinates by x and our current co-ordinates by y we have :

$$\frac{\partial g^{+}_{\alpha\beta}}{\partial x^4} = \frac{\partial}{\partial x^4}\left\{ \frac{\partial y^a}{\partial x^\alpha}\frac{\partial y^b}{\partial x^\beta} g^{+}_{ab}(y) \right\}$$

$$= \left\{ \frac{\partial}{\partial x^\alpha} n^a \right\} \frac{\partial y^b}{\partial x^\beta} g^{+}_{ab}(y) + \left\{ \frac{\partial}{\partial x^\beta} n^b \right\} \frac{\partial y^a}{\partial x^\alpha} g^{+}_{ab}(y) + \left\{ \frac{\partial}{\partial y^c} g^{+}_{ab}(y)\right\} n^c \frac{\partial y^b}{\partial x^\beta}\frac{\partial y^a}{\partial x^\alpha}$$

where we have used : $\frac{\partial}{\partial x^4}\left(\frac{\partial y^a}{\partial x^\alpha}\right) = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial y^a}{\partial x^4}\right)$, $\frac{\partial y^a}{\partial x^4} \equiv n^a$, $g^{+}_{ab}(y) = g\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right)$ ...(5.1)

where the x co-ordinates are $x^1 = t_+$, $x^2 = y_+$, $x^3 = z_+$ and x^4 is the normal co-ordinate in V^+, and the y co-ordinates are $y^1 = t_+$, $y^2 = y_+$ $y^3 = z_+$, $y^4 = r_+$.

We have that

$$\frac{\partial y^b}{\partial x^\beta} = \delta^b_\beta$$

furthermore since $n^a = \delta^a_4\, e^{-\lambda/2}$, the first of the three terms in (5.1) is zero (using $\partial y^4/\partial x^\alpha = 0$) and similarly, term two is zero. Finally, from the third term of (5.1)

$$\frac{\partial g^{+}_{\alpha\beta}}{\partial x^4} = \frac{\partial y^b}{\partial x^\beta}\frac{\partial y^a}{\partial x^\alpha} n^c \frac{\partial}{\partial y^c} g_{ab}$$

$$= e^{-\lambda/2} \frac{\partial}{\partial r_+} g^{+}_{\alpha\beta}(y)$$

Clearly, the above process holds in V^- and in computing the connection jumps, we have

$$\Gamma^{+4}_{\alpha\beta} = \Gamma^{-4}_{\alpha\beta} = 0 \qquad \Gamma^{+4}_{11} = \Gamma^{-4}_{11} = 0 , \qquad \alpha \neq \beta$$

and it is easily shown that

$$\frac{\partial g^{\pm}_{33}}{\partial x^4} = \frac{\partial g^{\pm}_{22}}{\partial x^4} = e^{-\overset{\pm}{\lambda}/2} \frac{\partial}{\partial \tau_{\pm}} g^{\pm}_{\alpha\beta}(y) = e^{-\overset{\pm}{\lambda}/2} \frac{\partial}{\partial \tau_{\pm}} e^{\omega} = P_{\pm}^{-1} \overset{\pm}{W} \left(\frac{\partial}{\partial \tau} e^{\overset{\pm}{\omega}/2}\right)^{-1} \frac{\partial}{\partial \tau}\left(e^{\overset{\pm}{\omega}/2}\right)^2 = 2e^{\overset{\pm}{\omega}/2} W_{\pm}/P_{\pm} = 2\varphi^{\pm} W_{\pm}/P_{\pm}$$

Now, indeed, $P^+ = P^-$ on (by choice of co-ordinates), and thence, $\alpha = \beta = 2,3,$

$$\begin{aligned} 8\pi S^*_{\alpha\beta} &= \Gamma^{+4}_{\alpha\beta} - \Gamma^{-4}_{\alpha\beta} \\ &= \tfrac{1}{2}\left[\frac{\partial g^-_{\alpha\beta}}{\partial x^4} - \frac{\partial g^+_{\alpha\beta}}{\partial x^4}\right] \\ &= \tfrac{1}{2}\,\frac{2}{P^2}\left[\varphi^- W^- - \varphi^+ W^+\right] \\ &= \frac{\varphi}{P^2}\left[W^- - W^+\right] \qquad (\varphi^+ = \varphi^- \text{ by m.c.c.}) \end{aligned}$$

$$S^*_{\alpha\beta} = 0 \qquad \alpha \neq \beta \text{ or } \alpha = \beta = 1$$

Absence of a surface layer implies that

$$W^+(r_0) = W^-(r_0)$$

Also $H^+(r_0) = \varphi^+(r_0,0) = \varphi^-(r_0,0) = H^-(r_0)$. We then have, since

$$0 = \left(\frac{\partial}{\partial t}\varphi(r_0)\right)^2 = W^{\pm 2}(r_0) - 1 + S^{\pm}(r_0)/\varphi(r_0)$$

that $S^+(r_0) = S^-(r_0)$. These are the matching conditions of Bonnor(1976):

$$W^+(r_0) = W^-(r_0), \quad S^+(r_0) = S^-(r_0), \quad H^+(r_0) = H^-(r_0)$$

The matching conditions to match a Schwarzchild solution to a Robertson-Walker background universe (as in the Swiss-Cheese models) follow from these.

By way of an example of the above, we seek the surface layer needed to match a Minkowski to a Schwarzchild solution. A Minkowski solution with metric in Szekeres form looks like

$$(ds)^2 = -(dt)^2 + \varphi^{+2}_{r}(dr)^2 + \varphi_+^2(d\theta^2 + \sin^2\theta\, d\bar{\varphi}^2)$$

with $\varphi^+ = \varphi^+(r)$, $W^+ = 1$, $S^+ = 0$ (e.g. $\varphi^+(r) = r$).

whilst a Schwarzchild metric is given in Szekeres form by choosing

$W^- = \bar{W}(r)$ (arbitrarily)

$S^- = 2M$ (M the Schwarzchild mass)

$\varphi_{-t}^2 = w_-^2 - 1 + 2M/\varphi_-$ (which gives meaning to t)

$P^- = 1$ $(a^- = c^- = 1/2, f^- = g^- = 0)$

(to see this, simply substitute the above in the Szekeres form of the metric, and recover the Schwarzchild form in polar co-ordinates)

To match :

$$\varphi_-(r_0, t) = \varphi_+(r_0)$$

$$(\varphi_-)_t(r_0, t) = \varphi_{+t}(r_0) = 0$$

$$0 = W_-^2 - 1 + 2M/\varphi_-(r_0, t)$$

$$W_- = (1 - 2M/\varphi_+(r_0))$$

picking $\varphi_+(r_0) = r_0$

$$W_- = (1 - 2M/r_0)^{1/2}$$

and hence :

$$8\pi\, S^*_{\alpha\beta} = r_0(1 - 2M/r_0)^{1/2} \quad \text{for } \alpha = \beta = 2,3$$

using an orthonormal basis set (to obtain the physical significance of S*)

$$\frac{1}{\varphi}\frac{\partial}{\partial y} \qquad \frac{1}{\varphi}\frac{\partial}{\partial z}$$

obtain

$$8\pi\, S^*\left(\frac{1}{\varphi}\frac{\partial}{\partial y}, \frac{1}{\varphi}\frac{\partial}{\partial z}\right) = \frac{8\pi}{\varphi^2}\, S^*\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)$$

$$= \frac{8\pi}{r_0^2}\,(1 - 2M/r_0)^{1/2}$$

$$\simeq 8\pi\left(\frac{1}{r_0^2} - \frac{M}{r_0^3}\right) \qquad (\text{for } r_0 \gg M)$$

If we choose r_0 large enough, then the jump in tidal forces implied by S* will become negligibly small, and effectively not detectable, being of say the order of interatomic collision impulses.

5.2 The impulsive plane wave

The case of the impulsive, purely gravitational plane wave has been already discussed in chapter two. We compute the components S^{*}_{ab}

In V^{-}, the connection co-oefficients are, ofcourse, zero. In V^{+}, the metric takes on the form with co-ordinates $(\underset{+}{x}^1, \underset{+}{x}^2, \underset{+}{x}^3, \underset{+}{x}^4) \equiv (\underset{+}{u}, \underset{+}{x}, \underset{+}{y}, \underset{+}{v})$

$$\{ \overset{+}{g}_{ab} \} = \begin{Bmatrix} 0 & 0 & 0 & -1 \\ 0 & (1+\underset{+}{v})^2 & 0 & 0 \\ 0 & 0 & (1-\underset{+}{v})^2 & 0 \\ -1 & 0 & 0 & 0 \end{Bmatrix}$$

and

$$\{ \overset{+}{g}^{ab} \} = \begin{Bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1/(1+\underset{+}{v})^2 & 0 & 0 \\ 0 & 0 & 1/(1-\underset{+}{v})^2 & 0 \\ -1 & 0 & 0 & 0 \end{Bmatrix}$$

We then have :

$$\overset{+}{\Gamma}{}^{4}_{\alpha\beta} = \tfrac{1}{2} g^{41} \left\{ \frac{\partial \overset{+}{g}_{\alpha 1}}{\partial x^{\beta}} + \frac{\partial \overset{+}{g}_{\beta 1}}{\partial x^{\alpha}} - \frac{\partial \overset{+}{g}_{\alpha\beta}}{\partial x^{1}} \right\} = 0$$

$$\overset{+}{\Gamma}{}^{4}_{4\alpha} = \tfrac{1}{2} g^{41} \left\{ \frac{\partial \overset{+}{g}_{41}}{\partial x^{\alpha}} + \frac{\partial \overset{+}{g}_{\alpha 1}}{\partial x^{4}} - \frac{\partial \overset{+}{g}_{4\alpha}}{\partial x^{1}} \right\} = 0$$

$$\overset{+}{\Gamma}{}^{4}_{44} = \tfrac{1}{2} g^{41} \left\{ \frac{\partial \overset{+}{g}_{41}}{\partial x^{4}} + \frac{\partial \overset{+}{g}_{41}}{\partial x^{4}} - \frac{\partial g_{44}}{\partial x^{1}} \right\} = 0$$

$$\overset{+}{\Gamma}{}^{\alpha}_{\gamma 4} = \tfrac{1}{2} g^{\alpha m} \left\{ \frac{\partial g_{\gamma m}}{\partial x^{4}} + \frac{\partial g_{4m}}{\partial x^{\gamma}} - \frac{\partial g_{4 4}}{\partial x^{1}} \right\} = \tfrac{1}{2} g^{\alpha\alpha} \left\{ \frac{\partial g_{\gamma\alpha}}{\partial x^{4}} \right\} \quad \text{(no sum on } \alpha\text{)}$$

from which it follows that if $\alpha \neq \gamma$ then $\Gamma^{+\alpha}_{\gamma 4} = 0$ and if $\alpha = \gamma$ then

$$\Gamma^{+2}_{24} = \frac{1}{2}\frac{1}{(1+V_+)^2}\frac{\partial}{\partial V_+}(1+V_+)^2 = \frac{1}{1+V_+} \Rightarrow \Gamma^{+2}_{24}\Big|_\Sigma = 1 \qquad \Sigma \equiv V_+ = V_- = 0$$

$$\Gamma^{+3}_{34} = \frac{1}{2}\frac{1}{(1-V_+)^2}\frac{\partial}{\partial V_+}(1-V_+)^2 = \frac{1}{V_+-1} \Rightarrow \Gamma^{+3}_{34}\Big|_\Sigma = -1$$

Finally, $\Gamma^{+\alpha}_{44} = \frac{1}{2} g^{\alpha m}\left(\frac{\partial g_{4m}}{\partial x^4} + \frac{\partial g_{4m}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^4} \right) = 0.$

It follows immediately that

$$S^*_{\alpha\beta} = 0$$

and that

$$S^*_{\alpha 4} = S^*_{4\alpha} = 0 = S^*_{44}$$

and there is no surface layer in Σ. The jump in the shear across Σ is

$$\sigma^- - \sigma^+ = \Gamma^{-2}_{24} - \Gamma^{+2}_{24} - \Gamma^{-3}_{34} + \Gamma^{+3}_{34} = -2.$$

---- oOo ----

We have shown that, given two C^k manifolds V^+ and V^-, with boundaries ∂V^+ and ∂V^-, and with metrics g^+ and g^- (C^k), there exists a C^k structure for $V = V^+ U_f V^-$ unique up to a C^k diffeomorphism with at most one C^{r+1} structure F for V such that the joined metric $g=g^+U_f g^-$ is defined and C^r with respect to F.

Based on the assumption, on physical grounds, that g is at least continuous (the satisfaction of the metric compatibility condition), so that a single C^1 structure for V has been singled out, we have derived the jump conditions across $\Sigma = \partial V^+ U_f \partial V^-$ implied by a functional form of the Einstein equations (with zero cosmological term). In particular, we have shown that in a physically reasonable Universe, if Σ is space-like, then g is at least C^1, if Σ is time-like, g fails to be C^1 only if Σ is the history of a surface layer (or shock front) of energy-momentum, whilst if Σ is null, g fails to be C^1 on V if either Σ is the history of a surface layer, or if a purely gravitational impulsive wave propogates in Σ.

We have also written the Bianchi identities in a functional form across Σ and shown that if creation and destruction of material is not to occur in V, i.e. if V is conservative, then g C^1 implies that the normal flow of energy momentum across Σ is continuous. Thence, we have deduced that if V^+ and V^- contain perfect fluids, in which the speed of sound is less than the speed of light, and if Σ does not represent the history of a surface layer, or of a purely gravitational impulsive wave, then the joint energy momentum tensor $T=T^+U_f T^-$ can be discontinuous across Σ only if Σ is comoving with the fluid, when the fluid pressure is continuous across Σ.

When g is continuous only, we have shown that the conservation of surface layer energy-momentum in Σ leads, via the functional Bianchi identities to an equation of motion for the surface layer, and to the continuity of the normal flow of energy-momentum across Σ.

It remains to study the implications of the functional Bianchi identities across Σ when g is continuous only and Σ is null, to study the nature and implications of a functional form of the Weyl tensor in V, and to extend the results of this thesis to the case when the boundaries of the manifolds to be joined are not smooth, as outlined in the introduction.

APPENDIX 1

INVERSE FUNCTION THEOREM (Kahn (1980) p 17)

Let $F : R^n \to R^n$ be a C^k map. Let p be in R^n and suppose that F is defined in an open neighbourhood of p, and is of full rank at p. Then there exists an open set O containing p and an open set O' containing F(p), such that

$$F|_O : O \to O'$$

is a C^k diffeomorphism on O onto O'.

Proof: see Kahn.

The two corollaries of this theorem regarding C^k maps and imbeddings on manifolds given by Kahn, may be extended somewhat to manifolds with boundaries :

COROLLARY 1

Let $f : W \to V$ be a C^k map from W to V, both n-dimensional manifolds (with boundary) and suppose that f is of full rank (n) at p in W; then there is a W neighbourhood O_p of p such that $f|_{O_p}$ is a C^k diffeomorphism with its image.

Proof:

Let (U,h) and (U',h') be p- and f(p)-charts on W and V. Without loss of generality, we may assume that $U' \subset f(W)$ (so that U' may be a neighbourhood of f(p) only in the V induced topology on f(W)). Let $F = h' \circ f \circ h^{-1} : R^n \rightarrow R^n$; F is defined on $h(U \cap f^{-1}(U'))$ and since f is C^k; it follows that there is an extension $\tilde{F}$ of F to an open neighbourhood of $h(U \cap f^{-1}(U'))$ in R^n, which is a C^k map (by the definition of differentiability). Apply the theorem to $\tilde{F}$ (which is of full rank, since f and hence F are of full rank at p and h(p) respectively). There exists an open neighbourhood O of h(p) in R^n on which $\tilde{F}$ is a diffeomorphism. It follows that F is a C^k diffeomorphism when restricted to $O \cap h(U \cap f^{-1}(U'))$, onto the image under F of this set. Notice that $O_p = h^{-1}(O \cap h(U \cap f^{-1}(U'))$ is open in W since O is open in R^n and since f is continuous onto its image. Then $F \circ h_{|O_p}$ is a diffeomorphism of O_p onto its image, and thus $f_{|O_p} = h'^{-1} \circ F \circ h$ is a C^k diffeomorphism of O_p onto its image $f(O_p)$.

It is not necessarily true that $f(O_p)$ is open in f(W), or even that f^{-1} is well defined on $f(O_p)$, although ${f_{|O_p}}^{-1}$ is - a good example of this is the map of a strip in R^2 onto the ordinary cylinder, by wrapping the strip around the cylinder several times.

COROLLARY 2

Let W,V be n-dimensional C^k differentable manifolds; suppose a C^k $f : W \rightarrow V$ is of full rank on W, an open map of W with its image in V, and one to one (i.e., f is an imbedding of W into V). Then f is a diffeomorphism of W with f(W) and the differentiable structure so defined on f(W) agrees with the structure induced on W by V - i.e. f(W) is a submanifold of V.

Proof

Let p be in W. To prove the result, we need to produce a chart for f(W) about f(p), in the structure induced by W on f(W), which is a restriction of an f(p)-chart in V.

Let (U,h) and (U',h') be p and f(p)-charts. Without loss of generality, we may assume that f(U) = U'$\cap$ f(W), since f is open onto its image. Let F = h' o f o h^{-1}; then F has rank n at h(p), and indeed on h(U). It then follows by the theorem, and by the definition of a C^k map, that there is an extension $\tilde{F}$ of F to an R^n (open neighbourhood of h(U), 0 say, containing h(U), with $\tilde{F}$ a C^k diffeomorphism on 0, to 0'; consider 0' $\cap$ h'(U'); this is open in R^n (or R^{n-1} x $[0,\infty)$ or R^{n-1} x $(-\infty,0]$) so that $(h')^{-1}$(0' $\cap$ h'(U')) = $\hat{U}$ is open in V and $\hat{U}$ $\cap$ f(W) = f(U). (This because rank f = n on all of U and because of a topological lemma (Munkres (1966) p 54) which states that a map on R^n which is a homeomorphism in a nhd of each point of a set is a homeomorphism on a nhd of the set).
Let $\hat{h}$ = $(\tilde{F})^{-1}$ o h'; $(\hat{U},\hat{h})$ is clearly a V chart and $\hat{h}_{|f(U)}$ = h o f^{-1}.

COROLLARY 3

Let W,V,f be as in corollary 2, excepting that dimension W = m < dim V = n, and obtain the same conclusion.

Proof

Again, let p be in W, (U,h) and (U',h') p- and f(p)-charts in W and V with (by the open and one to one nature of f) $f(U) = U' \cap f(W)$. Let $Y = \{\underline{y} \text{ in } R^n \mid y^1 = y^2 = \ldots = y^{n-m} = 0\}$; regard $R^m = \{x \text{ in } R^n \mid x^{n-m+1} = \ldots = x^n = 0\}$ for the moment. Then the function for $\underline{x}$ in R^m and $\underline{y}$ in Y defined by

$$f'(\underline{x},\underline{y}) = (h')^{-1} (h'(f(h^{-1}(\underline{x})) + \underline{y}))$$

is easily seen to be C^k and of rank n on h(U); it follows that f' has an extension $\tilde{f}'$ to a neighbourhood 0 of h(U) in R^n which is a C^k diffeomorphism. Proceeding as in corollary 2, let $\tilde{f}'(0) = 0'$; $0' \cap h'(U')$ is open in R^n so that $(h')^{-1} (0' \cap h'(U')) = \hat{U}$ is open in V, with $\hat{U} \cap f(W) = f(U)$; set $\hat{h} = (\tilde{f}')^{-1} \circ h'$ - a chart for V arises with

$$\hat{h}|_{f(U)} = h \circ f^{-1} \text{ as before.}$$

We now turn to some geometrical lemmas, adapted from Munkres (1966).

LEMMA 1

Let $V = V^+ \cup_f V^-$ be as in section 1.3; suppose F is any structure for V such that $V^{\pm}$ are imbedded into V by the inclusion maps which take them to themselves in V. Then there exists a neighbourhood N of Σ in V and a C^k diffeomorphism $P: N \to \Sigma \times (-1,1)$ such that

x in Σ implies $P(x) = (x,0)$

x in $V^+ \setminus \Sigma$ implies $P(x) = (x',r)$ and $r > 0$

x in $V^- \setminus \Sigma$ implies $P(x) = (x',r)$ and $r < 0$.

Proof

The proof depends on the following two results :

(a) if $g: M \rightarrow M'$ is a C^k imbedding of a non-bounded C^k mainfold M into a C^k manifold M', then there is a neighbourhood N of g(M) in M' and a C^k retraction $R:N \rightarrow g(M)$ (a retraction obeys the requirement that if x is in N, then $R(x) = x$).

For proof, see Munkres (1966) p 53.

(b) There is a C^k function mapping $V \rightarrow R$, r, which satisfies

i) $r(x) = 0$ for x in Σ

ii) $r(x) > 0$ for x in $V^+ \setminus \Sigma$

iii) $r(x) < 0$ for x in $V^- \setminus \Sigma$

iv) rank r = 1 on Σ

To prove (b) :

for each p in Σ, we may find (by the imbedding assumptions) a p-chart (U,h) such that if $h(q)=\underline{x}(q)=(x^1(q),\ldots,x^n(q))$, then $x^n(q) < 0$ if q is in $U \cap_f V^-$, $x^n(q) = 0$ if q is in $U \cap_f \Sigma$, and $x^n(q) > 0$, if q is in $U \cap_f V^+$. For p not in Σ we may find charts with this property, still.

Let $\{(U_i,h_i)\}$ be a locally finite covering of V by such charts. Let (B_i,φ_i) be a subordinate C^k partition of unity (see e.g. Sternberg (1964)) and if $h_i(q) = (x_i^1(q),\ldots x_i^n(q))$ for q in U_i, set

$$r(q) = \sum_i \varphi_i(q)\, x_i^n(q)$$

(this is always a finite sum).

r is clearly C^k; if q is in Σ etc. the conditions (i) to (iii) are satisfied. Further, if (U,h) is a p-chart for p in Σ, of the type described above, then,

$$\left(\frac{\partial}{\partial x^n}\right)_p [r] = \sum_i \left(\frac{\partial}{\partial x^n}\right)_p [\varphi_i]\, x_i^n(p) + \varphi_i(p) \left(\frac{\partial}{\partial x^n}\right)_p [x_i^n] > 0$$

(the first term is zero, since $x_i^n(p) = 0$ and the second is +ve since

$(\frac{\partial}{\partial x}n)_p[x_i^n] > 0$), so that rank r = 1.

The proof of the lemma is :

by (a), there exists a neighbourhood of Σ (which is imbedded in V by the assumptions), N say, and a C^k retraction R : $N \rightarrow \Sigma$ together with a C^k map r : $N \rightarrow R$ (by (b)); let $P_1 : N \rightarrow \Sigma \times R$ by

$$P_1(q) = (R(q), r(q)).$$

Rank P_1 = n on Σ; to see this, let (U,h) be a p-chart for p in Σ, of the kind described in the proof of (b). This induces a chart (U_Σ, h_Σ) on Σ in the obvious fashion, and hence a chart of $\Sigma \times R$, $(U_\Sigma \times R, k)$ by

$k(q,s) = (h_\Sigma(q), s)$ for q in U_Σ. As a linear operator, $(k \circ P_1 \circ h^{-1})_*$ is represented by the matrix:

$$\begin{pmatrix} I_{(n-1)x(n-1)} & \underline{b} \\ \underline{0} & \frac{\partial}{\partial x}n\ (r \circ h^{-1}) \end{pmatrix}$$

in terms of the co-ordinate bases in $T_p(V)$ and $T_{P_1(p)}(\Sigma \times R)$ ($\underline{b}$ is unspecified and immaterial).

It follows that P_1 is a homeomorphism on a neighbourhood of p in V (corollary 1 above) and hence, by a lemma of topology (used also in corollary 2 - see Munkres p 54) on a neighbourhood of Σ contained in N. Without loss of generality, we may assume this to be N.

There then exists a C^k function $\delta : \Sigma \rightarrow R$ such that if $|s| < \delta(x)$ then $P_1^{-1}(x,s)$ is in N (Munkres p 56) and $\delta(x) > 0$ on Σ; for q in N, if $P_1(q) = (q', s)$ set $P(q) = (q', s/\delta(q'))$.

<u>Note:</u> This lemma is a slight generalization of the theorem 5.9 in Munkres

quoted in theorem 1.1 of this thesis - with the obvious restrictions, it yields that theorem as a corollary, and is therefore included in its entirety for completeness.

LEMMA 2 (Munkres p 63)

Let M be a non-bounded (i.e. $\partial M = \phi$) C^r manifold; let W be a neighbourhood of M x $\{0\}$ in M x $[0,\infty)$. Let f be a C^r imbedding of W into M x $[0,\infty)$ which equals the identity on M x $\{0\}$. Then there is a C^r diffeomorphism f' of W onto f(W) which equals f in a neighbourhood of the complement of W in M x $[0,\infty)$, and equals the identity in a neighbourhood of M x $\{0\}$.

APPENDIX II

LEMMA 1

If K is compact in a manifold V, and (A_i) is a locally finite covering of V, then $K \cap A_i$ is non empty for at most finitely many A_i.

Proof

Since (A_i) is locally finite and a covering of V, to each p in K there may be assigned an open Neighbourhood N_p of p such that $A_i \cap N_p$ is empty for all but a finite number of A_i. N_p cover K. Select a finite subcover and hence a finite subcover of K by A_i.

The remainder of this appendix is devoted to a proof of Lemma 3.4. We shall prove the result only for a distribution of order q - the result for tensor distributions follows similarly. We need :

LEMMA 3.4.1

The space $C_o(V)$ is dense in $D^m(V)$.

Proof:

Let $\{(U_i, h_i)\}$ be a locally finite C atlas for V and let $\{(\psi_i, K_i)\}$ be a subordinate partition of unity, each ψ_i having compact support K_i, together defining $\| \|_{K,m}$ for each K in K(V).

Let φ be in $D^m(V)$, with support K; by lemma 1 above, $K \cap K_i \neq \emptyset$ for finitely many K_i, $K_1, \ldots K_N$ say, which we assume dominated by $(U_1, h_1), \ldots, (U_N, h_N)$

say. Set $\varphi_i = \psi_i \varphi$ so that $\varphi = \sum_1^N \varphi_i$; supp $\varphi_i = K_i \cap K$, which is compact. $h_i(K_i \cap K)$ is compact in R^n; without loss of generality (by reducing the support of ψ_i if necessary) we may assume that $h_i(K_i \cap K) \subset \text{Int } K_i' \subset h_i(U_i)$ for some compact K_i' in R^n, where $h_i(K_i \cap K) \subset \text{Int } K_i'$ (Int = topological interior), Set $K_i'' = h_i^{-1}(K_i')$ and K_i'' is in $K(V)$. Let $K'' = \bigcup_1^N K_i''$; then K'' is in $K(V)$ and φ is in $D^m_{K''}(V)$ since $K \subset K''$.

We show that every neighbourhood of φ in $D^m_{K''}(V)$ contains a $\tilde{\varphi}$ in $C_o(V)$ which is indeed in $D^m_{K''}(V)$. The lemma will then follow, since if O is open neighbourhood of φ in $D^m(V)$, $O \cap D^m_{K''}(V)$ is open in $D^m_{K''}(V)$ (lemma 3.3(iii)).

Given $\epsilon > 0$ there exists (cf Choquet-Bruhat (1977) p 354) a $\tilde{\varphi}_i$ in $D_{K''}(V)$ such that $\|\varphi_i - \tilde{\varphi}_i\|_{K'',m} = \|\varphi_i - \tilde{\varphi}_i\|_{K_i'',m} < \epsilon / N$.

Then :

$$\tilde{\varphi} = \sum_1^N \tilde{\varphi}_i \quad \in \quad D^m_{K''}(V) \quad \text{and}$$

$$\|\varphi - \tilde{\varphi}\|_{K'',m} \leq \sum_1^N \|\varphi_i - \tilde{\varphi}_i\|_{K',m}$$

$$< N \cdot \epsilon / N = \epsilon$$

q.e.d.

We need also the following derivative of the Hahn-Banach theorem ;

<u>EXTENSION THEOREM</u> (see Choquet Bruhat p 345)

Let X be given a topology defined by a family of semi-norms. Let $Y \subset X$ be dense in X in this topology, and let f be a linear functional on Y, continuous w.r.t. the X induced topology on Y. Then there is an unique linear extension of f to a functional on X, continuous on X.

Proof of Lemma 3.4

The topology of $D^q(V)$ which is a locally convex linear topological space can be defined by a family of seminorms (see for example Choquet-Bruhat pp 344 and 349) - set $X = D^q(V)$ in the extension theorem. Let $Y = C_0(V)$. A linear functional will be continuous on Y in the $D^q(V)$ induced topology iff it is continuous on $Y \cap D^q_K(V)$ for all K in K(V). Thus a distribution f of order q is continuous on Y in the $D^q(V)$ induced topology. It follows by the above extension theorem that there is a unique linear continuous extension of f to $D^q(V)$.

q.e.d.

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