GEOMETRY OF FINSLER SPACES
CONSIDERED AS
GENERALIZED MINKOWSKIAN SPACES

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Abstract of Thesis

FINSLER SPACES AS GENERALISED MINKOWSKIAN SPACES.

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To every regular problem in the Calculus of Variations there corresponds a Finsler space. A tangent Minkowskian space T_n is attached to each point of this space, so that the metric of the Finslerian phase-space is determined locally by the corresponding T_n. Cartan, Berwald and their successors reduce the Finslerian case to the Riemannian case by means of the introduction of an arbitrary field of line-elements, i.e. a "fibering". Here, on the contrary, we have to deal primarily with a detailed investigation of Minkowskian spaces (finite-dimensional Banach spaces).

It is convenient to use tensors in view of the affine structure of Minkowskian spaces. The most useful tool is the indicatrix of Carathéodory (unit sphere), so that the methods of the Theory of Convex Figures are applicable, although throughout a Euclidean background has been avoided scrupulously. With the aid of the function of support of the indicatrix we introduce a conjugate Minkowskian tangent-space T'_n, in which the figuratrix assumes the rôle of the unit sphere. There exists a one-one correspondence between the points of T_n and those of T'_n; this correspondence represents the raising and lowering of suffixes. It is due to this fact that the Minkowskian metric is suitable for the use of tensors. The figuratrix is closely related to the Hamiltonian function and the points of T'_n represent the canonical coordinates of the given problem in the Calculus of Variations. Simple geometrical results are easily derived for future use, and by defining the trigonometrical functions geometrically, irrespective of analytical convenience, the foundations of a Trigonometry of Minkowskian spaces are laid; for instance, addition formulae for sine and cosine and their series-expansions are evaluated. The angular measure, whose definition has been subject to discussion for some time, is here defined by the shortest arc-length on the indicatrix, and we may, if necessary, introduce a suitable normalisation-factor. By means of this measure we define the curvature of curves in a manner similar to the classical Euclidean case. In the theory of curvature of surfaces, when an appropriate definition for the curvature of curves on surfaces is used, the indicatrix of Dupin appears as a conic section, whereas the principal directions of curvature lose their significance. Furthermore, analogues of the theorems of Rodrigues and Meusnier are also to be found in Minkowskian Differential Geometry.

The study of Finsler spaces depends chiefly on investigations into the variation of the Minkowskian metric between two neighbouring points of the space. For a first approximation the so-called "generalised Christoffel symbols" are sufficient, by means of which a notion of absolute parallelism is defined. In order to determine the curvature-properties of the Finsler space, we investigate the relationship between two neighbouring geodesics and obtain a measure R of curvature, which expresses the "defect" between the Finsler space and the Minkowskian tangent-space at the same point. For example, in the case of two dimensions R can also be interpreted geometrically by the formula

$$R = \frac{6}{L(1)} \lim_{s \to 0} \left(\frac{L(s) - G(s)}{s^3} \right)$$

where $L(s)$ and $G(s)$ are the circumference of the Minkowskian circle and the geodesic circle of radius s at the same point respectively. Finally, a formula analogous to the classical Gauss-Bonnet Theorem is derived.

dedicated to the Memory of

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INTRODUCTION

The aim of the following investigations is principally a preparatory one. A geometrical framework which would be most suitable as a background for the Hamilton-Jacobi Theory is sought, so that we have restricted the present discussion to purely geometrical questions. The space under consideration is of the type first investigated systematically by Finsler (1)*, after whom it has been named. Finsler treated the geometrical properties of his space from the point of view of the Calculus of Variations; in fact, the notion of such spaces was derived directly from this field.

Since then there have been different approaches to the Geometry of Finsler Spaces, all of which, apart from the work of Busemann, have one important feature in common: namely extensive use of the Tensor Calculus, which was not used by Finsler. Firstly, Synge (2) and Taylor (3) gave definitions of parallel displacement, and these methods were developed by Cartan (4), Berwald and Funk (5), all of whom worked along somewhat different lines. An essential aspect of these methods lies in the fact that the authors distinguish a field of curves, so that at each point of the space there exists a special direction with respect to which all measurements at that point are made. Although this has the great advantage that it enables us to retain many methods of Riemannian Geometry with certain modifications, the results obtained depend on the choice of the field. This state of affairs is not entirely satisfactory from the geometrical point of view; this opinion has also recently been expressed by Busemann in an extremely illuminating historical review (6).

It was this that led me to attempt yet another geometrical theory of this kind which would also satisfy the demands of the Hamilton-Jacobi Theory. My definitions and problems have been guided throughout by a desire for geometrical clarity: nowhere have I allowed my approach to be influenced by analytical technique, nor did I seek a mere extension of Riemannian Geometry. From this point of view

*Numbers in brackets refer to the references at the end.

it seems more natural to regard Finsler spaces as locally Minkowskian, where it is to be noted that the Minkowskian space is of affine structure. This has been clearly recognised by Busemann, who carried out many investigations in this line, using, however, the methods of infinitesimal direct geometry, so that the problems treated by him are of an entirely different character.

Thus the first four chapters are devoted to a study of Minkowskian spaces. Our chief tool is the indicatrix, whose function of support is shown to be closely related to the Hamiltonian of the problem in the Calculus of Variations arising out of the function defining arc-length. It will be found that for the purely analytical technique it is convenient to use tensors: after all, the concepts of contra- and covariance are inherent to the affine tensor calculus irrespective of the metric. Furthermore, our investigations show that the introduction of a conjugate space by means of the function of support allows us to attribute a geometrical meaning to the process of raising and lowering of suffixes, which is generally a notion secondary to that of contra- and covariance, due to the fact that a specialised metric is required. Making use of the affine structure of the Minkowskian space, simple geometrical results are derived for future use, and the foundations of a Trigonometry of such spaces are laid. This enables us to discuss some of the basic theorems of classical Differential Geometry on surfaces embedded in a Euclidean space for the case of surfaces embedded in a Minkowskian space. The definitions of angle and area have been the subject of discussion for some time - new definitions which seem to satisfy the demands of the present theory are suggested here.

In chapter V we proceed to a study of Finsler spaces, using the indicatrix to obtain the Christoffel symbols and leading to a form of parallelism which is independent of an arbitrary field, in contrast to the definitions given by earlier authors. This is done by means of a covariant derivative and simplified by the introduction of what we shall call the "extended Christoffel symbols". A detailed study of Geodesic Deviation enables us to define a curvature tensor. This, however, is not the only purpose of this treatment, but it also provides the key to further investigations into problems "in the large". The latter results do not fit into the scope of the present work and will be published separately. With the aid of the curvature tensor the curvature of the Finsler space is derived: this depends only on the point under consideration and not on a direction through that point. This is shown to have very clear geometrical interpretations in a two-dimensional manifold, for which a theorem analogous to that of Gauss and Bonnet is finally derived.

Very little attention has been paid to questions concerning tensor characteristics and invariance, as it is my intention to introduce the Contact Transformation of Sophus Lie rather than the point-transformations into the theory of Finsler spaces, and to develop a corresponding Tensor Calculus which will be more suitable for a treatment of the Hamilton-Jacobi Theory.

Throughout we have strictly discarded any Euclidean background, although some of the most fundamental ideas are based on Minkowski's Theory of Convex Figures. This indicates the desirability of investigations into the possibility of a reconstruction of Minkowski's theory alonf such lines, especially the recent work of Fenchel and Jessen (7) who have discarded differentiability assumptions.

My best thanks are due to Prof. L. C. Young who first introduced me to the Calculus of Variations, and to Dr. C. Y. Pauc for his constant encouragement and many very valuable suggestions.

University of Cape Town,

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CHAPTER 1

PRELIMINARIES

§ 1. Connection with the Calculus of Variations.

In an n-dimensional Riemannian manifold the arc-length s measured along a curve represented by an arbitrary parameter t

$$x^i = x^i(t) \qquad (i = 1, 2, \ldots n) \tag{1}$$

is given by an integral of the form*

$$s = \int_{t_0}^{t} (g_{ij}\, x'^i x'^j)^{1/2}\, dt\,, \tag{2}$$

where the functions g_{ij} depend only on the position, and where

$$x'^i = \frac{dx^i}{dt}. \tag{3}$$

We shall consider a more general space of n dimensions, where the arc-length of the curve (1) is given by a function $F(x^i, x'^i)$, so that the integral (2) is replaced by

$$s = \int_{t_0}^{t} F(x^i, x'^i)\, dt\,. \tag{4}$$

The problem of finding the geodesics (i.e. curves of minimum length joining two points on it, provided they are sufficiently close to each other) of such a space is then reduced to the first problem of the Calculus of Variations for the function $F(x^i, x'^i)$: namely finding a curve which minimises the integral (4).

In accordance with the usual procedure we therefore make the following assumptions with regard to $F(x^i, x'^i)$:

(A): $F(x^i, x'^i)$ is analytic in the 2n variables x^i, x'^i.

(B): The integral (4) must be independent of the choice of the arbitrary parameter t.

The necessary and sufficient condition for this is that F be positively homogeneous of the first degree in the x'^i,

* Throughout the summation convention as regards repeated indices is used.

i.e.*

$$F(x^i, kx'^i) = k F(x^i, x'^i); \quad (k>0) \tag{5}$$

(C): The function $F(x^i, x'^i)$ is strictly convex in the x'^i.

We shall show that this is equivalent to the more well-known regularity condition, namely that the function $F_1(x^i, x'^i)$ of the Calculus of Variations, defined by

$$F_1(x^k, x'^k) = \frac{-1}{F^2(x^k, x'^k)} \begin{vmatrix} \dfrac{\partial^2 F(x^k, x'^k)}{\partial x'^i \partial x'^j}, & \dfrac{\partial F(x^k, x'^k)}{\partial x'^i} \\ \dfrac{\partial F(x^k, x'^k)}{\partial x'^j}, & 0 \end{vmatrix}$$

is everywhere positive and non-vanishing. (Mason and Bliss (10), Taylor (11)).

(D): $F(x^i, x'^i)$ is positive and non-vanishing throughout the regions under consideration.

This is not an essential restriction, as we may, in virtue of our previous assumptions always add an exact differential of the form

$$\frac{\partial \alpha(x^k)}{\partial x^i} x'^i$$

and thus obtain an "equivalent " problem in the Calculus of Variations. (Caratheodory (8), p. 197).

Spaces in which the metric is defined by (4) are known as Finsler spaces; the geodesics are then the extremals. A case of special importance occurs when F is a function of the direction x' only, for then the extremals are straight lines in the euclidean sense. (Bolza (9), p. 32). The space is then said to be Minkowskian, and we may use the ordinary notions of vector, plane, etc., i.e. it is an affine space.

§ 2. Analytical deductions.

The following analytical results will be found to be very useful later on. From Euler's Theorem on homogeneous functions we deduce from equation (5):

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*Caratheodory (8), p. 213; Bolza (9), p. 193.

$$\left.\begin{aligned} F(x^i, x'^i) &= \frac{\partial F(x^i, x'^i)}{\partial x'^k} x'^k \\ \frac{\partial^2 F(x^i, x'^i)}{\partial x'^h \partial x'^k} x'^k &= 0 \end{aligned}\right\} \tag{6}$$

We shall now put

$$\varphi(x, x') = \tfrac{1}{2}[F(x^i, x'^i)]^2 \tag{7}$$

so that φ is positively homogeneous of second degree in the x'^i, i.e. for $k > 0$ we have

$$\varphi(x, kx') = k^2 \varphi(x, x') \tag{8}$$

Differentiating (8) twice with respect to k, we find

$$\frac{\partial \varphi(x, kx')}{\partial x'^i} x'^i = 2k\varphi(x, x') \tag{9}$$

and

$$\frac{\partial^2 \varphi(x, kx')}{\partial x'^i \partial x'^j} x'^i x'^j = 2\varphi(x, x') \tag{10}$$

where it is to be noted that the left-hand side of (10) is independent of k.

Let us now introduce a transformation of the type which were called "extended point-transformation" by S. Lie ((12), p. 44) :

$$\bar{x}^i = \bar{x}^i(x^j) \tag{11}$$

whose Jacobian does not vanish at any point. The quantities x'^i as defined by (3) then transform according to the law

$$\bar{x}'^i = \frac{\partial \bar{x}^i(x^k)}{\partial x^j} x'^j \tag{12}$$

Since the function $F(x^i, x'^i)$ or $\varphi(x, x')$ is a measure of length, its intrinsic nature allows us to regard it as a scalar invariant, so that we have

$$\frac{\partial \bar{\varphi}(\bar{x}, \bar{x}')}{\partial \bar{x}'^i} = \frac{\partial \varphi(x, x')}{\partial x'^j} \cdot \frac{\partial x'^j}{\partial \bar{x}'^i}$$

But it follows from (12) that

$$\frac{\partial \bar{x}'^i}{\partial x'^j} = \frac{\partial \bar{x}^i}{\partial x^j}$$

and the inverse equation, so that we deduce immediately that the quantities $\partial\varphi(x, x')/\partial x'^i$ transform like the components of a covariant vector.

Similarly,

$$\frac{\partial^2 \bar{\varphi}(\bar{x}, \bar{x}')}{\partial \bar{x}'^i \partial \bar{x}'^j} = \frac{\partial^2 \varphi(x, x')}{\partial x'^h \partial x'^k} \cdot \frac{\partial x'^h}{\partial \bar{x}'^i} \cdot \frac{\partial x'^k}{\partial \bar{x}'^j}$$

from which it follows in the same way that the quantities $\partial^2\varphi(x,x')/\partial x'^h\partial x'^k$ transform like the components of a covariant tensor of rank 2. Similar results may be derived for derivatives of higher order. (These facts were first proved by Taylor (13) and Synge (14)).

We may thus write in accordance with the usual notation of the tensor calculus:

$$g_{ij}(x,x') = \frac{\partial^2\varphi(x,x')}{\partial x'^i\,\partial x'^j} \tag{13}$$

From (10) it follows that the g_{ij} are symmetrical in i and j and homogeneous of degree zero, so that

$$g_{ij}(x,kx') = g_{ij}(x,x') \tag{14}$$

Differentiating once more with respect to k and putting $k=1$, we find

$$\left.\begin{aligned} \frac{\partial g_{ij}(x,x')}{\partial x'^k}\,x'^k &= 0 \\ \text{and} \qquad \frac{\partial g_{ij}(x,x')}{\partial x'^k}\,x'^i &= 0 \end{aligned}\right\} \tag{15}$$

in consequence of the arbitraryness of the order of differentiation in (13). Also, differentiating (14) with respect to x^ℓ, we have

$$\frac{\partial g_{ij}(x,kx')}{\partial x^\ell} = \frac{\partial g_{ij}(x,kx')}{\partial x^\ell} \tag{16}$$

so that analogously to (15) we obtain

$$\left.\begin{aligned} \frac{\partial^2 g_{ij}(x,x')}{\partial x^\ell\,\partial x'^k}\,x'^k &= 0 \\ \frac{\partial^2 g_{ij}(x,x')}{\partial x^\ell\,\partial x'^k}\,x'^i &= 0 \end{aligned}\right\} \tag{15'}$$

§ 3. The Indicatrix.

From equations (4) and (7) we deduce that the element of distance is given by

$$ds^2 = 2\varphi(x,x')\,dt^2 \tag{17}$$

and using (10) and (13) it follows that

$$ds^2 = g_{ij}(x,x')\,dx^i\,dx^j \tag{18}$$

where it is to be noted that the argument in the g_{ij} must correspond to the direction of the infinitesimal displacement dx. We shall show later that in consequence of our convexity condition the quadratic form

$$g_{ij}(x, x')\,\xi^i \xi^j$$

is positive definite for all directions x'.

In Minkowskian spaces the g_{ij} are independent of the x^i, so that we may regard <u>the Finsler space as a space with a locally Minkowskian metric</u>. (Busemann (15)). Thus to each point of the Finsler space we can attach a "tangent" Minkowskian space T_n. The coordinates of any point (or vector) in this space will then be given by the n numbers x'^i.

At any point x^i (which we shall regard as fixed for the moment) the function $F(x^i, x'^i)$ has the following properties in T_n at x^i :

(a) $F(x^i, x'^i) > 0$ for $x'^i \neq 0$, $F(x^i, 0) = 0$.

(b) $F(x^i, k x'^i) = k\,F(x^i, x'^i)$, $(k > 0)$.

(c) $F(x^i, x'^i_{(1)} + x'^i_{(2)}) \leq F(x^i, x'^i_{(1)}) + F(x^i, x'^i_{(2)})$.

These follow directly from the assumptions made in §1 .

Then it can be shown* (Bonnesen und Fenchel (16), p. 22) that the set of points in T_n whose coordinates satisfy the inequality

$$F(x^i, x'^i) \leq 1$$

is a convex figure, so that the surface

$$F(x^i, x'^i) = 1 \tag{19}$$

is strictly convex and includes the point $x'^i = 0$ as an interior point. This surface is called the <u>Indicatrix</u>** at the point x^i, and may be regarded as the "unit sphere" in T_n as all the points on the surface (19) are equidistant from the origin O. If a vector $\overrightarrow{OP}$ cuts the indicatrix in Q and $\overrightarrow{OP} = m.\overrightarrow{OQ}$, it follows from (b) that the length of the vector $\overrightarrow{OP}$ is m. The set of vectors at O with the same **length** ℓ thus lie on the surface obtained from the indicatrix by a dilatation in the ratio ℓ:1 . In virtue of (b) this

*The proof given in Bonnesen and Fenchel presupposes a euclidean background, but the argument is valid for any affine space.

**The indicatrix was first introduced by Caratheodory ((17) and (18)).

surface is homothetic to the indicatrix and will be called the indicatrix of radius ℓ .

According to the usual procedure in affine geometry we may write the equation to the tangent-plane at a fixed point $x'_{(0)}$ of the surface (19) in the form

$$(x'^i - x'^i_{(0)}) \frac{\partial F(x^k, x'^k_{(0)})}{\partial x'^i} = 1,$$

or, with the aid of (6) and (19)

$$x'^i \frac{\partial F(x^k, x'^k_{(0)})}{\partial x'^i} = 1. \qquad (20)$$

Differentiating (7) with respect to x'^i, and taking into account (19) once more, this becomes

$$x'^i \frac{\partial \varphi(x, x'_{(0)})}{\partial x'^i} = 1 \qquad (20)'$$

Put $k = 1$ in (9) and differentiate with respect to x'^j :

$$\frac{\partial^2 \varphi(x, x')}{\partial x'^i \partial x'^j} x'^i + \frac{\partial \varphi(x, x')}{\partial x'^i} \delta^i_j = 2 \frac{\partial \varphi(x, x')}{\partial x'^j},$$

or, from (13)

$$g_{ij}(x, x') x'^i = \frac{\partial \varphi(x, x')}{\partial x'^j}. \qquad (21)$$

We may thus write the equation (20)' to the tangent-plane at $x'_{(0)}$ in the form

$$g_{ij}(x, x'_{(0)}) x'^i_{(0)} x^j = 1. \qquad (22)$$

Similarly, the equation of the tangent-plane at the point $x'_{(0)}$ of the indicatrix of radius ℓ is given by

$$g_{ij}(x, x'_{(0)}) x'^i_{(0)} x^j = \ell^2. \qquad (22)'$$

From the purely geometrical point of view it may seem strange that we use derivatives of the second order for the tangent-plane where it is customary to use only first order derivatives, but it will be seen that this has considerable analytical advantages.
Furthermore, it is clear that equation (22) also represents the tangent-plane at the point $x'^i_{(0)}$ to the quadric surface

$$g_{ij}(x, x'_{(0)}) x'^i x'^j = 1 \qquad (23)$$

This surface is called the <u>osculating indicatrix</u> corresponding to the direction $x'_{(0)}$ in T_n. This notion was first introduced by Finsler ((1), p. 42) - it defines a Riemannian metric in the direction $x'_{(0)}$. We shall show later that as a consequence of our convexity assumptions these surfaces are (n - 1)-dimensional ellipsoids.

§ 4. The Figuratrix.

An equation of the type $\alpha_i x'^i =$ const. represents a plane in T_n : it will be seen that the quantities α_i are of a covariant nature. We shall consider the expression $y_i x'^i$, and we define

$$H(y_i) = \text{least upper bound}\,(y_i x'^i) \tag{24}$$

where x'^i assumes all possible values subject to the condition

$$F(x^i, x'^i) \leq 1 .$$

Thus for all points interior to and on the indicatrix we have

$$H(y_i) \geqslant y_i x'^i \tag{25}$$

Since we have assumed strict convexity of the indicatrix, it follows that for any particular set of values $y_i^{(0)}$ of y_i $(i = 1, 2, \ldots, n)$ the equation

$$y_i^{(0)} x'^i = H(y_i^{(0)}) \tag{26}$$

represents a tangent-plane to the indicatrix (actually a plane of support ("Stützebene") if the indicatrix is not strictly convex) touching it, say, at the point $x'^i_{(0)}$. It is clear from (24) that we may introduce the normalisation condition

$$H(y_i^{(0)}) = 1 \tag{27}$$

if $x'^i_{(0)}$ lies on the indicatrix.

This enables us to establish a one-one correspondence between the points x'^i on the indicatrix and corresponding values of the n quantities y_i. It is natural, therefore, to introduce a second tangent space T'_n at each point x^i in the Finsler space, such that the points in T'_n are represented by the y_i. To each of the points y_i corresponds a function $H(x^i, y_i)$, defined by (24), which satisfies the following conditions in T'_n :

(a) $H(x^i, y_i) > 0$ for $y_i \neq 0$, $H(x, 0) = 0$.
(b) $H(x^i, k y_i) = k H(x^i, y_i)$.
(c) $H(x^i, y_i^{(1)} + y_i^{(2)}) \leq H(x, y_i^{(1)}) + H(x, y_i^{(2)})$

These properties may be proved as in Bonnesen und Fenchel (16). (Compare the first footnote of p. 8).

It is to be noted that the right-hand side of (27) is, in general, to be replaced by ℓ, if the corresponding point $x_{(0)}^{\prime i}$ lies on the indicatrix of radius ℓ.

Then, as a result of these conditions, it follows as before that the set of points in T_n' satisfying the inequality

$$H(x^i, y_i) \leq 1$$

is a convex figure, and the surface

$$H(x^i, y_i) = 1 \qquad (28)$$

is strictly convex and contains the point $y_i = 0$ as an interior point. This surface is called the <u>Figuratrix</u> at the point x^i of our Finsler space. The function H may be regarded as the function of distance in T_n', so that the figuratrix represents the "unit sphere" in T_n'.

<u>**Note**</u>: The figuratrix was first introduced by Minkowski ((19), (20)), and its significance with regard to the Calculus of Variations noted in Caratheodory ((8), p. 246) and Hadamard ((21), p. 92). These authors presuppose a euclidean background in their treatment, and show that in this case the figuratrix is the polar reciprocal of the indicatrix with respect to the euclidean unit sphere. ((16), p. 28).

In order to establish the law of correspondence between points in T_n on the indicatrix and points in T_n' on the figuratrix, we find, on comparison of the equations (22) and (26) in conjunction with (27) for the tangent-planes to the indicatrix,

$$y_i^{(0)} = g_{ij}(x, x'_{(0)}) x'^j_{(0)} \qquad (29)$$

This law of transformation is not only restricted to points on the indicatrix, but is perfectly general for all points in T_n (and in T_n'), in virtue of the fact that the figuratrix of radius ℓ is homothetic with respect to the surface (28).

Using (21), we deduce from (29) that

$$y_i^{(0)} = \frac{\partial \phi(x, x'_{(0)})}{\partial x'^j} \tag{30}$$

From §2 it therefore follows that the y_i transform like the components of covariant vectors, so that if $\bar{y}_i$ is the transform of y_i in a new coordinate system, we have

$$\bar{y}_i = \frac{\partial x^j}{\partial \bar{x}^i} y_j , \tag{31}$$

so that

$$\frac{\partial \bar{y}_i}{\partial y_j} = \frac{\partial x^j}{\partial \bar{x}^i} \tag{31'}$$

As in the case of T_n, we may now introduce the functions $\psi(x,y)$ and $g^{ij}(x,y)$ in T'_n; these functions being defined analogously by the equations

$$2\psi(x,y) = [H(x,y)]^2 \tag{32}$$

and

$$g^{ij}(x,y) = \frac{\partial^2 \psi(x,y)}{\partial y_i \, \partial y_j} \tag{33}$$

Again, in view of the natural invariance of the function ψ we have

$$\bar{g}^{ij}(\bar{x}, \bar{y}) = \frac{\partial^2 \bar{\psi}(\bar{x}, \bar{y})}{\partial \bar{y}_i \, \partial \bar{y}_j} = \frac{\partial^2 \psi(x,y)}{\partial y_h \, \partial y_k} \cdot \frac{\partial y_h}{\partial \bar{y}_i} \frac{\partial y_k}{\partial \bar{y}_j}$$

or, using the inverse of (31)' together with (33)

$$\bar{g}^{ij}(\bar{x}, \bar{y}) = g^{hk}(x,y) \frac{\partial \bar{x}^i}{\partial x^h} \cdot \frac{\partial \bar{x}^j}{\partial x^k} \tag{34}$$

showing that the g^{ij} are components of a contravariant tensor of rank 2. It is clear that the functions $\psi(x, y)$ and $g^{ij}(x,y)$ enjoy the same homogeneity properties with respect to the y_k as the functions $\phi(x, x')$ and $g_{ij}(x,x')$ with respect to the x'^k.
We thus have for the points y_i on the figuratrix of radius ℓ

$$H^2(x,y) = 2\psi(x,y) = g^{ij}(x,y) y_i y_j = \ell^2 \tag{35}$$

To find the inverse of the transformation (29), we multiply each of these equations by $x'^i_{(0)}$ and on adding the resulting equations we obtain addition to the fact that they may also be regarded as the covariant components of a vector whose contravariant components are the x^i when one adopts the usual point of view of affine geometry. Great care must however be excercised when transformation laws such as (29) and (37) are used: the arguments in the functions g_{ij} and g^{ij} must correspond to the vector under consideration. For this reason the arguments will be stated explicitly whenever these functions appear.

Furthermore, our equations show clearly the affinity between the function of support $H(x^i, y_i)$ and the Hamiltonian associated with the fundamental function $F(x^i, x'^i)$ of the Calculus of Variations. <u>The value of the figuratrix for a geometrical interpretation of the Hamilton-Jacobi Theory thus becomes clearly evident.</u>

$$x'^{i}_{(0)} y^{(0)}_i = \ell^2 \tag{36}$$

when we take into account equations (13), (10) and (7). Using (35), we immediately deduce

$$x'^{i}_{(0)} = g^{ij}(x, y^{(0)}) y^{(0)}_j \tag{37}$$

Also, substituting from equations (29) and (37) in (36) it follows that

$$x'^{i}_{(0)} y^{(0)}_i = g^{ij}(x, y^{(0)}) g_{ih}(x, x'_{(0)}) x'^{h}_{(0)} y^{(0)}_j ,$$

so that

$$g^{ij}(x, y^{(0)}) g_{ih}(x, x'_{(0)}) = \delta^j_h . \tag{38}$$

The functions $g^{ij}(x, y^{(0)})$ are thus the cofactors of the elements $g_{ij}(x, x'_{(0)})$ in the determinant $|g_{ij}(x, x'_{(0)})|$ and <u>the relation between the functions $F(x, x'_{(0)})$ and $H(x, y^{(0)})$ is therefore completely symmetrical.</u>

On differentiating equation (35) with respect to y_k, one immediately finds

$$2\frac{\partial \psi(x,y)}{\partial y_k} = \frac{\partial g^{ij}(x,y)}{\partial y_k} y_i y_j + 2 g^{ij}(x,y) y_i \delta^j_k$$

when it is observed that the g^{ij} are symmetrical in i and j. The first term on the right-hand side of this equation vanishes identically as a result of the homogeneity properties of the g^{ij}, so that

$$\frac{\partial \psi(x,y)}{\partial y_k} = g^{ik}(x,y) y_i \tag{39}$$

or, using(37)

$$x'^{k}_{(0)} = \frac{\partial \psi(x, y^{(0)})}{\partial y^k} \tag{40}$$

If, in particular, the vector $x'_{(0)}$ is a unit vector, these equations may be written in the form

$$x'^{k}_{(0)} = \frac{\partial H(x, y^{(0)})}{\partial y_k} \tag{41}$$

and

$$y^{(0)}_k = \frac{\partial F(x, x'_{(0)})}{\partial x'^k} \tag{42}$$

The functions y_k thus appear as the <u>canonical variables in the Calculus of Variations</u>, (Caratheodory, (8), p. 216), in

CHAPTER II

SIMPLE GEOMETRICAL RESULTS AND THE MEASURE OF ANGLE IN MINKOWSKIAN SPACES

§5. Definition of cosine.

In the following sections we shall be concerned solely with Minkowskian spaces, so that the g_{ij} depend only on the directional arguments. Most of the results derived in this chapter are of very little interest in themselves and are deduced for future use.

We shall now define the cosine corresponding to two arbitrary directions $\vec{\lambda}$ and $\vec{\mu}$. This function is not to be regarded as a measure of angle but only as a function of two directions. Throughout we shall represent the length of the vector $\vec{a}$ by a. Representing $\vec{\lambda}$ and $\vec{\mu}$ by the straight line OP, OQ respectively, we construct the indicatrices of radius λ and μ, each passing through one of the points P and Q, and being represented by the equations

$$\left.\begin{aligned} \lambda^2 &= g_{ij}(x')x'^i x'^j \\ \mu^2 &= g_{ij}(x')x'^i x'^j \end{aligned}\right\} \tag{1}$$

According to (22)', §3, the equation to the tangent-plane at P is

$$g_{ij}(\lambda)\lambda^i x'^j = \lambda^2. \tag{2}$$

Produce OQ to meet this plane at Q'. We then define

$$\cos(\vec{\lambda},\vec{\mu}) = \pm \frac{|OP|}{|OQ'|} \tag{3}$$

the negative sign being taken only when $\vec{\lambda}$ and $\vec{\mu}$ lie on opposite sides of the plane through O parallel to the plane (2). Representing OQ' by the vector $r\vec{\mu}$ where r is a scalar, it follows from (2) that r is given by

$$g_{ij}(\lambda)\lambda^i\mu^j = \frac{\lambda^2}{r} \tag{4}$$

and hence, using (3)

$$\cos(\lambda,\mu) = \frac{g_{ij}(\lambda)\lambda^i\mu^j}{\sqrt{g_{ij}(\lambda)\lambda^i\lambda^j \cdot g_{hk}(\mu)\mu^h\mu^k}} \tag{5}$$

Similarly, if we draw the tangent-plane to the second indicatrix at the point Q, we find

$$\cos(\mu,\lambda) = \frac{g_{ij}(\mu)\lambda^i\mu^j}{\sqrt{g_{ij}(\lambda)\lambda^i\lambda^j \cdot g_{hk}(\mu)\mu^h\mu^k}} \tag{6}$$

Thus only when the g_{ij} are independent of direction will the function $\cos(\lambda,\mu)$ be symmetric in $\vec{\lambda}$ and $\vec{\mu}$, i.e. only when we deal with a euclidean metric, in which case (5) reduces to a well-known form of the euclidean cosine.

In virtue of (14), §2 we see immediately that $\cos(\lambda,\mu)$ is independent of the lengths λ and μ. It is therefore simply a function of direction. If we introduce the Weierstrass $\mathcal{E}$-function for two directions x'^i, ξ'^i (Caratheodory (8), p. 223) defined by

$$\mathcal{E}(x^i, x'^i, \xi'^i) = F(x^k, \xi'^k) - \frac{\partial F(x^k, x'^k)}{\partial x'^j}\xi'^j$$

we have for two unit vectors $\vec{\lambda}$, $\vec{\mu}$ in virtue of (42) and (29) of §4

or

$$\mathcal{E}(\lambda,\mu) = 1 - \lambda_i\mu^i = 1 - g_{ij}(\lambda)\lambda^j\mu^i$$

or,

$$\mathcal{E}(\lambda,\mu) = 1 - \cos(\lambda,\mu) \tag{7}$$

This result was also obtained by Finsler ((1), p. 39) starting from a different definition of cosine. He then used this equation to deduce a measure of angle with the aid of the classical series expansion of the cosine. We shall not follow this procedure.

We may also derive immediately the well-known result of Caratheodory ((1), p. 244), where, however, the indicatrix is embedded in a Euclidean space.

For the unit vectors $\vec{\lambda}$ and $\vec{\mu}$ the corresponding points P,Q lie on the indicatrix. Draw QQ'' parallel to OP to meet PQ' in Q''. Then we have from (7) and (4)

$$\mathcal{E}(\lambda,\mu) = \frac{r-1}{r} = \frac{QQ'}{OQ'} = \frac{QQ''}{OP}$$

This is equivalent to the geometrical interpretation of Caratheodory of the $\mathcal{E}$-function.

Other definitions for the cosine were given by Berwald ((5), p. 217; (22), p. 56); Cartan (4) and Synge ((2), p. 65-67). Apart from one of the two definitions seggested by Synge, these are all identical and differ only from the equation (5) in so far as that the g_{ij} have the same directional argument, i.e. they are referred to the same osculating indicatrix.

In an analogous manner we may define the cosine with respect to the figuratrix: denoting the vectors in T'_n corresponding to the vectors $\vec{\lambda}$ and $\vec{\mu}$ by $\underline{\vec{\lambda}}$ and $\underline{\vec{\mu}}$, and the corresponding cosine by $\underline{\cos}(\lambda,\mu)$, we find as before

$$\underline{\cos}(\lambda,\mu) = \frac{g^{ij}(\underline{\lambda})\lambda_i\mu_j}{\lambda\mu} = \frac{\lambda^i\mu_i}{\lambda\mu}$$

It follows from (6) that

$$\underline{\cos}(\lambda,\mu) = \cos(\mu,\lambda) \quad . \tag{8}$$

and, conversely,

$$\underline{\cos}(\mu,\lambda) = \cos(\lambda,\mu) \tag{8'}$$

The difference between the indicatrix and figuratrix thus accounts for the lack of symmetry of the cosine.

§ 6. Transversality.

Any vector $\vec{x}'$ lying in the tangent-plane at the point $x'^i_{(0)}$ of the indicatrix is said to be <u>orthogonal with respect to</u> $\vec{x}'_{(0)}$. From (3) it follows that the cosine vanishes:

$$\left.\begin{aligned} g_{ij}(x'_{(0)})\,x'^i_{(0)}\,x'^j &= 0 \\ \text{or} \qquad y^{(0)}_i\,x'^i &= 0 \end{aligned}\right\} \tag{9}$$

in virtue of (29), §4. This is identical to the notion of <u>transversality</u> as given in Caratheodory ((8), p. 248).

The above terminology is justified by the fact that if the equation

$$y^{(0)}_i x'^i = \ell^2 \tag{10}$$

represents the tangent-plane to the indicatrix of radius ℓ at the point $x'^i_{(0)}$, the shortest distance from any point to this plane is measured along a direction parallel to the vector $\vec{x}'_{(0)}$. Geometrically this result is obvious and may be proved analytically as follows: Without loss of generality we may take this point to be the origin due to the affine structure of our space. If $P(x'^i)$ is any point on the plane, its distance from O is given by

$$x'^2 = g_{ij}(x')x'^i x'^j \tag{11}$$

According to the multiplier rule of Lagrange, we have for an extreme value of (11)

$$\frac{\partial}{\partial x'^k}\left[g_{ij}(x')x'^i x'^j + 2k(y^{(0)}_i x'^i - \ell^2)\right] = 0$$

where k is the undetermined multiplier. Denoting the roots (if any) by $x'^i_{(1)}$, we find

$$\frac{\partial g_{ij}(x'_{(1)})}{\partial x'^k} x'^i_{(1)} x'^j_{(1)} + 2g_{ij}(x'_{(1)})x'^i_{(1)}\delta^j_k + 2k y^{(0)}_j \delta^j_k = 0$$

where the first term vanishes by (15), §2. Thus

$$g_{ij}(x'_{(1)})x'^i_{(1)} = -k y^{(0)}_j \tag{12}$$

Multiplying this equation by $x'^j_{(1)}$ and summing over j , it follows from (10) and (11) that

$$x'^2_{(1)} = -k\ell^2$$

so that we may write (12) in the form

$$\frac{g_{ij}(x'_{(1)})x'^i_{(1)}}{x'^2_{(1)}} = \frac{g_{ij}(x'_{(0)})x'^i_{(0)}}{\ell^2}$$

Multiplying this equation by $x'^j_{(0)}$ and again summing over j , we thus obtain

$$g_{ij}(x'_{(1)})x'^i_{(1)}x'^j_{(0)} = x'^2_{(1)}$$

so that

$$x'^j_{(0)} = x'^j_{(1)}$$

i.e. the solution must be the orthogonal vector. Geometrically it is clear that a finite extreme value can only be a minimum.

It is to be observed from the definition of orthogonality that this is, in general, not a symmetrical property: when we write

$$\vec{\lambda} \perp \vec{\mu}$$

we imply that $\vec{\lambda}$ is orthogonal with respect to $\vec{\mu}$. On the figuratrix the position is once more reversed.

§ 8. Convexity of the Indicatrix.

On the indicatrix

$$g_{ij}(x')x'^i x'^j = 1 \tag{17}$$

we consider a fixed point $P(x'_{(0)})$; the tangent-plane at that point is then given by

$$y_i^{(0)} x'^i = 1 \tag{18}$$

where $\vec{y}^{(0)}$ is the vector in T_n' corresponding to $\vec{x}'_{(0)}$. We shall now calculate the distance p of a neighbouring point $Q(x'^i_{(0)} + dx'^i + \frac{1}{2}d^2x'^i)$ on the indicatrix from the plane (18). Differentiating (17) with respect to x', we find

$$\frac{\partial g_{ij}(x')}{\partial x'^k} x'^i x'^j dx'^k + 2g_{ij}(x')x'^i dx'^j = 0 \tag{19}$$

where the first term vanishes in virtue of (15), §2, so that the vector dx' is orthogonal with respect to x', i.e. it lies in the tangent-plane. Writing (19) in the form

$$y_i dx'^i = 0 \tag{19'}$$

a second differentiation gives

$$dy_i dx'^i + y_i d^2x'^i = 0 \tag{20}$$

Noting that $H(y_i) = 1$ for points on the figuratrix, we deduce from (16) and (18)

$$p = -[y_i^{(0)}(x'^i_{(0)} + dx'^i + \tfrac{1}{2}d^2x'^i)] + 1$$

or, using (19)' and (20)

$$p = \tfrac{1}{2} dy_i dx'^i \tag{21}$$

But from (29), §2 we have

$$dy_i = \frac{\partial g_{ij}(x')}{\partial x'^k} x'^j dx'^k + g_{ij}(x')dx'^j$$

where again the first term vanishes*. Thus

$$p = \tfrac{1}{2} g_{ij}(x'_{(0)}) dx'^i dx'^j \tag{21'}$$

This equation enables us to express the assumption of the convexity of the indicatrix analytically as follows:
In order that the indicatrix be strictly convex at all points ,
the quadratic form

* If y_i in T_n' corresponds to x'^i, the vector $\vec{dy}$ corresponds to $\vec{dx'}$ if and only if $\vec{dx'}$ has the same direction as $\vec{x'}$.

$$g_{ij}(x')\,\xi^i \xi^j \tag{22}$$

must be positive definite subject to the linear condition

$$y_i \xi^i = 0 \,. \tag{23}$$

From the theory of quadratic forms (Caratheodory (8), Ch. XI) we deduce that this requires the condition that the determinant

$$-\begin{vmatrix} g_{ij}(x') \,, & y_i \\ y_j \,, & 0 \end{vmatrix}$$

must be positive and non-vanishing. In virtue of equations (13), §2, and (42), §4, we see that this is equivalent to the result anticipated in assumption (C) (p. 5).

Furthermore, as the convexity of the indicatrix implies the convexity of the function $\phi(x')$, we may use a well-known theorem on convex functions (Hardy, Littlewood and Polya, (22), p. 80) to deduce that the quadratic form

$$\frac{\partial^2 \phi(x')}{\partial x'^i \, \partial x'^j}\, \xi^i \xi^j$$

be strictly positive, i.e.

$$g_{ij}(x')\,\xi^i \xi^j > 0 \tag{24}$$

for all ξ^i whatever the argument in the g_{ij} may be. Hence the determinant $|g|$* formed by the g_{ij} and the principal minors of $|g|$ are positive and non-vanishing for all directions x'. From (24) it follows that under the assumptions (A) - (D) the osculating indicatrices are all ellipsoids.

§ 9. Angular Measure.

We shall now introduce a further assumption commonly made in the Calculus of Variations:

(E) : The value of the integral

$$\int_{t_0}^{t_1} F(x^i, x'^i)\,dt$$

is independent of the direction of integration along the

*This fact was proved along different lines by Taylor ((11), p. 262).

curve under consideration.

Then, apart from condition (5), §1, we must have (Caratheodory (8), p. 214)

$$F(x^i, x'^i) = F(x^i, -x'^i)$$

so that for $k<0$ the relation

$$F(x^i, kx'^i) = -k\, F(x^i, x'^i)$$

is satisfied, and therefore

$$g_{ij}(x^i, x'^i) = g_{ij}(x^i, -x'^i) \tag{25}$$

This ensures the <u>symmetry of the indicatrix with respect to the origin</u>.

In the case of two-dimensional Minkowskian spaces we shall define the angle between two vectors $\vec{\lambda}$ and $\vec{\mu}$ of length ℓ to be the shorter of the two lengths of the indicatrix of radius ℓ cut off by the vectors $\vec{\lambda}$ and $\vec{\mu}$. Thus, if x'^i and $x'^i + dx'^i$ are two vectors, infinitesimally close together, the angle $d\theta$ between them is defined by

$$d\theta = \frac{1}{\ell}\left[g_{ij}(dx')\,dx'^i\,dx'^j\right]^{1/2} \tag{26}$$

In the case of an n-dimensional space the vector $\vec{x}'$ whose end-point passes along the indicatrix from $\vec{\lambda}$ to $\vec{\mu}$ must, for the sake of uniqueness, be constrained to lie in the two-dimensional sub-space defined by $\vec{\lambda}$ and $\vec{\mu}$; i.e. $\vec{x}'$ must satisfy a subsidiary relation of the form

$$\vec{x}'(t) = a(t)\,\vec{\lambda} + b(t)\,\vec{\mu} \qquad (\lambda = 1,\ \mu = 1) \tag{27}$$

where $a(t)$ and $b(t)$ are continuous functions of the parameter t. Also, since $\vec{x}'(t)$ remains on the indicatrix, we must introduce a normalisation factor for $a(t)$, $b(t)$. Thus, if we suppose that $t=0$ when $\vec{x}' = \vec{\lambda}$ and $t=1$ when $\vec{x}' = \vec{\mu}$, we have

$$\left.\begin{array}{ll} a(0) = 1, & b(0) = 0, \\ a(1) = 0, & b(1) = 1, \end{array}\right\}$$

and consequently a satisfactory form of (27) would be

$$x'^i(t) = \frac{(1-t)\lambda^i + t\mu^i}{|(1-t)\lambda^i + t\mu^i|} \qquad (\lambda = 1,\ \mu = 1) \tag{27'}$$

where the denominator is the length of the vector $(1-t)\vec{\lambda} + t\vec{\mu}$, which ensures that $\vec{x}'(t)$ is a unit vector. From (27)' we can calculate $dx'^i(t)/dt$ as a function of t, so that we can define the angle by the integral

$$\theta = \int_0^t \sqrt{g_{ij}\left(\frac{dx'}{dt}\right)\frac{dx'^i}{dt}\cdot\frac{dx'^j}{dt}}\; dt \qquad (28)$$

in which these values must be substituted.

We note the following remarks:

(1) The angle (26) is independent of the radius of the indicatrix. This follows immediately from equation (14) of §2.

(2) The postulate of additivity is fulfilled.

(3) The sum of the angles of a triangle is equal to half the length of the circumference of the unit indicatrix. This is easily verified by means of the symmetry condition (25).

(4) Normalisation: Since the length of the circumference of an indicatrix is proportional to its radius, we deduce that for each two-dimensional subspace there exists a constant

$$\frac{1}{2}\cdot\frac{\oint\sqrt{g_{ij}(dx')\,dx'^i\,dx'^j}}{\sqrt{g_{ij}(x')\,x'^i\,x'^j}} \qquad (29)$$

where the integral is to be taken around the indicatrix and the vector $\vec{x}'$ constrained to lie within the sub-space under consideration. This is the generalisation of the number π. It is clear that for different sub-spaces this number will vary; in particular, the angle corresponding to two parallel opposite directions is indeterminate unless a sub-space has been fixed previously for the purpose of measurement. Also, even in a two-dimensional Finsler space the value of (28) is fixed at each point, but will vary as we pass to a neighbouring point.

These disadvantages are not as serious as they might appear at first sight, as in Differential Geometry we are mainly concerned with small angles. Furthermore, we can introduce what we shall call the <u>normalised angle</u>, which is defined to be the length of the indicatrix-arc as defined by (28) subject to (27)', divided by the total length of the circumference of the indicatrix. Then the angle corresponding to two parallel opposite directions has the fixed value 1/2 throughout the Finsler Space. We shall use both notions of angle as defined here, but when the normalised angle is used this will be stated explicitly.

Note: Other definitions of angle are given by Finsler (1), Cartan (4), Bliss (24), Landsberg (25)*. Finsler's definition has been mentioned in §5; it does not satisfy the postulate of additivity. For small angles those given by Cartan and Landsberg depend only on the properties of the indicatrix in the immediate neighbourhood of the directions concerned: i.e. the geometrical interpretation is given by the osculating indicatrices, in contrast to the definition given here, where the measurement of an element of arc of the indicatrix in the neighbourhood of $\vec{x}'$ requires a comparison with the length of a vector othogonal with respect to $\vec{x}'$.

Finally, the definition as given by Bliss (given for two dimensions) is intrinsic, but his calculations involve the use of the classical euclidean background; but it will be clear that for two dimensions our definition as given above will coincide with that of Bliss.

Special mention must be made of the article by Busemann (27) on angular measure and integral curvature in metric manifolds. It is pointed out that although the angular measure must be additive for angles with the same vertex and that straight angles should have a fixed measure, "an insistence on particular measures ..... makes the concept of angular measure barren and unnatural". The results obtained by Busemann in this paper amply justify his opinion.

However, in the present work it has been found unavoidable to introduce a particular angular measure, in view of the fact

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*Compare the survey of St. Golab (26).

that the notion of angle has been used to define the curvature of curves both in Minkowskian and Finsler spaces. Both these concepts were necessary for the derivation of an analogue of the Gauss-Bonnet Theorem (Ch. VIII) for Finsler spaces, as well as other applications to Differential Geometry. The possibility of a derivation of a similar analogue of this theorem for a less restricted form of angular measure is not entirely out of the question and should be investigated. Finally, it may be noted that the normalised angle as defined here certainly satisfies the initial demands made by Busemann for general angular measure.

CHAPTER III

THE TRIGONOMETRY OF MINKOWSKIAN SPACES

§ 10. Definition of Sine.

In §5 we defined the cosine of two directions $\vec{\lambda}$ and $\vec{\mu}$; in this section we shall define the corresponding sine and proceed to a systematic study of these functions.

Referring to the notation of §5 we can measure the length of the vector PQ' by means of the vector $\vec{\nu}$ parallel to PQ' and passing through the origin, this vector being the diagonal of the parallelogram formed by the vectors $-\vec{\lambda}$ and $r\vec{\mu}$. Here it is necessary to observe that in view of our symmetry condition the vectors $\vec{\lambda}$ and $-\vec{\lambda}$ have the same length. We may then write

$$\vec{\nu} = r\vec{\mu} - \vec{\lambda} \tag{1}$$

so that

$$|PQ'|^2 = g_{ij}(\nu)(r\mu^i - \lambda^i)(r\mu_j - \lambda^j) \tag{2}$$

We then define the sine corresponding to the two directions $\vec{\lambda}$ and $\vec{\mu}$ by the equation

$$\sin(\vec{\lambda},\vec{\mu}) = \pm\frac{|PQ'|}{|OQ'|} \tag{3}$$

the positive sign being taken when the rotation from O to P to Q' is counter-clockwise. Equation (2) then gives

$$\sin^2(\vec{\lambda},\vec{\mu}) = \frac{1}{\mu^2}\left[g_{ij}(\nu)\mu^i\mu^j - \frac{2}{r}g_{ij}(\nu)\lambda^i\mu^j + \frac{1}{r^2}g_{ij}(\nu)\lambda^i\lambda^j\right] \tag{4}$$

Now from (4) and (5) of §5 we deduce

$$\frac{1}{r} = \frac{\mu\cos(\vec{\lambda},\vec{\mu})}{\lambda} \tag{5}$$

so that we finally have

$$\sin^2(\vec{\lambda},\vec{\mu}) = \frac{g_{ij}(\nu)\mu^i\mu^j}{\mu^2} - 2\frac{g_{ij}(\nu)\lambda^i\mu^j}{\lambda\mu}\cos(\vec{\lambda},\vec{\mu}) + \frac{g_{ij}(\nu)\lambda^i\lambda^j\cos^2(\vec{\lambda},\vec{\mu})}{\lambda^2} \tag{6}$$

Again, this expression is independent of the lengths of $\vec{\lambda}$ and $\vec{\mu}$;

if, for instance we replace $\vec{\mu}$ by $k\vec{\mu}$ it follows from (5) that r changes such that the direction of $\vec{\nu}$ remains unchanged since the function $\cos(\lambda,\mu)$ is independent of the length of μ. Thus the $g_{ij}(\nu)$ are unaffected (equation (14), §2) and our statement then follows directly from equation (6)

In the special case where the g_{ij} are independent of direction equation (6) simply reduces to

$$\sin^2(\vec{\lambda},\vec{\mu}) = 1-\cos^2(\vec{\lambda}.\vec{\mu})$$

and we may therefore regard it as a generalisation of this well-known identity of euclidean trigonometry. Also, when $\vec{\mu}$ is orthogonal with respect to $\vec{\lambda}$, it follows from (5) that $r\to\infty$ i.e. the direction of $\vec{\nu}$ tends to the direction of $\vec{\mu}$ and we can deduce from (6) that the orthogonality condition can be expressed in the form

$$\sin(\vec{\lambda},\vec{\mu}) = \pm 1$$

Finally we note that the sine as a function of two directions is not symmetrical.

§ 11. Addition formulae for sine and cosine.

The Minkowskian space being affine, we may make use of the euclidean notions of proportionality, especially for the case of proportional triangles. In general, however, one cannot transpose the proofs for formulae in Euclidean Geometry to the Minkowskian case, as the concept of orthogonality cannot be applied in a similar manner. We shall now deduce the addition formulae for sine and cosine purely geometrically, as an analytical deduction would be extremely difficult.

Without loss of generality, we may consider three vectors $\vec{\lambda}$, $\vec{\mu}$, $\vec{\nu}$, having the same length and lying in the same plane, and we shall represent them by OP, OQ, OR respectively (Figure 2). The order in which they are taken is essential and we shall regard them as being counter-clockwise.

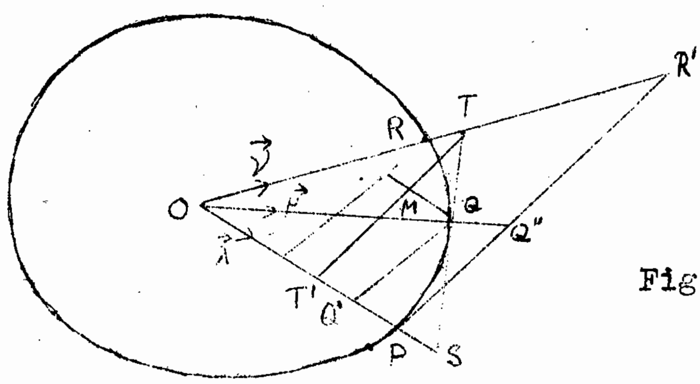

Fig. 2

Let OQ, OR produced meet the tangent to the indicatrix at P in Q'', R' respectively; similarly OP and OR meet the tangent at Q in S and T. Draw QQ',TT' parallel to PR' meeting OP in T', Q'. Then triangle OTT' is similar to triangle OR'P and we therefore have

$$\cos(\lambda,\nu) = \frac{OP}{OR'} = \frac{OT'}{OT} = \frac{OQ' - T'Q'}{OT}$$

If we draw QM parallel to PO to meet TT' in M, we can express this in the form

$$\cos(\lambda,\nu) = \frac{OQ'}{OT} - \frac{MQ}{OT} = \frac{OQ'}{OQ}\cdot\frac{OQ}{OT} - \frac{MQ}{OT}$$

From the similarity of triangles OQ'Q, OPQ'' it then follows that

$$\cos(\lambda,\nu) = \cos(\lambda,\mu)\cos(\mu,\nu) - \frac{MQ}{OT} \tag{7}$$

Also, since the triangles TMQ, QQ'S are similar, we have

$$\frac{MQ}{TQ} = \frac{Q'S}{QS}$$

and remembering that

$$\sin(\mu,\lambda) = \frac{QS}{OS}$$

we deduce

$$\frac{MQ}{OT} = \frac{Q'S}{OS}\cdot\frac{1}{\sin(\mu,\lambda)}\cdot\frac{TQ}{OT} = \frac{\sin(\mu,\nu)}{\sin(\mu,\lambda)}\cdot\frac{Q'S}{OS}$$

$$= \frac{\sin(\mu,\nu)}{\sin(\mu,\lambda)}\cdot\frac{OS - OQ'}{OS}$$

$$= \frac{\sin(\mu,\nu)}{\sin(\mu,\lambda)}\left(1 - \frac{OQ'}{OQ}\,\frac{OQ}{OS}\right)$$

$$= \frac{\sin(\mu,\nu)}{\sin(\mu,\lambda)}\left(1 - \cos(\lambda,\mu)\cos(\mu,\lambda)\right)$$

Using (7) we finally have the addition-formula

$$\cos(\lambda,\nu) = \cos(\lambda,\mu)\cos(\mu,\nu) - \frac{\sin(\mu,\nu)}{\sin(\mu,\lambda)}\left[1 - \cos(\lambda,\mu)\cos(\mu,\lambda)\right] \tag{8}$$

It is clear that when the sine and cosine are symmetrical this result will reduce to the classical euclidean formula.

To find the addition-formula for the sine, we proceed in a similar manner. We have

$$\sin(\lambda,\nu) = \frac{PR'}{OR'} = \frac{T'T}{OT} = \frac{T'M + MT}{OT}$$

$$= \frac{Q'Q}{OT} + \frac{MT}{OT}$$

$$= \frac{Q'Q}{OT} + \frac{MT}{QT}\cdot\frac{QT}{OT}$$

$$= \sin(\lambda,\mu)\cos(\mu,\nu) + \frac{MT}{QT}\sin(\mu,\nu) \tag{9}$$

But from the similarity of triangles TMQ,QQ'S we deduce

$$\frac{MT}{QT} = \frac{Q'Q}{SQ} = \frac{Q'Q}{OS}\cdot\frac{OS}{SQ} = \frac{Q'Q}{OS}\cdot\frac{1}{\sin(\mu,\lambda)}$$

$$= \frac{Q'Q}{OQ}\cdot\frac{OQ}{OS}\cdot\frac{1}{\sin(\mu,\lambda)}$$

$$= \frac{\sin(\lambda,\mu)\cos(\mu,\lambda)}{\sin(\mu,\lambda)}$$

so that in virtue of (9) we finally have

$$\sin(\lambda,\nu) = \sin(\lambda,\mu)\cos(\mu,\nu) + \frac{\sin(\lambda,\mu)}{\sin(\mu,\lambda)}\cos(\mu,\lambda)\sin(\mu,\nu) \tag{10}$$

or alternatively,

$$\sin(\lambda,\nu) = \frac{\sin(\lambda,\mu)}{\sin(\mu,\lambda)}\left[\sin(\mu,\lambda)\cos(\mu,\nu) + \cos(\mu,\lambda)\sin(\mu,\nu)\right] \tag{10'}$$

Again, as in the case of (9), this reduces to the ordinary euclidean formula when there is symmetry.

With the aid of equations (9) and (10) we can easily find generalisations of formulae of euclidean trigonometry; for instance, if we suppose $\vec{\nu}$ to be orthogonal with respect to $\vec{\mu}$ we would have

$$\sin(\lambda,\nu) = \frac{\sin(\lambda,\mu)}{\sin(\mu,\lambda)}\cos(\mu,\lambda)$$

corresponding to the identity sin(90 + A) = cosA .

If the vector $\vec{\delta\lambda}$ is such that $\vec{\lambda}+\vec{\delta\lambda}$ has the same length as $\vec{\lambda}$, we deduce from equation (18) of §8 that $\vec{\delta\lambda}$ is orthogonal with respect to $\vec{\lambda}$, i.e. $g_{ij}(\lambda)\lambda^i\,\delta\lambda^j$ is of the second order of smallness. In this case the last term in the expression (12) will be of the fourth order of smallness, so that we may write for vectors on the same indicatrix

$$\cos(\lambda,\lambda+\delta\lambda) = 1-\frac{1}{2\lambda^2}g_{ij}(\lambda)\delta\lambda^i\,\delta\lambda^j+\cdots\cdots \tag{13}$$

From this it follows that up to terms of second order

$$\cos(\lambda+\delta\lambda,\lambda) = \cos(\lambda,\lambda+\delta\lambda)$$

If $\delta\theta$ is the angle between the vectors, we have from the definition of angle

$$\lambda^2\,\delta\theta^2 = g_{ij}(\delta\lambda)\delta\lambda^i\,\delta\lambda^j \tag{14}$$

so that the classical formula

$$\cos(\delta\theta) = 1-\tfrac{1}{2}(\delta\theta)^2+\cdots\cdot$$

could only be transposed if $g_{ij}(\lambda)=g_{ij}(\delta\lambda)$; which brings out clearly the difference between our angle and that of Finsler, who assumed this expansion.

If, in the notation of §5, we denote the infinitesimal vector $\vec{PQ'}$ by $\vec{\Delta\nu}$, our definition (3) gives

$$\sin(\lambda,\lambda+\delta\lambda) = \frac{|\vec{\Delta\nu}|}{r\,|\vec{\lambda}+\vec{\delta\lambda}|} \tag{15}$$

It follows from (5) that when $\vec{\delta\lambda}\to 0$, i.e. when $\cos(\lambda,\lambda+\delta\lambda)\to 1$ that then $r\to 1$. In virtue of (2) we then have

$$\lim_{\vec{\delta\lambda}\to 0}\frac{|\vec{\Delta\nu}|}{|\vec{\Delta\lambda}|} = \lim_{r\to 1}\frac{\left[g_{ij}(\lambda(r-1)+r\delta\lambda)(\lambda^i(r-1)+r\delta\lambda^i)(\lambda^j(r-1)+r\delta\lambda^j)\right]^{\frac{1}{2}}}{|\vec{\delta\lambda}|}$$

$$= \frac{\left[g_{ij}(\delta\lambda)\delta\lambda^i\,\delta\lambda^j\right]^{1/2}}{|\vec{\delta\lambda}|}$$

$$= 1 \tag{16}$$

From (15) we deduce

$$\sin(\lambda,\lambda+\delta\lambda) = \frac{|\vec{\delta\lambda}|}{r\,|\vec{\lambda}+\vec{\delta\lambda}|}\cdot\frac{|\vec{\Delta\nu}|}{|\vec{\delta\lambda}|}$$

so that

$$\sin(\lambda,\lambda+\delta\lambda) = \delta\theta+\cdots\cdot \tag{17}$$

on account of (14). AAgain, for small angles the function

§ 12. Sine and Cosine of two neighbouring directions.

We shall consider two vectors $\vec{\lambda}$ and $\vec{\lambda}+\vec{\delta\lambda}$ such that the angle between them is of the first order of smallness, and we shall evaluate the corresponding expansions of the sine and cosine up to terms of at least the first order, from which we can derive the differential coefficients for the general sine and cosine.

From equation (5) of §5 we have

$$\cos(\lambda, \lambda+\delta\lambda) = \frac{g_{ij}(\lambda)\lambda^i(\lambda^j+\delta\lambda^j)}{\lambda(\lambda+\delta\lambda)} \tag{11}$$

where $\delta\lambda$ is the difference in length between the two vectors, (which must not be confused with the length of the vector $\vec{\delta\lambda}$, which we shall denote by $|\delta\lambda|$). Thus

$$\cos(\lambda, \lambda+\delta\lambda) = (\lambda^2 + g_{ij}(\lambda)\lambda^i\delta\lambda^j)\frac{1}{\lambda(\lambda+\delta\lambda)} \tag{11$'$}$$

It is clear that we must first obtain an expansion for $(\lambda+\delta\lambda)^{-1}$. From the definition

$$\frac{1}{\lambda} = (g_{ij}(\lambda)\lambda^i\lambda^j)^{-\frac{1}{2}}$$

we have

$$\frac{\partial}{\partial\lambda^k}\left(\frac{1}{\lambda}\right) = -\frac{1}{\lambda^3}\, g_{ik}(\lambda)\lambda^i$$

and

$$\frac{\partial^2}{\partial\lambda^h\,\partial\lambda^k} = \frac{3}{\lambda^5}\, g_{ik}(\lambda)\, g_{jh}(\lambda)\lambda^i\lambda^j - \frac{1}{\lambda^3}\, g_{hk}(\lambda)$$

where we have taken the homogeneity-condition (15) of §2 into account for each differentiation. We therefore obtain

$$\begin{aligned}\frac{1}{\lambda+\delta\lambda} &= \frac{1}{\lambda} - \frac{1}{\lambda^3}\, g_{ik}(\lambda)\lambda^i\delta\lambda^k + \frac{1}{2}\Big(-\frac{1}{\lambda^3}\, g_{hk}(\lambda)\,\delta\lambda^h\delta\lambda^k \\ &\qquad + \frac{3}{\lambda^5}\, g_{ik}(\lambda)\, g_{jh}(\lambda)\lambda^i\lambda^j\,\delta\lambda^k\delta\lambda^h\Big) + \dots \\ &= \frac{1}{\lambda}\Big[1 - \frac{1}{\lambda^2}\, g_{ij}(\lambda)\lambda^i\delta\lambda^j + \frac{1}{2}\Big(\frac{3}{\lambda^4}(g_{ij}(\lambda)\lambda^i\delta\lambda^j)^2 \\ &\qquad - \frac{1}{\lambda^2}\, g_{ij}(\lambda)\,\delta\lambda^i\delta\lambda^j\Big) + \dots\end{aligned}$$

Substituting this expression in equation (11)', we find after multiplication and simplification

$$\cos(\lambda, \lambda+\delta\lambda) = 1 - \frac{1}{2\lambda^2}\, g_{ij}(\lambda)\,\delta\lambda^i\delta\lambda^j + \frac{1}{2\lambda^4}(g_{ij}(\lambda)\lambda^i\delta\lambda^j)^2 + \dots \tag{12}$$

is symmetric in the two directions. The difference between (17) and the classical formula lies in the fact that here the second order term does not, in general, vanish identically.

§ 13. Differentiation of sine and cosine.

Due to the lack of symmetry of the trigonometrical functions, we have to specify which of the directional arguments has to be kept constant when we differentiate. Thus, for instance, we shall denote by $d_\mu(\cos(\lambda,\mu))$ the differential of $\cos(\lambda,\mu)$ when we suppose that μ is to be kept constant.

Consider three vectors $\vec{\lambda}$, $\vec{\mu}$, $\vec{\mu}+\vec{\delta\mu}$ in the same plane. Without loss of generality we may assume that their lengths are equal and fixed, and we shall regard $\vec{\mu}$ as a continuous function of a parameter s, so that the vector $\vec{\mu}+\vec{\delta\mu}$ is simply the vector $\vec{\mu}(s+\delta s)$.

From the addition-formula (8) we have

$$\cos(\lambda,\mu+\delta\mu) = \cos(\lambda,\mu)\cos(\mu,\mu+\delta\mu) - \frac{\sin(\mu,\mu+\delta\mu)}{\sin(\mu,\lambda)}\left\{1-\cos(\lambda,\mu)\cos(\mu,\lambda)\right\}$$

so that up to terms of the first order equations (13) and (17) allow us to express this relation in the form

$$\cos(\lambda,\mu+\delta\mu) = \cos(\lambda,\mu) - \frac{1-\cos(\lambda,\mu)\cos(\mu,\lambda)}{\sin(\mu,\lambda)}\,\delta\theta \qquad (18)$$

where $\delta\theta$ is the angle between the vectors $\vec{\mu}(s)$ and $\vec{\mu}(s+\delta s)$

Then

$$\frac{d_\lambda}{ds}\left[\cos(\lambda,\mu)\right] = \lim_{\delta s\to 0}\frac{\cos(\lambda,\mu+\delta\mu)-\cos(\lambda,\mu)}{\delta s}$$

i.e. in virtue of (18)

$$\frac{d_\lambda}{ds}\left[\cos(\lambda,\mu)\right] = -\lim_{\delta s\to 0}\left[\frac{1-\cos(\lambda,\mu)\cos(\mu,\lambda)}{\sin(\mu,\lambda)}\right]\frac{\delta\theta}{\delta s}$$

or

$$\frac{d_\lambda}{ds}\left[\cos(\lambda,\mu)\right] = -\left[\frac{1-\cos(\lambda,\mu)\cos(\mu,\lambda)}{\sin(\mu,\lambda)}\right]\frac{d\theta}{ds} \qquad (19)$$

This would once more reduce to the classical result if we had symmetry.

If, in the addition-formula (10), we interchange $\vec{\lambda}$ and $\vec{\nu}$, we have

$$\sin(\nu,\lambda) = \sin(\nu,\mu)\cos(\mu,\lambda) + \frac{\sin(\nu,\mu)}{\sin(\mu,\nu)}\cos(\mu,\nu)\sin(\mu,\lambda)$$

Putting $\nu = \mu + \delta\mu$ and using once more the relations (13) and (17) we have as a first approximation

$$\sin(\mu+\delta\mu,\lambda) = \delta\theta\cos(\mu,\lambda) + \sin(\mu,\lambda) \tag{20}$$

remembering that for small angles sine and cosine are symmetrical. We thus find

$$\frac{d_\lambda}{ds}\left[\sin(\mu,\lambda)\right] = \lim_{\delta s\to 0}\frac{\sin(\mu+\delta\mu,\lambda)-\sin(\mu,\lambda)}{\delta s}$$

$$= \lim_{\delta s\to 0}\cos(\mu,\lambda)\frac{\delta\theta}{\delta s}$$

or

$$\frac{d_\lambda}{ds}\left[\sin(\mu,\lambda)\right] = \cos(\mu,\lambda)\frac{d\theta}{ds} \tag{21}$$

as in the classical case.

In a similar manner we may derive the following results:

$$\frac{d_\mu}{ds}\left[\sin(\lambda,\mu)\right] = \frac{\sin(\lambda,\mu)}{\sin(\mu,\lambda)}\cos(\mu,\lambda)\frac{d\theta}{ds} \tag{22}$$

and

$$\frac{d_\mu}{ds}\left[\cos(\mu,\lambda)\right] = -\left[\sin(\mu,\lambda)\,g_{ij}(\mu)\frac{d\mu^j}{ds}\frac{d\mu^i}{ds}\right]\frac{d\theta}{ds} \tag{23}$$

With the aid of equations (19), (21), (22) and (23) it is now possible to derive a Taylor's Series for our trigonometrical functions. AS we shall be concerned mainly with small angles in the Differential Geometry of Minkowskian spaces such an expansion is not necessary for our purposes, and we shall therefore omit a discussion of this kind.

In the preceding sections we have laid the foundations of a Trigonometry of Minkowskian spaces and we have obtained the necessary tools from which further developments can be deduced analogously to the pattern of euclidean trigonometry.

§ 14. Area in a two-dimensional Minkowskian space.

Consider any two vectors $\vec{\lambda}$, $\vec{\mu}$ passing through the origin O. These vectors define a triangle obtained by joining their end-points. We introduce the function Δ of the two vectors by the equation

$$\Delta(\vec{\lambda},\vec{\mu}) = \frac{1}{2}|\vec{\lambda}|\cdot|\vec{\mu}|\cdot \sin(\vec{\lambda},\vec{\mu}) \tag{24}$$

where the oriented sine is used(§10), and where the order of the arguments in the sine is such that the positive alternative is chosen in (24). The orientation is such that there is only one such alternative. Thus, given any two vectors, this stipulation fixes the order of the arguments in the function Δ. If we complete the parallelogram, we find that the additional triangle is obtained by a translation of the triangle formed by the vectors $-\vec{\lambda}$, $-\vec{\mu}$ for which Δ is given by

$$\Delta(-\vec{\lambda},-\vec{\mu}) = \frac{1}{2}|\vec{\lambda}|\cdot|\vec{\mu}|\sin(-\vec{\lambda},-\vec{\mu}) = \frac{1}{2}|\vec{\lambda}|\cdot|\vec{\mu}|\sin(\vec{\lambda},\vec{\mu}) = \Delta(\vec{\lambda},\vec{\mu})^{*}$$

It is then plausible to define the function $A(\vec{\lambda},\vec{\mu})$ by the equation

$$A(\vec{\lambda},\vec{\mu}) = |\vec{\lambda}|\cdot|\vec{\mu}|\ \sin(\vec{\lambda},\vec{\mu}) \tag{25}$$

retaining the same convention as regards sign. We shall regard the functions Δ and A as measures of the areas of the triangle and parallelogram respectively.

In order to derive a certain property of the function A, we shall need the following simple result concerning triangles: Let OBC be any triangle, these points being taken in counter-clockwise rotation. Find a point A on OC such that the tangent to the indicatrix of center O and radius OA at the point A passes through B. By symmetry, the indicatrix of center C and radius CA will touch AB at A. By definition of oriented sine, we have

$$\frac{|AB|}{|BC|} = -\sin(\vec{CA},\vec{CB}) = -\sin(-\vec{OC},-\vec{BC}) \tag{26}$$

and

$$\frac{|AB|}{|OB|} = \sin(\vec{OC},\vec{OB}) \tag{27}$$

But from the symmetry of the indicatrix it follows that

$$\sin(-\vec{\lambda},-\vec{\mu}) = \sin(\vec{\lambda},\vec{\mu})^{*}$$

so that on dividing (27) by (26) we find that

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* The rotation of the vectors $-\vec{\lambda}$, $-\vec{\mu}$, $-(-\vec{\lambda})$ is counter-clockwise.

$$\frac{\sin(\overrightarrow{OC}, \overrightarrow{BC})}{|OC|} = - \frac{\sin(\overrightarrow{OC}, \overrightarrow{OB})}{|BC|} \tag{28}$$

Returning to our vectors $\vec{\lambda}$, $\vec{\mu}$ construct the parallelogram OABC where $\overrightarrow{OA} = \vec{\lambda}$, $\overrightarrow{OB} = \vec{\mu}$, such that the rotation from $\vec{\lambda}$ to $\vec{\mu}$ to $-\vec{\lambda}$ is counter-clockwise. Then the rotation from $\vec{\lambda}$ to $\overrightarrow{\lambda+\mu}$ to $-\vec{\lambda}$ is also counter-clockwise. Construct the parallelogram OADB formed by the vectors $\vec{\lambda}$, $\overrightarrow{\lambda+\mu}$. We then have

$$\sin(\vec{\lambda}, \vec{\mu}) = \sin(\overrightarrow{OA}, \overrightarrow{AB}) = -\sin(\overrightarrow{AO}, \overrightarrow{AB})$$

Applying formula (28) to the special case of triangle OAB, we find

$$\frac{\sin(\vec{\lambda}, \vec{\mu})}{|\overrightarrow{\lambda+\mu}|} = \frac{\sin(\vec{\lambda}, \overrightarrow{\lambda+\mu})}{|\vec{\mu}|} \tag{29}$$

After cross-multiplication and multiplication by $|\vec{\lambda}|$, we immediately deduce from the definition (25)

$$A(\vec{\lambda}, \vec{\mu}) = A(\vec{\lambda}, \overrightarrow{\lambda+\mu}) \tag{30}$$

Let us consider the segment of the indicatrix formed by the vectors $\vec{\lambda}$ and $\overrightarrow{\lambda+\delta\lambda}$ whose lengths are equal. Considering this segment as an infinitesimal triangle, we have in virtue of (17), §12

$$\Delta(\vec{\lambda}, \overrightarrow{\lambda+\delta\lambda}) = \tfrac{1}{2}|\vec{\lambda}|^2 \sin(\vec{\lambda}, \overrightarrow{\lambda+\delta\lambda}) = \tfrac{1}{2}|\vec{\lambda}|^2 \delta\theta + \cdots$$

so that if we denote the length of the circumference and the area of the indicatrix of radius $|\vec{\lambda}|$ by $L(I)$ and $A(I)$ respectively,

$$A(I) = \oint \lim_{\delta\lambda \to 0} \Delta(\vec{\lambda}, \overrightarrow{\lambda+\delta\lambda}) = \tfrac{1}{2}\oint |\vec{\lambda}|^2 d\theta = \tfrac{1}{2}|\vec{\lambda}|\, L(I) \tag{31}$$

using the definition of angle.

In order that we have a measure $A(I)$ which remains constant as we pass from one point to a neighbouring point in a Finsler space, we may introduce a <u>normalised area</u> defined by

$$A'(\vec{\lambda}, \vec{\mu}) = \frac{A(\vec{\lambda}, \vec{\mu})}{L_0(I)} \tag{32}$$

where $L_0(I)$ is the length of circumference of the unit indicatrix. This process is equivalent to replacing $\delta\theta$ by the normalised angle defined in §9.

<u>Note</u>: Probably the first definition of area was given by Bliss (28) which depends, however, on a distinguished field of curves. A more recent definition was suggested by Choquet (29), the significance of which was established beyond all doubt by Busemann (30).

In connection with area in affine spaces, special mention must be made of the discussion in Sperner (31), Chapter IV. In the case of a two-dimensional space a certain function $V(\vec{\lambda}, \vec{\mu})$ is attached to a system of two vectors $\vec{\lambda}$, $\vec{\mu}$ which is required to satisfy the following relations (p. 102):

(1) $V(\vec{\lambda}, \vec{\mu}) \geqslant 0$

(2) $V(\vec{\lambda}, \vec{\mu})$ must remain unchanged when any one vector is replaced by $\overrightarrow{\lambda + \mu}$. (Compare equation (30)).

(3) $V(\lambda, \mu)$ must become $kV(\vec{\lambda}, \vec{\mu})$ when one of $\vec{\lambda}$ or $\vec{\mu}$ is replaced by $k\vec{\lambda}$ or $k\vec{\mu}$.

(4) A normalisation condition.

A function $V(\vec{\lambda}, \vec{\mu})$ satisfying these conditions is interpreted as area. The area as defined by (32) clearly has similar properties: any differences that might arise are due to the lack of symmetry of the sine.

CHAPTER IV

THE DIFFERENTIAL GEOMETRY OF MINKOWSKIAN SPACES

§ 15. Curvature of Curves.

As we shall be concerned solely with Minkowskian spaces in this chapter, we may, without danger of ambiguity, denote the coordinates of a point simply by x^i. Let us consider a continuous curve

$$x^i = x^i(s) \tag{1}$$

in which the parameter s is the arc-length of the curve measured from some fixed point, so that

$$ds^2 = g_{ij}(dx)\, dx^i dx^j \tag{2}$$

where $\overrightarrow{dx}$ is an infinitesimal displacement along the curve. We then have

$$g_{ij}\left(\frac{dx}{ds}\right) \frac{dx^i}{ds} \cdot \frac{dx^j}{ds} = 1 \tag{3}$$

so that the vector $\vec{x}' = \overrightarrow{dx}/ds$ is a unit vector, which we shall regard as the tangent-vector to the curve (1). If $\vec{x}'(s)$, $\vec{x}'(s+\delta s)$ are the tangents at points on (1) corresponding to values s and $s+\delta s$ respectively, we shall denote the angle between them by $\delta\psi$, and we shall define the radius of curvature ρ at the point s by the equation

$$\frac{1}{\rho} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} \tag{4}$$

We can form the "spherical image" of Gauss on the indicatrix by constructing the locus on the indicatrix of end-points of unit vectors passing through the origin and being parallel to successive tangent-vectors of (1). If δs_1 is the element of arc-length along this curve, it follows from ourdefinition of angle (§9) that

$$\delta\psi = \delta s_1 \tag{5}$$

A n equivalent definition of curvature would then be

$$\frac{1}{\rho} = \lim_{\delta s \to 0} \frac{\delta s_1}{\delta s} \tag{4)'$$

Now, by the Mean-Value Theorem we have

$$\delta x^i = x^i(s+\delta s) - x^i(s) = \delta s\, x'^i(s + \theta_{(i)}\delta s)$$

where $0 < \theta_{(i)} < 1$, so that we may write

$$\left(\frac{ds_1}{ds}\right)^2 = \lim_{\delta s \to 0}\left[\frac{g_{ij}(\delta x')\,\delta x'^i\,\delta x'^j}{g_{ij}(x')\,\delta x^i\,\delta x^j}\right] = g_{ij}(x'')x''^i x''^j \tag{6}$$

and we consequently deduce from (4)' that

$$\rho = \left[g_{ij}(x'')x''^i x''^j\right]^{1/2} \tag{7}$$

A lso, differentiating (3) with respect to s, we find

$$0 = \frac{\partial g_{ij}(x')}{\partial x^k} x'^i x'^j x'^k + 2 g_{ij}(x') x'^i x''^j \tag{8}$$

where the first term on the right-hand side vanishes in virtue of (15),§2. Equation (8) then expresses the fact that the vector $\vec{x}''$ is orthogonal with respect to the tangent-vector, and we shall therefore call it the <u>normal-vector</u> to the curve (1). The unit vector in the direction of the normal is then $\rho\vec{x}''$.

As regards the definition of curvature it is worth noting that the angle between neighbouring normals is not necessarily equal to that between corresponding tangents. This would suggest an alternative definition of curvature, which, however, has not proved useful and has therefore been discarded. The only disadvantage of the first definition is that curves formed by the intersection of a plane through the origin with the indicatrix do not have constant curvature as in the euclidean case.

Starting from a different definition of curvature (for Finsler spaces) Taylor (3) obtained a generalisation of the Frenet-formulae - we have desisted from attempting such a generalisation as the notion of an ortho-normal set of vectors in an n-dimensional Minkowskian space appears to be a somewhat unnatural one in view of the lack of symmetry of the concept of orthogonality.

§ 16. Further Geometrical Interpretation of Curvature.

At any point O of the curve we choose our origin of coordinates and we suppose that $s = 0$ at O. We shall denote by $\vec{x}'_{(0)}$ and $\vec{x}''_{(0)}$ the tangent and normal at O respectively. Let P be another point on the curve whose arc-length from O measured along the curve is δs. If we denote the coordinates of P by δx^i, we have

$$\delta x^i = x'^i_{(0)}\,\delta s + x''^i_{(0)}\,\frac{\delta s^2}{2} + \cdots \tag{9}$$

Assuming firstly that the curve is a plane-curve, i.e. lies in a two-dimensional linear subspace, we may draw PH parallel to $\vec{x}''_{(0)}$ to meet $\vec{x}'_{(0)}$ in H. Let the length of OH be $\delta\sigma$. Then H has coordinates $x'^i_{(0)}\delta\sigma$, and the vector $\overrightarrow{PH}$ is given by

$$\overrightarrow{PH} = \vec{x}'_{(0)}\,\delta\sigma - \overrightarrow{\delta x'} \tag{10}$$

and we have

$$\frac{x'^i_{(0)}\,\delta\sigma - \delta x'^i}{x''^i_{(0)}} = \frac{x'^j_{(0)}\,\delta\sigma - \delta x'^j}{x''^j_{(0)}}$$

or

$$\delta\sigma\left[x'^i_{(0)}\,x''^j_{(0)} - x'^j_{(0)}\,x''^i_{(0)}\right] = x''^j_{(0)}\,\delta x'^i - x''^i_{(0)}\,\delta x'^j$$

Substituting for $\delta x'^i$ from (9) we find

$$\delta\sigma\left[x'^i_{(0)}\,x''^j_{(0)} - x'^j_{(0)}\,x''^i_{(0)}\right] = \delta s\left[x'^i_{(0)}\,x''^j_{(0)} - x'^j_{(0)}\,x''^i_{(0)}\right] + \frac{\delta s^2}{2}\left[x''^j_{(0)}x''^i_{(0)} - x''^i_{(0)}x''^j_{(0)}\right] + \cdots$$

so that

$$\delta\sigma = \delta s + \text{terms of } 3^{rd} \text{ order} + \cdots \tag{11}$$

From (9) and (10) we then deduce

$$\overrightarrow{HP} = \vec{x}''_{(0)}\left(\frac{\delta s^2}{2}\right) + \cdots \tag{12}$$

or, using (7)

$$|HP| = \frac{\delta s^2}{2\rho} + \cdots \tag{13}$$

This result is also true when the curve is not a plane-curve; this follows from the fact that in the limit PH is parallel to $\vec{x}''_{(0)}$ when P,H are points obtained by marking off equal distances δs along the curve and tangent respectively, as we then have, in virtue of (9) (13)'

$$\overrightarrow{PH} = \frac{1}{2}\left(\frac{\delta s^2}{\rho}\right)\vec{x}''_{(0)} + \cdots \tag{12'}$$

The first, more cumbersome argument has been given as we shall require later a knowledge of the approximation to which equation (11) is correct in the case of plane-curves when directions parallel to the normal are being considered. The osculating plane of the curve can be obtained just as in euclidean geometry.

§ 17. Curves on Surfaces.

We shall now consider any surface whose equation is given by an analytic function, and we propose to study the neighbourhood of an arbitrary point O on the surface, this point being chosen once more as the origin of our coordinates.

Let $\vec{\xi}$ denote the unit normal vector to the surface, i.e. for all curves on the surface through O we have

$$g_{ij}(\xi)\,\xi^i x'^j_{(0)} = 0 \tag{14}$$

where $\vec{x}'_{(0)}$ is the tangent-vector at O to any one of these curves.

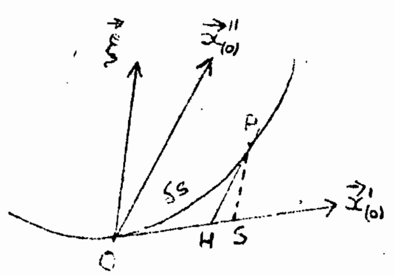

Fig. 3

We shall call curves on the surface whose osculating planes contain $\vec{\xi}$ "normal-cuts" of the surface, so that for such curves $\vec{x}''_{(0)}$, $\vec{x}'_{(0)}$, $\vec{\xi}$ lie in one plane. Contrary to the euclidean case the vectors $\vec{\xi}$ and $\vec{x}''_{(0)}$ do not in general coincide as the equations (8) and (14) are not equivalent. Consider a normal-cut OP (Fig. 3) where the arc-length of OP is again equal to δs. From P we drop PH, PS parallel to $\vec{x}''_{(0)}$ and $\vec{\xi}$ respectively to meet the tangent $\vec{x}'_{(0)}$ to the curve in H and S. In virtue of (14) we have $HS \perp PS$, so that it follows from our definition of cosine that

$$|PS| = |PH| \cos(\xi, x''_{(0)}). \tag{15}$$

From (13) we then have

$$|PS| = \frac{\delta s^2}{2\rho} \cos(\xi, x''_{(0)}) + \dots .$$

so that if we put

$$R = \frac{\rho}{\cos(\xi, x''_{(0)})} \tag{16}$$

we have

$$|PS| = \frac{\delta s^2}{2R} + \dots \tag{17}$$

We shall call $1/R$ the "secondary" curvature of the normal-cut, where it is to be noted that $\cos(\xi, x''_{(0)})$ varies for different cuts; in fact, we have a cone of directions generated by the vectors $\vec{x}''_{(0)}$. Note that $R \geqslant \rho$. Furthermore, since $\vec{x}''_{(0)}, \vec{x}'_{(0)}, \vec{\xi}$, lie in the osculating plane, there must exist a relation between them of the form

$$\rho x''^{i}_{(0)} = a \xi^i + b x'^{i}_{(0)} \tag{18}$$

where a , b are functions depending on the curve. Multiplying (18) by $g_{ij}(x''_{(0)}) x''^{j}_{(0)}$ and $g_{ij}(\xi)\xi^j$ in turn and summing over i in each case, we find

$$\left.\begin{aligned} 0 &= a \cos(x'_{(0)}, \xi) + b \\ \cos(\xi, x''_{(0)}) &= a \end{aligned}\right\}$$

when we take into account (8) and (14) and remembering that $\rho \vec{x}''_{(0)}$ is a unit vector. Equation (18) then becomes

$$x''^{i}_{(0)} = \frac{1}{R}\left[\xi^i - \cos(x'_{(0)}, \xi) x'^{i}_{(0)}\right] \tag{19}$$

which is the generalisation of the corresponding euclidean relation

$$x''^{i}_{(0)} = \frac{1}{R}\xi^i$$

§ 18. The Indicatrix of Dupin.

We shall now choose a special coordinate system as follows: Retaining the origin at O, let the tangent-plane to the surface at O be taken as the coordinate hyperplane $x^1 = 0$ and the straight lines which are parallel to $\vec{\xi}$ as the coordinate lines of the parameter x^1, this parameter measuring the lengths of the segments of these straight lines from the plane $x^1 = 0$. Since dx^1 is then the length of the vector parallel to $\vec{\xi}$ whose components are $(dx^1, 0, 0, \dots 0)$, we have

$$g_{11}(\xi) = 1 \tag{20}$$

By considering vectors in the tangent-plane wth components (0,1, 0,0,....0), (0,0,1,0....0), etc., we deduce from the orthogonality relation (14) that

$$g_{1\alpha}(\xi) = 0 \qquad (\alpha = 2, 3, \dots n) \tag{21}$$

(For the rest of this chapter Greek indices are to be summed from 2 to n, while Latin indices retain their original meaning).
The length of any vector $\vec{x}'$ in the tangent-plane will then be given by

$$|x'|^2 = g_{\alpha\beta}(x')x'^{\alpha}x'^{\beta} \tag{22}$$

Let the equation to our surface in these coordinates be

$$x^1 = x^1(x^\alpha) \tag{23}$$

This equation expresses the distance of a point on the surface from the plane $x^1 = 0$, this distance being measured in the direction of the normal vector $\vec{\xi}$. Then, if Δx^1 is the distance from the plane of a point on the surface whose coordinates are $(\Delta x^1, \delta x^\alpha)$ we have

$$\Delta x^1 = \frac{\partial x^1(0)}{\partial x^\alpha}\delta x^\alpha + \frac{1}{2}\frac{\partial^2 x^1(0)}{\partial x^\alpha \partial x^\beta}\delta x^\alpha \delta x^\beta + \dots \tag{24}$$

But for a normal-cut whose secondary curvature is R^{-1} and whose direction is given by

$$x'^{\alpha}_{(0)} = \frac{\delta x^\alpha}{\delta s}$$

it follows from (11) and (17) that

$$\Delta x^1 = \frac{\delta s^2}{2R} = \frac{g_{\alpha\beta}(x'_{(0)})\,\delta x^\alpha \delta x^\beta}{2R} \tag{25}$$

so that

$$\frac{\partial x^1(0)}{\partial x^\alpha} = \lim_{\delta s \to 0} \frac{\Delta x^1}{\delta x^\alpha} = \lim_{\delta s \to 0} \frac{\Delta x^1}{\delta s}\,\frac{\delta s}{\delta x^\alpha}$$

$$= \lim_{\delta s \to 0}\left(\frac{\delta s}{2R}\right)\frac{\delta s}{\delta x^\alpha}$$

in virtue of (25). Since the last limit is clearly zero, we find on comparison of (24) and (25)

$$\Delta x^1 = \frac{g_{\alpha\beta}(x'_{(0)})\,\delta x^\alpha \delta x^\beta}{2R} = \frac{1}{2}\frac{\partial^2 x^1(0)}{\partial x^\alpha \partial x^\beta}\delta x^\alpha \delta x^\beta \qquad (25)'$$

where it is to be noted that the coefficients in the last sum are independent of the direction $\vec{x}'_{(0)}$ of the vectors $\vec{\delta x}$.

In future we shall write

$$\frac{1}{2}\frac{\partial^2 x^1(0)}{\partial x^\alpha \partial x^\beta} = b_{\alpha\beta} \qquad (b_{\alpha\beta} = b_{\beta\alpha}). \qquad (26)$$

The equation

$$b_{\alpha\beta}\,\delta x^\alpha \delta x^\beta = \varepsilon^2 \qquad (27)$$

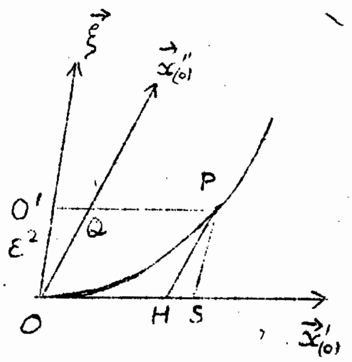

Fig. 4

where ε is a constant of the first order of smallness is then the equation of the intersection of a plane parallel to the tangent-plane at O and distant ε^2 from it with the surface. As in the Euclidean case, it is a central quadric ((n - 1)-dimensional). If we draw (Fig. 4) PO' parallel to $\vec{x}'_{(0)}$ to meet $\vec{\xi}$ in O' and $\vec{x}''_{(0)}$ in Q, we have

$$\delta s = OH = QP = O'P \pm O'Q$$

or

$$\delta s = O'P \pm HS \qquad (28)$$

and

$$OO' = SP = \varepsilon^2 \qquad (29)$$

Also, since

$$HS = PH \sin(\xi, x''_{(0)})$$

we have

$$HS = \frac{\delta s^2}{2\rho}\sin(\xi, x''_{(0)}) \qquad (30)$$

correct to the second order of smallness in virtue of (13). From (28) and (30) we thus obtain

$$|O'P|^2 = \delta s^2\left[1 \pm \frac{\delta s}{\rho}\sin(\xi, x''_{(0)}) + \frac{\delta s^2}{4\rho^2}\sin^2(\xi, x''_{(0)}) + \dots\right]$$

and therefore

$$|O'P|^2 = g_{\alpha\beta}(x'_{(0)})\,\delta x^\alpha \delta x^\beta \qquad (31)$$

correct to the second order of smallness, since $|\sin(\xi, x''_{(0)})| \leq 1$

From (25)' and (27) we deduce that for any radius-vector O'P of our intersection-quadric we have up to the second order

$$\frac{|O'P|^2}{2R} = \varepsilon^2 = \text{constant} \tag{32}$$

where R^{-1} is the secondary curvature corresponding to the normal-cut in the direction O'P. The secondary curvature is thus proportional to the square of the corresponding radius-vector of the quadric (27); and we are therefore justified in assuming the classical nomenclature and we shall thus call the quadric (27) the <u>Dupin Indicatrix</u>.

In euclidean differential geometry the great importance of the Dupin Indicatrix is due to the fact that not only does it represent the intersection of a plane parallel to the tangent-plane infinitesimally close to it with the surface, but it also provides us with a very clear picture of the distribution of the curvatures of normal-cuts. It is of interest to note that our definition of curvature, which is based ultimately on our notion of angle, allows us to preserve both these features of the Dupin Indicatrix for surfaces embedded in a Minkowskian space.

§ 19. Principal Directions.

We shall define <u>principal directions</u> on the surface at O as being those directions for which R assumes extreme values (if they exist). By putting

$$\frac{\delta x^\alpha}{\varepsilon} = x'^\alpha \tag{33}$$

we may write equation (27) for the Dupin Indicatrix in the form

$$b_{\alpha\beta}\, x'^\alpha x'^\beta = 1 . \tag{34}$$

Because of (31) and (32) our problem of finding the principal directions implies seeking the extreme values of the function

$$R = \tfrac{1}{2} g_{\alpha\beta}(x')\, x'^\alpha x'^\beta . \tag{35}$$

subject to the subsidiary condition (34). According to the multiplier rule of Lagrange, all directions $x'_{(1)}, x'_{(2)} \ldots , x'_{(k)} , \ldots$

which furnish us with extreme values, must satisfy the equation

$$\frac{\partial}{\partial x'^{\kappa}}\left[\tfrac{1}{2} g_{\alpha\beta}(x')x'^{\alpha}x'^{\beta} - \lambda(b_{\alpha\beta}x'^{\alpha}x'^{\beta}-1)\right] = 0$$

where λ is the undetermined multiplier. Taking into account the homogeneity condition (15) of §2 a possible solution $x'_{(k)}$ is given by

$$\tfrac{1}{2} g_{\alpha\beta}(x'_{(k)})x'^{\alpha}_{(k)} - \lambda_{(k)} b_{\alpha\beta} x'^{\beta}_{(k)} = 0 \tag{36}$$

where $\lambda_{(k)}$ is that value of λ corresponding to the direction $x'_{(k)}$. This is easily determined by multiplying (36) by $x'^{\alpha}_{(k)}$ and summing over α : we then find

$$\tfrac{1}{2} g_{\alpha\beta}(x'_{(k)})x'^{\alpha}_{(k)}x'^{\beta}_{(k)} - \lambda_{(k)} b_{\alpha\beta}x'^{\alpha}_{(k)}x'^{\beta}_{(k)} = 0$$

so that in vitue of (34) and (35) we obtain

$$\lambda_{(k)} = R_{(k)} \tag{37}$$

Since equation (36) is not linear in $x'_{(k)}$ we cannot make any conclusions as to the number of roots, or apply the classical theory of the reduction of quadratic forms. Consequently the great significance of Pricipal directions is confined to spaces with a locally euclidean metric, where these methods are directly applicable.

Assuming, however, the existence of two distinct solutions $x'_{(k)}$, $x'_{(h)}$ of equation (36) for which the corresponding secondary secondary curvatures $R^{-1}_{(k)}$, $R^{-1}_{(h)}$ are not zero, let us multiply (36) by $x'_{(h)}$ and interchange h and k in the resulting equation. We then obtain, with the aid of (37) the two equations

$$\frac{1}{2R_{(k)}} g_{\alpha\beta}(x'_{(k)})x'^{\alpha}_{(k)}x'^{\beta}_{(h)} - b_{\alpha\beta}x'^{\beta}_{(k)}x'^{\alpha}_{(h)} = 0,$$

$$\frac{1}{2R_{(h)}} g_{\alpha\beta}(x'_{(h)})x'^{\alpha}_{(h)}x'^{\beta}_{(k)} - b_{\alpha\beta}x'^{\beta}_{(h)}x'^{\alpha}_{(k)} = 0.$$

Because of the symmetry of $g_{\alpha\beta}$ and $b_{\alpha\beta}$ we find on subtrction

$$\frac{1}{R_{(k)}} g_{\alpha\beta}(x'_{(k)})x'^{\alpha}_{(k)}x'^{\beta}_{(h)} = \frac{1}{R_{(h)}} g_{\alpha\beta}(x'_{(h)})x'^{\alpha}_{(h)}x'^{\beta}_{(k)}$$

Using (35) and the definition of cosine, this equation reduces to

$$\frac{\cos(x'_{(k)}, x'_{(h)})}{R_{(k)}} = \frac{\cos(x'_{(h)}, x'_{(k)})}{R_{(h)}} \tag{38}$$

If the cosine were symmetrical this equation would imply $\cos(x'_{(k)}, x'_{(h)}) = 0$ for $R_{(k)} \neq R_{(h)}$, so that equation (38) may be regarded as the generalisation of the orthogonality properties of principal directions in euclidean geometry. It is at this point that the difference between the present Dupin Indicatrix and the euclidean analogue is marked most clearly. Certainly the notion of principal direction is inadequate to lead to a definition of Gaussian curvature of a surface. We shall develop this concept later along different lines for the more general case of a Finsler space: the methods to be used are, of course, directly applicable to surfaces embedded in a Minkowskian space. For the present we shall therefore leave this notion aside.

§ 20. Generalisation of the Formula of Rodrigues.

For the first part of this section we shall use general co-ordinates again, except that we still regard the point O on our surface as the origin. Again $\vec{\xi}$ is the unit normal to the surface and we shall consider a point P on the surface given by

$$\vec{OP} = \vec{\Delta x} = \vec{dx} + \tfrac{1}{2}\vec{d^2x} \tag{39}$$

Let PS be drawn once more parallel to $\vec{\xi}$ to meet the tangent-plane at O in S. Since $\vec{OS} \perp \vec{PS}$ we may write

$$|PS| = |PO| \cos(\xi, \Delta x)$$

or

$$|PS| = g_{ij}(\xi)\xi^i(dx^j + \tfrac{1}{2}d^2x^j) \tag{40}$$

in virtue of (39) and the fact that $\vec{\xi}$ is a unit vector. Also, since $\vec{dx}$ lies in the tangent-plane at O, we have

$$g_{ij}(\xi)\xi^i dx^j = 0 \tag{41}$$

from which we deduce by differentiation

$$g_{ij}(\xi)d\xi^i dx^j = -g_{ij}(\xi)\xi^i d^2x^j \tag{42}$$

taking into account the homogeneity condition (15) of §2. With the aid of equations (41), (42), equation (40) becomes

$$|PS| = -\tfrac{1}{2} g_{ij}(\xi)d\xi^i dx^j \tag{43}$$

Now let $\vec{\eta}$ be the vector on the figuratrix in T_n' correspon-ding to $\vec{\xi}$; so that

$$\eta_j = g_{ij}(\xi)\,\xi^i \tag{44}$$

and again taking into account equation (15) of §2

$$d\eta_i = g_{ij}(\xi)\,d\xi^i \tag{44)'$$

where we note once more that $\overrightarrow{d\eta}$, $\overrightarrow{d\xi}$ are not corresponding vectors in the same sense as $\vec{\eta}$ and $\vec{\xi}$. Thus (43) becomes

$$|PS| = -\tfrac{1}{2}\,d\eta_j\,dx^j \tag{45}$$

Let us now re-introduce the special coordinate system of §18. Since $\vec{\xi}$ is a unit vector

$$g_{ij}(\xi)\,\xi^i\,d\xi^j = 0$$

i.e. $\overrightarrow{d\xi}$ lies in the tangent-plane, and hence $d\xi^1 = 0$. From (20) and (21) we deduce

$$\left.\begin{aligned} d\eta_1 &= d\xi^1 = 0 \\ d\eta_\alpha &= g_{\alpha\beta}(\xi)\,d\xi^\beta \end{aligned}\right\} \tag{46}$$
$$(\alpha, \beta = 2, 3, \ldots n)$$

The first term in (45) will now disappear, and we may write it in the form

$$|PS| = -\tfrac{1}{2}\,d\eta_\alpha\,dx^\alpha \tag{47}$$

In the notation of §18 $|PS| = |OO'| = \varepsilon^2$, so that we have for all points P on the indicatrix (27)

$$b_{\alpha\beta}\,dx^\alpha dx^\beta = -\tfrac{1}{2}\,d\eta_\beta\,dx^\beta$$

Since here $b_{\alpha\beta}$ is independent of the direction and is symmetrical in α and β the condition for the application of the Rigorous Quotient Theorem is satisfied (Brown (32)), and we therefore obtain

$$\tfrac{1}{2}\,d\eta_\beta = -b_{\alpha\beta}\,dx^\alpha \tag{48}$$

If we rewrite this result in the form

$$d\xi^\alpha = -2\,b_{\beta\gamma}\,g^{\alpha\beta}(\xi)\,dx^\gamma \tag{48)'$$

we see that this equation represents the variation of the unit normal vector when we pass from O to a neighbouring point.

Let $\overrightarrow{dx_{(k)}}$ represent a principal direction: it then satisfies (36) and with the aid of (37) we deduce

$$\frac{g_{\alpha\beta}(dx_{(k)})\,dx^\alpha_{(k)}}{2\,R_{(k)}} = b_{\alpha\beta}\,dx^\alpha_{(k)} \tag{49}$$

If we denote by $\overrightarrow{dy}^{(k)}$ the vector in T_n' corresponding to $\overrightarrow{dx}_{(k)}$ and by $d\eta^{(k)}$ the vector corresponding to the resulting change $d\xi^{(i)}_{(k)}$ (as given by (46)), equations (48) and (49) enable us to write

$$R_{(k)} d\eta^{(k)}_{\beta} = - dy^{(k)}_{\beta} \tag{50}$$

In the case where the g_{ij} are independent of direction this would reduce to the classical result of O. Rodrigues.

In (50) great care must be taken as regards the covariant meaning of the expressions involved, and it is perhaps better to write it in the form

$$R_{(k)} g_{\alpha\beta}(\xi) d\xi^{\alpha}_{(k)} = - g_{\alpha\beta}(dx'_{(k)}) dx^{\alpha}_{(k)} \tag{50)'}$$

§ 21. <u>The Theorem of Meusnier.</u>

A theorem analogous to the one of Meusnier for surfaces in a euclidean space may be deduced, al though there is a small difference in the enunciation. In order to make this difference, which is mainly geometrical, entirely clear, we shall restrict this discussion to surfaces embedded in a three-dimensional Minkowskian space, although, as can be easily seen from the analytical character of the following deductions, that the theorem is valid also for the n-dimensional case.

At the point O of our surface we consider two curves on the surface with a common tangent $\vec{x}'_{(0)}$, one of which is a normal-cut, the other a plane curve whose osculating plane intersects the plane of the normal-cut along the vector $\vec{x}'_{(0)}$. We again construct the Dupin Indicatrix by forming the curve of intersection of the surface with a plane parallel to the tangent-plane at a distance ε^2 from O, intersecting $\vec{\xi}$ at O'. The osculating plane of the second curve intersects the Dupin Indicatrix in R and R'. (Fig. 5: <u>Note</u>: The figure given here represents the case where the surface is strictly convex at O - but the other case may be represented in the same manner with slight alterations in the formulation of the proof).

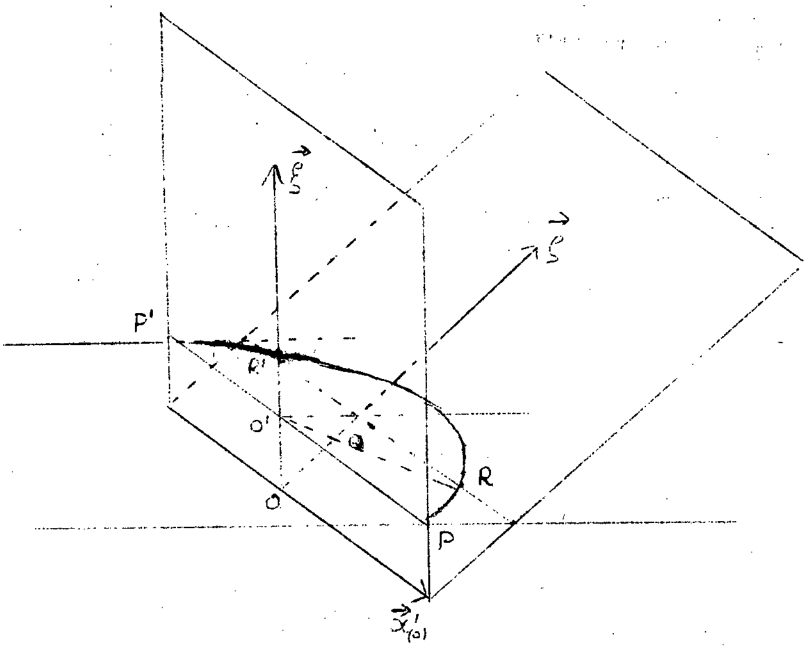

Fig. 5

Draw P'O'P parallel to $\vec x'_{(0)}$ to meet the normal-cut in P' and P, and form the diameter conjugate to PP' with respect to the Dupin Indicatrix. Let this diameter intersect RR' in Q. We shall regard the vector $\overrightarrow{OQ}$ as the "secondary" normal of the second curve, denoting it by $\vec\zeta$. Analogously to the normal-cuts we shall define the secondary curvature $(r_\theta)^{-1}$ of this curve by the equation

$$r_\theta = \frac{|QR|^2}{2|OQ|} \tag{51}$$

where θ is the angle between the vectors $\vec\xi$ and $\vec\zeta$. If R^{-1} is the secondary curvature of the normal-cut, we have from (17) and (11)

$$R = \frac{|O'P|^2}{2|OO'|} \tag{52}$$

Since O'Q lies in a plane parallel to the tangent-plane through O we have $\overrightarrow{O'Q} \perp \vec\xi$, i.e.

$$|OQ| = \frac{OO'}{\cos(\xi,\zeta)} = \frac{\varepsilon^2}{\cos(\xi,\zeta)} \tag{53}$$

and, denoting $\overrightarrow{O'Q}$ by $\vec q$

$$|q| = |OQ| \sin(\xi,\zeta)$$

or, from (53)

$$|q| = \varepsilon^2 \frac{\sin(\xi,\zeta)}{\cos(\xi,\zeta)} \tag{54}$$

The vector $\overrightarrow{QR}$ is, by construction, parallel to $\vec x'_{(0)}$, so that we may write $\overrightarrow{QR} = k\vec x'_{(0)}$, and putting $\overrightarrow{O'R} = \vec r$, we have

$$\vec r = k\vec x'_{(0)} + \vec q$$

This vector must satisfy equation (27) of the Dupin Indicatrix; we thus have

$b_{\alpha\beta} r^\alpha$ [redacted] ε^2 [redacted]

or

$$k^2 b_{\alpha\beta} x'^{\alpha}_{(0)} x'^{\beta}_{(0)} + 2k\, b_{\alpha\beta} x'^{\alpha}_{(0)} q^\beta + (b_{\alpha\beta} q^\alpha q^\beta - \varepsilon^2) = 0 \tag{55}$$

The two roots of k given by this quadratic represent the lengths of the vectors $\overrightarrow{QR}$ and $\overrightarrow{QR'}$: by our construction of conjugate diameters, since RR' is parallel to PP', these roots must be equal and opposite in sign. Their sum therefore vanishes, i.e.

$$b_{\alpha\beta} x'^{\alpha}_{(0)} q^\beta = 0 . \tag{56}$$

Assuming that $\cos(\xi, \zeta)$ is a finite number, it follows from (54) that q^α is a quantity of second order of smallness and therefore the term $b_{\alpha\beta} q^\alpha q^\beta$ is of fourth order. With this approximation, and together with (56) equation (55) reduces to

$$k^2 b_{\alpha\beta} x'^{\alpha}_{(0)} x'^{\beta}_{(0)} = \varepsilon^2 \tag{57}$$

up to the third order. But $\overrightarrow{O'P}$ is also parallel to $\vec{x}'_{(0)}$, and may therefore be represented in the form $k' \vec{x}'_{(0)}$; since P lies on the indicatrix, we also have

$$k'^2 b_{\alpha\beta} x'^{\alpha}_{(0)} x'^{\beta}_{(0)} = \varepsilon^2$$

so that from (57) we deduce $k^2 = k'^2$ up to the third order, i.e.

$$|O'P|^2 = |QR|^2 \tag{58}$$

neglecting quantities of the third order. It then follows from (51), (52) and (53) that in virtue of (58)

$$r_\theta = R \cos(\xi, \zeta) \tag{59}$$

which would reduce to the theorem of Meusnier in the euclidean case.

§ 22. Curvature of normal-cuts of the Indicatrix.

In the special case where the surface under consideration is the indicatrix, it follows from equation (21) of §8 that the distance Δx^4 (§18) of a point on the tangent-plane corresponding to a displacement $\vec{\delta x}$ from the surface is given by

$$\tfrac{1}{2} g_{ij}(\xi) \delta x^i \delta x^j + \dots$$

where $\vec{\xi}$ is once more the normal to the surface. Reverting to a general coordinate-system and using equation (25) we therefore deduce

$$R = \frac{g_{ij}(x')\, x'^i x'^j}{g_{ij}(\xi)\, x'^i x'^j} \tag{60}$$

for the secondary curvature of a normal-cut whose tangent-vector is $\vec{x}'$.

Thus the curvature of different normal-cuts at the same point on the indicatrix is not constant and equal to unity. This gives us a fair example of the "splitting" of properties when we compare the Indicatrix with the euclidean unit sphere.

CHAPTER V

PARALLEL DISPLACEMENT IN FINSLER SPACES

§ 23. Variation of Indicatrix and Figuratrix.

In order to proceed with an investigation into Finsler spaces using methods similar to those applied to Minkowskian spaces in the previous chapters, it is clear that we have to study the variation of the indicatrix and figuratrix as we pass from some point to a neighbouring point. As all our measurements at a point are made with respect to one of these figures, any change in such measurements as a result in the change of metric can then be easily deduced. The main difficulty lies in the fact that the Finsler space does not possess an affine structure, which simplified our methods in the Minkowskian space to such a great extent.

In order to isolate any changes due to the metric alone, we construct a vector-field consisting of vectors $\vec{x}'_{(0)}$ in the tangent-space T_n at each point of the Finsler space, such that the numerical value of the components of $\vec{x}'_{(0)}$ is the same when measured at any point with respect to a fixed parametrisation. The vectors $\vec{y}^{\,o}$ in T'_n corresponding to $\vec{x}'_{(0)}$ in T_n will not, in general, enjoy a similar property. Using the notation of §4 we may represent the indicatrix and the figuratrix at the point x^i by the functions $\varphi(x, x')$ and $\psi(x, y)$, where

$$\varphi(x, x'_{(0)}) = \tfrac{1}{2} g_{ij}(x, x'_{(0)}) x'^{i}_{(0)} x'^{j}_{(0)} \tag{1}$$

and

$$\psi(x, x'_{(0)}) = \tfrac{1}{2} g^{ij}(x, y^{(0)}) y^{(0)}_i y^{(0)}_j \tag{2}$$

If we differentiate the identity

$$g^{ij}(x, y^{(0)}) g_{kj}(x, x'_{(0)}) = \delta^i_k \tag{3}$$

obtained from (38) of §4 with respect to x^l, we have

$$g_{kj}(x, x'_{(0)}) \frac{\partial g^{ij}(x, y^{(0)})}{\partial x^l} = - g^{ij}(x, y^{0}) \frac{\partial g_{kj}(x, x'_{(0)})}{\partial x^l} \qquad (4)$$

the partial derivative on the left-hand side implying that $y^{(0)}$ is kept constant as well during the differentiation. Multiplying this equation by $g^{km}(x, y^{(0)})$ and summing over k, we find, in virtue of (3)

$$\frac{\partial g^{im}(x, y^{(0)})}{\partial x^l} = - g^{km}(x, y^{(0)}) g^{ij}(x, y^{(0)}) \frac{\partial g_{kj}(x, x'_{(0)})}{\partial x^l} \qquad (5)$$

Differentiating (2) with respect to x^l we then have

$$\frac{\partial \psi(x, y^{(0)})}{\partial x^l} = - \tfrac{1}{2} g^{km}(x, y^{(0)}) y^{(0)}_m g^{ij}(x, y^{(0)}) y^{(0)}_i \frac{\partial g_{kj}(x, x'_{(0)})}{\partial x^l}.$$

or, using (37) of §4

$$\frac{\partial \psi(x, y^{(0)})}{\partial x^l} = - \tfrac{1}{2} \frac{\partial g_{kj}(x, x'_{(0)})}{\partial x^l} x'^k_{(0)} x'^j_{(0)}.$$

We thus find on comparison after differentiating (1) with respect to x^l,

$$\frac{\partial \psi(x, y^{(0)})}{\partial x^l} = - \frac{\partial \varphi(x, x'_{(0)})}{\partial x^l} \qquad (6)$$

and, in particular, if $x'_{(0)}$ is a unit vector at a point x^i

$$\frac{\partial H(x, y^{(0)})}{\partial x^l} = - \frac{\partial F(x, x'_{(0)})}{\partial x^l} \qquad (6)'$$

Equation (6)' is well-known in the Calculus of Variations. It is to be noted, however, that in the partial differentiation in this equation both $x'_{(0)}$ and $y^{(0)}$ are kept constant. But from the geometrical point of view, we should let $y^{(0)}$ vary such that it corresponds to $x'_{(0)}$ at all points and not only at the particular point under consideration. We then have from (29) of §4

$$\frac{\partial y^{(0)}_i}{\partial x^l} = \frac{\partial g_{ij}(x, x'_{(0)})}{\partial x^l} x'^j_{(0)} \qquad (7)$$

Denoting differentiation of ψ under these revised circumstances by d, we find from (2)

$$\frac{d\psi(x, y^{0})}{dx^l} = \tfrac{1}{2} \frac{\partial g^{ij}(x, y^{(0)})}{\partial x^l} y^{(0)}_i y^{(0)}_j + \tfrac{1}{2} \frac{\partial g^{ij}(x, y^{(0)})}{\partial y_k} y^{(0)}_i y^{(0)}_j \frac{\partial y^{(0)}_k}{\partial x^l} + g_{ij}(x, y^{(0)}) \frac{\partial y^{(0)}_i}{\partial x^l} y^{(0)}_j \qquad (8)$$

From the homogeneity properties of $g^{ij}(x, y)$ with respect to y^i we deduce that the second term on the right-hand side of (8) vanishes identically, so that with the aid of (6) and (7) we may write taking into account (37) of §4

$$\frac{d\psi(x,y^0)}{dx^l} = -\frac{\partial\varphi(x,x'_{(0)})}{\partial x^l} + \frac{\partial g_{ij}(x,x'_{(0)})}{\partial x^l} x'^i_{(0)} x'^j_{(0)}$$

$$= -\frac{\partial\varphi(x,x'_{(0)})}{\partial x^l} + 2\frac{\partial\varphi(x,x'_{(0)})}{\partial x^l}$$

in virtue of (1), so that we finally have

$$\frac{d\psi(x,y^{(0)})}{dx^l} = \frac{\partial\varphi(x,x'_{(0)})}{\partial x^l} \tag{9}$$

where now only $x'_{(0)}$ is kept constant in contrast to (6)'. These equations allow us to deduce the variation of the figuratrix when the variation of the indicatrix is known.

§ 24. The Christoffel Symbols.

Retaining the special vector-field and the parametrisation of the previous section, we note that the length of the vector $\vec{x}'_{(0)}$ depends on the point x^i of the Finsler space at which the measurement is made. The variation in length, i.e. the variation of the indicatrix thus accounts for the change in φ. Let us consider two neighbouring points x^i and $x^i + dx^i$, where the vector $\vec{dx}$ has the same direction as $\vec{x}'_{(0)}$, and we denote the variation of $\varphi(x,x'_{(0)})$ due to the corresponding change of metric alone by $d_x(\varphi)$. We then have

$$d_x(\varphi) = \frac{1}{2}\frac{\partial g_{ij}(x,x'_{(0)})}{\partial x^k} x'^i_{(0)} x'^j_{(0)} x'^k_{(0)} ds \tag{10}$$

We shall write this relation in the form

$$d_x(\varphi) = [ik,j]_{(x,x'_{(0)})} x'^i_{(0)} x'^j_{(0)} x'^k_{(0)} ds \tag{11}$$

where the coefficients $[ik,j]_{(x,x'_{(0)})}$ depend on x^i and $x'^i_{(0)}$. Since (11) is a cubic form, we may arrange our terms such that $[ik,j]_{(x,x'_{(0)})}$ is symmetric in i and k. A direct comparison between the coefficients of (10) and (11) is not immediately possible, as the coefficients in (10) are symmetric in i and j. If, however, we write (11) in in the form

$$\tfrac{1}{2}\left\{ [ik,j]_{(x,x'_{(0)})} + [jk,i]_{(x,x'_{(0)})} \right\} x'^{i}_{(0)} x'^{j}_{(0)} x'^{k}_{(0)}\, ds$$

the coefficients are now symmetrical in i and j as well, and we therefore deduce immediately

$$[ik,j]_{(x,x'_{(0)})} + [jk,i]_{(x,x'_{(0)})} = \frac{\partial g_{ij}(x,x'_{(0)})}{\partial x^{k}} \tag{12}$$

Permuting the indices, we have

$$[ji,k]_{(x,x'_{(0)})} + [ki,j]_{(x,x'_{(0)})} = \frac{\partial g_{jk}(x,x'_{(0)})}{\partial x^{i}} \tag{12a}$$

and

$$[kj,i]_{(x,x'_{(0)})} + [ij,k]_{(x,x'_{(0)})} = \frac{\partial g_{ki}(x,x'_{(0)})}{\partial x^{j}} \tag{12b}$$

Subtracting (12) from the sum of (12a) and (12b) and taking into account our first symmetry-condition, we have

$$[ij,k]_{(x,x'_{(0)})} = \frac{1}{2}\left(\frac{\partial g_{jk}(x,x'_{(0)})}{\partial x^{i}} + \frac{\partial g_{ki}(x,x'_{(0)})}{\partial x^{j}} - \frac{\partial g_{ij}(x,x'_{(0)})}{\partial x^{k}} \right) \tag{13}$$

These symbols are therefore analogous to the Christoffel symbols in Riemannian Geometry - the only difference being that they depend on direction as well as position, so that we shall always attach the suffixes giving their arguments. It is clear from (11) that these symbols provide us with a measure of the "dilatation" of the indicatrix under the special circumstances described above when we undergo a change of position.

Now consider a vector-field consisting of vectors $\vec{y}^{(0)}$ in T'_n at each point of our space which is such that the components of $\vec{y}^{(0)}$ are the same at each point for the parametrisation used before. At the original point x^i the vectors $\vec{x}'_{(0)}$ and $\vec{y}^{(0)}$ correspond to each other. Denote the variation of $\psi(x,y^{(0)})$ as we pass from x^i to $x^i + dx^i$ by $d_x(\psi)$. It follows from (6) that under these circumstances the numerical value expressing this change must be minus that of $\varphi(x,x'_{(0)})$ in the reversed case considered previously, so that

$$d_x(\psi) = - d_x(\varphi) \tag{14}$$

or, in virtue of (11)

$$d_x(\psi) = -[ik,j]_{(x,x'_{(0)})} x'^{i}_{(0)} x'^{j}_{(0)} x'^{k}_{(0)}\, ds \tag{15}$$

When the arguments assume their values at the original point x^i, we put

$$g^{jk}(x, y^{(0)})\,[ik,j]_{(x,x'_{(0)})} = \left\{ {h \atop ik} \right\}_{(x,x'_{(0)})} \tag{16}$$

so that we may write (15) in the form

$$d_x(\psi) = -g^{ij}(x, y^{(0)}) \left\{ {h \atop ik} \right\}_{(x,x'_{(0)})} y^{(0)}_j y^{(0)}_k \, dx^k \tag{17}$$

This equation gives us the "dilatation" of the figuratrix due to the change of metric.

By a direct application of the laws of transformation it is easily seen that the Christoffel symbols as defined by (13) and (16) cannot be tensors. When the arguments in T_n and T'_n of all the terms correspond to each other, the Christoffel symbols satisfy the same identities as in Riemannian Geometry. We shall give them here for future reference:

$$[ij,k]_{(x,x')} = g_{hk}(x,x') \left\{ {h \atop ij} \right\}_{(x,x')} \tag{18}$$

$$\frac{\partial g^{ik}(x,y)}{\partial x^j} = -g^{hk}(x,y) \left\{ {i \atop hj} \right\}_{(x,x')} - g^{ih}(x,y) \left\{ {k \atop hj} \right\}_{(x,x')} \tag{18)'}$$

The deductions proceed along lines identical to the classical case.

§ 25. Geodesics.

Consider any continuous curve

$$x^i = x^i(s) \tag{19}$$

where s is the arc-length measured along the curve. Since we have

$$ds^2 = g_{ij}(x, dx)\, dx^i\, dx^j$$

for the curve, it follows that the vector

$$x'^i \equiv \frac{dx^i}{ds}$$

is a unit vector, which we shall regard as the tangent-vector to the curve. It follows that

$$\frac{d}{ds}\left[g_{ij}(x,x')\,x'^i x'^j\right] = 0$$

along the curve, i.e. if we consider two neighbouring points x^i and

$x^i + dx^i$ we have

$$\left.\begin{aligned}\frac{\partial g_{ij}(x,x')}{\partial x^k} x'^i x'^j dx^k + \frac{\partial g_{ij}(x,x')}{\partial x'^k} x'^i x'^j dx'^k + 2g_{ij}(x,x')x'^i dx'^j \\ = 0\end{aligned}\right\} \quad (20)$$

This equation enables us to find the tangent-vector $x'^i + dx'^i$ to the curve at the point $x^i + dx^i$. From equation (15) of §2 it follows that the second trem on the left-hand side of this equation vanishes, so that with the aid of (10) and (11) equation (20) can be written in the form

$$[i\,k,j]_{(x,x')} x'^i x'^j dx^k + g_{ij}(x,x')x'^i dx'^j = 0 \quad (21)$$

or, using (18)

$$g_{ij}(x,x')x'^j \left[dx'^i + \left\{ {i \atop h\,k} \right\}_{(x,x')} x'^h dx^k\right] = 0 \quad (21)'$$

This result is analogous to equation (8) (§15, p. 38) which we obtained for the Minkowskian space, and which partly led us to define the length of the vector $\vec{x}''$ as a measure of the curvature of (19). We shall therefore use as a preliminary measure of curvature the values of the system

$$\frac{dx'^i}{ds} + \left\{ {i \atop h\,k} \right\}_{(x,x')} \frac{dx^h}{ds} \cdot \frac{dx^k}{ds} \,{}^* \quad (22)$$

Curves for which the curvature vanishes everywhere are the "straight lines" of the Finsler space. They must therefore satisfy the condition

$$\frac{d^2x^i}{ds^2} + \left\{ {i \atop h\,k} \right\}_{(x,x')} \frac{dx^h}{ds} \cdot \frac{dx^k}{ds} = 0 \quad (23)$$

We shall call such curves the <u>geodesics</u> of our space: this terminology is justified in view of the fact that it can be shown that equations (23) are the Euler equations of the problem in the Calculus of Variations for the function $F(x^i, x'^i)$ (Synge (2)).

In order to obtain the equations (23) in terms of the vectors $\vec{y}$ in T_n' corresponding to $\vec{x}'$ in T_n, we note that we have

$$g^{ij}(x,y)y_i y_j = 1$$

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* We shall treat the question of curvature of curves in more detail in §30, where it is clear that (22) actually represents the components of a vector.

in T_n' along (19), and we find by differentiating and proceeding as before with the aid of equation (17)

$$\frac{dy_j}{ds} - \left\{ {h \atop j\ k} \right\}_{(x, x')} y_h x'^k = 0 \tag{23)'$$

The simplest method to obtain (23) or (23)' is to evaluate the first variation of the integral (4) (§1) from first principles.

§ 26. Parallel Displacement.

Several different definitions of parallel displacement in Finsler spaces have been given by Cartan (4), Berwald (5), (22), Synge (2) and Taylor (3), of which the latter two are identical. However, all these notions depend on the introduction of an arbitrary direction with respect to which parallelism is defined. More precisely, a distinguished, but arbitrary field of curves defines a field of osculating indicatrices, so that at each point of any curve C along which the vector is displaced parallel to itself, there exists an osculating indicatrix with respect to which all measurements are made. The great advantage of the definitions of these authors lies in the fact that under these conditions the length of a vector, or the cosine corresponding to two vectors remain unchanged when these vectors undergo parallel displacement. This is due to the fact that the introduction of a field of osculating indicatrices is virtually equivalent to the introduction of what we may call a "Riemannian strip" along the curve C.

We now propose to establish a notion of <u>absolute parallelism</u> i.e. a notion of parallelism depending only on the vector under consideration and on the change of position.

Consider two points $A(x^i)$ and $B(x^i + dx^i)$ and an arbitrary vector-field X^i in T_n at each point in the neighbourhood of A. There exists one and only one geodesic passing through A and B; its tangent at A will be $x'^i = dx^i/ds$ neglecting terms of the second order. At $x^i + dx^i$

the tangent-vector will be $x'^i + dx'^i$ as given by equation (23). We seek a vector $X^i + d^*X^i$ in T_n at B such that the following conditions are satisfied: Let $X^i + dX^i$ represent the numerical value of the components of the vector-field X^i at B and write

$$DX^i = dX^i - d^*X^i \tag{24}$$

Then we demand

(1) The scalar product of $X^i + d^*X^i$ with the tangent to the geodesic at B is equal to the scalar product of X^i with the tangent at A.

(2) $DX^i = 0$ if X^i is the tangent-vector to the geodesic.

(3) d^*X^i is linear in X^i.

(4) A condition analogous to Ricci's Lemma in Riemannian Geometry.

<u>Note</u>: The following remarks may help to clarify the meaning of these conditions.

The components X^i of the vector-field as measured with respect to the indicatrix at A undergo a change DX^i when we pass from A to B due to

(a) the "natural" variation in the vector-field denoted by dX^i where

$$dX^i = \frac{\partial X^i}{\partial x^k}\, dx^k$$

(b) the variation due to the change in metric, denoted by $-d^*X^i$.

We therefore have $DX^i = dX^i - d^*X^i$ or equation (24).

A natural approach to the problem of parallelism between two vectors at different points A and B would be to stipulate that a parallel vector at B as viewed from A would be identical (apart from length) to the corresponding vector at A. This requires, <u>analytically</u>, that the effect due to the change of metric as we pass from B to A is exactly outweighed by the change dX^i of the components of the vector, i.e. $DX^i = 0$; and <u>geometrically</u>, (i) that the projection of the two vectors at A and B onto the geodesic through A and B be the same (a condition that is certainly satisfied by the tangent-vectors to the geodesic), from which we deduce that d^*X^i is linear in X^i; and (ii) that the indicatrix at A when compared with that at B from an "internal" point of view is indistinguishable from that at B, which is the geometrical meaning of Ricci's Lemma in Riemannian Geometry.

The first condition is expressed by the equation

$$g_{ij}(x,x')x'^i X^j = g_{ij}(x+dx, x'+dx')(x'^i+dx'^i)(X^j+d^*X^j) \tag{25}$$

Considering only quantities of the first order, we have

$$g_{ij}(x+dx, x'+dx') = g_{ij}(x,x') + \frac{\partial g_{ij}(x,x')}{\partial x^k}dx^k + \frac{\partial g_{ij}(x,x')}{\partial x'^k}dx'^k \tag{26}$$

and equation (25) becomes

$$0 = \frac{\partial g_{ij}(x,x')}{\partial x^k}x'^i X^j dx^k + \frac{\partial g_{ij}(x,x')}{\partial x'^k}x'^i X^j dx'^k + g_{ij}(x,x')x'^i d^*X^j + g_{ij}(x,x')X^j dx'^i \tag{27}$$

Using the homogeneity-equation (15) of §2 and applying (12) and (23) to the first and last terms on the right-hand side of (27) respectively, we find

$$\left([ik,j]_{(x,x')} + [jk,i]_{(x,x')}\right)x'^i X^j dx^k - g_{ij}(x,x')\left\{{i \atop hk}\right\}_{(x,x')} X^j x'^h dx^k + g_{ij}(x,x')x'^i d^*X^j = 0$$

or, with the aid of (18)

$$([ik,j]_{(x,x')} + [jk,i]_{(x,x')})x'^i X^j dx^k - [hk,j]_{(x,x')} X^j x'^h dx^k + g_{ij}(x,x')x'^i d^*X^j = 0$$

Replacing the dummy-suffix h by i in the second expression, and applying (16) to the first

$$g_{ik}(x,x')x'^i\left[\left\{{k \atop jk}\right\}_{(x,x')} X^j dx^k + d^*X^k\right] = 0 \tag{28}$$

or

$$y_h\left[\left\{{h \atop jk}\right\}_{(x,x')} X^j dx^k + d^*X^h\right] = 0 \tag{28'}$$

where $\vec{y}$ in T'_n corresponds to $\vec{x}'$ in T_n. The introduction of the y_i is necessary as this enables us to see more clearly the kind of terms that will vanish automatically as a result of our homogeneity-conditions.

Thus, for a variation given by

$$d^*X^k = -\left\{{k \atop jk}\right\}_{(x,x')} X^j dx^k$$

condition (1) would be satisfied: but this is by no means the most general variation to do so. We have seen in §4 that as a result of the homogeneity-properties of the g^{ij} we have

$$y_k \frac{\partial g^{jk}(x,y)}{\partial x'^l} = g_{ik}(x,x')x'^i \cdot \frac{\partial g^{jk}(x,y)}{\partial x'^l} = 0 \qquad (29)$$

so that the addition of a factor of the form

$$f_k(x,x',X) \frac{\partial g^{hk}(x,y)}{\partial x'^l} dx'^l$$

to the above expression for d^*X^h will still allow condition (1) to be satisfied, where $f_k(x, x', X)$ is a linear function in X^i in accordance with condition (3). We shall therefore write

$$d^*X^i + \left\{ {i \atop h\,k} \right\}_{(x,x')} X^h dx^k = f_k(x,x',X) \frac{\partial g^{ik}(x,y)}{\partial x'^l} dx'^l \qquad (30)$$

and proceed to find the functions f_k from conditions (2) to (4).

As a particular case put $X^i \equiv x'^i$, where x'^i is the tangent-vector to the geodesic. Then from (24) we deduce that we can replace d^*X^i by dx'^i. It then follows that the left-hand side of (30) vanishes identically in virtue of the differential equation (23) for the geodesic, so that we have

$$f_k(x,x',x') \frac{\partial g^{ik}(x,y)}{\partial x'^l} dx'^l = 0 \qquad (31)$$

Equation (29) then suggests as a possible value for $f_k(x,x',x')$

$$f_k(x,x',x') = \chi(x,x') g_{mk}(x,x') x'^m$$

where $\chi(x,x')$ is another unknown scalar function to be determined by our conditions. Taking into account condition (3), we write tentatively

$$f_k(x,x',X) = \chi(x,x') g_{mk}(x,x') X^m \qquad (32)$$

Also, on differentiating the identity (3) of §23, we find

$$\frac{\partial g^{ik}(x,y)}{\partial x'^l} g_{mk}(x,x') = -g^{ik}(x,y) \frac{\partial g_{mk}(x,x')}{\partial x'^l} \qquad (33)$$

Thus on substituting for f_k from (32) in (30) and observing (33), we finally have

$$d^*X^i + \left\{ {i \atop h\,k} \right\}_{(x,x')} X^h dx^k + \chi(x,x') g^{ik}(x,y) \frac{\partial g_{mk}(x,x')}{\partial x'^l} X^m dx'^l = 0 \qquad (34)$$

noting that in this expression dx'^l is given by (23).

Apart from the function $\chi(x,x')$ which we shall derive in the next section, equation (34) supplies us with a value for d^*X^i which satisfies conditions (1) to (3). Hence, if in accordance with (24) we write

$$DX^i = dX^i + \left\{ {i \atop h\,k} \right\}_{(x,x')} X^h dx^k + \chi(x,x')\, g^{ik}(x,y) \frac{\partial g_{mk}(x,x')}{\partial x'^l} X^m dx'^l \quad (35)$$

we shall call DX^i the <u>covariant differential</u> of the vector X^i, anticipating for the moment a choice for $\chi(x,x')$ which will make DX^i a vector, thus justifying our nomenclature and notation. We shall then say that the vector X^i has undergone a <u>parallel displacement</u> from A to B if

$$DX^i = 0$$

It follows that this definition of parallel displacement depends only on the vector X^i and on the displacement dx^i from A to B.

A similar procedure may be applied to the covariant components Y_i of a vector in T'_n, and analogously to (34) we obtain (compare § 25)

$$d^*Y_i - \left\{ {h \atop i\,k} \right\}_{(x,x')} Y_h dx^k - \chi(x,x') \frac{\partial g_{ik}(x,x')}{\partial x'^l} g^{mk}(x,x') Y_m dx'^l \quad (36)$$

where it appears that the unknown function $\chi(x,x')$ is the same as that in equation (34).

§ 27. The Covariant Differential of a Tensor of Rank 2 and the determination of the function $\chi(x,x')$.

In order to be able to make use of condition (4) of §26 for the final determination of the function $\chi(x,x')$, it is clear that we must obtain a formula analogous to (35) for the covariant differential of a covariant tensor of renk 2, which must then be applied to the function $g_{ij}(x,x')$. We shall therefore consider a variable tensor-field $A_{ij}(x,x')$. For the sake of geometrical clarity we shall regard A_{ij} as being symmetrical in i and j - this restriction is, however, in no way essential from the analytical point of view, so that we shall take the result as being valid in any case.

An equation of the type

$$A(x^i) = A_{ij}(x, X) \tag{37}$$

where the x^i are the coordinates of a point in our Finsler space and the X^i are the components of a variable vector-field $\vec{X}$ in the tangent Minkowskian spaces T_n represents a surface in each T_n for which x^i and, in consequence, $A(x^i)$ is fixed. We obtain a similar surface in T_n at the neighbouring point $x^i + dx^i$ which may be regarded as having been obtained by "parallel displacement" of the surface (37) in T_n at x^i by a parallel displacement of all those vectors X^i in T_n at x^i to T_n at $x^i + dx^i$, whose end-points lie on the surface (37). Accordingly, we have as equation for this surface at $x^i + dx^i$

$$A(x) + DA(x) = (A_{ij} + dA_{ij})(X^i + d^*X^i)(X^j + d^*X^j) \tag{38}$$

where d^*X^i is given by (34) and dA_{ij} is the change in A_{ij} due to the change in the arguments x^i and X^i. Considering only quantities of the first order, we thus have

$$DA(x) = A_{ij} X^i d^*X^j + A_{ij} X^j d^*X^i + dA_{ij} X^i X^j \tag{39}$$

or, using (34),

$$DA(x) = dA_{ij} X^i X^j - A_{ij} X^i \left[\left\{ {j \atop hk} \right\}_{(x,x')} dx^k + \chi(x,x') g^{jk}(x,y) \frac{\partial g_{hk}(x,x')}{\partial x'^l} dx'^l \right] X^h$$
$$- A_{ij} X^j \left[\left\{ {i \atop hk} \right\}_{(x,x')} dx^k + \chi(x,x') g^{ik}(x,y) \frac{\partial g_{hk}(x,x')}{\partial x'^l} dx'^l \right] X^h$$

Interchanging our indices in order ot obtain the coefficients of $X^i X^j$, we find

$$DA(x) = X^i X^j \Big\{ dA_{ij} - A_{ih} \left[\left\{ {h \atop jk} \right\}_{(x,x')} dx^k + \chi(x,x') g^{hk}(x,y) \frac{\partial g_{jk}(x,x')}{\partial x'^l} dx'^l \right]$$
$$- A_{hj} \left[\left\{ {h \atop ik} \right\}_{(x,x')} dx^k + \chi(x,x') g^{hk}(x,y) \frac{\partial g_{ik}(x,y)}{\partial x'^l} dx'^l \right] \Big\} \tag{40}$$

We shall therefore write

$$DA_{ij} = dA_{ij} - A_{ih} \left[\left\{ {h \atop jk} \right\}_{(x,x')} dx^k + \chi(x,x') g^{hk}(x,y) \frac{\partial g_{jk}(x,x')}{\partial x'^l} dx'^l \right]$$
$$- A_{hj} \left[\left\{ {h \atop ik} \right\}_{(x,x')} dx^k + \chi(x,x') g^{hk}(x,y) \frac{\partial g_{ik}(x,x')}{\partial x'^l} dx'^l \right] \tag{41}$$

so that we may put equation (40) in the form

$$DA(x) = (DA_{ij}) X^i X^j \tag{42}$$

This equation actually represents the variation of the surface (37) as measured by means of vectors in T_n at the point x^i.

Thus, if we write in accordance with equation (40)

$-d^* A_{ij}$ = Variation in A_{ij} due to the change in metric as we pass from x^i to $x^i + dx^i$,

dA_{ij} = Variation in A_{ij} due to the change in arguments x^i and X^i (i.e. the "natural" variation),

DA_{ij} = "Total" variation in A_{ij} as measured by the indicatrix at x^i,

we must have, analogously to (24)

$$DA_{ij} = dA_{ij} - d^* A_{ij} \tag{43}$$

so that $d^* A_{ij}$ can be deduced immediately from equation (41). Again, the value of $X(x, x')$ will be determined so that DA_{ij} is a covariant tensor of rank 2 - in anticipation of this result we shall call DA_{ij}, as defined by (41), the <u>covariant</u> <u>differential</u> of the tensor A_{ij}.

If, in particular, the tensor A_{ij} is such that the "natural" variation dA_{ij} exactly outweighs the variation $d^* A_{ij}$ due to the change of metric, $DA_{ij} = 0$, and we may speak of a "parallel displacement" of the tensor. In this case the displaced surface in T_n at $x^i + dx^i$ as viewed by means of theindicatrix at x^i appears unchanged in vitue of equation (42).

As a special case of this, we may remark that an "internal observer" at x^i is unable to discern any difference between his indicatrix and that at the point $x^i + dx^i$, both appearing to him as "unit spheres". We may therefore now formulate condition (4) of §26 in the form

$$Dg_{ij}(x, x') = 0 \tag{44}$$

where again x' is the tangent-vector to the geodesic through the two points under consideration.

We thus have, on applying equation (41) to the case $A_{ij} \equiv g_{ij}$

$$\frac{\partial g_{ij}(x,x')}{\partial x^k}dx^k + \frac{\partial g_{ij}(x,x')}{\partial x'^l}dx'^l$$
$$= \left[g_{ih}(x,x')\left\{ {h \atop j\ k} \right\}_{(x,x')} + g_{hj}(x,x')\left\{ {h \atop i\ k} \right\}_{(x,x')} \right] dx^k$$
$$+ \chi(x,x')\left[g_{ih}(x,x')\, g^{hk}(x,y)\frac{\partial g_{jk}(x,x')}{\partial x'^l} + g_{hj}(x,x')\, g^{hk}(x,y)\frac{\partial g_{ik}(x,x')}{\partial x'^l} \right] dx'^l$$

In virtue of (18), §24 and (38), §4, this becomes

$$\frac{\partial g_{ij}(x,x')}{\partial x^k}dx^k + \frac{\partial g_{ij}(x,x')}{\partial x'^l}dx'^l = \left([jk,i]_{(x,x')} + [ik,j]_{(x,x')} \right) dx^k$$
$$+ \chi(x,x')\left[\frac{\partial g_{ij}(x,x')}{\partial x'^l} + \frac{\partial g_{ij}(x,x')}{\partial x'^l} \right] dx'^l$$

and finally, using (12) of §24 and the symmetry of the g_{ij} with respect to the indices i and j, we have

$$\frac{\partial g_{ij}(x,x')}{\partial x^k}dx^k + \frac{\partial g_{ij}(x,x')}{\partial x'^l}dx'^l = \frac{\partial g_{ij}(x,x')}{\partial x^k}dx^k + 2\chi(x,x')\frac{\partial g_{ij}(x,x')}{\partial x'^l}dx'^l$$

so that

$$\chi(x,x') = \tfrac{1}{2} \tag{45}$$

This value of χ completely determines our expressions for the covariant differentials.

A straight-forward calculation gives us the transformation-law of the Christoffel symbols, and putting $\chi = \frac{1}{2}$ in our expressions for DX^i and DA_{ij} the transformation-laws for these quantities shows that they are tensors as anticipated. This calculation is not given here, in view of the fact that it has already been carried out by Synge (2) and Taylor (3). The expressions for DX^i and DA_{ij} as given by these authors are similar to the present definitions - the difference being that the arguments $\vec{x}'$ in g_{ij} are the tangent-vectors to an arbitrary direction to which dx'^i also applies. Here, in contrast, x'^i and dx'^i apply to the geodesic through the points under consideration, thus depending solely on the displacement itself. Nevertheless, the calculation of the transformation-laws is almost identically the same in both cases.

CHAPTER VI

PROPERTIES OF THE COVARIANT DERIVATIVE

§ 28. The Extended Christoffel Symbols.

In equations (36), (37) and (41) of the previous chapter it is essential to observe that the quantities dx'^l actually refer to the geodesic through the points $A(x^i)$ and $B(x^i + dx^i)$, these being the points at which the vector-field X^i is studied. It follows from the differential equation (23) of §25 for the geodesics that dx'^l is given by

$$dx'^l = -\left\{ {l \atop j\,k} \right\}_{(x,x')} x'^j\, dx^k \tag{1}$$

and noting that $\chi = \frac{1}{2}$, the above-mentioned equations can be written in the final form

$$DX^i = dX^i + X^h\left[\left\{ {i \atop h\,k} \right\}_{(x,x')} - \frac{1}{2} g^{im}(x,y) \frac{\partial g_{hm}(x,x')}{\partial x'^l} \left\{ {l \atop j\,k} \right\}_{(x,x')} x'^j\right] dx^k \tag{2}$$

and

$$DY_i = dY_i - Y_h\left[\left\{ {h \atop i\,k} \right\}_{(x,x')} - \frac{1}{2} g^{hm}(x,y) \frac{\partial g_{im}(x,x')}{\partial x'^l} \left\{ {l \atop j\,k} \right\}_{(x,x')} x'^j\right] dx^k \tag{3}$$

and

$$\begin{aligned} DA_{ij} = dA_{ij} &- A_{ih}\left[\left\{ {h \atop j\,k} \right\}_{(x,x')} - \frac{1}{2} g^{hm}(x,y) \frac{\partial g_{jm}(x,x')}{\partial x'^l} \left\{ {l \atop n\,k} \right\}_{(x,x')} x'^n\right] dx^k \\ &- A_{hj}\left[\left\{ {h \atop i\,k} \right\}_{(x,x')} - \frac{1}{2} g^{hm}(x,y) \frac{\partial g_{im}(x,x')}{\partial x'^l} \left\{ {l \atop n\,k} \right\}_{(x,x')} x'^n\right] dx^k \end{aligned} \tag{4}$$

These equations suggest the following notation. If, for the sake of brevity we write

$$P^i_{hk}(x,x') = \left\{ {i \atop h\,k} \right\}_{(x,x')} - \frac{1}{2} g^{im}(x,y) \frac{\partial g_{hm}(x,x')}{\partial x'^l} \left\{ {l \atop j\,k} \right\}_{(x,x')} x'^j \tag{5}$$

equations (2), (3) and (4) may be expressed by

$$DX^i = dX^i + P^i_{hk}(x,x') X^h\, dx^k \tag{6}$$

and

$$DY_i = dY_i - P^h_{ik}(x,x') Y_h\, dx^k \tag{7}$$

and

$$DA_{ij} = dA_{ij} - A_{ih} P^{h}_{j\,k}(x,x')\,dx^k - A_{hj} P^{h}_{i\,k}(x,x')\,dx^k \tag{8}$$

In view of the close analogy between these equations and the corresponding results in Riemannian Geometry, we shall call the symbols $P^{i}_{hj}(x, x')$ the <u>extended Christoffel symbols</u> for the direction x' which determines the direction in which the parallel displacement takes place. Again, these symbols are not, in general, tensors as the notation might suggest at first sight. This is easily verified by a direct application of the transformation-laws.

In contrast to Riemannian Geometry, the extended Christoffel symbols are not symmetrical in the lower indices, and we shall therefore write

$$P^{i}_{hk} = p^{i}_{hk} + \pi^{i}_{hk} \tag{9}$$

where we have put

$$\left.\begin{aligned} p^{i}_{hk} &= \tfrac{1}{2}\left(P^{i}_{hk} + P^{i}_{kh}\right) \\ \pi^{i}_{hk} &= \tfrac{1}{2}\left(P^{i}_{hk} - P^{i}_{kh}\right) \end{aligned}\right\} \tag{10}$$

The p^{i}_{hk} are thus the symmetrical components of the P^{i}_{hk}; in order that the π^{i}_{hk} should vanish, so that we should obtain the desired symmetry, it would be necessary and sufficient that we should have for all directions x'

$$\frac{\partial^3 \varphi(x,x')}{\partial x'^k\,\partial x'^m\,\partial x'^l}\left\{\begin{matrix} l \\ j\;k \end{matrix}\right\}_{(x,x')} = \frac{\partial^3 \varphi(x,x')}{\partial x'^k\,\partial x'^m\,\partial x'^l}\left\{\begin{matrix} l \\ j\;h \end{matrix}\right\}_{(x,x')}$$

This condition is not satisfied in the general case.

We may also observe that the operator D satisfies the same laws as the ordinary differential operator d. Thus, if $\vec{X}_{(1)}$ and $\vec{X}_{(2)}$ are any two vectors in T_n at the point $A(x^i)$, we have

(a) $D(X^i_{(1)} + X^i_{(2)}) = DX^i_{(1)} + DX^i_{(2)}$

(b) $D(X^i_{(1)}\, X^j_{(2)}) = X^i_{(1)}\, DX^j_{(2)} + X^j_{(2)}\, DX^i_{(1)}$

The first of these is obvious in view of the linearity of X^i in equation (2); the second may be verified as follows:

If Y_i and Z_i are the covariant components of any two vectors in T_n, then $Y_i Z_j$ are the components of a covariant tensor of rank 2.

Applying equation (8) to $Y_i Z_j$, we have

$$D(Y_i Z_j) = d(Y_i Z_j) - Y_i Z_k P^k_{jk}\, dx^k - Y_k Z_j P^i_{kk}\, dx^k$$
$$= Y_i(dZ_j - Z_k P^h_{jk}\, dx^k) + Z_j(dY_i - Y_k P^h_{ik}\, dx^k)$$

or, noting (7)

$$D(Y_i Z_j) = Y_i DZ_j + Z_j DY_i$$

§ 29. Variation of Length and Angle.

In Riemannian Geometry one of the great advantages of the classical definition of parallel displacement is the preservation of the length of the displaced vector and the preservation of the cosine of the angle between two displaced vectors. Unfortunately, this is only possible in a Finsler space when our parallel displacement is taken with respect to an arbitrary direction and the lenghts and cosines are measured with respect to that osculating indicatrix corresponding to that direction. By having discarded such an approach we have lost this rather desirable simplification, although certain properties analogous to those of the classical parallel displacement are retained. Those that will be required later will be discussed here.

Firstly, the covariant derivative of some scalars, such as the square of the length of a vector, is simply the ordinary derivative of that scalar. For, if $\vec{X}$ is any vector, we have

$$D[g_{ij}(x,X)X^i X^j] = [Dg_{ij}(x,X)]X^i X^j + 2g_{ij}(x,X)X^j DX^i \tag{11}$$

in virtue of property (b) of the previous section and the symmetry of the g_{ij} . Using (6) and (8) of §28, we deduce

$$D[g_{ij}(x,X)X^i X^j] = dg_{ij}(x,X)X^i X^j - [g_{il}(x,X)P^l_{jk} + g_{hj}(x,X)P^h_{ik}]X^i X^j dx^k$$
$$+ 2g_{ij}(x,X)X^j[dX^i + P^i_{hk} X^h dx^k]$$

After suitable permutation of indices, this becomes

$$D[g_{ij}(x,X)X^iX^j] = dg_{ij}(x,X)X^iX^j + 2g_{ij}(x,X)X^j dX^i$$
$$-[g_{ih}(x,X)P^h_{jk}X^iX^j + g_{hi}(x,X)P^h_{jk}X^jX^i - 2g_{ih}(x,X)P^h_{jk}X^jX^i]dx^k$$

and again on permuting the indices in the last three terms on the right-hand side we finally obtain

$$D[g_{ij}(x,X)X^iX^j] = dg_{ij}(x,X)X^iX^j + 2g_{ij}(x,X)X^j dX^i \tag{12}$$

as was expected.

In particular, the covariant derivative of any constant scalar must vanish, as we may always associate with it a vector whose length is equal to the numerical value of that scalar.

If a vector undergoes parallel displacement from $A(x^i)$ to $B(x^i+dx^i)$ it follows from (6) that

$$dX^i = -P^i_{hk}(x,dx)X^h dx^k$$

The change $2d\varphi$ in the square of its length is given by (12):

$$2d\varphi = \frac{\partial g_{ij}(x,X)}{\partial x^k}X^iX^j dx^k + \frac{\partial g_{ij}(x,X)}{\partial x'^k}X^iX^j dX^k$$
$$-2g_{ij}(x,X)P^i_{hk}(x,x')X^jX^h dx^k$$

The second term on the right-hand side vanishes (equation (15), §2) and with the aid of (12), §24, we may write

$$2d\varphi = \{[ik,j]_{(x,x')} + [jk,i]_{(x,x')}\}X^iX^j dx^k - 2g_{ij}(x,X)P^i_{hk}(x,x')X^jX^h dx^k$$

or, on permutation of indices

$$d\varphi = \{[ik,j]_{(x,x')} - g_{ih}(x,X)P^h_{jk}(x,x')\}X^iX^j dx^k \tag{13}$$

This equation thus enables us to calculate the change in length of a vector undergoing parallel displacement in the direction of the geodesic given by the tangent vector $x'^k = dx^k/ds$. As regards the cosine of the vector with respect to the tangent-vector of the geodesic, we have

$$\cos(x',X) = \frac{g_{ij}(x,x')x'^iX^j}{(2\varphi)^{\frac{1}{2}}}$$

and since, by definition of parallel displacement (condition (1), §26), the numerator on the right-hand side remains constant for a parallel displacement along the geodesic, the change in the cosine is given by

$$d(\cos(x', X)) = -\frac{g_{ij}(x, x')x'^i X^j}{(2\varphi)^{3/2}}\, d\varphi = -\cos(x', X)\frac{d\varphi}{2\varphi} \qquad (14)$$

where $d\varphi$ is found from (13). Thus the cosine is preserved only when the length is preserved, and this, in general, is only the case when the displacement takes place in the direction of the vector. This is easily verified analytically by putting $\vec{X} = k\vec{x}'$ in (13) and using our homogeneity equations (§2), where k is a scalar.

In order to be able to compare angles at different points of our Finsler space, we may introduce the following <u>definition</u>: If θ is the angle between the vectors $\vec{X}_{(1)}$, $\vec{X}_{(2)}$ in T_n at $A(x^i)$, where $\vec{X}_{(1)}$, $\vec{X}_{(2)}$ are continuous vector-fields, we define $\theta + D\theta$ to be the angle, measured with respect to the indicatrix at A, between the vectors $\vec{X}_{(1)} + \vec{DX}_{(1)}$, $\vec{X}_{(2)} + \vec{DX}_{(2)}$ where $\vec{DX}_{(1)}$, $\vec{DX}_{(2)}$ are the covariant differentials of $\vec{X}_{(1)}$, $\vec{X}_{(2)}$ corresponding to the displacement to a neighbouring point $B(x^i + dx^i)$.

In the case of a parallel displacement, these covariant differentials vanish, so that then $D\theta = 0$.

§ 30. <u>Curvature of Curves.</u>

As in §25 we consider any continuous curve

$$C: \qquad x^i = x^i(s) \qquad (15)$$

in the Finsler space; where, as before, s denotes arc-length. Again the unit vector

$$x'^i = \frac{dx^i}{ds}$$

is regarded as the tangent-vector to C, so that it follows from the previous section that

$$\frac{D}{Ds}\left[g_{ij}(x, x')x'^i x'^j\right] = 0 \qquad (16)$$

along C. In view of the fact that we are considering infinitesimal displacements along this curve, whose tangent-vectors in T_n are the arguments in the g_{ij} in equation (16), we may apply equation (44) of §27 directly; carrying out the differentiation in (16), we deduce

$$g_{ij}(x,x')x'^i \frac{Dx'^j}{Ds} = 0 \tag{17}$$

i.e. the vector $\overrightarrow{Dx'}/Ds$ is orthogonal with respect to the tangent-vector to the curve in T_n. We shall therefore call it the <u>normal-vector</u> to the curve.

Let us consider two points A and B on the curve, distance ds apart. We denote the tangent-vector at A and B by $\vec{x}'_{(0)}$ and $\vec{x}'_{(0)} + \overrightarrow{dx'}$ respectively. We then form the vector $\vec{x}'_{(0)} + \overrightarrow{d^*x'}$ at B by means of a parallel displacement of the vector $\vec{x}'_{(0)}$ from A to B.

In order to measure the curvature of the curve at A, it is then necessary to measure the angle $D\theta$ between the vectors $\vec{x}'_{(0)} + \overrightarrow{dx'}$ and $\vec{x}'_{(0)} + \overrightarrow{d^*x'}$ with respect to the indicatrix in T_n at A. As seen from A, the first of these vectors is to be regarded as $\vec{x}'_{(0)} + D\vec{x}'$ (taking into account the change of metric as we pass from A to B); the second, in view of the nature of the parallel displacement is regarded as $\vec{x}'_{(0)}$. Making use of the definition given at the end of the last section, $D\theta$ is thus the angle between the vectors $\vec{x}'_{(0)}$ and $\vec{x}'_{(0)} + \overrightarrow{Dx'}$ as measured by the indicatrix at A.

It is clear from equation (17) that the vector $\vec{x}'_{(0)} + \overrightarrow{Dx'}$ is also a unit vector, so that we may use the indicatrix at A to form an analogue of the "spherical image" of Gauss. If A' and B' are the points on the indicatrix corresponding to A and B, let us denote the element of arc A'B' on the surface of the indicatrix by ds_1. Our definition of angle (§9) then allows us to write immediately

$$D\theta = ds_1 \tag{18}$$

As for the Minkowskian space we then define the <u>curvature</u> of the curve (15) by the equation

or
$$\frac{1}{\rho} = \frac{ds_1}{ds} \qquad (19)$$
$$\frac{1}{\rho} = \frac{D\theta}{Ds}$$

Using (28) of §9 we have therefore

$$\frac{1}{\rho} = \sqrt{g_{ij}\left(x, \frac{Dx'}{Ds}\right) \frac{Dx'^i}{Ds} \cdot \frac{Dx'^j}{Ds}} \qquad (20)$$

this expression being the length of the normal vector. If we write out the full expression for the components of this vector, we have in virtue of (2), §28,

$$\frac{Dx'^i}{Ds} = \frac{dx'^i}{ds} + \left\{ {i \atop h\ k} \right\}_{(x,x')} x'^h \frac{dx^k}{ds} - \frac{1}{2} g^{im}(x,x') \frac{\partial g_{hm}(x,x')}{\partial x'^l} \left\{ {l \atop j\ k} \right\}_{(x,x')} x'^h x'^j \frac{dx^k}{ds};$$

but since in this case the displacement dx^k is by definition identical to the vector $x'^k ds$, the last term on the right-hand side is identically equal to zero due to the presence of the factor

$$\frac{\partial g_{hm}(x,x')}{\partial x'^l} x'^h$$

which vanishes (equation (15), §2). For this particular case we therefore have the simple result

$$\frac{Dx'^i}{Ds} = \frac{dx'^i}{ds} + \left\{ {i \atop h\ k} \right\}_{(x,x')} x'^h x'^k \qquad (21)$$

which is the expression (22) as given in §25 as a preliminary measure of curvature. As anticipated, these expressions are components of vectors; also, the curvature of a geodesic vanishes identically in view of equation (23) of §25. (Here again it must be emphasized that although we chose our definition of geodesic in order to satisfy this requirement, the equation for the geodesic can equally well be derived from the Euler equations in the Calculus of Variations.)

<u>Note</u>: An interesting alternative to the notion of curvature was given by Finsler ((1), p.58 - 65), and it may be worth discussing the difference between the two approaches. By using a definition similar to the one introduced by Menger in his study of generalised metric spaces, Finsler defines the curvature by

the formula

$$\frac{1}{\varrho} = \sqrt{\lim_{s \to 0} \frac{24(s-\sigma)}{s^3}} \tag{22}$$

where s is the arc-length of a segment of the curve under consideration and σ is the length of the geodesic joining the end-points of the segment. A fairly long calculation shows that this is equivalent to the equation

$$\frac{1}{\varrho^2} = \frac{\partial^2 F(x,x')}{\partial x'^i \, \partial x'^j} (\ddot{x}^i - \ddot{\xi}^i)(\ddot{x}^j - \ddot{\xi}^j)[F(x,x')]^{-3} \tag{22)'$$

Here $\dot{x}$ and $\ddot{x}$ refer to the curve (15), while $\vec{\xi}'$ is the tangent-vector to a family of geodesics in which the curve is embedded in such a way that the geodesic of this family which passes through the point A at which the curvature of (15) is required touches this curve. The dots denote differentiation with respect to a parameter t, so that $ds/dt = F(x,\dot{x})$. Applying equation (23) of §25 to the geodesic, we have

$$\ddot{\xi}^i = -\left\{ {i \atop j\,k} \right\}_{(x,\xi')} \dot{\xi}^j \dot{\xi}^k = -\left\{ {i \atop j\,k} \right\}_{(x,\xi')} \xi'^j \xi'^k \, [F(x,\dot{x})]^2$$

so that

$$\ddot{x}^i - \ddot{\xi}^i = \left[\frac{dx'^i}{ds} + \left\{ {i \atop j\,k} \right\}_{(x,\xi')} \xi'^j \xi'^k \right] [F(x,\dot{x})]^2$$

Noting that according to this construction $\xi'^i = x'^i$ at the point A, equation (21) shows that

$$\ddot{x}^i - \ddot{\xi}^i = \frac{Dx'^i}{Ds} [F(x,\dot{x})]^2$$

showing that apart from a scalar factor the vector considered by Finsler in (22)' is the same as our normal vector. The essential difference between the two definitions lies in the fact that we use the coefficients

$$\frac{\partial^2 (\frac{1}{2} F^2(x, Dx'))}{\partial x'^i \, \partial x'^j}$$

instead of

$$F(x,x') \frac{\partial^2 F(x,x')}{\partial x'^i \, \partial x'^j}$$

in the quadratic form for the curvature.

Finsler shows (p. 67) that in the two-dimensional Riemannian case his definition is equivalent to the classical definition of geodesic curvature of a surface; and it is evident from equation (27) that in this case (for any number of dimensions) this is also true for the present definition.

§ 31. Geometrical Interpretation of Curvature.

Consider any point O with coordinates $x^i_{(0)}$ on the curve C (15) and let E be a geodesic touching C at O. We denote by P and H two points on E and C respectively, whose distances, measured along the respective curves from O is s. Let ξ^i and x^i represent the coordinates of P and H respectively. Both x^i and ξ^i are functions of s, so that we may write down the power-series expansions

$$x^i(s) = x^i_{(0)} + \frac{dx^i_{(0)}}{ds} s + \frac{1}{2} \frac{d^2x^i_{(0)}}{ds^2} s^2 + \cdots$$

$$\xi^i(s) = \xi^i_{(0)} + \frac{d\xi^i_{(0)}}{ds} s + \frac{1}{2} \frac{d^2\xi^i_{(0)}}{ds^2} s^2 + \cdots$$

Since the tangents of C and E coincide at O, the first two terms in both expansions are equal, so that

$$x^i(s) - \xi^i(s) = \frac{s^2}{2}\left[\frac{d^2x^i_{(0)}}{ds^2} - \frac{d^2\xi^i_{(0)}}{ds^2}\right] + \cdots \tag{23}$$

In view of the fact that E is a geodesic, we have from (23), §25

$$\frac{d^2\xi^i_{(0)}}{ds^2} = -\left\{ {i \atop j\,k} \right\}_{(x_{(0)},\, x'_{(0)})} x'^j_{(0)} x'^k_{(0)}$$

so that equation (23) becomes

$$x^i(s) - \xi^i(s) = \frac{s^2}{2}\left[\frac{dx'^i_{(0)}}{ds} + \left\{ {i \atop j\,k} \right\}_{(x_{(0)},\, x'_{(0)})} x'^j_{(0)} x'^k_{(0)}\right] + \cdots$$

or, using (21)

$$x^i(s) - \xi^i(s) = \frac{s^2}{2}\left[\frac{Dx'^i}{Ds}\right] + \cdots \tag{24}$$

It follows that $x^i - \xi^i$ will be of the second order of smallness as compared with s, and we may regard $\overrightarrow{HP}$ as an infinitesimal vector $\vec{\eta}$ in T_n at H, where the error involved in in identifying the Finsler space with the tangent space at H will be at most of the third order. (In fact, we may regard $\vec{\eta}$ as the tangent-vector to the geodesic joining H and P in T_n at H). Thus, putting

$$\eta^i(s) = x^i(s) - \xi^i(s) \tag{25}$$

it follows from (23) and (17) that in the limit the vector $\overrightarrow{HP}$ is orthogonal with respect to the tangent-vector to C in T_n at O. Also, we have

$$\lim_{s\to 0} \frac{\eta^2}{s^4} = \lim_{s\to 0} \frac{g_{ij}(x,\eta)\eta^i\eta^j}{s^4} = \frac{1}{4} g_{ij}\left(x, \frac{Dx'}{Ds}\right) \frac{Dx'^i}{Ds} \cdot \frac{Dx'^j}{Ds}$$

in virtue of (24), and we therefore deduce from equation (20) for the curvature that

$$\lim_{s\to 0} \frac{\eta^2}{s^4} = \frac{1}{4\rho^2} \tag{26}$$

Expressed geometrically, this means

$$|HP| = \frac{\delta s^2}{2\rho} + O(\delta s^3) \tag{27}$$

where δs represents a small displacement along the curves E and C (15).

This result expresses the most fundamental property of curvature of curves: a similar equation deduced in Chapter IV (p. 39) for Minkowskian spaces was used as the foundation on which the Differential Geometry as developed there was built up.

It is possible to extend our investigations as before to curves on surfaces embedded in a Finsler space in order to obtain the Dupin Indicatrix for such surfaces. We shall refrain from including such a discussion as the only difference contained in the arguments is found to be in tedious investigations into orders of magnitude of small variations which arise from the change of metric as we pass from some point to a neighbouring point. It has been found that since we deal only with small distances and angles, these effects could, ingeneral, be neglected, while the principal trend of the argument proceeded as in Chapter IV. It need hardly be pointed out that instead of using the "affine" parallelism of Minkowskian spaces, we made use of the notion of parallelism as developed in the previous chapter - for such purposes an absolute parallelism is naturally essential.

In conclusion it may be noted that the second part of Finsler's work (1) is devoted to similar questions of Differential Geometry, but as a result of the completely different approach the results obtained by him do not correspond with our present investigations.

CHAPTER VII

GEODESIC DEVIATION

§ 32. Curvature as "Defect".

In Riemannian spaces a well-known method of deriving a measure of the curvature of the space is the calculation of the change in a vector when it is displaced parallel to itself around an infinitesimal closed circuit. An analogous method was used by Berwald (22) in Finsler spaces, but the notion of parallelism employed by him depended on an arbitrary field of curves (compare § 26). When the same process is applied using our definition of parallelism, it is found that the final result is unsatisfactory due to the unavoidable presence of second order derivatives of a pair of families of curves introduced in order to obtain a circuit. The calculations are somewhat complicated and are not given here as we shall approach the subject of the curvature of a Finsler space from a completely different angle.

If we rember that the tangent Minkowskian space T_n at any point of our Finsler space is affine, i.e. a space of zero curvature, it is clear that any difference between T_n and the Finsler space at the point under consideration will give us some measure of the curvature of the Finsler space at that point. Such differences may be regarded as "defects" in comparison with the affine character of the local T_n.

Defects whose magnitudes are of the first order of smallness when compared with an infinitesimal displacement are adequately expressed with the aid of the extended Christoffel symbols; it is therefore evident that we shall have to seek defects whose magnitudes are of the second order of smallness. The following is a

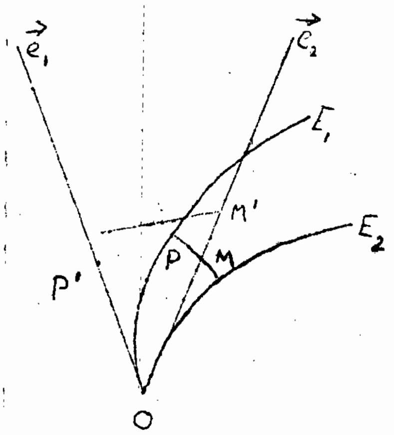

Fig. 6

brief summary of our line of approach to this problem:

In T_n at any point O of our Finsler space consider two directions $\vec{e}_1$, $\vec{e}_2$ making a small angle with each other. Mark off points P', M' along $\vec{e}_1$ and $\vec{e}_2$ respectively so that OP' = OM' = s. Then as a result of the affine structure of T_n we shall show that

$$\frac{d(\overrightarrow{P'M'})}{ds} = \text{const.}$$

and

$$\frac{d^2(\overrightarrow{P'M'})}{ds^2} = 0 \qquad (1)$$

If we consider the geodesics E_1 and E_2 passing through O and having as tangent-vectors $\vec{e}_1$, $\vec{e}_2$ we can construct points P and M on E_1 and E_2 respectively such that the arc-lengths OP and OM are equal to s, (Fig. 6). If s is small, the length of the segment PM of the geodesic through P and M will be at most of the second order of smallness in comparison with the angle between E_1 and E_2, so that we may regard $\overrightarrow{PM}$ as an infinitesimal vector in T_n at P. The calculations of this chapter will show that for the Finsler space

$$\frac{D^2(\overrightarrow{PM})}{Ds^2} \neq 0 \qquad (2)$$

so that in view of equation (1) the quantity $D^2(\overrightarrow{PM})$ will be a measure of a defect of second order.

Clearly the calculation of $D^2(\overrightarrow{PM})$ is simply the problem of Geodesic Deviation in Finsler spaces and we shall therefore devote this chapter to a study of this subject. We shall derive a notion of curvature, and it will be seen that in this way we are led to the equation of Jacobi which plays such an essential part in the investigation of the second Variation in the Calculus of Variations.

This enables us to see quite clearly why an approach along these lines is a more natural one for Finsler spaces, remebering that, after all, these spaces originate primarily from the Calculus of Variations.

Furthermore, our calculations will enable us to compare the indicatrix in T_n at 0 with the geodesic sphere at O. Geodesic spheres are defined to be those surfaces in the Finsler, space which are the locus of points on all geodesics through O and equidistant from O, i.e. the "unit spheres" in the Finsler space. Such defects will illustrate our notion of curvature very clearly. Finally, it need hardly be pointed out that Jacobi's equation enables us to study problems in the Calculus of Variations in the Large with the aid of our Finsler space, and we may expect that such investigations will lead to interesting results analogous to Bonnet's Inequalities for closed convex surfaces in a Euclidean space together with other well-known theorems of that kind. As far as I know, these problems have not yet been studied; but it is likely that the key to their solution is to be found in this and the following chapter. This view has been confirmed to some extent by the partial results obtained so far. For the present we shall desist from any such attempt and confine ourselves to questions of a purely local character.

§ 33. Calculation of $D(\overrightarrow{PM})$.

We shall consider two geodesics E_1 and E_2 issuing from O, and we shall denote the arc-length measured along E_1 and E_2 by s and σ respectively. The angle between E_1 and E_2 as measured by the indicatrix at O is small. The coordinates of points on E_1 are denoted by x^i and those on E_2 by y^i. For the kind of problem indicated in the last section it is necessary to obtain a more general view of the behaviour of two neighbouring geodesics so that we shall consider a wider law of correspondence between points P on E_1 and points M on E_2. Regarding x^i and ψ^i as functions of s and σ respectively, we must adopt some kind of functional relationship between s and σ , where s is regarded as the independent variable. We stipulate that this law of correspondence is such that the arc-length of the segment PM of the geodesic through P and M together with its rate of variation with respect to s is of the same order of smallness as the angle $d\varphi$ between E_1 and E_2. Denoting this length by ξ , and the tangent-vector to the geodesic PM in T_n at P by $\vec{\chi}$, we shall write

$$\xi^i = \xi \chi^i$$

so that the ξ^i and their first derivatives with respect to s are of the first order of smallness when s and σ are finite. We then have

$$\xi^i(s) = \psi^i(\sigma) - x^i(s) \qquad (3)$$

when we neglect the error involved in identifying the vector $\vec{\xi}$ in T_n at P with the displacement PM. In general, this error will be of the second order of smallness compared with $d\varphi$, but later, when we let $s \longrightarrow 0$ it will be of at most the third order. The correspondence between points P on E_1 and M on E_2 is best expressed in the form*

$$\frac{d\sigma}{ds} = 1 + \lambda(s) \qquad (4)$$

where $\lambda(s)$ is an arbitrary, continuous and differentiable function of s ; the only restriction on $\lambda(s)$ being contained in the above stipulation. We shall show that this entails that $\lambda(s)$ is also of the first order of smallness.

If we denote the tangents to E_1 and E_2 by $\vec{x}'$ and $\vec{\psi}'$ respectively, differentiation of (3) gives

$$\psi'^i(\sigma)\frac{d\sigma}{ds} = x'^i(s) + \frac{d\xi^i}{ds} \qquad (5)$$

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*This was suggested by a method used by Levi-Civita ((33), p.210) in his calculation of geodesic deviation in Riemannian spaces.

But on E_2 we have

$$g_{ij}(\psi,\psi')\psi'^i\psi'^j = 1$$

so that on substitution for ψ' from (5) we obtain

$$\left(\frac{d\sigma}{ds}\right)^2 = g_{ij}(\psi,\psi')\left(x'^i + \frac{d\xi^i}{ds}\right)\left(x'^j + \frac{d\xi^j}{ds}\right) \tag{6}$$

But neglecting quantities of the second order of smallness, we may write

$$g_{ij}(\psi,\psi') = g_{ij}(x,x') + \frac{\partial g_{ij}(x,x')}{\partial x^k}\xi^k + \frac{\partial g_{ij}(x,x')}{\partial x'^k}(\psi'^k - x'^k) \tag{7}$$

Also, on E_1 we have

$$g_{ij}(x,x')x'^i x'^j = 1 \tag{8}$$

and using these results we find that equation (6) then assumes the form

$$\left(\frac{d\sigma}{ds}\right)^2 = 1 + \frac{\partial g_{ij}(x,x')}{\partial x^k}x'^i x'^j \xi^k + \frac{\partial g_{ij}(x,x')}{\partial x'^k}(\psi'^k - x'^k)x'^i x'^j + 2g_{ij}(x,x')x'^i\frac{d\xi^j}{ds} \tag{9}$$

where we note that the third expression on the right-hand side vanishes in virtue of (15), §2.

Denoting by y_j the components of the vector in T'_n at P corresponding to the vector $\vec{x}'$ in T_n, we may, using equations (12), (18) of §24 express the second term as follows:

$$\begin{aligned}\frac{\partial g_{ij}(x,x')}{\partial x^k}x'^i x'^j \xi^k &= \left\{[ik,j]_{(x,x')} + [jk,i]_{(x,x')}\right\}x'^i x'^j \xi^k \\ &= \left[g_{jh}(x,x')\left\{{h \atop ik}\right\}_{(x,x')} + g_{ih}(x,x')\left\{{h \atop jk}\right\}_{(x,x')}\right]x'^i x'^j \xi^k \\ &= y_h\left\{{h \atop ik}\right\}_{(x,x')}x'^i\xi^k + y_h\left\{{h \atop jk}\right\}_{(x,x')}x'^j\xi^k \\ &= 2y_j\left\{{j \atop ik}\right\}_{(x,x')}x'^i\xi^k\end{aligned} \tag{10}$$

With the aid of this result equation (9) becomes

$$\left(\frac{d\sigma}{ds}\right)^2 = 1 + 2y_j\left[\frac{d\xi^j}{ds} + \left\{{j \atop ik}\right\}_{(x,x')}x'^i\xi^k\right] \tag{11}$$

The covariant differential of $\vec{\xi}$ in the direction of E_1 is given by ((2), §28)

$$D\xi^j = d\xi^j + \left\{{j \atop ik}\right\}_{(x,x')}\xi^k dx^i + \frac{1}{2}g^{jm}(x,x')\frac{\partial g_{mk}(x,x')}{\partial x'^l}\xi^k dx'^l$$

where again dx'^{l} refers to E_1. Thus

$$y_j D\xi^j = y_j [d\xi^j + \left\{ {j \atop i\,k} \right\}_{(x,x')} x'^i \xi^k ds] + \frac{1}{2} g_{jn}(x,x') x'^n g^{jm}(x,x') \frac{\partial g_{ml}(x,x')}{\partial x'^l} \xi^k dx'^l$$

Applying equation (3) of §23 to the last term on the right-hand side, we find

$$y_j D\xi^j = y_j [d\xi^j + \left\{ {j \atop i\,k} \right\}_{(x,x')} x'^i \xi^k ds] + \frac{1}{2} x'^m \frac{\partial g_{mk}(x,x')}{\partial x'^l} \xi^k dx'^l$$

so that again in virtue of (15), §2, the last term will vanish, and we finally have

$$y_j \frac{D\xi^j}{Ds} = y_j \left[\frac{d\xi^j}{ds} + \left\{ {j \atop i\,k} \right\}_{(x,x')} x'^i \xi^k \right] \qquad (12)$$

Comparing this result with equation (11), we have

$$\left(\frac{d\sigma}{ds}\right)^2 = 1 + 2 y_j \frac{D\xi^j}{Ds} \qquad (13)$$

Noting that we have stipulated that $D\xi^j/Ds$ is a quantity of the first order of smallness, it is clear that we have as a first approximation

$$\frac{d\sigma}{ds} \doteq 1 + y_j \frac{D\xi^j}{Ds} \qquad (13)'$$

Together with equation (4) this yields

$$\left. \begin{aligned} y_j \frac{D\xi^j}{Ds} &= \lambda(s) \\ g_{ij}(x,x') x'^i \frac{D\xi^j}{Ds} &= \lambda(s) \end{aligned} \right\} \qquad (14)$$

showing that as a result of our stipulation, $\lambda(s)$ is also a quantity of the first orler of smallness as we expected. Also, the geometrical meaning of $\lambda(s)$ becomes clear: $\lambda(s)$ is the projection of $\overrightarrow{D\xi}/Ds$ onto the geodesic E_1.

Also, it is worth noting that if we choose $\lambda(s) \equiv 0$, so that the law of correspondence between P and M would be given by $\sigma = s$, it follows from (14) that

$$g_{ij}(x,x') x'^i \frac{D\xi^j}{Ds} = 0 \qquad (14)'$$

which means that the vector $\overrightarrow{D\xi}/Ds$ is then orthogonal with respect to the tangent vector in T_n at P to the geodesic E_1. This result could have been expected from the purely geometrical point of view.

§ 34. Calculation of $d^2(\overrightarrow{PM})$.

If we substitute the value for $d\sigma/ds$ as given by (4) in equation (5), we obtain

$$\psi'^i(\sigma) = x'^i(s) + \frac{d\xi^i}{ds} - \psi'^i(\sigma)\lambda(s) \tag{15}$$

Also, differentiating (4) and (5) with respect to s, we have

$$\frac{d^2\sigma}{ds^2} = \frac{d\lambda}{ds} \tag{16}$$

and

$$\psi''^i(\sigma)\left(\frac{d\sigma}{ds}\right)^2 + \psi'^i(\sigma)\frac{d^2\sigma}{ds^2} = x''^i(s) + \frac{d^2\xi^i}{ds^2} \tag{17}$$

Expressing the second term on the left-hand side of (17) with the aid of (15) and (16) in the form

$$\psi'^i(\sigma)\frac{d^2\sigma}{ds^2} = \frac{d\lambda}{ds}\left[x'^i(s) + \frac{d\xi^i}{ds} - \psi'^i(\sigma)\lambda(s)\right]$$

we see that if we neglect quantities of the second order of smallness, we may write

$$\psi'^i(\sigma)\frac{d^2\sigma}{ds^2} = x'^i(s)\frac{d\lambda}{ds} \tag{18}$$

and equation (17) becomes

$$\psi''^i(\sigma)\left(\frac{d\sigma}{ds}\right)^2 = x''^i(s) + \frac{d^2\xi^i}{ds^2} - x'^i(s)\frac{d\lambda}{ds} \tag{19}$$

But it follows from equation (23) of §25 that we may express the equation to the geodesic E_2 in the form

$$\psi''^i(\sigma) = -\left\{{i \atop jh}\right\}_{(\psi,\psi')}\psi'^j\psi'^h \tag{20}$$

If we multiply (20) by $(d\sigma/ds)^2$ and substitute from (5) the values of ψ', we obtain

$$\left(\frac{d\sigma}{ds}\right)^2\psi''^i(\sigma) = -\left\{{i \atop jh}\right\}_{(\psi,\psi')}\left(x'^j + \frac{d\xi^j}{ds}\right)\left(x'^h + \frac{d\xi^h}{ds}\right)$$

and again on neglecting terms of the second order of smallness

$$\left(\frac{d\sigma}{ds}\right)^2 \psi''^i(\sigma) = -\left\{{i \atop j\,h}\right\}_{(\psi,\psi')} x'^j x'^h - 2\left\{{i \atop j\,h}\right\}_{(\psi,\psi')} x'^j \frac{d\xi^h}{ds} \qquad (21)$$

To the same degree of approximation we have also

$$\left\{{i \atop j\,h}\right\}_{(\psi,\psi')} = \left\{{i \atop j\,h}\right\}_{(x,x')} + \frac{\partial}{\partial x^k}\left\{{i \atop j\,h}\right\}_{(x,x')} \xi^k + \frac{\partial}{\partial x'^k}\left\{{i \atop j\,h}\right\}_{(x,x')} dx'^k$$

where

$$dx'^k = \psi'^k(\sigma) - x'^k(s) \qquad (22)$$

or, in virtue of (15)

$$dx'^k = \frac{d\xi^k}{ds} - \psi'^k(\sigma)\,\lambda(s) \qquad (22)'$$

Taking into account the equation

$$x''^i(s) + \left\{{i \atop j\,h}\right\}_{(x,x')} x'^j x'^h = 0$$

for the geodesic E_1, it follows from (21) that

$$\left(\frac{d\sigma}{ds}\right)^2 \psi''^i(\sigma) = x''^i(s) - \frac{\partial}{\partial x^k}\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j x'^h \xi^k - \frac{\partial}{\partial x'^k}\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j x'^h dx'^k - 2\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j \frac{d\xi^h}{ds}$$

again neglecting terms of the second order.

Comparing this result with equation (19), we find

$$\frac{d^2\xi^i}{ds^2} - x'^i(s)\frac{d\lambda}{ds} = -2\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j \frac{d\xi^h}{ds} - \frac{\partial}{\partial x^k}\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j x'^h \xi^k - \frac{\partial}{\partial x'^k}\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j x'^h dx'^k \qquad (23)$$

The last term on the right-hand side requires special reduction. We have, by equations (16), (18) and (13) of §24:

$$\frac{\partial}{\partial x'^k}\left\{{i \atop j\,h}\right\}_{(x,x')} = \frac{\partial}{\partial x'^k}\left[g^{im}(x,x')[j\,h, m]_{(x,x')}\right]$$

$$= \frac{\partial g^{im}(x,x')}{\partial x'^k}\, g_{mp}(x,x')\left\{{p \atop j\,h}\right\}_{(x,x')} + \frac{1}{2} g^{im}(x,x')\frac{\partial}{\partial x'^k}\left[\frac{\partial g_{hm}(x,x')}{\partial x^j} + \frac{\partial g_{jm}(x,x')}{\partial x^h} - \frac{\partial g_{jh}(x,x')}{\partial x^m}\right]$$

With the help of equation (4) of §23 we deduce

$$\frac{\partial}{\partial x'^k}\left\{{i \atop j\,h}\right\}_{(x,x')} x'^j x'^h dx'^k = -g^{im}(x,x')\frac{\partial g_{mp}(x,x')}{\partial x'^k}\left\{{p \atop j\,h}\right\}_{(x,x')} x'^j x'^h dx'^k + \frac{1}{2} g^{im}(x,x')\left[\frac{\partial^2 g_{hm}(x,x')}{\partial x'^k \partial x^j} + \frac{\partial^2 g_{jm}(x,x')}{\partial x'^k \partial x^h} - \frac{\partial^2 g_{jh}(x,x')}{\partial x'^k \partial x^m}\right] x'^j x'^h dx'^k \qquad (24)$$

According to equation (15)' of §2, each of these last three terms vanishes identically. Replacing for the moment - for the sake of brevity only - the term

$$\left\{ {p \atop j\,k} \right\}_{(x,x')} x'^j x'^k$$

by $-x''^p$ (in virtue of the equation to the geodesic E_1), and substituting for dx'^k from (22) we find

$$\frac{\partial}{\partial x'^k} \left\{ {i \atop j\,k} \right\}_{(x,x')} x'^j x'^k \, dx'^k = g^{im}(x,x') \frac{\partial g_{mp}(x,x')}{\partial x'^k} x''^p (\psi'^k - x'^k)$$

Using equation (15) of §2 we see that the last term on the right-hand side vanishes once more, and we are left with

$$\frac{\partial}{\partial x'^k} \left\{ {i \atop j\,k} \right\}_{(x,x')} x'^j x'^k \, dx'^k = g^{in}(x,x') \frac{\partial g_{mp}(x,x')}{\partial x'^k} x''^p \psi'^k \qquad (25)$$

If, instead of (22) we substitute from (22)' for dx'^k in equation (24), we find

$$\frac{\partial}{\partial x'^k} \left\{ {i \atop j\,k} \right\}_{(x,x')} x'^j x'^k \, dx'^k \qquad (26)$$
$$= g^{im}(x,x') \frac{\partial g_{mp}(x,x')}{\partial x'^k} x''^p \left[\frac{d\xi^k}{ds} - \psi'^k \lambda(s) \right]$$

and on multiplying equation (25) by $\lambda(s)$ and adding the result to equation (26), we obtain

$$(1+\lambda(s)) \frac{\partial}{\partial x'^k} \left\{ {i \atop j\,k} \right\}_{(x,x')} x'^j x'^k \, dx'^k = g^{im}(x,x') \frac{\partial g_{mp}(x,x')}{\partial x'^k} x''^p \frac{d\xi^k}{ds} \qquad (27)$$

Dividing by $(1+\lambda(s))$ and neglecting $[\lambda(s)]^2$, this becomes

$$\frac{\partial}{\partial x'^k} \left\{ {i \atop j\,k} \right\}_{(x,x')} x'^j x'^k \, dx'^k \qquad (27)'$$
$$= g^{im}(x,x') \frac{\partial g_{mp}(x,x')}{\partial x'^k} x''^p \left[\frac{d\xi^k}{ds} - \lambda(s) \frac{d\xi^k}{ds} \right]$$

It is to be noticed that equation (27) is exact in the sense that it has been deduced from previous equations in which no approximations have been made. When we substitute (27)' in (23) we can, however, neglect the term $\lambda(s)\, d\xi^k/ds$, and on replacing x''^p by its value along the geodesic E_1, equation (23) will then assume the form

$$\frac{d^2\xi^i}{ds^2} + 2\left[\left\{{i \atop j\;h}\right\}_{(x,x')} x'^j - \tfrac{1}{2} g^{im}(x,x') \frac{\partial g_{mk}(x,x')}{\partial x'^p}\left\{{p \atop j\;k}\right\}_{(x,x')} x'^j x'^k\right]\frac{d\xi^h}{ds}$$

$$= x'^i \frac{d\lambda}{ds} - \frac{\partial}{\partial x^k}\left\{{i \atop j\;h}\right\}_{(x,x')} x'^j x'^h \xi^k$$

where it is to be noted that the valid substitution

$$\frac{\partial g_{mp}(x,x')}{\partial x'^k} = \frac{\partial g_{mk}(x,x')}{\partial x'^p}$$

has been made, this involving only a change in the order of differentiation of the function $\varphi(x,x')$.

Introducing once more the "extended" Christoffel symbols by means of equation (5) of §28, we see that the last result can be expressed in the more convenient form

$$\frac{d^2\xi^i}{ds^2} + 2P^i_{kj}(x,x')x'^j \frac{d\xi^h}{ds} = x'^i \frac{d\lambda}{ds} - \frac{\partial}{\partial x^k}\left\{{i \atop j\;h}\right\}_{(x,x')} x'^j x'^h \xi^k \qquad (28)$$

For the sake of brevity we shall write for $P^i_{kj}(x,x')$ simply P^i_{hj} as there will be no possible danger of confusion as regards the arguments of these functions for the next few sections.

In order to derive an equation analogous to Jacobi's equation from (28) it is clear that we must find an expression for $D^2\xi^i/Ds^2$ which does not involve $d\xi^i/ds$ as in (28).

§ 35. Calculation of $D^2(\overrightarrow{PM})$.

Consider once more the covariant differential of the vector $\vec{\xi}$ as we pass from the point x^k on E_1 to a neighbouring point $x^k + dx^k$ also on E_1. We then have firstly from equation (6) of §28

$$\frac{D\xi^i}{Ds} = \frac{d\xi^i}{ds} + P^i_{hk}\xi^h x'^k$$

and secondly

$$\frac{D^2\xi^i}{Ds^2} = \frac{d}{ds}\left[\frac{d\xi^i}{ds} + P^i_{hk}\xi^h x'^k\right] + P^i_{hk}\left[\frac{d\xi^h}{ds} + P^h_{lj}\xi^l x'^j\right]x'^k$$

Carrying out the differentiation in the first term on the right-hand side and simplifying we find

$$\frac{D^2\xi^i}{Ds^2} = \frac{d^2\xi^i}{ds^2} + \frac{\partial P^i_{hk}}{\partial x'^l}\xi^h x'^k x''^l + \frac{\partial P^i_{hk}}{\partial x^j}\xi^h x'^k x'^j$$
$$+ 2P^i_{hj}x'^j\frac{d\xi^h}{ds} + P^i_{hk}\xi^h x''^k + P^i_{lk}P^l_{hj}\xi^h x'^j x'^k \qquad \ldots\ldots (29)$$

Using the differential equation for the geodesic E_1, we also have for the second last term on the right-hand side of (29):

$$P^i_{hk}\xi^h x''^k = -P^i_{hk}\xi^h\left\{{k \atop pj}\right\}_{(x,x')} x'^p x'^j$$
$$= -P^i_{hl}\left\{{l \atop jk}\right\}_{(x,x')}\xi^h x'^j x'^k$$

on interchange of dummy-suffixes.

Making use of this result, we compare the equations (28) and (29) by eliminating the term $d^2\xi^i/ds^2$. After factorisation and further interchange of dummy-suffixes we find

$$\frac{D^2\xi^i}{Ds^2} - x'^i\frac{d\lambda}{ds} = -\left[\frac{\partial}{\partial x^h}\left\{{i \atop jk}\right\}_{(x,x')} - \frac{\partial P^i_{hk}}{\partial x^j} + P^i_{hl}\left\{{l \atop jk}\right\}_{(x,x')} - P^i_{lk}P^l_{hj}\right]x'^j x'^k \xi^h \qquad (30)$$
$$- \frac{\partial P^i_{hk}}{\partial x'^l}\xi^h\left\{{l \atop pj}\right\}_{(x,x')} x'^p x'^j x'^k$$

It is to be noted that here we have succeeded in getting rid of the term involving $d\xi^h/ds$ and our results will only be useful because of this.

Equation (30) can be written in a more symmetrical form as follows: We have from (5), §28, that

$$\frac{\partial P^i_{jk}}{\partial x^h} = \frac{\partial}{\partial x^h}\left\{{i \atop jk}\right\} - \frac{1}{2}\frac{\partial g^{im}}{\partial x^h}\cdot\frac{\partial g_{mj}}{\partial x'^l}\left\{{l \atop pk}\right\}x'^p - \frac{1}{2}g^{im}\frac{\partial^2 g_{mj}}{\partial x^h\partial x'^l}\left\{{l \atop pk}\right\}x'^p$$
$$- \frac{1}{2}g^{im}\frac{\partial g_{mj}}{\partial x'^l}\frac{\partial}{\partial x^h}\left\{{l \atop pk}\right\}x'^p - \frac{1}{2}g^{im}\frac{\partial g_{mj}}{\partial x'^l}\left\{{l \atop pk}\right\}\frac{\partial x'^p}{\partial x^h}$$

where the arguments in the g_{ij} and the Christoffel symbols are (x,x') throughout. In virtue of (15) and (15)' of §2 we therefore find on multiplication of this equation by $x'^j x'^k \xi^h$

$$\frac{\partial P^i_{jk}(x,x')}{\partial x^h}x'^j x'^k\xi^h = \frac{\partial}{\partial x^h}\left\{{i \atop jk}\right\}_{(x,x')}x'^j x'^k\xi^h \qquad (31)$$

Furthermore,

$$P^{\ell}_{jk}(x,x')x'^j x'^k = \left\{ {\ell \atop j\,k} \right\}_{(x,x')} x'^j x'^k - \tfrac{1}{2} g^{\ell m}(x,x') \frac{\partial g_{jm}(x,x')}{\partial x'^p} \left\{ {p \atop h\,k} \right\}_{(x,x')} x'^h x'^j x'^k$$

where again the last term vanishes identically. Using this result together with equation (31) in (30), we finally obtain

$$\frac{D^2\xi^i}{Ds^2} - x'^i \frac{d\lambda}{ds} = -\Big[\frac{\partial P^i_{jk}(x,x')}{\partial x^h} - \frac{\partial P^i_{hk}(x,x')}{\partial x^j} + P^i_{h\ell}(x,x')P^{\ell}_{jk}(x,x') \tag{32}$$
$$- P^i_{\ell k}(x,x')P^{\ell}_{hj}(x,x') + \frac{\partial P^i_{hk}(x,x')}{\partial x'^{\ell}} \left\{ {\ell \atop p\,j} \right\}_{(x,x')} x'^p \Big] \xi^h x'^j x'^k$$

Since the left-hand side of this equation is a contravariant vector and $\xi^h x'^j x'^k$ are components of a contravariant tensor of rank 3, we shall put

$$R^i_{khj}(x,x') = \frac{\partial P^i_{jk}(x,x')}{\partial x^h} - \frac{\partial P^i_{hk}(x,x')}{\partial x^j} + P^i_{h\ell}(x,x')P^{\ell}_{jk}(x,x')$$
$$- P^i_{\ell k}(x,x')P^{\ell}_{hj}(x,x') + \frac{\partial P^i_{hk}(x,x')}{\partial x'^{\ell}} \left\{ {\ell \atop p\,j} \right\}_{(x,x')} x'^p \tag{33}$$

so that equation (32) can be expressed more simply by writing

$$\frac{D^2\xi^i}{Ds^2} - x'^i \frac{d\lambda}{ds} = -R^i_{khj}(x,x')\xi^h x'^j x'^k \tag{34}$$

We now observe that this equation is true for an arbitrary vector $\vec{\xi}$ - arbitrary in the sense that it depends on the arbitrary function $\lambda(s)$, so that $\vec{\xi}$ is by no means determined by $\vec{x}'$. It follows that the quantities

$$R^i_{khj}(x,x')x'^j x'^k \tag{35}$$

are the components of a mixed tensor: contravariant in i and covariant in h, so that we shall regard (35) as the components of the <u>curvature</u> <u>tensor</u> for the direction $\vec{x}'$ at the point P. It does not follow directly that R^i_{khj} is itself a tensor as there is the possibility of addition of terms which vanish when multiplied by $x'^j x'^k$, in view of the homogeneity relations (15) and (15)' of §2. The tensor-character of R^i_{khj} can onyl be tested by a direct transformation, but as we shall only be concerned with expressions involving (35) we have omitted such an investigation.

§ 36. Comparison with the Minkowskian Tangent-space.

In order to substantiate the remarks made in the introductory §32 we shall show briefly what can be obtained by applying the methods of the previous sections to two fixed directions $\vec{x}'$, $\vec{\psi}'$ in T_n at O. As before, we assume that the angle between the unit vectors $\vec{x}'$, $\vec{\psi}'$ is infinitesimal, i.e. $(x'^i - \psi'^i)$ is of the first order of smallness.

The coordinates of P', M' on $\vec{x}'$ and $\vec{\psi}'$ are given by

$$\psi^i = \sigma\psi'^i \quad ; \quad x^i = s x'^i$$

As in §33 we write

$$\frac{d\sigma}{ds} = 1 + \lambda(s) \tag{4'}$$

and putting $\vec{PM} = \vec{\xi}$, we have, since $\vec{OP} + \vec{PM} = \vec{OM}$

$$\xi^i = \sigma\psi'^i - s x'^i \tag{3'}$$

or, on differentiating with respect to s,

$$\frac{d\xi^i}{ds} = \frac{d\sigma}{ds}\psi'^i - s x'^i \tag{5'}$$

Now, since $\vec{x}'$ and $\vec{\psi}'$ are unit vectors, it follows from equation (13) of Chapter III that

$$\cos(x', \psi') = 1 - \tfrac{1}{2} g_{ij}(x, x')(x'^i - \psi'^i)(x'^j - \psi'^j) + \dots$$

so that

$$g_{ij}(x, x') x'^i \psi'^j = 1$$

up to the second order. Multiplying (5)' by $g_{ij}(x, x')x'^j$ it then follows that

$$\frac{d\sigma}{ds} = 1 + g_{ij}(x, x') x'^i \frac{d\xi^j}{ds}$$

or, using (4)'

$$\lambda(s) = y_j \frac{d\xi^j}{ds} \tag{14'}$$

as expected in view of the results of § 3. Also, differentiating equation (5)' with respect to S, and noting that x'^i, ψ'^i are constant, we have

$$\frac{d^2\xi^i}{ds^2} = \psi'^i \frac{d^2\sigma}{ds^2}$$

or, in virtue of (4)'

$$\frac{d^2\xi^i}{ds^2} - \psi'^i \frac{d\lambda}{ds} = 0$$

But since $(x'^i - \psi'^i)d\lambda/ds$ is of the second order of smallness, we can replace this last result by the equation

$$\frac{d^2\xi^i}{ds^2} - x'^i \frac{d\lambda}{ds} = 0 \qquad (34)'$$

Comparison with equation (34) shows clearly that the quantities R^i_{hkj} are a measure of the defect, especially as for the case $d\lambda/ds = 0$ the quantities $d^2\xi^i/ds^2$ in the Minkowskian T_n would vanish, whereas this is by no means so for $D^2\xi^i/Ds^2$ in the Finsler space.

Since the sum $y_j\xi^j$ is an invariant, our results in §29 show that we may write

$$\frac{d}{ds}(y_j\xi^j) = \frac{D}{Ds}\left[g_{ij}(x,x')x'^i\xi^j\right]$$
$$= \frac{Dg_{ij}(x,x')}{Ds}x'^i\xi^j + g_{ij}(x,x')\frac{Dx'^i}{Ds}\xi^j + y_j\frac{D\xi^j}{Ds}$$

Since the operator D here refers to a displacement along the tangent to the geodesic E_1, it follows from (44), §27, that the first two terms in this expression vanish, so that

$$\frac{d}{ds}(y_j\xi^j) = y_j\frac{D\xi^j}{Ds} \qquad (36)$$

(Actually, this result can be obtained equally well by direct calculation, using (12) of §33, without recourse to §27).

Using (14) of §33, this result becomes

$$\frac{d}{ds}(y_j\xi^j) = \lambda(s) \qquad (37)$$

Thus,

$$y_j\xi^j = \lambda s \quad \left(\frac{d\lambda}{ds} = 0\right) \qquad (38)$$

by direct integration, the constant vanishing since $\xi^i = 0$, when $s = 0$.

If we choose $\lambda \equiv 0$ for the function $\lambda(s)$, it follows that $s = \sigma$ and $\vec{\xi}$ is orthogonal with respect to the tangent-vector in

T_n at P to the geodesic E_1 .

Finally, since $y_j(D\xi^j/Ds)$ is also an invariant, we have, in the same manner as before,

$$\frac{d}{ds}\left(y_j \frac{D\xi^j}{Ds}\right) = y_j \frac{D^2\xi^j}{Ds^2} \tag{39}$$

so that in the case $\lambda = 0$ the three vectors

$$\vec{\xi}, \quad \frac{D\vec{\xi}}{Ds}, \quad \frac{D^2\vec{\xi}}{Ds^2}$$

are all orthogonal with respect to the tangent-vector of the geodesic at the point P.

CHAPTER VIII

CURVATURE OF A TWO-DIMENSIONAL FINSLER SPACE

§ 37. Polar Coordinates in a two-dimensional Finsler space.

In this chapter we shall consider the notion of curvature as derived in §35 applied to the case of a two-dimensional Finsler space. In order to simplify our calculations we shall introduce a coordinate system analogous to polar coordinates in a Euclidean or geodesic coordinates in a Riemannian space. Having chosen a fixed point O in the Finsler space as origin, then through a given point P, provided it is sufficiently close to O, there passes one and only one geodesic from O. This geodesic will make an angle φ, measured and normalised (compare §9) with respect to the indicatrix in T_n at O, with a geodesic that has previously been fixed. If s is the arc-length of the geodesic OP, we shall regard s, φ as the coordinates of the point P.

Let Q be the point ($s + ds$, $\varphi + d\varphi$): then, as before, we shall identify the small displacement PQ with the vector $\vec{\xi}$ in T_n at P. In our new coordinates we will then have

$$\xi^2 = \gamma_{11}(s, \varphi; \xi^1, \xi^2)\, ds^2 + 2\gamma_{12}(s, \varphi; \xi^1, \xi^2)\, ds\, d\varphi + \gamma_{22}(s, \varphi; \xi^1, \xi^2)\, (d\varphi)^2 \qquad (1)$$

where again the functions γ_{ij} depend on the direction of PQ as well as on the coordinates of P.

If R is a point on the same extremal as P and distant ds from P, it follows that the components in equation (1) will be (ds, 0) and we deduce that

$$ds^2 = \gamma_{11}(s, \varphi; ds, 0)\, ds^2$$

or

$$\gamma_{11} = 1 \qquad (2)$$

along any geodesic through O.

If $\vec{t}$ is the tangent-vector in T_n along the geodesic, we have

$$t^1 = 1, \quad t^2 = 0; \tag{3}$$

and from the differential equation (23) of §25 for the geodesic

$$\frac{d^2t^i}{ds^2} + \left\{ {i \atop j\ k} \right\}_{(s,\varphi;t)} t^j t^k = 0$$

it follows that

$$\left\{ {i \atop 1\ 1} \right\}_{(s,\varphi;t)} = 0 \qquad (i = 1, 2) \tag{4}$$

along any geodesic through O.

If, in particular, thepoint Q has coordinates (s , $\varphi + d\varphi$), it follows from equation (4) of §33 that we are dealing with the case $\lambda(s) \equiv 0$; (s variable, φ and $d\varphi$ fixed) so that we dedube from equation (38) of §36 that $\vec{\xi}$ is orthogonal with respect to the tangent-vector to the geodesic at P. Since this implies

$$\gamma_{ij}(s,\varphi;t)\, t^i \xi^j = 0$$

where now $\vec{\xi}$ has components

$$\xi^1 = 0, \quad \xi^2 = d\varphi \tag{5}$$

we have

$$\gamma_{12}(s;\varphi;t) = 0 \tag{6}$$

along any geodesic through O.

Also, on substituting the values (5) in equation (1), we find

$$\xi = +\sqrt{\gamma_{22}(s,\varphi;\xi)}\ d\varphi \tag{7}$$

where, as before, γ_{22} is homogeneous of order zero with respect to $\vec{\xi}$. (Compare §2). Now if the point P(s , φ) is given, the direction $\vec{\xi}$ orthogonal with respect to the tangent-vector at P (apart from sign) is fixed, and it follows that in equation (7) the function γ_{22} is a function of (s , φ) only, provided, of course, that we are measuring in the direction of $\vec{\xi}$. Accordingly, we denote the positive square root of this function by $f(s,\varphi)$, and write (7) in the form

$$\xi = f(s,\varphi)\, d\varphi \tag{8}$$

Keeping φ and $d\varphi$ fixed, we find by repeated differentiation with respect to s

$$\frac{\partial^2 \xi}{\partial s^2} = \frac{\partial^2 f(s,\varphi)}{\partial s^2}\, d\varphi \tag{9}$$

or, on eliminating $d\varphi$ between equations (8) and (9)

$$\frac{1}{\xi}\frac{\partial^2 \xi}{\partial s^2} = \frac{1}{f(s,\varphi)}\frac{\partial^2 f(s,\varphi)}{\partial s^2} \tag{10}$$

This result is clearly connected with our equation (34) of the previous chapter of the geodesic deviation when we have $\lambda(s) = 0$ and $n = 2$, and we shall establish this connection in the following section.

§ 38. Gaussian Curvature in Polar Coordinates.

Still using the notation of Chapter VII, let $\vec{x}$ be the unit vector in T_n at P whose direction coincides with $\vec{\xi}$. We then have

$$\xi^i = \xi X^i \qquad (i = 1, 2) \tag{11}$$

In view of the results of §29, covariant differentiation of this equation gives

$$\frac{D\xi^i}{Ds} = \xi \frac{DX^i}{Ds} + X^i \frac{\partial \xi}{\partial s} = \xi \frac{DX^i}{Ds} + \frac{\xi^i}{\xi}\frac{\partial \xi}{\partial s} \tag{12}$$

Multiplying this equation by y_i and summing over i, it follows from equations (14) and (38) of Chapter VII that

$$y_i \frac{DX^i}{Ds} = 0 . \tag{13}$$

Thus, either (a) the vector DX^i/Ds is orthogonal to the tangent-vector to the geodesic E_1 at P, or (b) it vanishes.

Let us assume first the former case. Then the direction of DX^i/Ds coincides with the vectors ξ^i and $D\xi^i/Ds$, since we have $n = 2$ and both vectors have been shown to be orthogonal with respect to the tangent-vector of E_1. Differentiating (12) once more with respect to s, we obtain

$$\frac{D^2\xi^i}{Ds^2} = \xi \frac{D^2X^i}{Ds^2} + 2\frac{DX^i}{Ds}\cdot\frac{\partial\xi}{\partial s} + X^i\frac{\partial^2\xi}{\partial s^2} \tag{14}$$

Comparing this with equation (34) of §35, we find

$$\xi\frac{D^2X^i}{Ds^2} + 2\frac{DX^i}{Ds}\frac{\partial\xi}{\partial s} + X^i\frac{\partial^2\xi}{\partial s^2} + R^i_{kh\,j}(x,x')x'^j x'^k\xi^h = 0 \tag{15}$$

Now, since $\vec{X}$ is a unit vector throughout, we have

$$\frac{D}{Ds}(X^iX_i) = X_i\frac{DX^i}{Ds} + X^i\frac{DX_i}{Ds} \tag{16}$$

But since the directions of $\vec{D}X/Ds$ and $\vec{X}$ coincide, we may write

$$X^i\frac{DX_i}{Ds} = g^{ij}(x,X)X_j\frac{DX_i}{Ds} = g^{ij}(x,\frac{DX}{Ds})\frac{DX_i}{Ds}X_j = X_j\frac{DX^j}{Ds},$$

so that (16) becomes

$$X_i\frac{DX^i}{Ds} = 0 \tag{16'}$$

Let us now return to our special coordinate system of the previous section. There we have

$$X^1 = \frac{DX^1}{Ds} = 0\ ;\ X^2 \neq 0$$

and equation (16)' becomes

$$X_2\frac{DX_2}{Ds} = \left[\gamma_{12}(s,\varphi;X)X^1 + \gamma_{22}(s,\varphi;X)X^2\right]\frac{DX^2}{Ds} = 0$$

and since $\gamma_{22}(s,\varphi;X)X^2 \neq 0$ we deduce (for all coordinate systems)

$$\frac{DX^i}{Ds} = 0$$

and

$$\frac{D^2X^i}{Ds^2} = 0 \tag{17}$$

so that case (a) automatically reduces to case (b).

Equation (15) now becomes

$$X^i\frac{\partial^2\xi}{\partial s^2} + R^i_{kh\,j}(x,x')x'^k x'^j\xi^h = 0$$

Multiplying this equation by X_i and summing over i, and taking into account equation (11), we find

$$\frac{\partial^2\xi}{\partial s^2} + \xi\left[g_{il}(x,X)R^i_{kh\,j}(x,x')x'^k x'^j X^l X^h\right] = 0 \tag{18}$$

Since the unit vector $\vec{X}$ is uniquely determined by the coordinates of P and the direction of the geodesic through P, we may write in the last expression

$$g_{il}(x, X)\, R^{i}_{\ hkj}(x, x')\, x'^{h} x'^{j}\, X^{l} X^{k} = K(x, x') \tag{19}$$

We shall call $K(x, x')$ the <u>Gaussian Curvature</u> of the Finsler space at the point P <u>for the direction $\vec{x}'$</u>. Since ξ (a length) is invariant, its second derivative with respect to s is also an invariant, so that it follows from the equation (18), which now assumes the form

$$\frac{1}{\xi}\frac{\partial^2 \xi}{\partial s^2} + K(x, x') = 0 \tag{20}$$

that $K(x, x')$ is an invariant. Later we shall give a definition of Gaussian Curvature which depends solely on the position of the point under consideration, i.e. a notion of curvature which is independent of direction.

Finally, it is clear that equation (20) is identical with equation (10) of the preceding section; we therefore have in polar coordinates

$$K(s, \varphi; t) = -\frac{1}{f(s, \varphi)} \cdot \frac{\partial^2 f(s, \varphi)}{\partial s^2} \tag{21}$$

§ 39. <u>Comparison of the length of the geodesic circle with the length of the local indicatrix.</u>

As indicated in §32 any difference between the geodesic circle at the point O and the indicatrix in T_n at O (for small "radii") will give us a measure of "defect". We shall now show that the difference in the lengths of these curves when the "radii" tend to zero can be expressed in terms of the curvature of the Finsler space at O. This will then lead us to define a measure of curvature depending only on the coordinates of O.

Since $\xi \to 0$ when $s \to 0$, it follows from (8) that if we keep φ and $d\varphi$ fixed

$$\lim_{s \to 0} f(s, \varphi) = 0 \tag{22}$$

We shall also assume that K does not vanish for any direction through 0, so that we have in virtue of (21)

$$\lim_{s \to 0} \frac{\partial^2 f(s, \varphi)}{\partial s^2} = 0 \tag{23}$$

Applying the well-known methods for indeterminate forms to the quotient on the right-hand side of (21), we have therefore

$$\lim_{s \to 0} K(s, \varphi; t) = - \lim_{s \to 0} \frac{\dfrac{\partial^3 f(s, \varphi)}{\partial s^3}}{\dfrac{\partial f(s, \varphi)}{\partial s}}$$

Since

$$\frac{\partial f(0, \varphi)}{\partial s} \neq 0$$

we thus have

$$\frac{\partial^3 f(0, \varphi)}{\partial s^3} = - \frac{\partial f(0, \varphi)}{\partial s} K(0, \varphi) \tag{24}$$

where we have written $K(0, \varphi)$ for $K(0, \varphi; t)$; $\vec{t}$ being the tangent-vector at 0 to the geodesic whose angular coordinate is φ.

Let L_0 be the length of the unit indicatrix at 0. If ξ is a small length of arc of the indicatrix of radius s, it will subtend an angle $d\varphi$ at 0 where

$$d\varphi = \frac{\xi}{s} (L_0)^{-1}$$

since $d\varphi$ is normalised (§9). This equation is exact in T_n, but in the Finsler space it is true only when we proceed to the limit $s = 0$ so that we have

$$d\varphi = \lim_{s \to 0} \frac{\xi}{s} (L_0)^{-1}$$

But in virtue of (22) we may write

$$\xi = f(s, \varphi)\, d\varphi = \frac{\partial f(0, \varphi)}{\partial s} s \, d\varphi \quad (s \to 0)$$

correct for the first order of smallness: thus

$$d\varphi = \frac{1}{L_0} \cdot \frac{\partial f(0, \varphi)}{\partial s} d\varphi$$

so that we finally have

$$\frac{\partial f(0,\varphi)}{\partial s} = L_0 \tag{25}$$

Equation (24) then becomes

$$\frac{\partial^3 f(0,\varphi)}{\partial s^3} = -L_0 K(0,\varphi) \tag{26}$$

Let us expand $f(s,\varphi)$ as a power-series in s, keeping φ fixed. We thus have

$$f(s,\varphi) = f(0,\varphi) + \frac{\partial f(0,\varphi)}{\partial s} s + \frac{1}{2}\frac{\partial^2 f(0,\varphi)}{\partial s^2} s^2 + \frac{1}{6}\frac{\partial^3 f(0,\varphi)}{\partial s^3} s^3 + \ldots .$$

and on substituting from equations (22), (23), (25) and (26) we deduce

$$f(s,\varphi) = L_0 s - \frac{1}{6} L_0 K(0,\varphi) s^3 + \ldots . \tag{27}$$

Denoting the lengths of the geodesic circle at O and the indicatrix in T_n at O by $G_0(s)$ and $L_0(s)$ respectively when the "radius" of both is s, we have, using equation (8)

$$G_0(s) = \oint f(s,\varphi)\, d\varphi$$

and

$$L_0(s) = \oint s L_0 \, d\varphi .$$

On integrating equation (27) with respect to φ while s is being kept constant, we therefore find

$$G_0(s) = L_0(s) - \frac{s^3}{6} L_0 \oint K(0,\varphi) d\varphi + \ldots . \tag{28}$$

We shall now put

$$\oint K(0,\varphi)\, d\varphi = R(0) \tag{29}$$

where $R(0)$ depends only on the position of O and represents a <u>mean of the Gaussian Curvatures for the various directions through the point O</u>. We shall call it the <u>Absolute Gaussian Curvature</u> of the Finsler space at O in view of the fact that it is a function of position only.

Equation (28) can now be written in the form

$$R(0) = \frac{6}{L_0} \lim_{s \to 0} \frac{L_0(s) - G_0(s)}{s^3} \tag{30}$$

This is the desired result. A similar equation for surfaces embedded in a three-dimensional Euclidean space was derived by Bertrand and Puiseux.*

§ 40. Comparison of the Area of the geodesic circle with that of the local Indicatrix.

The expansion (27) will enable us to give a further geometrical interpretation of the absolute Gaussian Curvature in terms of the areas enclosed in the geodesic circle at 0 and the indicatrix in T_n at 0.

Let P, Q be two points on the same geodesic through 0, where P lies on the geodesic circle of radius s, Q on the geodesic circle of radius $s + ds$. On the geodesic making an angle $d\varphi$ with the given geodesic we have corresponding points P', Q'. As $ds \to 0$ the area of the parallelogram PQQ'P' will be given by (§ 14)

$$dA = |PQ| \cdot |PP'| \sin(PQ, PP')$$

when measured with respect to the indicatrix at P. But PP' is orthogonal with respect to PQ (§ 36) so that the positive sine corresponding to these directions is unity; and hence the area will simply be

$$dA = \xi\, ds = f(s,\varphi)\, ds\, d\varphi \tag{31}$$

in virtue of equation (8). The area of the "geodesic ring" of radius s and thickness ds will therefore be

$$\oint [f(s,\varphi)\,ds]\,d\varphi = \int_0^1 [(L_0 s - \tfrac{1}{6} L_0 K(0,\varphi) s^3 + \dots)\,ds]\,d\varphi$$

in view of equation (27), so that the area $A_0(s)$ of the geodesic circle of radius s at 0 will be given by

*Journal de Mathématiques, vol. 13, (1848), p.83.

$$A_0(s) = \int_0^s \left(L_0 s - \frac{1}{6} L_0 R(0) s^3 + \cdots \right) ds.$$

when we take into account the definition (29) of $R(0)$.

Also, we have seen in §14 that the area $I_0(s)$ of the indicatrix of radius s in T_n at O is given by

$$I_0(s) = \frac{1}{2} L_0 s^2$$

so that we finally obtain

$$A_0(s) = I_0(s) - \frac{1}{24} L_0 R(0) s^4 + \cdots$$

or

$$R(0) = \lim_{s \to 0} \frac{24}{L_0} \left(\frac{I_0(s) - A_0(s)}{s^4} \right) \tag{32}$$

This equation is the desired result.

§ 41. Geodesic Triangles.

Let A, B, C be any three points in our Finsler space, these points being sufficiently close to each other in order that there exist unique geodesics joining them. Let the angles between these geodesics as measured at A, B, C be α, β, γ respectively. We choose A as the origin of our special system of polar coordinates, and through each point P on the geodesic BC we construct the geodesic circle of centre A (Fig. 7), and we denote the angle measured at P between BC and the geodesic AP by θ.

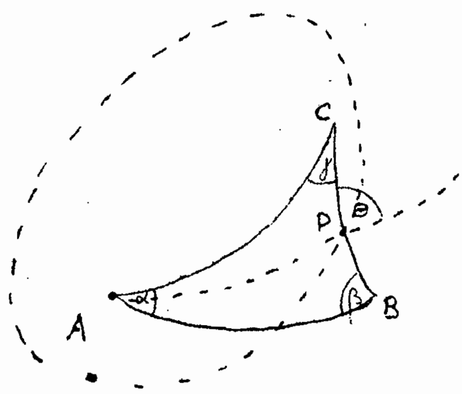

Fig. 7

We shall first calculate the increment $d\theta$ of this angle as we pass from P to a neighbouring point Q on BC. Let the coordinates

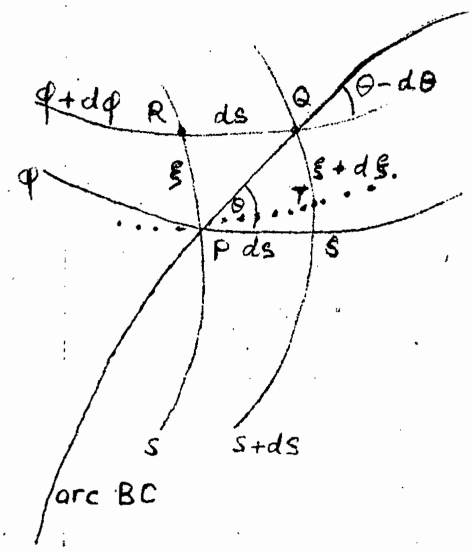

Fig. 8

of P and Q be (s, φ) and $(s + ds, \varphi + d\varphi)$ respectively, and let the geodesic circles of radii s and $s + ds$ cut the geodesic AQ in R and the geodesic AP in S. Then PR = ξ, and QS = $\xi + d\xi$ in our previous notation (Fig 8), where ξ is of the first order, while $d\xi$ is of the second order of smallness when compared with $d\varphi$ which we regard as being of the first order of smallness itself.

Along QS we mark off the point T such that QT = PR = ξ, so that TS = $d\xi$. Also, since $\vec{\xi}$ is everywhere orthogonal with respect to the geodesic AP (§36) we may write

$$PT = PS = ds$$

when we neglect terms of the second order. It then follows from §39 that if we measure the angle $\widehat{SPT}$ with respect to the indicatrix at P, we have

$$\widehat{SPT} = \frac{TS}{PS} = \frac{d\xi}{ds} \tag{33}$$

correct at least to the second order of smallness.

If, for a first approximation, we regard the infinitesimal parallelogram PTQR as a Minkowskian parallelogram, it follows immediately that

$$\widehat{SPT} = d\theta \tag{34}$$

where $\theta - d\theta$ is the angle between RQ and PQ.

Since ds is of the first order of smallness, the error involved will also be of the first order when applied to finite quantities; as, however, $d\theta$ is also of the first order, the error may be neglected. From (33) and (34) we deduce

$$d\theta = \frac{d\xi}{ds}$$

or, using (8) after differentiation

$$d\theta = \frac{\partial f(s, \varphi)}{\partial s}\, d\varphi \qquad (35)$$

Now, using equations (21) and (31) we find for the small area PRQS that

$$K(s, \varphi)\, dA = - \frac{\partial^2 f(s, \varphi)}{\partial s^2}\, ds\, d\varphi \qquad (36)$$

where $K(s, \varphi)$ is the curvature of the Finsler space at P in the direction of the geodesic AP. For the "triangular strip" Δ enclosed by the geodesics AP(φ) and AR($\varphi + d\varphi$) through the origin A, we have

$$\int_\Delta K(s, \varphi)\, dA = - \int_0^s \frac{\partial^2 f(s, \varphi)}{\partial s^2}\, ds\, d\varphi$$

regarding φ as constant over Δ. Thus

$$\int_\Delta K(s, \varphi)\, dA = - \left[\frac{\partial f(s, \varphi)}{\partial s} \right]_0^s d\varphi \qquad (37)$$

Since A is the origin, we have, as for equation (25)

$$\lim_{s \to 0} \frac{\partial f(s, \varphi)}{\partial s} = L_A$$

where L_A is the length of the <u>unit</u> indicatrix in T_n at A. In view of the fact that s is the geodesic distance of P from A, equation (37) thus gives when we integrate over the whole area of the geodesic triangle ABC

$$\iint K(s, \varphi)\, dA = \int_0^{\alpha'} L_A\, d\varphi - \int_0^{\alpha'} \frac{\partial f(s, \varphi)}{\partial s}\, d\varphi$$

where α' is the normalised angle α , i.e.

$$\alpha' = \frac{\alpha}{L_A}$$

Changing the variable in the last integral with the aid of (35) we find

$$\iint K(s, \varphi)\, dA = \alpha' L_A + \int_{\theta(B)}^{\theta(C)} d\theta$$

so that

$$\iint K(s,\varphi)\,dA = \alpha + \gamma - \Theta(B)$$

But from figure 7 it is clear that

$$\Theta(B) \doteq \tfrac{1}{2} L_B - \beta$$

thus we will finally have

$$\iint K(s,\varphi)\,dA = \alpha + \beta + \gamma - \tfrac{1}{2} L_B \qquad (38)$$

or, if we use normalised angles, denoted by dashes

$$\iint K(s,\varphi)\,dA = \alpha' L_A + \beta' L_B + \gamma' L_C - \tfrac{1}{2} L_B \qquad (38)'$$

This equation is analogous to the theorem of Gauss for geodesic triangles on a surface embedded in a euclidean space; it is to be noted that $\frac{1}{2} L_B$ is simply the value of "π" at the point B.

The fact that the length of the indicatrix at B appears in (38) is due firstly to our choice of origin as $K(s,\varphi)$ is taken with respect to geodesics through A, and secondly the sense in which the second integration was carried out: had we integrated from C to B instead, the value of the length of the indicatrix at C would have appeared in the last formula.

§ 42. Geodesic Polygons.

The last result can be extended immediately to the case of geodesic polygons of n sides, formed by the points $A_1, A_2, \ldots\ldots A_n$ and by joining successive points by geodesics, assuming that these points are so placed that none of the geodesics intersect. We subdivide this polygon into n geodesic triangles by constructing the geodesics $A_0A_1, A_0A_2, \ldots A_0A_n$, (Fig. 9) . Let us denote the interior angles of the μ^{th} triangle by $\alpha_\mu, \beta_\mu, \gamma_\mu$, and the exterior angle at the point A_μ by δ_μ . All these angles are measured by means of the indicatrices at the corresponding points and we shall write L_μ for the length of the unit indicatrix at A_μ

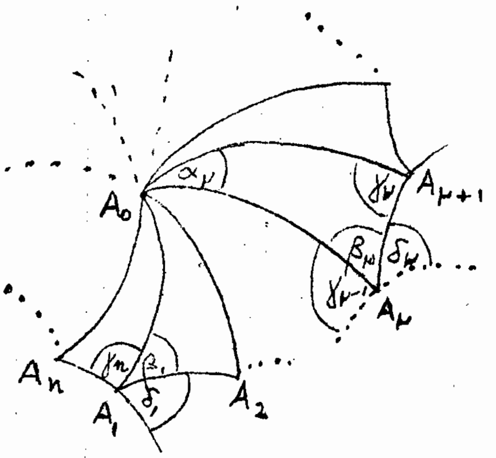

Fig. 9

Then, by taking the point A_0 as our origin, we integrate successively over each of these geodesic triangles: starting along $A_0 A_1$ and preserving the sense of integration of the previous section, we find

$$\iint K\,dA = \sum_{\mu=1}^{n} (\alpha_\mu + \beta_\mu + \gamma_\mu) - \frac{1}{2} \sum_{\mu=1}^{n} L_\mu \tag{39}$$

But we have

$$\sum_{\mu=1}^{n} \alpha_\mu = L_0 \tag{40}$$

and considering the angles at the point A_1,

$$\gamma_n + \beta_1 = \frac{1}{2} L_1 - \delta_1 \tag{41}$$

and for the point A_μ

$$\beta_\mu + \gamma_{\mu-1} = \frac{1}{2} L_\mu - \delta_\mu \qquad (\mu > 1) \tag{41'}$$

By writing equation (39) in the form

$$\iint K\,dA = \sum_{\mu=1}^{n} \alpha_\mu + \beta_1 + \sum_{\mu=2}^{n} \beta_\mu + \sum_{\mu=2}^{n} \gamma_{\mu-1} + \gamma_n - \frac{1}{2} L_1 - \frac{1}{2} \sum_{\mu=2}^{n} L_\mu$$

we find on substituting from (40), (41) and (41)'

$$\iint K\,dA = L_0 + \frac{1}{2} L_1 - \delta_1 + \sum_{\mu=2}^{n} \left(\frac{1}{2} L_\mu - \delta_\mu\right) - \frac{1}{2} L_1 - \frac{1}{2} \sum_{\mu=2}^{n} L_\mu$$

or finally

$$\iint K\,dA = L_0 - \sum_{\mu=1}^{n} \delta_\mu \tag{42}$$

where it is to be noted that the last term represents the sum of the exterior angles.

§ 43. The Gauss-Bonnet Theorem in Finsler spaces.

Let C be any closed curve bounding a simply-connected domain D. If ℓ is the length of C, it is possible to choose n points $A_1, A_2 \ldots$ $\ldots A_n$ on C such that the geodesic distance $\sigma(A_\mu, A_{\mu-1})$ between any two points A_μ, $A_{\mu-1}$ is such that

$$\sigma(A_\mu, A_{\mu-1}) \leq \frac{\ell}{n} \qquad (43)$$

if we remeber that the geodesic distance cannot be greater than the length measured along an arc of C. These points will form a geodesic polygon as in the previous section.

Now from our definition of curvature of curves (§30), if we denote the curvature of C (at A_μ) by ρ it follows directly that

$$\frac{1}{\rho} = \lim_{\sigma \to 0} \frac{\delta_\mu}{\sigma(A_\mu, A_{\mu-1})}$$

In view of (43) we see that $\sigma \to 0$ when $n \to \infty$, hence

$$\int_C \frac{ds}{\rho} = \lim_{n\to\infty}\left(\sum_{\mu=1}^{n} \sigma(A_\mu, A_{\mu-1}) \frac{\delta_\mu}{\sigma(A_\mu, A_{\mu-1})}\right) = \lim_{n\to\infty} \sum_{\mu=1}^{n} \delta_\mu \qquad (44)$$

Let us choose the origin A_0 at any point interior to the area bounded by C. Applying equation (42), we find, by letting $n \to \infty$

$$\iint_D K \, dA = L_0 - \int_C \frac{ds}{\rho} \qquad (45)$$

This equation then represents the Gauss-Bonnet formula in a Finsler space. It is to be noted that on both sides of (45) there are functions depending on the origin A_0. Finally, it may be observed that equation (45) is still valid when A_0 is a point on the curve C: this can be proved without difficulty by considering a geodesic polygon $A_0, A_1, A_2, \ldots\ldots A_n$ of n + 1 sides with A_0 as origin.

In conclusion we may add that the great significance of the Gauss-Bonnet Theorem in geometry cannot be stressed sufficiently: it establishes a firm link between the local and global properties of the space under consideration.

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