Aspects of Duality Theory for Spaces of Measurable Operators

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Abstract

It is well known that a commutative von Neumann algebra can be represented as a space of essentially bounded functions over a localizable measure space. In non-commutative integration theory, a von Neumann algebra takes over the role of the space of essentially bounded measurable functions. If the von Neumann algebra is semifinite, then there exists a faithful semifinite normal trace on it. Equipped with such a trace, a topology can be defined on the algebra, which in the commutative case is the familiar topology of convergence in measure. The completion of the algebra with respect to this topology yields an algebra of unbounded operators, the algebra of so-called measurable operators. In the first part of this thesis, the relationship between the nature of the lattice of projections of the von Neumann algebra and the properties of this topology, in particular its local convexity, is investigated.

In the duality theory for commutative Banach function spaces, one distinguishes between normal functionals and singular functionals. The study of the former leads to Köthe duality theory. A non-commutative Köthe duality theory already exists and a second aim of this thesis is to initiate a theory for singular functionals in the non-commutative setting. As a preparation for this, singular functionals are characterised in several ways in the commutative case and one of these is used as definition for singular functionals on Banach spaces of measurable operators. The known association between singular functionals and the subspace of elements with order continuous norm in a Banach function space is extended to the non-commutative setting.

Finally, duality for the space of measurable operators equipped with the measure topology is investigated. Its Köthe dual is first characterised, and then singular functionals on this space are investigated. In certain cases a full characterisation of the continuous dual is given.

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Introduction

Banach spaces of measurable functions, such as L_p-spaces, play an important role in analysis. There is a well developed general theory for these so-called Banach function spaces. The fact that they are also Banach lattices when equipped with a natural partial order plays an important part in this theory. In recent years, much progress has been made in the development of a generalisation of this theory, which also includes the theory of ideals of compact operators on a Hilbert space. Since a commutative von Neumann algebra can be represented as an L_∞-space, one may think of an arbitrary von Neumann algebra, $\mathcal{M}$, as a "non-commutative L_∞-space". If the von Neumann algebra can be equipped with a normal trace (as is certainly the case for semifinite von Neumann algebras), one may regard the trace as a generalised integral. This approach leads to a non-commutative integration theory. The trace can also be used to define a topology on $\mathcal{M}$ which in the commutative case may be identified with the topology of convergence in measure. The completion of the von Neumann algebra with this topology can be identified with an algebra of (in general) unbounded operators, $\widetilde{\mathcal{M}}$, the algebra of so-called measurable operators ([Nel74]). This algebra is the "home" of all the non-commutative Banach function spaces (in the sense that they are all contained in it). It has the additional advantage that one can associate with each measurable operator a non-negative function on the positive real line, called its generalised singular function, and that this function serves as a very useful link with the commutative theory. In the first part of the thesis we investigate $\widetilde{\mathcal{M}}$ and its measure topology in more detail, in particular, we determine under which circumstances the measure topology is locally convex. There exists a natural partial order on $\widetilde{\mathcal{M}}$, but in general $\widetilde{\mathcal{M}}$ is not a lattice. Therefore alternative methods are often required to those employed in developing the commutative theory.

There is a well developed duality theory for commutative Banach function spaces and appears for example in [Zaa67], [LZ71], [Zaa83] and [BS88]. One of the main features of this theory is that one can distinguish between two important types of (norm-) continuous linear functionals: the normal and singular functionals. Normal functionals are order-continuous and have integral representations; singular functionals are the ones that are disjoint from the set of normal functionals when one regards the dual as a vector lattice. These two types of functionals are the building blocks for all continuous linear functionals, in the sense that each continuous linear functional can be written uniquely as a sum of a normal and a singular functional. Normal functionals on non-commutative Banach function spaces have been investigated in detail in [DDP93]. No general theory for singular functionals has been established, though in the special case of von Neumann algebras an analogous theory to the commutative setting does exist ([Tak79]). A positive singular functional on a von Neumann algebra is defined as one that never majorizes a non-zero positive normal functional. (A normal functional sends nets of operators that decrease to zero to nets of scalars that decrease to zero.) One then considers a functional that is not necessarily positive, as an element of the predual of the bidual of the von Neumann algebra. It is known that an element of the predual of a von Neumann algebra has a unique decomposition into positive parts. Since the bidual of a von Neumann algebra is again a von Neumann algebra, a functional on a von Neumann algebra therefore has a unique decomposition into positive parts and is called singular if each of the positive parts is singular.

Positive singular functionals on Banach spaces of measurable operators, may also be defined as those functionals that never majorize a non-zero positive normal functional. If the functional is not necessarily positive, a problem arises. There does exist a decomposition for a functional on a real normed operator space into a difference of two positive functionals ([And62]), but this decomposition is not unique and it does not give an explicit reprensentation for the positive parts, as in the commutative case. The formula for the positive part of a functional on a Banach function space cannot be used in the non-commutative setting, since the proof that the positive part thus defined is an additive functional needs the Riesz decomposition property. This property is not available since there is in general no lattice structure on Banach spaces of measurable operators. We therefore find a characterisation for singular functions on Banach function spaces that does not depend on the lattice structure of the Banach function space and use it as the definition for singular functionals on Banach spaces of measurable operators. We are able to establish some properties of singular functionals using this definition, but many open problems still remain.

Singular functionals has not been studied extensively on spaces that are not normed spaces. Some work is done on Orlicz spaces that are not locally convex ([Now92]). The space of measurable operators, equipped with the topology of convergence in measure, is in general not a normed space. In this thesis we conclude with an investigation of duality, in particular of singular functionals, on $\widetilde{\mathcal{M}}$ equipped with the topology of convergence in measure.

We give a more detailed discussion of each of the chapters.

In Chapter 1 we introduce the space of τ-measurable operators, $\widetilde{\mathcal{M}}$. Most of the work of the earlier sections belong to others, but it is the basis for the remainder of the thesis and we therefore include the essential results without proofs. The subspace of τ-measurable operators whose generalised singular function decreases to zero, $\widetilde{\mathcal{M}_0}$, is discussed in detail as it plays an important role in further developments. Next we explore the local convexity of the topology of convergence in measure and devote a few sections to the structure of $\widetilde{\mathcal{M}}$. We see that $\widetilde{\mathcal{M}}$ equipped with the topology of convergence in measure can be written as a sum of a normed and a pseudonormed space. In certain settings $\widetilde{\mathcal{M}}$ can be written as a direct sum of reduced algebras. We shall return to many of these results when we investigate the dual of $\widetilde{\mathcal{M}}$ equipped with the measure topology.

The aim of Chapter 2 is to find an equivalent characterisation for singular functionals on Banach function spaces, that is useful (via a natural translation) as a definition for singular functionals on Banach spaces of measurable operators. We give a brief overview of the known results before proceeding to prove equivalent characterisations for singular functionals on Banach function spaces. Under certain conditions, singular functionals may be characterised as those functionals that vanish on every characteristic function of a set of finite measure. In the case of rearrangement invariant spaces the fundamental function is helpful in this regard.

In Chapter 3 we define singular functionals on Banach spaces of measurable operators. We also define when an element of a Banach space of measurable operators has order continuous norm and establish several equivalent characterisations for such an element, similar to the commutative setting. Contrary to the commutative setting though, it does not follow immediately that the set of elements with order continuous norm is norm closed nor a vector space. We generalise the commutative result that a singular functional vanishes on the set of elements with order continuous norm when the latter set equals the norm closure of the set of bounded operators with finite trace. We show that this characterisation for singular functionals is equivalent to the charaterisation that a singular functional vanishes on every projection with finite trace. To obtain the above mentioned generalisation, we need to consider induced spaces. (An induced space contains elements whose generalised singular functions are elements of a normed rearrangement invariant function space on the positive real line.)

When we consider $\widetilde{\mathcal{M}}$ equipped with the topology of convergence in measure in Chapter 4, we see that the set of elements on which the topology of convergence in measure is "order continuous", $\widetilde{\mathcal{M}}_a$, is always equal to the closure in the measure topology of the set of bounded operators with finite trace, $\widetilde{\mathcal{M}}_b$. (In fact, this set equals $\widetilde{\mathcal{M}}_0$, the set of elements whose generalised singular functions decrease to zero.) We therefore define a singular functional on $\widetilde{\mathcal{M}}$ equipped with the measure topology as one that vanishes on every projection with finite trace. We also characterise elements of the Köthe dual of $\widetilde{\mathcal{M}}$ and show that these always produce continuous linear functionals. If the associated von Neumann algebra, $\mathcal{M}$, contains only nonatomic (continuous) projections, we characterise the dual of $\widetilde{\mathcal{M}}$ equipped with the measure topology as the dual of the quotient space $\widetilde{\mathcal{M}}/(\widetilde{\mathcal{M}}_0 \cap \mathcal{M})$ with the induced quotient norm and see that this space consists of singular functionals only. If the associated von Neumann algebra $\mathcal{M}$ is atomic, and the traces of the projections are bounded away from zero, it is known that $\widetilde{\mathcal{M}} = \mathcal{M}$ and the topology of convergence in measure equals the topology induced by the norm on $\mathcal{M}$. Thus in this setting a duality theory already exists. We characterise the dual of $\widetilde{\mathcal{M}}$ in one of the remaining cases, but only in the case where $\mathcal{M}$ is commutative.

Index of notation

Von Neumann algebras and affiliated operators

Our conventions regarding von Neumann algebras and affiliated operators are the following:

$\mathcal{M}$	denotes a semifinite von Neumann algebra, with
$\mathcal{H}$	the underlying Hilbert space and
$\langle .,. \rangle$	the usual inner product in $\mathcal{H}$.
$\mathcal{B}(\mathcal{H})$	denotes the space of bounded linear operators on $\mathcal{H}$.
$\|\|.\|\|$ or $\|\|.\|\|_\infty$	denotes the operator norm on $\mathcal{M}$ and
τ	denotes a faithful semifinite normal trace on $\mathcal{M}$. Furthermore,
1	denotes the identity element in $\mathcal{M}$, and the set
$\mathcal{M}^p$	denotes the lattice of projections in $\mathcal{M}$.
$\widetilde{\mathcal{M}}$	is the space of τ-measurable operators affiliated with $\mathcal{M}$, equipped with
τ_{cm}	the topology of convergence in measure.
$\widetilde{\mathcal{M}}(\epsilon, \delta)$	is a basic neighbourhood of 0 for τ_{cm} on $\widetilde{\mathcal{M}}$ ($\epsilon, \delta > 0$). We write
$\widetilde{\mathcal{M}}(\epsilon)$	for $\widetilde{\mathcal{M}}(\epsilon, \epsilon)$.

Measurable operators

For the operator $x \in \widetilde{\mathcal{M}}$,

$\|x\|$	denotes the absolute value of x,
x^*	is the adjoint of x and
x^+, x^-	the positive parts in the decomposition for self-adjoint x. We use

$\mu(x)$	to denote the generalised singular function of x,
$\mu_\infty(x)$	for $\lim_{t\to\infty} \mu_t(x)$ and
$d(x)$	for the distribution function of x.
$\mathcal{D}(x)$	denotes the domain of x.
$\mathbf{n}(x)$	denotes the projection onto the kernel (or null space) of x and
$\mathbf{r}(x)$	is the right support of x and equals $1 - \mathbf{n}(x)$. The projection
$\mathsf{e}_B(x)$	is a spectral projection of self-adjoint x, where B is any Borel measurable subset of the complex field. For $t > 0$
$\mathsf{e}_t(x)$	denotes the spectral projection $\mathsf{e}_{(-\infty,t]}(x)$ or $\mathsf{e}_{[0,t]}(x)$ if $x \geq 0$. Moreover,
$\mathsf{e}_{(0,\infty)}(x)$	is called the support of x whenever $x \geq 0$, and in this case it equals $\mathbf{r}(x)$.

Banach function spaces

For Banach function spaces our notation is as follows:

(X, Σ, μ)	denotes a localizable measure space. The space
$L_\rho(X, \Sigma, \mu)$,	
$L_\rho(X)$ or L_ρ	denotes a Banach function space over (X, Σ, μ) where
ρ	is a function norm on
$L_0(X, \Sigma, \mu)$	the space of equivalence classes (modulo almost everywhere equivalence) of complex valued measurable functions.
$L_{\rho,r}$	denotes the collection of real valued functions in L_ρ.
m	denotes Lebesgue measure on the interval $(0, \infty)$. We write
$supp(f)$	for the support of $f \in L_0(X, \Sigma, \mu)$ and
χ_A	is the characteristic function for the set $A \in \Sigma$ and finally
Φ_ρ	denotes the fundamental function for the space L_ρ.

Subspaces of $\widetilde{\mathcal{M}}$

Important subspaces of $\widetilde{\mathcal{M}}$ are the following:

$L_1(\mathcal{M})$	is the set of elements in $\widetilde{\mathcal{M}}$ whose absolute values have finite trace.
$\mathcal{H}(\mathcal{M})$	is the space $L_1(\mathcal{M}) \cap \mathcal{M}$ and
$\mathcal{G}(\mathcal{M})$	the space $L_1(\mathcal{M}) + \mathcal{M}$.

$\widetilde{\mathcal{M}_{FS}}$ is the set of elements for which the trace of the right support is finite.

$\widetilde{\mathcal{M}}_0$ denotes the set of elements in $\widetilde{\mathcal{M}}$ whose generalised singular functions decrease to zero.

$\widetilde{\mathcal{M}}_a$ denotes the set $\{x \in \widetilde{\mathcal{M}} : |x| \geq x_\alpha \downarrow_\alpha 0 \Rightarrow \mu_t(x_\alpha) \to 0 \text{ for all } t > 0\}$.

$\widetilde{\mathcal{M}}_b$ is the closure of $\mathcal{H}(\mathcal{M})$ in the topology of convergence in measure.

The same type of notation will be used for Banach spaces $E \subseteq \widetilde{\mathcal{M}}$, for example

E_a denotes the set of elements of E that has order continuous norm and

E_b the closure of $\mathcal{H}(\mathcal{M}) \cap E$ in E-norm.

Spaces of continuous linear functionals on $\widetilde{\mathcal{M}}$

$\widetilde{\mathcal{M}}^*$ is the space of linear functionals on $\widetilde{\mathcal{M}}$ that are continuous with respect to the measure topology.

$\widetilde{\mathcal{M}}^{*n}$ denotes the set of normal functionals on $\widetilde{\mathcal{M}}$ and

$\widetilde{\mathcal{M}}^{*s}$ the set of singular functionals in $\widetilde{\mathcal{M}}^*$.

$\widetilde{\mathcal{M}}^\times$ denotes the Köthe dual of $\widetilde{\mathcal{M}}$ and

$\widetilde{\mathcal{M}_0}^\perp$ the annihilator of $\widetilde{\mathcal{M}_0}$ in $\widetilde{\mathcal{M}}^*$.

Similar notation will be used for subspaces of E^*, the continuous dual of E, where E is a Banach space of measurable operators.

Elements of the continuous dual of L_ρ

For the functional $\varphi \in L_\rho^*$,

φ_n denotes the normal part of φ.

φ_r, φ_{im} denote the real and imaginary parts of φ, respectively and

φ_r^+, φ_r^- the positive parts of φ_r.

Preliminaries

0.1 Von Neumann algebras and traces

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators from a Hilbert space $\mathcal{H}$ into itself. If $x \in \mathcal{B}(\mathcal{H})$ then we denote by $x^* \in \mathcal{B}(\mathcal{H})$ the adjoint of x. Recall that $x \in \mathcal{B}(\mathcal{H})$ is *self-adjoint* whenever $x = x^*$, *normal* if $xx^* = x^*x$ and *unitary* if $xx^* = x^*x = 1$. Every element $x \in \mathcal{B}(\mathcal{H})$ can be uniquely expressed as a linear combination of two self-adjoint elements, that is, $x = y + iz$ where $y = \frac{1}{2}(x + x^*)$ and $z = \frac{1}{2i}(x - x^*)$. Let $\langle .,. \rangle$ denote the inner product on $\mathcal{H}$. A self-adjoint operator $x \in \mathcal{B}(\mathcal{H})$ is called *positive*, written $x \geq 0$, if

$$\langle x\xi, \xi \rangle \geq 0 \text{ for all } \xi \in \mathcal{H}.$$

This defines a partial ordering on $\mathcal{B}(\mathcal{H})$ in the usual way. We say the net $(x_\alpha) \subseteq \mathcal{B}(\mathcal{H})$ increases to x,

$$x_\alpha \uparrow_\alpha x,$$

if (x_α) increases in the partial ordering and

$$\langle x\xi, \xi \rangle = \sup_\alpha \langle x_\alpha \xi, \xi \rangle$$

exists for all $\xi \in \mathcal{H}$. Similarly for $x_\alpha \downarrow_\alpha x$. Recall that the net (x_α) is *strong operator convergent* to $x \in \mathcal{B}(\mathcal{H})$ if

$$\|(x - x_\alpha)\xi\| \to_\alpha 0$$

for all $\xi \in \mathcal{H}$. The net $(x_\alpha) \subseteq \mathcal{B}(\mathcal{H})$ is *weak operator convergent* to $x \in \mathcal{B}(\mathcal{H})$ if

$$\langle (x - x_\alpha)\xi, \eta \rangle \to_\alpha 0$$

in $\mathcal{H}$ for all $\xi, \eta \in \mathcal{H}$. In general the weak operator topology is weaker (coarser) than the strong operator topology but they coincide on closures of a convex set of $\mathcal{B}(\mathcal{H})$.

Throughout our work $\mathcal{M}$ will denote a von Neumann algebra, that is, a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology. We shall assume that the von Neumann algebra $\mathcal{M}$ contains an identity element which we shall denote by 1. *A projection* in $\mathcal{M}$ is a self-adjoint operator p such that $\mathrm{p}^2 = \mathrm{p}$. The projection p satisfies $\mathrm{p} \geq 0$ and $\|\mathrm{p}\| = 1$ unless $\mathrm{p} = 0$. The set of projections in $\mathcal{M}$ is denoted by $\mathcal{M}^p$ and is a complete lattice, that is, every family of projections $\{\mathrm{p}_\alpha\}$ has a least upper bound and greatest lower bound, denoted by $\vee_\alpha \mathrm{p}_\alpha$ and $\wedge_\alpha \mathrm{p}_\alpha$, respectively. If $\mathrm{p}, \mathrm{q} \in \mathcal{M}^p$ we write $\mathrm{p} \vee \mathrm{q}$ for their upper bound and $\mathrm{p} \wedge \mathrm{q}$ for their lower bound. For a family of projections $\{\mathrm{p}_\alpha\} \subseteq \mathcal{M}^p$

$$\bigvee_\alpha (1 - \mathrm{p}_\alpha) = 1 - \bigwedge_\alpha \mathrm{p}_\alpha \text{ and } \bigwedge_\alpha (1 - \mathrm{p}_\alpha) = 1 - \bigvee_\alpha \mathrm{p}_\alpha$$

hold. An increasing net of projections (p_α) is strong operator convergent to $\vee_\alpha \mathrm{p}_\alpha$. If $\mathrm{p}, \mathrm{q} \in \mathcal{M}^p$ then $\mathrm{p} \leq \mathrm{q}$ if and only if $\mathrm{pq} = \mathrm{p}$. The projections $\mathrm{p}, \mathrm{q} \in \mathcal{M}^p$ are called *othogonal* if $\mathrm{pq} = 0$. If p and q commute, then $\mathrm{pq} = \mathrm{p} \wedge \mathrm{q}$.

We denote the positive operators in a von Neumann algebra $\mathcal{M}$ by $\mathcal{M}_+$. A *trace* on $\mathcal{M}$ is a function $\tau : \mathcal{M}_+ \to [0, \infty]$ such that

(i) $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathcal{M}_+$,

(ii) $\tau(\lambda x) = \lambda \tau(x)$ for all $\lambda \in \mathbf{R}_+$ and all $x \in \mathcal{M}_+$ and

(iii) $\tau(x^* x) = \tau(x x^*)$ for all $x \in \mathcal{M}$.

It follows from condition (i) that the trace is monotone and that if $x, y \in \mathcal{M}_+$ with $x \leq y$ and $\tau(x) < \infty$ then

$$\tau(y - x) = \tau(y) - \tau(x).$$

The projections $\mathrm{p}, \mathrm{q} \in \mathcal{M}$ are called *equivalent* if there exists $u \in \mathcal{M}$ such that

$$\mathrm{p} = u^* u \text{ and } \mathrm{q} = u u^*.$$

By (iii) we have that if two projections are equivalent then they have the same trace.

Using the Kaplansky formula, [KR86] Theorem 6.1.7, and the properties already mentioned, we have the following two properties that we shall use frequently:

(a) If $\mathrm{p}, \mathrm{q} \in \mathcal{M}$ and $\mathrm{p} \wedge \mathrm{q} = 0$ then $\tau(\mathrm{p}) \leq \tau(1-\mathrm{q})$,

(b) If $\mathrm{p}_1, \ldots, \mathrm{p}_n \in \mathcal{M}^p$, $n \in \mathbf{N}$, then $\tau\left(\bigvee_{i=1}^{n} \mathrm{p}_i\right) \leq \sum_{i=1}^{n} \tau(\mathrm{p}_i)$.

A trace τ is said to be *finite* if $\tau(1) < \infty$, or equivalently, $\tau(x) < \infty$ for all $x \in \mathcal{M}_+$. A trace is called *semifinite* if for every $0 < x \in \mathcal{M}_+$ there exists $0 < y \leq x$ such that $\tau(y) < \infty$.

If $(x_\alpha) \subseteq \mathcal{M}_+$ with $x_\alpha \uparrow_\alpha x$ in $\mathcal{M}$ implies that $\tau(x_\alpha) \uparrow_\alpha \tau(x)$, then the trace is called *normal.*

A trace τ is *faithful* whenever $x \in \mathcal{M}_+$ with $\tau(x) = 0$ implies that $x = 0$.

Examples 0.1.1 *(i) Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be the von Neumann algebra $B(\mathcal{H})$. Suppose $\{\mathrm{e}_\alpha\}$ is an orthonormal basis for $\mathcal{H}$. We define the* diagonal trace *(or* canonical trace*) on $B(\mathcal{H})$ by $\tau : B(\mathcal{H}) \to [0,\infty] : x \to \sum_\alpha \langle x\mathrm{e}_\alpha, \mathrm{e}_\alpha\rangle$. (The diagonal trace is independent of the orthonormal basis.) The diagonal trace is a faithful semifinite normal trace on $B(\mathcal{H})$ which is finite if and only if $\mathcal{H}$ is of finite dimension.*

(ii) Let X be a locally compact space and let μ be a positive Radon measure. The space $L_\infty(X,\mu)$ of essentially bounded μ-measurable functions over X is a commutative von Neumann algebra, when elements of $L_\infty(X,\mu)$ are regarded as operators acting on the Hilbert space $L_2(X,\mu)$ by multiplication. In fact, any commutative von Neumann algebra can be represented as a $L_\infty(X,\mu)$ space with X and μ as above (see [Tak79] Chapter III, Theorem 1.18). The integral over X is a trace, i.e. we define the trace $\tau : L_{\infty,+} \to [0,\infty]$ by $\tau(f) = \int_X f \, d\mu$. Then τ is a faithful, semifinite, normal (by the Monotone Convergence Theorem) trace. (The trace is finite if and only if μ is finite.)

(iii) In particular, the sequence space ℓ_∞ acting on the Hilbert space ℓ_2 by multiplication, that is, the set of operators

$$\mathbf{x} = (x_n) \in \ell_\infty : \ell_2 \to \ell_2 : (\xi_n) \to (\xi_n x_n)$$

form a von Neumann algebra. The trace defined by

$$\tau : \ell_{\infty,+} \to [0,\infty] : (x_n) \to \sum_{n=1}^{\infty} x_n$$

is a faithful semifinite normal trace.

A von Neumann algebra is semifinite if and only if it admits a faithful semifinite normal trace ([Tak79] Chapter V, Theorem 2.15).

Any unexplained concepts, notation or terminology can be found in [Tak79] and [KR86].

0.2 Reduced algebras

Suppose $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$. The commutant of $\mathcal{M}$, denoted by $\mathcal{M}'$, is the set of bounded operators on $\mathcal{H}$ that commute with every operator in $\mathcal{M}$. A von Neumann algebra can also be defined as a $*$-subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ such that $\mathcal{M} = \mathcal{M}''$. For $x \in \mathcal{M}$ and $\mathrm{q} \in \mathcal{M}^p$, we denote by x_{q} the restriction of $\mathrm{q}x$ to $\mathrm{q}\mathcal{H}$, and by $\mathcal{M}_{\mathrm{q}}$ the set $\{x_{\mathrm{q}} : x \in \mathcal{M}\}$. $\mathcal{M}_{\mathrm{q}}$ is called the *reduction* of $\mathcal{M}$ by q. The commutant of $\mathcal{M}$ is preserved under reductions, i.e.

$$(\mathcal{M}')_{\mathrm{q}} = (\mathcal{M}_{\mathrm{q}})'.$$

This was proved by [Dix81] I.2.1 and consequently we have that $\mathcal{M}_{\mathrm{q}}$ is a von Neumann algebra acting on the Hilbert space $\mathrm{q}\mathcal{H}$.

It follows from the definition that the reduced von Neumann algebra $\mathcal{M}_{\mathrm{q}}$ is isomorphic to the algebra $\mathrm{q}\mathcal{M}\mathrm{q}$. It is well known that the lattice of projections of $\mathcal{M}_{\mathrm{q}}$ is given by

$$(\mathcal{M}_{\mathrm{q}})^p = \{\mathrm{p}_{\mathrm{q}} : \mathrm{p} \in \mathcal{M}^p,\ \mathrm{p} \leq \mathrm{q}\}.$$

If τ is a faithful semifinite normal trace on $\mathcal{M}$ we define the reduction of τ to $\mathcal{M}_{\mathrm{q}}$, τ_{q}, on $\mathcal{M}_{\mathrm{q}}$ by setting $\tau_{\mathrm{q}}(x_{\mathrm{q}}) = \tau(\mathrm{q}x\mathrm{q})$ for all $x \in \mathcal{M}$. Then τ_{q} is a faithful semifinite normal trace on $\mathcal{M}_{\mathrm{q}}$.

0.3 Unbounded operators

We shall in general consider unbounded linear operators on a Hilbert space $\mathcal{H}$.

Let $x : \mathcal{D}(x) \to \mathcal{H}$ be a linear operator where $\mathcal{D}(x)$ denote the domain of x, a (linear) subspace of $\mathcal{H}$. With x we can associate its graph

$$\mathcal{G}(x) = \{(\xi, x\xi) : \xi \in \mathcal{D}(x)\}.$$

We say that x is *closed* if $\mathcal{G}(x)$ is closed. The operator x is *densely defined* if $\mathcal{D}(x)$ is dense in $\mathcal{H}$. If x is defined on all of $\mathcal{H}$ and its graph is closed, then by the Closed Graph Theorem x is bounded.

The closure $\overline{\mathcal{G}(x)}$ of $\mathcal{G}(x)$ is a linear subspace of $\mathcal{H} \oplus \mathcal{H}$. We say that y is an *extention* of x whenever $\mathcal{D}(x) \subseteq \mathcal{D}(y)$ and $y\xi = x\xi$ for every $\xi \in \mathcal{D}(x)$. It may be that $\overline{\mathcal{G}(x)}$ is the graph of an operator $\overline{x}$ but it need not be. If it is, then x extends to $\overline{x}$ and we say x is *preclosed*. We say $\overline{x}$ is the *closure* of x.

The *adjoint*, x^*, of the unbounded operator $x : \mathcal{D}(x) \to \mathcal{H}$, with $\mathcal{D}(x)$ dense in $\mathcal{H}$, is defined as follows: $\mathcal{D}(x^*)$ consists of those η in $\mathcal{H}$ such that for some ζ in $\mathcal{H}$ we have that

$$\langle x\xi, \eta \rangle = \langle \xi, \zeta \rangle$$

for all $\xi \in \mathcal{D}(x)$. For such η

$$x^*\eta = \zeta.$$

It follows that the adjoint x^* is a closed linear operator since x is densely defined. The operator x is called *self-adjoint* if $x = x^*$ and *normal* if $xx^* = x^*x$. A self-adjoint operator x is called *positive*, written $x \geq 0$ if it is densely defined and if

$$\langle x\xi, \xi \rangle \geq 0 \text{ for all } \xi \in \mathcal{D}(x).$$

We let $\mathcal{R}(x)$ denote the range of x and $\mathcal{N}(x)$ the kernel of x. The kernel of x is closed if x is closed. The projection onto $\mathcal{N}(x)$ is denoted by $\mathbf{n}(x)$. The *right support* of x is the projection

$$\mathbf{r}(x) = 1 - \mathbf{n}(x).$$

(The left support of x is the projection onto the closure of $\mathcal{R}(x)$, but we do not employ it.)

For details of the above we refer the reader to [KR86] Section 2.7.

0.4 Spectral Theory

In this section let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}^p$ the lattice of projections in $\mathcal{M}$. Let 1 denote the identity in $\mathcal{M}$. A *spectral measure* is a Boolean algebra homomorphism from the algebra $\mathcal{B}(\mathbf{C})$ of Borel measurable subsets of the complex field to $\mathcal{M}^p$,

$$\mathcal{B}(\mathbf{C}) \to \mathcal{M}^p : B \to \mathrm{e}_B$$

such that $\mathrm{e}_{\mathbf{C}} = 1$. A spectral measure is countably additive if for every disjoint sequence (B_n) in $\mathcal{B}(\mathbf{C})$ we have that

$$\sum_{n=1}^{\infty} \langle \mathrm{e}_{B_n}\xi, \xi \rangle = \langle \mathrm{e}_{\cup_{n=1}^{\infty} B_n}\xi, \xi \rangle$$

for every unit vector $\xi \in \mathcal{H}$. Let $\mathcal{M}(x)$ be the abelian von Neumann algebra generated by a self-adjoint operator x acting on a Hilbert space $\mathcal{H}$. (If x is self-adjoint, then the inverses of $x + i1$ and $x - i1$ are bounded. The von Neumann algebra $\mathcal{M}(x)$, generated by 1 and the inverses of $x + i1$ and $x - i1$, is the smallest von Neumann algebra with which x is affiliated and is known as the von Neumann algebra generated by x.) Then the spectrum of x, $\sigma(x)$, is real and there exists a countably additive spectral measure $\mathrm{e}_{.}(x)$ with range in $\mathcal{M}(x)$ that vanishes outside $\sigma(x)$. Define

$$\mathrm{e}_t(x) = \mathrm{e}_{(-\infty,t]\cap\sigma(x)}(x).$$

The family $\{\mathrm{e}_t(x) : t \in \mathbf{R}\}$ is called the *spectral family* for x or the *spectral resolution* of x. The spectral family for x has the following properties:

(a) If $t \leq s$ then $\mathrm{e}_t(x) \leq \mathrm{e}_s(x)$.

(b) The family is right continuous.

(c) $\mathrm{e}_t(x) \uparrow 1$ as $t \to \infty$.

(d) $\mathrm{e}_t(x) \downarrow 0$ as $t \to -\infty$.

(e) $x\mathrm{e}_t(x) \leq t\mathrm{e}_t(x)$ for t real.

(f) $t(1 - \mathrm{e}_t(x)) \leq x(1 - \mathrm{e}_t(x))$ for t real.

The spectral measure $e_{.}(x)$ is uniquely determined as explained in [KR86] Theorem 5.6.12 (iii) and the spectral family determines x in the following manner:

$$\mathcal{D}(x) = \{\xi \in \mathcal{H} : \int_{-\infty}^{\infty} t^2 d\|e_t(x)\xi\|^2 < \infty\}$$

and for $\xi \in \mathcal{D}(x)$ and $\eta \in \mathcal{H}$

$$\langle x\xi, \eta\rangle = \int_{-\infty}^{\infty} td\langle e_t(x)\xi, \eta\rangle.$$

If x is positive and closed, then the spectral projection $e_{(0,\infty)}(x)$ coincides with $\mathbf{r}(x) = 1 - \mathbf{n}(x)$, the right support projection of x.

If x is a bounded self-adjoint operator acting on $\mathcal{H}$ then $e_t(x) = 0$ for all $t \leq -\|x\|$, $e_t(x) = 1$ for all $t \geq \|x\|$ and

$$x = \int_{-\|x\|}^{\|x\|} t \, de_t(x)$$

in the sense of norm convergence of approximating Riemann sums. This means that x is the norm limit of finite linear combinations of orthogonal projections $e_s(x) - e_t(x) = e_{(t,s]\cap\sigma(x)}(x)$, $s > t$, with coefficients in the spectrum of x.

The above may be found in [KR86] Theorems 5.2.2, 5.6.12, 5.6.18 or [DS88] Theorem XII 2.3. Properties (e) and (f) of the spectral scale for x may be found in [SZ79] Exercises E9.10 and E9.12.

We shall need operational calculus for the polar decomposition of an unbounded operator and in Section 1.6. We therefore include the following which may be found in [DS88] Theorems XII 2.6 and 2.9 or [KR86] Theorem 5.6.26.

A closed, densely defined operator x in $\mathcal{H}$ with domain $\mathcal{D}(x)$ is said to be *affiliated* with $\mathcal{M}$ if $u^*xu = x$ for all unitary operators u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The preceding operator inequality is understood in the sense that u^*xu and x have the same domain so that $u(\mathcal{D}(x)) = \mathcal{D}(x)$. The collection of affiliated operators is a $*$-algebra with respect to strong sum (the closure of the algebraic sum), strong product (the closure of the algebraic product) and the adjoint operation. This algebra will be denoted by $[\mathcal{M}]$.

If $\mathcal{M}(x)$ is the abelian von Neumann algebra generated by a self-adjoint operator x acting on a Hilbert space $\mathcal{H}$ then there exists a homomorphism $f \to f(x)$ of the algebra of Borel measurable subsets of the spectrum of x, $\mathcal{B}(\sigma(x))$, into $[\mathcal{M}(x)]$ which maps the constant function 1 onto the identity operator 1 and the identity function $id_{\sigma(x)}$ onto x.

For $f \in \mathcal{B}(\sigma(x))$, the operator $f(x)$ has the following properties:

(a) $\mathcal{D}(f(x)) = \{\xi \in \mathcal{H} : \int_{-\infty}^{\infty} |f(t)|^2 d\|\mathrm{e}_t(x)\xi\|^2 < \infty\}$.

(b) For $\xi \in \mathcal{D}(f(x))$ and $\eta \in \mathcal{H}$

$$< f(x)\xi, \eta > = \int_{-\infty}^{\infty} f(t) d < \mathrm{e}_t(x)\xi, \eta > .$$

(c) $f(x)$ commutes with $\mathrm{e}_B(x)$ for any $B \in \mathcal{B}(\mathbf{C})$.

(d)

$$\|f(x)\| = \mathrm{e}_{.}(x) - \text{ess} \sup_{t \in \sigma(x)} |f(t)| = \inf_{\substack{B \in \mathcal{B}(\mathbf{R}) \\ \mathrm{e}_B(x)=1}} \sup_{t \in B \cap \sigma(x)} |f(t)|$$

and so $f(x) \in \mathcal{M}$ if and only if f is $\mathrm{e}_{.}(x)$-essentially bounded.

(e)

$$\sigma(f(x)) = \cap_{\substack{B \in \mathcal{B}(\mathbf{C}) \\ \mathrm{e}_B(x)=1}} \overline{f(B)}.$$

(f) If f is real valued then $f(x)$ is self-adjoint and

$$\mathrm{e}_B(f(x)) = \mathrm{e}_{f^{-1}(B)}(x)$$

for every $B \in \mathcal{B}(\mathbf{C})$.

(g) If $f = \chi_B$ where B is a Borel set, then $f(x) = \mathrm{e}_B(x)$.

0.5 The polar decomposition for an unbounded operator

Let $x : \mathcal{D}(x) \to \mathcal{H}$ be a positive linear operator. By applying the operational calculus in Section 0.4, it follows that x has a unique positive square root denoted by $x^{\frac{1}{2}}$, see [KR86] Remark 5.6.32.

Suppose that x is a closed, densely defined operator on a Hilbert space $\mathcal{H}$. Then

$$x = v(x^*x)^{\frac{1}{2}} = (xx^*)^{\frac{1}{2}}v$$

where v is a partial isometry with initial space the closure of $\mathcal{R}\left((x^*x)^{\frac{1}{2}}\right)$ and final space the closure of $\mathcal{R}(x)$. Restricted to the closures of $\mathcal{R}(x^*)$ and $\mathcal{R}(x)$, respectively, x^*x and xx^* are unitarily equivalent and v implements this equivalence. If

$$x = wy$$

where y is a positive operator and w is a partial isometry with initial space the closure of $\mathcal{R}(y)$, then

$$y = (x^*x)^{\frac{1}{2}} \text{ and } w = v.$$

In addition, the operator x is affiliated to $\mathcal{M}$ if and only if $v \in \mathcal{M}$ and $(x^*x)^{\frac{1}{2}}$ is affiliated to $\mathcal{M}$. We define

$$(x^*x)^{\frac{1}{2}} = |x| \text{ and } (xx^*)^{\frac{1}{2}} = |x^*|.$$

The decomposition

$$x = v|x|$$

is called the *polar decomposition* of x.

For details of the above we refer the reader to [KR86] Section 6.1.

If the operator x is self-adjoint, then its right support, $\mathbf{r}(x)$, equals the projection onto the closure of $\mathcal{R}(x)$ (its left support) and is called the support of x. It follows from the polar decomposition that if x is a self-adjoint linear operator on $\mathcal{H}$, there exists positive operators x^+ and x^- such that

$$x = x^+ - x^-$$

and

$$\mathbf{r}(x^+)\mathbf{r}(x^-) = 0$$

and these conditions determine the operators x^+ and x^- uniquely ([SZ79] Corollary 9.31).

0.6 Banach function spaces

Suppose (X, Σ, μ) is a localizable measure space, which means the measure μ is semifinite (or locally finite), that is, every set of positive measure has a subset of finite positive measure, and the supremum of any collection of μ-measurable sets with finite measure, exists ([LZ71] p.505). The set of equivalence classes (modulo a.e. equivalence) of (a.e. finite) complex valued μ-measurable functions on (X, Σ, μ) will be denoted by $L_0(X, \Sigma, \mu)$, or just L_0 when there is no danger of confusion. With the partial order $f \leq g$ in L_0 if $f(x) \leq g(x)$ μ-a.e. on X, L_0 is a vector lattice. An *ideal* $\mathcal{I}$ in $L_0(X, \Sigma, \mu)$ is a solid linear subspace of $L_0(X, \Sigma, \mu)$, that is, if $g \in \mathcal{I}$ and $f \in L_0(X, \Sigma, \mu)$ with $|f| \leq |g|$, then $f \in \mathcal{I}$.

The set of positive functions in $L_0(X, \Sigma, \mu)$ will be denoted by $L_0(X, \Sigma, \mu)_+$. A *function norm* ρ is a function $\rho : L_0(X, \Sigma, \mu)_+ \to [0, \infty]$ satisfying the following properties: For $f, g \in L_0(X, \Sigma, \mu)_+$

(i) $\rho(f) = 0$ if and only if $f = 0$ μ-a.e. ;

(ii) $\rho(\lambda f) = \lambda \rho(f)$ for every $\lambda \geq 0$;

(iii) $\rho(f + g) \leq \rho(f) + \rho(g)$;

(iv) If $f \leq g$ then $\rho(f) \leq \rho(g)$.

We extend the function norm ρ to $L_0(X, \Sigma, \mu)$ by putting $\rho(f) = \rho(|f|)$ for any $f \in L_0(X, \Sigma, \mu)$.

The set of functions for which $\rho(f) < \infty$ is a vector subspace of $L_0(X, \Sigma, \mu)$ and we denote it by $L_\rho(X, \Sigma, \mu)$, or L_ρ for short. The function norm ρ is a norm on L_ρ with the additional property that if $g \in L_\rho$ and $f \in L_0$ with $|f| \leq |g|$, then $f \in L_\rho$ and $\rho(f) \leq \rho(g)$. Thus L_ρ is an ideal in L_0. We say $L_\rho(X, \Sigma, \mu)$ is a *Banach function space* when L_ρ, equipped with the metric induced by the norm ρ, is complete. With the μ-almost everywhere ordering, L_ρ is a Banach lattice, that is, L_ρ is a vector lattice which is also a Banach space such that $f, g \in L_\rho$ with $0 \leq f \leq g$ implies that $\rho(f) \leq \rho(g)$. We write $f_\alpha \downarrow_\alpha 0$ in L_ρ when $f_\alpha(x) \downarrow_\alpha 0$ for μ-almost every $x \in X$.

We may associate with the Banach function space $L_\rho(X, \Sigma, \mu)$ its *Köthe dual space*

$$L_\rho^\times = \{f \in L_0 : \int_X |fg| \, d\mu < \infty \text{ for all } g \in L_\rho\}$$

with norm

$$\|f\|_\times = \sup\{\int_X |fg| \, d\mu : \rho(g) \leq 1\} = \sup\{\left|\int_X fg \, d\mu\right| : \rho(g) \leq 1\}.$$

We denote by L_ρ^* the set of ρ-continuous linear functionals on L_ρ. We say the functional $\varphi \in L_\rho^*$ is *σ-normal* (or *σ-order continuous*) if

$$f_n \downarrow_n 0 \text{ in } L_\rho \Rightarrow \varphi(f_n) \to_n 0.$$

Suppose the measure space (X, Σ, μ) is σ-finite. Then the set of σ-normal functionals on L_ρ is exactly the Köthe dual of L_ρ under the following correspondence: $\varphi \in L_\rho^*$ is σ-normal if and only if $\varphi(f) = \int_X fg \, d\mu$ for some $g \in L_\rho^\times$ and for all $f \in L_\rho$. (The elements φ and g determine each other uniquely.)

The above results with their proofs may be found in [Zaa67] Chapter 15. For a more general discussion the reader may consult [Fre74] Paragraph 65.

Contents

Chapter 1

Spaces of τ-measurable operators

We consider the space, $\widetilde{\mathcal{M}}$, of τ-measurable operators affiliated to a semifinite von Neumann algebra equipped with a distinguished trace, τ. In this chapter we investigate aspects of the structure of $\widetilde{\mathcal{M}}$ and the topology of convergence in measure on $\widetilde{\mathcal{M}}$. In particular, we investigate the local convexity of the topology of convergence in measure on $\widetilde{\mathcal{M}}$. This will prove useful when we consider continuous (with respect to the measure topology) linear functionals on $\widetilde{\mathcal{M}}$ in Chapter 4.

The results contained in Sections 1.1 through 1.3 are the work of others. We therefore state these preliminary results without proofs. In the first section we define τ-measurable operators, their generalised singular functions and give their properties. We then state the known results about the order structure in $\widetilde{\mathcal{M}}$, due to [DDP93]. In the following two sections we discuss subspaces of $\widetilde{\mathcal{M}}$ that are important in our further work. In particular, Section 1.4 discusses the subspace $\widetilde{\mathcal{M}_0}$ of $\widetilde{\mathcal{M}}$, consisting of τ-measurable operators whose generalised singular functions decrease to zero, in detail. Next we investigate the local convexity of the topology of convergence in measure on $\widetilde{\mathcal{M}}$ and in the final sections we indicate how the structure of $\widetilde{\mathcal{M}}$ depends on the nature of the lattice of projections on $\mathcal{M}$.

1.1 Introduction to τ-measurable operators

Let $\mathcal{M}$ be a semifinite von Neumann algebra (with underlying Hilbert space $\mathcal{H}$), equipped with a distinguished faithful semifinite normal trace τ.

An affiliated operator x is called *τ-measurable* if for every $\delta > 0$ there exists a projection $\mathrm{p} \in \mathcal{M}^p$ such that $\mathrm{p}\mathcal{H} \subseteq \mathcal{D}(x)$ and $\tau(1-\mathrm{p}) \leq \delta$. We denote by $\widetilde{\mathcal{M}}$ the set of all τ-measurable operators on $\mathcal{H}$. The set $\widetilde{\mathcal{M}}$ is a $*$-algebra with the sum and product operations defined as the respective closures of the algebraic sum and product. The trace τ extends naturally to the positive cone of $\widetilde{\mathcal{M}}$ so that the faithfulness, semifiniteness and normality of τ are preserved. See, for example [FK86] Theorem 3.5 (extention of the trace) or [DDP93] Section 3 (alternative approach to the extention of the trace from which the properties follow more directly). If x is τ-measurable then, as for bounded operators, x can be written as a linear combination of two self-adjoint operators. Thus since an unbounded self-adjoint linear operator is the difference of two positive operators (Section 0.5), we have that $x \in \widetilde{\mathcal{M}}$ can be written as a linear combination of four positive operators.

Let $\|.\|$ denote the operator norm in $\mathcal{M}$. The sets

$$\widetilde{\mathcal{M}}(\epsilon,\delta) = \{x \in \widetilde{\mathcal{M}} : \mathrm{p}\mathcal{H} \subseteq \mathcal{D}(x)\ ,\ \|x\mathrm{p}\| \leq \epsilon\ ,\ \tau(1-\mathrm{p}) \leq \delta \text{ for some } \mathrm{p} \in \mathcal{M}^p\},\ \ \epsilon,\delta > 0$$

form a base at zero for a metrizable Hausdorff topology on $\widetilde{\mathcal{M}}$ called the *measure topology*. Equipped with the measure topology, $\widetilde{\mathcal{M}}$ is a complete topological $*$-algebra in which $\mathcal{M}$ is dense. For proofs of these facts the reader may consult the papers of Nelson [Nel74], Terp [Ter81] or Fack and Kosaki [FK86].

We list properties of the basic neighbourhoods (see [Ter81] 1.26).

Lemma 1.1.1 *Let $\epsilon,\delta,\gamma,\nu > 0$ and $\lambda \in \mathbf{C}$. Then*

(i) $\widetilde{\mathcal{M}}(\epsilon,\delta)^* = \widetilde{\mathcal{M}}(\epsilon,\delta)$;

(ii) $\{|x| : x \in \widetilde{\mathcal{M}}(\epsilon,\delta)\} = \{x \in \widetilde{\mathcal{M}}(\epsilon,\delta) : x \geq 0\}$;

(iii) $\widetilde{\mathcal{M}}(|\lambda|\epsilon,\delta) = \lambda\widetilde{\mathcal{M}}(\epsilon,\delta)$;

(iv) $\widetilde{\mathcal{M}}(\epsilon,\delta) \subseteq \widetilde{\mathcal{M}}(\gamma,\nu)$ *whenever* $\epsilon \leq \gamma$ *and* $\delta \leq \nu$;

(v) $\widetilde{\mathcal{M}}(\epsilon \wedge \gamma, \delta \wedge \nu) \subseteq \widetilde{\mathcal{M}}(\epsilon, \delta) \cap \widetilde{\mathcal{M}}(\gamma, \nu)$;

(vi) $\widetilde{\mathcal{M}}(\epsilon, \delta) + \widetilde{\mathcal{M}}(\gamma, \nu) \subseteq \widetilde{\mathcal{M}}(\epsilon + \gamma, \delta + \nu)$.

We write $\widetilde{\mathcal{M}}(\epsilon)$ for $\widetilde{\mathcal{M}}(\epsilon, \epsilon)$. We identify $\widetilde{\mathcal{M}}$ in a few cases.

Examples 1.1.2 *(i) If $\tau(1) < \infty$ then $\widetilde{\mathcal{M}}$ consists of all densely defined closed linear operators affiliated with $\mathcal{M}$.*

(ii) If $\mathcal{M}$ is $B(\mathcal{H})$, the von Neumann algebra of all bounded linear operators in $\mathcal{H}$ equipped with the diagonal trace, then $\mathcal{M}$ coincides with $\widetilde{\mathcal{M}}$ and the measure topology coincides with the operator norm topology.

(iii) If $\mathcal{M}$ is commutative, we may represent $\mathcal{M}$ as $L_\infty(X, \Sigma, \mu)$ where (X, Σ, μ) is a localizable measure space. If the trace is the integral, that is, $\tau(f) = \int_X f \, d\mu$ for $f \in L_\infty(X, \Sigma, \mu)$, then $\widetilde{\mathcal{M}}$ is the set of all measurable complex functions on X that are bounded except on a set of finite measure, and the measure topology coincides with the usual topology of convergence in measure.

(iv) In particular, if $\mathcal{M} = \ell_\infty$ with the canonical trace, acting on $\mathcal{H} = \ell_2$ by multiplication in the usual way, then $\widetilde{\mathcal{M}} = \ell_\infty = \mathcal{M}$.

The structure of $\widetilde{\mathcal{M}}$ depends heavily on the nature of $\mathcal{M}^p$. We need

Definition 1.1.3 *We call a projection* atomic *if it has no nonzero subprojections and we say a projection is* nonatomic *(or* continuous*) if it has no atomic subprojections. $\mathcal{M}^p$ is called* atomic *if it contains no nonatomic projections and $\mathcal{M}^p$ is* nonatomic *(or* continuous*) if it possesses no atomic projections.*

In examples *(ii)* and *(iv)* above, we see that

$$\inf\{\tau(\mathrm{p}) : \mathrm{p} \text{ atomic}, \ \mathrm{p} \in \mathcal{M}^p\} > 0$$

and that $\widetilde{\mathcal{M}} = \mathcal{M}$. In fact, it was shown in [SW93] Examples 2.2(3) that the following statements are equivalent.

(a) $\widetilde{\mathcal{M}} = \mathcal{M}$.

(b) The topology of convergence in measure equals the norm topology.

(c) The infimum of the traces of the nonzero projections is greater than zero.

Suppose $x \in \widetilde{\mathcal{M}}$ and let $x = v|x|$ be the polar decomposition of x. By [Ter81] Proposition 21, the *distribution function* of x,

$$d_t(x) = \tau\left(\mathrm{e}_{(t,\infty)}(|x|)\right),$$

is eventually finite valued, and

$$d_t(x) \to 0 \text{ as } t \to \infty.$$

Hence for any $t > 0$ there exists $s > 0$ such that $d_s(x) \leq t$. Thus the *generalised singular function* of x (also known as the *decreasing rearrangement* of x),

$$\mu_t(x) = \inf\{s \geq 0 : d_s(x) \leq t\} = \inf\{s \geq 0 : \tau\left(\mathrm{e}_{(s,\infty)}(|x|)\right) \leq t\}$$

is finite valued. The generalised singular function is decreasing, right continuous, a.e. continuous (with respect to Lebesgue measure on the positive real line) and the above infimum is attained when choosing the spectral projection $\mathrm{e}_{\mu_t(x)}(|x|)$. An alternative characterisation for the generalised singular function of x is

$$\mu_t(x) = \inf\{\|x\mathrm{p}\| : \mathrm{p} \in \mathcal{M}^p,\ \tau(1-\mathrm{p}) \leq t\}.$$

Here we point out that

$$\|x\mathrm{e}_\epsilon(|x|)\| \leq \epsilon$$

always holds and we have equality when ϵ is in the range of $\mu(x)$. If $\mathrm{p} \in \mathcal{M}^p$ then

$$\mu_t(\mathrm{p}) = \chi_{(0,\tau(\mathrm{p}))}(t).$$

A basis of neighbourhoods at zero for the measure topology can be given in terms of the generalised singular function by the sets

$$\widetilde{\mathcal{M}}(\epsilon, \delta) = \{x \in \widetilde{\mathcal{M}} : \mu_\delta(x) \leq \epsilon\},\ \epsilon, \delta > 0.$$

We list properties of the generalised singular function (see [FK86] Lemma 2.5).

Proposition 1.1.4 *Suppose $x, y, z \in \widetilde{\mathcal{M}}$ and $\lambda \in \mathbf{C}$.*

(i) $\lim_{t \to 0+} \mu_t(x) = \|x\|$.

(ii) $\mu(x) = 0$ *if and only if* $x = 0$.

(iii) $\mu_t(\lambda x) = |\lambda| \mu_t(x)$ *for all* $t > 0$.

(iv) $\mu_t(x) \leq \mu_t(y)$ *for all* $t > 0$ *whenever* $|x| \leq |y|$.

(v) $\mu_{t+s}(x + y) \leq \mu_t(x) + \mu_s(y)$ *for all* $t, s > 0$.

(vi) $\mu_{t+s}(xy) \leq \mu_t(x)\mu_s(y)$ *for all* $t, s > 0$.

(vii) $\mu_t(xyz) \leq \|x\| \mu_t(y) \|z\|$ *for all* $t > 0$.

(viii) $|\mu_t(x) - \mu_t(y)| \leq \|x - y\|$ *for all* $t > 0$.

(ix) $\mu_{d_t(x)}(x) \leq t$ *for all* $t > 0$ *for which* $d_t(x)$ *is finite.*

(x) $d_{\mu_t(x)}(x) \leq t$ *for all* $t > 0$.

If the sequence (x_n) converge to $x \in \widetilde{\mathcal{M}}$ for the measure topology then $\mu_t(x_n) \to_n \mu_t(x)$ whenever $t > 0$ is a point of continuity of $\mu(x)$. In particular

$$\mu_t(x_n) \to_n \mu_t(x) \text{ a.e.}$$

with respect to Lebesgue measure on $(0, \infty)$, and if $x = 0$,

$$\mu_t(x_n) \to_n 0 \text{ for all } t > 0.$$

Conversely, $\mu_t(x_n) \to_n 0$ for all $t > 0$ implies that $x_n \to_n 0$ in the measure topology.

The above results regarding the generalised singular function are due to Fack and Kosaki and for details the reader is referred to [FK86].

1.2 The order structure in $\widetilde{\mathcal{M}}$

The order structure in $\widetilde{\mathcal{M}}$ plays an important part in what follows and we therefore give a short summary of the relevant facts.

If x is a self-adjoint linear operator on $\mathcal{H}$ and if $\langle .,. \rangle$ denotes the inner product in $\mathcal{H}$ then we have that $x \geq 0$ if and only if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. $\widetilde{\mathcal{M}}$ is an ordered vector space with respect to the partial ordering defined by setting

$$x \geq y \text{ if and only if } x - y \geq 0$$

where $x - y$ means the closure of the algebraic difference. Further we have that for $0 \leq x \in \widetilde{\mathcal{M}}$, $y^*xy \geq 0$ for all $y \in \widetilde{\mathcal{M}}$. The positive cone $\widetilde{\mathcal{M}}_+$ is closed for the measure topology.

$(\widetilde{\mathcal{M}}, \leq)$ is order complete in the sense that if $0 \leq x_\alpha \uparrow_\alpha \leq y$ holds in $\widetilde{\mathcal{M}}$ then $x = \sup_\alpha x_\alpha$ exists in $\widetilde{\mathcal{M}}$. Moreover, we have the following:

Proposition 1.2.1 *If $0 \leq x_\alpha \uparrow_\alpha x$ holds in $\widetilde{\mathcal{M}}$ then*

$$0 \leq y^*x_\alpha y \uparrow_\alpha y^*xy$$

holds in $\widetilde{\mathcal{M}}$ for all $y \in \widetilde{\mathcal{M}}$.

Proposition 1.2.2 *Let $0 \leq x \in \widetilde{\mathcal{M}}$. Then there exists a net $(x_\alpha) \subseteq \mathcal{M}$ with $\tau(x_\alpha) < \infty$ for each α such that $0 \leq x_\alpha \uparrow_\alpha x$ in $\widetilde{\mathcal{M}}$.*

Proposition 1.2.3 *If $0 \leq x_\alpha \uparrow_\alpha x$ in $\widetilde{\mathcal{M}}$ then*

$$\mu_t(x_\alpha) \uparrow_\alpha \mu_t(x)$$

holds for all $t > 0$.

The above results are due to Peter Dodds, Theresa Dodds and Ben de Pagter and for proofs and full details the reader is referred to [DDP93] Section 1.

1.3 The subspaces $\mathcal{H}(\mathcal{M})$ and $\mathcal{G}(\mathcal{M})$ of $\widetilde{\mathcal{M}}$

Recall that for a pair of Banach spaces E and F with respective norms $\|.\|_E$ and $\|.\|_F$, we may consider their intersection $E \cap F$ and their sum $E + F$ as follow: The intersection $E \cap F$ consists of elements common to E and F, and with the norm

$$\|x\|_{E\cap F} = \max\{\|x\|_E, \|x\|_F\} \text{ for } x \in E \cap F,$$

$E \cap F$ is a Banach space. The sum $E + F$ is also a Banach space equipped with the norm

$$||x||_{E+F} = \inf\{||y||_E + ||z||_F : x = y + z, y \in E, z \in F\} \text{ for } x \in E + F,$$

[KPS82] Chapter I, Section 3.1. In particular, we shall apply this to the pair of Banach spaces $L_1(0,\infty)$ and $L_\infty(0,\infty)$. (As usual $L_1(0,\infty)$ consists of all Lebesgue integrable functions on the half real line $(0,\infty)$ and $L_\infty(0,\infty)$ contains all essentially bounded Lebesgue measurable complex functions on $(0,\infty)$.) We define the non-commutative analogue of $L_1(0,\infty)$ by setting

$$L_1(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in L_1(0,\infty)\}$$

with the norm

$$||x||_{L_1(\mathcal{M})} = ||\mu(x)||_1 = \int_0^\infty \mu(x)\ dt,$$

the usual Lebesgue integral.

We define $L_\infty(\mathcal{M})$ and $||.||_{L_\infty(\mathcal{M})}$ similarly. Note that $x \in L_\infty(\mathcal{M})$ if and only if $\mu(x) \in L_\infty(0,\infty)$ if and only if $x \in \mathcal{M}$ and

$$||x||_{L_\infty(\mathcal{M})} = ||\mu(x)||_\infty = \lim_{t\to 0^+} \mu_t(x) = ||x||$$

by Proposition 1.1.4 (i), where $||.||_\infty$ denotes the essential supremum norm on $L_\infty(0,\infty)$. Hence

$$L_\infty(\mathcal{M}) = \mathcal{M}$$

as Banach spaces. We now define

$$x \in (L_1 \cap L_\infty)(\mathcal{M}) \text{ if and only if } \mu(x) \in L_1(0,\infty) \cap L_\infty(0,\infty)$$

with norm

$$||x||_{(L_1\cap L_\infty)(\mathcal{M})} = ||\mu(x)||_{(L_1\cap L_\infty)(0,\infty)} \quad .$$

where we write $(L_1 \cap L_\infty)(0,\infty)$ for $L_1(0,\infty) \cap L_\infty(0,\infty)$. Similarly we define the space $(L_1 + L_\infty)(\mathcal{M})$. As one would expect, we have that

$$(L_1 \cap L_\infty)(\mathcal{M}) = L_1(\mathcal{M}) \cap L_\infty(\mathcal{M}) = L_1(\mathcal{M}) \cap \mathcal{M}$$

and

$$(L_1 + L_\infty)(\mathcal{M}) = L_1(\mathcal{M}) + L_\infty(\mathcal{M}) = L_1(\mathcal{M}) + \mathcal{M}$$

as Banach spaces and we denote them by $\mathcal{H}(\mathcal{M})$ and $\mathcal{G}(\mathcal{M})$, respectively.

We will make use of the following characterisation of elements in $\mathcal{G}(\mathcal{M})$ which can be found in [DDP93] Proposition 2.6.

Proposition 1.3.1 *If $x \in \widetilde{\mathcal{M}}$ then the following statements are equivalent.*

(a) $x \in \mathcal{G}(\mathcal{M})$.

(b) $\int_0^\theta \mu_t(x)\, dt < \infty$ *for some* $\theta > 0$.

(c) $\int_0^\theta \mu_t(x)\, dt < \infty$ *for all* $\theta > 0$.

(d) $\mathrm{e}|x|\mathrm{e} \in L_1(\mathcal{M})$ *for all projections* e *with finite trace.*

1.4 The subspace $\widetilde{\mathcal{M}_0}$ of $\widetilde{\mathcal{M}}$

Suppose $x \in \widetilde{\mathcal{M}}$, then we know that there exists a $t > 0$ such that the generalised singular function of x at t is finite, that is, $\mu_t(x) < \infty$. Therefore since $\mu_t(x)$ is decreasing in t, $\lim_{t\to\infty} \mu_t(x)$ exists and we denote it by $\mu_\infty(x)$. Then μ_∞ is a $*$-algebra semi-norm on $\widetilde{\mathcal{M}}$, the details to be found in [SW93] Section 2. If $x_\alpha \to_\alpha 0$ in the topology of convergence in measure in $\widetilde{\mathcal{M}}$ then $\mu_t(x_\alpha) \to_\alpha 0$ for all $t > 0$ and hence $\mu_\infty(x_\alpha) \to_\alpha 0$. Thus μ_∞ is continuous at 0 and hence on $\widetilde{\mathcal{M}}$. We define $\widetilde{\mathcal{M}_0}$ to be the kernel of μ_∞, i.e.

$$\widetilde{\mathcal{M}_0} = \{x \in \widetilde{\mathcal{M}} : \mu_\infty(x) = 0\}.$$

We will see that $\widetilde{\mathcal{M}_0}$ plays an important role in the structure of $\widetilde{\mathcal{M}}$ and an integral part in the investigation of the dual of $\widetilde{\mathcal{M}}$.

It is easy to see that $\widetilde{\mathcal{M}_0}$ is a linear subspace of $\widetilde{\mathcal{M}}$. We notice that $\widetilde{\mathcal{M}_0}$ is solid in $\widetilde{\mathcal{M}}$ in the sense that if $x \in \widetilde{\mathcal{M}_0}, y \in \widetilde{\mathcal{M}}$ with $|y| \leq |x|$, then $y \in \widetilde{\mathcal{M}_0}$ since $|y| \leq |x|$ implies that $\mu_t(y) \leq \mu_t(x)$ for all $t > 0$. Hence $\lim\limits_{t\to\infty} \mu_t(y) \leq \lim\limits_{t\to\infty} \mu_t(x) = 0$ which means that $y \in \widetilde{\mathcal{M}_0}$.

By definition of $\widetilde{\mathcal{M}_0}$ it is clear that $x \in \widetilde{\mathcal{M}_0}$ if and only if $\tau(\mathrm{e}_{(t,\infty)}(|x|) < \infty$ for all $t > 0$. Therefore we know that any τ-measurable operator with finite trace is contained in $\widetilde{\mathcal{M}_0}$

and that $\mathcal{H}(\mathcal{M}) \subseteq \widetilde{\mathcal{M}_0}$. In particular, note that if $\mathrm{p} \in \widetilde{\mathcal{M}}$ then $\mu_t(\mathrm{p}) = \chi_{(0,\tau(\mathrm{p}))}(t)$ for all $t > 0$ and thus $\mathrm{p} \in \widetilde{\mathcal{M}_0}$ if and only if $\tau(\mathrm{p}) < \infty$.

We refer back to the Examples 1.1.2 we mentioned in Section 1.1.

Examples 1.4.1 *(i) If $\tau(1) < \infty$ then by definition of $\widetilde{\mathcal{M}_0}$ it equals the whole space $\widetilde{\mathcal{M}}$.*

(ii) If $\mathcal{M} = \mathcal{B}(\mathcal{H}) = \widetilde{\mathcal{M}}$ with the canonical trace then $\widetilde{\mathcal{M}_0} = \mathcal{C}(\mathcal{H})$, the set of compact operators on $\mathcal{H}$.

(iii) If $\mathcal{M} = L_\infty(X, \Sigma, \mu)$ and the trace the integral, then $\widetilde{\mathcal{M}}$ is the set of complex measurable functions that are bounded except on a set of finite measure and

$$\widetilde{\mathcal{M}_0} = \{f \in \widetilde{\mathcal{M}} : \mu\{t \in X : |f(t)| > s\} < \infty \text{ for all } s > 0\}.$$

(iv) If $\mathcal{M} = \ell_\infty = \widetilde{\mathcal{M}}$ with the canonical trace then $\widetilde{\mathcal{M}_0} = c_0$.

It is shown in [DDP93] Proposition 2.7 that $\widetilde{\mathcal{M}_0}$ is exactly the closure in the topology of convergence in measure of $\mathcal{H}(\mathcal{M})$ in $\widetilde{\mathcal{M}}$ and that the closure of $\mathcal{H}(\mathcal{M})$ in the space $\mathcal{G}(\mathcal{M})$ is $\widetilde{\mathcal{M}_0} \cap \mathcal{G}(\mathcal{M})$.

If we define the linear subspace $\mathcal{M}_0$ of $\mathcal{M}$ in a similar way to $\widetilde{\mathcal{M}_0}$, that is,

$$\mathcal{M}_0 = \{x \in \mathcal{M} : \mu_\infty(x) = 0\} = \widetilde{\mathcal{M}_0} \cap \mathcal{M}$$

then it was shown in [Wes93] Corollary 6.2.4 that $\mathcal{M}_0$ is the closed (in operator norm) two sided ideal in $\mathcal{M}$ generated by the projections with finite trace, i.e., $\mathcal{M}_0 = \overline{\mathcal{H}(\mathcal{M})}^{\|\cdot\|}$.

Note that as before, $\inf\{\tau(\mathrm{p}) : \mathrm{p} \text{ atomic}, \mathrm{p} \in \mathcal{M}^p\} > 0$ in Examples (ii) and (iv) above. We now have that the following four statements are equivalent (see [SW93] Examples 2.2(3)):

(a) $\widetilde{\mathcal{M}} = \mathcal{M}$.

(b) The topology of convergence in measure equals the norm topology.

(c) The infimum of the trace of the nonzero projections is greater than zero.

(d) $\widetilde{\mathcal{M}_0} = \mathcal{M}_0$.

By the definition of τ-measurability, any $x \in \widetilde{\mathcal{M}}$ can be decomposed as $x\mathrm{p} + x(1-\mathrm{p})$ where $\mathrm{p} \in \mathcal{M}^p$, $\|x\mathrm{p}\| < \infty$ and $\tau(1-\mathrm{p}) < \infty$. Hence $x(1-\mathrm{p}) \in \widetilde{\mathcal{M}_0}$ and $x\mathrm{p} \in \mathcal{M}$. We have that

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}_0} + \mathcal{M}.$$

It is interesting to note that each $x \in \widetilde{\mathcal{M}}$ admits a decomposition $x = x_0 + x_\infty$ ($x_0 \in \widetilde{\mathcal{M}_0}$ and $x_\infty \in \mathcal{M}$) such that $\|x_\infty\| = \mu_\infty(x)$. This decomposition is the best in the sense that it gives the smallest possible value for the norm of x_∞ and the smallest possible generalised singular function of x_∞, as can be seen from the following ([SW93] Remarks 3.2.2)

$$\|x_\infty\| \geq \mu_t(x_\infty) \geq \mu_\infty(x_\infty) = \mu_\infty(x) = \|x_\infty\| \text{ for all } t > 0.$$

The following result, which can be found in [DDP93] Lemma 3.5, shows that the topology of convergence in measure is "order continuous" on $\widetilde{\mathcal{M}_0}$.

Theorem 1.4.2 *If $x \in \widetilde{\mathcal{M}_0}$ and $x \geq x_\alpha \downarrow_\alpha 0$ in $\widetilde{\mathcal{M}}$ then $\mu_t(x_\alpha) \downarrow_\alpha 0$ for all $t > 0$, and so $x_\alpha \to_\alpha 0$ in the measure topology.*

If we assume that $\mathcal{M}^p$ is nonatomic (contains no atomic projections), then we can give another characterisation of the subspace $\widetilde{\mathcal{M}_0}$. This is a generalisation of the work of S.J. Dilworth and D.A. Trautman, [DT90] Lemma 3.2.

Lemma 1.4.3 *Let $\mathcal{M}$ be a semifinite Von Neumann algebra, τ a faithful semifinite normal trace on $\mathcal{M}$ and $\mathcal{M}^p$ nonatomic. Then*

$$\widetilde{\mathcal{M}_0} = \bigcap_{\epsilon,\delta>0} conv\widetilde{\mathcal{M}}(\epsilon,\delta)$$

where $\widetilde{\mathcal{M}}(\epsilon,\delta)$ is a basic neighbourhood of zero in $\widetilde{\mathcal{M}}$ and $conv\widetilde{\mathcal{M}}(\epsilon,\delta)$ denotes the convex hull of the neighbourhood $\widetilde{\mathcal{M}}(\epsilon,\delta)$.

Proof: A typical neighbourhood of zero in $\widetilde{\mathcal{M}_0}$ is

$$\widetilde{\mathcal{M}_0}(\epsilon,\delta) = \{x \in \widetilde{\mathcal{M}_0} : \mathrm{p}\mathcal{H} \subseteq \mathcal{D}(x), \|x\mathrm{p}\| \leq \epsilon, \tau(1-\mathrm{p}) \leq \delta \text{ for some } \mathrm{p} \in \mathcal{M}^p\},$$

where $\epsilon, \delta > 0$. We first show that

$$\widetilde{\mathcal{M}_0} = conv\widetilde{\mathcal{M}_0}(\epsilon, \delta)$$

for any $\epsilon, \delta > 0$. Therefore let ϵ, $\delta > 0$. Clearly $conv\widetilde{\mathcal{M}_0}(\epsilon,\delta) \subset \widetilde{\mathcal{M}_0}$. Conversely, let $x \in \widetilde{\mathcal{M}_0}$. Then $d_\epsilon(x) = \tau(e_{(\epsilon,\infty)}(|x|)) < \infty$ by [SW93] Theorem 2.1. Put $\mathrm{p} = e_\epsilon(|x|)$ and notice that $||x\mathrm{p}|| \leq \epsilon$ and $\tau(1-\mathrm{p}) < \infty$. Since $\mathcal{M}^{\mathrm{p}}$ is nonatomic, there exist disjoint $\mathrm{e}_1, \dots, \mathrm{e}_n \in \mathcal{M}^{\mathrm{p}}, n \in \mathrm{N}$ such that $1 - \mathrm{p} = \bigvee_{i=1}^n \mathrm{e}_i$ and with $\tau(\mathrm{e}_i) \leq \delta$ for all $i = 1, \dots, n$. Note that $\mathrm{p}(1-\mathrm{e}_i) = \mathrm{p}$ for all i since $\mathrm{p} = 1 - \bigvee_{i=1}^n \mathrm{e}_i = \bigwedge_{i=1}^n (1-\mathrm{e}_i)$.

Put $x_i = x\mathrm{p} + nx\mathrm{e}_i$ for $1 \leq i \leq n$.

Then $x_i \in \widetilde{\mathcal{M}_0}(\epsilon, \delta)$ for $1 \leq i \leq n$ since for each i, there exists a projection namely $1 - \mathrm{e}_i \in \mathcal{M}^{\mathrm{p}}$ such that

$$\begin{aligned} ||(x\mathrm{p} + nx\mathrm{e}_i)(1-\mathrm{e}_i)|| &= ||x\mathrm{p}(1-\mathrm{e}_i) + 0|| \\ &= ||x\mathrm{p}|| \\ &\leq \epsilon \end{aligned}$$

and $\tau(1-(1-\mathrm{e}_i)) = \tau(\mathrm{e}_i) \leq \delta$. Finally,

$$\begin{aligned} \sum_{i=1}^n \frac{1}{n} x_i &= \frac{1}{n}(x\mathrm{p} + nx\mathrm{e}_1 + x\mathrm{p} + nx\mathrm{e}_2 + \ldots + x\mathrm{p} + nx\mathrm{e}_n) \\ &= \frac{1}{n}(nx\mathrm{p} + nx(\mathrm{e}_1 + \mathrm{e}_2 + \ldots + \mathrm{e}_n)) \\ &= x\mathrm{p} + x(1-\mathrm{p}) \\ &= x. \end{aligned}$$

Thus $x \in conv\widetilde{\mathcal{M}_0}(\epsilon,\delta)$ and so $\widetilde{\mathcal{M}_0} = conv\widetilde{\mathcal{M}_0}(\epsilon, \delta)$ for any $\epsilon, \delta > 0$. Hence

$$\widetilde{\mathcal{M}_0} = \bigcap_{\epsilon,\delta>0} conv\widetilde{\mathcal{M}_0}(\epsilon,\delta) \subseteq \bigcap_{\epsilon,\delta>0} conv\widetilde{\mathcal{M}}(\epsilon,\delta).$$

Conversely, we will show that $x \in \widetilde{\mathcal{M}} \backslash \widetilde{\mathcal{M}_0}$ implies that $x \notin conv\widetilde{\mathcal{M}}(\epsilon, \delta)$ for some $\epsilon, \delta > 0$. Suppose that $x \in \widetilde{\mathcal{M}} \backslash \widetilde{\mathcal{M}_0}$. Then by [SW93] Theorem 2.1 there exists $M > 0$ such that $\tau\left(e_{(M,\infty)}(|x|)\right) = \infty$.

Let $0 < \epsilon < M$, let $\alpha_1, \dots, \alpha_n, n \in \mathrm{N}$ be positive real numbers with $\sum_{i=1}^n \alpha_i = 1$ and let $x_1, \dots, x_n \in \widetilde{\mathcal{M}}(\epsilon, \epsilon)$. Then for each $i = 1, \dots, n$ we have that

$$\alpha_i x_i \in \alpha_i \widetilde{\mathcal{M}}(\epsilon, \epsilon) = \widetilde{\mathcal{M}}(\alpha_i \epsilon, \epsilon)$$

by Lemma 1.1.1 (*iii*) and

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i x_i &\in \widetilde{\mathcal{M}}(\alpha_1\epsilon,\epsilon)+\ldots+\widetilde{\mathcal{M}}(\alpha_n\epsilon,\epsilon)\\ &\subseteq \widetilde{\mathcal{M}}(\alpha_1\epsilon+\ldots+\alpha_n\epsilon, n\epsilon)\\ &= \widetilde{\mathcal{M}}(\epsilon, n\epsilon)\end{aligned}$$

by Lemma 1.1.1 (*vi*). This means that $\mu_{n\epsilon}\left(\sum_{i=1}^{n}\alpha_i x_i\right)\leq\epsilon$ as noted above Proposition 1.1.4. When we take

$$\delta=\mu_{n\epsilon}\left(\sum_{i=1}^{n}\alpha_i x_i\right)=\inf\left\{\theta\geq 0:\tau\left(e_{(\theta,\infty)}\left(|\sum_{i=1}^{n}\alpha_i x_i|\right)\right)\leq n\epsilon\right\}$$

then since $\delta\leq\epsilon$ we have that

$$\tau\left(e_{(\epsilon,\infty)}\left(|\sum_{i=1}^{n}\alpha_i x_i|\right)\right)\leq n\epsilon$$

and since $M>\epsilon$

$$\tau\left(e_{(M,\infty)}\left(|\sum_{i=1}^{n}\alpha_i x_i|\right)\right)\leq n\epsilon,$$

but $\tau\left(e_{(M,\infty)}(|x|)\right)=\infty$ and it thus follows that x can not be written as a linear combination of elements in $\widetilde{\mathcal{M}}(\epsilon)$, that is, $x\notin conv\widetilde{\mathcal{M}}(\epsilon)$ which proves the result. □

We have seen in Theorem 1.4.2 that the measure topology is "order continuous" on $\widetilde{\mathcal{M}}_0$, in the sense that if a net decreases to zero in $\widetilde{\mathcal{M}}_0$ then the net converges to zero in the measure topology.

Definition 1.4.4 *We define*

$$\widetilde{\mathcal{M}}_a=\{x\in\widetilde{\mathcal{M}}:|x|\geq x_\alpha\downarrow 0\Rightarrow\mu_t(x_\alpha)\to 0\text{ for all }t>0\}.$$

Thus $\widetilde{\mathcal{M}}_a$ is the largest subspace of $\widetilde{\mathcal{M}}$ on which the measure topology is "order continuous". As is the case with $\widetilde{\mathcal{M}}_0$, $\widetilde{\mathcal{M}}_a$ is solid in $\widetilde{\mathcal{M}}$ in the sense that if $x\in\widetilde{\mathcal{M}}_a, y\in\widetilde{\mathcal{M}}$ with $|y|\leq|x|$, then $y\in\widetilde{\mathcal{M}}_a$. Indeed, if $|y|\geq y_\alpha\downarrow 0$ holds then $|x|\geq|y|\geq y_\alpha\downarrow 0\Rightarrow\mu_t(y_\alpha)\to 0$ for all $t>0$ since $x\in\widetilde{\mathcal{M}}_a$.

Proposition 1.4.5

$$\widetilde{\mathcal{M}}_a=\widetilde{\mathcal{M}}_0.$$

Proof: By Theorem 1.4.2 we have that

$$\widetilde{\mathcal{M}}_0 \subseteq \widetilde{\mathcal{M}}_a.$$

Conversely, suppose that $x \notin \widetilde{\mathcal{M}}_0$. Then there exists $c > 0$ such that $\lim_{t\to\infty} \mu_t(x) = c$ and hence $\tau\left(\mathrm{e}_{(t,\infty)}(|x|)\right) = \infty$ for $0 < t \leq c$. Denote $\mathrm{e}_{(c,\infty)}(|x|)$ by e.
Let $\{\mathrm{p}_\alpha\}_{\alpha\in A}$ be a family of projections maximal with respect to $\mathrm{p}_\alpha \neq 0$ for all $\alpha \in A$, $\tau(\mathrm{p}_\alpha) < \infty$ for all $\alpha \in A$ and $\sum_{\alpha\in F} \mathrm{p}_\alpha \leq \mathrm{e}$ for all finite $F \subset A$. Since $\mathcal{M}$ is semi-finite, $A \neq \emptyset$. Let $\mathcal{F} = \{F \subset A : F \text{ is finite }\}$ be ordered by inclusion. Then

$$\left(\sum_{\alpha\in F} \mathrm{p}_\alpha\right)_{F\in\mathcal{F}}$$

is an increasing net in the partial order, bounded above by e and hence by the Monotone Convergence theorem it increases to its supremum, say $\mathrm{p} \leq \mathrm{e}$. Note that

$$\tau\left(\sum_{\alpha\in F} \mathrm{p}_\alpha\right) \leq \sum_{\alpha\in F} \tau(\mathrm{p}_\alpha) < \infty$$

for every finite $F \subset A$. Suppose the projection $\mathrm{e} - \mathrm{p} > 0$. By faithfulness of the trace $\tau(\mathrm{e} - \mathrm{p}) > 0$ and then by semi-finiteness of $\mathcal{M}$ there exists $0 < \mathrm{q} \in \mathcal{M}^p$ such that $\mathrm{q} \leq \mathrm{e} - \mathrm{p}$ and $0 < \tau(\mathrm{q}) < \infty$. Hence q should be included in the above family contradicting its maximality. Thus $\mathrm{e} = \mathrm{p}$. Hence

$$\left(\sum_{\alpha\in F} \mathrm{p}_\alpha\right)_{F\in\mathcal{F}} \uparrow \mathrm{e} \text{ and } \mathrm{q}_F := \mathrm{e} - \sum_{\alpha\in F} \mathrm{p}_\alpha \downarrow 0$$

where $F \in \mathcal{F}$. Note that

$$\tau(\mathrm{q}_F) = \tau(\mathrm{e}) - \tau\left(\sum_{\alpha\in F} \mathrm{p}_\alpha\right) = \infty$$

for all $F \in \mathcal{F}$ since $\tau\left(\sum_{\alpha\in F} \mathrm{p}_\alpha\right) < \infty$. Now $|x| \geq |x|\mathrm{e}_{(c,\infty)}(|x|) = |x|\mathrm{e} \geq c\mathrm{e} \geq c\mathrm{q}_F \downarrow 0$ but $\mu_t(c\mathrm{q}_F) = c\mu_t(\mathrm{q}_F) = c\chi_{(0,\tau(\mathrm{q}_F))}(t) = c$ for all $t > 0$ since $\tau(\mathrm{q}_F) = \infty$ for all $F \in \mathcal{F}$. Hence we have found a net $(c\mathrm{q}_F)_{F\in\mathcal{F}}$ with $|x| \geq c\mathrm{q}_F \downarrow_{F\in\mathcal{F}} 0$ but $\mu_t(c\mathrm{q}_F) \not\to 0$ for all $t > 0$. Thus $x \notin \widetilde{\mathcal{M}}_a$ and $\widetilde{\mathcal{M}}_0 = \widetilde{\mathcal{M}}_a$. □

1.5 The local convexity of $\widetilde{\mathcal{M}}$

We investigate when the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is locally convex. The answer is dependent on the nature of the projections in the von Neumann algebra, and we shall differentiate between four different cases that we describe as we proceed.

Let us first discuss the case where the lattice of projections, $\mathcal{M}^p$, contains a nonatomic projection and we will consider the case where $\mathcal{M}^p$ is atomic later.

Theorem 1.5.1 *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and suppose that $\mathcal{M}^p$ has a nonatomic projection. Then the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is not locally convex.*

Proof: Suppose $\mathcal{M}^p$ contains a nonatomic projection p. We may assume that $\tau(\mathrm{p}) < \infty$ since $\mathcal{M}$ is semifinite. We show that for all δ with $0 < \delta < \frac{1}{2}\tau(\mathrm{p})$ there is no ϵ such that $0 < \epsilon < \delta$ and $\widetilde{\mathcal{M}}(\epsilon) \subseteq conv\widetilde{\mathcal{M}}(\epsilon) \subseteq \widetilde{\mathcal{M}}(\delta)$, i.e., we show that for $0 < \epsilon < \delta < \frac{1}{2}\tau(\mathrm{p})$ given, $\widetilde{\mathcal{M}}(\epsilon) \subseteq conv\widetilde{\mathcal{M}}(\epsilon) \not\subseteq \widetilde{\mathcal{M}}(\delta)$.

So let $0 < \epsilon < \delta < \frac{1}{2}\tau(\mathrm{p})$ be given. Choose $n \in \mathrm{N}$ such that $\frac{2\delta}{n} < \epsilon$ and find $\mathrm{p}' \leq \mathrm{p}$ such that $\tau(\mathrm{p}') = 2\delta$. Now since p$'$ is nonatomic, we can subdivide it into n disjoint projections $\mathrm{p}'_1, \ldots, \mathrm{p}'_n$ such that $\tau(\mathrm{p}'_k) < \epsilon$ for all $k = 1, \ldots, n$, i.e., $\sum_{k=1}^{n} \mathrm{p}'_k = \mathrm{p}'$.

For $k = 1, \ldots, n$ define

$$x_k = 2n\delta \mathrm{p}'_k.$$

Then each $x_k \in \widetilde{\mathcal{M}}(\epsilon)$ since for each $k = 1, \ldots, n$ $d_\epsilon(x_k) = \tau(\mathrm{p}'_k) < \epsilon$.

Define

$$\begin{aligned} x &= \sum_{k=1}^{n} \frac{1}{n} x_k \\ &= 2\delta \bigvee_{k=1}^{n} \mathrm{p}'_k \\ &= 2\delta \mathrm{p}'. \end{aligned}$$

Then $x \in conv\widetilde{\mathcal{M}}(\epsilon)$ but $d_\delta(x) = \tau(\mathrm{p}') = 2\delta > \delta$ so that $x \notin \widetilde{\mathcal{M}}(\delta)$. Hence

$\widetilde{\mathcal{M}}(\epsilon) \subseteq conv\widetilde{\mathcal{M}}(\epsilon) \not\subseteq \widetilde{\mathcal{M}}(\delta)$ and the topology of convergence in measure is not locally convex. □

Let us now consider $\widetilde{\mathcal{M}}$ with the projections all atomic, that is, $\mathcal{M}^p$ does not contain a nonatomic projection. We first consider the case where the traces of the projections are bounded away from zero. We have seen in Section 1.4 that this means that $\widetilde{\mathcal{M}}$ equals $\mathcal{M}$ and that the measure topology is the same as the operator norm topology on $\mathcal{M}$ which is locally convex. We therefore state the following theorem without proof.

Theorem 1.5.2 *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ, $\mathcal{M}^p$ is atomic and*

$$\inf\{\tau(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^p,\ \tau(\mathrm{p}) \neq 0\} > 0.$$

Then $\widetilde{\mathcal{M}} = \mathcal{M}$ and the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is locally convex and equal to the norm topology on $\mathcal{M}$.

We now turn to the case where $\mathcal{M}^p$ is atomic and the infimum of the trace of the nonzero projections equals zero. We subdivide it into the following two cases:

(a) There exists a constant $K > 0$ such that

$$\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p}\ \text{atomic}}} \tau(\mathrm{p}) < \infty;$$

(b) There is no such K.

Theorem 1.5.3 *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and suppose that $\mathcal{M}^p$ is atomic,*

$$\inf\{\tau(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^p,\ \tau(\mathrm{p}) \neq 0\} = 0$$

and there exists a constant $K > 0$ such that

$$\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p}\ \text{atomic}}} \tau(\mathrm{p}) < \infty.$$

Then the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is locally convex.

Proof: We show that for all $\delta > 0$ there exists $0 < \epsilon < \delta$ such that

$$\widetilde{\mathcal{M}}(\epsilon) \subseteq conv\widetilde{\mathcal{M}}(\epsilon) \subseteq \widetilde{\mathcal{M}}(\delta).$$

Suppose that $\delta > 0$ and let $K > 0$ be such that

$$\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p}\text{ atomic}}} \tau(\mathrm{p}) < \infty.$$

There are at most countably many non-zero terms in this sum since it is finite. We may assume that the terms have been arranged in descending order, say,

$$\tau(\mathrm{p}_n) \downarrow_n 0.$$

Since $\sum_{n=1}^{\infty} \tau(\mathrm{p}_n) < \infty$, we choose $N \in \mathbf{N}$ such that $\sum_{n=N}^{\infty} \tau(\mathrm{p}_n) < \delta$.

If $\delta < K$, since $\tau(\mathrm{p}_n) \downarrow_n 0$, we can find an $\epsilon > 0$ such that $\epsilon < \delta < K$ and $M > N$ such that $\tau(\mathrm{p}_M) > \epsilon \geq \tau(\mathrm{p}_{M+1})$. If $\delta \geq K$ we can find an $\epsilon > 0$ such that $\epsilon < K \leq \delta$ and $M > N$ such that $\tau(\mathrm{p}_M) > \epsilon \geq \tau(\mathrm{p}_{M+1})$.

Now let $x \in conv\widetilde{\mathcal{M}}(\epsilon)$. Then there exists $\alpha_1, \ldots, \alpha_n \geq 0$, $n \in \mathbf{N}$, $\sum_{i=1}^{n} \alpha_i = 1$, $x_1, \ldots, x_n \in \widetilde{\mathcal{M}}(\epsilon)$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i.$$

For $i = 1, \ldots, n$ there exists $\mathrm{q}_i \in \mathcal{M}^p$ such that $\mathrm{q}_i\mathcal{H} \subseteq \mathcal{D}(\mathrm{x}_i)$, $\|x_i \mathrm{q}_i\| \leq \epsilon$ and $\tau(1-\mathrm{q}_i) \leq \epsilon$. Hence for $i = 1, \ldots, n$ we have that $1 - \mathrm{q}_i \neq \mathrm{p}_k$ for $k = 1, \ldots, M$ since these p_k's have trace bigger than ϵ and also $1 - \mathrm{q}_i \neq \mathrm{p}$ for p atomic with $\tau(\mathrm{p}) \geq K$ since $K > \epsilon$. Therefore $1 - \mathrm{q}_i \leq \bigvee_{k=M+1}^{\infty} \mathrm{p}_k$.

Thus

$$\bigvee_{i=1}^{n} (1 - \mathrm{q}_i) \leq \bigvee_{k=M+1}^{\infty} \mathrm{p}_k.$$

Put $\mathrm{q} = \bigwedge_{i=1}^{n} \mathrm{q}_i$. Then

$$\mathrm{q}\mathcal{H} = \bigcap_{i=1}^{n} \mathrm{q}_i\mathcal{H} \subseteq \bigcap_{i=1}^{n} \mathcal{D}(x_i) = \mathcal{D}(x) ,$$

and since $\mathrm{q} \leq \mathrm{q}_i$

$$\|x\mathrm{q}\| = \|\sum \alpha_i x_i \mathrm{q}\|$$

$$
\begin{aligned}
&\leq \sum \alpha_i \|x_i \mathrm{q}\| \\
&= \sum \alpha_i \|x_i \mathrm{q}_i \mathrm{q}\| \\
&\leq \sum \alpha_i \|x_i \mathrm{q}_i\| \|\mathrm{q}\| \\
&\leq \epsilon \sum \alpha_i = \epsilon < \delta
\end{aligned}
$$

and lastly

$$
\begin{aligned}
\tau(1-\mathrm{q}) &= \tau\left(1 - \bigwedge_{i=1}^{n} \mathrm{q}_i\right) \\
&= \tau\left(\bigvee_{i=1}^{n} (1-\mathrm{q}_i)\right) \\
&\leq \tau\left(\bigvee_{k=M+1}^{\infty} \mathrm{p}_k\right) \\
&\leq \sum_{k=M+1}^{\infty} \tau(\mathrm{p}_k) < \delta
\end{aligned}
$$

since $M > N$. Thus $x \in \widetilde{\mathcal{M}}(\delta)$ and we have that

$$
\widetilde{\mathcal{M}}(\epsilon) \subseteq conv\widetilde{\mathcal{M}}(\epsilon) \subseteq \widetilde{\mathcal{M}}(\delta)
$$

which concludes the proof. □

We now consider the case where $\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p}\ \text{atomic}}} \tau(\mathrm{p}) = \infty$ for all $K > 0$. We need

Lemma 1.5.4 *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and suppose that $\mathcal{M}^{\mathrm{p}}$ is atomic and*

$$
\inf\{\tau(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^{\mathrm{p}},\ \tau(\mathrm{p}) \neq 0\} = 0.
$$

Then the following are equivalent.

(a) For all $K > 0$ we have that $\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p}\ \text{atomic}}} \tau(\mathrm{p}) = \infty$.

(b) There exists a sequence of atomic projections (p_n) in $\mathcal{M}^{\mathrm{p}}$ with $\tau(\mathrm{p}_n) \downarrow_n 0$ such that $\sum_{n=1}^{\infty} \tau(\mathrm{p}_n) = \infty$.

Proof: We first show that (b) implies (a). Suppose there exists a sequence of atomic projections (p_n) in $\mathcal{M}^p$ with $\tau(\mathrm{p}_n) \downarrow_n 0$ such that $\sum_{n=1}^{\infty} \tau(\mathrm{p}_n) = \infty$.

We show that for all $K > 0$ we have that $\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p} \text{ atomic}}} \tau(\mathrm{p}) = \infty$.

Let $K > 0$ be given. There exists $N \in \mathbf{N}$ such that $\tau(\mathrm{p}_n) < K$ for all $n \geq N$ since $\tau(\mathrm{p}_n) \downarrow_n 0$. Now $\sum_{n=N}^{\infty} \tau(\mathrm{p}_n) = \infty$ since $\sum_{n=1}^{\infty} \tau(\mathrm{p}_n) = \infty$ and hence

$$\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p} \text{ atomic}}} \tau(\mathrm{p}) \geq \sum_{n=N}^{\infty} \tau(\mathrm{p}_n) = \infty.$$

Conversely, to show (a) implies (b), let us suppose that for all $K > 0$ we have that $\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p} \text{ atomic}}} \tau(\mathrm{p}) = \infty$. Thus for all $n \in \mathbf{N}$ we have that $\sum_{\substack{\tau(\mathrm{p})<\frac{1}{n} \\ \mathrm{p} \text{ atomic}}} \tau(\mathrm{p}) = \infty$. Hence for every $n, k \in \mathbf{N}$ there exists a finite number of atomic projections each with trace less than $\frac{1}{n}$ such that their sum is larger than $\frac{1}{k}$ ([Pie72] Theorem 1.1.3). So for $n, k = 1$ we can find a finite number of terms less than 1, $\tau(\mathrm{p}_1), \ldots, \tau(\mathrm{p}_{n_1})$ such that $1 < \sum_{i=1}^{n_1} \tau(\mathrm{p}_i) < 2$. Looking at the sum over all terms less than $\frac{1}{2}$, i.e. $n = 2$, we can find a finite number of them, $\tau(\mathrm{p}_{n_1+1}), \ldots, \tau(\mathrm{p}_{n_2})$, excluding ones already used, such that $\frac{1}{2} < \sum_{i=n_1}^{n_2} \tau(\mathrm{p}_i) < 1$. Continue inductively to obtain a sequence of terms

$$\tau(\mathrm{p}_1), \ldots, \tau(\mathrm{p}_{n_1}), \tau(\mathrm{p}_{n_1+1}), \ldots, \tau(\mathrm{p}_{n_2}), \tau(\mathrm{p}_{n_2+1}), \ldots, \tau(\mathrm{p}_{n_3}), \ldots.$$

Then $\tau(\mathrm{p}_i) \downarrow_i 0$ by construction and

$$\begin{aligned}
\sum_{i=1}^{\infty} \tau(\mathrm{p}_i) &= \sum_{i=1}^{n_1} \tau(\mathrm{p}_i) + \sum_{i=n_1}^{n_2} \tau(\mathrm{p}_i) + \sum_{i=n_2}^{n_3} \tau(\mathrm{p}_i) + \ldots \\
&> 1 + \frac{1}{2} + \frac{1}{3} + \ldots \\
&= \infty
\end{aligned}$$

and hence $(\mathrm{p}_i)_{i=1}^{\infty}$ is the required sequence. □

We now impose a stronger condition than in the above setting by assuming that the sequence of projections with the given properties consists of mutually orthogonal projections. The following proof is very similar to the proof of Theorem 1.5.1.

Theorem 1.5.5 *Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and suppose that $\mathcal{M}^p$ is atomic,*

$$\inf\{\tau(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^p,\ \tau(\mathrm{p}) \neq 0\} = 0$$

and there exists a sequence of mutually orthogonal atomic projections (p_n) in $\mathcal{M}^p$ with $\tau(\mathrm{p}_n) \downarrow_n 0$ such that

$$\sum_{n=1}^{\infty} \tau(\mathrm{p}_n) = \infty .$$

Then the topology of convergence in measure on $\widetilde{\mathcal{M}}$ is not locally convex.

Proof: Suppose there exists a sequence of orthogonal atomic projections (p_n) in $\mathcal{M}^p$ with $\tau(\mathrm{p}_n) \downarrow_n 0$ such that $\sum_{n=1}^{\infty} \tau(\mathrm{p}_n) = \infty$.

We now prove that the measure topology is not locally convex by showing that for all $\delta > 0$ there is no $0 < \epsilon < \delta$ such that $conv\widetilde{\mathcal{M}}(\epsilon) \subseteq \widetilde{\mathcal{M}}(\delta)$ holds.

So suppose $0 < \epsilon < \delta$. We choose $M \in \mathbf{N}$ such that $\tau(\mathrm{p}_n) \leq \frac{\epsilon}{2}$ for all $n \geq M$ and since $\sum_{n=M}^{\infty} \tau(\mathrm{p}_n) = \infty$ choose $N \in \mathbf{N}$ such that $\sum_{n=M}^{M+N} \tau(\mathrm{p}_n) > \delta$. For $k = M, \ldots, M+N$ define

$$x_k = (N+1)2\delta \mathrm{p}_k .$$

Then each $x_k \in \widetilde{\mathcal{M}}(\epsilon)$ since for each $k = M, \ldots, M+N$

$$d_\epsilon(x_k) = \tau(\mathrm{p}_k) < \epsilon.$$

Hence

$$x = \sum_{k=M}^{M+N} \frac{1}{N+1} x_k = 2\delta \bigvee_{k=M}^{M+N} \mathrm{p}_k \in\ conv\widetilde{\mathcal{M}}(\epsilon)$$

but $x \notin \widetilde{\mathcal{M}}(\delta)$ since

$$d_\delta(x) = \tau(1 - \mathrm{e}_\delta(|x|)) = \tau(\bigvee_{k=M}^{M+N} \mathrm{p}_k) = \sum_{k=M}^{M+N} \tau(\mathrm{p}_k) > \delta$$

by orthogonality of the p_k's. Thus $conv\widetilde{\mathcal{M}}(\epsilon) \not\subseteq \widetilde{\mathcal{M}}(\delta)$ and the measure topology is not locally convex. □

The following example shows that it is not always possible to find a sequence of orthogonal projections that is required in the hypothesis of Theorem 1.5.5.

Example 1.5.6 *Let $M_2(\mathbf{C})$ be the von Neumann algebra of 2×2 matrices over the complex field $\mathbf{C}$. There exists a unique, up to multiplication by a positive constant, faithful finite normal trace, τ, on the factor $M_2(\mathbf{C})$, see [KR86] Proposition 8.5.3. Put*

$$\mathcal{M} = \oplus_{n=1}^{\infty} M_2(\mathbf{C})_n$$

where each $M_2(\mathbf{C})_n$ is a copy of $M_2(\mathbf{C})$ with trace $\tau_n = \frac{1}{2^n}\tau$. The trace on $\mathcal{M}$ is thus $\tau_{\mathcal{M}} = \oplus_{n=1}^{\infty}\tau_n$. Note that $\mathcal{M}^p$ is atomic, that

$$\inf\{\tau_{\mathcal{M}}(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^p,\ \tau_{\mathcal{M}}(\mathrm{p}) \neq 0\} = 0$$

and that for any $K > 0$

$$\sum_{\substack{\tau_{\mathcal{M}}(\mathrm{p})<K \\ \mathrm{p}\ \text{atomic}}} \tau_{\mathcal{M}}(\mathrm{p}) = \infty.$$

Yet, the trace of any family of orthogonal projections is finite since the trace on $\mathcal{M}$ is finite.

In the commutative setting we have a complete characterisation of the conditions under which the topology of convergence in measure is locally convex. The reason for this is that the atomic projections in this setting are mutually orthogonal.

Corollary 1.5.7 *Let (X, Σ, μ) be a localizable measure space and μ a semifinite measure. Then the topology of convergence in measure on $\widetilde{L_\infty}(X, \Sigma, \mu)$ is locally convex if and only if (X, Σ, μ) is atomic and $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) \neq 0\} > 0$, or (X, Σ, μ) is atomic, $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) \neq 0\} = 0$ and there exists $K > 0$ such that* $\sum_{\substack{\mu(A)<K \\ A\ \text{atomic} \in\Sigma}} \mu(A) < \infty.$

1.6 $\widetilde{\mathcal{M}}$ as a sum

Recall that the subspace $\mathcal{G}(\mathcal{M})$ of $\widetilde{\mathcal{M}}$ is the sum $\mathcal{M} + L_1(\mathcal{M})$ with the canonical sum norm (Section 1.3). In this section we show that $\widetilde{\mathcal{M}}$, equipped with the measure topology, may be regarded as the sum of a normed and a pseudonormed space.

We need the following "triangle inequality". The case where x and y are bounded was considered in [AAP82] Theorem 2.2. Using the arguments in the proofs of Lemma 4 and Corollary 5 of [Kos84], the inequality can be extended to unbounded operators in $\widetilde{\mathcal{M}}$:

Lemma 1.6.1 *Let x, $y \in \widetilde{\mathcal{M}}$. Then there exist partial isometries u, $v \in \mathcal{M}$ such that*

$$|x + y| \leq u|x|u^* + v|y|v^*.$$

Definition 1.6.2 *We define*

$$\widetilde{\mathcal{M}_{FS}} = \{x \in \widetilde{\mathcal{M}} : \tau(\mathbf{r}(x)) < \infty\},$$

where $\mathbf{r}(x) = 1 - \mathbf{n}(x)$ is the right support of x and $\mathbf{n}(x)$ is the projection onto the kernel of x.

Note that if x is τ-measurable, x is closed and therefore the kernel of x is closed.

It is easy to see that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}_{FS}} + \mathcal{M}$ as sets. Indeed, $\mathcal{M}$ and $\widetilde{\mathcal{M}_{FS}}$ are contained in $\widetilde{\mathcal{M}}$ and hence also their sum. Conversely, let $x \in \widetilde{\mathcal{M}}$. By the definition of τ-measurability there exists a projection $\mathrm{p} \in \mathcal{M}^p$ with its complement having finite trace and $x\mathrm{p} \in \mathcal{M}$. Now $x(1-\mathrm{p})(1-\mathrm{p}) = x(1-\mathrm{p})$ and so $\mathbf{r}\,(x(1-\mathrm{p})) \leq 1-\mathrm{p}$. Thus $\mathbf{r}\,(x(1-\mathrm{p}))$ has finite trace and $x(1-\mathrm{p})$ is an element of $\widetilde{\mathcal{M}_{FS}}$.

Recall the following definition. Let V be a vector space. A *pseudonorm* (or *F-seminorm*) is a map $\varrho : V \longrightarrow [0, \infty)$ with the properties:

(i) $\varrho(\lambda x) \leq \varrho(x)$ for all $x \in V$ and $|\lambda| \leq 1$,

(ii) $\lim_{n\to\infty} \varrho\left(\frac{1}{n}x\right) = 0$ for all $x \in V$,

(iii) $\varrho(x + y) \leq \varrho(x) + \varrho(y)$ for all $x, y \in V$.

If $\varrho(x) = 0$ implies that $x = 0$, then ϱ is said to be an $F - norm$.

A collection $\{\varrho_\alpha\}_{\alpha\in A}$ of pseudonorms induces a vector topology on V with a subbasis for neighbourhoods at zero being $\{\{x \in V : \varrho_\alpha(x) < \epsilon\} : \epsilon > 0, \alpha \in A\}$.

We define

$$\varrho(x) = \int_0^\infty \frac{\mu_t(x)}{1 + \mu_t(x)}\, dt, \quad x \in \widetilde{\mathcal{M}_{FS}}.$$

We now show that ϱ is a pseudonorm on $\widetilde{\mathcal{M}_{FS}}$.

Clearly $\varrho(x) \geq 0$ for all $x \in \widetilde{\mathcal{M}_{FS}}$. To show that $\varrho(x) < \infty$ for all $x \in \widetilde{\mathcal{M}_{FS}}$, we use operational calculus.

We define the function $f : [0, \infty) \to [0, \infty)$ by $f(t) = \frac{t}{1+t}$. Then f is continuous, concave, monotone increasing and $f(0) = 0$. Let $x \in \widetilde{\mathcal{M}_{FS}}$. Since $\varrho(x) = \varrho(|x|)$ we may assume that $x \geq 0$. Then

$$\begin{aligned} \varrho(x) &= \int_0^\infty \frac{\mu_t(x)}{1+\mu_t(x)}\, dt \\ &= \int_0^\infty f\left(\mu_t(x)\right)\, dt \\ &= \int_0^\infty \mu_t\left(f(x)\right)\, dt \end{aligned}$$

where the last equality follows by [FK86] Theorem 2.5 (iv) since f is continuous and increasing. It also follows by [FK86] Theorem 2.5 (iv) that

$$\int_0^\infty \mu_t\left(f(x)\right)\, dt = \tau\left(f(x)\right).$$

Since $f(t) \leq 1$ for all $t \geq 0$ we have that $f(x) \leq e_{(0,\infty)}(x) = \mathbf{r}(x)$ and thus by the above that

$$\varrho(x) = \tau\left(f(x)\right) \leq \tau(\mathbf{r}(x)) < \infty.$$

We verify the properties of a pseudonorm.

(i) Let $x \in \widetilde{\mathcal{M}_{FS}}$ and let λ be such that $|\lambda| \leq 1$. Then

$$\begin{aligned} \varrho\left(\lambda x\right) &= \int_0^\infty \frac{\mu_t(\lambda x)}{1+\mu_t(\lambda x)}\, dt \\ &= \int_0^\infty \frac{|\lambda|\mu_t(x)}{1+|\lambda|\mu_t(x)}\, dt \\ &\leq \int_0^\infty \frac{\mu_t(x)}{1+\mu_t(x)}\, dt \\ &= \varrho(x) \end{aligned}$$

since the function $f(t) = \frac{t}{1+t}$ is increasing. So $\varrho(\lambda x) \leq \varrho(x)$ for all $x \in \widetilde{\mathcal{M}_{FS}}$ and $|\lambda| \leq 1$.

(ii) Let $x \in \widetilde{\mathcal{M}_{FS}}$. We have that

$$\begin{aligned}\lim_{n\to\infty} \varrho\left(\frac{1}{n}x\right) &= \lim_{n\to\infty} \int_0^\infty \frac{\mu_t\left(\frac{1}{n}x\right)}{1+\mu_t\left(\frac{1}{n}x\right)}\, dt \\ &= \lim_{n\to\infty} \int_0^\infty \frac{\mu_t(x)}{n+\mu_t(x)}\, dt \\ &= 0\end{aligned}$$

by the Dominated Convergence Theorem. Hence $\lim_{n\to\infty} \varrho\left(\frac{1}{n}x\right) = 0$ for all $x \in \widetilde{\mathcal{M}_{FS}}$.

(iii) Let $0 \le x,\ y \in \widetilde{\mathcal{M}_{FS}}$.

As before, define $f : [0,\infty) \to [0,\infty)$ by $f(t) = \frac{t}{1+t}$. Then

$$\begin{aligned}\varrho(x+y) &= \int_0^\infty \frac{\mu_t(x+y)}{1+\mu_t(x+y)}\, dt \\ &= \int_0^\infty f\left(\mu_t(x+y)\right)\, dt \\ &= \int_0^\infty \mu_t\left(f(x+y)\right)\, dt \\ &= \tau\left(f(x+y)\right)\end{aligned}$$

as before. Since f is continuous, concave and $f(0) = 0$ we have by [BK90] Proposition 15 that

$$\tau\left(f(x+y)\right) \le \tau\left(f(x)\right) + \tau\left(f(y)\right).$$

Then it follows as before that

$$\begin{aligned}\tau\left(f(x)\right) + \tau\left(f(y)\right) &= \int_0^\infty \mu_t\left(f(x)\right)\, dt + \int_0^\infty \mu_t\left(f(y)\right)\, dt \\ &= \int_0^\infty f\left(\mu_t(x)\right)\, dt + \int_0^\infty f\left(\mu_t(y)\right)\, dt \\ &= \varrho(x) + \varrho(y).\end{aligned}$$

Hence

$$\varrho(x+y) \le \varrho(x) + \varrho(y)$$

for $0 < x, y \in \widetilde{\mathcal{M}_{FS}}$.

Now let $x,\ y \in \widetilde{\mathcal{M}_{FS}}$. By Lemma 1.6.1 there exist partial isometries $u,\ v \in \mathcal{M}$ such that

$$|x+y| \le u|x|u^* + v|y|v^*.$$

Hence

$$\mu_t(|x+y|) \leq \mu_t\left(u|x|u^* + v|y|v^*\right)$$

and since $f(t) = \frac{t}{1+t}$ is increasing we have

$$\varrho(x+y) = \varrho(|x+y|) \leq \varrho\left(u|x|u^* + v|y|v^*\right).$$

By the above we have

$$\varrho(x+y) \leq \varrho\left(u|x|u^*\right) + \varrho\left(v|y|v^*\right).$$

Now $\mu_t\left(u|x|u^*\right) \leq \mu_t\left(|x|\right) = \mu_t(x)$ and $\mu_t\left(v|y|v^*\right) \leq \mu_t(y)$

by [FK86] Theorem 2.5 and by using the monotonicity of f again we finally have that

$$\varrho(x+y) \leq \varrho(x) + \varrho(y)$$

for all $x, y \in \widetilde{\mathcal{M}_{FS}}$.

Define

$$\vartheta(x) = \inf\{\varrho(y) + \|z\| : x = y + z, y \in \widetilde{\mathcal{M}_{FS}}, z \in \mathcal{M}\}, \quad x \in \widetilde{\mathcal{M}}.$$

Then ϑ is also a pseudonorm on $\widetilde{\mathcal{M}}$ and our objective is to show that ϑ generates the topology of convergence in measure.

We first show that ϑ is indeed a pseudonorm.

Clearly $0 \leq \vartheta(x) < \infty$ for all $x \in \widetilde{\mathcal{M}}$.

(i) Let $x = y + z, y \in \widetilde{\mathcal{M}_{FS}}, z \in \mathcal{M}, |\lambda| \leq 1$. Then $\lambda x = \lambda y + \lambda z$. But

$$\varrho\left(\lambda y\right) + \|\lambda z\| \leq \varrho(y) + \|z\|.$$

Hence

$$\begin{aligned}
\vartheta(\lambda x) &\leq \inf\{\varrho\left(\lambda y\right) + \|\lambda z\| : x = y + z, y \in \widetilde{\mathcal{M}_{FS}}, z \in \mathcal{M}\} \\
&\leq \inf\{\varrho(y) + \|z\| : x = y + z, y \in \widetilde{\mathcal{M}_{FS}}, z \in \mathcal{M}\} \\
&= \vartheta(x).
\end{aligned}$$

(ii) Let $x \in \widetilde{\mathcal{M}}$ and let $\epsilon > 0$. Choose $y \in \widetilde{\mathcal{M}_{FS}}$ and $z \in \mathcal{M}$ such that $x = y + z$. So $\frac{1}{n}x = \frac{1}{n}y + \frac{1}{n}z$ for all $n \in \mathbb{N}$. Hence $\vartheta\left(\frac{1}{n}x\right) \leq \varrho\left(\frac{1}{n}y\right) + \|\frac{1}{n}z\|$. Choose $N \in \mathbb{N}$ such that $\varrho\left(\frac{1}{n}y\right) \leq \frac{\epsilon}{2}$ and $\|\frac{1}{n}z\| \leq \frac{\epsilon}{2}$ for all $n \geq N$. Hence $\vartheta\left(\frac{1}{n}x\right) \leq \epsilon$ for all $n \geq N$ and thus $\lim_{n\to\infty} \vartheta\left(\frac{1}{n}x\right) = 0$.

(iii) Let $x_1, x_2, \in \widetilde{\mathcal{M}}$ and $\epsilon > 0$. Then there exist $y_1, y_2 \in \widetilde{\mathcal{M}_{FS}}$ and $z_1, z_2 \in \mathcal{M}$ such that $x_1 = y_1 + z_1, x_2 = y_2 + z_2$ with $\varrho(y_1) + \|z_1\| \leq \vartheta(x_1) + \frac{\epsilon}{2}$ and $\varrho(y_2) + \|z_2\| \leq \vartheta(x_2) + \frac{\epsilon}{2}$. Now

$$\begin{aligned}
\vartheta(x_1 + x_2) &\leq \varrho(y_1 + y_2) + \|z_1 + z_2\| \\
&\leq \varrho(y_1) + \|z_1\| + \varrho(y_2) + \|z_2\| \\
&\leq \vartheta(x_1) + \vartheta(x_2) + \epsilon
\end{aligned}$$

which proves the triangle inequality.

We now prove

Theorem 1.6.3

$$\left(\widetilde{\mathcal{M}}, \tau_{cm}\right) = \left(\widetilde{\mathcal{M}_{FS}} + \mathcal{M}, \vartheta(.)\right)$$

where

$$\vartheta(x) = \inf\{\varrho(y) + \|z\| : x = y + z, y \in \widetilde{\mathcal{M}_{FS}}, z \in \mathcal{M}\}, \quad x \in \widetilde{\mathcal{M}}$$

and

$$\varrho(y) = \int_0^\infty \frac{\mu_t(y)}{1 + \mu_t(y)}\, dt, \quad y \in \widetilde{\mathcal{M}_{FS}}.$$

Proof: Let $\epsilon, \delta > 0$ be given. Without loss of generality we may assume that $0 < \delta < 1$.

We need to show that there exists a $\gamma > 0$ such that $\vartheta(x) < \gamma$ implies that $x \in \widetilde{\mathcal{M}}(\epsilon, \delta)$.

Choose $\eta > 0$ such that $\eta < \delta$ and define $\gamma = \min\{\frac{1}{4}\epsilon, \frac{1}{2}\frac{\epsilon}{2+\epsilon}(\delta - \eta)\}$.

Let $x \in \widetilde{\mathcal{M}_{FS}} + \mathcal{M}$ such that $\vartheta(x) \leq \gamma$. We want to show that $x \in \widetilde{\mathcal{M}}(\epsilon, \delta)$.

Using the definition of $\vartheta(.)$ there exist $y \in \widetilde{\mathcal{M}_{FS}}$ and $z \in \mathcal{M}$ such that $x = y + z$ with $\varrho(y) + \|z\| \leq \vartheta(x) + \gamma \leq 2\gamma$. Now $\|z\| \leq 2\gamma$ and hence $z \in \widetilde{\mathcal{M}}(2\gamma, \eta)$.

If $\gamma = \frac{1}{4}\epsilon$ then $\widetilde{\mathcal{M}}(2\gamma, \eta) = \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \eta)$ and if $\gamma = \frac{1}{2}\frac{\epsilon}{2+\epsilon}(\delta - \eta)$ then $\widetilde{\mathcal{M}}(2\gamma, \eta) \subseteq \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \eta)$ ($\frac{1}{2+\epsilon} < \frac{1}{2}$ implies $\frac{1}{2}\frac{\epsilon}{2+\epsilon}(\delta - \eta) < \frac{1}{2}\frac{\epsilon}{2}$ since $0 < \eta < \delta < 1$ and so $\delta - \eta < 1$).

Hence $z \in \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \eta)$.

Also $\varrho(y) \leq 2\gamma$.

We show that $y \in \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \delta - \eta)$. So suppose for a contradiction that $y \notin \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \delta - \eta)$. This means that $d_{\frac{1}{2}\epsilon}(|y|) = \tau\left(e_{(\frac{1}{2}\epsilon,\infty)}(|y|)\right) > \delta - \eta$. It follows by the definition of the generalised singular function of y that $m\left\{t \in (0,\infty) : \mu_t(y) > \frac{1}{2}\epsilon\right\} > \delta - \eta$, where m denotes Lebesgue measure on the positive real line. (We write $\left[\mu(y) > \frac{1}{2}\epsilon\right]$ for $\left\{t \in (0,\infty) : \mu_t(y) > \frac{1}{2}\epsilon\right\}$.) This implies that

$$\begin{aligned}
\varrho(y) &= \int_0^\infty \frac{\mu_t(y)}{1+\mu_t(y)}\, dt \\
&\geq \int_{[\mu(y)>\frac{1}{2}\epsilon]} \frac{\mu_t(y)}{1+\mu_t(y)}\, dt \\
&> \frac{\frac{1}{2}\epsilon}{1+\frac{1}{2}\epsilon}(\delta - \eta) \\
&= \frac{\epsilon}{2+\epsilon}(\delta - \eta).
\end{aligned}$$

Hence $\varrho(y) > \frac{\epsilon}{2+\epsilon}(\delta - \eta) \geq \frac{1}{2}\epsilon = 2\gamma$, contradicting the fact that $\varrho(y) \leq 2\gamma$.

Hence $y \in \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \delta - \eta)$ and

$$x = y + z \in \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \delta - \eta) + \widetilde{\mathcal{M}}(\frac{1}{2}\epsilon, \eta) \subseteq \widetilde{\mathcal{M}}(\epsilon, \delta).$$

Conversely, let $\gamma > 0$ be given. We need to show that there exists $\epsilon > 0$ such that $x \in \widetilde{\mathcal{M}}(\epsilon)$ implies that $\vartheta(x) \leq \gamma$.

Choose $\epsilon = \frac{1}{2}\gamma$.

If $x \in \widetilde{\mathcal{M}}(\epsilon)$ then $d_\epsilon(|x|) = \tau\left(e_{(\epsilon,\infty)}(|x|)\right) \leq \epsilon$. We can write $x = xe_{(\epsilon,\infty)}(|x|) + xe_{[0,\epsilon]}(|x|)$ with $xe_{(\epsilon,\infty)}(|x|) \in \mathcal{M}_{FS}$ since $\tau\left(\mathbf{r}\left(xe_{(\epsilon,\infty)}(|x|)\right)\right) \leq \tau\left(e_{(\epsilon,\infty)}(|x|)\right) \leq \epsilon$ and $\|xe_{[0,\epsilon]}(|x|)\| \leq \epsilon$.

Define $y = xe_{(\epsilon,\infty)}(|x|)$ and $z = xe_{[0,\epsilon]}(|x|)$. So

$$\varrho(y) = \int_0^\infty \frac{\mu_t(y)}{1+\mu_t(y)}\, dt$$

$$
\begin{aligned}
&= \int_0^{d_\epsilon(|x|)} \frac{\mu_t(y)}{1+\mu_t(y)}\, dt \\
&\leq \int_0^{d_\epsilon(|x|)} 1\, dt \\
&= d_\epsilon(|x|) \leq \epsilon.
\end{aligned}
$$

Hence $\vartheta(x) \leq \varrho(y) + \|z\| \leq 2\epsilon = \gamma$ and the two topologies coincide. □

1.7 $\widetilde{\mathcal{M}}$ as a direct sum

Let $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ and suppose that $\mathcal{M}^p$ is atomic, $\inf\{\tau(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^p,\ \tau(\mathrm{p}) \neq 0\} = 0$ and there exists a constant $K > 0$ such that $\sum_{\substack{\tau(\mathrm{p})<K \\ \mathrm{p}\ \text{atomic}}} \tau(\mathrm{p}) < \infty$. We have seen in Section 1.5 that the measure topology in this case is locally convex. We show that in this case there is a natural way of writing $\widetilde{\mathcal{M}}$ as a direct sum topology.

Put $\mathcal{P} = \{\mathrm{p} \in \mathcal{M}^p : \tau(\mathrm{p}) < K\}$ and put $\mathrm{q} = \bigvee_{\mathrm{p}\in\mathcal{P}} \mathrm{p}$. We show that q is central. For this it will suffice to show that $u^*\mathrm{q}u = \mathrm{q}$ for every unitary u in $\mathcal{M}$. Suppose $\mathrm{p} \in \mathcal{P}$. If u is unitary in $\mathcal{M}$, then $u^*\mathrm{p}u \in \mathcal{P}$ since $(u^*\mathrm{p}u)^2 = u^*\mathrm{p}u$ and the trace is unitarily invariant, that is, $\tau(u^*\mathrm{p}u) = \tau(\mathrm{p}) < K$. Similarly $u^*\mathrm{p}u \in \mathcal{P}$ for a unitary operator in $\mathcal{M}$ implies that $\mathrm{p} \in \mathcal{P}$. Thus for a unitary operator u in $\mathcal{M}$ we have that

$$
\mathrm{q} = \bigvee_{\mathrm{p}\in\mathcal{P}} \mathrm{p} = \bigvee_{\mathrm{p}\in\mathcal{P}} u^*\mathrm{p}u.
$$

Let u be a unitary operator in $\mathcal{M}$. Now $\mathrm{q} \geq \mathrm{p}$ for all $\mathrm{p} \in \mathcal{P}$ and hence we have that $u^*\mathrm{q}u \geq u^*\mathrm{p}u$ for all $\mathrm{p} \in \mathcal{P}$. Now suppose $\mathrm{e} \in \mathcal{M}^p$ also satisfies $\mathrm{e} \geq u^*\mathrm{p}u$ for all $\mathrm{p} \in \mathcal{P}$. Then $u\mathrm{e}u^* \geq uu^*\mathrm{p}uu^* = \mathrm{p}$ for all $\mathrm{p} \in \mathcal{P}$. Hence $u\mathrm{e}u^* \geq \mathrm{q}$ and thus $\mathrm{e} \geq u^*\mathrm{q}u$. By definition of suprema $u^*\mathrm{q}u = \bigvee_{\mathrm{p}\in\mathcal{P}} u^*\mathrm{p}u$ and thus by the above paragraph that

$$
\mathrm{q} = \bigvee_{\mathrm{p}\in\mathcal{P}} \mathrm{p} = \bigvee_{\mathrm{p}\in\mathcal{P}} u^*\mathrm{p}u = u^*\mathrm{q}u.
$$

This holds for all unitary $u \in \mathcal{M}$ and hence q is central.

We now have a direct sum decomposition

$$
\widetilde{\mathcal{M}} = \mathrm{q}\widetilde{\mathcal{M}}\mathrm{q} \oplus (1-\mathrm{q})\widetilde{\mathcal{M}}(1-\mathrm{q})
$$

by noticing that $q\widetilde{\mathcal{M}}(1-q)$ and $(1-q)\widetilde{\mathcal{M}}q$ are trivial since q is central.

We note a few results about reduced algebras. As for von Neumann algebras, we similarly define reduced algebras for $\widetilde{\mathcal{M}}$. If $x \in \widetilde{\mathcal{M}}$ and $q \in \mathcal{M}^p$ we denote by x_q the restriction of the closure of qx to $q\mathcal{H}$ and put $\widetilde{\mathcal{M}}_q = \{x_q : x \in \widetilde{\mathcal{M}}\}$. On the other hand, let $\widetilde{\mathcal{M}_q}$ denote the completion of the reduced semifinite von Neumann algebra $\mathcal{M}_q$ in the measure topology generated by the reduced trace τ_q. (The reduced trace is defined in the preliminaries, Section 0.2.) It was shown in [SW93] Theorem 2.3 that $\widetilde{\mathcal{M}}_q = \widetilde{\mathcal{M}_q}$ and in particular that $(\widetilde{\mathcal{M}}_0)_q = (\widetilde{\mathcal{M}_q})_0$.

Thus

$$\widetilde{\mathcal{M}} = q\widetilde{\mathcal{M}}q \oplus (1-q)\widetilde{\mathcal{M}}(1-q) = \widetilde{\mathcal{M}_q} \oplus \widetilde{\mathcal{M}_{1-q}}.$$

We now characterise the measure topology on $\widetilde{\mathcal{M}}$ as the product of the measure topology restricted to the reduced algebra $\widetilde{\mathcal{M}_q}$ and the norm topology on $\widetilde{\mathcal{M}_{1-q}}$. We have that

$$\tau_{cm}|_{\widetilde{\mathcal{M}}} = \tau_{cm}|_{\widetilde{\mathcal{M}_q}\oplus\widetilde{\mathcal{M}_{1-q}}} \supseteq \tau_{cm}|_{\widetilde{\mathcal{M}_q}} \times \tau_{cm}|_{\widetilde{\mathcal{M}_{1-q}}}.$$

The projections in $\left(\widetilde{\mathcal{M}_{1-q}}\right)^p$ are less than or equal to $1-q$ and we know that the $\inf\{\tau(p) : p \in \left(\widetilde{\mathcal{M}_{1-q}}\right)^p,\ \tau(p) \neq 0\} \geq K > 0$ and thus all operators in $\widetilde{\mathcal{M}_{1-q}}$ are bounded and the measure topology equals the norm topology on $\mathcal{M}_{1-q}$. Hence

$$\tau_{cm}|_{\widetilde{\mathcal{M}}} \supseteq \tau_{cm}|_{\widetilde{\mathcal{M}_q}} \times \|\cdot\|_\infty|_{\widetilde{\mathcal{M}_{1-q}}}.$$

Conversely, suppose $y_\alpha \to_\alpha 0$ in the measure topology in $\widetilde{\mathcal{M}_q}$ and $z_\alpha \to_\alpha 0$ in the norm topology in $\widetilde{\mathcal{M}_{1-q}}$. Extend both sequences to the whole space by putting them equal to zero on the appropriate parts, that is, for each α redefine y_α as $y_\alpha q + 0(1-q)$ and similarly for z_α. Put $x_\alpha = y_\alpha + z_\alpha$ and we need to show that $\mu_t(x_\alpha) \to_\alpha 0$ for all $t > 0$. So let $t > 0$ and $\epsilon > 0$. Choose α_0 such that $\|z_\alpha\|_\infty \leq \frac{\epsilon}{2}$ for all $\alpha \geq \alpha_0$. For each $\alpha \geq \alpha_0$ we have that $\mu_{\frac{t}{2}}(z_\alpha) \leq \frac{\epsilon}{2}$ since $\lim_{t\to 0+} \mu_t(z_\alpha) = \|z_\alpha\|$ and $\mu_.(z_\alpha)$ is decreasing in t. Since $y_\alpha = y_\alpha q + 0(1-q)$ there exists α_1 such that $\mu_{\frac{t}{2}}(y_\alpha) \leq \frac{\epsilon}{2}$ for all $\alpha \geq \alpha_1$. We have that

$$\begin{aligned}\mu_t(x_\alpha) &\leq \mu_{\frac{t}{2}}(y_\alpha) + \mu_{\frac{t}{2}}(z_\alpha)\\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

for all $\alpha \geq \max\{\alpha_0, \alpha_1\}$ which gives the result.

In the commutative setting, we can also characterise the measure topology on $\widetilde{\mathcal{M}}_q$.

Let us therefore identify $\mathcal{M}$ with $L_\infty(X, \Sigma, m)$ in the usual way where (X, Σ, m) is a localizable purely atomic measure space with $\inf\{m(A) : A \in \Sigma,\ m(A) \neq 0\} = 0$ such that there exists a $K > 0$ with $\sum_{\substack{m(A)<K \\ A \text{ atomic } \in\Sigma}} m(A) < \infty$.

As before there are countably many atoms A_n which we can arrange in descending order of measure, say $m(A_n) \downarrow_n 0$.

Put $E = \bigcup_{n=1}^\infty A_n$. By hypothesis $m(E) < \infty$.

Define $\Sigma_E = \{A \in \Sigma : m(A) < K\}$ and $m_E = m_{|E}$ and similarly for $\Sigma_{X\setminus E},\ m_{X\setminus E}$.

Then

$$\widetilde{L_\infty}(E, \Sigma_E, m_E) = L_0(E, \Sigma_E, m_E)$$

since $m(E) < \infty$ and

$$\widetilde{L_\infty}(X \setminus E, \Sigma_{X\setminus E}, m_{X\setminus E}) = L_\infty(X \setminus E, \Sigma_{X\setminus E}, m_{X\setminus E})$$

since

$$\inf\{m(A) : A \in \Sigma_{X\setminus E}, m(A) \neq 0\} \geq K > 0.$$

Theorem 1.7.1 *Let (X, Σ, m) be a localizable, purely atomic measure space such that there exists a $K > 0$ with $\sum_{\substack{m(A)<K \\ A \text{ atomic}\in\Sigma}} m(A) < \infty$. Then*

$$\left(\widetilde{L_\infty}(X, \Sigma, m), \tau_{cm}\right) = \left(L_0(E, \Sigma_E, m_E) \oplus L_\infty(X \setminus E, \Sigma_{X\setminus E}, m_{X\setminus E}),\ \nu_p \times \|.\|_\infty\right)$$

where ν_p denotes the pointwise topology on $L_0(E, \Sigma_E, m_E)$.

Proof: We write $L_0(E)$ for $L_0(E, \Sigma_E, m_E)$ and $L_\infty(X \setminus E)$ for $L_\infty(X \setminus E, \Sigma_{X\setminus E}, m_{X\setminus E})$. We know that

$$\tau_{cm}|_{\widetilde{L_\infty}(X)} = \tau_{cm}|_{L_0(E)\oplus L_\infty(X\setminus E)} = \tau_{cm}|_{L_0(E)} \times \tau_{cm}|_{L_\infty(X\setminus E)}.$$

We already have that $\tau_{cm}|_{L_\infty(X\setminus E)} = \|\cdot\|_\infty$ on $L_\infty(X \setminus E)$.

We show that $\tau_{cm}|_{L_0(E)}$ equals the pointwise topology on $L_0(E)$.

Let $\epsilon > 0$ be given.

A typical neighbourhood of zero for the measure topology on $L_0(E)$ is

$$\mathcal{N}(\epsilon) = \{f \in L_0(E) : m\{A_i \in \Sigma_E : |f(A_i)| > \epsilon\} \leq \epsilon\}.$$

and a neighbourhood of zero for the pointwise topology on $L_0(E)$ is

$$\bigcap_{i=1}^{N} \{f \in L_0(E) : |f(A_i)| \leq \epsilon\}, \; N \in \mathrm{N}, \; A_i \in \Sigma_E \text{ for } i = 1, \ldots, N.$$

Let (A_i) be the sequence of atoms such that $m(A_i) \downarrow_i 0$ and $\sum_{i=1}^{\infty} m(A_i) < \infty$. There exists an $N \in \mathrm{N}$ such that $\sum_{i=N}^{\infty} m(A_i) \leq \epsilon$. Thus

$$\bigcap_{i=1}^{N-1} \{f \in L_0(E) : |f(A_i)| \leq \epsilon\} \subseteq \mathcal{N}(\epsilon).$$

Conversely, let $\delta < \min\{\epsilon, m(A_1), \ldots, m(A_{N-1})\}$. Then

$$\mathcal{N}(\delta) \subseteq \bigcap_{i=1}^{N-1} \{f \in L_0(E) : |f(A_i)| \leq \epsilon\}$$

since $\mathcal{N}(\delta)$ does not include any $f \in L_0(E)$ such that $|f(A_i)| > \epsilon$ for $i = 1, \ldots, N-1$ since these A_i's have measure bigger than δ. □

Chapter 2

Singular functionals on Banach function spaces

The usual definition for singular linear functionals on a Banach function space uses the lattice structure of the space of all continuous linear functionals on the space. Singularity is first defined for positive functionals, and then extended to arbitrary functionals by requiring all the positive components in the standard decomposition of a continuous linear functional to be singular. In this chapter several equivalent characterisations of singular functionals on Banach function spaces are investigated, in particular characterisations that are independent of the lattice structure of the dual space. This is done in order to find a suitable definition for singular functionals in the non-commutative setting, where the lattice structure is in general not available.

We present known results in the literature augmented with some new results in Section 2.2. We commence by giving an overview of the known duality theory for Banach function spaces, with the emphasis on singular functionals. In Section 2.2 we give equivalent characterisations for singular functionals on Banach function spaces and in Section 2.3 we investigate the role which the subspace of functions with order continuous norm plays in characterising singular functionals.

2.1 The dual of a Banach function space

Suppose (X, Σ, μ) is a localizable measure space, ρ a function norm on $L_0(X, \Sigma, \mu)$ and $L_\rho(X, \Sigma, \mu)$ a Banach function space. When there is no danger of confusion we will write L_ρ for $L_\rho(X, \Sigma, \mu)$.

Recall that the set of ρ-continuous linear functionals on L_ρ is denoted by L_ρ^*.

Definition 2.1.1 *A functional $\varphi \in L_\rho^*$ is* normal *(or* order continuous *) if for every net*

$$f_\alpha \downarrow_\alpha 0 \text{ in } L_\rho \text{ we have that } \varphi(f_\alpha) \to_\alpha 0.$$

The set of normal functionals will be denoted by L_ρ^{*n}. A functional $\varphi \in L_\rho^*$ is σ-normal *(or* σ-order continuous*) if for every sequence*

$$f_n \downarrow_n 0 \text{ in } L_\rho \text{ we have that } \varphi(f_n) \to_n 0.$$

The set of σ-normal functionals will be denoted by $L_\rho^{*\sigma n}$.

Recall that a Riesz space is *order separable* if every nonempty subset that has a supremum contains an at most countable subset with the same supremum. For spaces that are order separable we have that the set of normal functionals on them equals the set of σ-normal functionals, [Zaa83] Theorem 84.4. If (X, Σ, μ) is σ-finite, then the space of equivalence classes of real valued μ-measurable functions on X over the real field with the μ almost everywhere ordering, $L_{0,r}$, is order separable by [LZ71] Chapter 4, Example 23.3 (iv). ($L_{0,r}$ is super Dedekind complete which means it is Dedekind complete and order separable.) Let $L_{\rho,r}$ denote the real vector subspace of real valued functions in L_ρ. We know that $L_{\rho,r}$ is an ideal in $L_{0,r}$ and is therefore itself order separable ([LZ71] Chapter 4, Theorem 25.2). Hence if (X, Σ, μ) is σ-finite then $L_{\rho,r}^{*n} = L_{\rho,r}^{*\sigma n}$.

For $\psi, \varphi \in L_\rho^*$ we write $\psi \leq \varphi$ meaning that $\psi(f) \leq \varphi(f)$ for every $f \in L_\rho$.

Definition 2.1.2 *A positive functional $\varphi \in L_\rho^*$ is called* singular *if*

$$0 \leq \psi \leq \varphi \text{ with } \psi \in L_\rho^{*n} \Rightarrow \psi \equiv 0,$$

that is, if the only positive normal functional φ majorizes, is the zero functional.

We may define "σ-singular functionals" by replacing normal functionals with σ-normal functionals in the above definition. Note that for order separable Riesz spaces it does not make a difference whether we use σ-normal or normal functionals in the definition for singular functionals. We will only consider singular functionals as defined above.

We decompose $\varphi \in L_\rho^*$ in the following manner. (For full details we refer the reader to [Zaa67] Section 50.) Given $f \in L_{\rho,r}$, we write

$$\varphi(f) = \varphi_r(f) + i\varphi_{im}(f),$$

where $\varphi_r(f)$ and $\varphi_{im}(f)$ are the real and imaginary part, respectively, of $\varphi(f)$. Then φ_r and φ_{im} are real linear bounded functionals on $L_{\rho,r}$. Define

$$\varphi_r^+(f) = \sup\{\varphi_r(g) : 0 \leq g \leq f, g \in L_{\rho,r}\}, \quad 0 \leq f \in L_{\rho,r}$$

and

$$\varphi_r^-(f) = \varphi_r(f) - \varphi_r^+(f), \quad 0 \leq f \in L_{\rho,r}.$$

Decompose φ_{im} similarly. The functionals $\varphi_r^+, \varphi_r^-, \varphi_{im}^+, \varphi_{im}^-$ are well-defined, additive and positive on the set of positive functions in $L_{\rho,r}$. The functional φ_r^+ defined on $L_{\rho,r}$, is extended to L_ρ by defining

$$\varphi_r^+(f) = \varphi_r^+(g) + i\varphi_r^+(h)$$

for $f = g + ih \in L_\rho$ with $g, h \in L_{\rho,r}$. Similarly for $\varphi_r^-, \varphi_{im}^+$ and φ_{im}^-. The extended functionals are positive, bounded linear functionals on L_ρ and

$$\varphi = \varphi_r^+ - \varphi_r^- + i\left(\varphi_{im}^+ - \varphi_{im}^-\right).$$

This is known as the standard decomposition of φ into positive parts.

Definition 2.1.3 *A functional $\varphi \in L_\rho^*$ is called* singular *if each of its positive parts in the standard decomposition is singular. The set of singular functionals is denoted by L_ρ^{*s}.*

Every functional on L_ρ can be written uniquely as the sum of a a normal and a singular functional, [Zaa67] Theorem 2. The normal part has an explicit representation. If φ_n denotes the normal part of $\varphi \in L_\rho$ then for $0 \leq f \in L_\rho$ we have that

$$\varphi_n(f) = \inf\{\lim_\alpha \varphi(f_\alpha) : 0 \leq f_\alpha \uparrow_\alpha f, (f_\alpha) \subseteq L_\rho\},$$

by [AB85] Theorem 4.6. As before, whenever the measure space is σ-finite, we may replace the nets by sequences in the above representation.

2.2 Characterisations for singular functionals

As before, let (X, Σ, μ) denote a localizable measure space, ρ a function norm on $L_0(X, \Sigma, \mu)$ and $L_\rho(X, \Sigma, \mu)$ a Banach function space of complex valued functions. In this section we show that a functional on $L_\rho(X, \Sigma, \mu)$ is singular if and only if it vanishes on an order dense ideal in $L_\rho(X, \Sigma, \mu)$. This result is known for Banach function spaces of real valued functions over the real field and when the measure space is σ-finite, [Zaa83] Section 90. In our proof we need to consider decompositions of functionals corresponding to decompositions of the underlying measure space.

Let $X = Y \oplus Z$, i.e. $X = Y \cup Z$ and $Y \cap Z = \emptyset$. ($Y, Z \neq \emptyset$.) Denote by $\Sigma_{|Y}$ the set of all $A \cap Y$ such that $A \in \Sigma$, and put $\mu_{|Y}(A) = \mu(A \cap Y)$ for $A \in \Sigma$. The set $\Sigma_{|Z}$ and measure $\mu_{|Z}$ are defined similarly. For a function $f \in L_\rho(Y, \Sigma_{|Y}, \mu_{|Y})$ we define

$$f_Y(x) = \begin{cases} f(x) & \text{if } x \in Y \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

and we define f_Z similarly. Then we have that

$$L_\rho(X, \Sigma, \mu) = L_\rho(Y, \Sigma_{|Y}, \mu_{|Y}) \oplus L_\rho(Z, \Sigma_{|Z}, \mu_{|Z}).$$

We use $L_\rho(Y)$ and $L_\rho(Z)$ for short.

Let $0 \leq \varphi \in L_\rho^*$. Define φ_Y on $L_\rho(Y)$ by $\varphi_Y(f) = \varphi(f_Y)$ for all $f \in L_\rho(Y)$.

Lemma 2.2.1 *Let $0 \leq \varphi \in L_\rho^*$. If φ is singular on $L_\rho(X)$ then φ_Y is singular on $L_\rho(Y)$.*

Proof: Let $0 \leq \psi \leq \varphi_Y$ where $\psi \in L_\rho^{*n}(Y)$. Extend ψ to $L_\rho^*(X)$ by defining

$$\psi_X(f) = \psi(f\chi_Y)$$

for all $f \in L_\rho(X)$, where χ_Y denotes the characteristic function of the set Y. Then ψ_X is linear and continuous on $L_\rho(X)$ since ψ is linear and continuous on $L_\rho(Y)$ and $\rho(f_n - f) \to_n 0$ implies that $\rho(f_n\chi_Y - f\chi_Y) \to_n 0$.

We show ψ_X is normal on $L_\rho(X)$. Let $f_\alpha \downarrow_\alpha 0$ in $L_\rho(X)$. Then $f_\alpha\chi_Y \downarrow_\alpha 0$ in $L_\rho(Y)$. By normality of ψ on $L_\rho(Y)$ we have that $\psi_X(f_\alpha) = \psi(f_\alpha\chi_Y) \to_\alpha 0$.

Also

$$0 \le \psi_X(f) = \psi(f\chi_Y) \le \varphi_Y(f\chi_Y) = \varphi\left((f\chi_Y)_Y\right) \le \varphi(f)$$

for all $f \in L_\rho(X)$ since $(f\chi_Y)_Y = f\chi_Y \le f$ for all $f \in L_\rho(X)$ and $\varphi \ge 0$. Since φ is singular on $L_\rho(X)$, $\psi_X(f) = \psi(f\chi_Y) = 0$ for all $f \in L_\rho(X)$. Thus $\psi \equiv 0$ on $L_\rho(Y)$ and so $\varphi_Y \in L_\rho^{*s}(Y)$. □

Recall that a Riesz subspace V of an Archimedean Riesz space L is said to be *order dense in* L if for every $0 < f \in L$ there exists $g \in V$ such that $0 < g \le f$. An equivalent condition is that for every $0 < f \in L$ there exists a net $(f_\alpha) \in V$ such that $0 \le f_\alpha \uparrow_\alpha f$ ([AB78] Theorem 1.14).

The element f in a Riesz space L has the *Egoroff property* if given any double sequence $(u_{nk})_{n,k=1}^{\infty}$ in L with

$$0 \le u_{nk} \uparrow_k |f| \text{ for } n = 1, 2, \ldots$$

there exists a sequence $0 \le v_m \uparrow_m |f|$ such that for every m and n there exists a $k(m,n) \in \mathbf{N}$ with

$$v_m \le u_{nk(m,n)}.$$

The space L has the *Egoroff property* if every element of L has the Egoroff property ([LZ71] Chapter 10, Definition 67.2). It follows from the definition of the Egoroff property that any ideal of a space with the Egoroff property will also posess the property.

Proposition 2.2.2 *Let (X, Σ, μ) be a σ-finite measure space and functions in L_ρ be real valued. Let $0 \le \varphi \in L_\rho^*$. If φ is singular, then there exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$.*

Proof: We first note that L_ρ has the Egoroff property. Indeed, the set of real valued functions in $L_0(X, \Sigma, \mu)$, $L_{0,r}$, has the Egoroff property by [LZ71] Chapter 10, Theorem 75.2, and since L_ρ is an ideal in $L_{0,r}$, it has the Egoroff property.

Let $0 \le \varphi \in L_\rho^*$. Suppose that φ is singular. Define

$$\mathcal{I} = \{f \in L_\rho : \varphi(|f|) = 0\}.$$

(The set $\mathcal{I}$ is called the *absolute kernel* of the functional φ.) It follows easily by positivity of φ that $\mathcal{I}$ is an ideal. We need to show that $\mathcal{I}$ is order dense in L_ρ. Let $0 < f \in L_\rho$.

The normal component of φ is zero, that is,

$$\varphi_n(f) = \inf\{\lim \varphi(f_k) : 0 \leq f_k \uparrow_k f\} = 0.$$

By [Zaa83] Chapter 12, Exercise 90.13 the infimum is attained since L_ρ has the Egoroff property. Thus there exists a sequence $0 \leq f_k \uparrow_k f$ such that $0 \leq \varphi(f_k) \uparrow_k \varphi(f) = 0$ and so $f_k \in \mathcal{I}$ for all k. Note that there exists k_0 such that $f_k \neq 0$ for all $k \geq k_0$ since $f > 0$. Thus $\mathcal{I}$ is order dense in L_ρ. □

Theorem 2.2.3 *Let (X, Σ, μ) be a localizable measure space and the functions in L_ρ be real valued. Let $\varphi \in L_\rho^*$. Then the following statements are equivalent.*

(i) φ is singular.

(ii) There exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$.

Proof: Let $\{A_\alpha\}$ be a maximal disjoint system of sets, each with finite measure. This is possible by using a routine Zorn's argument and the semifiniteness of μ. For each α the restricted space $L_\rho(A_\alpha)$ is a function space on a finite measure space. Embed $L_\rho(A_\alpha)$ as a subspace of $L_\rho(X)$ by putting

$$f_{A_\alpha}(x) = \begin{cases} f(x) & \text{if } x \in A_\alpha \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

for $f \in L_\rho(A_\alpha)$. Then $f_{A_\alpha} \in L_\rho(X)$. We do this for each function in $L_\rho(A_\alpha)$ and rename the extended space $L_\rho(A_\alpha)$. Then $L_\rho(A_\alpha)$ is an ideal of the Dedekind complete space $L_\rho(X)$ and thus also Dedekind complete by [LZ71] Chapter 4, Theorem 25.2.

Suppose φ is singular. Then $|\varphi|$ is singular since all the components of φ are singular by definition. For each α we have that $A_\alpha \oplus X \setminus A_\alpha = X$ and thus by Lemma 2.2.1 that $|\varphi|_{|A_\alpha}$ is singular. By Proposition 2.2.2 we have that for each α there exists an order dense ideal J_α in $L_\rho(A_\alpha)$ such that $|\varphi|_{A_\alpha}(J_\alpha) = \{0\}$. As before, embed J_α in $L_\rho(X)$. Then $|\varphi|(J_\alpha) = \{0\}$ for each α. Put $J = \bigcup_{\alpha \in A} J_\alpha$ and consider the ideal, $\mathcal{I}$, generated by J in $L_\rho(X)$,

$$\mathcal{I} = \{f \in L_\rho : \text{there exist } n \in \mathbf{N},\ f_1, \ldots, f_n \in J,\ \lambda_1, \ldots, \lambda_n \in \mathbf{R}^+ \text{ with } |f| \leq \sum_{i=1}^{n} \lambda_i |f_i|\}.$$

We show that $\varphi(\mathcal{I}) = 0$. Let $f \in \mathcal{I}$. Then there exist $f_1, \ldots, f_n \in J$, $\lambda_1, \ldots, \lambda_n \in \mathbf{R}^+$, $n \in \mathbf{N}$, with $|f| \leq \sum_{i=1}^n \lambda_i |f_i|$. Hence

$$|\varphi|(|f|) \leq \sum_{i=1}^{n} \lambda_i |\varphi|(|f_i|) = \sum_{i=1}^{n} \lambda_i.0 = 0$$

since $f_i \in \bigcup_{\alpha \in A} J_\alpha$ for $i = 1, \ldots, n$. Hence since $|\varphi(f)| \leq |\varphi|(|f|) = 0$ we have that $\varphi(f) = 0$.

We need to show that $\mathcal{I}$ is order dense in L_ρ. Let $0 < f \in L_\rho$. Then

$$0 < f = \sup_\alpha \sup_n f \wedge n\chi_{A_\alpha}.$$

Thus we have that $f \wedge n\chi_{A_\alpha} \neq 0$ for at least one n_0 and one α_0. Also $f \wedge n_0\chi_{A_{\alpha_0}} \leq f$ and hence $f \wedge n_0\chi_{A_{\alpha_0}} \in L_\rho(A_{\alpha 0})$. Since $J_{\alpha 0}$ is order dense in $L_\rho(A_{\alpha 0})$ there exists $0 < g \in J_{\alpha_0}$ such that $0 < g \leq f \wedge n_0\chi_{A_{\alpha_0}} \leq f$. Since $g \in J \subseteq \mathcal{I}$ we have that $\mathcal{I}$ is order dense.

Conversely, suppose there exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$. Let us first suppose that $0 \leq \varphi$. We will show that the normal component of φ is zero. Let $0 \leq f \in L_\rho$. The normal part of φ is

$$\varphi_n(f) = \inf\{\lim \varphi(f_\alpha) : 0 \leq f_\alpha \uparrow_\alpha f\}.$$

We know that $\varphi_n(f) \geq 0$ since $\varphi(f_\alpha) \geq 0$ for all $f_\alpha \geq 0$. Let $\epsilon > 0$ be given. Since $\mathcal{I}$ is order dense, there exists a net (f_α) in $\mathcal{I}$ such that $f_\alpha \uparrow_\alpha f$, and we have $\varphi(f_\alpha) = 0$ for all α. Hence $\lim_\alpha \varphi(f_\alpha) = 0$ and so $\varphi_n(f) = 0$. This holds for all $0 \leq f \in L_\rho$ and so φ is singular.

Now suppose that $\varphi \in L_\rho^*$ and let $0 \leq f \in \mathcal{I}$. We decompose φ into its positive parts,

$$\varphi = \varphi^+ - \varphi^-.$$

If $0 \leq g \leq f$ then $g \in \mathcal{I}$, since $\mathcal{I}$ is an order ideal, and hence $\varphi(g) = 0$. Thus

$$\varphi^+(f) = \sup\{\varphi(g) : 0 \leq g \leq f\} = 0.$$

Hence $\varphi^+(\mathcal{I}) = \{0\}$ and by the above φ^+ is singular (since $\varphi^+ \geq 0$). Now if $f \in \mathcal{I}$, then

$$\varphi^-(f) = \varphi^+(f) - \varphi(f) = 0 - 0 = 0,$$

and hence $\varphi^{-}(\mathcal{I}) = \{0\}$. Again by the above we have that φ^{-} is singular ($\varphi^{-} \geq 0$). Hence φ is singular. □

Suppose $L_\rho(X, \Sigma, \mu)$ is a complex Banach function space. We may think of L_ρ as the complexification of the real Banach function space $L_{\rho,r}$, that is, $L_\rho = L_{\rho,r} + iL_{\rho,r}$. The set of real elements in the ideal $\mathcal{I}$ will be denoted by $\mathcal{I}_r$. We have the following ([Zaa83] Chapter 12, Theorem 91.6)

Theorem 2.2.4 *Let $L_\rho(X, \Sigma, \mu)$ be a complex Banach function space. If $\mathcal{I}$ is an ideal in $L_{\rho,r} + iL_{\rho,r}$, then $\mathcal{I}_r$ is an ideal in $L_{\rho,r}$ and $\mathcal{I} = \mathcal{I}_r + i\mathcal{I}_r$. Conversely, if $\mathcal{J}$ is an ideal $L_{\rho,r}$, then $\mathcal{J} + i\mathcal{J}$ is an ideal in $L_{\rho,r} + iL_{\rho,r}$ and $(\mathcal{J} + i\mathcal{J})_r = \mathcal{J}$.*

Corollary 2.2.5 *Let $L_\rho(X, \Sigma, \mu)$ be a complex Banach function space. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals in L_ρ. Then*

$$(\mathcal{I} \cap \mathcal{J})_r = \mathcal{I}_r \cap \mathcal{J}_r.$$

Definition 2.2.6 *Let $L_\rho(X, \Sigma, \mu)$ be a complex Banach function space. We say an ideal $\mathcal{I}$ is* order dense *in L_ρ if $\mathcal{I}_r$ is order dense in $L_{\rho,r}$.*

Zaanen defined an ideal $\mathcal{I}$ to be order dense in L_ρ if the band generated by $\mathcal{I}$ is the whole space $L_{\rho,r} + iL_{\rho,r}$. This is equivalent to the above definition ([Zaa83] Chapter 12, Section 91).

Theorem 2.2.7 *Let (X, Σ, μ) be a localizable measure space and let the functions in L_ρ be complex valued. Let $\varphi \in L_\rho^*$. Then the following statements are equivalent.*

(i) φ is singular.

(ii) There exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$.

Proof: Let $\varphi \in L_\rho^*$ be singular and decompose it into its real parts

$$\varphi = \varphi_r + i\varphi_{im}.$$

Each of the positive parts in the standard decomposition of φ is singular by definition. It follows that $|\varphi_r|$ and $|\varphi_{im}|$ are singular on $L_{\rho,r}$. By Theorem 2.2.3 there exist an order dense ideal J_1 in $L_{\rho,r}$ such that $|\varphi_r|(J_1) = \{0\}$ and an order dense ideal J_2 in $L_{\rho,r}$ such that $|\varphi_{im}|(J_2) = \{0\}$. Put $\mathcal{J}_1 = J_1 + iJ_1$, $\mathcal{J}_2 = J_2 + iJ_2$ and $\mathcal{I} = \mathcal{J}_1 \bigcap \mathcal{J}_2$. By Theorem 2.2.4 $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals in L_ρ and thus $\mathcal{I}$ is an ideal in L_ρ.

We first show that $\varphi(\mathcal{I}) = \{0\}$. Let $f \in \mathcal{I}$. Thus $|f| \in \mathcal{I}_r$ by Theorem 2.2.4 and thus $|\varphi_r|(|f|) = |\varphi_{im}|(|f|) = 0$ by Corollary 2.2.5. Since $|\varphi_r(f)| \leq |\varphi_r|(|f|) = 0$ and $|\varphi_{im}(f)| \leq |\varphi_{im}|(|f|) = 0$ we have that

$$\varphi(f) = \varphi_r(f) + i\varphi_{im}(f) = 0.$$

We show that $\mathcal{I}$ is order dense in L_ρ by showing that $\mathcal{I}_r$ is order dense in $L_{\rho,r}$. Therefore let $0 < f \in L_{\rho,r}$. Since J_1 is order dense in $L_{\rho,r}$ there exists $g \in J_1$ such that $0 < g \leq f$. Since J_2 is order dense in $L_{\rho,r}$ there exists $h \in J_2$ such that $0 < h \leq g \leq f$. Since J_1 is an ideal, $h \in J_1$ and hence $h \in \mathcal{I}_r$ by Corollary 2.2.5.

Conversely, suppose there exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$. Then $\mathcal{I}_r$ is an order dense ideal in $L_{\rho,r}$ by definition and $\mathcal{I} = \mathcal{I}_r + i\mathcal{I}_r$ by Theorem 2.2.4. Hence $\{0\} = \varphi(\mathcal{I}) = \varphi(\mathcal{I}_r) + i\varphi(\mathcal{I}_r)$ and $(1+i)\varphi(\mathcal{I}_r) \subseteq \varphi(\mathcal{I}_r) + i\varphi(\mathcal{I}_r)$ which implies that $\varphi(\mathcal{I}_r) = \{0\}$. Decompose $\varphi = \varphi_r + i\varphi_{im}$. Let $f \in \mathcal{I}_r$. Then $0 = \varphi(f) = \varphi_r(f) + i\varphi_{im}(f)$. Hence $\varphi_r(\mathcal{I}_r) = \{0\}$ and $\varphi_{im}(\mathcal{I}_r) = \{0\}$. Since $\mathcal{I}_r$ is order dense in $L_{\rho,r}$ it follows by Theorem 2.2.3 that φ_r and φ_{im} are singular and therefore φ is singular. □

We give another equivalent characterisation for singular functionals. We denote by *supp*(f) the support of $f \in L_\rho$, that is, the set $\{x \in X : f(x) \neq 0\}$.

Theorem 2.2.8 *Let (X, Σ, μ) be a localizable measure space and the functions in L_ρ be complex valued. Let $\varphi \in L_\rho^*$. Then the following statements are equivalent.*

(i) There exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$.

(ii) For all $A \in \Sigma$ with $\mu(A) > 0$ and $\chi_A \in L_\rho$ there exists $B \subseteq A$ with $\mu(B) > 0$ such that $\varphi(\chi_B) = 0$.

Proof: Suppose there exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = 0$. Let $A \in \Sigma$ with $\mu(A) > 0$ and $\chi_A \in L_\rho$.

Suppose φ is positive. Since $\mathcal{I}$ is order dense there exists $0 < f \in \mathcal{I}$ such that $0 < f \leq \chi_A$. There exists a sequence of step functions with $supp(f_n) \subseteq supp(f) \subseteq A$ such that $f_n \uparrow_n f$. Choose one of them, say $f_{n0} = \sum_{i=1}^{k} \lambda_i \chi_{A_i}$ with say $\mu(A_k) > 0$. Then

$$A_k \subseteq supp(f_{n_0}) \subseteq supp(f) \subseteq A$$

and $0 < \lambda_k \chi_{A_k} \leq f_{n_0} \leq f$. Then

$$0 \leq \varphi(\lambda_k \chi_{A_k}) \leq \varphi(f) = 0$$

and hence by linearity $\varphi(\chi_{A_k}) = 0$.

Suppose $\varphi \in L_\rho^*$. Decompose it into its positive parts

$$\varphi = \varphi_r^+ - \varphi_r^- + i\left(\varphi_{im}^+ - \varphi_{im}^-\right).$$

By the above there exists A_1 with $\mu(A_1) > 0$ such that $A_1 \subseteq A$ and $\varphi_r^+(\chi_{A_1}) = 0$. Given A_1, by the above there exists A_2 with $\mu(A_2) > 0$ such that $A_2 \subseteq A_1 \subseteq A$ and $\varphi_r^-(\chi_{A_2}) = 0$. Note that also $\varphi_r^+(\chi_{A_2}) = 0$. Continuing, we can find A_3 and A_4, each with positive measure such that each positive functional vanishes on χ_{A_4}. Hence $\varphi(\chi_{A_4}) = 0$.

Conversely, suppose for all $A \in \Sigma$ with $\mu(A) > 0$ and $\chi_A \in L_\rho$ there exists $B \subseteq A$ with $\mu(B) > 0$ such that $\varphi(\chi_B) = 0$. Put

$$\mathcal{I} = \{f \in L_{\rho,r} : |\varphi|(|f|) = 0\}.$$

Then $\mathcal{I}$ is an ideal in $L_{\rho,r}$ and so $\mathcal{I} + i\mathcal{I}$ is an ideal in L_ρ. We need to show that $\mathcal{I} + i\mathcal{I}$ is order dense in L_ρ and that $\varphi(\mathcal{I} + i\mathcal{I}) = \{0\}$. Let $0 < f \in L_{\rho,r}$. As in the first part of the proof we can find $A_k \subseteq supp(f)$ with $\mu(A_k) > 0$ and a $\lambda_k > 0$ such that $0 < \lambda_k \chi_{A_k} \leq f$. By hypothesis there exists $B \subseteq A_k$ with positive measure such that $\varphi(\chi_B) = 0$. Hence

$$\varphi(\lambda_k \chi_B) = 0$$

and $0 < \lambda_k \chi_B \leq \lambda_k \chi_{A_k} \leq f$. So $\lambda_k \chi_B \in \mathcal{I}$ and hence $\mathcal{I}$ is order dense in $L_{\rho,r}$ and by definition $\mathcal{I} + i\mathcal{I}$ is order dense in L_ρ .

To show that $\varphi(\mathcal{I}) = \{0\}$, we decompose φ into its real parts. As in the first part of the proof of Theorem 2.2.3, it follows that each of the real parts of φ vanishes on $\mathcal{I}$. Thus $\varphi(\mathcal{I}) = \{0\}$ and hence $\varphi(\mathcal{I} + i\mathcal{I}) = \{0\}$. □

Corollary 2.2.9 *Let (X, Σ, μ) be a localizable measure space and let the functions in L_ρ be complex valued. Let $\varphi \in L_\rho^*$. Then the following statements are equivalent.*

(i) φ is singular.

(ii) There exists an order dense ideal $\mathcal{I}$ in L_ρ such that $\varphi(\mathcal{I}) = \{0\}$.

(iii) For all $A \in \Sigma$ with $\mu(A) > 0$ and $\chi_A \in L_\rho$ there exists $B \subseteq A$ with $\mu(B) > 0$ such that $\varphi(\chi_B) = 0$.

2.3 Singular functionals and elements with order continuous norm

We have seen that $\varphi \in L_\rho^{*s}$ if and only if it vanishes on an order dense ideal in L_ρ. If L_ρ is real then this ideal can be taken to be the absolute kernel of the functional. Hence for different functionals the order dense ideal on which it vanishes may be different. We show in this section that for a large class of spaces there is an order dense ideal on which all the singular functionals vanish.

In this section we assume that the measure space (X, Σ, μ) is σ-finite. Recall that the norm ρ on $L_0(X, \Sigma, \mu)$ is called *saturated* if for every set $A \in \Sigma$ there exists $B \subseteq A$ with positive measure such that $\rho(\chi_B) < \infty$. Without loss of generality we may assume that the norm ρ is saturated when we consider the Banach function space $L_\rho(X, \Sigma, \mu)$ (since the measure space is σ-finite, see [Zaa67] Chapter 15, Section 67). Recall that a function f in $L_\rho(X, \Sigma, \mu)$ has *order continuous norm* whenever $\rho(f_n) \downarrow_n 0$ for every sequence (f_n) in L_ρ with $|f| \geq f_n \downarrow_n 0$ μ-a.e. An equivalent condition is the following: $\rho(f\chi_{A_n}) \downarrow_n 0$ for every sequence (A_n) in Σ that decreases to a set of measure zero. Let $L_{\rho,a}$ denote the set of functions in L_ρ with order continuous norm. We say L_ρ has order continuous norm if $L_\rho = L_{\rho,a}$.

If $V \subseteq L_\rho$ then we define the *annihilator of* V as the set

$$V^\perp = \{\varphi \in L_\rho^* : \varphi(f) = 0 \text{ for all } f \in V\}.$$

We also need to introduce the *carrier* of an ideal in L_ρ (as was done by [Zaa67] Chapter 15, Section 72). Recall that $D \subseteq X$ is *disjoint* to the ideal $\mathcal{I}$ if $f(x) = 0$ μ–a.e. on D

for all $f \in \mathcal{I}$. For $F \in \Sigma$ with finite measure, put

$$\delta = \sup\{\mu(D) : D \subseteq F, D \text{ is disjoint to } \mathcal{I}\}.$$

By the definition of the supremum we can find an increasing sequence $(D_n) \subseteq F$ of subsets, each disjoint to $\mathcal{I}$ such that $\mu(D_n) \uparrow_n \delta$. Put

$$D = \bigcup_{n=1}^{\infty} D_n.$$

Then $\mu(D) = \delta$, D is a maximal subset of F disjoint to $\mathcal{I}$ and D is uniquely determined (except on a set with measure zero) by this property. Now by σ-finiteness of (X, Σ, μ), there exists $X_n \uparrow_n X$ with $\mu(X_n) < \infty$ for all n. Let $F_1 = X_1$, $F_{n+1} = X_{n+1} - X_n$ and let D_n be the maximal subset of F_n disjoint to $\mathcal{I}$ for all n. Then

$$D = \bigcup_{n=1}^{\infty} D_n$$

is the maximal subset of X disjoint to $\mathcal{I}$. The set

$$carrier(\mathcal{I}) = X \setminus D$$

is called the *carrier* of $\mathcal{I}$. By [Zaa83] Chapter 14, Theorem 102.8 we have that

$$L_\rho^{*s} \subseteq L_{\rho,a}^{\perp}.$$

Also by [Zaa67] Chapter 15, Theorem 72.6,

$$L_\rho^{*s} = L_{\rho,a}^{\perp}$$

if and only if the carrier of $L_{\rho,a} = X$. We show that $L_{\rho,a}$ is order dense in L_ρ if and only if the carrier of $L_{\rho,a} = X$.

Proposition 2.3.1 *Let (X, Σ, μ) be a σ-finite measure space. Then the following statements are equivalent.*

(a) $L_{\rho,a}$ is order dense in L_ρ.

(b) The carrier of $L_{\rho,a}$ equals X.

Proof: Let $L_{\rho,a}$ be order dense in L_ρ and suppose for a contradiction that the carrier of $L_{\rho,a}$ does not equal X. Let $A \in X \backslash carrier(L_{\rho,a})$ with $\mu(A) > 0$. Since ρ is saturated, there exists $B \subseteq A$ with positive measure such that $\rho(\chi_B) < \infty$. There exists $f \in L_{\rho,a}$ such that

$$0 < f \leq \chi_B$$

since $L_{\rho,a}$ is order dense in L_ρ. As in the proof of Theorem 2.2.8 we can find $A_k \subseteq supp(f) \subseteq B$ and $\lambda_k > 0$ with

$$0 < \lambda_k \chi_{A_k} \leq f.$$

Hence $\chi_{A_k} \in L_{\rho,a}$ and so $A_k \subseteq carrier(L_{\rho,a})$ which is a contradiction.

Conversely, suppose that the carrier of $L_{\rho,a}$ is X. Let $0 \leq f \in L_\rho$. As before we can find $A_k \subseteq supp(f)$ and $\lambda_k > 0$ with

$$0 < \lambda_k \chi_{A_k} \leq f.$$

Let $Y_n \uparrow_n X$ be such that $\chi_{Y_n} \in L_{\rho,a}$ for all n. This is possible by the definition of the carrier and the exhaustion theorem, [Zaa67] Chapter 15, Theorem 67.3. Hence

$$A_k \cap Y_n \uparrow_n A_k \cap X = A_k$$

and hence $A_k \cap Y_n \neq \emptyset$ from some n onwards, say $A_k \cap Y_{n_0} \neq \emptyset$. Now

$$0 < \lambda_k \chi_{A_k \cap Y_{n_0}} \leq \lambda_k \chi_{A_k} \leq f$$

and $\lambda_k \chi_{A_k \cap Y_{n_0}} \in L_{\rho,a}$. Thus $L_{\rho,a}$ is order dense in L_ρ. □

There are alternative conditions on L_ρ that will ensure that $L_\rho^{*s} = L_{\rho,a}^\perp$. The closure in L_ρ of the set of bounded functions whose supports have finite measure will be denoted by $L_{\rho,b}$. We have that

$$L_{\rho,b}^\perp \subseteq L_\rho^{*s}$$

by [dJ73] Theorem 2.5 . Together with

$$L_\rho^{*s} \subseteq L_{\rho,a}^\perp$$

we see that

$$L_\rho^{*s} = L_{\rho,a}^\perp = L_{\rho,b}^\perp$$

whenever $L_{\rho,a} = L_{\rho,b}$.

Let Σ_F denote the set of measurable sets in Σ with finite measure. Recall that *simple functions* are functions of the form $\sum_{n=1}^{k} \alpha_n \chi_{A_n}$ where $\alpha_n' s$ are scalars and $A_n \in \Sigma_F$ for $n = 1, 2, \ldots, k$ for some $k \in \mathbf{N}$. Note that $\varphi \in L_{\rho,b}^{\perp}$ is equivalent to $\varphi(\chi_A) = 0$ for all $A \in \Sigma_F$ since $L_{\rho,b}$ is the closure in L_ρ of the simple functions by [BS88] Chapter 1, Proposition 3.10. Thus if $L_{\rho,a} = L_{\rho,b}$ then $\varphi \in L_\rho^*$ is singular if and only if

$$\varphi(\chi_A) = 0 \text{ for all } A \in \Sigma_F.$$

We now have two equivalent characterisations for singular functionals in terms of characteristic functions whenever $L_{\rho,a} = L_{\rho,b}$.

Proposition 2.3.2 *Let (X, Σ, μ) be a σ-finite measure space. Suppose $L_{\rho,a} = L_{\rho,b}$ and $\varphi \in L_\rho^*$. Then the following statements are equivalent.*

(a) $\varphi \in L_\rho^*$ is singular.

(b) $\varphi(\chi_A) = 0$ for all $A \in \Sigma_F$.

(c) For every $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \subseteq A$ with $\mu(B) > 0$ such that $\varphi(\chi_B) = 0$.

The implication $(b) \Rightarrow (c)$ holds for semifinite measure spaces without the assumption that $L_{\rho,a} = L_{\rho,b}$. The example below shows that $(c) \Rightarrow (b)$ does not generally hold.

Example 2.3.3 *Consider the space of all bounded real valued sequences ℓ_∞ with the supremum norm over the measure space $(\mathbf{N}, \mathcal{P}(\mathbf{N}), \nu)$ where $\nu(n) = \frac{1}{2^n}$. Note that $\nu(\mathbf{N}) < \infty$. Define the continuous linear functional on the subspace, c, of all convergent sequences in ℓ_∞, by*

$$\varphi(\mathbf{x}) = \lim_{n \to \infty} x_n$$

where $\mathbf{x} = (x_n) \in c \subseteq \ell_\infty$. Extend φ to a continuous linear functional on ℓ_∞ by the Hahn-Banach Theorem. We denote the extention again by φ and the subspace of sequences that converge to zero by c_0. Note that $\ell_{\infty,a} = c_0 \neq \ell_\infty = \ell_{\infty,b}$ and that φ is an element of $c_0^{\perp}$, that is, $\varphi(\mathbf{x}) = 0$ whenever $\mathbf{x} \in c_0$. Now take $A = \mathbf{N}$ in $\Sigma = \mathcal{P}(\mathbf{N})$. (Note that $\chi_{\mathbf{N}}$ is the sequence $(1, 1, 1, \ldots)$). Then condition (c) of Proposition 2.3.2 is satisfied but not condition (b) since $0 < \nu(\mathbf{N}) = \sum_{n=1}^{\infty} \nu(n) = 1 < \infty$ yet $\varphi((1, 1, 1, \ldots)) = 1 \neq 0$.

A Banach function space $L_\rho(X, \Sigma, \mu)$ is called *rearrangement invariant* if $f \in L_0$ and $g \in L_\rho$ with $\mu(f) \leq \mu(g)$ implies that $f \in L_\rho$ and $\rho(f) \leq \rho(g)$. Let L_ρ be a rearrangement invariant Banach function space. Recall that the *fundamental function*, Φ_ρ of L_ρ, is defined as

$$\Phi_\rho(t) = \sup\{\rho(\chi_A) : \mu(A) \leq t;\ A \in \Sigma_F\}$$

for all $t \geq 0$. The function Φ_ρ is non-decreasing, right-continuous on $(0, \infty)$ and $\Phi_\rho(0) = 0$. For rearrangement invariant Banach function spaces we use the fundamental function to give us yet another condition to ensure that $L_{\rho,a} = L_{\rho,b}$, or equivalently, that the carrier of $L_{\rho,a} = X$.

Suppose (X, Σ, μ) is a continuous measure space and that $L_\rho(X, \Sigma, \mu)$ is rearrangement invariant. The fundamental function Φ_ρ being right-continuous at zero (we write $\Phi_\rho(0^+) = 0$) is equivalent to $L_{\rho,a} = L_{\rho,b}$, by [BS88] Chapter 2, Theorem 5.5(b). Hence

$$L_\rho^{*s} = L_{\rho,a}^\perp = L_{\rho,b}^\perp$$

whenever $\Phi_\rho(0^+) = 0$. We will extend this result to all measure spaces in the more general setting of induced Banach operator spaces in Section 3.4. Note that if L_ρ has order continuous norm, then $L_\rho = L_{\rho,a} = L_{\rho,b}$ and thus

$$L_\rho^{*s} = L_\rho^\perp = \{0\}.$$

We mention a few examples of spaces whose fundamental functions are right-continuous at zero.

Examples 2.3.4 *(i) Let (X, Σ, μ) be a σ-finite measure space. Then for the spaces $L_p(X, \Sigma, \mu)$, $(1 \leq p < \infty)$, with the usual L_p-norm we have that $\Phi_p(t) = t^{\frac{1}{p}}$ for $0 \leq t < \mu(X)$. Hence $L_{p,a}(X, \Sigma, \mu) = L_{p,b}(X, \Sigma, \mu) = L_p(X, \Sigma, \mu)$ and the only singular functional on L_p is the zero functional.*

(ii) The space $(L_1 + L_\infty)(0, \infty)$ with the canonical sum norm has fundamental function $\Phi_{L_1+L_\infty}(t) = \min\{t, 1\}$ for $t \in [0, \infty)$ and is therefore right-continuous at zero. Hence $(L_1 + L_\infty)_a = (L_1 + L_\infty)_b$ and $(L_1 + L_\infty)^{*s} = (L_1 + L_\infty)_a^\perp = (L_1 + L_\infty)_b^\perp$. Note that $(L_1 + L_\infty)_a \neq (L_1 + L_\infty)$ in this case.

(iii) Let $\psi : [0,\infty) \to [0,\infty)$ be a concave function such that $\psi(0) = 0$ ($\psi \not\equiv 0$). We define the Lorentz space

$$\Lambda_\psi = \{f \in L_0(0,\infty) : \int_0^\infty \mu_t(f)\, d\psi(t) < \infty\}$$

where the integral is an improper Stieltjes integral. Define the norm for $f \in \Lambda_\psi$ by

$$\|f\|_{\Lambda_\psi} = \int_0^\infty \mu_t(f)\, d\psi(t).$$

Then $(\Lambda_\psi, \|f\|_{\Lambda_\psi})$ is a rearrangement invariant Banach function space. The fundamental function of Λ_ψ equals ψ. These facts may be found in [KPS82] Chapter 2, Section 5.1. Now if $\psi(0^+) = 0$ then $\Lambda_{\psi,a} = \Lambda_{\psi,b}$ and

$$\Lambda_\psi^{*s} = \Lambda_{\psi,a}^{\perp} = \Lambda_{\psi,b}^{\perp}.$$

If in addition $\psi(\infty) = \infty$ then $\Lambda_\psi = \Lambda_{\psi,b}$ by [KPS82] Corollary 5.3, and thus in this case the set of singular functionals is trivial.

(iv) Let $\psi : [0,\infty) \to [0,\infty)$ be a concave function such that $\psi(0) = 0$ ($\psi \not\equiv 0$) as in the previous example. For $f \in L_0(0,\infty)$ define the function norm $\|.\|_{M_\psi}$ by

$$\|f\|_{M_\psi} = \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu_s(f)\, ds$$

and the Marcinkiewicz space

$$M_\psi = \{f \in L_0 : \|f\|_{M_\psi} < \infty\}.$$

$(M_\psi, \|.\|_{M_\psi})$ is a rearrangement invariant Banach function space and the fundamental function for M_ψ is $\frac{t}{\psi(t)}$, [KPS82] Chapter 2, Section 5.2. If the fundamental function is right-continuous at zero then $M_{\psi,a} = M_{\psi,b}$ and

$$(M_\psi)^{*s} = (M_{\psi,a})^{\perp} = (M_{\psi,b})^{\perp}.$$

(v) All sequence spaces over the measure space $(\mathbf{N}, \mathcal{P}(\mathbf{N}), c)$ where c denotes counting measure, have fundamental functions that are right-continuous at zero.

Suppose L_ρ is a rearrangement invariant Banach function space on a continuous measure space (X, Σ, μ). If $\Phi_\rho(0^+) > 0$ then by [Con76] Proposition 3.2.5 we have that $L_\rho \hookrightarrow L_\infty$.

By the discussion above, we also have that $L_{\rho,a} \neq L_{\rho,b}$ if and only if $\Phi_\rho(0^+) > 0$. In fact, we have that $L_{\rho,a} = \{0\}$ whenever $\Phi_\rho(0^+) > 0$. Indeed, if $\Phi_\rho(0^+) > 0$ then $carrier(L_{\rho,a})$ is a proper subset of the measure space X by [Con76] Proposition 3.2.2. We also have that $carrier(L_{\rho,a})$ equals either the whole space X or is empty, by [Con76] Corollary 3.2.4. Thus $carrier(L_{\rho,a}) = \emptyset$ and thus $L_{\rho,a} = \{0\}$. We give a few examples of spaces whose fundamental functions are not right-continuous at zero.

Examples 2.3.5 *(i) For $L_\infty(0,\infty)$ with the usual supremum norm and Lebesgue measure, the fundamental function is*

$$\Phi_\infty(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t \in (0,\infty) \end{cases}$$

and thus $\Phi_\infty(0^+) = 1 > 0$. We have that $L_{\infty,a} = \{0\} \neq L_{\infty,b} \neq L_\infty$.

(ii) Considering $(L_1 \cap L_\infty)(0,\infty)$ we have that

$$\Phi_{L_1\cap L_\infty}(t) = \begin{cases} 0 & \text{if } t = 0 \\ \max\{1,t\} & \text{if } t \in (0,\infty) \end{cases}$$

Therefore $(L_1 \cap L_\infty)_a = \{0\} \neq (L_1 \cap L_\infty)_b = L_1 \cap L_\infty$. (The latter equality can be found in [KPS82] Chapter 2, Section 3.1.)

(iii) Recall the Lorentz space from Example 2.3.4 (iii),

$$\Lambda_\psi = \{f \in L_0(0,\infty) : \int_0^\infty \mu_t(f)\, d\psi(t) < \infty\}$$

with the norm

$$\|f\|_{\Lambda_\psi} = \int_0^\infty \mu_t(f)\, d\psi(t)$$

for $f \in \Lambda_\psi$ and fundamental function ψ. If $\psi(0^+) > 0$ then

$$\Lambda_{\psi,a} = \{0\} \neq \Lambda_{\psi,b}.$$

(This equality also holds for the Marcinkiewicz space defined in Example 2.3.4 (iv), if its fundamental function is bounded away from zero.) If in addition $\psi(\infty) = \infty$, then we have that $\Lambda_{\psi,b} = \Lambda_\psi$ by [KPS82] Chapter 2, Corollary 5.3. As we have remarked above, if $\psi(0^+) > 0$ then $\Lambda_\psi \hookrightarrow L_\infty(0,\infty)$. If in addition $\psi(\infty) < \infty$ then $L_\infty(0,\infty) \hookrightarrow \Lambda_\psi$ and hence $\Lambda_\psi = L_\infty(0,\infty)$ ([KPS82] Chapter 2, Section 5.1), which can also be seen by the remark following the examples.

(iv) *Consider the space of all bounded sequences ℓ_∞ with the supremum norm over the measure space $(\mathrm{N}, \mathcal{P}(\mathrm{N}), \nu)$ where $\nu(n) = \frac{1}{2^n}$ for all $n \in \mathrm{N}$ (as in Example 2.3.3). Then $\Phi_{\ell_\infty}(0^+) = 1 > 0$ and $\ell_{\infty,a} = c_0 \neq \ell_{\infty,b} = \ell_\infty$.*

It is interesting to note the following. Let L_ρ be given and let Λ_{Φ_ρ} and M_{Φ_ρ} denote the Lorentz and Marcinkiewicz space respectively, where Φ_ρ is the least concave majorant of the original fundamental function on L_ρ. Then we always have that

$$\Lambda_{\Phi_\rho} \hookrightarrow L_\rho \hookrightarrow M_{\Phi_\rho}$$

by [KPS82] Theorems 5.5 and 5.7. Now if $\Phi_\rho(0^+) > 0$ then by [Con76] Proposition 3.2.5 we have that

$$\Lambda_{\Phi_\rho} \hookrightarrow L_\rho \hookrightarrow M_{\Phi_\rho} \hookrightarrow L_\infty.$$

If $\Phi_\rho(0^+) = K > 0$ and $\lim_{t\to\infty} \Phi_\rho(t) < \infty$ then using

$$\begin{aligned} \|f\|_{\Lambda_{\Phi_\rho}} &= \|f\|_\infty \Phi_\rho(0^+) + \int_0^\infty \mu_t(f)\Phi_\rho'(t)\,dt \\ &\leq \|f\|_\infty K + \|f\|_\infty \int_0^\infty \Phi_\rho'(t)\,dt \\ &= \|f\|_\infty (K + M) \end{aligned}$$

for some constants $K, M > 0$ and for all $f \in \Lambda_{\Phi_\rho}$, we have that

$$L_\infty \hookrightarrow \Lambda_{\Phi_\rho}.$$

Thus if $\Phi_\rho(0^+) > 0$ and $\lim_{t\to\infty} \Phi_\rho(t) < \infty$ we have that

$$L_\infty \hookrightarrow \Lambda_{\Phi_\rho} \hookrightarrow L_\rho \hookrightarrow M_{\Phi_\rho} \hookrightarrow L_\infty$$

and hence all these spaces are isomorphic to L_∞.

Chapter 3

Singular functionals on Banach spaces of measurable operators

A Köthe duality theory for normed rearrangement invariant operator spaces was developed in [DDP93], but a general theory for singular functionals on such spaces is still not available. The case of singular functionals on von Neumann algebras has been considered (see for example [Tak79]), and recently some work has also been done on the existence of special kinds of singular functionals (singular traces) on certain non-commutative Marcinkiewicz spaces ([DPSS97]).

In this chapter we give a possible definition for a singular functional on a Banach space of measurable operators and investigate some of its consequences. We also investigate a possible analogue of $L_{\rho,a}$ (the set of functions in L_ρ with order continuous norm) in the non-commutative setting. We show that for certain Banach spaces of measurable operators the annihilator of the set of operators with order continuous norm is exactly the set of singular functionals. We conclude with a classification of rearrangement invariant Banach operator spaces according to their fundamental functions, similar to the classification for Banach function spaces.

3.1 Definition for singular functionals on Banach spaces of measurable operators

We consider a normed subspace E of $\widetilde{\mathcal{M}}$. If the norm is complete, E is called a *Banach space of measurable operators.* When it is clear from the context that $E \subseteq \widetilde{\mathcal{M}}$, we shall also call E a *Banach operator space.* E is called *rearrangement invariant* whenever $x \in \widetilde{\mathcal{M}}$ and $y \in E$ with $\mu(x) \leq \mu(y)$ imply that $x \in E$ and $||x||_E \leq ||y||_E$. It follows by the definition that $x \in E$ if and only if $x^* \in E$ if and only if $|x| \in E$ with equal norms. If $x \in E$ and $0 \leq u, v \in \mathcal{M}$ then $uxv \in E$ and $||uxv||_E \leq ||u||_\infty ||x||_E ||v||_\infty$. If $E \subseteq \widetilde{\mathcal{M}}$ is a normed rearrangement invariant operator space, then the natural inclusion of E into $\widetilde{\mathcal{M}}$ is continuous, [DDP93] Proposition 2.2. (See [DDP93] Proposition 2.3 and Corollary 2.4 for conditions that will ensure that the norm is complete.)

The Banach spaces of measurable operators are therefore the natural analogues to Banach function spaces. We are in search of a definition for singular functionals in the non-commutative setting. If we consider positive functionals only, we may use an analogue of the definition for positive singular functionals on Banach function spaces. A problem arises when we consider functionals that are not necessarily positive. If E is a real normed operator space then we do have a decomposition of a functional into the difference of two positive functionals, using the fact that E is order complete by [DDP93] Proposition 1.1, and then [And62] Lemma 1 and Theorem 1. However, this decomposition is not unique and does not give an explicit representation of the positive parts. If we translate the explicit formula for the positive part of a functional on a Banach function space to the non-commutative setting, we cannot show that the resulting positive functional is additive. In the commutative setting we use the Riesz decomposition property, but we cannot use it here because of the lack of lattice structure. We recall the Riesz decomposition property for a Riesz space L: If $0 \leq x, y, z \in L$ and $z = x + y$ then there exists $0 \leq u, v \in L$ such that $u \leq x, v \leq y$ and $z = u + v$. Takesaki remarked that the Riesz decomposition property characterises the commutativity of a C^*-algebra, [Tak79] Chapter 1, Remark 7.6, and only an asymmetric decomposition property holds in general for nonabelian C^*-algebras, [Tak79] Chapter 1, Theorem 7.7.

Let $E \subseteq \widetilde{\mathcal{M}}$ be a normed operator space.

For $x, y \in \widetilde{\mathcal{M}}$ we say that x is *submajorized* by y, written $x \prec\prec y$, if

$$\int_0^\theta \mu_t(x)\, dt \leq \int_0^\theta \mu_t(y)\, dt$$

for all $\theta > 0$. A normed operator space $E \subseteq \widetilde{\mathcal{M}}$ is called *symmetric* whenever $x, y \in E$ and $x \prec\prec y$ imply that $\|x\|_E \leq \|y\|_E$. E is *fully symmetric* whenever $x \in \widetilde{\mathcal{M}}$, $y \in E$ with $x \prec\prec y$ imply that $x \in E$ and $\|x\|_E \leq \|y\|_E$.

We say that $E \subseteq \widetilde{\mathcal{M}}$ is *intermediate for the Banach couple* $(L_1(\mathcal{M}), L_\infty(\mathcal{M}))$ if

$$\mathcal{H}(\mathcal{M}) \subseteq E \subseteq \mathcal{G}(\mathcal{M})$$

with continuous imbeddings. E is called a *properly symmetric* normed operator space if E is a normed rearrangement invariant symmetric operator space that is intermediate for the Banach couple $(L_1(\mathcal{M}), L_\infty(\mathcal{M}))$.

We define normal functionals on a Banach operator space as in [DDP93] Definition 5.8.

Definition 3.1.1 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space. Then $\varphi \in E^*$ is called* normal *or* order continuous *if*

$$x_\alpha \downarrow_\alpha 0 \text{ in } E \text{ implies that } \varphi(x_\alpha) \to_\alpha 0.$$

The space of all normal linear functionals will be denoted by E^{*n}.

An analogue of one of the characterisations for singular functionals in the commutative setting is suitable to use as a definition for singular functionals on operator spaces.

Definition 3.1.2 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space. Then $\varphi \in E^*$ is called* singular *if for every $0 \neq \mathrm{p} \in E \cap \mathcal{M}^p$ there exists $0 \neq \mathrm{q} \leq \mathrm{p} \in \mathcal{M}^p$ such that $\varphi(\mathrm{q}) = 0$. We denote the set of all singular functionals by E^{*s}.*

With this definition, the set of positive singular functionals is closed under addition and scalar multiplication. It follows directly from the definition that the set of all singular functionals (not only the positive ones) is closed under scalar multiplication. For closure under addition let $0 \leq \varphi, \psi \in E^{*s}$ and $0 \neq \mathrm{p} \in E \cap \mathcal{M}^p$ be given. Since φ is singular, there exists $0 \neq \mathrm{q}_1 \leq \mathrm{p} \in \mathcal{M}^p$ such that $\varphi(\mathrm{q}_1) = 0$. Since ψ is singular there exists

$0 \neq \mathrm{q}_2 \leq \mathrm{q}_1 \leq \mathrm{p} \in \mathcal{M}^p$ such that $\psi(\mathrm{q}_2) = 0$. Now $0 \leq \varphi(\mathrm{q}_2) \leq \varphi(\mathrm{q}_1) = 0$ by positivity of φ and hence

$$(\varphi + \psi)(\mathrm{q}_2) = \varphi(\mathrm{q}_2) + \psi(\mathrm{q}_2) = 0$$

and so $\varphi + \psi$ is singular.

It follows by the definition of singularity of a functional that the positive functionals in E^{*s} is solid, in the sense that if $0 \leq \varphi \in E^*$ and $0 \leq \psi \in E^{*s}$ with $\varphi \leq \psi$, then $\varphi \in E^{*s}$.

Positive singular functionals on von Neumann algebras can be defined as those that majorize no non-zero normal functionals (as for singular functionals on Banach function spaces). By considering a functional as an element of the predual of the bidual (the bidual is also a von Neumann algebra), it posesses a unique decomposition into positive components. A functional on a von Neumann algebra is then called singular if each of its positive components is singular. This definition is shown to be equivalent to Definition 3.1.2, with $E = \mathcal{M}$, in [Tak59], [Ake67] Proposition II.1 and in [Tak79] Theorem 3.8. We show that a positive singular functional, as defined in Definition 3.1.2, also has the property that it majorizes no non-zero normal functional. We need

Definition 3.1.3 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach space of measurable operators. Then we define the* Köthe dual $E^\times$ *of E by setting*

$$E^\times = \{x \in \widetilde{\mathcal{M}} : xy \in L_1(\mathcal{M}) \text{ for all } y \in E\}.$$

The Köthe dual is a linear subspace of $\widetilde{\mathcal{M}}$. (For its properties see [DDP93] Section 5.) We shall return to the Köthe dual in Chapter 4.

Proposition 3.1.4 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a properly symmetric Banach operator space. Let $0 \leq \varphi \in E^*$. If φ is singular then $0 \leq \psi \leq \varphi$, $\psi \in E^{*n}$ implies that $\psi \equiv 0$ on E.*

Proof: Suppose $0 \leq \varphi \in E^*$ is singular and let $0 \leq \psi(x) \leq \varphi(x)$ for all $0 \leq x \in E$ and $\psi \in E^{*n}$. We want to show that $\psi \equiv 0$ on E.

Using [DDP93] Proposition 5.11 there exists $0 \leq a \in E^\times$ such that $\psi(x) = \tau(ax)$ for all $x \in E$. Suppose that $a \neq 0$. Put $\mathrm{e} = \mathrm{e}_{(0,\infty)}(a) = \mathbf{r}(a) \neq 0$, the (right) support of

a. By hypothesis there exists $0 \neq q_1 \leq e$ such that $\varphi(q_1) = 0$. Hence $\psi(q_1) = 0$. Now if $e - q_1 \neq 0$ then by hypothesis there exists $0 \neq q_2 \leq e - q_1$ such that $\varphi(q_2) = 0$ and thus $\psi(q_2) = 0$.

Put

$$\mathcal{F} = \{0 \neq q \leq e : \psi(q) = 0, q \in \mathcal{M}^p\}.$$

$\mathcal{F}$ is nonempty by the above. Suppose $q_\alpha \uparrow_\alpha \bigvee_\alpha q_\alpha = q_s$ with $q_\alpha \in \mathcal{F}$ for all α. Then $q_s \neq 0$ and $q_s - q_\alpha \downarrow_\alpha 0$. Since $\psi \in E^{*n}$ we have that

$$\psi(q_s) = \lim_\alpha \psi(q_\alpha) = 0.$$

Thus $q_s \in \mathcal{F}$ and by Zorn's Lemma there exists a maximal element, say q_0, in $\mathcal{F}$. Suppose $\psi(e - q_0) \neq 0$. Then $e - q_0 \neq 0$ and the hypothesis contradicts the maximality of q_0. Hence $\psi(e - q_0) = 0$ and thus

$$0 = \psi(q_0) = \psi(e) = \tau(ae) = \tau(a)$$

and so $a = 0$ and hence $\psi(x) = \tau(ax) = 0$ for all $x \in E$. □

3.2 Elements with order continuous norm in Banach spaces of measurable operators

In a commutative Banach function space L_ρ, the set of functions with order continuous norm, $L_{\rho,a}$, of L_ρ is important in that it is an order dense ideal on which every singular functional vanishes whenever $L_{\rho,a} = L_{\rho,b}$. We now investigate a possible analogue of $L_{\rho,a}$ in Banach spaces of measurable operators.

The following definition can be found in [DDP93], Section 5.

Definition 3.2.1 *The norm $\|.\|_E$ on a properly symmetric Banach operator space $E \subseteq \widetilde{\mathcal{M}}$ is said to be* order continuous *if for any positive net $(x_\alpha) \subseteq E$ that decreases to zero in E we have that*

$$\|x_\alpha\|_E \to_\alpha 0.$$

We give an equivalent condition for order continuity of the norm in Proposition 3.2.3. Lemma 3.2.2 and Proposition 3.2.3 are due to [DDP96], but we give an outline of their proofs for the sake of completeness.

Lemma 3.2.2 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a fully symmetric Banach operator space. If $x, y \in \widetilde{\mathcal{M}}$ and $x^*x, y^*y \in E$ then $x^*y \in E$ and*

$$\|x^*y\|_E \leq \|x^*x\|_E^{\frac{1}{2}} \, \|y^*y\|_E^{\frac{1}{2}}.$$

Proof: By [DDP89c] Corollary 3.3 there exists a fully symmetric Banach function space F on $(0, \infty)$ such that $E = F(\mathcal{M})$, where $F(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in F\}$ with norm $\|x\|_{F(\mathcal{M})} = \|\mu(x)\|_F$ for $x \in F(\mathcal{M})$. Hence, $x \in E$ if and only if $\mu(x) \in F$ and $\|\mu(x)\|_F = \|x\|_E$. Now we use the argument of [Sim81] Lemma 2.

Without loss of generality we may assume that $\|x^*x\|_E^{\frac{1}{2}} = \|y^*y\|_E^{\frac{1}{2}} = 1$. By [FK86] Theorem 4.2 (iii) and 2.5 (iv), we have that

$$\mu(x^*y) \prec\prec \mu(x^*)\mu(y) = \mu(x)\mu(y) \leq \frac{1}{2}\left(\mu(x^*x) + \mu(y^*y)\right).$$

Since F is fully symmetric it now follows that $\mu(x^*y) \in F$ and

$$\|x^*y\|_E = \|\mu(x^*y)\|_F \leq \frac{1}{2}\|\mu(x^*x) + \mu(y^*y)\|_F \leq 1.$$

□

Proposition 3.2.3 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a properly symmetric Banach operator space. Then the following statements are equivalent.*

(i) E has order continuous norm.

(ii) $\|x\mathrm{p}_\alpha\|_E \to_\alpha 0$ for every $0 \leq x \in E$ and any net $\mathrm{p}_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$.

Proof: We first show $(i) \Rightarrow (ii)$. We observe that order continuity of the norm on E implies that E is fully symmetric by using the same argument as in the commutative case given in [KPS82] Theorem 2.4.10, with the help of [DDP93] Propositions 4.9 and 5.13. The result now follows by Lemma 3.2.2, as was observed in [CS94] Proposition

2.5. (We use this same argument in the proof of Proposition 3.3.9 and therefore omit it here.)

For the implication *(ii)* $\Rightarrow$ *(i)* we note that *(ii)* implies that every continuous linear functional on E is completely additive and by [DDP93] Proposition 5.11 normal. By the remarks following [DDP93] Proposition 5.11, it follows that E has order continuous norm. $\square$

Chilin and Sukochev used sequences rather than nets in their definition for order continuity of the norm. They proved equivalences, similar to those in Proposition 3.2.3 above, for a rearrangement invariant operator space in $\widetilde{\mathcal{M}}$ with order continuous norm, but with the additional requirement that $\mathcal{M}^p$ be nonatomic, [CS94] Corollary 2.3.

We aim to define order continuity of the norm for a single element of a Banach operator space and then to explore the subset of elements with order continuous norm. Recall from Section 2.2 that a function f in a commutative Banach function space $L_\rho(X, \Sigma, \mu)$ has *order continuous norm* whenever $\rho(f_n) \downarrow_n 0$ for every sequence (f_n) in L_ρ with $|f| \geq f_n \downarrow_n 0$ μ-almost everywhere, and that an equivalent condition is $\rho(f\chi_{A_n}) \downarrow_n 0$ for every sequence (A_n) in Σ that decreases to a set of measure zero. We use the latter as motivation for the definition in the non-commutative setting.

Definition 3.2.4 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space. Then $x \in E$ is said to have* order continuous norm *if $\|\mathrm{p}_\alpha |x| \mathrm{p}_\alpha\|_E \to 0$ for all $(\mathrm{p}_\alpha) \subseteq \mathcal{M}^p$ with $\mathrm{p}_\alpha \downarrow 0$. The set of elements in E with order continuous norm will be denoted by E_a.*

Proposition 3.2.5 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space. Then*

$$E_a \subseteq \widetilde{\mathcal{M}_0}.$$

Proof: Suppose that $x \in E_a$ but $x \notin \widetilde{\mathcal{M}_0}$. Then as in the proof of Proposition 1.4.5, we can find a net of projections (q_α) and a positive constant c with $|x| \geq c\mathrm{q}_\alpha \downarrow_\alpha 0$ such that $\mu_t(c\mathrm{q}_\alpha) = c > 0$ for all $t > 0$ and all α. Hence since $\mathrm{q}_\alpha c \mathrm{q}_\alpha \leq \mathrm{q}_\alpha |x| \mathrm{q}_\alpha$ we have that for all $t > 0$ and all α,

$$0 < c = \mu_t(c\mathrm{q}_\alpha) = \mu_t(\mathrm{q}_\alpha c \mathrm{q}_\alpha) \leq \mu_t(\mathrm{q}_\alpha |x| \mathrm{q}_\alpha)$$

i.e. $(q_\alpha|x|q_\alpha)$ does not converge to zero in the measure topology. Since $E \hookrightarrow \widetilde{\mathcal{M}}$ continuously by [DDP93] Proposition 2.2, $(q_\alpha|x|q_\alpha)$ does not converge to zero in E-norm. Hence we found a net $q_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$ but $||q_\alpha|x|q_\alpha||_E \not\to 0$. So $x \notin E_a$, a contradiction. □

E_a is order solid in E in the sense that if $x \in E_a, y \in E$ with $|y| \leq |x|$ then $y \in E_a$.

The commutative analogue of the following may be found in [Zaa67] Chapter 15, Section 72, Lemma α.

Proposition 3.2.6 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space. Then $x \in E_a$ implies that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\tau(p) < \delta \Rightarrow ||p|x|p||_E < \epsilon$ for all $p \in \mathcal{M}^p$.*

Proof: Suppose for a contradiction that there exists $\epsilon_0 > 0$ such that no $\delta > 0$ satisfies $\tau(p) < \delta$, $p \in \mathcal{M}^p \Rightarrow ||p|x|p||_E < \epsilon_0$. Hence there exists $(p_n) \subseteq \mathcal{M}^p$ such that $\tau(p_n) < \frac{1}{2^n}$ and $||p_n|x|p_n||_E \geq \epsilon_0$ for all n. Then the sequence (e_n) where

$$e_n = \bigvee_{k=n}^{\infty} p_k$$

is decreasing. Note that $e_n \geq p_n$ and

$$\tau(e_n) = \tau(\bigvee_{k=n}^{\infty} p_k) \leq \sum_{k=n}^{\infty} \tau(p_k) < \sum_{k=n}^{\infty} \frac{1}{2^k} < \frac{1}{2^{n-1}}$$

for all n. Since (e_n) is decreasing and bounded below by 0, it converges to say e in partial order. So $e \leq e_n$ for all n and thus $\tau(e) \leq \tau(e_n) < \frac{1}{2^{n-1}}$ for all n. Hence $\tau(e) = 0$ and by faithfulness of the trace $e = 0$. Hence $e_n \downarrow 0$ and by hypothesis $||e_n|x|e_n||_E \to 0$. However

$$\epsilon_0 \leq ||p_n|x|p_n||_E = ||p_n e_n|x|e_n p_n||_E \leq ||e_n|x|e_n||_E$$

which contradicts $||e_n|x|e_n||_E \to 0$. □

The converse holds if $\tau(1) < \infty$.

Corollary 3.2.7 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space and suppose $\tau(1) < \infty$. Then $x \in E_a$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\tau(p) < \delta \Rightarrow ||p|x|p||_E < \epsilon$ for all $p \in \mathcal{M}^p$.*

Proof: Let $\mathrm{p}_\alpha \downarrow 0$, then $1 - \mathrm{p}_\alpha \uparrow 1$ and by normality of the trace $\tau(1 - \mathrm{p}_\alpha) \uparrow \tau(1)$, that is, $\tau(1) - \tau(1 - \mathrm{p}_\alpha) \downarrow 0$. Hence $\tau(\mathrm{p}_\alpha) = \tau(1) - \tau(1 - \mathrm{p}_\alpha) \downarrow 0$ since $\tau(1 - \mathrm{p}_\alpha) < \infty$ (because $\tau(1) < \infty$).

Let $\epsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$\tau(\mathrm{p}) < \delta \Rightarrow \|\mathrm{p}|x|\mathrm{p}\|_E < \epsilon$$

for all $\mathrm{p} \in \mathcal{M}^p$. Since $\tau(\mathrm{p}_\alpha) \downarrow 0$ there exists a α_0 such that $\tau(\mathrm{p}_\alpha) < \delta$ for all $\alpha \geq \alpha_0$ and hence by hypothesis $\|\mathrm{p}_\alpha|x|\mathrm{p}_\alpha\|_E < \epsilon$ for all $\alpha \geq \alpha_0$. □

It is Lemma 3.2.2 and the polar decomposition of an element that provide the following equivalent statements.

Proposition 3.2.8 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a fully symmetric Banach operator space. Let $x \in E$. Then the following statements are equivalent.*

(i) $x \in E_a$.

(ii) $\|x\mathrm{p}_\alpha\|_E \to_\alpha 0$ *for all* $\mathrm{p}_\alpha \downarrow_\alpha 0$ *in* $\mathcal{M}^p$.

(iii) $\||x|\mathrm{p}_\alpha\|_E \to_\alpha 0$ *for all* $\mathrm{p}_\alpha \downarrow_\alpha 0$ *in* $\mathcal{M}^p$.

(iv) $\|\mathrm{p}_\alpha|x|\|_E \to_\alpha 0$ *for all* $\mathrm{p}_\alpha \downarrow_\alpha 0$ *in* $\mathcal{M}^p$.

Proof: Suppose $\mathrm{p}_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$ and let $x \in E$ have polar decomposition $x = v|x|$.

$(i) \Leftrightarrow (ii)$: Suppose that $x \in E_a$. Note that $|x| \in E$ and $\mathrm{p}_\alpha|x|\mathrm{p}_\alpha \in E$. Using Lemma 3.2.2 we have that

$$\begin{aligned} \|x\mathrm{p}_\alpha\|_E &= \|v|x|\mathrm{p}_\alpha\|_E \\ &\leq \||x|\mathrm{p}_\alpha\|_E \\ &= \||x|^{\frac{1}{2}}\left(|x|^{\frac{1}{2}}\mathrm{p}_\alpha\right)\|_E \\ &\leq \||x|\|_E^{\frac{1}{2}}\,\|\mathrm{p}_\alpha|x|\mathrm{p}_\alpha\|_E^{\frac{1}{2}} \to_\alpha 0. \end{aligned}$$

Conversely, if $x \in E$ satisfies (ii), we have that

$$\begin{aligned} \|\mathrm{p}_\alpha|x|\mathrm{p}_\alpha\|_E &\leq \||x|\mathrm{p}_\alpha\|_E \\ &= \|v^*x\mathrm{p}_\alpha\|_E \\ &\leq \|x\mathrm{p}_\alpha\|_E \to_\alpha 0. \end{aligned}$$

The other equivalences follow similarly by using the polar decomposition of x and the fact that $||\mathrm{p}_\alpha|x|||_E = ||\,(\mathrm{p}_\alpha|x|)^*\,||_E$. □

Corollary 3.2.9 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a fully symmetric Banach operator space. Then E_a is closed in E.*

Proof: Suppose $||x - x_n||_E \to_n 0$ with $(x_n) \subseteq E_a$ and $x \in E$ and $\mathrm{p}_\beta \downarrow_\beta 0$ in $\mathcal{M}^p$. Let $\epsilon > 0$ be given. Then there exists an n_0 such that
$||(x - x_{n_0})\mathrm{p}_\beta||_E \leq ||x - x_{n_0}||_E \leq \epsilon$ for all β. Thus we have

$$\begin{aligned} ||x\mathrm{p}_\beta||_E &\leq ||(x - x_{n_0})\mathrm{p}_\beta||_E + ||x_{n_0}\mathrm{p}_\beta||_E \\ &\leq \epsilon + ||x_{n_0}\mathrm{p}_\beta||_E \end{aligned}$$

for all β and the result follows by Proposition 3.2.8 since $x_{n_0} \in E_a$. □

Corollary 3.2.10 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a fully symmetric Banach operator space. Then E_a is a vector space.*

Proof: Suppose $x, y \in E_a$ and λ is a scalar. Then it follows easily that $\lambda x \in E_a$. If $\mathrm{p}_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$, then

$$||(x + y)\mathrm{p}_\alpha||_E \leq ||x\mathrm{p}_\alpha||_E + ||y\mathrm{p}_\alpha||_E \to_\alpha 0$$

by Proposition 3.2.8 and so $x + y \in E_a$. □

If $\tau(1) < \infty$ and $\mathcal{H}(\mathcal{M})$ is continuously imbedded in E then we will show that E_a is a vector space without requiring E to be fully symmetric. We need the following result that was proved for bounded operators in a unital C^*-algebra in [AAP82] Theorem 4.2. As before (see Lemma 1.6.1), this inequality holds for $x, y \in \widetilde{\mathcal{M}}$.

Lemma 3.2.11 *Let $x, y \in \widetilde{\mathcal{M}}$. Then for every $\epsilon > 0$ there exist unitaries v and w in $\mathcal{M}$ such that*

$$|x + y| \leq v|x|v^* + w|y|w^* + \epsilon 1.$$

Proposition 3.2.12 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a rearrangement invariant Banach operator space. Suppose $\tau(1) < \infty$ and that $\mathcal{H}(\mathcal{M})$ is continuously imbedded in E. Then E_a is a vector space.*

Proof: Let $x, y \in E_a$, $\mathrm{p}_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$ and let $\epsilon > 0$ be given. Using Lemma 3.2.11 there exist unitaries $v, w \in \mathcal{M}$ such that

$$\begin{aligned}
\|\mathrm{p}_\alpha|x+y|\mathrm{p}_\alpha\|_E &\leq \|\mathrm{p}_\alpha v|x|v^*\mathrm{p}_\alpha\|_E + \|\mathrm{p}_\alpha w|y|w^*\mathrm{p}_\alpha\|_E + \epsilon\|\mathrm{p}_\alpha 1\mathrm{p}_\alpha\|_E \\
&= \|vv^*\mathrm{p}_\alpha v|x|v^*\mathrm{p}_\alpha vv^*\|_E + \|ww^*\mathrm{p}_\alpha w|y|w^*\mathrm{p}_\alpha ww^*\|_E + \epsilon\|\mathrm{p}_\alpha\|_E \\
&\leq \|v\|_\infty\|v^*\mathrm{p}_\alpha v|x|v^*\mathrm{p}_\alpha v\|_E\|v^*\|_\infty + \\
&\qquad \|w\|_\infty\|w^*\mathrm{p}_\alpha w|y|w^*\mathrm{p}_\alpha w\|_E\|w^*\|_\infty + \epsilon\|\mathrm{p}_\alpha\|_E \\
&\leq \|v^*\mathrm{p}_\alpha v|x|v^*\mathrm{p}_\alpha v\|_E + \|w^*\mathrm{p}_\alpha w|y|w^*\mathrm{p}_\alpha w\|_E + \epsilon\|\mathrm{p}_\alpha\|_E.
\end{aligned}$$

Since $\tau(1) < \infty$ we have that $\mathcal{M}^p \subseteq \mathcal{H}(\mathcal{M})$. Since $\mathcal{H}(\mathcal{M})$ is continuously imbedded in E and $\|\mathrm{p}_\alpha\| \leq 1$ for all α there exists a constant K such that $\|\mathrm{p}_\alpha\|_E \leq K$ for all α. Hence $\epsilon\|\mathrm{p}_\alpha\|_E \leq \epsilon K$ for all α. Since $\mathrm{p}_\alpha \downarrow_\alpha 0$ we have that $v^*\mathrm{p}_\alpha v \downarrow_\alpha 0$ and $w^*\mathrm{p}_\alpha w \downarrow_\alpha 0$ and these are nets of projections since v and w are unitaries. Hence

$$\|v^*\mathrm{p}_\alpha v|x|v^*\mathrm{p}_\alpha v\|_E \to_\alpha 0 \;\text{ and }\; \|w^*\mathrm{p}_\alpha w|y|w^*\mathrm{p}_\alpha w\|_E \to_\alpha 0.$$

Therefore $\|\mathrm{p}_\alpha|x+y|\mathrm{p}_\alpha\|_E \to_\alpha 0$ and $x + y \in E_a$. Hence E_a is a vector space. □

The idea of the proof of Proposition 3.2.13 is taken from a part of the proof of [CS94] Proposition 2.5.

Proposition 3.2.13 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a fully symmetric, rearrangement invariant Banach operator space. Suppose for $x \in E$ the following holds:*

$$|x| \geq x_\alpha \downarrow_\alpha 0 \Rightarrow \|x_\alpha\|_E \to_\alpha 0.$$

Then $x \in E_a$.

Proof: Let $x \in E$ and $\mathrm{p}_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$.

Put $y_\alpha = \mathrm{p}_\alpha|x|^{\frac{1}{2}}$. Note that $y_\alpha y_\alpha^* = \mathrm{p}_\alpha|x|\mathrm{p}_\alpha \in E$ for all α. Since E is rearrangement invariant and $\mu_t(y_\alpha y_\alpha^*) = \mu_t(y_\alpha^* y_\alpha)$ for $t > 0$ by [Yea75] we have that $y_\alpha^* y_\alpha = |x|^{\frac{1}{2}}\mathrm{p}_\alpha|x|^{\frac{1}{2}} \in E$ for all α. Now $|x| \geq |x|^{\frac{1}{2}}\mathrm{p}_\alpha|x|^{\frac{1}{2}} \downarrow_\alpha 0$ in E and so

$$\|\mathrm{p}_\alpha|x|\mathrm{p}_\alpha\|_E = \||x|^{\frac{1}{2}}\mathrm{p}_\alpha|x|^{\frac{1}{2}}\|_E \to_\alpha 0$$

by hypothesis. Hence $x \in E_a$. □

The converse can be shown to hold if we impose further conditions on E. This will be done in Section 3.3.

Definition 3.2.14 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a rearrangement invariant Banach operator space. Then we define E_b as the closure in E of the set*

$$\mathcal{H}(\mathcal{M}) \cap E.$$

Note that $E_b = \overline{\mathcal{H}(\mathcal{M}) \cap E}^{\|\cdot\|_E} \subseteq \overline{\mathcal{H}(\mathcal{M}) \cap E}^{\mathcal{T}_{cm}} \subseteq \overline{\mathcal{H}(\mathcal{M})}^{\mathcal{T}_{cm}} = \widetilde{\mathcal{M}_0}$ since $E \hookrightarrow \widetilde{\mathcal{M}}$ continuously by [DDP93] Proposition 2.2.

Lemma 3.2.15 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a rearrangement invariant Banach operator space. Then*

$$E_a \subseteq E_b.$$

Proof: Suppose $x \in E_a$ and let $x = v|x|$ be the polar decomposition of x. Then $x \in \widetilde{\mathcal{M}_0}$ by Proposition 3.2.5. Define $x_n = v|x|\mathrm{e}_{(\frac{1}{n},n]}(|x|)$.

Then $\tau(x_n) < \infty$ since $x \in \widetilde{\mathcal{M}_0}$ implies that $\tau\left(\mathrm{e}_{(\alpha,\infty)}(|x|)\right) < \infty$ for all $\alpha > 0$. We also have that $\|x_n\|_\infty \leq n < \infty$ for all n and so $x_n \in E_b$.

Since the spectral projections of $|x|$ commute with $|x|$ we have that

$$\begin{aligned}
\|x - x_n\|_E &= \|v|x| - x_n\|_E \\
&= \|v|x|\mathrm{e}_{(0,\infty)}(|x|) - v|x|\mathrm{e}_{(\frac{1}{n},n]}(|x|)\|_E \\
&\leq \|v\|_\infty \, \||x|\mathrm{e}_{(0,\frac{1}{n}]}(|x|)\|_E + \|v\|_\infty \, \||x|\mathrm{e}_{(n,\infty)}(|x|)\|_E \\
&\leq \|\mathrm{e}_{(0,\frac{1}{n}]}(|x|)|x|\mathrm{e}_{(0,\frac{1}{n}]}(|x|)\|_E + \|\mathrm{e}_{(n,\infty)}(|x|)|x|\mathrm{e}_{(n,\infty)}(|x|)\|_E \to_n 0
\end{aligned}$$

as $n \to \infty$ since $x \in E_a$, $\mathrm{e}_{(0,\frac{1}{n}]}(|x|) \downarrow 0$ and $\mathrm{e}_{(n,\infty)}(|x|) \downarrow 0$ as $n \to \infty$. Hence $x \in E_b$ since E_b is closed in E. □

Definition 3.2.16 *An element in E is called* simple *if it is a finite linear combination of projections with finite trace.*

Lemma 3.2.17 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a fully symmetric, rearrangement invariant Banach operator space. Suppose that $\mathcal{H}(\mathcal{M})$ is continuously imbedded in E. Then E_b is the closure in E of the simple operators in E.*

Proof: Suppose $x \in E_b = \overline{\mathcal{H}(\mathcal{M}) \cap E}^{\|\cdot\|_E} = \overline{\mathcal{H}(\mathcal{M})}^{\|\cdot\|_E}$. By definition x is the limit in E-norm of elements in $\mathcal{H}(\mathcal{M}) \cap E$.

So suppose that $0 \leq x \in \mathcal{H}(\mathcal{M}) \cap E = \mathcal{H}(\mathcal{M})$. Define $x_n = x\mathrm{e}_{(\frac{1}{n},\infty)}(x)$. Note that $\tau(x_n) < \infty$ since $x \in \mathcal{H}(\mathcal{M}) \subseteq \widetilde{\mathcal{M}_0}$ and thus $\tau\left(\mathrm{e}_{(\alpha,\infty)}(x)\right) < \infty$ for all $\alpha > 0$. We have that

$$\begin{aligned} \|x - x_n\|_\infty &= \|x\mathrm{e}_{[0,\frac{1}{n}]}(x)\|_\infty \\ &\leq \frac{1}{n} \to_n 0 \end{aligned}$$

as $n \to \infty$. Note that $\mu(x) \in L_1(0,\infty)$ since $x \in \mathcal{H}(\mathcal{M}) \subseteq L_1(\mathcal{M})$ and $\mu(x_n) \leq \mu(x)$. We have that x_n converge to x in the measure topology (see for example [DDP93] Proposition 2.7 (a)) and hence by the Non-commutative Dominated Convergence Theorem, [DDP93] Proposition 3.3 (ii),

$$\|x - x_n\|_1 \to_n 0$$

as $n \to \infty$. Hence $\|x - x_n\|_{\mathcal{H}(\mathcal{M})} \to_n 0$ and since $\mathcal{H}(\mathcal{M}) \hookrightarrow E$ continuously

$$\|x - x_n\|_E \to_n 0$$

as $n \to \infty$. Hence $0 \leq x \in \mathcal{H}(\mathcal{M})$ is the limit in E-norm of bounded operators with finite support. Since an arbitrary $x \in \mathcal{H}(\mathcal{M})$ can be written as a linear combination of positive elements, this holds for all $x \in \mathcal{H}(\mathcal{M})$.

Now suppose that $x \geq 0$ is a bounded operator with finite support in $\mathcal{H}(\mathcal{M})$. By the Spectral Theorem we can choose simple operators x_n such that $\|x - x_n\|_\infty \leq \frac{1}{n}$ and $\mathbf{r}(x_n) = \mathrm{e}_{(0,\infty)}(x_n) \leq \mathrm{e}_{(0,\infty)}(x) = \mathbf{r}(x)$. Also since $\tau(\mathbf{r}(x)) < \infty$,

$$\|x - x_n\|_1 \to_n 0$$

as $n \to \infty$. Hence $x_n \to_n x$ in $\mathcal{H}(\mathcal{M})$ as $n \to \infty$ and thus also in E-norm. As before this holds for any x that is bounded with finite support. We have the result. □

The following result may be found in the commutative setting in [BS88] Chapter 1, Theorem 3.13.

Proposition 3.2.18 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a fully symmetric, rearrangement invariant Banach operator space. Suppose that $\mathcal{H}(\mathcal{M})$ is continuously imbedded in E. Then the following statements are equivalent.*

(a) Every projection with finite trace has order continuous norm.

(b) $E_a = E_b$.

Proof: The implication $(b) \Rightarrow (a)$ is trivial since every projection with finite trace in E is in E_b.

We show $(a) \Rightarrow (b)$. We know from Lemma 3.2.15 that E_a is always contained in E_b.

If every projection in E with finite trace has order continuous norm then so does every simple operator in E since E_a is a vector space by Corollary 3.2.10. Now suppose that x is an element in E_b. Then by Lemma 3.2.17 x is the limit in E-norm of simple operators in E. Since E_a is closed, Corollary 3.2.9, we have that $x \in E_a$. □

3.3 Elements with order continuous norm in induced normed rearrangement invariant operator spaces

We now consider induced spaces of measurable operators $E(\mathcal{M})$ and show that it is possible to characterise an element with order continuous norm using its generalised singular function.

Recall that if $E(0,\infty)$ is a normed rearrangement invariant function space on $(0,\infty)$ with Lebesgue measure, then we define the induced space

$$E(\mathcal{M}) = \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E(0,\infty)\}$$

and for $x \in E(\mathcal{M})$ we define the norm

$$\|x\|_{E(\mathcal{M})} = \|\mu(x)\|_{E(0,\infty)}.$$

With this norm $E(\mathcal{M})$ is a normed rearrangement invariant operator space which is a Banach (rearrangement invariant, symmetric, fully symmetric) space whenever $E(0,\infty)$ is a Banach (rearrangement invariant, symmetric, fully symmetric) space. For details of these facts the reader may consult [DDP89b] and [DDP89a].

It was shown in [DDP93] Proposition 3.6 that if the norm on a rearrangement invariant symmetric Banach function space $E(0,\infty)$ is order continuous, then the norm on the space $E(\mathcal{M})$ is order continuous. We want to establish this type of characterisation for a single element with order continuous norm. Our idea is to show that x has order continuous norm in $E(\mathcal{M})$ if and only if $\mu(x)$ has order continuous norm in $E(0,\infty)$. As before, we denote the set of elements with order continuous norm in $E(\mathcal{M})$ by $E(\mathcal{M})_a$.

Lemma 3.3.1 *Let $E(0,\infty)$ be a fully symmetric, rearrangement invariant Banach function space. Then*

$$E(0,\infty)_a = E(0,\infty)_b \Rightarrow E(\mathcal{M})_a = E(\mathcal{M})_b.$$

Proof: Suppose $E(0,\infty)_a = E(0,\infty)_b$. As was noted in [DDP93] Section 2, it follows from [KPS82] Chapter II, Theorem 4.1 that if $E(0,\infty)$ is a symmetric rearrangement invariant Banach function space then $E(\mathcal{M})$ is intermediate for the Banach couple $(L_1(\mathcal{M}),\mathcal{M})$, that is, $\mathcal{H}(\mathcal{M}) \hookrightarrow E(\mathcal{M}) \hookrightarrow \mathcal{G}(\mathcal{M})$ continuously.

We show that every projection with finite trace has order continuous norm, from which it will follow that $E(\mathcal{M})_a = E(\mathcal{M})_b$ by Proposition 3.2.18.

So suppose that $\mathrm{p} \in E(\mathcal{M}) \cap \mathcal{M}^p$ with $\tau(\mathrm{p}) < \infty$. Then $\mu(\mathrm{p}) = \chi_{(0,\tau(\mathrm{p}))}$ and hence $\mu(\mathrm{p}) \in E(0,\infty)_b = E(0,\infty)_a$. Suppose $\mathrm{p} \geq x_\alpha \downarrow_\alpha 0$ in $E(\mathcal{M}) \cap \mathcal{M}^p$. Then since $\mathrm{p} \in \widetilde{\mathcal{M}_0}$,

$$\mu_t(\mathrm{p}) \geq \mu_t(x_\alpha) \downarrow_\alpha 0$$

for all $t > 0$ by Proposition 1.1.4 (iv) and Theorem 1.4.2. Since $\mu(\mathrm{p}) \in E(0,\infty)_a$ we have that

$$||x_\alpha||_{E(\mathcal{M})} = ||\mu(x_\alpha)||_{E(0,\infty)} \to_\alpha 0$$

and hence $\mathrm{p} \in E(\mathcal{M})_a$ by Proposition 3.2.13. This proves the result. □

Lemma 3.3.2 *Let $E(0,\infty)$ be a rearrangement invariant Banach function space. Suppose $E(0,\infty)_a = E(0,\infty)_b$. Then for $x \in E(\mathcal{M})$ the following conditions are equivalent.*

(i) $|||x|\mathrm{e}_{(\lambda,\infty)}(|x|)||_{E(\mathcal{M})} \to 0$ *as* $\lambda \to \infty$.

(ii) $||\mu(x)\chi_{(0,s)}||_{E(0,\infty)} \to 0$ *as* $s \to 0^+$.

Proof: Let $x \in E(\mathcal{M})$. For $\lambda > 0$ denote $|x|e_{(\lambda,\infty)}(|x|)$ by y_λ. We use the fact that $d_\lambda(x) \to 0$ as $\lambda \to \infty$ (see Section 1.1) and that $\mu_t\left(|x|e_{(\lambda,\infty)}(|x|)\right) = \mu_t(x)\chi_{(0,d_\lambda(x))}(t)$ for all $t > 0$.

First let us suppose that $\||x|e_{(\lambda,\infty)}(|x|)\|_{E(\mathcal{M})} \to 0$ as $\lambda \to \infty$. Then if $\lambda \to \infty$

$$\begin{aligned} \|\mu(x)\chi_{(0,d_\lambda(x))}\|_{E(0,\infty)} &= \|\mu(y_\lambda)\|_{E(0,\infty)} \\ &= \||x|e_{(\lambda,\infty)}(|x|)\|_{E(\mathcal{M})} \to 0. \end{aligned}$$

We consider two cases.

(a) Suppose $d_t(x) > 0$ for all $t > 0$. Let $\epsilon > 0$ be given. Then there exists a constant $M_\epsilon > 0$ such that $\|\mu(x)\chi_{(0,d_\lambda(x))}\|_{E(0,\infty)} \leq \epsilon$ and $d_\lambda(x) \leq \epsilon$ whenever $\lambda \geq M_\epsilon$. Choose $\delta = d_{M_\epsilon}(x)$. Note that $\delta > 0$. Then for $0 < s \leq \delta$ we have that

$$\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \leq \|\mu(x)\chi_{(0,d_{M_\epsilon}(x))}\|_{E(0,\infty)} \leq \epsilon.$$

(b) Suppose there exists a constant $M > 0$ such that $d_t(x) = 0$ for all $t \geq M$. Note that

$$\|\mu(x)\|_\infty = \lim_{t\to 0^+} \mu_t(x) = M < \infty.$$

Choose any constant $K > 0$. Then we have that $\mu(x)\chi_{(0,K)} \in E(0,\infty)_b$. By hypothesis $E(0,\infty)_b = E(0,\infty)_a$ and hence for $s < K$

$$\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} = \|\mu(x)\chi_{(0,K)}\chi_{(0,s)}\|_{E(0,\infty)} \to 0$$

as $s \to 0^+$ since $\mu(x)\chi_{(0,K)} \in E(0,\infty)_a$.

Conversely, let $\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \to 0$ as $s \to 0^+$.

Let $\epsilon > 0$ be given. Then there exists a $\delta_\epsilon > 0$ such that $\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \leq \epsilon$ whenever $0 < s \leq \delta_\epsilon$. Choose λ large enough such that $d_\lambda(x) \leq \delta_\epsilon$. Then

$$\begin{aligned} \||x|e_{(\lambda,\infty)}(|x|)\|_{E(\mathcal{M})} &= \|\mu(y_\lambda)\|_{E(0,\infty)} \\ &= \|\mu(x)\chi_{(0,d_\lambda(x))}\|_{E(0,\infty)} \\ &\leq \|\mu(x)\chi_{(0,\delta_\epsilon)}\|_{E(0,\infty)} \leq \epsilon \end{aligned}$$

and hence we have the result. □

Note that $e_{(\lambda,\infty)}(|x|) \downarrow 0$ as $\lambda \to \infty$. Hence condition (*i*) in Lemma 3.3.2 is necessary for x to have order continuous norm in $E(\mathcal{M})$, and (*ii*) is a necessary condition for $\mu(x)$ to have order continuous norm in $E(0,\infty)$. However, as we show below, when the trace is finite, these equivalent conditions are necessary and sufficient for x to have order continuous norm in $E(\mathcal{M})$.

Proposition 3.3.3 *Let $E(0,\infty)$ be a rearrangement invariant Banach function space. Suppose that $E(0,\infty)_a = E(0,\infty)_b$ and $\tau(1) < \infty$. Then*

$$x \in E(\mathcal{M})_a \text{ if and only if } \|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \to 0 \text{ as } s \to 0^+.$$

Proof: Suppose $x \in E(\mathcal{M})_a$. Then $e_{(\lambda,\infty)}(|x|) \downarrow 0$ as $\lambda \to \infty$ since $e_\lambda(|x|) \uparrow 1$ as $\lambda \to \infty$. Hence $\||x|e_{(\lambda,\infty)}(|x|)\|_{E(\mathcal{M})} = \|e_{(\lambda,\infty)}(|x|)|x|e_{(\lambda,\infty)}(|x|)\|_{E(\mathcal{M})} \to 0$ as $\lambda \to \infty$ by hypothesis and by Lemma 3.3.2 $\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \to 0$ as $s \to 0^+$.

Conversely, suppose that $\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \to 0$ as $s \to 0^+$ for $x \in E(\mathcal{M})$ and let $p_\alpha \downarrow_\alpha 0$ in $\mathcal{M}^p$. Since $\tau(p_\alpha) < \infty$ for all α, $\tau(p_\alpha) \downarrow_\alpha 0$. Then

$$\begin{aligned} \|p_\alpha|x|p_\alpha\|_{E(\mathcal{M})} &= \|\mu(p_\alpha|x|p_\alpha)\|_{E(0,\infty)} \\ &\leq \|\mu(x)\chi_{(0,\tau(p_\alpha))}\|_{E(0,\infty)} \to_\alpha 0 \end{aligned}$$

by hypothesis and hence we have the result. □

If the trace is infinite, then we will need an extra condition (also in terms of the generalised singular function) to ensure that an element has order continuous norm in $E(\mathcal{M})$. Notice that condition (*ii*) in Lemma 3.3.4 is automatically satisfied whenever the trace is finite.

Lemma 3.3.4 *Let $E(0,\infty)$ be a rearrangement invariant Banach function space. Then for $x \in E(\mathcal{M}) \cap \widetilde{\mathcal{M}_0}$ the following conditions are equivalent.*

(i) $\||x|e_{(0,\lambda]}(|x|)\|_{E(\mathcal{M})} \to 0$ *as* $\lambda \to 0^+$.

(ii) $\|\mu(x)\chi_{(s,\infty)}\|_{E(0,\infty)} \to 0$ *as* $s \to \infty$.

Proof: Let $x \in E(\mathcal{M}) \cap \widetilde{\mathcal{M}_0}$. Note that $\mu_\infty(x) = 0$ and suppose that $\lambda > 0$. Define $y_\lambda = |x|e_{(0,\lambda]}(|x|) = \int_0^\lambda t\, de_t(|x|)$. By calculating the spectral projections and distribution functions for y_λ it follows that $\mu_t(y_\lambda) = \mu_{t+d_\lambda(x)}(x)$ for all $t > 0$. If $\lambda \to 0^+$ then $d_\lambda(x) \uparrow$, say $d_\lambda(x) \uparrow M \leq \infty$.

Suppose that $\||x|e_{(0,\lambda]}(|x|)\|_{E(\mathcal{M})} \to 0$ as $\lambda \to 0^+$. Then by hypothesis if $\lambda \to 0^+$

$$\begin{aligned} \|\mu_{t+d_\lambda(x)}(x)\|_{E(0,\infty)} &= \|\mu_t(y_\lambda)\|_{E(0,\infty)} \\ &= \||x|e_{(0,\lambda]}(|x|)\|_{E(\mathcal{M})} \to_\lambda 0 \end{aligned}$$

Let $\epsilon > 0$ and $N < M$ be given. Then there exists a $\lambda_0 > 0$ such that $d_\lambda(x) \geq N$ and $\|\mu_{t+d_\lambda(x)}(x)\|_{E(0,\infty)} \leq \epsilon$ for all $0 < \lambda \leq \lambda_0$. Choose $K = d_{\lambda_0}(x)$. Note that $K < \infty$ since $x \in \widetilde{\mathcal{M}}_0$. Then for $s \geq K \geq N$ we have that

$$\|\mu(x)\chi_{(s,\infty)}\|_{E(0,\infty)} \leq \|\mu(x)\chi_{(K,\infty)}\|_{E(0,\infty)} \leq \epsilon.$$

Conversely, suppose that $\|\mu(x)\chi_{(s,\infty)}\|_{E(0,\infty)} \to 0$ as $s \to \infty$. Let $\epsilon > 0$ be given. Then there exists a $K > 0$ such that $\|\mu(x)\chi_{(s,\infty)}\|_{E(0,\infty)} \leq \epsilon$ whenever $s \geq K$.

If $d_\lambda(x) \uparrow \infty$ as $\lambda \to 0^+$ then there exists a $\lambda_0 > 0$ (take $\lambda_0 \leq \mu_K(x)$) such that $d_\lambda(x) \geq K$ for all $0 < \lambda \leq \lambda_0$. Then

$$\||x|e_{(0,\lambda]}(|x|)\|_{E(\mathcal{M})} = \|\mu_{t+d_\lambda(x)}(x)\|_{E(0,\infty)} = \|\mu_t(x)\chi_{(d_\lambda(x),\infty)}(t)\|_{E(0,\infty)} \leq \epsilon$$

for $0 < \lambda \leq \lambda_0$ which gives the result.

If $d_\lambda(x) \uparrow M < \infty$ as $\lambda \to 0^+$, note that $\mu_t(x) = 0$ for all $t \geq M$. For $\lambda > 0$ such that $d_\lambda(x) > 0$ we have that

$$\begin{aligned} \||x|e_{(0,\lambda]}(|x|)\|_{E(\mathcal{M})} &= \|\mu_{t+d_\lambda(x)}(x)\|_{E(0,\infty)} \\ &= \|\mu_t\chi_{(d_\lambda(x),\infty)}\|_{E(0,\infty)} \\ &= \|\mu_t\chi_{(d_\lambda(x),M]}\|_{E(0,\infty)} \\ &\leq \lambda\,\|\chi_{(0,M]}\|_{E(0,\infty)} \end{aligned}$$

and this tends to 0 as $\lambda \to 0^+$ which proves the result. □

Note that $e_{(0,\lambda)}(|x|) \downarrow 0$ as $\lambda \to 0^+$. Thus condition (*i*) in Lemma 3.3.4 is necessary for x to have order continuous norm in $E(\mathcal{M})$, and (*ii*) is a necessary condition for $\mu(x)$ to have order continuous norm in $E(0,\infty)$. We now characterise elements with order continuous norm in $E(\mathcal{M})$ in terms of their generalised singular function, with the assumption on $E(0,\infty)$ that $E(0,\infty)_a = E(0,\infty)_b$.

Proposition 3.3.5 *Let $E(0,\infty)$ be a fully symmetric, rearrangement invariant Banach function space and suppose that $E(0,\infty)_a = E(0,\infty)_b$. Then $x \in E(\mathcal{M})_a$ if and only if $x \in \widetilde{\mathcal{M}_0}$, $\|\mu(x)\chi_{(0,s)}\|_{E(0,\infty)} \to 0$ as $s \to 0^+$ and $\|\mu(x)\chi_{(r,\infty)}\|_{E(0,\infty)} \to 0$ as $r \to \infty$.*

Proof: Suppose $x \in E(\mathcal{M})_a$. Then $x \in \widetilde{\mathcal{M}_0}$ by Proposition 3.2.5. We also have $e_{(\lambda,\infty)}(|x|) \downarrow 0$ as $\lambda \to \infty$ (since $e_\lambda(|x|) \uparrow 1$ as $\lambda \to \infty$) and $e_{(0,\lambda)}(|x|) \downarrow 0$ as $\lambda \to 0^+$. By hypothesis $\||x|e_{(\lambda,\infty)}(|x|)\|_{E(\mathcal{M})} \to 0$ as $\lambda \to \infty$ and $\||x|e_{(0,\lambda]}(|x|)\|_{E(\mathcal{M})} \to 0$ as $\lambda \to 0^+$. Hence by Lemmas 3.3.2 and 3.3.4 we have the result.

Conversely, let $x \in \widetilde{\mathcal{M}_0}$. Put $x_n = |x|e_{(\frac{1}{n},n]}(|x|)$. Note that $\tau\left(e_{(\lambda,\infty)}(|x|)\right) < \infty$ for all $\lambda > 0$ since $x \in \widetilde{\mathcal{M}_0}$ and hence $x_n \in \mathcal{H}(\mathcal{M})$.

As was noted in the proof of Lemma 3.3.1, $\mathcal{H}(\mathcal{M}) \hookrightarrow E(\mathcal{M}) \hookrightarrow \mathcal{G}(\mathcal{M})$ continuously since $E(0,\infty)$ is a symmetric rearrangement invariant Banach function space. Hence $\mathcal{H}(\mathcal{M}) \hookrightarrow E(\mathcal{M})_b \subseteq E(\mathcal{M})$ continuously. By Lemma 3.3.1 we have that $E(\mathcal{M})_b = E(\mathcal{M})_a$. Thus $x_n \in \mathcal{H}(\mathcal{M}) \subseteq E(\mathcal{M})_b = E(\mathcal{M})_a$ and so

$$\begin{aligned} \||x| - x_n\|_{E(\mathcal{M})} &= \||x|\left(e_{(0,\infty)}(|x|) - e_{(\frac{1}{n},n]}(|x|)\right)\|_{E(\mathcal{M})} \\ &\leq \||x|e_{(0,\frac{1}{n}]}(|x|)\|_{E(\mathcal{M})} + \||x|e_{(n,\infty)}(|x|)\|_{E(\mathcal{M})} \to_n 0 \end{aligned}$$

as $n \to \infty$ by Lemmas 3.3.2 and 3.3.4. Since $E(\mathcal{M})_a$ is closed, $|x|$ and hence $x \in E(\mathcal{M})_a$. □

Corollary 3.3.6 *Let $E(0,\infty)$ be a fully symmetric, rearrangement invariant Banach function space. Suppose that $E(0,\infty)_a = E(0,\infty)_b$. Then*

$$x \in E(\mathcal{M})_a \text{ if and only if } \mu(x) \in E(0,\infty)_a.$$

Proof: Suppose $x \in E(\mathcal{M})_a$. By Lemma 3.3.1 we have that $E(\mathcal{M})_a = E(\mathcal{M})_b$. Then using Proposition 3.3.5 twice, first on $E(\mathcal{M})$ and then on $E(0,\infty)$ (noting that $\mu(\mu(x)) = \mu(x)$), we have that $\mu(x) \in E(0,\infty)_a$.

Conversely, suppose $\mu(x) \in E(0,\infty)_a$. Then the conditions of Proposition 3.3.5 are satisfied and hence $x \in E(\mathcal{M})_a$. □

The following example shows that $E(0,\infty)_a = E(0,\infty)_b$ is a necessary hypothesis in Corollary 3.3.6.

Example 3.3.7 *Let $E(0,\infty) = L_\infty(0,\infty)$ and $\mathcal{M} = \ell_\infty$ over $(\mathrm{N}, \mathcal{P}(\mathrm{N}), c)$ where c denotes counting measure, with the usual supremum norm. Then $E(\mathcal{M}) = \ell_\infty$ as defined above, and $E(\mathcal{M})_a = c_0$ but $E(0,\infty)_a = \{0\}$.*

Corollary 3.3.8 *Let $E(0,\infty)$ be a fully symmetric rearrangement invariant Banach function space and suppose that $E(0,\infty)_a = E(0,\infty)_b$. Then $E(\mathcal{M})_a$ is rearrangement invariant.*

Proof: Note that $E(\mathcal{M})_a = E(\mathcal{M})_b$ by Lemma 3.3.1. The result now follows immediately from Proposition 3.3.5. □

Proposition 3.3.9 *Let $E(0,\infty)$ be a fully symmetric rearrangement invariant Banach function space. Suppose that $E(0,\infty)_a = E(0,\infty)_b$. Then $x \in E(\mathcal{M})_a$ if and only if $\|x_\alpha\|_{E(\mathcal{M})} \to_\alpha 0$ whenever $|x| \geq x_\alpha \downarrow_\alpha 0$ in $E(\mathcal{M})$.*

Proof: The sufficiency is Proposition 3.2.13. For the necessity, we first show that $E(\mathcal{M})_a$ is a properly symmetric Banach operator space.

By hypothesis and the remark at the beginning of this section, $E(\mathcal{M})$ is a fully symmetric, rearrangement invariant Banach operator space.

$E(\mathcal{M})_a$ is a closed vector subspace of the Banach space $E(\mathcal{M})$ by Corollary 3.2.9 and Corollary 3.2.10 and therefore itself a Banach space. $E(\mathcal{M})_a$ is rearrangement invariant by Corollary 3.3.8. Since $E(\mathcal{M})$ is symmetric it follows immediately that $E(\mathcal{M})_a$ is symmetric.

As remarked in the proof of Proposition 3.3.5, since $E(0,\infty)$ is a symmetric rearrangement invariant Banach function space we have that $E(\mathcal{M})$ is intermediate for the Banach couple $(L_1(\mathcal{M}), \mathcal{M})$. Hence $\mathcal{H}(\mathcal{M}) \hookrightarrow E(\mathcal{M})_b = E(\mathcal{M})_a \subseteq E(\mathcal{M}) \hookrightarrow \mathcal{G}(\mathcal{M})$ continuously. Thus $E(\mathcal{M})_a$ is a properly symmetric Banach operator space. The result now follows by Propositions 3.2.3, 3.2.8 and Definition 3.2.1. □

Definition 3.3.10 *Let $E \subseteq \widetilde{\mathcal{M}}$ be a Banach operator space and $A \subseteq E$. Then $\varphi \in E^*$ is called* orthogonal to A *if $\varphi(x) = 0$ for all $x \in A$. The set $A'^{\perp}$ of all $\varphi \in E^*$ that are orthogonal to A is called the* annihilator of A.

The annihilator of $A \subseteq E \subseteq \widetilde{\mathcal{M}}$ is a linear subspace of E^*.

Recall from Definition 3.1.2 that a functional $\varphi \in E(\mathcal{M})^*$ is singular if for every $0 \neq \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})$ there exists $0 \neq \mathrm{q} \leq \mathrm{p} \in \mathcal{M}^p$ such that $\varphi(\mathrm{q}) = 0$. Under certain conditions on $E(\mathcal{M})$ we have that $E(\mathcal{M})^{*s} = (E(\mathcal{M})_a)^{\perp}$, similar to the commutative setting.

Lemma 3.3.11 *Let $E(0,\infty)$ be a fully symmetric rearrangement invariant Banach function space and suppose that $E(0,\infty)_a = E(0,\infty)_b$. Let $\varphi \in E(\mathcal{M})^*$. Then φ is singular if and only if $\varphi(\mathrm{p}) = 0$ for all $0 \neq \mathrm{p} \in \mathcal{M}^p$ with finite trace.*

Proof: The sufficiency is trivial since $\mathcal{M}$ is semifinite.

To show the necessity, suppose $\varphi \in E^*$ is singular and $0 \neq \mathrm{p} \in \mathcal{M}^p$ with $\tau(\mathrm{p}) < \infty$. Since $\mathcal{H}(\mathcal{M})$ is (continuously) imbedded in $E(\mathcal{M})$ (by hypothesis, as before) we have that $\mathrm{p} \in E(\mathcal{M})$. Put

$$\mathcal{F} = \{0 \neq \mathrm{q} \leq \mathrm{p} : \varphi(\mathrm{q}) = 0\}.$$

$\mathcal{F}$ is nonempty by hypothesis and partially ordered. Suppose $\mathrm{q}_\alpha \uparrow_\alpha \vee_\alpha \mathrm{q}_\alpha = \mathrm{q}_0$ with $\mathrm{q}_\alpha \in \mathcal{F}$ for all α. Then $\mathrm{q}_0 \neq 0$ and $\mathrm{q}_0 \geq \mathrm{q}_0 - \mathrm{q}_\alpha \downarrow_\alpha 0$. Since $\tau(\mathrm{q}_0) \leq \tau(\mathrm{p}) < \infty$ we have that $\mathrm{q}_0 \in E(\mathcal{M})_b = E(\mathcal{M})_a$ and hence by Proposition 3.3.9 since q_0 has order continuous norm, $\|\mathrm{q}_0 - \mathrm{q}_\alpha\|_{E(\mathcal{M})} \to_\alpha 0$. Since φ is continuous

$$\varphi(\mathrm{q}_0) = \lim_\alpha \varphi(\mathrm{q}_\alpha) = 0$$

and hence $\mathrm{q}_0 \in \mathcal{F}$. By Zorn's lemma there exists a maximal element q_m in $\mathcal{F}$; we must have that $\tau(\mathrm{p}) = \tau(\mathrm{q}_m) = 0$, otherwise it will contradict the hypothesis. □

Theorem 3.3.12 *Let $E(0,\infty)$ be a fully symmetric rearrangement invariant Banach function space and suppose that $E(0,\infty)_a = E(0,\infty)_b$. Then*

$$E(\mathcal{M})^{*s} = E(\mathcal{M})_a^{\perp}.$$

Proof: Let $\varphi \in E(\mathcal{M})^{*s}$. By Lemma 3.3.11 we have that $\varphi(\mathrm{p}) = 0$ for all $0 \neq \mathrm{p} \in \mathcal{M}^p$ with finite trace. Hence by linearity φ vanishes on every simple operator. Since $E(\mathcal{M})_b$ is the closure in E of the simple operators (Lemma 3.2.17), φ vanishes on $E(\mathcal{M})_b$. Hence φ vanishes on $E(\mathcal{M})_b = E(\mathcal{M})_a$ by Lemma 3.3.1.

Conversely, suppose that φ vanishes on $E(\mathcal{M})_a = E(\mathcal{M})_b$. Since $\mathcal{H}(\mathcal{M})$ is contained in $E(\mathcal{M})$ by hypothesis, we have that $\mathcal{H}(\mathcal{M}) \subseteq E(\mathcal{M})_b = E(\mathcal{M})_a$. Hence for every projection $0 \neq \mathrm{p} \in \mathcal{M}^p$ with finite trace, $\varphi(\mathrm{p}) = 0$. By Lemma 3.3.11 φ is singular. □

3.4 Classification of a normed rearrangement invariant operator space using the fundamental function

In this section we generalise the classification of commutative rearrangement invariant Banach function spaces according to the fundamental function, to the non-commutative setting. We define the fundamental function for Banach rearrangement invariant operator spaces in a similar way to the definition in the commutative case.

Definition 3.4.1 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a rearrangement invariant Banach operator space. The* fundamental function for E *is the function*

$$\Phi_E(t) = \sup\{\|\mathrm{p}\|_E : \tau(\mathrm{p}) \leq t,\ \mathrm{p} \in \mathcal{M}^p \cap E\}, \quad \text{for all } t \geq 0.$$

The fundamental function is non-decreasing, right-continuous on $(0, \infty)$ and $\Phi_E(0) = 0$. If $\mathcal{M}^p$ is nonatomic then $\Phi_E(t) > 0$ for all $t > 0$. If $\mathcal{M}^p$ is atomic and $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} = c > 0$ then $\Phi_E(t) = 0$ for all $0 \leq t < c$ and thus always right-continuous at 0. We write $\Phi_E(0^+)$ for $\lim_{t \to 0^+} \Phi_E(t)$.

We consider induced spaces, as we employ Proposition 3.3.9 in the proofs of Lemmas 3.4.2 and 3.4.3. We have noted in Section 3.3 that if $E(0, \infty)$ is a symmetric, rearrangement invariant Banach operator space, then so is $E(\mathcal{M})$ and furthermore, that under these conditions $E(\mathcal{M})$ is intermediate for the Banach couple $(L_1(\mathcal{M}), \mathcal{M})$. We will again use the fact that if $E(0, \infty)$ is fully symmetric then so is $E(\mathcal{M})$.

Put $\tau(\mathcal{M}^p) = \{\tau(\mathrm{p}) : \mathrm{p} \in \mathcal{M}^p\}$. Note that for $t > 0$

$$\begin{aligned} \Phi_{E(\mathcal{M})}(t) &= \sup\{\|\mathrm{p}\|_{E(\mathcal{M})} : \tau(\mathrm{p}) \leq t,\ , \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\} \\ &= \sup\{\|\chi_{(0,\tau(\mathrm{p}))}\|_{E(0,\infty)} : \tau(\mathrm{p}) \leq t,\ \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\} \end{aligned}$$

$$= \sup\{\Phi_{E(0,\infty)}(\tau(\mathrm{p})) : \tau(\mathrm{p}) \le t,\ \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\}$$
$$= \sup\{\Phi_{E(0,\infty)}(s) : s \le t,\ s \in \tau\,(\mathcal{M}^p \cap E(\mathcal{M}))\}.$$

In particular, $\Phi_{E(\mathcal{M})}(t) = \Phi_{E(0,\infty)}(t)$ if $t \in \tau\,(\mathcal{M}^p \cap E(\mathcal{M}))$. Hence if $\mathcal{M}^p$ is nonatomic, then the fundamental functions for $E(\mathcal{M})$ and $E(0,\infty)$ are the same. It also follows that $\Phi_{E(\mathcal{M})}(0^+) = 0$ if and only if $\Phi_{E(0,\infty)}(0^+) = 0$ whenever $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\} = 0$. Using this, Lemma 3.4.2 could be proved by considering the commutative space $E(0,\infty)$ only. We give the non-commutative version as it is straight forward.

Lemma 3.4.2 *Suppose $E(0,\infty)$ is a fully symmetric, rearrangement invariant Banach function space and $\mathcal{M}^p$ contains a nonatomic projection. If every projection in $E(\mathcal{M})$ with finite trace has order continuous norm, then $\Phi_{E(\mathcal{M})}$ is right-continuous at zero.*

Proof: Let p be the nonatomic projection in $\mathcal{M}^p$. There exists a (nonatomic) subprojection $\mathrm{q} \le \mathrm{p} \in \mathcal{M}^p$ with $\tau(\mathrm{q}) < \infty$. By hypothesis we have that $\mathrm{q} \in E_a \cap \mathcal{M}^p$. Without loss of generality we may assume that $\tau(\mathrm{q}) = 1$. Since q is nonatomic we can choose a decreasing sequence of subprojections of q,

$$\mathrm{q} \ge \mathrm{q}_1 \ge \mathrm{q}_2 \ge \cdots$$

such that

$$\tau(\mathrm{q}_n) = \frac{1}{n} \text{ for } \mathrm{n} = 1, 2, \ldots.$$

It follows that $\mathrm{q}_n \downarrow_n 0$ as $n \to \infty$. By Proposition 3.2.18 we know that $E(\mathcal{M})_a = E(\mathcal{M})_b$ and hence by Proposition 3.3.9 since $\mathrm{q} \in E(\mathcal{M})_a$ we have that

$$\|\mathrm{q}_n\|_{E(\mathcal{M})} \to_n 0$$

as $n \to \infty$. Let $\epsilon > 0$ be given. There exists an n_0 such that $\|\mathrm{q}_n\|_{E(\mathcal{M})} \le \epsilon$ for all $n \ge n_0$. If $\mathrm{p} \in \mathcal{M}^p$ such that $\tau(\mathrm{p}) \le \tau(\mathrm{q}_{n_0}) = \frac{1}{n_0}$ then

$$\mu_t(\mathrm{p}) = \chi_{[0,\tau(\mathrm{p}))}(t) \le \chi_{[0,\tau(\mathrm{q}))}(t) = \mu_t(\mathrm{q})$$

and hence $\|\mathrm{p}\|_{E(\mathcal{M})} \le \epsilon$ since $E(\mathcal{M})$ is rearrangement invariant. Thus $\Phi_{E(\mathcal{M})}(t) \le \epsilon$ for all $0 < t \le \tau(\mathrm{q}_{n_0}) = \frac{1}{n_0}$ and we have the result. □

Lemma 3.4.3 *Suppose $E(0,\infty)$ is a fully symmetric, rearrangement invariant Banach function space and $\mathcal{M}^p$ is atomic. If every projection in $E(\mathcal{M})$ with finite trace has order continuous norm then $\Phi_{E(\mathcal{M})}$ is right-continuous at zero.*

Proof: If $\mathcal{M}^p$ is atomic and $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} = c > 0$ then as noted above $\Phi_E(t) = 0$ for all $0 \leq t < c$ and thus always right-continuous at 0.

So suppose that $\mathcal{M}^p$ is atomic and $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} = 0$. There exists a sequence of atomic projections (e_n) in $\mathcal{M}^p$ such that $\tau(\mathrm{e}_n) \leq \frac{1}{2^{n+1}}$. For $k = 1, 2, \ldots$ put

$$\mathrm{q}_k = \bigvee_{n=k}^{\infty} \mathrm{e}_n.$$

Then (q_k) is a decreasing sequence of projections. For $\ell = k, k+1, \ldots$ put

$$\mathrm{q}_{k_\ell} = \bigvee_{n=k}^{\ell} \mathrm{e}_n$$

and note that

$$\tau(\mathrm{q}_{k_\ell}) = \tau\left(\bigvee_{n=k}^{\ell} \mathrm{e}_n\right) \leq \sum_{n=k}^{\ell} \tau(\mathrm{e}_n).$$

Since $\mathrm{q}_{k_\ell} \uparrow_\ell \bigvee_{n=k}^{\infty} \mathrm{e}_n = \mathrm{q}_k$ we have by normality of the trace that $\tau(\mathrm{q}_{k_\ell}) \uparrow_\ell \tau(\mathrm{q}_k)$ as $\ell \to \infty$. Now we have that

$$\tau(\mathrm{q}_k) = \lim_{\ell\to\infty} \tau(\mathrm{q}_{k_\ell}) \leq \lim_{\ell\to\infty} \sum_{n=k}^{\ell} \tau(\mathrm{e}_n) \leq \sum_{n=k}^{\infty} \frac{1}{2^{n+1}} \leq \frac{1}{2^k} \downarrow_k 0.$$

Thus $\mathrm{q}_k \downarrow_k 0$. The result now follows by the same method used in the proof of Lemma 3.4.2. □

The following theorem is known in the commutative setting for rearrangement invariant spaces on a σ-finite, continuous measure space. The equivalence $(i) \Leftrightarrow (ii)$ (in Theorem 3.4.4) appears for example in [BS88] Chapter 2, Theorem 5.5 and $(ii) \Leftrightarrow (iii)$ in [BS88] Chapter 1, Theorem 3.13.

Theorem 3.4.4 *Let $E(0,\infty)$ be a fully symmetric, rearrangement invariant Banach function space. Suppose that $\mathcal{M}^p$ is nonatomic. Then the following statements are equivalent.*

(i) $\Phi_{E(\mathcal{M})}(0^+) = 0$.

(ii) $E(\mathcal{M})_a = E(\mathcal{M})_b$.

(iii) Every projection with finite trace has order continuous norm.

(iv) $\Phi_{E(0,\infty)}(0^+) = 0$.

(v) $E(0,\infty)_a = E(0,\infty)_b$.

Proof: The equivalence *(ii)* $\Leftrightarrow$ *(iii)* is Proposition 3.2.18. We also have the implication *(iii)* $\Rightarrow$ *(i)*, by Lemmas 3.4.2 and 3.4.3. The equivalence *(i)* $\Leftrightarrow$ *(iv)* follows by the remark preceding Lemma 3.4.2 and *(iv)* $\Leftrightarrow$ *(v)* follows by the commutative theory noted above. Finally, *(v)* $\Rightarrow$ *(ii)* by Lemma 3.3.1. □

The following example shows that the hypothesis that $\mathcal{M}^p$ be nonatomic is necessary for the equivalent statements in Theorem 3.4.4 to hold, and that the condition $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\} = 0$ would not be enough.

Example 3.4.5 *Let* $E(0,\infty) = L_\infty(0,\infty)$ *and* $\mathcal{M} = \ell_\infty$ *as defined in Example 2.3.3. Then* $E(\mathcal{M}) = \ell_\infty$ *of the same example. Note that* $\mathcal{M}^p$ *is atomic and that* $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\} = 0$. *The equivalent statements of Theorem 3.4.4 do not hold, as was shown in Example 2.3.3.*

We mention a few examples of operator spaces for which the fundamental function is right-continuous at zero.

Examples 3.4.6 *(i) For the spaces* $L_p(0,\infty)$ *for* $1 \leq p < \infty$ *we have that the fundamental function* $\Phi_{L_p(0,\infty)}(0^+) = 0$. *If* $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p \cap L_p(\mathcal{M})\} = 0$, *in particular, if* $\mathcal{M}^p$ *is nonatomic, then the spaces* $L_p(\mathcal{M})$, $1 \leq p < \infty$ *have no singular functionals, that is,* $L_{p,a}(\mathcal{M}) = L_{p,b}(\mathcal{M}) = L_p(\mathcal{M})$ *and so*

$$L_p(\mathcal{M})^{*s} = L_{p,a}(\mathcal{M})^\perp = L_p(\mathcal{M})^\perp = \{0\}.$$

(ii) Similarly, if $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p \cap E(\mathcal{M})\} = 0$, *then the same results given in Examples 2.3.4 (iii) and (iv), hold for the non-commutative induced Lorentz and Marcinkiewicz spaces.*

If the fundamental function is not right-continuous at zero, then we will show that E is continuously embedded in $\mathcal{M}$. This was shown by [Con76] Proposition 3.2.5 in the commutative setting.

For a commutative analogue to Lemma 3.4.7 the reader may consult [BS88] Chapter 2, Proposition 1.3. Recall that the distribution function for $x \in \widetilde{\mathcal{M}}$ is defined as $d_t(x) = \tau\left(e_{(t,\infty)}(|x|)\right)$ for all $t > 0$.

Lemma 3.4.7 *Suppose* $0 \leq x_\alpha \uparrow_\alpha x$ *in* $\widetilde{\mathcal{M}}$. *Then*

$$d_t(x_\alpha) \uparrow_\alpha d_t(x)$$

for all $t > 0$.

Proof: Suppose $0 \leq x_\alpha \uparrow_\alpha x$ in $\widetilde{\mathcal{M}}$. By Proposition 1.2.3

$$\mu_t(x_\alpha) \uparrow_\alpha \mu_t(x)$$

for all $t > 0$. Since $\mu_t(x_\alpha) \uparrow_\alpha$ for all $t > 0$ we have that $d_t(x_\alpha) \uparrow_\alpha$ for all $t > 0$ by definition. We have to show that $d_t(x_\alpha) \uparrow_\alpha d_t(x)$ for all $t > 0$. Suppose for a contradiction that there exist some $t_0 > 0$ such that $d_{t_0}(x_\alpha) \not\uparrow_\alpha d_{t_0}(x)$. Then there exist an $\epsilon > 0$ such that

$$d_{t_0}(x) - d_{t_0}(x_\alpha) \geq \epsilon$$

for all α. Put $d_{t_0}(x) = s$. Note that $d_{t_0}(x_\alpha) \leq s - \epsilon$ for all α. By Proposition 1.1.4 (ix) we have that $\mu_{d_{t_0}(x_\alpha)}(x_\alpha) \leq t_0$ for all α. If $r \in (s-\epsilon, s)$, then $d_{t_0}(x_\alpha) < r$ for all α and since $\mu(x_\alpha)$ is decreasing

$$\mu_r(x_\alpha) \leq \mu_{d_{t_0}(x_\alpha)}(x_\alpha) \leq t_0$$

for all α. Finally note that

$$\mu(x) > t_0$$

on $(s-\epsilon, s)$, for if $\mu_r(x) \leq t_0$ for some $r \in (s-\epsilon, s)$ then

$$d_{\mu_r(x)}(x) \leq r < s = d_{t_0}(x)$$

which contradicts the fact that $d(x)$ is a decreasing function (the first inequality is Proposition 1.1.4(x)). This shows that

$$\mu(x_\alpha) - \mu(x) \geq \delta > 0$$

for some $\delta > 0$ and all α on $(s - \epsilon, s)$, which contradicts the fact that

$$\mu_t(x_\alpha) \uparrow_\alpha \mu_t(x)$$

for all $t > 0$. This proves the result. □

In the following we use the ideas from the proofs of the commutative results in [Con76] Propositions 3.1.10 (d) and 3.2.5 .

Lemma 3.4.8 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a rearrangement invariant Banach operator space. Then for $x \in E$*

$$\sup_{t>0} t\, \Phi_E \left(\tau(\mathrm{e}_{(t,\infty)}(|x|))\right) \leq \|x\|_E.$$

Proof: Suppose

$$x = \sum_{i=1}^{n} \alpha_i \mathrm{p}_i$$

is a simple operator in E with $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and the p_i's pairwise disjoint. Put $\alpha_0 = 0$ and

$$\mathrm{q}_i = \sum_{j=i}^{n} \mathrm{p}_j = \bigvee_{j=i}^{n} \mathrm{p}_j. \text{ Then } x = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) \mathrm{q}_i.$$

It follows from [DDP89b] Proposition 2.3 that

$$d_t(x) = \begin{cases} \tau(\mathrm{q}_k) & \text{if } \alpha_{k-1} \leq t < \alpha_k \text{ for } k = 1, 2, \ldots, n \\ 0 & \text{if } t \geq \alpha_n. \end{cases}$$

Hence

$$d_t(x) = \sum_{k=1}^{n} \tau(\mathrm{q}_k) \chi_{[\alpha_{k-1}, \alpha_k)}(t).$$

So

$$\begin{aligned} \sup_{t>0} t\, \Phi_E\left(d_t(x)\right) &= \sup_{0<t<\alpha_n} t\Phi_E \left(\sum_{k=1}^{n} \tau(\mathrm{q}_k) \chi_{[\alpha_{k-1}, \alpha_k)}(t) \right) \\ &= \max_{1\leq k\leq n} \sup_{\alpha_{k-1}\leq t<\alpha_k} t\, \Phi_E\left(\tau(\mathrm{q}_k)\right) \\ &= \max_{1\leq k\leq n} \alpha_k \Phi_E\left(\tau(\mathrm{q}_k)\right) \\ &= \max_{1\leq k\leq n} \|\alpha_k \mathrm{q}_k\|_E \\ &= \max_{1\leq k\leq n} \|\alpha_k(\mathrm{p}_k + \mathrm{p}_{k+1} + \ldots + \mathrm{p}_n)\|_E \\ &\leq \|\alpha_1 \mathrm{p}_1 + \alpha_2 \mathrm{p}_2 + \ldots + \alpha_n \mathrm{p}_n\|_E \\ &= \|x\|_E. \end{aligned}$$

Now suppose $x \in \mathcal{H}(\mathcal{M})$. Then by the spectral theorem there exists a net (x_α) of simple operators such that $0 \le x_\alpha \uparrow_\alpha |x|$. By Proposition 1.2.3 $\mu_t(x_\alpha) \uparrow_\alpha \mu_t(x)$ for all $t > 0$. By Lemma 3.4.7 we have that $d_t(x_\alpha) \uparrow_\alpha d_t(x)$ for all $t > 0$. Since Φ_E is continuous at $d_t(x)$ for all $t > 0$ we have that

$$\begin{aligned} \sup_{t>0} t\, \Phi_E\left(d_t(x)\right) &= \lim_\alpha \sup_{t>0} t\, \Phi_E\left(d_t(|x_\alpha|)\right) \\ &\le \lim_\alpha \|x_\alpha\|_E \\ &\le \|x\|_E. \end{aligned}$$

Now suppose $x \in E$. Then by Proposition 1.2.2 there exists a net $(x_\alpha) \in \mathcal{H}(\mathcal{M})$ such that $0 \le x_\alpha \uparrow_\alpha |x|$ and by Proposition 1.2.3 $\mu_t(x_\alpha) \uparrow_\alpha \mu_t(x)$ for all $t > 0$. The result now follows as in the preceding paragraph. □

Lemma 3.4.9 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a rearrangement invariant Banach operator space. Then for $x \in E$*

$$\sup_{t>0} \Phi_E(t)\mu_t(x) \le \sup_{t>0} t\, \Phi_E\left(\tau(e_{(t,\infty)}(|x|))\right).$$

Proof: Let $x \in E$ and suppose that $\sup_{t>0} t\, \Phi_E\left(d_t(x)\right) = K < \infty$. Fix $s > 0$ and let $\epsilon > 0$ be given. We may assume that $\Phi_E(s) > 0$.

Put $t = \frac{K+\epsilon}{\Phi_E(s)}$. Then $t > 0$ and

$$\Phi_E\left(d_t(x)\right) \le \frac{K}{K+\epsilon}\Phi_E(s) \le \Phi_E(s).$$

Since Φ_E is non-decreasing we have that $d_t(x) \le s$ and so

$$\begin{aligned} \Phi_E(s)\mu_s(x) &= \Phi_E(s) \inf\{\theta \ge 0 : d_\theta(|x|) \le s\} \\ &= \inf\{\alpha \ge 0 : d_{\frac{\alpha}{\Phi_E(s)}}(|x|) \le s\} \\ &\le K + \epsilon. \end{aligned}$$

This hold for arbitrary ϵ and so we have that

$$\sup_{s>0} \Phi_E(s)\mu_s(x) \le K = \sup_{t>0} t\, \Phi_E\left(d_t(x)\right).$$

□

Theorem 3.4.10 *Suppose $E \subseteq \widetilde{\mathcal{M}}$ is a rearrangement invariant Banach operator space and $\Phi_E(0^+) = c > 0$. Then E is continuously imbedded in $\mathcal{M}$.*

Proof: It follows from Lemmas 3.4.8 and 3.4.9 that for $x \in E$ we have that

$$\sup_{t>0} \Phi_E(t)\mu_t(x) \leq \|x\|_E$$

and thus

$$\mu_t(x) \leq \frac{1}{\Phi_E(t)}\|x\|_E \leq \frac{1}{c}\|x\|_E$$

for all $t > 0$ since $\Phi_E(.)$ is non-decreasing. Hence

$$\|x\|_\infty = \lim_{t\to 0+} \mu_t(x) \leq \frac{1}{c}\|x\|_E$$

for all $x \in E$. □

We give examples of operator spaces E for which $\Phi_E(0^+) > 0$. We use Theorem 3.4.4 to show that $E(\mathcal{M})_a \neq E(\mathcal{M})_b$ for induced spaces $E(\mathcal{M})$ for which $E(0,\infty)$ is Banach, fully symmetric and rearrangement invariant. Recall that a singular functional, φ, is by definition a functional such that for every $0 \neq \mathrm{p} \in E \cap \mathcal{M}^p$ there exists $0 \neq \mathrm{q} \leq \mathrm{p}$ such that $\varphi(\mathrm{q}) = 0$.

Examples 3.4.11 *(i) For the bounded linear operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$, we have that $\mathcal{B}(\mathcal{H})_a = \mathcal{C}(\mathcal{H})$, the compact operators on $\mathcal{H}$ which is not equal to $\mathcal{B}(\mathcal{H})_b$.*

(ii) Consider $\mathcal{H}(\mathcal{M})$ with $\mathcal{M}^p$ nonatomic. Then the fundamental function of $\mathcal{H}(\mathcal{M})$ is equal to the fundamental function of $(L_1 \cap L_\infty)(0,\infty)$ which equals $\max\{1,t\}$ *for $t > 0$. Thus*

$$\mathcal{H}(\mathcal{M})_a \neq \mathcal{H}(\mathcal{M})_b = \mathcal{H}(\mathcal{M}).$$

Note that $\mathcal{H}(\mathcal{M})_a = \{0\}$ by Corollary 3.3.6 and Example 2.3.5 (ii).

(iii) If $\mathcal{M}^p$ is nonatomic, then we have the same results as in Example 2.3.5 (iii) for the non-commutative Lorentz and Marcinkiewicz spaces.

Chapter 4

Duality for $\widetilde{\mathcal{M}}$

We investigated duality for Banach function spaces and Banach spaces of measurable operators in the preceding two chapters. In this chapter we investigate duality for $\widetilde{\mathcal{M}}$, and to a lesser degree $\widetilde{\mathcal{M}_0}$, both equipped with the measure topology, which are in general not Banach spaces.

It is natural to start with Köthe duality for $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}_0}$. Since these spaces are not necessarily contained in $\mathcal{G}(\mathcal{M})$, their Köthe dual may be trivial. In the first section we characterise the Köthe dual of $\widetilde{\mathcal{M}}$ and show that it produces continuous (with respect to the measure topology) linear functionals on $\widetilde{\mathcal{M}}$.

In the next section we define singular functionals on $\widetilde{\mathcal{M}}$ in such a way that the definition coincides with the definition for singular functionals on Banach operator spaces, for the case when $\widetilde{\mathcal{M}}$ equals $\mathcal{M}$.

We have seen that there are cases for which the measure topology is not locally convex, and we know that the Köthe dual of $\widetilde{\mathcal{M}}$ may be trivial. Yet, we shall show that if $\mathcal{M}^p$ is nonatomic (and the measure topology is not locally convex and the Köthe dual of $\widetilde{\mathcal{M}}$ is trivial) that the dual of $\widetilde{\mathcal{M}}$ is non-trivial and consist of singular functionals only. We describe the dual of $\widetilde{\mathcal{M}}$ in terms of the dual of $\mathcal{M}$ in this case.

If $\mathcal{M}^p$ is atomic and the traces of the projections in $\mathcal{M}$ are bounded away from zero, then $\widetilde{\mathcal{M}}$ equals $\mathcal{M}$ and a duality theory for this case exists (see [Tak79]). Finally, we characterise the dual of $\widetilde{\mathcal{M}}$ in the commutative setting in the case where $\mathcal{M}^p$ is atomic, $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p}, \mathrm{p} \in \mathcal{M}^p\} = 0$ and there exists a constant $K > 0$ such that $\sum_{\substack{\mathrm{p} \text{ atomic } \in \mathcal{M}^p \\ \tau(\mathrm{p}) < K}} \tau(\mathrm{p}) < \infty$.

4.1 The Köthe dual of $\widetilde{\mathcal{M}}$

A Köthe duality theory exists for rearrangement invariant normed operator spaces contained in $\mathcal{G}(\mathcal{M})$, due to [DDP93]. We extend this theory to spaces that are not necessarily contained in $\mathcal{G}(\mathcal{M})$ (and therefore the Köthe dual may be trivial), and that are not in general normed spaces (they are topological vector spaces equipped with the topology of convergence in measure). We shall concentrate on the Köthe duals of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}_0}$.

In considering the Köthe dual of $\widetilde{\mathcal{M}}$, we shall see that under different assumptions on $\mathcal{M}^p$, the Köthe dual of $\widetilde{\mathcal{M}}$ is either trivial, equal to $\mathcal{H}(\mathcal{M})$ or properly contained in $\mathcal{H}(\mathcal{M})$.

Definition 4.1.1 *If $A \subseteq \widetilde{\mathcal{M}}$ then the* Köthe dual of A, *denoted by* $\mathrm{A}^\times$, *is defined as the set of all $x \in \widetilde{\mathcal{M}}$ such that $\tau(|xy|) < \infty$ for all $y \in A$, i.e. the set of all $x \in \widetilde{\mathcal{M}}$ such that $xy \in L_1(\mathcal{M})$ for all $y \in A$.*

The Köthe dual $A^\times$ may be trivial, as we shall see for example in Corollary 4.1.9. However, if $A \subseteq \mathcal{G}(\mathcal{M})$, then $\mathcal{G}(\mathcal{M})^\times \subseteq A^\times$ and since $\mathcal{G}(\mathcal{M})^\times = \mathcal{H}(\mathcal{M})$, [DDP93] Proposition 5.2 (*ix*), we have that $\mathcal{H}(\mathcal{M}) \subseteq A^\times$.

Note that

$$\widetilde{\mathcal{M}}^\times \subseteq \widetilde{\mathcal{M}_0}^\times$$

since $\widetilde{\mathcal{M}_0} \subseteq \widetilde{\mathcal{M}}$. Also note that

$$\widetilde{\mathcal{M}}^\times \subseteq \mathcal{G}(\mathcal{M})^\times = \mathcal{H}(\mathcal{M})$$

since $\mathcal{G}(\mathcal{M}) \subseteq \widetilde{\mathcal{M}}$ and using [DDP93] Proposition 5.2 (*ix*).

The following properties of the Köthe dual of a module A in $\widetilde{\mathcal{M}}$ and proofs thereof, are similar to the properties of the Köthe dual of a properly symmetric Banach operator space in $\widetilde{\mathcal{M}}$ and their proofs, [DDP93] Proposition 5.2. We therefore state most of these without proof.

Lemma 4.1.2 *Let $A \subseteq \widetilde{\mathcal{M}}$ be a module over $\mathcal{M}$. Then the following properties hold.*

(i) If $y \in A^\times$ and $x \in \mathcal{M}$ then $xy, yx \in A^\times$.

(ii) $y \in A^\times$ if and only if $|y| \in A^\times$ if and only if $y^* \in A^\times$.

(iii) $A^\times = \{x \in \widetilde{\mathcal{M}} : \tau(|yx|) < \infty$ for all $y \in A\}$.

(iv) If $x \in A$ and $y \in A^\times$ then $\tau(xy) = \tau(yx)$.

(v) If $0 \leq x \in A$ and $0 \leq y \in A^\times$ then $x^{\frac{1}{2}}yx^{\frac{1}{2}}, y^{\frac{1}{2}}xy^{\frac{1}{2}} \in L_1(\mathcal{M})$ and $\tau(xy) = \tau(x^{\frac{1}{2}}yx^{\frac{1}{2}}) = \tau(y^{\frac{1}{2}}xy^{\frac{1}{2}}) \geq 0$.

Proof: Statement (i) uses the fact that A is a module over $\mathcal{M}$. Property (iv) follows by [Wes93] Theorem 6.7.6 and the proof of (v) is the latter part of the proof of [DDP93] Proposition 3.4. □

We shall use the following inequality (see [DDP93] Section 3) in what follows.

Lemma 4.1.3 *Let $x \in L_1(\mathcal{M})$ and $y \in \mathcal{M}$. Then*

$$|\tau(xy)| \leq \|y\| \; \tau(|x|).$$

Hence $L_1(\mathcal{M})$ is a module over $\mathcal{M}$. Let $\widetilde{\mathcal{M}}^*$ denote the set of continuous linear functionals on $\widetilde{\mathcal{M}}$ equipped with the topology of convergence in measure. We shall sometimes use $(\widetilde{\mathcal{M}}, \tau_{cm})^*$ to denote this set.

Definition 4.1.4 *Suppose φ is a linear functional on $\widetilde{\mathcal{M}}$. Then φ is called* order continuous *or* normal *if and only if $x_\alpha \downarrow_\alpha 0$ implies that $\varphi(x_\alpha) \rightarrow_\alpha 0$. The set of normal linear functionals that are also continuous on $\widetilde{\mathcal{M}}$ will be denoted by $\widetilde{\mathcal{M}}^{*n}$.*

It is easy to check that $\widetilde{\mathcal{M}}^{*n}$ is a vector subspace of $(\widetilde{\mathcal{M}}, \tau_{cm})^*$. Moreover, the continuous functionals on $\left(\widetilde{\mathcal{M}_0}, \tau_{cm}\right)$ are normal:

Proposition 4.1.5

$$\left(\widetilde{\mathcal{M}_0}, \tau_{cm}\right)^* = \widetilde{\mathcal{M}_0}^{*n}.$$

Proof: By definition, the space $\left(\widetilde{\mathcal{M}_0}, \tau_{cm}\right)^* \supseteq \widetilde{\mathcal{M}_0}^{*n}$.

Conversely, let $\varphi \in \left(\widetilde{\mathcal{M}_0}, \tau_{cm}\right)^*$ and let $x_\alpha \downarrow_\alpha 0$ in $\widetilde{\mathcal{M}_0}$. Then by Theorem 1.4.2 $\mu_t(x_\alpha) \downarrow_\alpha 0$ for all $t > 0$. Thus $\varphi(x_\alpha) \to 0$ since φ is continuous with respect to the topology of convergence in measure. Hence φ is normal. □

We show that an element in the Köthe dual of $\widetilde{\mathcal{M}}$ always produces a normal linear functional.

Proposition 4.1.6 *Let $x \in \widetilde{\mathcal{M}}^\times$. Then for $y \in \widetilde{\mathcal{M}}$*

$$\varphi_x(y) = \tau(xy)$$

defines a normal linear functional on $\widetilde{\mathcal{M}}$.

Proof: Suppose $x \in \widetilde{\mathcal{M}}^\times$. Then $\varphi_x(y) = \tau(xy)$, $y \in \widetilde{\mathcal{M}}$ is well-defined since by Lemma 4.1.3 we have that $|\tau(xy)| \leq \|1\|\, \tau(|xy|) < \infty$ for all $y \in \widetilde{\mathcal{M}}$. The functional φ_x is linear since the trace is linear on $L_1(\mathcal{M})$.

Suppose that $0 \leq x \in \widetilde{\mathcal{M}}^\times$ and let $y_\alpha \downarrow_\alpha 0$ in $\widetilde{\mathcal{M}}$. Then $x^{\frac{1}{2}} y_\alpha x^{\frac{1}{2}} \downarrow_\alpha 0$ in $\widetilde{\mathcal{M}}$. By Lemma 4.1.2 (v) we have that $x^{\frac{1}{2}} y_\alpha x^{\frac{1}{2}} \in L_1(\mathcal{M})$ for all α. Using this and normality of the trace, we have that $\tau\left(x^{\frac{1}{2}} y_\alpha x^{\frac{1}{2}}\right) \downarrow_\alpha 0$. Using Lemma 4.1.2 (v) once more, we have that

$$\varphi_x(y_\alpha) = \tau(x y_\alpha) = \tau\left(x^{\frac{1}{2}} y_\alpha x^{\frac{1}{2}}\right) \downarrow_\alpha 0.$$

Thus φ_x is normal.

If $x \in \widetilde{\mathcal{M}}^\times$, then we decompose x into a linear combination of positive parts. By the above each positive part defines a normal functional. A linear combination of normal functionals is again normal. □

Theorem 4.1.7 *Let $0 \neq x \in \widetilde{\mathcal{M}_0}^\times$. Then*

$$\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|x|)\} > 0.$$

Proof: We may assume that $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} = 0$, otherwise the result will be trivially true. Let $0 \neq x \in \widetilde{\mathcal{M}_0}^\times$ and suppose for a contradiction that $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|x|)\} = 0$. Without loss of generality we may assume that $0 < x \in \widetilde{\mathcal{M}_0}^\times$ and that $\tau\left(\mathrm{e}_{(0,\infty)}(x)\right) = \tau\left(\mathbf{r}(x)\right) \geq 1$. Note that $\mathrm{e}_{(0,\infty)}(x) = \mathbf{r}(x) > 0$ since $x \neq 0$.

Choose $0 \neq \mathrm{p}_n \leq \mathbf{r}(x)$ such that $\tau(\mathrm{p}_n) \leq \frac{1}{2^n}$ for $n \in \mathrm{N}$. As before by Lemma 4.1.2 (v) we have that $\tau(x\mathrm{p}_n) \geq 0$ for all n. Since $\mathrm{p}_n \leq \mathbf{r}(x)$ we have $\tau(x\mathrm{p}_n) > 0$ for all n. (Otherwise if $\tau(x\mathrm{p}_n) = 0$ for some n, then $x\mathrm{p}_n = 0$ and hence $x(\mathbf{r}(x) - \mathrm{p}_n) = x$ which contradicts that $\mathbf{r}(x)$ is the smallest projection such that $x\mathbf{r}(x) = x$.) Using Lemma 4.1.3 we have that $|\tau(x\mathrm{p}_n)| \leq \tau(|x\mathrm{p}_n|) < \infty$ for all n. Define for each $m \in \mathrm{N}$

$$y_m = \sum_{n=1}^{m} \frac{1}{\tau(x\mathrm{p}_n)}\mathrm{p}_n.$$

Each y_m is positive, bounded, has finite trace and (y_m) is increasing. Moreover, the sequence (y_m) is Cauchy in $\widetilde{\mathcal{M}}$. If $\epsilon > 0$ is given, there exists a $k > 0$ such that $\sum_{n=k+1}^{\infty} \tau(\mathrm{p}_n) \leq \epsilon$. Denoting the projection $1 - \bigvee_{n=k+1}^{\infty} \mathrm{p}_n = \bigwedge_{n=k+1}^{\infty} (1 - \mathrm{p}_n)$ by q, we have that

$$\tau(1 - \mathrm{q}) = \tau\left(\bigvee_{n=k+1}^{\infty} \mathrm{p}_n\right) \leq \sum_{n=k+1}^{\infty} \tau(\mathrm{p}_n) \leq \epsilon.$$

For $\ell > k$ we have that

$$\begin{aligned} \|(y_\ell - y_k)\mathrm{q}\| &= \left\|\left(\sum_{n=k+1}^{\ell} \frac{1}{\tau(x\mathrm{p}_n)}\mathrm{p}_n\right)\mathrm{q}\right\| \\ &= \left\|\sum_{n=k+1}^{\ell} \frac{1}{\tau(x\mathrm{p}_n)}\mathrm{p}_n\mathrm{q}\right\| \\ &\leq \sum_{n=k+1}^{\ell} \left\|\frac{1}{\tau(x\mathrm{p}_n)}\mathrm{p}_n\left((1 - \mathrm{p}_n)\mathrm{q}\right)\right\| \\ &= 0 \leq \epsilon. \end{aligned}$$

Therefore

$$y_\ell - y_k = \sum_{n=k+1}^{\ell} \frac{1}{\tau(x\mathrm{p}_n)}\mathrm{p}_n \in \widetilde{\mathcal{M}}(\epsilon, \epsilon).$$

Hence $(y_m) \subseteq \mathcal{M}$ is a Cauchy sequence in $\widetilde{\mathcal{M}_0}$. Following the proof of [Ter81] 1.28, there exists a τ-measurable operator y such that

$$y\xi = \lim_m y_m^*\xi = \lim_m y_m\xi$$

for all $\xi \in \mathcal{D}(y) = \bigcup_m \left(\bigvee_{i=m+1}^{\infty} \mathrm{p}_i\mathcal{H} \right)$. By the same proof we have that y is preclosed with closed extention $\overline{y} = y^*$ and that $y_m \to_m y^*$ for the measure topology. Hence since y is positive (each y_m is positive and (y_m) is increasing), we have that

$$\overline{y} = y^* = y,$$

that is, y is closed and $y \in \widetilde{\mathcal{M}}$.

Notice that $y \in \widetilde{\mathcal{M}_0}$ since $\widetilde{\mathcal{M}_0}$ is closed for the measure topology.

By continuity of the inner product and the fact that (y_m) is increasing, we have that

$$0 \leq y_m \uparrow_m y$$

in $\widetilde{\mathcal{M}_0}$. We write

$$y = \sum_{n=1}^{\infty} \frac{1}{\tau(x\mathrm{p}_n)} \mathrm{p}_n.$$

Note that $x^{\frac{1}{2}} y_m x^{\frac{1}{2}} \uparrow_m x^{\frac{1}{2}} y x^{\frac{1}{2}}$ by Proposition 1.2.1.

Using Lemma 4.1.2 (v), the continuity of multiplication in $\widetilde{\mathcal{M}}$, the normality, positive homogeniety and additivity of the trace and again Lemma 4.1.2 (v) we now have that

$$\begin{aligned}
\tau(xy) = \tau(x^{\frac{1}{2}} y x^{\frac{1}{2}}) &= \tau\left(x^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{1}{\tau(x\mathrm{p}_n)} \mathrm{p}_n \right) x^{\frac{1}{2}} \right) \\
&= \tau\left(\sum_{n=1}^{\infty} \frac{1}{\tau(x\mathrm{p}_n)} x^{\frac{1}{2}} \mathrm{p}_n x^{\frac{1}{2}} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{\tau(x\mathrm{p}_n)} \tau(x^{\frac{1}{2}} \mathrm{p}_n x^{\frac{1}{2}}) \\
&= \sum_{n=1}^{\infty} \frac{1}{\tau(x\mathrm{p}_n)} \tau(x\mathrm{p}_n) \\
&= \sum_{n=1}^{\infty} 1 = \infty
\end{aligned}$$

which contradicts $x \in \widetilde{\mathcal{M}_0}^{\times}$. Hence $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathbf{r}(x)\} > 0$. □

Corollary 4.1.8 *Let* $0 \neq x \in \widetilde{\mathcal{M}}^{\times}$. *Then*

$$\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|x|)\} > 0.$$

Proof: $\widetilde{\mathcal{M}}^{\times} \subseteq \widetilde{\mathcal{M}_0}^{\times}$. □

Corollary 4.1.9 *Suppose $\mathcal{M}^p$ is nonatomic. Then*

$$\widetilde{\mathcal{M}}^{\times} = \widetilde{\mathcal{M}_0}^{\times} = \{0\}.$$

Proof: Note that for $0 \neq x \in \widetilde{\mathcal{M}_0}$ or $0 \neq x \in \widetilde{\mathcal{M}}$ in this setting we always have that $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathbf{r}(x)\} = 0$. □

The following result together with Corollary 4.1.8 characterises a nonzero element in $\widetilde{\mathcal{M}}^{\times}$.

Theorem 4.1.10 *Suppose $0 \neq x \in \mathcal{H}(\mathcal{M})$ and*

$$\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|x|)\} = K > 0.$$

Then $x \in \widetilde{\mathcal{M}}^{\times}$.

Proof: Suppose $0 \neq x \in \mathcal{H}(\mathcal{M})$ and $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|x|)\} = K > 0$. Let $y \in \widetilde{\mathcal{M}}$. Put

$$\mathrm{q} = \bigvee_{\tau(\mathrm{p}) \geq K} \mathrm{p}.$$

Then as in Section 1.7, q is a central projection. Consider the reduced algebra $\widetilde{\mathcal{M}}_{\mathrm{q}}$. Since $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p_{\mathrm{q}}\} \geq K > 0$ we have that $\widetilde{\mathcal{M}}_{\mathrm{q}} = \mathcal{M}_{\mathrm{q}}$. Also note that $\mathrm{e}_{(0,\infty)}(|x|) \leq \mathrm{q}$ and thus $\mathrm{e}_{(0,\infty)}(|x|) = \mathrm{e}_{(0,\infty)}(|x|)\mathrm{q}$. Now

$$\begin{aligned}
|\tau(xy)| &= |\tau(x(1-\mathrm{q})y) + \tau(x\mathrm{q}y)| \\
&= \left|\tau\left(x\mathrm{e}_{(0,\infty)}(|x|)(1-\mathrm{q})y\right) + \tau(x\mathrm{q}y)\right| \\
&= \left|\tau\left(x\mathrm{e}_{(0,\infty)}(|x|)\mathrm{q}(1-\mathrm{q})y\right) + \tau(x\mathrm{q}y)\right| \\
&= |0 + \tau(x\mathrm{q}y\mathrm{q})| \\
&\leq \|\mathrm{q}y\mathrm{q}\|\, \tau(|x|) < \infty
\end{aligned}$$

by Lemma 4.1.3 since $\mathrm{q}y\mathrm{q} \in \widetilde{\mathcal{M}}_{\mathrm{q}} = \mathcal{M}_{\mathrm{q}}$ and $x \in \mathcal{H}(\mathcal{M})$. □

Corollary 4.1.11

$$0 \neq x \in \widetilde{\mathcal{M}}^{\times} \Leftrightarrow 0 \neq x \in \mathcal{H}(\mathcal{M}) \text{ and } \inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|x|)\} > 0.$$

We know the Köthe dual of $\widetilde{\mathcal{M}}$ is trivial when $\mathcal{M}^p$ is nonatomic. If $\mathcal{M}^p$ is atomic and $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} > 0$, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and $\mathcal{M}^\times = L_1(\mathcal{M})$. The following theorem shows that an element in the Köthe dual of $\widetilde{\mathcal{M}}$ always produces a continuous functional. The proof is very similar to the above proof of Theorem 4.1.10.

Theorem 4.1.12 *For $0 \neq a \in \widetilde{\mathcal{M}}^\times$ define $\varphi_a : \widetilde{\mathcal{M}} \to \mathbf{C}$ by*

$$\varphi_a(x) = \tau(ax).$$

Then $\varphi_a \in \widetilde{\mathcal{M}}^*$.

Proof: Suppose $0 \neq a \in \widetilde{\mathcal{M}}^\times$. By Corollary 4.1.11 we have that $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \leq \mathrm{e}_{(0,\infty)}(|a|)\} = K > 0$ for some constant K. Let $x_\alpha \to_\alpha 0$ in the measure topology in $\widetilde{\mathcal{M}}$. Put

$$\mathrm{q} = \bigvee_{\tau(\mathrm{p}) \geq K} \mathrm{p}.$$

Then as before we have that

$$|\tau(ax_\alpha)| \;\leq\; \|\mathrm{q}x_\alpha\mathrm{q}\|\,\tau(|a|) < \infty$$

by Lemma 4.1.3 since $\mathrm{q}x_\alpha\mathrm{q} \in \widetilde{\mathcal{M}}_\mathrm{q} = \mathcal{M}_\mathrm{q}$ for all α and $a \in \widetilde{\mathcal{M}}^\times \subseteq \mathcal{H}(\mathcal{M})$.

Now by continuity of multiplication in $\widetilde{\mathcal{M}}$ we have that $\mathrm{q}x_\alpha\mathrm{q} \to_\alpha 0$ in the measure topology in $\widetilde{\mathcal{M}}$. Since the measure topology equals the topology induced by the operator norm on $\widetilde{\mathcal{M}}_\mathrm{q} = \mathcal{M}_\mathrm{q}$, we have that $\|\mathrm{q}x_\alpha\mathrm{q}\| \to_\alpha 0$. Hence $|\tau(ax_\alpha)| \to_\alpha 0$ and so $\varphi_a(x_\alpha) \to_\alpha 0$. □

For the sequence space ℓ_∞, we have that the set of elements with order continuous norm is c_0, which equals $\ell_{\infty,0}$, the set of elements whose generalised singular functions decrease to zero, and that

$$\ell_{\infty,0}^* = c_0^* = \ell_1 = \ell_\infty^\times.$$

We also know that for $\mathcal{B}(\mathcal{H})$ with the operator norm and canonical trace, the set of elements with order continuous norm is $\mathcal{C}(\mathcal{H})$, the subspace of compact operators, which equals $\mathcal{B}(\mathcal{H})_0$, the set of elements in $\mathcal{B}(\mathcal{H})$ whose generalised singular functions decrease to zero. For this example we also have that

$$\mathcal{B}(\mathcal{H})_0{}^* = \mathcal{C}(\mathcal{H})^* = \mathcal{T}(\mathcal{H}) = \mathcal{B}(\mathcal{H})^\times$$

where $\mathcal{T}(\mathcal{H})$ is the set of trace class or nuclear operators, [Tak79] Chapter II, Theorems 1.6 and 1.8. We generalise this result in what follows, i.e., we show that $\mathcal{M}_0^* = \mathcal{M}^\times$ for von Neumann algebras for which the trace of the projections are bounded away from zero. (We conclude this after Corollary 4.1.16.)

Theorem 4.1.13 *Let $\mathcal{M}$ be a semifinite Von Neumann algebra with a faithful semifinite normal trace τ. Then*

$$\mathcal{M}_0{}^\times = \mathcal{M}^\times \; (= L_1(\mathcal{M}) = \mathcal{M}_*).$$

Proof: As noted above, $\mathcal{M}^\times \subseteq \mathcal{M}_0^\times$ since $\mathcal{M}_0 \subseteq \mathcal{M}$.

Conversely, let $x \in \mathcal{M}_0^\times = \{x \in \mathcal{M} : \tau(|xy|) < \infty \text{ for all } y \in \mathcal{M}_0\}$ and we show that $x \in \mathcal{M}^\times$ by showing that $\tau(|xy|) < \infty$ for all $y \in \mathcal{M}$. So suppose $y \in \mathcal{M}$.

Since $\mathcal{M}$ is semifinite, there exists a net $(\mathrm{p}_\alpha)_{\alpha \in A}$ in $\mathcal{M}^p$ such that $\mathrm{p}_\alpha \uparrow_\alpha 1$ in the strong operator topology with $\tau(\mathrm{p}_\alpha) < \infty$ for all $\alpha \in A$. By continuity of multiplication in the strong operator topology $|xy|^{\frac{1}{2}}\mathrm{p}_\alpha|xy|^{\frac{1}{2}} \uparrow_\alpha |xy|$ in the strong operator topology. Since the trace is normal $\tau(|xy|^{\frac{1}{2}}\mathrm{p}_\alpha|xy|^{\frac{1}{2}}) = \tau(|xy|\mathrm{p}_\alpha) \uparrow_\alpha \tau(|xy|)$. Let $xy = \mathrm{v}|xy|$ be the polar decomposition of xy. Hence $|xy| = \mathrm{v}^*xy$.

Define the linear functionals $\varphi_\alpha : \mathcal{M} \to \mathbf{C}$ by $\varphi_\alpha(z) = \tau(\mathrm{v}^*xz\mathrm{p}_\alpha)$ for $\alpha \in A$. Now for each $\alpha \in A$ and for every $z \in \mathcal{M}$ we have that $z\mathrm{p}_\alpha \in \mathcal{M}_0$ since by [FK86] 2.6 $\mu_t(z\mathrm{p}_\alpha) = 0$ for $t \geq \tau(\mathrm{p}_\alpha)$ and since $\tau(\mathrm{p}_\alpha) < \infty$ we have that $\mu_t(z\mathrm{p}_\alpha) \to 0$ as $t \to \infty$.

Now $\tau(|xz\mathrm{p}_\alpha|) < \infty$ for every $z \in \mathcal{M}$ and for each $\alpha \in A$ since $x \in \mathcal{M}_0^\times$. Using Lemma 4.1.3 we have that $|\tau(\mathrm{v}^*xz\mathrm{p}_\alpha)| \leq \tau(|xz\mathrm{p}_\alpha|)\|\mathrm{v}^*\|$ since $xz\mathrm{p}_\alpha \in L_1(\mathcal{M})$ and $\mathrm{v}^* \in \mathcal{M}$. Hence $|\varphi_\alpha(z)| = |\tau(\mathrm{v}^*xz\mathrm{p}_\alpha)| < \infty$ for every $z \in \mathcal{M}$ and for each $\alpha \in A$, that is, each functional is pointwise bounded. Since $\mathcal{M}$ is complete with respect to the operator norm, by the Uniform Boundedness Principle, [Hor66], the functionals are uniformly bounded in norm, i.e. $|\varphi_\alpha(z)| = |\tau(\mathrm{v}^*xz\mathrm{p}_\alpha)| \leq M\|z\|$ for all $z \in \mathcal{M}$ and $\alpha \in A$ where $M > 0$ is a constant.

Hence the net $(\tau(\mathrm{v}^*xy\mathrm{p}_\alpha)) = (\tau(|xy|\mathrm{p}_\alpha))$ is bounded from above (by $M\|y\|$) and hence its supremum $\tau(|xy|)$ is finite. This proves the theorem. □

Corollary 4.1.14 *Let* $\inf\{\tau(\mathrm{p}) : \tau(\mathrm{p}) \neq 0, \mathrm{p} \in \mathcal{M}^p\} > 0$. *Then*

$$\widetilde{\mathcal{M}_0}^\times = \mathcal{M}^\times = L_1(\mathcal{M}) = \mathcal{M}_*.$$

Proof: Note that under this hypothesis $\widetilde{\mathcal{M}} = \mathcal{M}$, $\widetilde{\mathcal{M}_0} = \mathcal{M}_0$. □

Theorem 4.1.15 *Let* $\mathcal{M}$ *be a semifinite Von Neumann algebra. Then*

$$\mathcal{M}_0{}^{\times} = \mathcal{M}_0{}^{*n}$$

Proof: It is easy to verify that $(\mathcal{M}_0, ||.||)$ is a properly symmetric Banach space contained in $\widetilde{\mathcal{M}}$. Then $\mathcal{M}_0{}^{*n} = \mathcal{M}_0{}^{\times}$ by [DDP93] Proposition 5.11. □

Note that we now have for a semifinite Von Neumann algebra $\mathcal{M}$ that

$$\mathcal{M}_0{}^{*n} = \mathcal{M}_0{}^{\times} = \mathcal{M}^{\times} = \mathcal{M}_*$$

by Theorems 4.1.13 and 4.1.15.

Corollary 4.1.16 *Let* $\inf\{\tau(\mathrm{p}) : \tau(\mathrm{p}) \neq 0, \mathrm{p} \in \mathcal{M}^p\} > 0$. *Then*

$$\widetilde{\mathcal{M}_0}^{\times} = \widetilde{\mathcal{M}_0}^{*n}$$

Proof: Note that under this hypothesis $\widetilde{\mathcal{M}} = \mathcal{M}$, $\widetilde{\mathcal{M}_0} = \mathcal{M}_0$ and $\tau_{cm} = ||\cdot||_\infty = ||\cdot||$ where the latter denotes the operator norm topology. □

Thus if $\inf\{\tau(\mathrm{p}) : \tau(\mathrm{p}) \neq 0, \mathrm{p} \in \mathcal{M}^p\} > 0$ we have that

$$\mathcal{M}_0{}^{*} = \widetilde{\mathcal{M}_0}^{*} = \widetilde{\mathcal{M}_0}^{*n} = \widetilde{\mathcal{M}_0}^{\times} = \widetilde{\mathcal{M}}^{\times} = \mathcal{M}^{\times}$$

by Proposition 4.1.5 and Corollaries 4.1.16 and 4.1.14. We know that

$$\left(\mathcal{M}^{\times}\right)^{*} = L_1(\mathcal{M})^{*} = \mathcal{M}$$

and hence we have that

$$\mathcal{M}_0{}^{**} = \left(\mathcal{M}^{\times}\right)^{*} = \mathcal{M}$$

if $\inf\{\tau(\mathrm{p}) : \tau(\mathrm{p}) \neq 0, \mathrm{p} \in \mathcal{M}^p\} > 0$.

4.2 Singular functionals on $\widetilde{\mathcal{M}}$

Recall from Chapter 3 that φ is a singular functional on a Banach operator space $E \subseteq \widetilde{\mathcal{M}}$ if for every $0 \neq \mathrm{p} \in \mathcal{M}^p$ there exists $0 \neq \mathrm{q} \leq \mathrm{p}$ such that $\varphi(\mathrm{q}) = 0$. When we consider induced spaces for which $E(\mathcal{M})_a = E(\mathcal{M})_b$, this definition for a singular functional φ on $E(\mathcal{M})$, simplifies to φ is singular if $\varphi(\mathrm{p}) = 0$ for every nonzero projection p with finite trace.

We need a definition for singular functionals on $\widetilde{\mathcal{M}}$, since $\widetilde{\mathcal{M}}$ equipped with the measure topology is not necessarily a Banach space. For the case where $\widetilde{\mathcal{M}}$ is in fact a Banach space, that is, when $\widetilde{\mathcal{M}} = \mathcal{M}$, the new definition has to coincide with the theory given in Chapter 3. In $\widetilde{\mathcal{M}}$ we have that $\widetilde{\mathcal{M}_0} = \widetilde{\mathcal{M}}_a$ by Proposition 1.4.5. We define $\widetilde{\mathcal{M}}_b$ as the closure of $\mathcal{H}(\mathcal{M})$ in $\widetilde{\mathcal{M}}$, that is, the closure in the measure topology. We know by [DDP93] Proposition 2.7 (a) that $\widetilde{\mathcal{M}}_b = \widetilde{\mathcal{M}_0}$ and hence

$$\widetilde{\mathcal{M}}_a = \widetilde{\mathcal{M}}_b = \widetilde{\mathcal{M}_0}.$$

The above paragraphs motivate the following definition for singular functionals on $\widetilde{\mathcal{M}}$.

Definition 4.2.1 *Suppose $\varphi \in \widetilde{\mathcal{M}}^*$. Then φ is called* singular *if $\varphi(\mathrm{p}) = 0$ for every $0 \neq \mathrm{p} \in \mathcal{M}^p$ with finite trace. The set of singular functionals on $\widetilde{\mathcal{M}}$ is denoted by $\widetilde{\mathcal{M}}^{*s}$.*

The set $\widetilde{\mathcal{M}}^{*s}$ is closed under addition and scalar multiplication and is therefore a vector subspace of $\widetilde{\mathcal{M}}^*$.

A functional $\varphi \in \widetilde{\mathcal{M}}^*$ is called *positive* if $\varphi(x) \geq 0$ for all $0 \leq x \in \widetilde{\mathcal{M}}$. If $0 \leq \varphi, \psi \in \widetilde{\mathcal{M}}^*$ we say $\varphi \leq \psi$ if and only if $\psi - \varphi$ is positive. The set of positive functionals in $\widetilde{\mathcal{M}}^{*s}$ is solid, in the sense that if $0 \leq \varphi \in \widetilde{\mathcal{M}}^{*s}$ and $0 \leq \psi \in \widetilde{\mathcal{M}}^{*s}$ with $\varphi \leq \psi$, then $\varphi \in \widetilde{\mathcal{M}}^{*s}$.

Proposition 4.2.2 *Suppose $\varphi \in \widetilde{\mathcal{M}}^*$. Then*

$$\widetilde{\mathcal{M}}^{*s} = \widetilde{\mathcal{M}_0}^{\perp}.$$

Proof: We first show that $\widetilde{\mathcal{M}_0}^{\perp} = \mathcal{H}(\mathcal{M})^{\perp}$.

We know that $\widetilde{\mathcal{M}_0}^{\perp} \subseteq \mathcal{H}(\mathcal{M})^{\perp}$ since $\mathcal{H}(\mathcal{M}) \subseteq \widetilde{\mathcal{M}_0}$. Conversely, suppose that $\varphi \in \mathcal{H}(\mathcal{M})^{\perp}$. Since $\widetilde{\mathcal{M}_0}$ is the closure of $\mathcal{H}(\mathcal{M})$ in the topology of convergence in measure in $\widetilde{\mathcal{M}}$ by [DDP93] Proposition 2.7 (a), $\varphi \in \widetilde{\mathcal{M}_0}^{\perp}$ and hence

$$\widetilde{\mathcal{M}_0}^{\perp} = \mathcal{H}(\mathcal{M})^{\perp}.$$

We now show that $\widetilde{\mathcal{M}}^{*s} = \mathcal{H}(\mathcal{M})^{\perp}$. Suppose $\varphi \in \widetilde{\mathcal{M}}^{*s}$. Let $x \in \mathcal{H}(\mathcal{M})$ be a self-adjoint operator. As in the proof of Lemma 3.2.17 there exists a sequence of simple operators (x_n) such that $\|x - x_n\| \to_n 0$ and $e_{(0,\infty)}(|x_n|) \leq e_{(0,\infty)}(|x|)$. Thus since $\mathcal{M}$ is continuously imbedded in $\widetilde{\mathcal{M}}$, Proposition 1.1.4 (*viii*), we have that $x_n \to_n x$ in the measure topology. Since φ vanishes on each projection with finite trace, it vanishes on every simple operator x_n and hence by continuity $\varphi(x) = 0$. Now let $x \in \mathcal{H}(\mathcal{M})$ and decompose it into its self-adjoint parts. By the above φ vanishes on each self-adjoint part and by linearity $\varphi(x) = 0$.

Conversely, suppose $\varphi(x) = 0$ for all $x \in \mathcal{H}(\mathcal{M})$. Since every projection with finite trace is in $\mathcal{H}(\mathcal{M})$ we have that φ is singular. Hence

$$\widetilde{\mathcal{M}}^{*s} = \mathcal{H}(\mathcal{M})^{\perp} = \widetilde{\mathcal{M}_0}^{\perp}.$$

□

Examples 4.2.3 *(i) If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is equipped with the canonical trace, then $\mathcal{M} = \widetilde{\mathcal{M}}$ and the measure topology is the topology induced by the operator norm. We have that $\mathcal{B}(\mathcal{H})_0 = \mathcal{C}(\mathcal{H})$, the subspace of compact operators and*

$$\mathcal{B}(\mathcal{H})^{*s} = \mathcal{C}(\mathcal{H})^{\perp}.$$

(ii) If $\tau(1) < \infty$ then $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}_0}$ and

$$\widetilde{\mathcal{M}}^{*s} = \widetilde{\mathcal{M}}^{\perp} = \{0\}.$$

(iii) If $\mathcal{M} = \ell_\infty = \widetilde{\mathcal{M}}$ and the topology of convergence in measure is the topology induced by the supremum norm, then $\ell_{\infty,0} = c_0$ and

$$\ell_\infty^* = \ell_1 \oplus c_0^{\perp} = \ell_\infty^{*n} \oplus \ell_\infty^{*s},$$

by [Köt69] Section 31.1 and Proposition 4.2.2.

4.3 The dual of $\widetilde{\mathcal{M}}$ for nonatomic $\mathcal{M}^p$

In this section we assume that $\mathcal{M}^p$ is nonatomic. We characterise the continuous dual of $\widetilde{\mathcal{M}}$ with the topology of convergence in measure, and see that it consists of singular functionals only. Recall from Section 1.4 the linear subspace of $\widetilde{\mathcal{M}}$ defined by

$$\mathcal{M}_0 = \widetilde{\mathcal{M}_0} \cap \mathcal{M}.$$

Note that $\mathcal{M}/\mathcal{M}_0$ is a Banach space with the canonical quotient norm. We show that the dual of $\widetilde{\mathcal{M}}$ with the measure topology is isomorphic to the dual of the quotient space $\mathcal{M}/\mathcal{M}_0$ in the sense of vector spaces. In fact, if we equip the dual of $\widetilde{\mathcal{M}}$ with a suitable norm, we can show that the isomorphism is isometric.

There are no continuous linear functionals on the subspace $\widetilde{\mathcal{M}_0}$ of $\widetilde{\mathcal{M}}$. We need this theorem to prove the main result.

Theorem 4.3.1 *Let $\mathcal{M}$ be a semifinite von Neumann algebra, τ a faithful semifinite normal trace and $\mathcal{M}^p$ nonatomic. Then*

$$(\widetilde{\mathcal{M}_0}, \tau_{cm})^* = \{0\}.$$

Proof: We have seen in Lemma 1.4.3 that $\widetilde{\mathcal{M}_0} = conv\widetilde{\mathcal{M}_0}(\epsilon, \delta)$ for any ϵ, $\delta > 0$.

Now suppose for a contradiction that $0 \neq \varphi \in (\widetilde{\mathcal{M}_0}, \tau_{cm})^*$. Then there exists a $0 \neq x \in \widetilde{\mathcal{M}_0}$ suct that $\varphi(x) \neq 0$. Let $\epsilon > 0$ be given. Since φ is continuous, there exists a $\delta_\epsilon > 0$ such that $|\varphi(y)| < \epsilon$ whenever $y \in \widetilde{\mathcal{M}_0}(\delta_\epsilon) := \widetilde{\mathcal{M}_0}(\delta_\epsilon, \delta_\epsilon)$.

Since $\widetilde{\mathcal{M}_0} = conv\widetilde{\mathcal{M}_0}(\delta_\epsilon)$ and $x \in \widetilde{\mathcal{M}_0}$, there exist positive real numbers $\alpha_1, \ldots, \alpha_n$ where $n \in \mathbf{N}$ with $\sum_{i=1}^n \alpha_i = 1$ and $x_1, \ldots, x_n \in \widetilde{\mathcal{M}_0}(\delta_\epsilon)$ such that

$$x = \sum_{i=1}^n \alpha_i x_i.$$

Now by linearity and continuity of φ we have that

$$\begin{aligned} |\varphi(x)| &= |\sum_{i=1}^n \alpha_i \varphi(x_i)| \\ &\leq \sum_{i=1}^n \alpha_i |\varphi(x_i)| \\ &< \sum_{i=1}^n \alpha_i \epsilon = \epsilon \end{aligned}$$

which contradicts the fact that $\varphi(x) \neq 0$. Thus we have proved the result. □

Corollary 4.3.2 *If* $\varphi \in (\widetilde{\mathcal{M}}, \tau_{cm})^*$ *then*

$$\varphi(\widetilde{\mathcal{M}_0}) = \{0\}, \text{ that is, } \varphi \in \widetilde{\mathcal{M}}^{*s}.$$

Proof: The restriction of φ to $\widetilde{\mathcal{M}_0}$ is continuous and it therefore follows by Theorem 4.3.1 that

$$\varphi(\widetilde{\mathcal{M}_0}) = \{0\}.$$

□

Recall from Section 1.4 that, for $x \in \widetilde{\mathcal{M}}$, $\mu_\infty(x) = \lim_{t\to\infty} \mu_t(x)$ and that μ_∞ is a $*$-algebra semi-norm on $\widetilde{\mathcal{M}}$ ([SW93] Section 2). It was shown in [SW93] Section 3 that $\mu_\infty(x + \widetilde{\mathcal{M}_0}) = \mu_\infty(x)$ defines a $*$-algebra norm on the quotient $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ and that $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ with its usual quotient topology derived from the measure topology, is isomorphic to $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)$. Note that we can identify the quotient norm on $\mathcal{M}/\mathcal{M}_0$ with the μ_∞ norm as defined above: In the proof of [SW93] Theorem 3.5, it was shown that $q(y + \mathcal{M}_0) = \inf_{z\in\mathcal{M}_0} \|y - z\| = \mu_\infty(y)$ for $y \in \mathcal{M}$. It follows that $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)$ is isometrically $*$-isomorphic to the Banach space $\mathcal{M}/\mathcal{M}_0$ with the canonical quotient norm, [SW93] Theorem 3.5. Therefore it follows that their dual spaces are isometrically isomorphic as Banach spaces, i.e., $(\mathcal{M}/\mathcal{M}_0, \text{ quotient norm })^* \cong \left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)^*$. We are now in a position to prove the main result, which is a generalisation of [DT90] Theorem 3.4.

Theorem 4.3.3 *Let* $\mathcal{M}^p$ *be nonatomic,* $\mathcal{M}$ *a semifinite von Neumann algebra and* τ *a faithful semifinite normal trace. Then*

$$\left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^* \text{ is isomorphic to } (\mathcal{M}/\mathcal{M}_0, \text{ quotient norm })^*.$$

Proof: By the above $(\mathcal{M}/\mathcal{M}_0, \text{ quotient norm })^*$ is isometrically isomorphic to $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)^*$. We now show that the dual of $\widetilde{\mathcal{M}}$ with the topology of convergence in measure is algebraically isomorphic to the dual of $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)$. Define

$$\Psi : \left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^* \to \left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)^*$$

by $\Psi(\varphi) = \widetilde{\varphi}$ where we put $\widetilde{\varphi}(x + \widetilde{\mathcal{M}_0}) = \varphi(x)$. We first show that

$$\widetilde{\varphi} \in \left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)^*.$$

(i) $\widetilde{\varphi}$ is well-defined:

Let $x + \widetilde{\mathcal{M}_0} = y + \widetilde{\mathcal{M}_0}$. Then $x - y \in \widetilde{\mathcal{M}_0}$ and $\varphi(x - y) = 0$ by Lemma 4.3.2, i.e. $\varphi(x) = \varphi(y)$ by linearity of φ and hence $\widetilde{\varphi}(x + \widetilde{\mathcal{M}_0}) = \widetilde{\varphi}(y + \widetilde{\mathcal{M}_0})$ by Definition of $\widetilde{\varphi}$.

(ii) $\widetilde{\varphi}$ is linear since φ is linear.

(iii) $\widetilde{\varphi}$ is continuous:

Note that $\varphi = \widetilde{\varphi} \circ j$ where $j : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ is the canonical quotient mapping. Let U be an open set in $\mathbf{C}$. Then $\varphi^{-1}(U)$ is open in $(\widetilde{\mathcal{M}}, \tau_{cm})$ since φ is continuous, i.e. $j^{-1}\left(\widetilde{\varphi}^{-1}(U)\right)$ is open in $(\widetilde{\mathcal{M}}, \tau_{cm})$. However j is an open mapping and therefore $\widetilde{\varphi}^{-1}(U)$ is open in $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)$, since $\left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right) \cong \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ with its usual quotient topology, as remarked in the paragraph preceding Theorem 4.3.3.

Now it remains to prove that Ψ is bijective and linear.

To show that Ψ is injective we suppose that $\Psi(\varphi_1) = \Psi(\varphi_2)$. This means $\widetilde{\varphi}_1 = \widetilde{\varphi}_2$, i.e. $\widetilde{\varphi}_1(x + \widetilde{\mathcal{M}_0}) = \widetilde{\varphi}_2(x + \widetilde{\mathcal{M}_0})$ for all $x + \widetilde{\mathcal{M}_0} \in \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$. Hence $\varphi_1(x) = \varphi_2(x)$ for all $x \in \widetilde{\mathcal{M}}$ and hence $\varphi_1 = \varphi_2$.

That Ψ is linear also follows easily for if α and β are scalars and $\varphi_1, \varphi_2 \in (\widetilde{\mathcal{M}}, \tau_{cm})^*$ then

$$\begin{aligned}
\Psi\left((\alpha\varphi_1 + \beta\varphi_2)(x)\right) &= (\widetilde{\alpha\varphi_1 + \beta\varphi_2})(x + \widetilde{\mathcal{M}_0}) \\
&= (\alpha\varphi_1 + \beta\varphi_2)(x) \\
&= \alpha\varphi_1(x) + \beta\varphi_2(x) \\
&= \alpha\widetilde{\varphi}_1(x + \widetilde{\mathcal{M}_0}) + \beta\widetilde{\varphi}_2(x + \widetilde{\mathcal{M}_0}) \\
&= \alpha\Psi(\varphi_1)(x) + \beta\Psi(\varphi_2)(x)
\end{aligned}$$

for all $x \in \widetilde{\mathcal{M}}$. We show the surjectivity of Ψ.

Suppose $\widetilde{\varphi} \in \left(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}, \mu_\infty\right)^*$. We use the canonical quotient map $j : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ to define $\varphi : \widetilde{\mathcal{M}} \to \mathbf{C}$ by setting

$$\varphi(x) = (\widetilde{\varphi} \circ j)(x) = \widetilde{\varphi}(x + \widetilde{\mathcal{M}_0}).$$

Then φ is linear since $\widetilde{\varphi}$ and j are. Also φ is continuous since if we suppose that U is an open set in $\mathbf{C}$, then by continuity of $\widetilde{\varphi}$, $\widetilde{\varphi}^{-1}(U)$ is open in $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ with the canonical quotient topology. Hence by continuity of j we have $j^{-1}\left(\widetilde{\varphi}^{-1}(U)\right) = \varphi^{-1}(U)$ is open in $\widetilde{\mathcal{M}}$. This proves the result. □

Our objective is now to define a norm on the dual of $\widetilde{\mathcal{M}}$ in such a manner as to obtain an isometry for the isomorphism Υ from $\left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ to $(\mathcal{M}/\mathcal{M}_0,\ \textit{quotient norm})^*$ obtained in Theorem 4.3.3. We know

$$\|\varphi\| = \|\Upsilon(\varphi)\|,\ \varphi \in \widetilde{\mathcal{M}}^*$$

defines a norm on $\widetilde{\mathcal{M}}^*$ (since Υ is an isomorphism). Using Theorem 4.3.3 we define Υ explicitly: For $\varphi \in (\widetilde{\mathcal{M}}, \tau_{cm})^*$

$$\Upsilon(\varphi) = \psi \text{ where } \psi(x + \mathcal{M}_0) = \varphi(x) \text{ for all } x \in \mathcal{M}.$$

Thus for $\varphi \in (\widetilde{\mathcal{M}}, \tau_{cm})^*$ we have that

$$\begin{aligned} \|\varphi\| &= \|\Upsilon(\varphi)\| \\ &= \sup_{\substack{\|x+\mathcal{M}_0\|\leq 1 \\ x\in\mathcal{M}}} |\varphi(x)| \\ &= \sup_{\substack{\mu_\infty(x)\leq 1 \\ x\in\mathcal{M}}} |\varphi(x)| \end{aligned}$$

since we know that the Banach space $(\mathcal{M}/\mathcal{M}_0,\ \textit{quotient norm})^*$ is isometrically isomorphic to $(\mathcal{M}/\mathcal{M}_0, \mu_\infty)^*$.

Let $\varphi \in \left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ and $x \in \widetilde{\mathcal{M}}$. Then

$$x = x_0 + x_\infty$$

with $x_0 \in \widetilde{\mathcal{M}_0}$ and $x_\infty \in \mathcal{M}$ (see Section 1.4). By linearity of φ and then by using Lemma 4.3.2 we have that

$$\varphi(x) = \varphi(x_0) + \varphi(x_\infty) = \varphi(x_\infty).$$

Also for $x \in \widetilde{\mathcal{M}}$ we have that

$$\mu_\infty(x) = \mu_\infty(x_\infty)$$

by construction of x_∞ when we put

$$x_\infty = x\mathrm{e}_{[0,\mu_\infty(x))}(x) + \mu_\infty(x)\mathrm{e}_{[\mu_\infty(x),\infty)}(x)$$

(since $\mu_\infty(x) = \mu_\infty(|x|)$, it suffices to consider $x \geq 0$ in $\widetilde{\mathcal{M}}$). Hence

$$\mu_\infty(x) \leq 1 \text{ if and only if } \mu_\infty(x_\infty) \leq 1$$

and so

$$\begin{aligned} \|\varphi\| &= \sup_{\substack{\mu_\infty(x)\leq 1 \\ x\in\mathcal{M}}} |\varphi(x)| \\ &= \sup_{\substack{\mu_\infty(x)\leq 1 \\ x\in\widetilde{\mathcal{M}}}} |\varphi(x)|. \end{aligned}$$

Thus if we equip $\left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ with the norm

$$\|\varphi\| = \sup_{\mu_\infty(x)\leq 1} |\varphi(x)|$$

for $\varphi \in \left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ then

$$\left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^* \cong (\mathcal{M}/\mathcal{M}_0, \text{ quotient norm })^*$$

as Banach spaces. We state this result in the following corollary.

Corollary 4.3.4 $\left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ *is isometrically isomorphic to* $(\mathcal{M}/\mathcal{M}_0,$ *quotient norm*$)^*$, *where the norm on* $\left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ *is defined as*

$$\|\varphi\| = \sup_{\mu_\infty(x)\leq 1} |\varphi(x)|$$

for $\varphi \in \left(\widetilde{\mathcal{M}}, \tau_{cm}\right)^*$ *and the norm on the dual space of the quotient is the natural norm induced by the quotient norm.*

4.4 The dual of $\widetilde{\mathcal{M}}$ for atomic $\mathcal{M}^p$

For atomic $\mathcal{M}^p$ we will first consider the case when $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} > 0$. Then we know that $\widetilde{\mathcal{M}} = \mathcal{M}$ and that the measure topology equals the norm topology on

$\mathcal{M}$. Therefore we apply the known theory on duality for von Neumann algebras. We give an outline.

The dual of $\mathcal{M}$ has a direct sum decomposition into a subspace of normal functionals and a subspace of singular functionals. We give a brief description of how these subspaces are constructed. For details we refer the reader to [Tak79] Chapter 3, Section 2.

We consider the von Neumann algebra $\mathcal{M}$ to be a C^*-algebra with the identity representation $\{\pi, \mathcal{H}\}$. (The identity representation π is the action of $\mathcal{M}$ onto the underlying Hilbert space $\mathcal{H}$.) The predual $\mathcal{M}_*$ of $\mathcal{M}$, is the Banach space of all σ-weakly continuous linear functionals on $\mathcal{M}$ and is a two-sided invariant subspace of the continuous dual $\mathcal{M}^*$ of $\mathcal{M}$ associated with $\{\pi, \mathcal{H}\}$. Then by [Tak79] Theorem III.2.7 $\mathcal{M}_*$ is of the form $\mathcal{M}^*\mathrm{z}$ where z is a central projection in the bidual $\mathcal{M}^{**}$ of $\mathcal{M}$. (This bidual is isometric to the universal enveloping von Neumann algebra of $\mathcal{M}$ and the projection z is called the support projection of $\mathcal{M}_*$ associated with π and is unique.) $\mathcal{M}^*\mathrm{z}$ is to be understood as an action of the central projection z on the dual $\mathcal{M}^*$ in the following way: if $\varphi \in \mathcal{M}^*$ then

$$\langle x, \varphi\mathrm{z}\rangle = \langle \mathrm{z}x, \varphi\rangle$$

for all $x \in \mathcal{M}^{**}$ where φ is seen as a σ-weakly continuous linear functional on $\mathcal{M}^{**}$ and thus $\langle \mathrm{z}x, \varphi\rangle$ makes sense. Now we write $\mathcal{M}$ as the direct sum

$$\mathcal{M}^*\mathrm{z} \oplus \mathcal{M}^*(1-\mathrm{z}) = \mathcal{M}_* \oplus \mathcal{M}^*(1-\mathrm{z})$$

and each functional in $\mathcal{M}_*$ is called *normal* and the functionals in $\mathcal{M}^*(1-\mathrm{z})$ are called *singular*. Then a singular functional is characterised as our Definition 3.1.2 for Banach operator spaces, i.e., φ is singular on $\mathcal{M}$ if for every $0 \neq \mathrm{p} \in \mathcal{M}^p$ there exists $0 \neq \mathrm{q} \leq \mathrm{p}$ suct that $\varphi(\mathrm{q}) = 0$, [Tak59] or [Ake67] Proposition II.1. However, as we noted in Section 4.2, in this setting we know that $\mathcal{M}_a = \mathcal{M}_b$. Hence by Lemma 3.3.11 we have that a singular functional vanishes on every projection with finite trace. This agrees with our Definition 4.2.1 for singular functionals on $\widetilde{\mathcal{M}}$.

We now consider the case when $\mathcal{M}^p$ is atomic, $\inf\{\tau(\mathrm{p}) : 0 \neq \mathrm{p} \in \mathcal{M}^p\} = 0$ and there exists $K > 0$ such that $\sum_{\substack{\mathrm{p}\text{ atomic} \\ \tau(\mathrm{p})<K}} \tau(\mathrm{p}) < \infty$. We characterise the dual of $\widetilde{\mathcal{M}}$ in the commutative setting only.

If (X, Σ, m) is a localizable, purely atomic measure space such that there exists a $K > 0$

with $\sum_{\substack{m(A)<K \\ A \text{ atomic} \in \Sigma}} m(A) < \infty$, then by Theorem 1.7.1 we have that

$$\left(\widetilde{L_\infty}(X, \Sigma, m), \tau_{cm}\right) = \left(L_0(E, \Sigma_E, m_E) \oplus L_\infty(X \setminus E, \Sigma_{X\setminus E}, m_{X\setminus E}), \nu_p \times \|.\|_\infty\right)$$

where ν_p denotes the pointwise topology. In the commutative setting atoms are mutually orthogonal and functions are constant on an atom. Therefore we may think of $L_0(E, \Sigma_E, m_E)$ as a weighted sequence space over a countable index set, say $\mathbf{N}$. We will write $L_0(\mathbf{N})$ for short. It follows that the dual of $L_0(\mathbf{N})$ with the pointwise topology is the space of all sequences that have finitely many nonzero coordinates, usually denoted by $\Phi(\mathbf{N}) = L_0(\mathbf{N})^*$ $(= L_0(\mathbf{N})^{*n})$.

Similarly we may think of $L_\infty(X \setminus E, \Sigma_{X\setminus E}, m_{X\setminus E})$ as the weighted sequence space $\ell_\infty(\Gamma)$ over an index set Γ with the usual supremum norm. Then we know that the dual of $(\ell_\infty(\Gamma), \|.\|_\infty)$ is $\ell_1(\Gamma) \oplus c_0(\Gamma)^\perp = \ell_\infty(\Gamma)^{*n} \oplus \ell_\infty(\Gamma)^{*s}$.

Thus the dual of the direct sum is the direct sum of the duals, [RR64] Proposition 2.6, since the product and direct sum topologies coincide when the product is taken over a finite number of spaces. We can therefore identify the dual of $\left(\widetilde{L_\infty}(X, \Sigma, m), \tau_{cm}\right)$ as $\Phi(\mathbf{N}) \oplus \ell_1(\Gamma) \oplus c_0(\Gamma)^\perp$.

4.5 Summary

We present a summary in the following table.

$\mathcal{M}^p$	Nonatomic	Atomic, $\inf_{\substack{p \in \mathcal{M}^p \\ \tau(p) \neq 0}} \tau(p) > 0$	Atomic, $\inf_{\substack{p \in \mathcal{M}^p \\ \tau(p) \neq 0}} \tau(p) = 0$, $\exists K > 0$: $\sum_{\substack{\tau(p)<K \\ p \text{ atomic}}} \tau(p) < \infty$	Atomic, $\inf_{\substack{p \in \mathcal{M}^p \\ \tau(p) \neq 0}} \tau(p) = 0$, $\exists (p_n) \subseteq \mathcal{M}^p$ with $\tau(p_n) \downarrow_n 0$: $\sum_{n=1}^{\infty} \tau(p_n) = \infty$
τ_{cm}	not locally convex	$= \|.\|$ locally convex	locally convex	(comm.- not locally convex)
$\widetilde{\mathcal{M}_0} =$	$\bigcap_{\epsilon,\delta>0} conv \widetilde{\mathcal{M}}(\epsilon, \delta)$	$\mathcal{M}_0$		
$\widetilde{\mathcal{M}_0}^{\times}$	$= \{0\}$	$= L_1(\mathcal{M}) = \mathcal{M}_*$ $= \mathcal{M}^{\times} = \mathcal{M}_0{}^*$ $= \mathcal{M}_0{}^{*n} = \mathcal{M}^{*n}$	$= \widetilde{\mathcal{M}}^{\times}$	$\supseteq \widetilde{\mathcal{M}}^{\times}$
$\widetilde{\mathcal{M}_0}^{*} = \widetilde{\mathcal{M}_0}^{*n}$	$= \{0\}$	$= L_1(\mathcal{M}) = \mathcal{M}_*$ $=$ etc.		
$\widetilde{\mathcal{M}}$		$= \mathcal{M}$	$= \widetilde{\mathcal{M}_q} \oplus \widetilde{\mathcal{M}_{1-q}}$ where $q = \bigvee_{\substack{\tau(p)<K \\ p \text{ atomic}}} p$	
$\widetilde{\mathcal{M}}^{\times}$	$= \{0\}$	$= L_1(\mathcal{M}) = \mathcal{M}_*$ $=$ etc.	$= \widetilde{\mathcal{M}_0}^{\times}$	$\subseteq \widetilde{\mathcal{M}_0}^{\times}$
$\widetilde{\mathcal{M}}^*$	$\simeq (\mathcal{M}/\mathcal{M}_0)^*$ (quotient norm) $= \mathcal{M}_0{}^{\perp}$	$=$ $\mathcal{M}^* z \oplus \mathcal{M}^*(1-z)$ $=$ $\mathcal{M}_* \oplus \mathcal{M}^*(1-z)$		

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