TOPICS OF ENTROPY IN LOCALLY COMPACT ABELIAN GROUPS

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Abstract: The present MSc thesis discusses some notions of abelian group theory in connection with recent topics of topological entropy of locally compact abelian groups. It has been used the reference of [D. J. S. Robinson, A Course in the Theory of Groups, Springer, 1996, New York], which is a classical textbook in group theory. A list of exercises, relevant to our purposes, has been selected, in order to introduce some recent aspects of topological entropy of locally compact abelian groups. It is worth to mention that many of the exercises, which have been solved in the present thesis, are subject to technicalities which require the application of theorems of decomposition for abelian groups. Therefore the logic of the solutions allows us to describe the topological entropy in presence of an appropriate factorization.

Contents

Introduction

In the present MSc thesis we discuss the notion of topological entropy in connection with abelian groups. Topological entropy has had a reasonable amount of time in literature and it has been discussed in many different settings, for instance, it was discussed in metric spaces, in general compact spaces (see [14]), and later taken into uniform spaces (see [13]).

In order to discuss properly of the notion of topological entropy for abelian groups, we spend some time reviewing classical concepts in group theory, relying heavily on a well known textbook in group theory, namely [24], and offering some solutions of technical exercises in this textbook.

It turns out to be very useful to have a number of facts on abelian groups, available for the formalization of the concepts which are illustrated in the final part of the present thesis. On the other hand, the choice of the exercises of [24] has independent interest due to their difficulty and to the strategy of solution which are used.

We describe the structure of the present thesis. In Chapter 1 we relied heavily on [24]; this chapter contained the basic concepts of group theory and classical concepts of free groups and group presentations, and these notions are thoroughly discussed in [24], and [24] is also where we extracted our list exercises that we discuss from Chapter 1 and 2. The reference [9] is a well known abelian group theory textbook, so it was of course used as a further reference in Chapter 2 together with [24] being the main reference.

When we introduced the basic concepts of measure theory in Chapter 3, we used [4] together with [15] and [23]; [4] is a well-known textbook in measure theory and [23] is an introductory textbook in general topology while [15] is a well-known textbook in algebraic topology.

The notion of entropy, which is central in Chapter 3, was first discussed in [2] for compact spaces and in uniform spaces in [13]. The basic definitions of entropy that appear in Section 3.2 were taken from [6]. The computation of the entropy of $\mathbb{R}^m$ given in Section 3.2, appears in [2, 3, 7, 16].

Dual to the notion of varieties of groups discussed in [21] and [24], we find the generalized case of topological groups in [17, 18, 19, 20]. We speculate that the notion of variety of groups for LCA groups (or more generally for topological groups, see also [12]) may help to identify classes of groups with finite entropy, but we do not touch this problem in the present thesis. Mostly we refer to [11] for compactly generated LCA groups and periodic locally compact groups, illustrating results in [6, 16, 22, 25] that the question of the entropy of these groups is still open in general.

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Notation

Symbols	Description
$\mathfrak{X}$	Classes of groups
$[x, y]$,	Commutator of two elements in a group; i.e.: $x^{-1}y^{-1}xy$
$H_1H_2\cdots H_n$	Product of subgroups in a group
$H_1 + H_2 + \cdots + H_n$	Sum of subgroups in a group
$\langle X_\lambda \mid \lambda \in \Lambda\rangle$	Subgroup generated by subsets X_λ of a group
$\langle X \mid R\rangle$	Group presented by generators X and relators R
$d(G)$	Minimum number of generators of the finitely generated group G
$\dim(G)$	$\max\{d(H) \mid H$ is a finitely generated subgroup of $G\}$
$r_p(G), r_0(G), r(G)$	p-rank, torsion-free rank (Prüfer) rank of an abelian group G
$\Omega^n(G)$	Subgroup of a group G consisting of all g^n with $g \in G$
$\Omega_n(G)$	Subgroup of a group G consisting of all $g \in G$ such that $g^n = 1$.
$\lvert G\rvert$, $o(g)$	Order of the group G, order of the element $g \in G$
$[G : H]$	Index of the subgroup H in the group G
H^G, H_G, $Z(G)$	Normal closure, core of H in G, centre of G
$\mathrm{Aut}\ G$, $\mathrm{End}\ G$	Group of automorphisms, ring of endomorphisms of the group G
$H \times K$, $H \oplus K$	Direct product, direct sum of the groups H and K
$G' = [G, G]$	Derived subgroup of a group G
$N \rtimes H$	Semidirect product with normal factor N and cofactor H in a group G
S_n, A_n	Symmetric, alternating groups of degree n
$\mathbb{Z}$, $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	Sets of integers, prime numbers, rational numbers, real numbers, complex numbers
$\mathbb{Z}(n) := \mathbb{Z}/n\mathbb{Z}$	Integers modulo n
$\mathbb{Z}_p, \mathbb{Q}_p$	p-adic integers, p-adic rationals
$\mathrm{GL}(n, F)$	Group of $n \times n$ nonsingular matrices over the field F
max, min	Maximal, minimal conditions on subgroups
LCA group	Locally compact abelian group
$\mathrm{comp}_0(G)$	Connected component of the identity in an LCA group G
$B(G)$	The largest compactly covered subgroup of G
$\mathrm{rk}_{\mathbb{Z}_p}(G)$, $r_p(G)$	$\mathbb{Z}_p$-rank, p-rank of an LCA group G

Chapter 1

Some classical results on free groups

1.1 Basic Concepts

Given two groups A and B with multiplicative operations, a map $f : A \to B$ is a homomorphism (of groups), if it preserves the operations of A and B, that is,

$$f(x\ y) = f(x)\ f(y), \quad \text{for all} \ \ x, y \in A. \tag{1.1.1}$$

It is easy to see that

$$\text{Ker}\ f = f^{-1}(1) = \{a \in A \ \mid \ f(a) = 1\}, \tag{1.1.2}$$

that is, the counterimage of the neutral element of B, is a subgroup of A and similarly one can see that

$$f(A) = \text{Im}\ f = \{f(a) \ \mid \ a \in A\}, \tag{1.1.3}$$

that is, the image of A via f, turns out to be a subgroup of B. Injective homomorphisms (of groups) are called *monomorphisms*, surjective homomorphisms (of groups) are called *epimorphisms*, bijective homomorphisms (of groups) are called *isomorphisms*. It is not difficult to check that a homomorphism f is injective if and only if Ker $f = 1$, and that f is surjective if and only if Im $f = B$.

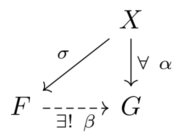
The classical theorems of homomorphism of groups can be found in [24, Theorems 1.4.3 to 1.4.5, p.19] and they describe some elementary facts on subgroups and quotients of groups in presence of homomorphisms.

Note that the symbol $\langle X \rangle$ denotes the *subgroup generated by* X, where X is just a set in a group G. Denoting by $\mathcal{L}(G)$ the set of all subgroups of G and according to [24, p.9], $\langle X \rangle$ is the smallest subgroup of G containing X, that is,

$$\langle X \rangle = \bigcap_{\substack{X \subseteq H \\ H \in \mathcal{L}(G)}} H \tag{1.1.4}$$

Of course, $X \subseteq \langle X \rangle$, but the equality holds if and only if X is a subgroup (not only a set), that is, X contains the neutral element and the inverse x^{-1} of each element $x \in X$. In the present chapter we present some notions on free groups. The definition is well known and is reported below.

Definition 1.1.1 (See [24], pp.44–48)**.** *Let F be a group, X a nonempty set, and $\sigma : X \to F$ a function. Then F is said to be free on X if for each function $\alpha : X \to G$ to group G, there exists a corresponding unique homomorphism $\beta : F \to G$ such that the following diagram is commutative:*



that is, such that $\alpha = \beta \circ \sigma$. Moreover, the rank of F is the cardinality $|X|$ of X.

Due to the relevance of the set X in the previous definition, it is also used the terminology *free group F of basis X* in Definition 1.1.1. Sometimes X is also said to be a finite *alphabet* for F if X is finite and its elements are also called *letters*. The *free semigroup* generated by X is the set of all finite sequences of letters with the operation of concatenation. Elements of this semigroup are the well known *words* on X. It is also well known that each word can be written in a unique way as a sequence of letters, so it is possible to define for each word w its length $|w|$:

$$w = x_1 x_2 \ldots x_k \quad x_i \in X \Longrightarrow |w| = k \in \mathbb{N}. \tag{1.1.5}$$

It is useful to introduce the formal *empty word* as the word of length 0. Any word w may simplified by omitting elements of the form xx^{-1}, or $x^{-1}x$, for a given $x \in X$. This operation is known as reduction, and it does not change the group element represented by the word. A *reduced word* is a word that contains no redundant pairs. Any word can be simplified to a reduced word by performing a sequence of reductions. The free semigroup on the finite alphabet X admits an equivalence relation, identifying two words up to the reduction, and one can see that the resulting quotient from the free semigroup is a concrete way to construct the free group F in Definition 1.1.1 when X is a finite set.

Note that free groups exist by the following result.

Theorem 1.1.2 (See [24], Theorem 2.1.1)**.** *If X is a nonempty set, then there exists a group F and a function $\sigma : X \to F$ such that F is free on X and F coincides with the subgroup $\langle \mathrm{Im}\ \sigma \rangle$ generated by* $\mathrm{Im}\ \sigma$.

Since free groups exist by Theorem 1.1.2, the problem of their uniqueness is answered by the following results.

Theorem 1.1.3 (See [24], Theorem 2.1.4)**.** *If F_1 is free on X_1, F_2 is free on X_2 and if $|X_1| = |X_2|$, then F_1 is isomorphic to F_2*

Theorem 1.1.3 shows that free groups on sets of equal cardinality turn out to be isomorphic. The converse of Theorem 1.1.3 also is true, in that if two free groups are isomorphic, then the sets on which they are free, have equal cardinality, and this turns out to be easy to be check.

Exercise 1.1.4 (See [24], Exercise 2.1.7)**.** *If F_i is free on X_i, $i = 1, 2$, and F_1 is isomorphic with F_2, then $|X_1| = |X_2|$.*

Proof. Let F_1 be free on X_1, and F_2 be free on X_2. Then consider the set (and therefore a group, since $\mathbb{Z}(2)$ denotes the abelian group with two elements) of all homomorphisms $\mathrm{Hom}(F_1, \mathbb{Z}(2))$ from F_1 to $\mathbb{Z}(2)$, and similarly $\mathrm{Hom}(F_2, \mathbb{Z}(2))$. If these are viewed as vector spaces over $\mathbb{Z}(2)$, then $\mathrm{Hom}(F_1, \mathbb{Z}(2))$ is a vector space over $\mathbb{Z}(2)$ with a basis X_1 and $\mathrm{Hom}(F_2, \mathbb{Z}(2))$ a vector space over $\mathbb{Z}(2)$ with a basis X_2. Let $\varphi : F_1 \to F_2$ be the given isomorphism, we have the isomorphism $f : \mathrm{Hom}(F_1, \mathbb{Z}(2)) \to \mathrm{Hom}(F_2, \mathbb{Z}(2))$ between vector spaces. Then $|X_1| = |X_2|$ since isomorphic vector spaces will have equal dimensions. □

Exercise 1.1.4 together with Theorem 1.1.3, as mentioned above, ensure that the definition of a rank given above makes sense.

The fact that Im σ generates F follows by Theorem 1.1.2, but it can be proved independently.

Exercise 1.1.5 (See [24], Exercise 2.1.8)**.** *If (F, σ) is free on a set X, then* Im σ *generates F.*

Proof. Let Y be another set such that $|Y| = |X|$ as in Exercise 1.1.4 above. If F_1 is a free group on Y, then $\varphi : F_1 \to F$ is an isomorphism and $f : Y \to X$ is a bijection. Let $w_1 \in F_1$, then w_1 can uniquely be expressed as $w_1 = y_1^{k_1} y_2^{k_2} \cdots y_r^{k_r}$, with $r \geq 0$, $0 \neq k_i \in \mathbb{Z}$ and $y_i \neq y_{i+1}$. But elements of $\langle Y^\varphi \rangle$ are of the form

$$\varphi(y_1)^{z_1} \varphi(y_2)^{z_2} \cdots \varphi(y_r)^{z_r}, \tag{1.1.6}$$

for some $z_i \in \mathbb{Z}$. As $\varphi(w_1) \in F$, we know it can be written uniquely as

$$\varphi(w_1) = \varphi(y_1^{k_1} y_2^{k_2} \cdots y_r^{k_r}) = \varphi(y_1)^{k_1} \varphi(y_2)^{k_2} \cdots \varphi(y_r)^{k_r}.$$

Therefore $\varphi(w_1)$ has the form of the elements in $\langle Y^\varphi \rangle$, hence $\varphi(w_1) \in F$ and $F \subseteq \langle Y^\varphi \rangle$. Since $\langle Y^\varphi \rangle \subseteq F$, we have $F = \langle Y^\varphi \rangle$ and so

$$F = \langle Y^\varphi \rangle = \langle f(Y) \rangle = \langle X \rangle = \langle \mathrm{Im}\ \sigma \rangle. \tag{1.1.7}$$

Note that $\sigma : X \to F$ is an injection so that X and Im σ are in bijection. □

It is more interesting to observe that a group G is *torsion-free*, if the only element $g \in G$ such that its k-power $g^k = 1$ turns out to be the trivial element $g = 1$ of G for any possible choice of the positive integer $k \in \mathbb{N}$. On the contrary, if there is a nontrivial element $g \in G$ such that $g^k = 1$ for some $k \in \mathbb{N}$, then g has *order* k (in symbols $o(g) = k$) and we say that G has *torsion*, that is, G has some elements of finite order.

In the following chapters we will see that there are torsion-free groups without torsion, torsion groups without torsion and groups which are neither torsion-free nor with torsion (the latter one are called *mixed groups*).

The following fact is mentioned in [24, Exercise 2.1.1] and we offer a solution.

Exercise 1.1.6 (See [24], Exercise 2.1.1)**.** *Free groups are torsion-free.*

Proof. We claim that the only element $w \in F$ such that its n-power $w^n = 1$ turns out to be $w = 1$ of F for any possible choice of the integer $n \in \mathbb{N}$. There is no loss of generality in assuming $w \in F$ reduced word and $n \in \mathbb{N}$. For $n = 1$ we have $w = 1$ and the result holds when $n = 1$. Assume that the hypothesis is true for some $k = n - 1$. Then $w^n = w^{k+1} = w^k w = 1w = w$ and this word equals one if and only if $w = 1$ that is if and only if w is trivial. Then the result follows.

□

From [24, p.26], the set

$$Z(G) = \{x \in G \mid xg = gx, \text{ for all } g \in G\} \tag{1.1.8}$$

is the *centre* of the group G. It is a well known fact that $Z(G)$ is always a subgroup of G, and it is always normal in G. Now, as $Z(G)$ is a subgroup of G, one sees that, informally speaking, the size of $Z(G)$ can somehow measure how far is G from being abelian. For instance, if $Z(G)$ is trivial, this means that only the neutral element commutes with the rest of the elements, and that if $Z(G) = G$, then every element of G commutes with every other element, that is, G is *abelian*. In particular, Exercise 1.1.6 shows that there are no finite abelian groups which are free.

Another aspect of free groups is that they are abelian only in a very special situation. As we will show in Exercise 1.1.7 below, it turns out that free groups of *rank* strictly bigger than 1 have trivial *centre*.

Note that the notion of rank in Definition 1.1.1 and Theorem 1.1.3 is discussed in Exercise 1.1.4 below.

Exercise 1.1.7 (See [24], Exercise 2.1.2)**.** *A free group has trivial center if and only if its rank is* > 1*.*

Proof. Let F be a free group of rank > 1. Then

$$Z(F) = \{u \in F : uw = wu, w \in F\} = \{u \in F : (uw)(uw) = (wu)(uw), w \in F\} \tag{1.1.9}$$

$$= \{u \in F : (uw)^2 = wu^2w, w \in F\} = \{u \in F : u^2w = wu^2, w \in F\}$$

$$= \{u \in F : u^2wu^{-2} = w \in F\}. \tag{1.1.10}$$

Now, since $u, w \in F$ are both reduced, and $u^2 \neq 1$, we must have $u = 1$ and the center is trivial. On the other hand, we may argue by negation: if the rank of F is one, then F is generated by powers of a single element, so it is abelian, $Z(F) = F$ and the proof is concluded. □

In particular, according to Definition 1.1.1, if F is free on X, then $F = \langle \mathrm{Im}\,\sigma \rangle$ Assume that F is abelian. Then $F = Z(F)$ implies $|X| = 1$ by Exercise 1.1.7 and so F is generated by the powers of a single element, that is, F is isomorphic to $\mathbb{Z}$. Viceversa one can easily check by Definition 1.1.1 that $\mathbb{Z}$ is free on a single generator. Therefore we have shown that

Exercise 1.1.8 (See [24], Exercise 2.1.3)**.** *A free group is abelian if and only if it is infinite cyclic.*

Let us look at some natural examples of free groups among linear groups. Define the following maps

$$\alpha : x \in \mathbb{C} \longmapsto \alpha(x) = x + 2 \in \mathbb{C}, \qquad \beta : x \in \mathbb{C} \longmapsto \beta(x) = \frac{x}{2x+1} \in \mathbb{C},$$

noting that both are bijective maps, but while α is a translation and so a linear map from the additive abelian group $\mathbb{C}$ of the complex numbers (in fact α is isomorphism of groups), β is a linear fractional map from $\mathbb{C}$ to $\mathbb{C}$.

Note that we may consider arbitrary constants $a, b, c, d \in \mathbb{C}$ and so the general form of a *linear fractional map* would be

$$\lambda : x \in \mathbb{C} \longmapsto \lambda(x) = \frac{ax+b}{cx+d} \in \mathbb{C}, \tag{1.1.11}$$

realizing for $a = 1$, $b = 2$, $c = 0$, $d = 1$ the map α, and for $a = d = 1$, $b = 0$, $c = 2$ the map β. Denoting by

$$\mathrm{Aut}(\mathbb{C}) = \{\varphi : \mathbb{C} \to \mathbb{C} \mid \varphi \text{ is isomorphism of groups}\} \tag{1.1.12}$$

we note that (1.1.12) has the structure of nonabelian group with respect to the composition of two isomorphisms. In particular, we can see that $\alpha \in \mathrm{Aut}(\mathbb{C})$ but $\beta \notin \mathrm{Aut}(\mathbb{C})$, even if β is bijective. On the other hand, we may define

$$\mathrm{LF}(\mathbb{C}) = \{\lambda : \mathbb{C} \to \mathbb{C} \mid \lambda \text{ is a linear fractional map}\}, \tag{1.1.13}$$

observe that it has the structure of nonbelian group with respect to the composition of maps and clearly we have that linear maps are special types of linear fractional maps. From the fact that $\mathrm{Aut}(\mathbb{C})$ is isomorphic to the group $\mathrm{GL}(2, \mathbb{C})$ of nonsingular square matrices 2-by-2 with complex coefficients, we may identify any element of $\mathrm{Aut}(\mathbb{C})$ via a matrix in $\mathrm{GL}(2, \mathbb{C})$, therefore $\mathrm{Aut}(\mathbb{C})$ embeds into $\mathrm{LF}(\mathbb{C})$ in the same way $\mathrm{GL}(2, \mathbb{C})$ embeds into $\mathrm{LF}(\mathbb{C})$, that is, via the homomorphism of groups

$$\Phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}) \longmapsto \Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \lambda \in \mathrm{LF}(\mathbb{C}). \tag{1.1.14}$$

In this way we may visualize geometrically the free group on two generators.

Exercise 1.1.9 (See [24], Exercise 2.1.4)**.** *Let a be a complex number such that $|a| \geq 2$. Then the free group F on two generators is the subgroup of* $\mathrm{GL}(2, \mathbb{C})$ *generated by the elements* $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

Proof. It is sufficient to see that the two matrices above do not commute and finitely many products of powers of them give rise to elements of $\mathrm{GL}(2,\mathbb{C})$ yet, so they represent a free group on two independent generators in $\mathrm{GL}(2,\mathbb{C})$. □

Incidentally Exercise 1.1.9 shows that groups of isomorphisms of the additive group of the complex numbers contain free groups. Exercise 1.1.9 can be modified without problems replacing the role of the complex field with that of the real field or of the rational field.

Now we discuss a few more facts on free groups and generators of free groups. An arbitrary intersection of normal subgroups is again a normal subgroup by [24, Theorem 1.3.16, p.16], and we may introduce for a nonempty subset X of a group G *normal closure* of X in G, denoted it by X^G, as the intersection of all normal subgroups of G containing X. Note also that the normal closure of X in G is the *smallest normal subgroup containing* X. In the opposite direction, we have the notion of *normal core* X_G of X in G as *the largest normal subgroup contained in* X, see [24, p.16]. It can be shown that for $X = H$ subgroup of G, we have always $H_G \subseteq H \subseteq H^G$, moreover

$$H^G = \langle g^{-1}Hg \mid g \in G\rangle, \quad \text{and} \quad H_G = \bigcap_{g\in G} g^{-1}Hg. \tag{1.1.15}$$

Exercise 1.1.4 below shows that a quotient group of a free group by the normal closure is free on some special sets.

Exercise 1.1.10 (See [24], Exercise 2.1.5)**.** *If F is a free group on a subset X and Y a nonempty subset of X, then F/Y^F is free on $X \setminus Y$.*

Proof. Let F be free on $X \subseteq F$ and Y nonempty subset of X. Then $\iota : X \to F$ is an injection, and for every G and $\alpha : X \to G$ there is a unique $\beta : F \to G$ such that $\alpha = \beta \circ \iota$ by Definition 1.1.1. For a nonempty set $Y \subseteq X \subseteq F$ we may consider the normal closure $N = Y^F$. Then we have the homomorphism

$$\overline{\beta} \ : \ w = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n} y_1^{\delta_1} y_2^{\delta_2} \cdots y_m^{\delta_m} \in F/N \tag{1.1.16}$$

$$\longmapsto \bar{\beta}(w) = \alpha(x_1)^{\epsilon_1}\alpha(x_2)^{\epsilon_2}\cdots\alpha(x_n)^{\epsilon_n}\alpha(y_1)^{\delta_1}\alpha(y_2)^{\delta_2}\cdots\alpha(y_m)^{\delta_m} \in G,$$

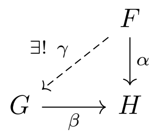
where $x_i \in X \setminus Y$, $y_j \in Y$. Define

$$\overline{\sigma} : x \in X \setminus Y \longmapsto \overline{\sigma}(x) = xN \in F/N \tag{1.1.17}$$

noting that a typical element xN is a juxtaposition of the reduced words in $X\setminus Y$ and reduced words in Y^F. Consider the restriction $\overline{\alpha} = \alpha \mid_{X\setminus Y}\colon\ X \setminus Y \to G$, here $\overline{\alpha}(x) = \alpha(x) = \beta(x)$ for all $x \in X \setminus Y$, and since $x^* \sim x'N$ for $x^* \in X$ and $x' \in X \setminus Y$, we may conclude that $(\overline{\beta} \circ \overline{\sigma})(x) = \beta(x)$ for $x \in X \setminus Y$, that is, $\overline{\alpha} = \overline{\beta} \circ \overline{\sigma}$. This shows the result via Definition 1.1.1. □

Under what circumstances is the ambient group of a free quotient G/N is the *direct product* of N and some other subgroup? The application of the Universal Projective Property of Free Groups, that is, of the following result answers this question in Exercise 1.1.12 below.

Theorem 1.1.11 (Universal Projective Property of Free Groups, see ([24], Theorem 2.1.6)**.** *Let F be a free group and let G and H be given groups. Assume that $\alpha : F \to H$ is a unique homomorphism and $\beta : G \to H$ an epimorphism. Then there is a homomorphism $\gamma : F \to G$ such that the following diagram is commutative*



that is $\alpha = \beta \circ \gamma$.

Note that given an arbitary group G and H, K two subgroups of G, the product of H and K is the set

$$HK = \{hk \mid h \in H, k \in K\}. \tag{1.1.18}$$

This product is generally not a group, but when $HK = KH$, it becomes a group. We say that G is *generated* by H and K if $G = HK$. In fact we have $G = \langle H, K \rangle = HK$ in this situation. In particular, G is a *direct product* of H and K, denoting $G = H \times K$, if G is generated by H and K, $H \cap K = 1$, and both H and K are normal in G. Note that H and K are called *direct factors* of G. If only H is normal in G, but still $G = HK$ and $H \cap K = 1$, we say that G is the *semidirect product* of H by K (with normal factor H and nonnormal factor K), denoting $G = H \rtimes K$. Of course, the notion of direct product is a special case of the notion of semidirect product and this is a special case of the notion of product of two groups. There are plenty of examples, showing that these notions are indeed different in general.

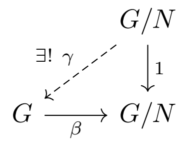
Note that in the case of abelian groups we replace these notions with their abelian analogues as the *sums of elements* and *direct sums* of groups. Note that "semidirect sums" are just direct sums since *every subgroup of an abelian group is normal.* Let $H, K \leq G$, the set

$$H + K = \{h + k \mid h \in H, k \in K\} \tag{1.1.19}$$

is called the *sum* of H and K, and $H + K = K + H$ since they are both subgroups of an abelian group. If in addition $H \cap K = 0$, where 0 represents the trivial group, then we have $G = H \oplus K$, which we call the *direct sum* of H and K. Direct sums are at the core of the abelian group theory as we will see later that the classification of the groups, which we will give, will be expressed in terms of direct sums.

Exercise 1.1.12 (See [24], Exercise 2.1.6)**.** *If N is a normal subgroup of a group G with free quotient G/N, then $G = N \rtimes H$ for some subgroup H of G.*

Proof. Let $F = G/N$ and let $\beta : G \longrightarrow G/N$ the canonical projection of G onto G/N. Theorem 1.1.11 implies we have the commutative diagram:



for unique $\gamma : G/N \longrightarrow G$ such that $\beta \circ \gamma = 1$. We want to show that $G = \ker \beta \rtimes \ker \gamma$. First of all, if $g \in G$, then $(\beta \circ (\gamma \circ \beta))(gg^{-1}) = 1$, and so $g \in \ker \beta \ \mathrm{Im}\ \gamma$, hence $G \subseteq \ker \beta \ \mathrm{Im}\ \gamma$. Viceversa, it is clear that $\ker \beta \ \mathrm{Im}\ \gamma$ turns out to be a subgroup of G. Therefore $G = \ker \beta \ \mathrm{Im}\ \gamma$. Of course, $\ker \beta$ is a normal subgroup of G, since kernels of homomorphisms are always normal subgroups in their domain. It remains to check that $\ker \beta$ intersects trivially $\mathrm{Im}\ \gamma$, but this follows from the condition $\beta \circ \gamma = 1$. The result follows. □

Exercise 1.1.12 says that normal subgroups N such that G/N is free may detect direct factors and decompositions in groups. We will see later in Theorem 2.2.5 the analog in the abelian case.

On the other hand, we may ask whether we may detect a decomposition for a free group in other ways, that is, looking at different properties of prescribed families of quotients. The answer is positive when we look at finite quotients but it requires some cautionary observations.

First of all, we need to recall that a group G is *residually finite* if for each non-identity element in the group, there is a normal subgroup of finite index not containing that element. Introducing the intersection $\mathrm{res}_{\mathfrak{F}}(G)$ of all subgroups H of finite index in G, that is,

$$\mathrm{res}_{\mathfrak{F}}(G) = \bigcap_{\substack{H \in \mathcal{L}(G) \\ |G:H|<\infty}} H, \text{and } \mathrm{res}_{\mathfrak{X}}(G) = \bigcap_{\substack{N \in \mathcal{N}(G) \\ G/N \in \mathfrak{X}}} N, \qquad (1.1.20)$$

one can check that $\mathrm{res}_{\mathfrak{F}}(G)$ is indeed a normal subgroup of G, called *finite residual of* G. Moreover G turns out to be residually finite if and only if (1.1.20) is trivial. Again one can check that the property of being residually finite for a group G is equivalent to embedded G in the direct product of a family of finite groups. In this sense, free groups satisfy an important structural property of decomposition, since they may be embedded in the direct product of finite groups.

Exercise 1.1.13 (See [24], Exercise 2.1.10)**.** *Free groups are residually finite.*

Proof. We claim that a free group F possessing a subgroup H of finite index has the property that every nontrivial subgroup of F intersects H nontrivially. Let H_F the normal core of H in F. Since $|F : H|$ is finite, then $|F : H_F| = m$ is finite. From $|F : H_F| = m$, we get that $|F/H_F| = |F : H_F| = m$. Let K

be a nontrivial subgroup of F and $1 \neq k \in K$. Then $1 \neq k^m \in K$ since F is torsion-free by Exercise 1.1.6. Define the canonical epimorphism

$$\varphi : k \in K \longmapsto \varphi(k) = kH_F \in F/H_F. \tag{1.1.21}$$

Then $\varphi(k^m) = k^m H_F = (kH_F)^m = H_F$, which implies that $k^m H_F = H_F$ and that $k^m \in H_F$. This also means that

$$k^m \in K \cap H_F \subseteq K \cap H \neq 1. \tag{1.1.22}$$

Hence $H \cap K \neq 1$ for all subgroups K of F, as claimed. □

Exercise 1.1.13 may be interpreted as saying that subgroups of free groups that are of finite index cannot be direct factors.

1.2 Presentations

Let F be a free group on X and G be any group, the epimorphism $\pi : F \to G$ is called a *free presentation* of G. Suppose $R = \text{Ker } \pi$, then $F/R \cong G$ by [24, Theorem 1.4.3, p.19]. The elements of R are those elements of F that reduce to a neutral element in G, and therefore call them *relations* and we write $G = \langle X \mid R\rangle$. Observe that, if $R = 1$, then G is itself a free group since $F/1 \cong F \cong G$. Now since $R \subseteq F$ and F is infinite, R might be infinite as well. Now choose $S \subseteq R$ such that $S^F = R$, that is R is a normal closure of S in F, then we call S a *set of defining relations* for G. X and S determines a set of generators and defining relators for G, and write

$$G = \langle X \mid S\rangle. \tag{1.2.1}$$

A presentation of G is said to be finite if both X and S are finite.

Exercise 1.2.1 (See [24], Exercise 2.2.6)**.** *Let A be an abelian group with generators $x_1, x_2, \cdots, x_n$ and defining relations consisting of $[x_i, x_j] = 1, i < j = 1, 2, \dots, n$, and r further relations. If $r < n$, then A is infinite.*

As a generalization of Exercise 1.2.1, we see in the following exercise that groups with less relations than generators are necessarily infinite. For example, the free group on a single generator is placed in this situation.

Exercise 1.2.2 (See [24], Exercise 2.2.7)**.** *Suppose that G is a group with n generators and r relations where $r < n$. Then G is infinite.*

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of generators of G, and let $\pi : F \to G$ be the presentation of G. Then, F is free on X and π is surjective, and hence $G \cong F/\ker\pi$. Let $\ker\pi = \{s_1, s_2, \dots, s_r\}$ be relators of G, where $r < n$. Assume that G is finite and $k \in \mathbb{N}$ the order of an arbitrary element $g \in G$. If $1 \neq f \in F \setminus \ker\pi$, then we may write $f = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ as product of powers of the elements of X for suitable $l_1, l_2, \ldots, l_n \in \mathbb{Z}$. Formally g is a coset modulo $\ker\pi$, that is, $g = f \ \ker\pi$, and we have

$$g^k = 1 \Longleftrightarrow (f \ \ker\pi)^k = 1 \Longleftrightarrow f^k \ \ker\pi = 1 \Longleftrightarrow (x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n})^k \ \ker\pi = \ker\pi \tag{1.2.2}$$

$$\Longleftrightarrow x_1^{l_1k} x_2^{l_2k} \cdots x_n^{l_nk} \ \ker\pi = \ker\pi \Longleftrightarrow x_1^{l_1k} x_2^{l_2k} \cdots x_n^{l_nk} \in \ \ker\pi \Longleftrightarrow f^k \in \ker\pi.$$

Then $\pi(f^k) = \pi(f^{k-1}f) = 1$, but we choose $f \neq 1$, and so $f^{-1} = f^{k-1}$, but $(f^{k-1})^{-1} = f^{1-k}$. This means that

$$f^{k-1}f = f^{1-k}f = f^{k-1} = f^{1-k} = f^{2k-2} = 1, \tag{1.2.3}$$

then $k = 1$ but f was chosen to be nontrivial and this is a contradiction. □

Exercise 1.2.3 (See [24], Exercise 2.2.8). *If G is a finitely presented group and let N be a normal subgroup which is finitely generated, then G/N is finitely presented.*

Proof. Let G be finitely presented and N a finitely generated normal subgroup. Let $R_1, \ldots, R_t$ be the relators of G and $X = \{x_1, \ x_2, \ \ldots, \ x_n\}$ be the minimal number of generators of G. Without loss of generality, assume that the first r (with $r < n$) elements of X constitute the minimal number of generators of N.

Claim: $\{x_1N, \ x_2N, \ \ldots, \ x_rN\}$ *is a generating set of G/N.* Let $l_i \in \mathbb{Z}$ and consider the product of cosets

$$(x_1N)^{l_1}(x_2N)^{l_2} \cdots (x_rN)^{l_r}(x_{r+1}N)^{l_{r+1}} \cdots (x_nN)^{l_n}. \tag{1.2.4}$$

From this we get that $x_1, x_2, \ldots, x_r \in N$ and note that N is the neutral element in the group G/N, hence

$$\begin{aligned}(x_1N)^{l_1}(x_2N)^{l_2} \cdots (x_rN)^{l_r}(x_{r+1}N)^{l_{r+1}} \cdots (x_nN)^{l_n} &= N(x_{r+1}N)^{l_{r+1}} \cdots (x_nN)^{l_n} \\ &= (x_{r+1}N)^{l_{r+1}} \cdots (x_nN)^{l_n} = \left(x_{r+1}^{l_{r+1}} x_{r+2}^{l_{r+2}} \cdots x_n^{l_n}\right) N.\end{aligned} \tag{1.2.5}$$

On the other hand, the product $x_{r+1}^{l_{r+1}} x_{r+2}^{l_{r+2}} \cdots x_n^{l_n}$ is of form of the elements of G so we have shown that any element of $G/N = \{gN \mid g \in G\}$ can be written as $g = x_{r+1}^{l_{r+1}} x_{r+2}^{l_{r+2}} \cdots x_n^{l_n}$ taking only $r + 1$ generators in X. This is sufficient to show that G/N is finitely generated.

Regarding the relators of G, they induce on G/N finitely many relators $S_1, S_2, \ldots, S_k$ with $k \leq t$ obtained by $R_1, R_2, \ldots, R_t$ modulo N. □

Exercise 1.2.3 is motivated by groups of the form of $S = \mathbb{Z}^{(\mathbb{N})} \rtimes \mathbb{Z} = N \rtimes H$, that is, the semidirect product of countably many copies of $\mathbb{Z}$ by $\mathbb{Z}$, where the nonnormal factor H of S acts on the normal factor N as the shifting automorphism which sends each factor in the successive one. The group S shows that Exercise 1.2.3 does not hold when N is not finitely generated, even if S/N is finitely presented. Note that S is not finitely presented. This example also shows that normal subgroups of finite index in finitely presented groups give rise to finitely presented quotients, but this is no longer true for normal subgroups of infinite index.

1.3 Varieties of Groups

We consider the *class* of groups which will be equationaly defined to form *varieties*. First we give a couple of definitions as given in [24].

Let $X = \{x_1, x_2, \ldots\}$ and F be a free group on X with $\emptyset \neq W \subseteq F$. Let w be a reduced word in W in its normal form, that is, $w = x_{i_1}^{l_1} x_{i_2}^{l_2} \cdots x_{i_r}^{l_r} \in W$ for $r \geq 0$, $l_r \in \mathbb{Z}$ and $x_i \neq x_{i+1}$. If $g_1, g_2, \ldots, g_r \in G$, then the *value* of w at $(g_1, g_2, \ldots, g_r)$ is defined to be

$$w(g_1, g_2, \ldots, g_r) = g_1^{l_1} g_2^{l_2} \cdots g_r^{l_r}. \tag{1.3.1}$$

Consider the subgroup of G generated by all values $w(g_1, g_2, \ldots) \in G$ of words $w \in W$. We call this subgroup the *verbal subgroup* of G determined by W, and we denote it by

$$W(G) = \langle w(g_1, g_2, \ldots) \mid g_i \in G, w \in W \rangle. \tag{1.3.2}$$

It is easy to see that verbal subgroups are *fully-invariant*. A subgroup H of G is said to be *fully-invariant* in G if $\alpha(H) \subseteq H$ for all $\alpha \in \mathrm{End}(G)$, that is, H is fixed by all endomorphisms of the group G. The converse here is generally false (see Exercise 1.3.4 below), but is true under certain conditions.

Theorem 1.3.1 (B.H. Neumann, see [24], Theorem 2.3.1)**.** *A fully-invariant subgroup of a free group is verbal.*

We consider a simple example of verbal subgroups. Let F be a free group on $\{x_1, x_2\}$ and $W \subseteq F$ be a singleton consisting of only $[x_1, x_2]$, that is, $W = \{[x_1, x_2]\}$. One can check that $W(G)$ is just the derived subgroup G' of G. In fact $W = \{[x_1, x_2]\} = \{x_1^{-1} x_2^{-1} x_1 x_2\}$ and $g_1, g_2 \in G$. Then the value of $w = x_1^{-1} x_2^{-1} x_1 x_2 \in W$ at (g_1, g_2) is $w(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2 \in G$, therefore the verbal subgroup determined by W is

$$W(G) = \langle w(g_1, g_2) \mid g_i \in G, w \in W \rangle = \langle g_1^{-1} g_2^{-1} g_1 g_2 \mid g_i \in G \rangle = G'. \tag{1.3.3}$$

In fact, derived subgroups are always fully-invariant. Given a normal subgroup N of a group G, we may consider words, where at a given position we omit an entry and the word does not change after this omission, that is, words such that

$$w(g_1, \ldots, g_{i-1}, g_i a, g_{i+1}, \ldots, g_r) = w(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_r) \tag{1.3.4}$$

for all $g_i \in G$, $a \in N$ and all $w(x_1, x_2, \ldots, x_r) \in W$. In this case we say that N is *W-marginal*.

The defining condition of a W-marginal subgroup is equivalent to requiring that:

$$g_i \equiv h_i \text{ mod } N, (1 \leq i \leq r) \implies w(g_1, \ldots, g_r) = w(h_1, \ldots, h_r). \qquad (1.3.5)$$

W-marginal subgroups of G, by Exercise 1.3.4 below, generate a W-marginal normal subgroup. We call it *the W-marginal subgroup of G* and is written as $W^*(G)$.

Marginal subgroups and verbal subgroups satisfy some forms of duality which we are going to explain here.

First of all we give a well known example of marginal subgroup. Take $W \subseteq F$ which is a singleton containing only $[x_1, x_2]$, where F is a free group on $X = \{x_1, x_2, \ldots\}$. One can check that $W^*(G) = Z(G)$, where $Z(G)$ is the centre of G.

Just looking at the examples which we gave: note that $G/W(G) = G/G'$ is abelian, and $W^*(G) = Z(G)$ is abelian. This is a first form of duality that one can have, that is, most of the times a property that we recognise for $G/W(G)$ is the same for $W^*(G)$ and viceversa. Another form of duality is related intuitively to the size of these two subgroups:

Theorem 1.3.2 (See [24], Lemma 2.3.2)**.** *Let W be a nonempty set of words in $x_1, x_2, \ldots$ in a group G. Then $W(G) = 1$ if and only if $W^*(G) = G$.*

Theorem 1.3.2 shows that large marginal subgroups correspond to small verbal subgroups and viceversa. There are more and more interesting relations between $W(G)$ and $W^*(G)$ and details can be found in [24].

A marginal subgroup is always characteristic. A subgroup H of G is said to be *characteristic* in G if $\alpha(H)$ is a subgroup of H for all $\alpha \in \text{Aut } G$, that is, if α is fixed by all automorphisms of the group. Marginal subgroups need not be fully invariant.

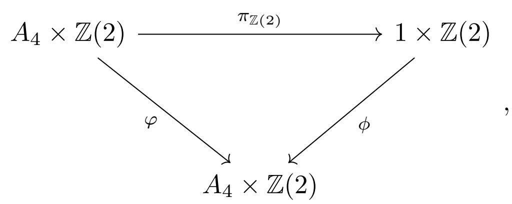
Exercise 1.3.3 (See [24], Exercise 1.5.9)**.** *There are marginal subgroups which are characteristic but not fully invariant.*

Proof. Consider $W^*(G) = Z(G)$ when $G = A_4 \times \mathbb{Z}(2)$, where A_4 is the alternating group. Here $Z(G) = Z(A_4 \times \mathbb{Z}(2)) = 1 \times \mathbb{Z}(2)$. Let $(1,2)(3,4) \in A_4$ be an order 2 element, so $\langle (1,2)(3,4) \rangle = H$ is a subgroup of A_4 isomorphic to $\mathbb{Z}(2)$. Then define $\phi : \ 1 \times \mathbb{Z}(2) \to \langle (1,2)(3,4) \rangle \times 0$ explicitly as follows:

$$\phi : \begin{cases} (1,0) \longmapsto (1,0) \\ (1,1) \longmapsto ((1,2)(3,4), 0) \end{cases},$$

and $\pi_{\mathbb{Z}(2)} : \ A_4 \times \mathbb{Z}(2) \to 1 \times \mathbb{Z}(2)$ be the projection on the second factor. Then define an endomorphism from $A_4 \times \mathbb{Z}(2)$ as follow: $\varphi = \phi \circ \pi_{\mathbb{Z}(2)} : \ A_4 \times \mathbb{Z}(2) \to$

$A_4 \times \mathbb{Z}(2)$ such that the following diagram



commutes. For instance, one can see that

$$\begin{aligned}\varphi(Z(G)) &= (\phi \circ \pi_{\mathbb{Z}(2)})(Z(G)) = \phi\left(\pi_{\mathbb{Z}(2)}(Z(G)\right) \\ &= \phi(Z(G)) = H \times 0\end{aligned} \tag{1.3.6}$$

and that $\phi(Z(G)) \cap Z(G)$ is the trivial element of G, showing that $\phi(Z(G))$ is not contained in $Z(G)$. □

We illustrate a few more facts on marginal subgroups, focusing on questions about the generation of marginal subgroups.

Exercise 1.3.4 (See [24], Exercise 2.3.1)**.** *A subgroup which is generated by W-marginal subgroups is itself W-marginal.*

Proof. Consider the subgroup H of a group G generated by all W-marginal subgroups N in G. Then by definition, H is normal in G. If $h \in H$, then

$$\begin{aligned}w(g_1, g_2, \ldots, g_{i-1}, g_i h, g_{i+1}, \ldots, g_r) &= w(g_1, g_2, \ldots, g_{i-1}, g_i a, g_{i+i}, \ldots, g_r) \\ &= w(g_1, g_2, \ldots, g_{i-1}, g_i, g_{i+i}, \ldots, g_r)\end{aligned} \tag{1.3.7}$$

for some $a \in N$ since h belongs to all the subgroups N which are W-marginal. Then H itself is a W-marginal subgroup of G. □

Note that in abelian group theory we have a precise meaning for certain marginal and verbal subgroups.

Exercise 1.3.5 (See [24], Exercise 2.3.2)**.** *If $W = \{x^p\}$ for p prime and G abelian, then identify $W^*(G)$ and $W(G)$.*

Proof. Let $a \in W^*(G)$ and $1 \in G$. Then $1 = w(1) = w(1a) = w(a) = a^p$. In this case, all the elements of $W^*(G)$ are of finite order that divides p. If $g \in G$, consider $w(ga)$. Then, $w(ga) = w(g)$, and $w(ag) = w(g)$, so that $w(ga) = w(g) = w(ag)$ implies $w(ga) = w(ag)$. But $gaga = (ga)^p = g^p 1 = g^p$ if and only if $gaga = g^p$. From this, we get $aga = g$ if and only if $ga = a^{-1}g = ag$, the last equality follows from the fact $a^p = 1$.
Now one sees that

$$W^*(G) = \{a \in G \mid a^p = 1\} = \Omega_p(G), \tag{1.3.8}$$

$$W(G) = \{a^p \in G \mid a \in G\} = \Omega^p(G). \tag{1.3.9}$$

□

The subgroups (1.3.8) and (1.3.9) will be discussed later on, when we focus on abelian groups. They play an important role in the decompositions of subgroups and actually one can find many relevant structural properties of abelian groups, which are related to (1.3.8) and (1.3.9).

If the orders of the elements of a group are finite and bounded, the group is said to have *finite exponent* and the *exponent* of the group is then the least common multiple of all the orders. According to this terminology, (1.3.8) consists of all elements of exponent p (prime).

One can go further on this: let $W = \{x^n\}$ for $n \in \mathbb{N}$, so not necessarily $n = p$ prime and consider $a \in W^*(G)$. Then

$$1 = w(1) = w(1a) = w(a) = a^n \Longleftrightarrow a^n = 1, \tag{1.3.10}$$

so again, elements in this subgroup $W^*(G)$ are of finite exponent n. Also if G is abelian, then

$$\underbrace{(ga)(ga)\cdots(ga)}_{n-\text{factors}} = (ga)^n \Longleftrightarrow \underbrace{(ga)(ga)\cdots(ga)}_{n-\text{factors}} = g^n a^n,$$

the last equality implies

$$W^*(G) = \{g \in G \mid g \text{ is of exponent } n\} = \Omega_n(G) \tag{1.3.11}$$

which is exactly (1.3.8) when $n = p$. Similarly one can define more generally for an abelian group G the corresponding subgroup

$$W(G) = \{g^n \in G \mid g \in G\} = \Omega^n(G). \tag{1.3.12}$$

We end with a few interesting examples always in the abelian case.

Exercise 1.3.6 (See [24], Exercise 4.3.3)**.** *Let G be the multiplicative group of all complex $(2^n)^{th}$ roots of unity, $n = 0, 1, 2, \ldots$. Then 1 and G are the only verbal subgroups of G, but that every subgroup is marginal. Moreover G has a fully invariant subgroup which is not verbal.*

Proof. One can see that the required group is $G \cong \mathbb{Z}(2^\infty)$, and so all its proper nontrivial subgroups are finite cyclic. $W(G)$ is not cyclic (not even abelian in general) since it is generated by the values, in G, of *all* words in W, so in this way, G cannot admit proper verbal subgroups. Any subgroup H of G is normal since G is abelian, so let H be a nonmarginal subgroup of G, that is, $w(g_1, g_2, \ldots, g_{j-1}, g_j h, g_{j+1}, \ldots, g_r) \neq w(g_1, g_2, \ldots, g_{j-1}, g_j, g_{j+1}, \ldots, g_r)$. This implies

$$w(g_1, g_2, \ldots, g_{j-1}, g_j h, g_{j+1}, \ldots, g_r) \neq w(g_1, g_2, \ldots, g_{j-1}, g_j, g_{j+1}, \ldots, g_r)$$

$$\Longrightarrow\ g_1^{l_1} g_2^{l_2} \cdots g_{j-1}^{l_{j-1}} (g_j h)^{l_j} g_{j+1}^{l_{j+1}} \cdots g_r^{l_r} \neq g_1^{l_1} g_2^{l_2} \cdots g_{j-1}^{l_{j-1}} (g_j^{l_j}) g_{j+1}^{l_{j+1}} \cdots g_r^{l_r} \tag{1.3.13}$$

$\implies h^{l_j} \neq 1$. Then we have $h^{l_j} \neq 1, l_j \neq 1$. This means that $\forall\ h \in H$ are of infinite order. This is false since H, as a subgroup of G, must be a finite cyclic subgroup, that is, every element is of finite order. It remains to check that G has a fully invariant subgroup which is not verbal. Due to the construction of G as direct limits of finite 2-groups $\mathbb{Z}(2^n)$, the first group in the direct limit construction is the unique minimal normal M subgroup of G (of order 2), and so it is fully invariant, since any endomorphism of G should preserve it. If this does not happen, then one can indeed argue against the minimality of M. The result follows. □

In order to see whether we are dealing with a variety of groups or not, the first step is to individualize a specific property of the groups, then we should look for an appropriate formulation via equations. This is in general quite difficult, and more often it can involve a branch of group theory, called *representation theory*, where one should look into appropriate embeddings of family of groups in linear groups and then use some matrix formulas in order to find appropriate properties via equations. We cannot describe this field of research here, because it requires a lot of details. See [24] for details.

On the other hand, there is a very efficient way to detect variety of groups looking at certain stability properties of class of groups. We describe briefly this approach.

Definition 1.3.7. *A class of groups $\mathfrak{X}$ satisfies always the following two conditions:*

(i) $\mathfrak{X}$ *contains the trivial subgroup;*

(ii) $\mathfrak{X}$ *is invariant under isomorphisms of groups.*

Then we say that

(iii) $\mathfrak{X}$ *is stable with respect to subgroups, if* $H \in \mathfrak{X}$ *for all* H *subgroup of* $G \in \mathfrak{X}$*;*

(iv) $\mathfrak{X}$ *is stable with respect to quotients, if* $G/N \in \mathfrak{X}$ *for all normal subgroups* N *of* $G \in \mathfrak{X}$*;*

(v) $\mathfrak{X}$ *is stable with respect to finite direct products, if* $G^{(I)} \in \mathfrak{X}$ *for all* $G_i \in \mathfrak{X}$ *and* $i \in I$ *with* I *finite index set.*

Looking at [24], we may offer examples which satisfy (iii) but not (iv) and (v), or more generally that satisfy one of the conditions above but not the others.

What happens when $\mathfrak{X}$ satisfies all the conditions above from (i) to (v) ? This is interesting: it leads to a variety of groups. Therefore we have an alternative and concrete way to look for varieties of groups via Birkhoff below. But before giving that, we make a mention of the notion of *residually* $\mathfrak{X}$ and give a result which is used in the proof of Birkhoff even though we do not give the proof here.

Having in mind [the first part of] (1.1.20), we may extend the notion of residually finiteness to a group of a certain class $\mathfrak{X}$ which is not necessarily the class $\mathfrak{F}$ of all finite groups.

For a class $\mathfrak{X}$, a group G is said to be a *residually* $\mathfrak{X}$-group if , for $1 \neq g \in G$, there exists a normal subgroup N_g in G such that $g \notin N_g$ and $G/N_g \in \mathfrak{X}$. This corresponds to (iv) above, and one can formulate this notion equivalently in terms of residual subgroups, that is, introducing the set of all normal subgroups $\mathcal{N}(G)$, which is indeed a sublattice of the lattice of all subgroups $\mathcal{L}(G)$, and [second part of] (1.1.20) which turns out to be a characteristic subgroup of G, called $\mathfrak{X}$-*residual* of G. It is not difficult to check that G is a residually $\mathfrak{X}$-group if and only if its $\mathfrak{X}$-*residual* is trivial, that is, $\mathrm{res}_{\mathfrak{X}}(G) = 1$.

Therefore we have the description in terms of (v) in the following result:

Theorem 1.3.8 (See [24], Theorem 2.3.3, p.59)**.** *A group G is a residually $\mathfrak{X}$ if and only if it is a finite direct product of groups $G_i \in \mathfrak{X}$.*

A quick look at the definition of a variety satisfy (i) to (v). Conversely, we have the following result due to Birkhoff and others:

Theorem 1.3.9 (Birkhoff, Kogalovskii, Sain, see [24], Theorem 2.3.5)**.** *Any class of groups which is stable with respect to subgroups, quotients and finite direct products is a variety of groups.*

For instance, all finite groups form a variety of groups, looking at Theorem 1.3.9. Abelian groups also form a variety of groups, looking at Theorem 1.3.9.

Let W be a set of words in $\{x_1, x_2, \ldots\}$. The class of all groups G such that $W(G) = 1$, or equivalently (see Theorem 1.3.2) $W^*(G) = G$, is called the *variety* $\mathfrak{B}(W)$ determined by W. We also call W a *set of laws* for the variety $\mathfrak{B}(W)$.

Therefore we can describe free groups in any variety. They exists and led to interesting theorems of decomposition, as we have seen before in case of free groups.

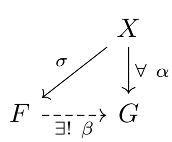
Exercise 1.3.10 (See [24], Exercise 2.3.5)**.** *Let $\mathfrak{B}$ be any variety. If G is a $\mathfrak{B}$-group with a normal subgroup N such that G/N is a free $\mathfrak{B}$-group, show that there is a subgroup H such that $G = HN$ and $H \cap N = 1$.*

Proof. This is the consequence of the Universal Projective Property of Free Groups 1.1.11 and is solved exactly as Exercise 1.1.12. □

Another very useful concept is that of freeness in a variety. From this notion we will see that free groups do exist in varieties of groups, while in general the problem of their existence may be questionable; and what we will see later on is in fact the fundamental properties of free groups in the variety of abelian groups.

The symbol $\mathfrak{B}$ denotes a variety and $F \in \mathfrak{B}$, and let $\sigma : X \to F$ be a set function from a nonempty set X.

We say that F is $\mathfrak{B}$-*free on* X if for all set functions $\alpha : X \to G$ with $G \in \mathfrak{B}$, there exists a unique homomorphism $\beta : F \to G$ such that



commutes.

Suppose now that $\mathfrak{B}$ is a variety of *all* groups, then, that F is $\mathfrak{B}$-free on X, coincides with Definition 1.1.1. $\mathfrak{B}$-free groups are described in terms of three groups.

Theorem 1.3.11 (See [24], Theorem 2.3.6)**.** *Let $\overline{F}$ a free group on X and $\mathfrak{B}$ a variety with a set of laws W. Then $F = \overline{F}/W(\overline{F})$ is a $\mathfrak{B}$-free group on X. Moreover, every group which is $\mathfrak{B}$-free on X is isomorphic to F.*

As we have mentioned earlier, every group is an appropriate homomorphic image of a free group and this continues to be true in the following sense: every $\mathfrak{B}$-group is an image of a free $\mathfrak{B}$-group :

Theorem 1.3.12 (See [24], Theorem 2.3.7)**.** *Let $\mathfrak{B}$ be a variety and $G \in \mathfrak{B}$. If G is generated by X, the group F is $\mathfrak{B}$-free on Y and $\alpha : Y \to X$ is a surjection, then α extends to an epimorphism from F to G. In particular, every $\mathfrak{B}$-group is an image of a free $\mathfrak{B}$-group.*

Chapter 2

Abelian Groups

Let G be group, if for all $g_1, g_2 \in G$, we have that $g_1 + g_2 = g_2 + g_1$, then we call G *abelian.* Note that our binary operation on G is now written additively and this is the standard practice in abelian group theory. All groups considered in this chapter are abelian and therefore a given group G is assumed to be abelian unless otherwise is said. Take $g \in G$, then the sum:

$$\underbrace{g + g + \cdots + g}_{n-\text{summands}} \tag{2.0.1}$$

is represented by ng and

$$\underbrace{-g - g - \cdots - g}_{n-\text{summands}} \tag{2.0.2}$$

is represented by $-ng$ for some $n \in \mathbb{N}$. If there exists some $n \in \mathbb{N}$ such that $ng = 0$, then the *smallest* such n is called *the order* of g, we denote it by $o(g)$. If there does not exist an $n \in \mathbb{N}$ such that $ng = 0$, we say that g is of infinite order. The cardinal number $|G|$ of the elements in G is called *the order of* G. G is said to be *finite, countable,* or *uncountable* if $|G|$ is finite (with 1 being the smallest possible order), countably infinite, or uncountable, respectively.

The most important examples of abelian groups that we will discuss in great detail below are: *infinite cyclic, finite cyclic, rational,* and *Prüfer* groups, denoted by $\mathbb{Z}$, $\mathbb{Z}(n)$, $\mathbb{Q}$, $\mathbb{Z}(p^\infty)$ respectively where n is a positive integer and p a prime. The latter group is called the *quasicyclic.*

With regards to $\mathbb{Z}(p^\infty)$, let $\mathbb{Z}(p_i)$ be a cyclic group of order p^i where p is fixed prime and $i \in \mathbb{N}$. The mapping

$$\alpha_i : x_i \in \mathbb{Z}(p^i) \longmapsto \alpha_i(x_i) = px_{i+1} \in \mathbb{Z}(p^{i+1}) \tag{2.0.3}$$

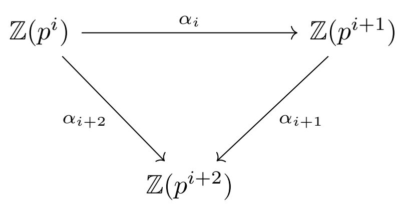
defines a monomorphism. It is clear that

$$\text{Ker } \alpha_i = \big\{x_i \in \mathbb{Z}(p^i) \mid \alpha_i(x_i) = 0\big\} = \big\{x_i \in \mathbb{Z}(p^i) \mid px_i = 0\big\} = 0, \tag{2.0.4}$$

implies α_i injective. Consider the composition, $\alpha_{i+1} \circ \alpha_i$. Let $x_i \in \mathbb{Z}(p^i)$. Then

$$(\alpha_{i+1} \circ \alpha_i)(x_i) = \alpha_{i+1}(\alpha_i(x_i)) = \alpha_{i+1}(px_{i+1}) = px_{i+2}, \tag{2.0.5}$$

which is exactly $\alpha_i(x_i) = px_{i+2}$. Therefore, $\alpha_{i+1} \circ \alpha_i = \alpha_i$, that is



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commutes. We call $\{\alpha_i : \mathbb{Z}(p^i) \to \mathbb{Z}(p^{i+1}) \mid i \in \mathbb{N}\}$ a *direct system*. By [24, Theorem 1.4.9], we can write the limit of the direct system as

$$\varinjlim_{i \in \mathbb{N}} \mathbb{Z}(p^i) = \bigcup_{i \geq 1} \mathbb{Z}(p^i) \tag{2.0.6}$$

and this group $\bigcup\limits_{i \geq 1} \mathbb{Z}(p^i)$ is called the Prüfer group, denoted by $\mathbb{Z}(p^\infty)$.

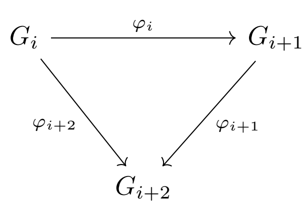
With regards to $\mathbb{Q}$ (see [24, Exercise 1.4.11]), let $G_1, G_2, \ldots$ be the sequence of groups (where $\forall i, G_i = \mathbb{Z}$), the mapping

$$x_i \in \varphi_i : G_i \longmapsto \varphi_i(x_i) = nx_{i+1} \in G_{i+1} \tag{2.0.7}$$

defines a monomorphism and it is clear that it is injective from

$$\text{Ker } \varphi_i = \{x_i \in G_i \mid \varphi_i(x_i) = 0\} = 0. \tag{2.0.8}$$

It is also clear that



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commutes and so let $\{\varphi_i : G_i \to G_{i+1} \mid i \in \mathbb{N}\}$ be a direct system, then by [24, Theorem 1.4.9], we have this direct limit

$$\varinjlim_{i \in \mathbb{N}} G_i = \bigcup_{i \geq 1} G_i \tag{2.0.9}$$

and $\bigcup\limits_{i \geq 1} \mathbb{Z}$ is called the rational group and denoted by $\mathbb{Q}$.

Now we pass to describe a dual situation, reverting arrows in the above diagrams and considering the notion of *inverse limit*, not anymore of direct limit, for abelian groups of primer power order. References for these constructions can be found in [9, 11]. Dual to (2.0.3), define

$$\varphi_i : \ x_{i+1} \in \mathbb{Z}(p^{i+1}) \longmapsto \varphi_i(x_{i+1}) = x_i \in \mathbb{Z}(p^i). \tag{2.0.10}$$

Then one easily verifies that φ_i is a homomorphism and therefore

$$\left\{\varphi_i : \ \mathbb{Z}(p^{i+1}) \to \mathbb{Z}(p^i) \mid i \in \mathbb{N}\right\} \tag{2.0.11}$$

is *inverse system*. Since both $\mathbb{Z}(p^{i+1})$ and $\mathbb{Z}(p^i)$ are finite and $\mathbb{Z}(p^i) \subseteq \mathbb{Z}(p^{i+1})$ and φ_i maps each element of $\mathbb{Z}(p^{i+1})$ to a unique element of $\mathbb{Z}(p^i)$, one sees that φ_i is surjective. Then we write the limit of the inverse system as

$$\varprojlim_{i \in \mathbb{N}} \mathbb{Z}(p^i) = \mathbb{Z}_p \tag{2.0.12}$$

and $\mathbb{Z}_p$ is called the group of p-adic integers. This group is relevant in topological group theory and its properties will be described properly in the final chapter of the present thesis.

2.1 Torsion Groups and Divisible Groups

Let G be a group and $\mathrm{tor}(G)$ be the set of all elements of finite order in G, these elements are called *torsion*, that is,

$$\mathrm{tor}(G) = \{g \in G \mid o(g) < \infty\},$$

then it is not difficult to see that $\mathrm{tor}(G)$ is the subgroup of G, called *the torsion* subgroup of G. G is called *torsion* if $G = \mathrm{tor}(G)$. If $\mathrm{tor}(G) = 0$, then G is called *torsion-free.* Similarly, let $g \in G$, if there is some $g_1 \in G$ such that $g = ng_1$, for $n \in \mathbb{Z}$ we say that g is *divisible* by n in G. If each $g \in G$ is divisible by n in G, then we call G *divisible* and write $G = \Omega^n(G)$, where

$$\Omega^n(G) = \{ng \mid g \in G\},$$

(see (1.3.12)). Dual to $\Omega^n(G)$, we have

$$\Omega_n(G) = \{g \in G \mid ng = 0\},$$

(see (1.3.8)) which is another subgroup of G, which will later be called a *bounded* subgroup (compare this with the notion of finite exponent in the previous chapter). The set of elements in G of p-power order is called a *p-primary component*, symbolically,

$$G_p = \{g \in G \mid o(g) = p^n \text{ for some positive integer } n\}, \tag{2.1.1}$$

and again, it is not difficult to show that this is a subgroup of G, in addition, if $G = G_p$, then G is called an *elementary p-primary* group, obviously G_p is torsion. Therefore, abelian groups are either *torsion, torsion-free* or *mixed,* it is called mixed when $\mathrm{tor}(G) \neq 0$ and $\mathrm{tor}(G) \neq G$. As we have said, p-components are torsion, it turns out that the p-components of a group are enough to describe torsion groups, in fact we have the following result.

Theorem 2.1.1 (The Primary Decomposition Theorem, See [24], Theorem 4.1.1)**.** *In an abelian group G the torsion-subgroup $\mathrm{tor}(G)$ is the direct sum of the primary components of G.*

Exercise 2.1.2 (See [24], Exercise 4.1.1)**.** *Prove that a group of type p^∞ has exactly one subgroup of each order p^i and this is cyclic. Show also that every proper subgroup is finite.*

Proof. As realized as direct limit in (2.0.6), we can write a group of type p^∞ as $\mathbb{Z}(p^\infty) = \bigcup_{i\in I} \langle x_i \rangle$ where each $\langle x_i \rangle$ is a cyclic of order p^i and p is a fixed prime. For the first part, all cyclic groups of the same order are isomorphic, hence there must be *one* subgroup (up to isomorphism) of the order p^i. Since $\langle x_i \rangle$ has order p^i for each i, we have that

$$G = \mathbb{Z}(p^i)^{(I)}$$

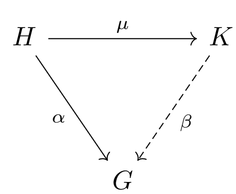
where G is a torsion group by Theorem 2.1.1. Suppose $g \in \bigcup_{i\in I} \langle x_i \rangle$ then,

$$g \in \bigcup_{i\in I} \langle x_i \rangle \implies \exists i \in I : g \in \langle x_i \rangle \implies g \in G$$

therefore we have $\bigcup_{i\in I} \langle x_i \rangle \subseteq G$. This means that every element of $\bigcup_{i\in I} \langle x_i \rangle$ is of finite order and hence every proper subgroup is of finite order. □

Now Exercise 2.1.2 above and Exercise 2.1.6 below say that group G whose all proper subgroups are cyclic of prime-power order is Prüfer, and conversely, an infinite group whose all proper subgroups are finite is Prüfer for some prime. Therefore, Prüfer groups give us an example of infinite torsion abelian group. Let's recall some classical results, before to go ahead.

As we know, all infinite cyclic groups are known to be isomorphic to $\mathbb{Z}$. It turns out that infinite groups of finite proper quotients are just $\mathbb{Z}$. Indeed, let $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ for some nonzero $n \in \mathbb{N}$, then $\mathbb{Z}/n\mathbb{Z}$ is just $\mathbb{Z}(n)$, a finite cyclic group of order n. See Exercise 2.1.6. Let G be a group. If for an injective homomorphism $\mu : H \to K$ and any homomorphism $\alpha : H \to G$, for any groups H and K, there exists a homomorphism $\beta : K \to G$ such that



commutes, then G is called *injective.* Baer in Theorem 2.1.3 says that if we have injective groups, then we have divisible groups, and vice-versa.

Theorem 2.1.3 (Baer, See [24], Theorem 4.1.2)**.** *An abelian group is injective if and only if it is divisible.*

The most important consequence of Theorem 2.1.3 is that divisible subgroups are direct summands.

Theorem 2.1.4 (See [24], Theorem 4.1.3)**.** *If D is a divisible subgroup of an abelian group G, then $G = D \oplus E$ for some subgroup E of G.*

Let also recall another decomposition, which we may encounter when we deal with divisible abelian groups.

Theorem 2.1.5 (The Structure of Divisible Abelian Groups, See [24], Theorem 4.1.5)**.** *An abelian group G is divisible if and only if it is a direct sum of isomorphic copies of $\mathbb{Q}$ and of quasicyclic groups $\mathbb{Z}(p^\infty)$.*

An application of the above classical result is given by the following description of abelian infinite groups which are plenty of finite quotients, or of finite subgroups.

Exercise 2.1.6 (See [24], Exercise 4.1.2 and 4.1.3)**.** *If G is an infinite abelian group all of whose proper subgroups are finite, then G is of type p^∞ for some prime p. On the other hand, if all proper quotients are finite, then G is isomorphic to $\mathbb{Z}$.*

Proof. Let G be an infinite abelian group whose all proper subgroups are finite. First note that $G \neq \mathbb{Z}$ since $n\mathbb{Z}$ is a proper infinite subgroup of $\mathbb{Z}$ for all $n \in \mathbb{Z}$. So since every proper subgroup of G is finite, that means that all elements of G have finite order, and hence by Theorem 2.1.1 G is a direct sum of its primary p-components. Then consider $\Omega^p(G)$, proper subgroup of $G = \bigoplus_{p\in\mathbb{P}} G_p$. So $\Omega^p(G)$ is either properly contained in G or it is G itself, in the latter case, $G = \Omega^p(G)$. If $\Omega^p(G)$ is properly contained in G, then $\Omega^p(G)$ is finite by how G is assumed to be. Then $G/\Omega^p(G)$ is infinite. So let $V = G/\Omega^p(G)$ be an infinite dimensional $\mathbb{Z}(p)$-vector space. Then homomorphism $\alpha: G \to V$, is an epimorphism and $\alpha^{-1}(W)$ is a proper infinite subgroup of G for a subspace $W \subseteq V$. This is a contradiction since G does not admit infinite proper subgroups. So $\Omega^p(G)$ is not finite, and $\Omega^p(G) = G$. We have the following relation

$$G = \bigoplus_{p\in\mathbb{P}} G_p = \Omega^p(G) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$$

and so $G = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$. Since G is the direct sum of $\mathbb{Z}(p^\infty)$ and each $\mathbb{Z}(p^\infty)$ is infinite, it must be the case that $G = \mathbb{Z}(p^\infty)$.

For the second part, let G be an infinite abelian group of whose proper quotient groups are finite. Then for some subgroup H of G,

$$G/H = \{g + H \mid g \in G\} \text{ with } |G/H| = n \tag{2.1.2}$$

for some $n \in \mathbb{N}$. Suppose G is not cyclic. Then:

$$\forall\, g \in G, \exists\, m \in \mathbb{N} : m(g+H) = mg + H = H \implies mg \in H.$$

This means that $\forall\, g \in G$, $\exists\, m \in \mathbb{N} : h = mg$, making H divisible. Then by Theorem 2.1.4, we have $G = H \oplus E$ for some subgroup E of G and so the Second Isomorphism Theorem [24, Theorem 1.4.4] implies

$$G/E = (H+E)/E \cong H/(H \cap E) = H. \tag{2.1.3}$$

Then H is divisible and finite, and this is impossible (see Theorem 2.1.5). □

Exercise 2.1.7 (See [24], Exercise 4.1.6)**.** *Let G be an abelian p-group such that $G/\Omega_p(G)$ is divisible. Prove that G is the direct sum of a divisible group and an elementary abelian p-group.*

Proof. Assume G is abelian and let $G/\Omega_p(G)$ be divisible. From Theorem 2.1.3, we have $G = A \oplus G_p$ where A is not divisible and G_p is a p-primary component of G. Then we have $A \cap G_p = 0$ and g can be uniquely expressed as $g = a + g_p$ with $a \in A$ and $g_p \in G_p$ with p^s order of g_p. Therefore the definition of divisibility implies,

$$\forall\, g + \Omega_p(G) \in G/\Omega_p(G), \exists\, m \in \mathbb{N} : g + \Omega_p(G) = m(g_1 + \Omega_p(G)) \tag{2.1.4}$$

for some $g_1 \in G$, since $G/\Omega_p(G)$ is divisible. From $g = a + g_p$, we get

$$\begin{aligned}
& g + \Omega_p(G) = (a + g_p) + \Omega_p(G) \\
\implies & m(g_1 + \Omega_p(G)) = (a + g_p) + \Omega_p(G) \\
\implies & mg_1 + \Omega_p(G) = (a + g_p) + \Omega_p(G) \\
\implies & (mg_1 - (a + g_p)) + \Omega_p(G) = \Omega_p(G) \\
\implies & (mg_1 - (a + g_p)) \in \Omega_p(G) \\
\implies & p^s\,(mg_1 - (a + g_p)) = p^s mg_1 - p^s(a + g_p) = 0 \\
\implies & p^s mg_1 - p^s a - p^s g_p = 0 \\
\implies & p^s mg_1 - p^s a = 0 \\
\implies & p^s a = p^s mg_1 \\
\implies & a = mg_1
\end{aligned}$$

which means that a is divisible,a contradiction. □

In particular, $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are divisible. Now, consider $\mathbb{Z}$ as subgroup of $\mathbb{Q}$. It is now easy to see that $\mathbb{Z}$ is not divisible even though it is a subgroup of a divisible group. It turns out that divisible groups exhaust *all* groups, in the sense of the following result:

Theorem 2.1.8 (See [24], Theorem 4.1.6)**.** *Every abelian group is isomorphic to a subgroup of a divisible abelian group.*

2.2 Direct Sums of Cyclic and Quasicyclic Groups

In this section we describe the structure of *finite* and *finitely generated* groups, the structure of groups with the *minimal condition* and those with the *maximal condition*. In fact, we will establish that the following conditions are equivalent:

(i) G is finitely generated (Theorem 2.2.8);

(ii) G is the direct sum of the finite number of cyclic groups (Theorem 2.2.8);

(iii) the subgroups of G satisfy the maximal condition (Theorem 2.2.18).

These groups are all expressed as the direct sums of cyclic and quasicyclic groups.

According to the terminology of Chapter 1, we record the result below as further description of free abelian groups.

Theorem 2.2.1 (Free Abelian Groups, See ([24], Theorem 2.3.8))**.** *If F is a free abelian group on a subset X, then F is the direct product of the infinite cyclic subgroups $\langle x \rangle$, $x \in X$. Conversely a direct product $\mathbb{Z}^{(X)}$ is free abelian on X.*

Theorem 2.2.1 as well as Exercise 1.1.8 essentially says that a group F is free on X if and only if $F = \mathbb{Z}^{(X)}$, where $|X|$ is the rank of F as in Definition 1.1.1.

Exercise 2.2.2 (See [24], Exercise 4.2.1)**.** *If G is a free abelian group on a set with n elements, then G cannot be generated by fewer than n elements.*

Proof. Let $X = \{x_1, x_2, \dots, x_n\}$ be the set on which G is free. Then any $g \in G$ is uniquely expressed in the form $g = m_1x_1 + m_2x_2 + \cdots + m_nx_n$ with $0 \neq x_i \in X$ all distinct and $m_i \in \mathbb{Z}$. If the set $\overline{X} = \{x_1, x_2, \dots, x_r\}$, with $r < n$, generates G, then $g = l_1x_1 + l_2x_2 + \cdots + l_rx_r$ and from $g - g = 0$ we get:

$$(m_1 - l_1)x_1 + (m_2 - l_2)x_2 + \cdots + m_nx_n = 0 \Longleftrightarrow m_i - l_i = 0, m_n = 0 \quad (2.2.1)$$

this means that all the $r + 1$ integers must be 0, violating the unique expression of g above.

□

Before we go further, let us discuss the notion of the rank on any group G which is not necessarily free. A subset $0 \neq X \subseteq G$ of nonzero elements is said to be *linearly independent* if

$$l_1x_1 + l_2x_2 + \cdots + l_nx_n = 0 \implies l_ix_i = 0, \quad (2.2.2)$$

where $x_i \in X$ and $l_i \in \mathbb{Z}$, (see [24, p.99]). Of course, that $l_ix_i = 0$, means that $l_i = 0$ if x_i is torsion-free, that is, $o(x_i) = \infty$; and $o(x_i)|l_i$ if x_i is torsion of order $o(x_i)$. Now let X be a linearly independent subset of G, then X is said to be a *maximal* linearly independent subset if it is not properly contained in any other linearly independent subset of G. We will refer, for short, to a maximal linearly independent subset as just *maximal independent* subset.

If $X \subseteq G$ is a maximal independent subset containing elements of infinite order and elements of prime-power order, we refer to the cardinal $|X|$ as *the rank* of G, denoted by $r(G)$. On the other hand, if X is a maximal independent subset containing elements of infinite order, then the cardinal $|X|$ is referred to as *the torsion-free rank* or *0-rank* of G, denoted by $r_0(G)$. Lastly, if X is a maximal independent subset containing elements of prime-power order, then the cardinal $|X|$ is referred to as *the p-rank* of G, denoted by $r_p(G)$. By the so called rank formula, the rank $r(G)$ of G, can be shown to be

$$r(G) = r_0(G) + \max\{r_p(G) \mid p \in \pi(G)\}, \tag{2.2.3}$$

where

$$\pi(G) = \left\{p \in \mathbb{P} \mid g \in \operatorname{tor}(G) \text{ and } o(g) = p^k \text{ for some } k \in \mathbb{N}\right\}. \tag{2.2.4}$$

The rank of a group G does not depend on the chosen maximal independent set, but only of G itself. Formally, we have:

Theorem 2.2.3 (See [24], Theorem 4.2.1)**.** *If G is an abelian group, two maximal independent subsets consisting of elements with order a power of the prime p have the same cardinality. The same is true for the maximal independent subsets consisting for the elements of infinite order. Thus $r_0(G)$, $r_p(G)$, and $r(G)$ are arithmetic invariants depending only on G.*

Let X be the maximal independent subset of a group G, then by Theorem 2.2.3, the rank of G is unique, hence we speak of *the rank* of G as is was the case in Chapter 1 in free groups by Theorem 1.1.3 and Exercise 1.1.4. If, in addition, X generates G, that is, $\langle X\rangle = G$, then we call X a *basis* of G, (compare this to Definition 1.1.1).

In particular, $r(G)$ equals $d(G)$ in a finitely generated group, where $d(G)$ is the cardinal of the minimum number of generators of G (see Exercise 2.2.2 and Exercise 2.2.9). Note that

$$d(\mathbb{Z}^{(X)}) = r(\mathbb{Z}^{(X)}) = r_0(\mathbb{Z}^{(X)}) = |X| \tag{2.2.5}$$

and

$$d(\mathbb{Z}(p)^{(X)}) = r_p(\mathbb{Z}(p)^{(X)}) = |X| \tag{2.2.6}$$

but

$$d(A_5^{(X)}) \neq r_p(A_5^{(X)}) \text{ (see [8])}. \tag{2.2.7}$$

For instance, we can ask as in [24, Exercise 4.2.12] how many isomorphism types are there of abelian groups of order p^n? We can start, for example, by letting $p = 2$ and consider the Table 2.1. We see that for 2^0, we have only one group; for 2^1, we have only one group; for 2^2, we have only two groups; for 2^3, we have only three groups; for 2^4, we have only five groups, and so on. We see

n	p^2	**Groups**
0	2^0	$\mathbb{Z} \cong 0$
1	2^1	$\mathbb{Z}(2)$
2	2^2	$\mathbb{Z}(4), \mathbb{Z}(2) \oplus \mathbb{Z}(2)$
3	2^3	$\mathbb{Z}(8), \mathbb{Z}(4) \oplus \mathbb{Z}(2), \lvert 2-\text{group}\rvert = 2^3$
4	2^4	$\mathbb{Z}(16), \mathbb{Z}(8) \oplus \mathbb{Z}(2), \mathbb{Z}(4) \oplus \mathbb{Z}(4), \mathbb{Z}(4) \oplus K_4, \lvert 2-\text{group}\rvert = 2^4$

Table 2.1: Isomorphic types of abelian groups of order p^n

that the number of groups is equal to the partition of n. We can see the abelian groups isomorphic to, for example, an abelian group of order 2^2 is 2 which is the integer partition of $n = 2$. So the abelian groups isomorphic to an abelian group of order p^n is given by the partition of n. See Table 2.1.

Exercise 2.2.4 (See [24], Exercise 4.2.4)**.** *If A and B are finitely generated abelian groups and B is torsion-free, show that $d(A \oplus B) = d(A) + d(B)$.*

Proof. Let $G = A \oplus B$, then G is finitely generated. Then

$$G \cong \left(\mathbb{Z}^n \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Z}(p_i^{\alpha_i})\right)\right) \oplus \mathbb{Z}^m \cong \mathbb{Z}^l \oplus \left(\bigoplus_{i=1}^{k} \mathbb{Z}(p_i^{\alpha_i})\right) \tag{2.2.8}$$

where $l = n + m$. Since

$$\mathbb{Z}^l \cap \left(\bigoplus_{i=1}^{k} \mathbb{Z}(p_i^{\alpha_i})\right) = 0, \tag{2.2.9}$$

we have $d(G) = l + k = d(A) + d(B)$. □

Exercise 1.1.12 says that normal subgroups N such that G/N is free may detect direct factors and decompositions in groups. In Theorem 2.2.5 below, we have the abelian analog.

Theorem 2.2.5 (See [24], Theorem 4.2.5)**.** *If G is an abelian group and if H is a subgroup of G such that G/H is free abelian, then $G = H \oplus K$ for some subgroup K of G.*

Below we present the most crucial theorem as the first of our classification theorems of abelian groups. Due to Frobenius-Stickelberger, Theorem 2.2.6 describes all the finite abelian groups. We also give a proof of this result based on the proof given in [24].

Theorem 2.2.6 (Frobenius-Stickelberger, See [24], Theorem 4.2.6)**.** *An abelian group G is finite if and only if it is a direct sum of finitely many cyclic groups with prime-power orders.*

Sketch of the proof. Assume that G is finite. Then by Theorem 2.1.1, we can write: $G = \bigoplus_{p \in \mathbb{P}} G_p$. From now on, we assume that G is a p-group, that is, $G = G_p$.

Claim: *Let A be any finite abelian p-group whose elements have bounded order $\leq p^n$ and take $a \in A$ an element of order exactly p^n. Then $\langle a \rangle$ is a direct summand of A.*

Saying that all the elements of A have bounded order means that for all the elements of A we have that the order is $\leq p^n$ (for a given $n \geq 1$).

Let $a \in A$ of exactly order p^n and consider $\langle a \rangle$ which will be cyclic of order p^n. Then by Zorn's Lemma there is a subgroup M which is maximal with respect to $M \cap \langle a \rangle = 0$.

Now we want to check that M is the direct summand that we are looking for. If $A = M + \langle a \rangle$, then $A = M \oplus \langle a \rangle$.

If $A \neq M + \langle a \rangle$, then necessarily $A \supset M + \langle a \rangle$ and assume that x is an element in $A \setminus (M + \langle a \rangle)$ of smallest order. We will see that this is not possible.

In fact A is a p-group, so the smallest possible order of x should be a multiple of p, but the choice of x imposes that any other element of order bigger than x is out from $A \setminus (M + \langle a \rangle)$, and so,

$$px \in M + \langle a \rangle \tag{2.2.10}$$

thus there are $l \in \mathbb{Z}$ and $y \in M$ such that

$$px = y + la. \tag{2.2.11}$$

Since a has maximal order in A, say p^n for instance, we have

$$p^{n-1}px = p^{n-1}y + p^{n-1}la \implies 0 = p^n x = p^{n-1}y + p^{n-1}la, \tag{2.2.12}$$

and so $(p^{n-1}l)a \in M \cap \langle a \rangle = 0$.

Consequently, $p^n \mid p^{n-1}l$ and $p \mid l$. Now we can write $l = pj$ for some $j \in \mathbb{Z}$, so that from $px = y + la$, we get

$$px - la = y \implies px - pja = y \implies p(x - ja) = y \in M, \tag{2.2.13}$$

while $(x - ja) \notin M$ since $x \notin (M + \langle a \rangle)$.

Since M is maximal (with respect to $M \cap \langle a \rangle = 0$),

$$\langle x - ja, M \rangle \cap \langle a \rangle \neq 0, \tag{2.2.14}$$

and this implies that there is a nontrivial element in $\langle a \rangle$ which can be written also as an element of $\langle x - ja, M \rangle$, that is, there are $k, m \in \mathbb{Z}$ and $y' \in M$ such that

$$0 \neq ka = m(x - ja) + y' \iff 0 \neq ka = mx - mja + y' \tag{2.2.15}$$

$$\iff ka + mja + (-y') = mx \iff (k + mj)a + (-y') = mx \implies mx \in M + \langle a \rangle.$$

Suppose that $p \mid m$. We have $p(x - ja) \in M$, because

$$p(x - ja) = px - pja = px - (pj)a = px - la = y \in M, \tag{2.2.16}$$

and it follows that

$$m(x - ja) \in M \implies ka = 0, \tag{2.2.17}$$

because

$$m(x - ja) + y' \in M, \text{ and } M \cap \langle a \rangle = 0. \tag{2.2.18}$$

Hence this cannot happen and we should have $\gcd(p, m) = 1$. However $px \in M + \langle a \rangle$, so $x \in M + \langle a \rangle$, which is a contradiction with the choice of x. The claim is proved.

Let us come back to our $G = G_p$ after we proved the claim. If $g \in G$ is of maximal order, then $G = \langle g \rangle \oplus G_1$ for some subgroup G_1 of G.

Since $|G_1| < |G|$, we choose $g_1 \in G_1$ of maximal order such that

$$G_1 = \langle g_1 \rangle \oplus G_2 \tag{2.2.19}$$

for some subgroup G_2 of G_1, such that

$$G = \langle g \rangle \oplus G_1 = \langle g \rangle \oplus \langle g_1 \rangle \oplus G_2. \tag{2.2.20}$$

The result follows this way, on induction on the finite order of G. The converse is clear. This completes the proof. □

A few facts on the rank in abelian p-groups are recalled here.

Exercise 2.2.7 (See [24], Exercise 4.2.2)**.** *If G is an abelian group, show that $r(G)$ is finite if and only if*

$$\dim(G) = \max\{d(H) \mid H \textit{ is a finitely generated subgroup of } G\} \tag{2.2.21}$$

is finite. In this case $r(G) = \max\ \{d(H)\}$.

Proof. The fact that $\dim(G) \le r(G)$ is clear. Conversely, suppose that $r(G) < \infty$. This means that the maximal independent subset X of G is finite. Let H be any finitely generated subgroup of G. Then we know that, any $g \in G$ can be expressed as

$$g = m_1x_1 + m_2x_2 + \cdots + m_rx_r, \tag{2.2.22}$$

where $|X| = r < \infty$ and $m_1, m_2, \ldots, m_r \in \mathbb{Z}$. We also know that, any $h \in H$ can be expressed as

$$h = n_1y_1 + n_2y_2 + \cdots + n_ly_l, \tag{2.2.23}$$

for $y_1, y_2, \ldots, y_l \in Y$, $n_1, n_2, \ldots, n_l \in \mathbb{Z}$ and Y is the generating set of H with $|Y| = l$. Since

$$h = k_1x_1 + k_2x_2 + \cdots + k_rx_r, \tag{2.2.24}$$

where $k_1, k_2, \ldots, k_r \in \mathbb{Z}$ and $x_1, x_2, \ldots, x_r \in X$, we have $Y \subseteq X$. So all the generating sets of these finitely generated subgroups H of G are contained in X.

Also, since X is finite, we must have that $\dim(G) < \infty$. Then assume that $\dim(G) < \infty$. This means that the largest minimal generating set of all the finitely generated subgroups H of G is finite. If Y is the minimal generating of H, then we have that $|Y| < \infty$ and Y forms a basis for H, so that $r(H) = |Y|$. Since the bases of the finitely generated subgroups of G are contained in X, we conclude that $r(G) \leq \dim(G)$. □

The generalization of Frobenius-Stickelberger (Theorem 2.2.6) for finite abelian groups is for finitely generated abelian groups. Formally:

Theorem 2.2.8 (Structure of Finitely Generated Groups, See [24], Theorem 4.2.10)**.** *An abelian group G is finitely generated if and only if it is a direct sum of finitely many cyclic groups of infinite or prime-power orders.*

The details of the proof can be found in [24, Theorem 4.2.10] and uses strongly Theorem 2.2.6. We omit it since it is a well known fact.

Exercise 2.2.9 (See [24], Exercise 4.2.3)**.** *If G is a finitely generated abelian group, show that $d(G) = r(G)$. Also $d(G) = r_0(G)$ if and only if G is torsion-free.*

Proof. Let G be finitely generated, then

$$G \cong \mathbb{Z}^n \oplus \left(\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i}) \right) \tag{2.2.25}$$

so $\mathbb{Z}$ and $\mathbb{Z}(p_i^{\alpha_i})$ are single-generator groups where $p_1, p_2, \ldots, p_k$ are primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers, so $\mathbb{Z}^n$ has n minimal number of generators and $\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i})$ with k minimal number of generators. So let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ be the minimal number of generators of $\mathbb{Z}^n$ and $\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i})$ respectively. Since $\mathbb{Z}^n \cap \left(\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i}) \right) = 0$, we have $X \cap Y = 0$ and so $|X \cup Y| = |X| + |Y| = n + k = d(G)$. $\mathbb{Z}$ is cyclic, so it has rank 1, or $\mathbb{Z}^n$ has rank n, also $\mathbb{Z}(p_i^{\alpha_i})$ is cyclic so it has rank 1 or $\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i})$ has rank k. $\mathbb{Z}^n$ is torsion-free and $\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i})$ is torsion, so $r(G) = r_0(\mathbb{Z}^n) + \max\limits_p r_p \left(\overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i}) \right) = n + k$.

Assume that G is torsion-free. Since G is finitely generated, $G \cong \mathbb{Z}^n$ and so

$$d(G) = d(\mathbb{Z}) = r(\mathbb{Z}^n) = r_0(G). \tag{2.2.26}$$

Assume $d(G) = r_0(G)$ with $r(G) = r_0(G) + \max\limits_p r_p(G)$. Since G is finitely generated, $G \cong \mathbb{Z}^n \bigoplus \overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i})$ and by the previous argument, we get

$$d \left(\mathbb{Z}^n \bigoplus \overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i}) \right) = r \left(\mathbb{Z}^n \bigoplus \overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i}) \right) = r_0 \left(\mathbb{Z}^n \right). \tag{2.2.27}$$

It means that

$$r\left(\mathbb{Z}^n \bigoplus \overset{k}{\underset{i=1}{\oplus}} \mathbb{Z}(p_i^{\alpha_i})\right) = r_0\left(\mathbb{Z}^n\right) \tag{2.2.28}$$

and $\max\limits_p r_0(G) = 0$. So G must be torsion-free. □

Combining Exercises 2.2.4, 2.2.7 and 2.2.9, we see that in an abelian group G, $\dim(G) = r(G) = d(G)$. In particular, Exercises 2.2.9 and 2.2.4 show that the function $r : G \in \mathfrak{B}(W) \mapsto r(G) \in \mathbb{N}$ is additive under appropriate assumptions. When this function is small enough, one can find interesting results of structure and the following is a classical situation describing groups of small rank.

Exercise 2.2.10 (See [24], Exercise 4.2.5)**.** *Prove that an abelian group has rank ≤ 1 if and only if it is isomorphic with a subgroup of $\mathbb{Q}$ or subgroup of $\mathbb{Z}(p^\infty)$.*

Proof. Let G be an abelian group. First assume that $r(G) \leq 1$. We know that $r(G) \leq 1$ if and only if G is *locally cyclic*. A group is called *locally cyclic* if and only its finitely generated subgroups are cyclic. Since locally cyclic groups are not mixed, then either G is torsion or torsion-free. Now $r(\mathbb{Q}) = 1$, then for $r(G) = 1$, we have $G \cong H$ is isomorphic to a subgroup of $\mathbb{Q}$ and for $r(G) = 0$ we must have $\max\limits_p r_p(G) = 0$ and $r_p(G) = 0$, so $G \cong H$ is isomorphic to a subgroup of $\mathbb{Z}(p^\infty)$. □

Another relevant result of structure for groups of bounded rank can be found in the following result.

Exercise 2.2.11 (See [24], Exercise 4.2.6)**.** *A group G is torsion-free abelian of $r(G) \leq r$ if and only if it is isomorphic to a subgroup of $\mathbb{Q}$-vector space of dimension r.*

Proof. Let G be a torsion-free abelian group. Already $r(G) = r_0(G)$. Assume that G is a torsion-free abelian group which has rank 1. Then, by Exercise 2.2.10, $G \cong H$ is isomorphic to a subgroup of $\mathbb{Q}$. If G is a torsion-free abelian group of rank 2, then $G = H \oplus K$ is isomorphic to a subgroup of $\mathbb{Q} \oplus \mathbb{Q} \cong \mathbb{Q}^2$, where $r_0(H) = 1$ and $r_0(K) = 1$. Assume this is true for $r = n-1$ and suppose $r = n$. Then

$$H = \bigoplus_{i=1}^{n-1} H_i \tag{2.2.29}$$

is isomorphic to a subgroup of $\mathbb{Q}^{n-1}$ and is of rank $n-1$, where $r_0(H_i) = 1$. Then $G \cong H \oplus K$ is isomorphic to a subgroup of $\mathbb{Q}^n \cong \mathbb{Q}^r$, where $r_0(K) = 1$.

Conversely, let V be a $\mathbb{Q}$-vector space of dimension r and let $G \cong H$ be isomorphic to a subgroup of V. Then every element $v \in V$ can be expressed uniquely as

$$v = q_1 v_1 + \cdots + q_r v_r, \tag{2.2.30}$$

where $v_i \in V$ are basis elements and $q_i \in \mathbb{Q}$. Then any $g \in G$ can be expressed uniquely as

$$g = q_1 g_1 + \cdots + q_s g_s, \tag{2.2.31}$$

where $s \leq r$. Then $\{g_i \mid i = 1, 2 \cdots, s\}$ is an independent (by the uniqueness of the representation of g) generating set of G. So the rank of G is $s \leq r$. If there is $n \in \mathbb{N}$ such that $ng = 0$, then

$$\begin{aligned} ng &= n(q_1 g_1 + \cdots + q_s g_s) = 0 \qquad (2.2.32) \\ &\Longrightarrow nq_1 g_1 + \cdots + nq_s g_s = 0 \\ &\Longrightarrow (nq_1) g_1 + \cdots + (nq_s) g_s = 0 \\ &\Longrightarrow nq_i = 0. \end{aligned}$$

As $n \neq 0$, that would mean that $q_i = 0$ for all i, so that g is trivial. Finally the only element in G of finite order is the trivial element. □

We continue to describe situations when the function of rank is additive, or is bounded. Apparently these considerations were made by Hirsch originally for solvable groups (see the notion of Hirsch rank, or Hirsch length in [24]).

Exercise 2.2.12 (See [24], Exercise 4.2.7)**.** *If H is a subgroup of an abelian group G, prove that the following properties are valid:*

(a) $r_0(H) + r_0(G/H) = r_0(G)$;

(b) $r_p(H) + r_p(G/H) \geq r_p(G)$ with inequality in general.

Proof of (a). Since $r_0(G)$ is the cardinality of the maximal independent subset of torsion-free elements of G, let $\{h_1, h_2, \ldots, h_s\}$ be an independent subset of torsion-free elements of H. We extend this independent subset to get an independent maximal subset

$$\{h_1, h_2, \ldots, h_s, g_1, g_2, \ldots, g_k\}, \tag{2.2.33}$$

of G. We want to show that $\{g_1 + H, g_2 + H, \ldots g_k + H\}$ is an independent maximal subset of torsion-free elements of G/H, so that

$$r_0(G/H) = k = (s + k) - s = r_0(G) - r_0(H) \tag{2.2.34}$$

or

$$r_0(G) = r_0(G/H) + r_0(H). \tag{2.2.35}$$

First, assume that $\sum_{i=1}^{k} m_i(g_i + H) = H$, for $m_i \in \mathbb{Z}$, then

$$m_1(g_1 + H) + m_2(g_2 + H) + \cdots + m_k(g_k + H) = H \tag{2.2.36}$$

$$\implies m_1g_1 + H + m_2g_2 + H + \cdots + m_kg_k + H = H$$
$$\implies (m_1g_1 + m_2g_2 + \cdots + m_kg_k) + H = H$$
$$\implies (m_1g_1 + m_2g_2 + \cdots + m_kg_k) \in H$$
$$\implies m_1g_1 + \cdots + m_kg_k = z_1h_1 + \cdots + z_sh_s,\ z_i \in \mathbb{Z}$$
$$\implies m_i = z_i = 0,\ \text{since } \{h_1, \cdots, h_s, g_1, \ldots, g_k\} \text{ is linearly independent,}$$

hence $\{g_1 + H, g_2 + H, \ldots g_k + H\}$ is linearly independent. Let us add another element to this linearly independent set, that is, $\{g_1+H, g_2+H, \ldots g_k+H, g+H\}$ for $0 \neq g \in G$ and consider:

$$n(g + H)+ \sum_{i=1}^{k} m_i(g_i + H) = H, \tag{2.2.37}$$

for some $n, m_i \in \mathbb{Z}$. Then we get

$$ng + (n_1g_1 + n_2g_2 + \cdots + n_kg_k - r_1h_1 - r_2h_2 - \cdots - r_sh_s) = 0 \tag{2.2.38}$$

which would mean that $ng = 0$ and $n_1g_1+n_2g_2+\cdots+n_kg_k-r_1h_1-r_2h_2-\cdots-r_sh_s = 0$. The former will force $g = 0$ which is a contradiction, so $\{g_1 + H, g_2 + H, \ldots g_k + H\}$ is also maximal, that is, it is a linearly independent maximal subset. Hence $r_0(G) = r_0(G/H) + r_0(H)$. □

Proof of (b). We first note that $r_p(G) = r(G_p)$ and $r_0(G) = r(G/\text{tor}(G))$, where G_p and $\text{tor}(G)$ are p-primary components and a torsion subgroup of G respectively. So $r_p(G) \leq r_p(H)+r_p(G/H)$ is equivalent to $r(G_p) \leq r(H_p)+r((G/H)_p)$, in particular, $r(\Omega_p(G)) \leq r(\Omega_p(H))+r(\Omega_p(G/H))$. Note that, all $\Omega_p(G)$, $\Omega_p(H)$ and $\Omega_p(G/H)$ are all $\mathbb{Z}(p)$-vector spaces where $\Omega_p(G) = \{g \in G \mid pg = 0\}$. Letting φ such that $\Omega_p(G) \to \Omega_p(G/H)$ be a canonical linear transformation (or a an epimorphism between two abelian groups) and observing that $\text{Ker}\ \varphi = \Omega_p(H)$ and $\text{Im}\ \varphi = \Omega_p(G/H)$. Thus, we get our result by applying the dimension-rank formula of linear transformations between vector spaces. □

We recall that we gave a very useful result in Chapter 1 (Theorem 1.1.11) in the form of the Universal Projective Property of general free groups, and here we give a similar account for free abelian case due to McLane:

Theorem 2.2.13 (McLane, See [24], Theorem 4.2.5)**.** *If G is an abelian group and if H is a subgroup of G such that G/H is free abelian, then $G = H \oplus K$ for some subgroup K.*

Exercise 2.2.14 (See [24], Exercise 4.2.9)**.** *An abelian group G is free if and only if it has the following property: if K is a subgroup of an abelian group H and H/K is isomorphic to G, then K is a direct summand of H.*

Proof. Let G be free and let H/K be isomorphic to G. Then $H = K \oplus E$ for some subgroup E of H by Theorem 2.2.13.

Conversely, let K be a subgroup of H and that H/K is isomorphic to G implies that $H = K \oplus E$ for some subgroup E of H. Let $\varepsilon :\ H \to H/K$

be a natural homomorphism which is known to be surjective. Also define $\beta : H/K \to H$ by $h + K \mapsto h$ with $1 : H/K \to H/K$ an identity homomorphism. Since $(\varepsilon \circ \beta)(h+K) = \varepsilon(h) = h+K$ then $\varepsilon \circ \beta = 1$, so that H/K is projective and hence free by MacLane, Theorem 2.2.13. □

Exercise 2.2.15 (See [24], Exercise 4.2.10)**.** *If G is a finitely generated abelian group, every surjective endomorphism of G is an automorphism.*

Proof. Let $\{x_1, x_2, \dots, x_k\}$ be the generating set of G and $\varphi : G \to G$ be a homomorphism such that $\varphi(G) = G$. For $g \in G$, we have $g = n_1x_1 + n_2x_2 + \cdots + n_kx_k$ for $n_i \in \mathbb{Z}$. As G is a $\mathbb{Z}$-module, we have that $\varphi(g) = n_1\varphi(x_1) + n_2\varphi(x_2) + \cdots + n_k\varphi(x_k)$. Assuming that Ker $\varphi \neq 0$, there is $0 \neq g \in$ Ker φ. Then $\varphi(g) = 0$. That is $\varphi(g) = n_1\varphi(x_1) + n_2\varphi(x_2) + \cdots + n_k\varphi(x_k) = 0$ this implies that $n_i = 0$ as $\varphi(x_i) \neq 0$. But if $n_i = 0$ for all i, then $g = 0$ contradicting our assumption that g was nontrivial. Showing that Ker $\varphi = 0$. □

Definition 2.2.16. *A group G (not necessarily abelian) is called Hopfian if every epimorphism $G \to G$ is an automorphism.*

In order to give a further description for the groups in Theorem 2.2.18 and Exercises 2.2.19 and 2.2.20 we give the following definition.

Definition 2.2.17. *Let X be a set and $\leq$ be a binary relation. Then we say $(X, \leq)$, or simply X, is partially ordered, or briefly a poset if:*

(i) $x \leq x$, for all $x \in X$ (that is $\leq$ is reflexive);

(ii) $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in X$ (that is $\leq$ is antisymmetric); and

(iii) $x_1 \leq x_2$ and $x_2 \leq x_3$, then $x_1 \leq x_3$, for all $x_1, x_2, x_3 \in X$ (that is $\leq$ is transitive)

Let X be a poset and $S \subseteq X$, then element $m \in X$ is said to be a *maximal* element of S if $s \leq m$ for all $s \in S$. Then if for each $S \subseteq X$, S has at least one maximal element, we say that X satisfies the *maximal condition.* The notion of the *minimal condition* is defined analogously. If a set satisfies the maximal condition (or the minimal condition), we say it satisfies the *max* (or the *min.*) Clearly, if G is a group, and $\mathcal{L}(G)$ the set of all subgroups of G, the $\mathcal{L}(G)$ together with the set inclusion $\subseteq$ is a partially ordered set. It turns out that groups that satisfy the max on their subgroups are special, in fact, apart from obvious examples of finite groups, we find infinite groups, described by the following theorem.

Theorem 2.2.18 (See [24], Theorem 3.1.6)**.** *A group satisfies max if and only if every subgroup is finitely generated.*

In particular (and of great import in abelian group theory), Theorem 2.2.18 says that G satisfies max on subgroups if and only if each subgroup of G is the normal closure of a finite set.

Exercise 2.2.19 (See [24], Theorem 3.1.7)**.** *If G has min, then* $\mathrm{tor}(G) = G$.

Proof. Let G be a group with a min condition but torsion free. Then for all $g \in G$, $\langle g \rangle$ is of infinite order. Now since for each $g \in G$, $\langle g \rangle$ corresponds to a subgroup of G, that means that none of the groups in the chain $1 \leq G_1 \leq G_2 \leq \cdots$ is finite. In particular, G_1 is infinite. If G_1 is infinite, then we can form another infinitely many nontrivial subgroups of G_1. This cannot happen since it will lead to an infinite descending chain, violating min. □

Note that the previous argument does not assume that the group is abelian, and so, it is in fact true for general groups.

Exercise 2.2.20 (See [24], Exercise 4.2.11)**.** *If G is an abelian group with min, every injective endomorphism of G is an automorphism.*

Proof. Let G be an abelian group that satisfies the min. Then Exercise 2.2.19 says that G is torsion, therefore $G = \bigoplus_{p \in \mathbb{P}} G_p$, by Theorem 2.1.1. Let $\varphi :\ G \to G$ be an injective endomorphism. Then by the First Isomorphism Theorem (see [24, Theorem 1.4.3]), we have that

$$\mathrm{Im}\ \varphi \cong G/\mathrm{Ker}\ \varphi \tag{2.2.39}$$

Suppose that φ is not surjective. Then $\mathrm{Im}\ \varphi$ is a proper subgroup of G and from (2.2.39), we get that $\mathrm{Im}\ \varphi \cong G$ is a proper subgroup of G, which is clearly false. Then φ must be surjective. □

Definition 2.2.21. *A group G (not necessarily abelian) is called co-Hopfian if every monomorphism $G \to G$ is an automorphism.*

We have seen that Hopfianity is inherited to finitely generated abelian groups and to groups with max, see Exercise 2.2.15 and Theorem 2.2.18 respectively. Dually, we have seen that co-Hopfianity is inherited to torsion abelian groups and to min, see Exercises 2.2.19 and 2.2.20 respectively. Categorical approaches are possible but we do not get into the details here.

We end this section with a few applications of Theorems 2.2.6 and 2.2.8.

Exercise 2.2.22 (See [24], Exercise 4.2.13)**.** *If G is a finite abelian group whose order is divisible by m, then G has both a subgroup and a quotient of order m.*

Proof. Let G be finite and $|G| = n$ such that $m|n$. Then there is $0 \neq q \in \mathbb{Z}$ such that $n = qm$ or $m = n/q$. If there is no $g \in G$ with $o(g) = m$, then for all $g \in G$, $mg \neq 0$. Then

$$\forall\ g \in G,\ mg \neq 0 \implies (n/q)\, g \neq 0 \implies (1/q)(ng) \neq 0. \tag{2.2.40}$$

But $ng = 0$ and so $(1/q)(ng)$ must be 0. Hence there is $g \in G$ such that $o(g) = m$, and therefore $\langle g \rangle$ is a subgroup of G with $|\langle g \rangle| = o(g) = m$.

Assume that there is no subgroup H of G such that the order of G/H is m. Then we must have that $m(g+H) \neq H$ for all $g \in G$. But

$$m(g+H) = mg + H = (n/q)g + H = (1/q)(ng) + H \tag{2.2.41}$$
$$= (1/q)(0) + H = 0 + H = H,$$

which is a contradiction. Then there must exist a subgroup H of G such that G/H has order m. □

Another interesting property of Hopfianity in abelian groups is the following.

Exercise 2.2.23 (See [24], Exercise 4.2.15)**.** *A finitely generated abelian group is residually finite.*

Proof. We begin to show that $\mathbb{Z}$ is residually-finite. Let $0 \neq d \in \mathbb{Z}$ with $o(d) = j < m$ for some $m \in \mathbb{Z}$. Assume that $d \in m\mathbb{Z}$, then there is $0 \neq a \in \mathbb{Z}$ such that $d = ma$. Multiply both sides by j to get

$$jd = jma \implies 0 = (jm)a \implies jm = 0 \tag{2.2.42}$$

now since neither j nor m is zero, the last equality derives a contradiction, and therefore $d \notin m\mathbb{Z}$. Then $[\mathbb{Z} : m\mathbb{Z}] < \infty$ by a well known fact of subgroups of $\mathbb{Z}$, so $\mathbb{Z}$ is residually finite.By the same argument, we can show that $\mathbb{Z} \oplus \mathbb{Z}$ is residually finite, and therefore $\mathbb{Z}^n$ is residually finite for some $n \in \mathbb{N}$. Any finite group is of course residually finite, so by the structure theorem of finitely generated abelian groups, that is, Theorem 2.2.8, we conclude the proof. □

2.3 Pure Subgroups and p-Groups

From Theorem 2.1.8, we know that a subgroup H of G may fail to be divisible even when G is divisible. What happens when the divisibility of G gives rise to the divisibility down in the subgroup H? We call a subgroup H of G *pure* in G if, every element $h \in H$ that is divisible by n in G is also divisible by n down in the subgroup H, symbolically, we have

$$\Omega^n(H) = \Omega^n(G) \cap H. \tag{2.3.1}$$

The first observation of the definition shows that *direct summands are pure.*

In this section we will discuss properties of pure subgroups, for instance in Exercises 2.3.3 and 2.3.4 which tell us which p-groups are divisible. A notion of *Basic* subgroups will be discussed, with their existence in Theorem 2.3.10, and their uniqueness in Theorem 2.3.11. We will also introduce the notion of a *bounded* subgroups and give its complete classification in Theorem 2.3.12.

We begin to illustrate that purity is transitive.

Exercise 2.3.1 (See [24], Exercise 4.3.1)**.** *If H is pure in K and K is pure in G, then H is pure in G.*

Proof. Let H be pure in K, that is, $\Omega^n(K) \cap H = \Omega^n(H)$, $\forall\, n \geq 0$ where $n \in \mathbb{Z}$, and K be pure in G, that is, $\Omega^m(G) \cap K = \Omega^m(K)$, $\forall\, m \geq 0$ where $m \in \mathbb{Z}$, clearly H is contained in K which is a subgroup of G. Suppose that H is not pure in G. This means that we have $h \in H$ that is divisible by l in G yet not divisible by l in H. Then $h = lg_1$ for some $g_1 \in G$, as h is not divisible by l in H, there is no $h_1 \in H$ such that $h = lh_1$. This means that there is no h_1 such that

$$lg_1 = lh_1 \implies g_1 = h_1. \tag{2.3.2}$$

But, $\Omega^n(K) \cap H = \Omega^n(H)$, for every $h \in H$, so, there is $g_1 \in G$ such that $h = ng_1$ and there also is $h_1 \in H$ such that $h = nh_1$. Then we have

$$h = h \implies ng_1 = nh_1 \implies g_1 = h_1 \tag{2.3.3}$$

contradicting that there is no $h_1 \in H$ such that $g_1 = h_1$ originally. □

In Chapter 1 we saw that an arbitrary intersection of normal subgroups is again normal. Is there anything that can be said about the arbitrary intersection of pure subgroups of a group G? A result similar to that of normal subgroups can be established but under prescribed restrictions.

Exercise 2.3.2 (See [24], Exercise 4.3.2)**.** *In a torsion-free abelian group the intersection of a family of pure subgroups is pure. However in a finite abelian group this may be false.*

Proof. Let G be a torsion-free abelian, and let $\{H_\lambda \mid \lambda \in \Lambda\}$ be the family of pure subgroups in G. Let

$$H = \bigcap_{\lambda \in \Lambda} H_\lambda. \tag{2.3.4}$$

By G being torsion-free, for all $n \in \mathbb{N}$, we have that

$$\forall\, g \in G, \exists\, g_1 \in G \text{ such that } ng = g_1. \tag{2.3.5}$$

If H is not pure in G, we have that every element $h \in H$ is divisible by n in G but not divisible by n in H. This means that there is no $h_1 \in H$ such that $h = nh_1$ for all $n \in \mathbb{N}$. But this contradicts with the fact that both h and h_1 are torsion-free. Let $G = \mathbb{Z}(4)$ and $H = \{0, 2\}$ be a subgroup. Now $G \cap H = H$ and for $n = 1,\ 1 \in \mathbb{Z}(4)$, thus we have $1 \times 1 \in G$ but $1 \times 1 \notin H$. So we have found a subgroup $H = \{0, 2\}$ that is not pure in $G = \mathbb{Z}(4)$. □

The fact that divisible subgroups were direct summands was shown in Theorem 2.1.4. In Exercise 2.3.3 below we show that the situation is the same even for pure subgroups of divisible groups. This is a direct consequence of the definition of purity and Theorem 2.1.4.

Exercise 2.3.3 (See [24], Exercise 4.3.3)**.** *The pure subgroups of a divisible abelian group are just the direct summands.*

Proof. Let G be a divisible abelian group, let H be a subgroup of G. We just need to show that H is divisible, so the conclusion will follow from Theorem 2.1.4. Since H is pure in G, we have, for all integers $n \geq 0$, $\Omega^n(G) \cap H = \Omega^n(H)$;

$$\forall\ h \in H, \exists\ h_1 \in H \text{ such that } h = nh_1 \implies \exists\ g_1 \in G \text{ such that } h = ng_1. \tag{2.3.6}$$

Then every element of H is divisible, hence H is divisible. □

The group $\mathbb{Z}(p^\infty)$ is an example of an infinite divisible p-group. All the proper subgroups of $\mathbb{Z}(p^\infty)$ are finite cyclic by Exercises 2.1.2 and 2.1.6 . There are no finite groups that are divisible (see Theorem 2.1.5). We generalize this fact below in Exercise 2.3.4.

Exercise 2.3.4 (See [24], Exercise 4.3.4)**.** *An abelian p-group is divisible if and only if it contains no nontrivial pure cyclic subgroups.*

Proof. Let G be a divisible p-group. Let g_c be a nontrivial element in G and assume that $\langle g_c \rangle$ is pure in G. So we have, $\Omega^n(G) \cap \langle g_c \rangle = \Omega^n(\langle g_c \rangle)$, for all integers $n \geq 0$. That is

$$\forall\ h \in \langle g_c \rangle, \exists\ g_1 \in G \text{ such that } h = ng_1 \implies \exists\ h_1 \in \langle g_c \rangle \text{ such that } h = nh_1. \tag{2.3.7}$$

Then there exist $l, l_1 \in \mathbb{Z}$ such that $h = lg_c$ and $h_1 = l_1 g_c$. If $o(g_1) = p^m$, then we have, from $h = ng_1$,

$$\begin{aligned} p^m(lg_c) &= p^m(ng_1) \qquad (2.3.8)\\ \implies (p^m l)g_c &= n(p^m g_1)\\ \implies (p^m l)g_c &= 0\\ \implies g_c &= 0 \text{ since } p^m l \neq 0. \end{aligned}$$

This is a contradiction as g_c was chosen to be nontrivial.

Conversely, assume that G is a p-group that contains no nontrivial pure cyclic subgroups, yet is not divisible. Then G is altogether not divisible by assumption. In this case, there is $g \in G$ and $0 \neq n \in \mathbb{Z}$ such that $g \neq nh$, for all $h \in G$. Let $o(g) = p^l$ and q be any positive integer. Then we get

$$g \neq nh \implies p^{l+q} g \neq p^{l+q}(nh) \implies (p^q/n)(p^l g) \neq p^{l+q} h \implies 0 \neq p^{l+q} h. \tag{2.3.9}$$

Now, that $p^{l+q}h \neq 0$ for all $q \in \mathbb{N}$, implies that h does not have an order which is some power of p. But $h \in G$ where G is a p-group. □

Abelian groups that satisfy the minimal condition have a decomposition in quasicyclic groups and cyclic groups of prime-power order because of Exercise 2.2.19 and Theorem 2.1.1. Formally, we have the following theorem.

Theorem 2.3.5 (Kurosh, See [24], Theorem 4.2.11)**.** *An abelian group G satisfies the minimal condition if and only if it is a direct sum of finitely many quasicyclic groups and cyclic groups of prime-power order.*

In particular, we can show an additional property.

Exercise 2.3.6 (See [24], Exercise 4.3.5)**.** *An abelian p-group has finitely many elements of each order if and only if it satisfies min.*

Proof. Let G be a p-group that satisfies the minimal condition and suppose that G has infinitely many elements of each order. Then we have descending chain of subgroups becomes constant after a finite length. It means that we can pick the smallest nontrivial subgroup H of G. Now let $g \in G$ and $o(g) = p$, where p is the smallest possible order of any nontrivial element in G. Hence $\langle g \rangle = H$. Then H has p elements of orders all at most p. By Lagrange's theorem (see [24, Theorem 1.3.6]), nontrivial elements of H all have order dividing p. This means that we have p elements of order p, and p is finite, and it contradicts with the assumption that there are infinitely many elements of each order.

Conversely, let G be a p-group that has finitely many elements of each order and yet does not admit the minimal condition. Then there does not exist the minimal subgroup of G. Now let $g \in G$ with $o(g) = p$, then $\langle g \rangle$ has p elements of order p. This means that we can find another nontrivial subgroup H such that $H \leq \langle g \rangle$ but by a well known fact from subgroups of $\mathbb{Z}$ (see [24, Theorem 1.3.10]), H is either 0 or $\langle g \rangle$. Since H is nontrivial, $H = \langle g \rangle$, then $\langle g \rangle$ is the minimal subgroup. □

If k is the maximal positive integer such that $g = p^k g_1$ for some $g_1 \in G$, we say that k is a *p-height* of g, if no such maximal k exists, g is said to be of *infinite* p-height. One then sees that the elements of infinite p-height in G are exactly the elements of the subgroup $\bigcap_{n\geq 1} \Omega^{p^n}(G)$.

Exercise 2.3.7 (See [24], Exercise 4.3.6)**.** *Every abelian p-group is an image of some direct sum of cyclic p-groups.*

Proof. Let H and K be cyclic p-groups, and set $G = H \oplus K$. Let N be a subgroup of G and consider G/N. We want to prove that G/N is a p-group so that it being an image of G is itself is a p-group. Let $g \in G$ such that $o(g) = d = \text{lcm}(o(h), o(k))$ where $d = p^m$ for some $0 < m \in \mathbb{Z}$, $h \in H$ and $k \in K$. Consider $\overline{g} = g + N$. Then

$$\overline{g} = g + N \implies d\overline{g} = d(g + N) \implies d\overline{g} = dg + N \implies d\overline{g} = N, \tag{2.3.10}$$

that is, $\overline{g} = g + N$ has an order that is a power of p. □

We end this section with an interesting application on presentation of groups that we have seen in Chapter 1, but for specialized abelian groups. First, if a group G does not contain any proper divisible subgroup, the G is called *reduced*, for example, finite groups are reduced since they fail to be divisible in the first place.

Theorem 2.3.8 (Prüfer, See [24], Theorem 4.3.15)**.** *A countable abelian p-group G is a direct sum of cyclic groups if and only if it contains no nontrivial elements of infinite height.*

Then we may show an interesting fact, where the assumption of $|G|$ is countable allows us to make a specific argument.

Exercise 2.3.9 (See [24], Exercise 4.3.7)**.** *Let G be generated by $x_1,\ x_2,\ \ldots$ subject to defining relations $px_1 = 0,\ p^i x_{i+1} = x_1$ and $x_i + x_j = x_j + x_i$. Prove that G is a countable reduced abelian p-group containing a nonzero element of infinite height.*

Proof. Let $X = \{x_1, x_2, \ldots\}$, so X is countable. Since $X \subseteq G$, then G is countable as each element $g \in G$ is uniquely expressed by

$$g = \sum_{j=1}^{k} z_j x_{i_j}, \tag{2.3.11}$$

where $z_j \in \mathbb{Z}$. Take $px_1 = 0$ and $p^i x_{i+1} = x_1$, then

$$px_1 = pp^i x_{i+1} \implies 0 = p^{i+1} x_{i+1}, \tag{2.3.12}$$

for all i. So each x_i has an order of some power of p. Hence $g = \sum_{j=1}^{k} z_j x_{i_j}$ has an order of some power of p. Therefore, G is a p-group. By Exercise 2.3.7, G is a direct sum of cyclic subgroups, so it being a p-group. Then by Theorem 2.3.8, G has no nontrivial elements of infinite height. Suppose that a subgroup H of G is divisible. Then H is a divisible p-group. Suppose H is nontrivial. Then for all $h \in H$, $h = nh_1$ for $n > 0$ and some $h_1 \in H$. If $o(h) = p^l$,

$$h = nh_1 \implies p^l h = p^l n h_1 \implies 0 = (p^l n) h_1 \implies h_1 = 0, \tag{2.3.13}$$

as $p^l n \neq 0$. Then h is trivial, that is, H is also trivial. That is, we have a contradiction. So G is reduced. □

Let G be torsion (these groups are known by Theorem 2.1.1.) A subgroup B of G is called *basic* if it satisfies the following properties:

(i) B is pure in G;

(ii) B is a direct sum of cyclic groups; and

(iii) G/B is divisible.

As not every group has pure subgroups, whether basic subgroups exist or not is a relevant question. Basic subgroups exist.

Theorem 2.3.10 (Kulikov, See [24], Theorem 4.3.4)**.** *Every abelian torsion group G has a basic subgroup.*

The existence of basic subgroups is dealt with in Theorem 2.3.10, now we deal with the uniqueness of basic subgroups, in fact: where basic subgroups exist in a group, we just have only one, hence we speak about *the* basic subgroup.

Theorem 2.3.11 (Kulikov-Fuchs, See [24], Theorem 4.3.6)**.** *If G is an abelian torsion group, then all basic subgroups of G are isomorphic.*

For instance (see [24, Exercise 4.3.13]), if $G = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots$ where $|x_i| = p^{n_i}$ and $n_1 < n_2 < \cdots$ and also B is the basic subgroup generated by all $\overline{x_i} = x_i - p^{n_{i+1}-n_i}x_{i+1}$. Then one sees from Theorem 2.3.11 that B is basic in G and $B \neq G$ for $x_i \notin B$.

If $G = \Omega_n(G)$, then we say that G is *n-bounded* or briefly *bounded*. It is obvious that bounded groups are torsion, so their structure is known by Theorem 2.1.1. In particular, bounded groups are completely classified.

Theorem 2.3.12 (Prüfer-Baer's Theorem, See [24], Theorem 4.3.5)**.** *An abelian group G is bounded if and only if it is a direct sum of cyclic groups with boundedly finite orders.*

In Theorem 2.1.4 we saw that divisible subgroups are direct summands, now it turns out that if a bounded subgroup is pure, then it is a direct summand (see Theorem 2.3.13). Divisible subgroups are infinite direct summands and pure bounded subgroups are torsion direct summands. Now in Exercise 2.3.14 we see that if a p-group decomposes into a divisible group and a bounded group, then it has a bounded basic subgroup, and the converse is also true.

Theorem 2.3.13 (See [24], Theorem 4.3.8)**.** *A pure bounded subgroup H of an abelian group G is a direct summand.*

Exercise 2.3.14 (See [24], Exercise 4.3.8)**.** *In particular, we can show that an abelian p-group has a bounded basic subgroup if and only if it is the direct sum of a divisible group and a bounded group.*

Proof. Assume that G is written as the direct sum $G = H \oplus K$ where, say, H is divisible and K is bounded. Then K has only basic subgroup following Theorem 2.3.11 which is itself. Hence K is the desired bounded basic subgroup.

Conversely, assume that B is a bounded basic subgroup of G. Then by Theorem 2.3.13, B is a direct summand as it is a pure bounded subgroup and so G is isomorphic to $B \oplus (G/B)$, where G/B is bounded by the definition of B being basic. □

Exercise 2.3.15 (See [24], Exercise 4.3.10)**.** *If G is an abelian group such that* Aut G *is finite, prove that G has finite torsion subgroup. If* End G *is finite, prove that G is finite.*

Proof. Let Aut G be finite. Assume for contradiction that G does not have the torsion subgroup. Then G is torsion-free. But, for instance, $\mathbb{R}^n$ is torsion-free yet Aut G is isomorphic to $\mathrm{GL}(n, \mathbb{R})$, which is not finite.

Let End G be finite. We recall that End G is a ring (see [24, Theorem 1.5.1]). Now assume that G is infinite, take for instance $\mathbb{Z}$. We know that End $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ (see [9, Chapter XV, Example 1.1]) which is infinite. Hence G cannot be infinite. □

We end with an interesting construction in abelian group theory.

Exercise 2.3.16 (See [24], Exercise 4.3.11)**.** *Let G be an abelian p-group with no nontrivial elements of infinite height. Let B be basic in G and write $B = B_1 \oplus B_2 \oplus \cdots$ where B_i is a direct sum of cyclic groups of order p^i. Define C_n to be the subgroup generated by $p^n G$ and $B_{n+1}, B_{n+2}, \ldots$*

(a) If $g \in G$, then there exist elements $b_i \in B_i$ such that $g = b_1 + \cdots + b_n \bmod C_n$ for all $n > 1$.

(b) Prove that the mapping $\theta : \ g \mapsto (b_1, b_2 ...)$ is a well-defined monomorphism from G to the torsion-subgroup $\overline{B}$ of the cartesian product $B^{\mathbb{N}}$ of the B_i.

(c) Show that $\theta(G)$ is pure in $\overline{B}$.

Proof of (a). Let $g \in G$ and assume that there are no $b_i \in B_i$ such that $g = b_1 + b_2 + \cdots + b_n \bmod C_n$ for all $n > 1$. This means that, for all $c \in C_n$, $c \nmid g-(b_1+b_2+\cdots+b_n)$. Which also means that $g-(b_1+b_2+\cdots+b_n) \neq lc, \forall\, l \in \mathbb{Z}$. If p^s is the order of $g \in G$, then we get

$$p^s g - p^s(b_1 + b_2 + \cdots + b_n) \neq p^s lc \tag{2.3.14}$$

$$\implies 0 \neq (b_1 + b_2 + \cdots + b_n) + lc. \tag{2.3.15}$$

Now let $n_1 = pp^2 \cdots p^n$, so multiplying (2.3.15) by n_1, we get

$$0 \neq pp^2 \cdots p^n(b_1 + b_2 + \cdots + b_n) + pp^2 \cdots p^n lc$$

$$\implies 0 \neq n_1 lc \implies 0 \neq lc, \tag{2.3.16}$$

(2.3.16) follows from $b_i p^i = 0$, but $c \in C_n$ is of finite order as C_n is generated by groups with elements of finite order. This shows a contradiction and hence such $b_i \in B_i$ must exist. □

Proof of (b). Let $g_1 = g_2 \in G$ and consider

$$\theta(g_1) - \theta(g_2). \tag{2.3.17}$$

Since $\theta(g_1)$ and $\theta(g_2)$ are torsion elements, let m_1, m_2 be their respective orders. Now multiplying (2.3.17) by $m_1 m_2$ yields

$$(m_1 m_2)\,(\theta(g_1) - \theta(g_2)) = m_2\,(m_1\theta(g_1)) - m_1\,(m_2\theta(g_2)) = 0, \tag{2.3.18}$$

which means that $\theta(g_1) = \theta(g_2)$, from which we conclude that θ is well-defined.

It is clear that θ is a homomorphism. Let $g_{1,}g_2 \in G$ with $\theta(g_1) = \theta(g_2)$, then

$$\theta(g_1) - \theta(g_2) = 0 \implies \theta(g_1 - g_2) = 0 \implies g_1 = g_2, \tag{2.3.19}$$

showing that θ is a monomorphism.

□

Proof of (c). Let $k \in n\overline{B} \bigcap \theta(G)$ and write $k = n\bar{b}$ for some $\bar{b} \in \overline{B}$. If $k \notin n\theta(G)$,

$$\text{there is no } \theta(g) \in \theta(G) \text{ such that } k = n\theta(g), \qquad (2.3.20)$$

for all $n \geq 0$. But for $n = 1$, $k = \theta(g)$ for some $g \in G$ as $k \in \theta(G)$. Then $k \in n\theta(G)$, showing that $n\overline{B} \bigcap \theta(G) = n\theta(G)$. □

If G and B are as in Exercise 2.3.16, then one can see that $|G| \leq |B|^{\aleph_0}$. In fact by part (b) in Exercise 2.3.16, we have that $\theta : G \to \overline{B}$, where $\overline{B}$ is a subgroup of $B^{\mathbb{N}}$, is injective and $\theta(G)$ is a proper subgroup of $\overline{B}$. If θ is surjective, then we would have $\theta(G)$ isomorphic to $\overline{B}$ which is isomorphic to a proper subgroup of $B^{\mathbb{N}}$. If θ is not surjective, then $|G| < |\overline{B}| < |B|^{\aleph_0}$.

In Theorem 2.3.13 we got a source of torsion direct summands. Theorem 2.3.17 below gives an example of the torsion subgroup that is not a direct summand.

Theorem 2.3.17 (See [24], Theorem 4.3.10)**.** *If C is the cartesian sum of cyclic groups of orders $p,\ p^2,\ p^3,\ \ldots,$ the torsion-subgroup T is not a direct summand of C.*

What are the direct summands of nontorsion-free abelian groups? Application of Theorem 2.3.13 above shows that these groups have nontrivial direct summands either cyclic or quasicyclcic, (see Theorem 2.3.18).

Theorem 2.3.18 (See [24], Theorem 4.3.11)**.** *If G is an abelian group which is not torsion-free, it has a nontrivial direct summand which is either cyclic or quasicyclic.*

Indecomposable abelian groups are those groups that cannot be expressed as $G = K \oplus H$ for nontrivial subgroups H and K. Theorem 2.3.19 classifies indecomposable nontorsion-free abelian groups as either a simple cyclic group of prime order, that is, $\mathbb{Z}(p)$ or the quasicyclic group $\mathbb{Z}(p^\infty)$.

Theorem 2.3.19 (See [24], Theorem 4.3.12)**.** *An indecomposable abelian group which is not torsion-free is either a cyclic p-group or a quasicyclic group.*

Now combining Theorem 2.3.18 and Theorem 2.3.19 we get a structure of abelian p-groups of finite p-rank.

Theorem 2.3.20 (See [24], Theorem 4.3.13)**.** *An abelian p-group G has finite p-rank if and only if it is a direct sum of finitely many cyclic and quasicyclic groups.*

By Theorem 2.3.19, we know that a nontorsion-free abelian group has a direct summand that is either cyclic or quasicyclic. Theorem 2.3.21 gives a criterion for an abelian p-group to be a direct sum of cyclic groups as follows:

Theorem 2.3.21 (Kulikov, See [24], Theorem 4.3.14)**.** *An abelian p-group G is a direct sum of cyclic groups if and only if there is an ascending chain of subgroups $G_1 \leq G_2 \leq \cdots \leq G_n \leq \cdots$ whose union is G such that the height of a nonzero element of G_n cannot exceed some positive integer $k(n)$.*

Let us now close this section by giving a useful application of Theorem 2.3.17 due to Kulikov (see Theorem 2.3.22).

Theorem 2.3.22 (Kulikov, See [24], Theorem 4.3.16)**.** *If G is a direct sum of cyclic groups, every subgroup of G is likewise a direct sum of cyclic groups.*

Chapter 3

Topological Entropy

In this chapter we discuss our main notion of the *Topological Entropy for Locally Compact Abelian Groups*. Before getting into the notion of entropy, we first recall a few definitions from topology to enable ourselves to define topological groups on which the entropy will be calculated. In the next section we first report classical definitions of the notions of *topology* and *topological space,* and then we report another classical definition of a σ-*algebra* which both will enable us to define the notions of *topological groups* and a *measure.*

3.1 Topological Groups

For a nonempty set X, a topology will be a subset of the powerset which will satisfy specific axioms, the elements of a topology will be called *open* sets, and X together with the specified topology will be called a *topological space.* The symbol $\mathcal{P}(X)$ will denote a powerset of a nonempty set X, that is, the collection of *all* subsets of X.

Definition 3.1.1 (See ([23], Definition 1.3.1))**.** *Let X be a set and let $\tau \subseteq \mathcal{P}(X)$ be a collection of subsets of X satisfying:*

(i) $\emptyset,\ X \in \tau$;

(ii) if $\{U_1,\ U_2,\ \ldots, U_n \mid U_i \in \tau\} \subseteq \mathcal{P}(X) \implies \bigcap_{i=1}^{n} U_i \in \tau$; *and*

(iii) if $\{U_\lambda \mid \lambda \in \Lambda, U_\lambda \in \tau\} \subseteq \mathcal{P}(X) \implies \bigcup_{\lambda\in\Lambda} U_\lambda \in \tau,$

then the collection τ is called a topology on X, and its elements are called open sets; the pair (X, τ) is called a topological space and its elements are called points. When there is no confusion that can arise, we will simply write X instead of (X, τ_X). and τ instead of τ_X.

For any nonempty set X, there are always two trivial examples of topologies that are always available on X, the first is where $\tau = \{\emptyset,\ X\}$, is called *indiscrete* topology; and the second is where $\tau = \mathcal{P}(X)$, is called the *discrete* topology. It is easy to see that for any topology τ on X, we have the containment:

$$\{\emptyset,\ X\} \subseteq \tau \subseteq \mathcal{P}(X). \tag{3.1.1}$$

The discrete topology will be very useful when we deal with finite topological groups.

Let X be a general topological space. X is said to be *disconnected* if there does not exist open disjoint subsets U and V such that X is the their disjoint union, that is, $X = U \cup V$. A subset $A \subseteq X$ is called connected if it is connected as the *subspace* of X. If the only connected subspaces of X are only the singletons, then X is said to be *totally disconnected*. If $x \in X$ for a topological space X, the largest connected subset containing x is called the *connected component* of x. If G is an LCA (see Definition 3.1.3) group, we denote by $c(G)$ its connected component and it is a subgroup (see [6]).

Let (X, τ_X) and (Y, τ_Y) be two topological spaces, a function $f:\ (X, \tau_X) \to (Y, \tau_Y)$ is said to be *continuous* (with respect to the topologies τ_X and τ_Y) if for every open set $V \in \tau_Y$, its counterimage $f^{-1}(V)$ is open in (X, τ_X), that is, $f^{-1}(V) \in \tau_X$. A function $f:(X, \mathcal{U}) \to (Y, \mathcal{V})$ between two *uniform spaces* (see [23, Definition 12.2.5] for detailed discussion of uniformities and uniform spaces) is *uniformly continuous* if for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $(f(x), f(y)) \in V$. Let $N \subseteq X$ and $x \in N$, so we call N a *neighbourhood* of x if there exists an open set U such that $x \in U \subseteq N$. N is called an *open neighbourhood* if N is open. If for every distinct points $x, y \in X$ there exist open neighbourhoods U_x, U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$, then X is called a *Hausdorff* topological space. In our definition of the topological groups, we will require our topological spaces to be Hausdorff.

Let X be a topological space, by a *cover* of a subset $S \subseteq X$ is meant a collection

$$\mathcal{C} = \{U_\lambda \mid \lambda \in \Lambda, U_\lambda \subseteq X\} \tag{3.1.2}$$

such that $S \subseteq \bigcup_{U_\lambda \in \mathcal{C}} U_\lambda$. When the indexing set Λ is finite, we call $\mathcal{C}$ a *finite cover*. $\mathcal{C}$ is called an *open* cover if each U_λ in $\mathcal{C}$ is open. The notion of an open cover gives rise to a notion of *compactness*.

Definition 3.1.2 (See [15], Definition 7.4)**.** *A topological space X is said to be compact if every open cover $\mathcal{C}$ has a finite subcover, that is, a finite subcollection of $\mathcal{C}$.*

Let G be an LCA (see Definition 3.1.3) group and consider

$$\mathcal{C} = \{G_\lambda \mid \lambda \in \Lambda,\ G \supseteq G_\lambda \text{ is compact}\}, \tag{3.1.3}$$

then G is said to be *compactly covered* if $G \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$. Denote *the* largest compactly covered subgroup of G by $B(G)$ (see [6]).

If $\mathbb{R}$ is equipped with its standard topology, its compact subsets are the closed and bounded intervals, of the form

$$[a,\ b] = \{a \leq x \leq b \mid a, b \in \mathbb{R}, a < b\}. \tag{3.1.4}$$

For a general topological space X, easy examples of compact subsets are those that are finite, hence X itself is compact if it is finite. Note that even though finiteness implies compactness, the converse is false. For instance, $[a,\ b] \subseteq \mathbb{R}$ is compact yet not finite. Therefore, we can think of compactness as a topological generalization of a set-theoretic finiteness. X is said to be *locally compact* if every point in X has a compact neighbourhood. We are now in a position to give our main definition of this section.

Definition 3.1.3 (See [11], Definition 1.1)**.** *Let G be an abelian group, τ a Hausdorff topology on G and put on $G \times G$ the product topology. Then G is a topological group if the operation of product in G*

$$\cdot : (g_1, g_2) \in G \times G \mapsto \cdot(g_1, g_2) = g_1 g_2 \in G$$

and the operation of inverse

$$(\ )^{-1} : g \in G \mapsto (g)^{-1} = g^{-1} \in G$$

are both continuous for all $g, g_1, g_2 \in G$.

A topological group G is said to be a *compact abelian group* if the underlying topology τ is compact, and a *locally compact abelian group* if the underlying topology τ is locally compact.

Let us give a few examples, in order to visualize the situation.

(1) Every abelian group with its discrete topology is a locally compact abelian group;

(2) The additive group $\mathbb{R}$ of reals is a locally compact abelian group with its standard topology;

(3) $\mathbb{Z}_p$ and $\mathbb{Z}(p^\infty)$ are compact abelian groups and so is $\mathbb{Q}/\mathbb{Z}$;

(4) $\mathbb{Q}_p$ is a locally compact group;

(5) All finite abelian groups with their discrete topology are locally compact abelian groups, in particular, $\mathbb{Z}(n)$ is locally compact.

Every compact group is locally compact, but a locally compact abelian group is not necessarily compact as can be seen in the case of $\mathbb{R}$ of example above. In the rest of this chapter we will be concerned with locally compact abelian groups, LCA groups for short. By a group now on, we mean a locally compact abelian group unless otherwise is said.

3.2 Topological Entropy

We start off this section by reporting a classical definition about σ*-algebras*.

Definition 3.2.1 (See [4], p.1)**.** *Let X be an arbitrary set. A collection $\sum$ of subsets of $\mathcal{P}(X)$ is a called a σ-algebra if:*

1. $\emptyset,\ X \in \sum$;
2. $A \in \sum \implies A^{\mathsf{c}} \in \sum$;
3. $\{A_1,\ A_2,\ \ldots \mid A_i \in \sum\} \subseteq \mathcal{P}(X) \implies \bigcup_{n\geq 1} A_n \in \sum$;
4. $\{A_1,\ A_2,\ \ldots \mid A_i \in \sum\} \subseteq \mathcal{P}(X) \implies \bigcap_{n\geq 1} A_n \in \sum$.

Let G be our group and τ be a topology on G. The smallest σ-algebra, denoted by $\sigma(\tau)$, on G that contains τ is called a *Borel σ-algebra*. The elements of $\sigma(\tau)$ are called *Borel measurable subsets*, or briefly *Borel sets*. See [4] for a detailed discussion of these notions. The only σ-algebra that we will be using for our purposes is the Borel σ-algebra.

With now Borel σ-algebra at our disposal, we are in the position to define a function on $\sigma(\tau)$ to the *extended positive real line* that we will call a *measure*. First, we wish to make a brief comment on what will be our codomain of this function. Let $\mathbb{R}$ be a real line, we call the subset $\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$, given by

$$\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\} \tag{3.2.1}$$

the positive real line. We now define,

$$[0, +\infty] = \mathbb{R}_{\geq 0} \cup \{+\infty\} \tag{3.2.2}$$

and call it the *extended positive real line*. Let $\mu : \sigma(\tau) \to [0, +\infty]$ be a function. μ is said to be the *Borel measure* on G (see [4]) if the following properties hold:

(i) $\mu(\emptyset) = 0$; and

(ii) if $A_1,\ A_2,\ \ldots$ is a sequence of measurable sets that are pairwise disjoint, that is, $A_i \cap A_j = \emptyset,\ \forall\ i \neq j$, then $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu\left(A_n\right).$

Let G be an LCA group and μ a Borel measure on G. A subset $A \in \sigma(\tau)$ is *inner regular* (with respect to μ) if

$$\mu(A) = \sup\{\mu(C) : C \subseteq A \text{ and } C \text{ compact }\} \tag{3.2.3}$$

and *outer regular* (with respect to μ), if

$$\mu(A) = \inf\{\mu(B) : A \subseteq B \text{ and } B \text{ open}\}. \tag{3.2.4}$$

The Borel measure μ is *regular* (see [4]), if it is both inner regular and outer regular for all $A \in \sigma(\tau)$. For an inner regular $A \in \sigma(\tau)$, one says that the measure of A can be approximated from below by compact subsets; and for an outer regular $A \in \sigma(\tau)$, one says that the measure of A can be approximated from above by open subsets. For a regular Borel measure we report the following definition:

Definition 3.2.2 (See [11], Definition 2.6). *Let G be an LCA group. A regular Borel measure μ on G is called a left Haar measure on G if*

(i) μ is not a zero measure;

(ii) if μ is finite on the compact measurable sets $A \in \sigma(\tau)$; and

(iii) μ is left invariant, that is, $\mu(x + A) = \mu(A)$, $\forall\ x \in G$ and $A \in \sigma(\tau)$.

Definition 3.2.2 does not say anything about the existence of a Haar measure on LCA groups and whether or not it is unique when it exists. On this regard, we cite a classical result on the existence and uniqueness on LCA groups.

Theorem 3.2.3 (See [11], Theorem 2.8). *Let G be an LCA group. There exists a unique, up to a multiplicative constant $\lambda \in \mathbb{R}$, left Haar measure on G. If G is not only locally compact but compact, then there is one and only one left Haar measure on G, that is, $\lambda = 1$.*

Let G be an LCA group, and $\mathcal{U}$ a left uniformity on G. Take $\varphi :\ G \to G$ as a uniformly continuous endomorphism, and $\mathcal{C}(G)$ the collection of all compact neighbourhoods of the neutral element $0 \in G$. Let μ be a left Haar measure on G (its existence is guaranteed by Theorem 3.2.3). Also for $n \in \mathbb{N}$ and $U \in \mathcal{C}(G)$, let

$$C_n(\varphi, U) = U \cap \varphi^{-1}(U) \cap \cdots \cap \varphi^{-n+1}(U) \in \mathcal{C}(G) \tag{3.2.5}$$

be the n^{th} *φ-cotrajectory* of U. By the *topological entropy* of U with respect to $U \in \mathcal{C}(G)$ we mean

$$H_{top}(\varphi, U) \coloneqq \limsup_{n \to +\infty} \frac{-\log \mu\left(C_n(\varphi, U)\right)}{n}, \tag{3.2.6}$$

and by the *topological entropy of φ*, we mean the number

$$h_{top}(\varphi) \coloneqq \sup \left\{H_{top}(\varphi, U) \mid U \in \mathcal{C}(G)\right\} \tag{3.2.7}$$

and by the *topological entropy of G* we mean the set

$$E_{top}(G) \coloneqq \left\{h_{top}(\varphi) \mid \varphi \in \text{End } G\right\}. \tag{3.2.8}$$

The computation of $E_{top}(G)$ (where $G = \mathbb{R}^m$) was made and shown to be finite in [2, Theorem 15], [22, Example 7] and [6, Remark 2.7]. Suppose $G = \mathbb{R}^m$ and let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be an automorphism, for a example, if we define $\varphi :\ v \mapsto Av$

where $A \in \mathrm{GL}(m, \mathbb{R})$, then it is well-known that this φ is uniformly continuous and bijective. In this situation, the compact neighbourhoods of the neutral element $0 \in \mathbb{R}^m$ have the form:

$$\mathcal{C}(\mathbb{R}^m) = \{[a,\ b] \subseteq \mathbb{R}^m \mid a \leq 0 \leq b,\ a, 0, b \in \mathbb{R}^m\}. \tag{3.2.9}$$

Let μ be the left Haar measure, for example let it be the Lebesgue measure, and for $n \in \mathbb{N}$, let

$$C_n(\varphi, U) = U \cap \varphi^{-1}(U) \cap \cdots \cap \varphi^{-n+1}(U) \in \mathcal{C}(\mathbb{R}^m) \tag{3.2.10}$$

be the n^{th} φ-cotrajectory of $U \in \mathcal{C}(\mathbb{R}^m)$. As lim sup always exists on the extended real line, our topological entropy of φ with respect to U is given by

$$H_{top}(\varphi, U) = \limsup_{n \to +\infty} \frac{-\log \mu\left(C_n(\varphi, U)\right)}{n}. \tag{3.2.11}$$

From this, one attaches to the entropy the interpretation that it is a measure of decay of $C_n(\varphi, U)$. Then our topological entropy of φ is given by

$$h_{top}(\varphi) = \sup\{H_{top}(\varphi, U) \mid U \in \mathcal{C}(\mathbb{R}^m)\} \tag{3.2.12}$$

and the topological entropy of $\mathbb{R}^m$ is thus given by

$$E_{top}(\mathbb{R}^m) = \{h_{top}(\varphi) \mid \varphi \in \mathrm{Aut}\ \mathbb{R}^m\} = \{h_{top}(\varphi) \mid \varphi \in \mathrm{GL}(m,\ \mathbb{R})\} \tag{3.2.13}$$

with last equality following from the fact that Aut $\mathbb{R}^m \cong \mathrm{GL}(m,\ \mathbb{R})$. For a complete computation which is very far from trivial, (see [7], [16], and [2, Theorem 15] and the comments in [6, Remark 2.7]), also see the Yuzuinski Formula in [3, Theorem 2.2].

In fact, as one can see from [6, Remark 2.7], for every $r \in \mathbb{R}_{\geq 0}$, there exists an automorphism of the topological group $\mathbb{R}$ with topological entropy r. The interest for the topological entropy is due to the following:

Open question: *LCA groups G with $E_{top}(G)$ close to zero may have structural decomposition.*

This question was asked under various perspectives in [6].

For instance, finite abelian groups have zero entropy: in fact there is a well known decomposition for them in cyclic subgroups. The structure of these groups is known by Theorem 2.2.6 and an immediate answer for the above question is possible: consider a finite abelian group; $\mathbb{Z}(p)^n$ as discrete compact group. Recall that $\mathbb{Z}(p)^n$ is an elementary abelian p-group of order p^n. Take $\varphi : \mathbb{Z}(p)^n \to \mathbb{Z}(p)^n$ as a uniformly continuous automorphism. Note that $\mathcal{C}(\mathbb{Z}(p)^n) = 0$ since $\mathbb{Z}(p)^n$ is Hausdorff and discrete. Now let μ be a counting measure. Then as $\log \mu(C_n(\varphi, U))$ is finite, the topological entropy of φ with respect to $U \in \mathcal{C}(\mathbb{Z}(p)^n)$, that is, $H_{top}(\varphi, U)$ is zero, and hence both $h_{top}(\varphi, U)$ and $E_{top}(\mathbb{Z}(p)^n)$ are zero.

The topological entropy $E_{top}(\mathbb{Z})$ can also easily be shown to be zero, that is, $\mathbb{Z} \in \mathfrak{G}_0$. In fact, one can see that this is true for any discrete LCA group (see [6, Remark 2.4]).

By recalling the notion of rank of a group from Chapters 1 and 2, it is useful to introduce an analogue for topological groups. The notion of rank allows us to characterize the structure of certain locally compact abelian p-groups (see Theorem 3.2.5 below).

For a finite abstract group, maximal p-subgroups always exist (and therefore p-subgroups), and they are the well known p-Sylow subgroups (see Sylow's Theorems in [24, Theorem 1.6.16]). It is possible to extend the notion of p-Sylow subgroup to LCA groups that might not be necessarily finite. For locally finite discrete groups, Sylow Theorems are known and they can be found also in [24]. The real issue is to give this notion for profinite groups, or LCA groups here. The corresponding Sylow's Theory also is known, but is very different (see [1, 10] for more details).

Let G be a locally compact group. Note that our construction of $\mathbb{Z}_p$ realizes a profinite group, in fact, it is even a procyclic group, that is, a projective limit (or inverse limit) of finite p-groups. For $g \in G$, the closure of the cyclic group generated by g, that is, $\overline{\langle g \rangle}$, is called a *monothetic* subgroup, and G itself monothetic if $G = \overline{\langle g \rangle}$. Denote by G_0 the identity component of G, and call the elements in

$$\mathrm{comp}(G) = \left\{ g \in G \mid \overline{\langle g \rangle} \text{ is compact} \right\} \tag{3.2.14}$$

compact elements of G.

Definition 3.2.4 ([10], Definition 1.13)**.** *An LCA group G is called periodic if $G = \mathrm{comp}(G)$ and $G_0 = 0$.*

Following [10], let G be an LCA group, if we take $g \in G$ such that $\overline{\langle g \rangle}$ is procyclic, then we let $\pi(g)$ be the set of all prime numbers such that $\overline{\langle g \rangle}$ has a nontrivial p-component, that is, the direct factors of $\overline{\langle g \rangle}$ are either $\mathbb{Z}(p^n)$ or $\mathbb{Z}_p$.

If G is finite, we set $\pi(G)$ to be the set of divisors of $|G|$. Now consider, σ, any set of primes. We call g a σ-element if $\pi(g) \subseteq \sigma$, and denote the set of all σ-elements by G_σ, we then call (see [10]) G a σ-group if G is, in addition, periodic and $\pi(G) = \bigcup_{g \in G} \pi(g) \subseteq \sigma$.

For a finite group H of order m, we construct $\pi(H) = \pi(h_1) \cup \cdots \cup \pi(h_m)$ where each $\pi(h_i)$ is a set of divisors of m. In this situation, we can set σ be the set of divisors of m and let $H_\sigma = \{h \mid |h| \text{ is prime}\}$, that is, H_σ contains h such that $\langle h \rangle$ is maximal in H.

A maximal σ-subgroup of G (see [10]) is called a σ-Sylow subgroup.

Considering our finite group H of order m again, now if $\sigma = \{p\}$, then we require that $\pi(H) = \pi(h_1) \cup \cdots \cup \pi(h_m) \subseteq \sigma = \{p\}$, that is, the only divisor of m is p, then we get $H_\sigma = \{h \mid |h| = p\}$, in this particular situation, we speak of a p-element and a p-Sylow subgroup.

Lastly, let G be a periodic LCA group and let G_p be the union of all p-subgroups (we know these are contained in a p-Sylow subgroup as seen in [10, Lemma 2.4]), then G_p is a closed subgroup and itself a p-Sylow subgroup (see [10, p.49]).

As we know, any abstract abelian group G can be viewed as a $\mathbb{Z}$-module. Similarly, a locally compact abelian p-group G can be viewed as a $\mathbb{Z}_p$-module. Then let G be $\mathbb{Z}_p$-module. Let $\mathrm{rk}_{\mathbb{Z}_p}(G)$ be its $\mathbb{Z}_p$-rank and $r_p(G)$ be its p-rank. Now, if both $\mathrm{rk}_{\mathbb{Z}_p}(G)$ and $r_p(G)$ are finite, then it is possible to define

$$\mathrm{rank}_p(G) = \mathrm{rk}_{\mathbb{Z}_p}(G) + r_p(G), \tag{3.2.15}$$

which is a natural replacement for the notion of rank, which we introduced in previous chapters when abstract groups were discussed without topological structure on themselves.

The following theorem, connecting decompositions of topological groups with the presence of small entropy via (3.2.15), was proved recently.

Theorem 3.2.5 (See [6], Theorem 1.4)**.** *A locally compact p-group G has $\mathrm{rank}_p(G) < \infty$ if and only if it is isomorphic to*

$$\mathbb{Z}_p^{n_1} \times \mathbb{Q}_p^{n_2} \times \mathbb{Z}(p^\infty)^{n_3} \times F_p$$

for some integers $n_1, n_2, n_3 \in \mathbb{N}$ and a finite p-group F_p with $r_p(F_p) = n_4 \in \mathbb{N}$. In particular,

$$\mathrm{rank}_p(G) = n_1 + n_2 + n_3 + n_4,$$

with $\mathrm{rk}_{\mathbb{Z}_p}(G) = n_1 + n_2$ and $r_p(G) = n_3 + n_4$.

The next theorem says that all locally compact abelian p-groups with finite rank_p have finite entropy, and the structure of those that are in $\mathfrak{G}_0$ can be derived from Theorem 3.2.5 above.

Theorem 3.2.6 (See [6], Theorem 1.5)**.** *If G is a locally compact abelian p-group with $\mathrm{rank}_p(G) < \infty$, then G has finite entropy.*

The reader can also refer to [5] for more details. Now we turn into a class of LCA groups that is characterized by the totally disconnectedness property. According to Definition 3.2.4, one can easily deduce that periodic groups are totally disconnected. In light of [6, Theorem 1.1], if an LCA group G is in $\mathfrak{G}_0$, then it was concluded that G necessarily is totally disconnected. With the help of our construction of maximal p-subgroups above, in the next result we start from a totally disconnected LCA group G and give necessary and sufficient conditions for G to be in $\mathfrak{G}_0$ or $\mathfrak{G}_{<\infty}$.

Theorem 3.2.7 (See [6], Theorem 1.2)**.** *Let G be a periodic locally compact abelian group. Then:*

(a) $G \in \mathfrak{G}_0$ if and only $G_p \in \mathfrak{G}_0$ for every $p \in \mathbb{P}$;

(b) $G \in \mathfrak{G}_{<\infty}$ if and only if $G_p \in \mathfrak{G}_{<\infty}$ for every $p \in \mathbb{P}$ and $G_p \in \mathfrak{G}_0$ for almost all $p \in \mathbb{P}$.

In view of Theorem 3.2.7, we see that in order to decide whether the $G \in \mathfrak{G}_0$ or $G \in \mathfrak{G}_{<\infty}$, it is sufficient to know the entropy of the p-Sylow subgroups G_p. This is, in part, due to [10, Lemma 2.8], which asserts that a periodic LCA group is generated by its p-Sylow subgroups, that is, due to the information about the entropy of the generators, the information about the entropy of the whole group is available.

We now move on to consider another interesting class of LCA groups that generalizes the ideas of Theorem 2.2.8 of abstract groups into a more general case of LCA groups which are known as *compactly* generated LCA groups.

Definition 3.2.8 (See [11])**.** *Let G be an LCA group and $K \subseteq G$ a compact subset, then we say that G is compactly generated if $\langle K \rangle = G$.*

As in the abstract case, finite groups are finitely generated, but finitely generated groups are not necessarily finite, for instance the infinite cyclic $\mathbb{Z}$. The situation is the same in LCA groups, compact groups are compactly generated, but compactly generated groups are not necessarily compact, for instance $\mathbb{R}$. Similar to Theorem 2.2.8, we have an analog in LCA groups, that is, compactly generated groups are classified as can be seen in the following result.

Theorem 3.2.9 (See, [11], Theorem 7.57)**.** *Every compactly generated LCA group is isomorphic to a group of the form $\mathbb{R}^n \times \mathbb{Z}^m \times K$ for some compact group K and some $n, m \geq 0$*

Now one sees that Theorem 2.2.8 is implied by Theorem 3.2.9. For instance, Theorem 2.2.8 says that finitely generated abstract abelian groups decomposes into a direct sum of a free abelian group and a finite abelian group. In Theorem 3.2.9, let $n = 0$ and K be any finite abelian group, in this way we have recovered Theorem 2.2.8.

Bibliography

[1] D. L. Armacost, *The Structure of Locally Compact Abelian Groups*, Marcel Dekker, 1981, New York. 55

[2] R. Bowen, Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.* **153** (1971), 401–414. 4, 53, 54

[3] A. G. Bruno and S. Virili, Algebraic Yuzinvinski Formula. *J. Alg.* **423** (2015), 114—147. 4, 54

[4] D. L. Cohn, *Measure Theory*, Birkhäuser, 1980, Boston. 4, 52, 53

[5] D. Dikranjan and M. Sanchis, Dimension and entropy in compact topological groups, *J. Math. Anal. Appl.* **476** (2019), 337—366. 56

[6] D. Dikranjan, A. G. Bruno and F. G. Russo, Finiteness of topological entropy for locally compact abelian groups, *Glasgow Math. J.* **361** (2020), 403—442. 4, 50, 53, 54, 55, 56

[7] D. Dikranjan, M. Sanchis and S. Virili, New and old facts about entropy in uniform spaces and topological groups. *Topology Appl.* **159** (2012), 1916-–1942. 4, 54

[8] A. Erfanian and J. Weigold, A note on growth sequences of finite simple groups, *Bull. Austral. Math. Soc.* **51** (1995), 495–499. 30

[9] L. Fuchs, *Infinite Abelian Groups*, Academic Press, 1970, New York. 4, 25, 45

[10] W. Herfort, K. H. Hofmann and F. G. Russo, *Periodic Locally Compact Groups*, de Gruyter, 2019, Berlin. 55, 56, 57

[11] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, de Gruyter, 2020, Berlin. 4, 25, 51, 53, 57

[12] K. H. Hofmann, S. A. Morris and M. Stroppel, Locally compact groups, residual Lie groups, and varieties generated by Lie groups. *Topology Appl.* **71** (1996), 63–91. 4

[13] B. M. Hood, Topological entropy and uniform spaces, *J. London Math. Soc.* **8** (1974), 633—641. 4

[14] A. G. Konheim, M. H. McAndrew and R. L. Adler, Topological entropy. *Trans. Amer. Math. Soc.* **114** (1965), 633—641. 4

[15] C. Konsiowski. *A First Course in Algebraic Topology.* Cambridge University Press, 1980. Cambridge. 4, 50

[16] D. A. Lind and T. Ward, Automorphisms of solenoids and p-adic entropy, *Ergodic Theory Dyn. Systems* **8** (1988), 411–419. 4, 54

[17] S. A. Morris. Varieties of topological groups II, *Bull. Austral. Math. Soc.* **2** (1970), 1–13. 4

[18] S. A. Morris, Varieties of topological groups III, *Bull. Austral. Math. Soc.* **2** (1970), 165–178. 4

[19] S. A. Morris, Varieties of topological groups: a survey, *Bull. Austral. Math. Soc.* **1** (1969), 145-–160. 4

[20] S. A. Morris, Varieties of topological groups: a survey, *Colloq. Math.* **46** (1982), 147–165. 4

[21] H. Neumann, *Varieties of groups*, Springer, 1976, Berlin. 4

[22] J. Peters, Entropy of automorphisms on LCA groups, *Pacific J. Math.* **96** (1981), 475—488. 4, 53

[23] T. Richmond, *An Introduction to General Topology*, de Gruyter, 2020, Berlin. 4, 49, 50

[24] D. J. S. Robinson, *A Course in the Theory of Groups.* Springer, 1996, New York. 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 55

[25] S. Virili, Entropy for endomorphisms of LCA groups, *Topology Appl.* **159** (2012), 2546—2556. 4