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The scale of a quasi-uniform space

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Contents

Abstract

Over the last forty years much progress has been made in the investigation of the scale of a uniform space. In particular, Bushaw, Kent, Ramsey and Richardson published several articles concerning the scale of a uniform space. The aim of this dissertation is to begin a similar investigation into the scale of a quasi-uniform space. It starts off with a summary of results obtained for the scale of a uniform space, which, has been investigated in the past. We conclude by commencing an investigation into the scale of a quasi-uniform space. Here several results obtained for the scale of a uniform space are generalized, and some original results for the scale of a quasi-uniform space are presented.

Chapter 1

Introduction

The concept of a "scale of a uniform space" was first introduced by D. Bushaw in 1967 [6] to support the generalization of Liapunov's "direct method" in stability theory to abstract systems. He investigated a stability concept analogous to the classical uniform stability as a relationship between a quasi-order and a uniformity on the same set. He showed that stability in this sense occurs if and only if there exists a Liapunov function taking values in a certain partially ordered uniform space associated with the given uniformity and called its retracted scale.

In 1967 D. C. Kent published a paper [16] defining an partial order in the retracted scale which makes it a complete distributive lattice and the canonical mapping from the scale to the retracted scale is order preserving. The lattice operations in the retracted scale are uniformly continuous. His main result is that the completion of a Hausdorff uniform space is a subspace of its retracted scale uniform space. Both the scale and retracted scale are complete.

In 1967 O. C. Ramsey [29] studied some properties of the scale of a uniform space. He defined a function called "écart" ρ on $X \times X$ into the scale P where P is the collection of all nonempty subsets α of $\mathcal{U}$ satisfying $U \in \alpha$ and $V \supseteq U$ implying $V \in \alpha$ and $(X,\mathcal{U})$ is a uniform space, and he showed that the uniform properties of $(X,\mathcal{U})$ can be written in terms of ρ and the neighborhoods of 0 in P without explicitly mentioning $\mathcal{U}$. These properties then resemble metric properties. Then ρ is used to prove that the following are equivalent: $(X,\mathcal{U})$ is pseudometrizable, $(P,\mathcal{U}')$ is pseudometrizable, and

$(P, \mathcal{U}')$ satisfies the first axiom of countability, where $\mathcal{U}'$ is a uniformity defined on the scale P. The écart ρ is also used to investigate the scales of two spaces which have the same topology. Necessary conditions are obtained for a scale to be separable or Lindelöf or to satisfy the second axiom of countability.

In 1972 G. C. Leslie and D. G. Kent [19] show that the scale of a uniform space is uniformly connected if and only if the original uniform space is bounded. D. C. Richardson [33] characterizes uniform connectedness for uniform subspaces of the scale of the real line and he also shows that the scale of any uniform space is locally compact if and only if the original uniformity $\mathcal{U}$ has a last element, see [32]. The scale is compact if and only if the original space X is finite or $\mathcal{U} = \{X \times X\}$. This statement remains true if compact is replaced by countably compact, totally bounded, Lindelöf, second countable, or separable.

In 1983 D. C. Kent introduced the notion of an order scale, see [19], and showed that the order topology is compact and T_2 in both scale and retracted scale of any uniform space $(X, \mathcal{U})$. If $(X, \mathcal{U})$ is T_2 and totally bounded, the Samuel compactification associated with $(X, \mathcal{U})$ can be obtained by uniformly embedding $(X. \mathcal{U})$ in its order retracted scale. This implies that every compact T_2 space is a closed subspace of a complete, infinitely distributive lattice in its order topology, and also a continuous, closed image of a closed subspace of a complete atomic Boolean algebra in its order topology [19].

In the light of the above, it is natural to start an investigation into the scale of a quasi-uniform space. This is our focus in this dissertation.
We define our scale of a quasi-uniform space motivated by the definition of the Hausdorff hyperspace quasi-uniformity of a quasi-uniform space as introduced in [3]. In Proposition 5.1.3 we show that if the original space is a uniform space then our scale is the same as the scale investigated by Bushaw and Kent, see [6] and [16]. In Corollary 5.6.1 we also show that the quasi-uniform space given by a Hausdorff hyperspace quasi-uniformity is quasi-uniformly embedded into the left-sided scale of a quasi-uniform space.

It is interesting to note the various resemblances between the scale of a uniform space and the scale of a quasi-uniform space such as that the scale and retracted scale of a uniform space are both complete and the scale and retracted scale of a quasi-uniform space are both bicomplete.

The scale of a uniform space and its retracted scale are pseudometrizable if and only if the original uniform space is pseudometrizable. Nearly the same applies to the quasi-uniform space (Theorem 3.5.2 and Theorem 5.4.2), where "pseudometrizable" must be replaced in the statement by "quasi-pseudometrizable". In addition to these coincidences, it is interesting to note that the same condition that is both necessary and sufficient for the scale of a uniform space to be totally bounded is also necessary and sufficient for the scale of a quasi-uniform space to be totally bounded (Proposition 4.3.3 and Theorem 5.3.4).

We then introduce two slight generalizations of the scale of a quasi-uniform space, which we call the prefilter space and the left-sided scale of a quasi-uniform space. We show that the prefilter space on $X \times X$, where $(X, \mathcal{U})$ is a quasi-uniform space and the left-sided scale of the quasi-uniform space $(X, \mathcal{U})$ are bicomplete (Proposition 5.6.2).

We also show that total boundedness is preserved by another modified scale called the two-sided scale of a quasi-uniform space (Proposition 5.7.3).

We bring it to the reader's attention that some of the most interesting new results on the scale of a quasi-uniform space obtained during this investigation are collected in [28] for possible publication. The proofs given in this dissertation and in [28] respectively may sometimes differ.

This dissertation starts with some preliminary definitions which are given in the next chapter. That chapter contains two separate sections on the Hausdorff hyperspace quasi-uniformity and the quasi-uniformity of quasi-uniform convergence on a multifunction space respectively, each of which lists some often used basic definitions and results needed to understand this dissertation.

Chapter 3 consists of a short summary of some uniform results about the scale of a uniform space.

Chapter 4 deals with the topological properties of the scale of a uniform space. It mentions some known results about connectedness in the scale of a uniform space and about the cardinality of the scale of a uniform space.

Chapter 5 starts our investigations into the scale of a quasi-uniform space. We show that total boundedness is preserved by the two-sided scale of a quasi-uniform space. The prefilter space of a quasi-uniform space is quasi-uniformly embedded into the left-sided scale of a quasi-uniform space. We also show that the scale of a quasi-uniform space, the prefilter space of a quasi-uniform space and the left-sided scale of a quasi-uniform space are bi-complete.

We conclude this dissertation by listing a few unsolved problems in Chapter 6.

Chapter 2

Basic definitions

In this chapter. firstly we recall the definition and some properties of quasi-uniform spaces and we summarize some results about order convergence. Secondly we also summarize facts about the Hausdorff hyperspace quasi-uniformity discussed by many authors (see e.g. [3], [23]) and thirdly we recall the notion of the quasi-uniform multifunction space discussed in [8]. We also introduce some notations used throughout this dissertation.

2.1 Definitions

In this section we recall the concept of a quasi-uniform space.

The following definitions can be found in the book [12].

Definition 2.1.1 *([12]) A* quasi-uniformity *$\mathcal{U}$ on the set X is a filter on $X \times X$ such that*
(1) *Each member U of $\mathcal{U}$ contains the diagonal $\triangle_X = \{(x, x) : x \in X\}$ of X;*
(2) *For each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$.*
(Here $V^2 = V \circ V = \{(x, z) \in X \times X$: there is $y \in X$ such that $(x, y) \in V$ and $(y, z) \in V\}$. Hence $\circ$ is the usual composition of binary relations.)
The pair $(X, \mathcal{U})$ is called a quasi-uniform space.

The members of $U \in \mathcal{U}$ are called the *entourages* of $\mathcal{U}$. The elements of X are called *points*.

If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1}$ is also a quasi-uniformity on X called the *conjugate* of $\mathcal{U}$.

Definition 2.1.2 *([12]) Given a quasi-uniform space* $(X,\mathcal{U})$ *we shall denote by* $\mathcal{U}^s$ *the coarsest uniformity finer than* $\mathcal{U}$ *and its conjugate* $\mathcal{U}^{-1}$ *(i.e.* $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$*). If* $U \in \mathcal{U}$ *we denote by* U^s *the entourage* $U \cap U^{-1}$ *in* $\mathcal{U}^s$*.*

It is easily seen that a quasi-uniformity $\mathcal{U}$ is a uniformity if and only if $\mathcal{U} = \mathcal{U}^{-1}$.

Definition 2.1.3 *If* $\mathcal{U}$ *is a quasi-uniformity on* X*,* $\mathcal{B}$ *is called a* base *for* $\mathcal{U}$ *if* $\mathcal{B} \subseteq \mathcal{U}$ *and for every* $U \in \mathcal{U}$ *there is a* $B \in \mathcal{B}$ *such that* $B \subseteq U$*. Furthermore* $\mathcal{B}$ *is called a subbase for* $\mathcal{U}$ *if* $\{\bigcap \mathcal{H} \mid \mathcal{H} \subseteq \mathcal{B},\ \mathcal{H}$ *is non-empty and finite*$\}$ *is a base for* $\mathcal{U}$*.*

Topologies induced by quasi-uniformities

Definition 2.1.4 *([12]) Each quasi-uniformity* $\mathcal{U}$ *on a set* X *induces a topology* $\tau(\mathcal{U})$ *as follows: For each* $x \in X$ *and* $U \in \mathcal{U}$ *set* $U(x) = \{y : (x,y) \in U\}$*. A subset* G *of* X *belongs to* $\tau(\mathcal{U})$ *if and only if for each* $x \in G$ *there exists* $U \in \mathcal{U}$ *such that* $U(x) \subseteq G$*.*

If τ is a topology on X, then $\mathcal{U}$ is said to be compatible with τ if $\tau(\mathcal{U}) = \tau$, and (X,τ) is said to admit $\mathcal{U}$.

Proposition 2.1.1 *([12]) If* $\mathcal{U}$ *and* $\mathcal{V}$ *are quasi-uniformities on a set* X *such that* $\mathcal{U} \subseteq \mathcal{V}$*, then* $\tau(\mathcal{U}) \subseteq \tau(\mathcal{V})$*.*

Proof. See [12, Proposition 1.29]. □

Quasi-uniform continuity

In the theory of quasi-uniform spaces the structure-preserving maps are the quasi-uniformly continuous maps.

Definition 2.1.5 *([12]) A map* $f : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$ *between two quasi-uniform spaces* $(X,\mathcal{U})$ *and* $(Y,\mathcal{V})$ *is called* quasi-uniformly continuous *provided that for each* $V \in \mathcal{V}$ *there is* $U \in \mathcal{U}$ *such that* $(f \times f)(U) \subseteq V$*. Here* $f \times f$ *is the product map from* $X \times X$ *to* $Y \times Y$ *defined by* $(f \times f)(x_1,x_2) = (f(x_1), f(x_2))$ $(x_1,x_2 \in X)$*. Moreover* f *is called a* quasi-uniform isomorphism *if it is one-to-one, onto, and* f *as well as its inverse are quasi-uniformly continuous.*

Proposition 2.1.2 *([12]) Suppose $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is quasi-uniformly continuous. Then $f : (X, \tau(\mathcal{U})) \longrightarrow (Y, \tau(\mathcal{V}))$ is continuous.*

Proof. See [12, Proposition 1.14]. $\square$

Cauchy filter

Definition 2.1.6 *([23]) A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called a $\mathcal{U}^s$-*Cauchy filter *if for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \times F \subseteq U$.*

Definition 2.1.7 *([34]) A quasi-uniform space $(X, \mathcal{U})$ is said to be* bicomplete *if each $\mathcal{U}^s$-Cauchy filter on X converges with respect to the topology $\tau(\mathcal{U}^s)$, i.e., if the uniform space $(X, \mathcal{U}^s)$ is complete.*

A bicompletion of a quasi-uniform space $(X, \mathcal{U})$ is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a $T(\mathcal{V}^s)$-dense subspace quasi-uniformly isomorphic to $(X.\mathcal{U})$.

We have the following uniqueness property for T_0 bicompletions.

Proposition 2.1.3 *([21]) Let $(X, \mathcal{U})$ be a quasi-uniform space, let $(Y, \mathcal{V})$ be a bicomplete quasi-uniform space T_0-space, let D be a dense subset of $(X, \tau(\mathcal{U}^s))$, and let $f : (D, \mathcal{U} \mid D) \rightarrow (Y, \mathcal{V})$ be a quasi-uniformly continuous map. Then there is a unique continuous extension $g : (X, \tau(\mathcal{U}^s)) \rightarrow (Y, \tau(\mathcal{V}^s))$ of f, and $g : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is quasi-uniformly continuous.*

Proof. See [21, Theorem 2.7.3]. $\square$

Remark 2.1.1 *([21]) Let $(X, \mathcal{U})$ be a quasi-uniform space that is a $\tau(\mathcal{V}^s)$-dense subspace of a bicomplete quasi-uniform T_0-space $(Y, \mathcal{V})$. Then $(Y, \mathcal{V})$ is isomorphic to the bicompletion of $(X, \mathcal{U})$ under an isomorphism that keeps the points of X pointwise fixed. When this identification is made, the minimal $\mathcal{U}^s$-Cauchy filters on X are the traces of the $\tau(\mathcal{V}^s)$-neighborhood filters of the points of Y.*

Totally bounded quasi-uniformity

Definition 2.1.8 *Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{U}$ is said to be* totally bounded *if for every $U \in \mathcal{U}$ there is a finite cover $\{A_i \mid 1 \leq i \leq n\}$ (for some integer n) of X such that for each $1 \leq i \leq n$, $A_i \times A_i \subseteq U$.*

Definition 2.1.9 *([15]) A quasi-uniform space $(X,\mathcal{U})$ is called* precompact *provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$.*

Definition 2.1.10 *([31]) Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(X,\mathcal{U})$ is* totally bounded *if and only if $(X,\mathcal{U}^s)$ is totally bounded.*

Remark 2.1.2 *It is not too difficult to see that every totally bounded quasi-uniform space is precompact, and that a uniformity is totally bounded if and only if it is precompact.*

Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\bigcap_{U\in\mathcal{U}}(U \cap U^{-1}) = \triangle$ if and only if $\tau(\mathcal{U}^s)$ is Hausdorff if and only if $(\bigcap_{U\in\mathcal{U}} U) \cap (\bigcap_{U\in\mathcal{U}^{-1}} U) = \triangle$ if and only if $\bigcap_{U\in\mathcal{U}} U(x) \cap \bigcap_{U\in\mathcal{U}} U^{-1}(x) = \{x\}$ whenever $x \in X$ if and only if $cl_{\tau(\mathcal{U}^{-1})}\{x\} \cap cl_{\tau(\mathcal{U})}\{x\} = \{x\}$ whenever $x \in X$ (see [21]).

Order convergence

We define the relevant lattice concepts, and summarize some known results about order convergence.

Definition 2.1.11 *([17]) Let L be a complete lattice. If $x \in L$, $A \subseteq L$, and $\mathcal{F}$ is a filter on L, then let:*

$$\uparrow x = \{y \in L : y \geq x\},\ \uparrow A = \bigcap\{\uparrow x : x \in A\},\ \uparrow \mathcal{F} = \bigcup\{\uparrow F : F \in \mathcal{F}\}.$$

The symbols $\downarrow x$, $\downarrow A$, $\downarrow \mathcal{F}$ designate the corresponding sets of lower bounds.

Definition 2.1.12 *([17]) Let L be a complete lattice. Let x, y be two elements on L. If $x \leq y$, then $[x, y] = \uparrow x \cap \downarrow y$ denotes the closed interval spanned by x and y.*

Definition 2.1.13 *([17]) A filter $\mathcal{F}$ on L order-converges to x if*

$$x = sup \downarrow \mathcal{F} = \inf \uparrow \mathcal{F}.$$

The order topology on L has for its closed sets those sets A which contain all of their order-convergence limit points. Order convergence does not always coincide with convergence in the order topology; when they do coincide, order convergence is said to be *topological*, and the resulting order topology is regular and T_1 (see [17]).

Definition 2.1.14 *([17]) An element x of L is* compact *if $A \subseteq L$ and $x \leq supA$ implies that there is a finite subset B of A such that $x \leq supB$.*

An element with the dual property is said to be *cocompact.*

Definition 2.1.15 *([17]) The complete lattice L is* compactly (cocompactly) *generated if each element of L is the supremum (infimum) of a set of compact (cocompact) elements; if L is both compactly and cocompactly generated, then L is said to be* bicompactly *generated.*

Definition 2.1.16 *([17]) A subset A of L with the inherited order is called a* subcomplete lattice *of L if, for every nonempty subset B of A,* $\sup_A B = \sup_L B$ *and* $\inf_A B = \inf_L B$.

Proposition 2.1.4 *([17]) (a) If L is bicompactly generated, then order convergence in L is topological, and if $x \in L$, then the neighborhood filter with respect to the order topology at x has an open base consisting of sets of the form $[a, b]$, where a is a compact lower bound of x and b is a cocompact upper bound of x. The order topology is totally disconnected.*
(b) If L is a subcomplete lattice of an atomic Boolean algebra, then L is bicompactly generated and the order topology is, in addition, compact (see [17]).

Proof. See [17, Proposition 1.3]. □

2.2 The Hausdorff hyperspace quasi-uniformity

The Hausdorff hyperspace quasi-uniformity of a quasi-uniform space was investigated by many authors (see e.g. [3], [22]). This section contains the basic definitions.

Definition 2.2.1 *([23]) Given a quasi-uniform space $(X,\mathcal{U})$ we shall consider the $\mathcal{U}$-*equivalence relation *$\sim$ on X that underlies its T_0-reflection: For $x, y \in X$ we have $x \sim y$ if and only if $(x,y) \in \bigcap \mathcal{U} \cap (\bigcap \mathcal{U}^{-1})$.*

In [22] Künzi and Ryser have used the following definition.

Definition 2.2.2 *([22]) Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{P}_0(X)$ be the set of nonempty subsets of X. For any $U \in \mathcal{U}$ let*

$$U_+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}$$

and

$$U_- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}.$$

Furthermore set $U_H = (U_-) \cap (U_+)$ whenever $U \in \mathcal{U}$. Then $\{U_- : U \in \mathcal{U}\}$ is a base for the *lower quasi-uniformity* $\mathcal{U}_-$ on $\mathcal{P}_0(X)$ and $\{U_+ : U \in \mathcal{U}\}$ is a base for the *upper quasi-uniformity* $\mathcal{U}_+$ on $\mathcal{P}_0(X)$. Moreover $\mathcal{U}_H = \mathcal{U}_+ \vee \mathcal{U}_-$ is the so-called *Hausdorff hyperspace quasi-uniformity or Bourbaki quasi-uniformity* of $(X,\mathcal{U})$ (see [21]).

Lemma 2.2.1 *([22, Lemma 1]) (a) Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $x \mapsto \{x\}$ is a quasi-uniform embedding of $(X,\mathcal{U})$ into $(\mathcal{P}_0(X),\mathcal{U}_H)$.*
(b) Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be quasi-uniform spaces and let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous map. Then the map $f : (\mathcal{P}_0(X),\mathcal{U}_H) \to (\mathcal{P}_0(Y),\mathcal{V}_H)$ defined by $f(A) := \{f(a) : a \in A\}$ is quasi-uniformly continuous, too.

Proof. See [22, Lemma 1]. $\square$

2.3 The quasi-uniformity of quasi-uniform convergence on a multifunction space

In this section we mention some basic definitions on the quasi-uniformity of quasi-uniform convergence on a function space used in this dissertation. The notation and conventions employed mostly correspond to those used in [8].

Definition 2.3.1 *([8]) A* multifunction (or multi-valued function) $F : X \to Y$ *is a point-to-set correspondence from X to Y such that $F(x) \neq \emptyset$ for each $x \in X$.*

Definition 2.3.2 *([8]) Let Y^{mX} be the set of all multifunctions from a nonempty set X to a quasi-uniform space $(Y, \mathcal{U})$. For each $U \in \mathcal{U}$, define*

$W(U) = \{(F, H) : H(x) \subseteq U(F(x))$ *and* $F(x) \subseteq U^{-1}(H(x))$ *for all* $x \in X\}$.

Moreover $\{W(U) : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{mX}$ on Y^{mX} called the quasi-uniformity of quasi-uniform convergence *on Y^{mX}. The topology induced by $\mathcal{U}_{mX}$ on Y^{mX} is called the* topology of quasi-uniform convergence.

We shall use the multifunction space in Section 5.7.

Chapter 3

Summary of some results about the scale of a uniform space

In this chapter we summarize the most often used fundamental results pertaining to the scale of a uniform space. We will first recall the concept of the scale of a filter. Secondly, we will recall the scale of a uniform space. In the third section we shall recall the construction of the retracted scale of a uniform space. The fourth section will recall the notion of stability in a quasi-ordered uniform space. In the last section, we shall summarize pseudometrizabilty, separability, the Lindelöf condition, and the first and second axioms of countability of the scale of a uniform space. A large portion of this chapter was obtained from [6] and [16]. In order to avoid repetitions it will sometimes happen that we state a result in this chapter but delay its proof until Chapter 5. There it is either proved for the more general case of quasi-uniformities, or derived from other results obtained in that chapter.

3.1 The scale of a filter

In this section, we recall the notions of a prefilter and the scale of a filter introduced by D. Bushaw [6].

Our prefilters are the "nonempty stacks" of some other authors (see e.g. Száz in [38]).

Definition 3.1.1 *([6]) By a* prefilter *on a nonempty set X we shall* *mean a nonempty collection α of nonempty subsets of X such that $A \in \alpha$ and $A \subseteq B \subseteq X$ imply $B \in \alpha$.*

For any nonempty subset A_0 of X, the collection $\{A \subseteq X : A \supseteq A_0\}$ is a prefilter; it will be denoted by $\langle A_0 \rangle$ and called the *principal* prefilter generated by A_0. We shall say that a prefilter α is proper if $\alpha \neq \{X\}$ or equivalently (since $X \in \alpha$ in any case) if α has at last two elements. A prefilter α is a filter if it is multiplicative: $\{A, B\} \subseteq \alpha$ implies $A \cap B \in \alpha$.

Definition 3.1.2 *([6]) Let $\mathcal{F}$ be a filter on a set X. We shall denote by $P(\mathcal{F})$, or simply by P where there is no risk of ambiguity, the set of all prefilters on the set X which are contained in $\mathcal{F}$.*

The set P may be partially ordered by the inverse of inclusion.

Definition 3.1.3 *([6]) For any $\{\alpha, \beta\} \subseteq P$ we shall say that $\alpha \leq \beta$ if and only if $\beta \subseteq \alpha$. The partially ordered set $(P, \leq)$ will be called the* scale *of a filter $\mathcal{F}$.*

Some significant information about the order-theoretic structure of the scale of a filter is provided by the following proposition.

Proposition 3.1.1 *([6]) The scale $(P, \leq)$ of a filter $\mathcal{F}$ has a last element 0, where $0 = \{X\}$. The subset $P^+ = P - \{0\}$ is directed downward (" filtrant à gauche") and has a last element if and only if $\mathcal{F}$ is a proper principal filter.*

Proof. See [6, Proposition 2.1]. □

3.2 Definition of the scale of a uniform space

We shall recall in this section the definition of the scale of a uniform space and we will introduce an example of the scale of a uniform space.

Definition 3.2.1 *([16]) Let $(X, \mathcal{U})$ be a uniform space, and let P be the set of all prefilters on $X \times X$ which are subsets of $\mathcal{U}$.*

Definition 3.2.2 *For any two elements α and β in P, $\alpha \leq \beta$ means that $\beta \subseteq \alpha$.*

The following definition is according to Kent (see [16]).

Definition 3.2.3 *([16]) The order structure of $(P, \leq)$ is called the* order of the scale *of $(X,\mathcal{U})$.*

Proposition 3.2.1 *For any uniform space $(X,\mathcal{U})$, the order scale $(P, \leq)$ is a complete distributive lattice whose suprema and infima are set intersections and set unions, respectively, and $\mathcal{U}$ is the last element and $\{X \times X\}$ is the greatest element.*

Proof. See [16, Proposition 1]. $\square$

In the following, we are going to define a uniformity on the scale of a uniform space.

Definition 3.2.4 *([16]) Let $(X,\mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$,*

$U' = \{(\alpha, \beta) \in P \times P :$ if $A \in \alpha$ then $U \circ A \in \beta$ and if $B \in \beta$ then $U \circ B \in \alpha\}$.

Note that $\{U' : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{U}'$ on P. Indeed the following proposition shows that the scale of a uniform space is a uniform space.

Proposition 3.2.2 *Let $(X,\mathcal{U})$ be a uniform space. Then $(P,\mathcal{U}')$ is a uniform space.*

Proof. (1) We show that $\mathcal{U}'$ is a filter on $P \times P$.

For each $U \in \mathcal{U}$, $U' \subseteq P \times P$ by definition of U'.

If U, V belong to $\mathcal{U}$ then it suffices to show that $(U \cap V)' \subseteq U' \cap V'$. Let $(\alpha, \beta) \in (U \cap V)'$, if $A \in \alpha$, $(U \cap V) \circ A \in \beta$ and if $B \in \beta$, $(U \cap V) \circ B \in \alpha$. Then $(U \cap V) \circ A \subseteq U \circ A \in \beta$ and $(U \cap V) \circ B \subseteq U \circ B \in \alpha$, where we use that α, β are prefilters. If $A \in \alpha$ then $U \circ A \in \beta$ and if $B \in \beta$ then $U \circ B \in \alpha$

implies $(\alpha, \beta) \in U'$. Similarly $(\alpha, \beta) \in V'$ so $(\alpha, \beta) \in U' \cap V'$.

(2) Each member of $\mathcal{U}'$ contains the diagonal.

(3) Let $U' \in \mathcal{U}'$, $U'^{-1} \in \mathcal{U}'$ because U' is symmetric for each $U \in \mathcal{U}$.

(4) Let $U \in \mathcal{U}$ then there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$. We show that $V'^2 \subseteq U'$.
Let $(\alpha, \beta) \in V'^2$. This implies there is a $\gamma \in P$ such that $(\alpha, \gamma) \in V'$ and $(\gamma, \beta) \in V'$. If $A \in \alpha$, $V \circ V \circ A = V^2 \circ A \subseteq U \circ A \in \beta$ and if $B \in \beta$, $V \circ V \circ B = V^2 \circ B \subseteq U \circ B \in \alpha$ thus $(\alpha, \beta) \in U'$. So $\{U' : U \in \mathcal{U}\}$ is a filter base generating the uniformity $\mathcal{U}'$. □

Proposition 3.2.3 *([16]) Let $(X,\mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$,*
$U'^2 = U^{2'}$.

Proof. By the previous argument, $U'^2 \subseteq U^{2'}$.

Conversely, let (α, β) be in $U^{2'}$, and let γ be the prefilter which contains all oversets of sets of the form $U \circ T$, for some T in α, or $U \circ V$, for some V in β. It follows easily that (α, γ) and (γ, β) are both in U'. □

The following definition is according to Kent in [16].

Definition 3.2.5 *([16]) The pair $(P,\mathcal{U}')$ is called the* scale *of the uniform space $(X,\mathcal{U})$.*

The next paragraph discusses the example of the discrete uniform space.

Example 3.2.1 *If $(X,\mathcal{U})$ is a discrete uniform space, then its scale $(P,\mathcal{U}')$ is a discrete uniform space.*

Indeed, $\mathcal{U} = \langle \triangle \rangle$ and $\alpha \in P$ if and only if α is a nonempty family of reflexive relations on X such that if $A \in \alpha$ and $A \subseteq B$ then $B \in \alpha$.

Let $(\alpha, \beta) \in \triangle'$, if $A \in \alpha$ implies $\triangle \circ A = A \in \beta$ and if $B \in \beta$ implies $\triangle \circ B = B \in \alpha$ thus $\alpha = \beta$.

3.3 Construction of the retracted scale of a uniform space

This is the section where we investigate the construction of the retracted scale of a uniform space. It will be constructed with a simple method that has been used by Kent in [16] by introducing an equivalence relation R on P.

Definition 3.3.1 *([16]) Let α, β in P. We define*

$$\alpha \ R \ \beta \ (or \ (\alpha, \beta) \in R) \ if \ (\alpha, \beta) \in U' \ for \ each \ U \in \mathcal{U}.$$

Lemma 3.3.1 *R is an equivalence relation on P.*

Proof. Let $\alpha \in P$, α R α because each U' contains the diagonal of $P \times P$.

If α R β implies for each $U \in \mathcal{U}$, $(\alpha, \beta) \in U'$, and U' is symmetric so $(\beta, \alpha) \in U'$, which implies β R α.

If α R β and β R γ, implies $(\alpha, \beta) \in U'$ and $(\beta, \gamma) \in U'$ for each $U \in \mathcal{U}$, if $A \in \alpha$, $U \circ U \circ A \in \gamma$ and if $C \in \gamma$, $U \circ U \circ C \in \alpha$ implies $(\alpha, \beta) \in U'$ whenever $U \in \mathcal{U}$, thus α R γ . □

The following notations are used by Kent in [16].

Notations. Let

$$\alpha_R = \{\beta \in P : \alpha \ \mathrm{R} \ \beta\}$$

and

$$P_R = \{\alpha_R : \alpha \in P\} \ .$$

Kent gave the following definition (see [16]).

Definition 3.3.2 *For each prefilter $\alpha \in P$, we associate prefilters α_0 and α^0 defined by :*

$$\alpha_0 = \{U \in \mathcal{U} : V \circ U \in \alpha \ for \ all \ V \in \mathcal{U}\}$$

and

$$\alpha^0 = \{U \in \mathcal{U} \colon V \circ A \subseteq U \text{ for some } V \in \mathcal{U} \text{ and some } A \in \alpha\}.$$

Before defining an equivalence relation in the retracted scale, we need the following proposition.

Proposition 3.3.1 *(a) For each $\alpha \in P$, $\alpha_R = \{\beta \in P : \alpha_0 \leq \beta \leq \alpha^0\}$.*

(b) $\quad$ *$\alpha \; R \; \beta$ implies $\alpha_0 = \beta_0$ and $\alpha^0 = \beta^0$.*

(c) $\quad$ *$\alpha_0 \subseteq \beta_0$ if and only if $\alpha^0 \subseteq \beta^0$.*

Proof. (a) Let $\beta \in \alpha_R$ which implies α R β. Let $B \in \alpha^0$. Then $U \circ A \subseteq B$ for some $U \in \mathcal{U}$ and $A \in \alpha$, since $U \circ A \in \beta$ we have $B \in \beta$, so $\alpha^0 \subseteq \beta$. Let $C \in \beta$. Then $U \circ C \in \alpha$ for each $U \in \mathcal{U}$ which implies $C \in \alpha_0$, so $\beta \subseteq \alpha_0$. Thus $\alpha_0 \leq \beta \leq \alpha^0$.

(b) Let $B \in \beta_0$. For each $U \in \mathcal{U}$, $B \subseteq U \circ B \in \beta$ implies that $\beta_0 \subseteq \beta$ since α R β by (a) $\alpha^0 \subseteq \beta \subseteq \alpha_0$, so $\beta_0 \subseteq \alpha_0$. By the symmetry of R and by a similar argument $\alpha_0 \subseteq \beta_0$ thus $\alpha_0 = \beta_0$.

Let $T \in \alpha^0$. Then $U \circ A \subseteq T$ for some $U \in \mathcal{U}$ and $A \in \alpha$ since α R β, $U \circ A \in \beta$ implies that $T \in \beta^0$ so $\alpha^0 \subseteq \beta^0$. Similarly $\beta^0 \subseteq \alpha^0$, thus $\alpha^0 = \beta^0$.

(c) Suppose $\alpha_0 \subseteq \beta_0$. We show that $\alpha^0 \subseteq \beta^0$. Indeed, $T \in \alpha^0$, for some $U \in \mathcal{U}$ and $A \in \alpha$, $U \circ A \subseteq T$, $A \subseteq U \circ A \in \beta_0$ implies $U \circ A \in \beta$. So $T \in \beta^0$. We obtain the reciprocal inclusion by a similar argument. $\square$

In the following we define an equivalence relation in the retracted scale.

Definition 3.3.3 *([16]) Let α_R, β_R in P_R. We define*

$$\alpha_R \leq \beta_R \text{ to mean } \beta_0 \subseteq \alpha_0.$$

Lemma 3.3.2 *The relation $\leq$ defined above is an partial order on P_R.*

Proof. We show that $\leq$ is well defined: If α R γ and β R δ such that $\beta_0 \subseteq \alpha_0$ then $\delta_0 \subseteq \gamma_0$. Let $B' \in \delta_0$. Since β R δ implies that $\beta_0 = \delta_0$ so $B' \in \alpha_0$ thus $B' \in \gamma_0$.

The relation $\leq$ is reflexive, antisymmetric and transitive by dual inclusion of prefilters. □

In [16] Kent used the following definition:

Definition 3.3.4 *([16]) The order structure* $(P_R, \leq)$ *is called* order retracted scale *of* $(X,\mathcal{U})$.

In the following, we are going to show that the canonical map from the scale to its retracted scale is order-preserving.

Lemma 3.3.3 *Let φ be the canonical map from P to P_R defined by for $\alpha \in P$, $\varphi(\alpha) = \alpha_R$. Then φ is order-preserving.*

Proof. Let α, β in P be such that $\alpha \leq \beta$. We show that $\alpha_R \leq \beta_R$. Indeed, $\alpha \leq \beta$ means that $\beta \subseteq \alpha$. Let $B \in \beta_0$. For each $U \in \mathcal{U}$, $U \circ B \in \beta$ implies that $U \in \mathcal{U}$, $U \circ B \in \alpha$ which means that $B \in \alpha_0$. So $\beta_0 \subseteq \alpha_0$ thus $\alpha_R \leq \beta_R$. □

Proposition 3.3.2 *([16]) Let $\{\alpha_i : i \in I\}$ and $\{\beta_i : i \in I\}$ be sets of prefilters with α_i R β_i, for all $i \in I$. Then $(\bigcup\{\alpha_i : i \in I\})$ R $(\bigcup\{\beta_i : i \in I\})$ and $(\bigcap\{\alpha_i : i \in I\})$ R $(\bigcap\{\beta_i : i \in I\})$.*

Proof. See [16, Proposition 4]. □

Proposition 3.3.3 *([16]) The order retracted scale of any uniform space is a complete distributive lattice. The operations sup and* inf *are preserved under the canonical map $\varphi : P \to P_R$ defined in Lemma 3.3.3.*

Proof. See [16, Proposition 6]. □

We next define the retracted scale uniformity $\mathcal{U}''$ on P_R.

Definition 3.3.5 *For each* $U \in \mathcal{U}$,
$U" = \varphi(U') = \{(\alpha_R, \beta_R) \in P_R \times P_R : (\alpha, \beta) \in U'\}$.

Definition 3.3.6 *([16]) The pair* $(P_R, \mathcal{U}")$ *is called the* retracted scale uniform space *associated with* $(X, \mathcal{U})$, *where* $\mathcal{U}"$ *is defined in the next line.*

One can show that the filter $\mathcal{U}"$ generated by $\{U" : U \in \mathcal{U}\}$ is a T_2 uniformity.

Theorem 3.3.1 *([16]) The retracted scale uniform space* $(P_R, \mathcal{U}")$ *associated with an arbitrary uniform space* $(X, \mathcal{U})$ *is Hausdorff and is the quotient uniform structure derived from* $(P, \mathcal{U}')$ *under the map* φ. *The lattice operations for the retracted scale are uniformly continuous with respect to* $\mathcal{U}"$.

Proof. For the first part we refer the reader to [16, Theorem 1].

We show that the lattice operations for the retracted scale are uniformly continuous.

(a) Given $U \in \mathcal{U}$, let $U" \vee U" = \{(\alpha_R \vee \beta_R, \delta_R \vee \gamma_R)$: $(\alpha_R, \delta_R) \in U", (\beta_R, \gamma_R) \in U"\}$, we shall verify that $U" \vee U" = U"$. It suffices to show that $U" \vee U" \subseteq U"$.
Let (α_R, δ_R), (β_R, γ_R) belong to $U"$ and choose αR δ, β R γ such that $\alpha \in \alpha_R$, $\delta \in \delta_R$, $\beta \in \beta_R$ and $\gamma \in \gamma_R$. If $A \in \alpha \vee \beta$ implies A belongs to α or A belongs to β, so $U \circ A \in \delta$ or $U \circ A \in \gamma$, and if $B \in \delta \vee \gamma$ implies B belongs to δ or B belongs to γ, and so $U \circ B \in \alpha$ or $U \circ B \in \beta$. We have : if $A \in \alpha \vee \beta$ then $U \circ A \in \delta \vee \gamma$ and if $B \in \delta \vee \gamma$ then $U \circ B \in \alpha \vee \beta$ means $(\alpha \vee \beta, \delta \vee \gamma) \in U'$.

(b) Given $U \in \mathcal{U}$, we can prove that $U" = U" \wedge U"$. Similarly to the "join ", where it follows that $U" \subseteq U" \wedge U"$. □

We shall now consider order convergence in the scale $(P, \leq)$ of an arbitrary uniform space $(X, \mathcal{U})$.

Lemma 3.3.4 *([16]) Each ultrafilter* ξ *on* P *order-converges to some element of* P.

Proof. See [16, Lemma 1]. A similar proof is given in Lemma 5.5.2. □

Kent needs the following lemma for his proof that the scale of a uniform space and its retracted scale are both complete uniform spaces.

Lemma 3.3.5 *([16]) Let $U \in \mathcal{U}$, $F \subseteq P$, and $F \times F \subseteq U'$. If α is an element of P such that inf $F \leq \alpha \leq$ sup F, then $(\alpha, \delta) \in U'$ for all $\delta \in F$.*

Proof. See [16, Lemma 2]. □

Theorem 3.3.2 *([16]) For any uniform space $(X, \mathcal{U})$, $(P, \mathcal{U}')$ and $(P_R, \mathcal{U}'')$ are both complete uniform spaces.*

Proof. See [16, Theorem 2]. The analogous results hold for quasi-uniformities (see Theorem 5.3.1). □

Lemma 3.3.6 *([16]) Let $(X, \mathcal{U})$ be a uniform space, and let a be any fixed element of X. Let $\upsilon_a : X \longrightarrow P$ be defined at each $x \in X$ by $\upsilon_a(x) = \{V \in \mathcal{U}: (a, x) \in V\}$. Then υ_a is a uniformly continuous map with respect to the scale uniformity on P.*

Proof. Choose U a symmetric entourage in $\mathcal{U}$, and let $(x, y) \in U$. We will show that $(\upsilon_a(x), \upsilon_a(y)) \in U'$. Indeed, $V \in \upsilon_a(x)$ implies $(a, x) \in V$. Since $(x, y) \in U$, we have $(a, y) \in U \circ V$, and hence $U \circ V \in \upsilon_a(y)$. The reciprocal argument establishes that $V \in \upsilon_a(y)$ implies $U \circ V \in \upsilon_a(x)$, and the proof is complete. □

Lemma 3.3.7 *([16]) If $(X, \mathcal{U})$ is a Hausdorff uniform space and υ_a as in the lemma above, then the map $\varphi \circ \upsilon_a$ is a uniform embedding of $(X, \mathcal{U})$ onto a subspace of $(P_R, \mathcal{U}'')$.*

Proof. See [16, Lemma 4]. □

Theorem 3.3.3 *([16]) The completion of a Hausdorff uniform space is a subspace of its retracted scale uniform space.*

Proof. Indeed, the completion of $(X, \mathcal{U})$ is simply the closure in P of the range of $\varphi \circ \upsilon_a$. for any chosen a in X. □

3.4 Stability in quasi-ordered uniform spaces

Many proposed abstract models for the study of control theory, differential equations with or without uniqueness of solutions, finite automata, etc. ([1], [7], [13], [26], [35] and [36] constitute a sample of this literature), although differing considerably from one another, have at last the following elements in common: a set E of "events", often interpreted as pairs (t, x), where t represents a time variable and x a "state" of some system; furthermore quasi-order (reflexive and transitive binary relation on E) ρ, so interpreted that $(e_1, e_2) \in \rho$ means that the event e_2 could follow e_1 in some feasible history of the system. In other words, for any $e \in E$, $\rho(e)$ is the set of all events which may follow, or are attainable from e.
We shall consider a concept of stability in this very general setting due to Bushaw. To formulate it we assume that there is given also a uniformity $\mathcal{U}$ on E. A set $A \subseteq E$ will be called *stable* relative to ρ and $\mathcal{U}$ if, for every $V \in \mathcal{U}$, there exists a $W \in \mathcal{U}$ such that $\rho W(A) \subseteq V(A)$, which in terms of the interpretation sketched above means that any event attainable from an event sufficiently close to A is itself arbitrarily close to A. As stability concepts go, it is rather simple.

Stability is usually defined only for *invariant* sets, i.e., sets A satisfying $\rho(A) \subseteq A$. The following observation shows that this discrepancy is not severe.

Proposition 3.4.1 *([6]) Let E be a uniform space. A set $A \subseteq E$ is stable relative to ρ and $\mathcal{U}$ if and only if its closure $\overline{A}$ in the uniform topology is stable; and any closed stable set is invariant.*

Proof. See [6, Proposition 4.1]. □

Liapunov functions and the stability criterion

Let $(M, \leq)$ be a partially ordered set with first element 0. The set $M - \{0\}$, assumed nonempty, will be denoted by M^+.

Definition 3.4.1 *([6]) Generalizing an existing definition (see [2]), Bushaw said that a map $v : N \longrightarrow M$, where N is some invariant uniform neighborhood of A, is a* Liapunov function *for A if it satisfies:*

(L_1) *for any* $e_1 \in N$, $(e_1, e_2) \in \rho$ *implies* $v(e_1) \geq v(e_2)$;

(L_2) *for any* $\lambda \in M^+$ *there exists a* $V \in \mathcal{U}$ *such that* $v(e)$ *is not* $\geq \lambda$ *if* $e \in V(A) \cap N$;

(L_3) *for any* $V \in \mathcal{U}$, *there exists a* $\lambda \in M^+$ *such that* $v(e) \geq \lambda$ *if* $e \in N - V(A)$.

Thus, informally, a Liapunov function is one which (L_1) decreases relative to ρ, (L_2) "approaches 0" uniformly as e approaches A, and (L_3) is bounded away from 0 outside any uniform neighborhood of A.

Proposition 3.4.2 *([6]) If* v *is a Liapunov function for* A, *then* $v(e) = 0$ *if and only if* $e \in \overline{A}$.

Proof. See [6, Proposition 5.1]. □

Liapunov's direct method in stability theory characterizes stability properties by the existence of Liapunov functions of corresponding types. It seems that before Bushaw started his investigations the partially ordered set $(M, \leq)$ had always been taken to be the set of non-negative reals with the standard order, or a very mild extension thereof, and the types of systems $(E, \rho, \mathcal{U})$ for which Liapunov stability criteria had been found had been restricted accordingly. The introduction of the (retracted) scale of $\mathcal{U}$ at this point wiped away the restrictions (see [6]).

Theorem 3.4.1 *([6]) A set* A *of a uniform space* $(X, \mathcal{U})$ *is stable if and only if there exists a Liapunov function for* A *into the retracted scale of* $\mathcal{U}$.

Proof. See [6, Theorem 1]. □

3.5 Pseudometrizability, separability, the Lindelöf condition, and the first and second axioms of countability of the uniform scale

Ramsey introduced in [29] a function which acts like a pseudometric on a uniform space, but which has as its range a subset of the uniform scale of that space. This function will then be used to investigate pseudometrizability, separability, the Lindelöf condition, and the first and second axioms of countability.

Definition 3.5.1 *([29]) Let X and M be nonempty sets, κ a function from $X \times X$ into M, t_0 a fixed point in M, and h_{t_0} a family of sets $\{X_\lambda : \lambda \in \wedge\}$ in M such that $t_0 \in X_\lambda$ for every $\lambda \in \wedge$. Then κ will be called an* écart *if the following conditions are satisfied.*

1. $\kappa(a,a) = t_0$ for every $a \in X$.

2. $\kappa(a,b) = \kappa(b,a)$ for every a and b in X.

3. h_{t_0} is submultiplicative, i.e., for every λ and λ' in $\wedge$, there exists a $\mu \in \wedge$, such that $X_\mu \subseteq X_\lambda \cap X_{\lambda'}$.

4. For every $\lambda \in \wedge$, there exists a $\mu \in \wedge$ such that $\kappa(a,b) \in X_\mu$ and $\kappa(b,c) \in X_\mu$ imply that $\kappa(a,c) \in X_\lambda$.

If κ is an écart on X, then a uniformity can be defined on X by using the following base:

$$W_\lambda = \{(a,b) \in X \times X : \kappa(a,b) \in X_\lambda\},\ \lambda \in \wedge.$$

The uniformity will be Hausdorff if and only if the following condition is satisfied.

5. $\kappa(x,a) \in \bigcap\{X_\lambda : \lambda \in \wedge\}$ implies that $a = x$.

Now, let $(X,\mathcal{U})$ be an arbitrary uniform space and $(P,\mathcal{U}')$ its uniform scale.

Definition 3.5.2 *([29]) Let 0 be the last element of P, and let $h_0 = \{U'(0) : U \in \mathcal{U}\}$. Define $\kappa : X \times X \longrightarrow P$ by*

$$\kappa(x, y) = \{U \in \mathcal{U} : (x, y) \in U \text{ and } (y, x) \in U\}$$

for every x and y in X.

Theorem 3.5.1 *([29]) Then κ is an écart, and the uniformity defined on X by κ is $\mathcal{U}$. Moreover, if $\mathcal{B}$ is a closed, symmetric base for $\mathcal{U}$, then for any $V \in \mathcal{B}$, $(a, b) \in V$ if and only if $\kappa(a, b) \in \mathcal{B}$.*

Proof. See [29, Theorem 2.1]. A similar proof is given in Theorem 5.4.1.□

Let $\varphi : P \longrightarrow P_R$ be the retraction map that is the (canonical map) as defined in Lemma 3.3.3. It is clear that if κ is an écart with values in P, then $\varphi\kappa$ is an écart with values in P_R. The following proposition implies that $\varphi\kappa$ also induces the uniformity $\mathcal{U}$ on X.

Theorem 3.5.2 *([29]) Let $(X,\mathcal{U})$ be a uniform space, $(P,\mathcal{U}')$ its scale and $(P_R,\mathcal{U}'')$ its retracted scale. Then the following statements are equivalent.*

1. $(X,\mathcal{U})$ is pseudometrizable.

2. $(P,\mathcal{U}')$ is pseudometrizable.

3. $(P,\mathcal{U}')$ satisfies the first axiom for countability.

4. $(P,\mathcal{U}')$ has a countable base for the neighborhood filter at 0.

5. There exists a map ϕ from P into the nonnegative real numbers such that $\phi(0) = 0$ and $\{\phi^{-1}([0, \varepsilon)) : \varepsilon > 0\}$ is a base for the neighborhood filter at 0.

Furthermore $(P,\mathcal{U}')$ can be replaced by $(P_R,\mathcal{U}'')$ in any of the statements (2) *through* (5).

Proof. See [29, Theorem 2.3]. A similar proof is given in Theorem 5.4.2.□

Questions (compare [29]): The écart κ maps $X \times X$ into the scale P. That suggests the following questions: Does there exist a uniformly continuous, order preserving map ϕ from P into the nonnegative real numbers and, if so, what can be said about the map $\phi\kappa : X \times X \longrightarrow \mathbb{R}^+$. The next series of propositions will be concerned with this problem.

First let $(X, d, \mathcal{U})$ be a pseudometric space, where $\mathcal{U}$ is the pseudometric uniformity. For $\varepsilon > 0$, let $U_\varepsilon = \{(x,y) \in X \times X : d(x,y) \leq \varepsilon\}$; note that this is not the usual meaning.

Lemma 3.5.1 *([29]) For any $\varepsilon, \delta \geq 0$, $U_\varepsilon^{'} \circ U_\delta^{'} \subseteq U_{\varepsilon+\delta}^{'}$.*

Proof. Here the proof is completed by using the composition of entourages.□

Now suppose that d is a bounded pseudometric.

Definition 3.5.3 *([29]) Define $\widetilde{d} : P \times P \longrightarrow \mathbb{R}^+$ by*

$$\widetilde{d}(\alpha, \beta) = \inf\{\varepsilon > 0 : (\alpha, \beta) \in U_\varepsilon^{'}\}.$$

(The fact that d is bounded ensures that $\widetilde{d}$ is well defined.)

Proposition 3.5.1 *([29]) Then $\widetilde{d}$ is a pseudometric on P, and the scale uniformity $\mathcal{U}'$ is the pseudometric uniformity induced by $\widetilde{d}$.*

Proof. First we show that $\widetilde{d}$ is pseudometric.

i. $\widetilde{d}(\alpha, \alpha) = 0$ for every $\alpha \in P$ since $(\alpha, \alpha) \in U_\varepsilon^{'}$ for every $\varepsilon > 0$.

ii. $\widetilde{d}(\alpha, \beta) = \widetilde{d}(\beta, \alpha)$ for every α and β in P by the symmetry of the $U_\varepsilon^{'}$'s.

iii. Let $\varepsilon > 0$, $\widetilde{d}(\alpha, \beta) = a$, and $\widetilde{d}(\beta, \gamma) = b$. Then $(\alpha, \beta) \in U_{a+\varepsilon}^{'}$ and $(\beta, \gamma) \in U_{b+\varepsilon}^{'}$ imply that $(\alpha, \gamma) \in U_{a+\varepsilon}^{'} \circ U_{b+\varepsilon}^{'} \subseteq U_{a+b+2\varepsilon}^{'}$. Therefore for any $\varepsilon > 0$, $\widetilde{d}(\alpha, \gamma) \leq \widetilde{d}(\alpha, \beta) + \widetilde{d}(\beta, \gamma) + 2\varepsilon$; hence $\widetilde{d}(\alpha, \gamma) \leq \widetilde{d}(\alpha, \beta) + \widetilde{d}(\beta, \gamma)$.

Thus $\widetilde{d}$ is a pseudometric. It is clear that $U_\varepsilon^{'} = \{(\alpha, \beta) : \widetilde{d}(\alpha, \beta) \leq \varepsilon\}$, so $\mathcal{U}'$ is the pseudometric uniformity. □

Remark 3.5.1 *Note that $\widetilde{d}(\alpha, \beta) = 0$ if and only if $(\alpha, \beta) \in U_\varepsilon^{'}$ for every $\varepsilon > 0$, and the latter is true if and only if α is equivalent to β. Thus the pseudometric becomes a metric on the retracted scale $(P_R, \mathcal{U}'')$.*

Lemma 3.5.2 *Let $(X,\mathcal{U})$ be a uniform space and $\alpha, \beta \in P$. If $\alpha \leq \beta$ and $\beta \in U'(0)$ for some $U \in \mathcal{U}$, then $\alpha \in U'(0)$.*

Proof. Let $\beta \in U'(0)$ for some $U \in \mathcal{U}$, which implies that for every $V \in \mathcal{U}$, $U \circ V \in \beta$. If $\alpha \leq \beta$, then $\beta \subseteq \alpha$, so $U \circ V \in \alpha$ for every $V \in \mathcal{U}$. Thus $\alpha \in U'(0)$. □

Proposition 3.5.2 *([29]) If $(X,\mathcal{U})$ is a pseudometrizable space and $(P,\mathcal{U}')$ its scale, then there exists a uniformly continuous, order preserving map ϕ from $(P,\mathcal{U}')$ into the nonnegative real numbers. (The latter set is assumed to be equipped with its standard metric uniformity.)*

Proof. Let d be a bounded pseudometric on X such that $\mathcal{U}$ is the pseudometric uniformity for d. Let $\widetilde{d}$ be the pseudometric on P defined by

$$\widetilde{d}(\alpha, \beta) = \inf\{\varepsilon > 0 : (\alpha, \beta) \in U_\varepsilon{}'\},$$

for all $\alpha, \beta \in P$. Then $\widetilde{d}$ is a pseudometric on P, and $\mathcal{U}'$ is its pseudometric uniformity by Proposition 3.5.1. Define $\phi : P \longrightarrow \mathbb{R}^+$ by $\phi(\alpha) = \widetilde{d}(\alpha, 0)$ for every $\alpha \in P$.

i. ϕ is order preserving: If $\alpha \leq \beta$, then by Lemma 3.5.2, for any $\varepsilon > 0$, $\beta \in U_\varepsilon{}'(0)$ implies that $\alpha \in U_\varepsilon{}'(0)$. Then $\widetilde{d}(\alpha, 0) \leq \widetilde{d}(\beta, 0)$, and so $\phi(\alpha) \leq \phi(\beta)$.

ii. ϕ is uniformly continuous:
Let $\varepsilon > 0$. If $(\alpha, \beta) \in U_\varepsilon{}'$, then $\widetilde{d}(\alpha, \beta) \leq \varepsilon$, so $\mid \phi(\alpha) - \phi(\beta) \mid = \mid \widetilde{d}(\alpha, 0) - \widetilde{d}(\beta, 0) \mid \leq \widetilde{d}(\alpha, \beta) \leq \varepsilon$. Therefore ϕ is uniformly continuous, and the proof is complete. □

If $(X,\mathcal{U})$ is not pseudometrizable, then a similar map ϕ can still be constructed. The next proposition gives the necessary tools to prove this statement.

Proposition 3.5.3 *([29]) Let d be a bounded pseudometric on X such that $\mathcal{U}_d \subseteq \mathcal{U}$, where $\mathcal{U}_d$ is the pseudometric uniformity defined by*

$$\widetilde{d}(\alpha, \beta) = \inf\{\varepsilon > 0 : (\alpha, \beta) \in U_\varepsilon{}'\}.$$

where U_ε is defined with respect to d. Then $\widetilde{d}$ is a pseudometric on P,

$$U_\varepsilon^{'} = \{(\alpha, \beta) : \widetilde{d}(\alpha, \beta) \leq \varepsilon\},$$

and $\mathcal{U}_{\widetilde{d}} \subseteq \mathcal{U}^{'}$, where $\mathcal{U}_{\widetilde{d}}$ is the uniformity generated on P by $\widetilde{d}$.

Proof. This is just like the proof of Proposition 3.5.1. We have $\{U_\varepsilon^{'} : \varepsilon > 0\} \subseteq \mathcal{U}^{'}$ since $\mathcal{U}_d \subseteq \mathcal{U}$, and it is clear that these sets form a base for a uniformity on P. □

Theorem 3.5.3 *([29]) If $(X,\mathcal{U})$ is any uniform space and $(P,\mathcal{U}^{'})$ its scale, then there exists a uniformly continuous, order preserving map ϕ from $(P,\mathcal{U}^{'})$ into the nonnegative real numbers.*

Proof. This is a consequence of Proposition 3.5.2 and Proposition 3.5.3. Then ϕ is order preserving and uniformly continuous with respect to $\mathcal{U}_{\widetilde{d}}$. Since $\mathcal{U}_{\widetilde{d}} \subseteq \mathcal{U}^{'}$, ϕ is also uniformly continuous with respect to $\mathcal{U}^{'}$. □

If $(X,\mathcal{U})$ is not pseudometrizable, then the sets $\{(x, y) \in X \times X : \phi\kappa(x, y) < \varepsilon\}$, $\epsilon > 0$, do not form a base for $\mathcal{U}$.

Proposition 3.5.4 *([29]) If $(X, d, \mathcal{U})$ is a bounded pseudometric space, $(P, \widetilde{d}, \mathcal{U}^{'})$ its scale, κ the écart of Theorem 3.5.1 , and ϕ the map in Proposition 3.5.2, then $d = \phi\kappa$.*

Proof. See [29, Proposition 2.10]. A similar proof is given in Proposition 5.4.5. □

3.6 The order scale

We start this section with some basic results on the order scale of a uniform space. This structure was introduced in [17].

Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}^{'})$ its scale uniform space. From Proposition 3.2.1 P is a complete distributive lattice with last element $\mathcal{U}$ (which we denote by 0) and greatest element $\{X \times X\}$(denoted by 1); infima and suprema in P are set unions and intersections, respectively, and from this it follows that P is infinitely distributive. Indeed, P is a subcomplete lattice of the power set of the power set of $X \times X$, with the dual of its usual ordering, which means that P is a subcomplete lattice of an atomic Boolean algebra, and thus bicompactly generated (see [17]).

Definition 3.6.1 *([17]) Let $(X,\mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$, we associate two prefilters σ_U and ρ_U as follows:*

$$\sigma_U = \{V \in \mathcal{U} : U \subseteq V\},$$

$$\rho_U = \{V \in \mathcal{U} : U \textit{ is not a subset of } V\}.$$

Note that $\sigma_U = \langle U \rangle$ as defined in Definition 3.1.1.

Proposition 3.6.1 *([17]) Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its scale.*
(a) Then $\alpha \in P$ is compact if and only if there are entourages $U_1, ..., U_n$ in $\mathcal{U}$ such that $\alpha = \rho_{U_1} \vee ... \vee \rho_{U_n}$.
(b) Then $\alpha \in P$ is cocompact if and only if there are entourages $U_1, ..., U_n$ in $\mathcal{U}$ such that $\alpha = \sigma_{U_1} \wedge ... \wedge \sigma_{U_n}$.

Proof. See [17, Proposition 2.1]. □

By Proposition 2.1.4, order convergence in P is topological; we denote the order topology by θ.

The following theorem summarizes properties of the order scale topology.

Theorem 3.6.1 *For any uniform space $(X,\mathcal{U})$, (P,θ) is a compact, T_2, totally disconnected topological space. If $\alpha \neq 0$, the θ-neighborhood filter $\mathcal{U}_\theta(\alpha)$ at α has an open subbase of sets of the form $[\rho_U, \sigma_V]$, where U does not belong to α and $V \in \alpha$. The θ-neighborhood filter of 0 has an open subbase of the form $[0, \sigma_V]$, $V \in \mathcal{U}$.*

Proof. This follows from Proposition 2.1.4 and Proposition 3.6.1. □

Since (P,θ) is compact and T_2, there is a unique uniformity ϑ for P which induces θ. We shall call ϑ the *order scale uniformity.*

Definition 3.6.2 *([17]) Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its uniform scale. The term* order scale *will be used ambiguously to mean either the topological space (P,θ) or the uniform space (P,ϑ).*

Theorem 3.6.2 *([17]) For any uniform space $(X,\mathcal{U})$, (P,ϑ) is a uniform lattice.*

Proof. See [17, Theorem 2.3]. □

Theorem 3.6.3 *([17]) The order scale (P,θ) is first countable (second countable, metrizable) in its order topology if and only if $(X,\mathcal{U})$ is finite or discrete.*

Proof. See [17, Theorem 2.5]. □

Kent showed that a uniformly continuous map between two uniform spaces can be "lifted" to a uniformly continuous map between their order scales (see [17]).

Theorem 3.6.4 *Let $(X_1,\mathcal{U}_1)$, $(X_2,\mathcal{U}_2)$ be two uniform spaces. If $f : (X_1,\mathcal{U}_1) \to (X_2,\mathcal{U}_2)$ is uniformly continuous, then so is $\widehat{f} : (P_1,\vartheta_1) \to (P_2,\vartheta_2)$. Let $\alpha \in P_1$. We define $\widehat{f}(\alpha) = [(f \times f)(\alpha)] \cap \mathcal{U}_2$, where $[(f \times f)(\alpha)]$ is the smallest prefilter on $X_2 \times X_2$ containing $\{(f \times f)(A) : A \in \alpha\}$. Then $\widehat{f}$ is also uniformly continuous. (As we shall see later, a similar result holds for the (quasi)-uniform scale (see Proposition 5.3.1.).)*

Proof. See [17, Theorem 2.6]. □

The relationship between the scale and order scale uniformities is summarized in the next proposition.

Proposition 3.6.2 *([17]) Let $(X,\mathcal{U})$ be a uniform space, $(P,\mathcal{U}')$ its uniform scale and (P,ϑ) its order scale.*
(a) $\mathcal{U}' \subseteq \vartheta$ if and only if $(X,\mathcal{U})$ is discrete.
(b) $\vartheta \subseteq \mathcal{U}'$ if and only if $(X,\mathcal{U})$ is finite or discrete.

Proof. See [17, Proposition 3.4]. □

Chapter 4

Topological properties of the scale of a uniform space

In this chapter, we shall summarise known results about the topological properties of the uniform scale of a uniform space. We shall first discuss the property of connectedness in the scale of a uniform space. These results were obtained by Leslie and Kent [19]. We also mention conditions due to Leslie and Kent that are necessary for local connectedness, connectedness and arcwise connectedness in the scale of a uniform space. In the last section of Chapter 4, we shall investigate other topological properties of the scale of a uniform space dealt with by Richardson [32].

4.1 Connectedness in the scale of a uniform space

In this section, we show that the scale of a uniform space is uniformly connected if and only if the uniform space is bounded [19]. Sufficient conditions are found on the uniform space which make the uniform scale connected and arcwise-connected, and the relationship between connectedness and local connectedness is also considered.

Let $(X,\mathcal{U})$ be a uniform space and $(P,\mathcal{U}')$ its scale uniform space. The last element of P is $\mathcal{U}$, denoted by 0; the greatest element is $\{X\times X\}$, denoted by 1, as mentioned before.

Definition 4.1.1 *([19]) Let $(X,\mathcal{U})$ be a uniform space; $(X,\mathcal{U})$ is said to be bounded if, for each $U \in \mathcal{U}$, there is a positive integer n such that $U^n = X \times X$.*

Definition 4.1.2 *([19]) Let $(X,\mathcal{U})$ be a uniform space; $(X,\mathcal{U})$ is said to be uniformly connected if every uniformly continuous map from the space into a discrete uniform space is a constant map.*

The following lemma shows that boundedness implies uniform connectedness.

Lemma 4.1.1 *(Mrówka and Pervin)([19]) A uniform space $(X,\mathcal{U})$ is uniformly connected if and only if, for each pair x, y of elements in X and each $U \in \mathcal{U}$, there is an integer n such that $(x, y) \in U^n$.*

Proof. See [19, Lemma 1]. □

Theorem 4.1.1 *([19]) The following statements are equivalent.*

(1) $(X,\mathcal{U})$ *is bounded.*

(2) $(P,\mathcal{U}')$ *is bounded.*

(3) $(P,\mathcal{U}')$ *is uniformly connected.*

Proof. That (1) implies (2) follows from Proposition 3.2.3. That (2) implies (3) is a consequence of Lemma 4.1.1. To show that (3) implies (1), assume that $(P,\mathcal{U}')$ is uniformly connected and let $U \in \mathcal{U}$. By Lemma 4.1.1. there is an integer n such that $(0, 1) \in U'^n$; furthermore we can assume by Proposition 3.2.3 that $U'^n = U^{n'}$. Thus for each $V \in \mathcal{U}$, $U^n \circ V = X \times X$, and in particular $U^{n+1} = X \times X$. So $(X,\mathcal{U})$ is bounded. □

A topological space is connected if and only if the only continuous map from the space into a discrete topological space is a constant map. Thus connectedness implies uniform connectedness in a uniform space, and boundedness in $(X,\mathcal{U})$ is necessary for connectedness in $(P,\mathcal{U}')$.

Definition 4.1.3 *([19]) For each $U \in \mathcal{U}$ and $\alpha \in P$, let*

$$\alpha_U = \{V \in \mathcal{U} : U \circ V \in \alpha\}$$

and

$$\alpha^U = \{V \in \mathcal{U} : \text{for some } A \in \alpha,\ U \circ A \subseteq V\}.$$

Lemma 4.1.2 *([19]) For each $U \in \mathcal{U}$ and for each $\alpha \in P$, $U^{'}(\alpha) = [\alpha_U, \alpha^U]$.*

Proof. Let $\beta \in U^{'}(\alpha)$. Let $V \in \alpha^U$, for some $A \in \alpha$, then $U \circ A \subseteq V \in \beta$ so $\alpha^U \subseteq \beta$. Let $B \in \beta$ which implies $U \circ B \in \alpha$ so $B \in \alpha_U$.

Conversely, let β be such that $\alpha^U \subseteq \beta \subseteq \alpha_U$. Let $V \in \alpha^U$. For some $A \in \alpha$, $U \circ A \subseteq V \in \beta$ and let $B \in \beta$ which implies $U \circ B \in \alpha$, so $(\alpha, \beta) \in U^{'}$. Thus $B \in U^{'}(\alpha)$. □

Proposition 4.1.1 *([19]) The collection $\{[\alpha_U, \alpha^U] : U \in \mathcal{U}\}$ forms a basic system of neighborhoods at α for the uniform topology on P generated by $\mathcal{U}^{'}$.*

Proof. See [19, Proposition 2]. □

We next define the composition of prefilters.

Definition 4.1.4 *([19]) Let M be a subset of P. Given $\alpha, \beta \in P$, we define their product as follows:*

$$\alpha \circ \beta = \{W \in \mathcal{U} : \text{for some } U \in \alpha \text{ and } V \in \beta, W \supseteq U \circ V\}.$$

Let

$$M \bullet \alpha = \{\beta \circ \alpha : \beta \in M\}. \text{ Let } M \vee \alpha = \{\beta \vee \alpha : \beta \in M\}$$

and

$$M \wedge \alpha = \{\beta \wedge \alpha : \beta \in M\}.$$

The lattice operations $\vee$ and $\wedge$ in P are uniformly continuous by Theorem 3.3.1. Also for each α the map $\beta \mapsto \beta \circ \alpha$ is uniformly continuous [19]. Then we obtain the following results about connectedness and local connectedness in the uniform scale space. They are immediate (see [19]).

Proposition 4.1.2 *([19]) If M is a connected (arcwise connected) subset of P, then the sets $M \bullet \alpha$, $M \vee \alpha$ and $M \wedge \alpha$ are likewise connected (arcwise connected) for any choice of α.*

Proposition 4.1.3 *([19]) If there is a connected (arcwise connected) subset C of P containing α and β, then $[\alpha \wedge \beta, \alpha \vee \beta]$ is connected (arcwise connected).*

Corollary 4.1.1 *([19]) The scale of a uniform space is connected (arcwise connected) if 0 and 1 belong to the same connected (arcwise connected) component.*

Corollary 4.1.2 *([19]) If the closed interval $[\alpha, \beta]$ in P is a subset of a connected (arcwise connected) set M, then $[\alpha, \beta]$ is connected (arcwise connected).*

Corollary 4.1.3 *([19]) The scale of a uniform space is locally connected (locally arcwise connected) if and only if there is a connected (arcwise connected) neighborhood for each point.*

The following theorem characterizes the local connectedness condition.

Theorem 4.1.2 *([19]) The scale $(P, \mathcal{U}')$ of a uniform space $(X, \mathcal{U})$ is locally connected (locally arcwise connected) if and only if there is a connected (arcwise connected) neighborhood of 0.*

We next study connectedness of the scale of a uniform space.

Theorem 4.1.3 *([19]) The scale $(P, \mathcal{U}')$ of a bounded uniform space $(X, \mathcal{U})$ is connected (arcwise connected) if and only if it is locally connected (arcwise connected).*

Proof. See [19, Theorem 3]. □

Boundedness in $(X, \mathcal{U})$ is not a necessary condition for local connectedness in the scale of a uniform space $(P, \mathcal{U}')$. In the next proposition, we cite a weaker condition found by Leslie and Kent which is necessary for local connectedness in the scale of a uniform space.

Proposition 4.1.4 *([19]) If $(P,\mathcal{U}')$ is locally connected, then there is an entourage $U \in \mathcal{U}$ such that, for every $V \in \mathcal{U}$, $U \subseteq V^n$ for some integer n.*

Proof. See [19, Proposition 5]. □

4.2 Connectedness conditions in the scale of a uniform space

In this section, we will now give conditions on $(X,\mathcal{U})$ which are sufficient for connectedness, local connectedness, and arcwise connectedness in $(P,\mathcal{U}')$.

In [19] Leslie and Kent used the following definitions.

Definition 4.2.1 *([19]) A uniform space $(X,\mathcal{U})$ satisfies Condition A if there is a base $\mathcal{B}$ for the uniform space which is closed under unions and finite compositions such that, given a chain $\{V_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{B}$ and $W \in \mathcal{U}$, there is an index $\lambda_0 \in \Lambda$ such that $W \circ V_{\lambda_0} \supseteq \cup_{\lambda\in\Lambda} V_\lambda$.*

Definition 4.2.2 *([19]) Let (X,d) be a pseudo-metric space, and for each $\epsilon > 0$ let O_ϵ be the entourage $\{(x,y) : d(x,y) < \epsilon\}$. If $O_\epsilon \circ O_\delta = O_{\epsilon+\delta}$ for all positive real numbers ϵ, δ then (X,d) is said to satisfy Condition B.*

The uniformity derived from a pseudo-metric which satisfies Condition B will surely satisfy Condition A, since $\mathcal{B}$ can be chosen to consist of $\{O_\epsilon : \epsilon > 0\}$.

Theorem 4.2.1 *([19]) If $(X,\mathcal{U})$ is a bounded uniform space which satisfies Condition A, then $(P,\mathcal{U}')$ is connected.*

Proof. Assume that there is a subset M of P which is both closed and open, distinct from P, and contains an element α. We shall show that 1 is in M. If N is the complement of M, then precisely the same argument establishes that $1\in N$, a contradiction.

Let $\mathcal{C} = \{V \in \mathcal{U} : \alpha^V \in M, V \in \mathcal{B}\}$, where $\mathcal{B}$ is the base for $\mathcal{U}$ specified in Condition A. Then $\mathcal{C}$ is non-empty since M is open. We shall partially order $\mathcal{C}$ by set inclusion and apply Zorn's Lemma to establish the existence of a maximal element U in $\mathcal{C}$. Indeed, if $\{V_i\}$ is a chain in $\mathcal{C}$, and $V = \cup V_i$, then Condition A is precisely what is needed to show that the net $\{\alpha^{V_i}\}$ converges to α^V. Since M is closed, α^V is in M, and $V \in \mathcal{C}$.

Since $\alpha^U \in M$ and M is open, there is $W \in \mathcal{B}$ such that $(\alpha^U)^W = \alpha^{WU} \in M$. Thus $WU \in \mathcal{C}$ (recall that $\mathcal{C}$ is closed under finite compositions) and so $WU = U$. The proof will be completed by showing that, necessarily, $U = X \times X$. Indeed, $W^2U = WWU = U$, and by induction $W^kU = U$ for all integers k. But $(X, \mathcal{U})$ is bounded, and so $W^n = X \times X$ for some integer n. Thus $U = X \times X$, and $\alpha^U = 1 \in M$, which completes the proof. □

If (X, d) satisfies Condition B, then boundedness in (X, d) is equivalent to boundedness in the associated uniform space, and hence this term as used in the next theorem can be interpreted in either sense.

Theorem 4.2.2 *([19]) If a pseudo-metric space (X, d) satisfies Condition B, then the scale $(P, \mathcal{U}')$ of the associated uniform space is locally arcwise connected. If, in addition, (X, d) is bounded, then $(P, \mathcal{U}')$ is arcwise connected.*

Proof. We can assume without loss of generality that d assumes values less than or equal to 1. Let f be the function mapping the unit interval $[0, 1]$ into $(P, \mathcal{U}')$ defined by $f(r) = \langle O_r \rangle$, where $\langle O_r \rangle$ denotes the prefilter consisting of all supersets of O_r; in particular let $f(0) = 0$. Condition B guarantees that f is uniformly continuous. Thus by Proposition 4.1.3, the interval $[0, \langle O_r \rangle]$ is arcwise connected, and it follows from Theorem 4.1.2 that $(P, \mathcal{U}')$ is locally arcwise connected. □

4.3 Other topological properties and the cardinality of the scale of a uniform space

In the following section we describe under which condition the scale of a uniform space is compact and we shall discuss the cardinality of the scale of a uniform space and its retracted scale.

The next proposition will be useful in proving that a necessary and sufficient condition for the scale to have a totally bounded neighborhood of 0 is that $\mathcal{U}$ has a last element.

Proposition 4.3.1 *([32]) Suppose that $(X,\mathcal{U})$ is a uniform space such that there exists a $U \in \mathcal{U}$ with the property that the complement of U is an infinite set. Then either $\mathcal{U}$ has a last element or there exists a countably infinite discrete subspace of $[0, \langle U \rangle]$.*

Proof. See [32, Proposition 3]. □

Proposition 4.3.2 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then $(P,\mathcal{U}')$ has a totally bounded neighborhood of 0 if and only if $\mathcal{U}$ has a last element.*

Proof. See [32, Proposition 4]. □

Theorem 4.3.1 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then the following are equivalent conditions:*

(a) *$(P,\mathcal{U}')$ has a totally bounded neighborhood of 0.*

(b) *$(P,\mathcal{U}')$ has a compact neighborhood of 0.*

(c) *$(P,\mathcal{U}')$ is locally compact.*

(d) *$\mathcal{U}$ has a last element.*

Proof. $(a) \Longrightarrow (b)$: Proposition 4.3.1 implies that $\mathcal{U}$ has a last element U. Then $[0, \langle U \rangle] = \{0\}$ is a compact neighborhood of 0. $(b) \Longrightarrow (c)$: Let $\alpha \in P$; then if U is the last element of $\mathcal{U}$, $U'(\alpha) \subseteq V'(\alpha)$ for each $V \in \mathcal{U}$, and so $U'(\alpha)$ is a compact neighborhood of α. That $(c) \Longrightarrow (d)$ and $(d) \Longrightarrow (a)$ follows from Proposition 4.3.1. □

Example 4.3.1 *([32]) It is possible for a uniformity to have a last element which is not the diagonal $\triangle$. Let $X = \mathbb{R}$ and let $\mathcal{U}$ be the set of all subsets of $\mathbb{R}^2$ that contain $\triangle \cup \{(-1,1),(1,-1)\}$. Then the set $\triangle \cup \{(-1,1),(1,-1)\}$ is the last element of $\mathcal{U}$.*

We now turn from local properties of the uniform scale space to its global properties.

Proposition 4.3.3 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then $(P,\mathcal{U}')$ is totally bounded if and only if either X is finite or $\mathcal{U} = \{X \times X\}$.*

Proof. See [32, Proposition 5]. We shall see later that the same results hold for quasi-uniformities (Theorem 5.3.4). □

Note that in Example 4.3.1 $(P,\mathcal{U}')$ has a totally bounded neighborhood of 0, but $(P,\mathcal{U}')$ is not totally bounded.

The following lemmas are immediate ([32]). They will be used to show that the statement "the scale of a uniform space is compact if and only if X is finite or $\mathcal{U} = \{X \times X\}$" remains true if compact is replaced by countably compact, totally bounded, Lindelöf, second countable, or separable.

Lemma 4.3.1 *([32]) Let $(X,\mathcal{U})$ be a uniform space and $V \in \mathcal{U}$ be such that $X - V(x) =: A$ for some $x \in X$, where the cardinality of A is infinite. Then P_R contains a closed discrete subset of cardinality 2^{cardA}.*

Lemma 4.3.2 *([32]) Suppose that $(X,\mathcal{U})$ is a uniform space where $card(X) = \kappa$, and V is a symmetric entourage such that for all $x \in X$, $card(X - V(x)) = \lambda < \kappa$ where κ and λ are infinite cardinal numbers. Then $V^2 = X \times X$.*

Remark 4.3.1 *([32]) The conclusion of the above lemma remains true if κ is infinite and $X - V(x)$ has finitely many elements for all $x \in X$.*

Theorem 4.3.2 *([32]) Let $(X,\mathcal{U})$ be a uniform space. Then the following conditions are equivalent:*

(a) *$(P,\mathcal{U}')$ is countably compact.*

(b) *$(P,\mathcal{U}')$ is totally bounded.*

(c) *$(P,\mathcal{U}')$ is Lindelöf.*

(d) *$(P,\mathcal{U}')$ is compact.*

(e) $(P, \mathcal{U}')$ *is second countable.*

(f) $(P, \mathcal{U}')$ *is separable.*

(g) *Either* X *is finite or* $\mathcal{U} = \{X \times X\}$.

Proof. If X is finite or if $\mathcal{U} = \{X \times X\}$, all the remaining conditions easily follow.
Suppose that $(P, \mathcal{U}')$ is countably compact, totally bounded, Lindelöf, compact, second countable, or separable. Suppose that X is not finite. We wish to show that $\mathcal{U} = \{X \times X\}$.
Assume that $\mathcal{U} \neq \{X \times X\}$. Then by Lemma 4.3.2 and Remark 4.3.1, there exist $V \in \mathcal{U}$ and $x \in X$ such that $X - V(x)$ is infinite. By Lemma 4.3.1, P_R contains a closed, discrete subset of cardinality $2^{card(X-V(x))}$. This fact is contrary to $(P, \mathcal{U}')$ being countably compact, totally bounded, Lindelöf, compact, or second countable.
Moreover, we claim that it is contrary to $(P, \mathcal{U}')$ being separable. Suppose that there is a countable dense subset E of P. Choose a symmetric $W \in \mathcal{U}$ such that $W^3 \subseteq V$. Let B be the uncountable subset of P which is defined in the proof of Lemma 4.3.1. By that proof for each $\alpha \in P$, $W'(\alpha)$ contains at most one element of B. Since E is a countable dense subset of P and B is an uncountable subset of P, there must exist distinct elements $\alpha, \beta \in B$ such that $W'(\alpha) \cap W'(\beta) \cap E \neq \emptyset$. This implies that $\beta \in W'(\alpha)$, a contradiction. We conclude that $\mathcal{U} = \{X \times X\}$, and the theorem is proved. □

Since the retracted scale P_R is made up of equivalence classes of elements of the scale P, the cardinality of P_R must be less than or equal to the cardinality of P. For certain uniform spaces the cardinalities are equal. Let T be a set. In the proof of the following theorem $\mathcal{P}(T)$ will denote the power set of T.

Theorem 4.3.3 *([32]) Let* X *be an infinite set and* $\mathcal{U}$ *a uniformity on* X *such that* $\mathcal{U} \neq \{X \times X\}$. *If* $card(X) = \kappa$, *then* $card(P_R) = 2^{2^\kappa} = card(P)$.

Proof. Since $P \subseteq \mathcal{P}(\mathcal{P}(X \times X))$, $card(P_R) \leq card(P) \leq 2^{2^\kappa}$. By Lemma 4.3.2 there exists $V \in \mathcal{U}$ such that $card(X - V(x)) = \kappa$ for some $x \in X$. Let $W \in \mathcal{U}$ be such that $W^3 \subseteq V$ and let $B = (X - V(x)) \times \{x\}$ where $x \in X$, and W and V are symmetric. Let $F(B)$ denote the collection of all filters on $X \times X$ which contain B. Then $card(F(B)) = 2^{2^\kappa}$. For each $\mathcal{F} \in F(B)$, let

$\alpha_{\mathcal{F}} = \langle W \rangle \cap \mathcal{F}$, and $A = \{\alpha_{\mathcal{F}} | \mathcal{F} \in F(B)\}$. Notice that if $\mathcal{F}, \mathcal{G} \in F(B)$ such that $\mathcal{F} \neq \mathcal{G}$, then $\alpha_{\mathcal{F}} \neq \alpha_{\mathcal{G}}$, which implies that $card(A) = 2^{2^\kappa}$. Also, note that $A \subseteq P$.

We claim that A is discrete. If $\alpha \in P$, and $\alpha_{\mathcal{F}}$, $\alpha_{\mathcal{G}} \in W'(\alpha)$, then $(\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}) \in (W^2)'$ since W is symmetric. If $\mathcal{F} \neq \mathcal{G}$, we may assume without loss of generality that there exists $F \in \mathcal{F} - \mathcal{G}$. For all $G \in \mathcal{G}$ the set $(X - F) \cap G$ is nonempty. Since $(\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}) \in (W^2)'$ and $W \cup F \in \alpha_{\mathcal{F}}$, $W^2 \circ (W \cup F) \in \alpha_{\mathcal{G}}$. Hence $W^2 \circ (W \cup F) \supseteq W \cup G$ for some $G \in \mathcal{G}$. Let (s, x) be a point in $G - F$. Then $(s, z) \in W \cup F$, and $(z, x) \in W^2$ for some $z \in X$. Now, (s, x) does not belong to $W^3 \subseteq V$ because $(s, x) \in G$ and G is contained in the complement of V. Therefore, $(s, z) \in F \subseteq B$, which implies that $z = x$ and $(s, x) \in F$. We have reached a contradiction to our choice of (s, x); hence, $W'(\alpha)$ contains at most one element of A. Therefore, A is a discrete subset of P, the closure of A in P is $\cup\{(\alpha_{\mathcal{F}})_R | \mathcal{F} \in F(B)\}$, and the image of the closure of A under φ is $\varphi(A)$.

Now P and P_R both contain discrete subsets of cardinality 2^{2^κ}, so $card(P)$ and $card(P_R)$ are at last 2^{2^κ}. Recalling that $card(P) \leq 2^{2^\kappa}$, we conclude that $card(P) = 2^{2^\kappa} = card(P_R)$. $\square$

Chapter 5

The scale of a quasi-uniform space

In this main chapter we introduce the scale of a quasi-uniform space. In [16] and [17], Kent studied the scale of a uniform space and its retracted scale. We summarized some of the results of his investigations in the first part of this thesis.

We will first define the scale of a quasi-uniform space and the retracted scale of a quasi-uniform space associated with it and we will establish the connection and similarities between our scale of a quasi-uniform space and the scale of a uniform space investigated by Kent [16] and Bushaw [6]. In particular we shall succeed in generalizing several results from the uniform to the quasi-uniform setting.

We will then define the prefilter space of a quasi-uniform space. That space is closely related to our (left-sided) scale of a quasi-uniform space. We will also show that the Hausdorff hyperspace quasi-uniform space is embedded into the prefilter space of a quasi-uniform space.

In the last section we will define the two-sided scale of a quasi-uniform space and we will show that total boundedness is preserved by this modified scale. When the results in this chapter are new, we shall give complete proofs.

5.1 Definition of the scale of a quasi-uniform space

In the following we shall start by defining the scale of a quasi-uniform space. First it is not clear which set of prefilters we should choose for a quasi-uniform space $(X,\mathcal{U})$: Prefilters of $\mathcal{U}$, of $\mathcal{U} \cup \mathcal{U}^{-1}$ or $\mathcal{U}^s$. We decided to make the following choice. Let $(X,\mathcal{U})$ be a quasi-uniform space and let $P_{\mathcal{U}^s} = \{\alpha : \alpha$ is prefilter on $X \times X$ and $\alpha \subseteq \mathcal{U}^s\}$. But see Section 5.6.

Definition 5.1.1 *Let $\alpha \in P_{\mathcal{U}^s}$. We define α^{-1} by*

$\alpha^{-1} = \{A^{-1} : A \in \alpha\}$. Of course $\alpha^{-1} \in P_{\mathcal{U}^s}$, too.

Definition 5.1.2 *For any two elements α and β in $P_{\mathcal{U}^s}$, $\alpha \leq \beta$ if $\alpha \supseteq \beta$.*

We next define the order of the scale.

Definition 5.1.3 *The order structure $(P_{\mathcal{U}^s}, \leq)$ is called the* order of the scale *of the quasi-uniform space $(X,\mathcal{U})$.*

Proposition 5.1.1 *(Compare Proposition 3.2.1) For any quasi-uniform space $(X,\mathcal{U})$, the order of the scale $(P_{\mathcal{U}^s}, \leq)$ is a complete distributive lattice whose suprema and infima are set intersections and set unions, respectively, and $\mathcal{U}^s$ is the last element and $\{X \times X\}$ is the greatest element.*

Proof. For the proof we refer to Proposition 3.2.1. □

Again it is not obvious how to define the quasi-uniformity of the scale of a quasi-uniform space. Our definition is motivated by the definition of the Hausdorff hyperspace quasi-uniformity of a quasi-uniform space.

We introduce the following notations:

Let X be a set. For any $R \subseteq X \times X$ and some prefilters α and β on $X \times X$.

We write $R \circ \alpha := \{R \circ A : A \in \alpha\}$, and

$$\alpha \circ R := \{A \circ R : A \in \alpha\}$$

Proposition 5.1.2 *Let* $(X, \mathcal{U})$ *be a quasi-uniform space and let* $P_{\mathcal{U}^s} = \{\alpha : \alpha$ *is prefilter on* $X \times X$ *and* $\alpha \subseteq \mathcal{U}^s\}$. *For each* $U \in \mathcal{U}$ *we set*

$$S(U)_+ = \{(\alpha, \beta) \in P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} : U \circ \alpha \subseteq \beta\}$$

and

$$S(U)_- = \{(\alpha, \beta) \in P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} : \beta^{-1} \circ U \subseteq \alpha^{-1}\}.$$

Furthermore set $S(U) = S(U)_- \cap S(U)_+$ *whenever* $U \in \mathcal{U}$.
Then $\{S(U)_- : U \in \mathcal{U}\}$ *is a base for the* negative *quasi-uniformity* $S(\mathcal{U})_-$ *on* $P_{\mathcal{U}^s}$ *and* $\{S(U)_+ : U \in \mathcal{U}\}$ *is a base for the* positive *quasi-uniformity* $S(\mathcal{U})_+$ *on* $P_{\mathcal{U}^s}$. *Moreover* $\{S(U) : U \in \mathcal{U}\}$ *is a base for the* scale quasi-uniformity $S(\mathcal{U})$.

Proof. For each $U \in \mathcal{U}$ and any $\alpha \in P_{\mathcal{U}^s}$, we have $(\alpha, \alpha) \in S(U)_+$ and similarly $(\alpha, \alpha) \in S(U)_-$ and $(\alpha, \alpha) \in S(U)$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $S(V)_+ \subseteq S(U)_+$ and similarly $S(V)_- \subseteq S(U)_-$, and $S(V) \subseteq S(U)$. Hence $\{S(U)_+ : U \in \mathcal{U}\}$, $\{S(U)_- : U \in \mathcal{U}\}$ and $\{S(U) : U \in \mathcal{U}\}$ are filter bases on $P_{\mathcal{U}^s} \times P_{\mathcal{U}^s}$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\alpha, \gamma) \in {S(V)_+}^2$. Then there is $\beta \in P_{\mathcal{U}^s}$ such that $(\alpha, \beta) \in S(V)_+$ and $(\beta, \gamma) \in S(V)_+$. Let $A \in \alpha$. Then $V \circ A \in \beta$. We have $V^2 \circ A \in \gamma$ implies $U \circ A \in \gamma$. We have shown that $(\alpha, \gamma) \in S(U)_+$. Similarly ${S(V)_-}^2 \subseteq S(U)_-$ and $S(V)^2 \subseteq S(U)$. We deduce that $S(\mathcal{U})_+$, $S(\mathcal{U})_-$ and $S(\mathcal{U})$ are quasi-uniformities on $P_{\mathcal{U}^s}$. □

The following lemma is similar to Proposition 3.2.3.

Lemma 5.1.1 *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. For each* $U \in \mathcal{U}$, $S(U^2) = S(U)^2$.

Proof. Let $(\alpha, \gamma) \in S(U)^2$ which implies that there exists $\beta \in P_{\mathcal{U}^s}$ such that $(\alpha, \beta) \in S(U)$ and $(\beta, \gamma) \in S(U)$. Then $U \circ \alpha \subseteq \beta$ implies $U^2 \circ \alpha \subseteq \gamma$

and $\gamma^{-1} \circ U \subseteq \beta^{-1}$ implies $\gamma^{-1} \circ U^2 \subseteq \alpha^{-1}$. Thus $(\alpha, \gamma) \in S(U^2)$.

Conversely, let (α, γ) be in $S(U^2)$ and let β be the prefilter which contains all oversets of sets of the form $U \circ T$, for some $T \in \alpha$, or $V^{-1} \circ U$, for some $V \in \gamma$. So (α, β) and (β, γ) are both in $S(U)$, thus $(\alpha, \gamma) \in S(U)^2$. □

We next define the scale of a quasi-uniform space.

Definition 5.1.4 *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. Then* $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ *is called the* quasi-uniform scale *of the quasi-uniform space* $(X, \mathcal{U})$.

The following definition is similar to Definition 4.1.4.

Definition 5.1.5 *Let* M *be a subset of* $P_{\mathcal{U}^s}$. *Given* $\alpha, \beta \in P_{\mathcal{U}^s}$, *we define their product as follows:*

$$\alpha \circ \beta = \{W \in \mathcal{U}^s : \textit{for some } U \in \alpha \textit{ and } V \in \beta, W \supseteq U \circ V\}.$$

Let

$$M \bullet \alpha = \{\beta \circ \alpha : \beta \in M\}. \textit{ Let } M \vee \alpha = \{\beta \vee \alpha : \beta \in M\}$$

and

$$M \wedge \alpha = \{\beta \wedge \alpha : \beta \in M\}.$$

The following proposition makes a connection between the scale of a quasi-uniform space and the scale of a uniform space investigated by Bushaw [6] and Kent [16] (see Chapter 3 of this thesis).

Proposition 5.1.3 *If* $(X, \mathcal{U})$ *is a uniform space then* $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ *gives the Kent-Bushaw scale.*

Proof. Suppose that $(X, \mathcal{U})$ is a uniform space. We show that $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is the scale of the uniform space $(X, \mathcal{U})$ of Kent-Bushaw. Note that for $U \in \mathcal{U}$, we have $S(U \cap U^{-1}) \subseteq S(U)$ and $U \cap U^{-1} \in \mathcal{U}$ so that $\{S(U) : U \in \mathcal{U}, U$ symmetric $\}$ is a base of the scale. Indeed $P = P_{\mathcal{U}^s}$. Let $(\alpha, \beta) \in S(U)$. Then $A \in \alpha$ implies $U \circ A \in \beta$ and if $B \in \beta$ then $B^{-1} \circ U \in \alpha^{-1}$, so $(B^{-1} \circ U)^{-1} = U^{-1} \circ B \in \alpha$. Indeed $\mathcal{U}$ is a uniformity so that for each $U \in \mathcal{U}$ with $U^{-1} = U$ thus $(B^{-1} \circ U)^{-1} = U \circ B \in \alpha$; therefore $(\alpha, \beta) \in U'$. □

Proposition 5.1.4 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(P_{\mathcal{U}^s},S(\mathcal{U})^{-1}) = (P_{\mathcal{U}^s}, S(\mathcal{U}^{-1}))$, where the latter space is the scale of $(X,\mathcal{U}^{-1})$.*

Proof. Let $U \in \mathcal{U}$. Let $(\alpha,\beta) \in S(U)_+{}^{-1}$ which implies $(\beta,\alpha) \in S(U)_+$. We have $U \circ \beta \subseteq \alpha$ implies $\beta^{-1}\circ U^{-1} \subseteq \alpha^{-1}$, so $(\alpha,\beta) \in S(U^{-1})_-$. Hence $S(U)_+{}^{-1} = S(U^{-1})_-$.

Let $(\alpha,\beta) \in S(U)_-{}^{-1}$ which implies $(\beta,\alpha) \in S(U)_-$. We have $\alpha^{-1} \circ U \subseteq \beta^{-1}$ implies $U^{-1}\circ\alpha \subseteq \beta$ so $(\alpha,\beta) \in S(U^{-1})_+$. Hence $S(U)_-{}^{-1} = S(U^{-1})_+$. Thus for each $U \in \mathcal{U}$, $S(U)^{-1} = S(U^{-1})$. □

Call two prefilters α and β on $X \times X$ equivalent if $(\alpha,\beta) \in \bigcap_{U\in\mathcal{U}} S(U) \cap S(U^{-1})$.

The following lemma shows when two prefilters are equivalent.

Lemma 5.1.2 *Let $(X,\mathcal{U})$ be a quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale. Then $(\alpha,\beta) \in \bigcap_{U\in\mathcal{U}} S(U) \cap S(U^{-1})$ if and only if $\mathcal{U}\circ\alpha = \mathcal{U}\circ\beta$ and $\mathcal{U}^{-1}\circ\alpha = \mathcal{U}^{-1}\circ\beta$. We also note that for each $\alpha \in P_{\mathcal{U}^s}$, α is equivalent to $\mathcal{U}\circ\alpha \cup \mathcal{U}^{-1}\circ\alpha$.*

Proof. Indeed, suppose that $(\alpha,\beta) \in \bigcap_{U\in\mathcal{U}} S(U) \cap S(U^{-1})$. Let $U \in \mathcal{U}$, we have $(\alpha,\beta) \in S(U)$ and $(\alpha,\beta) \in S(U^{-1})$. Then $U\circ\alpha \subseteq \beta$, $\beta^{-1}\circ U \subseteq \alpha^{-1}$ and $U^{-1}\circ\alpha \subseteq \beta$, $U\circ\beta \subseteq \alpha$. By conjugation $\alpha^{-1}\circ U \subseteq \beta^{-1}$, $U\circ\beta \subseteq \alpha$. We have for each $U \in \mathcal{U}$, $U^2\circ\alpha \subseteq U\circ\beta$ and $U^2\circ\beta \subseteq U\circ\alpha$. Hence $\mathcal{U}\circ\alpha = \mathcal{U}\circ\beta$. Similarly $\mathcal{U}^{-1}\circ\alpha = \mathcal{U}^{-1}\circ\beta$. The converse is trivial. Since for instance $\mathcal{U}\circ\beta \subseteq \beta$ and $\mathcal{U}^{-1}\circ\beta \subseteq \beta$. □

The following proposition introduces a family of quasi-uniformly continuous functions from the original quasi-uniform space to its scale quasi-uniform space. It should be compared with Lemma 3.3.6.

Proposition 5.1.5 *Let $(X,\mathcal{U})$ be a quasi-uniform space, and let a be any fixed element of X. Let $\eta_a : X \longrightarrow P_{\mathcal{U}^s}$ be defined at each x in X by $\eta_a(x) = \{V \in \mathcal{U}^s : (a,x) \in V\}$. Then η_a is a quasi-uniformly continuous map with respect to $\mathcal{U}$ on X and the scale quasi-uniformity $S(\mathcal{U})$ on $P_{\mathcal{U}^s}$.*

Proof. Choose an entourage U in $\mathcal{U}$, and let $(x, y) \in U$. We will show that $(\eta_a(x), \eta_a(y)) \in S(U)$.

Indeed, $V \in \eta_a(x)$ implies $(a, x) \in V$. Since $(x, y) \in U$, we have $(a, y) \in U \circ V$, and hence $U \circ V \in \eta_a(y)$. Let $W \in \eta_a(y)$ with $(a, y) \in W$, and thus $(y, a) \in W^{-1}$.

Since $(x, y) \in U$, we have $(x, a) \in W^{-1} \circ U$, and hence $W^{-1} \circ U \in \eta_a(x)^{-1}$.□

In the next lemma, we show that any T_0-quasi-uniform space can be quasi-uniformly embedded in its quasi-uniform scale space.

Lemma 5.1.3 *If $(X,\mathcal{U})$ is a T_0-quasi-uniform space and η_a is defined as in Proposition 5.1.5, then the map η_a is a quasi-uniform embedding.*

Proof. Indeed, from Proposition 5.1.5 η_a is quasi-uniformly continuous.

Let $x, y \in X$ be such that $\eta_a(x) = \eta_a(y)$. Let $U \in \mathcal{U}$. Since $U \circ \{(a, x)\} \in \eta_a(x) = \eta_a(y)$ we have $(a, y) \in U \circ \{(a, x)\}$. Thus $(x, y) \in U$. Similarly $(y, x) \in U$. Thus $(x, y) \in \bigcap_{U \in \mathcal{U}} U \cap \bigcap_{U \in \mathcal{U}} U^{-1} = \triangle$ and $x = y$.

Given $U \in \mathcal{U}$ there is $H \in \mathcal{U}$ such that $H^4 \subseteq U$. Suppose that for $x, y \in X$, $(\eta_a(x), \eta_a(y)) \in S(H)$. Then $H \circ \eta_a(x) \subseteq \eta_a(y)$ and $(\eta_a(y))^{-1} \circ H \subseteq (\eta_a(x))^{-1}$.

Consequently $(a, y) \in H \circ (H \cup \{(a, x)\})$ and $(a, x) \in (H \cup \{(y, a)\}) \circ H$. Then $(a, y) \in H^2$ and $(x, a) \in H^2$, or $(x, y) \in H$. In any case $(x, y) \in H^4 \subseteq U$. Hence $\eta_a{}^{-1} \mid \eta_a(X)$ is quasi-uniformly continuous as a map from $(\eta_a(X), S(\mathcal{U}) \mid \eta_a(X)) \to (X, \mathcal{U})$. □

We next give an example of a quasi-uniform scale space.

Example 5.1.1 *Let X be a set. If ρ is a reflexive and transitive relation on X, then $\mathcal{U} = \langle \{\rho\} \rangle$ is a quasi-uniformity on X. The scale $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is the quasi-uniform space on $P_{\mathcal{U}^s}$ generated by the preorder $S(\rho)$.*

Indeed $\alpha \in P_{\mathcal{U}^s}$ if and only if α is a family of relations that contain ρ^s and if $A \in \alpha$ and $A \subseteq B$ then $B \in \alpha$.

$(\alpha, \beta) \in S(\rho) \Longleftrightarrow$ *(if* $A \in \alpha$ *then* $\rho \circ A \in \beta$ *and if* $B \in \beta$ *then* $\rho^{-1} \circ B \in \alpha$*).*

$S(\rho)$ *is reflexive because it contains the diagonal.*

Let $(\alpha, \beta) \in S(\rho)^2$. *This implies there exists* $\gamma \in P_{\mathcal{U}^s}$ *such that* $(\alpha, \gamma) \in S(\rho)$ *and* $(\gamma, \beta) \in S(\rho)$. *If* $A \in \alpha$ *which implies* $\rho^2 \circ A \in \beta$ *by* $\rho^2 \subseteq \rho$ *then* $\rho \circ A \in \beta$. *Reciprocally if* $B \in \beta$ *which implies* $\rho^{-2} \circ B \in \alpha$ *then* $\rho^{-1} \circ B \in \alpha$. *Thus* $S(\rho)$ *is transitive, as expected.*

5.2 The retracted scale of a quasi-uniform space

Starting with the scale quasi-uniform space ($P_{\mathcal{U}^s}$,$S(\mathcal{U})$) of a quasi-uniform space $(X,\mathcal{U})$ in this section, we will define the retracted scale quasi-uniform space.
We recall that $\alpha \sim \beta$ if $\mathcal{U} \circ \alpha = \mathcal{U} \circ \beta$ and $\mathcal{U}^{-1} \circ \alpha = \mathcal{U}^{-1} \circ \beta$ by Lemma 5.1.2.

Definition 5.2.1 *Let* $\alpha \in P_{\mathcal{U}^s}$. *We set:*

$$\alpha_R = \{\beta \in P_{\mathcal{U}^s} : \alpha \sim \beta\}$$

and

$$(P_{\mathcal{U}^s})_R = \{\alpha_R : \alpha \in P_{\mathcal{U}^s}\}$$

We introduce an partial order on $(P_{\mathcal{U}^s})_R$ as follows:

Definition 5.2.2 *For each prefilter* $\alpha \in P_{\mathcal{U}^s}$, *we associate the prefilter* $\alpha^{\mathcal{U}}$ *defined by*

$$\alpha^{\mathcal{U}} = \mathcal{U} \circ \alpha.$$

We introduce an partial order on $(P_{\mathcal{U}^s})_R$ as follows:

Definition 5.2.3 *Let* α_R *and* β_R *in* $(P_{\mathcal{U}^s})_R$, $\alpha_R \leq \beta_R$ *means* $\beta^{\mathcal{U}} \cup \beta^{\mathcal{U}^{-1}} \subseteq \alpha^{\mathcal{U}} \cup \alpha^{\mathcal{U}^{-1}}$.

Lemma 5.2.1 *The relation* $\leq$ *defined above is an partial order on* $(P_{\mathcal{U}^s})_R$.

Proof. We show that $\leq$ is well defined. If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ such that ${\beta_1}^{\mathcal{U}} \cup {\beta_1}^{\mathcal{U}^{-1}} \subseteq {\alpha_1}^{\mathcal{U}} \cup {\alpha_1}^{\mathcal{U}^{-1}}$ then ${\beta_2}^{\mathcal{U}} \cup {\beta_2}^{\mathcal{U}^{-1}} \subseteq {\alpha_2}^{\mathcal{U}} \cup {\alpha_2}^{\mathcal{U}^{-1}}$.

The relation $\leq$ is reflexive, transitive by dual inclusion of prefilters and antisymmetric, since $\mathcal{U} \circ \alpha \cup \mathcal{U}^{-1} \circ \alpha = \mathcal{U} \circ \beta \cup \mathcal{U}^{-1} \circ \beta$ implies $\mathcal{U} \circ \alpha = \mathcal{U} \circ \beta$ and $\mathcal{U}^{-1} \circ \alpha = \mathcal{U}^{-1} \circ \beta$. □

Definition 5.2.4 *The order structure* $((P_{\mathcal{U}^s})_R, \leq)$ *is called the* order retracted scale *of the quasi-uniform space* $(X, \mathcal{U})$.

The next lemma shows that the canonical map is order-preserving.

Lemma 5.2.2 *Let* ψ *be the canonical map from* $P_{\mathcal{U}^s}$ *into* $(P_{\mathcal{U}^s})_R$ *defined by* $\psi(\alpha) = \alpha_R$, *for* $\alpha \in P_{\mathcal{U}^s}$. *Then* ψ *is order-preserving.*

Proof. Let α and β in $P_{\mathcal{U}^s}$ such that $\alpha \leq \beta$. We show that $\beta^{\mathcal{U}} \cup \beta^{\mathcal{U}^{-1}} \subseteq \alpha^{\mathcal{U}} \cup \alpha^{\mathcal{U}^{-1}}$.

Indeed, $\alpha \leq \beta$, which implies $\beta \subseteq \alpha$. Hence $\mathcal{U} \circ \beta \subseteq \mathcal{U} \circ \alpha$ and $\mathcal{U}^{-1} \circ \beta \subseteq \mathcal{U}^{-1} \circ \alpha$. Then we have $\mathcal{U} \circ \beta \cup \mathcal{U}^{-1} \circ \beta \subseteq \mathcal{U} \circ \alpha \cup \mathcal{U}^{-1} \circ \alpha$. □

Proposition 5.2.1 *For any pair of elements* α_R, β_R *in* $(P_{\mathcal{U}^s})_R$, *we have:*

$$\alpha_R \vee \beta_R = (\alpha \vee \beta)_R$$

and

$$\alpha_R \wedge \beta_R = (\alpha \wedge \beta)_R.$$

Proof. We leave it to the reader to check that these equalities make sense. □

The following proposition is similar to Proposition 3.3.3.

Proposition 5.2.2 *The order retracted scale of any quasi-uniform space is a complete distributive lattice. The operations* sup *and* inf *in* $(P_{\mathcal{U}^s}, \leq)$ *are preserved under the canonical map* ψ.

Proof. To see that the sup operation is invariant, let $A \subseteq P_{\mathcal{U}^s}$, $B = \psi(A)$, and $\alpha = \sup(A)$; we shall show that $\psi(\alpha) = \sup(B)$. Note $\psi(\alpha) \geq \gamma_R$, for all γ_R, since ψ is order-preserving. If $\beta_R \geq \gamma_R$ for all $\gamma_R \in B$, $\beta \in \beta_R$ and $\gamma \in A$ then $\beta \geq \gamma$ so $\beta \subseteq \gamma \subseteq \alpha$ thus $\beta_R = \psi(\beta) \geq \psi(\alpha)$. A similar argument establishes that $\psi(\inf(A)) = \inf(B)$. Since ψ is an onto map, $(P_{\mathcal{U}^s}, \leq)$ is a complete lattice. The fact that the retracted scale is distributive follows from Proposition 5.2.1. $\square$

We next define the retracted scale quasi-uniformity $S(\mathcal{U})_R$. We recall that with each $U \in \mathcal{U}$ we have an associated entourage $S(U)$ in the scale quasi-uniformity $S(\mathcal{U})$.

Definition 5.2.5 *Let $(X, \mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$,*

$$S(U)_R = \psi(S(U)) = \{(\alpha_R, \beta_R) : (\alpha, \beta) \in S(U)\}.$$

We next define the retracted scale quasi-uniform space.

Definition 5.2.6 *The quotient space $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ of $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ under the equivalence relation $\sim$ is called the* retracted scale quasi-uniform space *associated with the quasi-uniform space $(X, \mathcal{U})$.*

The next proposition observes that the topology $\tau(S(\mathcal{U})_R)$ of the retracted scale of a quasi uniform space $(X, \mathcal{U})$ is T_0.

Proposition 5.2.3 *The retracted scale quasi-uniform space $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ associated with an arbitrary quasi-uniform space $(X, \mathcal{U})$ is T_0 for the topology associated with $\tau(S(\mathcal{U})_R)$ and is the quotient quasi-uniform space structure derived from $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ under the map ψ.*

Proof. This is a consequence of Proposition 5.2.2. $\square$

The next definition should be compared with Section 3.4 of Chapter 3.

Definition 5.2.7 *Let $(X, \mathcal{U})$ be a quasi-uniform space and ρ a quasi-order (reflexive and transitive binary relation) on X. A set $A \subseteq X$ will be called* stable *relative to ρ and $\mathcal{U}$ if, for every $V \in \mathcal{U}$, there exists a $W \in \mathcal{U}$ such that $\rho(W(A)) \subseteq V(A)$.*

The next proposition shows that the composition of the canonical mapping ψ from the scale of a quasi-uniform space into the retracted scale and the quasi-uniformly continuous map v_a (defined below) from the original quasi-uniform space into its scale is a Liapunov function. A Liapunov function on a quasi-uniform space is defined analogously to the Definition 3.4.1.

Proposition 5.2.4 *Let $(X, \mathcal{U})$ be a quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its scale. If A is a stable subset of X then $\psi \circ v_a$ is a Liapunov function, where a is a fixed element of A, ψ is the canonical map of $P_{\mathcal{U}^s}$ into $(P_{\mathcal{U}^s})_R$, $v_a(x) = \{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\}$, whenever $x \in X$ and ρ is a quasi-order relation on X.*

Proof. We suppose that A is stable and N an invariant neighborhood of A. We take $N = X$ and for each $x \in X$, set $v(x) = \psi(\{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\})$ to simplify the notation.

We shall show that v is a Liapunov function for A.
Let $(x, y) \in \rho$ and $\rho(x) \subseteq V(a)$, by transitivity of ρ, $\rho(y) \subseteq \rho^2(x) = \rho(x)$, so $\rho(y) \subseteq V(a)$. Thus $\{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\} \subseteq \{V \in \mathcal{U}^s : \rho(y) \subseteq V(a)\}$, and the property of order-preserving of ψ gives $v(x) \subseteq v(y)$ so $v(y) \leq v(x)$.

Let $\gamma_R \in (P_{\mathcal{U}^s})_R - \{\mathcal{U}\}$ and let $V \in \mathcal{U} - \bigcup \gamma_R$. Choose $W \in \mathcal{U}$ so that $\rho W(a) \subseteq V(a)$. We suppose that $x \in W(a)$. If we had $v(e) \geq \gamma_R$, then we would have $V \in \{V \in \mathcal{U}^s : \rho(x) \subseteq V(a)\} \subseteq v(x) \subseteq \gamma_R$, in contradiction to the choice of V.

Let $V \in \mathcal{U}$. If $V(a) = X$, there exists a $\gamma_R \in (P_{\mathcal{U}^s})_R$ such that $v(x) \geq \gamma_R$ where $x \in N - V(a)$. So suppose that $V(a) \neq X$ and let $W^2 \subseteq V$. Put $\alpha_R = \bigcup \{v(x) : x \notin V(a)\}$. We shall show that $W \notin \alpha_R$. In the contrary case we would have $W \in v(x)$ for some $x \notin V(a)$. If $W \in v(x)$, then by definition of v, $\rho(x) \subseteq W(a) \subseteq W^2(a) \subseteq V(a)$. Since ρ is reflexive, this implies $x \in V(a)$, a contradiction. Since $W \notin \alpha_R$, we have $\alpha_R \neq 0$ and $\gamma_R = \psi(\alpha) \neq 0$. If $x \notin V(a)$, then $v(x) \subseteq \alpha \subseteq \gamma$, which means that $v(x) \geq \gamma_R$, and γ_R meets the requirements. $\square$

5.3 Properties of the scale of a quasi-uniform space

In this section we shall investigate some properties of the scale of a quasi-uniform space.

Bicompletion of the scale of a quasi-uniform space

We next state a lemma that turns out to be useful in the proof that any quasi-uniform scale space is indeed bicomplete (see Lemma 3.3.4).

Lemma 5.3.1 *Let Υ be an ultrafilter on $P_{\mathcal{U}^s}$. Then* $\cup_{Z\in\Upsilon}\cap_{\zeta\in Z}\zeta = \cap_{Z\in\Upsilon}\cup_{\zeta\in Z}\zeta$.

Proof. The proof is similar to the one of Lemma 3.3.4 (see also Lemma 5.5.2). □

The following theorem generalizes Theorem 3.3.2 from the theory dealing with uniform spaces.

Theorem 5.3.1 *For any quasi-uniform space $(X,\mathcal{U})$, $(P_{\mathcal{U}^s},S(\mathcal{U}))$ and $((P_{\mathcal{U}^s})_R,S(\mathcal{U})_R)$ are both bicomplete quasi-uniform spaces.*

Proof. (1) Let Ξ be a filter on $P_{\mathcal{U}^s}$ that is $S(\mathcal{U})^s$-Cauchy. We show that Ξ $\tau(S(\mathcal{U})^s)$-converges to $\gamma := \cup_{Z\in\Upsilon}\cap_{\zeta\in Z}\zeta = \cap_{Z\in\Upsilon}\cup_{\zeta\in Z}\zeta$ where Ξ is an ultrafilter on $P_{\mathcal{U}^s}$ containing Ξ. Let $U \in \mathcal{U}$. By assumption there is $G \in \Xi$ such that $G \times G \subseteq S(U)$. Let $\xi' \in G$ be arbitrary. Then $U \circ \xi' \subseteq \xi$ whenever $\xi \in G$ by definition of $S(U)$.

Thus $U \circ \xi' \subseteq \cap_{\xi\in G}\xi \subseteq \gamma$. So $U \circ \xi' \subseteq \gamma$. Let $W \in \gamma^{-1}$ be arbitrary. Then $W \in \cup_{\xi\in G}\xi^{-1}$, since $G \in \Upsilon$.

Thus there is $\xi \in G$ such that $W \in \xi^{-1}$. Furthermore $(\xi',\xi) \in S(U)$ and thus $W \circ U \in \xi^{-1} \circ U \subseteq (\xi')^{-1}$. We conclude that $\gamma^{-1} \circ U \subseteq (\xi')^{-1}$. Therefore $G \subseteq (S(U))^{-1}(\gamma)$ and Ξ $\tau(S(\mathcal{U}^{-1}))$-converges to γ.

Let $\xi' \in G$ be arbitrary. Then $U \circ \xi \subseteq \xi'$ whenever $\xi \in G$ by definition of $S(U)$. Thus $U \in \gamma \subseteq U\circ \cup_{\xi\in G}\xi \subseteq \xi'$ and so $U \circ \gamma \subseteq \xi'$. Let $W \in (\xi')^{-1}$ be arbitrary. We have that $(\xi, \xi') \in S(U)$ and thus $W \circ U \in (\xi')^{-1} \circ U \subseteq \xi^{-1}$.

Hence $W \circ U \in \cap_{\xi \in G} \xi^{-1} \subseteq \gamma^{-1}$. So $W \circ U \in (\xi')^{-1} \circ U \subseteq \gamma^{-1}$. Consequently $G \subseteq (S(U))(\gamma)$ and Ξ $\tau(S(\mathcal{U}))$-converges to γ. Thus $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is bicomplete.

Next, let Γ be a Cauchy filter on $(P_{\mathcal{U}^s})_R$. It follows from Proposition 5.2.2 that $\psi^{-1}(\Gamma)$ is a Cauchy filter on $P_{\mathcal{U}^s}$. If $\psi^{-1}(\Gamma)$ converge to α in the topology compatible with $S(\mathcal{U})$, then $\Gamma = \psi(\psi^{-1}(\Gamma))$ τ-converges to α_R by Proposition 5.2.2. □

The next proposition describes an induced map between two scales of quasi-uniform spaces when the original quasi-uniform spaces are connected by a quasi-uniformly continuous map.

Proposition 5.3.1 *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces and $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ a quasi-uniformly continuous map.*
If $\alpha \in P_{\mathcal{U}^s}$, then $(f \times f)(\alpha) \cap \mathcal{V}^s \in P_{\mathcal{V}^s}$. Furthermore the map $f_S : (P_{\mathcal{U}^s}, S(\mathcal{U})) \longrightarrow (P_{\mathcal{V}^s}, S(\mathcal{V}))$ defined by $f_S(\alpha) = (f \times f)(\alpha) \cap \mathcal{V}^s$ is quasi-uniformly continuous.

Proof. Let $V \in \mathcal{V}$. By quasi-uniform continuity of f there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Since $\alpha \in P_{\mathcal{U}^s}$, $\alpha \subseteq \mathcal{U}^s$ and then $(f \times f)(\alpha) \cap \mathcal{V}^s \in P_{\mathcal{V}^s}$.

It remains to be shown that f_S is quasi-uniformly continuous. Let $V \in \mathcal{V}$, as above, there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Consider $(\alpha, \beta) \in S(U)$, then $U \circ \alpha \subseteq \beta$ and $\beta^{-1} \circ U \subseteq \alpha^{-1}$. Consequently $V \circ ((f \times f)(\alpha) \cap \mathcal{V}^s) \subseteq (f \times f)(\beta) \cap \mathcal{V}^s$ and $(((f \times f)(\beta))^{-1} \cap \mathcal{V}^s) \circ V \subseteq ((f \times f)(\alpha))^{-1} \cap \mathcal{V}^s$. Hence the map f_S is quasi-uniformly continuous. □

The next theorem introduces a functor from the category of quasi-uniform spaces into itself.

Theorem 5.3.2 *Let QU be the category of quasi-uniform spaces where the quasi-uniformly continuous maps are the morphisms. If $(X, \mathcal{U})$ is a quasi-uniform space, we set $S((X, \mathcal{U})) := P_{\mathcal{U}^s}$ and $S(f) := f_S$ where f is a quasi-uniformly continuous map from $(X, \mathcal{U})$ into $(Y, \mathcal{V})$. Then S is a functor from QU into itself.*

Proof. Let $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \longrightarrow (Z, \mathcal{W})$ quasi-uniformly continuous maps. We must show that $S(g \circ f) = S(g) \circ S(f)$ and $S(id_{(X,\mathcal{U})}) = id_{S((X,\mathcal{U}))}$.

Indeed, let $\alpha \in P_{\mathcal{U}^s}$ then $(g_S \circ f_S)(\alpha) = g_S(f_S(\alpha)) = ((g \times g) \circ (f \times f))(\alpha) \cap \mathcal{W}^s = (g \circ f \times g \circ f)(\alpha) \cap \mathcal{W}^s = (g \circ f)_S(\alpha)$.

Let $\alpha \in P_{\mathcal{U}^s}$. Then $id_S(\alpha) = (id \times id)(\alpha) \cap \mathcal{U}^s = id_{P_{\mathcal{U}^s}}(\alpha)$. □

Lemma 5.3.2 *Let X and Y be sets and let $f : X \longrightarrow Y$ be a surjective map. Let A, B and U be subsets of $Y \times Y$. If $B \subseteq U \circ A$ then $(f \times f)^{-1}B \subseteq ((f \times f)^{-1}U) \circ ((f \times f)^{-1}A)$.*

Proof. Let $(x, y) \in (f \times f)^{-1}B$. We have $(f(x), f(y)) \in B$ which implies there exists $z \in Y$ such that $(f(x), z) \in A$ and $(z, f(y)) \in U$. By surjectivity of f there exists $c \in X$ such that $f(c) = z$. Hence $(x, c) \in (f \times f)^{-1}A$ and $(c, y) \in (f \times f)^{-1}U$ which implies $(x, y) \in ((f \times f)^{-1}U) \circ ((f \times f)^{-1}A)$. □

Proposition 5.3.2 *Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be quasi-uniform spaces and $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ be a quasi-uniformly continuous surjective map.*
If $\alpha \in P_{\mathcal{V}^s}$, we set $(f \times f)^{-1}(\alpha) :=$ prefilter coarser than $\mathcal{U}^s$ generated by $\{(f \times f)^{-1}(A) : A \in \alpha\}$. Then $(f \times f)^{-1}(\alpha) \in P_{\mathcal{U}^s}$. Furthermore the map $f_T : (P_{\mathcal{V}^s}, S(\mathcal{V})) \longrightarrow (P_{\mathcal{U}^s}, S(\mathcal{U}))$ defined by $f_T(\alpha) = (f \times f)^{-1}(\alpha)$ is quasi-uniformly continuous.

Proof. Let $V \in \mathcal{V}$. By quasi-uniform continuity of f there is $U \in \mathcal{U}$ such that $(f \times f)U \subseteq V$. Since $\alpha \in P_{\mathcal{V}^s}$ implies $\alpha \subseteq \mathcal{V}^s$, then $(f \times f)^{-1}(\alpha) \subseteq \mathcal{U}^s$.

It remains to be shown that f_T is quasi-uniformly continuous. Let $V \in \mathcal{V}$, as above. There exists $U \in \mathcal{U}$ such that $U \subseteq (f \times f)^{-1}V$. Consider $(\alpha, \beta) \in S(U)$. Then $U \circ \alpha \subseteq \beta$ and $\beta^{-1} \circ U \subseteq \alpha^{-1}$. Consequently $V \circ (f \times f)^{-1}(\alpha) \subseteq (f \times f)^{-1}(\beta)$ and $(f \times f)^{-1}(\beta^{-1}) \circ V \subseteq (f \times f)^{-1}(\alpha^{-1})$.□

Connectedness of the scale of a quasi-uniform space

In the next paragraphs we study connectedness of the quasi-uniform scale of a quasi-uniform space.

Let $(X,\mathcal{U})$ be a quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale. It will be convenient to introduce the following "closed interval" notation:

For $\alpha \leq \beta$, let $[\alpha,\beta] = \{\gamma \in P_{\mathcal{U}^s} : \alpha \leq \gamma \leq \beta\}$. For each $U \in \mathcal{U}$ and $\alpha \in P_{\mathcal{U}^s}$, let

$$\alpha^U = U \circ \alpha,$$

$$\alpha_{U^{-1}} = \{V \in \mathcal{U}^s : U^{-1} \circ V \in \alpha\}.$$

Lemma 5.3.3 *Let $U \in \mathcal{U}$. Then $S(U)(\alpha) = [\alpha_{U^{-1}}, \alpha^U]$ for each $\alpha \in P_{\mathcal{U}^s}$.*

Proof. Suppose that $\beta \in S(U)(\alpha)$. We show that $\alpha^U \subseteq \beta \subseteq \alpha_{U^{-1}}$.

Let $V \in \alpha^U$. Then for some $F \in \alpha$, $U \circ F \subseteq V \in \beta$. Let $G \in \beta$. Then $U^{-1} \circ G \in \alpha$ implies $G \in \alpha_{U^{-1}}$.

Conversely, if $V \in \alpha^U$ then for some $F \in \alpha$, $U \circ F \subseteq V \in \beta$. If $G \in \beta$ then $G \in \alpha_{U^{-1}}$ implies $U^{-1} \circ G \in \beta$. Thus $(\alpha, \beta) \in S(U)$. $\square$

The following proposition is similar to Proposition 4.1.1.

Proposition 5.3.3 *The collection $\{[\alpha_{U^{-1}}, \alpha^U] : U \in \mathcal{U}\}$ forms a basic system of neighborhoods at α for the quasi-uniform topology on $P_{\mathcal{U}^s}$ generated by $S(\mathcal{U})$.* $\square$

Consider the case $\alpha = 0$. Here $\mathcal{U}^s$ is the last element of $P_{\mathcal{U}^s}$. Note that $0_U = 0$. Recall that $\langle U \rangle$ denotes the prefilter consisting of all oversets of U. Since $0^V \leq \langle U \rangle \leq 0^U$ for $V^2 \subseteq U$, it follows from Proposition 5.3.3 that the collection $\{[0, \langle U \rangle] : U \in \mathcal{U}\}$ is a basic neighborhood system for 0.

One can show that the lattice operations $\vee$ and $\wedge$ in $P_{\mathcal{U}^s}$ are quasi-uniformly continuous (compare Theorem 3.3.1) and that for any $\alpha \in P_{\mathcal{U}^s}$ $\beta \mapsto \beta \circ \alpha$ is quasi-uniformly continuous.

Proposition 5.3.4 *If M is a connected (arcwise connected) subset of $P_{\mathcal{U}^s}$, then the sets $M \bullet \alpha$, $M \vee \alpha$ and $M \wedge \alpha$ are likewise connected (arcwise connected) for any choice of α.*

Proof. See [19, Proposition 3]. □

Proposition 5.3.5 *If there is a connected (arcwise connected) subset C of $P_{\mathcal{U}^s}$ containing α and β, then $[\alpha\wedge\beta, \alpha\vee\beta]$ is connected (arcwise connected).*

Proof. The sets $C \vee \alpha$ and $C \wedge \beta$ are connected sets which contain α, and $A = (C \vee \alpha) \cup (C \wedge \alpha)$ is a connected set which contains both $\alpha \vee \beta$ and $\alpha \wedge \beta$. If $\gamma \in [\alpha \wedge \beta, \alpha \vee \beta]$, $A \vee \gamma$ is a connected set which contains γ and $\alpha \vee \beta$. Thus

$$B = \cup\{A \vee \gamma : \gamma \in [\alpha \wedge \beta, \alpha \vee \beta]\}$$

is a connected set which includes $[\alpha \wedge \beta, \alpha \vee \beta]$ as a subset. But

$$[\alpha \wedge \beta, \alpha \vee \beta] = (B \vee (\alpha \wedge \beta)) \wedge (\alpha \vee \beta),$$

and the proof is complete. □

Corollary 5.3.1 *The scale of a quasi-uniform space is connected (arcwise connected) if 0 and 1 belong to the same connected (arcwise connected) component.*

Proof. This follows from Proposition 5.3.5. □

Corollary 5.3.2 *If the closed interval $[\alpha, \beta]$ in $P_{\mathcal{U}^s}$ is a subset of a connected (arcwise connected) set M, then $[\alpha, \beta]$ is connected (arcwise connected).*

Proof. This is a consequence of Corollary 5.3.1. □

Corollary 5.3.3 *The quasi-uniform scale of a quasi-uniform space $(X, \mathcal{U})$ is locally connected (locally arcwise connected) if and only if there is a connected (arcwise connected) neighborhood for each point of $(P_{\mathcal{U}^s}, \tau(S(\mathcal{U})))$.*

Proof. This follows from Corollary 5.3.2 and Proposition 5.3.3. □

Total boundedness of the quasi-uniform scale of a quasi-uniform space

We will need the following proposition to show under which condition the scale of a quasi-uniform space is totally bounded.

Proposition 5.3.6 *Let $(X,\mathcal{U})$ be a quasi-uniform space. If X is infinite and $\mathcal{U}$ distinct from $\{X\times X\}$, then there is $U\in\mathcal{U}$ such that $(X\times X)\backslash U^2$ is infinite.*

Proof. Indeed, otherwise for any $V\in\mathcal{U}$ and any $x,y\in X$, we have that both $X\backslash V^2(x)$ and $X\backslash V^2(y)$ are finite and thus there is $z\in V^2(x)\cap V^{-2}(y)$, since X is infinite. Hence $(x,y)\in V^4$. We conclude that $\mathcal{U}=\{X\times X\}$, a contradiction. □

Remark 5.3.1 *Let $\mathcal{U}$ be distinct from $\{X\times X\}$. By the assertion just proved there exists $U\in\mathcal{U}$ such that for each $n\in\omega$ we can choose $(a_n,b_n)\in(X\times X)\backslash U^2$ such that Case 1 $(a_n)_{n\in\omega}$ is injective or Case 2 $(b_n)_{n\in\omega}$ is injective.*

In Case 1 for each $n\in\omega$ set $\alpha_n=\langle U^s\cup\{(a_n,b_n)\}\rangle$, in case 2 for each $n\in\omega$ set $\beta_n=\langle U^s\cup\{(b_n,a_n)\}\rangle$. Observe that for each $n\in\omega$, $\alpha_n,\beta_n\in P_{\mathcal{U}^s}$.

In Case 1 we have for all $n,m\in\omega$ with $n\neq m$ that $U\circ\alpha_n$ is not a subset of α_m, since $(a_m,b_m)\notin U\circ(U^s\cup\{(a_n,b_n)\}$, because $a_m\neq a_n$ and $(a_m,b_m)\notin U^2$. In Case 2 for all $n,m\in\omega$ with $n\neq m$ we have that $U^{-1}\circ\beta_n$ is not a subset of β_m, since $(b_m,a_m)\notin U^{-1}\circ(U^s\cup\{(b_n,a_n)\})$, because $b_m\neq b_n$ and $(a_m,b_m)\notin U^2$. This remark will be used in the proof of Theorem 5.3.4.

The following theorem generalizes Proposition 4.3.3 from the theory of uniform spaces.

Theorem 5.3.3 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(P_{\mathcal{U}^s},S(\mathcal{U}))$ is totally bounded if and only if X is finite or $\mathcal{U}=\{X\times X\}$.*

Proof. Let X be finite. Then $P_{\mathcal{U}^s}$ is finite and thus $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is totally bounded. If $\mathcal{U} = \{X \times X\}$, then $S(\mathcal{U}) = \{P_{\mathcal{U}^s} \times P_{\mathcal{U}^s}\}$, hence $S(\mathcal{U})$ is totally bounded, too.

Conversely, suppose that X is infinite and $\mathcal{U}$ distinct from $\{X \times X\}$. By Proposition 5.3.6 there is $U \in \mathcal{U}$ such that $(X \times X)\backslash U$ is an infinite set of (a_n, b_n)'s where $a_n \neq a_m$ whenever $n \neq m$ or $b_n \neq b_m$ whenever $n \neq m$. For each $n \in \omega$ set (Case 1) $\alpha_n = \langle U^s \cup \{(a_n, b_n)\}\rangle$ and (Case 2) $\beta_n = \langle U^s \cup \{(b_n, a_n)\}\rangle$. (See Remark 5.3.1).

Hence in Case 1 for all $m, n \in \omega$ with $n \neq m$ we have $(\alpha_n, \alpha_m) \notin S(U)$. Similarly in Case 2 for all $m, n \in \omega$ with $n \neq m$ we have $(\beta_n, \beta_m) \notin S(U)$. Thus in either case we have found an infinite discrete subspace of $(P_{\mathcal{U}^s}, S(\mathcal{U}))$. Hence $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ is not totally bounded. □

Our next example shows that given a quasi-uniform space $(X, \mathcal{U})$ the quasi-uniformity $S(\mathcal{U})_+$ on $P_{\mathcal{U}^s}$ can be totally bounded under weaker conditions than those given in the theorem above.

Example 5.3.1 *Let $X = \omega$ and let F be a finite subset of ω. Set $V_F = [(X\backslash F) \times X] \cup \Delta_\omega$. Consider the quasi-uniformity $\mathcal{U}$ on X generated by the base $\{V_F : F \subseteq \omega$ and F is finite$\}$. We are going to show that $(P_{\mathcal{U}^s}, S(\mathcal{U})_+)$ is totally bounded.*

Proof. Fix $F \subseteq X$ finite. Let B be any reflexive binary relation on X. Set $B_F = \{x \in F : B(x) \subseteq F\}$. Then $V_F \circ B = [(X\backslash B_F) \times X] \cup (\bigcup_{x \in B_F} \{x\} \times B(x))$.

Indeed: Case 1, let $x \in X\backslash F$. Then $V_F = X$, so $(x, x) \in B$ yields the result $X = (V_F \circ B)(x)$.

Case 2, let $x \in F\backslash B_F$. Then there exists $b \in B(x)\backslash F$. Therefore $(x, b) \in B$ and $V_F(b) = X$. The assertion $X = (V_F \circ B)(x)$ follows.

Case 3, let $x \in B_F$. Then $B_F(x) \subseteq F$. Thus $(x, c) \in B$ and $(c, d) \in V_F$ implies that $c = d$. Therefore $(V_F \circ B)(x) = B(x)$.

Altogether for each $x \in X$ such that $B(x) \subseteq F$ we get that $(V_F \circ B)(x) = B(x)$ and $(V_F \circ B)(x) = X$ otherwise. We conclude that $\mathcal{M}_F = \{V_F \circ B : B$

is a reflexive relation on $X\}$ is finite.

For any nonempty collection $\mathcal{M}'$ of $\mathcal{M}_F$ we consider the prefilter $\gamma_{\mathcal{M}'} = \bigcup\{\langle M'\rangle : M' \in \mathcal{M}'\}$.

Let $\beta \in P_{\mathcal{U}^s}$ be arbitrary. Then $\mathcal{N} = \{V_F \circ B : B \in \beta\}$ is a nonempty finite subcollection of $\mathcal{M}_F$. Hence $\gamma_{\mathcal{N}} = \bigcup\{\langle V_F \circ B\rangle : B \in \beta\}$. Obviously $V_F \circ B \subseteq \gamma_{\mathcal{N}}$, since $\mathcal{N} \subseteq \gamma_{\mathcal{N}}$.

Furthermore $V_F \circ \gamma_{\mathcal{N}} \subseteq \beta$, since $C \in \gamma_{\mathcal{N}}$ implies that there is $B \in \beta$ such that $V_F \circ B \subseteq C$, and so $B \subseteq V_F \circ (V_F \circ B) \subseteq V_F \circ C$, because V_F is reflexive. We conclude that $S(\mathcal{U})_+$ is totally bounded. □

5.4 Quasi-pseudometrizability of the scale

We use the notion of an écart in a quasi-uniform space that was introduced in Chapter 1 for a uniform space. The definition of an écart is the same as in Definition 3.5.1, except that the condition (2) introducing the symmetry of the écart will be omitted.

Let $(X,\mathcal{U})$ be an arbitrary quasi-uniform space, and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale.

Definition 5.4.1 *Let 0 be the last element of $P_{\mathcal{U}^s}$, and let $h_0 = \{S(U)(0) : U \in \mathcal{U}\}$. Define $\sigma : X \times X \longrightarrow P_{\mathcal{U}^s}$ by*

$$\sigma(x,y) = \{U \in \mathcal{U}^s : (x,y) \in U\}$$

for every x and y in X.

Theorem 5.4.1 *Let σ be defined as above. Then σ is an écart, and the quasi-uniformity defined on X by σ is $\mathcal{U}$. Moreover, if $\mathcal{B}$ is a $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed base of $\mathcal{U}$ then for any $V \in \mathcal{B}$, $(a,b) \in V$ if and only if $\sigma(a,b) \in S(V)(0)$.*

Proof. We start by proving the second part. Let $\mathcal{B}$ be a $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed base of $\mathcal{U}$. Then for any $V \in \mathcal{B}$, we have the following: Let $\sigma(a,b) \in$

$S(V)(0)$ if and only if $(0, \sigma(a,b)) \in S(V)$ if and only if $V \circ U \in \sigma(a,b)$ if $U \in \mathcal{U}^s$ and $A^{-1} \circ V \in \mathcal{U}^s$ if $A \in \sigma(a,b)$ if and only if $(a,b) \in V \circ U$ for every $U \in \mathcal{U}^s$ if and only if for every $U \in \mathcal{U}^s$, $(a,c) \in U$, $(c,b) \in V$ for some $c \in X$ if and only if for every $U \in \mathcal{U}^s$, $V(a) \cap U^{-1}(b) \neq \emptyset$ if and only if $b \in cl_{\tau(\mathcal{U}^s)} V(a) = V(a)$ if and only $(a,b) \in V$. Thus the quasi-uniformity induced on X by σ is $\mathcal{U}$.

It will now be shown that conditions (1), (3) and (4) of the definition of an écart are satisfied.

1. That $\sigma(a,a) = 0$ for every $a \in X$ is clear.

2. Let $S(U)(0)$ and $S(V)(0)$ be in h_0. There exists a $W \in \mathcal{U}$ such that $W \subseteq U \cap V$. Then $S(W)(0) \subseteq S(U)(0) \cap S(V)(0)$, and $S(W)(0) \in h_0$.

3. For every $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then $\sigma(a,b) \in S(V)(0)$ and $\sigma(b,c) \in S(V)(0)$ which imply (a,b) and (b,c) are in V; hence (a,c) is in V^2. It follows that $\sigma(a,c) \in S(V^2)(0) \subseteq S(U)(0)$. This completes the proof. □

We recall the definition of the map $\psi : P_{\mathcal{U}^s} \longrightarrow (P_{\mathcal{U}^s})_R$ (see Lemma 5.2.2). It is clear that if σ is an écart with values in $P_{\mathcal{U}^s}$, then $\psi\sigma$ is an écart with values in $(P_{\mathcal{U}^s})_R$. The following proposition implies that $\psi\sigma$ also induces the quasi-uniformity $\mathcal{U}$ in X.

Proposition 5.4.1 *Let $V \in \mathcal{U}$ be $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed. Then $\alpha \in S(V)(0)$ if and only if $\psi(\alpha) \in S(V)(0)$.*

Proof. Let $\alpha \in S(V)(0)$, which implies $(0, \alpha) \in S(V)$. Let $\beta \in \psi(\alpha)$. Since $\alpha \sim \beta$ we have $V \circ U \in \psi(\alpha)$ for every $U \in \mathcal{U}^s$. Then $V \circ \mathcal{U}^s \subseteq \psi(\alpha)$. Furthermore $V^{-1} \circ \psi(\alpha) \subseteq \mathcal{U}^s$. Thus $(0, \psi(\alpha)) \in S(V)$. Similarly, if $\psi(\alpha) \in S(V)(0)$ then $(0, \alpha) \in S(V)$. □

The écart σ will now be used to establish some topological properties of the scale. The next theorem is similar to Theorem 3.5.2.

Theorem 5.4.2 *Let $(X, \mathcal{U})$ be a quasi-uniform space, $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its quasi-uniform scale and $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ its retracted scale. Then the follow-*

ing statements are equivalent:

(1). *$(X,\mathcal{U})$ is quasi-pseudometrizable.*

(2). *$(P_{\mathcal{U}^s},S(\mathcal{U}))$ is quasi-pseudometrizable.*

(3). *$(P_{\mathcal{U}^s},S(\mathcal{U}))$ satisfies the first axiom for countability.*

(4). *$(P_{\mathcal{U}^s},S(\mathcal{U}))$ has a countable base for the neighborhood filter at 0.*

(5). *There exists a map ϕ from $P_{\mathcal{U}^s}$ into the nonnegative real numbers such that $\phi(0) = 0$ and $\{\phi^{-1}([0,\varepsilon)) : \varepsilon > 0\}$ is a base for the neighborhood filter at 0.*

Moreover $(P_{\mathcal{U}^s},S(\mathcal{U}))$ can be replaced by $((P_{\mathcal{U}^s})_R,S(\mathcal{U})_R)$ in any of the statements (2) *through* (5).

Proof. (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4) are trivial.

If (4) is satisfied, then a countable, nested neighborhood base at $0 = M_1 \supseteq M_2 \supseteq \ldots$, can be constructed.
Define $\phi : P_{\mathcal{U}^s} \to \mathbb{R}^+$ by $\phi(\alpha) = 1/n$ if $\alpha \in M_n - M_{n+1}$ and 0 otherwise.
For each $\alpha \in P_{\mathcal{U}^s}$, ϕ is well defined since the M_i's are nested.
Furthermore $0 \in \bigcap_{i=1}^{\infty} M_i$, so $\phi(0) = 0$ and $\phi^{-1}\ ([0,\epsilon)) = M_n$, where n is the last integer greater than $1/\epsilon$. Thus ϕ satisfies (5).

We show that (5) $\Longrightarrow$ (1). Let $f = \phi\sigma$, where σ is the écart of Theorem 5.4.1 and ϕ is the given map. Consider the map $f : X \times X \to \mathbb{R}^+$. Let $B_r = \{\alpha : \phi(\alpha) < r\}$ for each $r > 0$. Finally let $\mathcal{B}$ be a $\tau(\mathcal{U}^s) \times \tau(\mathcal{U}^s)$-closed base of $\mathcal{U}$.

i. Given $U \in \mathcal{B}$, there exists an $r > 0$ such that $B_r \subseteq S(U)(0)$ that is, $f(x,y) < r$ implies $\sigma(x,y) \in B_r \subseteq S(U)(0)$, and hence $(x,y) \in U$. Therefore there exists an $r > 0$ such that $\{(x,y) \in X \times X : f(x,y) < r\} \subseteq U$.

ii. Let $\epsilon > 0$. There exists a $U \in \mathcal{B}$ such that $S(U)(0) \subseteq B_\epsilon$, and so $(x,y) \in U$ implies that $f(x,y) < \epsilon$. Therefore there exists a $U \in \mathcal{B}$ such that $U \subseteq \{(x,y) \in X \times X : f(x,y) < \epsilon\}$.

It follows that $\{f^{-1}([0,\epsilon)) : \epsilon > 0\}$ is a base for $\mathcal{U}$. Then $\{f^{-1}([0,1/n)) : n$ is a positive integer$\}$ is a countable base for $\mathcal{U}$, and $(X,\mathcal{U})$ is quasi-pseudometrizable.

We suppose that $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ is substituted for $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ in statements (2) through (5). Again it is clear that (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (5) are true. To prove that (5) implies (1), one must use the écart $\psi\sigma$ instead of σ. Then with the aid of Proposition 5.4.1, everything works as before. $\square$

Definition 5.4.2 *Let $(X, d, \mathcal{U}_d)$ be a quasi-pseudometric space. For $\epsilon > 0$, let $U_\epsilon = \{(x, y) \in X \times X : d(x, y) \leq \epsilon\}$.*

Lemma 5.4.1 *For any ϵ, $\delta \geq 0$, $S(U_\epsilon) \circ S(U_\delta) \subseteq S(U_{\epsilon+\delta})$.*

Proof. Let $(\alpha, \beta) \in S(U_\epsilon) \circ S(U_\delta)$. Then $(\alpha, \gamma) \in S(U_\delta)$ and $(\gamma, \beta) \in S(U_\epsilon)$ for some $\gamma \in P_{\mathcal{U}^s}$. Therefore $U_\delta \circ F \in \gamma$ for every $F \in \alpha$, $U_\varepsilon \circ H \in \beta$, and ${U_\delta}^{-1} \circ H \in \alpha$ for every $H \in \gamma$, ${U_\epsilon}^{-1} \circ G \in \gamma$ for every $G \in \beta$. Then $U_\epsilon \circ U_\delta \circ F \in \beta$ for every $F \in \alpha$ and ${U_\delta}^{-1} \circ {U_\epsilon}^{-1} \circ G \in \alpha$ for every $G \in \beta$. So $U_{\epsilon+\delta} \circ F \in \beta$ for every $F \in \alpha$ and ${U_{\varepsilon+\delta}}^{-1} \circ G \in \alpha$. Thus $(\alpha, \beta) \in S(U_{\epsilon+\delta})$. $\square$

Definition 5.4.3 *We suppose that d is a bounded quasi-pseudometric. Define $d_s : P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} \rightarrow \mathbb{R}^+$ by*

$$d_s(\alpha, \beta) = \inf\{\epsilon > 0 : (\alpha, \beta) \in S(U_\epsilon)\}.$$

The fact that d is bounded ensures that d_s is well defined.

The following proposition is similar to Proposition 3.5.3.

Proposition 5.4.2 *Let $(X, d, \mathcal{U}_d)$ be a bounded quasi-pseudometric space. Then d_s is a quasi-pseudometric on $P_{\mathcal{U}^s}$, and the scale quasi-uniformity $S(\mathcal{U})$ is the quasi-pseudometric quasi-uniformity of d_s.*

Proof. We shall show that d_s is a quasi-pseudometric.

i. $d_s(\alpha, \alpha) = 0$ for every $\alpha \in P_{\mathcal{U}^s}$, since $(\alpha, \alpha) \in S(U_\epsilon)$ for every $\epsilon > 0$.

ii. Let $\epsilon > 0$, $d_s(\alpha,\beta) = a$, and $d_s(\beta,\gamma) = b$. Then $(\alpha,\beta) \in S(U_{a+\epsilon})$ and $(\beta,\gamma) \in S(U_{b+\epsilon})$ imply that $(\alpha,\gamma) \in S(U_{b+\epsilon})\circ S(U_{a+\epsilon}) \subseteq S(U_{a+b+2\epsilon})$. Therefore for any $\epsilon > 0$, $d_s(\alpha,\gamma) \leq d_s(\alpha,\beta)+ d_s(\beta,\gamma)+ 2\epsilon$; hence $d_s(\alpha,\gamma) \leq d_s(\alpha,\beta)+ d_s(\beta,\gamma)$.

Thus d_s is a quasi-pseudometric. It is clear that $S(U_\epsilon) = \{(\alpha,\beta) : d_s(\alpha,\beta) \leq \epsilon\}$, so $S(\mathcal{U})$ is the quasi-pseudometric quasi-uniformity induced by d_s. □

Remark 5.4.1 *$d_s(\alpha,\beta) = 0$ if and only if $(\alpha,\beta) \in S(U_\epsilon)$ for every $\epsilon > 0$.*

Definition 5.4.4 *As before let $X = \mathbb{R}^+$ be the set of nonnegative real numbers. We define a quasi-pseudo-metric F on X by*
$F(x,y) = y - x$ if $y \geq x$ and $F(x,y) = 0$ if $y < x$.
For each $r > 0$, set $O_r = \{(x,y) \in X \times X : F(x,y) < r\}$. The filter on $X \times X$ generated by the base $\{O_r : r > 0\}$ is a quasi-uniformity and is called the standard quasi-pseudometric quasi-uniformity $\mathcal{U}_r$ induced by F on X.

The following proposition is similar to Theorem 3.5.3.

Proposition 5.4.3 *If $(X,\mathcal{U})$ is quasi-pseudometrizable space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its scale, then there exists a quasi-uniformly continuous, order-preserving map ϕ from $P_{\mathcal{U}^s}$ into the nonnegative real numbers.*

Proof. Let d be a bounded quasi-pseudometric on X such that $\mathcal{U}$ is the quasi-pseudometric quasi-uniformity for d. Let d_s be the quasi-pseudometric on $P_{\mathcal{U}^s}$ defined by

$$d_s(\alpha,\beta) = \inf\{\epsilon > 0 : (\alpha,\beta) \in S(U_\epsilon)\}$$

for all $\alpha,\beta \in P_{\mathcal{U}^s}$. Then d_s is a quasi-pseudometric on $P_{\mathcal{U}^s}$, and $S(\mathcal{U})$ is the quasi-pseudometric quasi-uniformity by Proposition 5.4.2. Define
$\phi : P_{\mathcal{U}^s} \to \mathbb{R}^+$ by $\phi(\alpha) = d_s(0,\alpha)$ for every $\alpha \in P_{\mathcal{U}^s}$.

For the proof that ϕ is quasi-uniformly continuous and order preserving we use the same argument as in the proof of Proposition 3.5.3(ii). □

Proposition 5.4.4 *Let d be a bounded quasi-pseudometric on X such that $\mathcal{U}_d \subseteq \mathcal{U}$, where $\mathcal{U}_d$ is the quasi-pseudometric quasi-uniformity defined by d. Let $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ be the scale of $(X, \mathcal{U})$, and define $d_s : P_{\mathcal{U}^s} \times P_{\mathcal{U}^s} \longrightarrow \mathbb{R}^+$ by*

$$d_s(\alpha, \beta) = \inf\{\epsilon > 0 : (\alpha, \beta) \in S(U_\epsilon)\},$$

where, U_ϵ is defined with respect to d. Then d_s is a quasi-pseudometric on $P_{\mathcal{U}^s}$, $S(U_\epsilon) = \{(\alpha, \beta) : d_s(\alpha, \beta) \leq \epsilon\}$, and $\mathcal{U}_{d_s} \subseteq S(\mathcal{U})$, where $\mathcal{U}_{d_s}$ is the quasi-uniformity generated on $P_{\mathcal{U}^s}$ by d_s.

Proof. It is shown in Proposition 5.4.2 that d_s is a quasi-pseudometric. $\{S(U_\epsilon) : \epsilon > 0\} \subseteq S(\mathcal{U})$ since $\mathcal{U}_d \subseteq \mathcal{U}$, and it is clear that these sets $S(U_\epsilon)$ form a base for a quasi-uniformity on $P_{\mathcal{U}^s}$. □

Theorem 5.4.3 *If $(X, \mathcal{U})$ is any quasi-uniform space and $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ its scale, then there exists a quasi-uniformly continuous, order-preserving map ϕ from $(P_{\mathcal{U}^s}, S(\mathcal{U}))$ into the nonnegative real numbers equipped with $\mathcal{U}_r$.*

Proof. This is a consequence of Proposition 5.4.3 and Proposition 5.4.4. □

Proposition 5.4.5 *If $(X, d, \mathcal{U})$ is a bounded quasi-pseudometric space, $(P_{\mathcal{U}^s}, d_s, S(\mathcal{U}))$ its quasi-uniform scale, σ the écart of Theorem 5.4.1, and ϕ the map in Proposition 5.4.3, then $d = \phi\sigma$.*

Proof. For any $x, y \in X$,
$d(x, y) = \inf\ \{\epsilon > 0 : (x, y) \in U_\epsilon\}$ and therefore
$d(x, y) = \inf\ \{\epsilon > 0 : \sigma(x, y) \in S(U_\epsilon)(0)\}$. Thus
$d(x, y) = d_s(0, \sigma(x, y))$ and hence
$d(x, y) = \phi(\sigma(x, y))$. □

5.5 The prefilter space of a quasi-uniform space

In this section, we will first introduce the prefilter space of a quasi-uniform space. Secondly we will establish a connection between the prefilter quasi-uniform space and the Hausdorff hyperspace quasi-uniform space. Furthermore we will show that total boundedness is preserved by the prefilter space of a quasi-uniform space. The prefilter space is a kind of generalization of the quasi-uniform scale with similar properties.

The following defines the set of all prefilters of a quasi-uniform space.

Definition 5.5.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. We define $\mathcal{PF}(X)$ to be the set of all prefilters on X.*

Definition 5.5.2 *(Compare Definition 5.1.2) We equip the set $\mathcal{PF}(X)$ with the following partial order: For $\alpha, \beta \in \mathcal{PF}(X)$ we set $\alpha \leq \beta$ if $\alpha \supseteq \beta$.*

The next proposition is similar to Proposition 5.1.1.

Proposition 5.5.1 *For any set X, the ordered prefilter space $(\mathcal{PF}(X), \leq)$ is a complete lattice with $\inf A = \bigcup_{\alpha \in A} \alpha$ and $\sup A = \bigcap_{\alpha \in A} \alpha$ whenever $A \subseteq \mathcal{PF}(X)$. For a nonempty set X, observe that $\{X\}$ is the largest element and $\mathcal{P}_0(X)$ is the smallest element of $\mathcal{PF}(X)$ (if $X = \emptyset$, then $\mathcal{PF}(X) = \emptyset$).*

Proof. This is obvious from the definition of the partial order $\leq$ in the prefilter space. □

Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$ and $A \subseteq X$, set $U(A) = \bigcup_{a \in A} U(a) = \{y \in X\colon \text{there is } a \in A \text{ such that } (a, y) \in U\}$.
Similarly as before, given a subset $\mathcal{S}$ of $\mathcal{P}_0(X)$, $\langle \mathcal{S} \rangle$ will denote the smallest prefilter containing $\mathcal{S}$ on X.

The following defines the $\mathcal{U}$-envelope and $\mathcal{U}$-roundness.

Definition 5.5.3 *Let $(X,\mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$ and each $\alpha \in \mathcal{PF}(X)$ we set $U(\alpha)$ equal to $\langle \{U(A) : A \in \alpha\} \rangle$, and call it the* U-hull *of α.*
Similarly we define $\mathcal{U}(\alpha)$ as the prefilter $\langle \{U(A) : A \in \alpha, U \in \mathcal{U}\} \rangle$. Let us note that $\mathcal{U}(\alpha)$ is usually called the $\mathcal{U}$-envelope *of α. A prefilter will be called* $\mathcal{U}$-round *if it is equal to its $\mathcal{U}$-envelope. Note that for any prefilter on X its $\mathcal{U}$-envelope is $\mathcal{U}$-round.*

Proposition 5.5.2 *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{PF}(X)$ be the set of all prefilters on X. For each $U \in \mathcal{U}$ we set*

$$U_{\oplus} = \{(\alpha,\beta) \in \mathcal{PF}(X) \times \mathcal{PF}(X) : U(\alpha) \subseteq \beta\}.$$

Then $\{U_{\oplus} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{\oplus}$ on $\mathcal{PF}(X)$ which will be called the positive quasi-uniformity. *For each $U \in \mathcal{U}$ we set*

$$U_{\ominus} = \{(\alpha,\beta) \in \mathcal{PF}(X) \times \mathcal{PF}(X) : U^{-1}(\beta) \subseteq \alpha\}.$$

Then $\{U_{\ominus} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{\ominus}$ on $\mathcal{PF}(X)$ which will be called the negative quasi-uniformity.

Furthermore for each $U \in \mathcal{U}$ set $U_{PF} = U_{\oplus} \cap U_{\ominus}$. Then $\{U_{PF} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_{PF}$ on $\mathcal{PF}(X)$ which will be called the prefilter quasi-uniformity.

Proof. Note first that for each $U \in \mathcal{U}$ and any $\alpha \in \mathcal{PF}(X)$, we have $(\alpha,\alpha) \in U_{\oplus}$, and similarly $(\alpha,\alpha) \in U_{\ominus}$ and $(\alpha,\alpha) \in U_{PF}$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $U_{\oplus} \subseteq V_{\oplus}$, $U_{\ominus} \subseteq V_{\ominus}$, and $U_{PF} \subseteq V_{PF}$. Hence $\{U_{\oplus} : U \in \mathcal{U}\}$, $\{U_{\ominus} : U \in \mathcal{U}\}$ and $\{U_{PF} : U \in \mathcal{U}\}$ are filter bases on $\mathcal{PF}(X) \times \mathcal{PF}(X)$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\alpha,\gamma) \in {V_{\oplus}}^2$ then there is $\beta \in \mathcal{PF}(X)$ such that $(\alpha,\beta) \in V_{\oplus}$ and $(\beta,\gamma) \in V_{\oplus}$. Let $A \in \alpha$, then $V(A) \in \beta$. We have $V^2(A) \in \gamma$ which implies $U(A) \in \gamma$. We have shown that $(\alpha,\gamma) \in U_{\oplus}$. Thus ${V_{\oplus}}^2 \subseteq U_{\oplus}$. Similarly ${V_{\ominus}}^2 \subseteq U_{\ominus}$, and thus $(V_{PF})^2 \subseteq U_{PF}$. We deduce that $\mathcal{U}_{\oplus}$, $\mathcal{U}_{\ominus}$ and $\mathcal{U}_{PF}$ are quasi-uniformities on $\mathcal{PF}(X)$. □

We next define the prefilter space of a quasi-uniform space.

Definition 5.5.4 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is called the* prefilter space *of the quasi-uniform space $(X,\mathcal{U})$.*

The next proposition shows that the lattice operations in the prefilter space of a quasi-uniform space are quasi-uniformly continuous.

Proposition 5.5.3 *The lattice operations on $(\mathcal{PF}(X),\mathcal{U}_{PF})$ are quasi-uniformly continuous (that is, $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is a quasi-uniform lattice).*

Proof. Let $U \in \mathcal{U}$. Consider $(\alpha,\beta),(\alpha',\beta') \in U_\oplus \cap U_\ominus$. Then $U(\alpha) \subseteq \beta$, $U^{-1}(\beta) \subseteq \alpha$, $U(\alpha') \subseteq \beta'$ and $U^{-1}(\beta') \subseteq \alpha'$. Thus $U(\alpha \cup \alpha') \subseteq \beta \cup \beta'$ and $U^{-1}(\beta \cup \beta') \subseteq \alpha \cup \alpha'$. Hence $(\alpha \cup \alpha', \beta \cup \beta') \in U_\oplus \cap U_\ominus$ and similarly $(\alpha \cap \alpha', \beta \cap \beta') \in U_\oplus \cup U_\ominus$. Therefore $\cup : (\mathcal{PF}(X) \times \mathcal{PF}(X), \mathcal{U}_{PF} \times \mathcal{U}_{PF}) \to (\mathcal{PF}(X), \mathcal{U}_{PF})$ and $\cap : (\mathcal{PF}(X) \times \mathcal{PF}(X), \mathcal{U}_{PF} \times \mathcal{U}_{PF})$ are quasi-uniformly continuous. □

The following lemma makes a connection between the prefilter space and the Hausdorff hyperspace quasi-uniform space of a quasi-uniform space.

Lemma 5.5.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space.*
(a) Then the Hausdorff hyperspace $(\mathcal{P}_0(X), \mathcal{U}_H)$ embeds quasi-uniformly via the map $A \longmapsto \langle A \rangle$ into $(\mathcal{PF}(X), \mathcal{U}_{PF})$.
(b) Let $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-uniformly continuous map. Then $\mathcal{PF}(f)(\alpha) = \langle f(\alpha) \rangle$ is a quasi-uniformly continuous map from $(\mathcal{PF}(X), \mathcal{U}_{PF})$ into $(\mathcal{PF}(Y), \mathcal{V}_{PF})$.

Proof. We leave the easy proof to the reader. □

The next proposition makes another connection between the prefilter space and the Hausdorff hyperspace quasi-uniform space.

Proposition 5.5.4 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}_{PF}$ is equal to $(\mathcal{U}_+)_- \bigvee (\mathcal{U}_-)_+ \mid \mathcal{PF}(X)$.*

Proof. First note the following fact: Let $\alpha \in \mathcal{PF}(X)$. Then α is a nonempty collection of nonempty subsets of X. Therefore $\alpha \subseteq \mathcal{P}_0(X)$, hence $\alpha \in \mathcal{P}_0(\mathcal{P}_0(X))$. So $\mathcal{PF}(X) \subseteq \mathcal{P}_0(\mathcal{P}_0(X))$.

Let $U \in \mathcal{U}$ and $\alpha, \beta \in \mathcal{PF}(X)$. Suppose $(\alpha,\beta) \in U_\oplus$. We have $U(\alpha) \subseteq \beta$ means for each $A \in \alpha$ that there exists $B \in \beta$ such that $B \subseteq U(A)$, hence for each $A \in \alpha$ there exists $B \in \beta$ such that $(A,B) \in U_+$. We have $\alpha \subseteq (U_+)^{-1}(\beta)$ means $(\beta,\alpha) \in ((U_+)_-)^{-1}$ so $(\alpha,\beta) \in (U_+)_-$. Similarly $(\alpha,\beta) \in U_\ominus$ is equivalent to $(\alpha,\beta) \in (U_-)_+$.
Therefore $U_\oplus \cap U_\ominus$ is equivalent to $(U_+)_- \cap (U_-)_+$ on $\mathcal{PF}(X)$, thus $\mathcal{U}_{PF}$ is equal to $(\mathcal{U}_+)_- \bigvee (\mathcal{U}_-)_+ \mid \mathcal{PF}(X)$. □

We next show that total boundedness is preserved by the prefilter space of a quasi-uniform space.

Proposition 5.5.5 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\mathcal{U}$ is totally bounded if and only if $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is totally bounded.*

Proof. This follows from Proposition 5.5.4 and Lemma 5.5.1 above, since $\mathcal{U}_H$ is known to be totally bounded if and only if $\mathcal{U}$ is totally bounded [22], and since total boundedness is preserved by quasi-uniform subspaces, as well as quasi-uniformly continuous images. $\square$

Lemma 5.5.2 *Let X be a set and let Υ be an ultrafilter on $\mathcal{PF}(X)$. Then $\bigcup_{Z\in\Upsilon}(\bigcap_{\zeta\in Z}\zeta) = \bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta)$.*

Proof. The inequality $\bigcup_{Z\in\Upsilon}(\bigcap_{\zeta\in Z}\zeta) \subseteq \bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta)$ is obvious: For any $Z_1, Z_2 \in \Upsilon$ we have that $\bigcap_{\zeta\in Z_1}\zeta \subseteq \bigcup_{\zeta\in Z_2}\zeta$, since Z_1 and Z_2 intersect. Therefore $\bigcup_{Z_1\in\Upsilon}(\bigcap_{\zeta\in Z_1}\zeta) \subseteq \bigcup_{\zeta\in Z_2}\zeta$ and the assertion follows.

Let $A \in \bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta)$. Then for each $Z \in \Upsilon$ there is $\zeta_Z \in Z$ such that $A \in \zeta_Z$. Let $E = \{\zeta_Z : Z \in \Upsilon\}$. Then $\zeta_Z \in E \cap Z$ whenever $Z \in \Upsilon$. Thus $E \in \Upsilon$, since Υ is an ultrafilter on $\mathcal{PF}(X)$. Furthermore $A \in \bigcap_{\zeta\in E}\zeta$ and $A \in \bigcup_{Z\in\Upsilon}\bigcap_{\zeta\in Z}\zeta$. Hence $\bigcap_{Z\in\Upsilon}(\bigcup_{\zeta\in Z}\zeta) \subseteq \bigcup_{Z\in\Upsilon}(\bigcap_{\zeta\in Z}\zeta)$ and the assertion is proved. $\square$

We next show that the prefilter space of a quasi-uniform space is bicomplete (compare with Theorem 3.3.2).

Proposition 5.5.6 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is bicomplete. (Note that the proof also shows that $(\mathcal{PF}(X),\mathcal{U}_\oplus)$ and $(\mathcal{PF}(X),\mathcal{U}_\ominus)$ are bicomplete.)*

Proof. Let Ξ be a filter on $\mathcal{PF}(X)$ that is $(\mathcal{U}_{PF})^s$-Cauchy. We show that Ξ $\tau((\mathcal{U}_{PF})^s)$-converges to $\gamma := \bigcap_{Y\in\Upsilon}(\bigcup_{\zeta\in Y}\zeta) = \bigcup_{Y\in\Upsilon}(\bigcap_{\zeta\in Y}\zeta)$ where Υ is an ultrafilter on $\mathcal{PF}(X)$ containing Ξ by Lemma 5.5.2.

Let $U \in \mathcal{U}$. By assumption there is $C \in \Xi$ such that $(C \times C) \subseteq (U_\oplus \cap U_\ominus)$. Let $\xi, \xi' \in C$ be arbitrary. Then $U(\xi') \subseteq \xi$ by definition of $U_\oplus$ and similarly $U^{-1}(\xi) \subseteq \xi'$ by definition of $U_\ominus$.

Thus for each $\xi' \in C$, $U(\xi') \subseteq \bigcap_{\xi\in C}\xi \subseteq \gamma$ and therefore $C \times \{\gamma\} \subseteq U_\oplus$. Also for each $\xi \in C$ we have $U(\gamma) \subseteq U(\bigcup_{\xi'\in C}\xi') \subseteq \xi$ and consequently

$\{\gamma\} \times C \subseteq U_\ominus$.

Analogously for each $\xi \in C$, $U^{-1}(\xi) \subseteq \bigcap_{\xi' \in C} \xi' \subseteq \gamma$ and therefore $\{\gamma\} \times C \subseteq U_\ominus$. Furthermore for each $\xi' \in C$, $U^{-1}(\gamma) \subseteq U^{-1}(\bigcup_{\xi \in C} \xi) \subseteq \xi'$. Therefore $C \times \{\gamma\} \subseteq U_\ominus$. We conclude that Ξ converges to γ in $(\mathcal{PF}(X),(\mathcal{U}_{PF})^s)$ and thus $(\mathcal{PF}(X),\mathcal{U}_{PF})$ is bicomplete. □

It follows from Proposition 5.5.7 and Lemma 5.5.1 (a) that the bicompletion of the T_0-reflection of $(\mathcal{P}_0(X), \mathcal{U}_H)$ is a subspace of the T_0-reflection of the prefilter space $(\mathcal{PF}(X),\mathcal{U}_{PF})$. From the results of [23] we conclude that the bicompletion can be identified with the subspace consisting of all 2-round doubly stable filters on $(X, \mathcal{U})$, compare [23] for more details.

5.6 The left-sided scale of a quasi-uniform space

In this section we revisit the quasi-uniform scale of a quasi-uniform space. In particular we shall extend the ground set of the quasi-uniform scale of a quasi-uniform space.

Definition 5.6.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. We define $\mathcal{R}(X)$ as the set of all reflexive binary relations on X.*

We next define the left-sided scale quasi-uniformity.

Proposition 5.6.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space.*
We define $\mathcal{PF}_r(X^2) = \{\alpha : \alpha$ prefilter on $X \times X$ such that $\alpha \subseteq \mathcal{R}(X)\}$.
For any $U \in \mathcal{U}$ we set

$$U_\uparrow = \{(\alpha, \beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) \colon U \circ \alpha \subseteq \beta\}$$

and

$$U_\downarrow = \{(\alpha, \beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) : U^{-1} \circ \beta \subseteq \alpha\}.$$

Furthermore set $U_{\updownarrow} = U_{\uparrow} \cap U_{\downarrow}$ *whenever* $U \in \mathcal{U}$. *Then* $\{U_{\downarrow} : U \in \mathcal{U}\}$ *is a base for the* down-quasi-uniformity $\mathcal{U}_{\downarrow}$ *on* $\mathcal{PF}_r(X^2)$ *and* $\{U_{\uparrow} : U \in \mathcal{U}\}$ *is a base for the* up-quasi-uniformity $\mathcal{U}_{\uparrow}$ *on* $\mathcal{PF}_r(X^2)$. *Moreover* $\{U_{\updownarrow} : U \in \mathcal{U}\}$ *is the base for the* left-sided scale quasi-uniformity $\mathcal{U}_{\updownarrow}$.

Proof. This is similar to the proof of Proposition 5.1.2. $\square$

Note that contrary to Bushaw and Kent we have defined the ground set of our left-sided scale as the set of all prefilters on $X \times X$ coarser than the filter $\mathcal{R}(X)$ and not as the set of all prefilters on $X \times X$ coarser than $\mathcal{U}^s$. This difference however seems of minor importance in the following investigations. Indeed in our context it is more convenient to work on the chosen larger set, since any quasi-uniformity $\mathcal{U}$ comes with closely related quasi-uniformities like $\mathcal{U}^{-1}$ and $\mathcal{U}^s$, and it is often inconvenient to change the ground set repeatedly according to varying needs.

We next define the left-sided scale of a quasi-uniform space.

Definition 5.6.2 *Let* $(X,\mathcal{U})$ *be a quasi-uniform space. Then* $(\mathcal{PF}_r(X^2),\mathcal{U}_{\updownarrow})$ *is called the* left-sided scale *of the quasi-uniform space* $(X,\mathcal{U})$.

We will need the next lemma in the following.

Lemma 5.6.1 *Let* M, N, A *be binary relations on a set* X. *Then* $M \circ A \circ N = \bigcup_{(a_1,a_2)\in A} N^{-1}(a_1) \times M(a_2)$.

Proof. Suppose $(x,y) \in M \circ A \circ N$. Then for some $(a_1,a_2) \in A$ we have $(x,a_1) \in N$, $(a_2,y) \in M$ and therefore $(a_1,x) \in N^{-1}$, $(a_2,y) \in M$. Hence with $(a_1,a_2) \in A$ we have $(x,y) \in N^{-1}(a_1) \times M(a_2)$. Thus $(x,y) \in \bigcup_{(a_1,a_2)\in A} N^{-1}(a_1) \times M(a_2)$. Conversely we use a similar argument. $\square$

Our next results discuss connections between the left-sided scale quasi-uniform space and the prefilter space.

Proposition 5.6.2 *Let* $(X,\mathcal{U})$ *be a quasi-uniform space. Then on* $\mathcal{PF}_r(X^2)$ *we have:*

$\mathcal{U}_\uparrow = (\mathcal{D}\times\mathcal{U})_\oplus \mid \mathcal{PF}_r(X^2)$,

$\mathcal{U}_\downarrow = (\mathcal{D}\times\mathcal{U})_\ominus \mid \mathcal{PF}_r(X^2)$ and

$\mathcal{U}_\updownarrow = (\mathcal{D}\times\mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2)$.

Here $\mathcal{D}$ denotes the discrete uniformity on X.

Furthermore $\mathcal{U}_\uparrow, \mathcal{U}_\downarrow, \mathcal{U}_\updownarrow$ are all bicomplete on $\mathcal{PF}_r(X^2)$.

Proof. The first statement follows from Lemma 5.6.1. For the second statement note that if in Lemma 5.5.2 Υ is an ultrafilter on $\mathcal{PF}_r(X^2)$, then $\gamma \in \mathcal{R}(X)$. The result follows from Proposition 5.5.7. □

The following corollary makes another connection between the prefilter space and the left-sided scale of a quasi-uniform space. It should be compared with Lemma 5.5.1. In particular we conclude that for any quasi-uniform space $(X,\mathcal{U})$, $(\mathcal{P}_0(X),\mathcal{U}_H)$ quasi-uniformly embeds into $(\mathcal{PF}_r(X^2),(\mathcal{D}\times\mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ via $A \mapsto \langle \triangle \cup (X \times A)\rangle$.

Corollary 5.6.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then we can embed $(\mathcal{PF}(X),\mathcal{U}_{PF})$ into $(\mathcal{PF}_r(X^2),(\mathcal{D}\times\mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$.*

Proof. For each $\alpha \in \mathcal{PF}(X)$ set $k(\alpha) = \langle\{\triangle \cup (X \times A) : A \in \alpha\}\rangle$.

We show that $k : (\mathcal{PF}(X),\mathcal{U}_{PF}) \to (\mathcal{PF}_r(X^2),(\mathcal{D}\times\mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ is a quasi-uniformly continuous map:

Let $(\alpha,\beta) \in U_\oplus$. Thus $U \circ \alpha \subseteq \beta$ and therefore for each $A \in \alpha$ there is $B \in \beta$ such that $B \subseteq U(A)$. Then for each $A \in \alpha$ there is $B \in \beta$ such that $\triangle \cup (X \times B) \subseteq U \circ (\triangle \cup (X \times A))$ if and only if $U \circ k(\alpha) \subseteq k(\beta)$. Similarly $(\alpha,\beta) \in U_\ominus$ implies that $U^{-1} \circ k(\beta) \subseteq k(\alpha)$. Thus $(k(\alpha),k(\beta)) \in U_\updownarrow$ and k is quasi-uniformly continuous. Note next that k is injective. Let $\alpha \neq \beta$, say α is not subset of β. Then there is $A_0 \in \alpha$ such for all $B \in \beta$, $B \setminus A_0 \neq \emptyset$. Then $\triangle\cup(X\times B)$ is not subset of $\triangle\cup(X\times A_0)$ whenever $B \in \beta$. Thus $k(\alpha) \neq k(\beta)$.

On the other hand suppose that $H \circ k(\alpha) \subseteq k(\beta)$. Then for each $A \in \alpha$ there exists $B \in \beta$ such that $B \subseteq H(A)$: Indeed let $B \in B$. Choose any

$a_0 \in A$. Then $(a_0, b) \in H \circ (X \times A)$ or $(a_0, b) \in H \circ \triangle = H$. We conclude that for some $a \in A$, we have $b \in H(a)$, or that $b \in H(a_0)$. Thus $B \subseteq H(A)$ in either case. Hence $H(\alpha) \subseteq \beta$. Similarly $H^{-1} \circ k(\beta) \subseteq k(\alpha)$ implies that $H^{-1}(\beta) \subseteq \alpha$. Therefore k is a quasi-uniform embedding. □

The following proposition discusses a connection between the left-sided scale of a quasi-uniform space and the scale of a quasi-uniform space defined in Definition 5.1.4.

Proposition 5.6.3 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the left-sided scale of a quasi-uniform space $(\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ coincides with the scale of a quasi-uniform space defined in Definition 5.1.4 on the subset $P_{\mathcal{U}^s}$ of $\mathcal{R}(X)$.*

Proof. Let $U \in \mathcal{U}$ and consider $(\alpha, \beta) \in (\triangle \times U)_{\mathcal{PF}}$, then $(\triangle \times U)(\alpha) \subseteq \beta$ and $(\triangle \times U)^{-1}(\beta) \subseteq \alpha$.
Suppose $A \in \alpha$ and let $(c_1, c_2) \in \bigcup_{(a_1,a_2)\in A} \triangle(a_1) \times U(a_2)$. By Lemma 5.6.1 we have $(c_1, c_2) \in U \circ A \circ \triangle = U \circ A$, hence $U \circ A \in \beta$. Similarly suppose $B \in \beta$ then $U^{-1} \circ B \in \alpha$. Thus $(\alpha, \beta) \in S(U)$. The converse is similar. □

Let us note that Corollary 5.6.1 also holds if we replace $(\mathcal{PF}_r(X^2),(\mathcal{D} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ by $(\mathcal{PF}_r(X^2),(\mathcal{U}^{-1} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$, which is of interest in the study below which deals with the two-sided scale.

5.7 The two-sided scale of a quasi-uniform space

Remark 5.7.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $U, V \in \mathcal{R}(X)$, and $Q \in \mathcal{U}$. Note that $(\langle U\rangle, \langle V\rangle) \in Q_{\updownarrow}$ if and only if $V \subseteq Q \circ U$ and $U \subseteq Q^{-1} \circ V$ if and only if $V(x) \subseteq Q(U(x))$ and $U(x) \subseteq Q^{-1}(V(x))$ whenever $x \in X$. The latter condition exactly means that $(U(x), V(x)) \in Q_H$ whenever $x \in X$ where here Q_H is the standard entourage related to Q of the Hausdorff quasi-uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$. Hence the scale quasi-uniformity of a quasi-uniform space yields on the set of prefilters generated by reflexive relations (understood as multifunctions) the quasi-uniformity of quasi-uniform convergence with respect to the Hausdorff quasi-uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$.*

We recall that the quasi-uniform space of multifunctions has been summarized in Chapter 2. Observe that the multifunction space can be embedded into $(\mathcal{PF}(X \times Y),(\mathcal{D}_X \times \mathcal{U})_{PF})$, where $\mathcal{D}_X$ denotes the discrete uniformity on X.

It is somewhat surprising that the definition by Bushaw is admittedly left-sided [6, p.105], although as we see above this definition becomes less unnatural if one considers the scale as a generalization of the multifunction space of a quasi-uniform space instead of considering it as a generalization of the prefilter space of a quasi-uniform space.

One possibly unwanted consequence of the surprising appearance of the discrete uniformity in the formula in Proposition 5.6.2 is that total boundedness is not preserved by the scale quasi-uniformity.

Below we modify the definition of the left-sided scale given in Proposition 5.6.2 by suggesting to work on $\mathcal{PF}_r(X^2)$ with the restrictions of the quasi-uniformities $(\mathcal{U}^{-1} \times \mathcal{U})_{\oplus}$, $(\mathcal{U}^{-1} \times \mathcal{U})_{\ominus}$ and $(\mathcal{U}^{-1} \times \mathcal{U})_{PF}$ instead.

Proposition 5.7.1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. On $\mathcal{PF}_r(X^2)$ for any $U \in \mathcal{U}$ we set*

$U_{\Uparrow} = \{(\alpha,\beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) : U \circ \alpha \circ U \subseteq \beta\}$ *and*

$U_{\Downarrow} = \{(\alpha,\beta) \in \mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2) : U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha\}$. *Furthermore for each $U \in \mathcal{U}$ set $U_{\Updownarrow} = U_{\Uparrow} \cap U_{\Downarrow}$.*

Then the two-sided scale *of the quasi-uniform space $(X,\mathcal{U})$ will be the quasi-uniform space $(\mathcal{PF}_r(X^2),\mathcal{U}_{\Updownarrow})$ where $\{U_{\Updownarrow} : U \in \mathcal{U}\}$ is the base of the quasi-uniformity $\mathcal{U}_{\Updownarrow}$. Similarly $\mathcal{U}_{\Uparrow}$ will be the quasi-uniformity generated by the base $\{U_{\Uparrow} : U \in \mathcal{U}\}$ and $\mathcal{U}_{\Downarrow}$ will be the quasi-uniformity generated by the base $\{U_{\Downarrow} : U \in \mathcal{U}\}$.*

Proof. Note first that for each $U \in \mathcal{U}$ and any $\alpha \in \mathcal{PF}_r(X^2)$, we have $(\alpha,\alpha) \in U_{\Uparrow}$, and similarly $(\alpha,\alpha) \in U_{\Downarrow}$ and $(\alpha,\alpha) \in U_{\Updownarrow}$. Observe also that $U, V \in \mathcal{U}$ with $U \subseteq V$ implies that $U_{\Uparrow} \subseteq V_{\Uparrow}$ and $U_{\Downarrow} \subseteq V_{\Downarrow}$, and $U_{\Updownarrow} \subseteq V_{\Updownarrow}$. Hence $\{U_{\Uparrow} : U \in \mathcal{U}\}$, $\{U_{\Downarrow} : U \in \mathcal{U}\}$ and $\{U_{\Updownarrow} : U \in \mathcal{U}\}$ are filter bases on

$\mathcal{PF}_r(X^2) \times \mathcal{PF}_r(X^2)$.

Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Let $(\alpha, \gamma) \in {V_\Uparrow}^2$. Then there is $\beta \in \mathcal{PF}_r(X^2)$ such that $(\alpha, \beta) \in V_\Uparrow$ and $(\beta, \gamma) \in V_\Uparrow$. Let $A \in \alpha$. Then $V \circ A \circ U \in \beta$. We have $V^2 \circ A \circ V^2 \in \gamma$ implies $U \circ A \circ U \in \gamma$. We have shown that $(\alpha, \gamma) \in U_\Uparrow$. Thus ${V_\Uparrow}^2 \subseteq U_\Uparrow$. Similarly ${V_\Downarrow}^2 \subseteq U_\Downarrow$, and thus $(V_\Updownarrow)^2 \subseteq U_\Updownarrow$. We deduce that $\mathcal{U}_\Uparrow$, $\mathcal{U}_\Downarrow$ and $\mathcal{U}_\Updownarrow$ are quasi-uniformities on $\mathcal{PF}_r(X^2)$. □

Our next proposition makes a connection between the prefilter quasi-uniformity and the two-sided quasi-uniformity.

Proposition 5.7.2 *Let $(X, \mathcal{U})$ be a quasi-uniform space and let $\alpha, \beta \in \mathcal{PF}_r(X^2)$. Then for each $U \in \mathcal{U}$, $(\alpha, \beta) \in (U^{-1} \times U)_{PF} \mid \mathcal{PF}_r(X^2)$ if and only if $U \circ \alpha \circ U \subseteq \beta$ and $U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha$.*
Therefore $\mathcal{U}_\Updownarrow = (\mathcal{U}^{-1} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2)$.

Proof. Let $U \in \mathcal{U}$ and suppose $(\alpha, \beta) \in (U^{-1} \times U)_{PF}$. We have $(U^{-1} \times U)(\alpha) \subseteq \beta$ and $(U^{-1} \times U)^{-1}(\beta) \subseteq \alpha$.
Let $A \in \alpha$. Then $(U^{-1} \times U)(A) \in \beta$ by Lemma 5.6.1. Thus $U \circ A \circ U \in \beta$. Hence $U \circ \alpha \circ U \subseteq \beta$, and similarly $U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha$.

Conversely for each $U \in \mathcal{U}$, we suppose $U \circ \alpha \circ U \subseteq \beta$ and $U^{-1} \circ \beta \circ U^{-1} \subseteq \alpha$. By Lemma 5.6.1 we have $(U^{-1} \times U)(\alpha) \subseteq \beta$ and $(U^{-1} \times U)^{-1}(\beta) \subseteq \alpha$. □

Our next result shows that total boundedness in the two-sided scale is preserved, which does not hold for the Bushaw-Kent version of a scale, even in the context of a uniform space.

Proposition 5.7.3 *The two-sided scale $(\mathcal{PF}_r(X^2), \mathcal{U}_\Updownarrow)$ of a totally bounded quasi-uniform space $(X, \mathcal{U})$ is joincompact, that is, the topology $\tau((\mathcal{U}_\Updownarrow)^s)$ is compact.*

Proof. For any quasi-uniform space $(X, \mathcal{U})$, $(\mathcal{PF}_r(X^2), (\mathcal{U}^{-1} \times \mathcal{U})_{PF} \mid \mathcal{PF}_r(X^2))$ is bicomplete. Hence $(\mathcal{PF}_r(X^2), \mathcal{U}_\Updownarrow)$ is bicomplete. It is totally bounded, since $\mathcal{U}^{-1} \times \mathcal{U}$ is totally bounded and the prefilter space preserves total boundedness. □

Corollary 5.7.1 *For any quasi-uniform space* $(X,\mathcal{U})$ *the two-sided scale* $(\mathcal{PF}_r(X^2),\mathcal{U}_{\Updownarrow})$ *is bicomplete.* $\square$

Corollary 5.7.2 *The two-sided scale* $(\mathcal{PF}_r(X^2),\mathcal{U}_{\Updownarrow})$ *of a quasi-uniform space* $(X,\mathcal{U})$ *is totally bounded if and only if* $(X,\mathcal{U})$ *is totally bounded.* $\square$

Chapter 6

Conclusion

In this dissertation, we have discussed the scale of a quasi-uniform space. We defined our scale of a quasi-uniform space and we made a connection between the scale of a quasi-uniform space and the scale of a uniform space introduced by Bushaw [6] and Kent [16]. We showed that the scale of a quasi-uniform space and its retracted scale are both bicomplete and that for any quasi-uniform space $(X,\mathcal{U})$ the associated hyperspace given by the Hausdorff quasi-uniformity is quasi-uniformly embedded into the left-sided scale of $(X,\mathcal{U})$.

We defined the prefilter space of a quasi-uniform space, and we made a connection between the Hausdorff hyperspace quasi-uniform space and our prefilter space. We also defined the two-sided scale of a quasi-uniform space and showed that total boundedness is preserved by our two-sided scale.

Our conclusion leads us to list some open problems encountered throughout the present investigation. We hope to study these questions in future work.

Problem 6.0.1 *Which quasi-uniform lattices are left-sided scales of a quasi-uniform space?*

Problem 6.0.2 *Under what conditions on the quasi-uniform space* $(X,\mathcal{U})$ *are the scale quasi-uniform space* $(P_{\mathcal{U}^s},S(\mathcal{U}))$ *and the retracted scale quasi-uniform space* $((P_{\mathcal{U}^s})_R, S(\mathcal{U})_R)$ *connected, locally connected, second countable, separable, locally compact, etc.?*

Problem 6.0.3 *Investigate the order scale of a quasi-uniform space (see Section 3.6)?*

Problem 6.0.4 *Are there simple conditions which characterize those quasi-uniform spaces which are quasi-uniformly isomorphic to the prefilter space or the two-sided scale quasi-uniform space of some quasi-uniform space?*

We will next list all those articles that we have consulted during the completion of this dissertation.

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