THE RATIO g_A/g_V IN CAVITY QCD

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ABSTRACT

BRS invariant quantum chromodynamics in a spherical cavity is developed using canonical quantization. The weak vector and axial form factors are defined, employing a classical external W^- -field. The Gell-Mann and Low theorem is extended to include non-diagonal matrix elements and degenerate perturbation theory.

The Sucher form of the Gell-Mann and Low theorem is employed to calculate corrections of order $G_F g^2$ in the weak and strong coupling constants to g_A and g_V for neutron beta decay. Up and down quarks are assumed massless.

The gauge-independent divergences from the loop diagrams cancel each other and can be regularized dimensionally, making renormalization unnecessary.

We find that the weak vector and axial current coupling constants are respectively:

$$g_V = 1.0000$$
$$g_A = 1.0883 + 0.2425\ \alpha_S,$$

where the preferred value of $\alpha_S = 2.2$ in the M.I.T. bag model gives $g_A = 1.62$.

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CONTENTS

INTRODUCTION

Since the confinement of Quantum Chromodynamics (QCD) was required by experiment, and appears to be a non-perturbative phenomenon currently outside theoretical reach, it was introduced by hand in the M.I.T. bag model (CE 74). The static spherical cavity breaks the translational, Lorentz and chiral symmetry of QCD. This causes a non-conservation of the axial current at the bag surface, which is a disturbing feature for the calculation of weak interaction form factors. The chiral bag model (CE 79) was developed where the discontinuity of the axial current on the bag surface was considered to be compensated by the emission of pseudoscalar mesons. Two non-renormalizable versions are available:

1. The "little" bag model, where mesons exist only outside the bag surface and exert a pressure on that surface so that the bag radius decreases, which fits well with the picture of nucleons in nuclei being non-overlapping ($R < 1$ fm).
2. The volume-type cloudy bag model, where mesons are allowed to exist inside the bag boundary (TS 88).

Buser *et al* (BU 88) have formulated QCD in a finite volume by imposing M.I.T. boundary conditions on the corresponding surface. These conditions have been shown to guarantee the conservation of BRS charge, ghost charge and quark number in the cavity, all of which are basic requirements for a consistent cavity field theory. Although a variety of boundary conditions can be chosen, the M.I.T. boundary conditions are used due to the relative success of the earlier classical M.I.T. bag model. Renormalization techniques were developed in (ST 90, 91).

A model QCD calculation can thus be performed in a cavity, although the confinement mechanism is still ill understood. We propose to calculate the weak vector coupling constant g_V, and the weak axial vector coupling constant g_A. This is justified by the precise measurement of these two quantities (although data fitting can only be done on the assumptions that no SU(3) symmetry breaking occurs in leading order in g_V, and a number of other assumptions, referred to as "Cabibbo-SU(3) fitting"). (MA 83) noted that the breaking of Lorentz invariance leads to a non-meaningful renormalization scheme in the large one-body term which occurred in the calculation for the magnetic moments of the nucleons, but did not lead to similar problems for the calculation of g_V and g_A. Due to the similarity of their formalism to ours, we can use this as a motivating factor for studying g_V and g_A instead of the nucleon magnetic moments, although we do not actually perform renormalization. If the quarks are made massive in subsequent developments of this thesis, renormalization will however be necessary.

Experimental data comes from two groups:

1. (PA 86) obtains $g_A/g_V = 1.254 \pm 0.006$;
2. (BO 83) carried out experimental studies of several hyperon semi-leptonic decays simultaneously and obtained $g_A/g_V = 1.239 \pm 0.009$. The authors argue against world averaging of existing data due to the delicacy of apparatus-dependent corrections.

In view of consensus that renormalized up and down quark masses should be small (~ 5 - 10 MeV) compared to the mass scale set by the cavity radius (~ 200 - 500 MeV), we neglect the quark masses. g_V and g_A are not dependent on the cavity radius R, which is set to 1 fm in the M.I.T. bag model. With the present experimental precision the use of a three-quark Cabibbo-Kobayashi-Maskawa matrix is not warranted, and we retain the traditional Cabibbo parametrization, $\cos\theta_c$. Centre of mass corrections have not been included, since the usual Peierl-Yoccoz method is not considered to be reliable.

Chapter 1 introduces QCD in free space and in a cavity. Chapter 2 includes the IVB model, the theory of nuclear beta decay and the restrictions to hadronic matrix elements generally considered to hold (e.g. the CVC hypothesis, and the non-existence of second-class currents).

Chapter 3 shows that a generalized form of the Gell-Mann and Low theorem can be consistently applied in the context of weak interactions by looking at an extension of the theorem to degenerate perturbation theory. The energy shift is expressed in terms of g_V and g_A, and these parameters are subsequently shown to be defined consistently with the literature.

In Chapter 4 the energy shift to first order in α_s is systematically generated, using the Sucher form of the Gell-Mann and Low theorem. Feynman diagrams relevant to the calculation of the weak coupling constants are reviewed, and their contributions to the energy shift calculated. Only the gauge-independent results are given (corresponding to the Feynman gauge, $\lambda = 1$) since (OC 90) demonstrated that gauge-dependent terms vanish.

Chapter 5 demonstrates the numerical methods used for evaluating expressions that were not simplified analytically, while the results are presented in Chapter 6. The appendices carry the bulk of the analytical work of the project, and is presented in a compact fashion.

Chapter 1

QUANTUM CHROMODYNAMICS

1.1 INTRODUCTION

> Yet all our labours were in vain. In the spring of 1971 Veltman informed us that his student Gerhardt 't Hooft has established the renormalizability of spontaneously broken gauge theory - Sheldon L. Glashow (GL 91)

The breakthrough which led to Quantum Chromodynamics (QCD) was put forward by (FR 73) when the property of asymptotic freedom of non-Abelian gauge theories was discovered. The classification of hadrons in terms of irreducible representations of SU(3), built up from a fundamental representation of quarks by (GE 64), initiated the search for them by applying a beam of particles without structure (e.g. leptons) at high momentum to the hadrons (called deep inelastic scattering). Bjorken (BJ 69) assumed that projectile electrons scatter off almost-free point-like constituents (called partons), and predicted a property called "Bjorken scaling" which was experimentally verified in 1968. According to this property, if the momentum transfer squared q^2 and the energy transfer ν of the electrons are very large, and the ratio q^2/ν is kept fixed, then the inelastic form factors depend only on the ratio q^2/ν rather than the two independent variables q^2 and ν. Other evidence for asymptotic freedom is that the e^+e^- cross-section corresponds to experiment although the final state quark interaction was neglected, which indicated a weak interaction at short distances.

Only Yang-Mills theory (YA 54) out of all the quantum field theories in four dimensional space-time was found, via the renormalization group equations, to have an ultraviolet fixed point at zero and hence a vanishing anomalous dimension, and so exhibits asymptotic freedom. The renormalization group equations parameterizes the change in scaling behaviour at high momenta of invariant amplitudes, due to the presence of an arbitrary momentum scale μ in the theory, which arises from the renormalization of the theory.

The BPHZ method, which uses the "forest formula" to construct recursion relations for the subtraction of overall divergences and subdivergences from the Feynman integrals (ZI 69), does not make recourse to the BRS symmetry of the QCD (the quantized version of the classical SU(3) locally gauge symmetry). Since the four interaction terms in QCD are related by this symmetry we can reduce the number of independent Green's functions, due to the generalized Ward-Takahashi identities satisfied by the generating functional. 't Hooft (TH 71) used this method to show that there are only four independent superficially divergent Feynman amplitudes: namely the gluon, ghost and quark self-energies, and the ghost-gluon vertex. The BPHZ recursion relations are then applied to these amplitudes to prove renormalizability of QCD to all orders. Recently a more elegant formulation of the Taylor-Slavnov identities has been found using the "background field" approach in non-Abelian gauge theories.

It was proposed (FR 73) that the extra symmetry of non-Abelian gauge theory can be identified with a new degree of freedom of quarks, called "colour". The gauge field then mediates the colour-force between the quarks, and is called the gluon field. The gluons themselves carry colour and hence interact with each other - an essential ingredient for having asymptotic freedom. The motivation for colour is as follows: (1) The Δ^{++} consists of three spin ½ up quarks in a spin-symmetric state, so the overall wavefunction is symmetric. But this is impossible by the Pauli principle for three fermions. Hence we need at least three additional degrees of freedom; (2) The experimental cross-section for e^+e^- annihilation into hadrons is three times larger than expected, being proportional to the number of colour degrees of freedom of quarks; (3) The experimental cross-section for $\pi^0 \rightarrow 2\gamma$ decays, which takes place through the anomaly in the divergence of axial-vector currents, is nine times larger than expected, being proportional to the square of the number of colours.

The non-observability of colour in nature is ensured by postulating that baryon states pick out the singlet representation out of the decomposition of the direct product of three quark triplets into irreducible representations. The meson states also choose the singlet representation of the product of one quark triplet and one antiquark complex conjugate triplet. A different way of stating this is that physical phenomena are invariant under SU(3) transformations in colour space, i.e. the symmetry is exact. Since only $q\bar{q}$ and qqq among the many low-lying configurations of quarks can belong to a colour singlet, we are in fact saying that quarks can only exist within baryons and mesons (called colour confinement). Confinement can possibly be explained as an infrared dynamical effect due to serious infrared divergences, which arise from massless gluons (WE 73). In the low-energy region (below a few GeV), the running coupling constant α_s becomes large and non-perturbative effects (e.g. instantons and gluon condensation) cannot be neglected. It is currently believed that the non-perturbative behaviour of QCD, when enhanced by the non-Abelian nature of the theory, leads to confinement.

Infrared divergences are of two types: (1) The soft divergences due to a massless gluon field and, (2) The collinear divergences due to the self-coupling of the massless gluon field (triple and quartic gluon coupling). The Block-Nordsieck theorem used in QED to show that low momentum radiative photon processes cancel infrared divergences, is not applicable in QCD for processes with nonleading twists. However, the general results for massless renormalized field theories (i.e. the Kinoshita-Poggio-Quinn theorem) is still believed to hold for QCD, and all recent papers confirm infrared divergence cancellation for some order of perturbation theory (e.g. KR 77).

Due to the small running coupling constant at small distances, we may safely use perturbation theory to discuss short-distance reactions. Using the operator product expansion one can safely extract the purely short-distance part of e.g. a deep inelastic cross-section (the operator product expansion provides an expansion of a singular product of operators in terms of regular operators and singular c-number coefficients). It was found that Bjorken scaling was violated logarithmically, which was later confirmed in deep inelastic muon-nucleon scattering.

QCD has predicted several results successfully, namely the Drell-Yan process (nucleon + nucleon → lepton + lepton + hadrons), the inclusive e^+e^- annihilation (e^+e^- → hadron + unobserved hadrons) and jet phenomena up to four jets.

In QCD the effective coupling constant at the experimental energy scales is not very small ($\alpha_s \sim 1/10$), so that the renormalization scheme dependence of perturbative calculations may have a non-negligible effect. This often limits the comparison of theory with data from experiments.

1.2 CANONICAL QUANTIZATION

The usual approach is via Feynman's path integral formalism (FE 48), or more recently the stochastic formalism (PA 81). Since it is not known how to use path-integral methods with cavity boundary conditions, we use the canonical operator formalism and closely follow (BU 88).

The QCD Lagrange density is given by

$$L = \overline{\psi}(i\gamma_\mu D^\mu - M_f)\psi - \tfrac{1}{2}i\partial_\mu(\overline{\psi}\gamma^\mu\psi) - \tfrac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu} \\ -\tfrac{1}{2}\lambda\partial_\mu\boldsymbol{A}^\mu\cdot\partial_\nu\boldsymbol{A}^\nu - i\partial_\mu\boldsymbol{\chi}\cdot D^\mu\boldsymbol{\omega} \quad , \qquad (1)$$

where ψ is the quark field and M_f is the (diagonal) mass matrix of the different flavours of quarks. $\boldsymbol{A}^\mu$ is the gauge field describing the gluons, and χ and ω are the Faddeev-Popov ghost fields. D^μ and D^μ are the covariant derivatives:

$$\mathrm{D}^\mu\psi = (\partial^\mu - \tfrac{1}{2}ig\boldsymbol{\lambda} \cdot \boldsymbol{A}^\mu)\psi \quad ; \quad D^\mu\boldsymbol{\omega} = \partial^\mu\boldsymbol{\omega} + g\boldsymbol{A}^\mu\times\boldsymbol{\omega} \quad , \tag{2}$$

with the vector dot product and cross product in colour space defined as

$$\boldsymbol{A}\cdot\boldsymbol{B} = \sum_{a=1}^{8} A_a B_b \quad ; \quad (\boldsymbol{A}\times\boldsymbol{B})_a = \sum_{b,c=1}^{8} f_{abc}A_b B_c \quad . \tag{3}$$

The $\boldsymbol{\lambda}$'s are the 8 Gell-Mann matrices, the f_{abc} are the structure constants of SU(3), and λ parameterizes the gauge.

We obtain eq. (1) by requiring the Lagrangean for a massive fermion field

$$L = \overline{\psi}(i\gamma_\mu\partial^\mu - M_f)\psi \quad , \tag{4}$$

to be locally gauge invariant under SU(3) transformations, since there is no reason why two space-like separated observers should agree on their phase convention for ψ. The Lagrangean thus obtained couples quarks to the gluon field. In analogy to QED we introduce a kinetic term for the gluons

$$L' = -\tfrac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu} \quad , \tag{5}$$

where we need eq. (5) to be locally gauge invariant. We obtain this in analogy to QED via the Bianchi identity

$$[\mathrm{D}^\mu, \mathrm{D}^\nu] = ig\frac{\boldsymbol{\lambda}}{2}\cdot\boldsymbol{F}^{\mu\nu} \quad , \tag{6}$$

from which we get

$$\boldsymbol{F}^{\mu\nu} = \partial^\mu\boldsymbol{A}^\nu - \partial^\nu\boldsymbol{A}^\mu + g\boldsymbol{A}^\mu\times\boldsymbol{A}^\nu \quad . \tag{7}$$

The Lagrangean density now obtained is not suitable for quantization because of a problem related to the gauge freedom

$$\boldsymbol{A}'_\mu = \boldsymbol{A}_\mu - \epsilon\boldsymbol{A}_\mu\times\boldsymbol{\omega} - \frac{\epsilon}{g}\partial_\mu\boldsymbol{\omega} \quad , \tag{8}$$

which can be seen by the fact that the canonical conjugate momentum to $\mathbf{A}^0$ vanishes. We add a covariant gauge-fixing term which is globally gauge invariant

$$L_{fix} = -\tfrac{1}{2}\lambda\partial_\mu\boldsymbol{A}^\mu\cdot\partial_\nu\boldsymbol{A}^\nu \quad , \tag{9}$$

but breaks the local gauge invariance. However, if the phase $\boldsymbol{\omega}$ satisfies the constraint

$$\partial_\mu D^\mu\boldsymbol{\omega} = 0 \quad , \tag{10}$$

we can restore the local gauge invariance in L_{fix}. This is done via the addition of the Faddeev-Popov ghost term (using the method of Lagrange multipliers) to obtain

$$L_{ghost} = i\boldsymbol{\chi}\cdot\partial_\mu D^\mu\boldsymbol{\omega} \quad . \tag{11}$$

The Lagrangean density is now that of eq. (1) with a new type of global gauge invariance, the Becchi-Rouet-Stora (BRS) invariance (BE 74).

The global Abelian U(1) transformation of the quark fields in eq. (1) leads to conservation of quark number. The global SU(3) symmetry of eq. (1) leads to colour conservation and the local SU(3) symmetry to a real Grassmann number, the conserved BRS charge. A global scale transformation on the ghost fields leads to the conservation of ghost charge.

The Euler-Lagrange equations can be derived from the Lagrangean. The Hamilton density can thus be defined and written as the sum of two terms, $H = H_o + H_{int}$. The first term H_o describes the noninteraction fields and is independent of the strong coupling constant g. It is given by

$$H_o = \overline{\psi}(-\tfrac{1}{2}i\gamma_k\overleftrightarrow{\partial}^k + M_f)\ \psi + \tfrac{1}{4}\ (\partial_k\mathbf{A}^l - \partial_l\mathbf{A}^k)\cdot(\partial_k\mathbf{A}^l - \partial_l\mathbf{A}^k) + \tfrac{1}{2}\mathbf{\Pi}^k\cdot\mathbf{\Pi}^k$$
$$- \tfrac{1}{2\lambda}\mathbf{\Pi}^o\cdot\mathbf{\Pi}^o + \mathbf{\Pi}^k\cdot\partial_k\mathbf{A}^o - \mathbf{\Pi}^o\cdot\partial_k\mathbf{A}^k - i\mathbf{\Omega}\cdot\mathbf{X} - \partial_k\boldsymbol{\chi}\cdot\partial_k\boldsymbol{\omega} \tag{12}$$

Here $\mathbf{\Pi}^\mu$, $\boldsymbol{\omega}$ and $\mathbf{X}$ are the canonically conjugate momenta to the gluon and two ghost fields respectively. The second term, H_{int}, depends linearly and quadratically on g and describes the interaction between the fields. This term can explicitly be written

$$H_{int} = -\tfrac{1}{2}g\overline{\psi}\gamma_\mu\boldsymbol{\lambda}\psi\cdot\mathbf{A}^\mu - \tfrac{1}{2}g(\partial_k\mathbf{A}^l - \partial_l\mathbf{A}^k)\cdot(\mathbf{A}^k\times\mathbf{A}^l) - g\mathbf{\Pi}^k\cdot(\mathbf{A}^k\times\mathbf{A}^o)$$
$$+ \tfrac{1}{4}g^2(\mathbf{A}^k\times\mathbf{A}^l)\cdot(\mathbf{A}^k\times\mathbf{A}^l) + g\mathbf{\Omega}\cdot(\mathbf{A}^o\times\boldsymbol{\omega}) + ig\partial_k\boldsymbol{\chi}\cdot(\mathbf{A}^k\times\boldsymbol{\omega}) \tag{13}$$

The different terms in the interaction Hamilton density H_{int} describe: (1) the interaction of two quarks and a gluon; (2) the elementary three-gluon vertex; (3) the elementary four-gluon vertex and; (4) the interaction of two ghosts and a gluon.

Quantization of the Hamilton density is performed by interpreting the classical fields ψ, $\mathbf{A}^\mu$, etc. as field operators. The following equal-time commutation and anti-commutation relations are imposed on the operators:

$$\{\psi_{cf\alpha}(\vec{x},t),\psi^{+}_{c'f'\beta}(\vec{y},t)\} = \delta_{cc'}\delta_{ff'}\delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}) \quad . \tag{14}$$

In the above the subscripts c, f and α on $\psi_{cf\alpha}$ denote the colour, flavour and spinor indices of the quark respectively. The commutation relations of the gluon fields are given by

$$[A^\mu_a(\vec{x},t),\Pi^\nu_b(\vec{y},t)] = ig^{\mu\nu}\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \quad . \tag{15}$$

The ghost fields originate from classical anti-commuting Grassmann fields and satisfies anti-commutation relations, although they have integral spin. The anti-commutators are:

$$\{\omega_a(\vec{x},t),\Omega_b(\vec{y},t)\} = \{\chi_a(\vec{x},t),X_b(\vec{y},t)\} = -i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \quad . \tag{16}$$

Now change the field operators and state vectors into the Dirac picture with the help of the time evolution operator $U(t,t_o)$, and denote quantities in this picture with a hat. The operator now satisfies the following differential equation:

$$\begin{aligned} U(t,t_o) &= e^{-iH_o(t-t_o)} \\ i\frac{\partial}{\partial t}U(t,t_o) &= \hat{H}_{int}(t)\,U(t,t_o) \end{aligned} \quad . \tag{17}$$

It is clear that $\hat{H}_{int}(t)$ satisfies the same differential equation as $H_{int}(t)$ if all arguments in eq. (13) are transformed into the Dirac picture. The field operators in the Dirac picture satisfy the following non-interacting field equations:

$$(i\gamma_\mu\partial^\mu - M_f)\hat{\Psi} = \bar{\hat{\Psi}}(i\gamma_\mu\overleftarrow{\partial}^\mu + M_f) = 0 \quad , \tag{18}$$

$$\Box\hat{A}^\mu + (\lambda-1)\partial^\mu\partial_\nu\hat{A}^\nu = 0 \quad , \tag{19}$$

$$\Box\hat{\omega} = \Box\hat{\chi} = 0 \quad . \tag{20}$$

1.3 CAVITY QCD

In the absence of a solution to the confinement problem, we put in the experimental fact of confinement by imposing boundary conditions by hand in a phenomenological way. The MIT bag model (CE 74) was the first such attempt, where it was done on an arbitrary space-like surface. The advantage of imposing confinement by hand in QCD is that we retain most of the properties of the underlying gauge theory. In fact we are merely doing field theory in a finite volume. The boundary conditions must be compatible with the field equations and we require that no quarks, gluons or ghosts may escape from the cavity, i.e. we demand that the conserved charges already mentioned remains conserved. We choose a static spherically symmetric cavity with a normalized space-like vector n_μ characterizing it ($n_\mu n^\mu = -1$ and $\hat{n}$ is the outward pointing normal of the surface S of the cavity). The conserved charge Q satisfies

$$Q = \int_V d^3x J^o(x) \qquad \frac{\partial Q}{\partial t} = \int_S d\Omega n_\mu J^\mu(x) \quad , \qquad (21)$$

so that if the conserved current $J^\mu(x)$ satisfies the boundary condition $n_\mu J^\mu(x)=0$ ($x\epsilon S$), then the charge Q will be time independent in the cavity.

The boundary conditions are simply those of the MIT group:

$$(in_k\gamma^k - 1)\Psi|_S = \bar{\Psi}(in_k\gamma^k + 1)|_S = 0 \quad , \qquad (22)$$

$$n_k(\partial^k\hat{\mathbf{A}}^\nu - \partial^\nu\hat{\mathbf{A}}^k)|_S = n_k\hat{\mathbf{A}}^k|_S = n_k\partial^k(\partial_\nu\hat{\mathbf{A}}^\nu)|_S = 0 \quad , \qquad (23)$$

$$n_k\partial^k\hat{\boldsymbol{\omega}}|_s = n_k\partial^k\hat{\boldsymbol{\chi}}|_s = 0 \quad . \qquad (24)$$

The solutions to eqs. (18) and (19) with the boundary conditions of eqs. (22) and (23), are respectively the well-known quark and gluon cavity modes which may be found in Appendix A.

1.4 THE QUARK AND GLUON PROPAGATORS

Field operators can be expanded in terms of a complete set of cavity modes (see Appendix A). The modes satisfy the same field equations and boundary conditions as the field operators in the Dirac picture.

The expansion of the quark field ψ in the complete set of cavity modes is given by

$$\Psi_{cf}(x) = \sum_{\substack{\kappa\mu \\ \nu>0}} [\hat{a}_{cfn}u_n(\vec{x})\,e^{-i\epsilon_n t} + \hat{b}^+_{cfn}u_{-n}(\vec{x})\,e^{i\epsilon_n t} \quad , \qquad (25)$$

where the radial, angular momentum and magnetic quantum numbers of the quark are characterized by $n \equiv (\nu, \kappa, \mu)$. The expansion coefficients $\hat{a}$ and $\hat{b}^+$ are the quark annihilation and antiquark creation operators respectively. Use is made of the fact that the spinors $\bar{u}_n(\vec{x})$ and $u_n(\vec{x})$ are solutions to the non-interacting field equations (18) with the MIT boundary conditions (i.e. the quark cavity modes), and ε_n is the energy of the cavity mode.

The quark propagator is found to be:

$$iS(x_1,x_2) = i\delta_{cc'}\delta_{ff'}\sum_n u_n(\vec{x}_1)\,\bar{u}_n(\vec{x}_2)\int\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t_1-t_2)}}{\omega - \varepsilon_n \pm i0} \quad . \tag{26}$$

In the above, both positive and negative radial quantum numbers are included in the sum over n. To make the contour integral well-defined the usual Feynman prescription for the poles should be employed, as indicated by the $\pm$ i0. In other words, poles with positive energy are given a small imaginary negative part, while the negative energy poles acquire a positive imaginary part.

The gluon fields may also be expanded in a complete set of cavity modes:

$$\hat{A}^{\mu}_{a}(\vec{x}) = \sum_{m\Sigma}\frac{1}{\sqrt{2\Omega^{\Sigma}_{m}}}\left[\hat{c}^{\Sigma}_{am}a^{\mu}_{m\Sigma}(\vec{x})\,e^{-i\Omega^{\Sigma}_{m}t} + \hat{c}^{\Sigma\dagger}_{am}a^{\mu *}_{m\Sigma}(\vec{x})\,e^{i\Omega^{\Sigma}_{m}t}\right] \quad . \tag{27}$$

In the above, $a^{\mu}_{m\Sigma}(\vec{x})$ are the gluon modes, where $m \equiv \{N,J,M\}$ denotes the radial, angular momentum and magnetic quantum numbers. The scalar, longitudinal, transverse magnetic and transverse electric polarizations of the gluon are respectively denoted by Σ = S,L,M,E.

Now define

$$\begin{aligned} q^{\Sigma} &\equiv (q^S, q^L, q^M, q^E) = (\omega, \Omega^{\Sigma}_{m}, 0, 0) \\ q_{\Sigma} &= (\omega, -\Omega^{\Sigma}_{m}, 0, 0) \end{aligned} \quad . \tag{28}$$

The vector, defined in this way, satisfies the requirement that

$$q^2 = \omega^2 - (\Omega_m^\Sigma)^2 \quad . \tag{29}$$

The cavity version of the propagator in an arbitrary gauge was derived by (ST 90) as

$$iD_{ab}^{\mu\nu}(x_1, x_2) = i\delta_{ab} \sum_{m\Sigma\Sigma'} a_{m\Sigma}^{\mu}(\vec{x}_1)\, a_{m\Sigma'}^{\nu *}(\vec{x}_2)$$
$$\int \frac{d\omega}{2\pi} \left[\frac{-g^{\Sigma\Sigma'}}{q^2} - \frac{1-\lambda}{\lambda} \frac{q^\Sigma q^{\Sigma'}}{q^4} \right] e^{i\omega(t_2 - t_1)} \quad . \tag{30}$$

It is well known that the gauge fields are associated with a Fock space of indefinite metric, since eq. (15) implies that

$$[c^\mu, c^{\nu +}] = - g^{\mu\nu} \quad , \tag{31}$$

where the gluon creation and annihilation operators are those of eq. (27). This leads, if we define $|1\rangle = c^{0+}|0\rangle$, to $\langle 1|1\rangle = -\langle 0|0\rangle$. This endangers the probabilistic interpretation of the quantum theory (this can be seen by acting with $|0\rangle$ on the left and right in eq. (31)). We also do not want spin zero fermions (ghosts) in physical space, since that violates the spin-statistics connection. In QED the Gupta-Bleuler condition discriminates between physical and non-physical states, but the generalisation of this condition to non-Abelian gauge theories is the condition

$$Q_B|\Psi_{phys}\rangle = 0 \quad , \tag{32}$$

where Q_B is the BRS charge (defined in (BU 88)). Physical states must always be chosen so that they do not contain scalar and longitudinal gluons and ghosts.

Chapter 2

WEAK INTERACTIONS

2.1 INTRODUCTION

Before the 1930's the electron released in nuclear beta decay was thought of as trapped inside a nucleus (because of a classical belief in particle number conservation), which contradicted the Heisenberg uncertainty principle. Since those days, quantum mechanics has led instead to generalized charge conservation, and the notion of a flavour-changing weak interaction. Fermi borrowed the idea of a current-current interaction from QED and postulated that the range of the weak interaction is zero, hence the idea of four fermion coupling. Later Feynman and Gell-Mann incorporated parity violation, although we shall refer to all theories essentially derived from Fermi as "Fermi theory". The Cabibbo hypothesis was formulated to explain patterns in magnitudes of strangeness-changing and strangeness-conserving weak interactions (e.g. the ratio between the kaon and pion decay constants are equal to $\tan^2\theta_c$, where θ_c is the Cabibbo angle). This is the idea of the mixing of down and strange quarks that produces a new state that interacts with the up quark:

$$d' = \cos\theta_c d + \sin\theta_c s \qquad J_\mu = d' \gamma_\mu (1 - \gamma_5) u \quad . \tag{33}$$

Decay amplitudes are calculated by assuming that the self-energy diagram in the path-integral quantized theory contains an imaginary part at $p^2 = m^2$, which is identified to be inversely proportional to the decay lifetime. If S is the square of the sum of the four-momenta of the incoming particles in a reaction, it is found that if $S \to \infty$ the cross-section for $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$ is proportional to S ($\sigma \propto S$), an intrinsic behaviour of the Fermi theory. Both from power counting arguments and the fact that an infinite number of independent parameters are necessary to absorb divergences, we see that it is non-renormalizable. However, unitarity considerations show that the theory can be applied consistently when $G_F S << 1$.

The Intermediate Vector Boson (IVB) Model, introduced to cure the pathologies of Fermi theory, incorporated the massive spin 1 fields W^+ and W^- (later discovered to both have mass 82 GeV (KA 87)) with Lagrangean

$$L = \overline{\psi}(i\partial\!\!\!/ - \underset{\sim}{M} - g\gamma^{\mu}(\underset{\sim}{C}W_{\mu} + \underset{\sim}{C}^{T} W_{\mu}^{*})(1 - \gamma_5))\psi - \tfrac{1}{2}W_{\mu\nu}^{*}W^{\mu\nu} + M_W^2 W_{\mu}^{*}W^{\mu} \quad , \tag{34}$$

where $\underset{\sim}{M}$ is the Fermion mass matrix, $\underset{\sim}{C}$ the Cabibbo-Kobayashi-Maskawa matrix, g the weak coupling constant and M_W the mass of the W^-particles. In the IVB model $\sigma \propto {}^1/_S$ for the process $\bar{\nu}_e + e^- \rightarrow \bar{\nu}_e + e^-$ as $S \rightarrow \infty$, which is encouraging; but for the process $W^- + W^+ \rightarrow \nu_e + \bar{\nu}_e$, $\sigma \propto S$ as $S \rightarrow \infty$, which brings back the problem of unitarity. The difficulty with the IVB model lies in the longitudinal polarization which is absent for a massless vector field. Power counting analysis confirms the verdict - the model is non-renormalizable.

Two mutually exclusive exit routes present themselves: (1) We introduce a heavy lepton partner to the electron, E^+, with new coupling strength g': the Georgi-Glashow model; (2) We introduce a weak neutral current, the Z_0: the Glashow-Weinberg-Salam (electroweak) model. The latter model turned out to be correct, and its development will be pursued briefly.

Heisenberg's idea of the mass of the proton and neutron being equal, modulo electromagnetic effects, leads to the notion of isotopic spin ("strong isospin") invariance where the nucleons are grouped in a SU(2) doublet. The corresponding fermion Lagrangean then exhibits global SU(2) symmetry. Yang and Mills said we should reserve the right to define the nucleon independently at each point, i.e. the Lagrangean should be locally SU(2) invariant. However, no massless field could be identified with the new gauge field and introducing a mass by hand for the gauge field amounted to destroying renormalizability. Massless Yang-Mills theory can be shown to be renormalizable by power counting. Once massless quark and lepton fields are introduced into pure Yang-Mills theory, we can write the IVB Lagrangean in a similar form (although not quite). This leads to the suspicion that we can introduce a notion of weak isospin similar to strong isospin where we postulate an underlying gauge group $SU(2)_L$, where the left-handed leptons transform as doublets and the right-handed leptons as singlets of SU(2). Starting from

$$L_0 = \overline{L}_L i\partial\!\!\!/ L_L + \overline{\ell}_R i\partial\!\!\!/ \ell_R \qquad L_L = \begin{pmatrix} \nu_\ell \\ \ell \end{pmatrix}_L \quad , \tag{35}$$

we apply a local $SU(2)_L$ transformation. In analogy to Gell-Mann's eight-fold way we then introduce the notion of weak hypercharge Y by defining $Q = T_3 + {}^{Y}/_{2}$. We note that our new Lagrangean still has a global U(1) invariance, which we associate with Y, and gauge it locally with $U(1)_Y$ to obtain the electroweak Lagrangean. The four gauge fields associated with the $SU(2)_L \times U(1)_Y$ group can then be linearly combined to yield the γ, W^+, W^- and Z_0 gauge bosons, provided

$$g\sin\theta_w = e \quad , \tag{36}$$

where $\theta_W = 0.23 \pm 0.02$ (GL 91) is called the Weinberg angle, and e is the electromagnetic coupling constant.

In order to provide masses to the fermions and gauge bosons in a way that preserves the gauge symmetry responsible for the renormalizability of the theory, we use the idea of spontaneous symmetry breaking (SSB). Goldstone showed that if a continuous global symmetry is broken in a field theory, there are always accompanying massless particles which appear. The Higgs mechanism is then used to give these "Goldstone bosons" masses. Thus we obtain massive electroweak theory, together with the prediction of a yet undetected particle, the Higgs boson.

2.2 THE FEYNMAN GELL-MANN UNIVERSAL V-A THEORY

In 1932 physicists believed that nuclear beta decay was the reaction $n \rightarrow p + e^-$, and since the nuclear recoil energy is negligible, it was predicted that the electron energy must be equal to the difference in mass energy between the proton and the neutron. Experiment showed that the decay electron energy was continuously distributed, and the change in nuclear spin was integral, while electron spin was non-integral. Both energy and angular momentum conservation seemed to be violated. In 1933 Pauli proposed a neutral spin ½ near-vanishing or zero mass particle (called the antineutrino) with no strong interactions, to solve the problem. The antineutrino was first observed by Reines and Coulan in inverse beta decay. Fermi formulated a four-fermion (or current-current) interaction in analogy to QED, for nuclear beta decay ($n \rightarrow p + e^- + \bar{\nu}_e$). The electron-antineutrino pair plays the role of the emitted photon, thus the following replacements are suggested from QED:

$$j_\alpha \rightarrow \bar{\Psi}_p \gamma_\alpha \Psi_n \qquad A^\alpha \rightarrow \bar{\Psi}_e \gamma^\alpha \Psi_\nu \quad , \tag{37}$$

and the charge -e is replaced by the Fermi coupling constant $G/\sqrt{2}$ which was determined by experiment as $G = 1.03 \times 10^{-5}\ m_p^{-2}$ where m_p is the mass of the proton, and the units are $\hbar = c = 1$. We obtain the Fermi beta decay Lagrangean

$$L = \frac{G}{\sqrt{2}} \bar{\Psi}_e(x) \gamma_\lambda \Psi_\nu J_o^\lambda(x) + \text{hermitean conjugate} \quad , \tag{38}$$

where $J_o^\mu(x)$ is the neutron proton current and is of the same form as the electron-antineutrino current, except for corrections due to the strong interaction. Eq. (38) is invariant under proper Lorentz transformations and spatial inversion (parity). Gamow and Teller (GA 36) noted that the same symmetries can be incorporated in a Lagrangean with vector x vector (VV) terms as in eq. (38), and in addition scalar x scalar (SS), tensor x tensor (TT), axial vector x axial vector (AA) and in addition, pseudoscalar x pseudoscalar terms. They showed that for beta decay the following nuclear spin-selection rules are obeyed:

For SS,VV $\Delta J = 0$

For AA,TT $\Delta J = 0, \pm 1$, but $J_i = 0 \nrightarrow J_f = 0$.

Thus at least an AA term was needed to account for $|\Delta J| = 1$ decays. In 1956 Lee and Yang discovered parity violation in that the K^+ - meson was found to decay into both odd and even parity final states. The universal V-A theory was proposed (FE 58) in which only VV and AA and in addition vector x axial (parity violating) terms appear. All the weakly interacting particles were given the same weak charges (thus we say the theory is "universal"), and the Lagrangean was of the form

$$L = \frac{G}{\sqrt{2}}(\overline{\psi}_e(x)\gamma_\lambda(1 - \gamma^5)\psi_\nu)J_o^\lambda(x) + \text{hermitean conjugate} \quad , \tag{39}$$

where $\frac{1}{2}(1 - \gamma^5)$ is the chirality projection operator, and projects out only the negative chirality electrons and positive chirality antineutrinos (since they are massless it means that all antineutrinos are right-handed or have positive helicity). Parity is hence "maximally violated" in the V-A theory. This theory breaks down at energies of the order of 82 GeV (the mass of the W^--particle).

Nuclear beta decay is a strangeness-conserving semileptonic decay, together with related processes like electron capture, and β^+-decay. Amongst the experimental triumphs of beta decay are: (1) The observation of asymmetry in beta emission from polarized Co^{60} by Wu in 1956, confirming parity violation; and (2) The placing of an upper limit on the probability of neutrinoless double beta decay, which confirms the conservation of lepton number. This is because lepton number differentiates between the neutrino and antineutrino so that the process $n \rightarrow p + e^- + \bar{\nu}_e$ and $\nu_e + n \rightarrow p + e^-$ cannot lead to the process $n + n \rightarrow p + p + e^- + e^-$.

2.3 RESTRICTIONS ON THE HADRONIC MATRIX ELEMENTS

From the two baryon momenta P_p and P_n we can construct two vectors $q = P_n - P_p$ and $p = P_p + P_n$. Because $\mathbf{1}$, γ^α, $\sigma^{\alpha\beta} = {}^i/_2\,[\gamma^\alpha, \gamma^\beta]$, $\gamma^\alpha \gamma^5$ and γ^5 from a basis for the vector space of 4 dimensional matrices we can write an arbitrary beta decay hadronic current in the form

$$J_o^\alpha = \cos\theta_c\, \overline{u}_p\big[f_1(q^2)\gamma^\alpha + if_2(q^2)\sigma^{\alpha\nu}q_\nu + f_3(g^2)q^\alpha + g_1(q^2)\gamma^\alpha\gamma^5 - ig_2(q^2)\sigma^{\alpha\nu}\gamma^5 q_\nu + g_3(q^2)\gamma^5 q^\alpha\big]u_n \tag{40}$$

The functions f_1, f_2, f_3, g_1, g_2 and g_3 must be Lorentz scalars, so they must depend on Lorentz scalars,
e.g. $q^2 = M_p^2 + M_n^2 - 2P_p \cdot P_n$, $p^2 = M_p^2 + M_n^2 + 2P_p \cdot P_n$,
$u_p \not{q} u_n = -(M_n + M_p)\, u_p u_n$, $u_p \not{p} u_n = (M_n - M_p)\, u_p u_n$.
We see that all scalars are either constants or related to q^2, so the functions depend only on q^2.

Time Reversal Invariance (T)

The transition rate for a reaction matrix is defined as

$$T_{fi} = \langle f|T|i\rangle = \frac{(2\pi)^4\delta^4(P_f - P_i)M}{\sqrt{\prod_i(2E_iV)\prod_f(2E_fV)}} \quad . \tag{41}$$

The corresponding operator T is hermitian, which implies $\langle f|T|i\rangle^* = \langle i|T|f\rangle$. Hence in M^* we have merely reversed the initial and final states, and not the momenta and spins of M, so M^* is equivalent to M. In the time reversed amplitude M', we also have initial and final states interchanged from those of M. We require time invariance of the theory, i.e. $M' = M^*$ (we use M^* instead of M because of the equivalence previously mentioned). It is well known that under time reversal the Dirac spinor u transforms to $i\gamma^1\gamma^3 u^*$. We find that under time reversal

$$M = \frac{G\cos\theta}{\sqrt{2}}\overline{u}_p\big[f_1\gamma^\alpha + if_2\sigma^{\alpha\nu}q_\nu + g_1\gamma^\alpha\gamma^5 - ig_2\sigma^{\alpha\nu}q_\nu\gamma^5 + g_3 q^\alpha\gamma^5\big]u_n\overline{u}_e\gamma_\alpha(1 - \gamma^5)u_\nu \quad , \tag{42}$$

transforms to

$$M' = \frac{G\cos\theta}{\sqrt{2}}\overline{u}_n\big[f_1\gamma^\alpha - if_2\sigma^{\alpha\nu}q_\nu + f_3 q^\alpha + g_1\gamma^\alpha\gamma^5 - ig_2\sigma^{\alpha\nu}q_\nu\gamma^5 - g_3 q^\alpha\gamma^5\big]u_p\overline{u}_\nu\gamma_\alpha(1 - \gamma^5)u_\nu \quad . \tag{43}$$

However, the complex conjugate of M is

$$M^* = \frac{G\cos\theta}{\sqrt{2}}\overline{u}_n\left[f_1^*\gamma^\alpha - if_2^*\sigma^{\alpha\nu}q_\nu + f_3^*q^\alpha + g_1^*\gamma^\alpha\gamma^5 - ig_2^*\sigma^{\alpha\nu}q_\nu\gamma^5 - g_3^*q^\alpha\gamma^5\right]u_p\overline{u}_\nu(1-\gamma^5)u_e \tag{44}$$

Hence $M^* = M'$ implies that f_1, f_2, f_3, g_1, g_2 and g_3 are real. Note that CP-violation has been found in kaon decay, but the effect is small and can safely be neglected (CP-violation would imply T-violation via the CPT theorem).

Conserved Vector Current Hypothesis (CVC) (FE 58)

The strangeness-conserving beta decay vector hadronic current of eq. (40) is

$$\begin{aligned} J_o^\alpha &= \cos\theta_c\ \overline{u}_p\left[f_1(q^2)\gamma^\alpha + if_2(q^2)\sigma^{\alpha\nu}q_\nu + f_3(q^2)q^\alpha\right]u_n \\ &= \cos\theta_c\ \overline{u}\ \left[f_1(q^2)\gamma^\alpha + if_2(q^2)\sigma^{\alpha\nu}q_\nu + f_3(q^2)q^\alpha\right]\frac{\tau_+}{2}u \end{aligned} \quad , \tag{45}$$

where u is an isospin doublet which contains the proton and neutron Dirac spinors, and $\tau_+ = \tau_x + i\tau_y$, $\tau_- = \tau_x - i\tau_y$, $\tau_3 = \tau_z$; where τ_x, τ_y and τ_z are the Pauli matrices for isospin. Taking the limit as $q^2 \to 0$

$$J_o^\alpha \to \overline{u}\gamma^\alpha f_1(0)\frac{\tau^+}{2}u \equiv g_v\overline{u}\gamma^\alpha\tau^+u \quad , \tag{46}$$

where g_v is called the vector coupling constant for nuclear beta decay. It is shown in (CO 75) that the electromagnetic proton-neutron current can be written with the use of the conservation of electromagnetic current and the idea of isospin, in terms of an isoscalar and an isovector term. It is shown that as $q^2 \to 0$ the isovector term becomes

$$j^\alpha_{EM,\ isov} \to C_p(0)\ \left[\overline{u}\gamma^\alpha\frac{\tau_3}{2}u\right] \quad , \tag{47}$$

where $C_p(0) = 1$. This reflects the fact that the charge of the proton is the same as that of a positron, which has no strong interactions. Experimentally it is found that $g_v \approx 1$, which is the same value as the vector coupling constant of muon decay, where there are no strong interactions. We say that $C_p(0)$ and g_v are unmodified by strong interaction. Eqs. (46) and (47) suggest that $J_o^{\alpha *}$, J_0^α and $j^\alpha_{EM,isov}$ are members of a single isotriplet vector of currents (CVC hypothesis). With this assumption the known conservation of electromagnetic current implies $q_\alpha J_o^\alpha = 0$ if $q^2 \to 0$. Now assuming this also holds if q^2 is larger we see from eq. (45) that this implies $f_3(q^2) = 0$.

G Parity

The strong interaction is invariant under a G-parity transformation (a compounded charge conjugation and rotation of 180° in isospin space around the I_2 axis). If we have no strong interaction, then the vector and axial vector terms in eq. (40) would take the leptonic form

$$\tilde{J}_V^\alpha = \cos\theta_c\, \overline{u}_p\gamma^\alpha u_n \qquad \tilde{J}_A^\alpha = \cos\theta_c\, \overline{u}_p\gamma^\alpha\gamma^5 u_n \quad . \tag{48}$$

As the strong interaction is turned on, the terms become the vector and axial vector terms in eq. (40), respectively. Now, since the strong interaction is G-invariant, the G-transformation properties of $\tilde{J}_V^\alpha$ ($\tilde{J}_A^\alpha$) should be the same as that of the vector (axial) term in eq. (40). We hence require the G-symmetry of terms in f_2 and f_3 to be the same as terms in f_1, and the G-symmetry of terms in g_2 and g_3 to be the same as that of terms in g_1. Under isospin transformations the six terms transfer in the same way, as isovectors (see eq. (45)). Thus the question of G-symmetry reduces to the question of charge conjugation (C) symmetry. As is familiar, under a C-transformation

$$\begin{array}{l} \overline{u}_p u_n \to -\overline{u}_n u_p;\ \overline{u}_p\gamma^5 u_n \to -\overline{u}_n\gamma^5 u_p;\ \overline{u}_p\gamma^\mu\gamma^5 u_n \to -\overline{u}_n\gamma^\mu\gamma^5 u_p; \\ \overline{u}_p\gamma^\mu u_n \to \overline{u}_n\gamma^\mu u_p;\ \overline{u}_p\sigma^{\mu\nu} u_n \to \overline{u}_n\sigma^{\mu\nu} u_p;\ \overline{u}_p\sigma^{\mu\nu}\gamma^5 u_n \to u_n\sigma^{\mu\nu}\gamma^5 u_p \end{array} \quad . \tag{49}$$

Thus terms in f_2 do, but terms in f_3 do not have the same G-symmetry as terms in f_1. Similarly, terms in g_1 and g_3 transform the same way, but terms in g_2 transform oppositely. We conclude that $f_3(q^2)$ and $g_2(q^2)$ are zero.

Goldberger - Treiman Relation

Goldberger *et al* (GO 58) shows how the usual Feynman rules can be used to show that the vertex diagrams contribute to terms proportional to g_1 in eq. (40), and that one-pion exchange diagrams contribute to terms proportional to g_3 (remember $g_2 = 0$ by G-parity). They obtain, after dropping higher-order exchange diagrams, that

$$J_A^\alpha \equiv \cos\theta_c\, \overline{u}_p \left[g_1(q^2)\gamma_\mu\gamma^5 + g_3(q^2)q_\mu\gamma^5\right] u_n$$
$$= \cos\theta_c\, \overline{u}_p \left[g_1(q^2)\gamma_\mu\gamma^5 - \frac{f_\pi g_o\sqrt{2}}{q^2 - M_\pi^2}F(q^2)q_\mu\gamma^5\right] u_n \quad , \tag{50}$$

where f_π is the pion decay constant and g_o is the pion-nucleon coupling constant. (GO 58) assumed that the current of eq. (50) is conserved if $M_\pi \to 0$ (called the partially conserved axial vector current hypothesis, PCAC). We see that

$$q_\alpha J_A^\alpha = 0 \quad as\ M_\pi \to 0 \quad => \quad g_1(q^2) = \frac{f_\pi g_o\sqrt{2}}{M_p + M_n}F(q^2) \quad . \tag{51}$$

Now assume that $F(q^2)$ is varying slowly, and make use of $F(q^2 = M_\pi^2) = 1$. We find that $F(q^2 = 0) = 1$. So

$$g_A \equiv g_1(0) \simeq \frac{f_\pi g_o\sqrt{2}}{M_p + M_n} \quad , \tag{52}$$

which gives a value of g_A agreeing to within 10% of the experimental value of 1.23. Using the definition of g_3 in eq. (50) we obtain by eqs. (50) and (51) that

$$g_3(q^2) \simeq \frac{2g_1(q^2)}{M_\pi^2 - q^2} \quad . \tag{53}$$

From this we see that we do not expect $g_3(q^2)$ to contribute to beta decay since $q^2 \simeq 0$ in beta decay.

2.4 NUCLEAR BETA DECAY

Up until now we have regarded each of the particles participating in neutron beta decay as plane waves with definite spin and momentum and we wrote (compare eq. (41)):

$$T_{fi} = \frac{G_F}{\sqrt{2}}\int\frac{d^4x}{\sqrt{2E_nV2E_pV2E_eV2E_\nu V}}\left[\overline{u}_p e^{iP_p\cdot x}J_o^\mu u_n e^{-iP_n\cdot x}\right] \times\left[\overline{u}_e e^{iP_e\cdot x}\gamma_\mu(1-\gamma^5)u_\nu e^{iP_\nu\cdot x}\right] \quad , \tag{54}$$

$$= \frac{(2\pi)^4\delta^4(P_p+P_e+P_\nu-P_n)M}{\sqrt{2E_nV2E_p2E_eV2E_\nu V}} \quad , \tag{55}$$

$$M = \frac{G_F}{\sqrt{2}}\overline{u}_p J_o^\mu u_n \overline{u}_e\gamma_\mu(1-\gamma^5)u_\nu$$

where J_o^μ is defined in eq. (40). We want to describe the particles participating in beta decay as spatial wavefunctions with energies still well defined. We assume V = 1 (just a convention) and change

$$\frac{1}{\sqrt{2E_y}}u_y e^{iE_yt}e^{-iP_y\cdot x} \to \psi_y(\boldsymbol{x})e^{+iE_yt} \quad , \quad y = n,\ p,\ e,\ \nu \quad . \tag{56}$$

Now assume that $f_3 = 0$ (CVC hypothesis) and define

$$j_\mu(x) = \overline{\psi}_e(\boldsymbol{x})\gamma_\mu(1-\gamma^5)\psi_\nu(\boldsymbol{x}) \quad . \tag{57}$$

We obtain

$$T_{fi} = \frac{G_F\cos\theta_c}{\sqrt{2}}\int dt\int d^3x\overline{\psi}_p(\boldsymbol{x})e^{iE_pt}\Big\{f_1(q^2)\gamma^\mu j_\mu(\boldsymbol{x})e^{iq_ot} + f_2(q^2)\sigma^{\mu\alpha}\frac{\partial}{\partial x^\alpha}\left[j_\mu(\boldsymbol{x})e^{iq_ot}\right] + g_1(q^2)\gamma^\mu\gamma^5 j_\mu(\boldsymbol{x})e^{iq_ot} - g_2(q^2)\sigma^{\mu\alpha}\gamma^5\frac{\partial}{\partial x^\alpha}\left[j_\mu(\boldsymbol{x})e^{iq_o\cdot t}\right] - ig_3(q^2)\gamma^5\frac{\partial}{\partial x_\mu}\left[j_\mu(x)e^{iq_ot}\right]\Big\}\psi_n(\boldsymbol{x})e^{-iE_nt} \quad . \tag{58}$$

This formula may be simplified considerably because the momentum imparted to the leptons is generally very small: The mass difference between the neutron and proton is of the order of 1 MeV/c^2. It is found that the recoil momentum of the neutron is in the keV range, so that it can be neglected. Using the usual decomposition of Dirac spinors, and noticing that $E_n \simeq E_p \simeq M_p \simeq M_n$, we obtain

$$\psi_y = \begin{pmatrix} \chi_y \\ \frac{-i\boldsymbol{\sigma}\cdot\nabla}{2}\chi_y \end{pmatrix} \quad y = p,\ n \quad . \tag{59}$$

The integration in t can be done in eq. (58) to yield

$$T_{fi} = \int dt\ e^{i(E_e + E_\nu - E_n + E_p)} U = 2\pi\delta(E_e + E_\nu - E_n + E_p)\, U \quad ; \tag{60}$$

where

$$U = \frac{G_F}{\sqrt{2}}\cos\theta_c \left[g_V \int \chi_p^+(\boldsymbol{x})\, \tau^+ \chi_n(\boldsymbol{x})\, d^3x j_o(0) + g_A \int \chi_p^+(\boldsymbol{x})\, \boldsymbol{\sigma} \cdot \tau^+ \chi_n(\boldsymbol{x})\, d^3x \boldsymbol{j}(0) \right] \quad . \tag{61}$$

Here we used the fact that the de Broglie wavelength of the lepton in beta decay is typically much larger than the nuclear radius, consequently the lepton wavefunction varies only slightly over the nuclear volume, so we discarded terms containing derivatives of lepton wavefunctions. We also neglected terms proportional to nuclear momenta.

Choosing the lepton wavefunctions as constant over the nuclear volume amounts to making them spherically symmetric, or with orbital angular momentum 0. So the total angular momentum carried off by the electron and antineutrino is just their total spin, which is either 1 or 0. S = 0 corresponds to a Fermi transition, S = 1 corresponds to a Gamow-Teller transition. We obtain the following selection rules for using conservation of angular momentum:

$$\begin{aligned} &\textit{Fermi:} && J_n = J_p \\ &\textit{Gamow-Teller:} && J_n = J_p,\ J_p \pm 1 \qquad J_n = 0 \not\Rightarrow J_p = 0 \end{aligned} \quad , \tag{62}$$

and since the leptons carry off no orbital angular momentum, the initial and final parities of the nucleons must be identical. The Fermi transitions need to be identified with the first term in eq. (61) and the Gamow-Teller transitions with the second term in eq. (61).

Forbidden beta decays do occur in violation of the rules in eq. (62), which must be accounted for in terms of the terms neglected in the derivation of eq. (61) and are generally orders of magnitude smaller.

The PCAC hypothesis and the assumed commutation relation for the chiral changes $[I^5_+, I^5_-] = 2I_3$ (which was derived in (CA 63)) have been employed independently by Adler and Weisberger (AD 65) to derive a numerical value for g_A which is in good agreement with experiment. Numerical integration of pion-nucleon cross-sections yield $g_A = 1.24$, in excellent agreement with the experimental value of $g_A = 1.23 \pm 0.01$.

In this report we attempt to calculate g_A and g_V as observable predictions of cavity QCD (introduced in Chapter 1).

Chapter 3

THE GELL-MANN AND LOW THEOREM

3.1 THE GELL-MANN AND LOW THEOREM

The usual formulation of the Gell-Mann and Low theorem in the interaction picture (Dirac picture) (GE 51) is through the form given in (FE 71), which we shall consider in this introductory section.

Theorem: Denote by

$$U^{\varepsilon}(-\infty,0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{0} dt_1 \ldots \int_{-\infty}^{0} dt_n \times T\left[\hat{H}^{\varepsilon}_{int}(t_1) \ldots \hat{H}^{\varepsilon}_{int}(t_n)\right]_c \quad , \tag{63}$$

the time evolution operator of a system, defined in terms of the adiabatically damped Hamiltonian

$$\hat{H}^{\varepsilon}_{int}(t) \equiv e^{-\varepsilon|t|}\hat{H}_{int}(t) \quad , \tag{64}$$

and the Wick time-ordered product T. If the following quantity exists to all orders in perturbation theory,

$$\frac{|\psi_o\rangle}{\langle\phi_o|\psi_o\rangle} \equiv \underset{\varepsilon\to 0}{Lim} \frac{\hat{U}_{\varepsilon}(O,\ -\infty)\,|\phi_o\rangle}{\langle\phi_o|\hat{U}_{\varepsilon}(O,\ -\infty)\,|\phi_o\rangle} \quad , \tag{65}$$

where the time-evolution operator is acting on the non-interacting ground state of the system, then the quantity in eq. (65) is an eigenstate of $\hat{H}$ (0):

$$\hat{H}(0)\ \frac{|\psi_o\rangle}{\langle\phi_o|\psi_o\rangle} \equiv E\ \frac{|\psi_o\rangle}{\langle\phi_o|\psi_k\rangle} \quad . \tag{66}$$

Outline of Proof: Using the fact that derivatives with respect to time inside a time-ordered product can be taken outside the time-ordered product, and taking care to take the limit as $\varepsilon \to 0$

only at the right time, (such that $\langle\phi_0|\psi_0\rangle$ acquires an infinite phase $^1/_\varepsilon$) we see that

$$(\hat{H}(0) - E)\frac{|\psi_o\rangle}{\langle\phi_o|\psi_o\rangle} = i\hbar\varepsilon g\frac{\partial}{\partial g}\left(\frac{|\psi_o\rangle}{\langle\phi_o|\psi_o\rangle}\right) \quad . \tag{67}$$

By the assumptions of the theorem the right-hand side is well-behaved as $\varepsilon \to 0$, and vanishes.

Multiplying eq. (66) from the left by the non-interacting ground state, we conclude that

$$\Delta E \equiv E - E_0 = \frac{\langle\phi_o|\hat{H}_{int}(0)|\psi_o\rangle}{\langle\phi_o|\psi_o\rangle} \quad , \tag{68}$$

where E is the energy of the interacting state at t = 0 and E_0 is the energy of the non-interacting ground state.

Note that the prescription of eq. (65) generates the eigenstate that develops adiabatically from an eigenstate of $\hat{H}_0$ as the interaction is turned on. We assume that this interacting eigenstate is in fact the interacting ground-state at t = 0, because the interacting ground-state is assumed to have a perturbative expansion in the strong coupling constant. The essential point of the theorem is that the denominator and the numerator in eq. (66) do not separately exist as $\varepsilon \to 0$. The subscript "c" in eq. (63) indicates that we only have to include connected diagrams.

For reasons that will soon emerge, the extension of the Gell-Mann and Low theorem to the case where the non-interacting ground-state is degenerate (ST 87), is especially relevant to the consistent formulation of the theorem in the context of weak beta decay.

3.2 THE FORMULATION OF THE PROBLEM

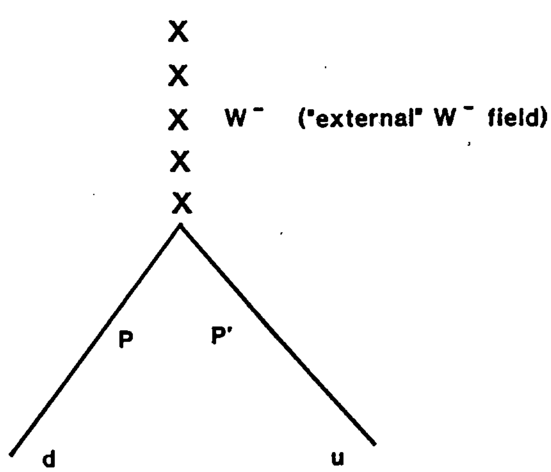
In the nuclear beta decay of the neutron, we have $n \rightarrow p + e^- + \bar{\nu}_e$. In terms of the quark constituents of the nucleon and the Feynman diagrams (in the IVB model) we have at the tree level that a down quark decays into an up quark and a W^-. The W^- then decays into an electron and antineutrino.

Now, any higher order corrections to first order should be in the strong coupling constant α_s, then in the electromagnetic coupling constant α and then in the weak coupling constant $^G/_{\surd 2}$, since $\alpha_s > \alpha > {}^G/_{\surd 2}$. This project concerns itself with first-order corrections in α_s. The usual approach, as outlined in the previous chapter, is to write the transition matrix element in terms of the electron-neutrino current

$$W^+_\mu(\boldsymbol{x}) = \overline{\psi}_{e^-}(\boldsymbol{x})\gamma_\mu(1 - \gamma^5)\psi_{\bar{\nu}_e}(\boldsymbol{x}) \qquad (69)$$

carried by the spin 1 W^- boson (which was first seen at CERN in 1983 with a mass of 82 GeV). The approach is to ignore our interest in the fact that $j_\mu(x)$ is an electron-neutrino current, and simply regard it as an "external" W^--current of adjustable space-time dependence. In fact we shall choose $j_0(x) = W_0$, $j_1(x) = j_2(x) = 0$, $j_3(x) = W_z$; where W_0 and W_z are constants.

We now have the following Feynman diagram:



The non-existence of external W^--fields does not restrict their use as a theoretical and computational tool.

3.3 EXTENSION OF THE GELL-MANN AND LOW THEOREM

If $|N\uparrow\rangle$ and $|P\uparrow\rangle$, respectively, denote the neutron and proton non-interacting ground state SU(6) wavefunctions, then according to the Gell-Mann and Low theorem, following (FE 71), if

$$\lim_{\varepsilon \to 0} \frac{\hat{U}_\varepsilon(O, -\infty)|N\uparrow\rangle}{\langle P\uparrow|\hat{U}_\varepsilon(O, -\infty)|N\uparrow\rangle} \tag{70}$$

exists, then eq. (70) has an eigenvalue E_N of $\hat{H}(0)$. Note that the normalization in eq. (70) is different from that of eq. (65). If we assume

$$\lim_{\varepsilon \to 0} \frac{\hat{U}_\varepsilon(O, -\infty)|P\uparrow\rangle}{\langle N\uparrow|\hat{U}_\varepsilon(O, -\infty)|P\uparrow\rangle} \tag{71}$$

exists, then eq. (71) has an eigenvalue E_P of $\hat{H}(0)$. To obtain an energy difference equation (the equivalent of eq. (68)) we need to assume E_{0_N} and E_{0_P} (the eigenvalues of $\hat{H}_0$ in the states (70) and (71), respectively) are equal (assumption I). This gives

$$\frac{\langle P\uparrow|\hat{H}_{int}(0)\,\hat{U}_\varepsilon(O, -\infty)|N\uparrow\rangle}{\langle P\uparrow|\hat{U}_\varepsilon(O, -\infty)|N\uparrow\rangle} = E_N - E_{O_P} = E_N - E_O \quad , \tag{72}$$

if $E_{0_N} = E_{0_P} = E_0$. In addition, if the energy at t = 0 is independent of whether we start with $|N\rangle$ or $|P\rangle$, (i.e. the degeneracy of the non-interacting ground states is not lifted in some order of perturbation theory) (assumption II) we see that

$$\frac{\langle N\uparrow|\hat{H}_{int}(0)\,\hat{U}_\varepsilon(O,\ -\infty)\,|P\uparrow\rangle}{\langle N\uparrow|\hat{U}_\varepsilon(O,\ -\infty)\,|P\uparrow\rangle} = E_P - E_{O_N} = E_P - E_O = E_N - E_O$$
$$= \frac{\langle P\uparrow|\hat{H}_{int}(0)\,\hat{U}_\varepsilon(O,\ -\infty)\,|N\uparrow\rangle}{\langle P\uparrow|\hat{U}_\varepsilon(O,\ -\infty)\,|N\uparrow\rangle} \quad , \qquad (73)$$

if $E_N = E_P = E$. To be consistent in this extended Gell-Mann and Low theorem we need eq. (73) to hold. This brings us to the justification of the two assumptions.

Assumption I: We show that $E_{0_N} = E_{0_P}$ with the Hamiltonian under consideration, and we show $E_{0_N} = \langle N\uparrow|H_o|N\uparrow\rangle$ is independent of flavour. Note that $H_o = \bar{\psi}\ (-\tfrac{1}{2}\ i\gamma \cdot \overleftrightarrow{\partial})\ \psi$ (BU 88) (neglecting gluon and ghost terms, since they do not contribute to E_{0_N}). Now H_o is symmetric under flavour exchange, since $M_u = M_d = 0$. In the next chapter it will be seen that the matrix elements like $\langle N\uparrow|H_o|N\uparrow\rangle$ contains all reference to colour and flavour in the colour-flavour matrix elements, which will be identical for both the proton and the neutron states if H_o is symmetric under flavour exchange. Hence $\langle N\uparrow|H_o|N\uparrow\rangle = \langle P\uparrow|H_o|P\uparrow\rangle$. Hence $E_{0_N} = E_{0_P}$. A more intuitive way of seeing this is to notice that the mass difference between the proton and neutron is of order α, and hence of electromagnetic origin. We are only concerned with the weak and strong interaction, so this is immaterial.

Assumption II: $E_N = E_P$ is justified in Appendix E, since the colour-flavour matrix elements are identical: $\langle N\uparrow|CF|P\uparrow\rangle = \langle P\uparrow|CF|N\uparrow\rangle$ to first order in $\bar{\alpha}_s$ (the order of the calculation).

At this stage it needs to be noted that although $\hat{H}(0)$ is hermitian, this does not imply that the energy difference $\Delta E = E - E_o$ is real. This is so since, in general

$$\Delta E^* = \frac{\langle P\uparrow|\hat{U}_\varepsilon^+(0,-\infty)\,\hat{H}_{int}(0)\,|N\uparrow\rangle}{\langle P\uparrow|\hat{U}_\varepsilon^+(0,-\infty)\,|N\uparrow\rangle} \neq \frac{\langle P\uparrow|\hat{H}_{int}(0)\,\hat{U}_\varepsilon(0,-\infty)\,|N\uparrow\rangle}{\langle P\uparrow|\hat{U}_\varepsilon(0,-\infty)\,|N\uparrow\rangle} = \Delta E \quad , \tag{74}$$

although equality is attained at the tree level where U = 1. We may hence expect complex contributions to the energy shift in higher order diagrams, which is in fact the case (see Appendix B.2). These complex contributions do not, however, contribute to g_A and g_V to be defined in eqs. (96) and (97).

Stoddart (ST 87) shows that if we deal with degeneracy of the initial unperturbed state, and the degeneracy is lifted in some order of the coupling constant λ, then the coefficient of λ will diverge. The Gell-Mann and Low eigenvector exists only for certain linear combinations of unperturbed states. We demonstrate that if we consider non-diagonal matrix elements of $\hat{H}_{int}(0)$, this problem does not occur.

In order to demonstrate this, we examine the simple case of a doubly degenerate eigenstate of $\hat{H}_o$, $|N\uparrow\rangle = |\phi_1\rangle$, $|P\uparrow\rangle = |\phi_2\rangle$. The energy shift as a power series in λ is given by

$$\begin{aligned}\Delta E &= \lim_{\varepsilon\to 0} \frac{\langle P\uparrow|\lambda\hat{H}_{int}(0)\,\hat{U}^\varepsilon(0,-\infty)\,|N\uparrow\rangle}{\langle P\uparrow|\hat{U}^\varepsilon(0,-\infty)\,|N\uparrow\rangle} \\ &= \lim_{\varepsilon\to 0} \lambda\langle P\uparrow|\hat{H}_{int}(t)\,|N\uparrow\rangle - i\lambda^2\int_{-\infty}^{0} dt e^{-\varepsilon|t|} \\ &\times \{\langle P\uparrow|\hat{H}_{int}(0)\,\hat{H}_{int}(t)\,|N\uparrow\rangle - \langle P\uparrow|\hat{H}_{int}(0)\,|N\uparrow\rangle\langle P\uparrow|\hat{H}_{int}(t)\,|N\uparrow\rangle\}\end{aligned} \tag{75}$$

Using the Heisenberg equation of motion for Dirac operators

$$i\frac{\partial}{\partial t}\hat{F}(\underset{\sim}{x},t) = \left[\hat{F}(\underset{\sim}{x},t)\,,\hat{H}_o\right] \quad , \tag{76}$$

we make the time dependence of $\hat{H}_{int}(t)$ explicit

$$\hat{H}_{int}(t) = e^{i\hat{H}_o t}\hat{H}_{int}(0)e^{-i\hat{H}_o t} \quad . \tag{77}$$

We simplify eq. (75) by inserting a complete set of eigenstates and performing the time integration. We obtain

$$\Delta E = \lambda\langle P\uparrow|\hat{H}_{int}(0)|N\uparrow\rangle - \lambda^2\sum_{i\neq 2}\frac{1}{E_i - E_1}\langle P\uparrow|\hat{H}_{int}(0)|\phi_i\rangle \times \langle\phi_i|\hat{H}_{int}(0)|N\uparrow\rangle \quad , \tag{78}$$

which has a singularity if i = 1, which is cancelled by the fact that $\langle\phi_1|\hat{H}_{int}(0)|N\uparrow\rangle = \langle N\uparrow|\hat{H}_{int}(0)|N\uparrow\rangle = 0$ for the weak interaction Hamiltonian. Hence ΔE does not contain singularities. Hence any linear combination of degenerate initial eigenstates can be used consistently.

3.4 THE HAMILTONIAN AND GROUND-STATE WAVEFUNCTIONS

For the energy shifts up to order Gg^2 we do not have gluon-gluon and ghost-gluon coupling, but only gluon-fermion, W^--fermion and W^+-fermion coupling. So we reduce the cavity $\hat{H}_{int}(t)$ and add the external weak current term to obtain

$$\hat{H}_{int}(t) = -\int d^3x\bar{\psi}(x)\left[g\frac{\lambda_a}{2}\gamma^\mu A_\mu^a(x) + \frac{G}{\sqrt{2}}\gamma^\mu(1-\gamma^5)\frac{\tau^+}{2}W_\mu^+ + \frac{G}{\sqrt{2}}\gamma^\mu(1-\gamma^5)\frac{\tau^-}{2}W_\mu\right]\psi(x) \quad , \tag{79}$$

where W_μ^+ and W_μ is interpreted as constant currents that can be taken out of the integral. From the Cabibbo hypothesis we have $G/\sqrt{2} = G_F/\sqrt{2}\cos\theta_c$, where θ_c is the Cabibbo angle, and experimentally $\theta_c \simeq 15°$ (CO 75). Note, if $\psi_{cf\alpha}$ is acted on by τ^+ or τ^-, we interpret $\psi_{cf\alpha}$ as a vector in flavour space and τ^+ and τ^- as matrices, i.e.

$$\frac{\tau^+}{2} = \frac{1}{2}(\tau_x + i\tau_y) = \frac{1}{2}\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad . \tag{80}$$

$$\frac{\tau^-}{2} = \frac{1}{2}(\tau_x - i\tau_y) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$A_a^{\mu}(x)$ is a vector in 8-dimensional colour space describing the colour potential of the gluons and the λ_a's are the Gell-Mann matrices. It might be thought that for the weak coupling in eq. (79) we need only the term in τ^+, but the term in τ^- is introduced to make $\hat{H}_{int}(t)$ hermitian:

$$\left(\frac{G}{\sqrt{2}}\bar{\psi}(x)\gamma^{\mu}(1-\gamma^5)\frac{\tau^+}{2}W_{\mu}^{+}(x)\psi(x)\right)^{+} = \frac{G}{\sqrt{2}}\psi^{+}(x)W_{\mu}(x)\frac{\tau^-}{2}(1-\gamma^5)\gamma^{\mu+}\gamma_o\psi(x)$$
$$= \frac{G}{\sqrt{2}}\bar{\psi}(x)\gamma^{\mu}(1-\gamma^5)\frac{\tau^-}{2}W_{\mu}(x)\psi(x) \quad , \tag{81}$$

where we used $\gamma_o\gamma^{\mu+}\gamma_o = \gamma^{\mu}$.

Note that

$$W_{\mu}^{+}(x) = \bar{\psi}_{e^-}(x)\ \gamma_{\mu}(1-\gamma^5)\ \psi_{\nu}(x)$$

as mentioned in paragraph 3.2, and taking the hermitian conjugate

$$W_{\mu}(x) = \bar{\psi}_{\nu}(x)\gamma_{\mu}(1-\gamma^5)\ \psi_{e^-}(x) .$$

So the weak part of eq. (79) is of the usual current-current form for the V-A theory

$$\hat{H}_{int} = -\frac{G}{\sqrt{2}}\int d^3x\left(J_{ud}^{\mu}(x)\,J_{\bar{e}\bar{\nu}\mu}^{+}(x) + J_{ud}^{\mu+}(x)\,J_{\bar{e}\bar{\nu}\mu}\right) \quad . \tag{82}$$

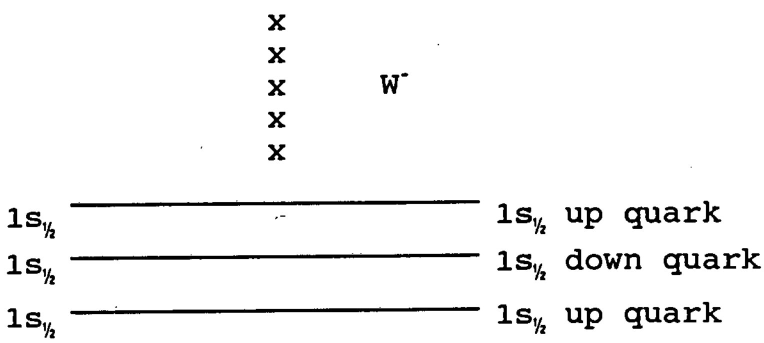
The wavefunctions $|P\uparrow\rangle$ and $|N\uparrow\rangle$ (mentioned in Appendix E.1 in second quantized form) can be constructed out of the cavity modes u_a (HA 84) by using the usual relation between wavefunctions and their second quantized form:

$$\hat{a}_a^+\hat{a}_b^+\hat{a}_c^+|0\rangle \equiv \frac{1}{\sqrt{6}}\sum_{\sigma} sgn\ \sigma\ u_{a\sigma(1)}\ u_{b\sigma(2)}\ u_{c\sigma(3)} \quad , \tag{83}$$

where σ is a permutation on the labels 1, 2, 3 of the quarks. The case with two spin-½ up-quarks in the ground-state does not violate the Pauli Principle due to differing colours. Agreement between Appendix E.1 and (HA 84) is obtained by using the anti-symmetry of the ϵ-tensor.

3.5 THE VECTOR AND AXIAL CURRENT COUPLING CONSTANTS

We obtain independently a relationship between ΔE of eq. (73) and g_V and g_A as defined in (CE 74). To zeroth order in α_s only a single quark interacts with the W^--particle.



Hence to zeroth order in α_s (noting that $|N\uparrow\rangle$ and $|P\uparrow\rangle$ are composed of individual quark wavefunctions), we see that

$$\Delta E = \langle N\uparrow|\hat{H}_{int}(0)|P\uparrow\rangle = -\int d^3x\bar{\psi}(\boldsymbol{x})\frac{G}{\sqrt{2}}\gamma^{\mu}(1-\gamma^5) \times \left(\frac{\tau^+}{2}W_\mu^+(\boldsymbol{x}) + \frac{\tau^-}{2}W_\mu(\boldsymbol{x})\right)\psi(\boldsymbol{x}) + \text{similar terms} \quad , \qquad (84)$$

where only the weak interaction part of $\hat{H}_{int}(0)$ remains, since the strong interaction part does not involve change of isospin. In addition, the $\psi(\mathbf{x})$ is regarded as an isospin doublet of actual cavity wavefunctions. In higher orders, $\psi(\mathbf{x})$ needs to be regarded as an operator, which brings in corrections to ΔE and hence g_A and g_V due to the Gell-Mann and Low theorem. Assuming $W_o(\mathbf{x}) = W_o$, $W_1(\mathbf{x}) = W_2(\mathbf{x}) = 0$, $W_3(\mathbf{x}) = W_z$ with W_o and W_z real constants, and using the representation of γ_3 in Appendix F, we find

$$\Delta E = -\frac{G}{\sqrt{2}}\left(\left(\int d^3x\psi^+(1-\gamma^5)\left(\frac{\tau^+}{2}+\frac{\tau^-}{2}\right)\psi\right)W_o + \left(\int d^3x\psi^+\sigma_3(1-\gamma^5)\left(\frac{\tau^+}{2}+\frac{\tau^-}{2}\right)\psi\right)W_z\right) + \text{similar terms} \quad . \tag{85}$$

Looking at an expansion of eq. (85) in terms of flavours, using the quark wavefunctions in Appendix A and noting n = n′ = $1s_{1/2}$, we obtain

$$\begin{aligned}\int d^3x\psi^+_{fn}(1-\gamma^5)\psi_{f'n'} &= \int d^3x\left\{g_n g_{n'}\chi_\kappa^{+\mu}\chi_{\kappa'}^{\mu'} - if_n g_{n'}\chi_\kappa^{+\mu}\sigma\cdot\hat{r}\chi_{\kappa'}^{\mu'}\right.\\ &\quad \left. + if_{n'}g_n\chi_\kappa^{+\mu}\sigma\cdot\hat{r}\chi_{\kappa'}^{\mu'} + f_n f_{n'}\chi_{\kappa'}^{+\mu}\sigma\cdot\hat{r}\hat{\sigma}\cdot\hat{r}\chi_{\kappa'}^{\mu'}\right\}\\ &= \int d^3x\left\{g_n^2 + f_n^2\right\}\chi_\kappa^{+\mu}\chi_{\kappa'}^{\mu'} = \int d^3x\psi^+_{fn}\psi_{f'n'} \quad ,\end{aligned} \tag{86}$$

where we used

$$\sigma\cdot\hat{\mathbf{r}}\,\sigma\cdot\hat{\mathbf{r}} = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1.$$

In a similar manner, with the addition of $\chi_k^\mu = \sum_S \langle 00\tfrac{1}{2}S|\mu\tfrac{1}{2}\rangle Y_{00}\,\chi_S = {}^1/_{\sqrt{4\pi}}\,\chi_\mu$ for the $1s_{1/2}$ state and $\mu = \mu'$ from momentum conservation, we obtain

$$\begin{aligned}\int d^3x\psi^+_{fn}\sigma_3(1-\gamma^5)\psi_{f'n'} &= \int d^3x\left\{g_n g_{n'}\chi_\kappa^{\mu^+}\sigma_3\chi_{\kappa'}^{\mu'} - if_n g_{n'}\chi_k^{\mu^+}\sigma_3\sigma\cdot\hat{r}\chi_{\kappa'}^{\mu'}\right.\\ &\quad \left. + if_{n'}g_n\chi_\kappa^{\mu^+}\sigma\cdot\hat{r}\sigma_3\chi_{\kappa'}^{\mu'} + f_n f_{n'}\chi_\kappa^{\mu^+}\sigma\cdot\hat{r}\sigma_3\sigma\cdot\hat{r}\chi_{\mu'}^{\kappa'}\right\}\\ &= \int d^3x\left\{g_n g_{n'}\chi_\kappa^{\mu^+}\sigma_3\chi_{\kappa'}^{\mu'} + f_n f_{n'}\chi_{\kappa'}^{\mu^+}\sigma\cdot\hat{r}\sigma_3\sigma\cdot\hat{r}\chi_{\mu'}^{\kappa'}\right\} = \int d^3x\psi^+_{fn}\sigma_3\psi_{f'n'} \quad ,\end{aligned} \tag{87}$$

where

$$\begin{aligned}\left(\chi_\mu^{\kappa^+}\sigma_3\sigma\cdot\hat{r}\chi_{\mu'}^{\kappa'} - \chi_\mu^{\kappa^+}\sigma\cdot\hat{r}\sigma_3\chi_{\mu'}^{\kappa'}\right) &= \left(\sigma_3\chi_\mu^\kappa\right)^+\sigma\cdot\hat{r}\chi_{\mu'}^{\kappa'} - \chi_\mu^{\kappa^+}\sigma\cdot\hat{r}\left(\sigma_3\chi_{\mu'}^{\kappa'}\right)\\ &= \frac{\mu}{\sqrt{4\pi}}\left\{\chi_\mu^{\kappa^+}\sigma\cdot\hat{r}\chi_\mu^\kappa - \chi_\mu^{\kappa^+}\sigma\cdot\hat{r}\chi_\mu^\kappa\right\} = 0 \quad .\end{aligned}$$

Note that in eq. (85) the terms in ΔE containing $^{\tau^+}/_2$ vanish, which can be seen in Appendix E. So

$$\langle N\uparrow|\hat{H}_{int}(0)|P\uparrow\rangle = -\frac{G}{\sqrt{2}}\left(\left(\int d^3x\psi_N^{+}(1-\gamma^5)\frac{\tau^-}{2}\psi_P\right)W_o + \left(\int d^3x\psi_N^{+}\sigma_3(1-\gamma^5)\frac{\tau^-}{2}\psi_P\right)W_z\right) \quad , \tag{88}$$

where ψ_N and ψ_p denotes a quark originating from $|N\uparrow\rangle$ and $|P\uparrow\rangle$ respectively. Now notice that $|N\uparrow\rangle$ can be changed to $|P\uparrow\rangle$ by flipping up and down labels in the wavefunctions, and that either of two cases will hold:

<u>**Case 1**</u>: $\psi_N = \binom{\phi}{0} = {}^{\tau+}/_2 \binom{0}{\phi} = {}^{\tau+}/_2\psi_{P'}$, where $\binom{0}{\phi}$ is associated with the proton since it is a flavour-flip of $\binom{\phi}{0}$. So in flavour-space

$$\begin{aligned}\int d^3x\psi_N\frac{\tau^-}{2}\psi_P &= \int d^3x(\frac{\tau^+}{2}\psi_{P'})^+\frac{\tau^-}{2}\psi_P \\ &= \int d^3x\psi_{P'}^+(\frac{\tau^-}{2})^2\psi_P = 0\end{aligned} \quad . \tag{89}$$

<u>**Case 2**</u>: $\psi_N = \binom{0}{\phi} = {}^{\tau-}/_2 \binom{\phi}{0} = {}^{\tau-}/_2\psi_{P'}$. So in flavour-space

$$\begin{aligned}\int d^3x\psi_N\frac{\tau^-}{2}\psi_P &= \int d^3x\psi_{P'}^+\frac{\tau^+}{2}\frac{\tau^-}{2}\psi_P = \int d^3x\psi_{P'}^+\left(\frac{\tau^-}{2}\frac{\tau^+}{2}+\tau_3\right)\psi_P \\ &= \int d^3x\psi_{P'}^+\tau_3\psi_P\end{aligned} \quad . \tag{90}$$

Since if

$$\psi_P = \binom{\phi}{0} \text{ then } {}^{\tau+}/_2\psi_P = 0, \text{ and if } \psi_P = \binom{0}{\phi}$$

then

$$\psi_{P'}^+ \; {}^{\tau+}/_2 \; {}^{\tau-}/_2 \; \psi_P = 0.$$

So eq. (88) becomes by eqs. (89) and (90)

$$-\frac{G}{\sqrt{2}}\left(\left(\int d^3x\psi_{P'}^+(1-\gamma^5)\tau_3\psi_P\right)W_o + \left(\int d^3x\psi_{P'}^+\sigma_3(1-\gamma^5)\tau_3\psi_P\right)W_z\right) \tag{91}$$

$$+ \text{ similar terms}$$

and by eqs. (86) and (87) the above becomes

$$-\frac{G}{\sqrt{2}}\left(\left(\int d^3x\psi^+_{P'}\tau_3\psi_P\right)W_o + \left(\int d^3x\psi^+_{P'}\sigma_3\tau_3\psi_P\right)W_z\right) \quad , \tag{92}$$
$$+ \text{ similar terms}$$

which is equal to

$$-\frac{G}{\sqrt{2}}\left(\langle P\uparrow|\int d^3x\psi^+\tau_3\psi|P\uparrow\rangle W_o + \langle P\uparrow|\int d^3x\psi^+\sigma_3\tau_3\psi|P\uparrow\rangle W_z\right) \quad . \tag{93}$$

So finally the energy shift can be written

$$\Delta E = \frac{G}{\sqrt{2}}(g_V W_o + g_A W_z) \quad , \tag{94}$$

where we define the weak vector current coupling constant as

$$g_V \equiv -\langle P\uparrow|\int d^3x\psi^+(\boldsymbol{x})\tau_3\psi(\boldsymbol{x})|P\uparrow\rangle \quad , \tag{95}$$

and the weak axial vector current coupling constant as

$$g_A \equiv -\langle P\uparrow|\int d^3x\psi^+(\boldsymbol{x})\sigma_3\tau_3\psi(\boldsymbol{x})|P\uparrow\rangle \quad . \tag{96}$$

We can also obtain estimates for g_V and g_A in other models. In SU(6) (the non-relativistic quark model) we take $|N\uparrow\rangle$ and $|P\uparrow\rangle$ as in section 3.4 and define

$$\frac{g_A}{g_V} = \frac{\langle P\uparrow|\sum_{i=1}^{3}\tau_i^+\sigma_i^z|N\uparrow\rangle}{\langle P\uparrow|\sum_{i=1}^{3}\tau_i^+|N\uparrow\rangle} \quad , \tag{97}$$

with τ_i^+ and σ_i^z operating on the i^{th} quark in $|N\uparrow\rangle$, where $\tau_i^+|u_j\rangle = 0$, $\tau_i^+|d_j\rangle = \delta_{ij}|u_j\rangle$, $\sigma_i^z|\uparrow\rangle = \frac{1}{2}\delta_{ij}|\uparrow\rangle$ and $\sigma_i^z|\downarrow\rangle = -\frac{1}{2}\delta_{ij}|\downarrow\rangle$.

After some tedious algebra (CL 79), we obtain $g_A/g_V = {}^5/_3$. A better estimate can be obtained in zero'th order in cavity QCD. For massless quarks we obtain (CE 77)

$$\frac{g_A}{g_V} = \frac{5}{3}\left(1 - \frac{2x_{1,-1} - 3}{3(x_{1,-1} - 1)}\right) = 1.09 \quad . \qquad (98)$$

This result for relativistic quarks differs because for these quarks $if_n(r)\ \chi^{\mu}_{-\kappa}\ (\hat{r})$ (Appendix A) is not zero, and has opposite spin orientation to $g_n(r)\ \chi^{\mu}_{\kappa}(\hat{r})$. The experimental value of $g_A/g_V = 1.23$.

3.6 THE SUCHER FORMULATION IN CAVITY QCD

There is an equivalent form of the Gell-Mann and Low theorem, due to Sucher (SU 57). The energy shift, given in terms of the dummy variable ξ, is

$$\Delta E \equiv E - E_o = \lim_{\substack{\xi \to 1 \\ \varepsilon \to 0}} \frac{i\varepsilon}{2} \frac{\dfrac{\partial \langle N\uparrow|S_{\varepsilon,\xi}|P\uparrow\rangle_c}{\partial \xi}}{\langle N\uparrow|S_{\varepsilon,\xi}|P\uparrow\rangle_c} \quad . \qquad (99)$$

$S_{\varepsilon,\xi}$ is the adiabatic S-matrix and is closely related to the time evolution operator of eq. (63). Expanding $S_{\varepsilon,\xi}$ in terms of ξ gives

$$S_{\varepsilon,\xi} = 1 + \sum_{n=1}^{\infty} S^{(n)}_{\varepsilon,\xi}$$

$$S^{(n)}_{\varepsilon,\xi} = \frac{(-i\xi)^n}{n!}\int_{\infty}^{-\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n e^{-\varepsilon|t_1|} \ldots e^{-\varepsilon|t_n|} T\left[\hat{H}_{int}(t_1) \ldots \hat{H}_{int}(t_n)\right]$$

(100)

The symmetry of the limits of integration makes it unnecessary in the expansion of eq. (99) to use time-ordered diagrams, which is a luxury. This form is preferred, although time integrations are often more tedious than in the canonical Gell-Mann and Low formalism. Following (OC 90) the energy shift to order Gg^2 can be obtained from eqs. (99), (100) and (79). Taking the limit $\xi \to 1$ we obtain

$$\Delta E = \lim_{\varepsilon \to 0} i\frac{\varepsilon}{2} \Big[\langle S_\varepsilon^{(1)} \rangle_c + 2\langle S_\varepsilon^{(2)} \rangle_c + 3\langle S_\varepsilon^{(3)} \rangle_c - \langle S_\varepsilon^{(1)} \rangle_c^2 + \langle S_\varepsilon^{(1)} \rangle_c^3 - 3\langle S_\varepsilon^{(1)} \rangle_c \langle S_\varepsilon^{(2)} \rangle_c \Big] \qquad (101)$$

Term 1 contains diagrams in G (the zeroth order diagram); term 2 contains diagrams in g^2, Gg, G^2; term 4 contains diagrams in g^2, Gg and G^2; term 5 contains diagrams in G^3. Since we require terms in Gg^2 we get (we shall later deal with the zeroth order diagram separately)

$$\Delta E = \lim_{\varepsilon \to 0} \frac{3i\varepsilon}{2} \Big[\langle S_\varepsilon^{(3)} \rangle_c - \langle S_\varepsilon^{(1)} \rangle_c \langle S_\varepsilon^{(2)} \rangle_c \Big] \quad . \qquad (102)$$

We concentrate on the first term, since the second term comes from the denominator of eq. (99), and is expected to cancel terms coming from the numerator of eq. (99).

Chapter 4

THE FEYNMAN DIAGRAMS

The vertex correction and self-energy graphs contain ultraviolet divergences (MU 87), which need to be regularized in order for the results to be physically meaningful. The only method that respects the generalized Ward-Takahashi identities in Yang-Mills theory is dimensional regularization. This has been employed by (OC 90) to regularize the divergences by using a free space calculation, and using the fact that ultraviolet divergences are a large momentum and thus a short distance phenomenon. The cavity radius is comparatively large at these scales, and you do not expect short distance phenomena to be affected by boundary conditions at the surface. It can thus be argued that the cavity and free space divergences are identical if a similar parameterization scheme is employed. No renormalization is performed, since it is found that the sum of vertex correction and self-energy diagrams lead to a cancellation of divergences. It has also been shown numerically that the gauge-dependent part of the diagrams vanish, so we shall not consider it here. The results in this section bear similarity to (OC 90) and commonly used results will only be quoted.

4.1 WICK THEOREM AND FEYNMAN DIAGRAMS

Referring to (FE 71) the T-product of fermionic operators $\hat{\psi}_\alpha(t,\mathbf{x})$ is defined by

$$T[\psi_\alpha(t,\mathbf{x})\,\psi_\beta(t',\mathbf{x}')] = \begin{cases} \psi_\alpha(t,\mathbf{x})\,\psi_\beta^\dagger(t',\mathbf{x}') & t > t' \\ -\,\psi_\beta^\dagger(t',\mathbf{x}')\,\psi_\alpha(t,\mathbf{x}) & t < t' \end{cases} \quad , \quad (103)$$

which orders the operators at the latest time on the left, and includes an additional factor of -1 for interchange of fermion operators.

With normal ordering all the annihilation operators are placed to the right of all the creation operators, again including a factor of −1 for exchange of fermion operators. If P is the number of permutations of fermion operators needed to rearrange the product $\hat{A}\hat{B}\hat{C}\hat{D}$ ... to $\hat{C}\hat{A}\hat{D}\hat{B}$ ... we have

$$\begin{aligned} T(\hat{A}\hat{B}\hat{C}\hat{D}\ldots) &= (-1)^P\, T(\hat{C}\hat{A}\hat{D}\hat{B}\ldots) \\ N(\hat{A}\hat{B}\hat{C}\hat{D}\ldots) &= (-1)^P\, N(\hat{C}\hat{A}\hat{D}\hat{B}\ldots) \\ T[(\hat{A}+\hat{B})(\hat{C}+\hat{D}\ldots)] &= T(\hat{A}\hat{C}\ldots) + T(\hat{A}\hat{D}\ldots) + T(\hat{B}\hat{C}\ldots) + T(\hat{B}\hat{D}\ldots) + \ldots \\ N[(\hat{A}+\hat{B})(\hat{C}+\hat{D}\ldots)] &= N(\hat{A}\hat{C}\ldots) + N(\hat{A}\hat{D}\ldots) + N(\hat{B}\hat{C}\ldots) + N(\hat{B}\hat{D}\ldots) + \ldots \end{aligned} \qquad (104)$$

A contraction of $\hat{U}$ and $\hat{V}$ is defined

$$\underbrace{\hat{U}\hat{V}} = T(\hat{U}\hat{V}) - N(\hat{U}\hat{V}) \quad . \qquad (105)$$

Most contractions are zero, except that some (the ones between a field $\hat{\psi}_\alpha$ and an adjoint field $\hat{\psi}^+_\beta$) are of the form

$$iG^o_{\alpha\beta}(t,\boldsymbol{x},t',\boldsymbol{x}'), \text{ where}$$

$$iG^o_{\alpha\beta}(t,\boldsymbol{x},t',\boldsymbol{x}') = \frac{\langle\Phi_o|T(\hat{\Psi}_\alpha(t,\boldsymbol{x})\,\hat{\Psi}^+_\beta(t',\boldsymbol{x}'))|\Phi_o\rangle}{\langle\Phi_o|\Phi_o\rangle} \quad , \qquad (106)$$

is the free field Green's function. The contractions are c-numbers in the occupation number Hilbert space.

<u>Theorem</u>: (Wick)

$$\begin{aligned} T(\hat{U}\hat{V}\hat{W}\ldots\hat{X}\hat{Y}\hat{Z}) &= N(\hat{U}\hat{V}\hat{W}\ldots\hat{X}\hat{Y}\hat{Z}) + \\ N(\underbrace{\hat{U}\hat{V}}\hat{W}\ldots\hat{X}\hat{Y}\hat{Z}) &+ N(\underbrace{\hat{U}\hat{V}\hat{W}}\ldots\hat{X}\hat{Y}\hat{Z}) + \ldots + N(\underbrace{\hat{U}\hat{V}\hat{W}\ldots\hat{X}\hat{Y}\hat{Z}}) \\ &= N(\hat{U}\hat{V}\hat{W}\ldots\hat{X}\hat{Y}\hat{Z}) + N\,(\text{sum of all possible pairs of contractions}) \quad . \end{aligned} \qquad (107)$$

<u>Outline of proof:</u> Consider a time ordering, and start moving creation parts to the left within this ordering. Each time a creation part fails to anti-commute, it generates an additional term, which is just a contraction. We may include <u>all</u> possible contractions, since the contraction vanishes if the creation part is already to the left of the destruction part.

Wick's theorem allows us to evaluate the energy shift, and the different terms in the expansion in g and G are associated with a picture called a Feynman diagram according to canonical rules.

The contribution of diagrams in the denominator of eq. (99) exactly cancels the contribution of disconnected diagrams in the numerator (ST 87), so that we only need the expansion into connected diagrams.

Concentrating on the first term in eq. (102), the energy shift is

$$\Delta E = -\lim_{\varepsilon \to 0} \frac{3i\varepsilon}{2} \frac{(-i)^3}{3!} \int d^4x_1 \int d^4x_2 \int d^4x_3 e^{-\varepsilon(|t_1|+|t_2|+|t_3|)}$$
$$\times \langle T\Big[(\bar{\psi}(g\tfrac{\lambda_a}{2}A\!\!\!/_a + \tfrac{G}{\sqrt{2}}(\tfrac{\tau^+}{2}W\!\!\!\!/^+ + \tfrac{\tau^-}{2}W\!\!\!\!/))\psi)_{x_1} (\bar{\psi}(g\tfrac{\lambda_b}{2}A\!\!\!/_b + \tfrac{G}{\sqrt{2}}(\tfrac{\tau^+}{2}W\!\!\!\!/^+ + \tfrac{\tau^-}{2}W\!\!\!\!/)\psi)$$
$$\times (\bar{\psi}(g\tfrac{\lambda_c}{2}A\!\!\!/_c + \tfrac{G}{\sqrt{2}}(\tfrac{\tau^+}{2}W\!\!\!\!/^+ + \tfrac{\tau^-}{2}W\!\!\!\!/)\psi)_{x_3}\Big]\rangle_c \quad . \tag{108}$$

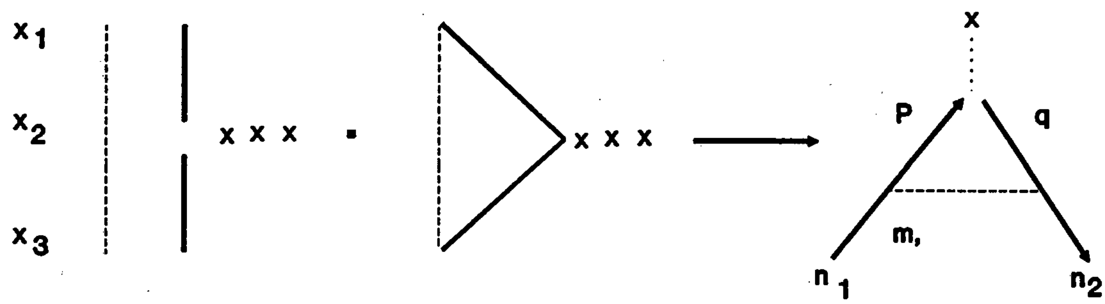
We note that eq. (108) contains three terms of $O(^G/_{\surd 2}g^2)$ which is of interest. It also contains one term of $O(g^3)$, one of $O((^G/_{\surd 2})^3)$ and three of $O((^G/_{\surd 2})^2 g)$. The $O(g^3)$ diagram must be zero since there are no external gluon lines, and the $O((^G/_{\surd 2})^3)$ diagrams are very small to the order in which we are interested, due to the weakness of the W^--interaction.

For the three terms that are of interest we see that they are equal because of symmetry under integration label exchange in eq. (108). So we only consider one such typical term and contract it, requiring that diagrams obtained must be connected to asymptotic three body $|N\uparrow\rangle$ and $|P\uparrow\rangle$ states. For four contractions we obtain vacuum diagrams which are not connected to the asymptotic states. For one contraction we obtain three-body diagrams which do not exist in $O(^G/_{\surd 2}g^2)$, for two contractions we obtain two-body diagrams, and for three contractions we obtain one-body diagrams. For zero contractions the contribution is zero, since there are no external gluon lines.

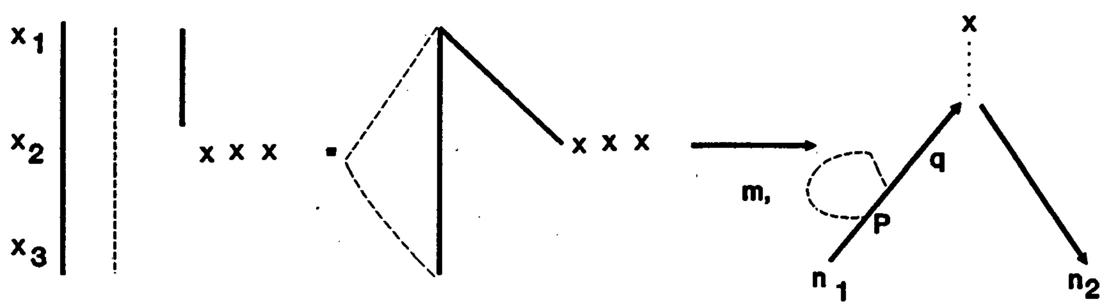
Looking at a typical term in eq. (108), one can heuristically write (note that only A-A, $\bar{\psi}$ - ψ and ψ - $\bar{\psi}$ contractions are non-zero):

$$\langle N[(\bar{\psi}A\psi)_{x_1} (\bar{\psi}W\psi)_{x_2} (\bar{\psi}A\psi)_{x_3}]\rangle \quad ,$$

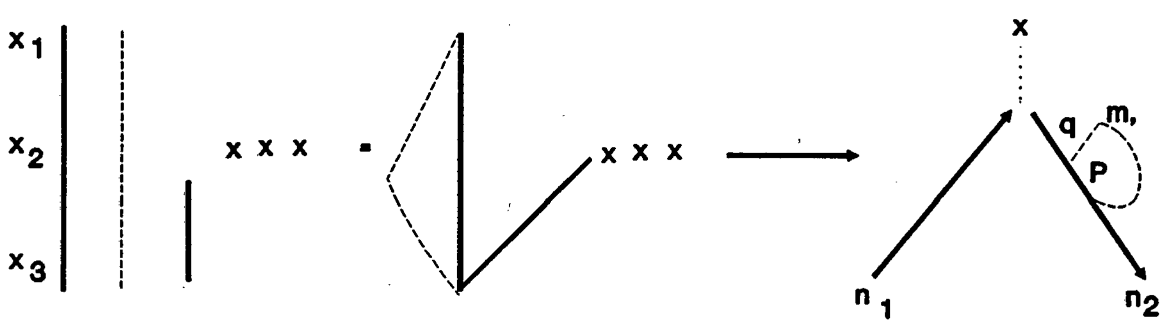
where we shall indicate by stripes the gluon-gluon contractions, and by solid lines the quark-adjoint quark contractions. The type of diagrams obtained are:



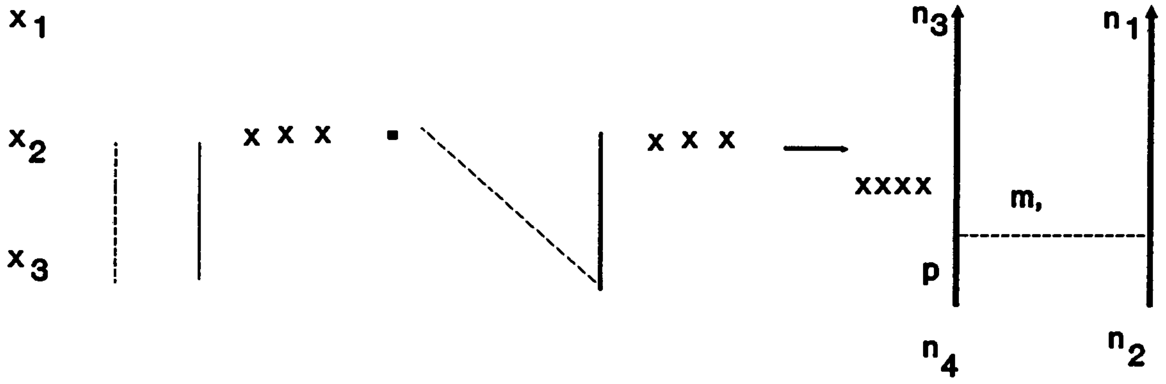
(1a) VERTEX CORRECTION



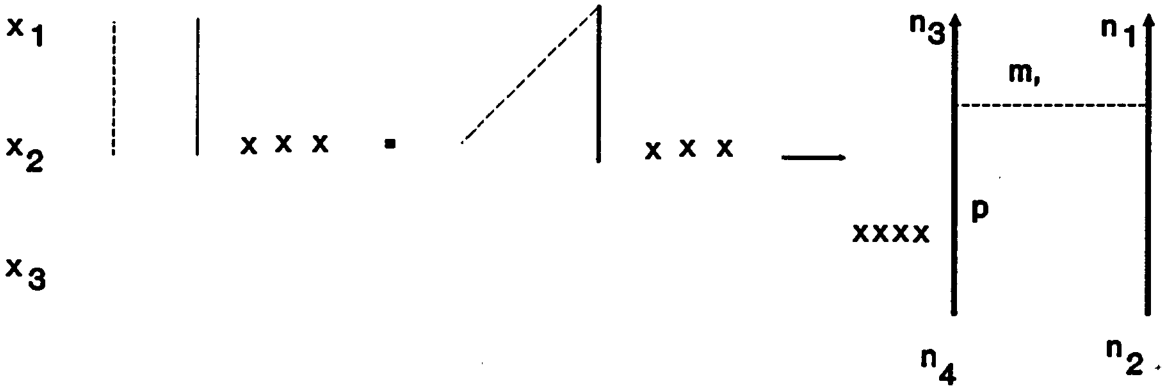
(1b) SELF-ENERGY



(1c) SELF-ENERGY

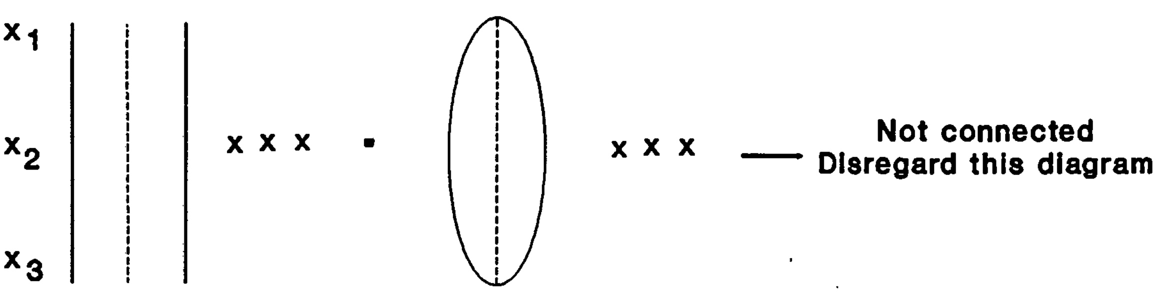


(2a) TWO-BODY

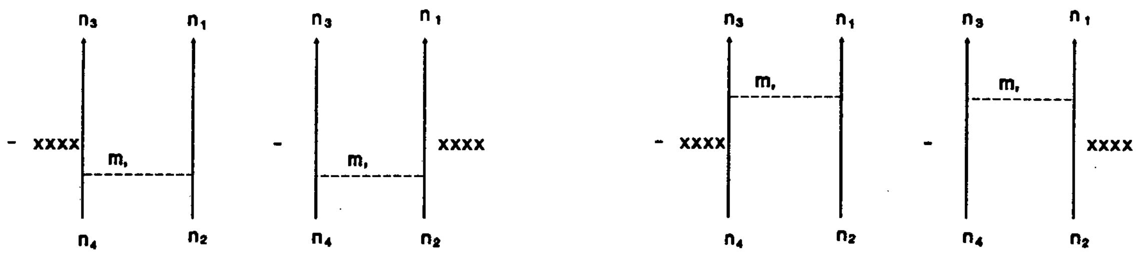


(2b) TWO-BODY

(109)



A more detailed analysis shows that there are actually two diagrams corresponding to each of the above diagrams. This will explicitly be demonstrated for the two-body case. We show explicitly the contraction and corresponding creation and annihilation operations. Note that $\psi_n \bar{\psi}_{n'}$ leads to the free space Green's functions, while $\bar{\psi}_n \psi_{n'}$ leads to **minus** the free space Green's functions. The numbering is merely for identification purposes and Q_{ij} indicates that the external W^--field acts between quarks labelled i and j:



2a(1) 2a(2) 2b(1) 2b(2)

$$2a(1)\quad \langle N[(\overset{1}{\bar\psi}G\overset{2}{\psi})(\bar\psi G\overset{4}{\psi})(\overset{3}{\bar\psi}E\psi)]\rangle \sim \langle N[a_1^+ a_2 Q_{34} a_4 a_3^+]\rangle = -\langle a_1^+ a_3^+ Q_{34} a_2 a_4\rangle$$

$$2a(2)\quad \langle N[(\bar\psi G\overset{2}{\psi})(\overset{3}{\bar\psi}G\overset{4}{\psi})(\overset{1}{\bar\psi}E\psi)]\rangle \sim \langle N[a_2 a_3^+ Q_{12} a_4 a_1^+]\rangle = -\langle a_1^+ a_3^+ Q_{12} a_2 a_4\rangle$$

$$2b(1)\quad \langle N[(\overset{1}{\bar\psi}G\overset{2}{\psi})(\overset{3}{\bar\psi}G\psi)(\bar\psi E\overset{4}{\psi})]\rangle \sim \langle N[a_1^+ a_2 Q_{34} a_3^+ a_4]\rangle = -\langle a_1^+ a_3^+ Q_{34} a_2 a_4\rangle$$

$$2b(2)\quad \langle N[(\overset{1}{\bar\psi}G\psi)(\overset{3}{\bar\psi}G\overset{4}{\psi})(\bar\psi E\overset{2}{\psi})]\rangle \sim \langle N[a_1^+ a_3^+ Q_{12} a_4 a_2]\rangle = -\langle a_1^+ a_3^+ Q_{12} a_2 a_4\rangle \qquad (110)$$

Now if we exchange the labels 1 ↔ 3, 2 ↔ 4 in 2a(2) we obtain 2a(1), and similarly from 2b(2) we obtain 2b(1). Hence diagrams 2a(1) and 2a(2) are identical and this is represented by 2a. Similarly, diagrams 2b(1) and 2b(2) are identical and this is represented by 2b. Hence, in addition to a factor of 3 due to integration label symmetry, there is a factor of 2 here, showing that we must multiply the energy shift due to a diagram by 6.

4.2 THE ZEROTH ORDER DIAGRAM

This is included as a concise summary and explicit demonstration of methods employed in this chapter in a sample case, and also because this diagram does in fact make the dominant contribution to g_A/g_V. Correspondence is obtained with the square-well potential model in eq. (98).

From the Sucher formulation in eqs. (99) and (100), we obtain

$$\Delta E = \lim_{\substack{\xi \to 1 \\ \varepsilon \to 0}} \frac{i\varepsilon}{2} \frac{\frac{\partial}{\partial \xi}\langle i - i\xi \int_{-\infty}^{\infty} dt e^{-\varepsilon|t|} T[\hat{H}_{int}(t)]\rangle_c}{\langle 1 - i\xi \int_{-\infty}^{\infty} dt e^{-\varepsilon|t|} T[\hat{H}_{int}(t)]\rangle_c}$$

$$= \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \langle \int_{-\infty}^{\infty} dt e^{-\varepsilon|t|} T[\hat{H}_{int}(t)]\rangle_c$$

(to the first order in $\frac{G}{\sqrt{2}}$, and zeroth order in g)

$$= \lim_{\varepsilon \to 0} - \frac{\varepsilon}{2}\frac{G}{\sqrt{2}} \int_{-\infty}^{\infty} dt d^3x e^{-\varepsilon|t|} \langle \overline{\psi}(x) \gamma^\mu (1 - \gamma^5) (\frac{\tau^+}{2} W^+_\mu(x) + \frac{\tau^-}{2} W_\mu(x)) \psi(x) \rangle$$

(where this is the only connected contribution; we do not make any contractions)

$$= \lim_{\varepsilon \to 0} \sum_{n,n'} - \frac{\varepsilon}{2}\frac{G}{\sqrt{2}} \int_{-\infty}^{\infty} dt e^{-\varepsilon|t|} e^{i\varepsilon(\varepsilon_n - \varepsilon_{n'})} \langle \hat{a}^+_{cfn} (\frac{\tau^+}{2} + \frac{\tau^-}{2})_{ff'} \hat{a}_{c'f'n'}\rangle$$
$$\times \int d^3x \overline{u}_n(\boldsymbol{x}) \gamma_\mu A^\mu_{ext}(\boldsymbol{x}) u_{n'}(\boldsymbol{x}) \qquad (111)$$

(where we used the expansion of the quark field in eq. (25) and used only the particle creation operators, since $|N\uparrow\rangle$ and $|P\uparrow\rangle$ contain no antiparticle operators).

The colours cannot change due to a W^--current since the current is colourless, so c = c′. Conservation of angular momentum and energy at the external vertex, together with the fact that the energy eigenvalues are discrete in the cavity, means that the radial and angular momentum quantum numbers must be the same on either side of the operator insert. Hence n′ = n. Another way to see this is to note that an explicit evaluation of the colour-flavour matrix element would lead to n′ = n = $1s_{1/2}$.

Since $\varepsilon_n = \varepsilon_{n'}$, the time integral in eq. (111) becomes

$$\int_{-\infty}^{\infty} dt e^{-\varepsilon|t|} = \frac{2}{\varepsilon} \quad , \tag{112}$$

and hence eq. (111) becomes

$$\Delta E = -\frac{G}{\sqrt{2}} \sum_{\mu} \sum_{c} \langle N\uparrow | \hat{a}^{+}_{cdn_1} \hat{a}^{+}_{cun_2} + \hat{a}^{+}_{cun_1} \hat{a}_{cdn_2} | P\uparrow \rangle M_{nn} \quad , \tag{113}$$

where $n = (1s_{1/2}, \mu)$ and we use section E.1 and B.2 to evaluate ΔE to obtain

$$\Delta E = \frac{G}{\sqrt{2}} \langle N\uparrow | CF | P\uparrow \rangle \; M_{nn} = -\frac{G}{\sqrt{2}} \sum_{\mu} \frac{1}{3} \, (\delta_{\mu\downarrow} - 4\delta_{\mu\uparrow})$$
$$\times \left\{ \int drr^2 S_{nn} W_o + \int drr^2 (g_n g_n A^{\mu}_{\kappa\kappa} + f_n f_n D^{\mu}_{\kappa\kappa}) \, W_z \right\} \quad , \tag{114}$$

where $A^{\mu}_{\kappa\kappa}$ and $D^{\mu}_{\kappa\kappa}$ are defined in eqs. B (22) and B (28), and S_{nn} in eq. B (9). Here $\langle N\uparrow | CF | P\uparrow \rangle$ refers to the colour-flavour matrix element in eq. (113) and is explicitly obtained by noting that $\langle N\uparrow | CF | P\uparrow \rangle$ is equal to eq. E (1), with the exception of the factor of $^4/_3$ in eq. E (5). Finally we obtain

$$\Delta E = \frac{G}{\sqrt{2}} \left\{ \int drr^2 S_{nn} W_o + \frac{5}{3} \left(\int drr^2 g_n g_n - \frac{1}{3} \int drr^2 f_n f_n \right) W_z \right\} \quad . \tag{115}$$

4.3 THE VERTEX CORRECTION DIAGRAM

Multiplying by a factor of 6 in eq. (108) and using the contractions for diagram (1a), we obtain

$$\Delta E = -\lim_{\varepsilon \to 0} \frac{3i\varepsilon}{2} (-i)^3 \int d^4x_1 \int d^4x_2 \int d^4x_3 \, e^{-\varepsilon(|t_1|+|t_2|+|t_3|)}$$
$$\times \langle N\Big[(\bar{\psi} g \frac{\lambda_a}{2} \not{A}_a \psi)_{x_1} (\frac{G}{\sqrt{2}} \bar{\psi} \gamma^{\mu} (1-\gamma^5) (\frac{\tau^+}{2} W^+_{\mu} + \frac{\tau^-}{2} W_{\mu}) \psi)_{x_2} (\bar{\psi} g \frac{\lambda_b}{2} \not{A}_b \psi)_{x_3} \Big] \rangle_c \tag{116}$$

Substituting the cavity mode expansion of the fields, and inserting the quark propagator and the gauge-independent part of the gluon propagator, we find that

$$\Delta E = \lim_{\varepsilon \to 0} \frac{-3i\varepsilon}{2} \frac{1}{(2\pi)^3} g^2 \frac{G}{\sqrt{2}} \sum_{\substack{f'fc'c \\ n_1 n_2}} \langle \hat{a}^{+}_{c'f'n_1} \left(\frac{\lambda^a}{2}\right)_{c'd} \left(\frac{\lambda^a}{2}\right)_{dc} \left(\frac{\tau^+}{2} + \frac{\tau^-}{2}\right) \hat{a}_{cfn_2} \rangle$$

$$\times \sum_{pqm\Sigma} g^{\Sigma\Sigma} \tilde{Q}^{m\Sigma}_{n_1 p} M_{pq} Q^{m\Sigma}_{qn_2} \int_{-\infty}^{\infty} dt_1\, dt_2 dt_3 \int_{-\infty}^{\infty} d\omega d\omega' d\omega'' ,$$

$$\times \frac{e^{-\varepsilon(|t_1|+|t_2|+|t_3|)} e^{it_1(\varepsilon_1 + \omega - \omega')} e^{it_2(\omega' - \omega'')} e^{it_3(\omega'' - \omega - \varepsilon_2)}}{(\omega' - \varepsilon_p)(\omega'' - \varepsilon_q)(\omega^2 - \Omega^2)} \tag{117}$$

where we have assumed $W^{+\mu} = W^{\mu}$. $\tilde{Q}^{m\Sigma}_{n_1 p'}$, $Q^{m\Sigma}_{qn_2}$, M_{pq} are defined in Appendix B, and the expectation value of the colour, isospin and creation/annihilation operators are given in Appendix E.

O' Connor (OC 90) performs contour integrals over ω', ω'', t_1, t_2, t_3, Wick rotates $\omega \to i\omega$, $\varepsilon_1 \to i\varepsilon_1$, $\varepsilon_2 \to i\varepsilon_2$; uses the Feynman parameterization, makes a shift of variables, performs a standard Gaussian integral over ω, rotates back to Minkowski space and then performs an integration over parameterization variables x and y to leave a non-integrable singularity in z. Defining $I^{m\Sigma}_{pq}$ as the part of eq. (117) that excludes $g^2\, {}^{G}/_{\sqrt{2}}$, the colour-flavour-matrix element, and the spin sum (defined in Appendix C), he obtains

$$I^{m\Sigma}_{pq} = \int_0^{\infty} dz I^{m\Sigma}_{pq}(z)\, \delta_{\varepsilon_1 \varepsilon_2} \tag{118}$$

$$I^{m\Sigma}_{pq}(z) = \frac{(\varepsilon_1 + \varepsilon_q)^2 - \Omega^2 + 2\varepsilon_1\varepsilon_p}{8\varepsilon_1^2(\varepsilon_q^2 - \varepsilon_p^2)\sqrt{\pi z}} e^{-ze_q^2} - \frac{(\varepsilon_1 + \varepsilon_p)^2 - \Omega^2 + 2\varepsilon_1\varepsilon_q}{8\varepsilon_1^2(\varepsilon_q^2 - \varepsilon_p^2)\sqrt{\pi z}} e^{-ze_p^2} - \frac{e^{-z\Omega^2}}{8\varepsilon_1^2\sqrt{\pi z}}$$

$$+ \frac{(\varepsilon_1^2 + \varepsilon_q^2 - \Omega^2 + 2\varepsilon_1\varepsilon_p)\left[(\varepsilon_1 + \varepsilon_q)^2 - \Omega^2\right]}{16\varepsilon_1^3(\varepsilon_q^2 - \varepsilon_p^2)} \left[e^{-z\Omega^2} \mathrm{nerf}(\sqrt{z}A_+) + e^{-z\varepsilon_q^2} \mathrm{nerf}(\sqrt{z}A_-)\right] ,$$

$$- \frac{(\varepsilon_1^2 + \varepsilon_q^2 - \Omega^2 + 2\varepsilon_1\varepsilon_p)\left[(\varepsilon_1 + \varepsilon_q)^2 - \Omega^2\right]}{16\varepsilon_1^3(\varepsilon_q^2 - \varepsilon_p^2)} \left[e^{-z\Omega^2} \mathrm{nerf}(\sqrt{z}B_+) + e^{-z\varepsilon_q^2} \mathrm{nerf}(\sqrt{z}B_-)\right] \tag{119}$$

where the normalized error function is defined

$$nerf(x) = \frac{2}{\sqrt{\pi}} e^{x^2}\int_0^x e^{-t^2}dt \quad , \tag{120}$$

and the following shorthand has been introduced:

$$\begin{array}{ll} A_+ = (\varepsilon_1^2 + \Omega^2 - \varepsilon_q^2)/2\varepsilon_1 & A_- = (\varepsilon_1^2 - \Omega^2 + \varepsilon_q^2)/2\varepsilon_1 \\ B_+ = (\varepsilon_1^2 + \Omega^2 - \varepsilon_p^2)/2\varepsilon_1 & B_- = (\varepsilon_1^2 - \Omega^2 + \varepsilon_p^2)/2\varepsilon_1 \end{array} \quad . \tag{121}$$

The apparent singularity in eq. (119) when $\varepsilon_p = \varepsilon_q$ can be found to be such that $I^{m\Sigma}{}_{pq}$ is identical to the expression in section 4.4 for that case, so it is not listed here. Note that $\delta_{\varepsilon_1\varepsilon_2}$ limits the incoming and outgoing quark energies to being identical.

4.4 THE SELF-ENERGY INSERT DIAGRAMS

The other two diagrams containing a divergence are the self-energy diagrams (1b) and (1c). After multiplying by 6 in eq. (108) and using the contraction of (1c) the energy shift for diagram (1c) is

$$\Delta E = -\lim_{\varepsilon \to 0} \frac{3i\varepsilon}{2} (-i)^3 \int d^4x_1 \int d^4x_2 \int d^4x_3 e^{-\varepsilon(|t_1|+|t_2|+|t_3|)}$$
$$\times \langle N\Big[(\tfrac{G}{\sqrt{2}}\bar{\psi}\gamma^\mu(1-\gamma^5)(\tfrac{\tau^+}{2}W_\mu^+ + \tfrac{\tau^-}{2}W_\mu)\psi)_{x_1} (\bar{\psi}g\tfrac{\lambda_a}{2}\not{A}_a\psi)_{x_2} (\bar{\psi}g\tfrac{\lambda_b}{2}\not{A}_b\psi)_{x_3}\Big]\rangle_c \quad , \tag{122}$$

which gives for the gauge-independent part:

$$\Delta E = \lim_{\varepsilon \to 0} \frac{-3i\varepsilon}{2} \frac{1}{(2\pi)^3} g^2 \frac{G}{\sqrt{2}} \sum_{\substack{f'fc'c \\ n_1n_2}} \langle \hat{a}^+_{c'f'n_1} (\frac{\lambda^a}{2})_{c'd} (\frac{\lambda^a}{2})_{dc} (\frac{\tau^+}{2} + \frac{\tau^-}{2}) \hat{a}_{cfn_2}\rangle$$
$$\times \sum_{pqm\Sigma} g^{\Sigma\Sigma} M_{n_1q} \tilde{Q}^{m\Sigma}_{qp} Q^{m\Sigma}_{pn_2} \int_{-\infty}^{\infty} dt_1\, dt_2 dt_3 \int_{-\infty}^{\infty} d\omega d\omega' d\omega'' \quad ,$$
$$\times \frac{e^{-\varepsilon(|t_1|+|t_2|+|t_3|)} e^{it_1(\varepsilon_1 - \omega')} e^{it_2(\omega' - \omega'' + \omega)} e^{it_3(\omega'' - \omega - \varepsilon_2)}}{(\omega' - \varepsilon_q)(\omega'' - \varepsilon_p)(\omega^2 - \Omega^2)} \tag{123}$$

where we have assumed $W^{+\mu} = W^\mu$.

O'Connor (OC 90) shows that the case $\varepsilon_q = \varepsilon_2$ leads to a logarithmically divergent expression when inserted into eq. (123), while the case $\varepsilon_q \neq \varepsilon_2$ remains finite. The divergent term is shown to acquire a factor of ½ relative to the finite term, which ultimately leads to the cancellation of divergences when the vertex correction and two self-energy insert diagrams are added. He obtains

$$I_{pq}^{m\Sigma}(z) = \begin{cases} \int_0^\infty dz K_{pq}^{m\Sigma}(z)\,\delta_{\varepsilon_1\varepsilon_2} \text{ if } \varepsilon_q \neq \varepsilon_2 \\ \int_0^\infty dz\; L_p^{m\Sigma}(z)\,\delta_{\varepsilon_1\varepsilon_2} \text{ if } \varepsilon_q = \varepsilon_2 \end{cases}$$

$$K_{pq}^{m\Sigma}(z) = \frac{1}{4\varepsilon_1(\varepsilon_2 - \varepsilon_q)\sqrt{\pi z}}\left(e^{-z\varepsilon_p^2} - e^{-z\Omega^2}\right)$$
$$+ \frac{(\varepsilon_p + \varepsilon_1)^2 - \Omega^2}{8\varepsilon_1^2(\varepsilon_2 - \varepsilon_q)}\left\{e^{zB_+^2 - z\Omega^2}\mathrm{erf}(\sqrt{z}B_+) + e^{zB_-^2 - z\varepsilon_p^2}\mathrm{erf}(\sqrt{z}B_-)\right\}$$

$$+ 2L_p^{m\Sigma}(z) = \frac{1}{16\varepsilon_1^4\sqrt{\pi z}}\left[(\varepsilon_1 + \varepsilon_p + \Omega)^2(\varepsilon_1 + \varepsilon_p - \Omega)^2 z + 2\varepsilon_1^2\right]e^{-z\Omega^2}$$
$$+ \frac{1}{16\varepsilon_1^4\sqrt{\pi z}}\left[\left((\varepsilon_1 + \varepsilon_p)^2 - \Omega^2\right)\left((\varepsilon_1^2 - 2\varepsilon_1\varepsilon_p - \varepsilon_p^2 + \Omega^2)z + 4\varepsilon_p\varepsilon_1^3 z - 2\varepsilon_1^2\right]e^{-z\varepsilon_p^2}$$
$$+ \frac{1}{32\varepsilon_1^5}\left[(\varepsilon_1^2 - \varepsilon_p^2 + \Omega^2)\left((\varepsilon_1 + \varepsilon_p)^2 - \Omega^2\right)z - 4\varepsilon_1^2\right]\left((\varepsilon_1 + \varepsilon_p)^2 - \Omega^2\right)$$
$$\times\left[e^{-z\Omega^2}\mathrm{nerf}(\sqrt{z}B_+) + e^{-z\varepsilon_p^2}\mathrm{nerf}(\sqrt{z}B_-)\right] \quad , \tag{124}$$

where $I_{pq}^{m\Sigma}$ is the part of eq. (123) excluding $g^2\, {}^G/_{2}$, the colour-flavour matrix element, and the spin sum. B_+ and B_- are defined in eq. (121) and the possibility $\varepsilon_q = \varepsilon_2$ is, as promised in section 4.3, actually a special case of eq. (124) for the vertex correction diagram. Once again $\delta_{\varepsilon_1\varepsilon_2}$ limits the incoming and outgoing quark energies to being identical. The energy shift for diagram (1c) turns out to be identical to that of diagram (1b). The latter will be incorporated by multiplying eq. (124) by two.

An intuitive argument to support that the energy shift for (1b) equals that for (1c) can be given. We note that the expression for ΔE (due to diagram (1c)) contains $\delta_{\varepsilon_1\varepsilon_2}$, and since in the spin sum we have $\delta_{\mu_1\mu_2}$, we conclude that the quantum numbers of the incoming and outgoing quarks are identical. Hence if we shift labels 1 $\leftrightarrow$ 2 externally, the energy shift should be the same. But this is just the energy shift due to (1b).

4.5 THE ONE-GLUON EXCHANGE DIAGRAMS

Multiplying by a factor of 6 in eq. (108) and using the contractions of diagram (2b), we obtain

$$\Delta E = -\lim_{\varepsilon \to 0} \frac{3i\varepsilon}{2} g^2 (-i)^3 \int d^4x_1 \int d^4x_2 \int d^4x_3 e^{-\varepsilon(|t_1|+|t_2|+|t_3|)}$$
$$\times \langle N\Big[(\overline{\psi}\tfrac{\lambda_a}{2}\not{A}_a\psi)_{x_1} (\overline{\psi}\tfrac{\lambda_b}{2}\not{A}_b\psi)_{x_2} (\tfrac{G}{\sqrt{2}}\overline{\psi}\gamma^\mu(1-\gamma^5)(\tfrac{\tau^+}{2}W_\mu^+ + \tfrac{\tau^-}{2}W_\mu)\psi)_{x_3}\Big]\rangle_c \tag{125}$$

After substituting the quark and gluon propagators, the gauge-independent part becomes

$$\Delta E = \lim_{\varepsilon \to 0} \frac{3i\varepsilon}{2} g^2 \frac{G}{\sqrt{2}} \sum_{\substack{c'd'cd \\ f'g'fg \\ n_1n_2n_3n_4}} \langle \hat{a}^+_{c'f'n_1} \hat{a}^+_{d'g'n_3} (\tfrac{\lambda^a}{2})_{c'c} (\tfrac{\lambda^a}{2})_{d'd} (\tfrac{\tau^+}{2} + \tfrac{\tau^-}{2}) \hat{a}_{cfn_2} \hat{a}_{dgn_4} \rangle$$
$$\times \sum_{pm\Sigma} g^{\Sigma\Sigma} \tilde{Q}^{m\Sigma}_{n_1n_2} Q^{m\Sigma}_{n_3p} M_{pn_4} \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \frac{d\omega'}{(2\pi)}$$
$$\times \frac{e^{-\varepsilon(|t_1|+|t_2|+|t_3|)} e^{it_1(\varepsilon_1 - \varepsilon_2 + \omega)} e^{it_2(\varepsilon_3 - \omega - \omega')} e^{it_3(\omega' - \varepsilon_4)}}{(\omega' - \varepsilon_p)(\omega^2 - \Omega^2)}, \tag{126}$$

where we used $W^{+\mu} = W^\mu$. Note that the sign of eq. (126) is opposite to that of the vertex correction in eq. (117) and the self-energy in eq. (123). This has its origin in the negative sign in eq. (110).

If we make $I_p^{m\Sigma}$ equal to the part of eq. (126) excluding $g^2\ {}^{G}/_{12}$, the colour-flavour matrix element and the spin sum, (OC 90) obtains

$$I_p^{m\Sigma} = \begin{cases} \dfrac{1}{(\varepsilon_4 - \varepsilon_p)\left[(\varepsilon_1 - \varepsilon_2)^2 - \Omega^2\right]} & \varepsilon_p \neq \varepsilon_4 \\ 0 & \varepsilon_p = \varepsilon_4 \end{cases} \quad . \qquad (127)$$

The energy shift for diagram (2b) turns out to be identical to that of diagram (2a). The latter will be incorporated by multiplying eq. (127) by two.

Chapter 5

NUMERICAL METHODS

5.1 PREPARATIONS FOR NUMERICAL CALCULATION

The expressions for the energy shift in Chapter 4 are now ready for numerical calculation. We briefly indicate the numerical methods connected to the vertex correction shift in eqs. (117) and (118). We calculate the colour-flavour matrix elements of eq. (117) in Appendix F (call it <CF>). We note that the sum over flavours and colours is done in <CF> since it does not occur in any other term in eq. (117). The sum over n_1 and n_2 in eq. (117) simplifies to a sum over μ, since <CF> restricts both the external quarks to be in the $1s_{1/2}$ state, and restricts $\mu_1 = \mu_2 \equiv \mu$.

After summing in eq. (117) over the spins of the intermediate gluons and quarks (done in section C.1), we obtain

$$\begin{aligned} \Delta E &= g^2 \frac{G}{\sqrt{2}} \sum_{\substack{f'fc'c \\ n_1 n_2}} \langle CF\rangle \int_0^\infty dz \sum_{pqM\Sigma} g^{\Sigma\Sigma} \tilde{Q}^{m\Sigma}_{n_1 p} M_{pq} Q^{m\Sigma}_{n_1 p} I^{m\Sigma}_{pq}(z) \\ &= \alpha_S \frac{G}{\sqrt{2}} \sum_{\mu} \sum_{\Sigma} g^{\Sigma\Sigma} \sum_{\kappa_p \kappa_q \nu_p \nu_q} \langle CF\rangle \Lambda^{m\Sigma}_{pq} \int_0^\infty dz I^{m\Sigma}_{pq}(z) \\ &\equiv \int_0^\infty dz \Delta E(z) \end{aligned} \qquad (128)$$

$$\Lambda^{m\Sigma}_{pq} \equiv 4\pi \sum_{\mu_p \mu_q M} \tilde{Q}^{m\Sigma}_{np} M_{pq} Q^{m\Sigma}_{np} \quad .$$

$\Lambda^{m\Sigma}_{pq}$ can be found explicitly in eqn. C (8) (where we use $j_p = |\kappa_p| - \frac{1}{2}$, $j_q = |\kappa_q| - \frac{1}{2}$). m, p, q are understood not to include M, μ_p, μ_q in eq. (128). The sum over μ is carried out externally ($\mu = \pm \frac{1}{2}$, since $j_1 = j_2 = \frac{1}{2}$). Only <CF> and $\Lambda^{m\Sigma}_{pq}$ in eq. (128) has an explicit dependence on μ. $I^{m\Sigma}_{pq}$ (z) does not depend on M, μ_p, μ_q and has been given in eq. (119). We understand that n_1 and n_2 in $\Lambda^{m\Sigma}_{pq}$

are chosen such that $j_1 = j_2 = \frac{1}{2}$ and $\epsilon_1 = \epsilon_2 = 2.04$, which implies that $\kappa_1 = \kappa_2 = -1$. This again implies that $\ell_1 = \ell_2 = 0$ and $\bar{\ell}_1 = \bar{\ell}_2 = 1$, by eqs. A (7) and A (8). The sums over ν_p and ν_q run over all non-zero integers: negative integers corresponding to negative energies and positive integers corresponding to positive energies. The spin sum in eq. C (8) restricts $\Lambda^{m\Sigma}_{pq}$ to be non-zero if $|\kappa_q| = |\kappa_p|$ for terms proportional to W_o, and $|\kappa_q| = |\kappa_p| \pm 1$ for terms proportional to W_z. Hence in eq. (128) we remain with an infinite sum in 5 dimensions: κ_p, ν_p, ν_q, N, J. Here $\nu_p, \nu_q = \ldots -3, -2, -1, 1, 2, 3, \ldots$; $N = 0, 1, 2, 3 \ldots$; $\kappa_p = \ldots -3, -2, -1, 1, 2, 3, \ldots$; and $J = 0, 1, 2 \ldots$ (where $J = 0$ is only allowed for the scalar and longitudinal gluon modes).

The infinite sum over five dimensions must be truncated at some point, and this is done by introducing an energy cutoff E_{max} such that all energies satisfy $|\epsilon_p|$, $|\epsilon_q|$, $|\Omega^{\Sigma}_m| \leq E_{max}$ in order to be included in the sum. This condition is chosen since the quark and gluon propagators become elevated to exponentials (as in eqs. (117) and (119)) via the regularization technique, and these exponentials become very small for high energies. For this calculation a cutoff of $E_{max} = 50$ is used. Higher values of E_{max} lead to a power law increase in the number of terms to be included in the sum, and hence in computing time.

The calculation of the quark energies are done by assuming the masses of the up and down quarks to be zero, as has been consistently done in section A.1. We then solve eq. A (9) to obtain typical values (CE 74) of $x_{\nu\kappa}$, e.g.

for $\kappa = -1$ $x_{1\,-1} = 2.04$, $x_{2\,-1} = 5.40$, $x_{-1\,-1} = -2.04$, $x_{-2\,-1} = -5.40.$,

for $\kappa = 1$ $x_{1\,1} = 3.81$, $x_{2\,1} = 7.00$, $x_{-1\,1} = -3.81$, $x_{-2\,1} = -7.00.$,

where ν classifies the order of the solution. For the gluon energies we do not have any negative energy solutions of eq. A (20).

We can check the accuracy of eq. (128) as a whole by using the sum rule (Appendix D). In eq. (128) we also perform the truncated sum before the integral over z is performed, in this way we obtain a smooth curve $\Delta E(z)$, a parametric representation of the energy shift ΔE.

We know from previous discussion that ΔE is singular, and that the functional form of this non-integrable singularity in the cavity should be exactly the same as the form derived in free space, which is derived here in a similar way to the derivation by (OC 90) of a related quantity, and is given by

$$\Delta E_S = -\frac{\alpha_S}{4\pi}\frac{G}{\sqrt{2}}M_{n_1n_2}\int_0^\infty dz\frac{e^{-z}}{z} = -2\frac{\alpha_S}{4\pi}\frac{G}{\sqrt{2}}M_{n_1n_2}\int_0^\infty dy\frac{e^{-y^2}}{y} \equiv \int_0^\infty dy\,\Delta E_S(y) \quad . \tag{129}$$

The step shown above is to shift the variable from z to y^2. The reason for this is that $I^{m\Sigma}_{pq}$ (z) in eq. (118) has an integrable singularity of the form $^1/_{\sqrt{z}}e^{-\eta z}$. Hence this term gives a finite contribution to ΔE, but if $\Delta E(z)$ is plotted it would still appear divergent near $z = 0$. Making the variable shift z to y^2 we find

$$\int_0^\infty \frac{1}{\sqrt{z}}e^{-\eta z}dz = \int_0^\infty 2e^{-\eta y^2}dy \quad , \tag{130}$$

and so $\Delta E(y)$ is regular at the origin $y = 0$, for the integrable singularity in ΔE.

We can now calculate $\Delta E(y)$ by making the shift $z \to y^2$ in eq. (128) such that $\Delta E \equiv \int_0^\infty dy\ \Delta E(y)$, and subtract from it $\Delta E_S(y)$. We obtain a regular function of y, which behaves regularly at the origin. In other words we have regularized $\Delta E(y)$ by subtracting from it the divergent contribution $\Delta E_S(y)$. Integrating the result over y this gives a finite contribution to the vertex correction. It is found that $\Delta E(y)$ shows discontinuous behaviour at very small y, which can be attributed to the truncation of the infinite sum over cavity modes at some energy cutoff E_{max}. We note from eq. (118) that the dominant contribution at high energies in typical terms like $e^{-\eta y^2}$ is when y is small (η is the square of an energy). Hence at small y we expect the discrepancy due to neglected high energies to be maximal. The part showing discontinuous behaviour can be neglected and replaced by an extrapolation based on the other known points.

5.2 NUMERICAL ROUTINES

Bessel Functions: In order to evaluate $j_\ell(x)$, it is found that the best balance between accuracy and speed is obtained by applying three different techniques in different regions of the (ℓ, x) space:

(1) $35 < x < \ell$: Due to inaccuracies with the other two techniques, a slower reverse recursion routine (GI 88) is utilized.

(2) $x > 0.32\ \ell^{1.25} + 3$: Forward recursion,

$$j_{\ell+1}(x) = \frac{2\ell}{x} j_\ell(x) - j_{\ell-1}(x) \quad .$$

(3) $x \leq 0.32\ \ell^{1.25} + 3$: Series expansion,

$$j_\ell(x) = \left(\tfrac{1}{2}x\right)^\ell \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{k!\Gamma(\ell + k + 1)} \quad .$$

Error Function: We define $\mathrm{erf}(x) = \int_0^x e^{-t^2}\, dt$ and use the cases:

(1) $0 \leq x < \sqrt{8}$: Series development,

$$erf(x) = \frac{e^{-x^2}x}{2} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2} + n)} x^{2n} \quad .$$

(2) $x \geq \sqrt{8}$: Continued fraction development,

$$erf(x) = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}x}{2} \left(\frac{1}{x^2+} \frac{\frac{1}{2}}{1+} \frac{1}{x^2+} \frac{\frac{3}{2}}{1+} \frac{2}{x^2+} \cdots \right) \quad .$$

(3) $x < 0$: $\mathrm{erf}(x) = -\mathrm{erf}(-x)$.

It is found that if energies are large in modulus, terms like $e^{-z\Omega^2}$ nerf $(\sqrt{z}\ A_+)$ + $e^{-z\epsilon_q^2}$ nerf $(\sqrt{z}\ A_-)$ in eq. (118) become very sensitive to the value of nerf$(\sqrt{z}\ A_+)$ or nerf$(\sqrt{z}\ A_-)$, which are then both 1 to the 16-digit accuracy available in double precision FORTRAN 77. We define serf(x) = $x^2 \int_x^\infty e^{-t^2}\,dt$, which has a smooth behaviour as $x \to \infty$, and does not die right away to zero for the large x used. We rewrite eq. (118) in terms of serf(x) and thus prohibit numerical subtraction errors. A continued fraction development is used, and serf(x) is used instead of erf(x) if $x > 2$.

Eigenenergies: The secant method of root finding is used, since the method assumes the approximate linearity of the function near the root, which is obeyed by Bessel functions. Under these conditions it is faster than the bisection method.

Integration: Gaussian quadratures have freedom for both the weights and abscissas, and hence the order of accuracy is twice that of the Newton-Cotes formula for the same number of points. More specifically we use Gauss-Legendre Quadrature. The radial integrals in eqs. B (5) to B (8) and B (30) are found to be strongly oscillatory for $\nu + \nu' \geq 18$. In this case we use 50 integration points. For other values of ν and ν', 30 integration points are found adequate to produce an accuracy of 11 digits. The integral $\Delta E = \int_0^\infty \Delta E(y)$ is also done with the above integration method, where the integral is done for $y \in [0,2]$ since $\Delta E(y) \sim 0$ for $y \in [2,\infty]$.

Main Program: The quark and gluon energies are stored on disk, and loaded in two arrays. The summation in order from the outer sum to the most inner sum is Σ, κ_p, κ_q, J, ν_p, ν_q, N. The radial quantum numbers are most internally summed over since they are not mentioned in the 3j- and 6j-symbols in the spin sum, which allows CPU time to be more effectively used. Note that for each of the above set of quantum numbers we calculate $\Delta E(y)$ for all values of y required by the integration routine, and since only $I_{pq}^{m\Sigma}(z)$ contains z we do not have to calculate $\Lambda_{pq}^{m\Sigma}$ for each z, incurring significant time savings. We have, however, not gone out of our way to prohibit multiple calculations of similar quantities.

Interpolation Polynomial: As mentioned in section 5.1, we need to extrapolate $\Delta E(y)$ in the region where y is very small. Polynomial interpolation is found adequate (fitting a second degree polynomial to the nearest 2 points) since ΔE has no poles (so rational function extrapolation is unnecessary) and we do not need the derivative of $\Delta E(y)$ to be smooth (so cubic spline extrapolation is unnecessary).

Chapter 6

RESULTS

6.1 CALCULATION

Note that the complex contributions to the energy shift in Appendix C are omitted in this calculation, since our method of defining g_V and g_A in eqs. (95) and (96) used a tree level approximation, in which these contributions vanish. This is not a loss since only leading order form factors can currently be deduced from experiment (TS 88).

For the vertex correction diagram, programming time is saved by noting that eq. C (8) gives the restrictions

$$|J - \tfrac{1}{2}| \le j_p \le J + \tfrac{1}{2}, \quad |J - \tfrac{1}{2}| \le j_q \le J + \tfrac{1}{2}, \text{ and}$$

$\kappa_p = \kappa_q$ or $\ell_p = \ell_q$ or $\bar{\ell}_p = \bar{\ell}_q$, as well as parity restrictions in the quark-gluon vertex integrals. The y-form $\Delta E(y)$ obtained in Fig. 6.1 and Fig. 6.3 can now be computed and compared with the singular function $\Delta E_S(y)$. The initial and final quarks were in a spin up state, and a energy cutoff of $E_{max} = 50$ was employed. The spin sum routine was called 2073313 times, and the calculation took 32 hours CPU time on an Apollo DN 5500 computer. Fig. 6.2 and Fig. 6.4 shows the regular function that results by subtracting from $\Delta E(y)$ the singular function $\Delta E_S(y)$. Integrating this regular function gives a finite contribution to the vertex correction. At the points $y \le 0.06$ we see sudden fluctuations that destroy the apparent continuity of $\Delta E(y) - \Delta E_S(y)$. These points are not plotted in the figures on the following pages, and were discarded in favour of the polynomial extrapolation mentioned in Chapter 5. Inaccuracies in the integration of the extrapolated regular energy shift were found to be sensitive to the type of extrapolation polynomial used, and the fact that integration was only performed in the interval $y\in[0,2]$ and not in the interval $y\in[0,\infty]$. Inaccuracies were at the level of the

Figure 6.1: Divergent vertex correction energy shift for GV

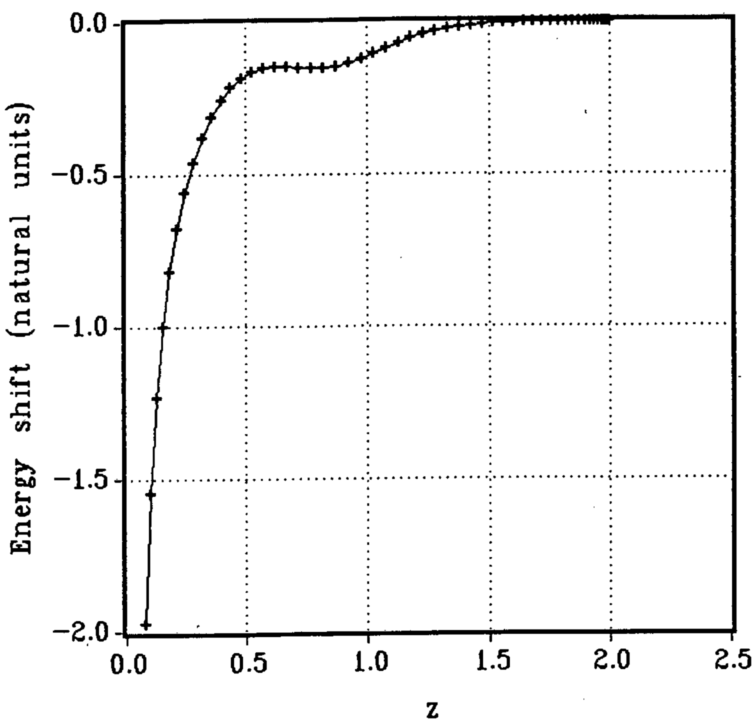


Figure 6.2: Regular vertex correction energy shift for GV

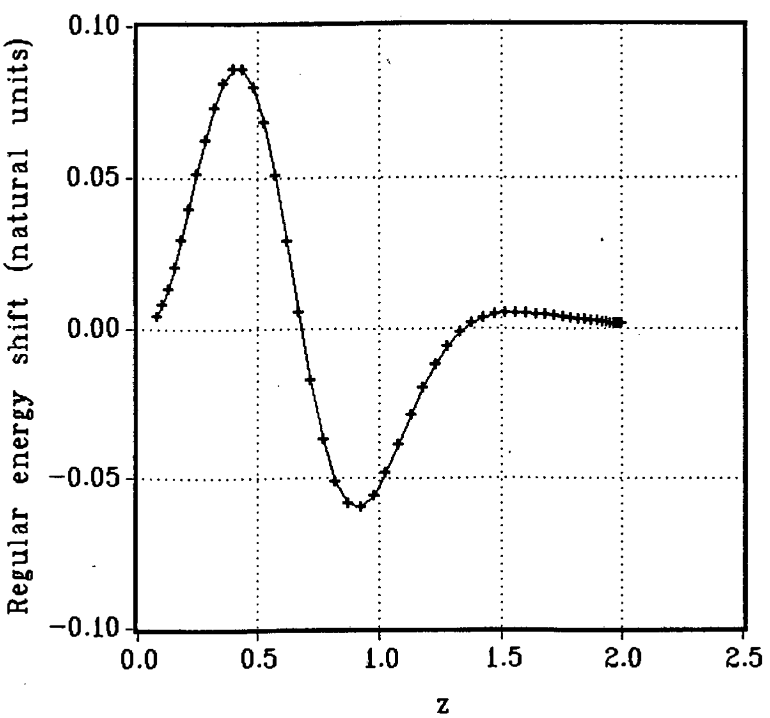


Figure 6.3: Divergent vertex correction energy shift for GA

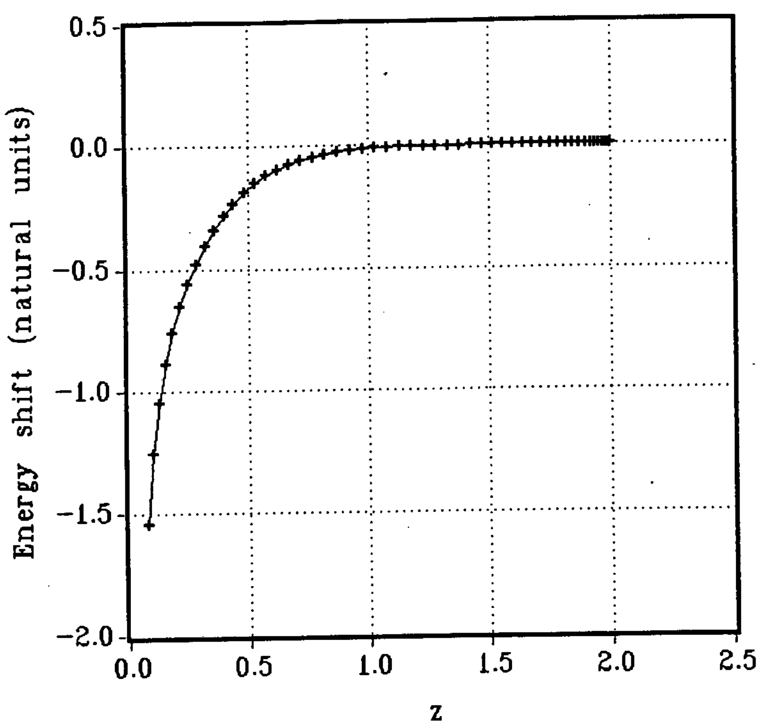
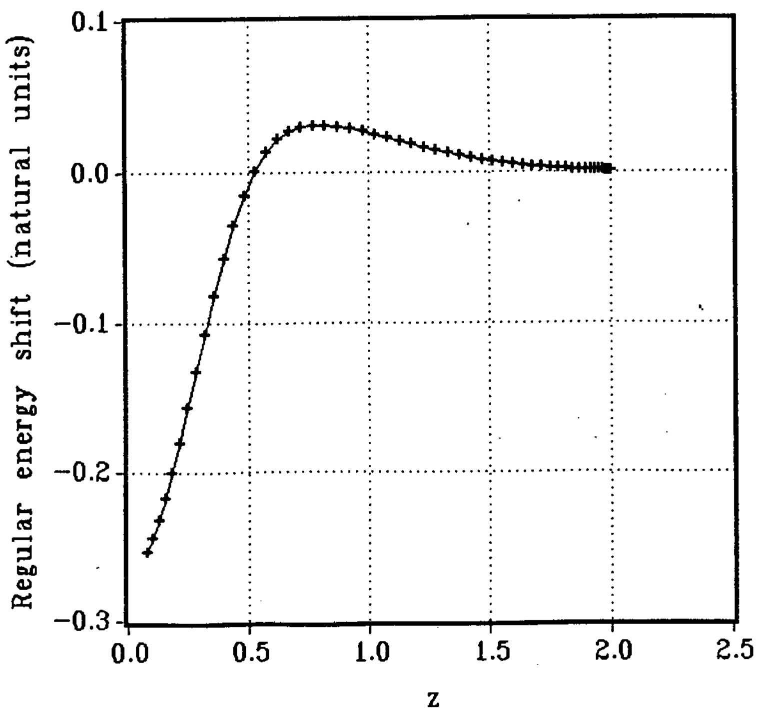


Figure 6.4: Regular vertex correction energy shift for GA



fifth digit in the value of g_V and g_A finally obtained, and are hence negligible. A 60-point Gaussian integration, with the first 7 points extrapolated using a quadratic polynomial for the calculation of g_V and g_A, is performed.

The self-energy insert diagrams turns out to be shorter to calculate than the vertex correction, and the spin sum imposes the conditions $j_q = \frac{1}{2}, |J - \frac{1}{2}| \leq j_p \leq |J + \frac{1}{2}|$, $\kappa_q = -1$; so that $A^{\mu_1}_{\kappa_1\kappa_q} = 2\mu_1$ and $D^{\mu_1}_{\kappa_1\kappa_q} = -\,^2/_3\mu_1$. A value of $E_{max} = 50$ was used, the spin sum routine was called 1453419 times, and the calculation took 21 hours CPU time on an Apollo DN 5500 computer. The divergent graphs are shown in Fig. 6.5 and Fig. 6.7 and the regular graphs in Fig. 6.6 and Fig. 6.8 are obtained after subtracting a singular term that is $-\frac{1}{2}$ times $\Delta E_S(y)$, whose origin is in the free space divergent contribution to the self-energy insert (OC 90). Similar remarks apply to Gaussian integration inaccuracies as given in the case of the vertex correction. The regular energy shift is integrated and multiplied by 2 (due to the two self-energy diagrams giving exactly the same contribution). The integration used 60 Gaussian integration points, 7 of which were extrapolated, using a quadratic polynomial for g_V and a linear polynomial for g_A.

In the calculation of the two-body diagrams the one-gluon exchange spin sum imposes the conditions:
$0 \leq J \leq 1$, $j_p = \frac{1}{2}, \,^3/_2$, $\mu_1 + \mu_3 = \mu_2 = \mu_4$ and $\kappa_p = -1$ or 2. This reduces the number of summations significantly, and in addition the sum is also absolutely convergent. In the spin sum if $\kappa_p = 1$ then $A^{\mu_4}_{\kappa_p\kappa_4} = 2\mu_4$ and $D^{\mu_4}_{\kappa_p\kappa_4} = -\,^2/_3\mu_4$, and if $\kappa_p = 2$ then $A^{\mu_4}_{\kappa_p\kappa_4} = 0$ and $D^{\mu_4}_{\kappa_p\kappa_4} = -\sqrt{^8/_9}$. All the energy shifts for the different allowed values of μ_1, μ_2, μ_3, μ_4 can be calculated within 5 minutes on an Apollo DN 5500 computer, when $E_{max} = 50$ is used.

6.2 RESULTS

The energy shifts for the vertex correction and self-energy insert, after subtracting the divergent contribution ΔE_S for the

Figure 6.5: Divergent self-energy energy shift for GV

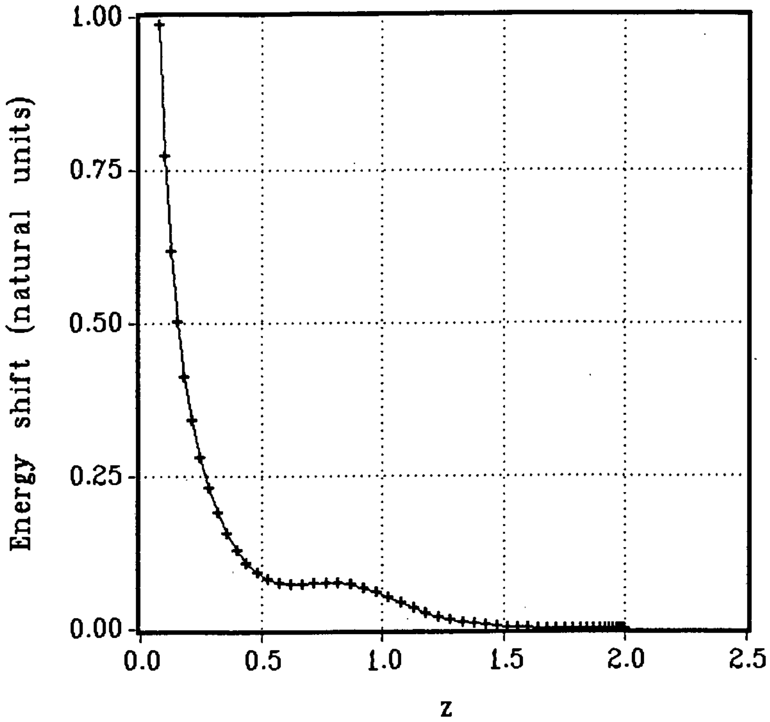


Figure 6.6: Regular self-energy energy shift for GV

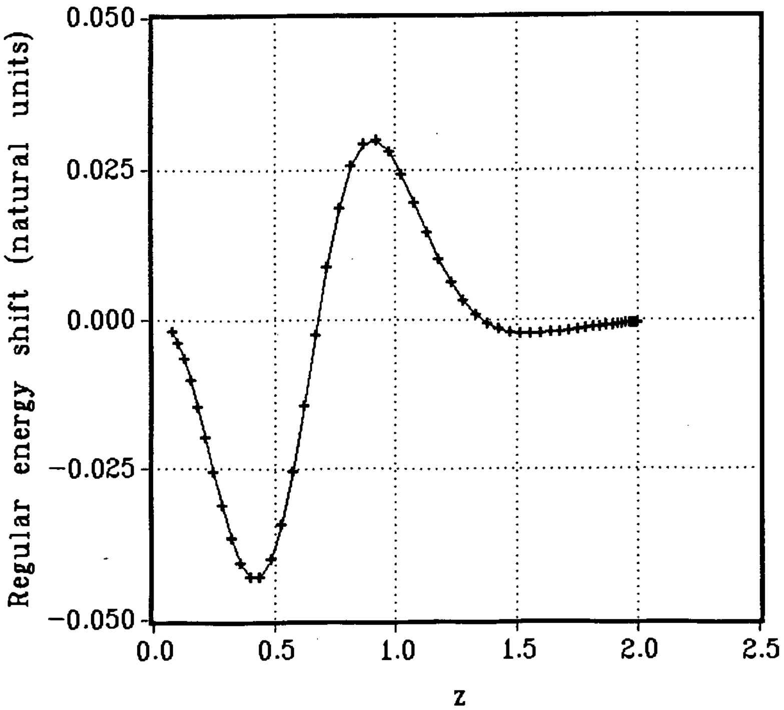


Figure 6.7: Divergent self-energy energy shift for GA

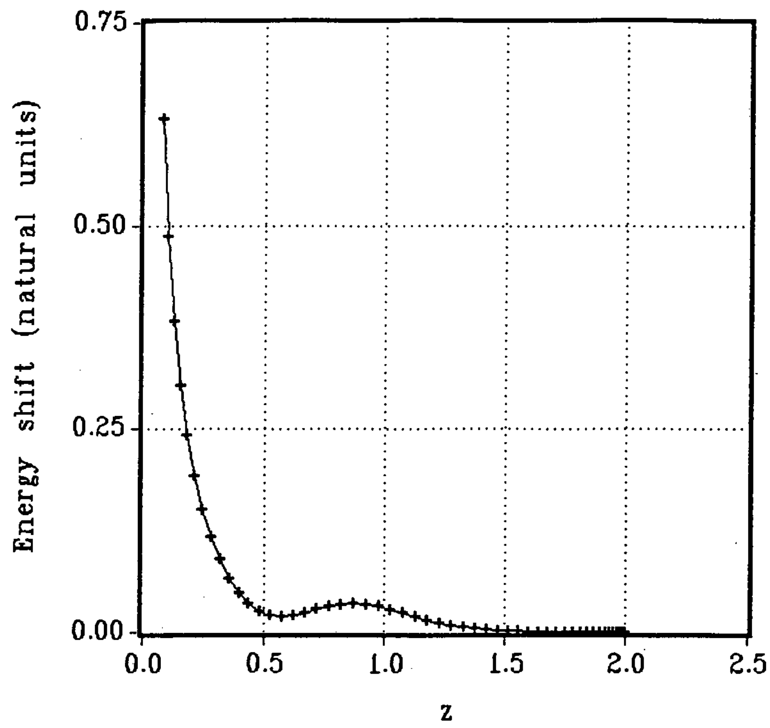
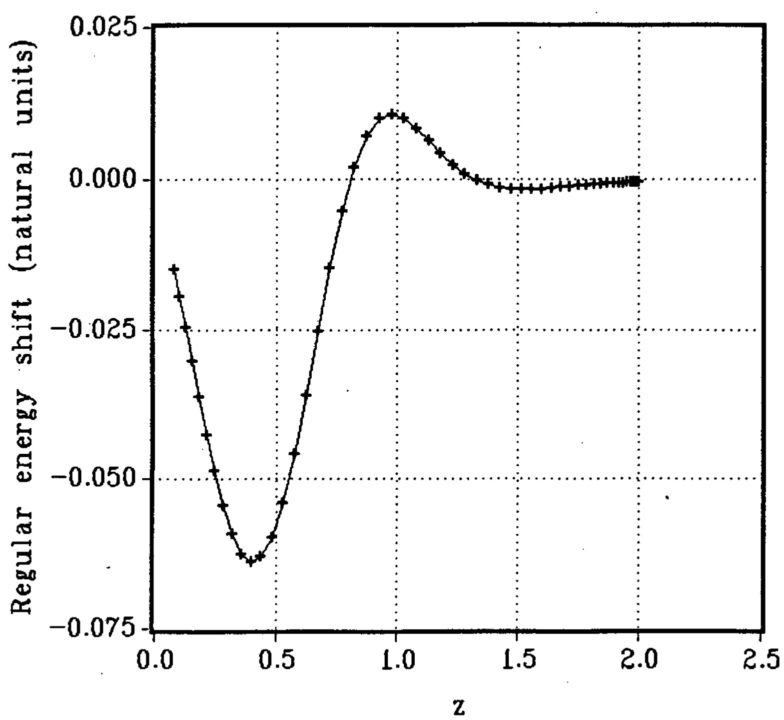


Figure 6.8: Regular self-energy energy shift for GA



external $1s_{1/2}$ quarks, are shown in table 6.1. The colour-flavour matrix element was not taken into account. The factor of 2 for the two self-energy diagrams was not included.

Diagram	**Contribution to**	μ	**Energy Shift**
VERTEX	g_V	↑	1.10497×10^{-2}
		↓	-1.10497×10^{-2}
	g_A	↑	-5.33513×10^{-2}
		↓	5.33513×10^{-2}
SELF-ENERGY	g_V	↑	-5.52484×10^{-3}
		↓	5.52484×10^{-3}
	g_A	↑	-2.75832×10^{-2}
		↓	2.75832×10^{-2}

Table 6.1: **Energy shifts due to radiative corrections to the quark-quark photon vertex in units of α_S. The spin projection $\mu \equiv \mu_1 = \mu_2$ of the external quarks is indicated by an arrow.**

Similarly, the energy shifts for the one-gluon exchange graphs due to $1s_{1/2}$ external quarks are shown in table 6.2, which tabulates external quarks satisfying the angular momentum projection conservation condition $\mu_1 + \mu_3 = \mu_2 + \mu_4$. The number of times the spin sum was called by the computer code is called "counter". The colour-flavour weighting of Appendix E.3 is shown, but the energy shift does not include this factor. The factor of 2, due to the 2 two-body diagrams was not included.

Weighting	μ_1	μ_2	μ_3	μ_4	Contribution to	Counter	Energy Shift
4	↑	↑	↑	↑	g_V	1350	-5.25647×10^{-15}
					g_A		2.19738×10^{-2}
1	↑	↓	↓	↑	g_V	900	-3.80585×10^{-15}
					g_A		5.01255×10^{-3}
1	↓	↑	↑	↓	g_V	900	-3.80585×10^{-15}
					g_A		-5.01255×10^{-3}
0	↓	↓	↑	↑	g_V	1350	-5.25647×10^{-15}
					g_A		-2.19738×10^{-2}
-2	↑	↑	↓	↓	g_V	1350	-1.45063×10^{-15}
					g_A		1.51269×10^{-2}
4	↓	↓	↑	↑	g_V	1350	-1.45063×10^{-15}
					g_A		-1.51269×10^{-2}

Table 6.2: **Energy shift due to one-gluon exchange. All spins of incoming and outgoing quarks are indicated by arrows.**

Note that the zero'th order contribution was calculated in eq. (115). We use the colour-flavour matrix elements in eqs. E (9) and E (17) in conjunction with table 6.1 and 6.2, to obtain table 6.3. No restoration of units is necessary since g_V and g_A are ordinary numbers independent of the cavity radius R.

	g_V	g_A
Zero'th order	1.0000	1.0883
Vertex correction	$-0.0147\alpha_s$	$0.1186\alpha_s$
Self-energy insert (2 diagrams)	$0.0147\alpha_s$	$0.1226\alpha_s$
One-gluon exchange (2 diagrams)	$0.0000\alpha_s$	$0.0013\alpha_s$
Sum of first order corrections	$0.0000\alpha_s$	$0.2425\alpha_s$
Experimental values	1.000	1.239

Table 6.3: **Final Results**

6.3 CONCLUSION

The weak vector and axial vector current coupling constants g_V and g_A are found to be:

$$g_V = 1.000$$
$$g_A = 1.0883 + 0.2425\ \alpha_s,$$

where it is remarkable that the vertex correction contribution cancels the self-energy contribution for g_V exactly, giving an independent verification of the correctness of the algebra in this thesis. The contributions beyond zero'th order are small compared to lowest-order values, supporting the underlying concept of the bag as a bounded perturbative region within a non-perturbative vacuum.

Of interest is to note that vertex and self-energy contributions are very nearly the same, in disagreement with papers where self-energy diagrams are omitted. For g_V the two-body term makes no contribution, so it depends only on one-body diagrams. We have thus verified the CVC hypothesis in eq. (47), which demands that g_V for a transition within an isomultiplet should not be influenced by meson clouds or gluonic exchange effects. The Ademollo-Gatto theorem demands that there should be no first-order SU(3)-breaking in the value of g_V, which is perfectly consistent with our results, since the value of g_V predicted by using perfect SU(3) symmetry is exactly 1. Ushio *et al* (US 84) demonstrated that by adding one-gluon exchange contributions to the M.I.T. bag model, a difficulty with the experimental fact $\mu_\Lambda > \mu_{\Xi^-}$ can be resolved. Tshushima *et al* (TS 88) included meson exchange effects in addition to gluon exchange effects, which changed his evaluation of g_A/g_V from 1.26 to 1.29. They found that one-gluon exchange effects had a negligible effect on the value of g^V, in agreement our results. Høgaasen *et al* (HO 88) confirms results due to (US 84) for one-gluon exchange corrections, and finds a correction of -0.0547 to g_A, which is of opposite sign to that obtained in this work. Their calculation was done in the M.I.T. bag model, and they stated that good agreement with experimental SU(6)-breaking corrections was obtained from the one-gluon exchange diagrams. Maxwell *et al* (MA 83) calculated the one-gluon corrections to g_A as 0.183. In the cavity QCD calculation the magnitude of the one-gluon exchange correction is much less than in the papers cited (see table 6.3), and has an insignificant influence on the value of g_A, although it agrees in sign with (MA 83), who used a similar formalism. They obtain a dominant correction to lowest-order results from two-body diagrams.

The value of the colour-electric coupling constant α_s, consistent with the hadron spectrum in the M.I.T. bag model with R ≈ 1 fm, is $\alpha_s = 2.2$. Using this we obtain $g_A = 1.62$. Perfect agreement with experiment would be obtained if we assumed $\alpha_s = 0.58$. Our results are hence not in good agreement with experiment, and this is mainly due to the size of the one-body corrections. (TS 88) obtained $g_A = 1.26$ in a volume-type cloudy bag model without introducing any adjustable parameters, in close agreement with experiment.

Centre of mass corrections were neglected in this calculation. Depending on which c.m. correction model is applied, agreement with experimental data is sometimes found to improve and sometimes found to deteriorate (see e.g. (TS 88) who found that the ratio g_A/g_V worsened when c.m. corrections were incorporated). Recoil corrections have also been neglected.

Our results do, however, recover the SU(3) result $g_A = 1.67$, which was lost in the M.I.T. bag model. This is a signature that massless quarks behave essentially in a non-relativistic way in a cavity field theory. This conclusion is supported by the vanishing spin-orbit splittings of cavity energies obtained by (ST 90) when the quark self-energies are added to the zero'th order energies for the 1p and 1d modes. They obtained that for $\alpha_s = 2.423$ for the 1p modes, and for $\alpha_s = 2.436$ for 1d the modes, the spin-orbit splitting vanishes completely. This signifies non-relativistic behaviour of quarks, due to the virtual gluon cloud around it. It can be seen that for $\alpha_s = 2.385$ we obtain the SU(3) value $g_A = 1.667$, in remarkable agreement with the values for the strong coupling constant obtained by (ST 90) in order to reproduce non-relativistic behaviour in the cavity field theory.

Suggested continuations of this work are: (1) The calculation of the ratio g_A/g_V for the strangeness-conserving decays: $\Sigma^- \rightarrow \Lambda$, $\Sigma^- \rightarrow \Sigma^0$, $\Xi^- \rightarrow \Xi^0$, as well as for the strangeness-changing decays, and (2) The inclusion of masses for the quarks, especially in decays where the strange quark is involved.

Appendix A

THE CAVITY MODES

The quark and gluon cavity modes given in (BU 88) and (OC 90) are briefly recalled for convenience.

A.1 THE QUARK CAVITY MODES

The cavity modes of the quarks are solutions of the Dirac equation, subject to the boundary conditions of a static spherical cavity as discussed in Chapter 1. The time-independent Dirac equation for a zero mass quark with flavour f is

$$(-i\vec{\gamma}\cdot\nabla)\, u_n(\vec{r}) = \varepsilon_n \gamma^o u_n(\vec{r}) \quad , \qquad \text{A (1)}$$

where ε_n is the quark energy. The solutions of this equation are given by the eigenmodes

$$u_n(\vec{r}) = \begin{pmatrix} g_n(r)\,\chi^{\mu}_{\kappa}(\hat{r}) \\ i f_n(r)\,\chi^{\mu}_{-\kappa}(\hat{r}) \end{pmatrix} \quad , \qquad \text{A (2)}$$

and the adjoint spinors by

$$\overline{u}_n(\vec{r}) = u_n^{+}(\vec{r})\,\gamma^o \quad . \qquad \text{A (3)}$$

The radial, Dirac and magnetic quantum numbers of the cavity mode are represented by n = $\{\nu, \kappa, \mu\}$. The usual two-component spherical spinor is $\chi^{\mu}_{\kappa}(\hat{r})$, and $\chi^{\mu}_{-\kappa}(\hat{r}) = \sigma\cdot\hat{r}\chi^{\mu}_{\kappa}(\hat{r})$. The radial wave functions $g_n(r)$ and $f_n(r)$ are defined by

$$g_n(r) = \frac{N_n}{R^{\frac{3}{2}}} j_\ell(p_n r) \quad , \qquad \text{A (4)}$$

$$f_n(r) = \frac{N_n}{R^{\frac{3}{2}}} sgn(\kappa)\, sgn(\nu)\, j_{\bar{\ell}}(p_n r) \quad , \qquad \text{A (5)}$$

where R is the cavity radius and $j_\ell(x)$ is the spherical Bessel function. The total angular momentum j and the orbital angular momentum ℓ are defined in terms of the Dirac quantum number κ as follows:

$$j(\kappa) = |\kappa| - \tfrac{1}{2} \qquad \text{A (6)}$$

$$\ell(\kappa) = j(\kappa) + \tfrac{1}{2}sgn(\kappa) \qquad \text{A (7)}$$

$$\bar{\ell}(\kappa) = j(\kappa) - \tfrac{1}{2}sgn(\kappa) \quad . \qquad \text{A (8)}$$

The quark momentum p_n is determined by the linear boundary conditions imposed on the quark fields at the surface of the cavity. This can be simplified to give that p_n is a solution of the equation

$$j_\ell(x_n) + sgn(\nu)\, sgn(\kappa)\, j_{\bar{\ell}}(x_n) = 0 \quad , \qquad \text{A (9)}$$

where the energy and momentum have been written in terms of the dimensionless quantities ω_n and x_n respectively. The equations are hence formulated to contain only dimensionless variables. We define

$$x_n = p_n R \qquad \text{A (10)}$$

$$\omega_n = \varepsilon_n R = sgn(\nu)\, x_n \quad . \qquad \text{A (11)}$$

Positive and negative energy solutions are characterised by $\nu > 0$ and $\nu < 0$ respectively. For calculational purposes we note that $\varepsilon_{-\nu,\kappa} = -\varepsilon_{\nu,-\kappa}$. The normalization constant N_n is:

$$N_n^2 = \frac{1}{2\omega_n(\omega_n + \kappa)}\left(\frac{x_n}{j_l(x_n)}\right)^2 \quad . \qquad A\ (12)$$

The eigenmodes, eq. A (2), form a complete and orthonormal set of states. The completeness relation is explicitly:

$$\sum_n u_n(\vec{r})\, u_N^+(\vec{r}) = \delta^{(3)}(\vec{r},\vec{r}')\, I \quad , \qquad A\ (13)$$

where I is the 4 x 4 unit matrix, and the orthonormality condition for the cavity modes is found to be

$$\int d^3r u_n^+(\vec{r})\, u_{n'}(\vec{r}) = \delta_{nn'} \quad . \qquad A\ (14)$$

A.2 THE GLUON CAVITY MODES

A similar treatment to that of the quarks is followed here for the gluons. The gluon modes $a^{\mu}_{m\Sigma}(\vec{r})$ are subject to the M.I.T. boundary conditions of Chapter 1, and so are solutions of the time-independent d'Alembert equations

$$(\nabla^2 + \Omega_m^2)\, a_m^{\mu}(\vec{r}) = 0 \quad . \qquad A\ (15)$$

Σ = S, L, M, E labels the solutions of these equations, and corresponds to the scalar, longitudinal, transverse magnetic and transverse electric polarisations respectively. The compact notation m = {N, J, M} denotes the radial, total angular momentum and magnetic quantum numbers of the eigenmodes respectively. In terms of spherical Bessel functions and vector spherical harmonics, the expansion of the cavity modes are explicitly:

$$a^{o}_{mS}(\vec{r}) = \frac{N_{mS}}{R^{\frac{3}{2}}} i j_J(\Omega^S_m r)\, Y_{JM}(\hat{r}) \qquad \text{A (16)}$$

$$\vec{a}_{mL}(\vec{r}) = \frac{N_{mL}}{\sqrt{R^3(2J+1)}}\left[\sqrt{J} j_{J-1}(\Omega^L_m r)\, \vec{Y}^{J-1}_{JM}(\hat{r}) + \sqrt{J+1} j_{J+1}(\Omega^L_m r)\, \vec{Y}^{J+1}_{JM}(\hat{r})\right] \qquad \text{A (17)}$$

$$\vec{a}_{mM}(\vec{r}) = \frac{N_{mM}}{R^{\frac{3}{2}}} j_J(\Omega^M_m r)\, \vec{Y}^{J}_{JM}(\hat{r}) \qquad \text{A (18)}$$

$$\vec{a}_{mE}(\vec{r}) = \frac{N_{mE}}{\sqrt{R^3(2J+1)}}\left[\sqrt{J+1} j_{J-1}(\Omega^E_m r)\, \vec{Y}^{J-1}_{JM}(\hat{r}) - \sqrt{J} j_{J+1}(\Omega^E_m r)\, \vec{Y}^{J+1}_{JM}(\hat{r})\right] \qquad \text{A (19)}$$

These equations are defined when the total angular momentum J is such that $J \geq 0$ for $\Sigma = S, L$ and $J \geq 1$ for $\Sigma = M, E$. The M.I.T. boundary conditions reduce to the following gluon energy eigenvalue conditions:

$$J j_J(\Omega^S_m R) - \Omega^S_m R j_{J+1}(\Omega^S_m R) = 0 , \quad \Sigma = S, L \qquad \text{A (20)}$$

$$(J+1) j_J(\Omega^M_m R) - \Omega^M_m R j_{J+1}(\Omega^M_m R) = 0 , \quad \Sigma = M \qquad \text{A (21)}$$

$$j_J(\Omega^E_m R) = 0 , \quad \Sigma = E \qquad \text{A (22)}$$

The energies of the scalar and longitudinal modes are the same, $\Omega^L_m = \Omega^S_m$. The gluon normalization constants are

$$N^{-2}_{mS} = N^{-2}_{mL} = \tfrac{1}{2} j^2_J(\Omega^S_m R)\left[1 - \frac{J(J+1)}{(\Omega^S_m R)^2}\right]$$

$$N^{-2}_{mM} = \tfrac{1}{2} j^2_J(\Omega^M_m R)\left[1 - \frac{J(J+1)}{(\Omega^M_m R)^2}\right] \qquad \text{A (23)}$$

$$N^{-2}_{mE} = \tfrac{1}{2} j^2_{J+1}(\Omega^E_m R)$$

For reference purposes we define the diagonal "metric" $g^{\Sigma\Sigma}$ as

$$g^{SS} = -g^{LL} = -g^{MM} = -g^{SS} = 1; \qquad g^{\Sigma\Sigma'} = 0 \text{ if } \Sigma \neq \Sigma' \qquad \text{A (24)}$$

Appendix B

VERTEX INTEGRALS

B.1 THE QUARK-GLUON VERTEX INTEGRAL

A brief recall of (BU 88) is given for quick reference. The interaction between quarks and gluons is described by the vertex integral

$$Q_{nn'}^{m\Sigma} = i\int d^3r\,\bar{u}_n(\vec{r})\,\gamma_\mu u_{n'}(\vec{r})\,a_{m\Sigma}^{\mu}(\vec{r}) \quad . \qquad \text{B (1)}$$

In an associated integral the gluon fields are replaced by their complex conjugate fields, and we define:

$$\tilde{Q}_{nn'}^{m\Sigma} = i\int d^3r\,\bar{u}_n(\vec{r})\,\gamma_\mu u_{n'}(\vec{r})\,a_{m\Sigma}^{\mu *}(\vec{r}) \;=\; -\,Q_{n'n}^{m\Sigma} \quad . \qquad \text{B (2)}$$

A separation of the radial and angular dependence of eq. B (1) yields that

$$\begin{aligned} Q_{nn'}^{m\Sigma} &= R^{-\frac{3}{2}}R_{nn'}^{m\Sigma}\int d\Omega\,\chi_\kappa^{\mu *}(\hat{r})\,Y_{JM}(\hat{r})\,\chi_{\kappa'}^{\mu'}(\hat{r}) \qquad \Sigma = S,L,E \\ Q_{nn'}^{mM} &= R^{-\frac{3}{2}}R_{nn'}^{mM}\int d\Omega\,\chi_\kappa^{\mu *}(\hat{r})\,Y_{JM}(\hat{r})\,\chi_{-\kappa'}^{\mu'}(\hat{r}) \qquad \Sigma = M \end{aligned} \quad . \qquad \text{B (3)}$$

The angular integral is readily done by expanding the spinors and spherical harmonics in a Clebsch-Gordan series (BU 88). The Wigner 3j-symbols is employed to give the result

$$\int d\Omega\,\chi_\kappa^{\mu *}(\hat{r})\,Y_{JM}(\hat{r})\,\chi_{\pm\kappa'}^{\mu'} = \frac{(-1)^{\mu+\frac{1}{2}}}{\sqrt{4\pi}}\;\frac{1 \pm (-1)^{\ell+J+\ell'}}{2}\,\hat{j}\hat{J}\hat{j}' \times \begin{pmatrix} j & J & j' \\ -\mu & M & \mu' \end{pmatrix} \begin{pmatrix} j & J & j' \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad , \qquad \text{B (4)}$$

where an abbreviation to shorten the lay-out of the equations has been introduced: $\hat{J} = \sqrt{2J+1}$. The notation of the 3j- symbols is that of (VA 88). The radial integrals are found to be

$$R_{nn'}^{mS} = -N_{mS}\int_0^R drr^2 j_J(\Omega_m^S r)\, S_{nn'}(r) \qquad \text{B (5)}$$

$$R_{nn'}^{mL} = -\frac{N_{mS}}{\Omega_m^S}\int_0^R drr\{\left[\Omega_m^S r j_{J+1}(\Omega_m^S r) - J j_J(\Omega_m^S r)\right]U_{nn'}(r) + (\kappa - \kappa')\, j_J(\Omega_m^S r)\, T_{nn'}(r)\} \qquad \text{B (6)}$$

$$R_{nn'}^{mL} = \frac{\kappa + \kappa'}{\sqrt{J(J+1)}} N_{mM}\int_0^R drr^2 j_J(\Omega_m^M r)\, T_{nn'}(r) \qquad \text{B (7)}$$

$$R_{nn'}^{mE} = \frac{N_{mE}}{\Omega_m^E\sqrt{J(J+1)}}\int_0^R drr\{J(J+1)\, j_J(\Omega_m^E r)\, U_{nn'}(r) + (\kappa - \kappa')\left[J j_J(\Omega_m^E r) - \Omega_m^E r j_{J-1}(\Omega_m^E r)\right]T_{nn'}(r)\} \quad . \qquad \text{B (8)}$$

Four further abbreviations have been introduced here for the radial parts of the quark wave functions:

$$\begin{aligned} S_{nn'} &= g_n g_{n'} + f_n f_{n'} \\ T_{nn'} &= g_n f_{n'} + f_n g_{n'} \\ U_{nn'} &= g_n f_{n'} - f_n g_{n'} \\ V_{nn'} &= g_n g_{n'} - f_n f_{n'} \end{aligned} \quad . \qquad \text{B (9)}$$

The phase factor from the angular integration of eq. B (4), which contains the parity selection rule, can be incorporated in the radial functions. This leads to simplifications in later work. We define

$$S_{nn'}^{m\Sigma} = \frac{1 \pm (-1)^{\ell+J+\ell'}}{2} R_{nn'}^{m\Sigma} \quad , \qquad \text{B (10)}$$

where the plus sign is used if Σ = S, L, E and the minus sign if Σ = M. The vertex integrals involving the scalar and longitudinal modes are related by the current conservation condition:

$$Q_{nn'}^{mL} = \frac{\varepsilon_{n'} - \varepsilon_n}{\Omega_m^S} Q_{nn'}^{mS} \quad . \qquad \text{B (11)}$$

B.2 THE QUARK-QUARK-EXTERNAL W^--VERTEX

If $W^\mu(x)$ is real, we can define the integral describing the interaction between the quark and the external field as

$$M_{nn'} = \int d^3r \overline{u}_n(\boldsymbol{r}) \gamma^\lambda (1 - \gamma^5) u_{n'}(\boldsymbol{r}) W_\lambda(\boldsymbol{r}) \quad . \qquad \text{B (12)}$$

By comparison to eq. (113) we see that $M_{nn'}$ is essentially the potential due to the W^--field. We note that $W^0 \neq 0$ since the W^--particle carries a charged current. Using eqs. A (2) and A (3), and the fact that in the Dirac representation

$$\gamma_o \gamma^\mu (1 - \gamma^5) = \begin{cases} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \text{if } \mu = 0 \\ \begin{pmatrix} -\sigma & \sigma \\ \sigma & -\sigma \end{pmatrix} & \text{if } \mu = 1,2,3 \end{cases} \quad , \qquad \text{B (13)}$$

and by definition $\chi^\mu_\kappa = \boldsymbol{\sigma} \cdot \hat{r}\, \chi^\mu_{-\kappa}$, with $\boldsymbol{\sigma}$ the Pauli matrix vector, so that

$$\begin{aligned} \chi^{\mu^+}_{-\kappa} \chi^{\mu'}_{\kappa'} &= \chi^{\mu^+}_{\kappa} \boldsymbol{\sigma}\cdot\hat{r} \chi^{\mu'}_{\kappa'} = \chi^{\mu^+}_{\kappa} \chi^{\mu'}_{-\kappa'} \\ \chi^{\mu^+}_{-\kappa} \chi^{\mu'}_{-\kappa'} &= \chi^{\mu^+}_{\kappa} \boldsymbol{\sigma}\cdot\hat{r}\boldsymbol{\sigma}\cdot\hat{r} \chi^{\mu'}_{\kappa'} = \chi^{\mu^+}_{\kappa} \chi^{\mu'}_{\kappa'} \quad , \\ \boldsymbol{\sigma}\cdot\hat{r}\boldsymbol{\sigma}\cdot\hat{r} &= 1 \end{aligned} \qquad \text{B (14)}$$

we obtain after some lengthy expansions

$$\begin{aligned} M_{nn'} = & \int d^3r \{ S_{nn'} \chi^{\mu^+}_{\kappa} \chi^{\mu'}_{\kappa'} - i U_{nn'} \chi^{\mu^+}_{\kappa} \chi^{\mu'}_{-\kappa'} \} W_o \\ & + \int d^3r \{ g_n g_{n'} \chi^{\mu^+}_{\kappa} \boldsymbol{\sigma} \chi^{\mu'}_{\kappa'} + i f_n g_{n'} \chi^{\mu^+}_{-\kappa} \boldsymbol{\sigma} \chi^{\mu'}_{\kappa'} \\ & - i g_n f_{n'} \chi^{\mu^+}_{\kappa} \boldsymbol{\sigma} \chi^{\mu'}_{-\kappa'} + f_n f_{n'} \chi^{\mu^+}_{-\kappa} \boldsymbol{\sigma} \chi^{\mu'}_{-\kappa'}) \cdot \boldsymbol{W} \ \} \quad . \end{aligned} \qquad \text{B (15)}$$

We choose $W_o(x) = W_o$ and $\boldsymbol{W}(x) = (0, 0, W_z)$, where W_o and W_z are real constants.

We have not used $\boldsymbol{\sigma}\cdot\hat{r}\boldsymbol{\sigma}\boldsymbol{\sigma}\cdot\hat{r} = -\boldsymbol{\sigma} + 2\boldsymbol{\sigma}\cdot\hat{r}\hat{r}$ to simplify terms proportional to W_o in eq. B (15), since the expansion would lead to a term which if followed through gives rise to a non-symmetric look to the spin sums: $\hat{r}_3 = \cos\theta = \sqrt{4\pi/3}\, Y_{10}$.

Now using

$$\chi_\kappa^{\mu *}(\hat{r}) = \sum_{m\sigma} \chi_\sigma Y_{\ell m}(\hat{r}) \langle \ell m \tfrac{1}{2}\sigma | jm \rangle \quad , \qquad \text{B (16)}$$

where χ_σ $(\sigma = -\tfrac{1}{2}, \tfrac{1}{2})$ are the normalized projection spin eigenstates, we obtain

$$\int d\Omega \chi_\kappa^{\mu *}(\hat{r}) \chi_{\kappa'}^{\mu'}(\hat{r}) = \delta_{\mu\mu'}\delta_{\kappa\kappa'} \quad , \qquad \text{B (17)}$$

where we used

$$\chi_\sigma^{+}\chi_{\sigma'} = \delta_{\sigma\sigma'} \quad , \quad \int d\Omega\, Y_{\ell m}^{*}(\hat{r})\, Y_{\ell' m'}(\hat{r}) = \delta_{\ell\ell'}\delta_{mm'} \quad . \qquad \text{B (18)}$$

Replacing $\kappa' \rightarrow -\kappa'$, and correspondingly $\ell' = j' + \tfrac{1}{2}\mathrm{sgn}\kappa' \rightarrow j + \tfrac{1}{2}\mathrm{sgn}\,(-\kappa') = \bar{\ell}'$, we obtain

$$\int d\Omega \chi_\kappa^{\mu'}(\hat{r}) \chi_{-\kappa'}^{\mu *}(\hat{r}) = \delta_{\mu\mu'}\delta_{\kappa-\kappa'} \quad . \qquad \text{B (19)}$$

Using eq. B (18) together with $\sigma_3\, \chi_\sigma = 2\sigma\chi_\sigma$ $(\sigma = -\tfrac{1}{2}, \tfrac{1}{2})$, we obtain

$$\int d\Omega \chi_\kappa^{\mu *} \sigma_3 \chi_{\kappa'}^{\mu'} = \delta_{\mu\mu'}\delta_{\ell\ell'} \sum_{\sigma = \frac{1}{2}, -\frac{1}{2}} 2\sigma \langle \ell\mu - \sigma \tfrac{1}{2}\sigma | j\mu \rangle \langle \ell\mu - \sigma \tfrac{1}{2}\sigma | j'\mu \rangle \qquad \text{B (20)}$$

and an explicit evaluation of the Clebsch-Gordan coefficients gives that eq. B (20) becomes

$$\delta_{\ell\ell'}\delta_{\mu\mu'}\begin{cases} \dfrac{\mu}{\sqrt{jj'}} & \text{if } j = \ell + \frac{1}{2},\ j' = \ell + \frac{1}{2} \\ \dfrac{-\sqrt{j^2 - \mu^2}}{\sqrt{j(j' + 1)}} & \text{if } j = \ell + \frac{1}{2},\ j' = \ell - \frac{1}{2} \\ \dfrac{-\sqrt{j'^2 - \mu^2}}{\sqrt{j'(j + 1)}} & \text{if } j = \ell - \frac{1}{2},\ j' = \ell + \frac{1}{2} \\ \dfrac{-\mu}{\sqrt{(j + 1)(j' + 1)}} & \text{if } j = \ell - \frac{1}{2},\ j' = \ell - \frac{1}{2} \end{cases} \qquad \text{B (21)}$$

This, together with eqs. A (6) to A (8) , and defining

$$A^{\mu}_{\kappa\kappa'} = \begin{cases} \dfrac{\mu}{\sqrt{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa < 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa - 1 > 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa - 1 < 0 \\ \dfrac{-\mu}{\sqrt{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa > 0 \end{cases} \qquad \text{B (22)}$$

gives that

$$\int d\Omega \chi^{\mu^*}_{\kappa} \sigma_3 \chi^{\mu'}_{\kappa'} = \delta_{\mu\mu'} A^{\mu}_{\kappa\kappa'} \quad . \qquad \text{B (23)}$$

Note that in eq. B (21) the angular momentum change is $\Delta j = j' - j = -1, 0, 1$. This is consistent through angular momentum conservation at the vertex with the photon carrying total angular momentum of 1, which is purely due to its spin of 1. The reason for this is that the electron and neutrino waves in β-decay are zero orbital angular momentum waves.

Replacing $\kappa' \rightarrow -\kappa'$ in eqs. B (22) and B (23), and defining

$$B^{\mu}_{\kappa\kappa'} = \begin{cases} \dfrac{\mu}{\sqrt{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa > 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa + 1 < 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa + 1 > 0 \\ \dfrac{-\mu}{\sqrt{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa < 0 \end{cases}, \qquad \text{B (24)}$$

we get:

$$\int d\Omega \chi^{\mu^*}_{\kappa} \sigma_3 \chi^{\mu'}_{-\kappa'} = \delta_{\mu\mu'} B^{\mu}_{\kappa\kappa'} \quad . \qquad \text{B (25)}$$

Replacing $\kappa \rightarrow -\kappa$ in eqs. B (22) and B (23), and defining

$$C^{\mu}_{\kappa\kappa'} = \begin{cases} \dfrac{\mu}{\sqrt{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa < 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa - 1 > 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa - 1 < 0 \\ \dfrac{-\mu}{\sqrt{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa > 0 \end{cases}, \qquad \text{B (26)}$$

we find that

$$\int d\Omega \chi^{\mu^*}_{-\kappa} \sigma_3 \chi^{\mu'}_{\kappa'} = \delta_{\mu\mu'} C^{\mu}_{\kappa\kappa'} \quad . \qquad \text{B (27)}$$

Replacing $\kappa \to -\kappa$, $\kappa' \to -\kappa'$ in eqs. B (22) and B (23), and defining

$$D^{\mu}_{\kappa\kappa'} = \begin{cases} \dfrac{\mu}{\sqrt{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = \kappa > 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 - |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa + 1 < 0 \\ -\sqrt{1 - \dfrac{\mu^2}{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa + 1 > 0 \\ \dfrac{-\mu}{\sqrt{\kappa^2 + |\kappa| + \frac{1}{4}}} & \text{if } \kappa' = -\kappa < 0 \end{cases}, \qquad \text{B (28)}$$

we find:

$$\int d\Omega \chi^{\mu *}_{-\kappa} \sigma_3 \chi^{\mu'}_{-\kappa'} = \delta_{\mu\mu'} D^{\mu}_{\kappa\kappa'} \quad . \qquad \text{B (29)}$$

Using eqs. B (15) and B (17) to B (29) we see

$$M_{nn'} = \int r^2 dr \{ (S_{nn'} \delta_{\kappa\kappa'} - i U_{nn'} \delta_{\kappa\, -\kappa'}) W_o$$
$$+ (g_n g_{n'} A^{\mu}_{\kappa\kappa'} - i g_n f_{n'} B^{\mu}_{\kappa\kappa'} + i f_n g_{n'} C^{\mu}_{\kappa\kappa'} + f_n f_{n'} D^{\mu}_{\kappa\kappa'}) W_z \} \delta_{\mu\mu'} \quad . \qquad \text{B (30)}$$

Appendix C

SPIN SUMS

The spin sums over the total angular projection of the intermediate particles, which can be done elegantly analytically, are presented in this appendix. The notation and phase conventions are that of (VA 88) which is consistent with (ED 57). For some steps it will be indicated on which page of (VA 88) the identity used can be cross-referenced, e.g. (VA p.57). In order to use the listed identities, use will be made of the symmetries of the Wigner symbols, the restrictions of the Clebsch-Gordan coefficients, the interchange of order of summation and sometimes the evaluation of Clebsch-Gordan coefficients, to bring the sum into the desired form.

C.1 VERTEX CORRECTION SPIN SUM

$$\begin{aligned} 4\pi \sum_{\mu_p\mu_q M} \tilde{Q}^{m\Sigma}_{n_1 p} M_{pq} Q^{m\Sigma}_{qn_2} = & - \sum_{\mu_p\mu_q M} R^{-3} S^{m\Sigma}_{pn_1} S^{m\Sigma}_{qn_2} (-1)^{\mu_p+\mu_q+1} \hat{J}^2 \hat{j}_1 \hat{j}_2 \hat{j}_p \hat{j}_q \\ & \times \begin{pmatrix} j_p & J & j_1 \\ -\mu_p & M & \mu_1 \end{pmatrix} \begin{pmatrix} j_p & J & j_1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} j_q & J & j_2 \\ -\mu_q & M & \mu_2 \end{pmatrix} \begin{pmatrix} j_q & J & j_2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \\ & \times \{ \{ \int dr r^2 \{ S_{pq} \delta_{\kappa_p \kappa_q} - i U_{pq} \delta_{\kappa_p - \kappa_q} \} \delta_{\mu_p \mu_q} W_\sigma \} \\ & + \{ \int dr r^2 \{ g_p g_q \int d\Omega \chi^{\mu_p +}_{\kappa_p} \sigma_3 \chi^{\mu_q}_{\kappa_q} - i g_p f_q \int d\Omega \chi^{\mu_p +}_{\kappa_p} \sigma_3 \chi^{\mu_q}_{-\kappa_q} \\ & + i f_p g_q \int d\Omega \chi^{\mu +}_{-\kappa} \sigma_3 \chi^{\mu'}_{\kappa'} + f_p f_q \int d\Omega \chi^{\mu +}_{-\kappa} \sigma_3 \chi^{\mu'}_{-\kappa'} \} \delta_{\mu_p \mu_q} W_z \} \end{aligned} \qquad \text{C (1)}$$

This is obtained by substitution of the expressions in Appendix B for $\tilde{Q}^{m\Sigma}_{n_1p}$, M_{pq}, $Q^{m\Sigma}_{n_2q}$. Here S_{pq}, U_{pq} are defined in Appendix B.1. Note that we have used eq. B (2).

Looking at the part of eq. C (1) proportional to W_o, and using (VA p.453(8)), we obtain

$$\sum_{\mu_p \mu_q M} (-1)^{\mu_p + \mu_q + 1} \begin{pmatrix} j_p & J & j_1 \\ -\mu_p & M & \mu_1 \end{pmatrix} \begin{pmatrix} j_q & J & j_2 \\ -\mu_q & M & \mu_2 \end{pmatrix} \delta_{\mu_p \mu_2} , \qquad \text{C (2)}$$

$$= \frac{(-1)^{j_1 + j_2 + 1}}{\hat{j}_1^2} \delta_{j_1 j_2} \delta_{u_1 u_2} \{j_1 j_p J\}$$

where we have defined

$$\{j_1 j_p J\} = \begin{cases} 1 \text{ if } j_1 + j_p + J = \text{integer and} \\ \quad |j_1 - j_p| \le J \le j_1 + j_p \\ 0 \text{ otherwise} \end{cases} . \qquad \text{C (3)}$$

Take the first term in eq. C (1) proportional to W_z, make use of eq. B (20) and change the Clebsch-Gordan coefficients into Wigner 3j-symbols to obtain

$$\sum_{\mu_p \mu_q M} (-1)^{\mu_p + \mu_q + 1} \begin{pmatrix} j_p & J & j_1 \\ -\mu_p & M & \mu_1 \end{pmatrix} \begin{pmatrix} j_q & J & j_2 \\ -\mu_q & M & \mu_2 \end{pmatrix} \int d\Omega \chi_{\kappa_p}^{\mu_p^*} \sigma_3 \chi_{\kappa_q}^{\mu_q} \delta_{\mu_p \mu_q} \qquad \text{C (4)}$$

$$= \sum_{\mu_p M m \sigma \mu_q} \delta_{\ell_p \ell_q} \hat{j}_p \hat{j}_q 2\sigma \begin{pmatrix} j_p & J & j_1 \\ -\mu_p & M & \mu_1 \end{pmatrix} \begin{pmatrix} j_q & J & j_2 \\ -\mu_q & M & \mu_2 \end{pmatrix} \begin{pmatrix} \ell_p & \frac{1}{2} & j_p \\ m & \sigma & -\mu_p \end{pmatrix} \begin{pmatrix} \ell_p & \frac{1}{2} & j_q \\ m & \sigma & -\mu_q \end{pmatrix}$$

Now use (VA p. 455 (10)), and after summing over all the indices except σ, we obtain

$$\sum_{\sigma} \hat{j}_p \hat{j}_q 2\sigma \delta_{\ell_p \ell_q} (-1)^{J + \ell_p + \mu_1 + \sigma - j_1 - j_2 + 1} \sum_{x = 0,\, 1} (-1)^x (2x+1) \qquad \text{C (5)}$$

$$\times \begin{pmatrix} j_1 & x & j_2 \\ \mu_1 & 0 & -\mu_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & x & \frac{1}{2} \\ -\sigma & 0 & \sigma \end{pmatrix} \begin{Bmatrix} j_1 & x & j_2 \\ j_q & J & j_p \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & x & \frac{1}{2} \\ j_q & \ell_p & j_p \end{Bmatrix}$$

Now summing first over σ, and writing down explicit expressions for the Wigner 3j- symbols containing σ (VA p.271), we find that only the x = 1 case is non-zero. Using

$$\sum_{\sigma} 2\sigma(-1)^{\sigma}\begin{pmatrix} \frac{1}{2} & x & \frac{1}{2} \\ -\sigma & 0 & \sigma \end{pmatrix} = \begin{cases} (-1)^{\frac{1}{2}}\sqrt{\frac{2}{3}} & \text{if } x = 1 \\ 0 & \text{if } x = 0 \end{cases} , \qquad \text{C (6)}$$

and summing over x, we find that eq. C (5) becomes

$$F(\ell_p, \ell_q) = \sqrt{6}\hat{j}_p\hat{j}_q\delta_{\ell_p\ell_q}(-1)^{J+\ell_p+\mu_1+j_1+j_2+\frac{1}{2}} \times \begin{pmatrix} j_1 & 1 & j_2 \\ \mu_1 & 0 & -\mu_2 \end{pmatrix}\begin{Bmatrix} j_1 & 1 & j_2 \\ j_q & 0 & -j_p \end{Bmatrix}\begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ j_q & \ell_p & j_p \end{Bmatrix} . \qquad \text{C (7)}$$

Note that ℓ_q only appears in eq. C (7) in the Kronecker delta.

Looking at the second, third and fourth terms in the part of eq. C (1) proportional to W_z, we replace:

for the second term : ℓ_q into $\bar{\ell}_q$ in eq. C (7);
for the third term : ℓ_p into $\bar{\ell}_p$ in eq. C (7);
for the fourth term : ℓ_q into $\bar{\ell}_q$ and ℓ_p into $\bar{\ell}_p$ in eq. C (7).

Finally we obtain

$$4\pi \sum_{\mu_p\mu_q M} \tilde{Q}^{m\Sigma}_{n_1 p} M_{pq} Q^{m\Sigma}_{qn_2} = -R^{-3} S^{m\Sigma}_{pn_1} S^{m\Sigma}_{qn_2} \hat{J}^2 \hat{j}_1 \hat{j}_2 \hat{j}_p \hat{j}_q$$

$$\times \begin{pmatrix} j_p & J & j_1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} j_q & J & j_2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}\Big\{\Big\{\int drr^2\{S_{pq}\delta_{\kappa_p\,\kappa_q} - iU_{pq}\delta_{\kappa_p\,-\kappa_q}\} \qquad \text{C (8)}$$

$$\times \frac{(-1)^{j_1+j_2+1}}{\hat{j}_1^2}\delta_{j_1 j_2}\delta_{\mu_1\mu_2}\{j_1 j_p J\}W_0\Big\} + \Big\{\int drr^2\{g_p g_q F(\ell_p, \ell_q)$$

$$- ig_p f_q F(\ell_p, \overline{\ell}_q) + if_p g_q F(\overline{\ell}_p, \ell_q) + f_p f_q F(\overline{\ell}_p, \overline{\ell}_q)\}W_z\Big\}\Big\} .$$

C.2 SELF-ENERGY INSERT SPIN SUM

$$4\pi\sum_{\mu_p\mu_q M} M_{n_1q}\tilde{Q}^{m\Sigma}_{qp}Q^{m\Sigma}_{pn_2} = -\sum_{\mu_p\mu_q M} R^{-3}s^{m\Sigma}_{pq}s^{m\Sigma}_{pn_2}(-1)^{\mu_p+\mu_p+1}\hat{J}^2\hat{j}_q\hat{j}_p^2\hat{j}_2$$

$$\times\begin{pmatrix} j_p & J & j_q \\ -\mu_p & M & \mu_q\end{pmatrix}\begin{pmatrix} j_p & J & j_q \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix} j_p & J & j_2 \\ -\mu_q & M & \mu_2\end{pmatrix}\begin{pmatrix} j_p & J & j_2 \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}$$

$$\times\left\{\left\{\int drr^2\{S_{n_1q}\delta_{\kappa_1\kappa_q} - iU_{n_1q}\delta_{\kappa_1-\kappa_q}\}\delta_{\mu_1\mu_q}W_o\right\}\right.$$

$$+\left\{\int drr^2\left\{g_{n_1}g_q\int d\Omega\chi^{\mu_p^+}_{\kappa_1}\sigma_3\chi^{\mu_q}_{\kappa_q} - ig_{n_1}f_q\int d\Omega\chi^{\mu_p^+}_{\kappa_1}\sigma_3\chi^{\mu_q}_{-\kappa_q}\right.\right.$$

$$\left.\left.\left.+ if_{n_1}g_q\int d\Omega\chi^{\mu_p^+}_{-\kappa_1}\sigma_3\chi^{\mu_q}_{\kappa_q} + f_{n_1}f_q\int d\Omega\chi^{\mu_p^+}_{-\kappa_1}\sigma_3\chi^{\mu_q}_{-\kappa_q}\right\}\delta_{\mu_1\mu_q}W_z\right\}\right\} \qquad \text{C (9)}$$

This is obtained in similar fashion to eq. C (1).

Looking at the part of eq. C (9) proportional to W_o, and using (VA p.453 (8)), we see

$$\sum_{\mu_p\mu_q M}(-1)^{\mu_p+\mu_p+1}\begin{pmatrix} j_p & J & j_q \\ -\mu_p & M & \mu_q\end{pmatrix}\begin{pmatrix} j_p & J & j_2 \\ -\mu_p & M & \mu_2\end{pmatrix}\delta_{\mu_1\mu_q} = \frac{1}{\hat{j}_q^2}\{j_qj_pJ\}\delta_{j_qj_2}\delta_{\mu_1\mu_2} \quad . \qquad \text{C (10)}$$

Take the first term in the part of eq. C (9) proportional to W_z, make use of eq. B (20), change the Clebsch-Gordan coefficients into Wigner 3j-symbols and use (VA p.453 (8)) to sum over μ_q and M. With the use of eq. B (23), the expression simplifies to:

$$\sum_{\mu_p\mu_q M}(-1)^{\mu_p+\mu_p+1}\begin{pmatrix} j_p & J & j_q \\ -\mu_p & M & \mu_q\end{pmatrix}\begin{pmatrix} j_p & J & j_2 \\ -\mu_p & M & \mu_2\end{pmatrix}\int d\Omega\chi^{\mu_1^+}_{\kappa_1}\sigma_3\chi^{\mu_q}_{\kappa_q}\delta_{\mu_1\mu_q} \qquad \text{C (11)}$$

$$= \frac{1}{\hat{j}_q^2}\{j_qj_pJ\}\delta_{j_qj_2}\delta_{\mu_1\mu_2}A^{\mu_1}_{\kappa_1\kappa_q}$$

Look at the second, third and fourth terms in the part of eq. C (9) proportional to W_z, and replace:

for the second term: κ_q into $-\kappa_q$ in eq. C (11);
for the third term: κ_1 into $-\kappa_1$ in eq. C (11);
for the fourth term: κ_1 into $-\kappa_1$ and κ_q into $-\kappa_q$ in eq. C (11).

Finally, we obtain using eqs. B (25), B (27) and B (29):

$$4\pi \sum_{\mu_p\mu_q M} M_{n_1 q}\tilde{Q}^{m\Sigma}_{qp}Q^{m\Sigma}_{pn_2} = -R^{-3}S^{m\Sigma}_{pq}S^{m\Sigma}_{pn_2}\hat{J}^2\frac{\hat{j}_p^2\hat{j}_2}{\hat{j}_q}\begin{pmatrix} j_p & J & j_q \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix} j_p & J & j_2 \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}\delta_{j_q j_2}\delta_{\mu_1\mu_2}$$

$$\times \left\{\left\{\int drr^2\left\{S_{n_1 q}\delta_{\kappa_1\kappa_q} - iU_{n_1 q}\delta_{\kappa_1 -\kappa_q}\right\}W_o\right\} + \right.$$

$$\left.\left\{\int drr^2\left\{g_{n_1}g_q A^{\mu_1}_{\kappa_1\kappa_q} - ig_{n_1}f_q B^{\mu_1}_{\kappa_1\kappa_q} + if_{n_1}g_q C^{\mu_1}_{\kappa_1\kappa_q} + f_{n_1}f_q D^{\mu_1}_{\kappa_1\kappa_q}\right\}W_Z\right\}\right\}$$

C (12)

C.3 ONE-GLUON EXCHANGE SPIN SUM

In this case the expression cannot be simplified, and no advantage is gained by summing last over σ, as in sections (C.1) and (C.2). We expand as usual:

$$4\pi \sum_{\mu_p M} \tilde{Q}^{m\Sigma}_{n_1 n_2}Q^{m\Sigma}_{n_3 p}M_{pn_4} = -\sum_{\mu_p M} R^{-3}S^{m\Sigma}_{n_2 n_1}S^{m\Sigma}_{n_3 p}\frac{(-1)^{\mu_2+\mu_3+1}}{4\pi}$$

$$\times \hat{J}^2\hat{j}_1\hat{j}_2\hat{j}_3\hat{j}_p\begin{pmatrix} j_2 & J & j_1 \\ -\mu_2 & M & \mu_1\end{pmatrix}\begin{pmatrix} j_2 & J & j_1 \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ -\mu_3 & M & \mu_p\end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix} \qquad \text{C (13)}$$

$$\times \left\{\left\{\int drr^2\left\{S_{pn_4}\delta_{\kappa_p\kappa_4} - iU_{pn_4}\delta_{\kappa_p -\kappa_4}\right\}\delta_{\mu_p\mu_4}W_o\right\} + \right.$$

$$\left\{\int drr^2\left\{g_p g_{n_4}\int d\Omega\chi^{\mu_p^+}_{\kappa_p}\sigma_3\chi^{\mu_4}_{\kappa_4} - ig_p f_{n_4}\int d\Omega\chi^{\mu_p^+}_{\kappa_p}\sigma_3\chi^{\mu_4}_{-\kappa_4}\right.\right.$$

$$\left.\left.\left. + if_p g_{n_4}\int d\Omega\chi^{\mu_p^+}_{-\kappa_p}\sigma_3\chi^{\mu_4}_{\kappa_4} + f_p f_{n_4}\int d\Omega\chi^{\mu_p^+}_{-\kappa_p}\sigma_3\chi^{\mu_4}_{-\kappa_4}\right\}\delta_{\mu_p\mu_4}W_z\right\}\right\}$$

This is obtained similarly to eq. C (1). Looking at the part of eq. C (13) proportional to W_o, we see

$$\sum_{\mu_p M}\begin{pmatrix} j_2 & J & j_1 \\ -\mu_2 & M & \mu_1\end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ -\mu_3 & M & \mu_p\end{pmatrix}\delta_{\mu_p\mu_4} = \sum_{M}\begin{pmatrix} j_2 & J & j_1 \\ -\mu_2 & M & \mu_1\end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ -\mu_3 & M & \mu_4\end{pmatrix} \qquad \text{C (14)}$$

Looking at the first term in the part of eq. C (13) proportional to W_z, and using eqs. B (20) and B (23), we see

$$\sum_{\mu_p M} \begin{pmatrix} j_2 & J & j_1 \\ -\mu_2 & M & \mu_1 \end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ -\mu_3 & M & \mu_p \end{pmatrix}\int d\Omega \chi_{\kappa_p}^{\mu_p^+}\sigma_3\chi_{\kappa_4}^{\mu_4}\delta_{\mu_p\mu_4}$$
$$= \sum_M \begin{pmatrix} j_2 & J & j_1 \\ -\mu_2 & M & \mu_1 \end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ -\mu_3 & M & \mu_4 \end{pmatrix} A_{\kappa_p\kappa_4}^{\mu_4} \quad . \qquad \text{C (15)}$$

Looking at the second term, third term and fourth term in the part of eq. C (13) proportional to W_z, we replace

for the second term: κ_4 into $-\kappa_4$ in eq. C (15);
for the third term: κ_p into $-\kappa_p$ in eq. C (15);
for the fourth term: κ_p into $-\kappa_p$ and κ_4 into $-\kappa_4$ in eq. C (15).

Finally we obtain, using eqs. B (25), B (27) and B (29):

$$4\pi \sum_{\mu_p M} \tilde{Q}_{n_1 n_2}^{m\Sigma} Q_{n_3 p}^{m\Sigma} M_{pn_4} = - R^{-3} s_{n_2 n_1}^{m\Sigma} S_{n_3 p}^{m\Sigma} (-1)^{\mu_2 + \mu_3 + 1}$$
$$\times \hat{J}^2 \hat{j}_1 \hat{j}_2 \hat{j}_3 \hat{j}_p \begin{pmatrix} j_2 & J & j_1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \sum_M \begin{pmatrix} j_2 & J & j_1 \\ -\mu_2 & M & \mu_1 \end{pmatrix}\begin{pmatrix} j_3 & J & j_p \\ -\mu_3 & 0 & \mu_4 \end{pmatrix} \qquad \text{C (16)}$$
$$\times \left\{\left\{\int drr^2 \left\{S_{pn_4}\delta_{\kappa_p\kappa_4} - iU_{pn_4}\delta_{\kappa_p - \kappa_4}\right\} W_o\right\} + \right.$$
$$\left.\left\{\int drr^2 \left\{g_p g_{n_4} A_{\kappa_p\kappa_4}^{\mu_4} - ig_p f_{n_4} B_{\kappa_p\kappa_4}^{\mu_4} + if_p g_{n_4} C_{\kappa_p\kappa_4}^{\mu_4} + f_p f_{n_4} D_{\kappa_p\kappa_4}^{\mu_4}\right\} W_z\right\}\right\} \quad .$$

Appendix D

SUM RULES

An outline of the derivation of a sum rule is presented in this appendix. It provides an independent method of checking the correctness of the computer code. It also gives one a handle on whether the energy truncation E_{max} used to truncate the infinite sum over the complete set of quark and gluon quantum numbers is adequate.

A "sum rule" is obtained analytically by summing over intermediate quarks and gluons, using the completeness relations of Appendix A, before doing the integrals over the cavity volume. It is found that a sum over all the quantum numbers (p, q and m, Σ) produces a delta-function in coordinate space, which will leave the sum rule badly defined. We hence choose to sum only over two of these (in this case p and q). It is also found that we can in addition sum over M, which has the advantage of making the sum rule even more sensitive to possible errors in the computer code. We could also have chosen other sets of two quantum numbers (p and m,Σ say).

The sum rules are too bulky to display and only an outline of the vertex correction sum rule will be given.

D.1 VERTEX CORRECTION SUM RULE

We define, retaining only vertex integrals in the expression for the energy shift in eq. (117):

$$V^{m\Sigma}_{nn'} = 4\pi g^{\Sigma\Sigma} \sum_{pqM} \tilde{Q}^{m\Sigma}_{np} M_{pq} Q^{m\Sigma}_{qn'} \quad , \qquad \text{D (1)}$$

where the sum includes a sum over the gluon spin projections. Using the definitions in eqs. B (12), B (1) and B (2) the above is expanded to give

$$V_{nn'}^{m\Sigma} = -4\pi g^{\Sigma\Sigma} \sum_{pqM} \int d^3x \bar{u}_n(\boldsymbol{x}) \gamma_\mu u_p(\boldsymbol{x}) a_{m\Sigma}^{\mu *}(\boldsymbol{x}) \quad \text{D (2)}$$
$$\times \int d^3y \bar{u}_p(\boldsymbol{y}) \gamma_\nu (1 - \gamma^5) u_q(\boldsymbol{y}) W^\nu(\boldsymbol{y}) \int d^3z \bar{u}_q(\boldsymbol{z}) \gamma_\rho u_{n'}(\boldsymbol{z}) a_{m\Sigma}^{\rho}(\boldsymbol{z}) \quad .$$

Use quark completeness in eq. A (14)

$$\sum_n u_n(\boldsymbol{x}) \bar{u}_n(\boldsymbol{x}') = \delta^3(\boldsymbol{x} - \boldsymbol{x}') \gamma_o \quad , \quad \text{D (3)}$$

to obtain the result

$$V_{nn'}^{m\Sigma} = -4\pi g^{\Sigma\Sigma} \sum_M \int d^3x \bar{u}_n(\boldsymbol{x}) \gamma_\mu \gamma_o \gamma_\nu (1 - \gamma^5) \gamma_o \gamma_\rho u_{n'}(\boldsymbol{x}) a_{m\Sigma}^{\mu *}(\boldsymbol{x}) W^\nu(\boldsymbol{x}) a_{m\Sigma}^{\rho}(\boldsymbol{x}) \quad \text{D . (4)}$$

Make use of the identities

$$\begin{aligned} \not{a}\, \gamma_o &= -\gamma_o \not{a} + 2a_o \qquad \not{a}^* \cdot \not{a} = a^* \cdot a \\ \not{a}\, \not{b} &= -\not{b}\, \not{a} + 2a \cdot b \qquad \not{a}\, \gamma_5 = -\gamma_5 \not{a} \quad , \end{aligned} \quad \text{D (5)}$$

to obtain after some Dirac algebra that

$$\begin{aligned} \gamma_o \not{a}^* \gamma_o \not{W} \gamma_o \not{a} &= a \cdot a^* (\gamma_o \not{W} - 2W_o) + 2(a_o W_0 + \boldsymbol{a} \cdot \boldsymbol{W}) \gamma_o \not{a}^* \\ \gamma_o \not{a}^* \gamma_o \not{W} \gamma^5 \gamma_o \not{a} &= \gamma^5 (\gamma_o \not{a}^* \gamma_o \not{W} \gamma_o \not{a}) \quad . \end{aligned} \quad \text{D (5)}$$

So eq. D (4) becomes

$$\begin{aligned} V_{nn'}^{m\Sigma} = -4\pi g^{\Sigma\Sigma} \sum_M \int d^3x u_n^+(\boldsymbol{x}) (1 - \gamma^5) [& a \cdot a^* (\gamma_o \not{W} - 2W_o) \\ &+ 2(a_o W_o + \boldsymbol{a} \cdot \boldsymbol{W}) \gamma_o \not{a}^*] \end{aligned} \quad \text{D (6)}$$

The explicit simplification of eq. D (6) in the Σ = L case is shown. If the potentials of the gluon fields (eqs. A (16) to

A (19) are used, the sum over the first term in eq. D (6) can be done using (VA 88)

$$\sum_{m=J}^{J} \mathbf{Y}_{JM}^{L*}(\mathbf{r}) \cdot \mathbf{Y}_{JM}^{L'}(\mathbf{r}) = \frac{2J+1}{4\pi}\delta_{LL'} \quad .$$

For the first term we have

$$4\pi g^{LL}\sum_{M} a \cdot a^{*} = (2J+1)\,\Phi_{mL}(r) \qquad \text{D (7)}$$

$$\Phi_{mL}(r) = N_{mL}^{2}((J+1)j_{J+1}^{2}(\Omega r) + Jj_{J-1}^{2}(\Omega r)) \quad .$$

The second term in eq. D (6) requires considerably more work. Noting that $\mathbf{W}$ $(\mathbf{r})$ = (0, 0, W_z), it can be seen that

$$2g^{LL}\sum_{M} \mathbf{a}_{mL}(\mathbf{r}) \cdot \mathbf{W}(\mathbf{r})\, \not{a}_{mL}^{*}(\mathbf{r}) = 2g^{LL}\sum_{M} (\mathbf{a}_{mL}(\mathbf{r}))_{z} W_{z} \not{a}_{mL}^{*}(\mathbf{r}) \quad , \qquad \text{D (8)}$$

and noting that $\hat{e}_z = \hat{e}_o$ (where $\hat{e}_o$ is the spherical basis vector, and $\hat{e}_z$ the cartesian basis vector), it will be observed that

$$(\mathbf{a}_{mL}(\mathbf{r}))_{z} = \mathbf{a}_{mL}(\mathbf{r}) \cdot \hat{e}_{o} \quad . \qquad \text{D (9)}$$

Making use of the gluon potentials in Appendix B.1, note (VA 88) that

$$\mathbf{Y}_{JM}^{L} \cdot \hat{e}_{o} = \langle LM10|JM\rangle Y_{LM} \qquad L = J-1 \text{ or } J+1 \quad , \qquad \text{D (10)}$$

from which we find:

$$(\mathbf{a}_{mL}(\mathbf{r}))_{o} = \frac{N_{mL}}{\sqrt{2J+1}}\Big[\sqrt{J}j_{J-1}(\Omega r)\langle J-1M10|JM\rangle Y_{J-1\,M}(\hat{r}) + \sqrt{J+1}\,j_{J+1}(\Omega r)\langle J+1M10|JM\rangle Y_{J+1\,M}(\hat{r})\Big] \quad . \qquad \text{D (11)}$$

Since $\not{a}_{mL}^{*}(\mathbf{r}) = -\gamma \cdot \mathbf{a}_{mL}^{*}(\mathbf{r})$, it can be seen that in order to calculate the sum in eq. D (8) we need to compute the quantity U(L,ℓ,J), which is defined by:

$$\mathbf{U}(L,\ell,J) \equiv \sum_{M} \langle LM10|JM\rangle Y_{LM}\mathbf{Y}_{JM}^{*\ell} \quad , \qquad \text{D (12)}$$

where L = J + 1 or J - 1
ℓ = J + 1 or J - 1.

An alternative is to make use of the recursion relation for the spherical harmonics (VA 88) to write eq. D (11) as:

$$(\boldsymbol{a}_{mL}(\boldsymbol{r}))_o = N_{mL}\Bigg[\sqrt{\frac{(J-M)(J+M)}{(2J+1)(2J-1)}}\,Y_{J-1\,M}(\hat{r}) \times (j_{J-1}(\Omega r) + j_{J+1}(\Omega r)) - j_{J+1}(\Omega r)\cos\theta\, Y_{JM}(\hat{r})\Bigg] \quad , \qquad \text{D (13)}$$

where the Clebsch-Gordan coefficients in eq. D (11) have been evaluated explicitly. This does not lead to significant simplification to what showed in eq. D (11), so we assume the form in eq. D (11) for the remainder of this appendix.

Now using $\mathbf{Y}^{*\ell}_{JM} = \mathbf{Y}^{\ell}_{J-M}$ (VA p.215), the expansion of the vector spherical harmonics in terms of spinors and spherical harmonics, and (VA p. 144):

$$Y_{LM}Y_{\ell\mu} = \sum_{Kk} \frac{\pounds\,\hat{\ell}}{\sqrt{4\pi}\,\hat{K}} \langle LM\ell\mu|Kk\rangle\, Y_{Kk} \quad , \qquad \text{D (14)}$$

it is found eq. D (12) is identical to

$$\boldsymbol{\sigma}(L,\ell,J) = \frac{\pounds\,\hat{\ell}}{\sqrt{4\pi}\,\hat{K}}\,(-1)^{J+\ell+1} \times \sum_{\mu\sigma MKk} \hat{e}_\sigma(-1)^M \langle LM10|JM\rangle\, \langle \ell\mu 1\sigma|J-M\rangle\, \langle LM\ell\mu|Kk\rangle\, Y_{Kk} \quad . \qquad \text{D (15)}$$

Now summing first over μ and M using (VA p.245) and (VA p.260) (16)), it is seen that eq. D (15) equals

$$\boldsymbol{\sigma}(L,\ell,J) = (-1)^{J+1}\,\frac{\mathcal{J}^2\,\pounds\,\hat{\ell}}{\sqrt{12\pi}} \sum_{K\sigma} (-1)^{K+\sigma} \times \langle K-\sigma 1\,0|1-\sigma\rangle \begin{Bmatrix} L & \ell & K \\ 1 & 1 & J \end{Bmatrix} \hat{e}_\sigma Y_{K-\sigma} \quad . \qquad \text{D (16)}$$

At this point we can evaluate the Clebsch-Gordan coefficients explicitly (VA p.271):

$$\langle K\ -\sigma\ 1\ 0|1\ -\ \sigma\rangle = \sqrt{1-\sigma^2} \quad \text{if } K = 0$$
$$= \frac{-\sigma}{2} \quad \text{if } K = 1 \quad , \qquad \text{D (17)}$$
$$= -\sqrt{\frac{4-\sigma^2}{10}} \quad \text{if } K = 2$$

as well as the 6j-symbols (VA p.311) in eq. D (16) and write a REDUCE 3.3 program to expand eq. D (16). However, we shall retain the form D (16) for the rest of this appendix. Now returning to the original aim of this exercise, we note that eq. A (8), when the gluon potentials are expanded, can be written

$$2g^{LL}\sum_M \boldsymbol{a}_{mL}(\boldsymbol{r})\cdot\boldsymbol{W}(\boldsymbol{r})\,\boldsymbol{a}^*_{mL}(\boldsymbol{r}) = \frac{2W_z N^2_{mL}}{\mathcal{J}}\gamma\Big[Jj^2_{J-1}(\Omega r)\,\boldsymbol{U}(J-1,J-1,J) \qquad \text{D (18)}$$
$$+\ 2\sqrt{J(J+1)}\,\boldsymbol{U}(J-1,J+1,J)\,j_{J+1}(\Omega r)\,j_{J-1}(\Omega r)$$
$$+\ (J+1)\,j^2_{J+1}(\Omega r)\,\boldsymbol{U}(J+1,J+1,J)\Big] \quad ,$$

where $\mathbf{U}$(J - 1, J + 1, J) = $\mathbf{U}$ (J + 1, J - 1, J) is used, which is obtained by noting that $\mathbf{U}$ (L, ℓ, J) as given in eq. D (16) is symmetric under exchange L ↔ ℓ since the 6j-symbol is symmetric under column exchange. The term in square brackets in eq. D (18) can be written, using eq. D (16) as

$$(-1)^{J+1}\frac{\mathcal{J}^2}{\sqrt{12}}\sum_{K\sigma}(-1)^{K+\sigma}\hat{e}_\sigma Y_{K-\sigma}\langle K-\sigma\ 10\ |1-\sigma\rangle\,\Psi_{mLK}(r)\,N^{-2}_{mL}$$

$$\Psi_{mLK}(r) = N^2_{mL}\Bigg[J(2J-1)\begin{Bmatrix}J-1 & J-1 & K\\ 1 & 1 & J\end{Bmatrix}j^2_{J-1}(\Omega r) \qquad \text{D (19)}$$
$$+\ 2\sqrt{J(J+1)(2J+3)(2J-1)}\begin{Bmatrix}J-1 & J+1 & K\\ 1 & 1 & J\end{Bmatrix}j_{J+1}(\Omega r)\,j_{J-1}(\Omega r)$$
$$+\ (J+1)(2J+3)\begin{Bmatrix}J+1 & J+1 & K\\ 1 & 1 & J\end{Bmatrix}j^2_{J+1}(\Omega r)\Bigg] \quad .$$

Finally, putting eq. D (19) into eq. D (18) and adding the previous term in eq. D (7) it is obtained that

$$V^{mL}_{nn'} = -\int d^3r u^+_n(\boldsymbol{r})\,(1-\gamma^5)\,(2J+1)\,\Phi_{m\Sigma}(r)\,(\gamma_o \not{W} - 2W_o)\, u_{n'}(\boldsymbol{r})$$
$$- \sqrt{\tfrac{4\pi}{3}}(-1)^{J+1} 2W_z \sum_{K\sigma} (-1)^{K+\sigma} \hat{e}_\sigma \langle K - \sigma 1 0 | 1 - \sigma\rangle \qquad \text{D (20)}$$
$$\times \int d^3r u^+_n(\boldsymbol{r})\,(1-\gamma^5)\,\gamma_o \boldsymbol{\gamma} Y_{K-\sigma}(\boldsymbol{r})\, u_{n'}(\boldsymbol{r})\, \Psi_{mLK}(r) \quad .$$

Now, the simplification of eq. D (20) is very straight forward. For the first term in eq. D (20) we see that it is very similar to eq. B (12), except for the additional radial function $\Phi_{m\Sigma}(r)$ which will appear in the radial part of the integral. For the second term in eq. D (20) we also have similarity to eq. B (12), with $\Psi_{mLK}(r)$ appearing additionally in the radial integral. The additional angular term $Y_{K-\sigma}(\hat{r})$ can be dealt with by using (VA p.148 (4)):

$$\int d\Omega \chi^{\mu^*}_{\kappa} \sigma_3 \chi^{\mu'}_{\kappa} Y_{K-\sigma} = \sum_{mm'\xi} 2\xi \frac{\hat{\ell}' \hat{K}}{\sqrt{4\pi}\,\hat{\ell}} \langle \ell' 0\; K 0 \,|\ell\, o\rangle \langle \ell' m' K - \sigma | \ell m\rangle \qquad \text{D (21)}$$
$$\times \langle \ell m \tfrac{1}{2}\xi | j\mu\rangle \langle \ell' m' \tfrac{1}{2}\xi | j'\mu'\rangle \quad .$$

The sum rule obtained in eq. D (20) can now be employed to check numerical errors.

Appendix E

COLOUR AND FLAVOUR MATRIX ELEMENTS

The matrix elements of the external operator, and quark creation and annihilation operators between the proton and neutron states, will be derived in this appendix. They were carried out using the symbolic manipulation package REDUCE 3.3 (a PL1-type language), taking about 9 minutes for the one-body and two hours for the two-body matrix elements on an Apollo DN 3500 computer. Because all external quarks have energy $1s_{1/2}$ we know that the requirement $\delta_{1s_{1/2},n'}$ should be part of the colour-flavour matrix element. The program employs non-commuting arrays depending on the colour, whether the operator annihilates or creates, the flavour and the spin projection. The colour sum (restricted by ϵ^{abc}) is large, but arithmetic is shortened by using the relation $<0\ |a^{+}_{cfn} = 0$ and $a_{cfn}\ |0> = 0$ during the evaluation. ϵ^{abc} is calculated by calculating the number of permutations necessary to bring it in a form $\epsilon^{a'b'c'}$ where $a' < b' < c'$.

E.1 ONE-BODY COLOUR AND FLAVOUR MATRIX ELEMENTS

In the main text the colour and flavour matrix elements for the one body interaction are given as

$$\sum_{f'fc'c} <N\uparrow|a^{+}_{c'f'n_1}\left(\frac{\lambda^a}{2}\right)_{c'd}\left(\frac{\lambda^a}{2}\right)_{dc}\left(\frac{\tau^{+}_{f'f}}{2} + \frac{\tau^{+}_{f'f}}{2}\right)a^{+}_{cfn_2}|P\uparrow> \quad , \qquad E\ (1)$$

where

$$\frac{\tau^{+}_{f'f}}{2} = \begin{cases} 1 \text{ if } f' = u,\ f = d \\ 0 \text{ otherwise} \end{cases} \quad , \qquad \frac{\tau^{-}_{f'f}}{2} = \begin{cases} 1 \text{ if } f' = d,\ f = u \\ 0 \text{ otherwise} \end{cases} \qquad \text{E (2)}$$

and the λ^a's are in the fundamental representation of SU(3) and in second quantization

$$|P\uparrow\rangle = \frac{\epsilon^{abc}}{\sqrt{18}}\left[a^{+}_{au1s\uparrow}a^{+}_{bd1s\downarrow} - a^{+}_{au1s\downarrow}\ a^{+}_{bd1s\uparrow}\right] a^{+}_{cu1s\uparrow}|0\rangle \qquad \text{E (3)}$$

$$|N\uparrow\rangle = \frac{\epsilon^{abc}}{\sqrt{18}}\left[a^{+}_{ad1s\uparrow}a^{+}_{bu1s\downarrow} - a^{+}_{ad1s\downarrow}\ a^{+}_{bu1s\uparrow}\right] a^{+}_{cd1s\uparrow}|0\rangle \quad . \qquad \text{E (4)}$$

Noting

$$\sum_d \left(\frac{\lambda^a}{2}\right)_{c'd}\left(\frac{\lambda^a}{2}\right)_{dc} = \frac{4}{3}\delta_{c'c} \quad , \qquad \text{E (5)}$$

we simplify

$$\sum_{f'fc'c} \langle N\uparrow|\hat{a}_{c'f'n_1}\left(\frac{\lambda^a}{2}\right)_{c'd}\left(\frac{\lambda^a}{2}\right)_{dc}\frac{\tau^{-}_{f'f}}{2}\hat{a}_{afn_2}|P\uparrow\rangle = \sum_c \frac{4}{3}\langle N\uparrow|\hat{a}^{+}_{cun_1}\hat{a}^{+}_{cdn_2}|P\uparrow\rangle \qquad \text{E (6)},$$

as well as

$$\sum_{f'fc'c} \langle N\uparrow|\hat{a}^{+}_{c'f'n_1}\left(\frac{\lambda^a}{2}\right)_{c'd}\left(\frac{\lambda^a}{2}\right)_{dc}\frac{\tau^{-}_{f'f}}{2}\hat{a}_{cfn_2}|P\uparrow\rangle = \sum_c \frac{4}{3}\langle N\uparrow|\hat{a}^{+}_{cdn_1}\hat{a}^{+}_{cun_2}|P\uparrow\rangle \qquad \text{E (7)}.$$

Note that the creation and annihilation operators arose from the expansion of the field operator ψ in terms of cavity modes and creation operators. To carry out eqs. E (6) and E (7) we use

$$\left\{a_{cfn}, a^{+}_{c'f'n'}\right\} = \delta_{cc'}\delta_{ff'}\delta_{nn'} \quad , \qquad \text{E (8)}$$

to anti-commute the annihilation operators to the right until they annihilate the vacuum. Similarly we anti-commute the creation operators to the left until they annihilate the vacuum.

We obtain that eq. E (6) equals zero, but eq. E (7) equals

$$\frac{4}{9}\left(\delta_{\mu_1\downarrow}\delta_{\mu_2\downarrow} - 4\delta_{\mu_1\uparrow}\delta_{\mu_2\uparrow}\right)\delta_{n,1s_{\frac{1}{2}}} \quad . \qquad \text{E (9)}$$

E.2 ONE-BODY SYMMETRY RELATIONS

Referring to eqs. E (6) and E (7), we find:

$$<N\uparrow|\hat{a}^{+}_{cf'n_1}\tau^{+}_{f'f}\hat{a}_{cfn_2}|P\uparrow> \; = \; <N\uparrow|\hat{a}^{+}_{cun_1}\hat{a}_{cdn_2}|P\uparrow> \quad . \qquad \text{E (10)}$$

Now because eq. E (10) is independent of flavour (refer to eq. E (9)), we can exchange the labels u and d inside eq. E (10) with the result that |N↑> changes into |P↑> , and vice versa. But then eq. E (10) becomes

$$<P\uparrow|\hat{a}^{+}_{cdn_1}\hat{a}_{cun_2}|N\uparrow> \; = \; <P\uparrow|\hat{a}^{+}_{cf'n_1}\tau^{-}_{f'f}\hat{a}_{cfn_2}|N\uparrow> \quad . \qquad \text{E (11)}$$

So eq. E (10) and E (11) yields, referring to the expressions for the energy shifts in Chapter 4 for the vertex correction, self-energy and two-body diagrams; that we get the same energy shifts if we use τ^{+} where |N↑> acts on the left and |P↑> on the right, and if we use τ^{-} where |P↑> acts on the left and |N↑> on the right. Similarly, we have the same energy shifts if we use τ^{-} where |N↑> acts on the left and |P↑> on the right, and if we use τ^{+} where |P↑> acts on the left and |N↑> on the right.

We also show that the colour-flavour matrix element with τ^{+} (eq. E (10)), in which |N↑> acts on the left and |P↑> on the right, is zero. Looking at the operators symbolically (where u and d refer to up and down flavours respectively), we obtain

$$\langle N\uparrow|u^+d|P\uparrow\rangle \sim \langle 0|dudu^+du^+d^+u^+|0\rangle$$
$$= -\langle 0|duu^+ddu^+d^+u^+|0\rangle = +\langle 0|duu^+ddu^+d^+u^+|0\rangle$$
$$= -\langle 0|dudu^+du^+d^+u^+|0\rangle \quad . \qquad \text{E (12)}$$

So

$$\langle N\uparrow|\hat{a}^+_{cun_1}\hat{a}_{cdn_2}|P\uparrow\rangle \equiv \langle N\uparrow|u^+d|P\uparrow\rangle = 0 \quad .$$

From the previous paragraph it follows that the colour-flavour matrix element with τ^- in which |P↑> acts on the left and |N↑> on the right must also be zero.

All the above (together with the expected result that if the colour-flavour matrix element is evaluated between two |N↑> states or two |P↑> states, then it is zero) has been verified in REDUCE 3.3. The above can be proven by a similar line of reasoning for the two-body colour-flavour matrix elements, and was also verified in REDUCE 3.3.

E.3 TWO-BODY COLOUR AND FLAVOUR MATRIX ELEMENTS

The two-body matrix elements are given by

$$\sum_{\substack{c'd'cd \\ f'g'fg}} \langle N\uparrow|\hat{a}^+_{c'f'n_1}\hat{a}_{d'g'n_3}\left(\frac{\lambda^a}{2}\right)_{c'c}\left(\frac{\lambda^a}{2}\right)_{d'd}\left[\left(\frac{\tau^+}{2}\right)_{g'g} + \left(\frac{\tau^-}{2}\right)_{g'g}\right] I_{f'f}\hat{a}_{cfn_2}\hat{a}_{dgn_4}|P\uparrow\rangle \quad \text{E (13)}$$

Now using the colour factor

$$\left(\frac{\lambda^a}{2}\right)_{c'c}\left(\frac{\lambda^a}{2}\right)_{d'd} = \frac{1}{2}\left(\delta_{c'd}\delta_{cd'} - \frac{1}{3}\delta_{c'c}\delta_{d'd}\right) \quad , \qquad \text{E (14)}$$

we restrict the colour changes along the quark lines. Noting the definitions of τ^+ and τ^- in eq. E (2), we obtain

$$\sum_{\substack{c'd'cd \\ f'g'fg}} \langle N\uparrow | a^{+}_{c'f'n_1} \hat{a}_{d'g'n_3} \left(\frac{\lambda^a}{2}\right)_{c'c} \left(\frac{\lambda^a}{2}\right)_{d'd} \left(\frac{\tau^+}{2}\right)_{g'g} I_{f'f} \hat{a}_{cfn_2} \hat{a}_{dgn_4} | P\uparrow \rangle \qquad \text{E (15)}$$

$$= \sum_{cdf} \left\{ \frac{1}{2} \langle N\uparrow | \hat{a}_{dfn_1} \hat{a}_{cun_3} \hat{a}_{cfn_2} \hat{a}_{ddn_4} | P\uparrow \rangle - \frac{1}{6} \langle N\uparrow | \hat{a}^{+}_{cfn_1} \hat{a}^{+}_{dun_3} \hat{a}_{cfn_2} \hat{a}_{ddn_4} | P\uparrow \rangle \right\} \quad ,$$

where the second d in the last annihilation operator refers to a down quark. Similarly

$$\sum_{\substack{c'd'cd \\ f'g'fg}} \langle N\uparrow | a^{+}_{c'f'n_1} \hat{a}^{+}_{d'g'n_3} \left(\frac{\lambda^a}{2}\right)_{c'c} \left(\frac{\lambda^a}{2}\right)_{d'd} \left(\frac{\tau^-}{2}\right)_{g'g} I_{f'f} \hat{a}_{cfn_2} \hat{a}_{dgn_4} | P\uparrow \rangle \qquad \text{E (16)}$$

$$= \sum_{cdf} \left\{ \frac{1}{2} \langle N\uparrow | \hat{a}^{+}_{dfn_1} \hat{a}^{+}_{cun_3} \hat{a}_{cfn_2} \hat{a}_{ddn_4} | P\uparrow \rangle - \frac{1}{6} \langle N\uparrow | \hat{a}^{+}_{cfn_1} \hat{a}^{+}_{dun_3} \hat{a}_{cfn_2} \hat{a}_{ddn_4} | P\uparrow \rangle \right\} \quad .$$

In a way similar to that described in section E.1, we calculate with REDUCE 3.3 that eq. E (15) equals zero, but eq. E (16) equals

$$\begin{aligned} -\frac{2}{9} \big(& 4\delta_{\mu_1\downarrow}\delta_{\mu_3\uparrow}\delta_{\mu_3\downarrow}\delta_{\mu_4\uparrow} + \delta_{\mu_1\downarrow}\delta_{\mu_3\uparrow}\delta_{\mu_2\uparrow}\delta_{\mu_4\downarrow} + \delta_{\mu_1\downarrow}\delta_{\mu_3\downarrow}\delta_{\mu_2\downarrow}\delta_{\mu_4\uparrow} \\ & - 2\delta_{\mu_1\uparrow}\delta_{\mu_3\downarrow}\delta_{\mu_2\uparrow}\delta_{\mu_4\downarrow} + 4\delta_{\mu_1\uparrow}\delta_{\mu_3\uparrow}\delta_{\mu_2\uparrow}\delta_{\mu_4\uparrow} \big) \\ & \times \delta_{n_1,\,1s\frac{1}{2}}\, \delta_{n_2,\,1s\frac{1}{2}}\, \delta_{n_3,\,1s\frac{1}{2}}\, \delta_{n_4,\,1s\frac{1}{2}} \quad . \end{aligned} \qquad \text{E (17)}$$

Appendix F

NATURAL UNITS AND CONVENTIONS

In this appendix commonly used conventions are explicitly stated and a brief discussion of the Natural unit system is given.

Throughout this thesis Natural units, i.e. $\hbar = c = 1$, are used. At the end of the calculation the reduced Planck constant and the speed of light can be restored by examining the dimensions of the quantities in the formulae. The following numerical values for $\hbar c$ and the proton mass are employed:

$\hbar c = 0.197\ 327$ GeV fm

$m_p = 0.938\ 272$ GeV/c^2.

The cavity radius R is set to equal 1 fm, and is the natural unit of length in the spherical cavity. This unit should also be restored if M·K·S· units are required. For example, if at the end of a calculation we want to change dimensionless energy ω into a dimensional energy ε, we note that energy has M·K·S· units of kgm^2s^{-2} which is the same as the units of $^{\hbar c}/_{R}$, so this quantity needs to be restored. We obtain $\varepsilon = {}^{\omega R}/_{\hbar c} = {}^{\omega}/_{0.197}$ GeV.

Bessel's differential equation is given by

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)\right] j_\nu(z) = 0 \quad .$$

The solutions are given by Bessel functions of the first kind, $j_\nu(x)$.

The convention for the flat Minkowski space metric is:

$g^{\mu\nu} = g_{\mu\nu} = \text{diag}\ \{+1, -1, -1, -1\}$,

and the Clifford algebra satisfied by the 4x4 Dirac γ matrices is

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

The γ matrices may be represented as

$$\gamma^{o} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^{\kappa} = \begin{pmatrix} 0 & \sigma^{\kappa} \\ -\sigma^{\kappa} & 0 \end{pmatrix} \quad ,$$

where the σ^{κ} are the 2x2 Pauli matrices:

$$\sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

Throughout this work we use the notation $G/\sqrt{2} = G_F/\sqrt{2} \cos \theta_c$, where G_F is the Fermi constant for weak interaction, and θ_c the Cabibbo weak mixing angle.

BIBLIOGRAPHY

AB 72 M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, 1972

AD 65 S.L. Adler, Phys. Rev. **140B** (1065) 736

BE 74 C. Becchi, A. Rouet, R. Stora, Phys. Lett. **52B** (1974) 344

BJ 69 J.D. Bjorken, Phys. Rev. **179** (1969) 1547

BO 83 M. Bourquin et. al., Z. Phys. **C21** (1983) 27

CE 79 G.E. Brown, M. Rho, Phys. Lett. **B82** (1979) 177

BU 88 R.F. Buser, R.D. Viollier, P. Zimak, International Journal of Theoretical Physics **27** (1988) 925

CA 63 N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531

CE 77 CERN 77-18, Proceedings of the 1977 CERN-JINR School of Physics, CERN, 1977

CE 74 A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, V.F. Weisskopf, Phys. Rev. **D9** (1974) 3471

CE 74 A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, Phys. Rev. **D10** (1974) 2599

CL 79 F.E. Close, An Introduction to Quarks and Partons, Academic Press, 1979

CO 75 E.G. Commins, Weak Interactions, McGraw-Hill, 1975

ED 57 A.R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton Univ. Press, 1957

FE 48 R.P. Feynman, Rev. Mod. Phys. **20** (1948) 367

FE 58 R.P. Feynman, M. Gell-Mann, Phys. Rev. **109** (1958) 193

FE 71 A.L. Fetter, J.D. Walecka, Quantum Theory of Many Particle Systems, McGraw-Hill, 1971

FR 73 H. Fritsch, M. Gell-Mann, H. Leutwyler, Phys. Lett. **B47**, (1973) 365

GA 36 G. Gamow, E. Teller, Phys. Rev. **49** (1936) 895

GE 51 M. Gell-Mann, F. Low, Phys. Rev. **84** (1951) 350

GE 64 M. Gell-Mann, Phys. Lett. **8** (1964) 214

GI 88 E. Gillman, H.R. Feibig, Computation in Physics Vol.**2** no.1 (1988) 62

GL 91 S.L. Glashow, The Charm of Physics, American Institute of Physics, 1991

GO 58 M.L. Goldberger, S.B. Treiman, Phys. Rev. **110** (1958) 1178; ibid, **111** (1958) 354

HA 84 F. Halzen, A.D. Martin, Quarks and Leptons : An Introductory Course in Modern Particle Physics, Wiley, 1984

HO 88 H. Høgaasen, F. Myhrer, Phys. Rev. **D37** (1988) 1950

IT 80 C. Itzykson, J. Zuber, Quantum Field Theory, McGraw-Hill, 1980

KA 87 G. Kane, Modern Elementary Particle Physics, Addison-Wesley, 1987

KR 77 F.G. Krausz, Phys. Lett. **66B** (1977) 251

OC 90 M.O. O'Connor, The Anomalous Magnetic Moment of the Nucleon in Cavity Quantum Chromodynamics, Ph.D. thesis, unpublished, University of Cape Town preprint UCT-TP 164/91

MA 83 O.V. Maxwell, V. Vento, Nucl. Phys. **A407**, (1983) 366

MU 87 T. Muta, Foundations of Quantum Chromodynamics, an Introduction to Perturbative Methods in Gauge Theories, World Scientific, 1987

PA 81 G. Parisi and Y. Wu, Sci. Sinica **24** (1981) 483

PA 86 Particle Data Group, Phys. Lett. **B170** (1986) 1

PR 86 W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, Numerical Recipes : The Art of Scientific Computing, Cambridge Univ. Press, 1986

ST 87 A.J. Stoddart, Perturbative Quantum Chromodynamics in a Cavity, M.Sc. thesis, unpublished, University of Cape Town preprint UCT-TP 83/87 (1987)

ST 90 A.J. Stoddart, Renormalization of Cavity Field Theories, Ph.D. thesis, unpublished, University of Cape Town preprint UCT-TP 139/90 (1990).

A.J. Stoddart, R.D. Viollier, Phys. Lett. **B236** (1990) 387;

ST 91 A.J. Stoddart, R.D. Viollier, Nucl. Phys. **A532** (1991) 657

SU 57 J. Sucher, Phys. Rev. **107** (1957) 1448

TH 71 G. 't Hooft, Nucl. Phys. **B33** (1971) 173

TS 88 K. Tsushima, T. Yamaguchi, Y. Kohyama, K. Kubodera, Nucl. Phys. **A489** (1988) 557

US 84 K. Ushio, H. Konashi, Phys. Lett. **B135** (1984) 468

VA 88 V.A. Varshalovich, A.N. Moskalev, V.K. Khersonskii, Quantum Theory of Angular Momentum, World Scientific, 1988

WE 73 S. Weinberg, Phys. Rev. Lett. **31** (1973) 494

YA 54 C.N. Yang, R.L. Mills, Phys. Rev. **96** (1954) 191

ZI 69 W. Zimmermann, Comm. Math. Phys. **15** (1969) 208