COUNTABLE INDUCTIVE LIMITS

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Countable Inductive Limits

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CONTENTS.

INTRODUCTION.

Inductive systems and inductive limits have by now become fairly well established in the general theory of topological vector spaces. It is a branch of Functional Analysis which is receiving a reasonable amount of attention by modern mathematicians. It is of course a very interesting subject of its own accord, but is also useful in solving problems and proving theorems which one does not suspect are intimately related to it. As an example we can consider the proof of the non-existence of a countably infinite dimensional metrisable barrelled space.

In section II the basic definitions and elementary results will be given which are necessary for the development of the theory which follows. It contains no new results although I have had to prove most of the statements without the help of existing literature in order that they fit in the given exposition.

C-convergence is defined and developed in the third section. It is probably the most self-contained section after the definitions. The theory of this section is not required for the latter work. Since the definition of C-convergence is original here, so is the subsequent theory. Chapter IV follows with a basic important theorem concerning the inductive limit in a barrelled or quasi-barrelled space.

Section V is the most important of the thesis. Theorem 5.3, which is original, was inspired by the two unproved theorems stated by Makarov [14]. A well known theorem (theorem 5.4) then follows as an easy corollary to this but the proof of theorem 5.3 is not a generalization of the usual proof of theorem 5.4. In fact the restriction of the general proof to the special case is more elegant and shorter than the conventional one. Following the theorem, an example is given showing that certain conditions in the statement in the forgoing theorem cannot be dropped. The section is concluded with a few interesting obsevations.

Bennett and Cooper state some interesting results on the weak topology in [3] without proofs. Section VI is concerned with proving some of these, and investigating an inductive system which arises naturally from the existing one.

In the paper on generalized inductive limits by Garling [7], it is found that theorem 5.3 of this thesis is almost sufficient to prove one of his basic results. In fact an independent proof to that involving theorem 5.3 is also given, as the original appears to have many gaps and some steps which seem almost unjustifiable. The sets he constructs appear not to fulfill their required purpose.

Chapter VIII is involved with proving an improvement of a theorem by Amemiya and Kōmura [1], which is followed by several interesting corollaries. The next section investigates a few particular inductive limits which are proved to be sequentially retractive, a concept defined by Floret [6].

In the final section, two inductive systems are investigated, and their equivalence is considered. The example is of particular interest here as it uses many of the results developed in the thesis.

I should like to take this opportunity to thank my supervisor Dr. J.H.Webb without whose constant encouragement and invaluable help at all times this project would never have materialised.

DEFINITIONS AND NOTATION.

The basic terminology is roughly that of Robertson, A.P. and W. [17] or Kelley, J.L. and Namioka, I. [9].

Consider a linear space E as the union of an increasing sequence of locally convex linear topological spaces E_n with topologies τ_n. Suppose that the injection maps

$$i_n : E_n \to E_{n+1}$$

are continuous, i.e. that $\tau_{n+1}|_{E_n} < \tau_n$.

Definition 2.1 The above will be called an inductive system.

Definition 2.2 Two inductive systems $E = \cup E_i$ and $F = \cup F_i$ will be called equivalent if, given an n, there exists an m such that $E_n \subset F_m$ and the inclusion map is continuous and, given any p, there exists a q such that $F_p \subset E_q$ with the inclusion map being continuous.

Definition 2.3 The inductive limit topology τ on E is the finest locally convex topology making each injection $j_n : E_n \to E$ continuous.

In other words τ is the finest locally convex topology such that $\tau|_{E_n} < \tau_n$ for each n. We will often refer to $E[\tau]$ as the inductive limit of the $E_n[\tau_n]$ and we will write

$$E[\tau] = \text{lim. ind. } E_n[\tau_n]$$

It follows directly from the above definitions that equivalent inductive systems yield the same inductive limit.

The set of all absolutely convex subsets V of E such that $V \cap E_n$ is a τ_n-neighbourhood for each n clearly forms a neighbourhood base of 0 for a locally convex topology τ (say). We therefore have the injection

$$j_n: E_n[\tau_n] \to E[\tau]$$

being continuous for each n. Conversely, if each j_n is continuous for some topology τ then, for each τ-neighbourhood U, $U \cap E_n$ must be a τ_n-neighbourhood. Thus, if we let τ be generated by all such U, τ is the finest locally convex topology making the injections continuous. This proves the existence of the inductive limit topology and gives us a frequently used characterization of its neighbourhoods.

We call an inductive system <u>strict</u> if $\tau_{n+1}|_{E_n} = \tau_n$. A strict inductive limit is then the inductive limit topology of a strict inductive system. We now establish that the relativisation to E_n of a strict inductive limit topology is exactly τ_n.

We need first the following:

<u>Remark.</u> Let E be a linear topological space and F a subspace. Let U be an absolutely convex neighbourhood in F (with the relative topology). Then there exists an absolutely convex neighbourhood V in E such that $V \cap F \subset U$. Let $W = \Gamma(U \cup V)$ (the

absolutely convex hull of U and V). Clearly $U \subset W \cap F$.

Conversely, let $x \in W \cap F$. Therefore

$$x = \alpha y + (1-\alpha)z \qquad \text{where } y \in V, \quad z \in U.$$

If $\alpha = 0$ then $x = z \in U$. If $\alpha \not\leq 0$, then

$$y = \frac{1}{\alpha}x - \frac{1-\alpha}{\alpha}z$$

$\frac{1}{\alpha}x \in F$ since it is a subspace and similarly, $\frac{1-\alpha}{\alpha}z \in F$ since $U \subset F$. Hence $y \in F$, thus $y \in F \cap V \subset U$. Hence $x \in U$ and we have $U = W \cap F$.

We therefore have, for any locally convex linear topological space E with subspace F, any absolutely convex neighbourhood U in F is exactly the restriction of an absolutely convex neighbourhood in E to F. We require this property in the following proof.

We know that $\tau_n > \tau|_{E_n}$. We therefore need only show the reverse inclusion to prove our earlier assertion.

Let U_n be any absolutely convex neighbourhood in E_n under τ_n. We prove that there is a τ-neighbourhood U in E such that $U \cap E_n = U_n$. The result then follows by the characterization given of an inductive limit topology to prove its existence.

Since τ_{n+1} induces τ_n on E_n, there exists, by our previous remark, an absolutely convex τ_{n+1}-neighbourhood U_{n+1} with $U_{n+1} \cap E_n = U_n$. Continuing in this way, we can define for each r, an absolutely convex τ_{n+r}-neighbourhood U_{n+r} such that $U_{n+r} \cap E_{n+s} = U_{n+s}$ for $0 \leq s \leq r$. The existence of the U_{n+r} is a consequence of the preceding remark.

Let $U = \bigcup_{r=0}^{\infty} U_{n+r}$. U is clearly absolutely convex. Also, by our construction, $U \cap E_{n+r} = U_{n+r}$, a neighbourhood of E_{n+r} for $r \geq 0$ and $U \cap E_m = U_n \cap E_m$ which is a neighbourhood of E_m for $m \leq n$. Hence U is a neighbourhood in E and we have proved our assertion.

The following is a useful characterization of the inductive limit topology τ. τ is the coarsest locally convex topology on E such that every linear mapping T from E into an arbitrary locally convex space F is continuous when the restriction $T|_{E_n} \to F$ is continuous, for each n.

We first show that this property is satisfied by the inductive limit topology. Let V be any neighbourhood in F. Suppose each restriction T_n of T to E_n is continuous. Hence $T_n^{-1}(V)$ is a neighbourhood in E_n, that is, $T^{-1}(V) \cap E_n$ is a neighbourhood in E_n for each n. Hence, by a previous remark, $T^{-1}(V)$ is a neighbourhood and T is therefore continuous.

To show that τ is the coarsest topology with the given property, let τ' be any other topology such that, for any F, a map $T:E \to F$ is continuous if each of its restrictions is continuous. In particular then, we can take F to be $E[\tau]$. Let T be the identity map: $E[\tau'] \to E[\tau]$. Its restrictions are clearly just the injections which are continuous by our definition of the inductive limit topology. Hence

T is continuous and therefore $\tau<\tau'$. Thus τ is the coarsest topology with the given property.

Common in the theory are abbreviations such as (LB)- and (LF)-spaces denoting the countable inductive limit of Banach (complete normed) and Frechet (complete metrisable) spaces respectively.

Definition 2.4 A space $E[\tau]$ will be called retractively bounded if, given any sequence $\{x_n\}$ in E, there exists a sequence of scalars $\lambda_n > 0$ such that $\lambda_n x_n \to 0$. It is equivalent to require that $\{\lambda_n x_n\}$ be bounded.

Mackey [13], on page 182, defines his first countability property. If, for every sequence of bounded sets $\{B_n\}$ in E, there exists a sequence $\lambda_n > 0$ such that $\cup \lambda_n B_n$ is bounded, then E has the first contability property. If E has the first countability property then clearly, E is retractively bounded

In the same paper Mackey uses the expression "retractively bounded" to mean something rather different from the above. His notation does not seem however to be standard.

Theorem 2.5 A metrisable topological vector space is retractively bounded.

Proof. By theorem 1.6.1 of Schaefer [19], there exists a function $x \to |x|$ on E into R (the reals) such that

(1) $|\lambda| \leqslant 1$ implies $|\lambda x| \leqslant |x|$ for all x in E.

(2) $|x+y| \leqslant |x| + |y|$

(3) $|x| = 0$ if and only if $x = 0$.

(4) The metric $(x,y) = |x-y|$ generates the topology τ.

For any integer $n > 0$, we have

$$|nx| = |x+x+x \ \ldots \ +x| \quad n \text{ times}$$
$$\leqslant |x| + |x| + \ \ldots \ + |x| \quad n \text{ times, by (2)}$$
$$\leqslant n|x|.$$

Hence for any $\lambda > 0$ we have $\lambda = k + \mu$ where k is a positive integer and $0 \leqslant \mu < 1$. Now

$$|\lambda x| = |kx + \mu x|$$
$$\leqslant |kx| + |\mu x| \quad \text{by (2)}$$
$$\leqslant k|x| + \mu|x| \quad \text{by above and (2)}$$
$$\leqslant \lambda|x|.$$

Suppose $\{x_n\}$ is any given sequence. Let

$$\lambda_i = \frac{1}{i|x_i|} \quad \text{if } |x_i| \neq 0$$
$$= 1 \quad \text{if } x_i = 0$$

Then $|\lambda_i x_i| \leqslant \lambda_i |x_i| \leqslant \frac{1}{i}$ for each i.

Hence $\lambda_i x_i \overset{\tau}{\to} 0$ since $|\ |$ defines a neighbourhood base at 0 for the topology τ.

We remark here that this theorem does not hold for general metrisable spaces which are vector spaces but whose topology is not necessarily compatible with the vector space structure. Consider any vector space with the discrete topology τ. The metric d defined by

$$d(x,y) = 1 \qquad \text{if } x \neq y$$
$$= 0 \qquad \text{if } x = y$$

defines the topology. Hence τ is metrisable. However a sequence $\{x_n\}$ cannot converge to 0 with respect to τ unless $x_n = 0$ for all n greater than some fixed k, and the same can therefore be said of $\lambda_n x_n$ for $\lambda_n > 0$.

Definition 2.6 A space $E[\tau]$ is said to satisfy Mackey's condition of convergence if every convergent sequence converges in the span of a bounded absolutely convex subset with its gauge.

Let $x_n \xrightarrow{\tau} 0$. Then $\{x_n\} \subset B$ (a bounded absolutely convex set). Let $E_B =$ span B and p_B be the semi-norm on E_B with B as unit ball. Then if E satisfies Mackey's condition of convergence,

$$p_B(x_n) \to 0$$

Köthe [12] refers to the above as "local convergence".

We say E has strict Mackey convergence if for every bounded absolutely convex set A, there exists a bounded absolutely convex set B such that $\tau_A = p_B|_A$ (p_B being the semi-norm with unit ball B). If E has strict Mackey convergence then E satisfies Mackey's condition of convergence since we can take for A the absolutely convex hull of the sequence $\{x_n\}$ which is bounded since x_n is convergent and E is a locally convex space.

It will be shown in the final chapter that if bounded sets are metrisable, the two properties coincide.

Proposition 2.7 A space $E[\tau]$ satisfies Mackey's condition of convergence if and only if, given any convergent sequence $x_n \to 0$, there exists a sequence of scalars $\lambda_n > 0$ tending to infinity, such that $\lambda_n x_n \to 0$.

Proof. Suppose, given any convergent sequence $x_n \to 0$, there exists a sequence $\lambda_n > 0$ with $\lambda_n \to \infty$ such that $\lambda_n x_n \to 0$. Let

$$B = \overline{\Gamma}\{\lambda_n x_n\}$$ the closed absolutely convex hull of $\{\lambda_n x_n\}$.

Then

$$\{x_n\} \subset E_B.$$

We also have $p_B(x_n) \leqslant \frac{1}{\lambda n} \to 0$. Hence τ satisfies Mackey's condition of convergence.

Now suppose τ satisfies Mackey's condition of convergence. Hence there exists a B such that

$$p_B(x_n) \to 0$$

Let

$$\lambda_n = \frac{1}{\sqrt{p_B(x_n)}} \qquad \text{if } p_B(x_n) \neq 0$$

$$= 1 \qquad \text{if } p_B(x_n) = 0$$

Then $$p_B(\lambda_n x_n) \to 0,$$

hence $$\lambda_n x_n \to 0, \quad \text{since } p_B > \tau|_{E_B}.$$

<u>Proposition 2.8</u> Every metrisable linear topological space satisfies Mackey's condition of convergence.

<u>Proof.</u> As in the proof of theorem 2.5, consider the function $x \to |x|$ of E into R. If $x_n \to 0$, then $|x_n| \to 0$. Let

$$\lambda_n = \frac{1}{\sqrt{|x_n|}} \qquad \text{if } |x_n| > 0$$

$$= n \qquad \text{if } x_n = 0.$$

Clearly then, $\lambda_n \to \infty$. But again, as in the proof of theorem 2.5,

$$|\lambda_n x_n| \leq \sqrt{|x_n|}$$
$$\to 0.$$

Hence $\lambda_n x_n$ converges to 0.

The converse statement is false. Mackey's condition of convergence is satisfied by $\varphi[\mu(\varphi,\omega)]$ which is not metrisable. (Here φ represents the set of all sequences with only a finite number of non-zero terms, ω the set of all sequences and μ the Mackey topology.) We can in fact consider any strict (LF)-space $F = \cup F_i$. If $F_i \subset F_{i+1}$ with proper inclusion for infinitely many i's, F is not metrisable as is shown at the end of section III. However, since F is sequentially retractive (see definition 2.9 and propoosition 9.1) any convergent sequence $\{x_n\}$ is contained in and converges in one of the F_i's. Hence, since F_i is metrisable we can find $\lambda_i \to \infty$ such that $\lambda_i x_i \to 0$ and F therefore satisfies Mackey's condition of convergence.

Definition 2.9 An inductive limit $E = \cup E_i$ is called sequentially retractive if each convergent sequence $x_n \to 0$ is contained in some E_n and converges to 0 with respect to τ_n.

This definition was originally given in Floret [6].

Definition 2.10 We say that $E[\tau]$ has a fundamental sequence of bounded sets $\{B_i\}$ $(i=1,2,\ldots)$ if the B_i are bounded and cofinal (ordering subsets by inclusion) for the class of all bounded sets.

The following definitions are well known but are inserted here for completeness.

Definition 2.11 A linear topological space E is called (quasi-) barrelled if every (bornivorous-) barrel is a neighbourhood.

Recall that a barrel is a closed absolutely convex absorbent set and that a bornivorous set is one which absorbs bounded sets. Clearly therefore, every barrelled space is quasi-barrelled.

<u>Definition 2.12</u> A linear topological space E is <u>bornological</u> if every absolutely convex bornivorous set is a neighbourhood of 0.

Every bornological space is clearly quasi-barrelled. It is not true that every barrelled space is bornological and hence also, that every quasi-barrelled space is bornological. One does however have relations such as E metrisable and $E'[\beta(E',E)]$ (the strong topology) being barrelled implying that $E'[\beta(E',E)]$ is bornological. Also, every sequentially complete quasi-barrelled space is barrelled.

The reader will note that for two topologies τ and μ on a space E, the notation $\mu < \tau$ implies that τ is finer than μ but that τ is not necessarily strictly finer than μ.

C-CONVERGENCE.

Given some arbitrary topology τ on a space E, we know there is a natural way in which to define a notion of convergence. The converse problem is not as straight forward.

In other words, what conditions must be imposed on sequences which we will call C-convergent (say) in order that they be the only and all the convergent sequences of some topology C (say). Kisyński [10] showed that the following properties K1-4 were sufficient.

K1: If $x_n = x$ for each n, then $x_n \overset{C}{\to} x$.

K2: If $x_n \overset{C}{\to} x$ then $x_{n_k} \overset{C}{\to} x$.

K3: If $x_n \overset{C}{\to} x$ and $x_n \overset{C}{\to} y$ then $x = y$.

K4: If each subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ has a subsubsequence $\{x_{n_{k_j}}\}$ such that $x_{n_{k_j}} \overset{C}{\to} x_o$, then $x_n \overset{C}{\to} x_o$.

We now consider a convergence in a strict inductive limit $E = \cup E_n$. We can in fact consider strict generalized inductive limits. This is discussed in more detail in section VII but briefly, it is the following. Given absolutely convex sets E_n whose union is the space E and whose topology μ_n is the restriction of a locally convex topology on span(E_n), the generalized inductive limit topology τ is the finest locally convex topology making the injections of E_n into E continuous. For the generalized inductive limit to be of interest in functional analysis, the E_n must satisfy the following conditions.

(i) $2E_n \subset E_{n+1}$

(ii) $\mu_n = \mu_{n+1}|_{E_n}$.

Let $x_n \xrightarrow{C} 0$ if there exists an m such that $\{x_n\} \subset E_m$ and $x_n \to 0$ in E_m, that is, with respect to $E_m[\tau|_{E_m}]$.

This convergence clearly satisfies K1-4. If τ is the inductive limit topology we have C-convergence implying τ-convergence. If the E_n are fundamental for the bounded sets, that is, if every τ-bounded set is contained in some E_n, we have the reverse implication.

<u>Definition 3.1</u> The <u>topology</u> <u>C</u> will consist of the open sets U such that $x_n \xrightarrow{C} x_o \in U$ implies $x_n \in U$ for all n greater than some m.

<u>Proposition 3.2</u> The inductive limit topology is weaker than C.

<u>Proof.</u> Let U be τ-open (where τ is the inductive limit topology). Let

$$x_n \xrightarrow{C} x \in U,$$

then

$$x_n \xrightarrow{\tau} x \in U,$$

hence there exists an N such that n>N implies

$$x_n \in U.$$

Thus U $\in$ C by the definition of C.

The above inclusion is in general strict. Note that

τ is always locally convex and a linear topology neither of which C need be. For a counter example, see Dudley, R.M. [5].

Now that this C-convergence notion has been established we could consider properties such as C-boundedness, C-compactness etc.

<u>Definition 3.3</u> We call a set B <u>C-bounded</u> if, given any sequence of scalars λ_n converging to 0, and $\{x_n\} \in B$, $\lambda_n x_n \overset{C}{\to} 0$.

Note that in a topological vector space this is equivalent to the usual definition of boundedness.

<u>Proposition 3.4</u> Every C-bounded subset of E is contained in some E_n.

<u>Proof.</u> We may assume without loss of generality that $4E_n \subset E_{n+1}$. (This can be done by creating a new inductive system by omitting some of the E_n's thus having an equivalent inductive system.) Assume B is bounded but that there does not exist an m such that $B \subset E_m$. We can therefore find a sequence $x_m \in B$ such that $x_m \in E_{m+1} - E_m$. (Set theoretic subtraction.) Let $\lambda_n = 2^{-n}$. Thus

$$\lambda_n x_n \overset{C}{\to} 0$$

Hence there exists an m such that

$$\lambda_n x_n \in E_m \qquad \text{for all } n.$$

therefore $\qquad x_n \in 2^n E_m \qquad$ for all n.

Hence $\qquad x_{2n} \in 2^{2n} E_m = 4^n E_m \qquad$ for all n.

In particular then, when n=m,

$$x_{2m} \in 4^m E_m \subset E_{2m}$$

which is a contradiction.

<u>Definition 3.5</u> A sequence $\{x_n\}$ is <u>C-Cauchy</u> if for each pair of increasing sequences of integers n_k and m_k, $x_{n_k} - x_{m_k} \xrightarrow{C} 0$.

If $\{x_n\}$ is C-convergent, $\{x_n\} \subset E_m$ and $x_n \xrightarrow{\tau_m} 0$ $(\tau_m = \tau|_{E_m})$. Hence given any n_k and m_k, $x_{n_k} - x_{m_k} \in E_{m+1}$ and $x_{n_k} - x_{m_k} \xrightarrow{\tau} 0$ (since τ is a linear topology), thus

$$x_{n_k} - x_{m_k} \xrightarrow{C} 0, \quad \text{hence } \{x_n\} \text{ is Cauchy.}$$

<u>Proposition 3.6</u> Every C-Cauchy sequence is C-bouded.

<u>Proof.</u> Suppose the C-Cauchy sequence $\{x_n\}$ is not contained in any E_n. Suppose also that $2E_n \subset E_{n+1}$ for each n. Then, without loss of generality, we may assume that $x_n \in E_{n+1} - E_n$. Let

$$n_k = k+2$$

and $\qquad m_k = k.$

Then $\qquad x_{n_k} - x_{m_k} = x_{k+2} - x_k \in E_m \qquad$ for all k and

for some fixed m, by our definition of C-convergence.

Hence $\qquad x_{n+2} \in E_m + x_n.$

Take n>m, then

$$x_{n+2} \in E_{n+1} + E_{n+1} \subset E_{n+2}$$ which is a contradiction.

Hence $\{x_n\} \subset E_m$ for some m. Thus $\{x_n\}$ is τ-Cauchy (by the definition of C-convergence). It is therefore τ-bounded since τ is a linear topology, hence it is C-bounded (by the definition of C-boundedness).

It is clear that if each E_n is sequentially complete, the same is true of E with the topology C. In what follows the word "sequentially" will be omitted in such notions as "sequentially complete", as it is obvious we are only interested in results derived from sequences in any case.

<u>Definition 3.7</u> We call B <u>C-compact</u> if for each sequence $\{x_n\}$ in B, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{C} x_o \in B$.

<u>Definition 3.8</u> A set A is <u>C-precompact</u> if for any sequence $\{x_n\}$ in A, there exists a subsequence $\{x_{n_k}\}$ which is C-Cauchy.

It is clear by a previous remark that C-compactness implies C-precompactness.

<u>Proposition 3.9</u> Every C-precompact set is C-bounded.

<u>Proof.</u> Suppose A is C-precompact. If there does not

exist an m such that $A \subset E_m$ one can, without loss of generality find a sequence $\{x_n\}$ in A such that

$$x_n \in E_n - E_{n-1}.$$

We may assume that $2E_{n-1} \subset E_n$ for each n. If any subsequence $\{x_{n_k}\}$ were C-Cauchy, we would have for any m_k and l_k

$$x_{m_k} - x_{l_k} \in E_p \qquad \text{for some fixed p.}$$

Let $l_k > p$ and $m_k > l_{k+1}$.

Hence $\qquad x_{m_k} \notin E_{l_k+1}$

thus $\qquad x_{m_k} \notin E_{l_k} + E_{l_k}$

and therefore $\qquad x_{m_k} \notin x_{l_k} + E_p.$

Thus $\qquad x_{m_k} - x_{l_k} \notin E_p$ which is a contradiction.

Hence $A \subset E_m$ for some m. Therefore A is precompact in $E_m[\tau_m]$. Hence it is τ-bounded (since τ is a linear topology and $\tau|_{E_m} = \tau_m$) and therefore C-bounded by definition.

<u>Corollary 3.10</u> Every C-compact set is C-bounded.

<u>Proof.</u> C-compactness implies C-precompactness which implies C-boundedness.

<u>Definition 3.11</u> Denote by $\overline{A}^s$ the C-closure of A defined by $\overline{A}^s = \{x : \exists\, x_n \in A : x_n \overset{C}{\to} x\}$.

We must bear in mind that we are omitting the adjective "sequential" in this section merely for convenience.

<u>Remark.</u> The C-closure of a set need not be closed.

Let $E = \cup E_n$ where each E_n is metrisable and properly contained in E_{n+1}.

Let $x_n \in E_{n+1} - E_n$, and $y_n \xrightarrow{C} 0$ with $\{y_n\} \subset E_1$ and $y_n \neq 0$. Suppose

$$A = \{y_n + \tfrac{1}{k}x_n : \text{for all } n \text{ and } k\}$$

Now $y_n \in \bar{A}^s$ since $\frac{1}{k}x_n \xrightarrow{C} 0$ as k tends to infinity since E_n is metrisable, and $y_n \xrightarrow{C} 0$. But $0 \notin \bar{A}^s$ otherwise we could find a subsequence

$$y_{n_r} + \frac{1}{k_r} x_{n_r} \xrightarrow{C} 0.$$

Hence there exists a p such that

$$\{y_{n_r} + \frac{1}{k_r} x_{n_r}\} \subset E_p$$

contradicting our choice of x_n.

This type of construction can be found in [20], in the discussion of almost closed sets. From this remark it follows that E is not metrisable and in particular, that a strict (LF)-space is not metrisable.

Note that the C-topology and the inductive limit topology do not necessarily coincide but they do contain the same convergent sequences. Hence if a strict (LF)-space were metrisable we could not find a set whose sequential closure was not sequentially closed and hence the same could be said for the C-topology.

BARRELLED AND QUASI-BARRELLED SPACES.

Many of the topological vector spaces one comes across are quasi-barrelled or even barrelled. It is for this reason that the following theorem is so useful.

It is proved in more generality by De Wilde, M. and Houet, C. [4] and also by Valdivia, M. [22]. The proof given here is due to my supervisor Dr. Webb.

Theorem 4.1 Let $E[\tau]$ be a quasi-barrelled space and suppose $E = \cup E_n$ with $E_{n-1} \subset E_n$. Suppose further that the E_n are fundamental for the bounded sets. Then $E[\tau] = \text{lim.ind.}\{E_n[\tau|_{E_n}]: n \in N\}$.

Proof. Let W be any absolutely convex subset of E such that $W \cap E_n$ is a $\tau|_{E_n}$-neighbourhood for each n.

There exists a τ-neighbourhood U_n in E such that

$$W \cap E_n \supset U_n \cap E_n$$

We can in fact choose absolutely convex U_n's.

Let

$$V_n = \overline{W \cap E_n} + \tfrac{1}{2} U_n.$$

Then V_n is an absolutely convex closed neighbourhood in E. We have

$$\begin{aligned} V_n &\subset W \cap E_n + \tfrac{1}{2}U_n + \tfrac{1}{2}U_n \\ &= W \cap E_n + U_n \end{aligned}$$

(since U_n is absolutely convex). Therefore

$$V_n \cap E_n \subset 2W \cap E_n. \qquad \text{(i)}$$

[If $x \in V_n \cap E_n$ then $x = y + z$ where $y \in W \cap E_n$, and $z \in U_n$. Therefore $z = x-y \in E_n$. Hence $z \in U_n \cap E_n \subset W \cap E_n$.] Now let

$$V = \bigcap_1^\infty V_n.$$

V is closed and absolutely convex since each V_n is. We must show that V is bornivorous.

Let B be an arbitrary bounded set in E. Then, since the E_n are fundamental for the bounded sets, there exists an m such that $B \subset E_m$.

Now $W \cap E_m$ is bornivorous in E_m, hence there exists a $\lambda > 0$ such that

$$\lambda B \subset W \cap E_m \qquad \subset W \cap E_{m+k}$$
$$\subset V_m \qquad \subset V_{m+k}.$$

Hence $\lambda B \subset V_n$ for all $n \geq m$. Choose $0 < \lambda_o \leq \lambda$ such that

$$\lambda_o B \subset V_i \qquad \text{for } 1 \leq i \leq m-1.$$

Hence

$$\lambda_o B \subset V$$

and therefore V is a bornivorous barrel.

Thus V is a neighbourhood of 0 in $E[\tau]$, since E is quasi-barrelled.

But by (i) $V \subset 2W$, hence W is a neighbourhood in E. Thus τ is the inductive limit topology determined by the $E_n[\tau|_{E_n}]$.

The proof of the following theorem is identical with the above if we replace the bounded set B by a single point

x. It is clear then that we do not require the E_n to be fundamental for the bounded sets.

<u>Theorem 4.2</u> Let $E[\tau]$ be a barrelled space and suppose $E = \bigcup_1^\infty E_n$ with $E_n \subset E_{n+1}$. Then $E[\tau] = \text{lim.ind.}\{E_n[\tau|_{E_n}]\}$.

It is not difficult to see that with only minor alterations to the proof, the theorem holds for the case where the E_n are not subspaces but absolutely convex subsets of E. It then turns out to be a generalized inductive limit. (See section VII.)

It will be shown in section VII by a counter example, that we cannot drop the requirement that the E_n be fundamental for the bounded sets in the statement of theorem 4.1.

BOUNDED SETS - REGULARITY.

In this section we are primarily interested in whether or not an inductive limit $E = \cup E_n$ is regular. By regular is meant, that the E_n are fundamental for the bounded sets. Theorem 5.4 is the best known in the theory but is rather restricted. It follows as a direct corollary from theorem 5.3.

Let E be the inductive limit of the subspaces E_n whose union is E. Let V_n be an absolutely convex closed neighbourhood of 0 in E_n and suppose that $\lambda_n V_n \subset V_{n+1}$ $(\lambda_n > 0)$.

Consider the following properties:

F1: Each V_n is closed in E and V_1 contains a non-trivial subspace of E.

F2: Each absolutely convex neighbourhood W_n of 0 in E_n, such that $W_n \subset V_n$, is closed in E_{n+1}.

Makarov [13] states, without proof, the following results, assuming one of F1 or F2 is satisfied.

<u>Theorem 5.1</u> If each E_n is separable, then E is separable, and each bounded set in E is contained in some E_n.

<u>Theorem 5.2</u> If each E_n is normed, then each set which is bounded in E is contained in and bounded in some E_n.

In an attempt to prove one of these theorems, the following theorem was discovered and proved.

<u>Theorem 5.3</u> Let E be the strict inductive limit of subspaces E_n. Suppose V_n is a balanced subset of E_n absorbing E_n and such that $V_n \subset V_{n+1}$ and V_n is closed in V_{n+1}. Then every bounded set B is contained in some E_n.

<u>Proof.</u> If $B \not\subset E_n$ for each n, choose $x_n \in B$ with x_n in $E_{n+1} - E_n$ (by taking a subsequence of the E_n's if necessary).

Note: We know that the inductive limit is not altered if a subsequence of the E_n's is taken. It remains to show that the property of V_n being closed in V_{n+1} is preserved. If A is closed in B, and B is closed in C, is A closed in C?

$$A \text{ closed in } B \iff \bar{A} \cap B = A$$

$$B \text{ closed in } C \iff \bar{B} \cap C = B$$

$$\begin{aligned} \text{Therefore } \bar{A} \cap C &= \bar{A} \cap (C \cap \bar{B}) \qquad \text{since } \bar{A} \subset \bar{B} \\ &= \bar{A} \cap B \\ &= A \iff A \text{ closed in } C. \end{aligned}$$

Choose $\lambda_n \to 0$ such that $y_n = \lambda_n x_n \in V_{n+1}$. Since B is bounded, $y_n \to 0$.

Now $y_n \notin kV_n$ for each k since $\bigcup_{k=1}^{\infty} kV_n = E_n$. Since V_n is closed in V_{n+1}, there exists a W_n closed in E such that $W_n \cap V_{n+1} = V_n$ for each n. Suppose $y_n \in kW_n$ for some k and n. Since $y_n \in kV_{n+1}$ (because V_{n+1} is balanced), we would have

$y_n \in kV_n \subset E_n$, a contradiction, hence $y_n \notin kW_n$ for each n and k. Furthermore kW_n is closed in E, hence we can find, for each n and k, a neighbourhood $U_{n,k}$ of 0 such that

$$y_n \notin kW_n + U_{n,k}$$

hence

$$y_n \notin kV_n + U_{n,k}.$$

Let

$$U = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} [kV_n + U_{n,k}].$$

We have

$$U \cap E_p = [\bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (kV_n + U_{n,k})] \cap E_p$$
$$= \bigcap_{n=1}^{p} \bigcup_{k=1}^{\infty} (kV_n + U_{n,k}) \cap E_p$$

since V_n is absorbent.

Thus $U \cap E_p$ is a neighbourhood of 0 in E_p since $kV_n + U_{n,k}$ is a neighbourhood of 0. Hence U is a neighbourhood of 0 for the strict inductive limit topology.

But $y_n \notin U$ for each n. Hence y_n does not converge to zero which is a contradiction. Hence B is contained in some E_n.

From this follows

Theorem 5.4 Let $E_n[\tau_n]$ be a strict inductive system with $E_n \subset E_{n+1}$ and E_n τ_{n+1}-closed in E_{n+1}. Let $E = \cup E_n$ and τ the inductive limit topology. If B is τ-bounded in E, then there exists an n such that $B \subset E_n$, and clearly then, B is bounded in E_n.

<u>Proof.</u> In the previous theorem take $V_n = E_n$.

Note that if we adapt the proof of theorem 5.3 for the more restricted case of theorem 5.4 we obtain a proof which is completely different from, and in fact more elegant, than the usual one for theorem 5.4.

The following example shows that the condition that E_n be closed in E_{n+1} cannot be dropped. The example constructed here is of a strict inductive limit $E = \text{lim.ind.} E_n$ in which a bounded set need not be contained in any of the E_n's. Makarov [15] gives an example of an (LB)-space with the same property. It is not difficult to show that this inductive limit cannot be strict.

Suppose ω = set of all sequences.

φ = set of all sequences with only a finite number of non-zero terms.

If σ, μ and β represent the weak, Mackey and strong topologies respectively then, $\sigma(\omega,\varphi) = \mu(\omega,\varphi) = \beta(\omega,\varphi)$ since $\omega[\sigma(\omega,\varphi)]$ is barrelled. It is in fact metrisable, [see the discussion of the inductive limit of weak topologies in section VI].

Clearly φ is properly contained in ω. Let x_i be chosen inductively such that

$$x_1 \notin \varphi$$
$$x_2 \notin \text{span}\{x_1, \varphi\}$$
$$x_n \notin \text{span}\{x_{n-1}, x_{n-2}, \ldots, x_1, \varphi\}$$

where $x_i \in \omega$.

Since φ is countable dimensional and ω has uncountable dimension, this is possible. The x_i thus chosen are obviously linearly independent. We can make $\{x_n\}$ bounded by multiplying by suitable scalars if necessary since $\omega[\sigma(\omega,\varphi)]$ is metrisable and in view of theorem 2.5.

Let F denote the span of $\{x_n\}$. Hence $F \subset \omega$ and $F \cap \varphi = \{0\}$. Let E be a complement of F, that is $E + F = \omega$, (the direct sum) such that $\varphi \subset E$. (See note.) Suppose $E_n = \text{span}\{F, x_1, x_2, \ldots, x_n\}$. (We cannot have E_n closed in E_{n+1} since φ is dense in ω, therefore $\bar{\varphi} = E$ and since $\varphi \subset E_n \subset E$, $\bar{E}_n = E$ for each n.)

Since ω is barrelled

$$\omega = \text{lim.ind.} E_n .$$

Clearly $\{x_n\} \not\subset E_p$ for any p although $\{x_n\}$ is bounded. ($x_{p+1} \notin E$ since $x_{p+1} \in F$. Hence $x_{p+1} \notin \text{span}\{E, x_1, x_2, \ldots x_p\}$ thus $x_{p+1} \notin E_p$.)

<u>Note:</u> We show the existence of a suitable complement E. Let X be the class of all subspaces E_α of ω such that $\varphi \subset E_\alpha$ and $E_\alpha \cap F = \{0\}$. Note that X is non-void since $\varphi \in X$. Order X by inclusion. If Y is a chain in X, then $\cup Y$ is clearly an upper bound, hence by Zorn's lemma, there exists a maximal element E in X. Suppose $E+F \neq \omega$, then there exists an x in ω with $x \notin E+F$.

Consider the space $E_1 = \text{span}\{E, x\}$, then $E_1 \cap F = \{0\}$, for otherwise there exists an $f \in F$ such that there exists an $e \in E$ and $0 \neq \lambda \in K$ with $f =$

$e + \lambda x$, hence $x = \frac{1}{\lambda}[f-e] \in E + F$, a contradiction. Clearly $\varphi \subset E_1$ hence $E_1 \in X$ and $E_1 \supset E$ with proper inclusion which contradicts the maximality of E. Hence $E+F = \omega$.

A few results are now stated which are a direct consequence of the previous theorems.

<u>Theorem 5.5</u> Let $E = \cup E_n$ be barrelled and suppose that the E_n are an expanding sequence of subspaces. Then, if B is bounded in E, there exists an n such that $B \subset \bar{\bar{E}}_n$.

<u>Proof.</u> Clearly $E = \cup\bar{\bar{E}}_n$. By theorem 4.2 $E = \text{lim.ind.}\bar{\bar{E}}_n$, where each $\bar{\bar{E}}_n$ has the relative topology. This is then obviously a strict inductive limit. Thus we satisfy the conditions of theorem 5.4 and hence $B \subset \bar{\bar{E}}_n$ for some n.

<u>Definition 5.6</u> A set B is <u>total</u> if, for each $f \in E'$ (the continuous linear funtionals on E), $f \neq 0$, there exists an x in B such that $f(x) \neq 0$.

<u>Proposition 5.7</u> A set B being total is equivalent to its span being dense in E.

<u>Proof.</u> Suppose spanB = S (say) is not dense in E. Then there exists an $x_o \in E$ and a neighbourhood U of 0 such that $(x_o + U) \cap S = \varphi$. Since E is locally convex we can

find a convex open neighbourhood $V \subset U$. Hence $(x_o + V) \cap S$ is empty and $x_o + V$ is open and we can therefore apply the Hahn-Banach theorem [see Treves [21], page 181] to obtain a closed hyperplane H containing S and not intersecting $x_o + V$. Let f be a non-zero continuous linear functional whose null space is H. For each $x \in H$, $f(x) = 0$. Hence for each x in B, $f(x) = 0$ which shows that B is not total.

Conversely, suppose that B is not total. Hence we can find a continuous non-zero linear functional f such that $f(x) = 0$ for each x in B. Let $N = \{x | f(x) = 0\}$. Now $N \neq E$ since $f \neq 0$ and N is closed since f is continuous. Also $B \subset N$ and hence $\text{span}(B) \subset N$ (since N is a subspace). Thus $\overline{\text{span}(B)} \subset N$ and hence span(B) is not dense in E.

<u>Corollary 5.8</u> (to theorem 5.5) If $E = \cup E_n$ is barrelled, and B is a bounded total set in E, then there exists an n such that $\overline{E}_n = E$.

<u>Corollary 5.9</u> A barrelled space E which contains a bounded total set is not the union of a strictly expanding sequence of closed subspaces.

This is reminiscent of the Baire category theorem. Note that a Baire space is barrelled but that the converse is not necessarily true (e.g. $\varphi[\mu(\varphi,\omega)]$).

Of interest in the context of fundamental sequences for bounded sets are the following theorems. We need first a lemma.

<u>Lemma 5.10</u> (Banach-Schauder) Let E and F be Frechet spaces and T a continuous map $T: E \to F$. If $\overline{T(E)} = F$ and $T(E)$ is not meagre, then $T(E) = F$.

<u>Proof.</u> See Köthe [12], page 170 theorems 12.1 and 12.2.

<u>Theorem 5.11</u> (Grothendieck) Let T be a continuous mapping of a Fréchet space E into an (LF)-space $F = \cup F_i$. Then there exists an m such that $T(E) \subset F_m$ and $T: E \to F_m$ is continuous.

<u>Proof.</u> Let $H_n = \{(x,Tx): x \in E \text{ and } Tx \in F_n\}$.

Then

$$H_n = \{(x,Tx): x \in E\} \cap (E \times F_n).$$

But $\{(x,Tx): x \in E\}$ is closed since T is continuous, hence H_n is closed in $E \times F_n$, so is complete since $E \times F_n$ is a Frechet space.

Define $P_n: H_n \to E$ by

$$P_n(x,Tx) = x.$$

Then $P_n(H_n) = \{x \in E: Tx \in F_n\}$ and $\cup P_n(H_n) = E$. Since E is Frechet, for some m, $P_m(H_m)$ is not meagre, and the interior of $\overline{P_m(H_m)}$ is not empty.

Hence

$$\overline{P_m(H_m)} = E, \qquad \text{thus by lemma 5.10}$$

$$P_m(H_m) = E.$$

Hence $Tx \in F_m$ for each x in E. Now H_m is closed in $E \times F_m$. But H_m is the graph of T. Hence, by the closed graph theorem, T is continuous.

We conclude this section with the following observations.

<u>Theorem 5.12</u> Let T be a bounded map of a space E into a space $F = \cup F_n$. Suppose that E is retractively bounded and F is the strict inductive limit of F_n and F_n is closed in F_{n+1}. Then there exists an m such that

$$T(E) \subset F_m.$$

<u>Proof.</u> Suppose, contrary to the statement, that there exists no m such that $T(E) \subset F_m$. Then we can find a sequence $\{x_n\} \subset E$ such that

$$T(x_n) \in F_{n+1} - F_n$$

(by taking a subsequence of the F_n's if necessary). Since E is retractively bounded, there exist $\lambda_n > 0$ such that

$$\lambda_n x_n \to 0.$$

Hence such that $\{\lambda_n x_n\}$ is bounded. Thus $T(\{\lambda_n x_n\})$ is bounded since T is and therefore, by theorem 5.4 we have

$$T(\lambda_n x_n) \in F_m \quad \text{for some m and all n.}$$

Hence in particular when $n = m$,

$$T(\lambda_m x_m) \in F_m$$

and since T is linear and F_m a linear space,

$T(x_n) \in F_n$ for n=m contrary to the choice of x_m.

Corollary 5.13 Suppose T is a bounded map of a metrisable space E into a strict (LF)-space $F = \cup F_i$, Then there exists an n such that

$$T(E) \subset F_n.$$

Proof. By theorem 5.12 and theorem 2.5.

This corollary is in fact a weak form of theorem 5.11 since strictness cannot be omitted for this method of proof.

THE WEAK TOPOLOGY.

In this section, the weak topologies on an (LF)-space and its generating subspaces are examined and we find that they form an inductive system. It is shown that the inductive limit topology of the generating subspaces with the weak topologies is not necessarily the weak topology of the original (LF)-space. To show that the weak topologies on the generating subspaces forms an inductive system, we prove the following.

Theorem 6.1 If $E = \cup E_n$ is a strict (LF)-space, where each E_n is a Frechet space, then

$$\sigma(E,E')|_{E_n} = \sigma(E_n,E_n').$$

This property and the following one are stated without proof in a paper by Bennett, G. and Cooper, J.B. [13]. In fact the proof given here is for an arbitrary strict inductive limit which is regular.

Proof. Let U be basic neighbourhood of 0 in $E[\sigma(E,E')]$. Hence

$$U = \{y_1, y_2, \ldots, y_n\}^{\circ} \quad \text{the polar of } \{\ \ \} \text{ where } y_i \in E'.$$

Thus

$$U \cap E_p = \{x \in E : |\langle x, y_i\rangle| \leq 1 \text{ for each } 1 \leq i \leq n\} \cap E_p$$

$$= \{x \in E_p : |\langle x, y_i\rangle| \leq 1 \text{ for each } 1 \leq i \leq p\}$$

where in particular, $y_i \in E_p'$ since $\tau_p = \tau|_{E_p}$.

Hence $\quad U \cap E_p = \{y_1, y_2, \ldots, y_n\}^{\circ}$ for $y_i \in E_p'$ and the polar taken in E_p. Thus

$$\sigma(E,E')|_{E_n} < \sigma(E_n, E_n').$$

Given any basic neighbourhood U of 0 in E_n with the topology $\sigma(E_n, E_n')$, we must find a neighbourhood W of 0 in $E[\sigma(E,E')]$ such that $U = W \cap E_n$.

Let

$$U = \{y_1, y_2, \ldots, y_p\}^{\circ} \quad \text{where } y_i \in E_n' \text{ and}$$

the polar is taken in E_n.

$$= \{x \in E_n : |\langle x, y_i \rangle| \leqslant 1 \text{ for each } 1 \leqslant i \leqslant p\}.$$

By the Hahn-Banach theorem we can find, for each i, a $y_i' \in E'$ such that $y_i' = y_i$ on E_n. Therefore

$$U = \{x \in E_n : |\langle x, y_i' \rangle| \leqslant 1 \text{ for each } 1 \leqslant i \leqslant p\}$$

$$= \{x \in E : |\langle x, y_i' \rangle| \leqslant 1 \text{ for each } 1 \leqslant i \leqslant p\} \cap E_n$$

$$= W \cap E_n$$

where W is a basic $\sigma(E,E')$-neighbourhood of 0. The theorem is therefore proved.

<u>Theorem 6.2</u> Let E and the E_n be as in the statement of theorem 6.1. If $x_k \to 0$ with respect to $\sigma(E,E')$ then there exists an m such that $\{x_k\} \subset E_m$ and $x_k \to 0$ with respect to $\sigma(E_m, E_m')$.

<u>Proof.</u> If $x_k \to 0$ with respect to (E,E'), then $\{x_k\}$ is $\sigma(E,E')$-bounded and hence τ-bounded. Hence, by theorem 5.4, $\{x_k\}$ is contained in some E_n. Therefore by theorem 6.1, $x_k \to 0$ with respect to $\sigma(E_n, E_n')$.

By theorem 6.1, the injection maps

$$E_{n-1}[\sigma(E_{n-1},E'_{n-1})] \to E_n[\sigma(E_n,E'_n)]$$

are continuous and hence form an inductive system. Let τ_o be the associated inductive limit topology. Then clearly $\tau_o > \sigma(E,E')$. The following example shows we can have a strict inequality.

Let $E = \varphi$ = set of infinite sequences with only a finite number of non-zoro terms.

$e_n = \{0,0,0,..,1,0,... \}$ with the 1 occupying the n'th coordinate.

$E_n = \text{span}\{e_1,e_2,....,e_n\}$.

Hence E_n is finite dimensional. Let $E' = \omega$ = set of all sequences. We can consider $E'[\sigma(E',E)]$ (i.e. $\omega[\sigma(\omega,\varphi)]$) as the countable product of copies of R (the reals). Since R is complete and metrisable which are productive properties (i.e. inherited by products), $E'[\sigma(E',E)]$ is complete and metrisable and hence barrelled. Bounded sets of R are relatively compact and hence by Tychonoff's theorem, the same is true of $E'[\sigma(E',E)]$ and thus it is semi-reflexive. Hence $E'[\sigma(E',E)]$ is reflexive which implies that $E[\mu(E,E')]$ is reflexive and therefore barrelled.

Hence, since $E = \cup E_n$, by theorem 4.2,

$$E[\mu(E,E')] = \text{lim.ind.}E_n[\mu(E,E')|_{E_n}].$$

However, on a finite dimensional space, all linear topologies coincide, hence

$$\sigma(E,E')|_{E_n} = \mu(E,E')|_{E_n}.$$

But

$$\begin{aligned} E[\mu(E,E')] &= \text{lim.ind.}E_n[\mu(E,E')|_{E_n}] \\ &= \text{lim.ind.}E_n[\sigma(E,E')|_{E_n}] \\ &= \text{lim.ind.}E_n[\sigma(E_n,E_n')] \\ &= \tau_o. \end{aligned}$$

But $\mu(E,E') \neq \sigma(E,E')$. To show this, consider the following. Since $E[\mu(E,E')]$ is barrelled, we have

$$\mu(E,E') = \beta(E,E').$$

Let $B = \{e_i\} \subset E'$. Then B is clearly $\sigma(E',E)$ bounded. Hence B^o is a $\beta(E,E')$ neighbourhood. If B^o were a $\sigma(E,E')$ neighbourhood we would have

$$B^o \supset \{a_1,a_2,\ldots,a_n\}^o \qquad \text{where } a_i \in E'.$$

Hence

$$\begin{aligned} \bar{\Gamma}B &= B^{oo} \\ &\subset \{a_1,a_2,\ldots,a_n\}^{oo} \\ &= \bar{\Gamma}\{a_1,a_2,\ldots,a_n\}. \end{aligned}$$

Thus

$$B \subset \bar{\Gamma}\{a_1,a_2,\ldots,a_n\}.$$

But the right hand side is clearly finite dimensional which B certainly is not. Hence we have a contradiction and $\sigma(E,E') \neq \mu(E,E')$.

This therefore shows that the inductive limit of a sequence of spaces with the weak topologies need not yield a weak topology.

GENERALIZED INDUCTIVE LIMITS.

In this chapter we are concerned primarily in simplifying and reproving some of the theory given in a paper on "Generalized inductive limits" by Garling, D.J.H. [7]. His proofs are incomplete and there are certain steps which I'm afraid I am unable to justify.

Recall that if we have a space E as the union of locally convex spaces $E_n[\tau_n]$, the inductive limit topology τ is the finest locally convex topology on E making the injections $i_n: E_n \to E$ continuous, hence $\tau|_{E_n} < \tau_n$ for each n. This situation can easily be generalized as follows. Consider E and E_n as above and let S_n be subsets of E_n. The generalized inductive limit topology τ' on E is the finest locally convex topology on E making each restriction $j_n: S_n \to E$ of i_n (to S_n) continuous. Although the notation here appears to be considerably simpler than in Garling [7], no generality is in fact lost. Let $\mu_n = \tau_n|_{S_n}$.

The conditions Garling imposes on the S_n for the theory to be of interest and value are as follows:

(i) $\quad \text{span}(\cup S_n) = E$.

(ii) $\quad 2S_n \subset S_{n+1}$ for each n.

(iii) $\quad \mu_n > \mu_{n+1}|_{S_n}$ for each n.

(iv) $\quad$ each S_n is absolutely convex.

From these it follows that

(v) $\quad \cup S_n = E$

(vi) Given $\alpha>0$ and $n\in N$, there exists an $m\in N$ such that $S_n \subset \alpha S_m$.

The reason we can simplify the notation is that Garling states that, without loss of generality, we may assume that his injections i_α are 1-1.

Let F_n= lin.span(S_n). Let τ'_n be the finest locally convex topology on F_n making the injections $i'_n: S_n \to F_n$ continuous. Garling shows that the usual inductive limit of $F_n[\tau'_n]$ is in fact the same as the generalized inductive limit of $S_n[\mu_n]$, (see proposition2.2). From this it seems reasonable to assume that we can restrict the previous conditions to include span(S_n) = E_n. We are now in a position to prove theorem 2 of Garling's third section in two stages.

<u>Theorem 7.1</u> If τ is the generalized inductive limit topology induced by $\{(E_n,\tau_n,i_n,S_n)\}$ (n=1,2,..), and if for each n, $\mu_n = \mu_{n+1}|_{S_n}$, then $\tau|_{S_n} = \mu_n$.

<u>Proof.</u> We can suppose, by condition (ii), that $3S_n \subset S_{n+1}$.

It is only necessary to show that, if W is a neighbourhood of 0 in S_o, in the topology induced by τ_o (i.e. μ_o), then there is a τ-neighbourhood U of 0 such that $U\cap S_o = W$ (since it makes no difference to the topology if a finite number of the sets S_i are ignored).

Since τ_o is a locally convex topology, we can find an absolutely convex τ_o-neighbourhood V_o say, in E_o such that $V_o \cap S_o \subset W$.

Since, by hypothesis, $\mu_n|_{S_{n-1}} = \mu_{n-1}$, the same holds for the restrictions to αS_{n-1} since μ_n is the restriction of a linear topology τ_n. Hence there is an absolutely convex τ_1-neighbourhood of 0 in E_1, V_1 say, such that

$$V_1 \cap (3S_o) \subset (\tfrac{1}{2}V_o) \cap (3S_o).$$

Hence, for each $n>0$, we can inductively find absolutely convex τ_n-neighbourhoods of 0 in E_n, V_n say, such that

$$V_n \cap (3S_{n-1}) \subset (\tfrac{1}{2}V_{n-1}) \cap (3S_{n-1}).$$

(Hence in particular, $V_n \cap (2S_{n-1}) \subset (\tfrac{1}{2}V_{n-1}) \cap (2S_{n-1})$ by intersecting both sides by $2S_{n-1}$.) Now let

$$U = \Gamma_{i=0}^{\infty} [(\tfrac{1}{2}V_i) \cap S_i].$$

But

$$\tfrac{1}{2}V_n \cap S_n \subset U \cap S_n \qquad \text{for each } n.$$

Since V_n is a τ_n-neighbourhood, $\tfrac{1}{2}V_n \cap S_n$ is a μ_n-neighbourhood and hence, $U \cap S_n$ is a μ_n-neighbourhood for each n. Thus U is a neighbourhood in the generalized inductive limit. We also have

$$U \subset U_1 = \Sigma_{i=0}^{\infty} [(\tfrac{1}{2}V_i) \cap S_i)$$

and it will be shown that $U_1 \cap S_o \subset W$.

Suppose that $z = z_o + z_1 + \ldots + z_n \in U_1 \cap S_o$, where $z_i \in (\tfrac{1}{2}V_i) \cap S_i$. Then, if $y_r = z_r + z_{r+1} + \ldots + z_n$, we have

$$y_r = z - (z_o + z_1 + \ldots + z_{r-1}) \in S_o + S_o + S_1 + \ldots S_{r-1}.$$

Since $S_{j+1} \supset 3S_j$, it follows that $y_r \in 2S_{r-1}$. In particular, since $z_n \in \frac{1}{2}V_n \subset V_n$,

$$z_n = y_n \in V_n \cap 2S_{n-1} \subset (\tfrac{1}{2}V_{n-1}) \cap 2S_{n-1}.$$

We now show by induction that

$$y_r \in V_r$$

and

$$y_r \in (\tfrac{1}{2}V_{r-1}) \cap 2S_{r-1}.$$

This induction starts at n and descends. Assume the above two equations hold for $r=k$, then

$$y_{k-1} = z_{k-1} + y_k$$

Hence

$$y_{k-1} \in \tfrac{1}{2}V_{k-1} + \tfrac{1}{2}V_{k-1}$$

(the second term arising from the induction hypothesis).

Hence

$$y_{k-1} \in V_{k-1}$$

But

$$y_{k-1} \in 2S_{k-2}$$

Thus

$$y_{k-1} \in V_{k-1} \cap 2S_{k-2} \subset (\tfrac{1}{2}V_{k-2}) \cap 2S_{k-2}$$

by our choice of V_k. This then completes the induction.

Then $z = z_o + y_1 \in \frac{1}{2}V_o + \frac{1}{2}V_o = V_o$ so that $z \in V_o \cap S_o \subset W$, and so $U_1 \cap S_o \subset W$, whence $U \cap S_o \subset W$ which shows that $\tau|_{S_o} = \mu_o$ since in any case $\tau|_{S_o} < \mu_o$.

<u>Theorem 7.2</u> If the conditions of theorem 7.1 are satisfied and if further, for each n S_n is closed in S_{n+1}, a set B in E is τ-bounded if and only if there exists an n such that $B \subset S_n$, and B is τ_n-bounded in E_n.

<u>Proof.</u> Suppose B is τ-bounded but not contained in any S_i. Then there exists a sequence $\{x_i\}$ of points of B such that, for each i, $x_i \notin S_{2.3^i}$. Since B is bounded, $3^{-i}x_i$ converges to 0 in the topology . Also since $3S_i \subset S_{i+1}$, $3^{-i}x_i \notin S_{3^i}$, so that the sequence $\{3^{-i}x_i\}$ is not contained in any one S_i.

It is therefore sufficient to show that, if $\{x_k\}$ is a sequence of points of E, not contained in any one S_i, then x_k does not converge to 0 in the topology .

It may be supposed, by choosing a subsequence of the $\{x_k\}$ and a subsequence of the S_i if necessary, that $x_k \in S_k$ and $x_k \notin S_{k-1}$. Note that the condition that S_n be closed in S_{n+1} is preserved when passing to a subsequence as is proved in the proof of theorem 5.3.

Since S_{k-1} is closed in S_k, there is an absolutely convex neighbourhood of 0 in E_k, W_k say, such that $[(x_k + W_k) \cap S_k] \cap S_{k-1}$ is empty. In other words $x_k \notin S_{k-1} + W_k$.

We now choose inductively, for each p, an absolutely convex neighbourhood of 0 in E_p, V_p say, in such a way that the following conditions are satisfied. To begin the induction, we can put $V_0 = \frac{1}{2}W_0$.

$$\text{I} \quad V_p \cap 3S_{p-1} \subset (\tfrac{1}{2}V_{p-1}) \cap 3S_{p-1}.$$

$$\text{II} \quad V_{p-1} \cap 2S_{p-1} \subset W_p \cap \tfrac{1}{2}W_{p-1}.$$

Condition I is the same as in the proof of theorem 7.1 and is possible for the same reasons as given there. Condition II is possible since W_{p-1} is a neighbourhood in E_{p-1} and therefore $W_p \cap \frac{1}{2}W_{p-1}$ is a neighbourhood in E_{p-1} (since $\mu_p|_{E_{p-1}} = \mu_{p-1}$).

Let $U_1 = \sum_{i=1}^{\infty}[(\frac{1}{2}V_i) \cap S_i]$ as before. Let $z = z_o + z_1 + \ldots z_n \in S_k \cap U$ where $z_i \in (\frac{1}{2}V_i) \cap S_i$. We may assume that $n > k+1$ by adding a suitable number of zeros if necessary.

Then

$$y_r = z - (z_o + \ \ldots \ + z_{r-1}) \in S_k + S_o + \ \ldots + S_{r-1}$$

so that

$$y_r \in 2S_k \qquad \text{if } r \leq k$$

$$y_r \in 3S_k \qquad \text{if } r = k+1$$

$$y_r \in 2S_{r-1} \qquad \text{if } r > k+1.$$

Now $y_n \in 2S_{n-1}$ since $n > k+1$ and we also have $z_n \in \frac{1}{2}V_n \subset V_n$. Hence

$$z_n = y_n \in V_n \cap 2S_{n-1} \subset (\tfrac{1}{2}V_{n-1}) \cap 2S_{n-1}$$

and as in the proof of theorem 7.1, we can show that

$$y_r \in V_r$$

and $$y_r \in (\tfrac{1}{2}V_{r-1}) \cap 2S_{r-1}$$

provided that $r > k+1$ since we require $y_r \in 2S_{r-1}$. Thus from the second equation we have

$$y_{k+2} \in \tfrac{1}{2}V_{k+1}$$

also $$z_{k+1} \in \tfrac{1}{2}V_{k+1}$$

therefore since

$$y_{k+1} = z_{k+1} + y_{k+2}$$

$$y_{k+1} \in \tfrac{1}{2}V_{k+1} + \tfrac{1}{2}V_{k+1} = V_{k+1}$$

and hence

$$y_{k+1} \in V_{k+1} \cap 3S_k \subset (\tfrac{1}{2}V_k) \cap 3S_k \qquad \text{by I}$$

therefore

$$y_{k+1} \in \tfrac{1}{2}V_k$$

Thus

$$\begin{aligned} y_k &= z_k + y_{k+1} \\ &\in (\tfrac{1}{2}V_k) \cap S_k + \tfrac{1}{2}V_k \\ &\subset V_k \end{aligned}$$

so that

$$y_k \in V_k \cap 2S_k. \qquad \text{(i)}$$

Also

$$z_o + z_1 + \ \dots \ + z_{k-2} \in S_o + S_1 + \ \dots \ + S_{k-2} \subset S_{k-1}$$

so that

$$z = (z_o + \ \dots \ + z_{k-2}) + z_{k-1} + y_k$$

(note that this is not true if $k>n+1$)

hence

$$z \in S_{k-1} + (\tfrac{1}{2}V_{k-1}) \cap S_{k-1} + V_k \cap 2S_k$$

the second term arises from the choice of z_{k-1} and the last from (i),

$$= S_{k-1} + \tfrac{1}{2}[V_{k-1} \cap 2S_{k-1}] + V_k \cap 2S_k$$

$$\subset S_{k-1} + \tfrac{1}{2}W_k + \tfrac{1}{2}W_k \qquad \text{by II twice}$$

$$= S_{k-1} + W_k.$$

That is, if $z \in S_k \cap U_1$, then $z \neq x_k$. In other words, $x_k \notin U_1$, for any values of k so $x_k \notin U$ (where U is a neighbourhood as in theorem 7.1) for any k and therefore cannot converge to 0.

It will be shown how theorem 5.3 can be applied to prove theorem 7.2 in a completely different way. In fact we prove only a weaker form as we can only show that the E_n , and not necessarily the S_n, are fundamental for the bounded sets.

In order to be able to apply theorem 5.3, we have to make the following observation.

Let $$\tau_n'' = \tau|_{E_n}$$

and suppose

$$E[\tau''] = \text{lim.ind.} E_n[\tau_n''].$$

By definition then, τ'' is the finest locally convex topology on E such that $\tau''|_{E_n} < \tau_n''$. But $\tau|_{E_n} < \tau_n''$ and therefore $\tau'' > \tau$.

However τ is the finest locally convex topology making each injection

$$S_n[\mu_n] \to E \quad \text{continuous.}$$

Hence the definition for τ is less restrictive than that for τ''. Hence $\tau > \tau''$. Thus

$$\tau = \tau''$$

Hence $E[\tau]$ is a <u>strict</u> inductive limit as required by theorem 5.3. On application of the theorem then, we find any bounded B contained in some E_n and clearly bounded in E_n since we have a strict inductive limit.

Let E be a non-barrelled normed space. As an example we could take the space of sequences with only a finite number of non-zero terms equipped with the supremum norm.

Let S be a barrel in E which is not a neighbourhood. Suppose that $E_n[\tau_n] = E[\tau]$ for each n and $S_n = 3^n S$. Then (E_n, S_n, τ_n) is a generalized inductive system, and S_n is closed in S_{n+1}. Let τ_o be the generalized inductive limit. Now $E[\tau] = \text{lim.ind.} E_n[\tau_n]$ obviously, since $E_n[\tau_n]$ is equal to $E[\tau]$ for each n. But $\tau < \tau_o$.

We show this by contradiction. Suppose $\tau = \tau_o$. Thus the unit ball B, being τ-bounded would be τ_o-bounded and hence by theorem 7.2, would be contained in some S_n. Thus $S \supset 3^{-n} B$ and therefore, S is a τ-neighbourhood contrary to its choice. Hence $\tau < \tau_o$ with strict inclusion.

In a remark after the proof of theorem 4.1, it was stated that one could not omit the condition requiring the S_n to be fundamental for the bounded sets to ensure that if $E = \cup S_n$ is quasi-barrelled, then

$$E[\tau] = \text{lim.ind.} S_n[\tau|_{S_n}]$$

which is obviously a generalized inductive limit. The above is in fact a counter example since a normed space is quasi-barrelled.

ON A RESULT OF AMEMIYA AND KŌMURA.

In this section we consider a result given by Amemiya and Kōmura [1]. The proof here involves inductive limits as opposed to the original. It also shows how theorem 5.3 enables us to pass from absolutely convex set properties to those of subspaces. It is from this sort of consideration that one is tempted not to investigate too deeply the more general inductive limit of subsets rather than subspaces.

The theorem here is for retractively bounded barrelled spaces which are less restrictive than the originals, which are metrisable and barrelled. It has been shown by Saxon, S. and Levin, M. [18], that merely barrelled spaces are sufficient, however, the relative simplicity of the following proof makes it worthwhile here.

We need a few preliminary theorems before the main theorem of the section can be proved.

Theorem 8.1 Let E be a barrelled and retractively bounded space. Then E is not expressible as a strictly increasing sequence of closed absolutely convex nowhere dense sets.

Proof. Assume that $E = \cup A_n$ where A_n is closed, absolutely convex nowhere dense and $A_n \subset A_{n+1}$. Let E_n be the span of A_n. If $E_n = E$ for any n, A_n would be aborbent. It is absolutely convex and closed and therefore a barrel, hence a neighbourhood, which is a contradiction. Therefore $E_1 \neq E$.

Hence there exists an n_2 such that

$$A_{n_2} - E_1 \neq \varphi$$

and inductively we can choose n_m so that

$$A_{n_m} - E_{n_{m-1}} \neq \varphi$$

Now, by theorem 4.2, since E is barrelled

$$E = \text{lim.ind.} E_{n_k}[\tau|_{E_{n_k}}] \qquad \text{(strict)}$$

Hence, since A_{n_k} is closed and contained in $A_{n_{k+1}}$, A_{n_k} absorbs E_{n_k} and the A_{n_k} are balanced, we satisfy the conditions of theorem 5.3.

By our choice of n_k, we can choose $x_m \in A_{n_m} - E_{n_{m-1}}$. Since E is retractively bounded, we can find λ_n such that $\{\lambda_n x_n\}$ is bounded. Hence, by theorem 5.3, we have $\{\lambda_m x_m\} \subset E_{n_k}$ for some k. Hence $\lambda_{k+1} x_{k+1} \in E_{n_k}$, thus $x_{k+1} \in E_{n_k}$ which is a contradiction.

<u>Theorem 8.2</u> If E is barrelled, and F is a subspace of finite codimension, then F is barrelled.

<u>**Proof.**</u> Without loss of generality we can clearly assume that codim.F = 1. Let U be a barrel in F. Consider $\overline{U}$, the closure in E. We have two cases.

If $\overline{U} \subset F$, then $\overline{U} = U$, since U is closed in F. Let

x_o be chosen so that

$$E = \text{span}\{F, x_o\} \qquad \text{hence } x_o \notin F$$

suppose

$$V = \{x+\lambda x_o : x \in U, \ |\lambda| \leq 1\}.$$

Then V is absolutely convex obviously. It is therefore a barrel if we can show that it is absorbent and closed.

Let $x \in E$

then

$$x = y + \mu x_o \qquad \text{where } y \in F.$$

Hence $y \in \alpha U$ since U is absorbent in F, thus $x \in \sup(\alpha,\mu)V$. Since the addition of a closed and a compact set is closed, V is a barrel in E. Hence V is a neighbourhood of 0 in E and $V \cap F = U$, thus U is a neighbourhood of 0 in F.

Now suppose $\bar{\bar{U}} \not\subset F$. Then $\bar{\bar{U}}$ is a barrel in E. That $\bar{\bar{U}}$ is absolutely convex and closed is obvious.

Let $x \in E$

then

$$x = y + \mu x_o$$

where $x_o \in \bar{\bar{U}} - F$ and $y \in F$, hence $y \in \alpha\bar{\bar{U}}$ for α sufficiently large and $\mu x_o \in \alpha U$. Thus $x \in 2\alpha\bar{\bar{U}}$. Hence $\bar{\bar{U}}$ is a barrel in E and therefore a neighbourhood of 0 in E. But $\bar{\bar{U}} \cap F = U$, hence U is a neighbourhood of 0 in F which shows that F is barrelled.

This proof was originally given by Kōmura, Y. [11].

We now have the means to prove the main theorem of this section. The proof given here follows the original quite closely.

<u>Theorem 8.3</u> A subspace G_o of countable codimension of a barrelled retractively bounded space G is also barrelled.

<u>Proof.</u> Assume G_o is not barrelled. Then there exists an absolutely convex closed absorbent nowhere dense subset V of G_o. Let $\bar{V}$ be the closure of V in G. Denote by H the linear subspace generated by $\bar{V}$.

If the codimension of H were finite, H would be barrelled by theorem 8.2. Hence it follows that we would have a contradiction since $\bar{V}$ would then be a neighbourhood of 0 in H. This cannot be since then $\bar{V} \cap G_o = V$ would be a neighbourhood of 0 in G_o. Hence the codimension of H must be infinite.

Let $\{x_n\}$ be a countable number of elements of G, such that G is spanned by G_o and $\{x_n\}$. By an inductive construction we get a sequence $\{A_n\}$ of subsets of G, such that $A_1 = \bar{V}$, and A_{n+1} is the absolutely convex hull of A_n and x_n. Each A_n is closed and also nowhere dense for otherwise, A_n would be a neighbourhood and $V = A_n \cap G_o$ would then be a neighbourhood in G_o which would contradict its choice.

But $G = \cup n A_n$ which then contradicts theorem 8.1. Hence G_o is a barrelled space.

The value of theorem 8.1 can also be seen in the proof of the the following theorem.

Theorem 8.4 A retractively bounded space has no closed subspaces of countably infinite codimension.

Proof. Suppose, to the contrary, that G_o is a closed subspace of G of countably infinite codimension. Choose linearly independent mod G_o $\{x_i\}$ (i.e. $\sum_1^m \lambda_i x_i \in G_o$ implies $\lambda_i = 0$ for each i) such that $\{G_o, x_i\}$ spans G.

Let $$X_o = G_o$$
and
$$X_n = \text{span}\{X_{n-1}, x_n\} \qquad \text{(closed)}$$

Clearly $X_n \subset X_{n+1}$ and $G = \cup X_n$. By theorem 8.1 there exists an n such that X_n contains an interior point. Hence $X_n = G$ contradicting the linear independence of the x_i. Hence G has no closed subspaces of countably infinite codimension.

Corollary 8.5 There does not exist a countably infinite dimensional retractively bounded barrelled space.

Proof. Directly from the above since $\{0\}$ is a closed subspace assuming E is Hausdorff.

Corollary 8.6 The existence of a countably infinite dimensional metrisable barrelled space is false.

Proof. From corollary 8.5 and theorem 2.5.

Bauer, W.R. and Benner, R.H. prove this in [2] for a Banach space without the use of category theory. It is however only with difficulty that one can show that a Banach space is barrelled without the use of category theory.

We give now an alternative direct proof of corollary 8.6.

Suppose E is barrelled, metrisable and countably infinite dimensional. Let $\{x_i\}$ be a base for E $(i=1,2,..)$. Let

$$E_o = \{0\}$$

and

$$E_n = \text{span}\{E_{n-1}, x_n\} \qquad n>0,$$

then

$$E = \cup E_n.$$

Each E_n is finite dimensional, hence homeomorphic to E^n, the euclidian n dimensional space and therefore complete. Since E is barrelled, we have by theorem 4.2 that E is the strict inductive limit of the E_n's. Therefore E is complete.

Each E_n is nowhere dense in E since E_n does not absorb E for any n. Thus E is of the first category (i.e. meagre). We would therefore have a contradiction to the Baire category theorem 9.4 of Kelley and Namioka [9].

SEQUENTIALLY RETRACTIVE INDUCTIVE LIMITS.

Recall definition 2.9 which states that an inductive limit $E = \cup E_i$ is sequentially retractive if each convergent sequence $x_n \overset{\tau}{\to} 0$ is contained in some E_n and converges with respect to τ_n.

Floret states without proof in [6] that the following three propositions hold.

Proposition 9.1 Strictly generated (LF)-spaces are sequentially retractive.

Proof. Let $E = \cup E_i$ where each E_i is a Fréchet space and $\tau_n|E_{n-1} = \tau_{n-1}$. Since E_n is complete, it is cloed in E_{n+1}. Thus by theorem 5.4, any bounded set B in E is contained in some E_n, hence in particular any convergent sequence is contained in some E_n and clearly converges with respect to τ_n since $\tau|_{E_n} = \tau_n$.

Proposition 9.2 Sequentially complete (LF)-spaces satisfying Mackey's condition of convergence are sequentially retractive.

Proof. If E is an (LF)-space, then given an absolutely convex bounded and sequentially complete set A, there exists an m such that $A \subset E_m$ and A is bounded in E_m. For a proof of this property, see Köthe [12] page 228, section 19, theorem 5.5.

Now let $x_n \to 0$ in E. Since Mackey's condition of convergence is satisfied, we can find $\lambda_n \to \infty$ such that $\lambda_n x_n \to 0$. Thus $\{\lambda_n x_n\}$ is bounded, hence $B = \overline{\Gamma}(\{\lambda_n x_n\})$ is absolutely convex, sequentially complete and bounded. Hence, by our previous remark there exists an m such that $B \subset E_m$ and B is bounded in E_m, thus

$$\frac{1}{\lambda_n}(\lambda_n x_n) \to 0 \qquad \text{in } E_m$$

hence

$$x_n \to 0 \qquad \text{in } E_m$$

Thus E is sequentially retractive.

Proposition 9.3 Bornological sequentially complete spaces in which there exists a fundamental sequence of bounded sets and satisfying Mackey's condition of convergence are sequentially retractive (LB)-spaces.

Proof. By Schaefer [19], theorem 8.4, E is the countable inductive limit of Banach spaces, thus E is an (LB)-space. To show that E is sequentially retractive, consider any sequence x_n converging to 0 in E. Then there exists a sequence λ_n such that $\lambda_n x_n \to 0$ since E satisfies Mackey's condition of convergence. Hence $\{\lambda_n x_n\}$ is contained in some element B of the fundamental sequence of bounded sets. Therefore $\{\lambda_n x_n\} \subset E_B$ (the span of B). But B is bounded in $E_B[\tau_B]$ since it is its unit ball.

Hence, since $\frac{1}{\lambda_n} \to 0$

$$\frac{1}{\lambda_n}(\lambda_n x_n) \to 0 \qquad \text{in } E_B$$

hence

$$x_n \to 0 \qquad \text{in } E_B$$

Thus E is sequentially retractive.

EQUIVALENT INDUCTIVE SYSTEMS.

Suppose $E[\tau]$ is a bornological space with a fundamental sequence of closed absolutely convex bounded sets $\{B_n\}$.

Let

$$E_n = \text{span}(B_n).$$

On E_n we have two natural topologies, namely

$$\tau_n = \tau|_{E_n}$$

and

$$\nu_n = \text{norm topology with unit ball } B_n.$$

Clearly $\tau_n < \nu_n$ for each n since B_n is τ_n-bounded. We have thus

(1) $E[\tau] = \text{lim.ind.} E_n[\nu_n]$ since E is bornological and has a fundamental sequence of bounded sets (see [19] theorem 8.5).

(2) $E[\tau] = \text{lim.ind.} E_n[\tau_n]$ since E is quasi-barrelled and has a fundamental sequence of bounded sets by theorem 4.1.

It seems reasonable therefore to ask the question: are these two inductive systems equivalent? We know that for any n, the injection map

$$E_n[\nu_n] \to E_n[\tau_n]$$

is continuous since $\tau_n < \nu_n$. However we also require, for each n, the existence of an $m \geqslant n$ such that the injection

$$E_n[\tau_n] \to E_m[\nu_m]$$

is also continuous. In other words, for each n, can we find an m such that

$$\nu_m|_{E_n} = \tau_n?$$

The answer to this question in general is no. A counter example follows later.

By our choice of E_n, the inductive system (2) is clearly sequentially retractive. For case (1) we know that if $E[\tau]$ satisfies Mackey's condition of convergence, E is sequentially retractive. It is clear that if case (1) is sequentially retractive, each convergent sequence is contained in one of the E_n's and converges to zero with respect to ν_n which shows that E satisfies Mackey's condition of convergence.

Hence, if we can find a space satisfying the above conditions but which does not satisfy Mackey's condition of convergence, case (1) is not sequentially retractive as opposed to (2) and the systems cannot be equivalent.

Is there such a space? The answer is in the affirmative. Let E be a Fréchet-Montel space. Hence in particular E is barrelled and every bounded set is relatively compact. Now $E'[\beta(E',E)]$ is reflexive and therefore barrelled (see Treves [19] page 376). It also has a fundamental sequence of bounded sets since we can take the polars of a countable number of neighbourhoods since E is metrisable. By theorem 6.6 of Schaefer, $E'[\beta(E',E)]$

is bornological. Hence we have a suitable example if we can find a Fréchet-Montel space which does not satisfy Mackey's condition of convergence.

Grothendieck shows in proposition 17 of [8] that a Fréchet-Montel space satisfying Mackey's condition of convergence is a Fréchet-Schwartz space and vice-versa. However, on page 118 he states the existence of a Fréchet-Montel space which is not quasi-normable and hence not Schwartz. Thus this space cannot satisfy Mackey's condition of convergence.

There is however still an open question. If $E[\tau]$ is bornological, has a fundamental sequence of bounded sets and satifies Mackey's condition of convergence, are the systems equivalent? I'm afraid I have been unable to resolve this question. If however we add the property that bounded sets are metrisable, the question is resolved if we consider the generalized inductive systems as the following remarks demonstrate.

In a topological vector space, neighbourhoods of the origin are sufficient to describe the topology on the whole space. The same is true also for absolutely convex sets, namely:

If $A\cap\Omega$ is a base at 0 in A ($A\subset E$, A absolutely convex) then $\{(x+B)\cap A : B\in\Omega\}$ is a base at x in A. For a related result see [17] page 102 lemma 1.

We prove now a lemma due to Grothendieck. The proof given here is an adaptation of a proof given by Matagne [16].

<u>Lemma 10.1</u> Let A be an absolutely convex set in a locally convex space E. Suppose τ is a topology on E which is metrisable on A. Let τ_n be a sequence of topologies on E. Then, if for each sequence $\{x_n\}$, $x_n \in A$ such that $x_n \overset{\tau}{\to} x$ there exists an m such that $x_n \overset{\tau_m}{\to} x$, we have

$$\tau_m < \tau \qquad (\text{on } A)$$

<u>Proof.</u> Let b_k be a base of absolutely convex $\tau|_A$-neighbourhoods at 0 in A. (By the previous remark we need not consider bases at arbitrary points x in A.) It can be chosen countably since τ is metrisable on A. We can further suppose that $b_k \subset b_{k-1}$ for each k.

Suppose $\{B^n\}$ is a base of absolutely convex neighbourhoods of 0 for the topology $\tau_n|_A$. We want to show that there exists an n such that for each α, we can find an integer $k(\alpha)$ such that

$$B^n \supset b_{k(\alpha)}.$$

We prove this by contradiction. The converse states that, for each n, there exists an α_n such that $B^n_{\alpha_n}$ does not contain any of the b_k.

In this case, for each n, we can construct a sequence $\{x^n_i\}$ such that

$$x_i^n \in b_i - B^n_{\alpha_n} \quad \text{for each } i. \text{ (Set theoretic subtraction.)}$$

By construction, the sequence $\{x_i^n\}$ converges to 0 in A with respect to τ but does not converge to 0 with respect to τ_n.

From this countably infinite number of sequences, we will construct a new sequence, converging to 0 in A with respect to τ but which does not converge to 0 with respect to any of the τ_n. This will contradict the hypothesis and the lemma will then be proved.

Let n_1 be the smallest integer such that

$$x_i^1 \in b_1 \qquad \text{for all } i \geq n_1.$$

Let n_2 be the smallest integer such that

$$x_i^1 \in b_2 \text{ and } x_i^2 \in b_2 \qquad \text{for all } i \geq n_2.$$

Clearly $n_2 \geq n_1$ since $b_2 \subset b_1$. Similarly, in general, let n_m be the smallest integer such that

$$x_i^1, x_i^2, \ldots, x_i^m \in b_m \qquad \text{for all } i \geq n_m.$$

We have then $n_1 \leq n_2 \leq n_3 \leq \ldots \leq n_m$. Construct a sequence in the following way. We take first the elements

$$x^1_{n_1}, x^1_{n_1+1}, \ldots, x^1_{n_2-1} \quad \text{in their order.}$$

We continue by adding

$$x^1_{n_2}, x^1_{n_2+1}, \ldots, x^1_{n_3-1}, x^2_{n_2}, \ldots, x^2_{n_3-1}$$

and similarly we add

$$x^1_{n_3}, x^1_{n_3+1}, \dots, x^1_{n_4-1}, x^2_{n_3}, \dots, x^2_{n_4-1}, x^3_{n_3}, \dots, x^3_{n_4-1}$$

and at the m'th stage, we add

$$x^1_{n_m}, \dots, x^1_{n_{m+1}-1}, x^2_{n_m}, \dots, x^2_{n_{m+1}-1}, \dots, \dots, \dots, x^m_{n_m}, \dots, x^m_{n_{m+1}-1}.$$

This sequence converges to 0 with respect to τ since, if we take any basic τ-neighbourhood b_p, all the terms added after the p-th stage (as above) are elements of b_p. On the other hand, each sequence $\{x_i^n\}$, except for a finite number of terms in each case, is contained in the new sequence. Consequently, the new sequence does not converge to 0 with respect to any τ_n, for otherwise the subsequence $\{x_k^n\}$, $k>n_k$ must converge with respect to τ_n which is contrary to the choice of $\{x_i^n\}$.

Let E have a fundamental system of bounded sets, be bornological and have the property that each bounded set is metrisable. Then if E satisfies Mackey's condition of convergence then E also has strict Mackey convergence.

Let τ be the topology of E and τ_n the norm topology of $E_n = \text{span}(B_n)$ (where $\{B_n\}$ is the fundamental sequence of bounded sets). If A is an absolutely convex metrisable bounded set, $x_n \in A$ and $x_n \overset{\tau}{\to} x$, since E satisfies Mackey's condition of convergence, there exists an m such that $x_n \overset{\tau_m}{\to} x$. Hence, by lemma 10.1, there exists an m such that $\tau_m|_A = \tau|_A$. Hence we have strict Mackey convergence on A.

If we take A to be one of the bounded sets B_n, we have the following.

Let E be a bornological locally convex space with a fundamental sequence of bounded sets $\{B_n\}$. Suppose further that the bounded sets of E are metrisable and E satisfies Mackey's condition of convergence. If, as before, ν_n represents the norm topology on $E_n = \text{span}(B_n)$ with unit ball B_n and $\tau|_{E_n} = \tau_n$, then the generalized inductive systems (E_n, B_n, ν_n) and (E_n, B_n, τ_n) are equivalent.

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