A Probabilistic Approach to a Classical Result of Ore

by

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Abstract

The subgroup commutativity degree $sd(G)$ of a finite group G was introduced almost ten years ago and deals with the number of commuting subgroups in the subgroup lattice $\mathrm{L}(G)$ of G. The extremal case $sd(G) = 1$ detects a class of groups classified by Iwasawa in 1941 (in fact $sd(G)$ represents a probabilistic measure which allows us to understand how far is G from the groups of Iwasawa). Among them we have $sd(G) = 1$ when $\mathrm{L}(G)$ is distributive, that is, when G is cyclic. The characterization of a cyclic group by the distributivity of its lattice of subgroups is due to a classical result of Ore in 1938. Therefore $sd(G)$ is strongly related to structural properties of $\mathrm{L}(G)$. Here we introduce a new notion of probability $gsd(G)$ in which two arbitrary sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$ are involved simultaneously. In case $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$, we find exactly $sd(G)$. Upper and lower bounds in terms of $gsd(G)$ and $sd(G)$ are among our main contributions, when the condition $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$ is removed. Then we investigate the problem of counting the pairs of commuting subgroups via an appropriate graph. Looking at the literature, we noted that a similar problem motivated the *permutability graph of non–normal subgroups* $\Gamma_N(G)$ in 1995, that is, the graph where all proper non–normal subgroups of G form the vertex set of $\Gamma_N(G)$ and two vertices H and K are joined if $HK = KH$. The graph $\Gamma_N(G)$ has been recently generalized via the notion of *permutability graph of subgroups* $\Gamma(G)$, extending the vertex set to all proper subgroups of G and keeping the same criterion to join two vertices. We use $gsd(G)$, in order to introduce the *non–permutability graph of subgroups* $\Gamma_{\mathrm{L}(G)}$; its vertices are now given by the set $\mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$, where $\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$ is the smallest sublattice of $\mathrm{L}(G)$ containing all permutable subgroups of G, and we join two vertices H, K of $\Gamma_{\mathrm{L}(G)}$ if $HK \neq KH$. We finally study some classical invariants for $\Gamma_{\mathrm{L}(G)}$ and find numerical relations between the number of edges of $\Gamma_{\mathrm{L}(G)}$ and $gsd(G)$.

Keywords and phrases

Subgroup Lattices
Subgroup commutativity degree
Permutability graph
Dihedral groups
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Dedication

I dedicate this thesis to my family and beloved friends for their patience, understanding and care during my studies.

Declaration

I, Seid Kassaw Muhie, hereby declare that the work on which this thesis is based is my original work (except where acknowledgements indicate otherwise) and that neither the whole work nor any part of it has been, is being, or is to be submitted for another degree in this or any other university. I authorise the University to reproduce for the purpose of research either the whole or any portion of the contents in any manner whatsoever.

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Contents

Introduction

In a finite group G, the set $\mathrm{L}(G)$ of all subgroups of G is partially ordered with respect to set inclusion, and forms a complete lattice. A sublattice $\mathrm{S}(G)$ of $\mathrm{L}(G)$ preserves the original structure of $\mathrm{L}(G)$ but it is evidently smaller than $\mathrm{L}(G)$. On the other hand, relevant sublattices are the sublattice $\mathrm{N}(G)$ of all normal subgroups of G, or the sublattice $\mathrm{SN}(G)$ of all subnormal subgroups of G, and their position inside $\mathrm{L}(G)$ as well their size may have a significant influence on the structure of the group. This fact is well known, not only at the level of group theory, but in wider context involving topology, combinatorics and algebra.

From [15, 61] we know that $\mathrm{L}(G)$ is *modular*, if all subgroups of G satisfy the *modular law*, and G is called *modular* whenever $\mathrm{L}(G)$ is modular (see [61, Section 2.1]). It is easy to check that all abelian groups are modular and a first example of modular sublattice is indeed $\mathrm{N}(G)$, when G is non-abelian. A classical result of Iwasawa (see [36, 37]) shows that the groups in which each pair of subgroups commutes are both nilpotent and modular (see [61, Theorem 2.4.14] and [61, Exercise 3, p.87]). This is one of the first time that has been related the notion of modularity and that of nilpotence in a group by a quantitative observation on the number of commuting subgroups. Few years before, Ore [58, 59] indicated that a special case of modular groups, namely finite groups with distributive lattice of subgroups, are characterized to be cyclic. Relevant structural information have been investigated, after the pioneering works of Iwasawa and Ore.

More recently it has been approached the study of modular groups, or of groups which are close to be modular, in a probabilistic way via the *subgroup commutativity degree* of G, that is,

$$sd(G) = \frac{|\{(X, Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid XY = YX\}|}{|\mathrm{L}(G)|\ |\mathrm{L}(G)|},$$

which measures the number of commuting subgroups (not of elements) in G. The formulation of subgroup commutativity degree follows from the notion of commutativity degree of G, replacing the role of elements of G with that of subgroups of G, which was originally suggested by Erdős and Turán [21], who founded the contemporary probabilistic group theory (almost 60 years ago).

Introducing the characteristic function

$$\chi : (H, K) \in \mathrm{S}(G) \times \mathrm{T}(G) \longmapsto \chi(H, K) = \begin{cases} 1, & \text{if } HK = KH, \\ 0, & \text{if } HK \neq KH, \end{cases}$$

for two arbitrary sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$, we investigate the *generalized subgroup commutativity degree*

$$gsd(G) = \frac{1}{|\mathrm{S}(G)|\ |\mathrm{T}(G)|} \sum_{(H,K)\in \mathrm{S}(G)\times \mathrm{T}(G)} \chi(H, K)$$

of a finite group G, getting for $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$ exactly $sd(G)$.

After studying $gsd(G)$ and its properties, we associate a suitable graph to G, following ideas in [16, 17, 18]. We call it the *non-permutability graph of subgroups* and devote a large part of the main results of this thesis, to connect the notion of generalized subgroup commutativity degree to interpret graph theoretical properties (and viceversa). In recent years similar approaches have been a topic of interest among algebraic graph theorists and this contributed significantly to solve problems of structure via methods of algebraic combinatorics.

In Chapter 1 , we give a brief overview of some basic concepts of lattice theory and subgroup lattices with examples and illustrated lattice diagrams. Chapter 2 deals with some results of Ore on the lattice theoretic characterization of the structure of group, while a probabilistic approach is discussed in Chapter 3. We describe the main results and their applications in Chapter 4 and Chapter 5.

It is appropriate to note that open questions are reported in Chapter 4 and Chapter 5, showing future directions of research on the topic.

Chapter 1

Basic Concepts of Lattice Theory

We begin to recall some definitions, notations and results from the literature that we shall use in the following chapters of the thesis. Most of them can be found in [15, 19, 53, 61]. We also provide some elementary examples, which are useful to understand certain behaviours of algebraic structures in the final part of the thesis.

Definition 1.0.1 (See [15], Definition 1.2)**.** Let P be a set. A partial order on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

(i). $x \leq x$,

(ii). $x \leq y$ and $y \leq x$ imply $x = y$,

(iii). $x \leq y$ and $y \leq z$ imply $x \leq z$.

The three above conditions are referred to, respectively, as reflexivity, antisymmetry and transitivity. When P is equipped with $\leq$ as in Definition 1.0.1, P is called *poset*.

Example 1.0.2. The set of natural numbers $\mathbb{N}$, that of integers $\mathbb{Z}$ and that of real numbers $\mathbb{R}$ are posets w.r.t. the usual order.

Example 1.0.3. For any set X, the powerset $\mathrm{P}(X)$ is a poset w.r.t. the set inclusion.

An element x of a poset P is said to be a *lower bound* for the subset S of P if $x \leq s$ for every $s \in S$. The element x is a greatest lower bound of S if x is a lower bound of S and

$y \leq x$ for any lower bound y of S. By Definition 1.0.1 (ii), such a greatest lower bound of S is unique if it exists; we denote it by $\inf S$. Similar definitions and remarks apply to the *upper bound* and to the least upper bound; the latter is denoted by $\sup S$.

If $x < y$ and there is no element $z \in P$ such that $x < z < y$, then we say that x is *covered* by y or that y *covers* x. We write O and I for the least and greatest elements of P, respectively (if they exist). An element of P that covers O is called *atom*, while an element that is covered by I is an *antiatom* of P. This terminology will be very useful for the description of the lattice of subgroups with few subgroups later on.

Two elements x and y in a poset P are *comparable*, if $x \leq y$ or $y \leq x$. A subset S of P is a *chain* if any two elements in S are comparable, S is an *antichain* if no two different elements of S are comparable. Of course, the length of a finite chain S is $|S| - 1$, and the poset P is said to be *of length* n(for some $n \in \mathbb{N}$) if there is a chain in P of length n and all chains in P are of length at most n. A poset P is of *finite length*, if it is of length n for some $n \in \mathbb{N}$. Similarly P is said to be of *width* n, if there is an antichain with n elements in P and all antichains in P have at most n elements.

Many important properties of an ordered set P are expressed in terms of the existence of certain upper or lower bounds for subsets of P. Two of the most important classes of ordered sets defined in this way are lattices and complete lattices. In the present thesis we are in fact interested to study lattices of subgroups of groups and eventually lattices of Lie subalgebras of Lie algebras.

Definition 1.0.4 (See [61], Page 2)**.** A *lattice* L is a poset in which every pair of elements has a least upper bound and a greatest lower bound. Moreover a poset in which every subset has a least upper bound and a greatest lower bound is called *complete*.

An elementary result is reported below for lattices.

Proposition 1.0.5 (See [61], Theorem 1.1.1)**.** *Let* $(L, \leq)$ *be a lattice and define the operations* $\cap$ *and* $\cup$ *by* $x \cap y = \inf\{x, y\}$ *and* $x \cup y = \sup\{x, y\}$. *Then the following properties hold for all* $x, y \in L$:

(i). $x \cap y = y \cap x$ *and* $x \cup y = y \cup x$,

(ii). $(x \cap y) \cap z = x \cap (y \cap z)$ *and* $(x \cup y) \cup z = x \cup (y \cup z)$,

(iii). $x \cap (x \cup y) = x$ *and* $x \cup (x \cap y) = x$.

Furthermore, we have $x \leq y$ *if and only if* $x = x \cap y$ *(or* $y = y \cup x$*). Conversely, let* L *be a set with two binary operations* $\cap$ *and* $\cup$ *satisfying (i), (ii), (iii) above and define the* *following relation $\leq$ on L: $x \leq y$ if and only if $x = x \cap y$. Then $(L, \leq)$ is a lattice with $x \cap y = \inf\{x, y\}$ and $x \cup y = \sup\{x, y\}$ for all $x, y \in L$.*

If G is any group, the set

$$\mathrm{L}(G) = \{H \mid H \text{ is a subgroup of } \ G\}$$

of all subgroups of G is a poset w.r.t. the set inclusion. Moreover any subset of $\mathrm{L}(G)$ has a greatest lower bound in $\mathrm{L}(G)$, the intersection of all its elements, and a least upper bound in $\mathrm{L}(G)$, the join of all its elements. Thus $\mathrm{L}(G)$ is a complete lattice, called the *subgroup lattice* of G. We denote the operations of this lattice by $\cap$ and $\cup$. So $X \cap Y = \inf\{X, Y\}$ and $X \cup Y = \langle X, Y\rangle = \sup\{X, Y\}$, where the notation $\langle X, Y\rangle$ is often preferred to denote the join of the subgroups X and Y of G. Of course, the trivial subgroup 1 is the least element in $\mathrm{L}(G)$ and G the greatest element. The minimal subgroups of G have a precise meaning: they are the atoms of $\mathrm{L}(G)$. The maximal subgroups of G are the antiatoms of $\mathrm{L}(G)$. New lattices can be created by forming sublattices, homomorphic images, and products of a prescribed lattice.

Definition 1.0.6 (See [61], Page 5)**.** A subset S of a lattice L is called *sublattice*, if it is closed w.r.t. $\cap$ and $\cup$ defined in L, that is, in Proposition 1.0.5.

Of course, a sublattice is a lattice with the induced operations by the original lattice.

Example 1.0.7. For $x, y \in L$ the set $S = \{x, y, \inf\{x, y\}, \sup\{x, y\}\}$ is a sublattice of L. Further relevant examples are the intervals: if $x \leq y$, the set

$$[y/x] = \{z \in L \mid x \leq z \leq y\},$$

called *interval* of L, is a sublattice of L.

Example 1.0.8. The set

$$\mathrm{N}(G) = \{H \mid H \text{ is a normal subgroup of } \ G\}$$

is a sublattice of $\mathrm{L}(G)$. Moreover, if H is a subgroup of G, then $\mathrm{L}(H)$ is always a sublattice of $\mathrm{L}(G)$.

We can construct more examples of sublattices of $\mathrm{L}(G)$. Details can be found in [61]. In particular, if H is a subgroup of G and $h \in H$, then

$$C_G(h) = \{g \in G \mid gh = hg\} \ \text{ and } \ \bigcap_{h \in H} C_G(h) = C_G(H)$$

denote the centralizer of h in G, and the centralizer of H in G, respectively. Of course, $C_G(H) \subseteq C_G(h)$ and they may be used to define the set

$$\mathrm{K}(G) = \{C_G(H) \mid H \text{ is a subgroup of } G\}$$

of all centralizers of subgroups of G. Note that $\mathrm{K}(G)$ is called *centralizer lattice* of G (see [61, Chapter 9]); its least element is the center $Z(G) = \{g \in G \mid gx = xg \ \forall x \in G\}$ of G; its greatest element is G. The interesting properties of $\mathrm{K}(G)$ motivated us to introduce two new sets in (5.0.2) and (5.0.3). However we will see that $\mathrm{K}(G)$ turns out to be very different from (5.0.2) and (5.0.3).

As noted before, another way to produce new lattices is via homomorphic images.

Definition 1.0.9 (See [61], Page 4)**.** Let L and $\bar{L}$ be two lattices. A map $\alpha : L \longrightarrow \bar{L}$ is called a homomorphism if α preserves both $\cap$ and $\cup$, that is, if $\alpha(x \cap y) = \alpha(x) \cap \alpha(y)$ and $\alpha(x \cup y) = \alpha(x) \cup \alpha(y)$ for all $x, y \in L$.

Of course, homomorphisms are isomorphisms when they are bijective, and an isomorphism of a lattice with itself is called automorphism. Specializing L to $\mathrm{L}(G)$, there is a wide literature which is devoted to study isomorphisms of lattices of subgroups. Following [61], if G and $\bar{G}$ are two groups, an isomorphism from $\mathrm{L}(G)$ to $\mathrm{L}(\bar{G})$ is called *projectivity* from G to $\bar{\mathrm{G}}$. We also say that G and $\bar{G}$ are *lattice isomorphic* if there exists a projectivity from G to $\bar{G}$.

Example 1.0.10. Consider the quaternion group Q_8 of order 8 and the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is cyclic of order two. Then $Q_8/Z(Q_8)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are lattice isomorphic groups, because $Q_8/Z(Q_8)$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isomorphic, but one can look at the cyclic groups $\mathbb{Z}_6$ and $\mathbb{Z}_{10}$, finding that there are lattice isomorphic groups which are not isomorphic as abstract groups.

As just seen, isomorphic abstract groups are lattice isomorphic, but the converse is not true, and Suzuki investigated closely the converse of such implication, under prescribed hypothesis (see Proposition 1.0.11 below). He discovered an interesting class of groups, which are going to mention here.

Recall from [61] that given a prime p, a finite p-group is a finite group in which the order of every element is a power of p. Elementary abelian p-groups of rank $r \geq 1$ are direct products of r-copies of $\mathbb{Z}_p$. In general, given an abelian group A (in principle finite or infinite), an abstract group homomorphism $\varphi : a \in A \to \varphi(a) \in A$ is a *power automorphism*, if there is some $m \in \mathbb{Z}$ such that $\varphi(a) = a^m$. Another notion that we must recall from [16] is that an arbitrary group G is the *semidirect product* of its subgroups

H and K, if simultaneously $HK = G$, $H \cap K = 1$ and H is normal in G.

Now if $n \geq 2$, we say that a group G *belongs to the class* $P(n,p)$, or that it is a P-*group*, if G is either elementary abelian of order p^n, or a semidirect product of an elementary abelian normal subgroup A of order p^{n-1} by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on A. That is, if $p = 2$ then the classes $P(n,2)$ only contain the elementary abelian groups of order 2^n and if $p \geq 3$ then $G = A\langle t \rangle$ with an elementary abelian p-group A and an element t of order q, and then there exist an integer r such that $t^{-1}at = a^r$ for all $a \in A$ and $r \not\equiv 1 \pmod p$ and $r^q \equiv 1 \pmod p$.

The following result of Suzuki shows conditions for detecting lattice isomorphic groups, which in principle are not isomorphic as abstract groups. Roughly speaking, it is the best possible generalization of the evidences in Example 1.0.10.

Proposition 1.0.11 (Suzuki, see [61], Theorem 2.2.6)**.** *Let p be a prime, n a natural number, G a group of order p^n, and suppose that α is a projectivity from G to some group $\bar{G}$. If $|G| \neq |\bar{G}|$ then either*

(i). G is cyclic and $\bar{G}$ is cyclic of order q^n where q is a prime different from p, or

(ii). G is elementary abelian, $n \geq 2$ and $\bar{G}$ is a non-abelian P-group of order $p^{n-1}q$ where q is a prime dividing $p-1$.

Direct products of lattices offer alternative ways of constructing new lattices.

Let $\{\mathrm{L}_\lambda \mid \lambda \in \Lambda\}$ be a family of lattices of subgroups for given groups G_λ and an arbitrary set of indices Λ. The direct product,

$$\mathrm{L} = \underset{\lambda\in\Lambda}{\mathrm{Dr}}\ \mathrm{L}_\lambda$$

may be endowed of the lattice structure if we consider the lexicographic order on L. If Λ is finite, say $\Lambda = \{1, 2, \cdots, n\}$, then

$$\mathrm{L} = \mathrm{L}_1 \times \mathrm{L}_2 \times \cdots \times \mathrm{L}_n$$

and we consider $(x_1, x_2, \cdots, x_n) \leq (y_1, y_2, \cdots, y_n)$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \cdots, n$. Of course, the case of Λ infinite can be treated in analogy. It turns out that this order induces a lattice structure on the direct product. Actually the method applies to general lattices, and not necessarily to lattices of subgroups.

We mentioned direct products, because we report briefly the structure of a group whose lattice of subgroups decomposes into a direct product. If $G = H \times K$, then in general

$\mathrm{L}(G) \not\simeq \mathrm{L}(H) \times \mathrm{L}(K)$. For example, we know that $\mathrm{L}(\mathbb{Z}_p \times \mathbb{Z}_p) \not\simeq \mathrm{L}(\mathbb{Z}_p) \times \mathrm{L}(\mathbb{Z}_p)$ for any prime p. On the other hand, it is possible to show that $\mathrm{L}(\mathbb{Z}_p \times \mathbb{Z}_q) \simeq \mathrm{L}(\mathbb{Z}_p) \times \mathrm{L}(\mathbb{Z}_q)$ for all primes p and q with $\gcd(p,q) = 1$. This suggests that we may reconstruct $\mathrm{L}(G)$ out of $\mathrm{L}(H) \times \mathrm{L}(K)$ and again we report a result of Suzuki, who was interested (compare Propositions 1.0.11 and 1.0.12) to investigate conditions of decomposability for the lattice of subgroups of finite groups.

Proposition 1.0.12 (Suzuki, see[61], Theorem 1.6.5)**.** *Let G be an arbitrary group (even infinite in principle) such that $\mathrm{L}(G) \simeq \underset{\lambda\in\Lambda}{Dr}\ \mathrm{L}_\lambda$, where $(\mathrm{L}_\lambda)_{\lambda\in\Lambda}$ is a family of lattices of subgroups of given groups, $|\Lambda| \geq 2$ for all $\lambda \in \Lambda$; write $\mathrm{L} = \underset{\lambda\in\Lambda}{Dr}\ \mathrm{L}_\lambda$ and suppose that $\sigma : \mathrm{L}(G) \longrightarrow \mathrm{L}$ is an isomorphism. For $\lambda \in \Lambda$ let O_λ be the least and I_λ be the greatest element of L_λ, define $f_\lambda \in \mathrm{L}$ by $f_\lambda(\mu) = O_\mu$ for $\lambda \neq \mu \in \Lambda$ and $f_\lambda(\mu) = I_\mu$ and, finally, let G_λ be subgroup of G with $\sigma(G_\lambda) = f_\lambda$. Then $G = \underset{\lambda\in\Lambda}{Dr}\ G_\lambda$, $1 = \gcd(g_\lambda, g_\mu)$ for all $g_\lambda \in G_\lambda$ and $g_\mu \in G_\mu$ and $\mathrm{L}(G_\lambda) \simeq \mathrm{L}_\lambda$ for all $\lambda \in \Lambda$.*

Note that the *Frattini subgroup* $\Phi(G)$ of an arbitrary group G is defined to be the intersection of all the maximal subgroups of G, with the stipulation that it shall equal G if G should have no maximal subgroups. It turns out to be a normal subgroup of G. Proposition 1.0.12 shows that for finite groups, $\mathrm{L}(G)$ is *directly decomposable* if and only if G is a nontrivial direct product of coprime groups. For finite groups the same property is inherited by the Frattini factor group $G/\Phi(G)$.

Proposition 1.0.13 (See [61], Theorem 1.6.9)**.** *Let G be a finite group. Then the following properties are equivalent.*

(i). $\mathrm{L}(G)$ *is directly decomposable.*

(ii). $G = H \times K$ *and* $\gcd(|H|, |K|) = 1$ *for two nontrivial subgroups H and K.*

(iii). $\mathrm{L}(G/\Phi(G))$ *is directly decomposable.*

Immediately we can see that cyclic groups of prime order do not fit Proposition 1.0.13. Now it is interesting to observe that cyclic groups of prime order have special symmetries in their lattices of subgroups. Their lattices are trivially distributive and modular and these two properties have profound consequences at the level of the algebraic structure. We will see this specifically in Proposition 2.0.5 later on.

Definition 1.0.14 (See [15], Definition 4.4)**.** Let L be an arbitrary lattice. L is *distributive* if for all $x, y, z \in L$ the distributive laws hold:

(i). $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$, (ii). $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

L is *modular*, if it satisfies the modular law: $x \leq z$ implies $x \cup (y \cap z) = (x \cup y) \cap z$.

It is not difficult to check that every distributive lattice is modular and it is easy to see that the only non-modular lattice with 5 or less elements is M_3, it is, however, not distributive. The lattice N_5 is non-modular and so also not distributive. Modularity and distributivity are preserved by sublattices, products and homomorphic images.

We have as yet no way of showing that the distributive law or the modular law is not satisfied except a random search for elements for which the law fails. The $M_3 - N_5$ Theorem remedies this in a most satisfactory way. It implies that it is possible to determine whether or not a finite lattice is modular or distributive from its diagram. The first part of the theorem is due to R. Dedekind and the second to G. Birkhoff.

Proposition 1.0.15 (The $M_3 - N_5$ Theorem, see [15], Theorem 4.10)**.** *The lattice L is non-modular if and only if it contains a sublattice isomorphic to N_5. Moreover, it is nondistributive if and only if it contains a sublattice isomorphic to M_3, or N_5.*

In particular, the first part can be rephrased in the context of groups as:

Corollary 1.0.16 (See [61], Theorem 2.1.2)**.** *For any G, the lattice $\mathrm{L}(G)$ is modular if and only if it does not contain a sublattice isomorphic to N_5.*

1.1 Some well known notions in group theory

Let G be a group and $x, y \in G$. Then the commutator of x and y is

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^{y}.$$

Now if X and Y are non-empty subsets of G. Then the commutator subgroup of X and Y is

$$[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle.$$

In general the commutator subgroup or derived subgroup of a group is the subgroup generated by all the commutators of the group, and denoted by $G' = [G, G]$.

The commutator subgroup is important because it is the smallest normal subgroup such that the quotient group through the original group is abelian. In other words, G/N is abelian if and only if N contains the commutator subgroup of G. Roughly speaking, G' provides a measure of how far the group is from being abelian; the "larger the commutator subgroup" is, the "less abelian" the group is (see [53]).

Now we may iterate and get the *derived series*

$$G = G^{[0]} \geq G^{[1]} = [G, G] \geq G^{[2]} = [G^{[1]}, G^{[1]}] \geq \ldots \geq G^{[n+1]} = [G^{[n]}, G^{[n]}] \geq \ldots$$

of an abstract group G consists of the subgroups defined by $G^{[n]} = (G^{[n-1]})'$ for $n \in \mathbb{N}$. A group G is *solvable* if $G^{[n]} = 1$ for some $n \in \mathbb{N}$, that is, if its derived series reaches the trivial subgroups after finitely many steps. Some elementary properties of solvable groups are recalled from [53]. Equivalently a group G is solvable, if it has an abelian series $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ in which each factor G_{i+1}/G_i is abelian and each G_i is normal in G_{i+1}.

Example 1.1.1. Every abelian group is solvable. The first example of a non-abelian solvable group is the symmetric group S_3 (see Figure 1.0.3). The smallest simple non-abelian group is A_5, (the alternating group of degree 5) it follows that every group with order less than 60 is solvable.

The *upper central series* $1 = Z_0(G) \subset Z_1(G) \subset Z_2(G) \subset \cdots$ of a group G consists of the sets defined by

$$Z_1(G) = Z(G),\ Z(G/Z_1(G)) = Z_2(G)/Z_1(G),\ Z(G/Z_2(G)) = Z_3(G)/Z_2(G),\ \ldots,$$

where $Z_i(G)$ is a normal subgroup of G (called *i-center* of G) and G is called *nilpotent* if $G = Z_n(G)$ for some $n \in G$. This means that the upper central series reaches G after finitely many steps and the smallest n such that $G = Z_n(G)$ is the *nilpotency class* of G.

By duality we may define the *lower central series*,

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots$$

of a group G as a series consisting of the sets defined by

$$\gamma_2(G) = [\gamma_1(G), G] = G', \gamma_3(G) = [\gamma_2(G), G] = [G', G] \text{ and so on.}$$

Alternatively we say that G is nilpotent if its lower central series reaches the trivial subgroup after finitely many steps.

It is not difficult to check that all the factors $Z_{n+1}(G)/Z_n(G)$ are abelian and also all the factors $\gamma_n(G)/\gamma_{n+1}(G)$ are abelian, so both the notion of central series and of derived series give a generalisation of non-abelian groups in terms of series with abelian factors. It is in fact clear that abelian groups are groups in which both the central series and the derived series stop after just one step, but one has to be careful since nilpotent groups are solvable but there are non-nilpotent solvable groups.

Example 1.1.2. All finite p-groups are in fact nilpotent. As shown from Example 1.2.3 the alternating group A_4 is a a normal subgroup of S_4 which is a classical example of a finite solvable group but not nilpotent, so the notion of nilpotence is stronger than the notion of solvability.

We can characterize nilpotency of a finite group in terms of Sylow subgroups. The following Proposition revealing some useful properties of nilpotency.

Proposition 1.1.3 (See [53], Page 130)**.** *The following statements are equivalent for a finite group G:*

(i). G is nilpotent,

(ii). If H is a proper subgroup of G, then H is a proper normal subgroup of the normalizer $N_G(H) = \{g \in G \mid g^{-1}Hg = H\}$,

(iii). Every maximal subgroup of G is normal,

(iv). G is the direct product of its Sylow subgroups.

The following observations point out some connection between metabelian, solvable and nilpotent groups, where a metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there is an abelian normal subgroup A such that the quotient group G/A is abelian. Metabelian groups are solvable, in fact, they are precisely the solvable groups of derived length at most 2. For instance, any dihedral group is metabelian, as it has a cyclic normal subgroup of index 2 and all nilpotent groups of class 3 or less are metabelian.

1.2 Hasse diagrams of some small groups

Every finite poset P, in particular every finite lattice, may be represented by a diagram in the usual plane. Represent each element of P by a point in the plane in such a way that the point p_y associated with an element y lies above the point associated with x whenever $x < y$. Then, whenever y covers x, connect the points p_x and p_y, by a line segment. This is a *Hasse diagam* for P.

We now give the following examples, since they illustrate $\mathrm{L}(G)$ and help to understand isomorphism between lattices of subgroups from their Hasse diagrams.

Example 1.2.1. Figure 1.0.1 shows the Hasse diagrams of the lattices M_3 and N_5 with five elements. These lattices are called the diamond and the pentagon respectively. Of course $M_3 \simeq \mathrm{L}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, and one can see without difficulties that for any prime $p \neq 2$ again $M_3 \simeq \mathrm{L}(\mathbb{Z}_p \times \mathbb{Z}_p)$. On the other hand, there is no group G such that $\mathrm{L}(G) \simeq N_5$ (see [15, Page 106]).

Figure 1.0.1: Hasse diagram of M_3 and N_5.

Example 1.2.2. The dihedral group of order 8 is $D_8 = \langle a, b \mid a^2 = b^4 = 1, a^{-1}ba = b^{-1}\rangle$ and has

$$\mathrm{L}(D_8) = \{\{1\}, \langle b\rangle, \langle b^2\rangle, \langle a\rangle, \langle ba\rangle, \langle b^2a\rangle, \langle b^3a\rangle, \{1, b^2, a, b^2a\}, \{1, b^2, ba, b^3a\}, D_8\}.$$

The normal subgroups are D_8, $\{1\}$, $B = \langle b\rangle$, $Z(D_8) = \langle b^2\rangle$, $M_1 = \{1, b^2, a, b^2a\}$ and $M_2 = \{1, b^2, ba, b^3a\}$. Notice that $H = \langle b^2a\rangle$ and $K = \langle a\rangle$ are contained in M_1, while $U = \langle ba\rangle$ and $V = \langle b^3a\rangle$ in M_2.

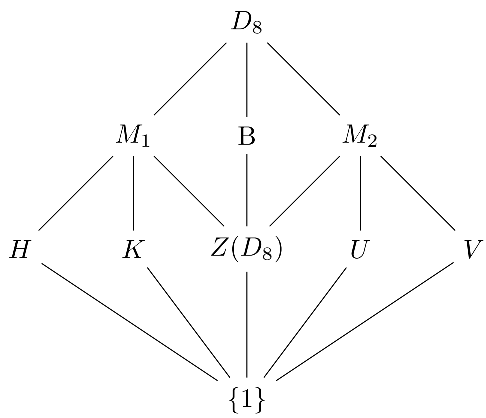

Figure 1.0.2: Hasse diagram of $\mathrm{L}(D_8)$

Example 1.2.3. The Symmetric group S_4 of order 24 is presented by

$$S_4 = \langle a, b, c \mid a^2 = b^3 = c^4 = abc = 1\rangle = \langle (12), (123), (1234)\rangle \text{ and has}$$

$$\mathrm{L}(S_4) = \{\{e\}, \langle 12\rangle, \langle 13\rangle, \langle 23\rangle, \langle 14\rangle, \langle 24\rangle, \langle 34\rangle, \langle (13)(24)\rangle, \langle (14)(23)\rangle,$$
$$\langle (12)(34)\rangle, \langle 123\rangle, \langle 124\rangle, \langle 134\rangle, \langle 234\rangle, \langle (12)(34), (13)(24)\rangle, \langle (13), (24)\rangle,$$
$$\langle (14), (23)\rangle, \langle (12), (34)\rangle, \langle (123), (12)\rangle, \langle (124), (12)\rangle, \langle (134), (13)\rangle,$$
$$\langle (234), (23)\rangle, \langle (1234), (13)\rangle, \langle (1243), (14)\rangle, \langle (1324), (12)\rangle, A_4, S_4\}.$$

Here we can observe that $\mathrm{L}(\langle (12)(34), (13)(24)\rangle) \simeq \mathrm{L}(\mathbb{Z}_2 \times \mathbb{Z}_2)$, $\mathrm{L}(\langle (123), (12)\rangle) \simeq \mathrm{L}(S_3)$ and $\mathrm{L}(\langle (1234), (13)\rangle) \simeq \mathrm{L}(D_4)$. The lattice diagrams of $\mathrm{L}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is shown in the Example 1.2.1, where the lattice diagrams of $\mathrm{L}(S_3)$, $\mathrm{L}(\langle (1234), (13)\rangle)$ and $\mathrm{L}(A_4)$ are shown in the Figures 1.0.3, 1.0.4 and 1.0.5 later on.

There are 30 elements in $\mathrm{L}(S_4)$ and these subgroups of S_4 are divided into 11 conjugacy classes and 9 isomorphism types. It is easy to visualize from Figure 1.0.6 below that there are 9 subgroups isomorphic to $\mathbb{Z}_2$ with 6 of them are in one class and the other 3 in an other different conjugacy class, 4 subgroups isomorphic to $\mathbb{Z}_3$, 3 subgroups isomorphic to $\mathbb{Z}_4$, 3 subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ with one of them is in different class, 4 subgroups isomorphic to S_3 and 3 subgroups isomorphic to D_4.

By definition, a normal subgroup is equal to all its conjugate subgroups, thus the four normal subgroups of S_4 are the ones in their own conjugacy class. Hence $\mathrm{N}(S_4) = \{\langle (1)\rangle, \langle (12)(34), (13)(24)\rangle, A_4, S_4\}$. Note that $S_3 \simeq S_4/\langle (12)(34), (13)(24)\rangle$ and then we have $S_4 = S_3 \ltimes \langle (12)(34), (13)(24)\rangle$ since $S_3 \cap \langle (12)(34), (13)(24)\rangle = \langle (1)\rangle$ as shown by Figure 1.0.6. Furthermore we observe that $\mathrm{L}(\langle (12)(34), (13)(24)\rangle)$, $\mathrm{L}(\langle (123), (12)\rangle)$, $\mathrm{L}(\langle (1234), (13)\rangle)$ and $\mathrm{L}(A_4)$ are some sublattices of $\mathrm{L}(S_4)$.

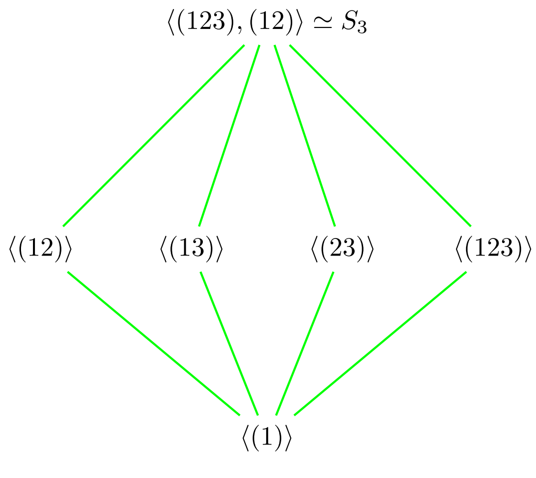

Figure 1.0.3: Hasse diagram of $\mathrm{L}(S_3)$.

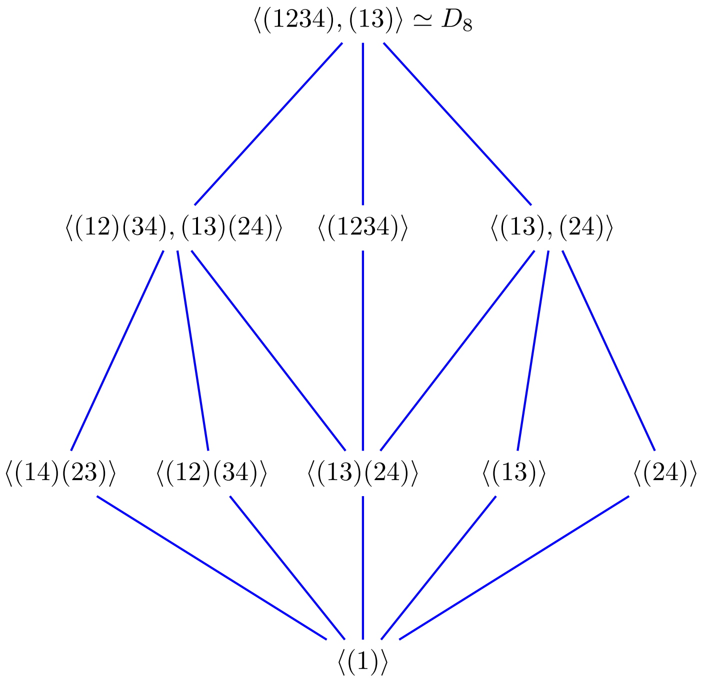

Figure 1.0.4: Hasse diagram of $\mathrm{L}(\langle(1234),(13)\rangle)$

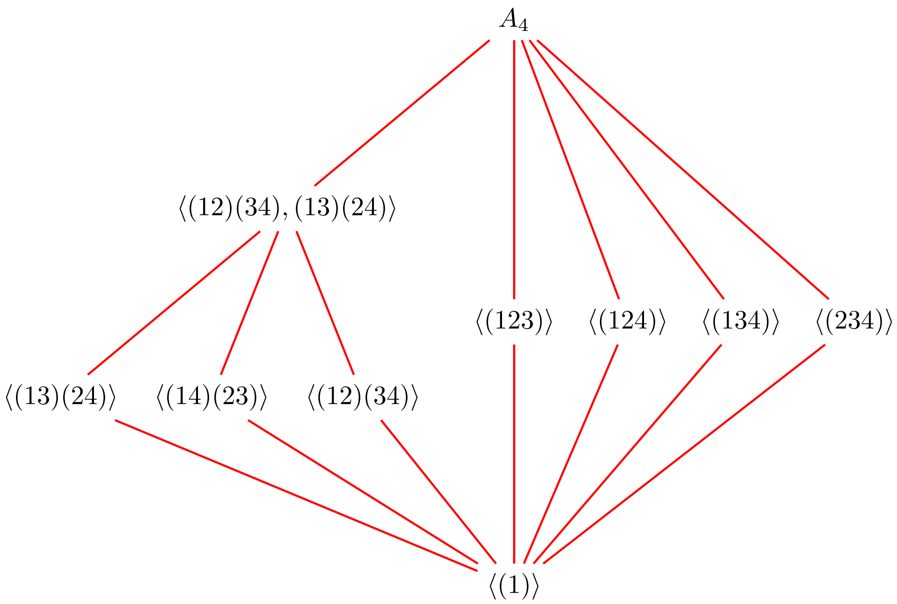

Figure 1.0.5: Hasse diagram of $\mathrm{L}(A_4)$.

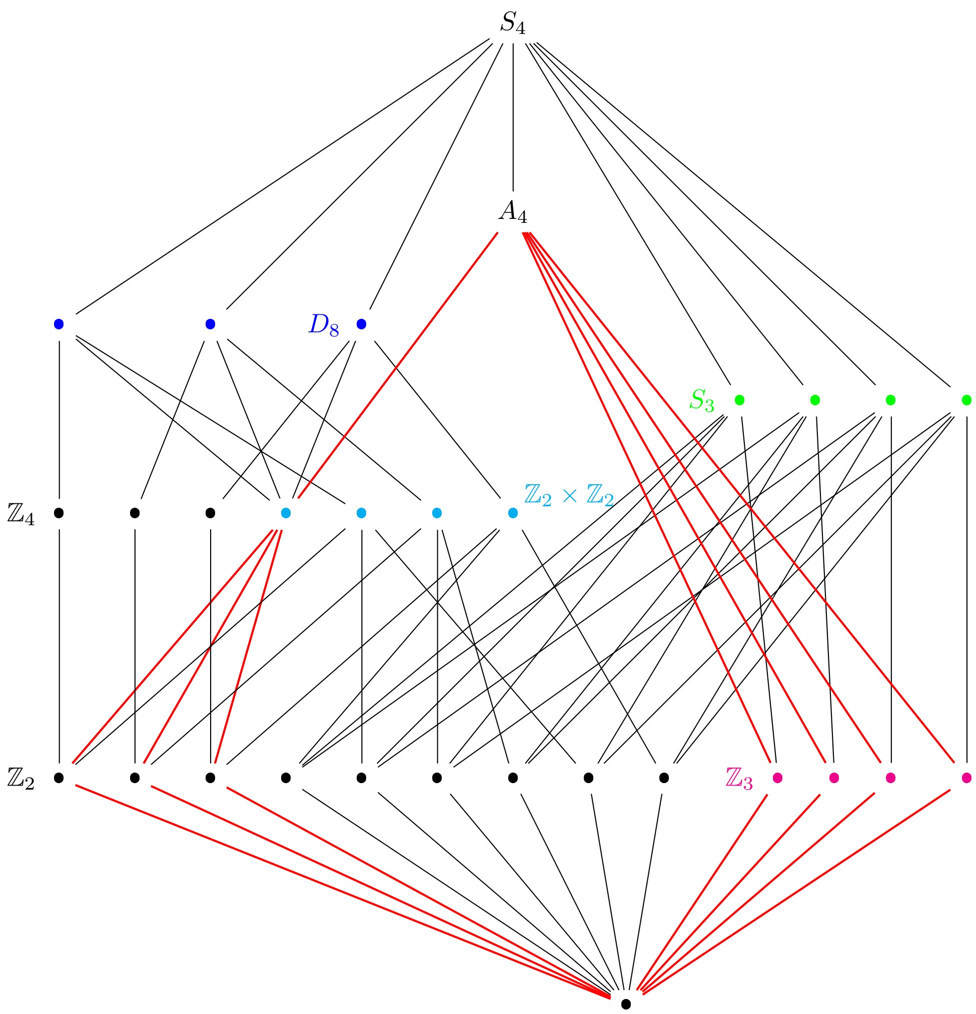

Figure 1.0.6: Hasse diagram of $\mathrm{L}(S_4)$.

Note that in the diagram above, the colours help to identify some sublattices, which are isomorphic to the lattice of subgroups of D_8, of S_3, of $\mathbb{Z}_2$, of $\mathbb{Z}_3$, of $\mathbb{Z}_2 \times \mathbb{Z}_2$, of $\mathbb{Z}_4$ and of A_4. They also appear in previous diagrams separately.

Chapter 2

A Short Discussion on a Classical Theorem of Ore

In this chapter, we report some classical results of O. Ore on lattice-theoretic characterizations of cyclic finite groups. In particular we will discuss the modular and distributive characterizations of certain classes of groups. This was at the origin of our researches and we will see that a series of generalizations are meaningful in different contexts.

We begin with the following definition:

Definition 2.0.1 (See [61], Page 43). We say that the element M of the lattice of subgroups $\mathrm{L}(G)$ of a group G is modular in $\mathrm{L}(G)$, if

(i). $\langle X, M \cap Z\rangle = \langle X, M\rangle \cap Z$ for all $X, Z \in \mathrm{L}(G)$ with $X \subseteq Z$; and

(ii). $\langle M, Y \cap Z\rangle = \langle M, Y\rangle \cap Z$ for all $Y, Z \in \mathrm{L}(G)$ with $M \subseteq Z$.

A subgroup M of G is *modular* in G, if M is modular in $\mathrm{L}(G)$. Modular elements were introduced by Kurosh in 1940, Zassenhaus in 1958 and largely used by Zacher and Napolitani (see [61]). In O. Ore [58] shows the following theorem, proving that the lattice of normal subgroups of an arbitrary group and the subgroup lattice of an abelian group are modular.

Proposition 2.0.2 (See [61], Theorem 2.1.4). *Let G be a group.*

(i). If N is normal in G, then $NH = HN$ for all $H \leq G$.

(ii). If $M \leq G$ such that $MH = HM$ for all $H \leq G$, then M is modular in G.

Since abelian groups are modular, we investigate non-abelian groups with modular subgroup lattices (the so called *M-groups*). If a subgroup M of G satisfies the hypothesis of (ii) of Proposition 2.0.2, we say that M is *permutable* in G. In fact (ii) of Proposition 2.0.2 may be rephrased by saying that permutable subgroups are modular in a group. Of course, normal subgroups are permutable; this is indeed (i) of Proposition 2.0.2, so (ii) of Proposition 2.0.2 can be seen as a weaker version than (i) of Proposition 2.0.2. These permutable subgroups were introduced by Ore [58, 59] who used the terminology *quasi-normal*. Proposition 2.0.2 actually shows that both normal subgroups and permutable subgroups are modular in G.

From the above discussion we observed that there are close connections between modular groups and abelian groups, but they do not form a nice class of groups, except for the case of abelian groups with distributive subgroup lattice. This observation was formalized by a classical result of O. Ore [59] in 1938.

Recall that a group G is called locally cyclic if every finite subset of G generates a cyclic subgroup. Equivalently we can say that $\langle a, b\rangle$ is cyclic for every pair a, b of elements of G. In particular, every locally cyclic group is abelian. The additive group a of rational numbers and the group $\mathbb{Q}/\mathbb{Z}$ of rational numbers modulo 1 are locally cyclic since finitely many rational numbers have a common denominator n and therefore are contained in the cyclic group generated by $\frac{1}{n}$. It is a familiar result that a group is locally cyclic if and only if it is isomorphic to a subgroup of $\mathbb{Q}$ or of $\mathbb{Q}/\mathbb{Z}$.

A Prüfer group $\mathbb{Z}(p^\infty)$ is an example of infinite abelian p-group which is locally cyclic, where the Prüfer p-group is defined as the Sylow p-subgroup of the quotient group $\mathbb{Q}/\mathbb{Z}$, consisting of those elements whose order is a power of p, that is

$$\mathbb{Z}(p^\infty) = \mathbb{Z}[1/p]/\mathbb{Z},$$

where $\mathbb{Z}[1/p]$ denotes the group of all rational numbers whose denominator is a power of p, using addition of rational numbers as group operation.

Proposition 2.0.3 (Ore's Theorem, See [61], Theorem 1.2.3)**.** *The subgroup lattice of a group G is distributive if and only if G is locally cyclic.*

Argument of Proof in [61], Pages 12-13. Suppose first that $\mathrm{L}(G)$ is distributive and let $a, b \in G$. We have to show that $\langle a, b\rangle$ is cyclic. Since $\langle a\rangle \cap \langle b\rangle \leq \langle a\rangle \leq C_G(\langle a\rangle)$ and $\langle a\rangle \cap \langle b\rangle \leq \langle b\rangle \leq C_G(\langle b\rangle)$, then $\langle a\rangle \cap \langle b\rangle$ is cyclic and abelian and $\langle a\rangle \cap \langle b\rangle \leq C_G(\langle a\rangle) \cap C_G(\langle b\rangle)$. Again since $Z(G) = \bigcap_{x\in G} C_G(\langle x\rangle) \leq C_G(\langle x\rangle)$, this gives $\langle a\rangle \cap \langle b\rangle$

is centralized by a and b, then $\langle a\rangle \cap \langle b\rangle \leq Z(G)$ and hence $\langle a\rangle \cap \langle b\rangle$ is normal. Also $\langle ab\rangle \cup \langle a\rangle = \langle a, b\rangle = \langle ab\rangle \cup \langle b\rangle$ and so by Definition 1.0.14(i),

$$\langle ab\rangle \cup (\langle a\rangle \cap \langle b\rangle) = (\langle ab\rangle \cup \langle a\rangle) \cap (\langle ab\rangle \cup \langle b\rangle) = \langle a, b\rangle.$$

Thus $\langle a, b\rangle/\langle a\rangle \cap \langle b\rangle \simeq \langle ab\rangle \cap (\langle a\rangle \cap \langle b\rangle)$ is cyclic and therefore $\langle a, b\rangle$ is abelian, as a cyclic extension of a central subgroup. By the structure of finitely generated abelian groups (see [53]) there exist $c, d \in G$ such that $\langle a, b\rangle = \langle c\rangle \times \langle d\rangle$. By what we have shown, $\langle c, d\rangle/\langle c\rangle \cap \langle d\rangle$ is cyclic. since $\langle c\rangle \cap \langle d\rangle = 1$, $\langle a, b\rangle = \langle c, d\rangle$ is cyclic.

Now suppose that G is locally cyclic and let $A, B, C \in \mathrm{L}(G)$. We need only show that the first distributive law holds, or, since G is abelian, that $A(B\cap C) = AB\cap AC$. Clearly $A(B\cap C) \leq AB\cap AC$, because for $ab \in A(B\cap C)$, we have $a \in A$ and $b \in B\cap C$ and this implies $ab \in AB$ and $ab \in AC$. Let $x \in AB\cap AC$, hence $x = ab = a'c$ with $a, a' \in A, b \in B$ and $c \in C$. Since G is locally cyclic, there exists $g \in G$ such that $\langle a, a', b, c\rangle = \langle g\rangle$. Thus $x = ab = a'c$ implies that $\langle g\rangle = (A\cap\langle g\rangle)(B\cap\langle g\rangle) = (A\cap\langle g\rangle)(C\cap\langle g\rangle)$. If one of the three subgroups $A\cap\langle g\rangle$, $B\cap\langle g\rangle$, $C\cap\langle g\rangle$ is trivial, then either $x = b = c \in B\cap C$ or $x \in A$. In both cases, $x \in A(B\cap C)$. So suppose that all these subgroups are nontrivial and let n, r, s e the respective indices of $A\cap\langle g\rangle$, $B\cap\langle g\rangle$, $C\cap\langle g\rangle$ in $\langle g\rangle$. Then $(n, r) = 1 = (n, s)$, hence $(n, rs) = 1$ and therefore

$$\langle g\rangle = \langle g^n\rangle\langle g^{rs}\rangle = (A\cap\langle g\rangle)(B\cap C\cap\langle g\rangle) \leq A(B\cap C).$$

Again it follows that $x \in A(B\cap C)$. Thus $AB\cap AC \leq A(B\cap C)$ as required. □

Ore's Theorem, and perhaps even more its consequences, is one of the most beautiful results and answered that:

1. Which class of groups is the class of all groups with a given lattice property? and, conversely,

2. Which lattice property characterizes a given class of groups?

An interesting class of lattices, the finite distributive lattices, belongs to a simple class of groups, the finite cyclic groups. Using Ore's theorem, however, it is not difficult to characterize the class of cyclic groups.

Proposition 2.0.4 (See [61], Theorem 1.2.5)**.** *The group G is cyclic if and only if its subgroup lattice $\mathrm{L}(G)$ is distributive and every ascending chain in $\mathrm{L}(G)$ has finite length.*

The following result shows that a finite cyclic group can be characterized via decompositions of $\mathrm{L}(G)$. This is in connection with the result of Suzuki in Proposition 1.0.12.

Proposition 2.0.5 (See [61], Theorem 1.2.7)**.** *Let* $n_1, n_2, \cdots, n_r \in \mathbb{N}$. *The group* G *is cyclic of order* $p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$ *with distinct primes* p_i *if and only if* $\mathrm{L}(G)$ *is a direct product of chains of lengths* $n_1, n_2, \cdots, n_r$.

There are a series of equivalent formulations of Ore's Theorem in different contexts. In fact, if we look at the argument of Proposition 2.0.3, the ideas and the methods can be easily generalized to rings, Lie algebras, modules, or even topological structures such as locally compact groups. Several generalizations of Proposition 2.0.3 appeared in different areas of pure mathematics. Some of them are very recent.

We report a result of Kolman [39], characterizing distributive Lie algebras over fields of any characteristic. The original ideas of "locally cyclic" in Ore's Theorem is replaced by the notion of " dimension one" (see [39] for details).

Proposition 2.0.6 (See [39], Proposition 2.1)**.** *Let* $\mathfrak{l}$ *be a Lie algebra over a field of any characteristic. Then the lattice of Lie subalgebras* $\mathrm{L}(\mathfrak{l})$ *of* $\mathfrak{l}$ *is distributive if and only if* $\mathfrak{l}$ *is one-dimensional.*

Proposition 2.0.6 is just an example of how the theory has been developed, in order to find characterizations of distributivity. Here we are interested to study Ore's result in terms of a probabilistic approach and we will see the details in the following chapters.

While finite groups with distributive subgroup lattice have been completely characterized by Proposition 2.0.3, one could ask whether similar classifications are possible for modular finite groups, which turn out to contain those with distributive subgroup lattice. The answer is positive and was given by Iwasawa [36, 37] long time ago.

Remark 2.0.7. Modular finite groups are nilpotent. Therefore they have a decomposition in Sylow p-subgroups, according to Proposition 1.1.3. This means by Proposition 1.0.12 that one can reduce the study of modular finite groups to that of finite p-groups.

We present Iwasawa's Theorem for finite modular p-groups.

Proposition 2.0.8 (Iwasawa's Theorem, see [61], Theorem 2.3.1)**.** *A finite* p*-group* G *is modular if and only if*

(i). G *is a direct product of a quaternion group* Q_8 *of order* 8 *with an elementary abelian* 2*-group, or*

(ii). G *contains an abelian normal subgroup* A *with cyclic factor group* G/A*; further*

there exists an element $b \in G$ with $G = A\langle b \rangle$ and a positive integer s such that $b^{-1}ab = a^{1+p^s}$ for all $a \in A$,with $s \geq 2$ in case $p = 2$.

We recall that a group is called *hamiltonian*, if it is non-abelian and all of its subgroups are normal. Moreover a group G is called *quasihamiltonian*, if it is non-abelian and all of its subgroups are permutable in G. Of course, quasihamiltonian groups are modular, and hamiltonian groups are in particular quasihamiltonian. The quaternion group Q_8 is an example of hamiltonian group, but those in (ii) of Proposition 2.0.8 may be quasi-hamiltonian and nonhamiltonian. Note that the direct product of a quaternion group and an elementary abelian 2-group of rank n is hamiltonian group.

However, D_8 and the non-abelian group of exponent p for $p > 2$ are generated by two elements of order p and are non-modular groups. On the other hand, finite groups of order p^3 where any two elements of order p commute and every subgroup of order p^2 is normal are modular groups. These are captured by (ii) of Proposition 2.0.8.

Recall that in a (finite or infinite) group G the quotient group H/K is called *section* of G, where K is normal in H and both H and K are subgroups of G. For an arbitrary p-group, the sections of order p^3 decide whether it has modular subgroup lattice or not. In fact [61, Lemma 2.3.3] shows that if G is a finite p-group, then G has modular subgroup lattice if and only if each of its sections of order p^3 does. Somehow this is a way to recognise modular p-groups from the size of its homomorphic images. Therefore if G is non-modular, then there exist subgroups H, K of G with K normal in G such that H/K is dihedral of order 8 or non-abelian of order p^3 and exponent p for $p > 2$. In other words, the presence of homomorphic images, which are isomorphic to D_8, or to a non-abelian p-group of order p^3 and exponent p, detects non-modular groups.

Additional information on modular infinite groups are reported below:

Proposition 2.0.9 (Iwasawa–Napolitani's Theorem, see [61], Theorem 2.4.14)**.** *Let p be a prime. The group G is a non-abelian locally finite $p-$group with modular subgroup lattice if and only if*

(i). G is a direct product of a quaternion group Q_8 of order 8 with an elementary abelian 2-group, or

(ii). G contains an abelian normal subgroup A of exponent p^k with cyclic factor group G/A of order p^m $(k, m \in \mathbb{N})$ and there exists an element $b \in G$ with $G = A\langle b \rangle$ and an integer s which is atleast 2 in case $p = 0$ such that $s < k \leq s + m$ and $b^{-1}ab = a^{1+p^s}$ for all $a \in A$.

From [61, Page 87], we can see that a group G (finite or infinite) is the direct product of its Sylow p-subgroups and these are locally finite p-groups with modular subgroup lattices if and only if G is a locally nilpotent M-group. Somehow one can see the infinite modular p-groups, at the level of structure, do not differ very much from the finite case.

It is simple to show that a modular subgroup of a group G is in general not permutable, and a permutable subgroup is in general not normal in G. So there are a series of examples and additional conditions, which the reader can find in [61], in order to detect when modular subgroups are permutable. For instance, this happens in the finite case when a subgroups is both modular and subnormal [61, Theorem 5.1.1].

To obtain a lattice-theoretic characterization of a class of groups we replace concepts appearing in the definition by lattice-theoretic concepts that are equivalent to them or nearly so. For example, we should try to replace "normal subgroup" by "modular subgroup", "cyclic factor group" by "distributive interval in the subgroup lattice", "abelian factor group" by "modular interval in the subgroup lattice", and so on, and see [61] for detail for the following lattice-theoretic characterization of a class of groups. It is instructive to report for this logic, the following fact.

Proposition 2.0.10 (See [61], Theorem 5.3.1)**.** *The finite group G is simple if and only if 1 and G are the only modular elements in $\mathrm{L}(G)$.*

The above result shows that the modularity of certain subgroups, instead of others, in a group G is strongly related to the complexity of $\mathrm{N}(G)$. In fact Proposition 2.0.10 may be rephrased by saying that $\mathrm{N}(G)$ becomes trivial as long as there are no modular subgroups in G, except for the trivial subgroups of G.

Actually a recent generalization, due to Herfort and others [32] may be formulated for locally compact groups and here the terminology can be found in [31]. Again we observe a characterization of the modularity, but this time both an algebraic and a topological structure are preserved at the level of topological lattices.

Proposition 2.0.11 (See [32], Theorem 1.1)**.** *The following statements for a locally compact abelian p-group G are equivalent:*

(i). G is topologically modular.

(ii). For U an open compact subgroup exclusively one of the following holds

(ii.1). U has finite p-rank. Then the torsion subgroup $\mathrm{tor}(G)$ of G is discrete and $G/\mathrm{tor}(G)$ has finite p-rank.

(ii.2). U has infinite p-rank. Then the largest divisible subgroup $\mathrm{div}(G)$ *of G is closed, G/U and* $\mathrm{div}(G)$ *both have finite p-rank, and, $G/\mathrm{div}(G)$ is compact.*

(iii). G is strongly topologically quasihamiltonian.

Chapter 3

Subgroup Commutativity Degrees and Generalizations

In this chapter we make a short review of the notions of commutativity degree and subgroup commutativity degree of a finite group. We change completely our approach to the problem of classification of groups, which are rich in permutable subgroups, stressing more on numerical restrictions and computational aspects. We also mention some generalizations of the subgroup commutativity degree, appeared [38, 40, 41, 42, 52, 57, 40, 41, 66]. Explicit formulas will be mentioned for some particular classes of finite groups. Some recent results are recalled from [40, 41, 42, 52, 57, 66, 67, 68, 69, 70].

One of the first question of the contemporary probabilistic group theory was posed by Erdős and Turán [21], who asked about the probability that two randomly chosen group elements are commuting in a finite group. Their answer is what is known today as commutativity degree of a finite group: it is a measure for abelianness of a group. For a finite group G,

$$d(G) = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G| \cdot |G|} \tag{3.0.1}$$

is the *commutativity degree* of G. This is also called *commuting probability* by some authors. Gustafson [28] and Gallagher [26] found that

$$d(G) = \frac{k(G)}{|G|} = \frac{|\mathrm{Irr}(G)|}{|G|},$$

where $k(G)$ is the number of conjugacy classes of G and $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G. Moreover Gustafson [28] showed that it was possible to generalize the above concept to infinite groups and recent progresses have been made by [33, 34, 54, 55, 51]

Remark 3.0.1. Given a finite group G, of course $d(G) = 1$ if and only if G is a abelian. Moreover a classical result [28] shows that $d(G) > 5/8$ implies G abelian. The numerical bound $5/8$ is achieved if and only if $G/Z(G)$ is 2-elementary abelian of rank two, so a structural condition is observed if and only if a precise numerical bound is reached.

Of course, isomorphic groups have the same commutativity degree, but actually we may require a less stringent condition, mentioned in [12, 24, 54, 55].

Definition 3.0.2 (See [54], Definition 1.1)**.** Let G_1, G_2 be two groups, H_1 a subgroup of G_1 and H_2 a subgroup of G_2. A pair (α, β) is said to be a relative n-isoclinism from (H_1, G_1) to (H_2, G_2) if the following holds:

(i) the map

$$\alpha^{n+1} : \left(\frac{H_1}{Z_n(G_1) \cap H_1}\right)^n \times \frac{G_1}{Z_n(G_1)} \longrightarrow \left(\frac{H_2}{Z_n(G_2) \cap H_2}\right)^n \times \frac{G_2}{Z_n(G_2)}$$

is an isomorphism;

(ii) β is an isomorphism from $[_nH_1, G_1]$ to $[_nH_2, G_2]$;

(iii) For all $h_1, ..., h_n \in H_1$, $k_1, ..., k_n \in H_2$, $g_1 \in G_1$, $g_2 \in G_2$ there exists a commutative diagram in which the map

$$\gamma(n, H_1, G_1) : ((h_1(Z_n(G_1)\cap H_1), ..., h_n(Z_n(G_1)\cap H_1), g_1Z_n(G_1))) \in \frac{H_1}{Z_n(G_1) \cap H_1} \times ...$$

$$... \times \frac{H_1}{Z_n(G_1) \cap H_1} \times \frac{G_1}{Z_n(G_1)} \mapsto [h_1, ..., h_n, g_1] \in [_nH_1, G_1]$$

and the map

$$\gamma(n, H_2, G_2) : ((k_1(Z_n(G_2)\cap H_2), ..., k_n(Z_n(G_2)\cap H_2), g_2Z_n(G_2))) \in \frac{H_2}{Z_n(G_2) \cap H_2} \times ...$$

$$... \times \frac{H_2}{Z_n(G_2) \cap H_2} \times \frac{G_2}{Z_n(G_2)} \mapsto [k_1, ..., k_n, g_2] \in [_nH_2, G_2],$$

can be composed by the rule

$$\gamma(n, H_2, G_2) \circ \alpha^{n+1} = \beta \circ \gamma(n, H_1, G_1).$$

We briefly say that G_1 and G_2 are *isoclinic* if Definition 3.0.2 is realized when $H_1 = G_1$, $H_2 = G_2$ and $n = 1$. Lescot [45] classified up to isoclinism all groups with $d(G) \geq \frac{1}{2}$, showing that G must be isoclinic either to the trivial group, or to an extraspecial 2-group, or to S_3. He also noted that for $d(G) > \frac{1}{2}$ the group G must be nilpotent. Solvability was studied by Lescot in [43, 44], finding that $d(G) > \frac{1}{12}$ implies the group is solvable. Guralnick and Robinson [29, Theorem 11] improved this bound, finding that if $d(G) > \frac{3}{40}$ then either G is solvable, or $G \simeq A_5 \times A$ for some abelian group A. Barry and others [9] proved that G must be supersolvable whenever $d(G) > \frac{1}{3}$, since $d(A_4) = \frac{1}{3}$, and this bound cannot be improved. Any group isoclinic to A_4 has commuting probability exactly equal to $\frac{1}{3}$ and the special role of A_4 among non-supersolvable groups with commuting probability greater than $\frac{5}{16}$ is illustrated below. Actually the papers of Lescot answered in fact some of the original questions of Erdős, Turán [21] and Gustafson, opening a new and productive line of research on the subject.

Proposition 3.0.3 (See [46], Theorems 1 and Theorems 4)**.** *Let G be a finite group. If $d(G) > \frac{5}{16}$, then either G is supersolvable, or G is isoclinic to A_4, $G/Z(G)$ is isoclinic to A_4. Moreover, if $d(G) > \frac{1}{s}$ for some integer $s > 1$ and G splits over an abelian normal nontrivial subgroup N, then G has a nontrivial conjugacy class inside N of size at most $s-1$.*

We have only mentioned a few evidences that numerical bounds for the commutativity degree have a precise meaning from the point of view of the structure of a finite group. This motivates a large interest in probabilistic group theory.

Erfanian and others [23] introduced the *relative commutativity degree*, weakening the original notion of commutativity degree and showing invariance up to weaker forms of isoclinism as in Definition 3.0.2. Recent conjectures were shown in [20, 33] so that the original line of research, indicated by Erdős, Turán and Joseph is well settled nowadays.

Tărnăuceanu [67, 68] introduced the *subgroup commutativity degree* of a finite group G in the perspective of lattice theory, defining

$$sd(G) = \frac{|\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid XY = YX\}|}{|\mathrm{L}(G)|\,|\mathrm{L}(G)|} \tag{3.0.2}$$

as the probability of commuting subgroups in $\mathrm{L}(G)$. This notion is the natural lattice theoretical perspective, which may allow us to intepret the number of commuting elements in G. The subgroup commutativity degree of a finite group G is in fact connected with permutable subgroups and modular subgroups of G.

Remark 3.0.4. In finite group G, we have $0 < sd(G) \leq 1$ and the equality $sd(G) = 1$

holds if and only if all subgroups of G are permutable.

Actually we can say more:

Proposition 3.0.5 (See,[67], Proposition 2.1)**.** *For finite group G we have $sd(G) = 1$ if and only if G is the direct product of its Sylow p-subgroups and these are all modular; or equivalently G is a nilpotent modular group.*

If G_1 and G_2 are two groups, in general we do not have $sd(G_1 \times G_2) = sd(G_1) \cdot sd(G_2)$, that is, that the probability of two independent events are independent. A sufficient condition in order to this equality holds is that G_1 and G_1 be of coprime orders, that is, hypotheses of Proposition 1.0.12 are satisfied.

Proposition 3.0.6 (See [67], Corollary 2.3)**.** *If G is a finite nilpotent group and G_i are the Sylow subgroups of G for $i = 1, 2, \ldots, k$, then $sd(G) = sd(G_1) \cdot sd(G_2) \cdot \ldots \cdot sd(G_k)$.*

If N is an arbitrary normal subgroup of G, one can correlate $sd(G)$ with $sd(N)$ and $sd(G/N)$.

Proposition 3.0.7 (See [67], Proposition 2.4)**.** *Let G be a finite group and N be a normal subgroup of G. Then*

$$sd(G) \geq \frac{1}{|\mathrm{L}(G)|^2} \cdot \Big((|\mathrm{L}(N)| + |\mathrm{L}(G/N)| - 1)^2 + (sd(N) - 1) \cdot |\mathrm{L}(N)|^2 + (sd(G/N) - 1) \cdot |\mathrm{L}(G/N)|^2 \Big) \tag{3.0.3}$$

which is only depending on G/N and N.

Suppose G is metabelian, then N and G/N are abelian, and this implies $sd(N) = sd(G/N) = 1$. Therefore (3.0.3) becomes (see [67, Corollary 2.5])

$$sd(G) \geq \Big(\frac{|\mathrm{L}(N)| + |\mathrm{L}(G/N)| - 1}{|\mathrm{L}(G)|} \Big)^2. \tag{3.0.4}$$

Moreover if N is of prime index, then (3.0.3) also becomes (see [67, Corollary 2.6])

$$sd(G) \geq \frac{1}{|\mathrm{L}(G)|^2} \big(sd(N) ||\mathrm{L}(N)|^2 + 2|\mathrm{L}(N)| + 1 \big)^2 \tag{3.0.5}$$

Again we can detect a significant result of structure for prescribed values of $sd(G)$, as observed above for $d(G)$, and the contributions [38, 40, 41, 42, 52, 57, 64, 65, 66, 68, 69] show advances in this direction.

More recently, the *cyclic subgroup commutativity degree*

$$csd(G) = \frac{|\{(H,K) \in \mathrm{L}_1(G) \times \mathrm{L}_1(G) \mid HK = KH\}|}{|\mathrm{L}_1(G)|^2}$$

of a finite group G has been studied in [41], where this quantity measures the probability of two random cyclic commuting subgroups of G. Here $\mathrm{L}_1(G)$ is the sublattice of cyclic subgroups of G. A criterion for a finite p-group to be an Iwasawa group with the structure of Proposition 2.0.8 is presented in terms of $csd(G)$ in [41]. Moreover the cyclic subgroup commutativity degree of the group $\mathbb{Z}_{2^n} \times Q_8, n \geq 2$, which is not captured by Proposition 2.0.8, tends to 1 when n tends to infinity (see [41, Theorem 4.1]). In addition Lazorec [41] shows that there is no constant $c \in (0,1)$ such that if $csd(G) > c$ then G satisfies Proposition 2.0.8. However, if the condition $csd(G) > c$ is replaced by the stronger condition $csd^*(G) > c$, where

$$csd^*(G) = \min\{csd(S) \mid S \text{ section of } G\}$$

then it is possible to see that:

Proposition 3.0.8 (See [41], Lemmas 4.3 and 4.4)**.** *Let G be a finite p-group such that $csd^*(G) > \frac{41}{49}$. Then G is modular. Furthermore if $csd^*(G) > \frac{19}{25}$, then G is nilpotent.*

Again and again we note numerical bounds which have a precise interpretation at the level of the group structure.

Another remarkable quantity associated to a finite group G is the *factorization number*

$$F_2(G) = |\{(H,K) \in \mathrm{L}(G)^2 \mid G = HK\}|,$$

which denotes the number of possible factorizations of G in the product of two subgroups H and K. There is a strong connection between $sd(G)$ and $F_2(G)$, due to Farrokhi and others [25, 40, 60, 70],

$$sd(G) = \frac{1}{|\mathrm{L}(G)|^2} \sum_{H \in \mathrm{L}(G)} F_2(H).$$

In [66], the *cyclic factorization number* of a finite group G,

$$CF_2(G) = |\{(H,K) \in \mathrm{L}_1(G)^2 \mid G = HK\}|$$

was introduced, obtaining an equivalent description for the cyclic factorization number;

$$csd(G) = \frac{1}{|\mathrm{L}_1(G)|^2} \sum_{H \in \mathrm{L}(G)} CF_2(H).$$

Note that a maximal factorizations of G appear also in [47] and have relations with the aforementioned invariants (see [40]).

Otera and Russo [52] introduced weaker forms of subgroup commutativity degree, namely the *subgroup S-commutativity degree* of a finite group, which measures the probability that subnormal subgroups commute with maximal subgroups in a finite groups. They showed the same lower bounds of $sd(G)$ via the subgroup S-commutativity degree. Note that Heineken and others [3, 10] describe the structure of finite groups in which the subnormal subgroups permute with all Sylow subgroups (the so-called PST–groups). Therefore the subgroup S-commutativity degree may be interpreted as a probabilistic approach to the study of groups which are close (or far) from being PST-groups.

One may replace the sublattice of subnormal subgroups $\mathrm{SN}(G)$ of $\mathrm{L}(G)$ and the smallest sublattice $\mathrm{M}(G)$ of $\mathrm{L}(G)$ containing all maximal subgroups of G with two arbitrary sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$. Therefore one can get a further generalization of the subgroup S-commutativity degree.

Russo [57] showed some results on the probability that a randomly picked pair (H, K) of subgroups of a finite group G satisfies $[H, K] = 1$. This notion was motivate by two facts: the first was the relative commutativity degree of H and K in G (see [23, 57])

$$d(H, K) = \frac{|\{(h, k) \in H \times K \mid [h, k] = 1\}|}{|H|\ |K|} = \frac{1}{|H|\ |K|} \cdot \sum_{h \in H} |C_K(h)|, \tag{3.0.6}$$

where H and K are two arbitrary subgroups of G. Of course, $d(G, G) = d(G)$ whenever $H = K = G$. The second motivation was that the condition $[H, K] = 1$ implies $HK = KH$, where $[H, K] = \langle [h, k] | h \in H, k \in K \rangle$. Conversely, $HK = KH$ does not imply $[H, K] = 1$, so it was meaningful to define the *strong subgroup commutativity degree* of G

$$ssd(G) = \frac{|\{(H, K) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid [H, K] = 1\}|}{|\mathrm{L}(G)|^2}, \tag{3.0.7}$$

noting that $ssd(G)$ denotes the probability that the subgroup $[H, K]$ of an arbitrarily chosen pair of subgroups H, K of G is equal to the trivial subgroup of G.

Remark 3.0.9. It is known that if a finite group G is modular, then $sd(G) = 1$, but $ssd(G)$ may be different from 1. However, a finite group G has $ssd(G) = 1$ if and only if it G is abelian by [57, Proposition 2.1].

Recall that a finite group G has *very strong centralizers*, if for all subgroups H of G

$$|C_G(H)| \geq |\{K \in \mathrm{L}(G) \ |HK = KH\}|.$$

This condition makes possible an interpolation with $sd(G)$ as shown below.

Proposition 3.0.10 (See [57], Theorem 2.6)**.** *Let H and K be two subgroups of a finite group G. If G has very strong centralizers, then*

$$ssd(G) \leq sd(G) \leq \frac{|G|^2}{|\mathrm{L}(G)|^2} \cdot \sum_{H,K \in \mathrm{L}(G)} d(H,K).$$

It is interesting to note that $ssd(G)$ can be reformulated via character theory. Now if H is a cyclic subgroup generated by $g \in G$ and $K = G$, then one can consider

$$ssd(\langle g \rangle, G) = \frac{|\{(X,Y) \in \mathrm{L}(\langle g \rangle) \times \mathrm{L}(K) \mid [X,Y] = 1\}|}{|\mathrm{L}(\langle g \rangle)||\mathrm{L}(K)|} \tag{3.0.8}$$

and has been observed that

Proposition 3.0.11 (See [57], Theorem 3.2)**.** *In a finite group G*

$$\xi(g) = |\{(X,Y) \in \mathrm{L}(\langle g \rangle) \times \mathrm{L}(K) \mid [X,Y] = 1\}|$$

is induced by a $\mathbb{Q}-$generalized character ξ of G for all $g \in G$.

We are now ready to appreciate the new notion introduced in the present thesis:

Given a finite group G and two sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$ and the power set $\mathrm{P}(G)$, we may consider the function

$$g : (H,K) \in \mathrm{S}(G) \times \mathrm{T}(G) \longmapsto g(H,K) = HK \in \mathrm{P}(G). \tag{3.0.9}$$

In principle we cannot say that g is symmetric, but when this happens $g(H,K) = g(K,H)$, that is, $HK = KH$ and this condition is equivalent to have $HK \in \mathrm{L}(G)$. We can rephrase the same idea via the characteristic function

$$\chi : (H,K) \in \mathrm{S}(G) \times \mathrm{T}(G) \longmapsto \chi(H,K) = \begin{cases} 1, & \text{if } HK = KH, \\ 0, & \text{if } HK \neq KH, \end{cases} \tag{3.0.10}$$

noting that $\chi(H,K) = 1$ if and only if (3.0.9) holds. This allows us to consider the special case $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$ and to get an equivalent formula

$$|\mathrm{L}(G)|\ |\mathrm{L}(G)|\ sd(G) = \sum_{(H,K)\in \mathrm{L}(G)\times \mathrm{L}(G)} \chi(H,K) \tag{3.0.11}$$

for (3.0.2), also available from [67, Page 2511].

Now it is possible to introduce a new terminology as follows:

Definition 3.0.12. The function g in (3.0.9) is a *probabilistic law* if it is symmetric.

In different words, g is a probabilistic law, if it is defined on $\mathrm{S}(G)\times \mathrm{T}(G)$ as product HK of it first variable $H \in \mathrm{S}(G)$ and of its second variable $K \in \mathrm{T}(G)$ and if $g(H,K) \in \mathrm{L}(G)$ for any choice of the pair (H,K). Therefore we may introduce

$$gsd(G) = \frac{|\{(H,K) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid g(H,K) = g(K,H)\}|}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|} \tag{3.0.12}$$

which will be called *generalized subgroup commutativity degree* of G and it generalizes those in [38, 40, 41, 42, 52, 67, 68].

Chapter 4

Main Results of Probabilistic Nature

In the present chapter we state and proof the main results of this thesis. Here we investigate upper and lower bounds in terms of $gsd(G)$ and $sd(G)$, based on the notion of probability $gsd(G)$ in which two arbitrary sublattices $\mathrm{S}(G)$ and $\mathrm{T}(G)$ of $\mathrm{L}(G)$ are involved simultaneously, which we have just mentioned in previous chapter. Further we improve the bound of $sd(G)$ given in [52, Lemma 2.6].

We may use the characteristic function to express the quantity

$$gsd(G) = \left(\frac{1}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|}\right) \sum_{(H,K)\in \mathrm{S}(G)\times \mathrm{T}(G)} \chi(H,K) \tag{4.0.1}$$

turns out to be an equivalent formulation for (3.0.12).

On the other hand, we noted an absence of literature on inequalities involving both the subgroup commutativity degree (or its generalizations) and the commutativity degree. Specific computations are known for prescribed classes of finite groups, but general bounds are not available at the moment.

4.1 Properties of measures and natural bounds

The formula (3.0.12) allows us to treat the problem from the point of view of the measure theory on groups. A computational advantage may be found in the calculation of $gsd(G_1 \times G_2)$, where G_1 and G_2 are two given groups. We observe that Proposition

1.0.12 shows that a group $G = G_1 \times G_2$ with coprime factors G_1 and G_2 induces a lattice decomposition of the form $\mathrm{L}(G) \simeq \mathrm{L}(G_1) \times \mathrm{L}(G_2)$. In general, this is no longer true if we replace $\mathrm{L}(G)$ with an arbitrary sublattice $\mathrm{S}(G)$ of $\mathrm{L}(G)$. Since we are going to focus mostly on solvable or nilpotent groups, we will avoid such situations. More precisely, if we have a group $G = G_1 \times G_2$ factorized in two subgroups G_1 and G_2 of coprime order and $\mathrm{S}(G)$ is a given sublattice of $\mathrm{L}(G)$, we say that $\mathrm{S}(G)$ *inherits the decomposition* of $\mathrm{L}(G)$ if $\mathrm{S}(G) \simeq \mathrm{S}(G_1) \times \mathrm{S}(G_2)$. Note that both the sublattice $\mathrm{N}(G)$ of normal subgroups of G and the sublattice $\mathrm{SN}(G)$ of subnormal subgroups of G inherit the decomposition of $\mathrm{L}(G)$, as illustrated in [61, Theorems 9.1.5 and 9.2.2].

Corollary 4.1.1. *Assume that $G = G_1 \times G_2$ has coprime factors G_1 and G_2. If $\mathrm{S}(G)$ and $\mathrm{T}(G)$ inherit the decomposition of $\mathrm{L}(G)$, then*

$$gsd(G_1 \times G_2) = gsd(G_1) \cdot gsd(G_2).$$

Proof. Using Proposition 1.0.13 (ii) the condition $\gcd(|G_1|, |G_2|) = 1$ implies $\mathrm{L}(G_1 \times G_2)$ is directly decomposable, this means $\mathrm{L}(G_1 \times G_2) \simeq \mathrm{L}(G_1) \times \mathrm{L}(G_2)$. From the assumptions $\mathrm{S}(G_1 \times G_2)$ and $\mathrm{T}(G_1 \times G_2)$ are directly decomposable, so $\mathrm{S}(G_1 \times G_2) \simeq \mathrm{S}(G_1) \times \mathrm{S}(G_2)$ and $\mathrm{T}(G_1 \times G_2) \simeq \mathrm{T}(G_1) \times \mathrm{T}(G_2)$. Therefore

$$|\mathrm{S}(G_1 \times G_2)| \cdot |\mathrm{T}(G_1 \times G_2)| \cdot gsd(G_1 \times G_2)$$

$$= \sum_{\substack{(Y_1,Y_2) \in \mathrm{T}(G_1 \times G_2) \\ (X_1,X_2) \in \mathrm{S}(G_1 \times G_2)}} \chi((X_1, X_2), (Y_1, Y_2)) = \sum_{\substack{(X_2,Y_2) \in \mathrm{S}(G_2) \times \mathrm{T}(G_2) \\ (X_1,Y_1) \in \mathrm{S}(G_1) \times \mathrm{T}(G_1)}} \chi(X_1 \times Y_1, X_2 \times Y_2)$$

$$= \left(\sum_{(X_1,Y_1) \in \mathrm{S}(G_1) \times \mathrm{T}(G_1)} \chi(X_1, Y_1) \right) \cdot \left(\sum_{(X_2,Y_2) \in \mathrm{S}(G_2) \times \mathrm{T}(G_2)} \chi(X_2, Y_2) \right)$$
$$= (|\mathrm{S}(G_1)| \cdot |\mathrm{T}(G_1)| \cdot gsd(G_1)) \ \cdot \ (|\mathrm{S}(G_2)| \cdot |\mathrm{T}(G_2)| \cdot gsd(G_2))$$

□

If G_1, G_2,...,G_n are groups such that $gcd(|G_i|, |G_j|) = 1$ for all $i, j \in \{1, ..., n\}$, then Corollary 4.1.1 may be generalized to

$$gsd(G_1 \times G_2 \times ... \times G_2) = gsd(G_1) \cdot gsd(G_2) \cdots gsd(G_n). \tag{4.1.1}$$

The proof is omitted because it is by analogy with that of Corollary 4.1.1.

A classical situation, where we can apply (4.1.1), is when G is abelian. Recall that an abelian group G of order $n = p_1^{n_1} p_2^{n_2} ... p_m^{n_m}$, where $p_1, p_2, \cdots, p_m$ are distinct primes and $n_1, n_2, ..., n_m$ are positive integers, has a canonical decomposition of the form $G = G_1 \times G_2 \times \cdots \times G_m$, where $G_1, G_2, \cdots, G_m$ are called p_i- primary components. It is well known from Proposition 1.0.13 that a nilpotent group G has

$$\mathrm{L}(G) = \mathrm{L}(G_1) \times \mathrm{L}(G_2) \times ... \times \mathrm{L}(G_m)$$

and we have $|\mathrm{L}(G)| = |\mathrm{L}(G_1)|\ |\mathrm{L}(G_2)|\ ...\ |\mathrm{L}(G_m)|$.

The following consequence of Corollary 4.1.1 reduces the study of $gsd(G)$ for a nilpotent group G to that of p-groups.

Corollary 4.1.2. *If G is nilpotent and we have sublattices of $\mathrm{L}(G)$ inheriting the decomposition of $\mathrm{L}(G)$, then*

$$gsd(G) = \prod_{i=1}^{m} gsd(G_i),$$

where G_i is a p_i-primary component of G.

Introducing the symbol $\mathrm{S}^{\perp}(G)$ for the sublattice of $\mathrm{L}(G)$ containing all subgroups X of G which are permutable with all $S \in \mathrm{S}(G)$, the following result is true.

Corollary 4.1.3. *In a group G we have $gsd(G) = 1$ if and only if $\mathrm{S}(G) \subseteq \mathrm{T}^{\perp}(G)$ or $\mathrm{T}(G) \subseteq \mathrm{S}^{\perp}(G)$.*

Proof. Assume $gsd(G) = 1$. Then

$$\sum_{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G)} \chi(X, Y) = |\mathrm{S}(G)| \cdot |\mathrm{T}(G)| = |\mathrm{S}(G) \times \mathrm{T}(G)| \tag{4.1.2}$$

and this means that all elements of $\mathrm{S}(G)$ permute with all the elements of $\mathrm{T}(G)$ or viceversa. Hence $\mathrm{S}(G) \subseteq \mathrm{T}^{\perp}(G)$ or $\mathrm{T}(G) \subseteq \mathrm{S}^{\perp}(G)$.

Conversely $\mathrm{T}^{\perp}(G) = \{Y \in G |\ g(X, Y) = g(Y, X),\ \forall X \in \mathrm{S}(G)\}$, and $\mathrm{S}(G) \subseteq \mathrm{T}^{\perp}(G)$, or $\mathrm{S}^{\perp}(G) = \{Y \in G |\ g(X, Y) = g(Y, X),\ \forall X \in \mathrm{T}(G)\}$, and $\mathrm{T}(G) \subseteq \mathrm{S}^{\perp}(G)$ imply (4.1.2) and the result follows. □

Recall from [61, Theorem 9.1.5] that the lattice $\mathrm{N}(G)$ of normal subgroups of a group G is directly decomposable if and only if $G = H \times K$ where $H \neq 1 \neq K$ and any two nontrivial central factors X/Y of H and S/T of K are coprime (recall that X/Y is a central factor of H if $Y \subset X \subset H$ and the commutator subgroup $[X, H] \subset Y$).

In particular $\mathrm{N}(G)$ is directly decomposable under the assumption that $G = H \times K$ with H and K nontrivial subgroups of coprime orders. In fact [61, Theorem 9.1.5] and Proposition 1.0.12 show that $\mathrm{N}(G)$ inherits the decomposition of $\mathrm{L}(G)$. However the converse is false: there are groups with $\mathrm{N}(G)$ decomposable and $\mathrm{L}(G)$ indecomposable; for instance $G = S_3 \times \mathbb{Z}_3$.

Now we will see some concrete situations for $\mathrm{S}(G)$ and $\mathrm{T}(G)$ mentioned in the proof of Corollary 4.1.1. This is to show that if $G = G_1 \times G_2$ with $|G_1|$ and $|G_2|$ are coprime, then $\mathrm{L}(G)$ transfers quite naturally its decomposition to important families of sublattices.

Consider the nilpotent group $G = D_8 \times \mathbb{Z}_9$ and let $\mathrm{S}_1(D_8) = \mathrm{N}(D_8)$ and $\mathrm{T}_1(D_8)$ be the lattice of normal subgroups of G forming a chain of length almost four. Referring to Example 1.2.2, we may consider

$$\mathrm{S}_1(D_8) = \big\{\{1\}, Z(D_8), M_1, M_2, B, D_8\big\} = \mathrm{N}(D_8)\ ;\ \mathrm{T}_1(D_8) = \big\{\{1\}, Z(D_8), M_1, D_8\big\},$$

where $\mathrm{T}_1(D_8)$ is a prescribed chain of normal subgroups of $\mathrm{L}(D_8)$. Then we pass to consider the following sublattices

$$\mathrm{S}_2(\mathbb{Z}_9) = \big\{\{0\}, \langle 3\rangle, \mathbb{Z}_9\big\} = \mathrm{N}(\mathbb{Z}_9)\ ;\ \mathrm{T}_2(\mathbb{Z}_9) = \big\{\{0\}, \langle 3\rangle, \mathbb{Z}_9\big\}$$

where $\mathrm{T}_2(\mathbb{Z}_9)$ is a chain of normal subgroups of $\mathrm{L}(\mathbb{Z}_9)$. For $i = 1, 2$ we may apply [61, Theorem 9.1.5], getting

$$\mathrm{S}_i(D_8) \times \mathrm{S}_i(\mathbb{Z}_9) \simeq \mathrm{S}_i(D_8 \times \mathbb{Z}_9).$$

On the other hand a direct computation shows the same for the remaining sublattices when $i = 1, 2$, that is,

$$\mathrm{T}_i(D_8) \times \mathrm{T}_i(\mathbb{Z}_9) \simeq \mathrm{T}_i(D_8 \times \mathbb{Z}_9).$$

By direct calculation we obtain $gsd(D_8 \times \mathbb{Z}_9) = 1 = gsd(D_8) \cdot gsd(\mathbb{Z}_9)$.

Recall also from [61, Theorem 9.2.2] that the sublattice $\mathrm{SN}(G)$ of a group G with a composition series is directly decomposable if and only if $G = H \times K$ where $H \neq 1 \neq K$ and no abelian composition factor of H is isomorphic to a composition factor of K.

Now if we look again at $G = D_8 \times \mathbb{Z}_9$ but with $\mathrm{S}_3(G) = \mathrm{SN}(G) = \mathrm{T}_3(G)$, then $\mathrm{S}_3(D_8) = \mathrm{T}_3(D_8) = \mathrm{SN}(D_8) = \mathrm{L}(D_8)$, $\mathrm{S}_3(\mathbb{Z}_9) = \mathrm{T}_3(D_8) = \mathrm{SN}(\mathbb{Z}_9) = \mathrm{L}(\mathbb{Z}_9)$ and again

$$\mathrm{S}_3(D_8 \times \mathbb{Z}_9) = \mathrm{T}_3(D_8 \times \mathbb{Z}_9) \simeq \mathrm{S}_3(D_8) \times \mathrm{S}_3(\mathbb{Z}_9) = \mathrm{T}_3(D_8) \times \mathrm{T}_3(\mathbb{Z}_9).$$

In particular the formula $gsd(D_8 \times \mathbb{Z}_9) = 1 = gsd(D_8) \cdot gsd(\mathbb{Z}_9)$ holds.

We show that the generalized subgroup commutativity degree of G is naturally upper bounded by the subgroup commutativity degree of G.

Lemma 4.1.4. *In a group G we have*

$$\frac{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|}{|\mathrm{L}(G)|^2} \cdot gsd(G) \leq \ sd(G)$$

and the bound is achieved, if $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$. *Viceversa, if the previous bound is exact, then*

$$\sum_{(X,Y)\in \mathrm{S}(G)\times \mathrm{T}(G)} \chi(X,Y) \geq \sum_{(X,Y)\in \mathrm{L}(G)\times \mathrm{L}(G)} \chi(X,Y).$$

Proof. Since $\mathrm{S}(G) \times \mathrm{T}(G) \subseteq \mathrm{L}(G)^2$, we have

$$\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid g(X,Y) = g(Y,X)\} \subseteq \ \{(X,Y) \in \mathrm{L}(G)^2 \mid g(X,Y) = g(Y,X)\}.$$

Then

$$|\mathrm{S}(G)|\ |\mathrm{T}(G)|\ gsd(G) = |\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid g(X,Y) = g(Y,X)\}|$$

$$\leq |\{(X,Y) \in \mathrm{L}(G)^2 \mid g(X,Y) = g(Y,X)\} = |\mathrm{L}(G)|^2\ sd(G)$$

therefore the bound follows.

Now suppose $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$. We have seen that (3.0.12) becomes (3.0.2) and the bound becomes trivially true. On the other hand, if the bound is exact, then

$$|\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid g(X,Y) = g(Y,X)\}|$$

$$= |\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid g(X,Y) = g(Y,X)\}|,$$

where the condition $\mathrm{S}(G) \times \mathrm{T}(G) \subseteq \mathrm{L}(G)^2$ shows that

$$|\{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G) \mid g(X,Y) = g(Y,X)\}|$$

$$\leq |\{(X,Y) \in \mathrm{L}(G) \times \mathrm{L}(G) \mid g(X,Y) = g(Y,X)\}|$$

is always satisfied. The result follows. □

Another basic property one could investigate is how to relate $gsd(G)$ to quotients and subgroups of G. We will investigate semidirect products $G = N \rtimes H$ with normal factor N and $H \simeq G/N$. A first result is the following.

Lemma 4.1.5. *If H is a subgroup of G such that $H \in \mathrm{S}(G) \cap \mathrm{T}(G)$, then*

$$gsd(G) \geq \frac{|\mathrm{S}(H)| \cdot |\mathrm{T}(H)|}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|} \cdot gsd(H).$$

Proof. $H \in \mathrm{S}(G) \cap \mathrm{T}(G)$ implies that $\mathrm{S}(H) \in \mathrm{S}(G) \cap \mathrm{T}(G)$ and $\mathrm{S}(H) \in \mathrm{T}(G) \cap \mathrm{T}(G)$ and hence $\mathrm{S}(H) \times \mathrm{T}(H) \subseteq \mathrm{S}(G) \times \mathrm{T}(G)$. Since

$$\mathrm{S}(G) \times \mathrm{T}(G) = \Big(\mathrm{S}(H) \times \mathrm{T}(H)\Big) \cup \Big((\mathrm{S}(G) - \mathrm{S}(H)) \times (\mathrm{T}(G) - \mathrm{T}(H))\Big),$$

we have that

$$|\mathrm{S}(G)| \cdot |\mathrm{T}(G)| \cdot gsd(G) = \sum_{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G)} \chi(X, Y)$$

$$= \sum_{(X,Y) \in \mathrm{S}(H) \times \mathrm{T}(H)} \chi(X, Y) + \sum_{(X,Y) \in (\mathrm{S}(G) - \mathrm{S}(H)) \times (\mathrm{T}(G) - \mathrm{T}(H))} \chi(X, Y)$$

$$\geq \sum_{(X,Y) \in \mathrm{S}(H) \times \mathrm{T}(H)} \chi(X, Y) = |\mathrm{S}(H)| \cdot |\mathrm{T}(H)| \cdot gsd(H).$$

Hence the result follows. □

In particular we have the following result for semidirect products.

Lemma 4.1.6. *If $G = N \rtimes H$ and $G/N \in \mathrm{S}(G/N) \cap \mathrm{T}(G/N)$, then*

$$gsd(G) \geq \frac{|\mathrm{S}(G/N)| \cdot |\mathrm{T}(G/N)|}{|\mathrm{S}(G)| \cdot |\mathrm{T}(G)|} \cdot gsd(G/N).$$

Proof. Note that $H \simeq G/N$ and apply Lemma 4.1.5. □

Most of the results which we have seen in this section will be applied to the proof of Theorem 4.2.7. In particular, the above lemma will play an important role.

4.2 Some new lower and upper bounds

Given a normal subgroup N of an arbitrary group G, we may always consider the following subsets:

$$A_1 = \{X \in \mathrm{S}(G) \mid N \subseteq X\}, A_2 = \{X \in \mathrm{S}(G) \mid X \subset N\} \tag{4.2.1}$$

$$B_1 = \{X \in \mathrm{T}(G) \mid N \subseteq X\}, \text{and} \quad B_2 = \{X \in \mathrm{T}(G) \mid X \subset N\} \tag{4.2.2}$$

and so we may introduce the quantities

$$gsd_1(G) = \frac{1}{|A_1 \cup A_2|^2} \sum_{(X,Y)\in(A_1\cup A_2)^2} \chi(X,Y) \tag{4.2.3}$$

depending on $\mathrm{S}(G)$ and N;

$$gsd_2(G) = \frac{1}{|B_1 \cup B_2|^2} \sum_{(X,Y)\in(B_1\cup B_2)^2} \chi(X,Y). \tag{4.2.4}$$

depending on $\mathrm{T}(G)$ and N;

$$gsd_3(G) = \frac{1}{|A_1| \cdot |B_1|} \sum_{(X,Y)\in A_1\times B_1} \chi(X,Y). \tag{4.2.5}$$

depending on $\mathrm{S}(G)$, $\mathrm{T}(G)$ and N. Then one can consider the quantity

$$\beta(\mathrm{S}(G),\mathrm{T}(G),N) = \frac{1}{|\mathrm{S}(G)|\ |\mathrm{T}(G)|} \left(|A_1|\ |B_1|\ gsd_3(G) + |B_1 - A_1| + |A_1 - B_1|\right) \tag{4.2.6}$$

depending on $\mathrm{S}(G)$, $\mathrm{T}(G)$ and N. Our first main result deals with new bounds for the generalized subgroup commutativity degree in terms of (3.0.2), (4.2.3), (4.2.4)and (4.2.5).

Theorem 4.2.1. *Assume* $\mathrm{S}(G) \cap \mathrm{T}(G) \subseteq \mathrm{N}(G)$ *in a group* G.

(i). If $\mathrm{S}(G) = \mathrm{T}(G)$, *then* $gsd_1(G) = gsd_2(G)$ *and*

$$|\mathrm{L}(G)|^2\ sd(G) \geq |\mathrm{S}(G)|^2\ gsd(G) \geq |A_1 \cup A_2|^2\ gsd_1(G).$$

(ii). If $\mathrm{S}(G) \neq \mathrm{T}(G)$ *and* $A_1 \times B_1 \subseteq (\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))) \times (\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)))$, *then*

$$|\mathrm{L}(G)|^2\ sd(G) \geq |\mathrm{S}(G)|\ |\mathrm{T}(G)|\ gsd(G) \geq |A_1|\ |B_1|\ gsd_3(G).$$

Proof. Case (i). We note that $(A_1 \cup A_2) \times (B_1 \cup B_2) \subseteq \mathrm{S}(G) \times \mathrm{T}(G)$, but $\mathrm{S}(G) = \mathrm{T}(G)$ implies $A_1 = B_1$, $A_2 = B_2$ so

$$\sum_{(X,Y)\in \mathrm{S}(G)\times \mathrm{S}(G)} \chi(X,Y) \geq \sum_{(X,Y)\in(A_1\cup A_2)\times(B_1\cup B_2)} \chi(X,Y) \tag{4.2.7}$$

$$= \sum_{(X,Y)\in(A_1\cup A_2)\times(A_1\cup A_2)} \chi(X,Y) = gsd_1(G)\ \cdot\ |A_1 \cup A_2|^2,$$

on the other hand

$$gsd_1(G) \ \cdot \ |A_1 \cup A_2|^2 = \sum_{(X,Y)\in(B_1\cup B_2)\times(B_1\cup B_2)} \chi(X,Y) \tag{4.2.8}$$

and $gsd_1(G) = gsd_2(G)$ follows. Note that (4.2.7) gives

$$|\mathrm{S}(G)| \cdot |\mathrm{S}(G)| \cdot gsd(G) \geq gsd_1(G) \ \cdot \ |A_1 \cup A_2|^2 \tag{4.2.9}$$

and now the result follows from Lemma 4.1.4.

Case (ii). We begin to write

$$\mathrm{S}(G) = \big(\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))\big) \cup (\mathrm{S}(G) \cap \mathrm{T}(G))$$

and, since the same is true also for $\mathrm{T}(G)$, we get

$$\Big(\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \cup (\mathrm{S}(G) \cap \mathrm{T}(G))\Big) \times \Big((\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \cup (\mathrm{S}(G) \cap \mathrm{T}(G))\Big) \tag{4.2.10}$$

$$= \Big((\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))) \times (\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)))\Big)$$

$$\cup\Big((\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))) \times (\mathrm{S}(G) \cap \mathrm{T}(G))\Big) \cup \Big((\mathrm{S}(G) \cap \mathrm{T}(G)) \times (\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)))\Big)$$

$$\cup\Big((\mathrm{S}(G) \cap \mathrm{T}(G)) \times (\mathrm{S}(G) \cap \mathrm{T}(G))\Big) = \mathrm{S}(G) \times \mathrm{T}(G).$$

Therefore

$$\sum_{(X,Y)\in \mathrm{S}(G)\times \mathrm{T}(G)} \chi(X,Y) = \sum_{\substack{X\in \mathrm{S}(G)-(\mathrm{S}(G)\cap \mathrm{T}(G)) \\ Y\in T(G)-(\mathrm{S}(G)\cap \mathrm{T}(G))}} \chi(X,Y) + \tag{4.2.11}$$

$$\sum_{\substack{X\in \mathrm{S}(G)-(\mathrm{S}(G)\cap \mathrm{T}(G)) \\ Y\in \mathrm{S}(G)\cap \mathrm{T}(G)}} \chi(X,Y) + \sum_{\substack{Y\in \mathrm{T}(G)-(\mathrm{S}(G)\cap \mathrm{T}(G)) \\ X\in \mathrm{S}(G)\cap \mathrm{T}(G)}} \chi(X,Y) + \sum_{\substack{X\in \mathrm{S}(G)\cap \mathrm{T}(G) \\ Y\in \mathrm{S}(G)\cap \mathrm{T}(G)}} \chi(X,Y)$$

Now since $\mathrm{S}(G) \cap \mathrm{T}(G) \subseteq \mathrm{N}(G)$ we may compute the quantities in (4.2.11), getting

$$= \Big(\sum_{\substack{X\in \mathrm{S}(G)-(\mathrm{S}(G)\cap \mathrm{T}(G)) \\ Y\in T(G)-(\mathrm{S}(G)\cap \mathrm{T}(G))}} \chi(X,Y)\Big) + |\mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))| \ |\mathrm{S}(G) \cap \mathrm{T}(G)| +$$

$$+|\mathrm{T}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))| \ |\mathrm{S}(G) \cap \mathrm{T}(G)| + |\mathrm{S}(G) \cap \mathrm{T}(G)|^2$$

$$= \Big(\sum_{\substack{X \in \mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \\ Y \in T(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))}} \chi(X,Y) \Big) + |\mathrm{S}(G) \cap \mathrm{T}(G)| \cdot \Big(|\mathrm{S}(G)| + |\mathrm{T}(G)| - |\mathrm{S}(G) \cap \mathrm{T}(G)| \Big) \tag{4.2.12}$$

$$\geq \sum_{\substack{X \in \mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \\ Y \in T(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))}} \chi(X,Y) \tag{4.2.13}$$

Therefore by our assumption,

$$\sum_{(X,Y) \in \mathrm{S}(G) \times \mathrm{T}(G)} \chi(X,Y) \geq \sum_{\substack{X \in \mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \\ Y \in T(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))}} \chi(X,Y) \geq \sum_{\substack{X \in A_1 \\ Y \in B_1}} \chi(X,Y)$$

Hence the result follows. □

We can not remove the additional condition of Theorem 4.2.1(ii) because of the following example.

Example 4.2.2. From Example 1.2.2 if we consider $\mathrm{S}(G) = \{\{1\}, H, K, M_1, Z(D_8), B, D_8\}$, $\mathrm{T}(G) = \{\{1\}, U, V, M_2, Z(D_8), B, D_8\}$, $A_1 = \{Z(D_8), M_1, B, D_8\}$, $B_1 = \{Z(D_8), M_2, B, D_8\}$, and then $\mathrm{S}(G) \cap \mathrm{T}(G) = \{Z(D_8), B, D_8\}$. Then one, and only one, of the following conditions can not happen;

$$A_1 \subseteq S(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \text{ and } B_1 \subseteq T(G) - (\mathrm{S}(G) \cap \mathrm{T}(G));$$

$$A_1 \subseteq S(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \text{ and } B_1 \subseteq \mathrm{S}(G) \cap \mathrm{T}(G);$$

$$A_1 \subseteq \mathrm{S}(G) \cap \mathrm{T}(G) \text{ and } B_1 \subseteq T(G) - (\mathrm{S}(G) \cap \mathrm{T}(G);$$

$$A_1 \subseteq \mathrm{S}(G) \cap \mathrm{T}(G) \text{ and } B_1 \subseteq \mathrm{S}(G) \cap \mathrm{T}(G).$$

There are more information in the proof of Theorem 4.2.1. In fact we introduce

$$\gamma(G) = \frac{|\mathrm{S}(G) \cap \mathrm{T}(G)| \cdot (|\mathrm{S}(G)| + |\mathrm{T}(G)| - |\mathrm{S}(G) \cap \mathrm{T}(G)|)}{|\mathrm{S}(G)|\ |\mathrm{T}(G)|}$$

and (4.2.12) shows:

Corollary 4.2.3. *Assume* $\mathrm{S}(G) \cap \mathrm{T}(G) \subseteq \mathrm{N}(G)$ *in a group* G. *If* $\mathrm{S}(G) \neq \mathrm{T}(G)$, *then*

$$gsd(G) - \gamma(G) = \frac{1}{|\mathrm{S}(G)|\ |\mathrm{T}(G)|} \sum_{\substack{X \in \mathrm{S}(G) - (\mathrm{S}(G) \cap \mathrm{T}(G)) \\ Y \in T(G) - (\mathrm{S}(G) \cap \mathrm{T}(G))}} \chi(X,Y).$$

Now we work on our second main result. In order to do this, we consider a normal subgroup N of a group G and introduce

$$\alpha(\mathrm{S}(G/N), \mathrm{S}(N)) = \frac{1}{|\mathrm{S}(G)|^2} \cdot \Big((|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1)^2 + (gsd(N) - 1) \cdot |\mathrm{S}(N)|^2 + (gsd(G/N) - 1) \cdot |\mathrm{S}(G/N)|^2 \Big), \quad (4.2.14)$$

which is only depending on $\mathrm{S}(G/N)$ and $\mathrm{S}(N)$.

Lemma 4.2.4. *If a group G has a normal subgroup $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$, then $gsd(G) \geq \alpha(\mathrm{S}(G/N), \mathrm{S}(N))$.*

Proof. Since $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$, [67, Proposition 2.4] gives a method to calculate $gsd_1(G)$:

$$|A_1 \cup A_2|^2 \, gsd_1(G) = \sum_{(X,Y) \in (A_1 \cup A_2)^2} \chi(X, Y) = \sum_{X,Y \in A_1 \cup A_2} \chi(X, Y) \quad (4.2.15)$$

$$= \sum_{X,Y \in A_1} \chi(X, Y) + \sum_{X,Y \in A_2} \chi(X, Y) + 2 \sum_{X \in A_1} \sum_{Y \in A_2} \chi(X, Y),$$

and so we evaluate the three terms separately:

$$\sum_{X,Y \in A_1} \chi(X, Y) = \sum_{(X,Y) \in A_1 \times A_1} \chi(X, Y) = gsd(G/N) \cdot |\mathrm{S}(G/N)|^2; \quad (4.2.16)$$

$$\sum_{X,Y \in A_2} \chi(X, Y) = \sum_{X,Y \in A_2 \cup \{N\}} \chi(X, Y) - 2 \sum_{X \in A_2 \cup \{N\}} \chi(X, N) + 1 \quad (4.2.17)$$

$$= gsd(N) \cdot |\mathrm{S}(N)|^2 - 2|\mathrm{S}(N)| + 1;$$

$$2 \sum_{X \in A_1} \sum_{Y \in A_2} \chi(X, Y) = 2 \, |A_1| \cdot |A_2| = 2 \, |\mathrm{S}(G/N)| \cdot (|\mathrm{S}(N)| - 1). \quad (4.2.18)$$

Therefore we may apply Theorem 4.2.1 (i), and we get the result. □

The bound of Lemma 4.2.4 is homogeneous, because it involves only the probability $gsd(G)$ in terms of subgroups and quotients, but the assumption $\mathrm{S}(G) = \mathrm{T}(G)$ is strong.

Now we focus on the case of a semidirect product.

Lemma 4.2.5. *Assume $G = N \rtimes H$ and $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$. If $\mathrm{S}(G) \neq \mathrm{T}(G)$, then $gsd(G) \geq \beta(\mathrm{S}(G), \mathrm{T}(G), N)$.*

Proof. Since $\mathrm{S}(G) \neq \mathrm{T}(G)$, the argument of Lemma 4.2.4 gives problems due to the application of Theorem 4.2.1 (i) in its final part. On the other hand,

$$gsd(G/N)\ |\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)| = \sum_{X,Y\in A_1\cup B_1} \chi(X,Y) \tag{4.2.19}$$

$$= \sum_{\substack{X\in A_1\\ Y\in B_1}} \chi(X,Y) + \sum_{\substack{X\in B_1-A_1\\ Y\in B_1}} \chi(X,Y) + \sum_{\substack{X\in A_1\cup B_1\\ Y\in A_1-B_1}} \chi(X,Y)$$

and the fact that $N \in A_1 \cup B_1$ and $N \in B_1$ imply

$$\sum_{\substack{X\in B_1-A_1\\ Y\in B_1}} \chi(X,Y) + \sum_{\substack{X\in A_1\cup B_1\\ Y\in A_1-B_1}} \chi(X,Y) \geq |B_1 - A_1| + |A_1 - B_1|$$

and we may conclude

$$gsd(G/N)\ |\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)| \geq |A_1|\ |B_1|\ gsd_3(G) + |B_1 - A_1| + |A_1 - B_1|.$$

Now the result follows from Lemma 4.1.6. □

Note that the proof of Lemma 4.2.5 shows a lower bound of independent interest in which the assumption of $G = N \rtimes H$ is not necessary.

Corollary 4.2.6. *If N is a normal subgroup of a group G and $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$ with $\mathrm{S}(G) \neq \mathrm{T}(G)$, then*

$$|\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)|\ gsd(G/N) \geq |A_1|\ |B_1|\ gsd_3(G) + |B_1 - A_1| + |A_1 - B_1|.$$

In particular, $|\mathrm{S}(G/N)|\ |\mathrm{T}(G/N)|\ gsd(G/N) \geq\ gsd_3(G)\ |A_1|\ |B_1|$.

Now we collect the result we obtained in Lemmas 4.2.4 and 4.2.5 and our second main result deals with semidirect products $N \rtimes H$ with normal factor N.

Theorem 4.2.7. *If $G = N \rtimes H$ and $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$, then*

$$gsd(G) \geq \max\{\alpha(\mathrm{S}(G/N), \mathrm{S}(N)), \beta(\mathrm{S}(G), \mathrm{T}(G), N)\}.$$

Proof. It follows from Lemmas 4.2.4 and 4.2.5. □

4.3 Applications

There are some interesting specializations of Theorem 4.2.7.

Corollary 4.3.1. *If a group $G = N \rtimes H$ has a normal subgroup $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$, and in addition $gsd(N) = gsd(G/N) = 1$, then*

$$gsd(G) \geq \max \left\{ \left(\frac{|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1}{|\mathrm{S}(G)|} \right)^2, \beta(\mathrm{S}(G), \mathrm{T}(G), N) \right\}.$$

A classical situation, in which Corollary 4.3.1 is applicable, is when $G = N \rtimes H$ is metabelian, that is, both N and H are abelian subgroups. Here if $N = G'$ and $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$, then we get exactly [67, Corollary 2.5].

Corollary 4.3.2. *If a group $G = N \rtimes H$ has a normal subgroup $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$, and if N is of prime index in G, then*

$$gsd(G) \geq \max \left\{ \frac{gsd(N) \cdot |\mathrm{S}(N)|^2 + 2|\mathrm{S}(N)| + 1}{|\mathrm{S}(G)|^2}, \beta(\mathrm{S}(G), \mathrm{T}(G), N) \right\}.$$

Proof. Since N is of prime index in G, G/N is an abelian group, hence $gsd(G/N) = 1$ independently on the choice of the family of subgroups in $\mathrm{L}(G/N)$. If $\mathrm{S}(G) = \mathrm{T}(G)$, then we apply Lemma 4.2.4 and get

$$gsd(G) \geq \alpha(\mathrm{S}(G/N), \mathrm{S}(N)) = \frac{gsd(N) \cdot |\mathrm{S}(N)|^2 + 2|\mathrm{S}(N)| + 1}{|\mathrm{S}(G)|^2}.$$

If $\mathrm{S}(G) \neq \mathrm{T}(G)$, then we apply Lemma 4.2.5 and get $gsd(G) \geq \beta(\mathrm{S}(G), \mathrm{T}(G), N)$. The result follows. □

Another application of Corollary 4.3.2 is given by [67, Corollary 2.6].

Corollary 4.3.3. *If a group $G = N \rtimes H$ has a normal subgroup $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$ with $\mathrm{S}(G) \neq \mathrm{T}(G)$, and if N is of prime index in G, then*

$$4 - (|B_1 - A_1| + |A_1 - B_1|) \geq gsd_3(G) \cdot |A_1| \cdot |B_1|$$

In particular, $4 \geq gsd_3(G) \cdot |A_1| \cdot |B_1|$.

Proof. Since N is of prime index in G, $gsd(G/N) = 1$. Now we look at the bound in Corollary 4.2.6 and note that the possible choices for the cardinalities of $\mathrm{S}(G/N)$ and $\mathrm{T}(G/N)$ are $|\mathrm{S}(G/N)|, |\mathrm{T}(G/N)| \in \{1, 2\}$. The widest range we can get is realised by the choice $|\mathrm{S}(G/N)| = |\mathrm{T}(G/N)| = 2$. This is described by the bound of the thesis. □

We now offer an example in which the conditions of Theorem 4.2.7 are satisfied.

Example 4.3.4. The symmetric group $G = S_3$ has a unique minimal normal subgroup $N = A_3$ and this is atomic, that is, it covers the identity element in $\mathrm{L}(S_3)$ (see [15, 61]). Here any choice of $\mathrm{S}(G)$ and $\mathrm{T}(G)$ satisfied the assumptions of Theorem 4.2.7 with $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$. On the other hand, if we consider $G = S_4$, then there is again a normal subgroup $N = A_4$, but it is well known that A_4 is not atomic in $\mathrm{L}(S_4)$, so for $G = S_4$ an appropriate choice of N, depending on a corresponding choice for $\mathrm{S}(G)$ and $\mathrm{T}(G)$ must be taken into account (because in general the condition $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$ might be false). More generally for an odd prime p and $r \geq 1$,

$$G = \langle x, a_1, a_2, \ldots, a_r \mid x^2 = a_1^p = a_2^p = \ldots = a_r^p = 1, x^{-1}a_ix = a_i^{-1},$$

$$[a_i, a_j] = 1, \ \forall \ i, j \in \{1, 2, ..., r\}\rangle$$

is of order $2p^r$ may be written in the form $G = N \rtimes H$, where N is an elementary abelian p-subgroup of rank r and $H = \mathbb{Z}_2 = \langle x \rangle$ is of order two acting on N by inversion. Here N turns out to be atomic in $\mathrm{L}(G)$. Here G satisfies the assumptions of Theorem 4.2.7, when $\mathrm{S}(G)$ and $\mathrm{T}(G)$ are chosen in such a way that $N \in \mathrm{S}(G) \cap \mathrm{T}(G)$.

The presence of atomic normal subgroups implies the following result.

Corollary 4.3.5. *If N is an atomic normal subgroup of $G = N \rtimes H$, $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$, then $gsd(G)$ is lower bounded by a function depending only on G/N:*

$$gsd(G) \geq \frac{1}{|\mathrm{S}(G)|^2} \cdot \Big(gsd_1(G/N).|\mathrm{S}(G/N)|^2 + 2|\mathrm{S}(G/N)| + 1\Big).$$

Proof. Assume $\mathrm{S}(G) = \mathrm{T}(G)$ and $N \in \mathrm{S}(G)$. We can calculate $gsd_1(G)$ by the argument in Lemma 4.2.4. Since N is atomic, $|\mathrm{S}(N)| = 2$ and $gsd_1(N) = 1$, and so we may apply Theorem 4.2.1 (i) and Lemma 4.2.4, getting the result. □

We shall compute $\alpha(\mathrm{S}(G/N), \mathrm{S}(N))$ and $\beta(\mathrm{S}(G), \mathrm{T}(G), N))$ explicitly if G has an abelian normal subgroup $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ with $1 \leq \alpha_1 \leq \alpha_2$ and prime p. This will involve a polynomial function which has been studied in [69].

Corollary 4.3.6. *Suppose a group $G = N \rtimes H$ has an abelian subgroup $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ with $1 \leq \alpha_1 \leq \alpha_2$ and p prime. If $N \in \mathrm{S}(G)$ and $\mathrm{S}(G) = \mathrm{T}(G)$ with $H \simeq G/N$ of prime order, then*

$$gsd(G) \geq \frac{1}{(p-1)^4 \cdot |\mathrm{S}(G)|^2} \cdot [(\alpha_2 - \alpha_1 + 1)p^{\alpha_1+2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1+1} - (\alpha_2 + \alpha_1 + 3)p$$

$$+(\alpha_2 + \alpha_1 + 2)]^2.$$

Proof. Assume $\mathrm{S}(G) = \mathrm{T}(G)$ and $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \in \mathrm{S}(G)$. From Lemma 4.2.4,

$$gsd(G) \geq \frac{1}{|\mathrm{S}(G)|^2} \cdot \Big((|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1)^2 + (gsd(N) - 1) \cdot |\mathrm{S}(N)|^2$$

$$+ (gsd(G/N) - 1) \cdot |\mathrm{S}(G/N)|^2 \Big).$$

Since $gsd(G/N) = gsd(N) = 1$ and $|\mathrm{S}(G/N)| = 2$, we obtain

$$gsd(G) \geq \Big(\frac{|\mathrm{S}(N)| + 1}{|\mathrm{S}(G)|} \Big)^2.$$

Now [69, Theorem 3.3] implies that:

$$|\mathrm{S}(N)| = \frac{1}{(p-1)^2} \cdot [(\alpha_2 - \alpha_1 + 1)p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1 + 1} - (\alpha_2 + \alpha_1 + 3)p$$

$$+ (\alpha_2 + \alpha_1 + 1)]$$

And then,

$$\Big(\frac{|\mathrm{S}(N)| + 1}{|\mathrm{S}(G)|} \Big)^2 = \frac{1}{(p-1)^4 \cdot |\mathrm{S}(G)|^2} \cdot [(\alpha_2 - \alpha_1 + 1)p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1 + 1}$$

$$- (\alpha_2 + \alpha_1 + 3)p + (\alpha_2 + \alpha_1 + 1) + 1]^2$$

Hence the result follows. □

Corollary 4.3.6 improves [52, Lemma 2.6], where specific choices of the sublattices are involved. Another generalization is reported separately for the subgroup commutativity degree.

Corollary 4.3.7. *In the same assumptions of Corollary 4.3.6,*

$$sd(G) \geq \frac{1}{(p-1)^4 \cdot |\mathrm{L}(G)|^2} \cdot [(\alpha_2 - \alpha_1 + 1)p^{\alpha_1 + 2} - (\alpha_2 - \alpha_1 - 1)p^{\alpha_1 + 1} - (\alpha_2 + \alpha_1 + 3)p$$

$$+ (\alpha_2 + \alpha_1 + 2)]^2.$$

Proof. See Lemma 4.1.4 and Corollary 4.3.6. □

Note that Corollary 4.3.7 improves the bound in [52, Theorem 2.8]. A classical source of examples, where Corollaries 4.3.6 and 4.3.7 may be verified, is given by the families of dihedral groups.

The real issue, which we leave open, is related to an approach in terms of characters and representation theory for the notions above. Therefore:

Question 4.3.8. Is it possible to formulate the notion of probabilistic law (and consequently (4.0.1)) in terms of generalized characters of G ?

Because if this is possible, then one can find important relations with the theory of the so-called T*-systems* in [27] and with corresponding problems on probabilities on words in [62].

Another problem we leave open, is motivated by [57, Corollary 3.4]:

Question 4.3.9. Is it possible to find lower and upper bounds, involving both the generalized subgroup commutativity degree and the commutativity degree ?

Chapter 5

An Approach via Graph Theory

In the present chapter we study the structure of the non-permutability graph of subgroups of non-hamiltonian group G and we determine the upper bounds of the size of edges of a subgraph induced by sublattice of subgroups of G. Further, we investigate the planarity of the non–permutability graph of subgroups of dihedral groups, which are well known in group theory.

The *non–commuting graph* of a group G is a graph with vertex set $V(G) = G - Z(G)$, where $Z(G)$ denotes the center of G, and two distinct non–central elements x and y are joined by an edge if and only if they do not commute, that is, $E(G) = \{(x, y) \in V(G) \times V(G) \mid x \sim y \ \Leftrightarrow \ xy \neq yx\}$. It turns out that the graph $(V(G), E(G))$ is undirected, simple and non–weighted. We will take inspiration from the theory of the non–commuting graph in [1, 2, 4, 5] so we will deal with graphs which are indeed undirected, simple and non–weighted. Note that the importance of the non–commuting graph became significant in various areas of group theory since it was proved that it allows to recognise simple groups (see [48, 56, 63]). There is also a growing interest for this graph in spectral graph theory and measure theory (see [8, 50]) in recent years.

The non–commuting graph can be related to the notion of *commutativity degree* $\mathrm{d}(G)$ of G. One important aspect, which connects the theory of the commutativity degree to that of the non-commuting graph, is due to the following counting formula

$$2\ |E(G)| = |G|^2\ (1 - \mathrm{d}(G)), \tag{5.0.1}$$

which is due to Erfanian and others [22].

Now we choose an $X \in \mathrm{L}(G)$ and consider the set

$$\mathfrak{C}_{\mathrm{L}(G)}(X) = \{Y \in \mathrm{L}(G) \mid XY = YX\} \tag{5.0.2}$$

of all subgroups of $\mathrm{L}(G)$ commuting with X and the intersection

$$\bigcap_{X \in \mathrm{L}(G)} \mathfrak{C}_{\mathrm{L}(G)}(X) = \{Y \in \mathrm{L}(G) \mid YX = XY, \quad \forall X \in \mathrm{L}(G)\} \tag{5.0.3}$$

turns out to be equal to the set of all permutable subgroups of G.

Of course, (5.0.3) is not a sublattice of $\mathrm{L}(G)$ because it is not stable with respect to joins and meets; for instance, a direct product of a non-abelian p-group of order p^3 and exponent p^2, and a cyclic group of order p^2 for any odd prime p shows that the intersection of two elements in (5.0.3) does not belong to (5.0.3) in general. On the other hand, a classical result of Ore (see [61, Theorem 5.1.1]) shows that permutable subgroups are always subnormal, hence (5.0.3) is always contained in the sublattice $\mathrm{SN}(G)$ of all subnormal subgroups of G. In view of this fact, we denote by $\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$ the smallest sublattice of $\mathrm{L}(G)$ containing (5.0.3), that is,

$$\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G)) = \bigcap_{\mathrm{S}(G) \supseteq \bigcap_{X \in \mathrm{L}(G)} \mathfrak{C}_{\mathrm{L}(G)}(X)} \mathrm{S}(G). \tag{5.0.4}$$

Thanks to (5.0.3), we may introduce an undirected non–weighted simple graph $\Gamma_{\mathrm{L}(G)}$ having vertices and edges

$$V(\mathrm{L}(G)) = \mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big); \tag{5.0.5}$$

$$E(\mathrm{L}(G)) = \{(X,Y) \in V(\mathrm{L}(G)) \times V(\mathrm{L}(G)) \mid X \sim Y \ \Leftrightarrow \ XY \neq YX\}, \tag{5.0.6}$$

respectively. The graph

$$\Gamma_{\mathrm{L}(G)} = (V(\mathrm{L}(G)), E(\mathrm{L}(G))) \tag{5.0.7}$$

will be called *non-permutability graph of subgroups* of G.

Of course, if we consider a smaller set of vertices $A(G) \subseteq V(\mathrm{L}(G))$ and we say that two elements of $A(G)$ are joined if they do not permute, this defines a subgraph of $\Gamma_{\mathrm{L}(G)}$. The real point is to see when it is possible to choose $A(G) = \mathrm{S}(G) - \mathfrak{C}_{\mathrm{S}(G)}(\mathrm{S}(G))$ for some appropriate sublattice $\mathrm{S}(G)$ of $\mathrm{L}(G)$. When we are in this situation, we replace $A(G)$ by $V(\mathrm{S}(G))$ and denote the corresponding subgraph by $\Gamma_{\mathrm{S}(G)}$. Details will be discussed in

Lemmas 5.0.7 and 5.0.8 below.

One of the main concepts, which we will investigate in the present chapter, is in fact (5.0.7), generalizing [13, 16, 17, 18] via (3.0.12).

It is useful to understand the behaviour of the dual context. Denote by $\Gamma^c_{\mathrm{L}(G)}$ the complement graph of $\Gamma_{\mathrm{L}(G)}$. This means that $\Gamma^c_{\mathrm{L}(G)}$ has vertex set $\mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$ and two vertices H and K are joined by an edge if and only if $HK = KH$, that is, $HK \in \mathrm{L}(G)$, that is, the sets of edges of $\Gamma^c_{\mathrm{L}(G)}$ is given by $\{(H,K) \in V(\mathrm{L}(G)) \times V(\mathrm{L}(G)) \mid HK = KH\}$.

Lemma 5.0.1. *If G is a quasihamiltonian group, then $\Gamma_{\mathrm{L}(G)}$ is a null graph.*

Proof. Since every subgroup in $\mathrm{L}(G)$ is permutable, $V(\mathrm{L}(G))$ is empty. □

Note that the sublattice $\mathrm{N}(G)$ of $\mathrm{L}(G)$ containing all normal subgroups of G satisfies

$$\mathrm{N}(G) \subseteq \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big) \subseteq \mathrm{SN}(G). \tag{5.0.8}$$

In fact $H \in \mathrm{L}(G)$ is *permutable, or quasinormal,* in G if $HK = KH$ for all $K \in \mathrm{L}(G)$ and permutable subgroups are subnormal (see [61, Page 43]). Note that permutable subgroups of G are contained in $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$. We explore some properties of (5.0.3).

Lemma 5.0.2. *$\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$ is the smallest sublattice of $\mathrm{L}(G)$ containing (5.0.3).*

Proof. This follows easily from definitions. □

Remark 5.0.3. Recall from [61, Theorem 2.3.1] that there are quasihamiltonian non-hamiltonian p-groups and an example is given by the p–group

$$M_{p^n} = \langle a, b \mid a^{p^{n-1}} = b^p = 1, b^{-1}ab = a^{1+p^{n-2}} \rangle$$

of order p^n for an odd prime p and $n \geq 3$. Here the subgroup $H = \langle b \rangle$ is quasinormal but non–normal in M_{p^n}. Therefore $H \in \mathfrak{C}_{\mathrm{L}(M_{p^n})}\big(\mathrm{L}(M_{p^n})\big) \setminus \mathrm{N}(M_{p^n})$.

We formally recall the graph, studied in [16, 17, 18].

Definition 5.0.4 (See [16])**.** For a group G, the *permutability graph of subgroups* of G is denoted by $\Gamma(G)$ and it is a graph with vertex set consists of all the proper subgroups of G and two vertices H and K of $\Gamma(G)$ are adjacent if $HK = KH$ (i.e.: $HK \in \mathrm{L}(G)$).

And that studied in [13]:

Definition 5.0.5 (See [13])**.** The *permutability graph of non-normal subgroups* of group G is the graph $\Gamma_N(G)$ whose vertex set is given by all the proper non-normal subgroups of G and two vertices H and K are adjacent if $HK = KH$.

The following remark allows us to see a first generalization of [13, 16, 17, 18] via (5.0.7).

Remark 5.0.6. Consider a group A such that $\mathrm{N}(A) = \mathfrak{C}_{\mathrm{L}(A)}\big(\mathrm{L}(A)\big)$. This means that we consider a group where all the permutable subgroups are normal. Then $\Gamma^c_{\mathrm{L}(A)} = \Gamma_N(A)$. Consider now a group B such that $\mathrm{N}(B) = \{\{1\}, B\} = \mathfrak{C}_{\mathrm{L}(B)}\big(\mathrm{L}(B)\big)$, that is, B is a simple group. Then $\Gamma^c_{\mathrm{L}(B)} = \Gamma(B)$. We may conclude that Definitions 5.0.4 and 5.0.5 realise two complement graphs of (5.0.7) under appropriate assumptions.

Let's study some subgraphs of $\Gamma_{\mathrm{L}(G)}$.

Lemma 5.0.7. *If* $\mathrm{S}(G)$ *is a sublattice of* $\mathrm{L}(G)$ *and* $H \in \mathrm{L}(G)$*, then*

$$\mathfrak{C}_{\mathrm{S}(G)}(H) \subseteq \mathfrak{C}_{\mathrm{L}(G)}(H).$$

Proof. If $K \in \mathfrak{C}_{\mathrm{S}(G)}(H)$, then $KH = HK$ with $K \in \mathrm{S}(G)$ and this is of course true when $K \in \mathrm{L}(G)$. Then $\mathfrak{C}_{\mathrm{S}(G)}(H) \subseteq \mathfrak{C}_{\mathrm{L}(G)}(H)$. □

Because of the above lemma, we have enough information for subgraphs.

Lemma 5.0.8. *If* $\mathrm{S}(G)$ *is a sublattice of* $\mathrm{L}(G)$ *containing* $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$*, then* $\Gamma_{\mathrm{S}(G)}$ *is the subgraph of* $\Gamma_{\mathrm{L}(G)}$ *with set of vertices* $V(\mathrm{S}(G)) = \mathrm{S}(G) - \mathfrak{C}_{\mathrm{S}(G)}(\mathrm{S}(G))$ *and edges*

$$E(\mathrm{S}(G)) = \bigcap_{H \in V(\mathrm{S}(G))} \Big(\big(V(\mathrm{S}(G)) \times V(\mathrm{S}(G))\big) - \big(\mathfrak{C}_{V(\mathrm{S}(G))}(H) \times \{H\}\big) \Big),$$

where $\mathfrak{C}_{V(\mathrm{S}(G))}(H) = \{A \in V(\mathrm{S}(G)) \mid AH = HA\}$.

Proof. From the definitions, $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$ is the smallest sublattice of $\mathrm{L}(G)$ containing (5.0.3) and $\mathfrak{C}_{\mathrm{S}(G)}\big(\mathrm{S}(G)\big)$ is the smallest sublattice of $\mathrm{S}(G) \subseteq \mathrm{L}(G)$ containing the set

$$\bigcap_{X \in \mathrm{S}(G)} \mathfrak{C}_{\mathrm{S}(G)}(X) = \{Y \in \mathrm{S}(G) \mid YX = XY, \quad \forall X \in \mathrm{S}(G)\},$$

i.e the set of all subgroups in $\mathrm{S}(G)$ permutable with subgroups in $\mathrm{S}(G)$. Then we have

$\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big) \subseteq \mathfrak{C}_{\mathrm{S}(G)}\big(\mathrm{S}(G)\big)$, hence

$$V(\mathrm{L}(G)) = \mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big) = \Big(\mathrm{L}(G) - \mathrm{S}(G)\Big) \cup \Big(\mathrm{S}(G) - \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)\Big)$$

$$\supseteq \Big(\mathrm{L}(G) - \mathrm{S}(G)\Big) \cup \Big(\mathrm{S}(G) - \mathfrak{C}_{\mathrm{S}(G)}\big(\mathrm{S}(G)\big)\Big).$$

This means $V(\mathrm{L}(G)) = \mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big) \supseteq \mathrm{S}(G) - \mathfrak{C}_{\mathrm{S}(G)}\big(\mathrm{S}(G)\big) = V(\mathrm{S}(G))$. Moreover

$$E(\mathrm{S}(G)) = \{(X,Y) \in V(\mathrm{S}(G)) \times V(\mathrm{S}(G)) \mid X \sim Y \ \Leftrightarrow \ XY \neq YX\}$$

$$= \big(V(\mathrm{S}(G)) \times V(\mathrm{S}(G))\big) - \bigcup_{H \in V(\mathrm{S}(G))} (\mathfrak{C}_{V(\mathrm{S}(G))}(H) \times \{H\})$$

$$= \bigcap_{H \in V(\mathrm{S}(G))} \Big(\big(V(\mathrm{S}(G)) \times V(\mathrm{S}(G))\big) - (\mathfrak{C}_{V(\mathrm{S}(G))}(H) \times \{H\})\Big).$$

Clearly $E(\mathrm{S}(G)) \subseteq E(\mathrm{L}(G))$ and hence $\big(V(\mathrm{S}(G)), E(\mathrm{S}(G))\big)$ turns out to be a subgraph $\Gamma_{\mathrm{S}(G)}$ of the graph $\Gamma_{\mathrm{L}(G)}$ individuated by the pair $\big(V(\mathrm{L}(G)), E(\mathrm{L}(G))\big)$ of first component given by the set of vertices and by second component given by the edges. □

It can be helpful to visualise Lemmas 5.0.1, 5.0.2, 5.0.7, 5.0.8 and Definitions 5.0.4, 5.0.5.

Example 5.0.9. From example 1.2.3 we have

$$\mathrm{L}(S_3) = \{\langle(1)\rangle, \langle(123)\rangle, \langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle, S_3\}$$

and we find that

$$\mathfrak{C}_{\mathrm{L}(S_3)}\big(\mathrm{L}(S_3)\big) = \{\langle(1)\rangle, \langle(123)\rangle, S_3\} = \mathrm{N}(S_3),$$

$$V(\mathrm{L}(S_3)) = \mathrm{L}(S_3) - \mathfrak{C}_{\mathrm{L}(S_3)}\big(\mathrm{L}(S_3)\big) = \{\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle\},$$

$$E(\mathrm{L}(S_3)) = \{(\langle(12)\rangle, \langle(13)\rangle), (\langle(12)\rangle, \langle(23)\rangle), (\langle(13)\rangle, \langle(23)\rangle)\}.$$

Therefore $\Gamma_{\mathrm{L}(S_3)} \simeq K_3$ so it is a cycle of length 3, called a triangle (see [6, Page 42]).

In addition, if we consider the sublattice

$$\mathrm{S}(S_3) = \{\langle(1)\rangle, \langle(123)\rangle, \langle(12)\rangle, \langle(13)\rangle, S_3\}$$

of $\mathrm{L}(S_3)$, then

$$\mathfrak{C}_{\mathrm{S}(S_3)}\big(\mathrm{S}(S_3)\big) = \{\langle(1)\rangle, \langle(123)\rangle, S_3\} = \mathfrak{C}_{\mathrm{L}(S_3)}(\mathrm{L}(S_3)) \subseteq \mathrm{S}(S_3).$$

Therefore, the graph $\Gamma_{\mathrm{S}(S_3)}$ with vertex set $V(\mathrm{S}(S_3)) = \{\langle(12)\rangle, \langle(13)\rangle\}$ and edge set

$$E(\mathrm{S}(S_3)) = \big(V(\mathrm{S}(G)) \times V(\mathrm{S}(G))\big) - \Big(\bigcup_{H \in V(\mathrm{S}(G))} (\mathfrak{C}_{V(\mathrm{S}(G))}(H) \times \{H\})\Big)$$

$$= \{(\langle(12)\rangle, \langle(13)\rangle), (\langle(13)\rangle, \langle(12)\rangle)\}$$

is a subgraph of $\Gamma_{\mathrm{L}(S_3)}$. The graph $\Gamma_{\mathrm{L}(S_3)}$ and its subgraph $\Gamma_{\mathrm{S}(S_3)}$ are shown below:

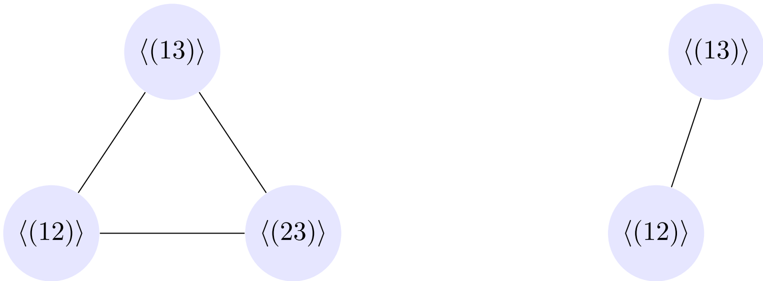

Figure 5.0.1: The graph $\Gamma_{\mathrm{L}(S_3)}$ and its subgraph $\Gamma_{\mathrm{S}(S_3)}$.

From Definition 5.0.4 and Remark 5.0.6, $\Gamma^c_{\mathrm{L}(S_3)} \simeq \Gamma_N(S_3)$ is the empty graph.

According to Definition 5.0.5, $\Gamma(S_3)$ is drawn below.

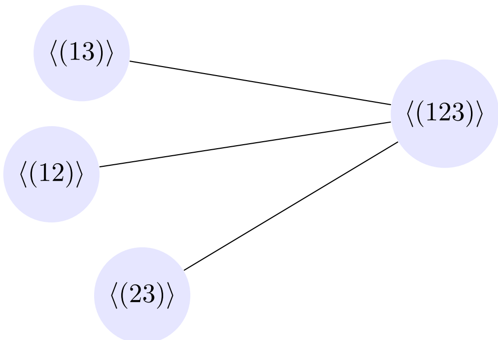

Figure 5.0.2 : The graph $\Gamma(S_3)$.

It is well known that S_3 can be seen also as group of symmetries of a triangle, that is, as the dihedral group D_6 of order six. So one can immediately note that the minimal non–abelian group has planar non–permutability graph of subgroups by Example 5.0.9. We will discuss the planarity of dihedral groups in one of our main results later on.

We pass to evaluate the size of the edges of (5.0.7).

Lemma 5.0.10. *According to* (3.0.12)*, if* $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big) \subseteq \mathrm{S}(G)$ *and* $\mathrm{T}(G) = \mathrm{S}(G) \subseteq \mathrm{L}(G)$*, then*

$$2\ |E(\mathrm{S}(G))| = |\mathrm{S}(G)|^2 \cdot (1 - gsd(G)).$$

Proof. From Lemma 5.0.8 we have,

$$E(\mathrm{S}(G)) = \bigcap_{H \in V(\mathrm{S}(G))} \Big(\big(V(\mathrm{S}(G)) \times V(\mathrm{S}(G))\big) - (\mathfrak{C}_{V(\mathrm{S}(G))}(H) \times \{H\})\Big),$$

and this is equivalent to

$$E(\mathrm{S}(G)) = \bigcap_{H \in \mathrm{S}(G)} \Big(\big(\mathrm{S}(G) \times \mathrm{S}(G)\big) - (\mathfrak{C}_{\mathrm{S}(G)}(H) \times \{H\})\Big).$$

Using (4.0.1) (see also [49, Definition 1.1 and Theorem 1.2]) we obtain

$$\Big|\bigcup_{H \in \mathrm{S}(G)} \big(\mathfrak{C}_{\mathrm{S}(G)}(H) \times \{H\}\big)\Big| = |\mathrm{S}(G)|^2 \cdot gsd(G), \text{ and since}$$

(K, H) and (H, K) in $E(\mathrm{S}(G))$ have the same edge, we count twice and this means

$$|E(\mathrm{S}(G))| = \frac{1}{2} \cdot \Big|\bigcap_{H \in \mathrm{S}(G)} \Big(\big(\mathrm{S}(G) \times \mathrm{S}(G)\big) - (\mathfrak{C}_{\mathrm{S}(G)}(H) \times \{H\})\Big)\Big|$$

$$= \frac{1}{2}\Big|\big(\mathrm{S}(G) \times \mathrm{S}(G)\big) - \bigcup_{H \in \mathrm{S}(G)} (\mathfrak{C}_{\mathrm{S}(G)}(H) \times \{H\})\Big| = \frac{1}{2}|\mathrm{S}(G)|^2 - \frac{1}{2}|\mathrm{S}(G)|^2 \cdot gsd(G).$$

Hence the result follows. □

We conclude this paragraph with the following observation.

Corollary 5.0.11. *Let H and K be two non-quasihamiltonian groups and $\mathrm{S}(H)$ and $\mathrm{S}(K)$ be sublattices of $\mathrm{L}(H)$ and $\mathrm{L}(K)$, respectively. If $|\mathfrak{C}_{\mathrm{S}(H)}\big(\mathrm{S}(H)\big)| = |\mathfrak{C}_{\mathrm{S}(K)}\big(\mathrm{S}(K)\big)| = 2$, then $H = K$.*

Proof. The case of quasihamiltonian groups is omitted in the assumptions because of Lemma 5.0.1. Now we look at the definitions and see that

$$\mathfrak{C}_{\mathrm{S}(H)}\big(\mathrm{S}(H)\big) = \bigcap_{X \in \mathrm{S}(H)} \mathfrak{C}_{\mathrm{S}(H)}(X) \supseteq \{1, H\},\ \mathfrak{C}_{\mathrm{S}(K)}\big(\mathrm{S}(K)\big) = \bigcap_{X \in \mathrm{S}(K)} \mathfrak{C}_{\mathrm{S}(K)}(X) \supseteq \{1, K\}.$$

Our assumptions imply $\mathfrak{C}_{\mathrm{S}(H)}\big(\mathrm{S}(H)\big) = \mathfrak{C}_{\mathrm{S}(K)}\big(\mathrm{S}(K)\big) = \{1, H\} = \{1, K\}$ and so $H = K$. Therefore the result follows. $\square$

If $\mathrm{S}(H)$ and $\mathrm{S}(K)$ are given in Corollary 5.0.11 in such a way that we can define properly $\Gamma_{\mathrm{S}(H)}$ and $\Gamma_{\mathrm{S}(K)}$ (see Lemma 5.0.8), then $|\mathfrak{C}_{\mathrm{S}(H)}\big(\mathrm{S}(H)\big)| = |\mathfrak{C}_{\mathrm{S}(K)}\big(\mathrm{S}(K)\big)| = 2$ implies immediately $\Gamma_{\mathrm{S}(H)} \simeq \Gamma_{\mathrm{S}(K)}$.

It may be useful to remind that for any two groups H and K such that $\mathrm{L}(H) \simeq \mathrm{L}(K)$, we cannot infer $\Gamma_{\mathrm{L}(H)} \simeq \Gamma_{\mathrm{L}(K)}$. For example, from Proposition 1.0.11 we have S_3 and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are lattice isomorphic groups, but $\Gamma_{\mathrm{L}(S_3)}$ and $\Gamma_{\mathrm{L}(\mathbb{Z}_3\times\mathbb{Z}_3)}$ are not graph isomorphic. Therefore the converse of Corollary 5.0.11 is false. On the other hand, the following problem can be interesting to investigate:

Question 5.0.12. Weakening of the hypotheses $|\mathfrak{C}_{\mathrm{S}(H)}\big(\mathrm{S}(H)\big)| = |\mathfrak{C}_{\mathrm{S}(K)}\big(\mathrm{S}(K)\big)| = 2$ in Corollary 5.0.11, in order to study the validity of the implication: $\Gamma_{\mathrm{S}(H)} \simeq \Gamma_{\mathrm{S}(K)} \implies \mathrm{S}(H) \simeq \mathrm{S}(K)$.

5.1 Main results of graph-theoretical nature and applications

Our first main result in this chapter offers a precise bound for (5.0.6).

Theorem 5.1.1. *If* $\mathrm{S}(G)$ *is a sublattice of* $\mathrm{L}(G)$ *containing* $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$*, then for any normal subgroup* N *of* G

$$2|E(\mathrm{S}(G))| \leq |\mathrm{S}(G)|^2 - \Big((|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1)^2 + (gsd(N) - 1) \cdot |\mathrm{S}(N)|^2 + (gsd(G/N) - 1) \cdot |\mathrm{S}(G/N)|^2\Big).$$

Proof. From Lemma 5.0.10, we have

$$|E(\mathrm{S}(G))| = \frac{|\mathrm{S}(G)|^2 \cdot (1 - gsd(G))}{2}$$

and this implies

$$\frac{|\mathrm{S}(G)|^2 - 2|E(\mathrm{S}(G))|}{|\mathrm{S}(G)|^2} = gsd(G) \tag{5.1.1}$$

Note from Lemma 5.0.2 that all normal subgroups are contained in $\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(G)\big)$ and so in $\mathrm{S}(G)$, and then for a given arbitrary normal subgroup N of a group G, from Lemma

4.2.4 we have

$$gsd(G) \geq \frac{1}{|\mathrm{S}(G)|^2} \cdot \Big((|\mathrm{S}(N)| + |\mathrm{S}(G/N)| - 1)^2 + (gsd(N) - 1) \cdot |\mathrm{S}(N)|^2$$

$$+(gsd(G/N) - 1) \cdot |\mathrm{S}(G/N)|^2\Big)$$

Therefore by replacing this quantity in to (5.1.1) we can obtain the required result. □

Recall from [14, Page 4] that a graph is *connected*, if any distinct two vertices are joined by a path. Another classical notion in graph theory comes from the notion of being *Hamiltonian*: this means that the graph contains a cycle having all the vertices of the graph (see [14, Page 12]). The *degree* of a vertex is the number of path joining the vertex (and counted only once). In 1960 ([14, Page 69]) Ore proved a classical result which ensures that a graph is hamiltonian, provided it is of order $n \geq 3$ and every pair u, v of distinct nonadjacent vertices satisfies the inequality $\deg(u)+\deg(v) \geq n$. This turns out be more efficient in the proofs, when we want to see that a graph is hamiltonian. Note that the numerical restriction of the following result is motivated by the behaviour of the dihedral case.

Theorem 5.1.2. *If a non-quasihamiltonian group G satisfies*

$$|\mathrm{L}(G)| + |\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))| \geq |\mathfrak{C}_{\mathrm{L}(G)}(H)| + |\mathfrak{C}_{\mathrm{L}(G)}(K)|$$

whenever H and K are non-adjacent vertices in $\Gamma_{\mathrm{L}(G)}$, then $\Gamma_{\mathrm{L}(G)}$ is Hamiltonian.

Proof. From Lemma 5.0.1 we shall assume that G is a non-quasihamiltonian group. If $\Gamma_{\mathrm{L}(G)}$ is complete, the statement is true. Assume $\Gamma_{\mathrm{L}(G)}$ is incomplete. Let H and K be non-adjacent vertices. Note that the degree of H in $\Gamma_{\mathrm{L}(G)}$ is equal to $|\mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}(H)|$ and $H, K \in \mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$. First of all, $\mathrm{L}(G) \supseteq \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$, $\mathfrak{C}_{\mathrm{L}(G)}(H) \supseteq \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$, $\mathfrak{C}_{\mathrm{L}(G)}(K) \supseteq \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))$ hence

$$|\mathrm{L}(G)| - |\mathfrak{C}_{\mathrm{L}(G)}(H)| - |\mathfrak{C}_{\mathrm{L}(G)}(K)| \geq -|\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))|$$

$$2|\mathrm{L}(G)| - |\mathfrak{C}_{\mathrm{L}(G)}(H)| - |\mathfrak{C}_{\mathrm{L}(G)}(K)| \geq |\mathrm{L}(G)| - |\mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))|.$$

On the other hand

$$\deg_{\Gamma_{\mathrm{L}(G)}}(H) + \deg_{\Gamma_{\mathrm{L}(G)}}(K) = |\mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}(H)| + |\mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}(K)|$$

$$= |\mathrm{L}(G)| - |\mathfrak{C}_{\mathrm{L}(G)}(H)| + |\mathrm{L}(G)| - |\mathfrak{C}_{\mathrm{L}(G)}(K)| = 2|\mathrm{L}(G)| - \Big(|\mathfrak{C}_{\mathrm{L}(G)}(H)| + |\mathfrak{C}_{\mathrm{L}(G)}(K)|\Big)$$

and it follows that

$$\deg_{\Gamma_{\mathrm{L}(G)}}(H) + \deg_{\Gamma_{\mathrm{L}(G)}}(K) \geq |\mathrm{L}(G) - \mathfrak{C}_{\mathrm{L}(G)}(\mathrm{L}(G))| = |V(\mathrm{L}(G))|.$$

By Ore's Theorem ([14, Theorem 2]) $\Gamma_{\mathrm{L}(G)}$ is Hamiltonian. □

5.2 Non–planarity of groups of symmetries on polygons

We examine some properties of the non–permutability graph of subgroups of dihedral, quaternion and quasi-dihedral groups, which are well known in group theory (see [35, Satz 14.9, Satz 13.10]). The dihedral group of order $2n$ $(n \geq 3)$ may be presented by

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle$$

and has $Z(D_{2n}) = 1$ when n is odd. It can be visualised as a group of symmetries of a regular polygon with n edges and n vertices. In addition, if n is odd prime, then $\mathrm{L}(D_{2n})$ is a diamond with $\mathrm{N}(D_{2n}) = \{1, N, D_{2n}\}$ which is a chain for a suitable normal subgroup N of $|N| = n$. The situation changes drastically when n is even, or not an odd prime.

In order to present some counting argument in some of the results of this thesis, it may be useful to recall that $\tau(n)$ denotes the number of divisors of n and that $\sigma(n)$ denotes the sum of all divisors of n. Let $n = p_1^{k_1} p_2^{k_2} ... p_t^{k_t}$, where $p_j's$ are primes, $t \geq 1$ and $k_j \geq 0$ with $j = 1, ..., t$, then [30, Theorems 273 and 274] show that

$$\tau(n) = (k_1 + 1) \cdots (k_t + 1) \tag{5.2.1}$$

and

$$\sigma(n) = \frac{(p_1^{k_1+1} - 1) \cdots (p_t^{k_t+1} - 1)}{(p_1 - 1) \cdots (p_t - 1)}, \tag{5.2.2}$$

and one can understand the relevance of such functions, noting that for all $n \geq 3$

$$|\mathrm{L}(D_{2n})| = \tau(n) + \sigma(n). \tag{5.2.3}$$

Now we have all that we need for the proof of the following lemma.

Lemma 5.2.1. *For any $n \geq 3$ we have $\mathfrak{C}_{\mathrm{L}(D_{2n})}(\mathrm{L}(D_{2n})) = \mathrm{N}(D_{2n})$. Moreover,*

$$|\mathfrak{C}_{\mathrm{L}(D_{2n})}(\mathrm{L}(D_{2n}))| = \begin{cases} \tau(n) + 1, & \text{if } n \text{ is odd}, \\ \tau(n) + 3, & \text{otherwise}, \end{cases}$$

$$|V(\Gamma_{\mathrm{L}(D_{2n})}| = \begin{cases} \sigma(n) - 1, & \textit{if } n \textit{ is odd,} \\ \sigma(n) - 3, & \textit{otherwise.} \end{cases}$$

Proof. We recall that the subgroups of D_{2n} can be described in one (and only one) of the following ways:

(i). Cyclic subgroups of the form $H_0^r = \langle a^{\frac{n}{r}} \rangle$ of order r, where r is a divisor of n;

(ii). Cyclic subgroups of the form $H_i^1 = \langle ba^{i-1} \rangle$ of order 2, where $i = 1, 2, ..., n$;

(iii). Dihedral subgroups of the form $H_i^r = \langle a^{\frac{n}{r}}, ba^{i-1} \rangle$ of order $2r$, where r is a divisor of n (r different from 1) and $i = 1, 2, ..., \frac{n}{r}$.

Since the number of subgroups listed in (i) is equal to the number of divisors of n, the number of subgroups listed in (ii) is equal to n, and the number of subgroups listed in (iii) is equal to $\frac{n}{r}$. We may use (5.2.3) and check that the permutable subgroups of D_{2n} are only the subgroups H_0^r in (i) and D_{2n} when n is odd. On the other hand, if n is even, then we have more permutable subgroups, in fact there are again H_0^r in (i) and D_{2n}, but this time we have also $H_i^{\frac{n}{2}}$ of index 2 in (iii).

All these permutable subgroups of D_{2n} are normal, hence

$$\mathfrak{C}_{\mathrm{L}(D_{2n})}(\mathrm{L}(D_{2n})) = \mathrm{N}(D_{2n}). \tag{5.2.4}$$

Now we show the second part of the thesis. Note that the lattice of cyclic normal subgroups of D_{2n} is isomorphic to the sublattice $\mathrm{S}(\langle a \rangle)$ of $\mathrm{L}(D_{2n})$ obtained picking $a \in D_{2n}$ and $\langle a \rangle \in \mathrm{N}(D_{2n})$. This implies $|\mathrm{S}(\langle a \rangle)| = |\mathrm{L}(H_0^n)| = \tau(n)$, so we apply (5.2.3), (5.0.5), (5.2.4) and the result follows. □

We can make a local analysis for dihedral subgroups of D_{2n}. In fact we are going to show an exact formula, which will help to compute the degree of each vertex of the non-permutability graph of subgroups.

Lemma 5.2.2. *According to the notation in the proof of Lemma 5.2.1 (ii) and (iii), if* $H_i^r \in \mathrm{L}(D_{2n})$*, then*

$$|\mathfrak{C}_{\mathrm{L}(D_{2n})}(H_i^r)| = x_i^r + \tau(n),$$

where

$$x_i^r = \begin{cases} \sum_{s|n} \frac{lcm(r,s)}{s} = r \sum_{s|n} \frac{1}{\gcd(r,s)}, & \text{if } n \textit{ is odd}, \\ 2^{u+2} - 2u + 2\alpha - 5, & \text{if } n = 2^{\alpha-1}, \alpha \geq 3, \textit{ where } r = 2^u, \\ & 0 \leq u \leq \alpha - 1, \\ (2^{\alpha+1} - 1)x_i^{r'}, & \text{if } n = 2^\alpha n', \ n' \textit{ is odd } \alpha \geq 1, \ r = 2^\beta r', \\ & r'|n', \ \beta = \alpha, \\ (2^{\beta+2} - 2\beta + 2\alpha - 3)x_i^{r'}, & \text{if } n = 2^\alpha n', \ n' \textit{ is odd } \alpha \geq 1, \ r = 2^\beta r', \\ & r'|n', \ \beta<\alpha. \end{cases}$$

Proof. Consider the subgroups H_i^r and H_j^s in $\mathrm{L}(D_{2n})$, where r and s are the divisor of n, $i \in \{1, 2, ..., \frac{n}{r}\}$, $j \in \{1, 2, ..., \frac{n}{s}\}$ as in the proof of Lemma 5.2.1 (ii) and (iii). One can see that

$$H_i^r H_j^s = H_j^s H_i^r \iff a^{2(i-j)} \in \langle a^{\frac{n}{\gcd(r,s)}} \rangle \iff \frac{n}{\gcd(r,s)} \mid 2(i-j)$$

$$\iff i \equiv j \mod \frac{n}{\gcd(r,s)}. \tag{5.2.5}$$

Now for a fixed divisor r of n, and $i \in \{1, 2, ..., \frac{n}{r}\}$, let x_i^r satisfying (5.2.5). This means x_i^r is the number of subgroups in $\mathrm{L}(D_{2n})$ commute with H_i^r except the cyclic subgroups of the form H_0^r and hence $|\mathfrak{C}_{\mathrm{L}(D_{2n})}(H_i^r)| = x_i^r + \tau(n)$. The value of x_i^r is described explicitly in [67], getting to the expression in the thesis. □

Noting that a connected (finite) graph is *Eulerian* if it contains a closed path which contains every edge of the graph exactly once (see [14, Page 14]). It is well known that an equivalent condition to be *Eulerian* is that every vertex of the graph has even degree ([14, Theorem 10]). Now we can describe the degrees of the vertices of $\Gamma_{\mathrm{L}(D_{2n})}$.

Theorem 5.2.3. *Let $n \geq 3$ be an integer. Then*

$$\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r) = \sigma(n) - x_i^r,$$

where x_i^r is given in Lemma 5.2.2. In particular for any $n \geq 3$, if $\Gamma_{\mathrm{L}(D_{2n})}$ is connected and $\sigma(n) - x_i^r$ even, then $\Gamma_{\mathrm{L}(D_{2n})}$ is Eulerian.

Proof. For each divisor r of n and $i \in \{1, 2, ..., \frac{n}{r}\}$, the number of dihedral subgroups of

D_{2n} permutes with H_i^r is x_i^r and

$$x_i^r = |\mathfrak{C}_{\mathrm{L}(D_{2n})}(H_i^r)| - |\mathrm{L}(H_0^n)| = |\mathfrak{C}_{\mathrm{L}(D_{2n})}(H_i^r)| - \tau(n).$$

This implies

$$\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r) = |\mathrm{L}(D_{2n})| - |\mathfrak{C}_{\mathrm{L}(D_{2n}}(H_i^r)| = (\sigma(n) + \tau(n)) - (x_i^r + \tau(n)) = \sigma(n) - x_i^r,$$

where x_i^r is given in Lemma 5.2.2. Now for any $n \geq 3$, $\Gamma_{\mathrm{L}(D_{2n})}$ is Eulerian, since all $\sigma(n) - x_i^r$ are even and the graph is connected. □

It may be useful to introduce an appropriate arithmetic function

$$g : k \in \mathbb{N} \longmapsto g(k) = r \sum_{r|k, s|k} \frac{1}{\gcd(r, s)} \in \mathbb{N}$$

as made in [52, 67], where r and s are divisors of k. It can be found in [67, Proof of Theorem 3.1.1, Case 1] that

$$k \text{ odd} \implies g(k) = \sum_{r|k} \sum_{i=1}^{\frac{k}{r}} x_i^r.$$

Moreover g is a multiplicative function such that

$$g(p^\alpha) = \frac{(2\alpha+1)p^{\alpha+2} - (2\alpha+3)p^{\alpha+1} + p + 1}{(p-1)^2} \tag{5.2.6}$$

for any prime p and $\alpha \in \mathbb{N}$.

Note that the equations (5.2.6), (5.2.7) and the following equation (5.2.8) are not new: they can be found in [52, 67]. They are recalled for convenience of the reader in the present context of investigation.

In particular, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ is a factorization of n in distinct odd primes p_j and $\alpha_j \geq 1$, then

$$g\left(\prod_{j=1}^{l} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}\right) = \prod_{j=1}^{l} \frac{(2\alpha_j+1)p_j^{\alpha_j+2} - (2\alpha_j+3)p_j^{\alpha_j+1} + p_j + 1}{(p_j-1)^2}. \tag{5.2.7}$$

Applying Lemma 5.2.2 we conclude that

$$g(n) = \begin{cases} \sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r, & \text{if } n \text{ is odd,} \\ \sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r = (\alpha-2)2^{\alpha+3}+9, & \text{if } n \ = 2^{\alpha-1}, \alpha \geq 3, \text{ where } r = 2^u, \\ & 0 \leq u \leq \alpha - 1, \\ \sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r = \big((\alpha-1)2^{\alpha+3}+9\big)g(n'), & \text{if } n = 2^{\alpha}n', \ n' \text{ is odd } \alpha \geq 1, \ r = 2^{\beta}r', \\ & r'|n', \beta \leq \alpha. \end{cases} \tag{5.2.8}$$

We recalled the previous notions, in order to formulate the following result.

Corollary 5.2.4. *Let $n \geq 3$ be an integer and $g(n)$ denotes the arithmetic function in (5.2.8). Then the following statements are true:*

(i). If n is odd, then

$$2|E(\mathrm{L}(D_{2n}))| = \sigma(n)^2 - g(n),$$

(ii). If $n = 2^{\alpha}, \alpha \geq 2$, then

$$2|E(\mathrm{L}(D_{2^{\alpha}}))| = (\alpha + 2^{\alpha} - 1)^2 - 2^{\alpha+2}(\alpha-2) - 2^{\alpha+1}\alpha - (\alpha-1)^2 - 8,$$

(iii). If $n = 2^{\alpha}n', \alpha \geq 1$ with n' odd, then

$$2|E(\mathrm{L}(D_{2n}))| = \sigma(n)^2 - \big((\alpha-1)2^{\alpha+3}+9\big)g(n').$$

Proof. Application of Lemma 5.0.10 with $\mathrm{S}(G) = \mathrm{T}(G) = \mathrm{L}(G)$ and (5.2.8). □

We can describe accurately the subgraphs of (5.0.7) for diehedral groups.

Corollary 5.2.5. *Let $n \geq 6$ and $H_i^r = \langle a^{\frac{n}{r}}, ba^{i-1}\rangle$ be a non-abelian subgroup of $\mathrm{L}(D_{2n})$ of order $2r$, where r is a proper divisor of n and $i = 1, 2, ..., \frac{n}{r}$, then $\Gamma_{\mathrm{L}(H_i^r)}$ is a subgraph of $\Gamma_{\mathrm{L}(D_{2n})}$ and isomorphic to the graph $\Gamma_{\mathrm{L}(D_{2r})}$.*

Proof. Since $\mathfrak{C}_{\mathrm{L}(D_{2n})}\big(\mathrm{L}(D_{2n})\big) \nsubseteq \mathrm{L}(H_i^r)$, $\mathfrak{C}_{\mathrm{L}(D_{2n})}\big(\mathrm{L}(D_{2n})\big) = \mathrm{I}_1(D_{2n}) \cup \mathrm{I}_2(D_{2n})$ where $\mathrm{I}_1(D_{2n}) \subseteq \mathrm{L}(H_i^r)$ and $\mathrm{I}_2(D_{2n}) = \mathfrak{C}_{\mathrm{L}(D_{2n})}\big(\mathrm{L}(D_{2n})\big) - \mathrm{I}_1(D_{2n}) \subseteq \mathrm{L}(D_{2n})$. Then $\mathrm{I}_1(D_{2n}) \subseteq$

$\mathfrak{C}_{\mathrm{L}(G)}\big(\mathrm{L}(H_i^r)\big)$ implies

$$V(\mathrm{L}(D_{2n})) = \mathrm{L}(D_{2n}) - \mathfrak{C}_{\mathrm{L}(D_{2n})}\big(\mathrm{L}(D_{2n})\big) = \mathrm{L}(D_{2n}) - \big(\mathrm{I}_1(D_{2n}) \cup \mathrm{I}_2(D_{2n})\big)$$

$$= \big(\mathrm{L}(D_{2n}) - \mathrm{L}(H_i^r) - \mathrm{I}_2(D_{2n})\big) \cup \big(\mathrm{L}(H_i^r) - \mathrm{I}_1(D_{2n})\big)$$

$$\supseteq \big(\mathrm{L}(D_{2n}) - \mathrm{L}(H_i^r) - \mathrm{I}_2(D_{2n})\big) \cup \big(\mathrm{L}(H_i^r) - \mathfrak{C}_{\mathrm{L}(H_i^r)}\big(\mathrm{S}(H_i^r)\big).$$

Thus $\mathrm{L}(D_{2n}) - \mathfrak{C}_{\mathrm{L}(D_{2n})}\big(\mathrm{L}(D_{2n})\big) \supseteq \mathrm{L}(H_i^r) - \mathfrak{C}_{\mathrm{L}(H_i^r)}\big(\mathrm{L}(H_i^r)\big)$ and then $V(\mathrm{L}(D_{2n})) \supseteq V(\mathrm{L}(H_i^r))$. Hence the result follows. □

Note that $\mathfrak{C}_{\mathrm{L}(D_{2n})}(\mathrm{L}(D_{2n})) = \mathrm{N}(D_{2n})$ shows that the complement of the graph $\Gamma_N(D_{2n})$ is $\Gamma_{\mathrm{L}(D_{2n})}$ and the structure of $\Gamma_N(D_{2n})$ for some values of n agrees with the description, given in [16, Theorem 4.5]. In opposition to this wealthy of information for the non-permutability graphs of dihedral groups, we discover easily that these graphs are almost always non–planar. The following example shows that we can easily draw the non-permutability graph of the subgroups of the dihedral groups by using the above results.

Example 5.2.6. If $n = 2^4$, then the dihedral group D_{16} of order 16 is defined by $D_{16} = \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ and we find that

$$\mathrm{L}(D_{16}) = \{1, \langle a^4\rangle, \langle a^2\rangle, \langle a\rangle, \langle b\rangle, \langle ba\rangle, \langle ba^2\rangle, \langle ba^3\rangle, \langle ba^4\rangle, \langle ba^5\rangle, \langle ba^6\rangle, \langle ba^7\rangle, \langle b, a^4\rangle, \langle ba, a^4\rangle,$$

$$\langle ba^2, a^4\rangle, \langle ba^3, a^4\rangle, \langle b, a^2\rangle, \langle ba, a^2\rangle, D_{16}\},$$

has $|\mathrm{L}(D_{16})| = 19$. In particular we find that

$$\mathfrak{C}_{\mathrm{L}(D_{16})}(\mathrm{L}(D_{16})) = \{1, \langle a^4\rangle, \langle a^2\rangle, \langle a\rangle, \langle b, a^2\rangle, \langle ba, a^2\rangle, D_{16}\},$$

$$V(\mathrm{L}(D_{16})) = \{\langle b\rangle, \langle ba\rangle, \langle ba^2\rangle, \langle ba^3\rangle, \langle ba^4\rangle, \langle ba^5\rangle, \langle ba^6\rangle, \langle ba^7\rangle, \langle b, a^4\rangle, \langle ba, a^4\rangle, \langle ba^2, a^4\rangle,$$

$$\langle ba^3, a^4\rangle, \langle b, a^2\rangle, \langle ba, a^2\rangle\},$$

and if $H, K \in V(\mathrm{L}(D_{16}))$ are non-adjacent, then $|\mathfrak{C}_{\mathrm{L}(D_{16})}(H)|$ (resp. $|\mathfrak{C}_{\mathrm{L}(D_{16})}(K)|$) either is equal to 11, or is equal to 13. These values are obtained looking at Fig. 5.2.1 and removing from $\mathrm{L}(D_{16})$ the number 8 (resp. 6), because each vertex (in Fig. 5.2.1) is connected eight times (resp. six times). Then it is easy to check that

$$|\mathrm{L}(D_{16})| + |\mathfrak{C}_{\mathrm{L}(D_{16})}(\mathrm{L}(D_{16}))| \geq |\mathfrak{C}_{\mathrm{L}(D_{16})}(H)| + |\mathfrak{C}_{\mathrm{L}(D_{16})}(K)|$$

is always satisfied when H and K are non-adjacent. This evidence justifies the numerical restriction in Theorem 5.1.2. Note also that Corollary 5.2.4 implies

$$|E(\mathrm{L}(D_{2^4})| = \frac{1}{2}\cdot\big((4-1)+2^4\big)^2 - \frac{1}{2}\Big((4-2)2^{4+2}+4\cdot 2^{4+1}+(4-1)^2+8\Big) = 44.$$

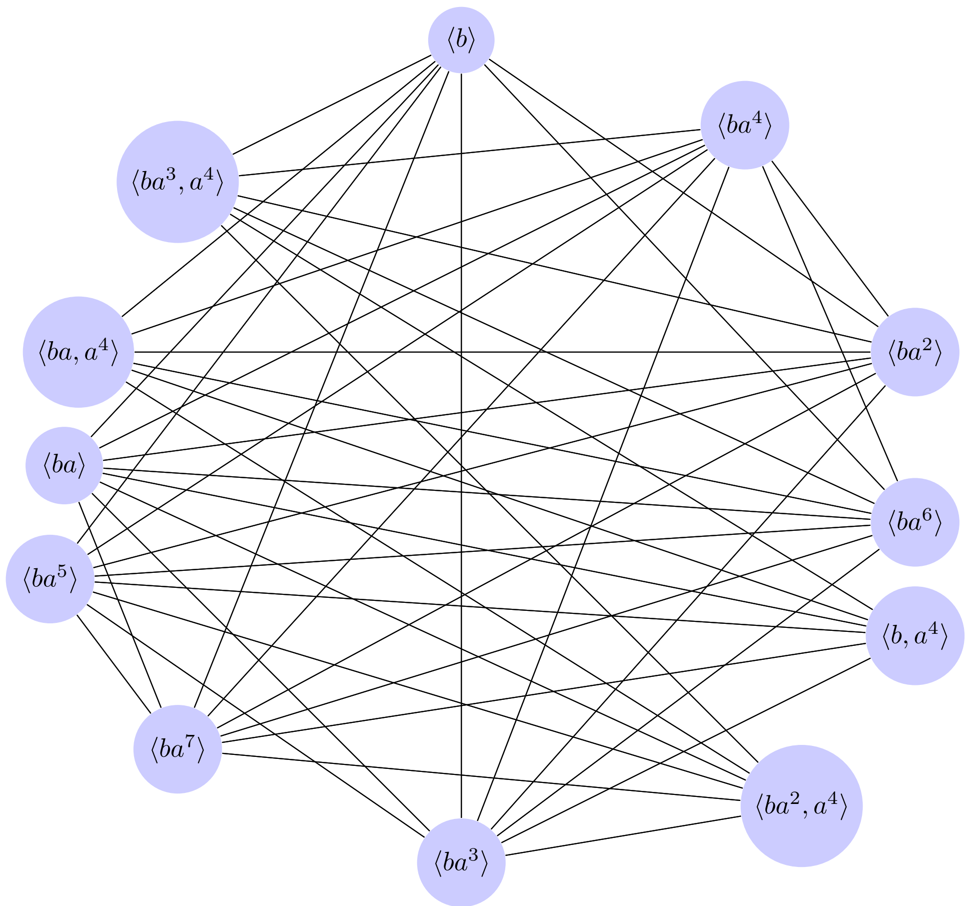

Figure 5.2.1: The graph $\Gamma_{\mathrm{L}(D_{16})}$

From Fig. 5.2.1 consider two non-adjacent vertices, say $H = \langle b\rangle$ and $K = \langle b, a^4\rangle$. Here H is contained in K and we check that $\mathcal{N}_{\mathrm{L}(D_{16})}(H) \supset \mathcal{N}_{\mathrm{L}(D_{16})}(K)$. Again if $H = \langle b\rangle$ and $K = \langle ba^4\rangle$, here either H is not contained in K (or K is not contained in H) and $(H, K) \in E(\mathrm{L}(G))$, then

$$\mathcal{N}_{\mathrm{L}(D_{16})}(H) = \mathcal{N}_{\mathrm{L}(D_{16})}(K) \text{ and } |\mathcal{N}_{\mathrm{L}(D_{16})}(H) \cap \mathcal{N}_{\mathrm{L}(D_{16})}(K)| \geq 2.$$

This information justifies that the intersection of the neighborhoods of two arbitrary non-adjacent vertices is non-empty.

In Fig. 5.2.1 we may observe an interesting fact. Let $H = \langle b, a^2\rangle$ be a maximal dihedral subgroup of D_{16}. Then

$$\mathrm{L}(H)=\{\{1\},\langle a^4\rangle,\langle a^2\rangle,\langle b\rangle,\langle ba^2\rangle,\langle ba^6\rangle,\langle ba^4\rangle,\langle b,a^4\rangle,\langle ba^2,a^4\rangle,\langle b,a^2\rangle\}$$

is a sublattice of $\mathrm{L}(D_{16})$ and $V(\mathrm{L}(H))=\{\langle b\rangle,\langle ba^2\rangle,\langle ba^6\rangle,\langle ba^4\rangle\}$. Clearly $\Gamma_{\mathrm{L}(H)}$ is a subgraph of $\Gamma_{\mathrm{L}(D_{16})}$ and we can easily observe that it is isomorphic to the graph $\Gamma_{\mathrm{L}(D_8)}\simeq C_4$. The existence of dihedral subgraphs of small size in $\Gamma_{\mathrm{L}(D_{2n})}$ is used to characterise the planarity in our final main result in this thesis.

Theorem 5.2.7. *Let $n\geq 3$ be an integer. Then $\Gamma_{\mathrm{L}(D_{2n})}$ is planar if and only if $n=3$ or $n=4$.*

Proof. By Examples 5.0.9 and 5.2.6 it is easy to see that if G is either D_6 or D_8, then $\Gamma_{\mathrm{L}(G)}$ is planar. Conversely suppose $\Gamma_{\mathrm{L}(D_{2n})}$ is planar and let $n=p_1^{k_1}p_2^{k_2}...p_t^{k_t}$ a factorization in product of distinct primes $p_j's$, $t\geq 1$ and $k_j\geq 0$ with $j=1,...,t$. We will show that all vertices of $\Gamma_{\mathrm{L}(D_{2n})}$ have degree at least 6 except for the cases $n=3$ or $n=4$. Because of the proof of Lemma 5.2.1, we should remove H_0^r and H_i^n for any value of n and r divisor of n from $V(\Gamma_{\mathrm{L}(D_{2n})})$. In fact H_0^r and H_i^n are normal in D_{2n}. We can also remove $H_i^{\frac{n}{2}}$ for the same reason when n is even (but not when n is odd). Therefore we focus on two situations, which are motivated by the fact that planar graphs have minimum degree <6 (see [14, Chapter 1, §4, Theorems 11, 12, 13]).

Case 1. For $r=1$ we have $\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r)=\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^1)\geq 6$ for all $i=1,...,n$, where H_i^1 are described in the proof Lemma 5.2.1 (ii).

Let $n=p^\alpha$ with p odd prime and $\alpha\geq 1$. If $r=1$, then by Lemma 5.2.2

$$x_i^1=\tau(n)=\alpha+1\geq 2,$$

$i=1,2,...,n$ and Theorem 5.2.3 gives

$$\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^1)=\sigma(n)-x_i^1=\sigma(p^\alpha)-\tau(p^\alpha)=\sum_{t=0}^{\alpha}p^t-\alpha-1\geq 6\quad\text{for all }\alpha\text{ if }\ n\geq 7$$

or

$$\sum_{t=0}^{\alpha}p^t-\alpha-1=4\quad\text{if and only}\quad n=5.$$

If $n=2^{\alpha-1},\alpha\geq 3$, and $r=1$, then Lemma 5.2.2 implies $1=r=2^u$ and this gives $u=0$

and $x_i^1 = 2\alpha - 1$. Therefore by Theorem 5.2.3

$$\deg_{\Gamma_{L(D_{2n})}}(H_i^1) = \sigma(2^{\alpha-1}) - x_i^1 = (2^\alpha - 1) - (2\alpha - 1) = 2^\alpha - 2\alpha \geq 6 \text{ if and only if } \alpha \geq 4.$$

If n is odd and $t \geq 2$, then (5.2.1) and (5.2.2) imply

$$\deg_{\Gamma_{L(D_{2n})}}(H_i^1) = \sigma(n) - x_i^1 = \sigma(n) - \tau(n) = \frac{(p_1^{k_1+1}-1)\cdots(p_t^{k_t+1}-1)}{(p_1-1)\cdots(p_t-1)} - (k_1+1)\cdots(k_t+1). \tag{5.2.9}$$

It is however required to find the lower bound of $\deg_{\Gamma_{L(D_{2n})}}(H_i^1)$. Since for all prime p_j

$$p_j^{k_j} + p_j \geq 2 \implies p_j^{k_j+1} + p_j^{k_j} + p_j \geq p_j^{k_j+1} + 2 \implies p_j^{k_j+1} - 1 \geq (p_j^{k_j} - 1)(p_j - 1), \tag{5.2.10}$$

this implies

$$\frac{(p_1^{k_1+1}-1)\cdots(p_t^{k_t+1}-1)}{(p_1-1)\cdots(p_t-1)} - (k_1+1)\cdots(k_t+1) \geq (p_1^{k_1}-1)\cdots(p_t^{k_t}-1) - (k_1+1)\cdots(k_t+1). \tag{5.2.11}$$

Now $1 = p_j^0 \leq \sqrt{2}(k_j+1)+1$ gives us

$$0 = \log_{p_j} {p_j}^0 \leq \log_{p_j}(\sqrt{2}(k_j+1)+1) \leq \log_{p_j} {p_j}^{k_j} = k_j \implies p_j^{k_j} \geq \sqrt{2}(k_j+1)+1$$

hence

$$p_j^{k_j} - 1 \geq \sqrt{2}(k_j+1). \tag{5.2.12}$$

Since $t \geq 2$, at least two $k_j, k_h \geq 1$, where $j, h \in \{1, ..., t\}$. Then we may lower bound the quantities in (5.2.11) in the following way.

$$\geq \sqrt{2}(k_1+1)\cdots\sqrt{2}(k_t+1) - (k_1+1)\cdots(k_t+1) \geq (\sqrt{2}^t - 1)(k_1+1)\cdots(k_t+1) \geq 6 \text{ for all } p_j's.$$

Therefore (5.2.9) gives $\deg_{\Gamma_{L(D_{2n})}}(H_i^1) \geq 4$ for all $p_j's$. Finally let n be even with $p_1 = 2$, $k_1 \geq 1$ and $p_j's$ odd primes for $t \geq 1$, $j = 2, ..., t$, this means $n = 2^{k_1}n'$ where $n' = p_2^{k_2}\cdots p_t^{k_t}$. Now if $r = 1$, then $1 = r = 2^\beta r'$, where $\beta \leq k_1$ and $r' | p_2^{k_2}\cdots p_t^{k_t}$, implies $\beta = 0$ and $r' = 1$ and then by (5.2.1) and Lemma 5.2.2

$$x_i^1 = (2k_1+1)x_i^{r'} = (2k_1+1)\left((k_2+1)\cdots(k_t+1)\right).$$

Then by (5.2.2) and Theorem 5.2.3

$$\deg_{\Gamma_{L(D_{2n})}}(H_i^1) = \sigma(n) - x_i^1 = \frac{(2k_1+1-1)(p_2^{k_2+1}-1)\cdots(p_t^{k_t+1}-1)}{(2-1)(p_2-1)\cdots(p_t-1)}$$
$$-(2k_1+1)((k_2+1)\cdots(k_t+1))$$

$$= \frac{(2^{k_1+1}-1)(p_2^{k_2+1}-1)\cdots(p_t^{k_t+1}-1)}{(p_2-1)\cdots(p_t-1)} - (2k_1+1)(k_2+1)\cdots(k_t+1). \tag{5.2.13}$$

Since n is even and n' is odd, along with arguments in (5.2.10) and (5.2.12), we repeat the steps for (5.2.13) because the first factor of n does not give a perfect analogy. So,

$$\frac{(2^{k_1+1}-1)(p_2^{k_2+1}-1)\cdots(p_t^{k_t+1}-1)}{(p_2-1)\cdots(p_t-1)} - (2k_1+1)(k_2+1)\cdots(k_t+1)$$

$$\geq (2^{k_1+1}-1)(p_2^{k_2}-1)\cdots(p_t^{k_t}-1) - (2k_1+1)(k_2+1)\cdots(k_t+1)$$

$$\geq (2^{k_1+1}-1)\sqrt{2}(k_2+1)\cdots\sqrt{2}(k_t+1) - (2k_1+1)(k_1+1)\cdots(k_t+1)$$

$$= \left(\sqrt{2}^{t}(2^{k_1+1}-1) - (2k_1+1)\right)(k_2+1)\cdots(k_t+1)$$

$$= \left(\sqrt{2}^{t-1}(2^{k_1+1}-2k_1-2)\right)(k_2+1)\cdots(k_t+1) \geq 6 \text{ for all } k_j's \text{ and } t \geq 1.$$

Now in all cases we have either $\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^1) = 4$ if and only if $n = 5$ and hence $\Gamma_{\mathrm{L}(D_{2n})} \simeq K_5$ or $\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^1) \geq 6$ for all $n \geq 6$. This shows the claim in Case 1.

Case 2. For $r \geq 2$ we have $\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r) \geq 6$ for all $i = 1, ..., \frac{n}{r}$ and r divisor of n, where H_i^r are described in the proof of Lemma 5.2.1 (iii).

Note that H_i^n, and $H_i^{\frac{n}{2}}$ of index two with n even should be removed, as observed earlier. From Lemma 5.2.1 we know that the structure of a dihedral group can be always written as $H_i^r = H_0^r H_i^1$ with H_0^r normal in H_i^r and H_i^1 subgroup of order two in H_i^r. We know from Case 1 above that there are at least 6 subgroups $A_1, A_2, \ldots, A_6 \in V(\Gamma_{\mathrm{L}(D_{2n})})$ such that $A_j H_i^1 \neq H_i^1 A_j$ for $j = 1, 2, \ldots, 6$. Because of the structure $H_i^r = H_0^r H_i^1$, we have $A_j H_i^r \neq H_i^r A_j$ for these 6 subgroups. In fact, if it would be true $A_j H_i^r = H_i^r A_j$, then $A_j H_i^1 = H_i^1 A_j$ against the choice of A_j. Therefore $\deg_{\Gamma_{\mathrm{L}(D_{2n})}}(H_i^r) \geq 6$. Our claim of Case 2 follows and the proof is complete. □

Note that in Case 1 above the inequality $\sigma(n) - \tau(n) \geq 6$ when $n > 5$ is odd and has at least two distinct prime factors can be proved differently. In fact $\tau(n) \geq 4$ in this case and $\sigma(n)$ is the sum of $\tau(n)$ distinct positive divisors, hence $\sigma(n) \geq 1 + 2 + ... + \tau(n) \geq 1 + 2 + 3 + \tau(n) = 6 + \tau(n)$ and the inequality follows. This could simplify the above argument. On the other hand, the above proof describes exactly the degree of the vertices in a case-by-case analysis and follows an approach which is familiar in the main literature [67, 68, 69] on the topic. We add a few applications of the previous result.

Corollary 5.2.8. *Any homomorphic image of a group in a dihedral group D_{2n} has planar non-permutability graph of subgroups if and only if $n \in \{3, 4\}$.*

Proof. We claim that for any group G for which there exists a homomorphism $G \xrightarrow{\varepsilon} D_{2n}$ has planar $\Gamma_{\mathrm{L}(G/\ker\varepsilon)}$ if and only if $n = 3$ or $n = 4$. It is enough to note that $G/\ker\varepsilon \simeq \mathrm{Im}(\varepsilon)$ either is abelian or a dihedral group of order smaller than $2n$ (see [35]). Now the result follows by Lemma 5.0.1 and Theorem 5.2.7. □

Thanks to Corollary 5.2.8, generalized quaternion groups and quasi-dihedral groups present (or do not present) planarity for their non-permutability graph of subgroups, just looking at their central quotients. In fact these groups [35, Satz 14.9] have central quotients which are isomorphic to dihedral groups.

We end with an open problem, which requires different techniques of investigation.

Question 5.2.9. Study the planarity of the non–permutability graph of subgroups in larger families of groups (with a special emphasis on permutation groups).

Due to the fact that $D_{2n} = N \rtimes A$ is the semidirect product of a normal cyclic subgroup N of $|N|$ by a cyclic group A of $|A| = 2$ acting by inversion on N, one can further observe that (for n odd) $N = Fit(D_{2n})$, where $Fit(G)$ denotes the largest normal nilpotent subgroup of a finite group G (called *Fitting subgroup* of G) and in addition $C_{D_{2n}}(N) = N$. Structural properties of splitting in semidirect products have been largely studied in the context of permutation groups. For instance, a classical result of O'Nan–Scott Theorem (see [7, Section 13]) illustrates what happens when we replace N with the so called *generalised Fitting subgroup*, introduced by Bender [11]. See also Structure Theorem for Primitive Permutation Groups in [7, Theorem 11.1]. The presence of a sophisticated structure in the lattice of subgroups of primitive permutation groups motivates us to believe that a study of the non–permutability graph of subgroups will require a completely different approach when we would like to focus on larger classes of groups containing dihedral groups.

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