Some Stationary and Evolution Problems Governed by Various Notions of Monotone Operators

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Abstract

The purpose of this work is to explore some notions of monotonicity for operators between Banach spaces and the applications to the study of boundary value problems (BVPs) and initial boundary value problems (IBVPs) for partial differential equations (PDEs), with the possibility in the end to examine new problems and provide some solutions. Variational approach will be used to reformulate these problems into stationary equations (in the case of BVPs) and evolution equations (in the case of IBVPs), where the underlined operators constructed as realizations of those problems in appropriate function spaces. This is known as weak formulation, which allows us to find weak solutions of the problems in a larger functions space rather than classical solutions that are sufficiently smooth. The theory of monotone and pseudomonotone operators will be applied to find existence theorems for stationary equations and evolution equations. In addition, the existence theorem for evolution equations with locally monotone operator will also be presented as a generalisation of the one with monotone operators. Another type of monotonicity so-called strict p-quasimonotonicity, which is defined in term of Young measures. This type of weaker, integrated version of monotonicity is directly applied in the study of elliptic and parabolic system of PDEs, the difficulty arises from dealing with this monotonicity is overcome by the theory of Young measures. The application of these monotonicity in the study of variational inequality will also be discussed. In particular, there is a new setting for strict p-quasimonotonicity in a particular type of elliptic variational inequalities, the proof of the new existence theorem will also be presented. Some open problems on the application of strict p-quasimonotonicity in the study of parabolic variational inequalities will also be discussed. Finally, we mention the theory of monotone and pseudomonotone operators in the study of second order evolution equations. A new setting of the local monotonicity in the second order evolution equations will be presented as well as the new existence theorem.

Contents

Introduction

The theory of partial differential equations (PDEs) has been significantly developed during the 20th century. This growth can be attributed to the successful development of its supporting mathematical fields (such as measure theory, functional analysis and function spaces) and ever-increasing demand for its application in physics, mechanics, dynamics, biology, chemistry (see [20, 32, 33, 69]). The aim in this thesis is to find the solutions to elliptic boundary value problems (EBVPs) and parabolic initial boundary value problems (PIBVPs) where the underlined operators are of the monotone type. Through the weak formulation, the problems can be formulated as abstract equations for which the abstract theory for monotone type of operators can be applied to establish the existence theorem. For example, consider the following nonlinear elliptic equation of the form:

$$-\sum_{j=1}^{n} D_j[a_j(x, u(x), Du(x))] + a_0(x, u(x), Du(x)) = f(x) \text{ for } x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain and $D_j = \frac{\partial}{\partial x_j}$, with the homogeneous Dirichlet boundary condition

$$u = 0 \text{ on } \partial\Omega. \tag{2}$$

We shall see later that (see Chapter 3) if the functions a_j and f satisfy some conditions, through the process of the weak formulation, we can rewrite the above equation as the following abstract operator equation:

$$A(u) = F, \tag{3}$$

where $A, F : W_0^{1,p}(\Omega) = V \to V^*$ are nonlinear operators, and

$$\langle A(u), v\rangle = \sum_{j=1}^{n} \int_\Omega a_j(x, u(x), Du(x)) D_j v dx + \int_\Omega a_0(x, u(x), Du(x)) v dx;$$

$$\langle F, v\rangle = \int_\Omega f v dx.$$

The existence of a weak solution $u \in V$ depends on the properties of the operator A. The study of the existence theorem for the above abstract equation (3) where A is a monotone type of operator plays an important role because it can be applied to solve various class of PDEs.

It turns out that the theory of monotone type of operators can also be applied to the study of variational inequalities (VIs), which have been recognised as mathematical tools that dealing with problems arising in different fields such as optimization theory, economic equilibrium and mechanics, etc., (see e.g., [37, 29, 51]). The initial existence results were established by Jacques-Louis Lions and Guido Stampacchia in 60's (see [79, 56]). After these pioneering works, enormous number of researches have been done in the theory of VIs by many mathematicians. We will also apply the abstract theory of monotone type of operators in the study of existence theorem for VIs in this work.

In this thesis, we will present various notions of monotonicity in the literature, namely, monotonicity, pseudomonotonicity, strict p-quasimonotonicity and local monotonicity. Then we will apply these notions to abstract equations (e.g., stationary equations and evolution equations), EBVPs, PIBVPs and VIs.

The fundamental step in the theory of monotone operators was made by George Minty in 1960s (see [62, 61]) where the first substantial results concerning monotone operators were published. Then Felix Earl Browder studied the properties of monotone operators systematically and applied them in the study of quasilinear elliptic PDEs (see [13, 14]). The theory of monotone operators was first applied in the study of VIs by G.Minty (see [63]). In Chapter 3, we are going to study these existence results for abstract equations, EBVPs, PIBVPs and VIs governed by monotone operators.

There are many PDEs having their corresponding abstract operator equations (e.g., (3)) governed by non monotone operators. In 1968, Haïm Brézis [11] introduced another vast class of operators which are called pseudomonotone operators. In 1977, F.E.Browder applied the theory of pseudomonotone operators in the study of quasilinear elliptic PDEs and proved the corresponding existence result (see [15]). The notion of pseudomonotonicity has also been applied in the study of VIs by various authors (see [78, 75, 90, 17, 74, 44]). In Chapter 4, we will study various existence results for variational problems such that the underlined operators are pseudomonotone.

Unlike the other notions of monotonicity in this work, the notion of strictly p-quasimonotonicity is not defined as an operator from a Banach space to its dual. The strictly p-quasimonotone function is defined in term of Young measures. This was introduced by Norbert Hungerbühler (see [35, 24]). N.Hungerbühler applied this notion in the study of elliptic and parabolic systems of partial differential equations. The existence theorems for these problems will be presented in Chapter 5. This notion has not yet been applied to VIs in literatures, in this work, we will apply this notion in the study of elliptic variational inequalities (EVIs) where the operator involves a strictly p-quasimonotone function. Then we will show the existence theorem for this new particular type of variational inequalities.

The class of locally monotone operators was first introduced by Wei Liu in 2010 (see [58]) in his study of stochastic evolution equations, this notion allows a generalisation of existence theorem under classical monotone operators. W.Liu [57] studied the analogous existence result for evolution equations governed by locally monotone operators. The main theorem in his work is a generalisation of the existence theorem for evolution equations under classical monotone operators. The importance of the theorem is that it can be applied to a wider class of PDEs and it allows a weaker growth condition on the operators. However, the theorem does require an additional compactness condition on the embedding $V \subset H$ for an evolution triple $V \subset H \subset V^*$. In Chapter 6, there are more precise details about this. While in the last Chapter 7, the notion of local monotonicity will be applied to the study of second order evolution equations, this has not yet been done in literatures.

The thesis is structured as follows:
In Chapter 1, we will present some preliminary results from various mathematical fields such as functional analysis, measure theory and function spaces. In particular, we focus on Sobolev spaces and Bochner-Sobolev spaces. The notion of Sobolev spaces plays a fundamental role in the weak formulation of the variational problems and the notion of Bochner-Sobolev spaces is crucial in the study of evolution equations. Many crucial theorems that will be frequently used throughout the thesis are presented in this chapter.

While in Chapter 2, we cover the abstract theory for monotone type of operators, we will first introduce definitions of monotone and B-pseudomonotone (in sense of Brézis, see Definition 2.2.1) operators and their properties. The pseudomonotonicity we will use throughout this work is bounded B-pseudomonotonicity, we will call this pseudomonotone from now on. We show that a monotone hemicontinuous and bounded operator is also pseudomonotone. We will also show that the prototype of any pseudomonotone operator is the sum of a strongly continuous operator and a monotone operator, hence the theory of pseudomonotone operators unifies both compactness and monotonicity arguments. Then we will introduce other notions of pseudomonotonicity introduced by various authors. The first one (see Definition 2.2.14) introduced by S.Karamardian [42], this type of pseudomonotonicity is a weaker notion than classical monotonicity, and it is mainly applied in the theory of variational inequalities. F.E.Browder also introduces a notion of pseudomonotonicity which is equivalent to the one in sense of Brézis under boundedness condition. In recent papers [36, 25], another notion of pseudomonotonicity so-called C-pseudomonotone (see Definition 2.2.16) was introduced as a weaker notion of B-pseudomonotone, is applied in variational inequality. The existence theorem for variational inequality with C-pseudomonotone is a generalization of many existence theorem for variational inequalities (see Theorem 15 in [36]). Then we will introduce the notion of local monotonicity which is defined in the context of an evolution triple (see [57]). Local monotonicity can be applied in the study of evolution equations to include a wider class of function than classical monotonicity. We show that a locally monotone, hemicontinuous and bounded operator with an additional assumption is also pseudomonotone. Lastly, some brief abstract theory about Young measures will be presented in order to define the strict p-quasimonotonicity. Young measures was introduced by Laurence Chisholm Young [87] to give description of limits of minimizing sequences in the Calculus of Variation and further in optimal control (see [88, 59]), which enable us to analyse the problems where the minimizers do not exist in the classical sense. The fundamental theorem on Young measures will be presented here to show the importance of Young measures in understanding limiting behaviours of a sequence of measurable functions under compositions with continuous functions. Young measures was also developed as a powerful tool in analysis oscillation effects and characterisation of oscillating sequences under compositions of continuous functions. However, Young measures completely ignore the concentrations effects. The use of Young measures in the analysis of possible oscillations of solutions of partial differential equations was first introduced by L.Tartar (see [80, 81, 82]). In the note [35], the theory of Young measures is applied in the analysis of quasilinear elliptic and parabolic system of equations. Young measures is also applied in the variational of problem in continuum mechanics and micro-structure of crystals (see e.g., [7, 8, 19, 45]).

In Chapter 3, we first show the existence theorems for abstract stationery and evolution equations with monotone operators, the proof we will use is Galerkin's approximation method, which consists of three main steps: the first step is to define the form of Galerkin's approximation and solve the problem in the finite dimension, in the stationary equations, the existence is obtained through projection and Browder fixed-point theorem; while in the evolution equations, the system is transferred into a system of ordinary differential equations, the existence of finite dimensional solution is obtained by Carathéodory existence theorem. The second step is to find the prior estimate by using the coercive condition, then it follows that the approximating sequence admits a convergent subsequence because of reflexivity. The last step is to verify that the weak limit is a solution using the monotonicity. Then we will formulate the type of EBVPs and PIBVPs to be solved, through weak formulation of these problems, we may reformulate them as stationary and evolution equations respectively. Now, the existence results for stationery and evolution equations can be applied to find the existence of the EBVPs and PIBVPs. Finally, some existence results regarding to elliptic variational inequalities will be presented.

In the subsequent Chapter 4, we deal with problems introduced in Chapter 3 with pseudomonotone operators. In addition, we study the existence theorem for parabolic variational inequalities using the method of Rothe. Rothe's method is a very powerful tool in the analysis of evolution problems. It consists of three main steps: the first step is discretize the time interval, so the parabolic problems can be transformed into elliptic problems on each time subinterval. The existence follows from the existence result on elliptic problems. The second step is find the prior estimate using coercive and monotonicity assumption. The last step is to construct Rothe's function and show that the Rothe's function converge to the solution of the original problem. The idea will be illustrated in more precise details in Section 4.3.2. At the end of this chapter, we will also brief mention the application of the existence result for variational inequalities in the study of obstacle problems.

In Chapter 5, we will first formulate the homogeneous Dirichlet problems for elliptic and parabolic system of equations that needed to be solved (see [35]). Then we will apply variational approach to show the existence results of these problems, the difficulty of showing compactness of the approximating sequence arises from dealing with the strict p-quasimonotonicity, which is a weaker, integrated version of monotonicity. We will see how Young measures are used to overcome this difficulty. After this, we will apply the notion of p-quasimonotonicity in the study of variational inequalities, which has not yet been done in literatures. We will first set up a particular type of elliptic variational inequalities where the operator A is defined as (5.22) and V is only taken to be a subspace of $W^{1,p}(\Omega)$, then we will prove the new existence result for this problem, the proof consists of two parts, the first part is to show the existence of the finite dimensional solution, this part is inspired by the standard approach of elliptic problems, which is projecting the problem onto Hilbert space and using Browder fixed-point theorem to show the existence. The second part of the proof is find the prior estimate and show that the weak limit of approximating sequence is a solution with the tools of Young measures, this part is inspired by N.Hungerbühler ([35, Chapter 3]). We will also formulate a stronger coercive condition (5.35) such that the existence result for the VIs enables us to find the solution for non-homogeneous Dirichlet problems for elliptic system of equations (see Theorem 5.3.5 and Remark 5.3.6). In the last part of this chapter, we will set up the open problems on application of strict p-quasimonotonicity in the study of a particular type of parabolic variational inequalities. The difficulty of applying Rothe's method will be explained in this section.

In Chapter 6, we will show the existence theorem for evolution equations under locally monotonicity, this is a generalization of existence result for monotone operators with relaxed growth and coercive conditions which include a larger class of functions. Then an example such that the underlined operator is locally monotone but not monotone will be presented. In the last part of this chapter, we will briefly introduce the use of the locally monotonicity to the stochastic evolution equations.

In the last Chapter 7, we briefly mention the use of monotone and pseudomonotone operators in the second order evolution equations. We list some existence theorems for monotone and pseudomonotone operators from literatures [77, 90]. Then we will apply the notion of local monotonicity in the second order evolution problems, which has not yet been done in literatures. We will first set up the problem, then we will prove the new existence theorem, the proof is inspired by standard Galerkin's method in solving first order evolution problem, e.g., Theorem 3.1.3. At the end of this chapter, we briefly mention the application of these theorem in the study of hyperbolic partial differential equations.

Chapter 1

Preliminary and Notations

1.1 Functional Analysis

Let $(V, \|\cdot\|_V)$, and $(W, \|\cdot\|_W)$ be two normed spaces. Denote $\mathcal{L}(V, W)$ as the set of all linear continuous mappings from V to W with the norm

$$\|A\|_{\mathcal{L}(V,W)} := \sup_{\|v\|_V \neq 0} \frac{\|A(v)\|_W}{\|v\|_V} = \sup_{\|v\|_V \leq 1} \|A(v)\|_W = \sup_{\|v\|_V = 1} \|A(v)\|_W .$$

The normed space $(\mathcal{L}(V, W), \|\cdot\|_{\mathcal{L}(V,W)})$ is a Banach space when W is a Banach space.

Definition 1.1.1. V is said to be embedded into W if $V \subset W$ and the identity map $id : V \to W$ is linear and continuous.

Definition 1.1.2. A linear mapping $L : V \to W$ is compact if for any bounded sequence $(u_k)_k$ in V, the sequence $(L(u_k))_k$ has a convergent subsequence.

Definition 1.1.3. V is said to be compactly embedding into W if $V \subset W$ and the identity map $id : V \to W$ is compact.

In the particular case where we have $W = \mathbb{R}$, then the linear space $\mathcal{L}(V; \mathbb{R})$ is also denoted by V^* and called dual space of V, which consists of all linear continuous mappings from V to $\mathbb{R}$. Similar to above, it has norm

$$\|A\|_{V^*} := \sup_{\|v\|_V \neq 0} \frac{|A(v)|}{\|v\|_V} = \sup_{\|v\|_V \leq 1} |A(v)| = \sup_{\|v\|_V = 1} |A(v)| .$$

Definition 1.1.4. The bilinear form $\langle \cdot, \cdot \rangle_{V^* \times V} : V^* \times V \to \mathbb{R}$ is called canonical duality pairing, sometimes briefly denoted as $\langle \cdot, \cdot \rangle$. For an operator $f \in V^*$, we will simply write $\langle f, v \rangle$ instead of $f(v)$ with the meaning that the value of f at v.

Proposition 1.1.5. The duality pairing is continuous if $V^* \times V$ equipped with norm $\times$ norm, weak$^*\times$ norm or norm $\times$ weak topology. i.e., $\lim_{k\to\infty} \langle f_k, u_k \rangle = \langle f, u \rangle$ if one of the following holds: (1) $f_k \to f$ strongly in V^* and $u_k \to u$ strongly in V; (2) $f_k \rightharpoonup f$ weakly* in V^* and $u_k \to u$ strongly in V; (3) $f_k \to f$ strongly in V^* and $u_k \rightharpoonup u$ weakly in V.
The duality pairing is continuous separately if $V^* \times V$ equipped with weak$^*\times$ weak topology. i.e., $\lim_{k\to\infty} \lim_{l\to\infty} \langle f_k, u_l \rangle = \langle f, u \rangle$ and $\lim_{l\to\infty} \lim_{k\to\infty} \langle f_k, u_l \rangle = \langle f, u \rangle$ if $f_k \rightharpoonup f$ weakly* in V^* and $u_k \rightharpoonup u$ weakly in V. (see Definitions 1.1.6 and 1.1.7 for weak convergence and weak* convergence respectively)

Similarly, we can define $V^{**} = (V^*)^*$ as the bidual space of V, which consists of all linear continuous mappings from V^* to $\mathbb{R}$ with the norm

$$\|A\|_{V^{**}} = \sup_{\|v\|_{V^*} \neq 0} \frac{|A(v)|}{\|v\|_{V^*}} = \sup_{\|v\|_{V^*} \leq 1} |A(v)| = \sup_{\|v\|_{V^*} = 1} |A(v)| .$$

There is a canonical embedding ϕ from V into its bidual V^{**} defined by

$$\langle \phi(v), f\rangle = \langle f, v\rangle \text{ for any } v \in V,\ f \in V^*.$$

It can be checked that ϕ is linear and isometry. If ϕ is also onto, i.e., there is an isometric isomorphism between V and V^{**}, then we say that the Banach space V is reflexive.
Now we will introduce the definitions of weak convergence and weak* convergence and their properties.

Definition 1.1.6 (Weak convergence). A sequence $(v_k)_k$ in V is said to converge weakly to $v \in V$, denoted as $v_k \rightharpoonup v$. If for any $f \in V^*$, we have

$$\langle f, v_k\rangle \to \langle f, v\rangle .$$

Definition 1.1.7 (Weak* convergence). A sequence $(f_k)_k$ in V^* is said to converge weakly* to $f \in V^*$, denoted as $f_k \overset{*}{\rightharpoonup} f$. If for any $v \in V$, we have

$$\langle f_k, v\rangle \to \langle f, v\rangle .$$

Proposition 1.1.8. Let $(v_k)_k$ be a sequence in V and let $(f_k)_k$ be a sequence in V^*, then
(i) If v_k converges strongly to $v \in V$, then v_k converges weakly to $v \in V$.
(ii) If f_k is weakly convergent to $f \in V^*$, then f_k is weakly* convergent to $f \in V^*$.

Remark 1.1.9. Above proposition shows that strong convergence $\Rightarrow$ weak convergence $\Rightarrow$ weak* convergence. In particular,
(i) when V is finite dimensional, then strong convergence is equivalent to weak convergence;
(ii) when V is reflexive Banach space, then weak* convergence and weak convergence are equivalent.

Proposition 1.1.10. Let V be a Banach space and sequence $(v_k)_k \subset V$. Then the following hold:
(1) If $v_k \rightharpoonup v$ weakly in V, then $(v_k)_k$ is bounded in V and $\|v\|_V \leq \liminf_{k\to\infty} \|v_k\|_V$.
(2) If $v_k \rightharpoonup v$ weakly in V and $f_k \to f$ strongly in V^*, then $\langle f_k, v_k\rangle \to \langle f, v\rangle$.

Remark 1.1.11. Above proposition also holds for 'weak* convergence'.

The following theorem shows that we may extract a strongly convergent sequence from a weakly convergent sequence that converges to the same limit.

Theorem 1.1.12 (Mazur). Every weakly convergent sequence in a Banach space has a sequence of convex combinations of its members that converges strongly to the same limit.

Definition 1.1.13 (Weak sequential compactness). A subset K of a Banach space V is said to be weakly sequentially compact (in short, w.s.c.) if for any sequence $(u_k)_k \subset K$, there exists a weakly convergent subsequence with the limit in K, i.e., there exists a numerical sequence (k_l) and $u \in K$ such that $u_{k_l} \rightharpoonup u$ weakly as $l \to \infty$. We can also similarly define weakly* sequential compactness.

Now we will introduce the Banach-Alaoglu theorem and its variants which will be frequently used to show 'weak compactness' in reflexive Banach spaces, this is very crucial in the qualitative study of PDEs.

Theorem 1.1.14 (Banach-Alaoglu). Let V be a normed space. Then any closed bounded subset of V^* is relative compact with respect to the weak* topology.

Remark 1.1.15. Note that above theorem does not assert relative sequential compactness. In this thesis, we care more about the sequential compactness result.

Theorem 1.1.16 (Sequential Version of Banach-Alaoglu). Let V be a separable normed space. Then any bounded sequence in V^* has a weakly* convergent subsequence.

Corollary 1.1.17. Let V be a separable, reflexive Banach space. Then any bounded sequence in V has a weakly convergent subsequence.

Theorem 1.1.18 (Eberlein-Šmulian). Let K be a subset of Banach space V. Then K is relatively weakly compact if and only if K is relatively weakly sequentially compact.

Corollary 1.1.19 (Sequential Version of Banach-Alaoglu 2). Let V be a reflexive Banach space. Then any bounded sequence in V has a weakly convergent subsequence.

Now we mention the fixed point theorem which will be used later in finding solutions of elliptic problems.

Theorem 1.1.20 (Browder). Any continuous mapping from a compact convex set in $\mathbb{R}^n$ to itself has a fixed point.

The following Schauder fixed point is an extension of above Browder's fixed point theorem to infinite dimensional Banach spaces.

Theorem 1.1.21 (Schauder). Any continuous mapping from a compact convex subset of a Banach space to itself has a fixed point.

1.2 Function Spaces

Let Ω be an open subset of Euclidean space $\mathbb{R}^n$, $n \geq 1$. Various type of function spaces will be introduced in this section.
Let $C_b(\Omega)$ be the set of bounded continuous functions $u : \Omega \to \mathbb{R}$. If it is endowed with the norm

$$\|u\|_\infty = \sup_{x\in\Omega} |u(x)|,$$

then $(C_b(\Omega), \|\cdot\|_\infty)$ is a Banach space.
Further, for positive integers $k \geq 1$, let C^k be the set of functions having continuous derivatives up to k-th order, i.e.,

$$C^k(\Omega) := \{u \in C_b(\Omega) : D^\alpha u \in C_b(\Omega) \text{ for all } |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \cdots \alpha_n)$ is a multi-index with $\alpha_i \in \mathbb{N} \cup \{0\}$ and $|\alpha| = \sum\limits_{i=1}^{n} \alpha_i$, and

$$D^\alpha u = \frac{\partial^{\alpha_1+\cdots\alpha_n} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}.$$

If they are endowed with the norm

$$\|u\|_{C^k(\Omega)} := \|u\|_\infty + \sum_{|\alpha|\leq k} \|D^\alpha u\|_\infty,$$

then they become Banach spaces.
Define $C^\infty(\Omega) := \bigcap\limits_{k=1}^{\infty} C^k(\Omega)$ the space of infinitely differentiable functions, which are also called smooth functions.

Definition 1.2.1. The support of a function $u : \Omega \to \mathbb{R}$ is defined as the closure of the set $\{x \in \Omega : u(x) \neq 0\}$.

We denote $C_c^k(\Omega)$ as the space of functions in $C^k(\Omega)$ with compact support in Ω. Similarly, we have the definition of $C_c^\infty(\Omega)$.
Now we will introduce the space of Lebesgue measurable functions. Let Ω be an open subset of $\mathbb{R}^n$ and let μ be the Lebesgue measure on $\mathbb{R}^n$. Let $L^p(\Omega; \mathbb{R})$ be the set of all equivalence classes of measurable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_{L^p(\Omega)} < \infty$ where we say that u and v belong to the same equivalence class if $u = v$ a.e. on Ω, and

$$\|u\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| = \inf_a \{a \in \mathbb{R} : \mu(\{|u(x)| > a\}) = 0\} & \text{for } p = \infty, \end{cases}$$

with Euclidean norm $|\cdot|$ on $\mathbb{R}$. The normed space $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space, also called Lebesgue space.
By apply the Hölder's inequality, we have the following theorem when Ω is bounded:

Theorem 1.2.2 (An embedding theorem for L^p space). Suppose that $\mu(\Omega) = \int_\Omega 1dx < \infty$ and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_{L^p(\Omega)} \leq (\mu(\Omega))^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q(\Omega)} .$$

Hence, we have the following embedding

$$L^q(\Omega) \to L^p(\Omega).$$

The following theorems characterize the duality, reflexivity and separability of L^p space.

Theorem 1.2.3 (Riesz representation theorem for L^p, $1 < p < \infty$). Let $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and let $\phi \in (L^p(\Omega))^*$. Then there exists a unique function $u \in L^{p'}(\Omega)$ such that

$$\langle \phi, f\rangle = \int_\Omega u(x)f(x)dx \quad \forall f \in L^p, \text{ with } \|u\|_{L^{p'}(\Omega)} = \|\phi\|_{(L^p(\Omega))^*} .$$

Remark 1.2.4. Above Theorem 1.2.3 says that any linear continuous functional on L^p $(1 < p < \infty)$ has an integral representation. The mapping $\phi \mapsto u$ is a linear surjective isometry, which allows us to identify $(L^p)^*$ with $L^{p'}$, i.e., $(L^p)^* = L^{p'}$.

Theorem 1.2.5 (Riesz representation theorem for L^1). Let $\phi \in (L^1(\Omega))^*$. Then there exists a unique function $u \in L^\infty(\Omega)$ such that

$$\langle \phi, f\rangle = \int_\Omega u(x)f(x)dx \quad \forall f \in L^1(\Omega), \text{ with } \|u\|_{L^\infty(\Omega)} = \|\phi\|_{(L^1(\Omega))^*} .$$

Remark 1.2.6. Above Theorem 1.2.5 allows us to identify $(L^1)^*$ with L^∞.

Theorem 1.2.7. $L^p(\Omega)$ is separable for $1 \leq p < \infty$, and $L^p(\Omega)$ is reflexive for $1 < p < \infty$.

The following types of convergence from measure theory will be used later.

Definition 1.2.8. Recall that a sequence $u_k : \Omega \to \mathbb{R}$ converges in measure to u if for all $\epsilon > 0$,

$$\lim_{k\to\infty} \mu\left(\{x \in \Omega : |u_k(x) - u(x)| \geq \epsilon\}\right) = 0.$$

Definition 1.2.9. Recall that a sequence u_k converges to u a.e. on Ω if

$$\mu\big(\{x \in \Omega : u(x) \neq \lim_{k\to\infty} u_k(x)\}\big) = 0.$$

Proposition 1.2.10.
(1) Suppose that $\mu(\Omega) < \infty$, then any sequence converging a.e. also converges in measure.
(2) Any sequence converging in measure has a subsequence that converges almost everywhere.
(3) For $0 < p \leq \infty$, any sequence converging in $L^p(\Omega)$ converges in measure.

Theorem 1.2.11 (Monotone Convergence Theorem). Let $(u_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ be an increasing sequence of non-negative functions that converges pointwise to u, then $u \in L^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega u_k dx = \int_\Omega u dx.$$

Lemma 1.2.12 (Fatou's Lemma). Let $(u_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ be a sequence of non-negative functions such that $\liminf_{k\to\infty} \int_\Omega u_k(x)dx < \infty$. Then the function $x \mapsto \liminf_{k\to\infty} u_k(x)$ is integrable and

$$\int_\Omega \liminf_{k\to\infty} u_k(x)dx \leq \liminf_{k\to\infty} \int_\Omega u_k(x)dx.$$

Theorem 1.2.13 (Lebesgue Dominated Convergence Theorem). Let $(u_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ be a sequence that converges a.e. to u and there exists a positive integrable function $v \in L^1(\Omega)$ such that $|u_k(x)| \leq v(x)$. Then $u \in L^1(\Omega)$ with

$$\lim_{k\to\infty} \int_\Omega u_k(x)dx = \int_\Omega u(x)dx \quad \text{and} \quad \lim_{k\to\infty} \int_\Omega |u_k(x) - u(x)|\, dx = 0.$$

Theorem 1.2.14. Let $(u_k)_k$ be a sequence that converges to u in $L^p(\Omega)$ where $1 \leq p \leq \infty$, then there exists a subsequence $(u_{k_l})_l$ of $(u_k)_k$ and a function $f \in L^p(\Omega)$ such that

$$u_{k_l}(x) \to u(x) \text{ a.e. } x \text{ in } \Omega \quad \text{and} \quad |u_{k_l}(x)| \leq f(x) \text{ for all } l \text{ for a.e. } x \text{ in } \Omega.$$

Theorem 1.2.15 (Dunford and Pettis). Let $E \subset L^1(\Omega)$ be a bounded subset, then the following statements are equivalent:
(i) E is relatively weakly compact in $L^1(\Omega)$.
(ii) E is uniformly integrable, i.e.,

$$\forall \epsilon > 0 \ \exists K \in \mathbb{R}^+ : \ \sup_{u\in E} \int_{\{x\in\Omega: |u(x)|\geq K\}} |u(x)|\, dx \leq \epsilon.$$

(iii) E is equi-absolutely-continuous, i.e.,

$$\forall \epsilon > 0 \ \exists \delta > 0 : \ \sup_{u\in E} \sup_{|A|<\delta} \int_A |u(x)|\, dx \leq \epsilon.$$

The following theorem is a generalization of the Dominated Convergence Theorem 1.2.13. It characterizes the convergence of L^p in term of pointwise almost everywhere convergence and a condition related to uniform integrability.

Theorem 1.2.16 (Vitali). Let $\mu(\Omega) < \infty$ and let $(u_k)_{k\in\mathbb{N}} \subset L^1(\Omega)$ be a sequence that converges a.e. to u, and let $1 \leq p < \infty$. Then $u \in L^p(\Omega)$ and $u_k \to u$ in $L^p(\Omega)$ if and only if $(|u_k|^p)_{k\in\mathbb{N}}$ is uniformly integrable.

Theorem 1.2.17 (Fubini). Considering two Lebesgue measurable sets $\Omega_1 \subset \mathbb{R}^{n_1}$ and $\Omega_2 \subset \mathbb{R}^{n_2}$, if $g \in L^1(\Omega_1 \times \Omega_2)$, then the following identity holds and each of the following double integrals does exist:

$$\int_{\Omega_1\times\Omega_2} g(x_1, x_2)dx_1 \otimes dx_2 = \int_{\Omega_1} \left(\int_{\Omega_2} g(x_1, x_2)dx_2 \right) dx_1 = \int_{\Omega_2} \left(\int_{\Omega_1} g(x_1, x_2)dx_1 \right) dx_2.$$

Now we will introduce the concept of Sobolev spaces which plays a crucial role in the weak formulation of PDEs.

1.2.1 Sobolev spaces

In this section, we will introduce the notion of Sobolev spaces, which is named after Russian mathematician *Sergei Sobolev*. Sobolev spaces plays a crucial role in the theory of partial differential equations. In fact, only certain specific type of partial differential equations can be solved directly to find solutions in classical sense. Sobolev spaces allows us to find a solution (what we call as weak solution or generalized solution) in a wider class of functions rather than in the space of continuous functions with the derivatives understood in the classical sense. Even if we are looking for a classical solution, we will first find a weak solution, then we prove the weak solution is sufficiently smooth.
To study Sobolev spaces, we need the notion of weak derivatives. Let Ω be an open subset of $\mathbb{R}^n$ and let $C_c^\infty(\Omega)$ be the space of infinitely differentiable functions $\varphi : \Omega \to \mathbb{R}$ with compact support in Ω. We call $\varphi \in C_c^\infty(\Omega)$ a test function. We denote $L^1_{loc}(\Omega)$ by the space of locally integrable functions which satisfy

$$\int_K |u|\, dx < \infty \text{ for all compact subset } K \text{ of } \Omega.$$

Definition 1.2.18 (Weak derivative). Suppose that $u \in L^1_{loc}(\Omega)$ and α is a multi-index. If there exists $v_\alpha \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi dx \ \text{ for all test functions } \varphi \in C_c^\infty(\Omega),$$

then we say that v_α is the α^{th}-weak partial derivative of u, written as $D^\alpha u = v_\alpha$.

With the notion of weak derivatives, we are ready to define Sobolev spaces.

Definition 1.2.19 (Sobolev spaces). For any positive integer k and $1 \le p \le \infty$, for open subset $\Omega \subset \mathbb{R}^n$ with C^1 boundary, we can define the Sobolev spaces $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le k\},$$

when equipped with norm

$$\|u\|_{k,p} = \begin{cases} \left(\sum_{0\le|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \|u\|_{k,\infty} = \max_{0\le|\alpha|\le k} \|D^\alpha u\|_\infty & \text{if } p = \infty. \end{cases}$$

$(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is a Banach space. We denote the Sobolev space $W_0^{k,p}(\Omega)$ as the space of functions $u \in W^{k,p}(\Omega)$ such that $u = 0$ on $\partial\Omega$. $W_0^{k,p}(\Omega)$ is also the closure of $C_c^\infty(\Omega)$ in $W^{k,p}$ with respect to the norm $\|\cdot\|_{k,p}$.

Theorem 1.2.20. The Sobolev space $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is a Banach space for any p and any positive integer k.

Theorem 1.2.21 (Separability, Reflexivity). For any positive integer k, $W^{k,p}(\Omega)$ is separable for $1 \le p < \infty$, while $W^{k,p}(\Omega)$ is reflexive for $1 < p < \infty$.

Now we introduce some results on Sobolev embeddings, which are useful in the analysis of PDEs.

Theorem 1.2.22 (Sobolev embedding). Let $\Omega \subset \mathbb{R}^n$ be an open boundary subset with C^1 boundary, then we have the following embedding

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$$

holds provided the exponent p^* is defined as

$$p^* := \begin{cases} \frac{np}{n-p} & \text{for } p < n, \\ \text{an arbitrary large real number} & \text{for } p = n, \\ +\infty & \text{for } p > n. \end{cases} \tag{1.1}$$

Remark 1.2.23. If Ω is bounded, then apply Theorem 1.2.2, for $q \le p^*$ where p^* defined as (1.1), we have the following embedding

$$W^{1,p}(\Omega) \subset L^q(\Omega).$$

Theorem 1.2.24 (Rellich-Kondrachov). Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with C^1 boundary, then the following compact embedding holds for p^* defined as (1.1)

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*-\epsilon}(\Omega), \ \epsilon \in (0, p^* - 1].$$

For higher-order Sobolev spaces $W^{k,p}(\Omega)$, apply theorem 1.2.22 k times and we will obtain the following corollary.

Corollary 1.2.25 (Higher order Sobolev embeddings).
(i) If $kp < n$, for any $\epsilon \in (0, \frac{np}{n-kp} - 1]$, there hold the continuous embedding

$$W^{k,p}(\Omega) \subset L^{np/(n-kp)}(\Omega),$$

and the compact embedding

$$W^{k,p}(\Omega) \hookrightarrow L^{np/(n-kp)-\epsilon}(\Omega).$$

(ii) If $kp = n$, for any $q < \infty$, then the following compact embedding holds

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega).$$

(iii) For $kp > n$, one has the compact embedding

$$W^{k,p}(\Omega) \hookrightarrow C(\overline{\Omega}).$$

Theorem 1.2.26 (Kondrachov embedding theorem). On a compact topological space with C^1 boundary, the Kondrachov embedding theorem states that if $k > l$ and $k - \frac{n}{p} > l - \frac{n}{q}$, then the Sobolev embedding

$$W^{k,p}(\Omega) \subset W^{l,q}(\Omega) \text{ is compact.}$$

Theorem 1.2.27 (Poincaré's inequality). Let Ω be an open bounded subset of $\mathbb{R}^n$, $1 \leq p < \infty$ and $u \in W_0^{1,p}(\Omega)$. Then we have the following estimate

$$\|u\|_{L^q(\Omega)} \leq C \|Du\|_{L^p(\Omega)} \quad \text{for each } q \in [1, p^*],$$

where the constant C depends only on p, q, n and Ω.

Remark 1.2.28. The above inequality implies that the norm $\|Du\|_{L^p(\Omega)}$ is equivalent to $\|u\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$ whenever Ω is bounded.

1.3 Bochner Spaces

The space of functions from a bounded interval $I \subset \mathbb{R}$ into in a Banach space V, which was introduced by Bochner, is an essential tool for the study of evolution problems. Throughout the section, we set $I = [0, T]$.
We say that $u : [0, T] \to V$ is simple if it takes only a finite number of values $v_i \in V$ and $A_i = u^{-1}(v_i)$ is Lebesgue measurable. If u is simple function, then we define the Bochner integral of u from 0 to T as the following:

$$\int_0^T u(t)dt = \sum_{i=1}^n \text{meas}(A_i)v_i.$$

We say that $u : [0, T] \to V$ is Bochner measurable if it is the pointwise limit of a sequence $(u_k)_{k \in \mathbb{N}}$ of simple functions in V, i.e., $u_k(t) \to u(t)$ in V for a.e. $t \in [0, T]$. In other words, $u : [0, T] \to V$ is Bochner measurable if $\lim_{k \to \infty} \int_0^T \|u_k(t) - u(t)\|_V \, dt = 0$, so we can define the Bochner integral of u from 0 to T as the following:

$$\int_0^T u(t)dt := \lim_{k \to \infty} \int_0^T u_k(t)dt,$$

this limit exists and is independent of the choice of the sequence $(u_k)_k$.
We say that $u : [0, T] \to V$ is absolutely continuous if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{n=1}^K \|u(t_n) - u(s_n)\|_V \leq \epsilon,$$

whenever $t_{n-1} \leq s_n \leq t_n \leq T$ for $n = 1, \cdots, K \in \mathbb{N}$, $t_0 = 0$ and $\sum_{n=1}^K (t_n - s_n) \leq \delta$.

Theorem 1.3.1 (Pettis). Let V be a separable Banach space. Then $u : [0, T] \to V$ is said to be Bochner measurable if and only if it is weakly measurable in the sense that $t \mapsto \langle v^*, u(t) \rangle$ is Lebesgue measurable for any $v^* \in V^*$.

Theorem 1.3.2 (Bochner). Let V be a separable Banach space. Then a Bochner measurable function $u : [0, T] \to V$ is Bochner integrable if and only if $t \mapsto \|u(t)\|_V$ is Lebesgue measurable.

For $1 \leq p < \infty$, the Bochner space $L^p(0, T; V)$ is the linear space (of equivalence classes) of Bochner integrable functions $u : [0, T] \to V$ satisfying $\int_0^T \|u(t)\|_V^p \, dt < \infty$. This space is a Banach space with the norm

$$\|u\|_{L^p(0,T;V)} := \left(\int_0^T \|u(t)\|_V^p \, dt \right)^{\frac{1}{p}}.$$

For $p = \infty$, we have the norm

$$\|u\|_{L^\infty(0,T;V)} := \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_V.$$

The following proposition shows that the density result on Bochner integrable functions.

Proposition 1.3.3. If $1 \le p < \infty$, then the set

$$\left\{ v : [0,T] \to V;\ \exists k \in \mathbb{N}, \forall 1 \le n \le 2^k : v|_{((n-1)\tau, n\tau)} \text{ is constant},\ \tau = 2^{-k}T \right\} \text{ is dense in } L^p(0,T;V).$$

In particular, if $p \in [1,\infty)$ and V is separable, then $L^p(0,T;V)$ is separable too.

The following proposition shows the result on the dual space of Bochner space $L^p(0,T;V)$.

Proposition 1.3.4. If $p \in [1,\infty)$, the dual space to $L^p(0,T;V)$ always contains $L^{p'}(0,T;V^*)$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. The equality holds when V is separable, and the duality pair can be given by the following formula:

$$\langle f, u \rangle_{L^{p'}(0,T;V^*) \times L^p(0,T;V)} = \int_0^T \langle f(t), u(t) \rangle_{V^* \times V}\, dt.$$

Thus, if $p \in (1,\infty)$ and V is reflexive and separable, then $L^p(0,T;V)$ is reflexive.

Theorem 1.3.5. If V is reflexive and $u : [0,T] \to V$ is absolutely continuous, then u is strongly differentiable a.e. on $[0,T]$ and

$$u(t) = u(0) + \int_0^t u'(s) ds,\ \ u' \in L^1(0,T;V).$$

Proposition 1.3.6 ([89, Proposition 23.9]). Let V be a Banach space and let $1 < p, p' < \infty$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $0 \le t \le T < \infty$. Then we have the following:
(a) if $u \in L^p(0,T;V)$, then

$$\left\langle v^*, \int_0^t u(\tau) d\tau \right\rangle = \int_0^t \langle v^*, u(\tau) \rangle\, d\tau \text{ for all } v^* \in V^*.$$

(b) if $u \in L^{p'}(0,T;V^*)$, then

$$\left\langle \int_0^t u(\tau) d\tau, v \right\rangle = \int_0^t \langle u(\tau), v \rangle\, d\tau \text{ for all } v \in V.$$

(c) if $u_n \to u$ in $L^p(0,T;V)$ as $n \to \infty$, then

$$\int_0^t u_n(\tau) d\tau \to \int_0^t u(\tau) d\tau \text{ in } V \text{ as } n \to \infty.$$

We need the following concept of evolution triple and Bochner-Sobolev space for the study of evolution problems and initial boundary value problems.

Evolution Triple and Bochner-Sobolev Space $W_p^1(0,T;V,H)$

Let H be a real Hilbert space, from the Riesz representation theorem, we can identify H with its own dual H^*. Let V be a subspace of H such that the embedding $V \subset H$ is continuous and V is a dense subset. Let $i : V \to H$ be the canonical embedding (inclusion map), and its adjoint operator $i^* : H \equiv H^* \to V^*$ defined by the identity:

$$\langle i^* u, v \rangle_{V^* \times V} = \langle u, iv \rangle_{H^* \times H} \quad \text{for any } u \in H^* \equiv H,\ v \in V.$$

It is easy to show that i^* is continuous and linear. i^* is also injective, i.e., suppose $i^* u_1 = i^* u_2$ in V^*, then $\langle i^* u_1 - i^* u_2, v \rangle_{V^* \times V} = 0$ for all $v \in V$, by the definition of adjoint and linearity of i^*, this is equivalent to $\langle u_1 - u_2, iv \rangle_{H^* \times H} = 0$ for all $v \in V$. Hence $u_1 = u_2$ in H.
So we can identify $i^* u$ with u if $u \in H$. Therefore, we may consider the duality pairing between V^* and V as a continuous extension of the inner product on H, denoted by $(\cdot,\cdot)$, i.e., for $u \in H$ and $v \in V$, we have

$$(u, v) \overset{\text{I}}{=} \langle u, v \rangle_{H^* \times H} \overset{\text{II}}{=} \langle u, iv \rangle_{H^* \times H} \overset{\text{III}}{=} \langle i^* u, v \rangle_{V^* \times V} \overset{\text{IV}}{=} \langle u, v \rangle_{V^* \times V},$$

where the first equality I follows from the identification of H with H^*, the second equality II follows from the definition of inclusion map i, the third equality III follows from the definition of i^*, and the last equality IV follows from the identification of $i^* u$ with u.

Definition 1.3.7 (Evolution triple). Let V be a real separable reflexive Banach space, and let H be a real separable Hilbert space identified with its dual H^*. V is embedded continuously and densely into H. Then the triple $V \subset H \subset V^*$ is called an evolution triple or a Gelfand triple.

Remark 1.3.8. Indeed, it follows that H is also continuously and densely embedded into V^*.

Example 1.3.9. Let $V = W_0^{1,p}(\Omega),\ H = L^2(\Omega),\ V^* = W^{-1,p'}(\Omega)$ for $p \geq 2$, then $V \subset H \subset V^*$ is an evolution triple.

Definition 1.3.10 (Generalized derivative). Let $V \subset H \subset V^*$ be an evolution triple, $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $u \in L^p(0,T;V)$, if there exists $w \in L^{p'}(0,T;V^*)$ such that

$$\int_0^T u(t)\varphi^{'}(t)dt = -\int_0^T w(t)\varphi(t)dt \ \text{ for all } \varphi \in C_c^\infty(0,T).$$

Then w is called the generalized derivative of u, and it is denoted by $u^{'}$ or u_t.

Remark 1.3.11. In the above equality, $u(t) \in V$ is considered as an element of V^*, and $w \in L^{p'}(0,T;V^*)$ is unique up to a set of measure 0.

The following proposition shows the characterization of generalized derivatives.

Proposition 1.3.12. Let $V \subset H \subset V^*$ be an evolution triple, $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $u \in L^p(0,T;V)$ possesses a generalized derivative $u^{'}$ if and only if there exists a function $w \in L^{p'}(0,T;V^*)$ such that

$$\int_0^T (u(t),v)_H\varphi^{'}(t)dt = -\int_0^T \varphi(t)\left\langle w(t),v\right\rangle_{V^*\times V} dt \ \text{ for all } \varphi \in C_c^\infty(0,T) \text{ and } v \in V.$$

Proof. Using (b) in Proposition 1.3.6, bilinearity of duality pair and $(u(t),v)_H = \langle u(t),v\rangle_{V^*\times V}$, the result follows easily. ∎

With the definition of generalized derivatives, we are able to introduce the following Bochner-Sobolev space $W_p^1(0,T;V,H)$.

Definition 1.3.13. For $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and let $V \subset H \subset V^*$ be an evolution triple, we define the following Bochner-Sobolev space

$$W_p^1(0,T;V,H) := \{u \in L^p(0,T;V) : u^{'} \in L^{p'}(0,T;V^*)\},$$

with the norm

$$\|u\|_{W_p^1(0,T;V,H)} := \|u\|_{L^p(0,T;V)} + \left\|u^{'}\right\|_{L^{p'}(0,T;V^*)}.$$

We state the following theorem which is crucial in the study of evolution problems.

Theorem 1.3.14 ([89, Proposition 23.23]). Let $V \subset H \subset V^*$ be an evolution triple and let $0 < T < \infty$. Let $p,\ p'$ be such that $1 < p, p' < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

(a) $W_p^1(0,T;V,H)$ is a Banach space with the norm given above.

(b) $W_p^1(0,T;V,H)$ is continuously embedded into $C(0,T;H)$ in the sense for every $u \in W_p^1(0,T;V,H)$, there is a uniquely determined $\tilde{u} \in C(0,T;H)$ such that $u(t) = \tilde{u}(t)$ for a.e. $t \in [0,T]$ and

$$\|\tilde{u}\|_{C(0,T;H)} \leq \text{const}\, \|u\|_{W_p^1(0,T;V,H)}.$$

(c) The integration by parts formula holds for any $u, v \in W_p^1(0,T;V,H)$, and $0 \leq s < t \leq T$

$$(u(t),v(t))_H - (u(s),v(s))_H = \int_s^t \left\langle u^{'}(\tau),v(\tau)\right\rangle_{V^*\times V} + \left\langle v^{'}(\tau),u(\tau)\right\rangle_{V^*\times V} d\tau,$$

where the values $u(t), v(t), u(s), v(s)$ are the values of continuous functions $u, v : [0,T] \to H$ in sense of (b).

Remark 1.3.15. In the case where $v = u \in W_p^1(0,T;V,H)$, then we obtain

$$\|u(t)\|_H^2 - \|u(s)\|_H^2 = 2\int_s^t \left\langle u^{'}(\tau),u(\tau)\right\rangle_{V^*\times V} d\tau.$$

Lions-Aubin Theorem

The following theorem was introduced by Jacques Louis Lions and Jean Pierre Aubin in 1963 (see [3]). It provides compactness criterion in the study of evolution problems. Note that

$$W^{1,p,q}(0,T;V,W) = \{u : u \in L^p(0,T;V) \text{ and } u' \in L^q(0,T;W)\},$$

with the norm

$$\|u\|_{W^{1,p,q}} = \|u\|_{L^p(0,T;V)} + \|u'\|_{L^q(0,T;W)},$$

is also a Banach space. It is reflexive if V and W are reflexive Banach spaces.

Theorem 1.3.16 (Lions-Aubin Theorem [3]). Let $V_0,\ V,\ V_1$ be Banach spaces with $V_0 \subset V \subset V_1$, assume $V_0 \subset V$ is compact and $V \subset V_1$ is continuous. Let $1 < p, q < \infty$ and let $V_0,\ V_1$ be reflexive Banach spaces, then the inclusion $W^{1,p,q}(0,T;V_0,V_1) \hookrightarrow L^p(0,T;V)$ is compact.
Before we prove the theorem, we need the following lemma.

Lemma 1.3.17. Under the assumptions of above theorem, for any $\delta > 0$, there exists C_δ such that

$$\|v\|_V \le \delta \|v\|_{V_0} + C_\delta \|v\|_{V_1}, \quad \text{for any } v \in V_0.$$

proof of the lemma. Suppose, to the contrary, that there exists a $\delta > 0$ and a sequence $(v_n)_n$ (without loss of generality) satisfying $\|v_n\|_{V_0} = 1$ and

$$\|v_n\|_V > \delta \|v_n\|_{V_0} + n \|v_n\|_{V_1}, \ n \ge 1. \tag{1.2}$$

Then $\|v_n\|_V = \frac{\|v_n\|_V}{\|v_n\|_{V_0}} \le$ const., so v_n is bounded in V. Hence we get from (1.2) that $v_n \to 0$ in V_1. Since $\|v_n\|_{V_0} = 1$ and the embedding $V_0 \subset V$ is compact, there exists a subsequence $(v_{n_m})_m$ of $(v_n)_n$ such that $(v_{n_m})_m$ is convergent in V, since $V \subset V_1$ is continuous, then $v_{n_m} \to 0$ in V as $m \to \infty$, which contradicts to (1.2). ∎

proof of the theorem. Let $(u_n)_n$ be a bounded sequence in $W^{1,p,q}(0,T;V_0,V_1)$, since $W^{1,p,q}$ is reflexive, then there exists a subsequence, which is again denoted by $(u_n)_n$, such that

$$u_n \rightharpoonup u \text{ in } L^p(0,T;V_0) \text{ and } u_n' \rightharpoonup u' \text{ in } L^q(0,T;V_1).$$

We need to show that $u_n \to u$ in $L^p(0,T;V)$. Without loss of generality, we can assume $u = 0$.
We will first show that $u_n(t) \to 0$ strongly in V_1. For any $t \in [0,T]$,

$$u_n(t) = u_n(t+s) - \int_t^{t+s} u_n'(\sigma)d\sigma.$$

Integrate s from 0 to δ and multiply by $\frac{1}{\delta}$, we obtain

$$u_n(t) = \underbrace{\frac{1}{\delta}\int_0^\delta u_n(t+s)ds}_{a_n} \underbrace{-\frac{1}{\delta}\int_0^\delta \int_t^{t+s} u_n'(\sigma)d\sigma ds}_{b_n}.$$

a_n :
Note that $T : u_n \mapsto \frac{1}{\delta}\int_0^\delta u_n(t+s)ds$ is a linear continuous mapping from $L^p(0,T;V_0)$ to V_0 and $u_n \rightharpoonup 0$ in $L^p(0,T;V_0)$, so $T(u_n) = a_n \rightharpoonup 0 = T(0)$ in V_0. Since $V_0 \subset V$ is compact, then we obtain $a_n \to 0$ in V. From $V \subset V_1$, we get that $a_n \to 0$ in V_1.
b_n :
From Fubini's theorem, we get

$$\int_0^\delta \int_t^{t+s} u_n'(\sigma)d\sigma ds = \int_t^{t+\delta}\int_{\sigma-t}^{\delta} u_n'(\sigma)dsd\sigma = \int_t^{t+\delta}(\delta + t - \sigma)u'(\sigma)d\sigma$$

Observe that $|\delta + t - \sigma| \le \delta$ for $\sigma \in (t, t+\delta)$, so we get

$$\begin{aligned}\|b_n\|_{V_1} &\le \frac{1}{\delta}\int_t^{t+\delta} |\delta + t - \sigma| \, \|u'(\sigma)\|_{V_1} d\sigma \le \frac{1}{\delta}\int_t^{t+\delta} \delta \|u'(\sigma)\|_{V_1} d\sigma \\ &\le \left(\int_t^{t+\delta} \|u_n'(\sigma)\|_{V_1}^q d\sigma\right)^{\frac{1}{q}} \left(\int_t^{t+\delta} 1^{q'} d\sigma\right)^{\frac{1}{q'}} \le \|u_n'\|_{L^q(0,T;V_1)} (\delta)^{\frac{1}{q'}=1-\frac{1}{q}} \le k\delta^{1-\frac{1}{q}}\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'}=1$ and a constant $k>0$, the last inequality follows from the boundedness of u_n' in $L^q(0,T;V_1)$. So for $\epsilon>0$, we can choose $\delta>0$ so small such that $\|b_n\|_{V_1}<\epsilon$. Hence, $u_n(t)\to 0$ in V_1 for all t.
From the embedding $W^{1,p,q}(0,T;V_0,V_1)\subset C(0,T;V_1)$, which shows that $(\|u_n\|_{C([0,T];V_1)})_n$ is bounded, so $\|u_n(t)\|_{V_1}\le C$ for all n and all $t\in[0,T]$. Hence, by Vitali convergence Theorem 1.2.16, we have

$$u_n\to 0 \text{ in } L^p(0,T;V_1).$$

From Lemma 1.3.17, it follows that for any $\alpha>0$,

$$\|u_n\|_{L^p(0,T;V)}\le\alpha\|u_n\|_{L^p(0,T;V_0)}+C_\alpha\|u_n\|_{L^p(0,T;V_1)}.$$

Since u_n is bounded in $L^p(0,T;V_0)$ and $u_n\to 0$ in $L^p(0,T;V_1)$, we pass the limit as $\alpha\to 0^+$ and obtain

$$u_n\to 0 \text{ in } L^p(0,T;V).$$

■

Remark 1.3.18. J.Simon [76] improved the theorem without the *reflexive* assumption on Banach space V_0 and V_1.

1.4 Ordinary Differential Equations

In this section, we will introduce Carathéodory existence theorem which is crucial in the study of evolution problems. Set $I=[t_0,t_0+a]$, $K=\{x\in\mathbb{R}^n:|x-x_0|\le b\}$ for some $a,b>0$. We will often see the initial-value problem for the system of n ordinary differential equations in the form:

$$\begin{cases}\dfrac{du}{dt}=f(t,u(t)) \text{ for a.e. } t\in I,\\ u(t_0)=x_0,\end{cases}\tag{1.3}$$

where $f:I\times K\to\mathbb{R}^n$ is a Carathéodory mapping, i.e., for all $j=1,2,\cdots n$, $t\mapsto f_j(t,x)$ is measurable on I for all $x\in K$ and $x\mapsto f_j(t,x)$ is continuous on K for a.e. $t\in I$. By a solution on time interval $[t_0,t_1]$, we mean an absolutely continuous mapping $u:[t_0,t_1]\to\mathbb{R}^n$ such that (1.3) holds for a.e. $t\in[t_0,t_1]$.

Theorem 1.4.1 (Carathéodory, [21, p.43]). Assume $f:I\times K\to\mathbb{R}^n$ is a Carathéodory mapping and there exists a Lebesgue integrable function $M(t)\in L^1(I)$ such that

$$|f_j(t,x)|\le M(t) \text{ for all } j=1,\cdots,n. \text{ and for all } x\in K, \text{ a.e. } t\in I.$$

Then there exists an absolutely continuous function satisfying the initial-value problem

$$\xi_j'(t)=f_j'(t,\xi(t)) \text{ a.e. in a neighbourhood of } t_0 \text{ and } \xi(t_0)=x_0,$$

where $\xi(t)=(\xi_1(t),\xi_2(t),\cdots,\xi_n(t))$.

Chapter 2

Various types of Monotonicity for Operators

Since the pioneering work of G.Minty in the '60s in connection with in the study of electronic networks (see [61]), the theory of monotone operators and its variants had played a crucial role in the study of variational problems. We can mention among others the work of F.E.Browder [13, 14] for PDEs which together with the contributions [62, 63] by G. Minty represent the first crucial works in the theory of monotonicity. Monotone operators (see Definiton 2.1.2) also have other applications such as algorithm, subgradients, etc.[70, 71]

Also, several types of pseudomonotonicity were introduced in the literatures ([11, 42, 15, 25, 36]). H.Brézis [11] introduced the B-pseudomonotonicity (see Definition 2.2.1) which appeared mainly on the reflexive Banach space and is used for solving boundary value problems and variational inequalities, etc., (see e.g., [78, 75, 90, 17, 74, 44]). On the other hand, S.Karamardian [42] introduced a type of pseudomonotonicity, which is mainly applied in variational inequalities (see [86, 85, 22, 23]). F.E.Browder [15] defined the bounded pseudomonotone operators in a different way from H.Brézis, and it can be shown that under the boundedness assumption, these two definitions are equivalent (see Remark 2.2.4). In the papers ([36, 25]), another type of pseudomonotonicity called C-pseudomonotonicity was introduced, which is a weaker notion of B-pseudomonotonicity, this notion is essentially applied in variational inequalities. The existence theorem for variational inequalities with C-pseudomonotone operators is a generalization of many existence theorems for variational inequalities (see [36, Theorem 15]).

The notion of locally monotonicity was introduced in the study of evolution equations as a generalisation of classical monotonicity to include a wider class of functions (see [58, 57]).

One type of quasimonotonicity is defined as a weaker notion of K-pseudomonotonicity and is used in [4] to establish existence results for some variational inequalities. For more details on this type of quasimonotonicity, we reference these literatures (see e.g., [50, 30, 31]).

In this thesis, we are interested in another type of quasimonotonicity so called strict p-quasimonontonicity. This notion was introduced by N.Hungerbühler to study initial and boundary value problems for quasilinear elliptic and parabolic system. The notion is a weak, integrated version of monotonicity compared to the classical pointwise monotonicity. The definition of strictly p-quasimonotonicity is phrased in term of Young measures (see section 2.4).

In this work, our interest is to apply the notions of monotonicity, local monotonicity, B-pseudomonotonicity and strict p-quasimonotonicity in the study of abstract equations, elliptic boundary value problems, parabolic initial boundary value problems and variational inequalities.

2.1 Monotonicity

Throughout the chapter, let V be a reflexive Banach space and V^* be its dual space with the norm $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ respectively. Let A be an operator which maps V to V^*.

Definition 2.1.1. The operator A is said to be bounded if it maps any bounded subset of V to a bounded subset of V^*.

Definition 2.1.2 (Monotonicity).
(a) A is said to be monotone if $\langle A(u) - A(v), u - v\rangle \geq 0$ for all $u, v \in V$.
(b) A is said to be strictly monotone if for $u \neq v$, then $\langle A(u) - A(v), u - v\rangle > 0$.
(c) A is said to be strongly monotone if there exists a constant $C > 0$ such that for all $u, v \in V$,

$$\langle A(u) - A(v), u - v\rangle \geq C \|u - v\|_V^2 .$$

(d) A is said to be uniformly monotone if

$$\langle A(u) - A(v), u - v\rangle \geq \zeta(\|u - v\|_V) \cdot \|u - v\|_V ,$$

for a strictly increasing function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ and $\lim_{t\to\infty} \zeta(t) = \infty$.

Remark 2.1.3. From the definitions above, it easily follows that

$$\text{strongly monotone} \Rightarrow \text{uniformly monotone} \Rightarrow \text{strictly monotone} \Rightarrow \text{monotone.}$$

Example 2.1.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ by $f(u) = |u|^{p-2} u$, then
(1) if $p > 1$, then f is strictly monotone;
(2) if $p \geq 2$, then f is uniformly monotone;
(3) if $p = 2$, then f is strongly monotone.

The following definition of coercivity is essential in the study of variational problems.

Definition 2.1.5 (Coercivity). The operator A is said to be coercive if

$$\lim_{\|u\|\to\infty} \frac{\langle A(u), u\rangle}{\|u\|} = \infty.$$

The operator A is said to be weakly coercive if

$$\lim_{\|u\|\to\infty} \langle A(u), u\rangle = \infty.$$

Remark 2.1.6. The operator A is uniformly monotone $\Rightarrow$ A is coercive.

2.2 Pseudomonotonicity

In this section, we will introduce the definitions of various type of pseudomonotone operators and some comparison results.

For pseudomonotonicity let us first start from the notion of pseudomonotone operators in the sense of Brézis.

Definition 2.2.1 (B-pseudomonotone). The operator $A : V \to V^*$ is B-pseudomonotone if $u_k \rightharpoonup u$ in V and $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0$, then

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \langle A(u), u - v\rangle \text{ for all } v \in V. \tag{2.1}$$

The following proposition follows from the above definition.

Proposition 2.2.2. If $A : V \to V^*$ is B-pseudomonotone, then $u_k \rightharpoonup u$ in V weakly implies that

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \geq 0.$$

Proof. Suppose that the contrary holds, i.e., $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle < 0$, then we may find a subsequence $(u_{k_j})_j$ of $(u_k)_k$ such that $u_{k_j} \rightharpoonup u$ and

$$\lim_{j\to\infty} \left\langle A(u_{k_j}), u_{k_j} - u\right\rangle < 0. \tag{2.2}$$

Then from B-pseudomonotonicity of A, we get

$$\liminf_{j\to\infty} \left\langle A(u_{k_j}), u_{k_j} - v\right\rangle \geq \langle A(u), u - v\rangle \text{ for all } v \in V.$$

In particular, take $v = u$ and we obtain

$$\liminf_{j\to\infty} \left\langle A(u_{k_j}), u_{k_j} - u\right\rangle \geq 0,$$

which contradicts to (2.2) and so the result follows. ■

F.E.Browder [15] introduced the following definition of bounded pseudomonotone operators.

Definition 2.2.3. The bounded operator $A : V \to V^*$ is pseudomonotone (in the sense of Browder) if $u_k \rightharpoonup u$ in V and $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0$ imply

$$A(u_k) \rightharpoonup A(u) \text{ in } V^* \text{ and } \lim_{k\to\infty} \langle A(u_k), u_k\rangle = \langle A(u), u\rangle\,. \tag{2.3}$$

Remark 2.2.4. The above definition of pseudomonotonicity in sense of Browder is equivalent to the bounded B-pseudomonotonicity. i.e., Assume $u_k \rightharpoonup u$ in V and $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0$, then

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \langle A(u), u - v\rangle \text{ for all } v \in V \Leftrightarrow A(u_k) \rightharpoonup A(u) \text{ in } V^* \text{ and } \lim_{k\to\infty} \langle A(u_k), u_k\rangle = \langle A(u), u\rangle\,.$$

Proof. ($\Rightarrow$) From the assumptions, we can easily get that A is bounded and hence there exists a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, such that $A(u_k) \rightharpoonup f$ in V^*. This implies that

$$0 \geq \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle = \limsup_{k\to\infty} \langle A(u_k), u_k\rangle - \langle f, u\rangle\,.$$

From the assumptions and the above inequality, we obtain that for all $v \in V$,

$$\langle A(u), u - v\rangle \leq \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \leq \limsup_{k\to\infty} \langle A(u_k), u_k\rangle - \langle f, v\rangle \leq \langle f, u - v\rangle\,. \tag{2.4}$$

Hence, $A(u) = f$. From the uniqueness of limit $A(u)$, again we can use the Cantor's trick to show that $A(u_k) \rightharpoonup f = A(u)$ in V^* for the whole sequence. i.e.,
Suppose that the contrary holds, i.e., there exists $\epsilon > 0$, a subsequence u_l of u_k and $v \in V$ such that

$$|\langle A(u_l) - A(u), v\rangle| \geq \epsilon \quad \text{for all } l. \tag{2.5}$$

Since $A(u_l)$ is bounded, apply the above argument again, we can obtain a subsequence of u_l, again denoted by u_l, such that $A(u_l) \rightharpoonup g = A(u)$ in V^*, which contradicts to (2.5). Hence the whole sequence $A(u_k) \rightharpoonup A(u)$ in V^*.
Setting $v = 0$ in (2.4), we get

$$\langle A(u), u\rangle \leq \liminf_{k\to\infty} \langle A(u_k), u_k\rangle \leq \limsup_{k\to\infty} \langle A(u_k), u_k\rangle \leq \langle A(u), u\rangle\,,$$

and hence $\lim_{k\to\infty} \langle A(u_k), u_k\rangle = \langle A(u), u\rangle$.
($\Leftarrow$) The other direction is trivial. For all $v \in V$, we have

$$\begin{aligned}\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \liminf_{k\to\infty} \langle A(u_k), u_k\rangle - \limsup_{k\to\infty} \langle A(u_k), v\rangle &= \lim_{k\to\infty} \langle A(u_k), u_k\rangle - \lim_{k\to\infty} \langle A(u_k), v\rangle \\ &= \langle A(u), u - v\rangle\,.\end{aligned}$$

■

Remark 2.2.5. The pseudomonotone operators we will use throughout the thesis are bounded B-pseudomonotone operators (or equivalently bounded pseudomonotone operators in sense of Browder), from now on, we will call them as pseudomonotone operators.

We list below some useful properties of pseudomonotone operators. Quite often, the study of PDEs is carried through an operator that is written as sum of two pseudomonotone operators. The natural question that arises is to find out whether that sum is also pseudomonotone.

Lemma 2.2.6. The sum of pseudomonotone operators is still pseudomonotone. i.e., if A_1 and A_2 are pseudomonotone, the $u \mapsto [A_1 + A_2](u)$ is pseudomonotone.

Proof. The boundedness of $[A_1 + A_2]$ is trivial. Let $(u_k)_k \subset V$ be such that $u_k \rightharpoonup u$ in V and

$$\limsup_{k\to\infty} \langle [A_1 + A_2](u), u_k - u\rangle \le 0,$$

we need to show that

$$\liminf_{k\to\infty} \langle [A_1 + A_2](u_k), u_k - v\rangle \ge \langle [A_1 + A_2](u), u - v\rangle \quad \text{for any } v \in V.$$

To this aim, we will first prove that

$$\limsup_{k\to\infty} \langle A_i(u_k), u_k - u\rangle \le 0 \quad \text{for } i = 1, 2. \tag{2.6}$$

Without loss of generality, suppose that the contrary holds, i.e., $\limsup_{k\to\infty} \langle A_1(u_k), u_k - u\rangle = \epsilon > 0$, then we may extract a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, such that $\lim_{k\to\infty} \langle A_1(u_k), u_k - u\rangle = \epsilon$. By our assumption, this implies for such a subsequence $(u_k)_k$,

$$\limsup_{k\to\infty} \langle A_2(u_k), u_k - u\rangle \le -\epsilon < 0 \tag{2.7}$$

From the pseudomonotonicity of A_2 and (2.7), we obtain that $\liminf_{k\to\infty} \langle A_2(u_k), u_k - v\rangle \ge \langle A_2(u), u - v\rangle$ for all $v \in V$. In particular, take $v = u$ and we obtain that $\liminf_{k\to\infty} \langle A_2(u_k), u_k - u\rangle \ge 0$, which contradicts to (2.7). Then (2.6) holds and by the pseudomonotonicity of A_1 and A_2, we obtain that

$$\begin{aligned}\liminf_{k\to\infty} \langle [A_1 + A_2](u_k), u_k - v\rangle &\ge \liminf_{k\to\infty} \langle A_1(u_k), u_k - v\rangle + \liminf_{k\to\infty} \langle A_2(u_k), u_k - v\rangle \\ &\ge \langle A_1(u), u - v\rangle + \langle A_2(u), u - v\rangle = \langle [A_1 + A_2](u), u - v\rangle \quad \text{for any } v \in V.\end{aligned}$$

∎

The following lemma shows that a shift of any pseudomonotone operator is still pseudomonotone. This lemma will be used in the study of non-homogeneous boundary value problems.

Lemma 2.2.7. A pseudomonotone operator remains pseudomonotone under a shift. i.e., if A is pseudomonotone mapping, then $u \mapsto A(u + w)$ is pseudomonotone for any fixed $w \in V$.

Proof. Boundedness of the mapping $u \mapsto A(u + w)$ is trivial. Let $(u_k)_k \subset V$ be such that $u_k \rightharpoonup u$ in V and $\limsup_{k\to\infty} \langle A(u_k + w), u_k - u\rangle \le 0$ for any fixed $w \in V$, then it follows easily that $u_k + w \rightharpoonup u + w$ in V and $\limsup_{k\to\infty} \langle A(u_k + w), (u_k + w) - (u + w)\rangle \le 0$. By pseudomonotonicity of A, we obtain that

$$\liminf_{k\to\infty} \langle A(u_k + w), (u_k + w) - (v + w)\rangle \ge \langle A(u + w), (u + w) - (v + w)\rangle \quad \text{for any } v \in V,$$

$$\Leftrightarrow \liminf_{k\to\infty} \langle A(u_k + w), u_k - v\rangle \ge \langle A(u + w), u - v\rangle \quad \text{for all } v \in V.$$

∎

Quite often, the pseudomonotonicity is obtained as a combination of the classical monotonicity with some of the following notions of continuity.

Definition 2.2.8 (Various Continuity Modes).
(a) The operator $A : V \to V^*$ is said to be hemicontinuous if the function $\lambda \mapsto \langle A(u_1 + \lambda u_2), v\rangle$ is continuous for any $u_1, u_2, v \in V$ and $\lambda \in \mathbb{R}$.
(b) The operator $A : V \to V^*$ is said to be demicontinuous if for any $v \in V$, the functional $u \mapsto \langle A(u), v\rangle$ is continuous. i.e., if $u_n \to u$ in V, then $A(u_n) \rightharpoonup A(u)$ in V^*.
(c) The operator $A : V \to V^*$ is said to be weakly continuous if for any $v \in V$, the functional $u \mapsto \langle A(u), v\rangle$ is weakly continuous. i.e., if $u_n \rightharpoonup u$ in V, then $A(u_n) \rightharpoonup A(u)$ in V^*.
(d) The operator $A : V \to V^*$ is said to be strongly (or totally) continuous if $u_k \rightharpoonup u$ in V, then $A(u_k) \to A(u)$ in V^*.

The following propositions establish the connection between the notions of continuity defined above and pseudomonotonicity.

Proposition 2.2.9. If a bounded operator $A : V \to V^*$ is pseudomonotone, then A is demicontinuous.

Proof. Assume $u_k \to u$ strongly in V, it follows that $\|u_k\|_V$ is bounded in V for all k and therefore $\|A(u_k)\|_{V^*}$ is a bounded sequence in the reflexive Banach space V^*. Hence we can extract a subsequence of u_k, which is denoted again by u_k, such that $A(u_k) \rightharpoonup f$ in V^* and

$$\begin{aligned}\limsup_{k\to\infty} \langle A(u_k), u_k\rangle - \langle f, u\rangle &= \limsup_{k\to\infty} \langle A(u_k), u_k\rangle - \lim_{k\to\infty} \langle A(u_k), u\rangle \\ &= \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \le \limsup_{k\to\infty} \|A(u_k)\|_{V^*} \|u_k - u\|_V = 0.\end{aligned}$$

From pseudomonotonicity of A, it follows from the above inequality that

$$\langle A(u), u - v\rangle \le \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \le \limsup_{k\to\infty} \langle A(u_k), u_k - v\rangle \le \langle f, u - v\rangle \text{ for all } v \in V.$$

Hence, $A(u) = f$ and

$$A(u_k) \rightharpoonup A(u) \text{ in } V^* \text{ (up to a subsequence of } (u_k)_k). \tag{2.8}$$

By using Cantor's trick, we can prove that above (2.8) also holds for the whole sequence $(u_k)_k$. Hence, A is demicontinuous. ■

Proposition 2.2.10. If a bounded operator $A : V \to V^*$ is strongly continuous, then A is pseudomonotone.

Proof. Let $(u_k)_k \subset V$ be such that $u_k \rightharpoonup u$ in V and $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \le 0$, strong continuity of A implies that $A(u_k) \to A(u)$ strongly in V^*. Hence, for any $v \in V$,

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle = \lim_{k\to\infty} \langle A(u_k), u_k - v\rangle = \langle A(u), u - v\rangle .$$

■

The following proposition establishes the connection between monotone operators and pseudomonotone operators.

Proposition 2.2.11. If an operator $A : V \to V^*$ is bounded, hemicontinuous and monotone, then A is pseudomonotone.

Proof. Let $(u_k)_k \subset V$ be such that

$$u_k \rightharpoonup u \text{ in } V \text{ and } \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \le 0, \tag{2.9}$$

we need to show that

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \ge \langle A(u), u - v\rangle \text{ for any } v \in V. \tag{2.10}$$

Since A is monotone, we have $\langle A(u_k) - A(u), u_k - u\rangle \ge 0$ for all $k \in \mathbb{N}$, which implies

$$\liminf_{k\to\infty} \langle A(u_k), u_k - u\rangle \ge \liminf_{k\to\infty} \langle A(u), u_k - u\rangle = 0,$$

together with (2.9), we obtain that

$$\lim_{k\to\infty} \langle A(u_k), u_k - u\rangle = 0. \tag{2.11}$$

For any $v \in V$, set $w = (1-\lambda)u + \lambda v,\ \lambda \in (0,1]$, we obtain by monotonicity of A that for all $k \in \mathbb{N}$,

$$\begin{aligned}\langle A(u_k) - A(w), u_k - w\rangle = (1-\lambda)\langle A(u_k), u_k - u\rangle + \lambda \langle A(u_k), u_k - v\rangle \\ - (1-\lambda)\langle A(w), u_k - u\rangle - \lambda \langle A(w), u_k - v\rangle \geq 0.\end{aligned}$$

Taking liminf in above inequality and using (2.9) and (2.11), we get

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \liminf_{k\to\infty} \langle A(w), u_k - v\rangle = \langle A(w), u - v\rangle \quad \text{for all } \lambda \in (0,1].$$

Now pass to the limit as $\lambda \to 0^+$, we obtain by hemicontinuity of A that

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \langle A(u), u - v\rangle \quad \text{for any } v \in V.$$

■

Remark 2.2.12. Note that in the above proposition, we can indeed drop the boundedness condition, then we have that monotonicity and hemicontinuity imply B-pseudomonotonicity.

The following proposition shows that the prototype of a pseudomonotone operator is sum of a strongly continuous operator and a monotone, hemicontinuous operator.

Proposition 2.2.13. Let $A,\ B : V \to V^*$ be bounded operators such that A is monotone, hemicontinuous and B is strongly continuous, then $A + B$ is a pseudomonotone operator.

Proof. The result follows directly from Lemma 2.2.6, Propositions 2.2.10 and 2.2.11.

The following definition of pseudomonotonicity was introduced by S.Karamardian [41].

Definition 2.2.14 (K-pseudomonotone). An operator $A : V \to V^*$ is K-pseudomonotone if for any $u, v \in V$, then

$$\langle A(v), u - v\rangle \geq 0 \Rightarrow \langle A(u), u - v\rangle \geq 0.$$

Remark 2.2.15. It is trivial to show that monotone $\Rightarrow$ K-pseudomonotone.

Note that in Remark 2.2.12, we have monotonicity and hemicontinuity imply the B-pseudomonotonicity. What happened if we assume K-pseudomonotonicity instead of monotonicity? This motivates the following notion of C-pseudomonotonicity. [36]

Definition 2.2.16 (C-pseudomonotone). The operator $A : V \to V^*$ is C-pseudomonotone if $u_k \rightharpoonup u$ in V and $\langle A(u_k), u_k - [(1-t)u + tv]\rangle \leq 0$ for all $t \in [0,1],\ k \in \mathbb{N}$ and $v \in V$, then

$$\langle A(u), u - v\rangle \leq 0.$$

Remark 2.2.17.
1. All of above operators can be defined on a closed convex subset K of V, i.e., $A : K \to V^*$.
2. B-pseudomonotone $\Rightarrow$ C-pseudomonotone.
Precisely, let $(u_k) \subset V$ be such that $u_k \rightharpoonup u$ in V and $\langle A(u_k), u_k - [(1-t)u + tv]\rangle \leq 0$ for all $t \in [0,1],\ k \in \mathbb{N}$ and $v \in V$. In particular, set $t = 0, 1$, we obtain for all $k \in \mathbb{N}$ and $v \in V$ that

$$\langle A(u_k), u_k - u\rangle \leq 0 \text{ and } \langle A(u_k), u_k - v\rangle \leq 0.$$

Hence,

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0 \text{ and } \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \leq 0.$$

By using B-pseudomonotonicity, we obtain

$$\langle A(u), u - v\rangle \leq \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \leq 0.$$

The following theorem shows the comparison between K-pseudomonotone and C-pseudomonotone, the original topological proof can be found in [25].

Theorem 2.2.18 ([25, Theorem 1]). If the mapping $A : V \to V^*$ is K-pseudomonotone and hemicontinuous, then it is C-pseudomonotone.

Proof. . We will use a sequential argument to prove the theorem.
Assume that $u_k \rightharpoonup u$ in V and $\langle A(u_k), u_k - [(1-t)u + tv]\rangle \le 0$ for all $t \in [0,1],\ k \in \mathbb{N},\ v \in V$, then

$$\langle A(u_k), [(1-t)u + tv] - u_k\rangle \ge 0 \text{ for all } t \in [0,1],\ k \in \mathbb{N}.$$

By K-pseudomonotonicity, this implies that

$$\langle A((1-t)u + tv), [(1-t)u + tv] - u_k\rangle \ge 0 \text{ for all } t \in [0,1],\ k \in \mathbb{N}.$$

By weak convergence of u_k, we pass the limit as $k \to \infty$ and obtain

$$\langle A((1-t)u + tv), v - u\rangle \ge 0 \text{ for all } t \in [0,1].$$

Now pass the limit as $t \to 0^+$, we obtain by using hemicontinuity of A that

$$\langle A(u), v - u\rangle \ge 0.$$

■

2.3 Local Monotonicity

The notion of local monotonicity was first introduced in the study of stochastic evolution equations to include a larger class of functions than the classical monotonicity (see [58]). Then the notion was also applied in the study of existence theorem for evolution equations (see [57]).
Locally monotonicity is defined in the context of an evolution triple. Recall in Definition 1.3.7, $V \subset H \equiv H^* \subset V^*$ is an evolution triple if H is a real separable Hilbert space that identified with its dual space H^* by Riesz's map, V is a real reflexive Banach space such that it is continuously and densely embedded into H. In an evolution triple, we have $\langle u, v\rangle = (u, v)$ for $u \in H$ and $v \in V$.

Definition 2.3.1 (Locally Monotone Operators). Let $V \subset H \subset V^*$ be an evolution triple and let the operator A that maps from V to its dual V^*. A is said to be locally monotone if

$$\langle A(u) - A(v), u - v\rangle \ge -(C + \rho(u) + \eta(v))\, \|u - v\|_H^2\,,$$

where $C > 0$ is a constant and $\rho,\ \eta : V \to [0,\infty)$ are measurable functions and locally bounded in V, that is, for any $x \in V$, there exists a neighbourhood U of x such that $|\rho(y)|$ is bounded for any $y \in U$.

Remark 2.3.2. It is trivial that monotonicity $\Rightarrow$ local monotonicity.

Remark 2.3.3. There is also another type of local monotonicity that is defined in term of Fréchet derivative, which is used to study Euler equation in fluid dynamics (see Section 29.12 in [90]).

From Proposition 2.2.11, we have boundedness, monotonicity and hemicontinuity imply the pseudomonotonicity. The natural question now is what will happen if we replace the monotonicity by local monotonicity? The following lemma gives the answer to the question.

Lemma 2.3.4. If the embedding $V \subset H$ is compact, the operator $A : V \to V^*$ is bounded, locally monotone and hemicontinuous, then A is pseudomonotone.

Proof. Assume $u_k \rightharpoonup u$ in V and

$$\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \le 0, \tag{2.12}$$

we need to show

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \ge \langle A(u), u - v\rangle \ \text{ for all } v \in V. \tag{2.13}$$

For any $v \in V$ and the constant C in Definition 2.3.1, we set

$$M := \|v\|_V + \|u\|_V + \sup_{k\in\mathbb{N}} \|u_k\|_V \,;\; C_1 = \inf_{\|x\|_V, \|y\|_V \le 2M} -(C + \rho(x) + \eta(y)).$$

Since $V \subset H$ is compact and $u_k \rightharpoonup u$ in V, then we have $u_k \to u$ in $H \equiv H^* \subset V^*$, and therefore

$$\langle C_1 u, u - v\rangle = \lim_{k\to\infty} \langle C_1 u_k, u_k - v\rangle \,. \tag{2.14}$$

Recall in Lemma 2.2.6, the sum of pseudomonotone operators is pseudomonotone, to prove (2.13), it is enough to show that

$$\liminf_{k\to\infty} \langle A_0(u_k), u_k - v\rangle \ge \langle A_0(u), u - v\rangle \;\text{ for all } v \in V,$$

where $A_0 = A - C_1 I$, I is identity operator.
Then from locally monotonicity of A and definition of C_1, we have

$$\begin{aligned}&\langle A_0(u_k) - A_0(u), u_k - u\rangle = \langle A(u_k) - A(u), u_k - u\rangle - C_1 \langle u_k - u, u_k - u\rangle\\ \ge\, &-(C + \rho(u_k) + \eta(u)) \|u_k - u\|_H^2 - C_1 \|u_k - u\|_H^2 = -(C + \rho(u_k) + \eta(u) + C_1) \|u_k - u\|_H^2 \ge 0.\end{aligned}$$

Taking liminf on both side of the above inequality and using $u_k \rightharpoonup u$ in V, we obtain

$$\liminf_{k\to\infty} \langle A_0(u_k), u_k - u\rangle \ge \liminf_{k\to\infty} \langle A_0(u), u_k - u\rangle = 0.$$

By (2.12), (2.14) and above inequality, we obtain

$$\lim_{k\to\infty} \langle A_0(u_k), u_k - u\rangle = 0. \tag{2.15}$$

Now set $z = u + \lambda(v - u)$ with $\lambda \in (0, \frac{1}{2})$, then $\|z\|_V \le \frac{3}{2}\|u\|_V + \frac{1}{2}\|v\|_V \le 2M$ and by local monotonicity, we obtain that

$$\begin{aligned}\langle A_0(u_k) - A_0(z), u_k - z\rangle &= \langle A(u_k) - A(z), u_k - z\rangle - \langle C_1 I(u_k) - C_1 I(z), u - z\rangle\\ &\ge -(C + \rho(u_k) + \eta(z)) \|u_k - z\|_H^2 - C_1 \|u_k - z\|_H^2 \ge 0,\end{aligned} \tag{2.16}$$

where the last inequality followed by the definition of C_1, since $\|u_k\|_V \le 2M$ and $\|z\|_V \le 2M$, so

$$-C_1 = \sup_{\|x\|_V, \|y\|_V \le 2M} C + \rho(x) + \eta(y) \ge C + \rho(u_k) + \eta(z).$$

Above inequality (2.16) is equivalent to

$$\lambda \langle A_0(z), u - v\rangle - (1 - \lambda) \langle A_0(u_k), u_k - u\rangle \le \lambda \langle A_0(u_k), u_k - v\rangle - \langle A_0(z), u_k - u\rangle \,.$$

Taking the liminf on both side of the above inequality, using (2.15) and $u_k \rightharpoonup u$ in V, we obtain

$$\langle A_0(z), u - v\rangle \le \liminf_{k\to\infty} \langle A_0(u_k), u_k - v\rangle \,.$$

Sending $\lambda \to 0^+$ and by hemicontinuity of A, we obtain

$$\langle A_0(u), u - v\rangle \le \liminf_{k\to\infty} \langle A_0(u_k), u_k - v\rangle \,.$$

Therefore, A is pseudomonotone and hence the lemma is proved. ■

2.4 p-strict quasimonotonicity and Young measures

Since the notion of p-strict quasimonotonicity is defined through the tool of Young measures, we will first start this section with a brief account on the theory of Young measures. Young measures were introduced by L.C.Young [87] to give descriptions of limits of minimizing sequences in the calculus of variation and further in the optimal control (see [88, 59]), which enable us to analyse the calculus of variation problems where the

minimizers do not exist in the classical sense. The approach we will use in this work to regard Young measures as elements of $L^\infty_{w^*}(\Omega;\mathcal{M}(\mathbb{R}^m))$ - the space of weakly* measurable functions $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ that are essentially bounded. The fundamental theorem of Young measures will be presented to show that the importance of Young measures in understanding the limiting behaviour of a sequence of measurable functions under composition with continuous functions (or Carathéodory functions). It turns out Young measures are powerful tools in the analysis of the oscillation effects and the characterization of limits of oscillating sequences under composition of a continuous functions (or Carathéodory functions). However, Young measures completely ignore the concentration effects, in other words, two sequences may share the same Young measure with one of them shows the concentration effect and the other one not. L.Tartar developed Young measures as tools for the analysis of possible oscillations of solutions of partial differential equations (see [80, 81, 82]). Young measures were also applied in mechanics and the study of microstructures of crystals. (see [7, 8, 19])
In this section, some brief theory about Young measures such as the fundamental theorem of Young measures and its applications will be presented, for more details on Young measures (see [83, 5]), followed by the introduction of the notion of strict p-quasimonotonicity, which will be used later to solve elliptic and parabolic partial differential equations in divergence form. Let Ω be a bounded open subset of $\mathbb{R}^n$ and we denote $\mathscr{B}(\Omega)$ as the σ-algebra of all Lebesgue measurable subsets of Ω. Let $\mathbb{M}^{m\times n}$ denote as the space of real $m \times n$ matrices. Before we introduce the fundamental theorem of Young measures, we need the concept of Radon measures and the representation theorem for duality of the space of continuous functions vanishing at infinity.

Radon Measures
Let $E \in \mathscr{B}(\mathbb{R}^n)$. A Radon measure on E is a measure over $\mathscr{B}(E)$ with values in $[-\infty,\infty]$ such that every compact subset of E has a finite measure. For every Radon measure μ, we can define a positive Radon measure $|\mu|$ called the total variation measure:

$$|\mu|(B) = \inf\left\{\sum_{i\in\mathbb{N}} |\mu(B_i)|\,,\ (B_i)_{i\in\mathbb{N}} \subset \mathscr{B}(E) \text{ with } B \subset \bigcup_i B_i\right\}.$$

The Banach space of all Radon measures on E with $\|\mu\|_{\mathcal{M}} := |\mu|(E) = \int_E d|\mu| < \infty$ is denoted by $\mathcal{M}(E)$.

Definition 2.4.1. For $E \subset \mathbb{R}^n$, we define the space of continuous functions that vanish at the infinity as

$$C_0(E) = \{f : E \to \mathbb{R} \text{ continuous and bounded with } \lim_{|\lambda|\to\infty} f(\lambda) = 0\}.$$

$C_0(E)$ endowed with the supremum norm $\|\cdot\|_\infty$, which becomes a separable Banach space.

Theorem 2.4.2 (Riesz-Alexandrov). $\mathcal{M}(E)$ is isometrically isomorphic to the dual space of $(C_0(E))^*$, with the duality pair given by

$$\langle \mu, f\rangle = \int_E f(\lambda)d\mu(\lambda), \quad \text{for } \mu \in \mathcal{M}(E) \text{ and } f \in C_0(E).$$

Definition 2.4.3. We say the mapping $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ is weakly* measurable if the mapping $x \mapsto \langle \nu(x), f\rangle$ is measurable for all $f \in C_0(\mathbb{R}^m)$.

2.4.1 Fundamental theorem on Young measures and its refinement

In this section, we shall introduce the fundamental theorem of Young measures. Ball proves the following fundamental theorem directly via duality rather than disintegration of measures on a product space (see [6]).

Theorem 2.4.4 (Ball [6, Theorem 1]). Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$, let K be a closed subset of $\mathbb{R}^m$, and let $u_k : \Omega \to \mathbb{R}^m,\ k \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_k \to K$ in measure as $k \to \infty$, i.e., given any open neighbourhood U of K in $\mathbb{R}^m$, one has

$$\lim_{k\to\infty} |\{x \in \Omega : u_k \notin U\}| = 0.$$

Then there exists a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, and a family $(\nu_x)_{x\in\Omega}$ of positive measures on $\mathbb{R}^m$, depending measurably on x, such that

(i) $\|\nu_x\|_{\mathcal{M}} := \int_{\mathbb{R}^m} d\nu_x \leq 1$ for a.e. $x \in \Omega$;
(ii) $\operatorname{supp}(\nu_x) \subset K$ for a.e. $x \in \Omega$, and
(iii) $f(u_k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda)$ in $L^\infty(\Omega)$ for each continuous function $f \in C_0(\mathbb{R}^m)$ vanishing at the infinity.

Suppose further that $(u_k)_k$ satisfies the following boundedness condition

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R : |u_k(x)| \geq L\}| = 0,\ \forall R > 0,\ \text{where } B_R = B_R(0). \tag{2.17}$$

Then ν_x is a probability measure on $\mathbb{R}^m$ for a.e. x in Ω. i.e.,

$$\|\nu_x\|_{\mathcal{M}} = 1 \text{ for a.e. } x \in \Omega. \tag{2.18}$$

For any measurable subset A of Ω and any continuous function $f : \mathbb{R}^m \to R$ such that $(f(u_k))_k$ is sequentially weakly relatively compact in $L^1(A)$, we have

$$f(u_k) \rightharpoonup \langle \nu_x, f\rangle \text{ in } L^1(A). \tag{2.19}$$

Remark 2.4.5.
1. The weakly* measurable mapping $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ is called the **Young measure** generated by the sequence $(u_k)_k$. In fact, any ν satisfies (i) is generated by some sequence $(u_l)_l$.
2. A family of measures $(\nu_x)_{x\in\Omega}$ is said to be **homogeneous** if it does not depend on x, i.e., $\nu_x = \nu$ for all $x \in \Omega$.

Proof. From the above Riesz-Alexandrov representation Theorem 2.4.2, there is an isometric isomorphism between the dual space $(C_0(\mathbb{R}^m))^*$ and the Banach space $\mathcal{M}(\mathbb{R}^m)$ of bounded Radon measures on $\mathbb{R}^m$. For each u_k, we assign the mapping $\nu^k : \Omega \to \mathcal{M}(\mathbb{R}^m)$ defined by

$$\nu_x^k = \delta_{u_k(x)}.$$

Note that $\|\nu^k\|_{\infty,\mathcal{M}} = \operatorname{ess\,sup}_{x\in\Omega} \|\nu_x^k\|_{\mathcal{M}} = 1$ for all $k \in \mathbb{N}$. For any $f \in C_0(\mathbb{R}^m)$ and for all $k \in \mathbb{N}$, u_k are measurable functions and f is continuous, so we have the following measurable mapping

$$x \mapsto \langle \nu_x^k, f\rangle = \int_{\mathbb{R}^m} f(\lambda) d\delta_{u_k(x)}(\lambda) = f(u_k(x)).$$

Hence, ν^k belongs to the space $L^\infty_{w^*}(\Omega; \mathcal{M}(\mathbb{R}^m))$ of equivalence class of weak* measurable mappings that are essentially bounded. Under the norm $\|\cdot\|_{\infty,\mathcal{M}}$, $L^\infty_{w^*}(\Omega; \mathcal{M}(\mathbb{R}^m))$ is a Banach space. Since $C_0(\mathbb{R}^m)$ is separable, from Proposition 1.3.4, there is an isometrically isomorphism between the dual space $(L^1(\Omega; C_0(\mathbb{R}^m)))^*$ and $L^\infty_{w^*}(\Omega; \mathcal{M}(\mathbb{R}^m))$ obtained by associating each $\mu \in L^\infty_{w^*}(\Omega; \mathcal{M}(\mathbb{R}^m))$ with the following linear functional on $L^1(\Omega; C_0(\mathbb{R}^m))$

$$\psi \mapsto \int_\Omega \langle \mu(x), \psi(x,\cdot)\rangle\, dx. \tag{2.20}$$

Since $C_0(\mathbb{R}^m)$ is separable, so is $L^1(\Omega; C_0(\mathbb{R}^m))$. By the sequential version of Banach-Alaoglu Theorem 1.1.16, we obtain that there exists a subsequence of ν^k, again denoted by ν^k, and an element $\nu \in L^\infty_{w^*}(\Omega; \mathcal{M}(\mathbb{R}^m))$ such that $\nu^k \overset{*}{\rightharpoonup} \nu$ in $L^\infty_{w^*}(\Omega; \mathcal{M}(\mathbb{R}^m))$. By (2.20), this implies that for every $\psi \in L^1(\Omega; C_0(\mathbb{R}^m))$

$$\begin{aligned}\int_\Omega \psi(x, u_k(x))dx &= \int_\Omega \int_{\mathbb{R}^m} \psi(x,\lambda) d\delta_{u_k(x)}(\lambda) dx = \int_\Omega \langle \nu_x^k, \psi(x,\cdot)\rangle\, dx \\ &= \langle \nu^k, \psi\rangle \to \langle \nu, \psi\rangle = \int_\Omega \langle \nu_x, \psi(x,\cdot)\rangle\, dx \quad \text{as } k\to\infty.\end{aligned} \tag{2.21}$$

In particular, writing $\psi(x,\lambda) = \phi(x) f(\lambda)$, where $\phi \in L^1(\Omega)$ and $f \in C_0(\mathbb{R}^m)$, then we have

$$f(u_k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle \text{ in } L^\infty(\Omega) \text{ for every } f \in C_0(\mathbb{R}^m),$$

which is (iii). By weak* lower semicontinuity of the norm,

$$\|\nu\|_{\infty,\mathcal{M}} \leq \liminf_k \|\nu_k\|_{\infty,\mathcal{M}} = 1,$$

which is (i).
To prove (ii), assume that $K \neq \mathbb{R}^m$ and let us denote $C_{0,K}(\mathbb{R}^m) := \{g \in C_0(\mathbb{R}^m) : g|_K = 0\}$. Now for $\epsilon > 0$ and $f \in C_{0,K}(\mathbb{R}^m)$, let U_ϵ be an open neighbourhood of K defined by $U_\epsilon = \{z \in \mathbb{R}^m : |f(z)| < \epsilon\}$. From the assumption, it follows from $u_k \to K$ in measure that

$$\lim_{k\to\infty} |\{x \in \Omega : u_k(x) \notin U_\epsilon\}| = 0,$$

by definition of U_ϵ, this is equivalent to

$$\lim_{k\to\infty} |\{x \in \Omega : |f(u_k(x))| \geq \epsilon\}| = 0.$$

Since $\epsilon > 0$ is arbitrary, this means that

$$f(u_k(\cdot)) \to 0 \text{ in measure.}$$

Since f is bounded, by Dominated Convergence Theorem 1.2.13 and (iii), we obtain that

$$\int_\Omega \phi(x) \langle \nu_x, f\rangle \, dx = \lim_{k\to\infty} \int_\Omega \phi(x) f(u_k(x)) dx = 0 \quad \text{for every } \phi \in L^1(\Omega).$$

It follows that for a.e. $x \in \Omega$, we have

$$\langle \nu_x, f\rangle = 0 \text{ for all } f \in C_{0,K}(\mathbb{R}^m).$$

i.e., $\operatorname{supp}(\nu_x) \subset K$. This proves (ii).
Now suppose that (2.17) holds, we define $h^r \in C_0(\mathbb{R}^m)$ by

$$h^r(\lambda) = \begin{cases} 1 & \text{for } |\lambda| \leq r, \\ 1 + r - |\lambda| & \text{for } r \leq |\lambda| \leq r+1, \\ 0 & \text{for } |\lambda| \geq r+1. \end{cases}$$

Then

$$\|h^r\|_\infty = 1 \quad \text{for all } r.$$

If $E \subset \Omega$ is bounded and measurable, by (iii) we obtain that

$$\lim_{k\to\infty} ⨍_E h^r(u_k(x)) dx = ⨍_E \langle \nu_x, h^r\rangle \, dx \leq ⨍_E \|\nu_x\|_{\mathcal{M}} \, dx, \qquad (\text{take } \frac{\chi_E}{|E|} \in L^1(\Omega)). \tag{2.22}$$

But

$$⨍_E (1 - h^r(u_k(x))) dx \leq \frac{|\{x \in E : |u_k(x)| \geq r\}|}{|E|}. \tag{2.23}$$

So passing the limit as $r \to \infty$ in (2.23), we obtain from (2.17) and (2.22) that

$$|E| \leq \int_E \|\nu_x\|_{\mathcal{M}} \, dx.$$

Since we proved (i) above that $\|\nu_x\|_{\mathcal{M}} \leq 1$ a.e. and E is arbitrary, so we have that

$$\|\nu_x\|_{\mathcal{M}} = 1 \text{ for a.e. } x \text{ in } \Omega.$$

Suppose further that $f : \mathbb{R}^m \to \mathbb{R}$ is continuous and that $\{f(u_k)\}_k$ is sequentially weakly relatively compact in $L^1(\Omega)$. Set $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, so that $f = f^+ - f^-$. By the Dunford Pettis Theorem 1.2.15, we get that $f^+(u_k)$ and $f^-(u_k)$ are both sequentially weakly relatively compact in $L^1(\Omega)$. Hence it is sufficient to prove (2.19) for the case $f \geq 0$.
Define $f^r \in C_0(\mathbb{R}^m)$ by $f^r(\lambda) = h^r(\lambda) f(\lambda)$, with h^r given above. Let A be a measurable subset of Ω and let $\phi \in L^\infty(A)$. We first claim that

$$\int_A \phi f^r(u_k) dx \to \int_A \phi f(u_k) dx \quad \text{as } r \to \infty \text{ uniformly in } k. \tag{2.24}$$

Indeed, we first note that

$$\left|\int_A \phi(f^r(u_k)-f(u_k))dx\right| \le \text{const.}\int_{\{|u_k(x)|\ge r\}} f(u_k)dx, \tag{2.25}$$

and by the Dunford-Pettis Theorem 1.2.15, $f(u_k)$ is equiintegrable in $L^1(A)$, i.e., for any $\epsilon>0$, there exists $M>0$ such that

$$\sup_k \int_{\{f(u_k(x))\ge M\}} f(u_k)dx < \epsilon.$$

Set

$$F := \{x\in A : f(u_k(x))\ge r\} \quad\text{and}\quad G := \{x\in A : f(u_k(x))\ge M\}.$$

From (2.17), for sufficiently large r and for all k, we have

$$\begin{aligned}\sup_k \int_F f(u_k)dx &\le \sup_k \int_{F\cap G} f(u_k(x))dx + \sup_k \int_{F\cap G^c} f(u_k(x))dx\\ &\le \epsilon + M\sup_k |F| \le 2\epsilon,\end{aligned}$$

which together with (2.25) proves the claim. Hence, we have that

$$\lim_{r\to\infty}\int_A \phi f^r(u_k)dx = \int_A \phi f(u_k)dx \quad \text{uniformly in } k.$$

On the other hand, choosing $\phi\ge 0$ and noting that $f^r\uparrow f$, using Monotone Convergence Theorem 1.2.11, it follows from (iii) that

$$\lim_{k\to\infty}\int_A \phi f(u_k)dx = \lim_{r\to\infty}\lim_{k\to\infty}\int_A \phi f^r(u_k)dx = \lim_{r\to\infty}\int_A \phi\langle\nu_x, f^r\rangle dx = \int_A \phi\langle\nu_x,f\rangle dx.$$

Hence $f(u_k)\rightharpoonup\langle\nu_x,f\rangle$ in $L^1(A)$. ■

Remark 2.4.6. (a) The condition in (2.17) is equivalent to the following *tightness* condition:
For any $R>0$, there exists a continuous non-decreasing function $g_R:[0,\infty)\to\mathbb{R}$ with $\lim_{t\to\infty} g_R(t)=\infty$ such that

$$\sup_{k\in\mathbb{N}}\int_{\Omega\cap B_R} g_R(|u_k(x)|)dx < \infty. \tag{2.26}$$

Proof. Suppose that (2.26) holds, since g_R is non-decreasing, we obtain that

$$\sup_k |\{x\in\Omega\cap B_R : |u_k(x)|\ge t\}|\, g_R(t) \le \sup_k\int_{\Omega\cap B_R} g_R(|u_k(x)|)dx < \infty.$$

Since $\lim_{t\to\infty} g_R(t)=\infty$, we get from above inequality that

$$\lim_{t\to\infty}\sup_k |\{x\in\Omega\cap B_R : |u_k(x)|\ge t\}| = 0.$$

Conversely, if (2.17) holds, we may choose strict increasing sequence $(t_k)_k\subset\mathbb{R}^+$ with $\lim_{k\to\infty} t_k=\infty$ such that

$$\sup_k |\{x\in\Omega\cap B_R : |u_k(x)|\ge t_j\}| \le j^{-3},$$

and let

$$\bar g_R(t) = \begin{cases} 0, & \text{if } t\in[0,t_1),\\ j, & \text{if } t\in[t_j,t_{j+1}).\end{cases}$$

Then

$$\sup_k\int_{\Omega\cap B_R}\bar g_R(|u_k(x)|)dx = \sup_k\sum_{j=1}^{\infty} j\,|\{x\in\Omega\cap B_R : t_j<|u_k(x)|<t_{j+1}\}| \le \sum_{j=1}^{\infty} j^{-2} < \infty$$

Choosing a suitable continuous non-decreasing function $g_R \leq \bar{g}_R$, we obtain (2.26).

(b) The same argument as in the proof above shows that under hypothesis (2.17), for any measurable $A \subset \Omega$,

$$f(\cdot, u_k) \rightharpoonup \langle \nu_x, f(x, \cdot) \rangle \text{ in } L^1(A)$$

for every Carathéodory function $f : A \times \mathbb{R}^m \to \mathbb{R}$ such that $\{f(\cdot, u_k)\}$ is sequentially weak relatively compact in $L^1(A)$. In fact, this is equivalent to (2.17), (2.18) and (2.19).

(c) In the above proof (see (2.21)), we also showed that if u_k generates the Young measure ν_x, then for $\psi \in L^1(\Omega; C_0(\mathbb{R}^m))$,

$$\lim_{k\to\infty} \int_\Omega \psi(x, u_k(x))dx = \int_\Omega \langle \nu_x, \psi(x, \cdot) \rangle \, dx.$$

N.Hungerbühler [34] introduces the following theorem which is the refinement of Ball's theorem.

Theorem 2.4.7 (Refinement of Ball's theorem). Let Ω, u_k and ν_x be as in Ball's theorem. Then (2.17), (2.18) and (2.19) are equivalent.

Proof. From Ball's theorem, we know that (2.17) $\Rightarrow$ (2.18) and (2.17) $\Rightarrow$ (2.19), so it is sufficient to show that (2.18) $\Rightarrow$ (2.17) and (2.19) $\Rightarrow$ (2.18).
Let us first show that (2.18) $\Rightarrow$ (2.17):
Suppose the contrary holds, i.e., there exists $R > 0$ and $\epsilon > 0$ with the following property: there exist a sequence $L_i \to \infty$ and integers k_i such that

$$|\{x \in \Omega \cap B_R : |u_{k_i}(x)| > L_i\}| > \epsilon \quad \text{for all } i \in \mathbb{N}.$$

For $r > 0$, consider the function $h^r \in C_0(\mathbb{R}^m)$ defined above. Hence, apply the first part of the Ball's theorem, we obtain

$$\lim_{k\to\infty} \int_\Omega h^r(u_k)\chi_{B_R} dx = \int_\Omega \int_{\mathbb{R}^m} h^r(\lambda) d\nu_x(\lambda) \chi_{B_R} dx. \tag{2.27}$$

Note that u_{k_i} is a subsequence of u_k, $k_i \to \infty$ as $i \to \infty$. For i large enough such that $L_i \geq r + 1$, we find

$$|\Omega \cap B_R| - \epsilon \geq |\{x \in \Omega \cap B_R : |u_{k_i}(x)| \leq L_i\}| = \int_{\{\Omega \cap B_R : |u_{k_i}(x)| \leq L_i\}} dx \geq \int_{\Omega \cap B_R} h^r(u_{k_i}) dx = \int_\Omega h^r(u_{k_i})\chi_{B_R} dx,$$

together with (2.27) imply that as $i \to \infty$

$$|\Omega - B_R| - \epsilon \geq \int_\Omega \int_{\mathbb{R}^m} h^r(\lambda) d\nu_x(\lambda) \chi_{B_R} dx. \tag{2.28}$$

On the other hand, by Monotone Convergence Theorem 1.2.13, we pass the limit as $r \to \infty$ and obtain the right hand side of (2.28) converges to

$$\int_\Omega \int_{\mathbb{R}^m} d\nu_x \chi_{B_R} dx = \int_\Omega \|\nu_x\|_{\mathcal{M}} \chi_{B_R} dx = |\Omega \cap B_R| \,,$$

where the last equality followed by (2.18). This contradicts to (2.28), so we have (2.18) $\Rightarrow$ (2.17).
Now it remains to show that (2.19) $\Rightarrow$ (2.18):
Fix $R > 0$ and let $f \equiv 1$ on $\mathbb{R}^m$, then $f(u_k)$ is sequentially weakly compact in $L^1(\Omega \cap B_R)$, so (2.19) implies

$$|\Omega \cap B_R| = \int_{\Omega \cap B_R} f(u_k)\chi_{B_R} dx \to \int_{\Omega \cap B_R} \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda) \chi_{B_R} dx = \int_{\Omega \cap B_R} \|\nu_x\|_{\mathcal{M}} \, dx.$$

Since $\|\nu_x\|_{\mathcal{M}} \leq 1$ by (i) in the above Ball's theorem, the above inequality implies that

$$\|\nu_x\|_{\mathcal{M}} = 1 \text{ for a.e. } x \in \Omega \cap B_R.$$

Since R is arbitrary, the result follows. ∎

2.4.2 Application of the fundamental theorem on Young measures

In this section, we are going to introduce some useful results which are applications of the above Ball's theorem and the refinement of Ball's theorem.

Proposition 2.4.8. If $|\Omega| < \infty$ and ν_x is the Young measure generated by a sequence u_k, then there holds

$$u_k \to u \text{ in measure } \Leftrightarrow \nu_x = \delta_{u(x)} \text{ for a.e. } x \in \Omega.$$

Proof. ($\Rightarrow$): Suppose that $u_k \to u$ in measure, i.e., for any $\epsilon > 0$, we have

$$\lim_{k\to\infty} |\{|u_k - u| > \epsilon\}| = 0. \tag{2.29}$$

For $\phi \in C_c^\infty(\mathbb{R}^m)$ and $\zeta \in L^1(\Omega)$, we have

$$\left|\int_\Omega \zeta(\phi(u_k) - \phi(u))dx\right| \le \underbrace{\left|\int_{|u_k-u|>\epsilon} \zeta(\phi(u_k) - \phi(u))dx\right|}_{I} + \underbrace{\left|\int_{|u_k-u|\le\epsilon} \zeta(\phi(u_k) - \phi(u))dx\right|}_{II}.$$

II: By choosing ϵ small enough, we can make II as small as we want, since

$$\text{II} \le \epsilon \|D\phi\|_{L^\infty(\Omega)} \|\zeta\|_{L^1(\Omega)}.$$

I: We have

$$\text{I} \le 2\|\phi\|_{L^\infty(\Omega)} \int_{|u_k-u|>\epsilon} |\zeta|\, dx,$$

which converges to 0 as $k \to \infty$, by absolute continuity of the integral and (2.29). Since $C_c^\infty(\mathbb{R}^m)$ is dense in $C_0(\mathbb{R}^m)$, then we conclude that for all $\phi \in C_0(\mathbb{R}^m)$,

$$\phi(u_k) \rightharpoonup \langle \delta_{u(x)}, \phi\rangle \text{ in } L^\infty(\Omega).$$

i.e., $\nu_x = \delta_{u(x)}$.
($\Leftarrow$) : Now, assume that $\nu_x = \delta_{u(x)}$. So (2.18) is satisfied.
First step: we will consider the case that u_k is bounded in $L^\infty(\Omega)$, then by (2.19) in the refined Ball's theorem, for $\phi(x) := |x|^2$ and $\psi \equiv 1$ on Ω, we obtain that

$$\|u_k\|_{L^2(\Omega)}^2 = \int_\Omega \phi(u_k)\psi dx \to \int_\Omega \phi(u)\psi dx = \|u\|_{L^2(\Omega)}^2 \quad \text{as } k \to \infty. \tag{2.30}$$

On the other hand, by choosing $\phi = \text{id}$, we can similarly find that $u_k \rightharpoonup u$ weakly in $L^2(\Omega)$, which together with (2.30) implies that $u_k \to u$ in $L^2(\Omega)$. Since $|\Omega| < \infty$, this implies that $u_k \to u$ in $L^1(\Omega)$. Hence we have that for all $\alpha > 0$,

$$\alpha |\{|u_k - u| \ge \alpha\}| \le \int_{\{|u_k-u|\ge\alpha\}} |u_k - u|\; dx \le \int_\Omega |u_k - u|\; dx \to 0 \quad \text{ as } k \to \infty,$$

and hence $u_k \to u$ in measure.
Second step:
We will show that if u_k generated the Young measure $\delta_{u(x)}$, then $T_R(u_k) \to T_R(u)$ in measure, where T_R denotes the truncation $T_R(x) := x\min\{1, \frac{R}{|x|}\}$, for fixed $R > 0$.
For $f \in C_0(\mathbb{R}^m)$, we have $f \circ T_R$ is continuous and $\{f(T_R(u_k))\}_k$ is uniformly bounded in k. Hence, $\{f(T_R(u_k))\}_k$ is equiintegrable, and by Dunford-Pettis Theorem 1.2.15, it is sequentially weakly precompact in $L^1(\Omega)$. Since (2.18) is satisfied, we obtain by (2.19) that

$$\int_\Omega \zeta f(T_R(u_k))dx \to \int_\Omega \zeta f(T_R(u))dx \quad \text{for } \zeta \in L^\infty(\Omega).$$

This implies that $T_R(u_k)$ generated the Young measure $\delta_{T_R(u(x))}$ and by the first step, the claim follows.
Third step:
We show that $u_k \to u$ in measure. Let $\epsilon > 0$ be given, then we have

$$|\{|u_k - u| > \epsilon\}| \leq \underbrace{|\{|u_k - u| > \epsilon, |u| \leq R, |u_k| \leq R\}|}_{\text{I}} + \underbrace{|\{|u| > R\}|}_{\text{II}} + \underbrace{|\{|u_k| > R\}|}_{\text{III}}.$$

II can be arbitrarily small by choosing $R > 0$ large enough. By (2.18) in the refined Ball's theorem, we have (2.17), which implies that III is arbitrarily small uniformly in k by taking R large enough. And finally, by the second step, I $\to 0$ as $k \to \infty$. ■

Remark 2.4.9. The above Proposition 2.4.8 shows that when u_k converges to u in measure, the Young measure generated by u_k is given by the graph of function u.

The following proposition is about the product measures.

Proposition 2.4.10. Let $|\Omega| < \infty$, if the sequences $u_k : \Omega \to \mathbb{R}^m$ and $v_k : \Omega \to \mathbb{R}^l$ generate the Young measures $\delta_{u(x)}$ and ν_x respectively, then (u_k, v_k) generates the Young measure $\delta_{u(x)} \otimes \nu_x$.

Proof. It is sufficient to show for all $\phi \in C_c^\infty(\mathbb{R}^m \times \mathbb{R}^l)$ which is a dense subset of $C_0(\mathbb{R}^m \times \mathbb{R}^l)$, there holds

$$\phi(u_k, v_k) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^k} \phi(u(x), \lambda) d\nu_x(\lambda) \quad \text{in } L^\infty(\Omega).$$

For $\zeta \in L^1(\Omega)$, we have

$$\left| \int_\Omega \zeta \left(\phi(u_k, v_k) - \int_{\mathbb{R}^l} \phi(u(x), \lambda) d\nu_x(\lambda) \right) dx \right| \leq \underbrace{\left| \int_{|u_k - u| < \epsilon} \zeta(\phi(u_k, v_k) - \phi(u, v_k)) dx \right|}_{\text{I}}$$

$$+ \underbrace{\left| \int_{|u_k - u| \geq \epsilon} \zeta(\phi(u_k, v_k) - \phi(u, v_k)) dx \right|}_{\text{II}} + \underbrace{\left| \int_\Omega \zeta \left(\phi(u, v_k) - \int_{\mathbb{R}^l} \phi(u, \lambda) d\nu_x(\lambda) \right) dx \right|}_{\text{III}}.$$

Since I $\leq \epsilon \|\zeta\|_{L^1(\Omega)} \|D\phi\|_{L^\infty}$, the first term is arbitrarily small for ϵ small enough. For ϵ fixed, we have, for $k \to \infty$, that

$$\text{II} \leq 2 \|\phi\|_{L^\infty} \int_{|u_k - u| \geq \epsilon} |\zeta| \, dx \to 0,$$

since u_k converges to u in measure by the previous Proposition 2.4.8. Since $\zeta \in L^1(\Omega)$, then the function $\zeta(x)\phi(u(x), \cdot)$ is in $L^1(\Omega; C_0(\mathbb{R}^l))$ and hence III $\to 0$ as $k \to \infty$ by (c) in Remark 2.4.6. Hence the result follows. ■

The following *Fatou's type* lemma will be the key to the limit passage for problems in Chapter 5.

Lemma 2.4.11 (*Fatou's type*). Let $|\Omega| < \infty$. Let $F : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m \times n} \to \mathbb{R}$ be a Carathéodory function and $u_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that $u_k \to u$ in measure and Du_k generates the Young measure ν_x with $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$. If the negative part $F^-(x, u_k(x), Du_k(x))$ is equiintegrable, then

$$\liminf_{k \to \infty} \int_\Omega F(x, u_k(x), Du_k(x)) dx \geq \int_\Omega \int_{\mathbb{M}^{m \times n}} F(x, u, \lambda) d\nu_x(\lambda) dx. \tag{2.31}$$

Proof. We may assume that limit inferior on the left hand side of (2.31) agrees with the limit and is not equal to ∞.
Set $F_R(x, u, p) = \min\{R, F(x, u, p)\}$ for $R > 0$, then F_R is Carathéodory function and $F_R^- = F^-$. For fixed $R > 0$, the sequence $\{F_R(x, u_k(x), Du_k(x))\}_k$ is equiintegrable. We have that for all k and $R > 0$,

$$\int_\Omega F_R(x, u_k(x), Du_k(x)) dx \leq \int_\Omega F(x, u_k(x), Du_k(x)) dx \leq C < \infty.$$

From assumption, $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in \Omega$, using (b) in Remark 2.4.6 for $g \equiv 1 \in L^\infty(\Omega)$, we obtain that for all $R > 0$,

$$\lim_{k\to\infty} \int_\Omega F_R(x, u_k(x), Du_k(x))dx = \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)d\nu_x(\lambda)dx \leq C. \tag{2.32}$$

By the fact that $F_R^- = F^-$ and using Monotone Convergence Theorem 1.2.11 for F_R^+, we obtain as $R \to \infty$ that

$$\lim_{R\to\infty} \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)d\nu_x(\lambda)dx = \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)d\nu_x(\lambda)dx \leq C < \infty. \tag{2.33}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega F(x, u_k(x), Du_k(x))dx - \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)d\nu_x(\lambda)dx \\
&= \underbrace{\int_\Omega F(x, u_k(x), Du_k(x))dx - \int_\Omega F_R(x, u_k(x), Du_k(x))dx}_{\mathrm{I}_k} \\
&\quad + \underbrace{\int_\Omega F_R(x, u_k(x), Du_k(x)) - \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)d\nu_x(\lambda)dx}_{\mathrm{II}_k} \\
&\quad + \underbrace{\int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)d\nu_x(\lambda)dx - \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)d\nu_x(\lambda)dx}_{\mathrm{III}}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\mathrm{I}_k \geq 0, \text{ by definition of } F_R, \\
&\mathrm{II}_k \to 0 \text{ as } k \to \infty \text{ for any fixed } R > 0, \text{ this is from (2.32)}, \\
&\mathrm{III} \to 0 \text{ as } R \to \infty, \text{ by (2.33)}.
\end{aligned}$$

Hence the result follows. ■

Remark 2.4.12. Although $F(x, u_k, Du_k)$ is not sequentially weakly relatively compact in $L^1(\Omega)$, the Young measure generated by (u_k, Du_k) somehow still provides some information about the limiting behaviour.

2.4.3 Gradient Young measures

Recall that by the fundamental theorem of Young measures, suppose a sequence $f_k : \Omega \to \mathbb{R}^m$ generates a Young measure ν, then the Young measure gives the description of weak limit of any continuous function φ composed with the sequence f_k, i.e., the weak limit of $\varphi(f_k)$. Hence in order to find ν, it is sufficient to compute the limit

$$\lim_{k\to\infty} \int_B \varphi(f_k(x))dx \quad \text{for any } B \in \mathcal{B}(\Omega) \text{ and } \varphi \in C_0(\mathbb{R}^m).$$

In this section, we are interested in the gradient Young measures generated by bounded sequences of gradients in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$, which are crucial in the definition of strict p-quasimonotonicity and dealing nonlinear PDEs in Chapter 5. First, let us consider $f_k \in L^\infty(\Omega;\mathbb{R}^m)$ with $f_k \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega;\mathbb{R}^m)$. By Ball's Theorem 2.4.4, the Young measure generated by f_k is a family of probability measures $(\nu_x)_{x\in\Omega}$ such that for each Carathéodory function ϕ on $\Omega \times \mathbb{R}^m$, one has

$$\phi(x, f_k(x)) \overset{*}{\rightharpoonup} \bar{\phi}(x) = \int_{\mathbb{R}^m} \phi(x,\lambda)d\nu_x(\lambda) \text{ in } L^\infty(\Omega).$$

Now we want to apply the above idea into the case where f_k is bounded in $L^p(\Omega;\mathbb{R}^m)$ for some $p \in [1,\infty)$ and $f_k = Du_k$ for $u_k \in W^{1,p}(\Omega;\mathbb{R}^m)$. Suppose that $f_k \in L^p(\Omega;\mathbb{R}^m)$ and $(f_k)_k$ is a bounded sequence, let Ω be an open bounded subset of $\mathbb{R}^n$. Then by Ball's Theorem 2.4.4 on Young measures, there exists a family of probability measures $(\nu_x)_{x\in\Omega}$ such that for a subsequence of $(f_k)_k$, again denoted by $(f_k)_k$, and a continuous function $\varphi \in C(\mathbb{R}^m)$ such that if $\{\varphi(f_k)\}_k$ is sequentially weakly relatively compact in $L^1(\Omega)$, then we have

$$\varphi(f_k) \rightharpoonup \bar{\varphi}(x) = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu_x(\lambda) \text{ in } L^1(\Omega). \tag{2.34}$$

For instance, if φ satisfies

$$|\varphi(\lambda)| \leq C(1 + |\lambda|^q), \text{ for } \lambda \in \mathbb{R}^m \text{ and } 1 \leq q < p, \tag{2.35}$$

then from Dunford-Pettis Theorem 1.2.15 and Hölder's inequality, we have $\{\varphi(f_k)\}_k$ is sequentially weakly relatively compact, since $\|f_k\|_p$ is bounded together with $|\Omega|$ is finite imply that $|f_k|^q$ is equiintegrable for $1 \leq q < p$. Therefore, the conclusion (2.34) holds. However, for the sake of application, we are more interested in the case that $q = p$ here, to find ν generated by the sequence f_k, we need to decide when it identifies the weak limit. It is clear that (2.34) does not hold for any bounded sequence in $L^p(\Omega; \mathbb{R}^m)$, so an additional assumption on the sequence or restriction of notion of Young measure as a characterization of oscillatory behaviour is needed in order to identify the weak limit. This motivate the notion of p-Young measures.
We set

$$E^p(W) := \{\psi \in C(W) : \lim_{|A|\to\infty} \frac{\psi(A)}{1 + |A|^p} \text{ exists}\},$$

and

$$X^p(W) = \{\psi \in C(W) : |\psi(A)| \leq C(1 + |A|^p) \text{ for all } A \in W\}.$$

E^p is isomorphic to the continuous functions on the Alexandrov one-point compactification of $\mathbb{M}^{m\times n}$ and is separable. X^p is suggested by (2.35) and non-separable.
If, in addition, $|f_k|^p$ converges weakly in $L^1(\Omega)$, then by Dunford-Pettis Theorem 1.2.15, it is equiintegrable and so is $\varphi(f_k)$. Hence (2.34) holds. This lead to the notion of p-Young measures.

Definition 2.4.13 ($p-$Young measures). A family of measures $\nu_x, x \in \Omega$, is called a $p-$Young measure, if there is a sequence $f_k \in L^p(\Omega; \mathbb{R}^m)$, $1 \leq p < \infty$, and $g \in L^1(\Omega)$ such that
(i) $|f_k|^p \rightharpoonup g$ in $L^1(\Omega)$,
(ii) $\varphi(f_k) \rightharpoonup \overline{\varphi}$ in $L^1(\Omega)$, where

$$\overline{\varphi}(x) = \int_{\mathbb{R}^m} \psi(\lambda) d\nu_x(\lambda) \text{ in } \Omega \text{ a.e. for } \varphi \in E^p(\mathbb{R}^m).$$

Now, let u_k be a bounded sequence in $W^{1,p}(\Omega, \mathbb{R}^m)$, suppose that $f_k = Du_k$ generates the gradient $W^{1,p}$ Young measure ν. Then it is characterized by the following theorem [47, Theorem 1.1].

Theorem 2.4.14 ($W^{1,p}$ Gradient Young Measure). Let ν_x, $x \in \Omega$, be a family of probability measures in $C(\mathbb{M}^{m\times n})^*$. Then $(\nu_x)_{x\in\Omega}$ is a $W^{1,p}$ gradient Young measure if and only if
(i) there is a $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ such that

$$Du(x) = \int_{\mathbb{M}^{m\times n}} A d\nu_x(A) \text{ a.e. in } \Omega.$$

(ii) Jensen's inequality

$$\psi(Du(x)) \leq \int_{\mathbb{M}^{m\times n}} \psi(A) d\nu_x(A)$$

holds for all $\psi \in X^p(\mathbb{M}^{m\times n})$ quasiconvex, and
(iii) the function

$$\varphi(x) = \int_{\mathbb{M}^{m\times n}} |A|^p \, d\nu_x(A) \in L^1(\Omega).$$

Note that we say that $\psi \in C(\mathbb{M}^{m\times n})$ is quasiconvex if

$$\psi(A) \leq \frac{1}{|\Omega|} \int_{\Omega} \psi(A + D\zeta) dx \text{ for all } \zeta \in C_0^\infty(\Omega; \mathbb{R}^m) \text{ and } A \in \mathbb{M}^{m\times n}.$$

2.4.4 Strictly p-quasimonotonicity

Now, with the concept of gradient Young measures, we can define the strictly p-quasimonotone function.

Definition 2.4.15. A function $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is said to be strictly p-quasimonotone if

$$\int_{\mathbb{M}^{m\times n}} (\eta(\lambda) - \eta(\bar{\lambda})) : (\lambda - \bar{\lambda}) d\nu(\lambda) > 0,$$

for all homogeneous $W^{1,p}$ gradient Young measures ν with centre of mass $\bar{\lambda} = \langle \nu, id \rangle$ which are not a single Dirac mass.

In this chapter, we have seen various type of monotonicity and some of their properties. Now, we are ready to study some applications of them in some variational problems. In the next chapter, we will first look at boundary value problems, initial boundary value problems and variational inequalities governed by the monotone operators.

Chapter 3

Variational Problems Governed by Monotone Operators

In this chapter, we will study some monotone operators arising from the study of variational problems such as elliptic boundary value problems, parabolic initial boundary value problems and variational inequalities.
The approach here will be to consider a specific type of variational problems that are formulated in an abstract form involving a certain operator which will turn out to be monotone and then apply the abstract properties of monotonicity to establish an existence theorem for the variational problems and possibly some properties of the solutions.
We begin this chapter by first showing existence theorems for the stationary problems and evolution problems.

3.1 Existence theorem for abstract equations

3.1.1 Stationary problems

Let V be a reflexive and separable Banach space. Let the operator A be a mapping from V to V^*. For any $F \in V^*$, we want to find $u \in V$ such that the following operator equation on V^* holds

$$A(u) = F. \tag{3.1}$$

We introduce the following existence theorem for stationary problems with monotone operators.

Theorem 3.1.1. Let V be a reflexive separable Banach space and let the operator $A : V \to V^*$ be bounded, hemicontinuous, monotone and coercive. Then for any $F \in V^*$, there exists a solution $u \in V$ such that $A(u) = F$. If A is strictly monotone, then the solution is unique.

Proof. The existence follows from Theorem 4.1.2 and Proposition 2.2.11.
So it is sufficient to show the uniqueness. Let $u_1, u_2 \in V$ be such that $A(u_i) = F$ for $i = 1, 2.$, then we have

$$0 = \langle A(u_1) - A(u_2), v\rangle \;\; \forall v \in V.$$

Take $v = u_1 - u_2$, the result $u_1 = u_2$ follows from strict monotonicity of A. ■

Remark 3.1.2. If A is uniformly monotone, i.e., there exists a strictly increasing function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ and $\zeta(t) \to \infty$ as $t \to \infty$ such that

$$\langle A(u_1) - A(u_2), u_1 - u_2\rangle \geq \zeta(\|u_1 - u_2\|_V)\, \|u_1 - u_2\|_V \quad \text{for all } u_1, u_2 \in V,$$

which implies

$$\|A(u_1) - A(u_2)\|_{V^*} \geq \zeta(\|u_1 - u_2\|_V). \tag{3.2}$$

In this case, the solution u of $A(u) = f$ is unique and depends continuously on $f \in V^*$, i.e., Suppose $A(u_i) = f_i$ for $i = 1, 2.$, then by (3.2), we can get

$$\|u_1 - u_2\|_V \leq \zeta^{-1}(\|f_1 - f_2\|_{V^*}),$$

where $\zeta^{-1} : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $\zeta^{-1}(0) = 0$.

3.1.2 Evolution problems

Let $V \subset H \subset V^*$ be an evolution triple, let $1 < p, p' < \infty$ be such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and let $T > 0$. For fixed $t \in [0,T]$, let the operator $\tilde{B}(t)$ be a mapping from V to V^*. Define the operator $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ by

$$[B(u)](t) = [\tilde{B}(t)](u(t)) \quad \text{for any } u \in L^p(0,T;V).$$

Given $u_0 \in H$, for any $F \in L^{p'}(0,T;V^*)$, we want to find $u \in W^1_p(0,T;V,H)$ such that the following evolution equation on $L^{p'}(0,T;V^*)$ holds

$$u' + B(u) = F, \tag{3.3}$$

with the initial condition

$$u(0) = u_0.$$

By Theorem 1.3.14 (b), the initial condition is the sense that $\tilde{u}(0) = u_0$. This problem is also known as Cauchy problem.
We have the following existence theorem to the above evolution equation when $\tilde{B}(t)$ is monotone.

Theorem 3.1.3. Assume that for all fixed $t \in [0,T]$, $\tilde{B}(t) : V \to V^*$ is monotone, hemicontinuous and bounded in the sense of there exist a constant c_1 and a function $k_1 \in L^{p'}(0,T)$ such that

$$\left\|[\tilde{B}(t)](v)\right\|_{V^*} \le c_1 \|v\|_V^{p-1} + k_1(t) \quad \text{for all } v \in V,\ t \in [0,T]. \tag{3.4}$$

$\tilde{B}(t)$ is coercive in the sense of there exist a constant $c_2 > 0$ and a function $k_2 \in L^1(0,T)$ such that

$$\left\langle [\tilde{B}(t)](v), v \right\rangle \ge c_2 \|v\|_V^p - k_2(t) \quad \text{for all } v \in V, t \in [0,T]. \tag{3.5}$$

And finally, the function

$$t \mapsto \left\langle [\tilde{B}(t)](u), v \right\rangle,\ t \in [0,T] \text{ is measurable for any } u, v \in V. \tag{3.6}$$

Then for any arbitrary $F \in L^{p'}(0,T;V^*)$ and $u_0 \in H$, there exists a unique solution of (3.3) with the operator B defined by $[B(u)](t) = [\tilde{B}(t)](u(t))$.

Proof. Though the proof is quite standard, we write it in details here for reader's convenience. In fact, it is our view that this will guide better for readers while going through some keys results in the following chapter.
The proof is based on Galerkin's approximation method. Since V is separable, there exists a countable set of linear independent elements $z_1, z_2, \cdots, z_k, \cdots$ such that their finite linear combinations are dense in V.
1. (Galerkin's approximation)
Let V_k be the span of $\{z_1, \cdots, z_k\}$, since V is dense in H, so we let $(u_{k0})_k$ be a sequence in V_k such that $u_{k0} \to u_0$ in H. We are looking for the k-th Galerkin's approximation $u_k \in V_k$ of a solution u in the form $u_k = \sum_{i=1}^{k} a_{ik} z_i$ such that the following are satisfied:

$$\left\langle u_k'(t), z_j \right\rangle + \left\langle [\tilde{B}(t)](u_k(t)), z_j \right\rangle = \langle F(t), z_j \rangle \quad \text{for a.e. } t \in [0,T],\ \text{for all } j = 1, 2, \cdots, k., \tag{3.7}$$

$$u_k(0) = u_{k0} \in V_k. \tag{3.8}$$

(3.7) is a system of ordinary differential equations for a_{ik} because it has the form

$$\sum_{i=1}^{k} a_{ik}'(t)(z_j, z_i) + \left\langle \tilde{B}(t)\left[\sum_{i=1}^{k} a_{ik}(t) z_k\right], z_j \right\rangle = \langle F(t), z_j \rangle\,. \tag{3.9}$$

And (3.8) is equivalent to

$$a_{jk}(0) = \alpha_{j0} \quad \text{for } j = 1, 2, \cdots, k., \quad \text{where } u_{k0} = \sum_{j=1}^{k} \alpha_{j0} z_j. \tag{3.10}$$

This system of ODEs can be transformed to explicit system of ODEs since the the $\det(z_j, z_i) \neq 0$ (Gram determinant is non-zero) because $z_1, z_2, \cdots, z_k$ is set of linearly independent vectors. i.e., Set:

$$a_k(t) = \begin{pmatrix} a_{1k}(t) \\ \vdots \\ a_{kk}(t) \end{pmatrix}, \; a_k^{'}(t) = \begin{pmatrix} a_{1k}^{'}(t) \\ \vdots \\ a_{kk}^{'}(t) \end{pmatrix},$$

$$\left\langle \tilde{B}(t)[y], z \right\rangle = \begin{pmatrix} \left\langle \tilde{B}(t)[y], z_1 \right\rangle \\ \vdots \\ \left\langle \tilde{B}(t)[y], z_k \right\rangle \end{pmatrix}, \; \langle F(t), z \rangle = \begin{pmatrix} \langle F(t), z_1 \rangle \\ \vdots \\ \langle F(t), z_k \rangle \end{pmatrix}.$$

Let M be a $k \times k$ matrix with ji-th entry (z_j, z_i). Then (3.9) can be written as

$$Ma_k'(t) + \left\langle \tilde{B}(t) \left[\sum_{i=1}^{k} a_{ik} z_i \right], z \right\rangle = \langle F(t), z \rangle .$$

Since the Gram matrix M has non-zero determinant, we can obtain the explicit form

$$a_k'(t) = M^{-1} \Big(\langle F(t), z \rangle - \left\langle \tilde{B}(t) \left[\sum_{i=1}^{k} a_{ik} z_i \right], z \right\rangle \Big).$$

2 (Existence of Galerkin's approximation solution u_k)
Now, we want to use Theorem of Carathéodory 1.4.1 to prove the existence of a_{ik} in (3.9).
Set

$$b_j(t, w) = b_j(t, w_1, w_2, \cdots, w_k) = \left\langle \tilde{B}(t) \left[\sum_{i=1}^{k} w_i z_i \right], z_j \right\rangle \quad j = 1, 2, \cdots, k.$$

From (3.6), we get $b_j(t, w)$ are measurable in t for fixed w.
$\tilde{B}(t)$ is monotone, hemicontinuous and bounded. $\Rightarrow \tilde{B}(t)$ is pseudomonotone. $\Rightarrow \tilde{B}(t)$ is demicontinuous. i.e., $y_k \to y \Rightarrow \tilde{B}(t)[y_k] \overset{*}{\rightharpoonup} \tilde{B}(t)[y]$ in V^*. Therefore, for fixed t, $b_j(t, w)$ are continuous in w for all j.
From (3.4), we obtain

$$|b_j(t, w)| = \left| \left\langle \tilde{B}(t)(wz), z_j \right\rangle \right| \leq \left\| \tilde{B}(t)(wz) \right\|_{V^*} \|z_j\|_V \leq \|z_j\|_V \left(c_1 |w| \, \|z\|_V^{p-1} + k_1(t) \right) = K(t).$$

$K(t)$ is an integrable function. So apply Theorem of Carathéodory 1.4.1, there exists an absolute continuous function a_{ik} that is a solution in (3.9) in a neighbourhood of 0. i.e., $[0, T_k)$ where T_k depends on k.
3 (boundedness of solution u_k)
Now we want to find the prior estimate for the solution u_k, which allows us to extend the solution a_{ik} in $[0, T_k)$ to the whole interval $[0, T]$ and pass to the limit later. If u_k satisfies (3.7) in a neighbourhood of 0, then multiply (3.7) by $a_{jk}(t)$ and summing over j, we obtain that

$$\left\langle u_k^{'}(t), u_k(t) \right\rangle + \left\langle \tilde{B}(t)[u_k(t)], u_k(t) \right\rangle = \langle F(t), u_k(t) \rangle . \tag{3.11}$$

Integrate above equation over an interval $(0, t)$, $t \in [0, T]$. By Remark 1.3.15, we obtain

$$\frac{1}{2} \|u_k(t)\|_H^2 - \frac{1}{2} \|u_k(0)\|_H^2 + \int_0^t \left\langle \tilde{B}(\tau)[u_k(\tau)], u_m(\tau) \right\rangle d\tau = \int_0^t \langle F(\tau), u_k(\tau) \rangle \, d\tau.$$

By coercivity assumption in (3.5) and Hölder's inequality, we get

$$\frac{1}{2} \|u_k(t)\|_H^2 + c_2 \int_0^t \|u_k(\tau)\|_V^p \, d\tau \leq \frac{1}{2} \|u_k(0)\|_H^2 + \int_0^T k_2(\tau) d\tau + \|F\|_{L^{p'}(0,T;V^*)} \left(\int_0^t \|u_k(\tau)\|_V^p \, d\tau \right)^{\frac{1}{p}}.$$

As $c_2 > 0$ and $0 < \frac{1}{p} < 1$, above inequality implies

$$\int_0^t \|u_k(\tau)\|_V^p \, d\tau \le \text{const}, \quad \text{for any } t \in [0,T]. \tag{3.12}$$

and hence

$$\|u_k(t)\|_H^2 \le \text{const}, \quad \text{for any } t \in [0,T]. \tag{3.13}$$

Therefore, $a_{jk}(t)$ (defined in a neighbourhood $[0, T_k)$ of 0) can be estimated by a constant which is independent from t. So the solution a_{jk} can be extended to the whole interval $[0,T]$ by iterating the step above, i.e., apply Carathéodory Theorem 1.4.1 and use $a_{jm}(T_k) = \lim\limits_{t\to T_k^-} a_{jk}(t)$ as initial condition, we can find a_{jk} on the neighbourhood $[T_k, T_k+\epsilon)$ where $\epsilon > 0$.
4 (limit passage)
Now we are ready to pass the limit and we need the following lemma first.
From (3.12) and (3.13), we get $\|u_k\|_{L^p(0,T;V)}$ and $\sup\limits_{t\in[0,T]} \|u_k(t)\|_H$ are uniformly bounded by estimates that are independent of k. Therefore $\|B(u_k)\|_{L^{p'}(0,T;V^*)}$ is bounded too, because B is a bounded operator by (3.4). Since $L^p(0,T;V)$, $L^{p'}(0,T;V^*)$ and H are reflexive, there exists a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, and $u \in L^p(0,T;V)$, $w \in L^{p'}(0,T;V^*)$, $z \in H$ such that

$$u_k \rightharpoonup u \text{ in } L^p(0,T;V), \; B(u_k) \rightharpoonup w \text{ in } L^{p'}(0,T;V^*), \; u_k(T) \rightharpoonup z \text{ in } H. \tag{3.14}$$

Lemma 3.1.4. Let $V \subset H \subset V^*$ be an evolution triple and let $1 < p < \infty$. Assume u_k satisfies (3.7), $u_k \rightharpoonup u$ in $L^p(0,T;V)$, $B(u_k) \rightharpoonup w$ in $L^{p'}(0,T;V^*)$, $u_k(0) \to u_0$ in H and $u_k(T) \rightharpoonup z$ in H. Then $u \in W_p^1(0,T;V,H)$ and

$$u'(t) + w(t) = F(t), \; u(0) = u_0, \; u(T) = z. \tag{3.15}$$

proof of lemma. Let $\psi \in C^\infty(0,T)$ be an arbitrary function and $v \in V$ an arbitrary element. Since $\overline{\bigcup_{l=1}^{\infty} V_l} = V$, we may choose $v_l \in V_l$ such that

$$v_l \to v \text{ in } V. \tag{3.16}$$

Since $\psi v_l \in W_p^1(0,T;V,H), u_k \in W_p^1(0,T;V,H)$, by (3.7) and integration by parts formula

$$\begin{aligned}(u_k(T), \psi(T)v_l) - (u_k(0), \psi(0)v_l) &= \int_0^T \left\langle u_k'(t), \psi(t)v_l \right\rangle + \left\langle \psi'(t)v_l, u_k(t) \right\rangle dt \\ &= \int_0^T \left\langle F(t) - \tilde{B}(t)[u_k(t)], \psi(t)v_l \right\rangle + \left\langle \psi'(t)v_l, u_k(t) \right\rangle dt,\end{aligned}$$

by assumption of lemma, we obtain as $k \to \infty$ that

$$(z, \psi(T)v_l) - (u_0, \psi(0)v_l) = \int_0^T \langle F(t) - w(t), \psi(t)v_l \rangle + \left\langle \psi'(t)v_l, u(t) \right\rangle dt.$$

From (3.16), we pass the limit as $l \to \infty$ and get that

$$(z, \psi(T)v) - (u_0, \psi(0)v) = \int_0^T \langle F(t) - w(t), \psi(t)v \rangle + \left\langle \psi'(t)v, u(t) \right\rangle dt. \tag{3.17}$$

If $\psi \in C_c^\infty(0,T)$, we get from (3.17) that

$$\int_0^T \langle F(t) - w(t), v \rangle \, \psi(t) dt = -\int_0^T \langle u(t), v \rangle \, \psi'(t) dt,$$

this implies there exists $u' \in L^{p'}(0,T;V^*)$ with $u'(t) = F(t) - w(t)$. This means that

$$u \in W_p^1(0,T;V,H) \text{ and } u' = F - w. \tag{3.18}$$

Then by (3.17), (3.18) and integration by part formula, we obtain that for all $v \in V$,

$$(u(T), \psi(T)v) - (u(0), \psi(0)v) = \int_0^T \left\langle u^{'}(t), \psi(t)v \right\rangle + \left\langle \psi^{'}(t)v, u(t) \right\rangle dt = (z, \psi(T)v) - (u_0, \psi(0)v),$$

by choosing $\psi \in C^\infty(0,T)$ with $\psi(T) = 1, \psi(0) = 0$ and $\psi(T) = 0, \psi(0) = 1$ respectively, we get $u(T) = z$ and $u(0) = u_0$. Hence the lemma follows.

From (3.7), (3.8) and above lemma, we have

$$u \in W_p^1(0,T;V,H), \;\; u^{'}(t) + w(t) = F(t), \;\; u(0) = u_0, \;\; u(T) = z.$$

Now we need to show that $B(u) = w$ in $L^{p'}(0,T;V^*)$ by showing that $\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \leq 0$ and B is pseudomonotone.
Show that $\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \leq 0$:
By (3.11), we have

$$\int_0^T \left\langle [\tilde{B}(t)][u_k(t)], u_k(t) \right\rangle dt = \int_0^T \langle F(t), u_k(t)\rangle \, dt + \frac{1}{2} \|u_k(0)\|_H^2 - \frac{1}{2} \|u_k(T)\|_H^2 . \tag{3.19}$$

From (3.14), taking the limsup in above equation, we obtain

$$\limsup_{k\to\infty} \int_0^T \left\langle [\tilde{B}(t)][u_k(t)], u_k(t) \right\rangle dt = \int_0^T \langle F(t), u(t)\rangle \, dt + \frac{1}{2} \|u(0)\|_H^2 - \frac{1}{2} \liminf_{k\to\infty} \|u_k(T)\|_H^2 . \tag{3.20}$$

By (3.14) and (3.15), we have $u_k(T) \rightharpoonup u(T)$ in Hilbert space H, which implies

$$\|u(T)\|_H \leq \liminf_{k\to\infty} \|u_k(T)\|_H .$$

Then by (3.15), (3.20) and Remark 1.3.15, we obtain

$$\limsup_{k\to\infty} \langle B(u_k), u_k\rangle \leq \langle F, u\rangle + \frac{1}{2} \|u(0)\|_H^2 - \frac{1}{2} \|u(T)\|_H^2 = \left\langle u^{'}, u \right\rangle + \langle w, u\rangle + \frac{1}{2} \|u(0)\|_H^2 - \frac{1}{2} \|u(T)\|_H^2 = \langle w, u\rangle .$$

From (3.14), above inequality is equivalent to

$$\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \leq \langle w, u\rangle - \langle w, u\rangle = 0.$$

Show that $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is pseudomonotone:
It is easy to show that $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is bounded and monotone from (3.4) and $\tilde{B}(t)$ is monotone for fixed t. By using hemicontinuity of $\tilde{B}(t)$, (3.4) and Dominated Convergence Theorem 1.2.13, we can prove that B is also hemicontinuous. So by Proposition 2.2.11, B is pseudomonotone. Consequently, by (3.14) and $\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \leq 0$, we get $B(u) = w$, which shows the existence of the solution.
The uniqueness of solution follows from the monotonicity of B. Assume that u_1, u_2 are the solution of (3.3), we have for all $t \in [0,T]$,

$$\int_0^t \left\langle u_i^{'}(\tau), u_1(\tau) - u_2(\tau) \right\rangle d\tau + \langle B(u_i), u_1 - u_2\rangle = \langle F, u_1 - u_2\rangle \text{ for } i = 1,2., \text{ with } u_1(0) = u_2(0) = u_0.$$

Hence,

$$\int_0^t \left\langle u_1^{'}(\tau) - u_2^{'}(\tau), u_1(\tau) - u_2(\tau) \right\rangle d\tau + \langle B(u_1) - B(u_2), u_1 - u_2\rangle = 0.$$

The second term on the left hand side of above equation is non-negative, thus by Remark 1.3.15, $\|u_1(t) - u_2(t)\|_H^2 - \|u_1(0) - u_2(0)\|_H^2 \leq 0$, which implies

$$\|u_1(t) - u_2(t)\|_H^2 \leq 0 \text{ for each } t.$$

Therefore, $u_1 = u_2$. And the theorem is complete. ∎

Remark 3.1.5. 1. The idea of the above proof is using Galerkin method to find the solution u_k in finite dimensional space V_k. This problem is transformed into a systems of ODEs which has an explicit form. Theorem of Carathéodory allows a local solution u_k for each k. Coercivity allows the uniform boundedness in $[0, T]$ of $\|u_k(t)\|_H$ and $\|u_k\|_{L^p(0,T;V)}$, therefore the local solution u_k can be extended to the whole interval $[0, T]$. We also obtain from reflexivity that $u_k \rightharpoonup u$ in $L^p(0, T; V)$ and $B(u_k) \rightharpoonup w$ in $L^{p'}(0, T; V^*)$. The lemma in the proof shows that the weak limits u and w satisfy the abstract equation. Finally, the pseudomonotonicity allows us to show that the weak limit $B(u) = w$ and we will get the existence of the solution.
In the proof above, we find Galerkin's approximating sequence u_k has a subsequence that converges weakly to u, indeed, from the uniqueness of the solution u, the whole sequence converges to u weakly by using Cantor's trick.
2. If $\tilde{B}(t)$ in the above theorem is uniformly monotone in the sense that $\zeta(\|v_1 - v_2\|) = C\|v_1 - v_2\|^{p-1}$ for some constant $C > 0$ in Definition 2.1.2, i.e.,

$$\left\langle [\tilde{B}(t)](v_1) - [\tilde{B}(t)](v_2), v_1 - v_2 \right\rangle \geq C\|v_1 - v_2\|_V^p \quad \text{for all } v_1, v_2 \in V \text{ and for all } t \in [0, T]. \tag{3.21}$$

Then the solution of (3.3) depends on F and u_0 continuously.
Suppose u_j are solutions of (3.3) with $F = F_j, u_0 = u_{0j}, j = 1, 2$. Then for all $t \in [0, T]$, we can obtain

$$\begin{aligned} \|u_1(t) - u_2(t)\|_H^2 - \|u_{01} - u_{02}\|_H^2 + 2\int_0^t \left\langle \tilde{B}(\tau)[u_1(\tau)] - \tilde{B}(\tau)[u_2(\tau)], u_1(\tau) - u_2(\tau) \right\rangle d\tau \\ = 2\int_0^t \langle F_1(\tau) - F_2(\tau), u_1(\tau) - u_2(\tau)\rangle\, d\tau. \end{aligned}$$

Apply the uniform monotonicity (3.21) and Hölder's inequality, we obtain

$$\|u_1(t) - u_2(t)\|_H^2 - \|u_{01} - u_{02}\|_H^2 + 2C\|u_1 - u_2\|_{L^p(0,t;V)}^p \leq 2\|F_1 - F_2\|_{L^{p'}(0,t;V^*)} \|u_1 - u_2\|_{L^p(0,t;V)}.$$

Apply Young's inequality with a sufficiently small $\epsilon > 0$, i.e., choose $\epsilon < C$, then we can obtain

$$\begin{aligned} \|u_1(t) - u_2(t)\|_H^2 + C\|u_1 - u_2\|_{L^p(0,t;V)}^p &\leq \tilde{C}(\epsilon)\|F_1 - F_2\|_{L^{p'}(0,t;V^*)}^{p'} + \|u_{01} - u_{02}\|_H^2 \\ &\leq \tilde{C}(\epsilon)\|F_1 - F_2\|_{L^{p'}(0,T;V^*)}^{p'} + \|u_{01} - u_{02}\|_H^2. \end{aligned}$$

3. If $\tilde{B}(t)$ is uniformly monotone in the sense of (3.21) above for $t \in [0, T]$, the whole Galerkin's approximation solution sequence (u_k) converges to u strongly in $L^p(0, T; V)$. i.e., from (3.21),

$$\begin{aligned} C\int_0^T \|u_k(t) - u(t)\|_V^p dt &\leq \int_0^T \left\langle \tilde{B}(t)[(u_k(t))] - \tilde{B}(t)[u(t)], u_k(t) - u(t) \right\rangle dt \\ &= \langle B(u_k) - B(u), u_k - u\rangle = \langle B(u_k), u_k - u\rangle - \langle B(u), u_k - u\rangle \to 0, \end{aligned}$$

by (3.14), $\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \leq 0$ and pseudomontonicity of B.

3.2 Application into variational problems

In this section, we shall apply the existence theorems that are established in the previous section to show the existence of solutions to boundary value problems and initial boundary value problems.

3.2.1 Elliptic boundary value problems

Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary. Let $1 < p, p' < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and let $V = W_0^{1,p}(\Omega)$. For $j = 0, 1, \cdots n.$, functions $a_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We consider the following boundary value problems:
given $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ such that

$$-\sum_{j=1}^n D_j[a_j(x, u(x), Du(x))] + a_0(x, u(x), Du(x)) = f \text{ in } \Omega, \tag{3.22}$$

(where $D_j = \frac{\partial}{\partial x_j}$ and $Du = (D_1 u, D_2 u, \cdots, D_n u)$), with the boundary condition

$$u|_{\partial\Omega} = \varphi. \tag{3.23}$$

Definition 3.2.1 (Strong Solution). The strong solution to the above problem is defined as $u \in C^2(\bar{\Omega})$ such that (3.22) and (3.23) hold.

Weak Formulation: Now suppose that u is the strong solution to the above problem, then we multiply (3.22) by a test function $v \in C_c^1(\Omega)$ and integrate over Ω, using Gauss's theorem, we obtain that

$$\sum_{j=1}^{n} \int_\Omega a_j(x, u(x), Du(x)) D_j v(x) dx + \int_\Omega a_0(x, u(x), Du(x)) v(x) dx = \int_\Omega f(x) v(x) dx. \tag{3.24}$$

Now if $f \in L^{p'}(\Omega)$ and a_j satisfy a certain growth condition such that $a_j \in L^{p'}(\Omega)$ for any $u \in W^{1,p}(\Omega)$, then (3.24) holds for any test function $v \in W_0^{1,p}(\Omega)$, since $W_0^{1,p}(\Omega)$ is the closure of $C_c^1(\Omega)$ with respect to the norm $W^{1,p}$.
We reach to the following definition of the weak solution to the above problem.

Definition 3.2.2 (Weak Solution). The weak solution to the above problem is defined as $u \in W^{1,p}(\Omega)$ such that (3.24) holds for all $v \in W_0^{1,p}(\Omega)$ and (3.23) holds where $u|_{\partial\Omega}$ stands for the trace of $u \in W^{1,p}(\Omega)$. In particular, if $\varphi = 0$ (homogeneous boundary condition), the weak solution is defined as $u \in W_0^{1,p}(\Omega)$ such that (3.24) for all $v \in W_0^{1,p}(\Omega)$.

We first want to find the weak solution of the problem (3.22) with the homogeneous boundary condition $\varphi = 0$. In this case, we are looking for $u \in V$ such that (3.24) holds for any $v \in V$. Assume functions a_j satisfy the following conditions:
(E1) (Carathéodory condition): a_j are Carathéodory functions, i.e., for all $j = 0, 1, \cdots, n.$,
for a.a. fixed $x \in \Omega$, $\xi \to a_j(x, \xi)$ are continuous for all $\xi \in \mathbb{R}^{n+1}$;
for any fixed $\xi \in \mathbb{R}^{n+1}$, $x \to a_j(x, \xi)$ are measurable for all $x \in \Omega$.
(E2) (Growth condition): there exist a constant $c_1 > 0$ and a nonnegative function $k_1 \in L^{p'}(\Omega)$ such that for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^{n+1}$

$$|a_j(x, \xi)| \leq c_1 |\xi|^{p-1} + k_1(x). \tag{3.25}$$

(E3) (Monotonicity condition): for a.e. $x \in \Omega$ and any $\xi, \xi' \in \mathbb{R}^{n+1}$, there holds

$$\sum_{j=0}^{n} [a_j(x, \xi) - a_j(x, \xi')](\xi - \xi') \geq 0. \tag{3.26}$$

(E4) (Coercive condition): there exists a constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$ such that for a.a $x \in \Omega$ and any $\xi \in \mathbb{R}^{n+1}$,

$$\sum_{j=0}^{n} a_j(x, \xi)\xi \geq c_2 |\xi|^p - k_2(x). \tag{3.27}$$

With the growth condition (E2), we may define the operator $A : W_0^{1,p}(\Omega) = V \to V^*$ by

$$\langle A(u), v\rangle = \sum_{j=1}^{n} \int_\Omega a_j(x, u(x), Du(x)) D_j v(x) dx + \int_\Omega a_0(x, u(x), Du(x)) v(x) dx, \quad \text{for } v \in V. \tag{3.28}$$

For $f \in L^{p'}(\Omega)$, we can define $F \in V^*$ as

$$\langle F, v\rangle = \int_\Omega f(x) v(x) dx. \tag{3.29}$$

Then (3.24) may be rewritten as

$$\langle A(u), v\rangle = \langle F, v\rangle \quad \text{for any } v \in V,$$

or equivalently the operator equation on V^*

$$A(u) = F.$$

To show the existence of $u \in V$, all we need to do now is to show that the operator A defined as (3.28) satisfies the assumptions in Theorem 3.1.1.

Proposition 3.2.3. Under conditions (E1) and (E2), the operator A defined as in (3.28) is bounded and hemicontinuous.

Proof. The boundedness of A directly follows from (E1), (E2) and Hölder's inequality.
It remains to show that A is hemicontinuous. Fix $u_1, u_2, v \in V$, consider the function

$$\lambda \mapsto \langle A(u_1 + \lambda u_2), v \rangle, \ \ \lambda \in \mathbb{R}.$$

Let $(\lambda_k)_k \subset \mathbb{R}$ be such that $\lim_{k\to\infty} \lambda_k = \lambda$, from (E1), we have for $j = 0, 1, \cdots, n.$, and for a.e. $x \in \Omega$, there holds

$$\lim_{k\to\infty} a_j(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2) = a_j(x, u_1 + \lambda u_2, Du_1 + \lambda Du_2).$$

From (E2) and boundedness of λ_k, we obtain for $j = 0, 1, \cdots, n.$,

$$\begin{aligned}|a_j(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^{p'} &\leq \text{const}[|(u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^p + k_1(x)^{p'}] \\ &\leq \text{const}[|u_1|^p + |u_2|^p + |Du_1|^p + |Du_2|^p + k_1(x)^{p'}].\end{aligned}$$

Hence we obtain by Young's inequality that for $j = 1, 2, \cdots, n.$,

$$|a_j(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2) D_j v| \leq \text{const}[|u_1|^p + |u_2|^p + |Du_1|^p + |Du_2|^p + k_1(x)^{p'}] + \text{const}\,|D_j v|^p.$$

Similarly, we can get the boundedness for $|a_0(x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)v|$. Then by Dominated Convergence Theorem 1.2.13, we have $\lim_{k\to\infty} \langle A(u_1 + \lambda_k u_2), v \rangle = \langle A(u_1 + \lambda u_2), v \rangle$, i.e., A is hemicontinuous. ■

It is obvious to see that (E3) and (E4) imply that A is monotone and coercive respectively. Under conditions (E1)-(E4), the assumptions of Theorem 3.1.1 hold, so there exists a weak solution $u \in V$ of the problem (3.22) with the homogeneous boundary condition.

Remark 3.2.4. For the non-homogeneous boundary condition $u|_{\partial\Omega} = \psi$, we can reduce it to the homogeneous case by translation. i.e., define $A_0(u) = A(u + \psi)$, and it is easy to see that A_0 satisfies the assumptions (monotone, bounded, hemicontinuous and coercive) in Theorem 3.1.1.

3.2.2 Parabolic initial boundary value problems

In this section, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with sufficiently smooth boundary. Let $1 < p' \leq 2 \leq p < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and let $T > 0$. Let $V = W_0^{1,p}(\Omega)$ and $H = L^2(\Omega)$, then $V \subset H \subset V^*$ is an evolution triple. Set $Q_T = (0, T) \times \Omega$ and $\Gamma_T = [0, T) \times \partial\Omega$. For $j = 0, 1, \cdots, n.$, functions $b_j : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We consider the following initial boundary value problem:
find $u : [0, T] \times \Omega \to \mathbb{R}$ such that

$$u_t - \sum_{j=1}^{n} D_j[b_j(t, x, u(t, x), Du(t, x))] + b_0(t, x, u(t, x), Du(t, x)) = f \ \text{ in } Q_T, \tag{3.30}$$

with the homogeneous boundary condition

$$u|_{\Gamma_T} = 0 \text{ on } \Gamma_T, \tag{3.31}$$

and the initial condition

$$u(0, x) = h(x), \ \forall x \in \Omega. \tag{3.32}$$

Definition 3.2.5 (Strong solution). A function $u \in C^{1,2}(\overline{Q_T})$ (continuously differentiable with respect to t and twice continuously differentiable with respect to x in Q_T) satisfying above (3.30) - (3.32) is called a strong solution.

Weak formulation: In order to define the weak solution of (3.30) with the boundary condition (3.31) and the initial condition (3.32). Similar to the previous section, assume b_j satisfy a certain growth condition such that for a.e. fixed $t \in [0,T]$, $x \mapsto b_j(t,x,u(t,x),Du(t,x)) \in L^{p'}(\Omega)$ if $x \mapsto u(t,x) \in W_0^{1,p}(\Omega)$, and assume $x \mapsto f(t,x) \in L^{p'}(\Omega)$ for all fixed $t \in [0,T]$. Then we multiply the equation (3.30) by a test function $v \in W_0^{1,p}(\Omega)$ and then integrate over Ω, we obtain from Gauss formula that for $t \in [0,T]$,

$$\int_\Omega u_t v dx + \sum_{j=1}^n \int_\Omega b_j(t,x,u,Du) D_j v dx + \int_\Omega b_0(t,x,u,Du) v dx = \int_\Omega f v dx. \tag{3.33}$$

So we reach to the following definition of the weak solution to the above problem (3.30) - (3.32).

Definition 3.2.6 (Weak solution). The weak solution is defined as $u \in W_p^1(0,T;V,H)$ such that (3.33) holds for all $v \in W_0^{1,p}(\Omega)$ and (3.32) holds.

Remark 3.2.7. The statement, (3.33) holds for all $v \in W_0^{1,p}(\Omega)$, implies that

$$\int_{Q_T} (D_t u) v dx dt + \sum_{j=1}^n \int_{Q_T} b_j(t,x,u,Du) D_j v dx dt + \int_{Q_T} b_0(t,x,u,Du) v dx dt = \int_{Q_T} f v dx dt. \tag{3.34}$$

Assume functions $b_j,\ j \in \{0,1,\cdots n\}$ satisfy the following:

(A1) (Carathéodory condition): Functions $b_j : Q_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy the Carathéodory condition, i.e.,
for a.e. fixed $(t,x) \in Q_T = (0,T) \times \Omega$, $\xi \mapsto b_j(t,x,\xi), \xi \in \mathbb{R}^{n+1}$ is continuous,
for each fixed $\xi \in \mathbb{R}^{n+1}$, $(t,x) \mapsto b_j(t,x,\xi)$, $(t,x) \in Q_T$ is measurable.

(A2) (Growth condition): There exist a constant $c_1 > 0$ and a function $k_1 \in L^{p'}(Q_T)$ such that for a.e. $(t,x) \in Q_T$ and all $\xi \in \mathbb{R}^{n+1}$,

$$|b_j(t,x,\xi)| \le c_1 |\xi|^{p-1} + k_1(t,x).$$

(A3) (Monotonicity condition): For a.e. $(t,x) \in Q_T$ and all $\xi, \xi^* \in \mathbb{R}^{n+1}$

$$\sum_{j=0}^n [b_j(t,x,\xi) - b_j(t,x,\xi^*)](\xi_j - \xi_j^*) \ge 0.$$

(A4) (Coercive condition): There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(Q_T)$ such that for a.e. $(t,x) \in Q_T$ and all $\xi \in \mathbb{R}^{m+1}$

$$\sum_{j=0}^n b_j(t,x,\xi)\xi_j \ge c_2 |\xi|^p - k_2(t,x).$$

To show the existence of u, we first need to write (3.33) into an evolution equation. For fixed $t \in [0,T]$, set

$$u(t) := x \mapsto u(t,x) \quad \text{for } x \in \Omega.$$

For $u \in L^p(0,T;V)$ and $v \in V$, from the growth condition (A2), we can define the operator $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ and $\tilde{B}(t) : V \to V^*$ by

$$\langle [B(u)](t), v \rangle = \left\langle [\tilde{B}(t)](u(t)), v \right\rangle = \sum_{j=1}^n \int_\Omega b_j(t,x,u,Du) D_j v dx + \int_\Omega b_0(t,x,u,Du) v dx. \tag{3.35}$$

Assume $x \mapsto f(t,x) \in V^*$, we can define $F(t) \in V^*$ by

$$\langle F(t), v \rangle = \int_\Omega f(t,x) v(x) dx.$$

Assume $x \mapsto u_t(t,x) \in V^*$, define $u'(t) \in V^*$ by

$$\langle u'(t), v \rangle = \int_\Omega u_t(t,x) v(x) dx.$$

Then for each fixed $t \in [0,T]$,

$$[B(u)](t) \in V^*,\ u'(t) \in V^*,\ F(t) \in V^*,$$

so (3.33) can be written as the following evolution equation on V^*:

$$u'(t) + [B(u)](t) = F(t),\ t \in [0,T]. \tag{3.36}$$

This is equivalent to the following evolution equation on $L^{p'}(0,T;V^*)$:

$$u' + B(u) = F.$$

It remains to show that $\tilde{B}(t)$ satisfies the assumptions in Theorem 3.1.3.

Proposition 3.2.8. Assume that (A1) and (A2), then the operator $\tilde{B}(t) : V \to V^*$ defined as in (3.35) satisfies for all fixed $t \in [0,T]$, $\tilde{B}(t)$ is hemicontinuous and bounded in the sense of there exist a suitable constant $c_1 > 0$ and a function $g_1 \in L^{p'}(0,T)$ such that

$$\left\|[\tilde{B}(t)](u)\right\|_{V^*} \le c_1 \|u\|_V^{p-1} + g_1(t) \quad \text{for all } u \in V.$$

Proof. The proof is similar to the proof of Proposition 3.2.3. For fixed $t \in [0,T]$, we may write $u(x)$ instead of $u(t,x)$, we have $x \mapsto b_j(t,x,u(x),Du(x))$ is measurable for any $u \in V$. Note that from (A2), for fixed t, we have $k_1(t,\cdot) \in L^{p'}(\Omega)$, set $g_1(t) = \left(\int_\Omega |k_1(t,x)|^{p'}\, dx\right)^{\frac{1}{p'}}$, then $g_1(t) \le C$ for some constant $C > 0$, and we obtain that for all $j = 0, 1, \cdots, n.$,

$$\int_\Omega |b_j(t,x,u(x),Du(x)|^{p'}\, dx \le \ \text{const}\left[\int_\Omega |u(x),Du(x)|^p\, dx + \int_\Omega |k_1(t,x)|^{p'}\, dx\right] \le \ \text{const}[\|u\|_V^p + (g_1(t))^{p'}].$$

By using the Hölder's inequality, we obtain

$$\left|\left\langle [\tilde{B}(t)](u), v\right\rangle\right| \le \int_\Omega \left|\sum_{j=1}^n b_j(t,x,u,Du)D_jv + b_0(t,x,u,Du)v\right| dx$$

$$\le \sum_{j=1}^n \left(\int_\Omega |b_j(t,x,u,Du)|^{p'}\, dx\right)^{\frac{1}{p'}} \|D_jv\|_{L^p} + \left(\int_\Omega |b_0(t,x,u,Du)|^{p'}\, dx\right)^{\frac{1}{p'}} \|v\|_{L^p} \le \ \text{const}[\|u\|_V^{\frac{p}{p'}} + g_1(t)]\, \|v\|_V\,.$$

Note that $\frac{p}{p'} = p - 1$, so

$$\left\|[\tilde{B}(t)](u)\right\|_{V^*} \le \ \text{const}[\|u\|_V^{p-1} + g_1(t)].$$

Hence, the boundedness of $\tilde{B}(t)$ follows.
Now we show that $\tilde{B}(t) : V \to V^*$ is hemicontinuous for fixed $t \in [0,T]$. For any $u_1, u_2, v \in V$, we need to show that the function $\lambda \mapsto \left\langle [\tilde{B}(t)](u_1 + \lambda u_2), v\right\rangle$ is continuous in $\lambda \in \mathbb{R}$.
Assume $\lim_{k\to\infty} \lambda_k = \lambda$ for a sequence $(\lambda_k)_k$ in $\mathbb{R}$. Then $|\lambda_k|$ is bounded for all $k \in \mathbb{N}$.
By (A1), for fixed $t \in [0,T]$, for a.e. $x \in \Omega$, for $j = 0, 1, \cdots n.$,

$$\lim_{k\to\infty} b_j(t,x,u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2) = b_j(t,x,u_1 + \lambda u_2, Du_1 + \lambda Du_2),$$

further by (A2), write $k_1(x)$ instead of $k_1(t,x)$, we have

$$\begin{aligned} |b_j(t,x,u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^{p'} &\le \ \text{const}[|(u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^p + |k_1(x)|^{p'}] \\ &\le \ \text{const}[|u_1|^p + |u_2|^p + |Du_1|^p + |Du_2|^p + |k_1(x)|^{p'}], \end{aligned}$$

where the last inequality follows from the boundedness of $|\lambda_k|$. And using Young's inequality, we get for all $j = 1, 2, \cdots n.$,

$$\begin{aligned} |b_j(t,x,u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)D_jv| &\le \ \text{const}\,|b_j(t,x,u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)|^{p'} + \ \text{const}\,|D_jv|^p \\ &\le \ \text{const}[|u_1|^p + |u_2|^p + |Du_1|^p + |Du_2|^p + |k_1(x)|^{p'}] + \ \text{const}\,|D_jv|^p\,, \end{aligned}$$

i.e., it is dominated by a integrable function in $L^1(\Omega)$ because of $u_1, u_2, v \in V = W_0^{1,p}(\Omega)$ and $|k_1(x)|^{p'} \in L^1(\Omega)$. We can obtain the similar result for $|b_0(t, x, u_1 + \lambda_k u_2, Du_1 + \lambda_k Du_2)v|$. Then by Dominated Convergence Theorem 1.2.13, we have

$$\lim_{k\to\infty} \left\langle [\tilde{B}(t)](u_1 + \lambda_k u_2), v \right\rangle = \left\langle [\tilde{B}(t)](u_1 + \lambda u_2), v \right\rangle.$$

Hence $\tilde{B}(t)$ is hemicontinuous. ■

Proposition 3.2.9. If (A3) is satisfied, then for all fixed $t \in [0, T]$, $\tilde{B}(t)$ defined above is monotone.

Proof. For fixed $t \in [0, T]$, we obtain by definition of $\tilde{B}(t)$ and (A3) that for any $u_1, u_2 \in V$,

$$\left\langle [\tilde{B}(t)](u_1) - [\tilde{B}(t)](u_2), u_1 - u_2 \right\rangle = \int_\Omega \sum_{j=1}^n [b_j(t, x, u_1(x), Du_1(x)) - b_j(t, x, u_2(x), Du_2(x))] D_j(u_1(x) - u_2(x))$$
$$+ [b_0(t, x, u_1(x), Du_1(x)) - b_0(t, x, u_2(x), Du_2(x))](u_1(x) - u_2(x))dx \geq 0.$$

■

Proposition 3.2.10. Assume (A4) is satisfied, then for all fixed $t \in [0, T]$, $\tilde{B}(t)$ is coercive in the sense that there exists $c_2 > 0$ and $g_2 \in L^1(0, T)$ such that

$$\left\langle [\tilde{B}(t)](u), u \right\rangle \geq c_2 \|u\|_V^p - g_2(t) \quad \text{for all } u \in V.$$

Proof. By (A4), we can obtain that

$$\left\langle [\tilde{B}(t)](u), u \right\rangle = \int_\Omega \sum_{j=1}^n b_j(t, x, u, Du) D_j u + b_0(t, x, u, Du) u dx \geq c_2 \int_\Omega |u, Du|^p \, dx - \int_\Omega k_2(t, x) dx$$
$$\geq c_2' \|u\|_V^p - g_2(t),$$

where $g_2(t) = \int_\Omega k_2(t, x) dx \in L^1(0, T)$ since $k_2(t, x) \in L^1(Q_T)$. ■

The following theorem can be obtained directly from above Proposition 3.2.8, Proposition 3.2.9, Proposition 3.2.10 and Theorem 3.1.3.

Theorem 3.2.11. Assume that (A1) - (A4) are satisfied, then the operators $B : L^p(0, T; V) \to L^{p'}(0, T; V^*)$ and $\tilde{B}(t) : V \to V^*$ defined as in (3.33), satisfies the assumptions of Theorem 3.1.3. Thus, for any $F \in L^{p'}(0, T; V^*)$ and $u_0 \in H = L^2(\Omega)$, there exists a unique solution of

$$u'(t) + [B(u)](t) = F(t) \quad \text{for all } t \in [0, T], \quad \text{with } u(0) = u_0.$$

We formulate conditions on b_j for which the underlined operator $\tilde{B}(t)$ is uniformly monotone.

Proposition 3.2.12. For all $j = 0, 1, \cdots, n.$, assume that functions b_j satisfy (A1), and for a.e. $(t, x) \in Q_T$, the functions $\xi \mapsto b_j(t, x, \xi)$ are continuously differentiable and the matrix

$$\left(\frac{\partial b_j(t, x, \xi)}{\partial \xi_k} \right)_{j,k=0}^n \quad \text{is positive semidefinite.} \tag{3.37}$$

Then (A3) holds, and therefore $\tilde{B}(t)$ is monotone.

Proof. For fixed t, x, ξ, ξ^*, for $j = 0, 1, \cdots, n.$, define the function:

$$h_j(\tau) = b_j(t, x, \xi^* + \tau(\xi - \xi^*)), \ \tau \in \mathbb{R}.$$

By differentiability of $\xi \mapsto b_j(t, x, \xi)$, we get

$$h_j(1) - h_j(0) = \int_0^1 h_j'(\tau) d\tau,$$

which is equivalent to

$$b_j(t,x,\xi) - b_j(t,x,\xi^*) = \int_0^1 \sum_{k=0}^n \frac{\partial a_j}{\partial \xi_k}(t,x,\xi^* + \tau(\xi - \xi^*))(\xi_k - \xi_k^*)d\tau. \tag{3.38}$$

Hence we obtain by (3.37) that

$$\sum_{j=0}^n [b_j(t,x,\xi) - b_j(t,x,\xi^*)](\xi_j - \xi_j^*) = \int_0^1 \sum_{j,k=0}^n \frac{\partial b_j}{\partial \xi_k}(t,x,\xi^* + \tau(\xi - \xi^*))(\xi_k - \xi_k^*)(\xi_j - \xi_j^*)d\tau \geq 0.$$

■

Proposition 3.2.13. Assume that the assumptions in Proposition 3.2.12 above are satisfied and that

$$\sum_{j,k=0}^n \frac{\partial a_j}{\partial \xi_k}(t,x,\xi)\eta_j\eta_k \geq c_3 \sum_{j=0}^n |\xi_j|^{p-2} |\eta_j|^2 \text{ for a.e. } (t,x) \in Q_T \text{ and for every } \xi,\ \eta \in \mathbb{R}^{n+1}, \tag{3.39}$$

with $p \geq 2$ and some constant $c_3 > 0$.
Then for some $\tilde{c}_3 > 0$, we have for a.e. $(t,x) \in Q_T$ and for every $\xi,\ \xi^* \in \mathbb{R}^{n+1}$ that

$$\sum_{j=0}^n [b_j(t,x,\xi) - b_j(t,x,\xi^*)](\xi_j - \xi_j^*) \geq \tilde{c}_3 \sum_{j=0}^n |\xi_j - \xi_j^*|^p. \tag{3.40}$$

Proof. From (3.38) and (3.39), we obtain

$$\begin{aligned}\sum_{j=0}^n [b_j(t,x,\xi) - b_j(t,x,\xi^*)](\xi_j - \xi_j^*) &= \int_0^1 \sum_{j,k=0}^n \frac{\partial b_j}{\partial \xi_k}(t,x,\xi^* + \tau(\xi - \xi^*))(\xi_k - \xi_k^*)(\xi_j - \xi_j^*)d\tau \\ &\geq \int_0^1 c_3 \sum_{j=0}^n |\xi_j^* + \tau(\xi_j - \xi_j^*)|^{p-2} |\xi_j - \xi_j^*|^2 d\tau. \end{aligned} \tag{3.41}$$

In order to show (3.40), it is sufficient to show that there exists $c_4 > 0$ (depending only on p) such that

$$\int_0^1 |\xi_j^* + \tau(\xi_j - \xi_j^*)|^{p-2} d\tau \geq c_4 |\xi_j - \xi_j^*|^{p-2}. \tag{3.42}$$

If $\xi_j - \xi_j^* = 0$, then above inequality holds. For $\xi_j - \xi_j^* \neq 0$, we have

$$\int_0^1 |\xi_j^* + \tau(\xi_j - \xi_j^*)|^{p-2} d\tau = |\xi_j - \xi_j^*|^{p-2} \int_0^1 \left| \frac{\xi_j^*}{\xi_j - \xi_j^*} + \tau \right|^{p-2} d\tau.$$

Set $d = \frac{\xi_j^*}{\xi_j - \xi_j^*}$, we will show that there exists $c_4 > 0$ (independent of d) such that

$$\int_0^1 |d + \tau|^{p-2} d\tau \geq c_4. \tag{3.43}$$

If $d \geq 0$ or $d \leq -1$, then $d + \tau$ has the same sign for $\tau \in [0,1]$ and $|d + \tau| \geq |\tau|$. So $p - 2 \geq 0$ implies the following

$$\int_0^1 |d + \tau|^{p-2} d\tau \geq \int_0^1 |\tau|^{p-2} d\tau = \frac{1}{p-1}.$$

In the case where $-1 < d < 0$, we have

$$\int_0^1 |d + \tau|^{p-2} d\tau = \int_0^{-d} (-d - \tau)^{p-2} d\tau + \int_{-d}^1 (d + \tau)^{p-2} d\tau = \frac{(-d)^{p-1}}{p-1} + \frac{(d+1)^{p-1}}{p-1} \geq \frac{1}{2^{p-2}(p-1)}.$$

So (3.43) is true and therefore (3.42) holds. We obtain from (3.41) that

$$\sum_{j=0}^n [b_j(t,x,\xi) - b_j(t,x,\xi^*)](\xi_j - \xi_j^*) \geq c_3 c_4 \sum_{j=0}^n |\xi_j - \xi_j^*|^p,$$

which completes the proof. ■

Remark 3.2.14. Above proposition implies that if we assume that the assumptions in Proposition 3.2.12 and Proposition 3.2.13 hold instead of (A3) in Theorem 3.2.11, then for each fixed $t \in [0, T]$, $\tilde{B}(t)$ is uniformly monotone in the sense that for all $u, v \in V$,

$$\left\langle [\tilde{B}(t)](u) - [\tilde{B}(t)](v), u - v \right\rangle \geq \tilde{c}_3 \, \|u - v\|_V^p .$$

Therefore, the solution u to

$$u'(t) + [B(u)](t) = F(t) \text{ for all } t \in [0, T], \; u(0) = u_0,$$

is unique and depends continuously on f and u_0 according to 2 and 3 in Remark 3.1.5. Moreover, the approximating sequence $(u_k)_k$ constructed by Galerkin's method converges strongly to u in $L^p(0, T; V)$.

3.3 Abstract Elliptic Variational Inequality

3.3.1 Existence Theorem

In this section, we will consider the abstract elliptic variational inequalities and present some existence results. Let V be a reflexive Banach space and let V^* be its dual space. Denote $\langle \cdot, \cdot \rangle$ as the duality pair between V^* and V. Let $K \subset V$ be a closed and convex subset. Let operator A be a nonlinear operator from K to V^*.
Consider the following problem:
Given $F \in V^*$, find $u \in K$ such that

$$\langle A(u), v - u \rangle \geq \langle F, v - u \rangle \;\text{ for all } v \in K. \tag{3.44}$$

Sometimes, we only consider the particular case where $F = 0$.

Remark 3.3.1. In the particular case where $K = V$, then (3.44) is equivalent to the stationary equation, i.e., given $F \in V^*$, find $u \in V$ such that $A(u) = F$.

Before we introduce the existence result, we need the following definition.

Definition 3.3.2. We say the mapping $A : K \to V^*$ is continuous on finite dimensional subspaces if for any finite dimensional subspace $L \subset V$, the restriction of A on $K \cap L$ is weakly continuous, i.e., for $(u_k)_k \subset K \cap L$ with $\lim_{k\to\infty} u_k = u \in K \cap L$, we have $\lim_{k\to\infty} \langle A(u_k), v\rangle \to \langle A(u), v\rangle$ for all $v \in V$.

To introduce the existence result on V, we also need the following Minty's lemma (see [62]) and the existence result for variational inequalities on $\mathbb{R}^n$ (first theorem on variational inequalities).

Lemma 3.3.3 (Minty). Let $A : K \to V^*$ be monotone and continuous on finite dimensional subspaces. Then the following two statements are equivalent: u satisfies

$$u \in K : \langle A(u), v - u \rangle \geq 0 \text{ for all } v \in K; \tag{3.45}$$

$$u \in K : \langle A(v), v - u \rangle \geq 0 \text{ for all } v \in K. \tag{3.46}$$

Theorem 3.3.4. Let $K \subset \mathbb{R}^n$ be a compact and convex subset. Let $A : K \to (\mathbb{R}^n)^*$ be continuous. Then there exists $u \in K$ such that

$$\langle A(u), v - u \rangle \geq 0, \;\text{ for all } v \in K.$$

Now we introduce the following existence theorem. The proof is based on the Minty's lemma and the above existence theorem on $\mathbb{R}^n$.

Theorem 3.3.5 ([49, Theorem 1.4, Chapter III]). Let $K \subset V$ be a closed bounded and convex subset. Let the operator $A : K \to V^*$ be monotone and continuous on finite dimensional subspaces. Then there exists $u \in K$ such that

$$\langle A(u), v - u \rangle \geq 0, \;\text{ for all } v \in K. \tag{3.47}$$

If the boundedness condition of K is removed, then the following equivalent condition is needed on the operator A for the existence result.

Theorem 3.3.6 ([49, Theorem 1.7, Chapter III]). Let $K \subset V$ be a closed and convex subset. Let the operator $A : K \to V^*$ be monotone and continuous on finite dimensional subspaces. Then there exists $u \in K$ of the variational inequality

$$\langle A(u), v - u \rangle \geq 0, \text{ for all } v \in K,$$

if and only if there exists $R > 0$ such that $u_R \in K_R := K \cap \{v : \|v\|_V \leq R\}$ and

$$\langle A(u_R), v - u_R \rangle \geq 0 \text{ for all } v \in K_R.$$

In the next chapter, we will prove the existence Theorem 4.3.6 with a weaker assumption of the operator A being pseudomonotone. By Proposition 2.2.11, we know if A is bounded, hemicontinuous and monotone, then A is pseudomonotone. Hence, we can state the following theorem.

Theorem 3.3.7. Let V be a reflexive Banach space and let $K \subset V$ be a closed and convex subset. Let the operator $A : K \to V^*$ be bounded, hemicontinuous, monotone and coercive in the following sense: there exists $v_0 \in K$ such that

$$\lim_{\|v\|_V \to \infty} \frac{\langle A(v), v - v_0 \rangle}{\|v\|_V} = \infty, \ v \in K.$$

Then for any $F \in V^*$, there exists $u \in K$ such that

$$\langle A(u), v - u \rangle \geq \langle F, v - u \rangle \text{ for all } v \in K.$$

Chapter 4

Variational Problems Governed by Pseudomonotone Operators

The goal with this chapter is to study those variational problems with this time the underlined operators being rather pseudomonotone operators in sense of Brézis. In addition to this, we will introduce the abstract parabolic variational inequality and prove the existence theorem.

4.1 Existence theorems for abstract equations

4.1.1 Stationary problems

We aim to show the existence theorem for the stationary problems that were introduced in Section 3.1.1. We first need the following lemma which is a result of Browder's Fixed Point Theorem 1.1.20.

Lemma 4.1.1. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and suppose that there exists $\rho > 0$ such that

$$\langle g(\xi), \xi \rangle \geq 0 \text{ for } |\xi| = \rho. \tag{4.1}$$

Then there exists $\xi_0 \in \mathbb{R}^n$ such that

$$g(\xi_0) = 0 \text{ with } |\xi_0| \leq \rho. \tag{4.2}$$

Proof. Suppose that the contrary holds, i.e., $g(\xi) \neq 0$ for $|\xi| \leq \rho$. Set $h(\xi) = -\rho \frac{g(\xi)}{|g(\xi)|}$, $|\xi| \leq \rho$. $h(\xi)$ is a continuous function mapping from the convex compact ball $B_\rho = \{\xi \in \mathbb{R}^n : |\xi| \leq \rho\}$ into itself. So by Browder's Fixed Point Theorem 1.1.20, h has a fixed point ξ^*. i.e.,

$$h(\xi^*) = \xi^* \text{ with } |\xi^*| = \rho.$$

Then we have

$$\langle h(\xi^*), \xi^* \rangle = \langle \xi^*, \xi^* \rangle = \rho^2 > 0. \tag{4.3}$$

On the other hand, from (4.1), we also have

$$\langle h(\xi^*), \xi^* \rangle = \left\langle -\rho \frac{g(\xi^*)}{|g(\xi^*)|}, \xi^* \right\rangle = \frac{-\rho}{|g(\xi^*)|} \langle g(\xi^*), \xi^* \rangle \leq 0,$$

which contradicts to (4.3) above, hence (4.2) holds. ■

Recall that in Theorem 4.1.2, we have the operator A is monotone, bounded and hemicontinuous. Now, we will show the existence with a weaker assumption on the operator A, that is, A is pseudomonotone.

Theorem 4.1.2. Let V be a reflexive and separable Banach space. Assume that the operator $A : V \to V^*$ is bounded, pseudomonotone and coercive. Then for any $F \in V^*$, there exists a solution $u \in V$ of such that the equation $A(u) = F$ holds.

Proof. The proof is based on Galerkin's method.
Since V is separable, there exists $z_1, z_2, \cdots$ of linear independent elements of V such that their linear combinations are dense in V. Denote V_k to be the linear span of $\{z_1, z_2, \cdots, z_k\}$.
1 (Galerkin's approximation)
We define the k-th Galerkin's approximation $u_k \in V_k$ to the solution $u \in V$ of $A(u) = F$ by the following:

$$\text{for all } v \in V_k, \ \ \langle A(u_k), v\rangle = \langle F, v\rangle \,, \tag{4.4}$$

or equivalently

$$\langle A(u_k), z_j\rangle = \langle F, z_j\rangle \quad \text{for } j = 1, 2, \cdots, k.$$

2 (existence of the approximation solution u_k)
Let $g = (g_1, g_2, \cdots, g_k)$ defined by

$$g_j(\xi_1, \cdots, \xi_k) = \langle A(\xi_1 z_1 + \cdots + \xi_k z_k), z_j\rangle - \langle F, z_j\rangle \ \text{ for } j = 1, \cdots, k. \text{ and } \xi \in \mathbb{R}^k.$$

Since A is pseudomonotone, then A is demicontinuous by Proposition 2.2.9, this means that g_j are continuous for all j. Set $z = \sum\limits_{j=1}^{k} \xi_j z_j$ and assume that $z \neq 0$, then we have

$$\langle g(\xi), \xi\rangle = \left\langle A\left(\sum_{j=1}^{k} \xi_j z_j\right), \sum_{j=1}^{k} \xi_j z_j\right\rangle - \left\langle f, \sum_{j=1}^{k} \xi_j z_j\right\rangle = \Big(\frac{\langle A(z), z\rangle}{\|z\|_V} - \frac{\langle f, z\rangle}{\|z\|_V}\Big)\|z\|_V \geq \Big(\frac{\langle A(z), z\rangle}{\|z\|_V} - \|f\|_{V^*}\Big)\|z\|_V \,.$$

Since A is coercive, we have for $\|z\|_V$ sufficiently large, which is equivalent to $|\xi|$ sufficiently large, we have

$$\frac{\langle A(z), z\rangle}{\|z\|_V} - \|f\|_{V^*} \geq 0.$$

i.e., $\langle g(\xi), \xi\rangle \geq 0$ for $|\xi| = \rho$. Therefore, by the above Lemma 4.1.1, there exists $\xi \in \mathbb{R}^k$ with $|\xi| \leq \rho$ such that $g(\xi) = 0$, i.e., we have a solution $u_k = \sum\limits_{j=1}^{k} \xi_j z_j$ on V_k.
If V is finite dimensional, then the theorem is proved. If V is not, then we have a sequence $(u_k)_k$ of elements satisfying (4.4).
3 (boundedness of the Galerkin's approximation u_k)
We will show that u_k is uniformly bounded, assume that $\|u_k\|_V$ is not bounded, we can find a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, such that $\lim\limits_{k\to\infty} \|u_k\|_V = \infty$ and

$$0 = \langle A(u_k), u_k\rangle - \langle f, u_k\rangle \geq \Big(\frac{\langle A(u_k), u_k\rangle}{\|u_k\|_V} - \|f\|_{V^*}\Big)\|u_k\|_V \,.$$

Since A is coercive, then we have as $k \to \infty$ that

$$\Big(\frac{\langle A(u_k), u_k\rangle}{\|u_k\|_V} - \|f\|_{V^*}\Big)\|u_k\|_V \to \infty,$$

which is a contradiction. Hence, $\|u_k\|_V \leq C$ for all $k \in \mathbb{N}$.
4 (limit passage)
Since $(u_k)_k$ is bounded in the reflexive Banach space V, there exist a subsequence, again denoted by $(u_k)_k$, and $u \in V$ such that $u_k \rightharpoonup u$ and

$$\langle A(u_k), v_m - u_k\rangle = \langle f, v_m - u_k\rangle \quad \text{for any } k \geq m, \ v_m \in V_m \subset V_k. \tag{4.5}$$

By the density of $\bigcup\limits_{k\in\mathbb{N}} V_k$ in V, we can choose a sequence $(v_k)_k$ such that $v_k \to u$. Then we obtain by (4.5) that

$$\begin{aligned} &\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle = \limsup_{k\to\infty}(\langle A(u_k), u_k - v_k\rangle + \langle A(u_k), v_k - u\rangle) \\ &\leq \limsup_{k\to\infty}(\langle f, u_k - v_k\rangle + \|A(u_k)\|_{V^*}\|v_k - u\|_V) = \lim_{k\to\infty}(\langle f, u_k - v_k\rangle + \|A(u_k)\|_{V^*}\|v_k - u\|_V) = 0. \end{aligned} \tag{4.6}$$

Since A is bounded, so $(A(u_k))_k$ is bounded in V^*. Hence, 'limsup' is a limit and the last equality holds. By the pseudomonotonicity of A, we get

$$\forall v \in V, \ \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \langle A(u), u - v\rangle . \tag{4.7}$$

On the other hand, we obtain by (4.5) that

$$\forall v \in \bigcup_{m\in\mathbb{N}} V_m : \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle = \lim_{k\to\infty} \langle f, u_k - v\rangle = \langle f, u - v\rangle . \tag{4.8}$$

From (4.7) and (4.8), we obtain $\langle A(u), u - v\rangle \leq \langle f, u - v\rangle$ for any v ranging over a dense subset $\bigcup_{m\in\mathbb{N}} V_m$ of V, which shows that $A(u) = f$. ∎

Remark 4.1.3. In the proof above, the Galerkin's approximating $(u_k)_k$ contains a subsequence which converges weakly to a solution u of the problem. If A is strictly monotone as in Theorem 3.1.1, in this case the solution is unique, so we can apply Cantor's trick to show that the whole Galerkin's approximating sequence converges weakly to the solution. If A is uniformly monotone, then the Galerkin's approximating sequence u_k converge strongly to the solution u of $A(u) = f$. i.e., A is uniformly monotone, which implies that as $k \to \infty$

$$\zeta(\|u_k - u\|_V)\, \|u_k - u\|_V \leq \langle A(u_k) - A(u), u_k - u\rangle = \langle A(u_k), u_k - u\rangle + \langle A(u), u_k - u\rangle \to 0.$$

The second term $\langle A(u), u_k - u\rangle$ converges to 0 since $u_k \rightharpoonup u$. The first term $\langle A(u_k), u_k - u\rangle$ converges to 0, which follows from $u_k \rightharpoonup u$, (4.6) and the pseudomonotonicity of A. Hence

$$\lim_{k\to\infty} \zeta(\|u_k - u\|_V)\, \|u_k - u\|_V = 0,$$

by definition of ζ, this is true if and only if

$$\lim_{k\to\infty} \|u_k - u\|_V = 0.$$

4.1.2 Evolution problems

Recall in Theorem 3.1.3, we prove the existence theorem for abstract evolution equations when the operator $\tilde{B}(t)$ is bounded, monotone, hemicontinuous and coercive. Note that in the proof of Theorem 3.1.3, we did not use the the monotonicity and hemicontinuity of $\tilde{B}(t)$ directly, we only used them to show that $\tilde{B}(t) : V \to V^*$ is demicontinuous and $B : L^p(0, T; V) \to L^{p'}(0, T; V^*)$ is pseudomonotone. Hence, we can assume that $\tilde{B}(t)$ is demicontinuous and B is pseudomonotone in Theorem 3.1.3 instead of $\tilde{B}(t)$ is monotone and hemicontinuous. From above comment, we can state the following theorem.

Theorem 4.1.4. Let $V \subset H \subset V^*$ be an evolution triple, let $1 < p < \infty$ and $0 < T < \infty$. Assume that for all fixed $t \in [0, T]$, $\tilde{B}(t) : V \to V^*$ is demicontinuous and bounded in the sense of there exist a constant $c_1 > 0$ and a function $k_1 \in L^{p'}(0, T)$ such that

$$\left\| [\tilde{B}(t)](v) \right\|_{V^*} \leq c_1 \|v\|_V^{p-1} + k_1(t) \ \text{ for all } v \in V, \ t \in [0, T].$$

$\tilde{B}(t)$ is coercive in the sense that there exist a constant $c_2 > 0$ and a function $k_2 \in L^1(0, T)$ such that

$$\left\langle [\tilde{B}(t)](v), v \right\rangle \geq c_2 \|v\|_V^p - k_2(t) \ \text{ for all } v \in V, \ t \in [0, T].$$

And for any $u, v \in V$, the function

$$t \mapsto \left\langle [\tilde{B}(t)](u), v \right\rangle, \ t \in [0, T] \text{ is measurable.}$$

Finally, the operator $B : L^p(0, T; V) \to L^{p'}(0, T; V^*)$ with $[B(u)](t) = [\tilde{B}(t)](u(t))$ is pseudomonotone. Then for any arbitrary $F \in L^{p'}(0, T; V^*)$ and $u_0 \in H$, there exists a solution of (3.3).

It turns out that the existence theorem for abstract evolution equations is also true for the following weaker form of pseudomonotonicity.

Definition 4.1.5 (Pseudomonotone with respect to $W_p^1(0,T;V,H)$). Let $V \subset H \subset V^*$ be an evolution triple and let $p > 1$. A bounded operator $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is called pseudomonotone with respect to $W_p^1(0,T;V,H)$ if for $(u_k)_k \subset W_p^1(0,T;V,H)$ with

$$u_k \rightharpoonup u \text{ weakly in } L^p(0,T;V),\ u_k^{'} \rightharpoonup u^{'} \text{ weakly in } L^{p'}(0,T;V^*) \text{ and } \limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \leq 0,$$

imply that

$$\liminf_{k\to\infty} \langle B(u_k), u_k - v\rangle \geq \langle B(u), u - v\rangle \text{ for all } v \in L^p(0,T;V),$$

or equivalently

$$\lim_{k\to\infty} \langle B(u_k), u_k - u\rangle = 0 \text{ and } B(u_k) \rightharpoonup B(u) \text{ weakly in } L^{p'}(0,T;V^*).$$

With the notion introduced above, we have the following theorem.

Theorem 4.1.6. Let $V \subset H \subset V^*$ be an evolution triple, let $1 < p < \infty$ and $0 < T < \infty$. Assume that for fixed $t \in [0,T]$, $\tilde{B}(t) : V \to V^*$ is demicontinuous and bounded such that for a suitable constant $c_1 > 0$ and a function $k_1 \in L^{p'}(0,T)$

$$\left\|[\tilde{B}(t)](v)\right\|_{V^*} \leq c_1 \|v\|_V^{p-1} + k_1(t) \text{ for all } v \in V,\ t \in [0,T]. \tag{4.9}$$

$\tilde{B}(t)$ is coercive such that for a suitable constant $c_2 > 0$ and a function $k_2 \in L^1(0,T)$,

$$\left\langle [\tilde{B}(t)](v), v\right\rangle \geq c_2 \|v\|_V^p - k_2(t) \text{ for all } v \in V,\ t \in [0,T]. \tag{4.10}$$

And for arbitrary fixed $u, v \in V$, the function

$$t \mapsto \left\langle [\tilde{B}(t)](u), v\right\rangle,\ t \in [0,T] \text{ is measurable.} \tag{4.11}$$

Finally, the operator $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ with $[B(u)](t) = [\tilde{B}(t)](u(t))$ is pseudomonotone with respect to $W_p^1(0,T;V,H)$. Then, for any $F \in L^{p'}(0,T;V^*)$ and $u_0 \in H$, there exists a solution u of (3.3).

Proof. We will follow the proof of Theorem 3.1.3 up to the limit passage where we will use the pseudomonotonicity in Definition 4.1.5 to show that $B(u_k) \rightharpoonup B(u)$ in $L^{p'}(0,T;V^*)$, we need to show that for a subsequence of Galerkin's approximating sequence $(u_k)_k$, again denoted by $(u_k)_k$, $u_k^{'} \rightharpoonup u^{'}$ in $L^{p'}(0,T;V^*)$. This can be done by showing $(u_k^{'})_k$ is bounded in reflexive Banach space $L^{p'}(0,T;V^*)$.
We know that $(u_k)_k$ satisfy

$$\langle u_k'(t), z_j\rangle + \left\langle \tilde{B}(t)(u_k(t)), z_j\right\rangle = \langle F(t), z_j\rangle \text{ for } j = 1, \cdots, k.,$$

$$u_k(0) = u_{k0} \in V_k := \text{span}\{z_1, \cdots, z_k\}, \text{ where } u_{k0} \to u_0 \text{ in } H.$$

Multiply the above equation by functions $b_{jk} \in L^p(0,T)$, $j = 1, 2, \cdots, n.$, and integrate over $[0,T]$, we obtain the sum of these equations:

$$\left\langle u_k^{'}, w\right\rangle + \langle B(u_k), w\rangle = \langle F, w\rangle, \tag{4.12}$$

where

$$w(t) = \sum_{j=1}^{k} b_{jk}(t) z_j \text{ and } w \in L^p(0,T;V). \tag{4.13}$$

Then by Hölder's inequality, (4.9) and (4.12), we get

$$|\langle u_k', w\rangle| \leq |\langle F, w\rangle| + |\langle B(u_k), w\rangle| \leq \left(\|F\|_{L^{p'}(0,T;V^*)} + \|B(u_k)\|_{L^{p'}(0,T;V^*)}\right) \|w\|_{L^p(0,T;V)} \leq C \|w\|_{L^p(0,T;V)},$$

where C is independent of k and w. The functions w of the form (4.13) are the dense in $L^p(0,T;V)$ because linear combinations of z_j are dense in V, therefore

$$\left|\left\langle u_k^{'}, w\right\rangle\right| \leq C \|w\|_{L^p(0,T;V)}$$

holds for all $w \in L^p(0,T;V)$. Hence, $(u_k^{'})$ is bounded with respect to the norm of $L^{p'}(0,T;V^*)$, which completes the proof of the theorem. ∎

4.2 Application to variational problems

In this section, we want to apply the existence theorems for abstract equations in the previous section to show the existence of boundary value problems and initial boundary value problems.

4.2.1 Elliptic boundary value problems

Now we want to apply Theorem 4.1.2 to solve the elliptic boundary value problem that was introduced in Section 3.2.1. Previously, we formulate the conditions (E1) - (E4) on a_j so that the underlined operator A defined as (3.28) is bounded, monotone and hemicontinuous, this imply that A is also pseudomonotone. The goal here to formulate another condition other than (E3) so that the underlined operator A is still pseudomonotone.
Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary. Let $1 < p, p' < \infty$ be such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and let $V = W_0^{1,p}(\Omega)$. For $j = 0, 1, \cdots n.$, functions $a_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. We assume the following condition on a_j:
(E3') There exists $c_3 > 0$ such that for a.e. $x \in \Omega$, all $\eta \in \mathbb{R}$ and all $\zeta, \zeta^* \in \mathbb{R}^n$

$$\sum_{j=1}^{n} [a_j(x, \eta, \zeta) - a_j(x, \eta, \zeta^*)](\zeta_j - \zeta_j^*) \geq c_3 \left| \zeta - \zeta^* \right|^p .$$

Theorem 4.2.1. Assume that (E1), (E2) and (E3') hold, then the operator A defined as in (3.28) is pseudomonotone.

Proof. The boundedness of A follows from Proposition 3.2.8. Now, let $(u_k)_k \subset V$ be a sequence that $u_k \rightharpoonup u$ in V and $\limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leq 0$, we need to show that

$$\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle \geq \langle A(u), u - v\rangle \text{ for any } v \in V. \tag{4.14}$$

From compact embedding Theorem 1.2.24, $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$, since u_k is bounded in V which is a closed subset of $W^{1,p}(\Omega)$, there exists a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, such that

$$u_k \to u \text{ in } L^p(\Omega), \tag{4.15}$$

by Proposition 1.2.10, we obtain (up to a further subsequence) that

$$u_k \text{ converges to } u \text{ a.e.}. \tag{4.16}$$

Note that $\|D_j u_k\|_{L^p} \leq \|u_k\|_V$, i.e., $(D_j u_k)_k$ is bounded in $L^p(\Omega)$, we may extract a subsequence (again denoted by u_k) such that

$$D_j u_k \rightharpoonup D_j u \text{ in } L^p(\Omega),\ j = 1, \cdots, n. \tag{4.17}$$

Now, we will show that $D_j u_k \to D_j u$ in $L^p(\Omega)$. Observe that

$$\langle A(u_k), u_k - u\rangle = \underbrace{\int_\Omega a_0(x, u_k, Du_k)(u_k - u)dx}_{\text{I}} + \underbrace{\sum_{j=1}^{n} \int_\Omega [a_j(x, u_k, Du_k) - a_j(x, u_k, Du)](D_j u_k - D_j u)dx}_{\text{II}}$$

$$+ \underbrace{\sum_{j=1}^{n} \int_\Omega a_j(x, u_k, Du)(D_j u_k - D_j u)dx}_{\text{III}} .$$

The first term I converges to 0 by (4.15), (E2) and Hölder's inequality; the last term III also converges to 0, since (4.17) and we claim

$$a_j(x, u_k, Du) \to a_j(x, u, Du) \text{ in } L^{p'}(\Omega).$$

To prove the claim, set $b_j^k = |a_j(x, u_k, Du) - a_j(x, u, Du)|^{p'}$. From (E2), we get $b_j^k \in L^1(\Omega)$ for all $j = 1, 2, \cdots, n.$ and all $k \in \mathbb{N}$. From (E1) and (4.16), we know that $b_j^k \to 0$ as $k \to \infty$. From (E2) and boundedness of u_k in

$L^p(\Omega)$, we get $(b_j^k)_k$ is uniformly integrable. Hence, the claim follows from Vitali's Theorem 1.2.16. Therefore, we have

$$\limsup_{k\to\infty}\sum_{j=1}^{n}\int_\Omega [a_j(x,u_k,Du_k)-a_j(x,u_k,Du)](D_ju_k-D_ju)dx \le 0. \tag{4.18}$$

From (E3'), we obtain that

$$\lim_{k\to\infty}\int_\Omega |Du_k-Du|^p\,dx = 0, \tag{4.19}$$

and by Proposition 1.2.10, we get for a subsequence (again denoted by u_k) that

$$Du_k \to Du \text{ a.e. in } \Omega. \tag{4.20}$$

Now, using (E1), (E2), (4.15), (4.16), (4.19) and (4.20), applying Vitali's Theorem 1.2.16 again, we will get

$$a_j(x,u_k,Du_k)\to a_j(x,u,Du) \text{ in } L^{p'}(\Omega) \text{ for all } j=0,1,\cdots,n. \tag{4.21}$$

This implies from Hölder's inequality that

$$A(u_k)\rightharpoonup A(u) \text{ weakly in } V^*. \tag{4.22}$$

Also from (4.21), (4.15) and (4.19), one has

$$\lim_{k\to\infty}\langle A(u_k),u_k-u\rangle = 0. \tag{4.23}$$

Note that (4.22) and (4.23) hold for a subsequence $(u_k)_k$, by using Cantor's trick (see the proof for equivalence of two pseudomonotonicity in Remark 2.2.4), we obtain that (4.22) and (4.23) hold for the original sequence. Therefore, we have that for any $v\in V$,

$$\begin{aligned}\liminf_{k\to\infty}\langle A(u_k),u_k-v\rangle &\ge \liminf_{k\to\infty}(\langle A(u_k),u_k-u\rangle+\langle A(u_k),u-v\rangle)\\ &= \lim_{k\to\infty}\langle A(u_k),u_k-u\rangle+\lim_{k\to\infty}\langle A(u_k),u-v\rangle = \langle A(u),u-v\rangle\,.\end{aligned}$$

■

From the above theorem, we know that if functions a_j satisfy (E1), (E2), (E3') and (E4), then the operator A defined as (3.28) satisfies the assumptions of Theorem 4.1.2. Therefore, there exists a solution to the problem (3.22) with homogeneous boundary condition. The non-homogeneous case can be reduced by homogeneous case (see remark 3.2.4) since pseudomonotone operator remains pseudomonotone under a shift by Lemma 2.2.7.

4.2.2 Parabolic initial boundary value problems

We want to apply Theorem 4.1.6 to show the existence of a solution to the parabolic initial boundary value problem (3.30) - (3.32) that was introduced in Section 3.2.2. The goal here is to formulate conditions on b_j so that the underlined operator B defined as (3.35) is pseudomonotone with respect to $W_p^1(0,T;V,H)$.
In this section, let $\Omega\subset\mathbb{R}^n$ be a bounded open domain with sufficiently smooth boundary. Let $1<p'\le 2\le p<\infty$ be such that $\dfrac{1}{p}+\dfrac{1}{p'}=1$ and let $T>0$. Let $V=W_0^{1,p}(\Omega)$ and $H=L^2(\Omega)$, then $V\subset H\subset V^*$ is an evolution triple. Set $Q_T=(0,T)\times\Omega$ and $\Gamma_T=[0,T)\times\partial\Omega$. For $j=0,1,\cdots,n.$, functions $b_j:[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$.
Instead of (A3), we assume
(A3') There exists a constant $c_3>0$ such that for a.e. $(t,x)\in Q_T$, all $\eta\in\mathbb{R}$ and all $\zeta,\zeta^*\in\mathbb{R}^n$,

$$\sum_{j=1}^{n}[b_j(t,x,\eta,\zeta)-b_j(t,x,\eta,\zeta^*)](\zeta_j-\zeta_j^*)\ge C_2\,|\zeta-\zeta^*|^p\,.$$

We first prove that $\tilde{B}(t)$ is pseudomonotone for $t\in[0,T]$.

Theorem 4.2.2. Assume that (A1) (A2) and (A3') hold. Then the operator $\tilde{B}(t) : V \to V^*$ defined as (3.35) above is bounded and pseudomonotone for $t \in [0, T]$.

Proof. According to Proposition 3.2.8, $\tilde{B}(t)$ is bounded for $t \in [0, T]$. Fixed $t \in [0, T]$, it follows from the proof of Theorem 4.2.1 that $\tilde{B}(t)$ is pseudomonotone.

Now we will show that the operator B is pseudomonotone with respect to $W^1_p(0, T; V, H)$ by using Lions-Aubin Theorem 1.3.16.

Theorem 4.2.3. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\partial\Omega$ is sufficiently smooth and (A1), (A2), (A3'), (A4) hold, then the operator B satisfies all conditions of Theorem 4.1.6.

Proof. The boundedness of $\tilde{B}(t) : V \to V^*$, (4.9) and (4.11) of Theorem 4.1.6 follow from Proposition 3.2.8, and Proposition 3.2.10 implies the coercive condition (4.10) of Theorem 4.1.6. The above Theorem 4.2.2 above shows that $\tilde{B}(t)$ is pseudomonotone for a.e. $t \in [0, T]$, then by Proposition 2.2.9, we obtain that $\tilde{B}(t) : V \to V^*$ is demicontinuous.
Now, it remains to show that B is pseudomonotone with respect to $W^1_p(0, T; V, H)$. Assume that

$$u_k \rightharpoonup u \text{ weakly in } L^p(0, T; V),\ u'_k \rightharpoonup u' \text{ weakly in } L^{p'}(0, T; V^*),\ \limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \le 0. \tag{4.24}$$

From Theorem 1.2.24, $W^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$. Since Ω is bounded, $p \ge 2 \ge p'$ and the continuity of embedding $L^2(\Omega) \subset (W^{1,p}(\Omega))^*$, it follows that $L^p(\Omega) \subset (W^{1,p}(\Omega))^*$ is continuous. From Lions-Aubin Theorem 1.3.16, there exists a subsequence of $(u_k)_k$, which is again denoted by $(u_k)_k$, such that

$$u_k \to u \text{ in } L^p(0, T; L^p(\Omega)) = L^p(Q_T) \text{ where } Q_T = [0, T) \times \Omega. \tag{4.25}$$

$(u_k)_k$ is bounded in $L^p(0, T; V)$, so $(D_j u_k)_k$ is bounded in reflexive Banach space $L^p(Q_T)$, we may assume for a further subsequence, again denoted by $(u_k)_k$ such that

$$D_j u_k \rightharpoonup D_j u \text{ weakly in } L^p(Q_T),\ j = 1, 2, \cdots, n.. \tag{4.26}$$

Moreover,

$$\langle B(u_k), u_k - u\rangle = \underbrace{\int_{Q_T} b_0(t, x, u_k, Du_k)(u_k - u)dtdx}_{\text{I}} + \underbrace{\sum_{j=1}^{n} \int_{Q_T} b_j(t, x, u_k, Du)(D_j u_k - D_j u)dtdx}_{\text{II}}$$
$$+ \underbrace{\sum_{j=1}^{n} \int_{Q_T} [b_j(t, x, u_k, Du_k) - b_j(t, x, u_k, Du)](D_j u_k - D_j u)dtdx}_{\text{III}}.$$

The first term I on the right hand side tends to 0 by (4.25) and Hölder's inequality. The second term II converges to 0 by (4.26) because (4.25), (A1), (A2) and Vitali's Theorem 1.2.16 imply that

$$b_j(t, x, u_k, Du) \to a_j(t, x, u, Du) \text{ in } L^{p'}(Q_T).$$

Set $h^k_j = |b_j(t, x, u_k, Du) - b_j(t, x, u, Du)|^{p'}$, then $h^k_j \to 0$ as $k \to \infty$ by (4.25) and (A1). From (A2),

$$h^k_j \le \text{const}[|u_k|^p + |u|^p + |Du|^p + 1].$$

So $\int_{Q_T} |h^k_j|\, dxdt$ is bounded by a constant which is independent of k, this imply that $(h^k_j)_k$ are equiintegrable and $h^k_j \in L^1(Q_T)$. From Vitali's Theorem 1.2.16, we get

$$b_j(t, x, u_k, Du) \to b_j(t, x, u, Du) \text{ in } L^{p'}(Q_T).$$

Consequently, we obtain from (4.24) that

$$\limsup_{k\to\infty} \sum_{j=1}^{n} \int_{Q_T} [b_j(t, x, u_k, Du_k) - b_j(t, x, u_k, Du)](D_j u_k - D_j u)dtdx \le 0.$$

From (A3'), we get

$$\lim_{k\to\infty}\int_{Q_T}|Du_k - Du|^p\,dtdx = 0, \tag{4.27}$$

and (for a subsequence)

$$Du_k \to Du \text{ a.e. in } Q_T. \tag{4.28}$$

Similar to above, by (A1), (A2), (4.25), (4.27), (4.28) and Vitali's theorem, we obtain

$$b_j(t,x,u_k,Du_k) \to b_j(t,x,u,Du) \text{ in } L^{p'}(Q_T), j = 0,1,\cdots,n. \tag{4.29}$$

Then by Hölder's inequality, one can easily show that

$$B(u_k) \rightharpoonup B(u) \text{ weakly in } L^{p'}(0,T;V^*). \tag{4.30}$$

Finally, from (A2), (4.25), (4.27), (4.28) and Hölder's inequality, we get

$$\lim_{k\to\infty}\langle B(u_k), u_k - u\rangle = 0. \tag{4.31}$$

Since (4.30) and (4.31) hold for a subsequence of $(u_k)_k$, by using Cantor's trick, we get that (4.30) and (4.31) also hold for the original sequence. ■

Now we will formulate the following more general assumptions (A3") and (A4") on functions b_j.
(A3") For a.e. $(t,x)\in Q_T$, all $\eta\in\mathbb{R}$, and $\zeta,\zeta^*\in\mathbb{R}^n$ with $(\zeta_1,\cdots,\zeta_n)=\zeta\neq\zeta^*=(\zeta_1^*,\cdots,\zeta_n^*)$, we have

$$\sum_{j=1}^n[b_j(t,x,\eta,\zeta)-b_j(t,x,\eta,\zeta^*)](\zeta_j-\zeta_j^*)>0.$$

(A4") There exist a constant $c_2>0$ and a function $k_2\in L^1(Q_T)$ such that for a.e. $(t,x)\in Q_T$, and for all $\xi=(\eta,\zeta)\in\mathbb{R}^{n+1}$ (let $\xi_0=\eta\in\mathbb{R}$)

$$\sum_{j=0}^n b_j(t,x,\eta,\zeta)\xi_j \geq c_2|\zeta|^p - k_2(t,x).$$

Note that (A3') and (A4) imply (A3") and (A4") respectively.

Theorem 4.2.4. Assume that (A1), (A2), (A3") and (A4") hold. Then for a.e. $t\in[0,T]$, the operator $\tilde{B}(t):V\to V^*$ defined by (3.35) with an arbitrary (possibly unbounded) domain $\Omega\subset\mathbb{R}^n$, is pseudomonotone.

Proof. Assume that for fixed $t\in[0,T]$, $u_k,u\in V$ such that

$$u_k \rightharpoonup u \text{ weakly in } V \text{ and } \limsup_{k\to\infty}\left\langle[\tilde{B}(t)](u_k(t)), u_k-u\right\rangle\leq 0. \tag{4.32}$$

Note that u_k,u can be regarded as elements of $L^p(0,T;V)$ by setting $u_k(t)=u_k$ and $u(t)=u$ for $t\in[0,T]$. We need to show that

$$\lim_{k\to\infty}\left\langle\tilde{B}(t)(u_k(t)),u_k-u\right\rangle=0 \text{ and } [\tilde{B}(t)](u_k(t))\rightharpoonup[\tilde{B}(t)](u(t)) \text{ weakly in } V^*. \tag{4.33}$$

We will show that above (4.33) holds for a suitable subsequence of u_k, by Cantor's trick, we will obtain that (4.33) holds for the original sequence u_k too.
Assume that Ω_m is a sequence of bounded domains with sufficiently smooth boundary $\partial\Omega_m$ such that $\Omega_m\subset\Omega_{m+1}$ and $\Omega=\bigcup_{m=1}^{\infty}\Omega_m$. Then for any fixed m, by Theorem 1.2.24, $W^{1,p}(\Omega_m)$ is compactly embedded into $L^p(\Omega_m)$, there exists a subsequence of $(u_k)_k$ which is convergent in $L^p(\Omega_m)$ and so a subsequence of this subsequence converges a.e. to u in Ω_m. By using a 'diagonal process', one can obtain that a subsequence of $(u_k)_k$ which converges to u a.e. in Ω. We will denote this subsequence again by $(u_k)_k$, so we have

$$u_k\to u \text{ a.e. in } \Omega. \tag{4.34}$$

Now we are going to show that

$$Du_k \to Du \text{ a.e. in } \Omega. \tag{4.35}$$

Set

$$p_k(x) = \sum_{j=1}^{n}[b_j(t,x,u_k,Du_k) - b_j(t,x,u,Du)](D_j u_k - D_j u) + [b_0(t,x,u_k,Du_k) - b_0(t,x,u,Du)](u_k - u), \tag{4.36}$$

then

$$\left\langle [\tilde{B}(t)](u_k(t)) - [\tilde{B}(t)](u(t)), u_k - u \right\rangle = \int_\Omega p_k(x)dx,$$

and by (4.32), we have

$$\limsup_{k\to\infty} \int_\Omega p_k(x)dx \le 0. \tag{4.37}$$

From (4.36), we have

$$p_k(x) = \sum_{j=1}^{n} b_j(t,x,u_k,Du_k)D_j u_k + b_0(t,x,u_k,Du_k)u_k - g_k(x), \tag{4.38}$$

where

$$\begin{aligned} g_k(x) = &\left[\sum_{j=1}^{n} b_j(t,x,u,Du)(D_j u_k - D_j u) + b_0(t,x,u,Du)(u_k - u)\right] \\ &+ \left[\sum_{j=1}^{n} b_j(t,x,u_k,Du_k)D_j u + b_0(t,x,u_k,Du_k)u\right]. \end{aligned} \tag{4.39}$$

By (A2),

$$\begin{aligned} |g_k(x)| \le &c_4[|u|^{p-1} + |Du|^{p-1} + k_1(t,x)][|u_k| + |Du_k| + |u| + |Du|] \\ &+ c_5[|u_k|^{p-1} + |Du_k|^{p-1} + k_1(t,x)][|u| + |Du|], \end{aligned} \tag{4.40}$$

integrate above inequality with respect to x over Ω, apply Holder's inequality using the fact that $k_1(t,\cdot) \in L^{p'}(\Omega)$ and $u_k, Du_k, u, Du \in L^p(\Omega)$ for fixed $t \in [0,T]$, we obtain that the sequence (g_k) is equiintegrable.
And, by using Young's inequality with ϵ in (4.40), we obtain for arbitrary $\epsilon > 0$, there exists a constant $c(\epsilon)$ and $k_4(t,\cdot) \in L^1(\Omega)$ such that

$$|g_k(x)| \le \epsilon\, |Du_k|^p + c(\epsilon)[|u_k|^p + |u|^p + |Du|^p + k_4(t,x)]. \tag{4.41}$$

Choosing sufficiently small $\epsilon > 0$ such that $\epsilon \le \frac{c_2}{2}$, one obtains from (A4"), (4.38) and (4.41)

$$p_k(x) \ge c_2\, |Du_k|^p - k_2(t,x) - |g_k(x)| \ge \frac{c_2}{2}\, |Du_k|^p - c_6[|u_k|^p + |u|^p + |u|^p + k_5(t,x)], \tag{4.42}$$

with some constant c_6 and $k_5(t,\cdot) \in L^1(\Omega)$. Let

$$p_k^+(x) = \max\{p_k(x), 0\},\ p_k^-(x) = -\min\{p_k(x), 0\},$$

then by (4.42)

$$0 \le p_k^-(x) \le k_2(t,x) + |g_k(x)|\,,$$

where $k_2(t,\cdot) \in L^1(\Omega)$ and g_k is equiintegrable, hence the sequence

$$(p_k^-)_{k\in\mathbb{N}} \text{ is equiintegrable.} \tag{4.43}$$

Now we will show that p_k^- converges to 0 a.e. in Ω. Indeed, p_k can be written in the following form:

$$p_k(x) = q_k(x) + r_k(x) + s_k(x), \tag{4.44}$$

where

$$q_k(x) = \sum_{j=1}^{n} [b_j(t, x, u_k, Du_k) - b_j(t, x, u_k, Du)](D_j u_k - D_j u),$$

$$r_k(x) = \sum_{j=1}^{n} [b_j(t, x, u_k, Du) - b_j(t, x, u, Du)](D_j u_k - D_j u),$$

$$s_k(x) = [b_0(t, x, u_k, Du_k) - b_0(t, x, u, Du)](u_k - u).$$

Let χ_k be the characteristic function of the set $\{x : p_k^-(x) > 0\}$ then

$$-p_k^- = \chi_k q_k + \chi_k r_k + \chi_k s_k. \tag{4.45}$$

By (4.42),

$$\frac{c_2}{2} |Du_k|^p \leq c_6[|u_k|^p + |u|^p + |Du|^p + k_5(t, x)], \text{ if } p_k(x) < 0,$$

hence by (4.34), we get $|u_k|^p$ is bounded for a.e. $x \in \Omega$, which implies that $(\chi_k Du_k)_k$ is bounded for a.e. $x \in \Omega$. Therefore, from (4.34), (A1) and (A2), we obtain that

$$\chi_k r_k \to 0 \text{ a.e. and } \chi_k s_k \to 0 \text{ a.e.}.$$

(A3") implies that $\chi_k q_k \geq 0$ a.e., it follows from (4.45) that

$$p_k^- \to 0 \text{ a.e.}. \tag{4.46}$$

Then by (4.43), (4.46) and Vitali's theorem

$$\lim_{k \to \infty} \int_\Omega p_k^- \, dx = 0. \tag{4.47}$$

Since $0 \leq p_k^+ = p_k + p_k^-$, we obtain from (4.37) and (4.47) that

$$\lim_{k \to \infty} \int_\Omega p_k^+ \, dx = 0. \tag{4.48}$$

Therefore, by (4.47) and (4.48), it follows that $\lim_{k \to \infty} \int_\Omega p_k = 0$ and by (4.36) we obtain that as $k \to \infty$

$$\begin{aligned} \left\langle [\tilde{B}(t)](u_k(t)), u_k - u \right\rangle &= \left\langle [\tilde{B}(t)](u_k(t)) - [\tilde{B}(t)](u(t)), u_k - u \right\rangle + \left\langle [\tilde{B}(t)](u(t)), u_k - u \right\rangle \\ &= \int_\Omega p_k(x) dx + \left\langle [\tilde{B}(t)](u(t)), u_k - u \right\rangle \ \to 0. \end{aligned}$$

So the first part of (4.33) is proved.
By (4.48) and Proposition 1.2.10, we get

$$p_k^+ \to 0 \text{ a.e. for a subsequence (which is denoted again by } p_k^+).$$

Hence, (4.46) implies that

$$p_k \to 0 \text{ for a.e. } x \in \Omega. \tag{4.49}$$

Then (4.34) and (4.42) imply the sequence $(Du_k)_k$ is bounded that for a.e. $x \in \Omega$.
Now, we will prove that (4.35) by contradiction. For fixed $x \in \Omega$, assume that (4.35) is not valid, then we have a subsequence of $(Du_k(x))_k$, (again denoted by $(Du_k(x))_k$, for simplicity), which converges to some $\zeta \neq Du(x)$. Since

$$u_k(x) \to u(x), \ r_k(x) \to 0, \ s_k(x) \to 0,$$

we obtain from (A1) that

$$0 = \lim_{k \to \infty} p_k(x) = \sum_{j=1}^{n} [b_j(t, x, u(x), \zeta) - b_j(t, x, u(x), Du(x)](\zeta_j - D_j u(x)).$$

So we get by (A3") that $\zeta = Du(x)$, which is a contradiction. Hence, we have shown (4.35).
To show the second part of (4.33). For arbitrary fixed $v \in V$, consider

$$\left\langle [\tilde{B}(t)](u_k(t)), v \right\rangle = \sum_{j=1}^{n} \int_\Omega b_j(t,x,u_k,Du_k)D_j(v)dx + \int_\Omega b_0(t,x,u_k,Du_k)vdx,$$

where the sequence of integrands above is equiintegrable by (A2) and Hölder's inequality, further, the sequence of integrands converges a.e. by (A1), (4.34) and (4.35). Therefore, by using Vitali's theorem, we have as $k \to 0$

$$\begin{aligned}\left\langle [\tilde{B}(t)](u_k(t)), v \right\rangle &= \sum_{j=1}^{n} \int_\Omega b_j(t,x,u_k,Du_k)D_j(v)dx + \int_\Omega b_0(t,x,u_k,Du_k)vdx \\ &\to \sum_{j=1}^{n} \int_\Omega b_j(t,x,u,Du)D_j(v)dx + \int_\Omega b_0(t,x,u,Du)vdx = \left\langle [\tilde{B}(t)](u(t)), v \right\rangle.\end{aligned}$$

Hence, for a.e. fixed $t \in [0,T]$, $\tilde{B}(t) : V \to V^*$ is pseudomonotone. ∎

Theorem 4.2.5. Assume (A1), (A2), (A3") and (A4) hold. Let the operator $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ defined by (3.35) satisfy the assumptions in Theorem 4.1.6. Then for any $F \in L^{p'}(0,T;V^*)$ and $u_0 \in H = L^2(\Omega)$, there is a solution $u \in W_p^1(0,T;V,H)$ satisfying

$$u' + B(u) = F, \; u(0) = u_0.$$

In the case when $V = W_0^{1,p}(\Omega)$, it is sufficient to assume (A4") instead of assuming (A4), since (A4") imply coercivity. (By Poincaré's inequality, $\|u\|_{W^{1,p}(\Omega)}$ is equivalent to $\|Du\|_{L^p(\Omega)}$. See Remark 1.2.28)

Proof. The boundedness of $\tilde{B}(t) : V \to V^*$, (4.9) and (4.11) of Theorem 4.1.6 follow from Proposition 3.2.8, and Proposition 3.2.10 implies the coercive condition (4.10) of Theorem 4.1.6. Note that (A4) implies that (A4"), so the above Theorem 4.2.4 shows that $\tilde{B}(t)$ is pseudomonotone for a.e. $t \in [0,T]$, then by Proposition 2.2.9, we obtain that $\tilde{B}(t) : V \to V^*$ is demicontinuous.
Now it remains to show that $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is pseudomonotone with respect to $W_p^1(0,T;V,H)$. According to Definition 4.1.5, we assume

$$(u_k)_k \subset W_p^1(0,T;V,H), \; u_k \rightharpoonup u \text{ weakly in } L^p(0,T;V), \tag{4.50}$$

$$u_k' \rightharpoonup u' \text{ weakly in } L^{p'}(0,T;V^*), \tag{4.51}$$

$$\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \le 0, \tag{4.52}$$

we have to show

$$\lim_{k\to\infty} \langle B(u_k), u_k - u\rangle = 0 \text{ and } B(u_k) \rightharpoonup B(u) \text{ weakly in } L^{p'}(0,T;V^*). \tag{4.53}$$

Since Ω is bounded and $\partial\Omega$ is sufficiently smooth, Theorem 1.2.24 implies that V is compactly embedded into $L^p(\Omega)$. By Theorem 1.3.16, we obtain that the embedding

$$W_p^1(0,T;V,H) \subset L^p(0,T;L^p(\Omega)) = L^p(Q_T) \text{ is compact.}$$

From (4.50) and (4.51), we obtain the boundedness of u_k in $W_p^1(0,T;V,H)$. Hence there is a subsequence of $(u_k)_k$, which is denoted again by $(u_k)_k$, with the properties

$$u_k \to u \text{ in } L^p(Q_T) \text{ and (for a further subsequence) a.e. in } Q_T. \tag{4.54}$$

The proof of (4.53) is almost the same as (4.33) in the proof of the above Theorem 4.2.4. Set

$$p_k(t,x) = \sum_{j=1}^{n} [b_j(t,x,u_k,Du_k) - b_j(t,x,u,Du)](D_ju_k - D_ju) + [b_0(t,x,u_k,Du_k) - b_0(t,x,u,Du)](u_k - u).$$

Then

$$\langle B(u_k) - B(u), u_k - u\rangle = \int_{Q_T} p_k(t,x)dtdx,$$

we have by (4.50), (4.51) and (4.52) that

$$\limsup_{k\to\infty} \int_{Q_T} p_k(t,x)dtdx \le 0.$$

Using a similar arguments of the proof of the above Theorem 4.2.4, we obtain

$$\lim_{k\to\infty} \int_{Q_T} p_k(t,x)dtdx = 0, \tag{4.55}$$

and

$$p_k \to 0 \text{ a.e. in } Q_T \text{ (for a further subsequence).} \tag{4.56}$$

(4.55) implies that the first part of (4.53). Moreover, by using a similar argument of the proof of above Theorem 4.2.4, (4.56), (4.54) and (A3") imply

$$Du_k \to Du \text{ a.e. in } Q_T. \tag{4.57}$$

Finally, from (4.54), (4.57), (A1), (A2) and Vitali's theorem, we obtain the second part of (4.53), which completes the proof of the theorem. In the particular case where $V = W_0^{1,p}(\Omega)$, we can use the same argument above to show the result follows. ■

4.3 Abstract Variational Inequalities

4.3.1 Abstract Elliptic Variational Inequality

In this section, we will show some existence results regarding to elliptic variational inequalities with pseudomonotone operators in the sense of Brézis. Let V be a reflexive Banach space and let K be a closed convex subset of V, let A be a nonlinear operator mapping from K to V^*. Denote $\mathbb{L}$ as the set of all finite dimensional subspace L of V such that $L \cap K \neq \emptyset$. We denote $K_L = K \cap L$.

Definition 4.3.1. The operator A is said to be compatible with finite-dimensional subspace of V if for any $L \in \mathbb{L}$ there exists a solution $u_L \in K_L$ to the following variational inequality:

$$\langle A(u_L), v - u_L\rangle \ge 0 \text{ for all } v \in K_L.$$

Theorem 4.3.2. Let $A : K \to V^*$ be pseudomonotone and compatible with finite-dimensional subspaces of V, then there exists a solution $u \in K$ to the following variational inequality:

$$\langle A(u), v - u\rangle \ge 0 \text{ for all } v \in K.$$

Proof. For each $L \in \mathbb{L}$, we consider the variational inequality on K_L, i.e., find $u_L \in K_L$ such that

$$\langle A(u_L), v - u_L\rangle \ge 0 \text{ for all } v \in K_L. \tag{4.58}$$

Since A is compatible with finite-dimensional subspaces of V, we conclude that such solution u_L exists.
Now for each $Y \in \mathbb{L}$, we denote U_Y as the set of all $\hat{u} \in K$ such that there exists a subspace $L \supset Y$ with the property that $\hat{u} \in K_L$ and

$$\langle A(\hat{u}), v - \hat{u}\rangle \ge 0,\ \forall v \in K_L.$$

From (4.58), we know that $U_Y \neq \emptyset$ since $u_Y \in U_Y$. Moreover, $\{\overline{U_Y}\}_{Y\in\mathbb{L}}$ has a finite intersection property, where $\overline{U_Y}$ is the closure of U_Y. Note that by definition of U_Y, if $Y_1 \subset Y_2$, then $U_{Y_2} \subset U_{Y_1}$. Indeed, taking any $L_1, \cdots, L_n \in \mathbb{L}$ and letting $M = \text{span}\{L_1, \cdots, L_n\}$, we have

$$M \in \mathbb{L} \text{ and } U_M \subset \bigcap_{i=1}^{n} U_{L_i}.$$

This implies that

$$\emptyset \neq U_M \subset \bar{U}_M \subset \overline{\bigcap_{i=1}^{n} U_{L_i}} \subset \bigcap_{i=1}^{n} \overline{U_{L_i}}.$$

Since $\bar{U}_Y \subset K$ and K is a compact set, the finite intersection property of $\{\bar{U}_Y\}_{Y\in\mathbb{L}}$ implies

$$\bigcap_{Y\in\mathbb{L}} \bar{U}_Y \neq \emptyset.$$

Hence, there exists a point u_0 such that $u_0 \in \bar{U}_Y$ for all $Y \in \mathbb{L}$. For any $v \in K$, take $Y \in \mathbb{L}$ with the property that Y contains v and u_0, there exists a sequence $u_k \in U_Y$ such that $u_k \to u_0$. By definition of U_Y, one has

$$\langle Au_k, w - u_k\rangle \geq 0 \ \forall w \in K_Y.$$

In particular, we have for all $k \in \mathbb{N}$,

$$\langle Au_k, u_k - u_0\rangle \leq 0 \text{ and } \langle Au_k, u_k - v\rangle \leq 0.$$

Hence,

$$\limsup_{k\to\infty} \langle Au_k, u_k - u_0\rangle \leq 0.$$

By the pseudomonotonicity of A, we obtain

$$\langle Au_0, u_0 - v\rangle \leq \liminf_{k\to\infty} \langle Au_k, u_k - v\rangle \leq 0.$$

Thus we have shown that

$$\langle Au_0, v - u_0\rangle \geq 0 \text{ for any arbitrary } v \in K.$$

This means that u_0 is a solution of the variational inequality. ■

Corollary 4.3.3. Let $A : K \to V^*$ be pseudomonotone and $0 \in K$. Assume further that, for each $v \in K$, the mapping $u \mapsto \langle Au, u - v\rangle$ is lower bounded on each compact subset of K and

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = \infty. \tag{4.59}$$

Then for each $f \in V^*$, there exists $u_0 \in K$ such that

$$\langle A(u_0) - f, v - u_0\rangle \geq 0 \text{ for all } v \in K. \tag{4.60}$$

Proof. For $f \in V^*$, define a mapping $A_f : K \to V^*$ by the formula $A_f(u) = A(u) - f$. It is easy to show that A_f is pseudomonotone. Choose a number $\gamma > \|f\|$ and let $\alpha = \gamma - \|f\|$. Then it follows from (4.59) that for every $u \in K$ with $\|u\| \geq \gamma$, one has

$$\langle Au - f, u\rangle \geq (\gamma - \|f\|)\,\|u\| \geq \alpha\gamma.$$

Since $0 \in K$, it follows that if u_0 satisfies (4.60) and put $v = 0$, then we know $\|u_0\| < \gamma$ from the above inequality.
Set $K_\gamma = K \cap \overline{B(0,\gamma)}$ and consider the following variational inequality:

$$\langle Au_0 - f, v - u_0\rangle \geq 0, \text{ for all } v \in K_\gamma. \tag{4.61}$$

From the above Theorem 4.3.2, we know that such u_0 exists and $\|u_0\| < \gamma$.
Now fix any $v \in K$ and note that $\|u_0\| < \gamma$, we have for sufficiently small $\lambda > 0$ that

$$u_0 + \lambda(v - u_0) \in K_\gamma.$$

Put $v = u_0 + \lambda(v - u_0)$ in (4.61), we obtain

$$\langle A(u_0) - f, \lambda(v - u_0)\rangle \geq 0.$$

Hence,

$$\langle A(u_0) - f, v - u_0\rangle \geq 0,$$

and the result follows since v is arbitrary. ■

Remark 4.3.4. In the special case where $K = V$, above variational inequality for the operator $A : V \to V^*$ is equivalent to the stationary problem

$$A(u) = f, \quad \text{for } f \in V^*.$$

Now we are going to show the existence theorem where no compatibility assumption is needed. First, we show the following theorem where K is a bounded subset.

Theorem 4.3.5. Let K be a bounded, closed and convex subset of V. Assume that $A : K \to V^*$ is bounded and pseudomonotone, then for any $f \in V^*$, there exists $u \in K$ such that the following variational inequality holds:

$$\langle A(u), v - u\rangle \geq \langle f, v - u\rangle \text{ for all } v \in K. \tag{4.62}$$

Proof. Since V is separable, then we can find

$$V_1 \subset V_2 \cdots \subset V_m \subset \cdots \text{ with } \overline{\bigcup_{i=1}^{\infty} V_i} = V.$$

Define $K_m = K \cap V_m$, then K_m is closed, bounded and convex, K_m also satisfies that

$$K_1 \subset K_2 \cdots \subset K_m \subset \cdots \text{ with } \overline{\bigcup_{i=1}^{\infty} K_i} = K.$$

We will first find finite dimensional solution of the problem (4.62), i.e., for any $f \in V^*$, find $u_m \in K_m$ such that

$$\langle Au_m, v - u_m\rangle \geq \langle f, v - u_m\rangle \text{ for any } v \in K_m. \tag{4.63}$$

In finite dimensional Banach space V_m, we can define scalar product $(\cdot, \cdot)$ on V_m such that it generates an equivalent norm to the original norm on V_m. If $g \in V^*$, then g is also a continuous linear functional on Hilbert space V_m, i.e., $g \in V_m^*$. Since

$$w \mapsto \langle g, w\rangle$$

is continuous for any $w \in V_m$, by Riesz representation theorem, there exists a linear and continuous operator $B : V^* \to V_m$ such that for any $w \in V_m$, we have

$$\langle g, w\rangle = (Bg, w).$$

Hence, the problem (4.63) can be written in the following form:

$$(BA(u_m), v - u_m) \geq (Bf, v - u_m) \text{ for all } v \in K_m.$$

This is equivalent to

$$(u_m, v - u_m) \geq (u_m - BA(u_m) + Bf, v - u_m) \text{ for all } v \in K_m. \tag{4.64}$$

Let P_m be the operator of projecting V_m on to the closed, convex set K_m with respect to the scalar product $(\cdot, \cdot)$. Then above inequality (4.64) is equivalent to

$$u_m = P_m(u_m - BA(u_m) + Bf).$$

Define the operator $Q_m : K_m \to K_m$ by

$$Q_m(v) = P_m(v - BA(v) + Bf).$$

We will show that Q_m is continuous. Indeed, it is sufficient to show weak continuity since K_m is of finite dimension. Assume $v_k \to v$ strongly in K_m. Since A is pseudomonotone and bounded, therefore it is demicontinuous, so we have

$$A(v_k) \rightharpoonup A(v) \text{ weakly in } V^*.$$

The weak continuity of Q_m follows easily from the continuity of B and P_m. Therefore Q_m is continuous and from Brouwer's fixed point theorem, Q_m admits a fixed point, i.e., there exists solution $u_m \in K_m \subset K$ of (4.63). Since K is bounded in reflexive Banach space V, it follows that

$$\|u_m\|_V \leq C,$$

and there exists a subsequence (again denoted by $(u_m)_m$) such that

$$u_m \rightharpoonup u \text{ weakly in } V. \tag{4.65}$$

Since K is convex and closed in reflexive Banach space V, so it is weakly compact, i.e., $u \in K$.
Now, we will show that

$$\limsup_{m\to\infty} \langle Au_m, u_m - u\rangle \le 0. \tag{4.66}$$

Since $\bigcup_{i=1}^{\infty} K_i$ is dense in K, for any $\epsilon > 0$, we can find a $u_0 \in \bigcup_{i=1}^{\infty} K_i$ such that

$$\|u - u_0\| < \epsilon. \tag{4.67}$$

Note that for sufficiently large m, $u_0 \in K_m$, so we have

$$\langle A(u_m), u_m - u_0\rangle \le \langle f, u_m - u_0\rangle,$$

it follows from (4.67) and boundedness of u_m, A that

$$\langle Au_m, u_m - u\rangle = \langle Au_m, u_m - u_0\rangle + \langle Au_m, u_0 - u\rangle \le \langle f, u_m - u_0\rangle + C\epsilon.$$

Taking the limsup in the above inequality as $m \to \infty$ and using (4.65), (4.67), we get for arbitrary $\epsilon > 0$ that

$$\limsup_{m\to\infty} \langle Au_m, u_m - u\rangle \le C'\epsilon.$$

Now let $\epsilon \to 0^+$, then (4.66) follows.
Now, we will show that the weak limit u of u_m is a solution to the variational inequality (4.62). For any $v \in \bigcup_{i=1}^{\infty} K_i$, $v \in K_m$ for sufficiently large m, thus we have

$$\langle Au_m, u_m - v\rangle \le \langle f, u_m - v\rangle.$$

Taking the liminf as $m \to \infty$ on both side and using (4.65), (4.66) and pseudomonotonicity of A, we have

$$\langle Au, u - v\rangle \le \liminf_{m\to\infty} \langle Au_m, u_m - v\rangle \le \liminf_{m\to\infty} \langle f, u_m - v\rangle = \langle f, u - v\rangle.$$

i.e.,

$$\langle Au, v - u\rangle \le \langle f, v - u\rangle \text{ for any } v \in \bigcup_{i=1}^{\infty} K_i. \tag{4.68}$$

Since $\bigcup_{i=1}^{\infty} K_i$ is dense subset of K, above inequality (4.68) holds for any $v \in K$, which completes the proof for the theorem. ∎

We can remove the boundedness condition on K by adding a coercive condition on A, which allows the finite dimensional solutions u_k are uniformly bounded.

Theorem 4.3.6. Assume that K is a closed and convex of V, assume that the operator $A : K \to V^*$ is bounded, pseudomonotone and satisfies the following coercive condition: there exists $v_0 \in K$ such that

$$\frac{\langle A(v), v - v_0\rangle}{\|v\|_V} \to \infty \text{ as } \|v\|_V \to \infty. \tag{4.69}$$

Then for any $f \in V^*$, there exists a solution $u \in K$ of (4.62).

Proof. Set $B_R = \{v \in V : \|v\|_V \le R\}$, $R \in \mathbb{N}$ and $K_R = B_R \cap K$. Since K_R is bounded, closed and convex, by the previous Theorem 4.3.5, we obtain that there exists $u_R \in K_R$ with

$$\langle A(u_R), v - u_R\rangle \ge \langle f, v - u_R\rangle \text{ for any } v \in K_R. \tag{4.70}$$

For sufficiently large R, i.e., $R > \|v_0\|_V$, we can apply $v = v_0$ in above inequality, i.e.,

$$\langle A(u_R), u_R - v_0\rangle \le \langle f, u_R - v_0\rangle \le \|f\|_{V^*} \|u_R - v_0\|_V ,$$

so we have

$$\frac{\langle Au_R, u_R - v_0\rangle}{\|u_R\|_V} \le \|f\|_{V^*} \frac{\|u_R - v_0\|_V}{\|u_R\|_V} \le \|f\|_{V^*} \frac{\|u_R\|_V + \|v_0\|_V}{\|u_R\|_V},$$

where the right hand side of above inequality is bounded for $\|u_R\|_V \ge 1$, from the coercive condition (4.69), we get $\|u_R\|_V$ is bounded for all sufficiently large R. Hence, there exists a subsequence $(u_{R_k})_k$ of $(u_R)_R$ and $u \in V$ such that

$$u_{R_k} \rightharpoonup u \text{ weakly in } V \text{ with } R_k \to \infty \text{ as } k \to \infty. \tag{4.71}$$

Then $u \in K$ since $u_{R_k} \in K_{R_k} \subset K$ and K is a closed, convex subset of reflexive Banach space. For sufficiently large k such that $R_k \ge \|u\|_V$, applying $v = u$ in (4.70), we get

$$\langle Au_{R_k}, u_{R_k} - u\rangle \le \langle f, u_{R_k} - u\rangle ,$$

taking limsup in the above inequality as $k \to \infty$ and from (4.71), we obtain

$$\limsup_{k\to\infty} \langle Au_{R_k}, u_{R_k} - u\rangle \le 0. \tag{4.72}$$

Hence from (4.71), (4.72) and pseudomonotonicity of A, we get

$$\liminf_{k\to\infty} \langle Au_{R_k}, u_{R_k} - v\rangle \ge \langle Au, u - v\rangle \text{ for any } v \in V. \tag{4.73}$$

Now, we show that the weak limit u is a solution of (4.62). For any $v \in K$, take sufficiently large k such that $R_k \ge \|v\|_V$, we obtain

$$\langle Au_{R_k}, u_{R_k} - v\rangle \le \langle f, u_{R_k} - v\rangle .$$

Taking liminf in the above inequality as $k \to \infty$, we obtain by (4.73) and (4.71) that

$$\langle Au, u - v\rangle \le \liminf_{k\to\infty} \langle Au_{R_k}, u_{R_k} - v\rangle \le \liminf_{k\to\infty} \langle f, u_{R_k} - v\rangle = \langle f, u - v\rangle .$$

i.e., u is a solution of (4.62). Hence the theorem is complete. ■

4.3.2 Abstract Parabolic Variational Inequality

In this section, we will formulate the parabolic variational inequality and then present some existence results. We will first consider the case where the operator $\tilde{B} : V \to V^*$ does not depend on t. Consider the following problem:

Let $V \subset H \subset V^*$ be an evolution triple, let $K \subset V$ be a closed and convex subset, let $p \ge 2$ and p' be its conjugate. Let $\mathcal{K} := \{v \in L^p(0,T;V) : v(t) \in K \text{ for a.e. } t \in [0,T]\}$. Given $f \in L^{p'}(0,T;V^*)$ and $u_0 \in K \cap H$, we are interested in finding $u \in \mathcal{K}$ with $u' \in L^{p'}(0,T;V^*)$ such that for a.e. $t \in [0,T]$

$$\langle u'(t), v(t) - u(t)\rangle + \left\langle \tilde{B}(u(t)), v(t) - u(t)\right\rangle \ge \langle f(t), v(t) - u(t)\rangle \quad \text{for any } v \in \mathcal{K}, \tag{4.74}$$

with the initial condition

$$u(0) = u_0. \tag{4.75}$$

Definition 4.3.7. A solution $u \in \mathcal{K}$ such that (4.74) and (4.75) holds is called the strong solution to the parabolic variational inequality.

Weak Formulation of Parabolic Variational Inequality

Take $v \in \mathcal{K}$ with $v' \in \mathcal{V}^*$, then (4.74) is equivalent to

$$\langle v'(t), v(t) - u(t)\rangle + \left\langle \tilde{B}(u(t)), v(t) - u(t)\right\rangle - \langle f(t), v(t) - u(t)\rangle \ge \langle v'(t) - u'(t), v(t) - u(t)\rangle .$$

Integrate above inequality with respect to t from 0 to T, using Remark 1.3.15, if $v(0) = u_0$, we obtain that

$$\int_0^T \left\langle v'(t) + \tilde{B}(u(t)) - f(t), v(t) - u(t)\right\rangle dt \ge \frac{1}{2}\|v(T) - u(T)\|_H^2 - \frac{1}{2}\|v(0) - u(0)\|_H^2 \ge -\frac{1}{2}\|v(0) - u(0)\|_H^2 = 0.$$

Definition 4.3.8. We say that $u \in \mathcal{K}$ is a weak solution to the parabolic variational inequality (4.74) and (4.75) if

$$\int_0^T \left\langle v'(t) + \tilde{B}(u(t)) - f(t), v(t) - u(t) \right\rangle dt \geq 0, \tag{4.76}$$

for any $v \in \mathcal{K}$ with $v' \in L^{p'}(0, T; V^*)$ and $v(0) = u_0$.

Before we introduce the existence theorem, we state following results that will be used.

Lemma 4.3.9 ([67, Lemma A.2]). Set

$$U := \{v \in L^p(0, T; V) : v' \in L^p(0, T; V) \text{ and } v(t) \in K \text{ for a.e. } t \in [0, T] \text{ with } v(0) = u_0\},$$

and

$$W := \{v \in L^p(0, T; V) : v' \in L^{p'}(0, T; V) \text{ and } v(t) \in K \text{ for a.e. } t \in [0, T] \text{ with } v(0) = u_0\}.$$

Then U is dense subset in W.

Lemma 4.3.10 ([67, Lemma A.3]). For $u \in L^p(0, T; V)$ and $u_0 \in K$, let v_j be such that

$$\begin{cases} \frac{1}{j} v_j' + v_j = v, \\ v_j(0) = u_0. \end{cases}$$

Then $v_j, \ v_j' \in L^p(0, T; V)$ and $v_j \to v$ strongly in $L^p(0, T; V)$ as $j \to \infty$.

Theorem 4.3.11. Assume that the operator $\tilde{B} : V \to V^*$ satisfies the following growth and coercivity condition:

$$\left\langle \tilde{B}w, w \right\rangle \geq \alpha_0 \|w\|_V^p - \alpha_1 \text{ and } \left\| \tilde{B}w \right\|_{V^*} \leq \beta_0 \|w\|_V^{p-1} + \beta_1 \quad \forall w \in V, \tag{4.77}$$

for some positive constants $\alpha_0, \alpha_1, \beta_0, \beta_1$ and $p \geq 2$. The operator $B : L^p(0, T; V) \to L^{p'}(0, T; V^*)$ defined by

$$[B(u)](t) = \tilde{B}(u(t))$$

is pseudomonotone. Given for any $f \in L^{p'}(0, T; V^*)$ and $u^0 \in K$, there exists a function $u \in \mathcal{K}$ such that

$$\int_0^T \langle v' + B(u) - f, v - u \rangle \, dt \geq 0, \tag{4.78}$$

for any $v \in \mathcal{K}$ with $v' \in L^{p'}(0, T; V^*)$ and $v(0) = u^0$.

Proof. In order to show the existence result, we will use finite difference in time. i.e., we discretize the time interval. Divide $[0, T]$ into n equidistant intervals with length $h = \frac{T}{n}$, for $i = 1, 2, \cdots, n.$, we define

$$f_i = \frac{1}{h} \int_{(i-1)h}^{ih} f(t) dt. \tag{4.79}$$

Set $u_0 = u^0$, for each time step $1 \leq i \leq n$, we recursively seek for a solution $u_i \in K$ of the following variational inequality:

$$\left\langle \frac{u_i - u_{i-1}}{h}, v - u_i \right\rangle + \left\langle \tilde{B}u_i, v - u_i \right\rangle \geq \langle f_i, v - u_i \rangle \text{ for all } v \in K. \tag{4.80}$$

Note that u_{i-1} and f_i are known for each time step i. Above variational inequality can be written as an elliptic variational inequality of the following form:

$$\frac{1}{h} \langle u_i, v - u_i \rangle + \left\langle \tilde{B}u_i, v - u_i \right\rangle \geq \left\langle f_i + \frac{u_{i-1}}{h}, v - u_i \right\rangle.$$

Define $B_h : K \to V^*$ by

$$\langle B_h u, v \rangle = \frac{1}{h} \langle u, v \rangle + \left\langle \tilde{B}u, v \right\rangle,$$

i.e.,

$$B_h = \frac{1}{h} I + \tilde{B}.$$

It is easy to see that I is monotone, hemicontinuous and bounded. The pseudomonotonicity of B implies that $\tilde{B}$ is pseudomonotone. Therefore, B_h is pseudomonotone for each $h > 0$.
Now we show that B_h is coercive in the sense of (4.69). For $v_0 \in K$, we obtain from (4.77) and $V \subset H \subset V^*$ that

$$\frac{\langle B_h u, u - v_0 \rangle}{\|u\|_V} = \frac{\frac{1}{h}\langle u, u - v_0\rangle + \left\langle \tilde{B}u, u - v_0 \right\rangle}{\|u\|_V} \geq \frac{\frac{1}{h}\|u\|_H^2 - \frac{1}{h}\|u\|_{V^*}\|v_0\|_V + \alpha_0\|u\|_V^p - \alpha_1 - (\beta_0 \|u\|_V^{p-1} + \beta_1)\|v_0\|_V}{\|u\|_V}$$
$$\geq -C\|v_0\|_V + \alpha_0 \|u\|_V^{p-1} - \beta_0 \|u\|_V^{p-2}\|v_0\|_V - \frac{\alpha_1 + \beta_1 \|v_0\|_V}{\|u\|_V} \to \infty \text{ as } \|u\|_V \to \infty.$$

Hence we obtain the existence of u_i from Theorem 4.3.6. We have the following lemma for the prior estimate:

Lemma 4.3.12. The solutions u_i satisfy the following prior estimate which is independent of j and h:

$$\|u_j\|_H^2 + h\alpha_0 \sum_{i=1}^{j} \|u_i\|_V^p \leq \lambda, \text{ for all } j = 1, 2, \cdots n, \tag{4.81}$$

where λ depends only on initial data f, u^0 and constants $p, p', \alpha_0, \alpha_1, \beta_0$ and β_1.
proof of the lemma. We set $v = u_0 = u^0 \in K$ in (4.80) and set $w_i = u_i - u_0$, we obtain

$$\frac{1}{h}\langle w_i - w_{i-1}, w_i \rangle + \left\langle \tilde{B}u_i, u_i - u_0 \right\rangle \leq \langle f_i, w_i \rangle. \tag{4.82}$$

Since $w_i - w_{i-1} \in K \subset V$, we have the following

$$\langle w_i - w_{i-1}, w_i \rangle = (w_i - w_{i-1}, w_i) = \frac{1}{2}(w_i, w_i) + \frac{1}{2}(w_i, w_i) - (w_i, w_{i-1}) + \frac{1}{2}(w_{i-1}, w_{i-1}) - \frac{1}{2}(w_{i-1}, w_{i-1})$$
$$= \frac{1}{2}(\|w_i\|_H^2 + \|w_i - w_{i-1}\|_H^2 - \|w_{i-1}\|_H^2).$$

Multiply (4.82) by $2h$ and using above inequality, we get

$$\|w_i\|_H^2 + \|w_i - w_{i-1}\|_H^2 - \|w_{i-1}\|_H^2 + 2h(\alpha_0\|u_i\|_V^p - \alpha_1) - 2h(\beta_0\|u_i\|_V^{p-1} + \beta_1)\|u_0\|_V \leq 2h\|f_i\|_{V^*}\|w_i\|_V.$$

Using the triangular inequality for w_i and using $\|w_i - w_{i-1}\|_H^2 \geq 0$, we obtain

$$\|w_i\|_H^2 - \|w_{i-1}\|_H^2 + 2h\alpha_0\|u_i\|_V^p$$
$$\leq 2h\|f_i\|_{V^*}\|u_i\|_V + 2h\beta_0\|u_i\|_V^{p-1}\|u_0\|_V + 2h\|f_i\|_{V^*}\|u_0\|_V + 2h\beta_1\|u_0\|_V + 2h\alpha_1.$$

Applying the Young's inequality for terms on the right-hand side, we will get

$$\|w_i\|_H^2 - \|w_{i-1}\|_H^2 + 2h\alpha_0\|u_i\|_V^p \leq 2h\left(\frac{\alpha_0}{4}\|u_i\|_V^p + \frac{4^{p'/p}}{p'(p\alpha_0)^{p'/p}}\|f_i\|_{V^*}^{p'}\right) + 2h\left(\frac{\alpha_0}{4}\|u_i\|_V^p + \frac{\beta_0^p 4^{p/p'}}{p(p'\alpha_0)^{p/p'}}\|u_0\|_V^p\right)$$
$$+ 2h\left(\frac{1}{p}\|u_0\|_V^p + \frac{1}{p'}\|f_i\|_{V^*}^{p'}\right) + 2\beta_1 h\left(\frac{1}{p}\|u_0\|_V^p + \frac{1}{p'}\right) + 2h\alpha_1$$
$$= h\alpha_0\|u_i\|_V^p + 2h\left(\frac{4^{p'/p} + (p\alpha_0)^{p'/p}}{p'(p\alpha_0)^{p'/p}}\|f_i\|_{V^*}^{p'} + \frac{\beta_0^p 4^{p/p'} + (\beta_1 + 1)(p'\alpha_0)^{p/p'}}{p(p'\alpha_0)^{p/p'}}\|u_0\|_V^p\right) + 2h\left(\alpha_1 + \frac{\beta_1}{p'}\right).$$

Hence,

$$\|w_i\|_H^2 - \|w_{i-1}\|_H^2 + h\alpha_0\|u_i\|_V^p \leq d_1 h\|f_i\|_{V^*}^{p'} + h(d_2\|u_0\|_V^p + d_3).$$

Summing the above inequalities for $i = 1, \cdots, j$. and noting $w_0 = 0$, we obtain

$$\|w_j\|_H^2 + h\alpha_0 \sum_{i=1}^{j}\|u_i\|_V^p \leq d_1 h \sum_{i=1}^{n}\|f_i\|_{V^*}^{p'} + T(d_2\|u_0\|_V^p + d_3), \text{ for } 1 \leq j \leq n.$$

Using Hölder's inequality, we will obtain

$$h\sum_{i=1}^{n}\|f_i\|_{V^*}^{p'} = h\sum_{i=1}^{n}\left\|\frac{1}{h}\int_{(i-1)h}^{ih} f(t)dt\right\|_{V^*}^{p'} \leq h^{1-p'}\sum_{i=1}^{n}\left[\left(\int_{(i-1)h}^{ih}\|f(t)\|_{V^*}^{p'}\,dt\right)^{\frac{1}{p'}}\cdot\left(\int_{(i-1)h}^{ih}1dt\right)^{\frac{1}{p}}\right]^{p'}$$
$$= h^{1-p'+\frac{p'}{p}}\sum_{i=1}^{n}\int_{(i-1)h}^{ih}\|f(t)\|_{V^*}^{p'}\,dt = \|f\|_{L^{p'}(0,T;V^*)}^{p'}.$$

Hence,

$$\|w_j\|_H^2 + h\alpha_0\sum_{i=1}^{j}\|u_i\|_V^p \leq d_1\,\|f\|_{L^{p'}(0,T;V^*)}^{p'} + T(d_2\,\|u_0\|_V^p + d_3),\ 1\leq j\leq n. \tag{4.83}$$

Since $\|u_i\|_H^2 \leq (\|w_i\|_H + \|u_0\|_H)^2 \leq 2(\|w_i\|_H^2 + \|u_0\|_H^2)$, we obtain the desired result

$$\|u_j\|_H^2 + 2h\alpha_0\sum_{i=1}^{j}\|u_i\|_V^p \leq 2d_1\,\|f\|_{L^{p'}(0,T;V^*)}^{p'} + 2T(d_2\,\|u_0\|_V^p + d_3) + 2\,\|u_0\|_H^2\,,\quad \text{for all } 1\leq j\leq n.$$

This complete the proof for the lemma.

Now we construct the approximation u_n of u by setting

$$u_n(x,t) = u_i(x) \text{ for } t\in((i-1)h, ih],\quad \text{where } 1\leq i\leq n.$$

We can easily see that the approximation u_n belongs to $\mathcal{K}$. And from the estimate (4.81) in the above lemma and (4.77), we see that

$$\{u_n\}_n \text{ is a bounded set of } L^\infty(0,T;H)\cap L^p(0,T;V),$$
$$\{B(u_n)\}_n \text{ is a bounded set of } L^{p'}(0,T;V^*).$$

Hence, we can extract a subsequence, which we denoted again by u_n, such that

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;V) \text{ and weakly}^* \text{ in } L^\infty(0,T;H), \tag{4.84}$$

$$Bu_n \rightharpoonup \varphi \text{ weakly in } L^{p'}(0,T;V^*). \tag{4.85}$$

Since K is convex and closed, it follows that the weak limit $u\in K$.

Now we prove that this limit function u is a solution of the variational inequality (4.77).
For test function $v\in\mathcal{K}$ with $v'\in L^{p'}(0,T;V^*)$ and $v(0)=u_0$. We define two approximation v_n and $\tilde{v}_n$ of v. On each time step $[(i-1)h, ih]$, $i=1,2,\cdots n$, we set

$$v_n(t) = v(ih) \text{ i.e., the step funcion approximation,} \tag{4.86}$$

$$\tilde{v}_n(t) = v((i-1)h) + \frac{t-t_{i-1}}{h}(v(ih)-v((i-1)h)) \text{ i.e., the linear interpolate approximation.} \tag{4.87}$$

We claim that $D_t\tilde{v}_n(t)\to D_tv = v'$ strongly in $L^{p'}(0,T;V^*)$.
proof of the claim. Observe that

$$\|D_t\tilde{v}_n(t)\|_{L^{p'}(0,T;V^*)}^{p'} = \sum_{j=0}^{n-1}h\left\|\frac{v((j+1)h)-v(jh)}{h}\right\|_{V^*}^{p'} = \sum_{j=0}^{n-1}h^{1-p'}\left\|\int_{jh}^{(j+1)h}v'(s)ds\right\|_{V^*}^{p'}$$

$$\leq \sum_{j=0}^{n-1}h^{1-p'}\left[\left(\int_{jh}^{(j+1)h}\|v'(s)\|_{V^*}^{p'}\,ds\right)^{\frac{1}{p'}}\cdot h^{\frac{1}{p}}\right]^{p'} = \sum_{j=0}^{n-1}h^{1-p'+\frac{p'}{p}}\int_{jh}^{(j+1)h}\|v'(s)\|_{V^*}^{p'}\,ds = \|v'(s)\|_{L^{p'}(0,T;V^*)}^{p'},$$

where the inequality follows from Hölder as above. It is easy to see that the map $v\mapsto D_t\tilde{v}_n$ is linear, so it is a linear and continuous mapping from $\mathcal{W} := \{v\in L^p(0,T;V) : v'\in L^{p'}(0,T;V^*)\}$ into $L^{p'}(0,T;V^*)$. Hence, it is sufficient to show the convergence result for a dense subset of $\mathcal{W}$, i.e., take the dense subset as $C^2([0,T];V^*)$. Take $v \in C^2([0,T];V^*)$, we have

$$\|D_t\tilde{v}_n - D_t v\|^{p'}_{L^{p'}(0,T;V^*)} = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \left\| \frac{v((j+1)h) - v(jh)}{h} \chi_{(jh,(j+1)h]}(s) - D_t v(s) \right\|^{p'}_{V^*} ds.$$

From the mean value theorem, we have

$$v((j+1)h) = v(jh) + hD_t v(t_h) \text{ for } t_h \in (jh,(j+1)h),$$

and hence

$$\|D_t\tilde{v}_n - D_t v\|^{p'}_{L^{p'}(0,T;V^*)} = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \|D_t v(t_h) - D_t v(t)\|^{p'}_{V^*} dt.$$

Finally, applying the mean value theorem again, we get for $\bar{t}_h \in (jh,(j+1)h)$ that

$$\begin{aligned}
\|D_t\tilde{v}_n - D_t v\|^{p'}_{L^{p'}(0,T;V^*)} &= \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \left\|D_t^2 v(\bar{t}_h)(\bar{t}_h - t)\right\|^{p'}_{V^*} dt \\
&\le \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \max_{t\in[jh,(j+1)h]} (\left\|D_t^2 v(t)\right\|_{V^*} \cdot |t_h - t|)^{p'} dt \\
&\le \|v\|_{C^2([0,T];V^*)} \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} h^{p'} dt = \|v\|_{C^2([0,T];V^*)} Th^{p'} \to 0
\end{aligned}$$

as $h = \frac{T}{n} \to 0$, this is equivalent to $n \to \infty$. Hence, the claim follows, i.e.,

$$\lim_{n\to\infty} \|D_t\tilde{v}_n - D_t v\|_{L^{p'}(0,T;V^*)} = 0. \tag{4.88}$$

For $f \in L^{p'}(0,T;V^*)$, we define

$$f_n = \sum_{i=1}^{n} f_i \chi_{((i-1)h,ih]} \text{ step function on each time step,}$$

with f_i defined as in (4.79).
We are going to approximate the variational inequality (4.78) as follows:
Set $v = v(ih)$ in (4.80) and we obtain

$$\left\langle \frac{u_i - u_{i-1}}{h}, v(ih) - u_i \right\rangle + \left\langle \tilde{B}u_i, v(ih) - u_i \right\rangle \ge \langle f_i, v(ih) - u_i \rangle,$$

multiply above inequality by h and rewrite above inequality as follows:

$$\langle u_i - u_{i-1}, v(ih) - u_i \rangle + h \left\langle \tilde{B}u_i, v(ih) - u_i \right\rangle - h \langle f_i, v(ih) - u_i \rangle \ge 0. \tag{4.89}$$

Let $y_i = v(ih) - u_i$ and add the term $\langle y_i - y_{i-1}, y_i \rangle$ to each side of above inequality (4.89), then we sum for $i = 1, 2, \cdots, n.$, to obtain that

$$\sum_{i=1}^{n} [\langle v(ih) - v((i-1)h), v(ih) - u_i \rangle + h \left\langle \tilde{B}u_i, v(ih) - u_i \right\rangle - h \langle f_i, v(ih) - u_i \rangle] \ge \sum_{i=1}^{n} \langle y_i - y_{i-1}, y_i \rangle.$$

This is equivalent to

$$\int_0^T \langle \tilde{v}_n', v_n - u_n \rangle dt + \int_0^T \langle Bu_n, v_n - u_n \rangle dt - \int_0^T \langle f_n, v_n - u_n \rangle dt \ge \sum_{i=1}^{n} \langle y_i - y_{i-1}, y_i \rangle.$$

Using the fact that $y_i = 0$ and summing the inequality $\langle y_i - y_{i-1}, y_i\rangle \geq \|y_i\|_H^2 - \|y_{i-1}\|_H^2$ from $i = 1$ to n, we have

$$\sum_{i=1}^{n} \langle y_i - y_{i-1}, y_i\rangle \geq 0.$$

Therefore, we have

$$\int_0^T \langle \tilde{v}_n', v_n - u_n\rangle \, dt + \int_0^T \langle Bu_n, v_n - u_n\rangle \, dt \geq \int_0^T \langle f_n, v_n - u_n\rangle \, dt. \tag{4.90}$$

Note as $n \to \infty$, we have the following results:

$$\tilde{v}_n' \to v' \text{ strongly in } L^{p'}(0,T;V^*);\ f_n \to f \text{ strongly in } L^{p'}(0,T;V^*);\ v_n \to v \text{ strongly in } L^p(0,T;V).$$

Taking the limsup as $n \to \infty$ in (4.90), we obtain

$$\limsup_{n\to\infty} \int_0^T \langle Bu_n, u_n\rangle \, dt \leq \int_0^T \langle v', v - u\rangle \, dt + \int_0^T \langle \varphi, v\rangle \, dt - \int_0^T \langle f, v - u\rangle \, dt. \tag{4.91}$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty} \int_0^T \langle Bu_n, u_n - u\rangle \, dt &\leq \limsup_{n\to\infty} \int_0^T \langle Bu_n, u_n\rangle - \liminf_{n\to\infty} \int_0^T \langle Bu_n, u\rangle \, dt \\ &\leq \int_0^T \langle v', v - u\rangle \, dt + \int_0^T \langle \varphi, v - u\rangle \, dt - \int_0^T \langle f, v - u\rangle \, dt. \end{aligned} \tag{4.92}$$

To show right hand side of above inequality ≤ 0, we use the following approximation z_j of u:

$$\frac{1}{j} z_j' + z_j = u \text{ and } z_j(0) = u_0.$$

By lemma 4.3.10, $z_j, z_j' \in L^p(0,T;V)$ and $z_j \to u$ in $L^p(0,T;V)$ as $j \to \infty$. Therefore, we have

$$\int_0^T \langle z_j', z_j - u\rangle \, dt = -j \int_0^T \|z_j - u\|_H^2 \, dt \leq 0,$$

and by Hölder's inequality, as $j \to \infty$

$$\int_0^T \langle f, z_j - u\rangle \, dt \to 0 \text{ and } \int_0^T \langle \varphi, z_j - u\rangle \, dt \to 0.$$

We can let $v = z_j$ in (4.92) and let $j \to \infty$ to get

$$\limsup_{n\to\infty} \int_0^T \langle Bu_n, u_n - u\rangle \, dt \leq 0. \tag{4.93}$$

Since B is pseudomonotone, we obtain that

$$\liminf_{n\to\infty} \int_0^T \langle Bu_n, u_n - w\rangle \, dt \geq \int_0^T \langle Bu, u - w\rangle \, dt,\ \forall w \in L^p(0,T;V). \tag{4.94}$$

Setting $w = 0$ in above inequality (4.94) and using (4.91), we obtain u satisfies the following inequality:

$$\int_0^T \langle Bu, u\rangle \, dt + \int_0^T \langle f, v - u\rangle \, dt \leq \int_0^T \langle v', v - u\rangle \, dt + \int_0^T \langle \varphi, v\rangle \, dt. \tag{4.95}$$

It is left to show $\varphi = Bu$. From (4.94), we have for any $w \in L^p(0,T;V)$,

$$\begin{aligned}\int_0^T \langle Bu, u - w\rangle \, dt &\leq \liminf_{n\to\infty} \int_0^T \langle Bu_n, u_n - w\rangle \, dt \leq \limsup_{n\to\infty} \int_0^T \langle Bu_n, u_n - w\rangle \, dt \\ &\leq \limsup_{n\to\infty} \int_0^T \langle Bu_n, u_n - u\rangle \, dt + \limsup_{n\to\infty} \int_0^T \langle Bu_n, u - w\rangle \, dt.\end{aligned}$$

Using (4.93) and (4.85), we get

$$\int_0^T \langle Bu, u-w\rangle\, dt \le \int_0^T \langle \varphi, u-w\rangle\, dt \text{ for all } w \in L^p(0,T;V),$$

which shows that $\varphi = Bu$. Hence (4.95) can be rewritten as:

$$\int_0^T \langle v', v-u\rangle\, dt + \int_0^T \langle Bu, v-u\rangle\, dt \ge \int_0^T \langle f, v-u\rangle\, dt, \tag{4.96}$$

for all $v \in \mathcal{K}_1 = \{v \in L^p(0,T;V) : v' \in L^p(0,T;V) \text{ and } v(t) \in K \text{ for a.e. } t \in [0,T] \text{ with } v(0) = u_0\}$, this is the dense subset of $\{v \in \mathcal{K},\ v' \in L^{p'}(0,T;V^*) \text{ with } v(0) = u_0\}$ by Lemma 4.3.9. Hence, the theorem is complete. ∎

We now introduce the existence result concerning with the strong solution of parabolic variational inequality (4.74).

Theorem 4.3.13. Suppose that the operator $\tilde{B} : V \to V^*$ is pseudomonotone and coercive in the sense of

$$\lim_{\|v\|_V \to\infty} \frac{\left\langle \tilde{B}v, v-v_0\right\rangle}{\|v\|_V} = \infty \text{ for some } v_0 \in K,$$

and suppose that there exists $z_0 \in H$ satisfying

$$(z_0, v-u_0) + \left\langle \tilde{B}u_0, v-u_0\right\rangle \ge \langle f(0), v-u_0\rangle \ \forall v \in V \text{ for the initial data } u_0 \in H. \tag{4.97}$$

Suppose also that $f : [0,T] \to H$ is Lipschitz. Then there exists a unique $u \in L^\infty(0,T;V) \cap C([0,T];H)$ with $u' \in L^\infty(0,T;H)$ such that

$$\langle u'(t), v(t)-u(t)\rangle + \left\langle \tilde{B}(u(t)), v(t)-u(t)\right\rangle \ge \langle f(t), v(t)-u(t)\rangle\,,$$

for all $v \in \mathcal{V}$ and a.e. $t \in [0,T]$, with $u(0) = u_0 \in H$.

Proof. The proof is based on Rothe's method, see [40, Theorem 2], where the author J.Kačur proves this theorem for a more general operator B.

We will briefly introduce the Rothe's method, also known as method of semidiscretization or method of line, which is a very powerful tool for analysis of evolution problem. First step is to discretize time so that the above parabolic variational inequality is transformed into an elliptic variational inequality, i.e., divide time interval $[0,T]$ into n equidistant intervals with width $= \frac{T}{n}$, on each time interval (t_{i-1}, t_i) $(i = 1, \cdots, n)$, considering the following elliptic variational inequality:

$$\left\langle \frac{u_i - u_{i-1}}{h}, v-u_i\right\rangle + \left\langle \tilde{B}u_i, v-u_i\right\rangle \ge \langle f_i, v-u_i\rangle\,,$$

where u_{i-1} and f_i are known. (we can use f_i as above in (4.79) or just $f_i = f(t_i)$.) As it is in above proof of the theorem, we have the pseudomonotonicity and coercivity of $B_h(u) = \frac{1}{h}u + \tilde{B}u$, this ensures that solution u_i exists on each step. Define the Rothe's function $u_n(x,t) = \sum_{i=1}^{n}(u_{i-1}(x) + \frac{t-t_{i-1}}{h}(u_i(x) - u_{i-1}(x)))\chi_{(t_{i-1},t_i]}$.

The second step is to find the prior estimate, i.e., we need to find the following estimate

$$\|u_i\|_V \le C \text{ and } \left\|\frac{u_i - u_{i-1}}{h}\right\|_H \le C,$$

where C is independent of n. This step is the most difficult step, where monotonicity assumption is need for $\tilde{B}$ and (4.97) will be used. Define $\overline{u}_n(t) = \sum_{i=1}^{n} u_{i-1} \cdot \chi_{(t_{i-1},t_i]}$.

The last step is rewrite above the elliptic variational inequality as

$$\langle u_n'(t), v(t) - \overline{u}_n(t)\rangle + \left\langle \tilde{B}(\overline{u}_n(t)), v(t) - \overline{u}_n(t) \right\rangle \geq \langle f_i, v(t) - \overline{u}_n(t)\rangle \quad \text{for all } v \in \mathcal{K} \text{ and for a.e. } t \in [0,T].$$

Integrate with respect to t and pass to the liminf as $n \to \infty$ to show that the limit u of u_n is a solution, using the prior estimate.

Remark 4.3.14. If the operator $\tilde{B}$ depends on t, we need $\tilde{B}(t) : V \to V^*$ is maximal monotone for all $t \in [0,T]$ and additional assumption that there exists $B'(t)u$ and $B''(t)u$ in V^* with respect to t such that

$$\|B'(t)u\|_{V^*} + \|B''(t)u\|_{V^*} \leq C_1 + C_2 r(\|u\|_V),$$

where $C_1, C_2 > 0$ and $r(x)$ is non-decreasing and satisfies $\lim_{x\to\infty} r(x) = \infty$. For more details, see [40, 38].

4.3.3 Application to the obstacle problem

In this section, we briefly introduce the idea of applying the existence theorems for abstract variational inequalities to solve obstacle problems.
Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary. Let $1 < p, p' < \infty$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and let $V = W_0^{1,p}(\Omega)$. For $j = 0, 1, \cdots n.$, functions $a_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Consider the following problem:
Given that $f \in (W_0^{1,p}(\Omega))^*$ and $\varphi \in W_0^{1,p}(\Omega)$, find $u \in W_0^{1,p}(\Omega)$ such that $u \geq \varphi$ and

$$\begin{aligned}\sum_{j=1}^{n} \int_\Omega a_j(x, u(x), Du(x))(Dv(x) - Du(x))dx + \int_\Omega a_0(x, u(x), Du(x))(v(x) - u(x))dx \\ \geq \int_\Omega f(x)(v(x) - u(x))dx \quad \text{for any } v \in W_0^{1,p}(\Omega) \text{ with } v \geq \varphi. \qquad (4.98)\end{aligned}$$

Recall that in the section 4.2.1, we formulated the conditions (E1), (E2), (E3') and (E4) on a_j such that the operator $A : V \to V^*$ is well defined by

$$\langle A(u), v\rangle = \sum_{j=1}^{n} \int_\Omega a_j(x, u(x), Du(x))D_j v(x)dx + \int_\Omega a_0(x, u(x), Du(x))v(x)dx, \quad \text{for } v \in V.$$

And the operator A is pseudomonotone. Define $F : V \to V^*$ by

$$\langle F, v\rangle = \int_\Omega f v dx.$$

Let $K := \{v \in V : v \geq \varphi\}$, then K is a closed and convex subset. K represents the imposed constraint determined by the obstacle φ. Now apply Theorem 4.3.6, by (E2) and (E4), we know that A is coercive in the sense of (4.16). Therefore, there exists a solution to the above problem.

Chapter 5

Variational problems with strictly p-quasimonotone function

Recall that in the previous two chapters, we first presented some existence theorems for abstract equations. Then we applied those results to solve elliptic boundary value problems (EBVPs) and parabolic initial boundary value problems (PIBVPs). Due to that the notion of the strict p-quasimonotonicity is not defined through an operator from a Banach space to its dual (see Definition 2.4.15), we will directly find the existence of solutions of EBVPs and PIBVPs here. In the first part of this chapter, we will first formulate EBVPs and PIBVPs that need to be solved. Then we will use variational approach to show the existence theorems for these problems. The existence results of these problems are established through Galerkin's approximation method. Unlike the monotonicity condition which is a pointwise condition, e.g., see (E3), the strict p-quasimonotonicity is a weaker, integrated version of monotonicity. The main difficulty in dealing with the strict p-quasimonotonicity to solve these problems is to show that the weak limit of the approximating sequence is a solution to the problem. This difficulty is overcome here with the use of the tools of Young measures. The details of how the Young measures are used to show the compactness of the approximating sequence will be given in the proof of Theorem 5.2.2. In the second part of this chapter, we will first apply the notion of strict p-quasimonotonicity in the study of elliptic variational inequalities (EVIs) where the operator A here is defined as (5.22) which involves a strictly p-quasimonotone function, and the Banach space V is only taken to be a subspace of $W^{1,p}(\Omega)$. Notice that the notion of strict p-quasimonotonicity has not yet been applied in the literatures for the study of EVIs. So we have in this chapter a new setting and a new existence result for the EVIs. The proof of existence Theorem 5.3.4 for EVIs consists of two parts: the first part is the standard approach of elliptic problems, which is projecting the problem on the finite dimensional space and finding the finite dimensional solution of the problem. The second part of the proof is to use the coercive condition to find a prior estimate for the finite dimensional solution sequence, then with the use of Young measures, we can show that the weak limit of the finite dimensional solution sequence is a solution to the problem. This part is inspired by the work in [35]. In the last section of this chapter, we will set up some open problems of applying the notion of strict p-quasimonotonicity in the study of the PVIs, the difficulty when applying Rothe's method to find the existence will also be pointed out.

5.1 Elliptic boundary value problems

Let Ω be a bounded open subset of $\mathbb{R}^n$ with sufficiently smooth boundary, let p and p' be such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.
We consider the following boundary value problem for elliptic system:
For $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$, find $u : \Omega \to \mathbb{R}^m$ such that

$$-\operatorname{div} a(x, u(x), Du(x)) = f \text{ on } \Omega, \tag{5.1}$$

with the boundary condition

$$u = 0 \text{ on } \partial\Omega. \tag{5.2}$$

Denote $\mathbb{M}^{m\times n}$ as the real vector space of $m \times n$ matrices equipped with inner product $M : N = \sum_{i,j=1}^{n} M_{ij}N_{ij}$.
We state our assumptions on function a.

(W1) (Carathéodory condition): $a : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $x \mapsto a(x, w, F)$ is measurable for every $(w, F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(w, F) \mapsto a(x, w, F)$ is continuous for a.e. $x \in \Omega$.
(W2) (Growth condition): Assume that there exist $c_1 \geq 0$, $\lambda_1 \in L^{p'}(\Omega)$ and $0 < q \leq n\frac{p-1}{n-p}$ such that

$$|a(x, w, F)| \leq \lambda_1(x) + c_1(|u|^q + |F|^{p-1}).$$

(W3) (Coercive condition): There exist $c_2 > 0$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{\frac{p}{\alpha}'}(\Omega)$ and $0 < \alpha < p$ such that

$$a(x, w, F) : F \geq -\lambda_2(x) - \lambda_3(x) |w|^\alpha + c_2 |F|^p .$$

(W4) (Monotonicity condition): $a(x, w, F)$ is strictly p-quasimonotone (see Definition 2.4.15) in F.
With these assumptions, we have the following theorem.

Theorem 5.1.1 ([35, Theorem 3.2]). Assume that a satisfies the condition (W1) - (W4), then there exists a weak solution $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ to the problem (5.1) and (5.2) for every $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$.

Proof. The idea of the proof is similar to the proof of theorem 4.1.2. First, using Galerkin's method and fixed point theorem to find finite dimensional solutions, then coercive condition provides the uniformly boundedness of finite dimensional solutions. Finally, with the tools of Young measures, we can show the weak limit of approximating sequence is a solution.

Remark 5.1.2. In the coercive condition (W3), it is enough to have $|Du|^p$ because we are working in the space $W_0^{1,p}$ where the norm $\|u\|_{W_0^{1,p}}$ is equivalent to $\|Du\|_{L^p}$ for $u \in W_0^{1,p}(\Omega)$.

5.2 Parabolic initial boundary value problems

In this section, we will formulate the abstract parabolic partial differential equations of divergence type, then we will present the existence theorem and we will demonstrate how Young measures are used in the limit passage. Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Let $p \in (\frac{2n}{n+2}, \infty)$ and let p' be its conjugate. $\mathbb{M}^{m\times n}$ denoted as the real vector space of $m \times n$ matrices equipped with the inner product $M : N = \sum_i \sum_j M_{ij} N_{ij}$. Given $f \in L^{p'}(0, T; W^{-1,p'}(\Omega; \mathbb{R}^m))$ and $u_0 \in L^2(\Omega; \mathbb{R}^m)$, we will consider the following initial boundary value problem:
find $u : \Omega \to \mathbb{R}^m$ such that

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x, t, u(x, t), Du(x, t)) = f \text{ on } \Omega \times [0, T]; \tag{5.3}$$

$$u(x, t) = 0 \text{ on } \partial\Omega \times [0, T]; \tag{5.4}$$

$$u(x, 0) = u_0(x) \text{ on } \Omega. \tag{5.5}$$

We assume a satisfies the following conditions:
(P1) (Carathéodory condition): $a : \Omega \times (0, T) \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $(x, t) \mapsto a(x, t, w, F)$ is measurable for any $(w, F) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and $(w, F) \mapsto a(x, t, w, F)$ is continuous for almost every $(x, t) \in \Omega \times (0, T)$.
(P2) (Growth condition): There exist $c_1 \geq 0$ and $\lambda_1 \in L^{p'}(\Omega \times (0, T))$ such that

$$|a(x, t, w, F)| \leq \lambda_1(x, t) + c_1(|w|^{p-1} + |F|^{p-1}).$$

(P3) (Coercivity condition): There exist $c_2 > 0$, $\lambda_2 \in L^1(\Omega \times (0, T))$ and $\lambda_3 \in L^{(\frac{p}{\alpha})'}(\Omega \times (0, T))$ for $0 < \alpha < p$ such that

$$a(x, t, w, F) : F \geq -\lambda_2(x, t) - \lambda_3(x, t) |w|^\alpha + c_2 |F|^p .$$

(P4) (Monotonicity condition): $a(x, t, w, F)$ is strictly p-quasimonotone in F (see Definition 2.4.15).

Remark 5.2.1. The Carathéodory condition (P1) ensures the function $a(x, t, u(x, t), Du(x, t)$ is measurable on $\Omega \times [0, T)$. The growth and coercive conditions (P2) (P3) are standard comparing with those in Section 3.2.2. The condition (P4) here states that we only have a weak, integrated version of monotonicity in the argument F instead of (w, F) in previous Section 3.2.2.

Theorem 5.2.2. If a satisfies (P1)-(P4) for some $p \in (\frac{2n}{n+2}, \infty)$, then the parabolic system (5.3) $-$ (5.5) has a weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ for every $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and for every $u_0 \in L^2(\Omega;\mathbb{R}^m)$.

Proof. We will use Galerkin's method to prove the existence of solution.
Galerkin's base:
Let $s \geq 1 + n(\frac{1}{2} - \frac{1}{p})$, by Kondrachov embedding Theorem 1.2.26, we have the compact embedding

$$W_0^{s,2}(\Omega;\mathbb{R}^m) \subset W_0^{1,p}(\Omega;\mathbb{R}^m).$$

For $\zeta \in L^2(\Omega;\mathbb{R}^m)$, we consider the bounded linear functional φ on Hilbert space $W_0^{s,2}(\Omega;\mathbb{R}^m)$

$$\varphi : v \mapsto (\zeta, v)_{L^2}, \text{ where } (\cdot,\cdot)_{L^2} \text{ denotes the inner product of } L^2.$$

By Riesz representation theorem, we may define the map $K : L^2 \to L^2$ such that $K\zeta \in W_0^{s,2}(\Omega;\mathbb{R}^m)$ is uniquely defined by

$$\varphi(v) = (\zeta, v)_{L^2} = (K\zeta, v)_{W^{s,2}} \text{ for all } v \in W_0^{s,2}(\Omega;\mathbb{R}^m).$$

The map K is linear, symmetric, bounded and compact (due to the embedding $W_0^{s,2} \subset L^2$ is compact). Moreover, since

$$(\zeta, K\zeta)_{L^2} = (K\zeta, K\zeta)_{W^{s,2}} \geq 0,$$

K is strictly positive. Hence, there exists an L^2-orthonormal base $W := \{w_1, w_2, \cdots\}$ of eigenvectors of K and positive real eigenvalues λ_i with $Kw_i = \lambda_i w_i$. This means that $w_i \in W_0^{s,2}(\Omega;\mathbb{R}^m)$ for all i and for all $v \in W_0^{s,2}(\Omega)$

$$\lambda_i (w_i, v)_{W^{s,2}} = (Kw_i, v)_{W^{s,2}} = (w_i, v)_{L^2}. \tag{5.6}$$

Hence, the functions w_i are orthogonal with respect to the inner product of $W^{s,2}(\Omega)$, since for $i \neq j$, choose $v = w_j$ in (5.6), we obtain

$$0 = \frac{1}{\lambda_i}(w_i, w_j)_{L^2} = (w_i, w_j)_{W^{s,2}}.$$

Choose $v = w_i$ in (5.6), we get

$$1 = \|w_i\|_{L^2}^2 = (w_i, w_i)_{L^2} = (Kw_i, w_i)_{W^{s,2}} = \lambda_i (w_i, w_i)_{W^{s,2}} = \lambda_i \|w_i\|_{W^{s,2}}^2 .$$

Therefore, $\tilde{W} := \{\tilde{w}_1, \tilde{w}_2, \cdots\}$ with $\tilde{w}_i = \sqrt{\lambda_i} w_i$, is an orthonormal set for $W_0^{s,2}(\Omega;\mathbb{R}^m)$. $\tilde{W}$ is actually a basis for $W_0^{s,2}(\Omega;\mathbb{R}^m)$. For any $v \in W_0^{s,2}(\Omega;\mathbb{R}^m)$,

$$s_n(v) := \sum_{i=1}^{n} (\tilde{w}_i, v)_{W^{s,2}} \tilde{w}_i$$

converges to some $\tilde{v}$ in $W_0^{s,2}(\Omega;\mathbb{R}^m)$. On the other hand, since W is a basis and by definition of $s_n(v)$, we also have that

$$s_n(v) = \sum_{i=1}^{n} (w_i, v)_{L^2} w_i$$

converges to v in $L^2(\Omega)$. By uniqueness of limit, we have $\tilde{v} = v$.
We defined the L^2-orthonormal projector $P_k : L^2 \to L^2$ onto the k dimensional subspace generated by $\{w_1, \cdots, w_k\}$, $k \in \mathbb{N}$, as follow:

$$P_k(u) = \sum_{i=1}^{k} (w_i, u)_{L^2} w_i,$$

with $\|P_k\|_{\mathcal{L}(L^2,L^2)} = 1$. Note that for $u \in W^{s,2}(\Omega)$,

$$P_k(u) = \sum_{i=1}^{k} (w_i, u)_{L^2} w_i = \sum_{i=1}^{k} (\tilde{w}_i, u)_{W^{s,2}} \tilde{w}_i,$$

so $\|P_k\|_{\mathcal{L}(W^{s,2},W^{s,2})} = 1$.
Galerkin's approximation:
Define the following approximating solution of (5.3) – (5.5):

$$u_k(x,t) = \sum_{i=1}^{k} c_{ki}(t) w_i(x),$$

where $c_{ki} : [0,T] \to \mathbb{R}$ are supposed to be measurable functions. The boundary condition (5.4) is taken into consideration by construction in the sense that $u_k \in L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^m))$. The initial condition (5.5) is taken care by choosing the initial coefficients $c_{ki}(0) = (u_0, w_i)_{L^2}$ such that

$$u_k(\cdot,0) = \sum_{i=1}^{k} c_{ki}(t) w_i(\cdot) \to u_0 \text{ in } L^2(\Omega;\mathbb{R}^m) \text{ as } k \to \infty. \tag{5.7}$$

We are looking for u_k such that the following system of ordinary differential equations hold

$$(\partial_t u_k, w_j)_{L^2} + \int_\Omega a(x,t,u_k,Du_k) : Dw_j dx = \langle f(t), w_j\rangle \quad \text{for all } j = 1,2,\cdots k. \tag{5.8}$$

For fixed $k \in \mathbb{N}$, let $0 < \epsilon < T$ and $J = [0,\epsilon]$. We choose $r > 0$ large enough such that the set $B_r(0) \subset \mathbb{R}^k$ contains the vector $(c_{k1}(0), \cdots, c_{kk}(0))$ and we set $K = \overline{B_r(0)}$.
Observe that by (P1), the function

$$F : J \times K \to \mathbb{R}^k$$
$$(t, c_1, \cdots, c_k) \mapsto \left(\langle f(t), w_j\rangle - \int_\Omega a(x,t,\sum_{i=1}^{k} c_i w_i, \sum_{i=1}^{k} c_i Dw_i) : Dw_j dx\right)_{j=1,\cdots,k}$$

is a Carathéodory function. And each component F_j may be estimated on $J \times K$ by

$$|F_j(t,c_1,\cdots,c_k)| \le \|f(t)\|_{W^{1,p}} \|w_j\|_{W_0^{1,p}} + \left(\int_\Omega \left|a(x,t,\sum_{i=1}^{k} c_i w_i, \sum_{i=1}^{k} c_i Dw_i)\right|^{p'} dx\right)^{\frac{1}{p'}} \left(\int_\Omega |Dw_j|^p dx\right)^{\frac{1}{p}}. \tag{5.9}$$

Using the growth condition in (P2), the right hand side of the above inequality (5.9) can be estimated by the following

$$|F_j(t,c_1,\cdots,c_k)| \le C(r,k) M(t) \quad \text{uniformly on } J \times K, \tag{5.10}$$

where $C(r,k)$ is a constant depends on r and k, and $M(t) \in L^1(J)$ (independent of j,k and r). Hence, applying the Carathéodory Theorem 1.4.1 to the ODEs system, for all $j \in \{1,2,\cdots,k\}$

$$\begin{cases} c_j'(t) = F_j(t, c_1(t), \cdots, c_k(t)); \\ c_j(0) = c_{kj}(0). \end{cases} \tag{5.11}$$

There exists an absolutely continuous solution c_j (depending on k) of the system (5.11) on a time interval $[0,\epsilon')$, where $\epsilon' > 0$, a prior estimate which may depend on k. Moreover, the corresponding integral equation

$$c_j(t) = c_j(0) + \int_0^t F_j(\tau, c_1(\tau), \cdots, c_k(\tau)) d\tau \quad \text{holds on } [0,\epsilon'). \tag{5.12}$$

Then $u_k := \sum_{j=1}^{k} c_j(t) w_j$ is the desired (short time interval) solution of (5.8) with the initial condition (5.7).
Now, we want to show that the local solution constructed above can be extended to the whole interval $[0,T)$ independent of k. We will first show that the coefficients $|c_{ki}(t)|$ is uniformly bounded:
(5.8) is linear in w_j, so we can use u_k as a test function in equation (5.8) in place of w_j. For any time τ in the existence interval, we have

$$\underbrace{\int_0^\tau (\partial_t u_k, w_j)_{L^2} dt}_{\text{I}} + \underbrace{\int_0^\tau \int_\Omega a(x,t,u_k,Du_k) : Dw_j dx dt}_{\text{II}} = \underbrace{\int_0^\tau \langle f(t), w_j\rangle \, dt}_{\text{III}}.$$

From integral by part formula 1.3.15, we obtain

$$\mathrm{I} = \frac{1}{2}\left\|u_k(\cdot,\tau)\right\|^2_{L^2(\Omega;\mathbb{R}^m)} - \frac{1}{2}\left\|u_k(\cdot,0)\right\|^2_{L^2(\Omega;\mathbb{R}^m)}.$$

Using the coercive condition in (P3) for the second term, we obtain

$$\mathrm{II} \geq -\left\|\lambda_2\right\|_{L^1(\Omega\times(0,T))} - \left\|\lambda_3\right\|_{L^{(\frac{p}{\alpha})'}(\Omega\times(0,T))}\left\|u_k\right\|^{\alpha}_{L^p(\Omega\times(0,\tau))} + c_2\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))}.$$

From Hölder's inequality, we get

$$\mathrm{III} \leq \left\|f\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))}\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))}.$$

Combine above three estimates, we obtain

$$\begin{aligned}\frac{1}{2}\left\|u_k(\cdot,\tau)\right\|^2_{L^2(\Omega;\mathbb{R}^m)} - \frac{1}{2}\left\|u_k(\cdot,0)\right\|^2_{L^2(\Omega;\mathbb{R}^m)} - \left\|\lambda_2\right\|_{L^1(\Omega\times(0,T))} - \left\|\lambda_3\right\|_{L^{(\frac{p}{\alpha})'}(\Omega\times(0,T))}\left\|u_k\right\|^{\alpha}_{L^p(\Omega\times(0,\tau))}\\ +c_2\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))} \leq \left\|f\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))}\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))},\end{aligned}$$

which we can re-arrange the terms and get

$$\begin{aligned}&\frac{1}{2}\left\|u_k(\cdot,\tau)\right\|^2_{L^2(\Omega;\mathbb{R}^m)} + c_2\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))}\\ \leq& \frac{1}{2}\left\|u_k(\cdot,0)\right\|^2_{L^2(\Omega;\mathbb{R}^m)} + \left\|\lambda_2\right\|_{L^1(\Omega\times(0,T))} + \left\|\lambda_3\right\|_{L^{(\frac{p}{\alpha})'}(\Omega\times(0,T))}\left\|u_k\right\|^{\alpha}_{L^p(\Omega\times(0,\tau))}\\ &+\left\|f\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))}\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))}.\end{aligned}$$

We claim that

$$|(c_{ki}(\tau))_{i=1,\cdots,k}|^2_{\mathbb{R}^k} = \left\|u_k(\cdot,\tau)\right\|^2_{L^2(\Omega)} \leq \bar{C},$$

for a constant $\bar{C}$ which is independent of τ and k.
Note that $u(\cdot,0) \to u_0(\cdot)$ in $L^2(\Omega;\mathbb{R}^m)$, then $\left\|u_k(\cdot,0)\right\|_{L^2(\Omega;\mathbb{R}^m)}$ is bounded. Suppose $\left\|u_k\right\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}$ is not bounded, i.e., $\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))} = \infty$, then we get

$$c_2\left\|u_k\right\|^p_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))} \leq c_3 + c_4\left\|u_k\right\|^{\alpha}_{L^p(\Omega\times(0,\tau);\mathbb{R}^m)} + c_5\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega;\mathbb{R}^m))}.$$

Since $\left\|u_k\right\|_{L^p(\Omega\times(0,\tau);\mathbb{R}^m)} \leq c\left\|u_k\right\|_{L^p(0,\tau;W_0^{1,p}(\Omega))}$ by poincaré inequality, and $\alpha < p$, the above inequality is impossible. Hence, $\left\|u_k\right\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}$ is bounded. Then the right hand side of above inequality is bounded and thus $\left\|u_k(\cdot,\tau)\right\|^2_{L^2}$ is bounded by a constant which is independent of τ and k.
Now, let

$$U := \{t \in [0,T) : \text{there exists a weak solution of the system (5.11) on } [0,t)\}.$$

U is non-empty since we have proved the local existence above. We will show that U is both open and closed which shows the local solutions can be extended to the whole interval.
Show U is open:
To see this, let $t \in U$ and $0 < \tau_1 < \tau_2 \leq t$. Since $\tau \mapsto c_{kj}(\tau)$ is absolutely continuous. Hence, we can apply the above argument again, solve (5.8) on $(t, t+\epsilon)$ with the initial data $\lim_{\tau\to t^-} u_k(\tau)$ and get a solution of (5.11) on $[0, t+\epsilon)$. Thus, U is open.
Show U is closed:
We consider the increasing sequence $(\tau_i)_i \subset U$ such that $\tau_i \uparrow t$. Let $c_{kj,i}$ denotes the solution of the system (5.11) we constructed on $[0,\tau_i]$ and define

$$\tilde{c}_{kj,i} := \begin{cases} c_{kj,i}(\tau) & \text{if } \tau \in [0,\tau_i] \\ c_{kj,i}(\tau_i) & \text{if } \tau \in (\tau_i, t), \end{cases}$$

then the sequence $\{\tilde{c}_{kj,i}\}_i$ is bounded and equicontinuous on $[0,t)$, as it is shown above. Hence, by the Arzela-Ascoli theorem, a subsequence (again denoted by $\tilde{c}_{kj,i}$) converges uniformly in τ on $[0,t)$ to a continuous function

$c_{kj}(\tau)$. Then by Dominated Convergence Theorem 1.2.13 in (5.12), it is now easy to see that $c_{kj}(\tau)$ solves (5.11) on $[0,t]$. Hence, $t\in U$ and thus U is closed. It follows that $U=[0,T)$, which shows that all local solutions can be extended to the whole interval.

Compactness of the Galerkin approximation:

Due to the linearity, we can substitute w_j by u_k in (5.8), and by a similar calculation as above, we will obtain that the sequence $\{u_k\}_k$ is bounded in

$$L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))\cap L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)).$$

Therefore, there exists a subsequence, which is again denoted by u_k, such that

$$u_k \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m));$$

$$u_k \rightharpoonup u \text{ in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)).$$

We will use the Aubin-Lions Theorem 1.3.16 to show the compactness of $(u_k)_k$ in a suitable space.
Set $B_0:=W_0^{1,p}(\Omega;\mathbb{R}^m)$, $B:=L^q(\Omega;\mathbb{R}^m)$ (for q with $2<q<p^*:=\frac{np}{n-p}$ if $p<n$ and $2<q<\infty$ if $p\geq n$) and $B_1:=(W_0^{s,2}(\Omega))^*$. Then we have the following chain of continuous injection:

$$B_0 \overset{i}{\hookrightarrow} B \overset{i_0}{\hookrightarrow} L^2(\Omega) \overset{\gamma}{\equiv} (L^2(\Omega))^* \overset{i_1}{\hookrightarrow} B_1.$$

For $i:B_0\to B$, we take the inclusion mapping and for $j:B\to B_1$, we take $j:=i_1\circ\gamma\circ i_0$.
From above, $\{u_k\}_k$ is a bounded sequence in $L^p(0,T;B_0)$. From (5.8), the time derivative $\frac{d}{dt}(j\circ i\circ u_k)$ is given by

$$\begin{aligned}\frac{d}{dt}(j\circ i\circ u_k):\ &[0,T)\to B_1=(W_0^{s,2}(\Omega))^*\\ &t\mapsto\Big(\psi\mapsto-\int_\Omega a(x,t,u_k,Du_k):D(P_k\psi)dx+\langle f(t),P_k\psi\rangle\Big).\end{aligned}$$

Indeed, $\{\partial_t j\circ j\circ u_k\}_k$ is a bounded sequence in $L^{p'}(0,T;(W_0^{s,2}(\Omega))^*)$:
we have by the growth condition in (P2) that

$$\left|-\int_0^T\int_\Omega a(x,t,u_k,Du_k):D(P_k\psi)dxdt+\langle f,P_k\psi\rangle\right|$$

$$\leq\left[C(\|\lambda_1\|_{L^{p'}((0,T)\times\Omega)}+\|u_k\|^{p-1}_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))})+\|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))}\right]\|P_k\psi\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}$$

and

$$\|P_k\psi\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))}\leq C'\,\|P_k\psi\|_{L^p(0,T;W_0^{s,2}(\Omega;\mathbb{R}^m))}\leq C'\,\|\psi\|_{L^p(0,T;W_0^{s,2}(\Omega;\mathbb{R}^m))},$$

where the last inequality follows since $\|P_k\|_{\mathcal{L}(W^{s,2},W^{s,2})}=1$.
Hence, from the Lions-Aubin Theorem 1.3.16, we may conclude that there exists a subsequence, which again denoted by u_k, having the property that

$$u_k\to u \text{ in } L^p(0,T;L^q(\Omega)) \text{ for all } q<p^* \text{ and in measure on } \Omega\times(0,T).$$

Note that in order to have strong convergence simultaneously for all $q<p^*$, we use the usual diagonal process. Since $W_0^{s,2}(\Omega;\mathbb{R}^m)\subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\partial_t\,(j\circ i\circ u)$ is in $L^{p'}(0,T;(W_0^{s,2}(\Omega;\mathbb{R}^m))^*)$, we can conclude that $\partial_t u$ (or rather $\partial_t(j\circ i\circ u)$) is an element of $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$.
Recall that from Theorem 1.3.14 the space

$$\{u\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)):\partial_t(j\circ i\circ u)\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)))\}$$

is continuously embedded in

$$C([0,T];L^2(\Omega)).$$

Hence, we have that $u\in C([0,T],L^2(\Omega;\mathbb{R}^m))$ after possible modification of u on a Lebesgue zero-set of $[0,T]$. Hence, $u(\cdot,t)\in L^2(\Omega;\mathbb{R}^m)$ and $u(\cdot,t)$ attains its initial value $u(\cdot,0)$ continuously in $L^2(\Omega;\mathbb{R}^m)$.

By a very similar proof of Lemma 3.1.4, we can prove that $u_k(\cdot,T) \rightharpoonup u(\cdot,T)$ weakly in $L^2(\Omega)$ and $u(\cdot,0)=u_0$. Since $(u_k)_k$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, it is easy to see that for a subsequence

$$u_k(\cdot,T) \rightharpoonup z \text{ weakly in } L^2(\Omega;\mathbb{R}^m),$$

and we will show that $z=u(\cdot,T)$.
Note that (up to possible choice of a further subsequence)

$$-\text{div } a(x,t,u_k,Du_k) \rightharpoonup \chi \text{ weakly in } L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)).$$

Now, for arbitrary $\psi \in C^\infty([0,T])$ and $v \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, we claim that

$$\int_\Omega z\psi(T)vdx - \int_\Omega u_0\psi(0)vdx = \langle f-\chi,\psi v\rangle + \int_0^T\int_\Omega \psi' vudx. \tag{5.13}$$

Since $\bigcup_{n\in\mathbb{N}} \text{span}\{w_1,\cdots,w_n\}$ is dense in $W_0^{1,p}(\Omega)$, it is sufficient to show (5.13) for $v \in \text{span}\{w_1,\cdots,w_n\}$. Then, by testing (5.8) by $v\psi$, we have for $m \geq n$ that

$$\int_0^T\int_\Omega \partial_t u_m v\psi dxdt + \int_0^T\int_\Omega a(x,t,u_m,Du_m) : Dv\psi dxdt = \langle f,\psi v\rangle,$$

from integral by parts, we get

$$\int_0^T\int_\Omega \partial_t u_m v\psi dxdt = \int_\Omega u_m(T)\psi(T)vdx - \int_\Omega u_m(0)\psi(0)vdx - \int_0^T\int_\Omega u_m v\psi' dxdt.$$

Now, (5.13) follows by letting m tend to infinity. In particular, choosing $\psi(0)=\psi(T)=0$ in (5.13), we have

$$\langle f-\chi,\psi v\rangle = -\int_0^T\int_\Omega \psi' vudxdt = \int_0^T \psi vu' dt,$$

and hence

$$u' + \chi = f.$$

Using the above equation and (5.13), we have, on the other hand, that

$$\int_\Omega z\psi(T)vdx - \int_\Omega u_0\psi(0)vdx = \langle u',\psi v\rangle + \int_0^T\int_\Omega \psi' vudxdtv = \int_\Omega (u\psi v)|_0^T dx = \int_\Omega u(T)\psi(T)vdx - \int_\Omega u(0)\psi(0)vdx.$$

Choosing $\psi(T)=1$ and $\psi(0)=0$, we get $u(\cdot,0)=u_0$.
Choosing $\psi(0)=1$ and $\psi(T)=0$, we get $u(\cdot,T)=z$.

The Young measure generated by the sequence of Galerkin's approximation:
The sequence (at least up to a subsequence) of gradient Du_k generated a Young measure $\nu_{(x,t)}$, and since u_k converges to u in measure on $\Omega\times(0,T)$, the sequence $(u_k,Du_k)_k$ generates the Young measure $\delta_{u(x,t)}\otimes\nu_{(x,t)}$ by Propositions 2.4.8 and 2.4.10. We are going to state some facts about the gradient Young measure ν generated by $(Du)_k$ in the following proposition.

Proposition 5.2.3 ([35, Proposition 4.3]). The Young measure $\nu_{(x,t)}$ generated by the sequence $\{Du_k\}_k$ has the following properties:
(1) $\nu_{(x,t)}$ is a probability measure on $\mathbb{M}^{m\times n}$ for almost all $(x,t)\in\Omega\times(0,T)$;
(2) $\nu_{(x,t)}$ satisfies $Du(x,t)=\langle\nu(x,t),id\rangle$ for almost every $(x,t)\in\Omega\times(0,T)$;
(3) $\nu(x,t)$ is a homogeneous Young measure for almost all $(x,t)\in\Omega\times(0,T)$.

parabolic div-curl inequality:
We also need the following parabolic version of the div-curl inequality, which will be the key to pass the limit in the approximating equation and to prove the weak limit u of the Galerkin's approximating sequence u_k is indeed a solution of (5.3) – (5.5).

Lemma 5.2.4 (Parabolic Div-Curl Inequality). The Young measure $\nu_{(x,t)}$ generated by the Gradients Du_k of the Galerkin's approximation u_k has the property that

$$\int_0^\Omega \int_{\mathbb{M}^{m\times n}} (a(x,t,u,\lambda) - a(x,t,u,Du)) : (\lambda - Du) d\nu_{(x,t)} dxdt \leq 0. \tag{5.14}$$

Proof. Let us consider the sequence

$$I_k := (a(x,t,u_k,Du_k) - a(x,t,u,Du)) : (Du_k - Du)$$

and show that its negative part I_k^- is equiintegrable on $\Omega \times (0,T)$:
To do this, we write I_k^- in the form

$$I_k = \underbrace{a(x,t,u_k,Du_k) : Du_k}_{\mathrm{II}_k} - \underbrace{a(x,t,u_k,Du_k) : Du}_{\mathrm{III}_k} - \underbrace{a(x,t,u,Du) : Du_k}_{\mathrm{IV}_k} + \underbrace{a(x,t,u,Du) : Du}_{\mathrm{V}_k}.$$

Note

$$\begin{aligned} I_k^- = \max\{-I_k, 0\} &\leq \max\{-a(x,t,u_k,Du_k) : Du_k, 0\} + \max\{a(x,t,u_k,Du_k) : Du, 0\} \\ &\quad + \max\{a(x,t,u,Du) : Du_k, 0\} + \max\{a(x,t,u,Du) : Du, 0\} \\ &\leq \mathrm{II}_k^- + |\mathrm{III}_k| + |\mathrm{IV}_k| + \mathrm{V}_k^-. \end{aligned}$$

We will show that $\mathrm{II}_k^-, \mathrm{V}_k^-, \mathrm{III}_k$ and IV_k are equiintegrable.
Show II_k^- is equiintegrable:
by the coercive condition in (P3), we have

$$\mathrm{II}_k^- = \max\{-a(x,t,u_k,Du_k) : Du_k, 0\} \leq \max\{\lambda_2(x,t) + \lambda_3(x,t)|u_k|^\alpha - c_2|Du_k|^p, 0\},$$

which implies the equiintegrability of II_k^- by boundedness of $|u_k|$ and $|Du_k|$. Similarly, we can obtain the equiintegrability of V_k^-.
Show III_k is equiintegrable:
take a measurable subset $S \subset \Omega \times (0,T)$, from the growth condition (P2), using Hölder's inequality, we obtain

$$\begin{aligned} \int_S |a(x,t,u_k,Du_k) : Du|\, dxdt &\leq \left(\int_S |a(x,t,u_k,Du_k|^{p'}\, dxdt)^{\frac{1}{p'}} (\int_S |Du|^p\, dxdt\right)^{\frac{1}{p}} \\ &\leq C \underbrace{\left(\int_S |\lambda_1(x,t)|^{p'} + |u_k|^p + |Du_k|^p\, dxdt\right)^{\frac{1}{p'}}}_{\mathrm{I}} \cdot \underbrace{\left(\int_S |Du|^p\, dxdt\right)^{\frac{1}{p}}}_{\mathrm{II}}. \end{aligned}$$

The first integral I is uniformly bounded in k, and the second integrand II is constant. Choose S with the measure of S small enough, we can get the equiintegrability of III_k. A similar argument can be used to prove the equiintegrability of IV_k.
Hence, I_k^- is equiintegrable. We may use 'Fatou's type' Lemma 2.4.11, which gives that

$$X := \liminf_{k\to\infty} \int_0^T \int_\Omega I_k dxdt \geq \int_0^T \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt. \tag{5.15}$$

On the other hand, we will now see that $X \leq 0$:
From Mazur's Theorem 1.1.12, there exists a sequence v_k in $L^p(0,T;W_0^{1,p}(\Omega))$ such that each v_k is a convex linear combination of $\{u_1, \cdots, u_k\}$ such that $v_k \to u$ in $L^p(0,T;W_0^{1,p}(\Omega))$. In particular, $v(t,\cdot) \in \mathrm{span}\{w_1, \cdots, w_k\}$ for all $t \in [0,T]$.
Now we have

$$X = \liminf_{k\to\infty} \int_0^T \int_\Omega a(x,t,u_k,Du_k) : (Du_k - Du) dxdt \tag{5.16}$$

$$= \liminf_{k\to\infty} \left(\int_0^T \int_\Omega a(x,t,u_k,Du_k) : (Du_k - Dv_k) dxdt + \int_0^T \int_\Omega a(x,t,u_k,Du_k) : (Dv_k - Du) dxdt\right)$$

$$\leq \liminf_{k\to\infty} \left(\langle f, u_k - v_k\rangle - \int_0^T \int_\Omega (u_k - v_k)\partial_t u_k dxdt + \left(\int_0^T \int_\Omega |a(x,t,u_k,Du_k)|^{p'}\, dxdt\right)^{\frac{1}{p'}} \|v_k - u\|_{L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^m))}\right),$$

where in the last inequality, we use (5.8) since $u_k - v_k \in \operatorname{span}\{w_1, \cdots, w_k\}$. The first term in above inequality (5.16)

$$\langle f, u_k - v_k\rangle$$

converges to 0, since $u_k - v_k \rightharpoonup 0$ in $L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^m)$.
The first factor in the last term of above inequality (5.16)

$$\left(\int_0^T \int_\Omega |a(x,t,u_k,Du_k|^{p'}\, dxdt\right)^{\frac{1}{p'}}$$

is uniformly bounded in k by the growth condition (P2) and the boundedness for u_k in $L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^m))$.
The second factor

$$\|v_k - u\|_{L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)}$$

converges to zero as $k\to\infty$ by construction of sequence v_k. Hence, the last term vanishes in the limit.
For the second term, we have

$$\begin{aligned}-\int_0^T\int_\Omega (u_k - v_k)\partial_t u_k dxdt &= -\int_0^T\int_\Omega \frac{1}{2}\partial_t u_k^2 dxdt + \int_0^T\int_\Omega v_k\partial_t u_k dxdt \\ &= -\frac{1}{2}\|u_k(\cdot,T)\|^2_{L^2(\Omega;\mathbb{R}^m)} + \frac{1}{2}\|u_k(\cdot,0)\|^2_{L^2(\Omega;\mathbb{R}^m)} + \int_0^T\int_\Omega v_k\partial_t u_k dxdt. \qquad (5.17)\end{aligned}$$

Consider the last term in (5.17), we claim that for $k\to\infty$, we have

$$\int_0^\Omega v_k\partial_t u_k dxdt \to \int_0^T\int_\Omega u\partial_t u dt = \frac{1}{2}\|u(\cdot,T)\|^2_{L^2(\Omega;\mathbb{R}^m)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega;\mathbb{R}^m)}. \qquad (5.18)$$

To see this, let $\epsilon>0$, then there exists M such that for all $l\ge m\ge M$, we have
(i) $\left|\int_0^T\int_\Omega (u-v_m)\partial_t u dxdt\right| \le \epsilon$. This is possible by the Hölder inequality, $\partial_t(j\circ i\circ u)\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and $v_m\to u$ in $L^p(0,T:W^{1,p}(\Omega,\mathbb{R}^m)$;
(ii) $\left|\int_0^T\int_\Omega (v_l-v_m)\partial_t u_t dxdt\right| \le \epsilon$. This is possible by (5.13) since $v_l - v_m\in\operatorname{span}\{w_1,\cdots,w_l\}$ for all fixed $t\in(0,T)$.
Now, we can fix $m\ge M$ and choose $m_0\ge m$ such that for all $l\ge m_0$,

$$\left|\int_0^T\int_\Omega v_m(\partial_t u - \partial_t u_l)dxdt\right| \le \epsilon.$$

This is possible, since $\partial_t u_l \overset{*}{\rightharpoonup} \partial_t u$ in $L^{p'}(0,T;(W_0^{s,2}(\Omega;\mathbb{R}^m))^*)$.
Combination of above, we get for $l = l(\epsilon)\ge m(\epsilon)$,

$$\begin{aligned}\left|\int_0^T\int_\Omega v_l\partial_t u_l dxdt - \int_0^T\int_\Omega u\partial_t u dxdt\right| &\le \left|\int_0^T (v_l - v_m)\partial_t u_l dxdt\right| + \left|\int_0^T\int_\Omega v_m(\partial_t u_l - \partial_t u)dxdt\right| \\ &\quad + \left|\int_0^T\int_\Omega (v_m - u)\partial_t u dxdt\right| \le 3\epsilon,\end{aligned}$$

which shows (5.18).
On the other hand, since $(u_k)_k$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, there exists a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$, such that

$$u_k(\cdot,T)\rightharpoonup u(\cdot,T) \text{ in } L^2(\Omega;\mathbb{R}^m).$$

Then by the weak lower semicontinuity of norm, we have

$$\liminf_{k\to\infty}\|u_k(\cdot,T)\|_{L^2} \ge \|u(\cdot,T)\|_{L^2}. \qquad (5.19)$$

And by the construction of u_k, we have

$$\lim_{k\to\infty} \|u_k(\cdot,0)\|_{L^2} = \|u_0\|_{L^2}. \tag{5.20}$$

Using (5.18), (5.19) and (5.20) in (5.17), we obtain

$$\begin{aligned}\liminf_{k\to\infty} -\int_0^T\int_\Omega (u_k - v_k)\partial_t u_k dxdt &= \liminf_{k\to\infty}\left(-\frac{1}{2}\|u_k(\cdot,T)\|_{L^2}^2 + \frac{1}{2}\|u_k(\cdot,0)\|_{L^2}^2 + \int_0^T\int_\Omega v_k\partial_t u_k dxdt\right)\\ &= -\liminf_{k\to\infty}\frac{1}{2}\|u_k(\cdot,T)\|_{L^2}^2 + \lim_{k\to\infty}\frac{1}{2}\|u_k(\cdot,0)\|_{L^2}^2 + \lim_{k\to\infty}\int_0^T\int_\Omega v_k\partial_t u_k dxdt\\ &= -\liminf_{k\to\infty}\frac{1}{2}\|u_k(\cdot,T)\|_{L^2}^2 + \frac{1}{2}\|u_0\|_{L^2}^2 + \frac{1}{2}\|u(\cdot,T)\|_{L^2}^2 - \frac{1}{2}\|u_0\|_{L^2}^2 \le 0.\end{aligned}$$

This shows that $X \le 0$ and from (5.15), hence the result follows. ■

Limit passage:
Suppose that $\nu_{(x,t)}$ is not a Dirac mass on a set $(x,t) \in S \subset \Omega \times (0,T)$ of positive Lebesgue measure $|S| > 0$. Then, by the strict p-quasimonotonicity of $a(x,t,u,\cdot)$ and the fact that $\nu_{(x,t)}$ is a homogeneous $W^{1,p}$ gradient Young measure for all $(x,t) \in \Omega \times (0,T)$, we have for a.e. $(x,t) \in S$,

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) > \int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda) : \bar\lambda d\nu_{(x,t)}(\lambda) &= \int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) : \bar\lambda\\ &= \int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) : Du(x,t).\end{aligned}$$

Integrating over $\Omega \times (0,T)$ and using above div-curl Lemma 5.2.4, we obtain

$$\begin{aligned}\int_0^T\int_\Omega\int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) : Du(x,t)dxdt &\ge \int_0^T\int_\Omega\int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt\\ &> \int_0^T\int_\Omega\int_{\mathbb{M}^{m\times n}} a(x,t,u,\lambda)d\nu_{(x,t)}(\lambda) : Du(x,t)dxdt,\end{aligned}$$

which is a contradiction. Hence, we have $\nu_{(x,t)}$ is Dirac and $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t) \in \Omega \times (0,T)$. From this, it follows by Proposition 2.4.8 that $Du_k \to Du$ on $\Omega \times (0,T)$ in measure for $k \to \infty$, and thus, $a(x,t,u_k,Du_k) \to a(x,t,u,Du)$ almost everywhere on $\Omega \times (0,T)$ (up to extraction of a further subsequence). By growth condition (P2), $a(x,t,u_k,Du_k)$ is equiintegrable. It follows that $a(x,t,u_k,Du_k) \to a(x,t,u,Du)$ in $L^1(\Omega \times (0,T))$ by Vitali's Theorem 1.2.16.
Now, we take a test function $w \in \text{span}_{i\in\mathbb{N}}\{w_1,\cdots,w_i\}$ and $\varphi \in C_0^\infty([0,T])$ in (5.8), integrate it over the interval $(0,T)$ and pass to the limit as $k \to \infty$, we obtain that for arbitrary $w \in \cup_{i\in\mathbb{N}}\text{span}\{w_1,\cdots,w_i\}$ and $\varphi \in C_0^\infty([0,T])$,

$$\int_0^T\int_\Omega \partial_t u(t,x)\varphi(t)w(x)dxdt + \int_0^T\int_\Omega a(x,t,u,Du) : Dw(x)\varphi(t)dxdt = \langle f, \varphi w\rangle.$$

Note that the set

$$\{\varphi w : \varphi \in C_0^\infty([0,T]);\ w \in \cup_{i\in\mathbb{N}}\text{span}\{w_1,\cdots,w_i\}\} \text{ is dense in } L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)),$$

it follows that u is in fact a weak solution and the theorem follows. ■

Remark 5.2.5. In the parabolic div-curl inequality, we need the assumption that $(u_k)_k$ are the solution of the equality (5.8). Indeed, the proposition also holds for $(u_k)_k$ satisfy the following inequality:

$$(\partial_t u_k, v - u_k)_{L^2} + \int_\Omega a(x,t,u_k,Du_k) : (Dv - Du_k)dx \ge \langle f(t), v - u_k\rangle,$$

for all $v \in \text{span}\{w_1,\cdots,w_k\}$.

5.3 Elliptic Variational Inequality

In this section, we aim to apply the notion of strict p-quasimonotonicity into elliptic variational inequalities which have not been done in the literatures. We will formulate a particular type of elliptic variational inequalities where the operator A involves a strictly p-quasimonotone function (see (5.22)). Then we will prove the existence theorem. The proof consists of two parts, the first part is to find the solution of the problems in the finite dimensional subspace, this follows the standard approach. The second part is to show that the weak limit of finite dimensional solution sequence is a solution of the problem, we need the tool of Young measures, this part is inspired by the proof for Theorem 5.2.2 in the previous section.
Let Ω be an open bounded subset of $\mathbb{R}^n$. Let $1 < p < \infty$ and p' be its conjugate, let $V = W_0^{1,p}(\Omega; \mathbb{R}^m)$, then V is reflexive and separable. Let $K \subset V$ be a closed and convex subset. We will consider the following problem: For any $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$, find $u \in K$ such that

$$\int_\Omega a(x, u(x), Du(x)) : (Dv - Du)dx \geq \int_\Omega f(x)(v(x) - u(x))dx \text{ for all } v \in K, \tag{5.21}$$

where $a : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfies (W1) - (W4) in Section 5.1.
Given $f \in W^{-1,p'}(\Omega; \mathbb{R}^m)$, we define $F : V \to V^*$ by

$$\langle F, v\rangle = \int_\Omega f(x)v(x)dx \quad \text{for any } v \in V.$$

And we define the operator $A : V \to V^*$ by

$$\langle A(u), v\rangle = \int_\Omega a(x, u(x), Du(x)) : Dvdx \quad \text{for any } v \in V. \tag{5.22}$$

Then A is well defined, linear and bounded. Hence (5.21) is equivalent to for any $F \in V^*$, find $u \in K$ such that

$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle\,, \text{ for all } v \in K. \tag{5.23}$$

We will first prove the case where K is bounded. Before we give the theorem, we need the following lemma.

Lemma 5.3.1. Let Ω be a bounded domain and let u_k be a bounded sequence in $W^{1,p}(\Omega; \mathbb{R}^m)$ for $p > 1$, then the Young measure ν_x generated by (at least a subsequence of) Du_k has the following properties:
(1) ν_x is a probability measure;
(2) ν_x is a homogeneous $W^{1,p}$ gradient Young measure;
(3) ν_x satisfies $\langle \nu_x, id\rangle = Du(x)$.

Proof. The proof of the lemma can be found in [24], where a more general case is proved.

Theorem 5.3.2. Let $K \subset V = W_0^{1,p}(\Omega; \mathbb{R}^m)$ be a bounded, closed, convex subset. Let A be defined as in (5.22) such that a satisfies (W1) (W2) and (W4) in Section 5.1. Then for any $F \in V^*$, there exists $u \in K$ to the following variational inequality:

$$\langle A(u), v - u\rangle \geq \langle F, v - u\rangle \text{ for all } v \in K. \tag{5.24}$$

Proof. The first part of proof is similar to the proof in Theorem 4.3.5, i.e., projecting the problem onto finite dimensional subspace and using fixed point theorem to find solution.
Since $V = W_0^{1,p}(\Omega; \mathbb{R}^m)$ is separable for $1 < p < \infty$, there exists k dimensional subspace V_k of V such that

$$V_1 \subset V_2 \cdots \subset V_k \subset \cdots \text{ and } \overline{\bigcup_{k=1}^\infty V_k} = V.$$

Set $K_k = V_k \cap K$, then K_k is a closed, convex and bounded subset of V,

$$K_1 \subset K_2 \cdots \subset K_k \subset \cdots \text{ and } \overline{\bigcup_{k=1}^\infty K_k} = K.$$

First, we wish to find $u_k \in K_k$ such that

$$\langle A(u_k), v - u_k \rangle \geq \langle F, v - u_k \rangle \text{ for all } v \in K_k. \tag{5.25}$$

Define scalar product $(\cdot, \cdot)$ on the finite dimensional space V_k generating a norm that is equivalent to the original norm on V_k. For $g \in V^*$ and $w \in V_k$, we have the linear functional

$$w \mapsto \langle g, w \rangle$$

is continuous on the Hilbert space V_k, so by Riesz representation theorem, there exists a linear and continuous operator $B : V^* \to V_k$ such that

$$\langle g, w \rangle = (Bg, w) \text{ for all } w \in V_k.$$

Hence, (5.25) can be written in the form

$$(BA(u_k), v - u_k) \geq (BF, v - u_k) \text{ for all } v \in K_k,$$

which is equivalent to

$$(u_k, v - u_k) \geq (u_k - BA(u_k) + BF, v - u_k) \text{ for all } v \in K_k. \tag{5.26}$$

Let P_k be the operator projecting V_k to the closed convex set K_k with respect to the scalar product $(\cdot, \cdot)$. Then (5.26) is equivalent to

$$u_k = P_k(u_k - BA(u_k) + BF). \tag{5.27}$$

Define $Q_k : K_k \to K_k$ by

$$Q_k(v) = P_k(v - BA(v) + BF). \tag{5.28}$$

We want to show that Q_k is continuous on finite dimensional space K_k. It is sufficient to show the weak continuity of $A(v)$. Assume $v_k \to v$ strongly in K_k, for any $w \in V$, we have

$$\langle A(v_k), w \rangle = \int_\Omega a(x, v_k(x), Dv_k(x)) : Dw dx = \int_\Omega \sum_{1 \leq i \leq m, 1 \leq j \leq n} a_{ij}(x, v_k(x), Dv_k(x))(Dw)_{ij} dx.$$

From (W1), we have, for all $1 \leq i \leq m, 1 \leq j \leq n$, that

$$a_{ij}(x, v_k(x), Dv_k(x)) \to a_{ij}(x, v(x), Dv(x)) \text{ a.e. in } \Omega.$$

From the growth condition (W2), we have $|a_{ij}(x, v_k(x), Dv_k(x))|^{p'}$ is uniformly integrable. Hence, by Vitali's theorem, we have

$$a_{ij}(x, v_k(x), Dv_k(x)) \to a_{ij}(x, v(x), Dv(x)) \text{ in } L^{p'}(\Omega).$$

Hence, we have as $k \to \infty$

$$\langle A(v_k), w \rangle \to \langle A(v), w \rangle,$$

i.e., $A(v_k) \rightharpoonup A(v)$ weakly in V^*. So

$$BA(v_k) \rightharpoonup BA(v) \text{ weakly in } K_k.$$

Since P_k is contractive, so it is continuous and we have

$$P_k(v_k - BA(v_k) + BF) \to P_k(v - BA(v) + BF) \text{ as } k \to \infty.$$

From Brouwer's fixed point theorem, there exists $u_k \in K_k$ such that

$$Q_k(u_k) = u_k.$$

i.e., u_k is solution of (5.27). Hence, for any $F \in V^*$, there exists $u_k \in K_k$ such that

$$\langle A(u_k), v - u_k \rangle \geq \langle F, v - u_k \rangle \text{ for all } v \in K_k.$$

This is equivalent to

$$\int_\Omega a(x, u_k(x), Du_k(x)) : (Dv - Du_k)dx \geq \langle F, v - u_k \rangle . \tag{5.29}$$

Since u_k is bounded, there exists a subsequence of $(u_k)_k$, again denoted by $(u_k)_k$ such that

$$u_k \rightharpoonup u \text{ weakly in } V,$$

and

$$u_k \to u \text{ in measure and in } L^r(\Omega),$$

where $0 < r < \frac{np}{n-p} = p^*$. Since the sequence of gradients Du_k is bounded, it generate Young measure ν_x. By Lemma 5.3.1, ν_x satisfies the following:
(1) ν_x is a probability measure;
(2) ν_x is a homogeneous $W^{1,p}$ gradient Young measure;
(3) ν_x satisfies $\langle \nu_x, id \rangle = Du(x)$.
Since u_k converge in measure to u, then from Proposition 2.4.8 and Proposition 2.4.10, we have (u_k, Du_k) generates the Young measure $\delta_{u(x)} \otimes \nu_x$.

There is also an elliptic version of div-curl inequality (see Section 3.4 in [35]), since $(u_k)_k$ satisfy the variational inequalities, by Remark 5.2.5, we have the following lemma:

Lemma 5.3.3. The Young measure generated by the gradients Du_k has the property that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : \lambda d\nu_x(\lambda)dx \leq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : Dud\nu_x(\lambda)dx.$$

Now we will show that Du_k converges to Du in measure from the strict p-quasimonotonicity, this is done by showing that ν_x is a Dirac measure. Suppose that ν_x is not a Dirac measure on an arbitrary set M of Lebesgue measure $|M| > 0$. Then by p-quasimonotonicity of $a(x, u, \cdot)$ and the fact that ν_x is a homogeneous $W^{1,p}$ gradient Young measure, we have for a.e. $x \in M$

$$\int_{\mathbb{M}^{m\times n}} (a(x, u, \lambda) - a(x, u, \overline{\lambda})) : (\lambda - \overline{\lambda})d\nu_x(\lambda) > 0,$$

which is equivalent to

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : \lambda d\nu_x(\lambda) > \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : \overline{\lambda} d\nu_x(\lambda) &= \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda)d\nu_x(\lambda) : \overline{\lambda} \\ &= \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda)d\nu_x(\lambda) : Du(x).\end{aligned}$$

Integrate over Ω and by above Lemma 5.3.3, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : Dud\nu_x(\lambda)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : \lambda d\nu_x(\lambda)dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} a(x, u, \lambda) : Dud\nu_x(\lambda)dx,$$

which is a contradiction. So we have $\nu_x = \delta_{Du(x)}$ for a.e. $x \in \Omega$. It follows from Proposition 2.4.8 that

$$Du_k \to Du \text{ in measure for } k \to \infty,$$

and thus

$$a(x, u_k, Du_k) \to a(x, u, Du) \text{ (up to a further subsequence) for a.e. } x \in \Omega.$$

From the growth condition (W2), using similar argument as above, it follows by Vitali's Theorem 1.2.16 that

$$a_{ij}(x, u_k, Du_k) \to a_{ij}(x, u, Du) \text{ in } L^{p'}(\Omega).$$

Since $v - u_k \rightharpoonup v - u$ in V, we can take the limit in (5.29) as $k \to \infty$ and get

$$\int_\Omega a(x, u(x), Du(x)) : (Dv - Du)dx \geq \langle F, v - u \rangle \quad \text{for } v \in \bigcup_{k\in\mathbb{N}} V_k.$$

i.e.,

$$\langle A(u), v-u\rangle \geq \langle F, v-u\rangle \text{ agrees on dense subset of } V.$$

Hence, the result follows. ∎

Now, we remove the boundedness condition of K, and we assume the coercive condition (W3).

Theorem 5.3.4. Let K be a closed and convex subset of $V = W_0^{1,p}(\Omega;\mathbb{R}^m)$. Let A be defined as in (5.22) and a satisfies (W1) - (W4) in Section 5.1. Then for any $F \in V^*$, there exists $u \in K$ such that the following variational inequality is satisfied:

$$\langle A(u), v-u\rangle \geq \langle F, v-u\rangle \text{ for all } v \in K. \tag{5.30}$$

Proof. Set $B_R = \{v \in V : \|v\| \leq R\}$ and $K_R = B_R \cap K$, then K_R is bounded, closed and convex subset, by the above Theorem 5.3.2, there exists $u_R \subset K_R$ satisfying

$$\langle A(u_R), v-u_R\rangle \geq \langle F, v-u_R\rangle \text{ for any } v \in K_R. \tag{5.31}$$

From the coercive condition (W3) and growth condition (W2), by using a similar calculation in the proof of Theorem 4.3.11, we can get for any $v_0 \in K$,

$$\frac{\langle A(u_R), u_R - v_0\rangle}{\|u_R\|_V} \to \infty \text{ as } \|u_R\|_V \to \infty. \tag{5.32}$$

Note that we have used that on $V = W_0^{1,p}(\Omega)$, $\|u_R\|_V$ is equivalent to $\|Du_R\|_{L^p}$.
Choose $R \geq v_0$, we obtain from (5.31) that

$$\langle A(u_R), v-u_R\rangle \geq \langle F, v-u_R\rangle \geq -\|F\|_V \|v-u_R\|_V ,$$

which implies that

$$\frac{\langle A(u_R), v_0 - u_R\rangle}{\|u_R\|_V} \leq \|F\|_{V^*} \frac{\|v_0-u_R\|_V}{\|u_R\|_V} \leq \|F\|_{V^*} \frac{\|v_0\|_V + \|u_R\|_V}{\|u_R\|_V}.$$

The right hand side of above inequality is bounded when $\|u_R\|_V > 1$. Hence, we get from (5.32) that the $\|u_R\|$ is uniformly bounded for all R. So there exists a sequence R_k converging to ∞ such that

$$u_{R_k} \rightharpoonup u \text{ weakly in } V.$$

Since $u_{R_k} \in K_{R_k} \subset K$, so we have $u \in K$.
Now for any $v \in K$, we can find R_k such that $R_k \geq \|v\|_V$. From above, there exists u_{R_k} such that

$$\langle A(u_{R_k}), v-u_{R_k}\rangle \geq \langle F, v-u_{R_k}\rangle . \tag{5.33}$$

Using similar argument as in the above Theorem 5.3.2, we obtain that

$$Du_{R_k} \to Du \text{ in measure.}$$

and

$$\sigma(x, u_{R_k}, Du_{R_k}) \to \sigma(x,u,Du) \text{ in } L^{p'}(\Omega). \tag{5.34}$$

From $v - u_{R_k} \rightharpoonup v-u$ and (5.34), we pass the limit in (5.33) and obtain

$$\langle A(u), v-u\rangle \geq \langle F, v-u\rangle \text{ for any } v \in K.$$

∎

To prove (5.32), we need growth condition (W2), coercive condition (W3) and the equivalence of norm $\|u\|_V$ and $\|Du\|_{L^p}$ on $V = W_0^{1,p}(\Omega)$. Now, if we want to extend the above existence Theorem 5.3.4 to the case where V is a linear closed subspace such that $V \subset W^{1,p}(\Omega;\mathbb{R}^m)$. We do need the following stronger coercive condition.
(W3') (Coercive condition): There exist $c_2 > 0$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{\frac{p}{\alpha}'}(\Omega)$ and $0 < \alpha < p$ such that

$$a(x,w,F) : F \geq -\lambda_2(x) - \lambda_3(x)|w|^\alpha + c_2 |(w,F)|^p . \tag{5.35}$$

We will obtain similarly the following theorem:

Theorem 5.3.5. Let V be a linear closed subspace such that $V \subset W^{1,p}(\Omega;\mathbb{R}^m)$, let K be a closed and convex subset of V. Let A be defined as in (5.22) and a satisfies (W1) (W2) (W4) in Section 5.1 and (W3'). Then for any $F \in V^*$, there exists $u \in K$ such that the following variational inequality is satisfied:

$$\langle A(u), v-u\rangle \geq \langle F, v-u\rangle \text{ for all } v \in K. \tag{5.36}$$

Remark 5.3.6. The above theorem enables us to find the existence for non-homogeneous boundary value problem (5.1) with $u|_{\partial\Omega} = \varphi$. i.e., Setting $K = \varphi + W_0^{1,p}(\Omega;\mathbb{R}^m)$, $V = W^{1,p}(\Omega)$ and apply the above theorem, we obtain that for $F \in V^*$, there exists $u \in K$ such that

$$\langle A(u), v-u\rangle \geq \langle F, v-u\rangle \quad \text{for all } v \in K,$$

which is equivalent to

$$\langle A(u), w\rangle = \langle F, w\rangle \quad \text{for all } w \in W_0^{1,p}(\Omega;\mathbb{R}^m).$$

This imply the existence of the solution to the problem (5.1) with $u|_{\partial\Omega} = \varphi$.

5.4 Open problems on parabolic variational inequalities

In this section, we will set up the variational problems with strictly p-quasimonotone function that can be written as a particular type of parabolic variational inequalities.

Let Ω be an open bounded subset of $\mathbb{R}^n$. Let $2 \leq p < \infty$ and p' be its conjugate. Let $V = W_0^{1,p}(\Omega;\mathbb{R}^m)$ and let $H = L^2(\Omega;\mathbb{R}^m)$, then V is reflexive, separable and $V \subset H \subset V^*$ is an evolution triple. Let $K \subset V$ be a closed and convex subset. Define $\mathcal{K} := \{v \in L^p(0,T;V) : v(t) \in K \text{ for a.e. } t \in [0,T]\}$, then $\mathcal{K}$ is a closed convex subset of $L^p(0,T;V)$. Given $f \in L^{p'}(0,T;V^*)$ and $u_0 \in K \cap H$, we are interested in finding $u \in \mathcal{K}$ with $u' \in L^{p'}(0,T;V^*)$ such that for a.e. $t \in [0,T]$

$$\langle u'(t), v(t)-u(t)\rangle + \int_\Omega a(x,u(t,x),Du(t,x)) : (Dv-Du)dx \geq \langle f(t), v(t)-u(t)\rangle \quad \text{for all } v \in \mathcal{K}, \tag{5.37}$$

where $a : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$, and

$$u(0) = u_0. \tag{5.38}$$

We assume function a satisfies (W1) - (W4) in Section 5.1. Then we can define the operator $A : V \to V^*$ by

$$\langle A(w), v\rangle = \int_\Omega a(x,w(x),Dw(x)) : Dvdx, \text{ for any } v,w \in V. \tag{5.39}$$

The weak solution of the problem (4.74) (4.75) is defined as $u \in \mathcal{K}$ such that

$$\int_0^T \langle v'(t), v(t)-u(t)\rangle + \langle A(u(t)), v(t)\rangle \, dt \geq \langle f(t), v(t)-u(t)\rangle, \tag{5.40}$$

for any $v \in \mathcal{K}$ with $v' \in \mathcal{V}^*$ and $v(0) = u^0$.

If we follow Rothe's method in the proof of Theorem 4.3.11, we can construct the approximating sequence u_n, v_n, $\tilde{v}_n$ and f_n of (5.40). We can obtain the following

$$\int_0^T \langle \tilde{v}_n', v_n-u_n\rangle \, dt + \int_0^T \langle A(u_n), v_n-u_n\rangle \, dt \geq \int_0^T \langle f_n, v_n-u_n\rangle \, dt. \tag{5.41}$$

Note that as $n \to \infty$, we have

$$\tilde{v}_n' \to v' \text{ strongly in } L^{p'}(0,T;V^*); \;\; f_n \to f \text{ strongly in } L^{p'}(0,T;V^*);$$

$$u_n \rightharpoonup u \text{ weakly in } L^p(0,T;V); \;\; v_n \to v \text{ strongly in } L^p(0,T;V).$$

The difficult here is showing $A(u_n) \rightharpoonup A(u)$ in $L^{p'}(0,T;V^*)$. To obtain this, we need to show that

$$a(x,u_n,Du_n) \to a(x,u,Du) \;\text{ in } L^{p'}((0,T)\times\Omega). \tag{5.42}$$

From the continuity condition (W1), the growth condition (W2) and Vitali's Theorem 1.2.16, it is sufficient to show

$$u_n \to u \text{ and } Du_n \to Du \text{ in measure.}$$

$Du_n \to Du$ in measure can be obtain through the strict p-quasimonotonicity (W4). The difficulty is showing that u_n converges to u in measure. We have $u_n(x,t) = u_i(x)$ for $t \in ((i-1)h, ih]$ and $u_n \rightharpoonup u$ in $L^p(0,T;V)$, u_n is step function on $(0,T)$, so u_n does not admit time derivative u_n', and then Lions-Aubin theorem cannot be applied here.

Remark 5.4.1. One may also consider the following more complicated case. Define $A : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ by

$$\langle A(u), v\rangle = \int_0^T \int_\Omega a(t,x,u(t,x),Du(t,x)) : Dv(t,x)dxdt.$$

For fixed $t \in [0,T]$, we may define $\tilde{A}(t) : V \to V^*$ as

$$\left\langle \tilde{A}(t)(w), v \right\rangle = \int_\Omega a(t,x,w(x),Dw(x)) : Dv(x)dx \quad \text{for } w, v \in V.$$

So for $u \in L^p(0,T;V)$, we have $[\tilde{A}(t)](u(t)) = [A(u)](t)$ on V^* for $t \in [0,T]$.
Then we consider the following problem:
given $f \in L^{p'}(0,T;V^*)$ and $u_0 \in K \cap H$, find $u \in \mathcal{K}$ with $u' \in \mathcal{V}^*$ such that for a.e. $t \in [0,T]$, we have

$$\langle u'(t), v(t) - u(t)\rangle + \langle A(t)(u(t)), v(t) - u(t)\rangle \geq \langle f(t), v(t) - u(t)\rangle, \text{ for any } v \in \mathcal{K},$$

with

$$u(0) = u_0.$$

The function $a : (0,T) \times \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfy (P1) - (P4) and some additional conditions.

Chapter 6

Variational Problems Governed by Locally Monotone Operator

In this chapter, we aim to prove an existence result for evolution problems with the operator being locally monotone. This existence theorem, which is a generalization of the existence Theorem 3.1.3 for monotone operators, can be applied to establish the existence and uniqueness of solutions for a wider class of nonlinear evolution equations. It is not easy to formulate the conditions on the abstract initial boundary value problems so that the assumptions of the existence theorem are satisfied. (e.g. like what we did in Section 3.2.2, 4.2.2) However, an example of application of the existence theorem to the initial boundary value problem where the underlined operator is only locally monotone but not monotone will be presented. Finally, we will briefly introduced the use of local monotonicity in stochastic evolution equations.

6.1 Existence Theorem

Let Ω be an open bounded subset of $\mathbb{R}^n$, let $p > 1$ and p' be its conjugate. Let $V \subset H \subset V^*$ be an evolution triple. Let the operator B be a mapping from $L^p(0,T;V)$ to $L^{p'}(0,T;V^*)$. Given $f \in L^{p'}(0,T;V^*)$ and $u_0 \in H$, we are interested in finding $u \in W^{1,p}(0,T;V,H)$ such that the following evolution equation holds

$$u' + B(u) = f, \quad \text{with } u(0) = u_0. \tag{6.1}$$

For fixed $t \in [0,T]$, define $\tilde{B}(t) : V \to V^*$ by $[\tilde{B}(t)](u(t)) = [B(u)](t)$.
Recall in Theorem 3.1.3, we have $\tilde{B}(t)$ is monotone, bounded, hemicontinuous and coercive. In this section, instead of assuming $\tilde{B}(t)$ is monotone, we will assume $\tilde{B}(t)$ is locally monotone (see (B2)). And we will assume $\tilde{B}(t)$ is bounded in a weaker sense that (3.4) in Theorem 3.1.3. In addition, we required the embedding $V \subset H$ to be compact. We will establish the existence and uniqueness result when $\tilde{B}(t)$ is locally monotone, bounded, hemicontinuous and coercive.
Now we formulate the conditions that $\tilde{B}(t)$ satisfies, assume that for $p > 1$ and $\beta \geq 0$, there exist constants $\delta > 0$, C (may differ from (B1) to (B4)) and a positive function $g \in L^1(0,T;\mathbb{R})$ such that the following conditions hold for all $t \in [0,T]$ and $u, v, w \in V$.
(B1) (Hemicontinuity): The map $\lambda \mapsto \left\langle [\tilde{B}(t)](u + \lambda v), w \right\rangle$ is continuous on $\mathbb{R}$.
(B2) (Local monotonicity): $\tilde{B}(t)$ satisfies the following:

$$\left\langle [\tilde{B}(t)](u) - [\tilde{B}(t)](v), u - v \right\rangle \geq -(C + \rho(u) + \eta(v)) \left\| u - v \right\|_H^2,$$

where $\rho, \eta : V \to [0, \infty)$ are measurable functions and locally bounded in V.
(B3) (Coercivity):

$$2 \left\langle [\tilde{B}(t)](u), u \right\rangle \geq \delta \left\| u \right\|_V^p - C \left\| u \right\|_H^2 - g(t).$$

(B4) (Growth):

$$\left\| \tilde{B}(t)(u) \right\|_{V^*} \leq (C \left\| u \right\|_V^{p-1} + g(t)^{\frac{p-1}{p}})(1 + \left\| u \right\|_H^{\beta}).$$

(B5) (Measurability): The function $t \mapsto \left\langle [\tilde{B}(t)](u), v \right\rangle$ is measurable on $[0, T]$.

Remark 6.1.1. (1) (B4) is a weaker form of growth condition, i.e., if $\beta = 0$, then (B4) become (3.4) in Theorem 3.1.3.
(2) If $C = 0$ in (B2) and (B3), $\rho = \eta \equiv 0$, $\beta = 0$. Then (B1) - (B5) are exactly the assumptions in Theorem 3.1.3.

With all the conditions above, we can show the existence theorem for locally monotone operators.

Theorem 6.1.2. Suppose that $V \subset H \subset V^*$ is an evolution triple with the embedding $V \subset H$ being compact, and suppose $\tilde{B}(t)$ satisfies (B1) - (B5) for all $t \in [0, T]$. The operator $B : L^p(0, T; V) \to L^{p'}(0, T; V^*)$ is defined by $[B(u)](t) = [\tilde{B}(t)](u(t))$. Then for any $u_0 \in H$ and $f \in L^{p'}(0, T; V^*)$, (6.1) has a solution $u \in W^{1,p}(0, T; V, H)$ such that

$$(u(t), v) - (u_0, v) + \int_0^t \langle [B(u)](s), v \rangle \, ds = \int_0^t \langle f(s), v \rangle \, ds, \text{ for all } t \in [0, T] \text{ and } v \in V.$$

Moreover, if

$$\int_0^T \rho(u(s)) + \eta(v(s)) ds < \infty \text{ for all } u, v \in L^p(0, T; V), \tag{6.2}$$

then the solution u is unique.

Proof. We follow Galerkin's approximation method:
Since V is dense in H, we may find $\{z_1, z_2, \cdots\} \subset V$ that is an orthonormal basis for both V and H. Let $H_m := \text{span}\{z_1, \cdots, z_m\}$. Define $P_m : V^* \to H_m$ by

$$P_m y = \sum_{i=1}^m (y, z_i) z_i, \quad \text{for } y \in V^*.$$

P_m is the orthogonal projection onto H_m and we have for any $t \in [0, T]$,

$$\left\langle P_m [\tilde{B}(t)](u(t)), v \right\rangle = (P_m [\tilde{B}(t)](u(t)), v) = \left\langle [\tilde{B}(t)](u(t)), v \right\rangle \text{ for } u \in L^p(0, T; V) \text{ and } v \in H_m.$$

Let the sequence $(u_{m0})_m$ be such that $u_{m0} = P_m u_0 \to u_0$ in H as $m \to \infty$.
For each $m \in \mathbb{N}$, we will consider the following evolution equation on H_m.

$$u_m'(t) + P_m [\tilde{B}(t)](u(t)) = P_m f(t) \text{ for } t \in [0, T], \; u_m(0) = u_{m0} = P_m u_0 \in H_m. \tag{6.3}$$

It is easy to show that for all $t \in [0, T]$, $P_m [\tilde{B}(t)]$ is locally monotone and coercive on H_m. According to the classical result of Krylov (see [52]), there exists a unique solution u_m to (6.3) such that

$$u_m \in L^p(0, T; H_m) \cap C(0, T; H_m) \text{ with } u' \in L^{p'}(0, T; H_m).$$

We need the following lemma for the prior estimate on u_m.

Lemma 6.1.3. Under assumptions of the above Theorem 6.1.2, there exists a constant $K > 0$ (independent of m) such that

$$\|u_m\|_{L^p(0,T;V)} + \sup_{t \in [0,T]} \|u_m\|_H + \|B(u_m)\|_{L^{p'}(0,T;V^*)} \le K, \text{ for all } m \ge 1.$$

Proof. By integration by parts formula and the coercive condition (B3), we have

$$\|u_m(t)\|_H^2 - \|u_m(0)\|_H^2 = 2 \int_0^t \left\langle u_m'(s), u_m(s) \right\rangle ds = 2 \int_0^t \left\langle P_m f(s) - P_m [\tilde{B}(s)](u_m(s)), u_m(s) \right\rangle ds$$

$$= 2 \int_0^t \left\langle f(s) - [\tilde{B}(s)](u_m(s)), u_m(s) \right\rangle ds \le \int_0^t -\delta \|u_m(s)\|_V^p + C \|u_m(s)\|_H^2 + g(s) + \|f(s)\|_{V^*} \|u_m(s)\|_V \, ds$$

$$\le \int_0^t -\frac{\delta}{2} \|u_m(s)\|_V^p + C \|u_m(s)\|_H^2 + g(s) + C_1 \|f(s)\|_{V^*}^{p'} \, ds,$$

where the last inequality follows from Young's inequality and C_1 is a constant depending on p, p' and δ. Hence for $t \in [0,T]$, we have

$$\|u_m(t)\|_H^2 + \frac{\delta}{2}\int_0^t \|u_m(s)\|_V^p \, ds \le \|u_m(0)\|_H^2 + C\int_0^t \|u_m(s)\|_H^2 \, ds + \int_0^t g(s) + C_1 \|f(s)\|_{V^*}^{p'} \, ds. \tag{6.4}$$

Since $u_m(0) \to u_0$ in H, then $\|u_m(0)\|_H$ is bounded for all $m \in \mathbb{N}$. Therefore, we get

$$\|u_m(t)\|_H^2 \le \|u_m(0)\|_H^2 + C\int_0^t \|u_m(s)\|_H^2 \, ds + \int_0^t g(s) + C_1 \|f(s)\|_{V^*}^{p'} \, ds \le C\int_0^t \|u_m(s)\|_H^2 \, ds + C_2,$$

for some positive constant $C_2 = \sup_{m\in\mathbb{N}} \|u_m(0)\|_H^2 + \int_0^T g(s) + C_1 \|f(s)\|_{V^*}^{p'} \, ds$.
Then by Grönwall's inequality, we have

$$\int_0^t \|u_m(t)\|_H^2 \, dt \le e^{Ct}\left(\|u_m(0)\|_H^2 + \int_0^t C_2 ds\right),$$

which means $\int_0^t \|u_m(t)\|_H^2 \, dt$ is bounded for $t \in [0,T]$ and $m \ge 1$. So we get the right hand side in (6.4) is bounded, so there exists $C_3 > 0$ such that

$$\|u_m\|_{L^p(0,T;V)} + \sup_{t\in[0,T]} \|u_m(t)\|_H \le C_3 \text{ for all } m \ge 1. \tag{6.5}$$

By (B4) and (6.5), we have

$$\begin{aligned}\|B(u_m)\|_{L^{p'}(0,T;V^*)} = \int_0^T \|[B(u_m)](t)\|_{V^*}^{p'} \, dt &= \int_0^T \left\|[\tilde{B}(t)](u_m(t))\right\|_{V^*}^{p'} dt \\ &\le \text{ const}\left[\int_0^T g(t)dt + \int_0^T \|u_m(t)\|_V^p \, dt + 1\right],\end{aligned}$$

the right hand side of above inequality is bounded by a constant $C_4 > 0$. Hence the lemma is proved.

Since $L^p(0,T;V)$, $L^{p'}(0,T;V^*)$ and H are reflexive Banach space, the above lemma implies that there exists a subsequence, again denoted by (u_m), such that as $m \to \infty$

$$u_m \rightharpoonup u \text{ in } L^p(0,T;V), \; B(u_m) \rightharpoonup w \text{ in } L^{p'}(0,T;V^*), \; u_m(T) \rightharpoonup z \text{ in } H,$$

and $u_m(0) = u_{m0} \to u_0$ in H.
From Lemma 3.1.4, we have

$$u'(t) + w(t) = f(t) \text{ for all } t \in [0,T], \; u(0) = u_0, \; u(T) = z.$$

To show $w(t) = [\tilde{B}(t)](u(t))$, we need the following lemma.

Lemma 6.1.4. Under assumption of Theorem 6.1.2, suppose that

$$\limsup_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t)\right\rangle dt \le \int_0^T \langle w(t), u(t)\rangle \, dt, \tag{6.6}$$

then for any $v \in L^p(0,T;V)$, we have

$$\int_0^T \left\langle [\tilde{B}(t)](u(t)), u(t) - v(t)\right\rangle dt \le \liminf_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - v(t)\right\rangle dt. \tag{6.7}$$

Proof. Since $W_p^1(0,T;V,H) \subset C(0,T;H)$ is a continuous embedding, we have $u_m(t) \rightharpoonup u(t)$ weakly in H for all $t \in [0,T]$. Hence, $u_m(t) \rightharpoonup u(t)$ weakly in V for all $t \in [0,T]$ because $V \subset H \subset V^*$ is an evolution triple.
Claim: For all $t \in [0,T]$, we have

$$\liminf_{m\to\infty} \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - u(t)\right\rangle \ge 0. \tag{6.8}$$

Proof of the claim. Suppose that the contrary holds, ie.e, there exists t_0 such that

$$\liminf_{m\to\infty} \left\langle [\tilde{B}(t_0)](u_m(t_0)), u_m(t_0) - u(t_0) \right\rangle < 0.$$

Then we can extract a subsequence such that

$$\lim_{j\to\infty} \left\langle [\tilde{B}(t_0)](u_{m_j}(t_0)), u_{m_j}(t_0) - u(t_0) \right\rangle < 0.$$

From Lemma 2.3.4, we know that $\tilde{B}(t)$ is pseudomonotone for any $t \in [0, T]$, note that $u_{m_j}(t_0) \rightharpoonup u(t_0)$ weakly in V, so we have

$$\liminf_{j\to\infty} \left\langle [\tilde{B}(t_0)](u_{m_j}(t_0)), u_{m_j}(t_0) - v \right\rangle \geq \left\langle [\tilde{B}(t_0)](u(t_0)), u(t_0) - v \right\rangle \text{ for all } v \in V.$$

In particular, we take $v = u(t_0)$ and obtain

$$\liminf_{j\to\infty} \left\langle [\tilde{B}(t_0)](u_{m_j}(t_0)), u_{m_j}(t_0) - u(t_0) \right\rangle \geq 0,$$

which contradicts to our assumption. Hence, the claim follows.
Then by (6.6), (6.8) and Fatou's Lemma 1.2.12, we obtain

$$\begin{aligned} 0 &\geq \limsup_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) \right\rangle dt - \int_0^T \langle w(t), u(t) \rangle \, dt = \limsup_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - u(t) \right\rangle dt \\ &\geq \liminf_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - u(t) \right\rangle dt \geq \int_0^T \liminf_{m\to\infty} \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - u(t) \right\rangle dt \geq 0. \end{aligned}$$

Hence,

$$\lim_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - u(t) \right\rangle dt = 0. \tag{6.9}$$

Claim: There exists a subsequence $(u_{m_j})_j$ such that

$$\lim_{j\to\infty} \left\langle [\tilde{B}(t)](u_{m_j}(t)), u_{m_j}(t) - u(t) \right\rangle = 0 \text{ for a.e. } t \in [0, T].$$

Define $g_m(t) = -\left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - u(t) \right\rangle$ for $t \in [0, T]$, then we obtain by (6.9) and (6.8) that

$$\lim_{m\to\infty} \int_0^T g_m(t) dt = 0 \text{ and } \limsup_{m\to\infty} g_m(t) \leq 0 \text{ for all } t \in [0, T].$$

Then by Dominated Convergence Theorem 1.2.13

$$\lim_{m\to\infty} \int_0^T g_m^+(t) dt = 0, \quad \text{where } g_m^+(t) = \max\{g_m(t), 0\}.$$

Note that $|g_m(t)| = 2g_m^+(t) - g_m(t)$, therefore, we have

$$\lim_{m\to\infty} \int_0^T |g_m(t)| \, dt = 0.$$

Hence, we can extract a subsequence $(g_{m_j}(t))_j$ such that

$$\lim_{j\to\infty} g_{m_j}(t) = 0 \text{ for a.e. } t \in [0, T].$$

So the claim follows.
Therefore, for any $v \in L^p(0, T; V)$, we can choose a subsequence $(u_{m_j})_j$ such that

$$\begin{cases} \lim_{j\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_{m_j}(t)), u_{m_j}(t) - v(t) \right\rangle dt = \liminf_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - v(t) \right\rangle dt; \\ \lim_{j\to\infty} \left\langle [\tilde{B}(t)](u_{m_j}(t)), u_{m_j}(t) - u(t) \right\rangle = 0. \end{cases} \tag{6.10}$$

From pseudomonotonicity of $\tilde{B}(t)$, we have

$$\left\langle [\tilde{B}(t)](u(t)), u(t) - v(t) \right\rangle \leq \liminf_{j\to\infty} \left\langle [\tilde{B}(t)](u_{m_j}(t)), u_{m_j}(t) - v(t) \right\rangle, \ t \in [0,T].$$

By Fatou's Lemma 1.2.12 and (6.10), we obtain

$$\int_0^T \left\langle [\tilde{B}(t)](u(t)), u(t) - v(t) \right\rangle dt \leq \int_0^T \liminf_{j\to\infty} \left\langle [\tilde{B}(t)](u_{m_j}(t)), u_{m_j}(t) - v(t) \right\rangle dt$$
$$\leq \liminf_{j\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_{m_j}(t)), u_{m_j}(t) - v(t) \right\rangle dt = \liminf_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - v(t) \right\rangle dt,$$

which completes the proof of the lemma. ■

Now from integration by parts formula, we have

$$\|u_m(T)\|_H^2 - \|u_m(0)\|_H^2 = 2\int_0^T \left\langle f(t) - [\tilde{B}(t)](u_m(t)), u_m(t) \right\rangle dt;$$

$$\|u(T)\|_H^2 - \|u(0)\|_H^2 = 2\int_0^T \langle f(t) - w(t), u(t) \rangle \, dt.$$

Since $u_m(T) \rightharpoonup z$ in H, by lower semicontinuity of $\|\cdot\|_H$, we have

$$\liminf_{m\to\infty} \|u_m(T)\|_H^2 \geq \|z\|_H^2 .$$

Hence, we have

$$\limsup_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)(u_m(t)), u_m(t) \right\rangle dt \leq \int_0^T \langle f(t), u(t) \rangle \, dt - \frac{1}{2}(\|u(t)\|_H^2 - \|u(0)\|_H^2) = \int_0^T \langle w(t), u(t) \rangle \, dt.$$

By Lemma 6.1.4, we get for any $v \in L^p(0,T;V)$,

$$\int_0^T \left\langle [\tilde{B}(t)](u(t)), u(t) - v(t) \right\rangle dt \leq \liminf_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - v(t) \right\rangle dt$$
$$\leq \limsup_{m\to\infty} \int_0^T \left\langle [\tilde{B}(t)](u_m(t)), u_m(t) - v(t) \right\rangle dt \leq \int_0^T \langle w(t), u(t) \rangle \, dt - \int_0^T \langle w(t), v(t) \rangle \, dt = \int_0^T \langle w(t), u(t) - v(t) \rangle \, dt.$$

Since $v \in L^p(0,T;V)$ is arbitrary, we have

$$[\tilde{B}(t)](u(t)) = w(t) \text{ in } V^* \text{ for any } t \in [0,T].$$

So u is a solution to (6.1).
To prove the uniqueness, suppose u and v are the solutions to (6.1) with $u(0) = u_0$ and $v(0) = v_0$, then by integration by parts formula, we have for $t \in [0,T]$

$$\|u(t) - v(t)\|_H^2 = \|u_0 - v_0\|_H^2 - 2\int_0^T \left\langle [\tilde{B}(s)]u(s) - [\tilde{B}(s)]v(s), u(s) - v(s) \right\rangle ds$$
$$\leq \|u_0 - v_0\|_H^2 + 2\int_0^T (C + \rho(u(s)) + \eta(v(s))) \, \|u(s) - v(s)\|_H^2 \, ds.$$

From the assumption (6.2), we know

$$\int_0^T C + \rho(u(s)) + \eta(v(s)) ds < \infty \text{ for } t \in [0,T].$$

Then by Gronwall's inequality, we have for $t \in [0,T]$,

$$\|u(t) - v(t)\|_H^2 \leq \|u_0 - v_0\|_H^2 \exp\left\{ 2\int_0^T C + \rho(u(s)) + \eta(v(s)) ds \right\}.$$

If $u_0 = v_0$, this implies the uniqueness of the solution u. And therefore the theorem is proved. ■

6.2 Application of the existence theorem

Now, we are going to present an example to which the above existence Theorem 6.1.2 can be applied.
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded subset with smooth boundary. We will use D_i to denote the partial derivative $\frac{\partial}{\partial x_i}$. First, we will give an important lemma for verifying (B2).

Lemma 6.2.1. Consider the evolution triple

$$V := W_0^{1,2}(\Omega) \subset H := L^2(\Omega) \subset V^* := W^{-1,2}(\Omega),$$

and define the operator $A : V \to V^*$ by

$$A(u) = -\Delta u + \sum_{i=1}^{n} f_i(u) D_i u,$$

where f_i $(i = 1, 2, \cdots, n)$ are Lipchitz functions on $\mathbb{R}$.
(i) if $n < 3$, then there exists a constant $K > 0$ such that

$$\langle A(u) - A(v), u - v\rangle \geq \frac{1}{2}\|u - v\|_V^2 - (K + K\|u\|_{L^4}^4 + K\|v\|_V^2)\|u - v\|_H^2, \quad \text{for } u, v \in V.$$

In particular, if f_i are bounded functions for $i = 1, \cdots, n.$, then we have

$$\langle A(u) - A(v), u - v\rangle \geq \frac{1}{2}\|u - v\|_V^2 - (K + K\|v\|_V^2)\|u - v\|_H^2, \quad \text{for } u, v \in V.$$

(ii) For $n = 3$, for $u, v \in V$ and some constant $K > 0$, we have

$$\langle A(u) - A(v), u - v\rangle \geq \frac{1}{2}\|u - v\|_V^2 - (K + K\|u\|_{L^4}^8 + K\|v\|_V^4)\|u - v\|_H^2.$$

In particular, if f_i are bounded functions for $i = 1, 2, 3.$, then we have

$$\langle A(u) - A(v), u - v\rangle \geq \frac{1}{2}\|u - v\|_V^2 - (K + K\|v\|_V^4)\|u - v\|_H^2, \quad \text{for } u, v \in V.$$

(iii) If f_i are bounded measurable functions on Ω and independent of u for $i = 1, 2, \cdots n.$, i.e.,

$$A(u) = -\Delta u + \sum_{i=1}^{n} f_i D_i u,$$

then for any $n \geq 1$, we have

$$\langle A(u) - A(v), u - v\rangle \geq \frac{1}{2}\|u - v\|_V^2 - K\|u - v\|_H^2, \quad \text{for } u, v \in V.$$

Proof. (i) For $n < 3$, we will use the following estimate (see Lemma 2.1 in [60])

$$\|u\|_{L^4}^4 \leq 2\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \leq 2\|u\|_H^2 \|u\|_V^2, \ u \in W_0^{1,2}(\Omega). \tag{6.11}$$

Since all f_i are Lipchitz, then they all have at most linear growth. i.e.,

$$|f_i(u)| \leq C|u| + |f_i(0)| \leq C(|u| + 1). \tag{6.12}$$

From (6.11), (6.12) and Hölder's inequality, we obtain for $u, v \in V$,

$$\begin{aligned}
\langle A(u) - A(v), u - v\rangle &= -\int_\Omega \mathrm{div}(\nabla(u-v))(u-v)dx + \sum_{i=1}^n \int_\Omega [f_i(u)D_iu - f_i(v)D_iv](u-v)dx \\
&= \int_\Omega (\nabla(u-v))^2 dx + \sum_{i=1}^n \int_\Omega [f_i(u)(D_iu - D_iv) + D_iv(f_i(u) - f_i(v))](u-v)dx \\
&= \|u-v\|_V^2 + \sum_{i=1}^n \int_\Omega f_i(u)(D_i(u) - D_i(v))(u-v) + D_iv(f_i(u) - f_i(v))(u-v)dx \\
&\geq \|u-v\|_V^2 - \sum_{i=1}^n \Big[\Big(\int_\Omega |D_iu - D_iv|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |f_i(u)(u-v)|^2\,dx\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_\Omega |D_iv|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega (f_i(u) - f_i(v))^2(u-v)^2 dx\Big)^{\frac{1}{2}}\Big] \\
&\geq \|u-v\|_V^2 - K\|u-v\|_V\Big(\int_\Omega (1+|u|^4)dx\Big)^{\frac{1}{4}}\Big(\int_\Omega |u-v|^4\,dx\Big)^{\frac{1}{4}} - K\|v\|_V\Big(\int |u-v|^4\,dx\Big)^{\frac{1}{2}} \\
&\geq \|u-v\|_V^2 - K\|u-v\|_V^{\frac{3}{2}}\|u-v\|_H^{\frac{1}{2}}(1+\|u\|_{L^4}) - K\|v\|_V\|u-v\|_V\|u-v\|_H \\
&\geq \frac{1}{2}\|u-v\|_V^2 - (K + K\|v\|_V^2 + K\|u\|_{L^4}^4)\|u-v\|_H^2
\end{aligned}$$

$K > 0$ is a constant that may change from line to line, and last inequality follows from Young's inequality with $p_1 = \frac{4}{3},\ q_1 = 4$ for the second term and $p_2 = 2,\ q_2 = 2$ for the last term.
Now, if f_i are bounded function for all i, we can modify above proof and get the desired estimate.

(ii) For $n = 3$, we will use the following estimate (see [60])

$$\|u\|_{L^4}^4 \leq 4\|u\|_{L^2}\|\nabla u\|_{L^2}^3 \leq 4\|u\|_H\|u\|_V^3,\ u \in W_0^{1,2}(\Omega). \tag{6.13}$$

Similar to the proof above, apply above estimate (6.13), we have for any $u, v \in V$,

$$\begin{aligned}
\langle A(u) - A(v), u - v\rangle &\geq \|u-v\|_V^2 - \sum_{i=1}^n \Big[\Big(\int_\Omega |D_iu - D_iv|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |f_i(u)|^2|u-v|^2\,dx\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_\Omega |D_iv|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |f_i(u) - f_i(v)|^2|u-v|^2\,dx\Big)^{\frac{1}{2}}\Big] \\
&\geq \|u-v\|_V^2 - K\|u-v\|_V\Big(\int_\Omega 1+|u|^4\,dx\Big)^{\frac{1}{4}}\Big(\int_\Omega |u-v|^4\,dx\Big)^{\frac{1}{4}} - K\|v\|_V\Big(\int_\Omega |u-v|^4\,dx\Big)^{\frac{1}{2}} \\
&\geq \|u-v\|_V^2 - K\|u-v\|_V^{\frac{7}{4}}\|u-v\|_H^{\frac{1}{4}}(1+\|u\|_{L^4}) - K\|v\|_V\|u-v\|_V^{\frac{3}{2}}\|u-v\|_H^{\frac{1}{2}} \\
&\geq \frac{1}{2}\|u-v\|_V^2 - \big(K + K\|u\|_{L^4}^8 + K\|v\|_V^4\big)\|u-v\|_H^2,
\end{aligned}$$

$K > 0$ is a constant that may change line from line, and the last inequality follows from Young's inequality with $p_1 = \frac{8}{7},\ q_1 = 8$ for the second term and $p_2 = \frac{4}{3},\ q_2 = 4$ for the last term.
(iii) This follows easily using similar proof in (i). ■

Remark 6.2.2. From (iii), if all f_i are bounded, then local monotonicity (B2) also implies coercivity (B3).

We present the following example where the underlined operator is only locally monotone but not monotone.

Example 6.2.3. Let Ω be a bounded subset of $\mathbb{R}^n$ with smooth boundary. Consider the following initial boundary value problem:
Given $h(t) \in W^{-1,2}(\Omega)$ and $u_0 \in L^2(\Omega)$, find $u : [0,T] \times \Omega \to \mathbb{R}$ such that for all $t \in [0,T]$,

$$u'(t) - \Delta u(t) + \sum_{i=1}^n f_i(u)D_iu(t) + g(u)(t) = h(t), \tag{6.14}$$

with the boundary condition

$$u = 0 \text{ on } [0,T) \times \partial\Omega,$$

and the initial condition

$$u(0) = u_0 \text{ on } \Omega.$$

Suppose the following conditions hold for some constant $C > 0$:
(i) f_i are bounded Lipchitz functions on $\mathbb{R}$ for $i = 1, 2, \cdots, n$.
(ii) g is a continuous function on $\mathbb{R}$ such that for $x, y \in \mathbb{R}$,

$$g(x)x \geq -C(|x|^2 + 1), \quad |g(x)| \leq C(|x|^r + 1),$$

$$(g(x) - g(y))(x - y) \geq -C(1 + |x|^\alpha + |y|^\alpha)(x - y)^2,$$

where $r, \alpha \geq 1$ are some constants.
Then we have
(a) If $n = 2,\ r = \frac{7}{3}$ and $\alpha = 2$, then (6.14) has a unique solution $u \in W_2^1(0,T; W_0^{1,2}(\Omega), L^2(\Omega))$.
(b) If $n = 3,\ r = \frac{7}{3}$ and $\alpha \leq 3$, then (6.14) has a solution $u \in W_2^1(0,T; W_0^{1,2}(\Omega), L^2(\Omega))$. Moreover, if $\alpha = \frac{4}{3}$, $f_i (i = 1, 2, 3)$ are bounded measurable function on Ω and independent of u, then the solution of (6.14) is unique.

Proof. We define the Evolution triple

$$V := W_0^{1,2}(\Omega) \subset H := L^2(\Omega) \subset V^* := W^{-1,2}(\Omega),$$

and the operator $B : L^2(0,T;V) \to L^2(0,T;V^*)$ by

$$B(u) = -\Delta u + \sum_{i=1}^{n} f_i(u) D_i u + g(u).$$

Fixed $t \in [0,T]$, define $\tilde{B}(t) : V \to V^*$ by

$$[\tilde{B}(t)](u(t)) = [B(u)](t).$$

From continuity of f and g, we get that $\tilde{B}(t)$ is hemicontinuous.
By assumption (ii), we have

$$\langle g(u) - g(v), u - v \rangle \geq -C(1 + \|u\|_{L^{2\alpha}}^\alpha + \|v\|_{L^{2\alpha}}^\alpha) \|u - v\|_{L^4}^2 .$$

Then from (6.11), (6.13) and Lemma 6.2.1, we have for $n = 2$,

$$\left\langle [\tilde{B}(t)](u) - [\tilde{B}(t)](v), u - v \right\rangle \geq \frac{1}{2} \|u - v\|_V^2 - K(1 + \|v\|_V^2 + \|u\|_{L^{2\alpha}}^{2\alpha} + \|v\|_{L^{2\alpha}}^{2\alpha}) \|u - v\|_H^2 ;$$

and for $n = 3$,

$$\left\langle [\tilde{B}(t)](u) - [\tilde{B}(t)](v), u - v \right\rangle \geq \frac{1}{2} \|u - v\|_V^2 - K(1 + \|v\|_V^4 + \|u\|_{L^{2\alpha}}^{4\alpha} + \|v\|_{L^{2\alpha}}^{4\alpha}) \|u - v\|_H^2 .$$

So (B2) holds.
Note that

$$\langle g(u), u \rangle \geq -C(1 + \|u\|_H^2), \ \forall u \in V.$$

Then from Lemma 6.2.1 and Remark 6.2.2, (B3) holds with $p = 2$.
From Sobolev embedding theorem, when $n = p = 2$, we have

$$W_0^{1,2}(\Omega) \subset W^{1,2}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [1, \infty);$$

when $2 = p < n = 3$, we have

$$W_0^{1,2}(\Omega) \subset L^q(\Omega) \text{ for } q \in [1, p^* = \frac{np}{n-p} = 6].$$

Hence for $n = 2, 3$, we have $W_0^{1,2}(\Omega) \subset L^6(\Omega)$ and therefore

$$|\langle g(u), v\rangle| \le \int_\Omega |g(u)|\, v dx \le \left(\int_\Omega |g(u)|^{\frac{6}{5}}\, dx\right)^{\frac{5}{6}} \left(\int_\Omega |u|^6\, dx\right)^{\frac{1}{6}} = \left(\int_\Omega |g(u)|^{\frac{6}{5}}\, dx\right)^{\frac{5}{6}} \|u\|_{L^6}$$
$$\le C \left(\int_\Omega |g(u)|^{\frac{6}{5}}\, dx\right)^{\frac{5}{6}} \|u\|_V .$$

Hence,

$$\|g(u)\|_{V^*} \le C\left(1 + \|u\|^r_{L^{\frac{6}{5}r}}\right).$$

For $r = \frac{7}{3}$, we have by interpolation theorem that

$$\|u\|_{L^{\frac{6}{5}r}} \le \|u\|^{\frac{4}{7}}_{L^2} \|u\|^{\frac{3}{7}}_{L^6} .$$

Then

$$\|g(u)\|_{V^*} \le C(1 + \|u\|^r_{L^{\frac{6}{5}r}}) \le C(1 + \|u\|^{\frac{4}{3}}_H \|u\|_V),\ u \in V.$$

Hence, (B4) holds. Therefore, the result follows by Theorem 6.1.2.
In particular, if $n = 3$ and f_i $(i = 1, 2, 3)$ are bounded measurable functions on Ω and independent of u, then we have

$$\left\langle [\tilde{B}(t)](u) - [\tilde{B}(t)](v), u - v \right\rangle \ge \frac{1}{2}\|u - v\|^2_V - K(1 + \|u\|^{4\alpha}_{L^{2\alpha}} + \|v\|^{4\alpha}_{L^{2\alpha}})\, \|u - v\|^2_H , \quad \text{for all } u, v \in V.$$

Since $\alpha = \frac{4}{3}$,

$$\|u\|_{L^{2\alpha}} \le \|u\|^{\frac{5}{8}}_{L^2} \|v\|^{\frac{3}{8}}_{L^6} ,\ u \in V.$$

Therefore,

$$\|u\|^{4\alpha}_{L^{2\alpha}} \le C\, \|u\|^{\frac{10}{3}}_H \|u\|^2_V ,\ u \in V.$$

Hence, the solution of (6.14) is unique.

Remark 6.2.4. The classical existence theorem for monotone operators (see Theorem 3.1.3) cannot be applied to the above example. In the classical case, we require g is monotone and has at most linear growth. However, in the above example, we have g is locally monotone and has some polynomial growth, for which Theorem 6.1.2 can be applied.

For more details about applications of Theorem 6.1.2, such as Burger equation, Naiver-Stokes equation, see Section 3 in [57].

6.3 Locally Monotone Operators in Stochastic Differential Equations

In this section, we briefly introduce the use of locally monotone operators in the study of stochastic evolution equations. Let V be a reflexive Banach space and $V \subset H \subset V^*$ be a Gelfand triple. We assume H is a separable Hilbert space. Let $\{W_t\}_{t\ge 0}$ be a cylindrical Wiener process on H with respect to a completed filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Let the space of all Hilbert-Schmidt operators from U to H denoted by $(L^2(U; H), \|\cdot\|_2)$. Consider the following stochastic evolution equation:

$$dX_t = A(t, X_t)dt + B(t, X_t)dW_t, \tag{6.15}$$

where $A : [0, T] \times V \times \Omega \to V^*$ and $B : [0, T] \times V \times \Omega \to L_2(U; H)$ are progressively measurable, i.e., for any $t \in [0, T]$, these maps restrict to $[0, t] \times V \times \Omega$ are $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$ measurable ($\mathcal{B}$ stands for Borel σ-algebra). We briefly write $A(t, v)$ as the mapping $\omega \to A(t, v, \omega)$, similarly for $B(t, v)$. For more information on definition of cylindrical Wiener process, Hilbert-Schmidt operator and the setting up of the problem, see Chapter 1 in [68].

Similar to the evolution equation with monotone operators (see Theorem 3.1.3), there is the existence theorem for stochastic evolution equation with monotone operators. We state the following existence theorem from [53]. Suppose there exist constants $p > 1,\ \theta > 0, K$ and a positive adapted process $f \in L^1([0,T] \times \Omega; dt \times \mathbb{P})$ such that the following conditions hold for $v, v_1, v_2 \in V$ and $(t,\omega) \in [0,T] \times \Omega$.
(A1) (Hemicontinuity) The map $s \mapsto \langle A(t, v_1 + sv_2, \omega), v\rangle$ is continuous for $s \in \mathbb{R}$.
(A2) (Growth)

$$\|(A(t,v)\|_{V^*} \leq f_t^{\frac{p-1}{p}} + K\|v\|_V^{p-1}.$$

(A3) (Monotonicity)

$$2\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle + \|B(t,v_1) - B(t,v_2)\|_2^2 \leq K\|v_1 - v_2\|_H^2.$$

(A4) (Coercive)

$$2\langle A(t,v), v\rangle + \|B(t,v)\|_2^2 + \theta\|v\|_V^p \leq f_t + K\|v\|_H^2.$$

Theorem 6.3.1. Suppose that (A1) - (A4) hold, then for any $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (6.15) has a unique solution $\{X_t\}_{t\in[0,T]}$ satisfying

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t\|_H^2 < \infty.$$

Moreover, we have the following Itô formula

$$\|X_t\|_H^2 = \|X_0\|_H^2 + \int_0^t (2\langle A(s,X_s), X_s\rangle + \|B(s,X_s)\|_2^2)ds + \int_0^t \langle X_s, B(s,X_s)dW_s\rangle,$$

for $t \in [0,T]$ and $\mathbb{P}$-almost surely.

The notion of local monotonicity allows a generalisation of above theorem which can be applied into a larger class of equation such as stochastic Burgers equation, stochastic 2-D Navier-Stokes equation. We state the existence theorem regarding to the stochastic evolution equations with locally monotone operators (see [58]). Instead of (A2) and (A3), for $\alpha \geq 0$, we assume the following weaker conditions:
(A2') (Growth)

$$\|(A(t,v)\|_{V^*}^{\frac{p}{p-1}} \leq (f_t + K\|v\|_V^p)(1 + \|v\|_H^\alpha).$$

(A3') (Local Monotonicity)

$$2\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle + \|B(t,v_1) - B(t,v_2)\|_2^2 \leq (K + \rho(v_2))\|v_1 - v_2\|_H^2,$$

where $\rho : V \to [0,\infty)$ is measurable and locally bounded in V.

Remark 6.3.2. (1) If $\rho \equiv 0$ and $\alpha = 0$, then (A2') and (A3') become (A2) and (A3), i.e., the classical monotone case mentioned above.
(2) In (A2'), we allow some polynomial growth which includes many semilinear type equations (see Example 3.3 in [58]).

Theorem 6.3.3. Suppose (A1),(A2'),(A3') and (A4) hold for $f \in L^{\frac{q}{2}}([0,T] \times \Omega, dt \otimes \mathbb{P})$ with some $q \geq \alpha + 2$, and there exists a constant C such that for $t \in [0,T]$ and $v \in V$

$$\|B(t,v)\|_2^2 \leq C(f_t + \|v\|_H^2) \text{ and } \rho(v) \leq C(1 + \|v\|_V^p)(1 + \|v\|_H^\alpha).$$

Then for any $X_0 \in L^q(\Omega; H; \mathcal{F}_0, \mathbb{P})$, (6.15) has a unique solution $\{X_t\}_{t\in[0,T]}$ and satisfies

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|X_t\|_H^q + \int_0^T \|X_t\|_V^p\, dt\right) < \infty.$$

Chapter 7

Second order evolution equations

Various type of monotonicity have been introduced in this work so far. In this chapter, we briefly mention the use of these monotonicity in the study of second order evolution equations. We first present the existence theorems for second order evolution equations where the underlined operators are monotone or pseudomonotone. The details of the proof will not be given, we will give the main idea for the proof instead. In the last part of this chapter, we will try to adapt the idea and prove the existence result for second order evolution equations with locally monotone operators. We have in this part also a new setting and a new existence Theorem 7.0.5.
Let $V \subset H \subset V^*$ is an evolution triple and let $T > 0$. Let $p > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let operators $B,\ Q : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ be such that for any $u \in L^p(0,T;V)$ and $t \in [0,T]$,

$$[B(u)](t) = [\tilde{B}(t)](u(t)) \quad \text{and} \quad [Q(u)](t) = \tilde{Q}(u(t)), \tag{7.1}$$

where $\tilde{B}(t),\ \tilde{Q}(t) : V \to V^*$.
Given $F \in L^{p'}(0,T;V^*)$, we want to find the solution $u \in C(0,T;V)$ with $u' \in W^1_p(0,T;V,H)$ such that the following equation holds

$$u''(t) + \tilde{B}(t)(u'(t)) + \tilde{Q}u(t) = F(t), \tag{7.2}$$

with the initial conditions

$$u'(0) = u(0) = 0. \tag{7.3}$$

From (7.1), we obtain (7.2) is also equivalent to the following equation on V^*:

$$u'' + B(u') + Q(u) = F. \tag{7.4}$$

We state the following existence theorem with monotone operators.

Theorem 7.0.1 ([91, Theorem 33.A]). Let $V \subset H \subset V^*$ be an evolution triple. Assume that for each $t \in [0,T]$,
(1) the operator $\tilde{B}(t) : V \to V^*$ is monotone and hemicontinuous.
(2) $\tilde{B}(t)$ is coercive in the sense that there exists constants $c_1 > 0$ and $c_2 \geq 0$ such that for all $v \in V$,

$$\left\langle \tilde{B}(t)(v), v \right\rangle \geq c_1 \|v\|_V^p - c_2.$$

(3) $\tilde{B}(t)$ is bounded in the sense that there exists $c_3 \in L^{p'}(0,T)$ and $c_4 \geq 0$ such that for all $v \in V$,

$$\left\| \tilde{B}(t)v \right\|_{V^*} \leq c_3(t) + c_4 \|v\|_V^{\frac{p}{q}}.$$

(4) The map $t \to \tilde{B}(t)$ is weakly measurable, i.e., for $u, v \in V$, the function

$$t \mapsto \left\langle \tilde{B}(t)u, v \right\rangle \text{ is measurable on } [0,T].$$

The operator $\tilde{Q} : V \to V^*$ is linear, symmetric and strictly monotone. Then for any $F \in L^{p'}(0,T;V^*)$, there exists a solution $u \in C([0,T];V)$ with $u' \in W^1_p(0,T;V,H)$ such that (7.2) and (7.3) hold. The solution is unique if $\tilde{B}(t)$ is strictly monotone.

Remark 7.0.2. The assumptions above also imply that $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is monotone, hemicontinuous, coercive and bounded. As well as the operator $Q : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is linear, symmetric and strictly monotone.

In general, there are two methods of approaching the above problem. The first method is to reduce the order of the differential equation by setting $v = u'$, define the operator $S : L^p(0,T;V) \to L^p(0,T;V)$ by

$$Sv(t) = \int_0^t v(s)ds.$$

Then S is a linear continuous operator and $u = Sv$. Hence the problem (7.4) is equivalent to the following first-order evolution equation:
find $v \in W_p^1(0,T;V,H)$ such that

$$v' + Bv + QSv = F$$

with the initial condition $v(0) = 0$. So we may apply the existence result for first order evolution equations if $B + QS$ satisfies certain properties.

The other method is to apply the Galerkin approximation directly. Since V is separable, let $\{w_1, w_2, \cdots\}$ be a basis for V. We are looking for $u_n \in C([0,T],V_n)$ with $u_n' \in W_p^1(0,T;V_n,H_n)$ of the following form

$$u_n(t) = \sum_{k=1}^{n} c_{kn}(t)w_k. \tag{7.5}$$

For $j = 1, \cdots, n$ and a.e. $t \in [0,T]$, u_n satisfy the following

$$\langle u_n''(t), w_j\rangle + \left\langle \tilde{B}(t)u_n'(t), w_j \right\rangle + \left\langle \tilde{Q}u_n(t), w_j \right\rangle = \langle F(t), w_j\rangle,$$

with

$$u_n(0) = u_n'(0) = 0,$$

Substitute (7.5) into above, we get for a.e. $t \in [0,T]$ and for $j = 1, 2, \cdots, n.$,

$$\sum_{k=1}^{n} c_{kn}^{''}(t)(w_k, w_j) + \left\langle \tilde{B}(t)\sum_{k=1}^{n} c_{kn}^{'}(t)w_k, w_j \right\rangle + \left\langle \tilde{Q}\sum_{k=1}^{n} c_{kn}(t)w_k, w_j \right\rangle = \langle F(t), w_j\rangle, \tag{7.6}$$

with the initial conditions

$$c_{jn}(0) = c_{jn}^{'}(0) = 0. \tag{7.7}$$

Since $w_1, \cdots, c_n$ are independent, the Gram determinant $((w_k, w_j))$ is non-zero. (7.6) is a second-order system of ordinary differential equations, the solution can be found through numerical method in numerical analysis (see [2, Section 3.1]). Then, similar to first order evolution equations, we need find the prior estimate and pass the limit using the properties of operators.

L.Simon [77] shows the following existence results with pseudomonotone operators.
Define $L : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ by

$$\langle Lu, v\rangle = \int_0^T \langle u'(t), v(t)\rangle\, dt, \text{ for any } u \in D(L),\ v \in L^p(0,T;V),$$

where $D(L) = \{u \in L^p(0,T;V), u' \in L^{p'}(0,T;V^*) \text{ with } u(0) = 0\}$. Then L is a maximal monotone operator. Assume $B : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ is bounded, demicontinuous, pseudomonotone with respect to $D(L)$, i.e., if $u_k \rightharpoonup u$ weakly in $D(L)$, $Lu_k \rightharpoonup Lu$ weakly in $L^{p'}(0,T;V^*)$ and $\limsup_{k\to\infty} \langle B(u_k), u_k - u\rangle \le 0$, one has

$$\liminf_{k\to\infty} \langle B(u_k), u_k - v\rangle \ge \langle B(u), u - v\rangle \text{ for any } v \in L^p(0,T;V).$$

B is coercive in the sense that there exist $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$\langle B(u), v\rangle \ge c_1 \|v\|_{L^p(0,T;V)}^p - c_2 \text{ for any } v \in L^p(0,T;V).$$

Assume $Q : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ defined by $Qu(t) = \tilde{Q}(u(t))$, where $\tilde{Q} : V \to V^*$ is a linear, symmetric and continuous operator with

$$\left\langle \tilde{Q}v, v \right\rangle \geq 0 \ \forall v \in V.$$

Theorem 7.0.3. With the assumptions above on B and Q. For any $F \in L^{p'}(0,T;V^*)$, there exists $u \in C(0,T;V)$ such that $u' \in L^p(0,T;V)$ and $u'' \in L^{p'}(0,T;V^*)$ and

$$u'' + Au' + Qu = F$$

with

$$u'(0) = u(0) = 0.$$

In the previous chapter 6, we have seen the application of locally monotone operators in the study of first order evolution equations. The natural question now is to consider second order evolution equations with locally monotone operators.
Let $V \subset H \subset V^*$ be an evolution triple and let $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$. In addition, let the embedding $V \subset H$ be compact. Assume that for a.e. $t \in [0,T]$, $\tilde{B}(t)V \to V^*$ satisfies the following:
(B1) $\tilde{B}(t)$ is hemicontinuous.
(B2) $\tilde{B}(t)$ is locally monotone, i.e., there exists a constant $C > 0$ such that for any $u, v \in V$,

$$\left\langle [\tilde{B}(t)](u) - [\tilde{B}(t)](v), u - v \right\rangle \geq -(C + \rho(u) + \eta(v)) \left\| u - v \right\|_H^2,$$

where $\rho, \eta : V \to [0,\infty)$ are measurable functions and locally bounded in V with $\rho(0) = \eta(0) = 0$ and $\tilde{B}(t)(0) = 0$.
(B3) $\tilde{B}(t)$ is coercive in the sense that there exist constants $C_1 > 0$ and $C_2 \in \mathbb{R}$, $k_1 \in L^1(0,T)$ such that

$$\left\langle \tilde{B}(t)v, v \right\rangle \geq C_1 \left\| v \right\|_V^p - C_2 \left\| v \right\|_H^2 - k_1(t).$$

(B4) $\tilde{B}(t)$ is bounded in the following sense that for $C_3 > 0$, $k_2 \in L^{p'}(0,T)$ and $\beta > 0$, we have

$$\left\| \tilde{B}(t)v \right\|_{V^*} \leq (C_3 \left\| v \right\|_V^{p-1} + k_2(t))(1 + \left\| v \right\|_H^\beta).$$

(B5) The map $t \mapsto \left\langle \tilde{B}(t)u, v \right\rangle$ is measurable in $[0,T]$.
Assume that $\tilde{Q} : V \to V^*$ is linear, symmetric, strictly monotone and strongly continuous with $\tilde{Q}(0) = 0$. i.e., $\tilde{Q}$ is strictly monotone in the sense that for constant $c > 0$,

$$\left\langle \tilde{Q}(u) - \tilde{Q}(v), u - v \right\rangle \geq c \left\| u - v \right\|_V^p.$$

$B, Q : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ are defined through (7.1).

Remark 7.0.4. Note that there is an extra condition on $\tilde{B}(t)$ compared with first order evolution equations, that is, $\tilde{B}(t)(0) = 0$. And to adapt the method we introduced above, we assume $\tilde{Q}$ satisfies some stronger conditions.

With above assumptions, we have the following theorem.

Theorem 7.0.5. If the above assumptions are satisfied, for any $F \in L^{p'}(0,T;V^*)$, then there exists a solution $u \in C(0,T;V)$ with $u' \in W_p^1(0,T;V,H)$ such that (7.4) and (7.3) hold.

Proof. We are using the Galerkin's approximation. Since V is separable, let $\{w_1, w_2, \cdots\}$ be a countable linear independent set and its finite linear combinations form dense in V. We find the n-th Galerkin's approximation of the solution u in the form

$$u_n(t) = \sum_{k=1}^{n} c_{kn}(t) w_k \text{ with some } c_{kn} \in W^{2,p'}(0,T) \tag{7.8}$$

such that u_n satisfy

$$\langle u_n''(t), w_j\rangle + \left\langle [\tilde{B}(t)](u_n'(t)), w_j \right\rangle + \left\langle \tilde{Q}u_n(t), w_j \right\rangle = \langle F(t), w_j\rangle \text{ for } j = 1, 2, \cdots, n, \tag{7.9}$$

with

$$u_n'(0) = u_n(0) = 0. \tag{7.10}$$

The form of u_n ensures $u_n \in L^p(0,T;V)$ and $u_n' \in W_p^1(0,T;V,H)$. Substitute u_n into (7.9) and (7.10), we obtain

$$\sum_{k=1}^{n} c_{kn}''(t)(w_k, w_j) + \left\langle [\tilde{B}(t)] \left(\sum_{k=1}^{n} c_{kn}'(t) w_k \right), w_j \right\rangle + \left\langle \tilde{Q} \left(\sum_{k=1}^{n} c_{kn}(t) w_k \right), w_j \right\rangle = \langle F(t), w_j \rangle, \tag{7.11}$$

$$c_{kn}(0) = c_{kn}'(0) = 0. \tag{7.12}$$

(7.11) is a system of second order differential equation and can be transformed into explicit form

$$c_j''(t) = g(t, c_n(t), c_n'(t)),$$

because the determinant of $((w_k, w_j)_{kj})$ is non zero. Recall that from Lemma 2.3.4, we have compact embedding of $V \subset H$, (B1) and (B2) imply that $\tilde{B}(t) : V \to V^*$ is pseudomonotone, and therefore $\tilde{B}(t)$ is demicontinuous. (B4) implies that $|g(t, c_n(t), c_n'(t)|$ can be estimated locally by an integrable function $M(t)$. So g satisfies the Carathéodory condition, so there exists a solution of (7.11) in a neighbourhood of 0.
To show that the local solution c_{jn}, c_{jn}' can be extended to $[0,T]$, we need the following estimate. Multiplying (7.9) by $c_{jn}'(t)$ and summing with respect to j, we get

$$\langle u_n''(t), u_n'(t)\rangle + \left\langle [\tilde{B}(t)](u_n'(t)), u_n' \right\rangle + \left\langle \tilde{Q}(u_n(t)), u_n'(t) \right\rangle = \langle F(t), u_n'(t)\rangle. \tag{7.13}$$

Integrating above equation over the interval $(0,t)$ for $t \in [0,T]$, by Remark 1.3.15, we get

$$\frac{1}{2} \|u_n'(t)\|_H^2 = \int_0^t - \left\langle [\tilde{B}(s)]u_n'(s), u_n'(s) \right\rangle - \left\langle \tilde{Q}u_n(s), u_n'(s) \right\rangle + \langle F(s), u_n'(s)\rangle \, ds. \tag{7.14}$$

By (B3), we have

$$- \left\langle [\tilde{B}(s)]u_n'(s), u_n'(s) \right\rangle \le -C_1 \|u_n'(s)\|_V^p + C_2 \|u_n'(s)\|_H^2 + k_1(s). \tag{7.15}$$

From the symmetry of $\tilde{Q}$, we obtain

$$\frac{d}{dt} \left\langle \tilde{Q}v(t), v(t) \right\rangle = \left\langle \tilde{Q}v'(t), v(t) \right\rangle + \left\langle \tilde{Q}v(t), v'(t) \right\rangle = 2 \left\langle \tilde{Q}v(t), v'(t) \right\rangle.$$

By the strict monotonicity and the above equality, we obtain

$$2 \int_0^t \left\langle \tilde{Q}u_n(s), u_n'(s) \right\rangle ds = \left\langle \tilde{Q}(u_n(t)), u_n(t) \right\rangle \ge c \|u_n(t)\|_V^p \ge 0. \tag{7.16}$$

From (7.14) - (7.16), using Young's inequality and Hölder's inequality, we get

$$\frac{1}{2} \|u_n'(t)\|_H^2 \le \int_0^t -\frac{C_1}{2} \|u_n'(s)\|_V^p + C_2 \|u_n'(s)\|_H^2 + k_1(s) + \alpha \|F(s)\|_{V^*}^{p'} \, ds, \tag{7.17}$$

this is equivalent to

$$\frac{1}{2} \|u_n'(t)\|_H^2 + \int_0^t \frac{C_1}{2} \|u_n'(s)\|_V^p \, ds \le C_2 \int_0^t \|u_n'(s)\|_H^2 \, ds + \int_0^t k_1(s) + \alpha \|F(s)\|_{V^*}^{p'} \, ds. \tag{7.18}$$

By Gronwall's lemma, we obtain

$$\|u_n'(t)\|_H^2 \le \text{const, for all } t \in [0,T] \text{ and all } n, \tag{7.19}$$

and hence

$$\|u_n'(s)\|_{L^p(0,t;V)} \leq \text{const, for all } t \in [0,T] \text{ and all } n. \tag{7.20}$$

Now from (B2), (7.19) and $\tilde{B}(s)(0) = 0$ we obtain

$$\int_0^t \langle Bu_n'(s), u_n'(s)\rangle\, ds \geq -\int_0^t (C + \rho(u_n(s)))\, \|u_n'(s)\|_H^2\, ds \geq \text{ const.} \tag{7.21}$$

From (7.14), (7.16), (7.20), (7.21) and Hölder's inequality, we get

$$c\,\|u_n(t)\|_V^p \leq \int_0^t -\langle Bu_n'(s), u_n'(s)\rangle + \langle F(s), u_n'(s)\rangle\, ds \leq \text{ const,} \tag{7.22}$$

for all $t \in [0,T]$ and n. From (7.20) and (7.22), we get the uniform boundedness of c_{jn}, c_{jn}', and therefore they can be extended to $[0,T]$ for all j. By boundedness of $\tilde{B}(t)$ and linear continuity of $\tilde{Q}$, we also have

$$\|Bu_n'\|_{L^{p'}(0,T;V^*)} \leq \text{const} \;\text{ and }\; \|Qu_n\|_{L^{p'}(0,T;V^*)} \leq \text{const}.$$

Therefore, we have

$$u_n \rightharpoonup u \text{ in } L^p(0,T;V); \tag{7.23}$$

$$u_n' \rightharpoonup u' \text{ in } L^p(0,T;V); \tag{7.24}$$

$$Bu_n' \rightharpoonup z \text{ in } L^{p'}(0,T;V^*); \tag{7.25}$$

$$Qu_n \rightharpoonup y \text{ in } L^{p'}(0,T;V^*). \tag{7.26}$$

Now we will claim that

$$u'' + z + y = F. \tag{7.27}$$

For $v \in V$, let $v_l \in V_l = \text{ span}\{w_1, \cdots, w_l\}$ be such that $v_l \to v$ and let $\varphi \in C^\infty[0,T]$. For $n \geq l$, apply φv_l into (7.13), using integral by parts formula from Theorem 1.3.14, we obtain

$$\begin{aligned}(u_n'(T), \varphi v_l) &= \int_0^T \langle u_n''(s), \varphi(s)v_l\rangle + \langle u_n'(s), \varphi'(s)v_l\rangle\, ds\\ &= \int_0^T -\langle Bu_n', \varphi(s)v_l\rangle - \langle Qu_n, \varphi(s)v_l\rangle + \langle F(s), \varphi(s)v_l\rangle + \langle u_n'(s), \varphi'(s)v_l\rangle\, ds.\end{aligned}$$

First pass the limit as $n \to \infty$, then as $l \to \infty$, we obtain

$$(u'(T), \varphi v) = \int_0^T -\langle z, \varphi(s)v\rangle - \langle y, \varphi(s)v\rangle + \langle F(s), \varphi(s)v\rangle + \langle u'(s), \varphi'(s)v\rangle\, ds.$$

In the case where $\varphi \in C_0^\infty[0,T]$, above equality becomes

$$\int_0^T -\langle z, \varphi(s)v\rangle - \langle y, \varphi(s)v\rangle + \langle F(s), \varphi(s)v\rangle\, ds = -\int_0^T \langle u'(s), \varphi'(s)v\rangle\, ds.$$

This implies that $u'' = -z - y + F$ which proves the claim.
Now we show that $y = Qu$.
From our construction (7.8) for u_n above, we take $c_{kn} \in W^{2,p'}(0,T)$, by the Sobolev embedding on the interval I, $W^{2,p'}(I) \in C^1(\bar{I})$, then we can obtain that $u_n' \in C(0,T;V) \subset L^{p'}(0,T;V^*)$. Since the embedding $W_p^1(0,T;V,H) \subset C([0,T];H)$ is linear and continuous, we have $u_n(t) \rightharpoonup u(t)$ weakly in H for $t \in [0,T]$. Therefore, $u_n(t) \rightharpoonup u(t)$ weakly in V for $t \in [0,T]$. From the strong continuity of $\tilde{Q}$, we get for $t \in [0,T]$,

$$\tilde{Q}(u_n(t)) \to \tilde{Q}u(t) \;\text{ in } V^*. \tag{7.28}$$

For any $v \in L^p(0,T;V)$, consider

$$\langle Qu_n, v\rangle - \langle Qu, v\rangle = \int_0^T \left\langle \tilde{Q}(u_n(t)) - \tilde{Q}(u(t)), v(t)\right\rangle dt,$$

setting $h_n(t) = \left\langle \tilde{Q}(u_n(t)) - \tilde{Q}(u(t)), v(t) \right\rangle$, this is a family of integrable function in $L^1(0,T)$ with pointwise limit 0. By linearity and continuity of $\tilde{Q}$ and Hölder's inequality, $|h_n(t)|$ can be estimated by an integrable function, hence by Dominated Convergence Theorem 1.2.13, we have $y = Qu$.
Now it remains to show that $z = Bu'$.
We first show that $\limsup_{n\to\infty} \langle Bu_n', u_n' \rangle \leq \langle z, u' \rangle$. Note that by the above argument, we can obtain that $u_n'(t) \rightharpoonup u'(t)$ in V and H. From integral by parts formula, (7.14) and (7.27), we have

$$\frac{1}{2} \|u_n'(T)\|_H^2 = \int_0^T - \langle Bu_n'(s), u_n'(s) \rangle - \left\langle \tilde{Q}u_n(s), u_n'(s) \right\rangle + \langle F(s), u_n'(s) \rangle \, ds, \tag{7.29}$$

$$\frac{1}{2} \|u'(T)\|_H^2 = \int_0^T - \langle z, u'(s) \rangle - \left\langle \tilde{Q}u(s), u'(s) \right\rangle + \langle F(s), u'(s) \rangle \, ds. \tag{7.30}$$

Since $u_n'(T) \rightharpoonup u'(T)$ in H, we have by the lower semicontinuity of norm that

$$\|u'(T)\|_H \leq \liminf_{n\to\infty} \|u_n'(T)\|_H \,. \tag{7.31}$$

Subtract (7.30) from (7.31), we obtain by strong continuity of $\tilde{Q}$ that

$$\limsup_{n\to\infty} \int_0^T \langle Bu_n'(s), u_n'(s) \rangle \, ds \leq \int_0^T \langle z(s), u'(s) \rangle \, ds.$$

Noting that $[\tilde{B}(t)](u_n'(t)) = Bu_n'(t)$, we obtain by using Lemma 6.1.4 that for any $v \in L^{p'}(0,T;V^*)$,

$$\begin{aligned} \int_0^T \left\langle [\tilde{B}(s)](u'(s)), u'(s) - v(s) \right\rangle ds &\leq \liminf_{n\to\infty} \int_0^T \left\langle [\tilde{B}(s)](u_n'(s)), u_n'(s) - v(s) \right\rangle ds \\ \leq \limsup_{n\to\infty} \int_0^T \left\langle [\tilde{B}(s)](u_n'(s)), u_n'(s) - v(s) \right\rangle ds &\leq \int_0^T \langle z(s), u'(s) \rangle \, ds - \int_0^T \langle z(s), v(s) \rangle \, dt \\ &= \int_0^T \langle z(s), u'(s) - v(s) \rangle \, ds. \end{aligned}$$

Hence we have $z(t) = [\tilde{B}(t)](u'(t)) = Bu'(t)$ for $t \in [0,T]$. And the theorem is proved. ■

Remark 7.0.6. The theorem 7.0.5 above is quite weak, since the strong continuity assumption on $\tilde{Q}$ might be too much. One may consider a generalisation of the above theorem with weaker assumption on $\tilde{Q}$.

Application to hyperbolic partial differential equations

The existence theorems we presented above can be applied to solve partial differential equations of hyperbolic type. Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary, let $T > 0$. Denote Γ_T as $[0,T) \times \partial\Omega$ and denote Q_T as $(0,T) \times \Omega$. Let $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $V = W^{2,p}(\Omega)$ and $H = L^2(\Omega)$, then $V \subset H \subset V^*$ is an evolution triple. For $j = 0, 1, \cdots, n.$, functions $b_j : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. For $j,k = 1, \cdots, n.$, functions $q_{jk} : \Omega \to \mathbb{R}$ and the function $q : \Omega \to \mathbb{R}$. Consider the following homogeneous initial boundary value problem:
given $f \in L^{p'}(0,T;V^*)$, find $u : [0,T] \times \Omega \to \mathbb{R}$ such that

$$u'' - \sum_{j=1}^{n} D_j[b_j(t,x,u',Du')] + b_0(t,x,u',Du') - \sum_{j,k=1}^{n} D_j(q_{jk}(x)D_k u) + q(x)u = f(t,x) \text{ on } Q_T, \tag{7.32}$$

where $u' = \frac{du}{dt}$ and $D_j = \frac{d}{dx_j}$, with homogeneous initial and boundary conditions

$$u(0,x) = 0 \text{ and } u'(0,x) = 0 \text{ on } \Omega, \tag{7.33}$$

$$u|_{\Gamma_T} = 0. \tag{7.34}$$

For $u, v, w \in L^p(0, T; V)$, assume that functions b_j, q_{jk} and q some appropriate conditions, we may define the following operators:

$$\langle B(v), w\rangle = \int_0^T \left\langle [\tilde{B}(t)](v(t)), w(t)\right\rangle dt = \int_{Q_T} \sum_{j=1}^n b_j(t, x, v, Dv)D_j w + b_0(t, x, v, Dv)wdxdt, \tag{7.35}$$

$$\langle Qu, w\rangle = \int_0^T \left\langle \tilde{Q}(u(t)), w(t)\right\rangle dt = \int_{Q_T} \sum_{j,k=1}^n q_{jk}(x)D_k u D_j w + q(x)uwdxdt, \tag{7.36}$$

$$\langle F, w\rangle = \int_{Q_T} fwdxdt. \tag{7.37}$$

Through the weak formulation of above problem (7.32) - (7.34), with the operators introduced above, the problem can be rewritten as the following initial value problem:
Find $u \in C([0, T], V)$ with $u' \in W_p^1(0, T; V, H)$ such that for almost all $t \in [0, T]$, we have

$$u''(t) + \tilde{B}(t)(u'(t)) + \tilde{Q}u(t) = F(t) \tag{7.38}$$

with

$$u'(0) = u(0) = 0. \tag{7.39}$$

The boundary condition (7.34) is taken into consideration by setting $V = W_0^{1,p}(\Omega)$ and (7.3) is essentially (7.33). Note that (7.38) is also equivalent to

$$u'' + B(u') + Q(u) = F. \tag{7.40}$$

For conditions on functions b_j so that the underlined operator is monotone or pseudomonotone, see Section 3.2.2 and Section 4.2.2. For example such that the underlined operator is locally monotone, see section 3 in [57].
For the underlined operator $\tilde{Q} : V \to V^*$ to be strongly continuous, one may consider the case where $q_{jk}(x) = 0$ for all j, k. And $q(x)$ satisfies the growth condition in the sense that there exists $g(x) \in L^{p'}(\Omega)$

$$|q(x)z| \le C(g(x) + |z|^{p-1}) \text{ for all } z \in V.$$

Chapter 8

Appendix

8.1 Calculus fact

C^k boundary

Definition 8.1.1 (C^k boundary). Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $k \in \mathbb{N}$. We say that, the boundary of Ω, $\partial\Omega$ is C^k if for each point $x^0 \in \partial\Omega$, there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon relabelling and reorienting the coordinates axes if necessary, we have

$$U \cap B(x^0, r) = \{x \in B(x^0, r) | x_n > \gamma(x_1, \cdots, x_{n-1}\}.$$

Likewise, $\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for all $k = 1, 2, ...$, and $\partial\Omega$ is analytic if the mapping γ is analytic.

Remark 8.1.2. Another way to understand above definition is the following:
If $\partial\Omega$ is C^k, then the boundary is locally the image of a C^k embedding of $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$.
In the definition above, for any point on the boundary, there exist a neighbourhood of that point, after reorienting the coordinate system if necessary, all points on the boundary within the neighbourhood can be express as

$$(x_1, x_2, ..., x_{n-1}, \gamma(x_1, ..., x_{n-1})).$$

Consider the function

$$f : (x_1, x_2, ..., x_{n-1}) \to (x_1, x_2, ..., x_{n-1}, \gamma(x_1, ..., x_{n-1})),$$

then f is a C^k embedding from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$.

Definition 8.1.3. (i) If $\partial\Omega$ is C^1, then we can defined the outward pointing unit norm vector field along $\partial\Omega$ as:

$$\boldsymbol{\nu} = (\nu^1, ..., \nu^n).$$

The unit normal vector at any point $x^0 \in \partial\Omega$ is $\boldsymbol{\nu}(x^0) = (\nu^1(x^0), ..., \nu^n(x^0))$.
(ii) Let $u \in C^1(\bar{U})$. We call

$$\frac{\partial u}{\partial \nu} := \boldsymbol{\nu} \cdot Du$$

the (outward) normal derivative of u.

similarly, we have the definition of Lipschitz boundary where we change C^k into Lipschitz function.

Gauss-Green

We assume, in this section, that Ω is bounded, open subset of $\mathbb{R}^n$ as usual, and $\partial\Omega$ is C^1.

Integration by part formula

Let u, v be two continuously differentiable function on the closure of Ω, i.e., $u, v \in C^1(\bar{\Omega})$, then

$$\int_\Omega u_{x_i} v dx = \int_{\partial\Omega} uv\nu^i dS - \int_\Omega uv_{x_i} dx,$$

where $\boldsymbol{\nu}$ is the outward pointing unit normal vector field to $\partial\Omega$, and ν^i is the i-th component of $\boldsymbol{\nu}$.

8.2 Inequality

Young's inequality

Let $1 < p, p' < \infty$ be conjugate, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, then for any $a, b > 0$, we have that

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Proof. By the convexity of the map $x \mapsto e^x$, it follows that

$$ab = e^{\log a + \log b} = e^{\frac{1}{p}\log a^p + \frac{1}{p'}\log b^{p'}} \leq \frac{1}{p} e^{\log a^p} + \frac{1}{p'} e^{\log b^{p'}} = \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Remark 8.2.1. If we set $ab = ((\epsilon p)^{1/p} a) \cdot (\frac{b}{(\epsilon p)^{1/p}})$ and apply the above Young's inequality, we get the Young's inequality with ϵ

$$ab \leq \epsilon a^p + C(\epsilon) b^{p'} \text{ where } \epsilon > 0 \text{ and } C(\epsilon) = (\epsilon p)^{-\frac{p'}{p}} p'^{-1}.$$

Hölder's inequality

Assume that $1 \leq p, p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, we have

$$\int_\Omega |uv| \, dx \leq \|u\|_{L^p(\Omega)} \cdot \|v\|_{L^{p'}(\Omega)} \cdot$$

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