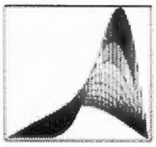
EFFICIENT NUMERICAL METHODS FOR THE VALUATION OF AMERICAN BARRIER OPTIONS



Mkhululi Dlamini
September 2002

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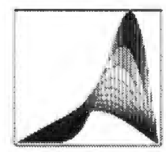

Abstract

The barrier option is the most popular exotic option traded today. Because such options have a discontinuous payoff pattern, their accurate valuation is a particular challenge. Most popular in the OTC market, a lack of a liquid secondary market in these products has meant that very often, an early exercise feature is added to the contract. This makes it of particular interest to study efficient numerical methods for the valuation of American barrier options. This thesis considers three methods that have been developed to price such options; the Ritchken Trinomial Method (RTM), the Finite Difference Method (FDM) and the Finite Element Method (FEM). First an account is given of the barrier option pricing problem accompanied by a description of the behavior of barrier option price and delta curves. Then the theory and implementation of each method is described in turn. Finally a detailed computational analysis is given where the three methods are compared in pricing and hedging applications, with concluding remarks on the performance results.

Appendix A provides information on how to use the code provided on the accompanying disk to reproduce the results quoted in this thesis.

Acknowledgements

I'd like to thank my supervisor, my family and my friends. But most of all I'd like to thank those brave souls who have contributed to our understanding of that great and mighty beast that is the financial market.

September, 2002

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0.0 Introduction

The popularity of path dependent options has been growing rapidly in recent years. These are options whose payoff depends on the movement in the price of the underlying during the life of the option. The most popular of these exotic options is the barrier option. This is an option that is either nullified, activated or exercised if the price of the underlying asset breaches some barrier level during the life of the option[1]. In fact, barrier option trading is responsible for approximately 50% of the volume of all exotic traded options, and 10% of the volume of all traded derivative securities (Lin, *et al* (1999)).

The valuation of barrier options dates back to Merton (1973) who derived a closed formula for a down-and-out call option. Their valuation in the Black and Scholes (1973) model has been much studied in recent years. Indeed barrier options have appealing properties, allowing investors to incorporate their personal beliefs about future asset price behavior. For example, for a double barrier knock-out option, the holder gets the corresponding vanilla option if the asset prices stays within the barriers or nothing if the asset price should stray out of the band. This added constraint, the possibility of a zero payout, makes barrier options cheaper than their vanilla counterparts and an attractive alternative for investors.

The discontinuity of the option payoff makes barriers difficult to price. As Boyle and Lau (1994) pointed out, the traditional binomial tree model performs poorly in the pricing of barrier options. It leads to a specification error where the effective barrier in the model is not the same as the true barrier. This leads to very slow and erratic, saw-tooth like convergence to the closed form option price. This saw-tooth effect becomes even worse for American barrier options. Many authors have addressed this problem and suggested refinements. Examples include Derman, Kani, Ergener and Bardhan (1995), Ritchken (1995), Cheuk and Vorst (1996), Rogers and Stapleton (1997), Boyle and Tian (1999) and Ahn, Figlewski and Gao (1999).

However all these approaches are either binomial or trinomial tree techniques. Each of them may be viewed as an explicit finite difference method to solve the underlying PDE. Little attention has been given in the extant literature to implicit finite difference methods, which have superior convergence and stability properties while also allowing for greater flexibility in the

[1] It is useful to think of barriers as a *feature* of the option contract since barriers may be incorporated into any type of option. Here we consider plain vanilla options with barrier features.

construction of the spatial grid. This is particularly important for options with barriers. Within the PDE approach one has two competing methods; finite differences and finite elements. Zvan, Vetzal and Forsyth (1997) describe a general PDE framework for the valuation of European and American barrier options and show their methods to obtain superior accuracy in fewer time steps.

As derivative products get more complex, fast, reliable and robust numerical methods become a necessity. This is even more important for the valuation of large portfolios and fast reaction in volatile markets. The aim of this thesis is to investigate and compare the trinomial tree, finite difference and finite element methods for the accurate pricing of American barrier options[2]. Algorithms are compared on the basis of numerical accuracy, convergence behavior, computational speed[3], and the ability to price options when the underlying is very close to the barrier.

The rest of the thesis is organized in the following way. Chapter 1 reviews basic option pricing theory, Chapter 2 describes the barrier option pricing problem, Chapters 3-5 outline the numerical methods chosen for investigation, Chapter 6 discusses other valuation methods for American barrier options, Chapter 7 gives the numerical results and Chapter 8 concludes. Appendix A provides information on using the Matlab and C executables provided on the accompanying disk.

[2] A note of caution: the price of a barrier option is extremely sensitive to the volatility. Although we will assume the same constant volatility model for all the algorithms, this does not impair our ability to compare them numerically, even though they may not be giving the 'correct' market prices.

[3] Lyuu (1998) argues that the only objective method for comparing algorithms is on the total time taken to achieve comparable results. This is the principal approach taken here.

1.0 Quick Review of Option Pricing Theory

Modern option pricing theory is based on the assumption that any financial instrument with a guaranteed non-negative payout must also have a non-negative price, or else it would present an *arbitrage opportunity*. Such an arbitrage opportunity would allow investors to make guaranteed profits at zero cost. However it is reasonable to assume that in an economy that is in equilibrium, any arbitrage opportunities would quickly disappear as increased investor demand would cause prices to rise. Inherent in this assumption is that there are no transaction costs i.e. buy price = sell price. In reality this is not true. However large institutional investors experience very little transaction costs and are thus able to take advantage of all arbitrage opportunities and so drive markets to the equilibrium that would exist if transaction costs were indeed absent.

In order to derive some fundamental results in option theory, we consider a continuous trading economy with a finite trading interval $[0,T]$. Uncertainty is modeled by the probability space (Ω, F, P). Prices $S(t)$ of traded assets can be modeled via Ito processes described by the stochastic differential equation (SDE):

$$dS(t) = \mu(t,w)dt + \sigma(t,w)dW$$

where W is a one-dimensional Brownian motion.
A *trading strategy* is a predictable n-dimensional stochastic process:

$$\beta(t,w) = (\beta_1(t,w),\ldots,\beta_N(t,w))$$

where $\beta_n(t,w)$ denotes the time t holding in asset n. The value of this strategy is then given by $U(\beta,t) = \sum_{n=1}^{N} \beta_n(t) S_n(t)$.
A trading strategy is *self-financing* if it has the property:

$$U(\beta,t) = U(\beta,0) + \sum_{n=1}^{N} \int_0^t \beta_n(s) dS_n(s)$$

$\forall t \in [0,T]$, where it is assumed that the gains from trading can be modeled as Ito integrals. Such a self financing strategy neither requires nor generates funds between $[0,T]$. It then follows that an *arbitrage opportunity* is a self-financing trading strategy β with:

$$P(U(\beta,T)\geq 0)=1 \text{ and } U(\beta,0)<0.$$

A *derivative security* or *option* is an F_T-measurable random variable $H(T)$ which describes the uncertain payoff of the derivative security at time T. If we can find a self financing trading strategy β such that $U(\beta,T)=H(T)$ with probability one, then the derivative is said to be *attainable* and β is called a *replicating strategy*.

If in an economy, all derivative securities are attainable then the market is said to be *complete*. Then in the absence of arbitrage opportunities and transaction costs the value of a replicating strategy at time t gives a *unique value* for the attainable derivative $H(T)$. This allows us to determine the value of derivative securities by the value of the replicating strategies, in a process called *pricing by arbitrage*.
This is the fundamental principle underlying the Black and Scholes (1973) option pricing model.

1.1 Derivation of Black-Scholes PDE

Many of the greatest advances in modern science have been based on discovering the underlying PDE for the process in question, and finance is no exception. The Black-Scholes (1973) option pricing model revolutionized financial markets. It describes a two asset economy with a risk less money-market account B with $B(0)=1$ and a stock S. B earns a constant rate of interest r and S follows a geometric Brownian motion (GBM) with drift μ and volatility σ:[4]

$$\begin{aligned} dB &= rBdt \\ dS &= \mu Sdt + \sigma SdW \end{aligned} \qquad (1.01)$$

The value U of a financial instrument contingent on stock S is completely determined at every instant t by the asset price $S(t)$ (markov property). By ito's lemma, this $U(t,S)$ follows the SDE:

$$dU = \left(U_t + \mu SU_S + \frac{1}{2}\sigma^2 S^2 U_{SS}\right)dt + \sigma SU_S dW \qquad (1.02)$$

[4] In general, the price of any path-dependent option (e.g. a barrier option) is extremely sensitive to the type of underlying process specified (see Boyle and Tian (1999)). Here we will always assume a constant volatility.

we can replicate this financial derivative $U(t,S)$ with a self-financing trading strategy β such that:

$$U(t,S) = \beta_S(t)S(t) + \beta_B B(t) \tag{1.03}$$

In differential form this is:

$$dU = \beta_S dS + \beta_B dB \tag{1.04}$$

substituting (1.01) into (1.04) and using (1.03) we get:

$$dU = (r(U - \beta_S S) + \mu S \beta_S)dt + \sigma S \beta_S dW \tag{1.05}$$

equating (1.02) and (1.05) gives:

$$\left(U_t - r(U - \beta_S S) + \mu S(U_S - \beta_S) + \frac{1}{2}\sigma^2 S^2 U_{SS}\right)dt + \sigma S(U_S - \beta_S)dW = 0 \tag{1.06}$$

choosing $\beta_S \equiv V_S$ eliminates the random term s.t.,

$$U_t + rSU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} - rU = 0 \tag{1.07}$$

If the underlying asset pays a continuous dividend yield δ, the holder of the asset gains $\delta S\,dt$ in dividends. To exclude arbitrage the asset price must fall by the same amount, hence the drift of the asset changes from r to $r-\delta$. Thus (1.07) becomes:

$$U_t + (r-\delta)SU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} - rU = 0 \tag{1.08}$$

This is the famous Black-Scholes backward parabolic partial differential equation for an asset paying a continuous dividend yield δ.

We consider two different approaches to option pricing in this paper. The PDE methods obtain the option price by solving the underlying PDE (1.08) subject to the boundary conditions of the option pricing problem. The tree methods on the other hand, price the option by simulating the continuous asset price process (1.01) by a discrete random walk and then using dynamic programming to solve for the asset price.

2.0 The Barrier Option Pricing Problem

Barrier options essentially come in two types; knock-in and knock-out options. Knock-in options come into existence if the underlying price hits a barrier level, else they expire worthless. Conversely, knock-out options become worthless when the barrier is hit. In some cases when a barrier is hit, an amount known as a rebate may be paid to the option holder, almost as a 'consolation prize' for when the option knocks out. For knock-in options, an additional premium may be due when the new option is established. A second distinction may be made regarding whether the barrier is crossed from above or below. Then we speak of up-barriers and down-barriers, up-and-in and down-and-out barrier options and so on. Capped options, which are automatically exercised when the underlying asset price reaches a certain level are special cases of the barrier theme. Double barrier options have both an up and a down barrier and have a payoff that depends on whether or not the underlying price stays within the band. Thereafter there are several variants of the barrier theme, partial barrier options are those for which the barrier only exists for an initial period in the option's life[5] (for forward starting barriers, the barrier period is at the end of the option's life), Parisian barrier options only knock-in or knock-out if the barrier is breached for a pre-specified length of time[6], for outside barriers, the barrier level is linked to some process other than the underlying, e.g. an exchange rate, and may be time-dependent or moving, and discrete barriers are when the barrier is not monitored continuously, but rather daily or weekly. Here we shall assume continuous monitoring[7] and non-moving inside barriers.

2.1 Applications of Barrier Options

The appealing property of barrier options is that they allow investors to incorporate their personal beliefs about the future price behavior of the underlying asset. They are popular with speculators wishing to make a directional play on an underlying. Investors expecting a market to rise buy up-and-in or down-and-out calls. This way they get the same payoff as with a plain vanilla option if they are right, but less if they are wrong. Because adding a barrier to an option introduces an additional constraint, barrier options are much cheaper than their plain vanilla counterparts, adding to their popularity. Hedgers may use barrier options to obtain

[5] In practice the payoff may then be a function of the spot price when the barrier is deactivated.

[6] This can reduce the effect of possible manipulation by traders to trigger a knock-in/out through moving the market briefly.

[7] Note that in reality, barrier options can only be discretely monitored.

insurance protection above or below a certain level of the underlying asset. Writers of options may like reverse barrier options, which have a knock-out barrier in-the-money or a knock-in barrier out-of-the-money. They offer interesting ways to manage risk, for example, for the writer of an up-and-out call, his liability disappears when losses reach a certain level. This makes barrier option very useful hedging instruments for risk management strategies.

2.2 Valuation of American Barrier Options

The pricing of American options has been of considerable interest to researchers. From Merton (1973) we know that it is never optimal to exercise an American call option on a non-dividend paying asset before maturity. In the case of American barrier options there is an optimal exercise policy even for calls, as the holder may avoid the possible loss if the barrier is hit. We state that in general, there is an early exercise incentive for out call options just before the barrier, provided the option is in the money. It is never optimal to exercise down-and-in and up-and-in calls before expiry. The argument for put options is the same, but with the additional caveat that it may be optimal to exercise whenever the asset price is very small.

The out option is constantly threatened by the possibility of expiring worthless while there's still a chance it might expire in the money. Because of the possibility that the option may expire worthless there tends to be a decline in the option value as the underlying asset price approaches the barrier. This is true for European barrier options, there is a constant "tug of war" between the influence of the barrier level versus the influence of the strike price. For American barrier options, the early exercise feature removes this particular conflict although the presence of the barrier does cause the option price to be depressed closer to maturity.[8] Consider the American up and out call option. If we assume that investors act optimally then we can say that with rising prices of the underlying, the barrier level for the up and out American call option coincides with the optimal exercise boundary.

We can formalize a description of the optimal exercise strategy as follows, again consider the American up-and-out call option. The payoff is given by:

$$U(S,\tau)=[S(\tau)-K]^{+}1_{\{\tau<\tau_h\}} \tag{2.01}$$

[8] Particular cases of barrier option price behavior are discussed in Section 2.5.

if the option is exercised at some time before maturity $\tau \le T$. Here h is the barrier level $(h>0)$ and we define $\tau_h := \inf\{\tau \ge 0 \mid S(\tau) > h\}$. The American early-exercise constraint is:

$$U(S,\tau) \ge [S(\tau)-K]^+ 1_{\{\tau<\tau_h\}} \qquad (2.02)$$

Since the option can be exercised at any time before maturity, it has the possibility of *optimal* early exercise, The option's time zero value may then be written as an expectation:

$$U(S,0) = \sup_{\tau \in T_{0,T-t}} E\left(e^{-r\tau}[S(\tau)-K]^+ 1_{\{\tau<\tau_h\}} \middle| F_\tau\right) \qquad (2.03)$$

where the supremum is over all stopping times $\tau \le T$. Finding the optimal stopping policy is equivalent to finding the points (S,t) for which early exercise is optimal. The boundary which separates the early exercise region from the continuation region is called the *optimal exercise boundary* (OEB). Since the OEB is not known at time zero, the solution to (2.03) must be found numerically. In Chapters 3-5, we investigate multinomial tree (or lattice) and PDE approaches to the solution of (2.03).

2.3 Barrier Option Hedge Parameters

The existence of a barrier level, which determines the option value at expiry, means that barrier options have a discontinuous payoff pattern. The jump that is experienced when the underlying price hits a barrier, due to the nullification or activation of the option, has a considerable impact on the hedge parameters of such options.

The option delta measures the rate of change in option value with respect to changes in the price of the underlying. Since the option value has a jump discontinuity at the barrier, its delta becomes very large when the price of the underlier is near that barrier. If the underlying has high volatility, the delta is even larger since this means that there is a higher probability of a knock-out. This makes barrier options highly leveraged instruments near barrier levels, more so than vanilla options.

The gamma of an option measures the sensitivity of the delta to underlying asset price changes, in other words how quickly or severely a delta hedge comes out of balance. The possibility of extreme change means that as the underlying gets closer and closer to the barrier, gamma grows larger and larger, without bound. This makes it very difficult to hedge barrier options near the barrier level because there is a very large difference in the amount of the underlying asset that has to be held, depending on whether or not the barrier is crossed. The delta and gamma curves of an up and out call (near maturity) are shown in Figure 1.0.

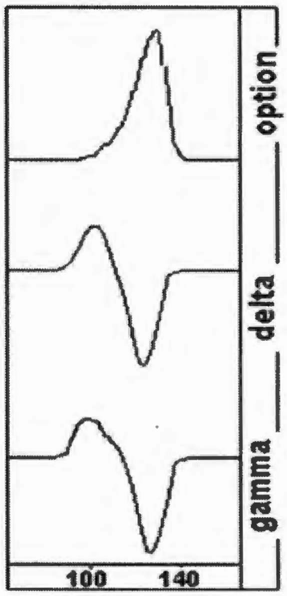

Figure 1.0: Barrier option Greeks

Here we can clearly see the opposing forces of moving further into the money, while the probability of a knock out increases at the same time. This is shown by the option value rising after 100 and then falling rapidly back to 0 by the time the barrier level of 140 is reached. The delta and gamma both become very large near the barrier level.[9]

2.4 Static Options Replication

Static options replication is a popular strategy for hedging barrier options where the option value is matched at some boundary. The key result is that if the replicating portfolio matches the barrier option position at some boundary then the option is also matched at all interior points of the boundary. Shorting that portfolio then provides the hedge. The portfolio is then unwound when any part of the chosen boundary is reached. This method was first described by Derman, Ergener and Kani (1995). The static replication approach offers the hedger

[9] Particular cases of hedge parameter behavior are discussed in a Section 2.5.

a lot of flexibility in the choice of boundary to match as well as what options to use in the replicating portfolio. It also doesn't require frequent rebalancing like dynamic options replication approaches such as delta hedging and is therefore less risky and less costly[10].

2.5 Barrier Price and Delta Curve Behavior

In exploring the behavior of barrier option value and delta parameters, we assume a test bed of several different barrier options to illustrate important aspects of barrier option behavior. For completeness, we consider European as well as American barrier options[11]. For each option we plot three dimensional mesh plots of the value and delta curves for varying spot prices and times to maturity. For generality all options have a strike price of 100, a rebate of zero and are assumed to be on an non-dividend paying underlying with 15% volatility in an economy with a 5% riskless interest rate.

2.5.1 Price and Delta Curves

The price curves for barrier options behave differently from those of their vanilla counterparts. As discussed in Section 2.2, for European barrier options, there is a constant "tug of war" between the influence of the barrier versus the influence of the strike price. In the case of American barrier options, this effect may be cancelled out by the possibility of early exercise as intrinsic value increases, although a decrease in time to maturity will still depress the option's value, as it does for vanilla options. To understand the shape of the price curves, it is worth noting some important factors governing the price of a barrier option:

- Intrinsic value: this is the payoff from exercising the option immediately. For a call option the intrinsic value increases with increasing spot price $(S-K)$, for a put option it increases with decreasing spot price $(K-S)$.
- Knock out/in probability: as the price of the underlying moves towards the barrier the probability of knocking in or out increases, thus affecting the price of the option[12].

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[10] In theory a dynamic replicating portfolio must be rebalanced continuously. Since this is impractical, rebalancing is typically done at discrete intervals, which causes errors that increase with larger intervals. Additionally, rebalancing incurs transaction costs which impact the profit margin of the portfolio. As a result, dynamic replication requires a compromise between accuracy and cost.

[11] The graphs for European options are included to provide additional insight into barrier option behavior.

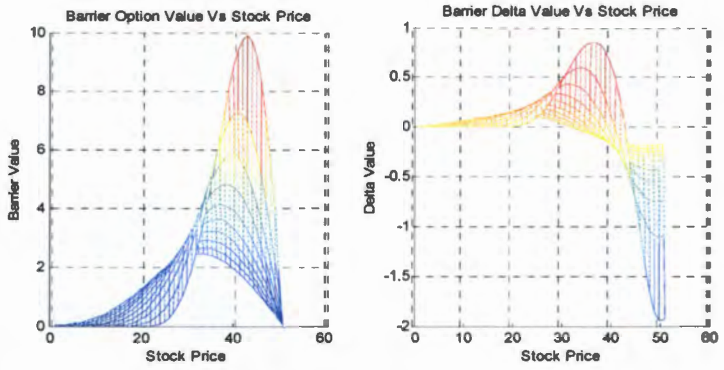
[12] We use the terms option price and option value interchangeably.

- Time to maturity: for times close to maturity the probability of large moves in the underlying stock price decreases, causing the probability of crossing the barrier or the strike to decrease, thus affecting the value of the option.
- Early exercise feature: its presence may diminish the effect of downward pressure on the option price exerted by the threat of knocking out.

The option delta curve follows as the gradient curve of the option price curve.

2.5.2 Up and Out Call (120)[13]

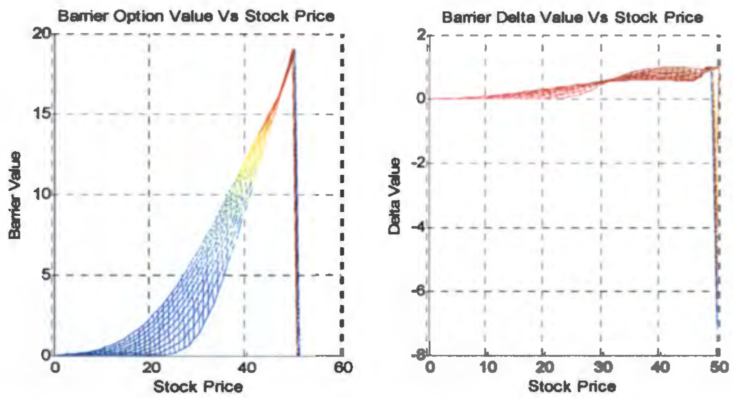
European:



Here we can clearly see the opposing effects of increasing intrinsic value and a corresponding increase in knock-out probability. As the option goes deeper into the money, its value continues to increase up until relatively close to the barrier. At some point before the barrier is reached the force of the barrier begins to exert downward pressure on the option price, forcing it to fall precipitously to zero, even as its intrinsic value continues to rise. We also see that this effect is more pronounced closer to maturity, simply because as expiration nears, the probability of knocking out (at a given level of the spot price) decreases and therefore where the option is in the money, its value increases.

[13] Please note that the up and down barriers have been scaled back by 70 and 80 respectively. This means that an up barrier of 120 corresponds to 50 on the graph's stock axis and a down barrier of 80 corresponds to 0.

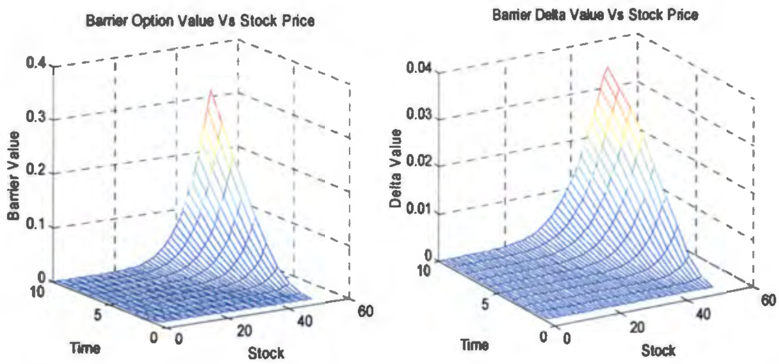
American:



In contrast to the European option, the downward pressure effect of the barrier is effectively nullified since the option may be exercised just before the barrier is hit. We can also see the evidence of a high American option premium as the American option price is larger than the corresponding European price above, for a given level of stock. The option delta behaves much like that of the vanilla American option, except that at the barrier, the delta gets very large and negative as the barrier option value falls rapidly to zero.

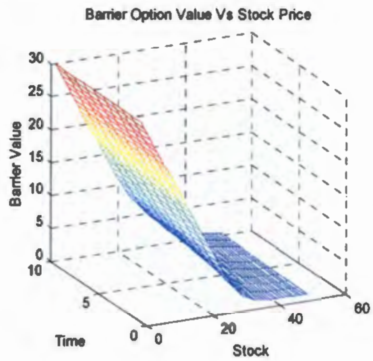
2.5.3 Up and In Put (120)

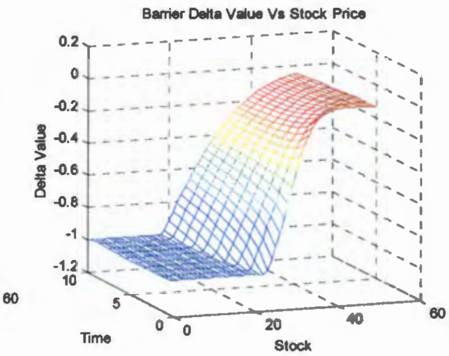
European:



The behavior of this option illustrates one of the possible effects of the barrier level being in the region where the option is out of the money. As the stock price moves towards the barrier from below the option moves further out of the money. The probability of knocking in increases while the probability of expiring in the money decreases. The intrinsic value of the option remains zero. Closer to maturity, the value of the out of the money option is less than it is further from maturity since there is a lower probability that the stock price will move favorably in the short time. Thus the option is worth more further from maturity, rising to the value of the corresponding vanilla put value at the knock in barrier. When the option is knocked in, the option-holder receives an out of the money put option whose value then begins to fall from the peak like that of the corresponding vanilla put[14].

American:

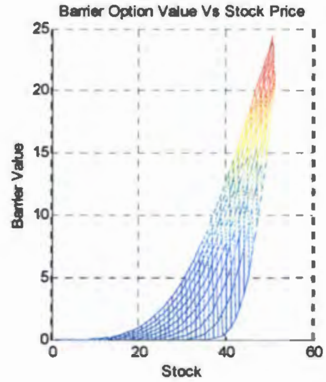


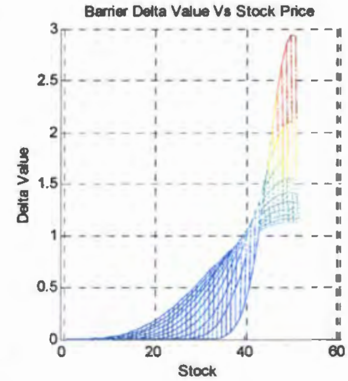


The American version of the contract illustrates how the early-exercise feature can nullify the adverse effect of the barrier. This option behaves much like the plain vanilla American option, the effect of the barrier is minimized since the option may be exercised prior to expiry.

2.5.4 Up and In Call (120)

European:

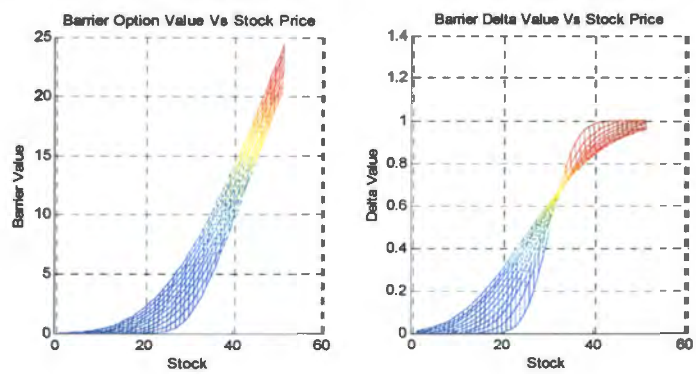




[14] The price curve is symmetrical about the barrier. It could not be shown here as none of the valuation methods tested could value the option *across* the sharp peak at the barrier.

Here we see how the value of a European up and in call is depressed from the corresponding vanilla option's value due to the possibility of expiring worthless if the barrier is not crossed (we can see this since near the barrier, the delta is above that of a deep in the money vanilla call of 1). Closer to the barrier and closer to expiry the price curve steepens sharply due to the downward pressure of the barrier. This leads to the high delta of nearly 3 near expiry which reflects how at this point, even small moves in the underlying stock price would cause large moves in the option value since the option has the possibility of going from having a probability of expiring worthless to being a deep-in-the-money call if it is knocked in.

American:



Here we see that the value of an up and in American call is much the same as that of the corresponding vanilla European or American call. The early exercise feature removes the depressing effect of the barrier and the option behaves much the same as the vanilla American option (Figure 2.0).

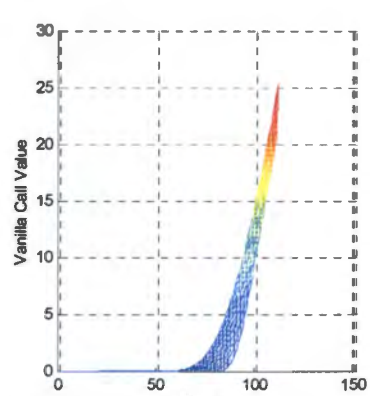


Figure 2.0: Vanilla American call value

2.5.5 Double Barrier Options

The double barrier out call option behaves in the same way as the up and out single barrier call option and the double barrier put option behaves like the single barrier down and out put. This follows from the fact that for double barrier *calls*, it is the *up* barrier that is important since it is the one that lies in the region where the option is in the money and similarly with the down barrier for double out put options.

European:

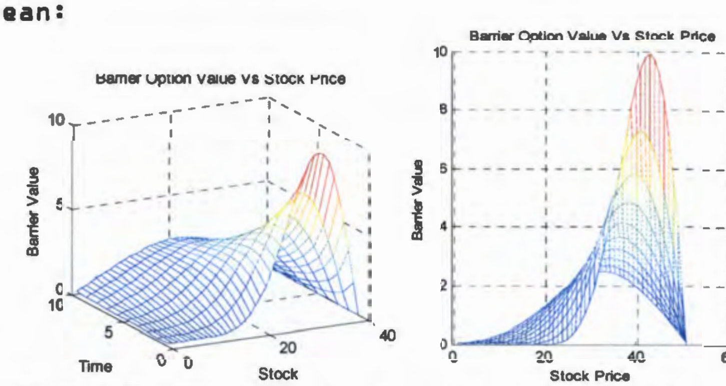


Figure 3.0: European double out call vs. European up and out call

American:

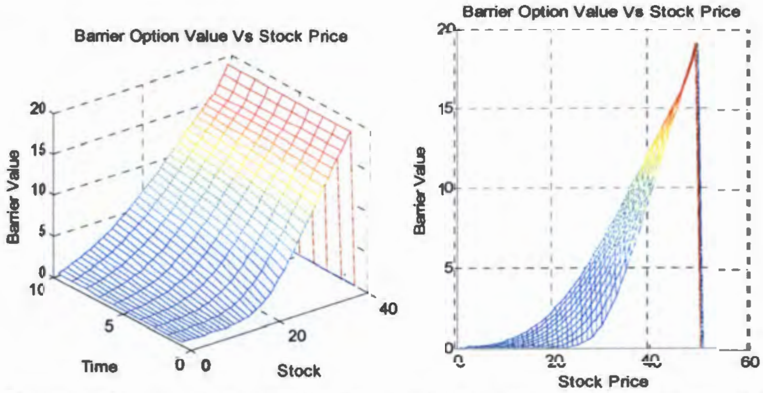


Figure 4.0: American double out call vs. American up and out call

Similarly for double in options.

European:

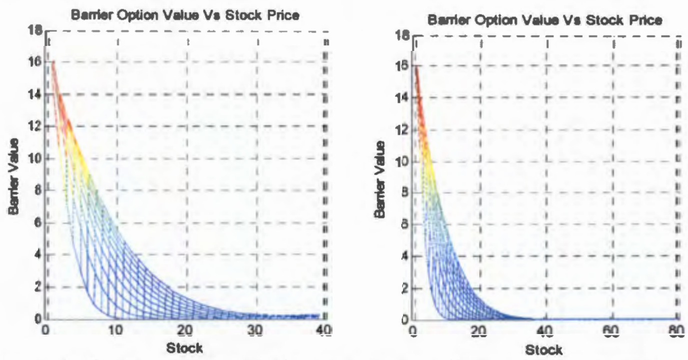


Figure 5.0: European double in put vs. European down and in put

American:

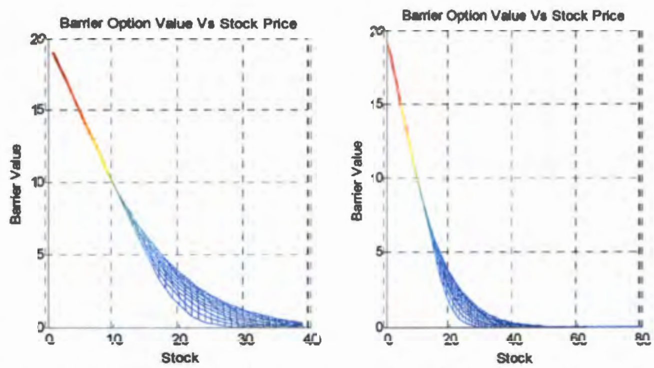


Figure 6.0: American double in put vs. American down and in put

3.0 Multinomial Tree Methods

Binomial schemes a very popular in the financial community and are used for the valuation of a wide variety of option models. They are popular because of their relative simplicity and intuitiveness as well as for having monotonic convergence. However they are limited in their applicability to complex option pricing. Boyle and Lau (1994) showed that the unadjusted binomial method performs unsatisfactorily in the pricing of European barrier options. This is because of the intrinsic limitation of binomial methods in that they do not easily allow for the incorporation of boundary conditions, such as the presence of a barrier price. Boyle and Lau account for this problem by choosing the number of time steps such that the barrier coincided with a horizontal layer of nodes on the tree. Other researchers have also addressed this problem. Ritchken (1995) introduced a trinomial lattice approach to value a wide range of barriers such as double barriers, rainbow barriers and curved barriers. He used the extra degree of freedom offered by the trinomial method (the free parameter λ) to ensure that tree nodes lined up with barriers. Ritchken's method was further developed by Cheuk and Vorst (1996) who introduced a time-dependent shift in the trinomial lattice. Another modification to the binomial method was suggested by Derman, Kani, Ergener and Bardhan (1995). They use an interpolation scheme to improve the pricing error of the standard binomial tree. Ahn, Figlewski and Gao (1999) suggested a method for improving the resolution of lattices near barriers by using an adaptive mesh model where a fine mesh is used for regions near the barrier and is then grafted onto a coarser mesh which is used in the rest of the tree. This is the same concept as spatial grid transformations in PDE methods.

3.1 The Problem With the Binomial Method

As demonstrated by Boyle and Lau (1994) the binomial method performs poorly in the pricing of barrier options. The inaccuracy of the unmodified binomial method comes from a misspecification of the barrier level by the model itself. Consider the case of a down-and-out call with barrier level H. For number of time steps N, let n_H denote the index such that:

$$S_0 d^{n_H} \geq H > S_0 d^{n_H+1} \qquad (3.01)$$

But since N is fixed, the backward recursion scheme will give the same answer for any barrier level between $S_0 d^{n_H}$ and $S_0 d^{n_H+1}$. Derman, Kani, Ergener and Bardhan (1995) call this the 'specification error'. They note two sources of error for binomial methods in a barrier option setting. First the 'quantization error' which comes from discretizing a continuous asset price process (and hence applies to the pricing of vanilla options as well) and the specification error described above. Because once a lattice has been constructed the possible values the asset price may take are fixed, if the barrier level H does not coincide with one of the available stock prices the model effectively 'moves' the barrier to the nearest available stock price (Figure 7).

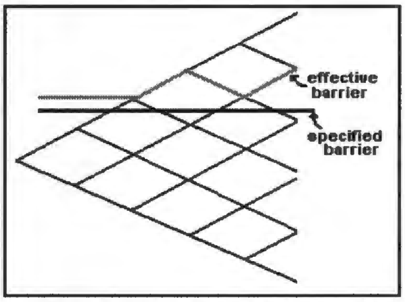


Figure 7: Specification error of CRR tree

This is what causes the inaccuracy and erratic convergence (Figure 8). An explanation of this behavior lies in the fact that increasing the refinement results in a new shape for the discretized model (tree) since the up and down shocks actually depend on the refinement Δt.[15] As the refinement of the tree is increased, the barrier will at times lie exactly on a row of nodes (thus giving the correct price) and at times between a row of nodes (giving an incorrect price). As the distance between rows of nodes decreases with increasing refinement of the tree, the CRR price slowly converges to the continuous price.

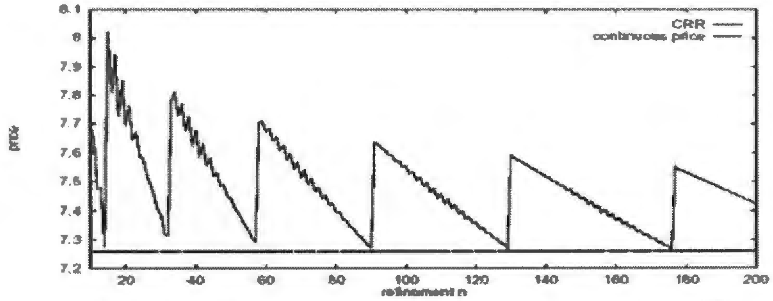


Figure 8: Saw-tooth convergence of CRR tree (22)

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[15] $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u$

One way to remedy this erratic convergence, as done in the approach by Boyle and Lau (1994), is to choose N such that the barrier coincides with a horizontal layer of nodes on the tree. Another way is to feed the algorithm with the correct barrier level. This is the approach taken by Derman, Kani, Ergener and Bardhan (1995) and by Ritchken (1995).

In this paper we shall investigate Ritchken's method[16].

3.2 Ritchken (1995) Method

The first tree method for the valuation of options with early exercise features was the binomial method introduced by Cox, Ross, and Rubinstein (1979) (CRR hereafter). It provides a method by which dynamic programming can be used to price American-style options by arbitrage in a complete economy. Since the market is complete, the option is a redundant asset that may be replicated by a combination of traded assets and the money market account as shown in Section 1. Because of the complete market assumption, the binomial method only provided for a two jump process (up or down) for the asset price. In an attempt to improve accuracy and performance, Boyle (1989) relaxed this constraint by allowing for a horizontal move in the asset price, creating a three jump process. This abandoned the assumptions of the CRR model since the market was no longer complete and there was now the possibility of arbitrage[17]. Unfortunately the transition probabilities in Boyle's model involved an exponential function of Δt. Kamrad and Ritchken (1991) obtained much simpler transition probabilities by working with the logarithm of the asset price distributions. They obtained:

$$\begin{aligned} p_u &= \frac{1}{2}\left(\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\nu}{\sigma}\right)\right) \\ p_d &= \frac{1}{2}\left(\frac{1}{\lambda^2} - \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{\nu}{\sigma}\right)\right) \\ p_m &= 1 - \frac{1}{\lambda^2}, \quad \lambda \geq 1 \end{aligned} \tag{3.02}$$

where $\nu = r - \delta - \frac{1}{2}\sigma^2$. The so-called "stretch factor" λ

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16 It is the opinion of the author that Ritchken's is the most versatile tree method for pricing many types of barrier options without being too logically and computationally complex.

17 The absence of arbitrage is not essential. If the distribution of the approximating discrete process converges to the continuous distribution, the option price given by the discrete model will converge to the continuous price.

appears as a free parameter of the geometry of the tree. Different values of λ give different convergence characteristics. λ essentially controls the size of the gap between layers of prices on the lattice. Note that for $\lambda = 1$, the trinomial method reduces to the binomial method. As in CRR (1979) the up and down factors are:

$$u = e^{\lambda\sigma\sqrt{\Delta t}} \text{ and } d = \frac{1}{u} . \qquad (3.03)$$

The idea of the Ritchken method is to choose the free parameter λ such that the barrier is always hit exactly. Then convergence will be the same as for standard options. This is done in the following way; first choose n_H such that:

$$n_H = \left[\frac{\ln\left(\frac{S_0}{H}\right)}{\sigma\sqrt{\Delta t}} \right] \qquad (3.04)$$

where $[q]$ denotes the greatest integer $\leq q$. Then,

$$\lambda = \frac{1}{n_H} \frac{\ln\left(\frac{S_0}{H}\right)}{\sigma\sqrt{\Delta t}} \qquad (3.05)$$

3.3 Implementation of Ritchken (1995) Method

Implementation of the Ritchken method is fairly straightforward, as it is simply a flat 3-node trinomial tree (Figure 9) with an additional precomputation of the stretch parameter λ and an adjustment for the existence of a barrier level, above or below which the option is only worth the contractual rebate.

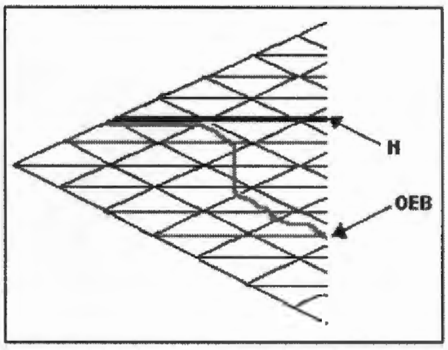


Figure 9: Trinomial lattice with barrier and optimal exercise boundary

3.3.1 Numerical Algorithm

In the algorithm, we first check how close the asset price is to the barrier. We need to ensure that at the given number of time steps, the resulting time step size is small enough to allow for the barrier to be hit exactly. If it is too close, then the barrier cannot be hit exactly at the given number of time steps and this number must be increased, or else the algorithm fails. This is the biggest disadvantage of this method. We then compute λ as defined in (3.05) to set a level of tree nodes at the barrier. Up and down factors are computed as in (3.03). The risk neutral probabilities are calculated as given in (3.02). The dynamic programming routine begins by specifying the terminal values of the option at expiry and computing option values at the other nodes via a backward cycle from time step N to zero. At each node the approximate value of an American-type option is given as the maximum between exercising that option at that node (intrinsic value) and the present value of not exercising but holding onto the option until the next time step. Since the tree is flat, the intrinsic values at all nodes can be computed at the beginning. This saves on computation complexity. The tree is split into two parts, in one part the barrier is active and the option is worth only the contractual rebate and in the other part the barrier is inactive and the option is valued as an ordinary American option. The numerical algorithm (generalized for either 'out' or 'in' barriers) is summarized as follows:

Algorithm 1.0: Ritchken (1995) Method

```
get user defined parameters (N,T, etc.)
check if asset price too close to barrier
calculate λ to set row of nodes along barrier
initialize up and down factors
calculate risk-neutral probabilities
calculate intrinsic and terminal values
for i=N:-1:0
   if node on one side of barrier
      option value = rebate
   else node on other side of barrier
      option value=max(exercising, PV of continuing)
   end if
end for
return price
```

4.0 Finite Difference Methods

In science and engineering, problems that can be modeled by differential equations are often solved using finite difference methods. These methods were first applied to option valuation problems by Brennan and Schwartz (1978), and extended by Courtadon (1982) and Geske and Shastri (1985). The key advantage of finite difference methods over trinomial methods is that they can easily accommodate boundary conditions and offer more flexibility in terms of improving performance by refining grid information near critical prices. More importantly, they offer better stability properties. For example, the Crank-Nicolson implicit method is unconditionally stable and converges quadratically whereas trinomial methods are stable on the condition that the time step is sufficiently small and only converge monotonically[18].

Finite difference methods obtain the option price by solving the underlying backward parabolic PDE (1.08) subject to the boundary conditions of the option pricing problem. We first transform (1.08) to canonical form as a forward parabolic PDE with constant coefficients by performing the change of variables $t \rightarrow T-t$, $x=\ln S$ and setting $u(t,x)=U\left(T-t,e^{x}\right)$. Then we have:

$$u_t + \left(r-\delta-\frac{\sigma^2}{2}\right)u_x + \frac{1}{2}\sigma^2 u_{xx} - ru = 0 \qquad (4.01)$$

The methodology behind finite difference methods is to solve (4.01) subject to the boundary conditions of the pricing problem by replacing the partial derivatives with finite difference approximations in time, $[0,T]$ and in space, $[-l,l]$. These finite differences are defined on a grid (Figure 10). The grid system is then solved numerically to obtain an approximate solution to the analytic equation. This is valid on the basis that as we make the grid finer and finer, the approximate solution given by the difference equations converges to the solution of the approximated PDE. The type of approximations (or discretizations) used for the partial derivatives determines the form of the finite difference scheme, e.g. explicit, Crank-Nicolson implicit, and fully implicit schemes. Each scheme has different consistency, stability and convergence characteristics. Here we shall

[18] Brennan and Schwartz (1978) show the equivalence of the trinomial lattice method to the explicit finite difference scheme. They both share the stability requirement that $\Delta t \leq \frac{\Delta x^2}{2}$.

implement the θ-method (Wilmot, Dewynne and Howison (1993), Lamberton and Lapeyre (1997)). The θ-method allows one to use either an explicit, Crank-Nicolson implicit or fully implicit finite difference scheme through the choice of parameter θ.[19]

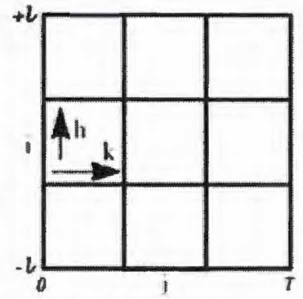


Figure 10: Typical finite difference grid

Once implemented, the finite difference scheme may then be solved using methods of solution such as Gauss factorization, Cryer's algorithm, projected successive over relaxation (PSOR), conjugate gradient (CG), biconjugate gradient stabilized (BCGST) and quasi-minimum residual (QMR) solvers. In a matrix based development environment such as Matlab, the scheme can be solved with a simple matrix inversion. This makes coding simpler but execution is rather slow. With non-matrix based development tools such as C, we have to resolve to one of the former solution techniques, but gain significant advantage in terms of execution speed. Here we shall investigate solution algorithms based upon Gauss factorization.

4.1 Option Pricing Problem Formulation

Consider the valuation of an up-and-out American put option. Since the option is American, its value can be formulated in terms of the following variational inequality:

$$\begin{cases} \max\left(\phi - u, \quad u_t + \left(r - \delta - \frac{\sigma^2}{2}\right)u_x + \frac{1}{2}\sigma^2 u_{xx} - ru\right) = 0, \ (t,x) \quad in \quad [0,T[\times\Omega \\ u(T,x) = \phi(x) \quad (t,x) \end{cases}$$

(4.02)

[19] Only the unconditionally stable Crank-Nicolson implicit scheme is investigated here.

with a Dirichlet boundary condition $u = R$ on $[0,T] \times \partial\Omega$ where $\Omega =]x-l, H[$, ϕ is the option payoff function, H is the barrier level and R is the rebate (which is often zero). The formulation of American option optimal stopping problems as variational inequalities was studied by Jaillet, Lamberton and Lapeyre (1990). Intuitively, expression (4.02) states that:

- The option value u cannot fall below the intrinsic value or payoff ϕ or else it is exercised.
- If the option value is above the intrinsic value then the option price is described by the Black-Scholes PDE as if it were a European option.
- The option value at expiry is its payoff ϕ which may be equal to the option's intrinsic value or the rebate.

To find a solution, we discretize (4.02) in space and time using a θ-method and then solve it using Gauss factorization.

4.2 θ-Method

First define the discretization in space as:

$$x_i := x - l + \frac{2i}{M}, for \quad 1 \le i \le M-1 \tag{4.03}$$

and choose M depending on how fine we want the discretization to be. For a uniform grid, we will choose the number of space steps M to be equal to the number of time steps N, e.g. $M = N = 100$. l defines the localization interval for the scheme. It is chosen via a probabilistic estimate and is approximately equal to 2 to 3 times the current spot price. We then discretize the differential operator A_h where:

$$A_h u_h(t,x_i) = \frac{\sigma^2}{2} u_{h_{xx}}(t,x_i) + \left(r - \delta - \frac{\sigma^2}{2}\right) u_{h_x}(t,x_i) - r u_h(t,x_i) \tag{4.04}$$

with:

$$\begin{aligned} u_{h_{xx}}(t,x_i) &= \frac{1}{h^2}\left(u_h(t,x_{i+1}) - 2u_h(t,x_i) + u_h(t,x_{i-1})\right) \\ u_{h_x}(t,x_i) &= \frac{1}{2h}\left(u_h(t,x_{i+1}) - u_h(t,x_{i-1})\right) \end{aligned} \tag{4.05}$$

where the functions $u_h(t,\cdot)$ are defined on the spatial grid covered by (4.03).

Then we may discretize (4.02) as:

$$\begin{cases} \max\left(\phi_h - u_h, \quad \dfrac{u_h^{n+1} - u_h^n}{k} + A_h\left(u_h^{n+1} + \theta\left(u_h^n - u_h^{n+1}\right)\right)\right) = 0 \quad for \quad 0 \le n \le N-1 \\ u_h^N = \phi_h \end{cases}$$

(4.06)

Where k is a fixed time discretization step such that $T = N * k$.
The boundary conditions (for our single barrier up-and-out put) are:

$$\begin{cases} u_h^n(x-l) = \phi(x-l) \\ u_h^n(H) = R \end{cases} \qquad (4.07)$$

The θ-method can be thought of as a θ-weighted average of the explicit and fully implicit finite difference methods. When $\theta = 0$, it gives the explicit method, when $\theta = 0.5$ it gives Crank-Nicolson and when $\theta = 1$ it gives the fully implicit method.

4.3 Implementation of θ-Method

When $0 < \theta \le 1$ we get an implicit method. We then have to solve at each time step, a linear system of the form:

$$\mathbf{M} u_{k,h}(jk,\cdot) = \mathbf{N} u_{k,h}((j+1)k,\cdot) \qquad (4.08)$$

where $\mathbf{M}$ and $\mathbf{N}$ are tridiagonal matrices of the form:

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & 0 & 0 & \cdots & a_M & b_M \end{pmatrix} \qquad (4.09)$$

where we derive $\mathbf{M}$ as:

$$a_i = \theta k\left(\frac{b}{2h} - \frac{\sigma^2}{2h^2}\right), \quad b_i = 1 + \theta k\left(r + \frac{\sigma^2}{h^2}\right), \quad c_i = -\theta k\left(\frac{b}{2h} + \frac{\sigma^2}{2h^2}\right) \qquad (4.10)$$

and $\mathbf{N}$ as:

$$a_i = (1-\theta)k\left(-\frac{b}{2h}+\frac{\sigma^2}{2h^2}\right),\quad b_i = 1-(1-\theta)k\left(r+\frac{\sigma^2}{h^2}\right),\quad c_i = (1-\theta)k\left(\frac{b}{2h}+\frac{\sigma^2}{2h^2}\right) \tag{4.11}$$

where all the pivots b_i are non-zero and $b = r-\delta-\frac{\sigma^2}{2}$.

We notice that (4.08) reduces to a linear system:

$$\mathbf{M}u = v \tag{4.12}$$

where u and v are $\mathbf{M}$-dimensional vectors, hence we may find a solution using Gauss factorization.

Gauss factorization is based on the fact that matrix $\mathbf{M}$ can be decomposed into a lower triangular matrix $\mathbf{L}$ and an upper triangular matrix $\mathbf{U}$.

$$\mathbf{M} = \mathbf{LU} \tag{4.13}$$

Then to find a solution to (4.12), we solve a linear system of the form $\mathbf{LU}z = v$ by decomposing it into $\mathbf{L}y = v$ and $\mathbf{U}z = y$ and then solving these easier problems separately.

4.3.1 Numerical Algorithm

The numerical algorithm may be described in the following way:

Algorithm 2.0: θ-method with Gauss Factorization

```
define time step
define integration domain determined by barrier
define space step
initialize lhs of (4.08)
initialize rhs of (4.08)
initialize terminal/payoff values
set Dirichlet boundary condition on barrier
for i=N:-1:0
   solve linear system (4.08) using Gauss factorization
   where option value=max(payoff,sol. to linear system)
end for
return price
```

5.0 Finite Element Methods

Finite difference methods have advantages in accuracy, stability and convergence over trinomial tree methods. However even for these methods, if the spot price is close to the barrier level, the discontinuity due to the incompatibility between the initial condition (option payoff) and the boundary condition (rebate) typically results in a loss in accuracy. The Crank-Nicolson method, although unconditionally stable, may suffer from finite oscillations at points of discontinuity, due to inaccuracies stemming from the approximation of the time derivative.[20] Furthermore, although this case isn't investigated here, if the barrier is time-dependent or not straight, finite difference methods again lose accuracy, as any misalignment between nodes and the barrier will cause errors. So there is a need for a method that naturally allows for greater freedom in the placement of grid nodes so as to enable us to concentrate nodes near barriers and if needed, place nodes to track barriers of any shape. A finite element method (FEM) provides us with such flexibility.

There are ways to improve the performance of finite difference methods, such as by using coordinate transformations. As described in Tavella and Randall (2000), these are used to warp the spatial grid in such a way that critical prices such as strike or barrier prices fall exactly on grid points. They may also be used to concentrate grid points near critical prices and even generate grids that can track moving barriers. These extensions to the standard finite difference methods essentially use finite differences in time and finite elements in space and as a result give improved accuracy only at the expense of a very large number of discretization points. We would be more interested in a *space-time* finite element method which provides greater accuracy at low additional computational cost.

The basic ideas of the finite element method as it is known today were presented in Turner, *et al.* (1956) in the context of aeronautical engineering. The method is used extensively in structural mechanics and computational fluid dynamics but isn't very well known in the finance literature. Essentially the finite element method is a procedure for approximating the solution to an entire domain from local approximations to that solution on smaller sub-domains called elements. These local approximations are often simple linear functions. With a continuity requirement that these approximating functions

[20] For this reason, we explore an oscillation-free fully implicit finite element method here, although it is only first order accurate in time, while the Crank-Nicolson method is second order accurate. The sacrifice in speed is small relative to the improvement in accuracy.

match each other at element nodes, the solution is approximated over the entire domain. It is easy to see that the size of the elements influences the convergence of the solution, if the elements are small, then the final solution is expected to be more accurate. A common strategy in applying the finite element method is to first get an integral formulation of the problem, subject to a stationary requirement (e.g. minimization) over the solution domain. This is called a variational form and the integral equation is called a functional. In the finite difference method a differential equation is approximated over a set of discrete points using finite differences, in the finite element method we obtain a solution by minimizing the associated functional over a set a discrete domains or elements.

Finite difference methods are characterized by grids made up of equally spaced rectangular elements. In contradistinction, the space-time finite element method used here involves trapezoidal elements that are unequally placed in space. This makes it easier to place nodes such that barriers are met exactly as well as refine meshes near critical prices such as strikes and barriers (Figure 11). Such a method is suggested in Busca (1998) for vanilla American options.

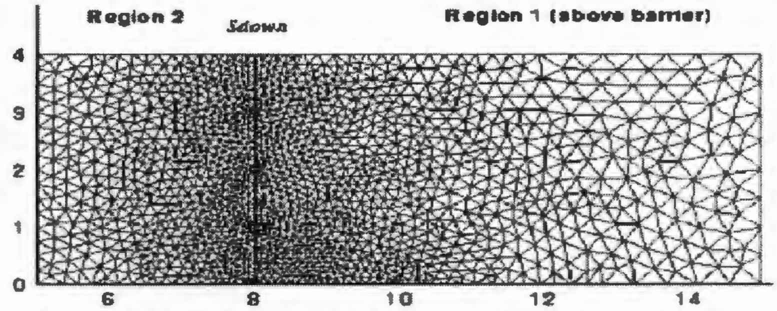


Figure 11: Typical FE scheme with element refinement near barrier (38)

5.1 Option Pricing Problem Formulation

We present the same option pricing problem as in the finite difference case of an up-and-out American put option. Since the option is American, its value can be formulated in terms of the following variational inequality:

$$\begin{cases} \min\left(u-\phi, \quad -u_t - \left(r-\delta-\frac{\sigma^2}{2}\right)u_x - \frac{1}{2}\sigma^2 u_{xx} + ru \right) = 0, \ (t,x) \quad in \quad [0,T[\times\Omega \\ u(T,x)=\phi(x) \quad (t,x) \end{cases}$$

(5.01)

with a Dirichlet boundary condition $u=R$ on $[0,T]\times\partial\Omega$ where $\Omega=]x-l,H[$, ϕ is the option payoff function, H is the barrier level and R is the rebate (which is often zero).

The numerical algorithm for the solution of (5.01) using space-time finite elements is identical to the finite difference method, except for the way in which we discretize (5.01).

5.2 Finite Element Method

In the finite element method one does not approximate the derivatives occurring in (5.01). Instead, we obtain a weak integral equivalent (variational formulation) of the variational inequality (5.01) and then approximate the integrals over the solution domain Ω. From the calculus of variations we know that the solution will be given by the function that minimizes the functional in the variational formulation.

5.3 Implementation of Finite Element Method

Our fixed localization interval for the problem is defined above as $\Omega=]x-l,H[$. The localized solution is close to the true solution when $H-(x-l)$ is large compared to some relevant scale of the problem. It is customary in option pricing to take l to be 2 or 3 times the value of x, or to use some probabilistic estimate such that the probability that x lies outside the interval at some time $s\in[0,T]$ is less than some small number ε. To get the weak variational form of the parabolic equation (5.01) we transfer the differentiation from the dependent variable u to the trial function or approximate solution $\widetilde{u}=u\phi$, where ϕ is a regular test function that satisfies the boundary conditions of the problem (i.e. ϕ vanishes on $\{x=-l\}$ and $\{x=H\}$). We then integrate on the set $C_n=[t^n,t^{n+1}]\times[x-l,H]$, which covers the domain of the problem to get:

$$
\begin{aligned}
&-\int_{C_n} u\partial_t\phi+\frac{\sigma^2}{2}\int_{C_n}\partial_x u\partial_x\phi+\left(r-\frac{\sigma^2}{2}\right)\int_{C_n} u\partial_x\phi+r\int_{C_n} u\phi \\
&+\int_{x-l}^{H} u(t^{n+1},x)\phi(t^{n+1},x)dx-\int_{x-l}^{H} u(t^n,x)\phi(t^n,x)dx=0
\end{aligned}
\tag{5.02}
$$

The next step is to discretize the functional (5.02). To do that first define a mesh made up of non-uniform quadrilateral elements E_i^n. We choose the time step $k = \Delta t = t^{n+1} - t^n$. The quadrilateral elements making up our mesh have four corners $P_i^n, P_{i+1}^n, P_{i+1}^{n+1}, P_i^{n+1}$. For any point $P \in E_i^n$ inside an element we use the coordinates $(\xi, \eta) \in [0,1]^2$ defined by: $\xi = \frac{t - t^n}{k}$ (fractional passage in time) and $\eta = \frac{x - x_i^{n+\xi}}{x_{i+1}^{n+\xi} - x_i^{n+\xi}}$ (time-dependent, proportional change in the space direction), where $x_i^{n+\xi} = (1-\xi)x_i^n + \xi x_i^{n+1}$ lies on the line segment between x_i^n and x_i^{n+1} in the time direction. Using this notation a point $P = (t,x) \in E_i^n$ with coordinates (ξ, η) is written as $P = x_{i+\eta}^{n+\xi}$. For every function φ defined in one of the arbitrary quadrilateral elements E_i^n, we can associate an equivalent function $\hat{\varphi}$ defined on the square $[0,1]^2$ such that $\hat{\varphi}(\xi,\eta) = \varphi(t,x)$.

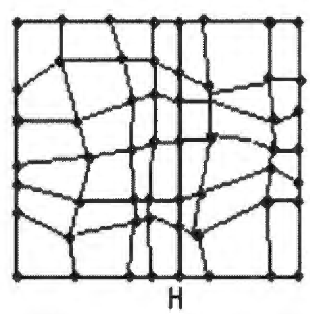


Figure 12: Trapezoidal element mesh

Now in order to transform the volume and boundary integrals appearing in (5.02) from the (t,x) space of arbitrarily shaped quadrilateral elements (Figure 12) to (ξ,η) space where the square $[0,1]^2$ with vertices $(P_s)_{1 \le s \le 4}$ is defined, we introduce the following quadrature rules[21]:

For the volume integral,

$$I_{E_i^n}(\varphi) = \int_{E_i^n} \varphi(t,x)dt = \int_{[0,1]^2} \hat{\varphi}(\xi,\eta)\hat{J}_{in}(\xi,\eta)d\xi d\eta = \frac{1}{4}\sum_{s=1}^{4} \hat{\varphi}\hat{J}_{in}(P_s) \qquad (5.03)$$

[21] Quadrature is the process of finding a square equal in area to a given area of arbitrary shape. We use this here as the finite element sub-domains are unevenly spaced as shown in Figure 12.

where the Jacobian term,

$$\hat{J}_{in} = \frac{\partial x}{\partial \eta}\frac{\partial t}{\partial \xi} - \frac{\partial x}{\partial \xi}\frac{\partial t}{\partial \eta}$$

and for the boundary integral,

$$I_{t^n}(\varphi) = \int_{x_i^n}^{x_{i+1}^n} \varphi(t^n, x)dx = \left(\frac{x_{i+1}^n - x_i^n}{2}\right)\left(\varphi(t^n, x_i^n) + \varphi(t^n, x_{i+1}^n)\right) \qquad (5.04)$$

We can then use these notations to rewrite the volume and boundary integrals in (5.02) to get:

$$\sum_{i=0}^{N}\left(\begin{array}{l} -I_{E_i^n}u\partial_t\phi + \frac{\sigma^2}{2}I_{E_i^n}\partial_x u\partial_x\phi + \left(r - \frac{\sigma^2}{2}\right)I_{E_i^n}u\partial_x\phi + rI_{E_i^n}u\partial_x\phi \\ + I_{t^{n+1}}\left(u(t^{n+1},x)\phi(t^{n+1},x)\right) - I_{t^{n+1}}\left(u(t^n,x)\phi(t^n,x)\right)\end{array}\right) = 0 \qquad (5.05)$$

We then choose linear interpolation functions $\hat{\varphi}$ which are an approximation to the solution u, *restricted* to each element E_i^n. These functions belong to the space of test functions: $\left\{\phi(t,x)\middle|\hat{\phi}(\xi,\eta) = \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi_\eta\right\}$. The interpolation functions satisfy a continuity requirement between elements: $\hat{\varphi}_{in}(\xi,\eta) = \xi(1-\eta)$ in E_i^n, $\hat{\varphi}_{in}(\xi,\eta) = \xi\eta$ in E_{i-1}^n and $\hat{\varphi} = 0$ elsewhere.

We can then compute the derivative expressions of the linear interpolation functions $\hat{\varphi}$ for $(t,x) \in E_i^n$:

$$\partial_t\hat{\varphi}_{in} = \frac{1}{k}(1-\eta) + \frac{\xi}{k}\frac{x_{i+\eta}^{n+1} - x_{i+\eta}^n}{x_{i+1}^{n+\xi} - x_i^{n+\xi}}, \text{ and}$$

$$\partial_x\hat{\varphi}_{in} = -\frac{\xi}{x_{i+1}^{n+\xi} - x_i^{n+\xi}}$$ and similarly for $(t,x) \in E_{i-1}^n$:

$$\partial_t\hat{\varphi}_{in} = \frac{\eta}{k} + \xi\left(-\frac{1}{k}\frac{x_{i-1+\eta}^{n+1} - x_{i-1+\eta}^n}{x_i^{n+\xi} - x_{i-1}^{n+\xi}}\right),$$ and $$\partial_x\hat{\varphi}_{in} = -\frac{\xi}{x_i^{n+\xi} - x_{i-1}^{n+\xi}}.$$ For the Jacobian term we have: $\hat{J}_{in} = k\left(x_{i+1}^{n+\xi} - x_i^{n+\xi}\right)$ in E_i^n and $\hat{J}_{in} = k\left(x_i^{n+\xi} - x_{i-1}^{n+\xi}\right)$ in E_{i-1}^n.

We can then use these to evaluate the $I_{E_i^n}, I_{E_{i-1}^n}, I_{t^n}, I_{t^{n+1}}$ terms in (5.05), after applying the quadrature rules (5.03) and (5.04).

For example:

$$I_{E_i^n}(u\partial_x\varphi_{in}) = \frac{1}{4}\sum_{s=1}^{4}\hat{u}\partial_x\hat{\varphi}_{in}\hat{J}_{in}(P_s) = \frac{1}{4}\sum_{\xi,\eta\in\{0,1\}} u_{i+\eta}^{n+\xi}(-\xi)k = -\frac{k}{4}\left(u_i^{n+1} - u_{i+1}^{n+1}\right) \tag{5.06}$$

and,

$I_{E_{i-1}^n}(u\partial_x\varphi_{in}) = \frac{1}{4}\sum_{\xi,\eta\in\{0,1\}} u_{i-1+\eta}^{n+\xi}(\xi)k = \frac{k}{4}\left(u_{i-1}^{n+1} - u_i^{n+1}\right)$, hence the volume integral $I_{C_n}(u\partial_x\varphi_{in}) = I_{E_i^n} + I_{E_{i-1}^n} = -\frac{k}{4}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right)$.

Similarly for the other integrals, one gets the following expressions:

$$I_{C_i^n}(u\varphi_{in}) = \frac{k}{4}u_i^{n+1}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right)$$

$$I_{C_i^n}(u\partial_x\varphi_{in}) = -\frac{k}{4}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right)$$

$$I_{C_i^n}(\partial_x u\partial_x\varphi_{in}) = -\frac{k}{2}\left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{x_{i+1}^{n+1} - x_i^{n+1}} - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{x_i^{n+1} - x_{i-1}^{n+1}}\right)$$

$$I_{t^{n+1}}(u\varphi_{in}) = \frac{1}{2}u_i^{n+1}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right), \quad I_{t^n}(u\varphi_{in}) = 0$$

$$I_{C_i^n}(u\partial_t\varphi_{in}) = \frac{1}{4}\left(u_i^n\left(x_{i+1}^n - x_{i-1}^n\right) + u_{i+1}^{n+1}\left(x_{i+1}^{n+1} - x_{i+1}^n\right) + u_i^{n+1}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right) - u_i^{n+1}\left(x_{i+1}^{n+1} - x_{i-1}^n\right)\right) \tag{5.07}$$

We then substitute (5.07) into (5.05) to get the finite element discretization of the underlying PDE:

$$u_i^n\left[\frac{1}{4}\left(x_{i+1}^n - x_{i-1}^n\right)\right] = u_{i-1}^{n+1}\left[\left(x_{i-1}^{n+1} - x_{i-1}^n\right) + \frac{\sigma^2 k}{4}\left(\frac{1}{x_i^{n+1} - x_{i-1}^{n+1}}\right) + \frac{k}{4}\left(r - \frac{\sigma^2}{2}\right)\right]$$
$$+ u_i^{n+1}\left[\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right) + \frac{1}{x_{i+1}^{n+1} - x_i^{n+1}} + \frac{1}{x_i^{n+1} - x_{i-1}^{n+1}} + \frac{1}{2}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right) + \frac{kr}{4}\left(x_{i+1}^{n+1} - x_{i-1}^{n+1}\right)\right]$$
$$- u_{i+1}^{n+1}\left[\left(x_{i+1}^n - x_{i-1}^n\right) + \frac{\sigma^2 k}{4}\left(\frac{1}{x_{i+1}^{n+1} - x_i^{n+1}}\right) - \frac{k}{4}\left(r - \frac{\sigma^2}{2}\right)\right] \tag{5.08}$$

For $i = 1, 2, \ldots N-1$ (5.08) reduces to a system of linear algebraic equations in the matrix form:

$$\mathbf{M}u = v \qquad (5.09)$$

where u and v are $\mathbf{M}$-dimensional vectors, therefore we can pursue a solution to (5.09) using the same Gauss pivoting algorithm as for the finite difference scheme.

5.3.1 Numerical Algorithm

The numerical algorithm for the finite element method is the same Gauss pivoting algorithm as for the finite difference method.

```
---------------------------------------------------
Algorithm 3.0: FEM-method with Gauss Factorization
---------------------------------------------------
define time step
define integration domain determined by barrier
define space step
initialize lhs of (5.05)
initialize rhs of (5.05)
initialize terminal/payoff values
set Dirichlet boundary condition on barrier
for i=N:-1:0
   solve linear system (5.05) using Gauss factorization
   where option value=max(payoff,sol. to linear system)
end for
return price
```

6.0 Other Valuation Methods[22]

The set of methods discussed above is not exhaustive, there are other valuation methods for barrier options that fall outside of the categories investigated in this dissertation. Duan, *et al* (1999) propose a method for pricing discretely monitored constant and time varying barrier options using a Markov chain. Their method reduces the pricing of American and European barrier options to simple matrix operations and in their paper they claim that this method is fast, flexible and easy to implement. Lyuu (1998) presents fast and simple algorithms for the valuation of European-style barrier options using combinatorial methods. Gao, Huang and Subrahmanyam (1999) price American barrier options using a decomposition technique where they separate the European option value from the early exercise premium to obtain an analytical representation of the option price and hedge parameters. Also worth noting is the work by Haug (2000) where he develops closed-formulae for the analytical valuation of certain types of American barrier options.

[22] These could not be included in the study because of time constraints.

7.0 Comparative Analysis of Algorithms

The principal means for comparing algorithms investigated here is the method where algorithms are compared on the basis of how long they take to achieve results of comparable accuracy. Lyuu (1998) makes a strong case for this approach. Following in the spirit of the work by Patterson and Hennessy (1994), Lyuu illustrates the pitfalls of the conventional method of comparing algorithms on the basis of the n at which they converge. In the context of this project, n may be the number of time partitions in a trinomial lattice for example. This is the most common method used in the extant literature, with a few exceptions such as Broadie and Detemple (1996). The flaw of this technique stems from the fact that it ignores computational efficiency; an algorithm may take a larger n to converge to the accurate value but may do so in a shorter time period than the method that takes a smaller n to converge. So clearly the only objective method of comparison is the time taken to achieve comparable accuracy, regardless of the n at which that accuracy is achieved. This is because convergence to the accurate value at lower n doesn't *necessarily* translate to convergence to the accurate value in the fastest time. For completeness, we will provide statistics for both the convergence rate and the convergence time. To compare the numerical accuracy of algorithms we compute the absolute relative error e for each approximate value given by an algorithm at the various values of n, the number of time partitions. We define the absolute relative error as:

$$e = \frac{\left|\overline{U} - U\right|}{U} \tag{7.01}$$

where $\overline{U}$ is the approximate option value given by the algorithm and U is the 'accurate value'. Where a closed-form solution is not available, the accurate value is taken as the approximate value given by the benchmark method. The benchmark method is chosen as the method with the least e in the pricing of European style barriers, where the accuracy can be reliably compared to the value given by a closed formula.

In addition to the 'computational analysis' of investigating the numerical accuracy, convergence behavior and speed of the three methods, we also perform a 'sensitivity analysis' where we assess the sensitivity of each method to the magnitude of the American option premium and also to the distance of the underlying price to the barrier level. The American option premium is the difference between the benchmark American and European option values. Testing the methods for cases where the American option premium is large gives us a good idea of how the algorithms cope with the presence of the American early exercise feature. For the case of the distance of the underlying price to the barrier, we note that in general, pricing and hedging barrier options very close to the barrier is problematical and consequently a matter of keen research interest. We will investigate how each algorithm performs in terms of giving price as well as delta values for situations where the underlying spot price is very close to the barrier.

7.1 Computational Analysis[23]

7.1.1 Pricing European Barrier Options: Establishing a Benchmark

To establish the benchmark method, we look at the absolute relative errors for each method in the pricing of an Up and Out European put option with the following parameters:

Spec 1

Spot Price = 100
Strike Price = 100
Volatility = 0.15 per year
T = 1 year
r = 0.1 per year
Barrier Level = 110

The accurate value as given by the closed form evaluation is 3.20136.

Table 1

Time Part-itions (n)	(Tree) Answer	(Tree) Abs. Rel. Error	(Tree) CPU Time (s)	(FDM) Answer	(FDM) Abs. Rel. Error	(FDM) CPU Time (s)	(FEM) Answer	(FEM) Abs. Rel. Error	(FEM) CPU Time (s)
50	3.20084	1.6180	0	3.20532	12.379	0.12	3.20110	7.9292	0
100	3.20174	1.1921	0	3.20086	1.5578	0.02	3.20163	8.2986	0.01
200	3.20120	0.4864	0	3.20111	0.7764	0	3.20144	2.5846	0.02
400	3.20143	0.2484	0	3.20132	0.1169	0.01	3.20137	0.2198	0.20
800	3.20139	0.1047	0	3.20136	0.0038	0.06	3.20136	0.1115	0.35
1000	3.20143	0.2342	0.01	3.20136	0.0097	0.08	3.20136	0.1352	0.55
1500	3.20139	0.0972	0.02	3.20135	0.0323	0.19	3.20136	0.0509	1.27
2000	3.20138	0.0759	0.04	3.20134	0.0500	0.33	3.20136	0.0743	2.38
2500	3.20137	0.0383	0.07	3.20135	0.0405	0.53	3.20136	0.0871	3.81
5000	3.20136	0.0042	0.20	3.20135	0.0420	2.21	3.20136	0.0829	17.44
7500	3.20136	0.0117	0.47	3.20135	0.0438	5.02	3.20136	0.0848	41.09

From Table 1 we see that the finite difference method (FDM) is the most accurate method, having the lowest relative error of 0.0038 at the accurate value of 3.20136. Second is the Ritchken tree method (RTM) with a relative error of 0.0042 and lastly the finite element

[23] All computations done on an AMD K6 500 MHz processor running Windows 2000.

method (FEM) with 0.0509. The FDM method converges fastest, converging to the accurate value at 800 time partitions. Second is the FEM method, and the RTM method converges slowest. In terms of computational speed the FDM is again fastest, reaching the accurate value in 0.06 seconds followed by the RTM at 0.2 seconds and the FEM at 1.27 seconds. Judging from its superiority to the other methods in computational speed, accuracy and rate of convergence, the FDM method is selected as the benchmark (using at least 7500 time steps).

7.1.2 Pricing American Barrier Options

7.1.2.1 Single Barriers

With the FDM method as the benchmark we then investigate the numerical accuracy, computational speed and convergence rate of each method in pricing the American version of the option in Spec 1. The 'accurate value' given by the benchmark is 3.687.

Table 2

Time Partitions (n)	(Tree) Answer	(Tree) Abs. Rel. Error	(Tree) CPU Time (s)	(FDM) Answer	(FDM) Abs. Rel. Error	(FDM) CPU Time (s)	(FEM) Answer	(FEM) Abs. Rel. Error	(FEM) CPU Time (s)
50	3.688	2.743	0	3.682	1.374	0	3.718	8.385	0.01
100	3.689	5.191	0.01	3.682	1.389	0	3.703	4.319	0
200	3.687	1.059	0	3.684	0.710	0	3.695	2.142	0.03
400	3.687	0.822	0	3.686	0.352	0.02	3.691	1.047	0.46
800	3.687	0.238	0.01	3.686	0.191	0.06	3.689	0.505	0.46
1000	3.687	0.279	0.02	3.686	0.162	0.12	3.688	0.394	0.71
1500	3.687	0.011	0.03	3.687	0.123	0.23	3.688	0.245	1.61
2000	3.687	0.086	0.90	3.687	0.104	0.41	3.688	0.177	2.94
2500	3.687	0.164	0.10	3.687	0.091	0.67	3.688	0.133	4.96
5000	3.687	0.290	0.40	3.687	0.067	2.80	3.687	0.045	20.11
7500	3.687	0.327	0.97	3.687	0	6.57	3.687	0.016	52.00

From the results in Table 2, the RTM method performs better at accommodating the early exercise feature of the option in it's algorithm. It converges to the accurate value at the lowest number of time steps and does so in the shortest amount of time. The FDM method takes longer to converge (in terms of elapsed time and number of time steps) and the FEM method converges very slowly.

7.1.2.2 Double Barriers

We then compare the performance of the algorithms in pricing an American double barrier out option. Such an option expires worthless if either of two barrier levels enclosing the current spot price is crossed. Here we test with a double barrier put option with a lower barrier of 80 and an upper barrier of 120. All other spec parameters are as in Spec 1. The accurate value of such an option is 4.203.

Table 3

Time Part-itions (n)	(Tree) Answer	(Tree) Abs. Rel. Error	(Tree) CPU Time (s)	(FDM) Answer	(FDM) Abs. Rel. Error	(FDM) CPU Time (s)	(FEM) Answer	(FEM) Abs. Rel. Error	(FEM) CPU Time (s)
50	4.458	0.061	0	4.191	0.003	0	4.243	0.010	0
100	17.88	3.255	0	4.198	0.002	0	4.223	0.005	0.05
200	4.556	0.083	0	4.201	0.001	0.01	4.213	0.002	0.03
400	4.658	0.108	0	4.202	0.0003	0.01	4.208	0.001	0.11
800	4.560	0.085	0.01	4.203	0	0.07	4.206	0.001	0.44
1000	4.565	0.086	0	4.203	0	0.14	4.205	0.001	0.68
1500	4.537	0.079	0.01	4.203	0	0.23	4.204	0	1.55
2000	4.572	0.088	0.02	4.203	0	0.41	4.204	0	2.84
2500	4.509	0.073	0.04	4.203	0	0.65	4.204	0	4.47
5000	4.495	0.070	0.11	4.203	0	2.78	4.204	0	18.76
7500	4.432	0.054	0.11	4.203	0	7.74	4.203	0	49.02
500000	4.255	0.012	123.00	-	-	-	-	-	-

The RTM method fails to price the double barrier option at a reasonable accuracy even for an extremely large number of time steps. Its computation of the delta (Figure 13), given that the price values are inaccurate, is also poor.

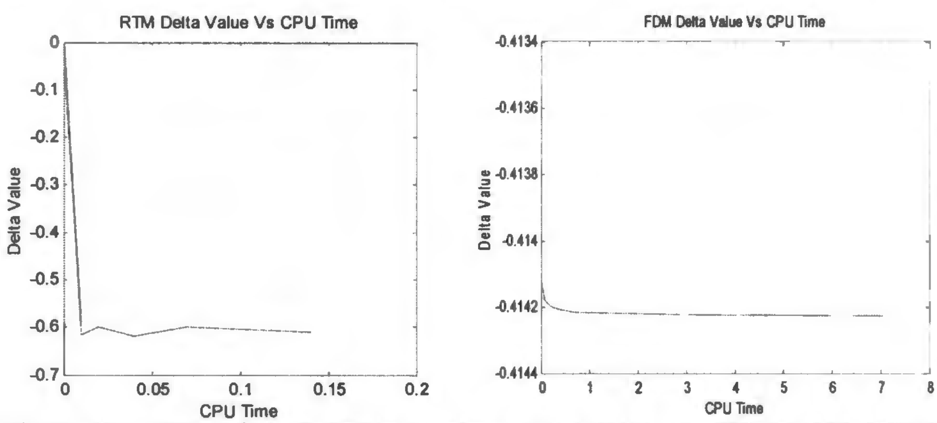


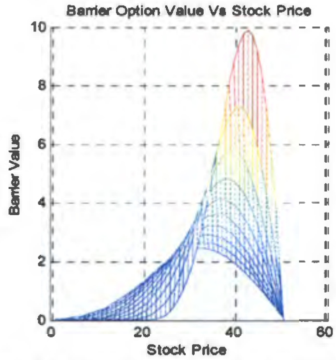
Figure 13: Computation of double barrier option delta for RTM and FDM methods

The FEM method performs better for double barriers than for the previous case of the single barrier option. It matches the FDM method for numerical accuracy but not on the level of computational speed.

7.2 Sensitivity Analysis

7.2.1 Sensitivity to American Option Premium

In performing our sensitivity analysis we first look at how each algorithm performs when the American early exercise premium is particularly high. For this purpose we investigate the pricing of an up and out American call option that is deep in the money (i.e. the spot price is much higher than the strike). For such an option, there is a big difference between the American option price and the corresponding European price. Because the option is deep in the money and has a high risk of knocking out, the American early exercise premium is very high. We can see this from the payoff patterns of the American and European versions of such an option (Figure 14).



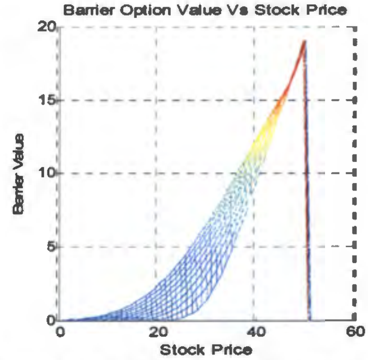


Figure 14: American option premium on an up and out call (barrier=50)

From Figure 14 we can see that at a stock price of 45, with a barrier level of 50, the European option only has a value of 8 while the American option has a price of 15.

To test the algorithms we will use the same Spec 1 as before, but for an American up and out call option that is deep in the money (spot price = 105). The accurate value given by the benchmark is 8.0.

Table 4

Time Part-itions (n)	(Tree) Answer	(Tree) Abs. Rel. Error	(Tree) CPU Time (s)	(FDM) Answer	(FDM) Abs. Rel. Error	(FDM) CPU Time (s)	(FEM) Answer	(FEM) Abs. Rel. Error	(FEM) CPU Time (s)
50	6.8	0.1517	0	5.8	0.2692	0	5.8	0.2694	0
100	7.3	0.0883	0	6.5	0.1925	0	6.5	0.1906	0.01
200	7.5	0.0586	0	6.9	0.1352	0.01	6.9	0.1361	0.02
400	7.8	0.0301	0	7.2	0.0946	0.01	7.2	0.0955	0.13
800	7.9	0.0163	0.01	7.5	0.0648	0.06	7.5	0.0653	0.43
1000	7.9	0.0118	0.01	7.5	0.0569	0.12	7.5	0.0573	0.68
1500	8.0	0.0029	0.04	7.6	0.0444	0.23	7.6	0.0446	1.54
2000	8.0	0.0008	0.06	7.7	0.0367	0.41	7.7	0.0367	2.79
2500	8.0	0.0024	0.12	7.7	0.0312	0.65	7.7	0.0314	4.44
5000	8.0	0.0084	0.45	7.9	0.0174	2.77	7.9	0.0175	18.9
7500	8.0	0.0113	1.07	8.0	0	6.47	8.0	0.0111	49.4

The RTM method again emerges as the best performer as it proves to be the least sensitive to the magnitude of the American early-exercise premium. It converges quickest to the accurate value, doing so at only 2000 time steps with a very low relative error of 0.0008 (Figure 15). It is also the fastest method when we compare the times taken to achieve comparable accuracy, taking 0.06 seconds compared to 6.47 and 49.4 seconds for the FDM and FEM methods respectively. The slow converging FEM method and the FDM method in particular, show significant falls in their rates of convergence and computational speeds in the presence of a high American early exercise premium. The

FDM method converged at 800 time steps for the European option (0.06 seconds) but at 7500 for the American option with a high premium (6.47 seconds).

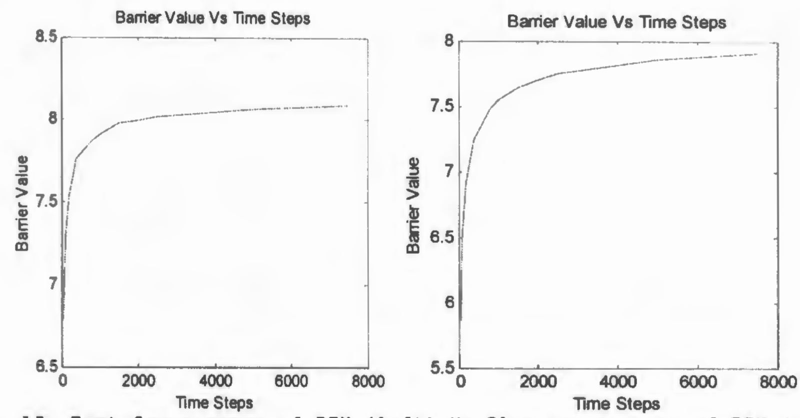


Figure 15: Fast Convergence of RTM (left) Vs Slow convergence of FEM (right)

7.2.2 Sensitivity to Distance from the Barrier

Perhaps the most interesting concern when evaluating barrier option pricing algorithms is how they perform when the spot price is very close to the barrier. From the price and delta curves in Section 2, we saw that the price curve is non-smooth across the barrier and the delta values get extremely large. To test algorithm performance close to the barrier we use the same Spec 1 as before and price an up and out American put with a barrier of 110 and a spot price of 109.5. The accurate value for the option is 0.1454.

<u>Table 5: Price Performance near Barrier</u>

Time Part-itions (n)	(Tree) Answer	(Tree) Abs. Rel. Error	(Tree) CPU Time (s)	(FDM) Answer	(FDM) Abs. Rel. Error	(FDM) CPU Time (s)	(FEM) Answer	(FEM) Abs. Rel. Error	(FEM) CPU Time (s)
50	?	?	?	0.1477	0.0155	0	0.1478	0.0164	0
100	?	?	?	0.1458	0.0027	0	0.1462	0.0055	0.01
200	?	?	?	0.1455	0.0006	0	0.1458	0.0028	0.03
400	?	?	?	0.1454	0	0.02	0.1456	0.0014	0.13
800	?	?	?	0.1454	0	0.07	0.1455	0.0006	0.45
1000	?	?	?	0.1454	0	0.13	0.1455	0.0006	0.70
1500	0.1454	0.0001	0.04	0.1454	0	0.23	0.1455	0.0004	1.64
2000	0.1454	0.0001	0.06	0.1454	0	0.42	0.1454	0.0003	2.92
2500	0.1454	0.0001	0.12	0.1454	0	0.73	0.1454	0.0003	4.64
5000	0.1454	0.0001	0.40	0.1454	0	3.12	0.1454	0.0002	19.69
7500	0.1454	0.0001	0.95	0.1454	0	8.43	0.1454	0.0001	51.27

The RTM method fails to price the American barrier option close to the barrier when the number of time partitions is less than 1500. This is because if the spot price is very close to the barrier, ensuring that a row of nodes is hit precisely will require a very small time step. For the time step size to be small, a very large number of time partitions is required[24]. In this case, the minimum number is 1500. As the trinomial method is merely an explicit finite difference scheme (Brennan and Schwartz (1978)), it is subject to a stability requirement governing the size of the time step i.e. $\Delta t \leq \frac{\Delta x^2}{2}$. Figure 17 shows an example of the unpredictable behavior of the algorithm when the stability requirement is not met. Even then the RTM method computes the accurate value in a relatively short 0.04 seconds, although the FDM method does so in half the time at 0.02 seconds. The speed and convergence of the FDM seems to be better closer to the barrier, compared to the results in Table 4. The FEM method continues to perform poorly by comparison, taking the longest time to achieve an accuracy comparable to that of the other RTM and FDM methods (Figure 16).

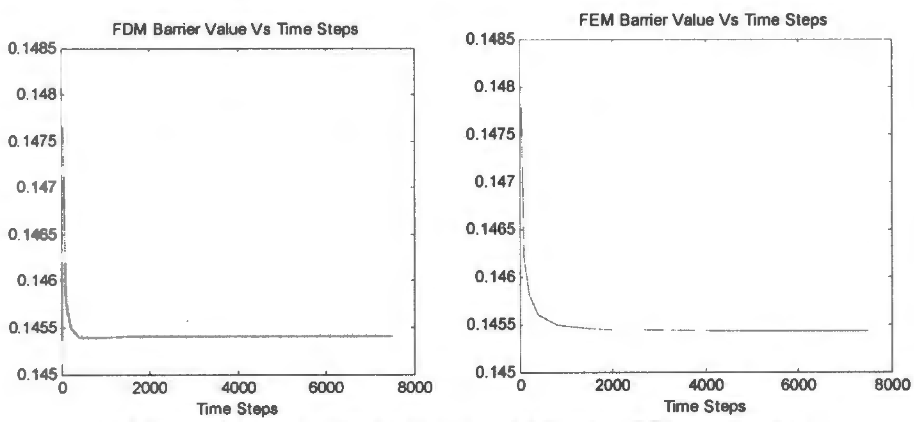


Figure 16: Convergence of FDM and FEM methods

Table 6: Delta Performance near the Barrier

Time Part-itions (n)	(Tree) Answer	(Tree) Abs. Rel. Error	(Tree) CPU Time (s)	(FDM) Answer	(FDM) Abs. Rel. Error	(FDM) CPU Time (s)	(FEM) Answer	(FEM) Abs. Rel. Error	(FEM) CPU Time (s)
50	?	?	?	-.2032	0.3082	0	0.3995	0.3596	0
100	?	?	?	-.2526	0.1403	0	-.2945	0.0025	0.01
200	?	?	?	-.2937	0.0005	0	-.2939	0.0003	0.03
400	?	?	?	-.2937	0.0002	0.02	-.2936	0.0008	0.13
800	?	?	?	-.2938	0	0.06	-.2939	0.0003	0.45
1000	?	?	?	-.2938	0	0.12	-.2938	0.0001	0.72
1500	-.2938	0.0001	0.03	-.2938	0	0.23	-.2939	0.0002	1.60
2000	-.2938	0.0009	0.06	-.2938	0	0.42	-.2937	0.0003	2.92
2500	-.2938	0.0008	0.12	-.2938	0	0.67	-.2938	0.0001	4.62
5000	-.2938	0.0010	0.40	-.2938	0	2.79	-.2938	0.0001	19.70
7500	-.2938	0.0009	0.95	-.2938	0	7.06	-.2938	0.0001	51.25

The results for the computation of the delta are much the same as for the option price, except that the RTM method edges out the other methods in term of computational

[24] $\Delta t = t/N$

speed. Here we see the wisdom of comparing on the basis of computational speed rather than the n at which convergence is achieved. Compared to the other methods, the RTM method converges at a larger n of 1500, but does so in the shortest time (0.03 seconds).

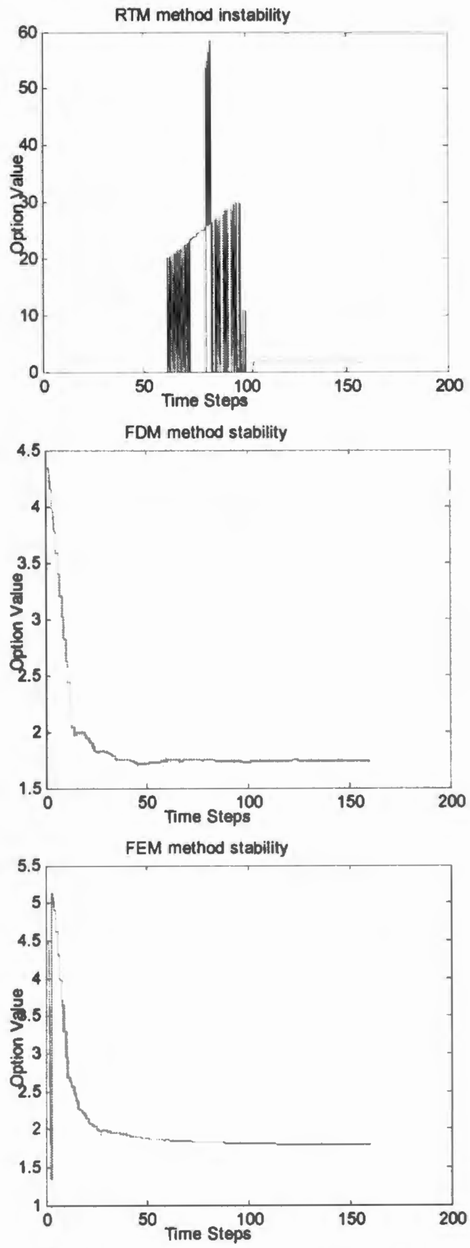


Figure 17: Conditional stability of RTM method vs. unconditionally stable FDM and FEM methods

8.0 Conclusions

The barrier option is the most popular exotic option traded today. Because such an options has a near zero 'expected payoff' near the barrier its accurate valuation is a particular challenge. Most popular on in the OTC market, a lack of a liquid secondary market in these products has meant that very often, an early exercise feature is often added to barrier contracts. This makes it of particular interest to study efficient numerical methods for the valuation of American barrier options. This is the cause that was taken up in this project. We studied Ritchken trinomial method (RTM), The Crank-Nicolson finite difference method (FDM), and the finite element method (FEM) and implemented their algorithms for comparison in terms of producing both pricing and hedging parameters. We performed a computational and sensitivity analysis on all three algorithms, investigating their numerical accuracy, computational speed, convergence behavior, sensitivity to the magnitude of the American option premium, as well as the sensitivity to the distance between the spot price and the barrier level of the option. Our results were slightly surprising. Firstly the FEM method performed very poorly in all applications. It had the slowest computational speed and slowest rate of convergence of all three methods. Secondly, although slightly handicapped by an upper limit on the time step size, the RTM method outperformed both the FDM and FEM methods in the single barrier case. Outperformed by the FDM method in the valuation of European barrier options, it handled the American options much better, being fastest to price them accurately and also being less affected by the presence of a high early exercise premium. However, it performed poorly at a fixed number of time steps, in the pricing of options close to the barrier. Even then, at a higher number of time steps it was able to match the FDM method in terms of computational speed and accuracy in the computation of both the price and the delta. However in the valuation of price and delta parameters for double barriers, the FDM method is clearly preferred. Overall, the FDM method provides the best tradeoff between stability, speed and accuracy.

Appendix A: Using the Implementation Code

The implementation code used in this project is provided on the accompanying disk. The algorithms were coded in C and prepared as Matlab MEX files and compiled to run in a Matlab environment as MEX DLLS[25]. This allows the C coded algorithms to be called from within Matlab as if they were built-in Matlab functions, providing significant speed advantages to coding the algorithms in the Matlab language itself. All executables were compiled using the Microsoft Visual Studio 6 C/C++ compiler as Matlab 6.0 MEX files and tested successfully in the Matlab 6.0 environment under Windows 2000. The reader must note that their performance on platforms that differ from the above specification is not known and therefore not guaranteed[26].

To test the pricing methods, the reader is provided with the following files:

- *payoffAlgo.m* :draws three-dimensional mesh plots of barrier options specified by the user.
- *priceAlgo.m* :prices user defined barrier options and compares the prices, delta values and execution times of each of the three methods investigated here.

To query the syntax for using each method DLL individually simple type:

```
>>help methodsyntax
```

at the Matlab prompt.

[25] To view the DLLs on a windows machine make sure folder options are set to *show all files*. Then copy all the contents of the disk to your Matlab work folder.

[26] Nevertheless complete source code is provided on the disk to recompile the DLLs if necessary, by using the following command at the Matlab prompt:
>>mex *filename.c*
in the Matlab directory where the source files are stored.

References

(1) **Ahn, D., Figlewski, S., Gao, B.**, 1999, "Pricing Discrete Barrier Options with an Adaptive Mesh Model," *Journal of Derivatives*, 1:33-43.

(2) **Black, F., Scholes, M.**, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81:637-654.

(3) **Boyle, P., Lau, S.**, "Bumping Up Against the Barrier with the Binomial Method," *Journal of Derivatives*, 1:6-14.

(4) **Boyle, P., Tian, Y.**, 1998, "An Explicit Finite Difference Approach to the Pricing of Barrier Options," *Applied Mathematical Finance*, 17-43.

(5) **Boyle, P., Tian, Y.**, 1999, "Pricing Lookback and Barrier Options Under the CEV Process," *Journal of Financial Economics and Quantitative Analysis*, 34:241-264.

(6) **Broadie, M., Detemple, J.**, 1996, "American Option Valuation: New Bounds, Approximations and a Comparison of Existing Methods," *The Review of Financial Studies*, 9:4(1211-1250).

(7) **Brennan, M., Schwartz, E.**, 1978, "Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: a Synthesis," *Journal of Financial Economics and Quantitative Analysis*, 13:462-474.

(8) **Busca, J.**, 1998, "A Finite Element Method for the Valuation of American Options," *Working Paper.*

(9) **Cheuk, T., Vorst, T.**, 1996, "Complex Barrier Options," *Journal of Derivatives*, 4:8-22.

(10) **Courtadon, G.**, 1982, "A More Accurate Finite Difference Approximation for the Valuation of Options," *Journal of Financial Economics and Quantitative Analysis*, 18:697-703.

(11) **Derman, E., Kani, I., Ergener, D., Bardhan, I.**, 1995, "Enhanced Numerical Methods for Options with Barriers," *Financial Analysts Journal* (51),6:65-74.

(12) **Derman, E., Kani, I., Ergener, D.**, 1995, "Static Options Replication," *Journal of Derivatives*, 4:78-95.

(13) **Duan, J., Dudley, E., Gauthier, G., Simonato, J.**, 1999, "Pricing Discretely Monitored Barrier Options by a Markov Chain," *Cirano Scientific Series.*

(14) **Eckel, B.**, 2000, "Thinking in C++," Second Edition, Prentice Hall, Upper Saddle River, New Jersey.

(15) **Gao, B., Huang, J., Subrahmanyam, M.**, 1999, "The Valuation of American Barrier Options using the Decomposition Technique," *Working Paper.*

(16) **Geske, R., Shastri, K.**, 1985, "Valuation by Approximations: A Comparison of Alternative Option Valuation Techniques," *Journal of Financial Economics and Quantitative Analysis*, 20:45-71.

(17) **Haug, E., G.**, 2000, "Closed-form Valuation of American Barrier Options," *Working Paper*.
(18) **Hull, J.**, 2000, "Options, Futures and Other Derivatives," Fourth Edition, Prentice Hall, Upper Saddle River, New Jersey.
(19) **Jaillet, P., Lamberton, D., Lapeyre, B.**, 1990, "Variational Inequalities and the Pricing of American Options," *Acta Applicandae Mathematicae*, 21:263:289.
(20) **Kwok, Y., W.**, 1998, "Mathematical Models of Financial Derivatives," Springer Finance.
(21) **Lamberton, D., Lapeyre, B.**, 1997, "Introduction to Stochastic Calculus Applied to Finance," First Edition, Chapman & Hall, 2-6 Boundary Row, London.
(22) **Liesen, D.**, 1999, "Valuation of Barrier Options in the Black-Scholes Setup with Jump Risk," *European Finance Review*, 3:319-342.
(23) **Lin, C., Markopoulos, D., Ng, K., Habl, S., Tang, S., Lin, Y.**, 1999, "Pricing and Hedging Barrier Options," *Stanford University/Chase Manhattan Bank, Working Paper*.
(24) **Lyuu, Y.**, 1998, "Very Fast Algorithms for Barrier Option Pricing and the Ballot Problem," *Working Paper*, National Taiwan University.
(25) **MathWorks Inc.**, 2000, "Matlab Version 6," *http://www.mathworks.com*.
(26) **Martini, C., Zanette, A.**, 1999, "Premia Documentation, Version 2," *I.N.R.I.A/C.E.R.M.I.C.S*
(27) **Merton, R.**, 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4:141-183.
(28) **Microsoft Corporation**, 1998, "Microsoft Visual C++ 6.0 Enterprise," *http://www.microsoft.com*.
(29) **Odegaard, B., A.**, 1999, "Financial Numerical Recipes in C++," *Working Paper*.
(30) **Patterson. D., A., Hennessy, J., H.**, 1994, "Computer Organization and Design: The Hardware/Software Interface," San Mateo, California: Morgan Kauffmann.
(31) **Reimer, M., Sandman, K.**, 1995, "A Discrete Time Approach for European and American options," *Working Paper, University of Bonn*.
(32) **Ritchken, P.**, 1995, "On Pricing Barrier Options," *Journal of Derivatives*, 3:19-28.
(33) **Rogers, L., Stapleton, E.**, 1997, "Fast Accurate Binomial Pricing," *Finance and Stochastics* 2:3-37.
(34) **Tavella, D., Randall, C.**, 2000, "Pricing Financial Instruments: The Finite Difference Method," John Wiley & Sons, New York.
(35) **Turner, M., Clough, R., Martin, H., Topp, L.**, 1956, "Stiffness and Deflection Analysis of Complex Structures," *Journal of Aeronautical Sciences*, 23:805-824.

(36) **VectorSpace**, 2000, "Workbook for VectorSpace C++ Library: Finite Element Method Primer," *Documentation and Source Code*.
(37) **Wilmot, P., Dewynne, J., Howison, J.**, 1993, "Option Pricing: Mathematical Models and Computation," Oxford Financial Press, Oxford.
(38) **Zhu, Z., Stokes, N.**, 1999, "A Finite Element Platform for Pricing Path-Dependent Exotic Options," *Working Paper, CSIRO Mathematical and Information Sciences*
(39) **Zvan, R., Vetzal, K., Forsyth, P.**, 1997, "PDE Methods for Pricing Barrier Options," *Technical Report, University of Waterloo*.