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Geometric Asian Option: Geometric Ornstein-Uhlenbeck Process

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Sen Lin Zhou (ZHXSEN001)

Supervised by Dr Sure Mataramvura

Abstract

Asian options, also known as average value options, are exotic options whose payoffs are dependent on the average prices of the underlying assets over the life of the options. The Asian options are very popular among the market participants when dealing with thinly traded commodities because the average property of the Asian options makes it very difficult to manipulate the payoffs of the options. Another reason for the popularity of Asian options is that they are cheaper than the corresponding portfolio of standard options to hedge the same exposure. The pricing of Asian options has been the subject of continuous studies. In previous studies, Asian options have been priced based on the assumption that the underlying asset follows a geometric Brownian motion. This dissertation, however, assumes that the underlying asset follows a geometric Ornstein-Uhlenbeck process and provides an explicit formula for the geometric Asian options. The geometric Ornstein-Uhlenbeck process is more economically appropriate than the geometric Brownian motion for modelling commodity prices, exchange rates and interest rates due to its mean-reverting property.

Keywords: Asian options, geometric Asian option, pricing options, geometric Brownian motion, geometric Ornstein-Uhlenbeck process

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1 Introduction

Financial derivatives are financial instruments whose values are derived from the values of the underlying assets such as stocks, bonds, commodities, indexes and other financial derivatives. The final payoff of a financial derivative is dependent on the underlying asset's future behaviour. The financial derivatives are used for a number of purposes, including risk management, hedging, arbitrage between markets and speculation.

An option, one of the main categories of financial derivatives, gives the owner of the option the right but not the obligation, to buy or sell a specified asset for a pre-specified price (strike price) at a pre-specified time (maturity). In return, the owner of the option pays a premium to the writer of the option for this right. The option can either be a call, which gives the owner the right to buy the specified asset, or it can be put, which gives the owner the right to sell the specified asset. The option can be either European, which can only be exercised at the maturity, or it can be American, which can be exercised at any time before the maturity.

There are two main categories of options, namely standard or vanilla options and exotic options. A standard option is a normal call or put option that has standardized terms and no special or unusual features. Black and Scholes (1973) derived the explicit formula for the prices of both the standard European call and put options.

Given the spot price of the underlying asset S_t , the strike price of the option K , the volatility σ , the risk-free interest rate r and the maturity date of the option T , the Black-Scholes formula for standard European call and

put options are given by:

$$C_t^{BS} = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$$

$$P_t^{BS} = K e^{-r(T-t)} \Phi(-d_2) - S_t \Phi(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz$$

There is no explicit formula for calculating the price of the standard American option. There is, however, a number of models, including the binomial model (Cox et al., 1979) and Black's approximation (Black, 1975), that can be used to approximate the price of the standard American option.

Exotic options are more complicated options due to their unusual terms and features. There are broadly two types of exotic options, namely correlation options and path-dependent options (Wiklaud, 2012). A correlation option is an option whose payoff is dependent on more than one underlying asset. Examples of correlation options include rainbow options and exchange options. A path-dependent option is an option whose payoff is dependent on the path of the underlying asset price. Examples of path-dependent options include lookback options, barrier options and Asian options. The focus of this dissertation is the Asian option.

Asian options were first issued in 1987 when the Bankers Trust's Tokyo office used them for pricing average options on the crude oil contracts and hence the name "Asian options" (Chen & Lyuu, 2007). Asian options are path-dependent options whose payoffs are dependent on the average price of the underlying assets over the life of the options. They are part of the

most popular class of exotic options and are almost always traded over-the-counter. The most common underlying assets for Asian options are typically commodities such as oil and gold, foreign exchange rates and interest rates (Floor, 2010).

There are many varieties of Asian options other than the usual call/put variations and European/American variations. An Asian option can either have a fixed strike, which means the payoff is the difference between the average price of the underlying asset and a fixed strike price if positive, or it can have a floating strike, which means the payoff is the difference between the price of the underlying asset at maturity and the average price of the underlying asset if positive.

There are also variations in terms of how the average is defined. The average can be defined as arithmetic or geometric. The average may be taken over a discrete set of prices or a continuous set of prices. The average may be a weighted average or an unweighted average. The averaging period may start at the beginning of the option or it may start at a future date. The former is known as an in-progress Asian option while the latter is known as a forward-start Asian option.

Asian options are extremely popular when dealing with thinly traded commodities as they lessen the risk of price manipulation. Since the payoff of an Asian option depends on the average price of the underlying asset, this makes it far more difficult for market participants to manipulate the payoff of an Asian option in comparison to the payoff of a standard option.

Another reason that makes Asian options popular is that an Asian option is cheaper than the corresponding portfolio of standard options due to the fact that the volatility of the average price of the underlying asset is lower than the volatility of the price of the underlying asset. Zhang (2009) provided

complete mathematical proof of this property of Asian options under certain constraints.

The pricing of Asian options has been the subject of continuous investigation by academics. Under the traditional assumption that the underlying asset follows a geometric Brownian motion, there are no explicit pricing formula for the arithmetic Asian options exists. This is because the sum of log-normally distributed random variables (e.g. underlying asset price) is no longer log-normally distributed. Various methodologies have been designed and used to approximate the value of the arithmetic Asian options. These methods generally fall within the following five main categories:

- Monte Carlo Simulation. This methodology is the most popular and most commonly used. The method tends to provide fairly accurate results and is very flexible with different types of Asian options. The method, however, lacks efficiency because it is very time-consuming to run the simulations. Kemna and Vorst (1990), Boyle et al. (1997), Addea (2006) and Floor (2010) are some of the authors that used this method in their work with the arithmetic Asian options.
- Partial Differential Equation (PDE) Approach. This methodology involves getting a numerical solution to the PDE via finite difference methods. This approach is fast but very complicated to implement and may be inaccurate in many cases. Rogers and Shi (1995), Zhang (2001) and Addea (2006) used this method to price arithmetic Asian options.
- Density Approximation. This methodology involves replacing the density of a sum of log-normal random variables with a more tractable density which leads to a closed-form solution. This method is fast and

easy to implement but it tends to be inaccurate for large volatilities and maturities. Turnbull and Wakeman (1991), Levy (1991) and Jacques (1996) used this method to price arithmetic Asian options.

- **Binomial Tree.** This methodology involves tracing the evolution of the underlying asset in discrete time by means of a binomial tree. The method is not commonly used in pricing the arithmetic Asian options due to several practical difficulties. However, Neave and Turnbull (1993) and Hsu and Lyuu (2003) used this method despite the problems.
- **Analytical Representation.** This methodology involves representing the option price in terms of infinite series and integral formulae which often require numerical methods to retrieve the price. Geman and Yor (1993) expressed the option value as a triple integral and used a Laplace transform inversion algorithm to obtain the price of the option. However they experienced numerical problems for pricing options with low volatilities and short maturities.

The price of the geometric Asian option, on other hand, can be calculated explicitly since the product of log-normally distributed random variables (e.g. underlying asset price) is still log-normally distributed. Kemna and Vorst (1990) provided an explicit solution for the price of the geometric Asian option under the Black-Scholes framework.

The goal of this dissertation is to find an explicit solution for the price of the geometric Asian option with fixed strike based on the assumption that the underlying asset follows a geometric Ornstein-Uhlenbeck process instead of the traditional geometric Brownian motion. The geometric Ornstein-Uhlenbeck process is more appropriate than the geometric Brownian motion

when modelling commodity prices, exchange rates and interest rates because the geometric Ornstein-Uhlenbeck process is a mean-reverting process which better suits the economic principles behind these assets.

This dissertation is organised as follows: Section 2 provides mathematical definitions of the payoffs of the Asian options. Section 3 shows the inequality between the arithmetic Asian option and the geometric Asian option. Section 4 discusses the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process. Section 5 provides the Black-Scholes PDE for the geometric Asian option with a fixed strike. Section 6 provides a proof of the pricing formula derived by Kemna and Vorst (1990). Section 7 is the key section of this dissertation that provides the derivation of the explicit formula for the price of the geometric Asian option with a fixed strike assuming that the underlying asset follows a geometric Ornstein-Uhlenbeck process. Section 8 discusses Monte Carlo simulation. It also compares the prices of the geometric Asian options obtained by Monte Carlo simulations with the prices obtained by the explicit formula derived in section 7. Section 9 is devoted to conclusions and discussion.

2 Payoffs of the Asian Options

The payoff of the Asian option will depend on how the average is defined. N denotes the number of averaging dates for the discretely sampled options, S_{t_i} denotes the evolution of the underlying asset observed at different instants t_i where $i=1,2,\dots,N$, t_N is equivalent to T and denotes the final maturity date for the option, t_1 is equivalent to t and denotes the start of the averaging period while K denotes fixed strike price. We can write payoffs of the Asian options as follow:

- arithmetic call or put Asian option with fixed strike:
 - Continuous case:

$$\left(\frac{1}{T-t} \int_t^T S_u du - K\right)^+ \text{ or } \left(K - \frac{1}{T-t} \int_t^T S_u du\right)^+$$
 - Discrete case:

$$\left(\frac{1}{N} \sum_{i=1}^N S_{t_i} - K\right)^+ \text{ or } \left(K - \frac{1}{N} \sum_{i=1}^N S_{t_i}\right)^+$$
- arithmetic call or put Asian option with floating strike:
 - Continuous case:

$$\left(S_T - \frac{1}{T-t} \int_t^T S_u du\right)^+ \text{ or } \left(\frac{1}{T-t} \int_t^T S_u du - S_T\right)^+$$
 - Discrete case:

$$\left(S_T - \frac{1}{N} \sum_{i=1}^N S_{t_i}\right)^+ \text{ or } \left(\frac{1}{N} \sum_{i=1}^N S_{t_i} - S_T\right)^+$$
- geometric call or put Asian option with fixed strike:
 - Continuous case:

$$\left(e^{\frac{1}{T-t} \int_t^T \ln S_u du} - K\right)^+ \text{ or } \left(K - e^{\frac{1}{T-t} \int_t^T \ln S_u du}\right)^+$$
 - Discrete case:

$$\left(\left(\prod_{i=1}^N S_{t_i}\right)^{\frac{1}{N}} - K\right)^+ \text{ or } \left(K - \left(\prod_{i=1}^N S_{t_i}\right)^{\frac{1}{N}}\right)^+$$
- geometric call or put Asian option with floating strike:
 - Continuous case:

$$\left(S_T - e^{\frac{1}{T-t} \int_t^T \ln S_u du}\right)^+ \text{ or } \left(e^{\frac{1}{T-t} \int_t^T \ln S_u du} - S_T\right)^+$$
 - Discrete case:

$$\left(S_T - \left(\prod_{i=1}^N S_{t_i}\right)^{\frac{1}{N}}\right)^+ \text{ or } \left(\left(\prod_{i=1}^N S_{t_i}\right)^{\frac{1}{N}} - S_T\right)^+$$

If t_0 denotes the current time, then the Asian option is termed an in-progress Asian option if $t=t_0$ and $t_1=t_0$. The Asian option is termed a forward-start Asian option if $t \geq t_0$ and $t_1 \geq t_0$.

3 Inequality Between the Arithmetic Asian Option and the Geometric Asian Option

Proposition 3.1: The following inequality between the arithmetic mean and the geometric mean holds:

$$\frac{1}{N} \sum_{i=1}^N S_{t_i} \geq \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}}$$

Proof: Since $f(x) = \ln(x)$ is a concave function, by Jensen's inequality the following result holds:

$$\begin{aligned} \ln\left(\frac{1}{N} \sum_{i=1}^N S_{t_i}\right) &\geq \frac{1}{N} \sum_{i=1}^N \ln(S_{t_i}) \\ &= \frac{1}{N} \ln\left(\prod_{i=1}^N S_{t_i}\right) \\ &= \ln\left(\prod_{i=1}^N S_{t_i}\right)^{\frac{1}{N}} \\ \frac{1}{N} \sum_{i=1}^N S_{t_i} &\geq \left(\prod_{i=1}^N S_{t_i}\right)^{\frac{1}{N}} \end{aligned}$$

Proposition 3.2: (fixed strike) Let \mathbb{Q} be the risk neutral measure and F_t be the filtration at time t . Let $C_{A,t}^{fixed}$ and $C_{G,t}^{fixed}$ be the respective call prices of the arithmetic and geometric Asian options with a fixed strike at time t and let $P_{A,t}^{fixed}$ and $P_{G,t}^{fixed}$ be the respective put prices of the arithmetic and geometric Asian option with a fixed strike at time t . The following inequalities hold:

$$\begin{aligned} C_{A,t}^{fixed} &\geq C_{G,t}^{fixed} \\ P_{A,t}^{fixed} &\leq P_{G,t}^{fixed} \end{aligned}$$

Proof: Since

$$\frac{1}{N} \sum_{i=1}^N S_{t_i} \geq \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}}$$

Therefore

$$\left(\frac{1}{N} \sum_{i=1}^N S_{t_i} - K \right)^+ \geq \left(\left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} - K \right)^+$$

and

$$\left(K - \frac{1}{N} \sum_{i=1}^N S_{t_i} \right)^+ \leq \left(K - \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} \right)^+$$

The price of the option at time t under the risk neutral measure is the expected discounted payoff. Therefore

$$\begin{aligned} C_{A,t}^{fixed} &= E_Q[e^{-r(T-t)} \left(\frac{1}{N} \sum_{i=1}^N S_{t_i} - K \right)^+ | F_t] \\ &\geq E_Q[e^{-r(T-t)} \left(\left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} - K \right)^+ | F_t] \\ &= C_{G,t}^{fixed} \end{aligned}$$

and

$$\begin{aligned} P_{A,t}^{fixed} &= e^{-r(T-t)} E_Q \left[\left(K - \frac{1}{N} \sum_{i=1}^N S_{t_i} \right)^+ | F_t \right] \\ &\leq e^{-r(T-t)} E_Q \left[\left(K - \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} \right)^+ | F_t \right] \\ &= P_{G,t}^{fixed} \end{aligned}$$

Proposition 3.3: (floating strike) Let $C_{A,t}^{floating}$ and $C_{G,t}^{floating}$ be the respective call prices of the arithmetic and geometric Asian option with a floating strike at time t and let $P_{A,t}^{floating}$ and $P_{G,t}^{floating}$ be the respective put prices of

the arithmetic and geometric Asian options with a floating strike at time t .

The following inequalities hold:

$$\begin{aligned} C_{A,t}^{floating} &\leq C_{G,t}^{floating} \\ P_{A,t}^{floating} &\geq P_{G,t}^{floating} \end{aligned}$$

Proof: Since

$$\frac{1}{N} \sum_{i=1}^N S_{t_i} \geq \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}}$$

Therefore

$$\left(S_T - \frac{1}{N} \sum_{i=1}^N S_{t_i} \right)^+ \leq \left(S_T - \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} \right)^+$$

and

$$\left(\frac{1}{N} \sum_{i=1}^N S_{t_i} - S_T \right)^+ \geq \left(\left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} - S_T \right)^+$$

The price of the option at time t under the risk neutral measure is the expected discounted payoff. Therefore

$$\begin{aligned} C_{A,t}^{floating} &= E_Q[e^{-r(T-t)}(S_T - \frac{1}{N} \sum_{i=1}^N S_{t_i})^+ | F_t] \\ &\leq E_Q[e^{-r(T-t)}(S_T - \left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}})^+ | F_t] \\ &= C_{G,t}^{floating} \end{aligned}$$

and

$$\begin{aligned} P_{A,t}^{floating}(t, s) &= E_Q[e^{-r(T-t)}(\frac{1}{N} \sum_{i=1}^N S_{t_i} - S_T)^+ | F_t] \\ &\geq E_Q[e^{-r(T-t)}(\left(\prod_{i=1}^N S_{t_i} \right)^{\frac{1}{N}} - S_T)^+ | F_t] \\ &= P_{G,t}^{floating} \end{aligned}$$

4 Dynamics of the Underlying Asset

This section discusses two dynamics of the underlying asset, namely the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process. The geometric Brownian motion is the most commonly used stochastic process to model asset prices in option pricing due to its statistical properties. Black and Scholes (1973) used the geometric Brownian motion to obtain the explicit solutions for the prices of the European standard options. Kemna and Vorst (1990) also used the geometric Brownian motion to obtain the explicit solutions for the prices of the geometric Asian options.

The geometric Ornstein-Uhlenbeck process is a mean-reverting process. The essential feature of a mean-reverting process is that its dynamic is tied to equilibrium long-term mean and that the process is constantly given inertia to its equilibrium long-term mean (Ewald and Young, 2007). This feature of the geometric Ornstein-Uhlenbeck process satisfies the behaviour of assets such as commodity prices, exchange rates and interest rates.

4.1 Geometric Brownian Motion

Assume that there is an underlying probability space (Ω, F, Q) with a natural filtration $\{F_t\}$ then under the risk-neutral measure Q , the underlying asset S_t follows a geometric Brownian motion that satisfies the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$$

where r is the constant drift and it is also the risk-free rate. σ is the constant volatility whereas \tilde{B}_t is standard Brownian motion under the risk neutral measure Q .

Proposition 4.1.1: For $u > t$, the solution of S_u conditional on S_t is given by:

$$S_u = S_t e^{(r - \frac{1}{2}\sigma^2)(u-t)} + \sigma \int_t^u d\tilde{B}_s$$

Proof: Let $Y_t = \ln S_t$

By Ito's formula:

$$\begin{aligned} dY_t &= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2 \\ &= rdt + \sigma d\tilde{B}_t - \frac{1}{2}\sigma^2 dt \\ &= (r - \frac{1}{2}\sigma^2)dt + \tilde{B}_t \end{aligned}$$

Integrate from t to u :

$$\begin{aligned} Y_u - Y_t &= \int_t^u (r - \frac{1}{2}\sigma^2) ds + \int_t^u \sigma d\tilde{B}_s \\ \ln S_u - \ln S_t &= \int_t^u (r - \frac{1}{2}\sigma^2) ds + \sigma \int_t^u d\tilde{B}_s \\ S_u &= S_t e^{(r - \frac{1}{2}\sigma^2)(u-t)} + \sigma \int_t^u d\tilde{B}_s \end{aligned}$$

Proposition 4.1.2: $\ln S_u$ conditional on S_t is normally distributed with

$$E[\ln S_u | S_t] = \ln S_t + (r - \frac{1}{2}\sigma^2)(u - t)$$

$$\text{Var}[\ln S_u | S_t] = \sigma^2(u - t)$$

Hence S_u conditional on S_t is log-normally distributed

Proof:

$$\ln S_u = \ln S_t + (r - \frac{1}{2}\sigma^2)(u - t) + \sigma \int_t^u d\tilde{B}_s$$

Since $\ln S_t + (r - \frac{1}{2}\sigma^2)(u - t)$ is deterministic and $\sigma \int_t^u d\tilde{B}_s$ is an Ito's integral

therefore $\ln S_u$ conditional on S_t is normally distributed with

$$\begin{aligned} E[\ln S_u | S_t] &= \ln S_t + (r - \frac{1}{2}\sigma^2)(u - t) \\ \text{Var}[\ln S_u | S_t] &= \text{Var}[\sigma \int_t^u d\tilde{B}_s | S_t] \\ &= \sigma^2 \int_t^u ds \quad \text{By Ito's isometry} \\ &= \sigma^2(u - t) \end{aligned}$$

Since $\ln S_u$ conditional on S_t is normally distributed, therefore S_u conditional on S_t is log-normal distributed.

4.2 Geometric Ornstein-Uhlenbeck Process

Assume that there is an underlying probability space (Ω, F, Q) with a natural filtration $\{F_t\}$ then under the risk-neutral measure Q , the underlying asset S_t follows a geometric Ornstein-Uhlenbeck process that satisfies the following stochastic differential equation:

$$dS_t = \lambda(\theta - \beta \ln S_t)S_t dt + \sigma S_t d\tilde{B}_t$$

where λ, β, θ and σ are all positive constants, $e^{\frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta}}$ is the equilibrium long-term mean of the underlying asset price S_t (see Proposition 4.2.3) and λ is the speed of adjustment. If $\beta \ln S_t < r$, then the positive coefficient λ will make the drift term positive and thus the underlying asset price will be pulled upward towards $e^{\frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta}}$. If $\beta \ln S_t > r$, then the drift term will be negative and thus the underlying asset price will be pulled downward towards $e^{\frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta}}$. σ is the constant volatility. \tilde{B}_t is standard Brownian motion under the risk neutral measure Q .

Proposition 4.2.1: For $u > t$, the solution of S_u conditional on S_t is given by:

$$S_u = \exp[e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta}(\lambda\theta - \frac{\sigma^2}{2})(1 - e^{-\lambda\beta(u-t)}) + e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s]$$

Proof: Let $Y_t = \ln S_t$

By Ito's formula:

$$\begin{aligned} dY_t &= \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} (dS_t)^2 \\ &= \frac{1}{S_t} (\lambda(\theta - \beta Y_t) S_t dt + \sigma S_t d\tilde{B}_t) - \frac{1}{2S_t^2} \sigma^2 S_t^2 dt \\ &= \lambda(\theta - \beta Y_t) dt + \sigma d\tilde{B}_t - \frac{\sigma^2}{2} dt \\ &= (\lambda(\theta - \beta Y_t) - \frac{\sigma^2}{2}) dt + \sigma d\tilde{B}_t \\ &= \lambda\theta dt - \lambda\beta Y_t dt - \frac{\sigma^2}{2} dt + \sigma d\tilde{B}_t \\ dY_t + \lambda\beta Y_t dt &= \lambda\theta dt - \frac{\sigma^2}{2} dt + \sigma d\tilde{B}_t \\ d(e^{\lambda\beta t} Y_t) &= e^{\lambda\beta t} [(\lambda\theta - \frac{\sigma^2}{2}) dt + \sigma d\tilde{B}_t] \end{aligned}$$

Integrate from t to u :

$$\begin{aligned} e^{\lambda\beta u} Y_u - e^{\lambda\beta t} Y_t &= \int_t^u e^{\lambda\beta s} (\lambda\theta - \frac{\sigma^2}{2}) ds + \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s \\ e^{\lambda\beta u} Y_u &= e^{\lambda\beta t} Y_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (e^{\lambda\beta u} - e^{\lambda\beta t}) + \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s \\ Y_u &= e^{-\lambda\beta(u-t)} Y_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)}) + e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s \\ \ln S_u &= e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)}) + e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s \\ S_u &= \exp[e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)}) + e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s] \end{aligned}$$

Proposition 4.2.2: $\ln S_u$ conditional on S_t is normally distributed with

$$E[\ln S_u | S_t] = e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)})$$

$$Var[\ln S_u | S_t] = \frac{\sigma^2}{2\lambda\beta} (1 - e^{-2\lambda\beta(u-t)})$$

Hence S_u conditional on S_t is log-normally distributed

Proof:

$$\ln S_u = e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)}) + e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s$$

Since $e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)})$ is deterministic and $e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s$ is an Ito's Integral, therefore $\ln X_u$ conditional on the value X_t is normally distributed with:

$$E[\ln S_u | S_t] = e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2}) (1 - e^{-\lambda\beta(u-t)})$$

$$Var[\ln S_u | S_t] = Var[e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s | S_t]$$

$$= \sigma^2 e^{-2\lambda\beta u} \int_t^u e^{2\lambda\beta s} ds \quad \text{By Ito's isometry}$$

$$= \frac{\sigma^2}{2\lambda\beta} (1 - e^{-2\lambda\beta(u-t)})$$

Since $\ln S_u$ conditional on S_t is normally distributed, S_u conditional on S_t is log-normally distributed.

Proposition 4.2.3: The equilibrium long-term mean of the geometric Ornstein-Uhlenbeck process is:

$$\lim_{u \rightarrow \infty} E[S_u | S_t] = e^{\frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta}}$$

Proof: The mean of S_u conditional on S_t is:

$$\begin{aligned}
E[S_u|S_t] &= e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta} (\lambda\theta - \frac{\sigma^2}{2})(1 - e^{-\lambda\beta(u-t)}) \\
&\quad + \frac{\sigma^2}{4\lambda\beta} (1 - e^{-2\lambda\beta(u-t)}) \\
&= \exp[e^{-\lambda\beta(u-t)} \ln S_t + \frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta} - \frac{\theta}{\beta} e^{\lambda\beta(u-t)} \\
&\quad + \frac{\sigma^2}{2\lambda\beta} e^{-\lambda\beta(u-t)} - \frac{\sigma^2}{4\lambda\beta} e^{2\lambda\beta(u-t)}]
\end{aligned}$$

The equilibrium long-term mean is obtained by letting $u \rightarrow \infty$:

$$\begin{aligned}
\lim_{u \rightarrow \infty} E[S_u|S_t] &= \exp[\lim_{u \rightarrow \infty} e^{-\lambda\beta(u-t)} \ln S_t + \frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta} - \lim_{u \rightarrow \infty} \frac{\theta}{\beta} e^{\lambda\beta(u-t)} \\
&\quad + \lim_{u \rightarrow \infty} \frac{\sigma^2}{2\lambda\beta} e^{-\lambda\beta(u-t)} - \lim_{u \rightarrow \infty} \frac{\sigma^2}{4\lambda\beta} e^{2\lambda\beta(u-t)}] \\
&= e^{\frac{\theta}{\beta} - \frac{\sigma^2}{2\lambda\beta}}
\end{aligned}$$

5 Black-Scholes PDE for the Geometric Asian Option

The Black-Scholes PDE for geometric Asian option governs the price of the geometric Asian option over time. The key idea behind the PDE is that one can perfectly hedge the option by buying and selling the underlying in just the right way. The PDE can be solved to deduce the explicit formula for the price of the geometric Asian option. It is however, very complicated to solve the PDE. This dissertation provides a alternative way to derive the explicit pricing formula.

The Black-Scholes PDE for the geometric Asian option will differ depending on the underlying asset dynamics. This section provides the Black-Scholes PDE's for the geometric Asian option with fixed strike based on two

underlying asset dynamics namely, the geometric Brownian motion and the geometric Ornstein-Uhlenbeck process.

Proposition 5.1: Based on the assumption that the underlying asset follows a geometric Brownian motion, the price of the geometric Asian option with fixed strike satisfies the following Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \ln S_t \frac{\partial V}{\partial Y_t} = rV$$

and the final boundary condition:

$$V(T, S_T, Y_T) = \begin{cases} (e^{\frac{1}{T-t}(Y_T - Y_t)} - K)^+ & \text{for geometric Asian call option} \\ (K - e^{\frac{1}{T-t}(Y_T - Y_t)})^+ & \text{for geometric Asian put option} \end{cases}$$

Proposition 5.2: Based on the assumption that the underlying asset follows a geometric Ornstein-Uhlenbeck process, the price of the geometric Asian option with fixed strike satisfies the following Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \lambda(\theta - \beta \ln S_t) S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \ln S_t \frac{\partial V}{\partial Y_t} = rV$$

and the final boundary condition:

$$V(T, S_T, Y_T) = \begin{cases} (e^{\frac{1}{T-t}(Y_T - Y_t)} - K)^+ & \text{for geometric Asian call option} \\ (K - e^{\frac{1}{T-t}(Y_T - Y_t)})^+ & \text{for geometric Asian put option} \end{cases}$$

Proof 5.1 & 5.2:

Given the underlying asset dynamic

$$dS_t = \eta(t, S_t)dt + \sigma S_t d\tilde{B}_t$$

where

$$\eta(t, S_t) = \begin{cases} rS_t & \text{for geometric Brownian motion} \\ \lambda(\theta - \beta \ln S_t) S_t & \text{for geometric Ornstein-Uhlenbeck process} \end{cases}$$

Define

$$Y_t = \int_0^t \ln S_u du$$

thus the stochastic differential equation for Y_t is

$$dY_t = \ln S_t dt$$

and

$$dY_t dS_t = dY_t dY_t = dY_t dt = 0$$

Under the risk-neutral measure, Q , the value of the geometric Asian option at time t , $V(t, S_t, Y_t)$, is given by:

$$V(t, S_t, Y_t) = E_Q[e^{-r(T-t)} V(T, S_T, Y_T) | F_t]$$

which implies

$$e^{-rt} V(t, S_t, Y_t) = E_Q[e^{-rT} V(T, S_T, Y_T) | F_t]$$

is a martingale under the risk-neutral measure, Q .

By Ito's formula:

$$\begin{aligned} d(e^{-rt} V) &= (-re^{-rt} V + e^{-rt} \frac{\partial V}{\partial t}) dt + e^{-rt} \frac{\partial V}{\partial S_t} dS_t \\ &\quad + \frac{1}{2} e^{-rt} \frac{\partial^2 V}{\partial S_t^2} (dS_t)^2 + e^{-rt} \frac{\partial V}{\partial Y_t} dY_t \\ &= (-re^{-rt} V + e^{-rt} \frac{\partial V}{\partial t}) dt + e^{-rt} \frac{\partial V}{\partial S_t} (\eta(t, S_t) dt + \sigma S_t d\tilde{B}_t) \\ &\quad + \frac{1}{2} e^{-rt} \frac{\partial^2 V}{\partial S_t^2} (\sigma^2 S_t^2 dt) + e^{-rt} \frac{\partial V}{\partial Y_t} (\ln S_t dt) \\ &= e^{-rt} (-rV + \frac{\partial V}{\partial t} + \eta(t, S_t) \frac{\partial V}{\partial S_t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} \\ &\quad + \ln S_t \frac{\partial V}{\partial Y_t}) dt + e^{-rt} \frac{\partial V}{\partial S_t} \sigma S_t d\tilde{B}_t \end{aligned}$$

Since $e^{-rt}V(t, S_t, Y_t)$ is a martingale under risk-neutral measure, the dt term of $d(e^{-rt}V(t, S_t, Y_t))$ must equal to zero:

$$e^{-rt}\left(-rV + \frac{\partial V}{\partial t} + \eta(t, S_t)\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \ln S_t \frac{\partial V}{\partial Y_t}\right) = 0$$

which implies:

$$\frac{\partial V}{\partial t} + \eta(t, S_t)\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \ln S_t \frac{\partial V}{\partial Y_t} = rV$$

Hence the Black-Scholes PDE based on geometric Brownian motion is

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \ln S_t \frac{\partial V}{\partial Y_t} = rV$$

and the Black-Scholes PDE based on geometric Ornstein-Uhlenbeck process is

$$\frac{\partial V}{\partial t} + \lambda(\theta - \beta \ln S_t)S_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + \ln S_t \frac{\partial V}{\partial Y_t} = rV$$

Since the payoff of the geometric Asian option with fixed strike is

$$V(T, S_T, Y_T) = \begin{cases} (e^{\frac{1}{T-t} \int_t^T \ln S_u du} - K)^+ & \text{for geometric Asian call option} \\ (K - e^{\frac{1}{T-t} \int_t^T \ln S_u du})^+ & \text{for geometric Asian put option} \end{cases}$$

therefore the final boundary condition is

$$V(T, S_T, Y_T) = \begin{cases} (e^{\frac{1}{T-t}(Y_T - Y_t)} - K)^+ & \text{for geometric Asian call option} \\ (K - e^{\frac{1}{T-t}(Y_T - Y_t)})^+ & \text{for geometric Asian put option} \end{cases}$$

6 Geometric Asian Option Pricing Formula by Kemna and Vorst (1990)

Kemna and Vorst (1990) provided an explicit formula for the prices of the geometric Asian call and put options with fixed strike:

$$\begin{aligned} C_t^{KV} &= S_t e^{-\frac{1}{2}(T-t)(r + \frac{1}{6}\sigma^2)} \Phi(d_1^*) - K e^{-r(T-t)} \Phi(d_2^*) \\ P_t^{KV} &= K e^{-r(T-t)} \Phi(-d_2^*) - S_t e^{-\frac{1}{2}(T-t)(r + \frac{1}{6}\sigma^2)} \Phi(-d_1^*) \end{aligned}$$

where

$$d_2^* = \frac{\ln(\frac{S_t}{K}) + \frac{1}{2}(r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{\frac{1}{3}(T - t)}}$$
$$d_1^* = d_2 + \sigma \sqrt{\frac{1}{3}(T - t)}$$
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz$$

The formula is derived under the following assumptions:

- The markets are arbitrage-free.
- The markets are complete.
- The markets are frictionless.
- Assets can be brought/sold at any time and in any amount.
- Short-selling is permitted.
- The yield curve is flat at deterministic interest rate r .
- Unlimited risk-free borrowing and lending is allowed.
- The underlying asset follows a geometric Brownian motion.

Proof: From Proposition 4.1.2, $\ln S_u$ conditional on S_t is:

$$\ln S_u = \ln S_t + (r - \frac{1}{2}\sigma^2)(u - t) + \sigma \int_t^u d\tilde{B}_s$$

Therefore $\frac{1}{T-t} \int_t^T \ln S_u du$ conditional on S_t is:

$$\begin{aligned}
\frac{1}{T-t} \int_t^T \ln S_u du &= \frac{1}{T-t} \int_t^T \ln S_t du + \frac{1}{T-t} \int_t^T (r - \frac{1}{2}\sigma^2)(u-t) du \\
&+ \frac{1}{T-t} \int_t^T \sigma \int_t^u d\tilde{B}_s du \\
&= \ln S_t + \frac{1}{T-t} \int_t^T (r - \frac{1}{2}\sigma^2) u du - \frac{1}{T-t} \int_t^T (r - \frac{1}{2}\sigma^2) t du \\
&+ \frac{\sigma}{T-t} \int_t^T \int_s^T du d\tilde{B}_s \\
&= \ln S_t + \frac{1}{2(T-t)}(T^2 - t^2)(r - \frac{1}{2}\sigma^2) - t(r - \frac{1}{2}\sigma^2) \\
&+ \frac{\sigma}{T-t} \int_t^T (T-s) d\tilde{B}_s \\
&= \ln S_t + \frac{1}{2}(T-t)(r - \frac{1}{2}\sigma^2) \\
&+ \frac{\sigma}{T-t} \int_t^T (T-s) d\tilde{B}_s
\end{aligned}$$

Since $\ln S_t + \frac{1}{2}(T-t)(r - \frac{1}{2}\sigma^2)$ is deterministic and $\frac{\sigma}{T-t} \int_t^T (T-s) d\tilde{B}_s$ is an Ito's integral, therefore $\frac{1}{T-t} \int_t^T \ln S_u du$ conditional on S_t is normally distributed with:

$$\begin{aligned}
E[\frac{1}{T-t} \int_t^T \ln S_u du | S_t] &= \ln S_t + \frac{1}{2(T-t)}(T^2 - t^2)(r - \frac{1}{2}\sigma^2) - t(r - \frac{1}{2}\sigma^2) \\
\text{Var}[\frac{1}{T-t} \int_t^T \ln S_u du | S_t] &= \text{Var}[\frac{\sigma}{T-t} \int_t^T (T-s) d\tilde{B}_s | S_t] \\
&= (\frac{\sigma}{T-t})^2 \int_t^T (T-s)^2 ds \quad \text{By Ito's isometry} \\
&= (\frac{\sigma}{T-t})^2 \int_t^T T^2 - 2Ts + s^2 ds \\
&= (\frac{\sigma}{T-t})^2 \int_t^T \frac{1}{3}(T^3 - 3T^2t + 3Tt^2 - t^3) ds
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\sigma}{T-t}\right)^2 \left(\frac{1}{3}(T-t)^3\right) \\
&= \frac{\sigma^2}{3}(T-t)
\end{aligned}$$

Now consider the geometric average of the underlying asset price, $G(T)$, from time t to time T , which can be expressed as:

$$\begin{aligned}
G(T) &= \exp\left(\frac{1}{T-t} \int_t^T \ln S_u du\right) \\
&= \exp\left[\ln S_t + \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right) + \frac{\sigma}{T-t} \int_t^T (T-s) d\tilde{B}_s\right] \\
&= \exp\left[\ln S_t + \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right) + \sigma\sqrt{\frac{1}{3}(T-t)}z\right]
\end{aligned}$$

where

$$z \sim N(0,1)$$

Rewrite the above equation with z as the subject:

$$\begin{aligned}
z &= \frac{\ln\left(\frac{G(T)}{S_t}\right) - \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\frac{1}{3}(T-t)}} \\
&= -\left(\frac{\ln\left(\frac{S_t}{G(T)}\right) + \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\frac{1}{3}(T-t)}}\right)
\end{aligned}$$

The payoff of the geometric Asian call option with fixed strike, $V_C(T)$, is:

$$\begin{aligned}
V_C(T) &= \begin{cases} G(T) - K & \text{for } G(T) \geq K \\ 0 & \text{for } G(T) < K \end{cases} \\
&= \begin{cases} G(T) - K & \text{for } z \geq -\frac{\ln\left(\frac{S_t}{K}\right) - \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\frac{1}{3}(T-t)}} \\ 0 & \text{for } z < -\frac{\ln\left(\frac{S_t}{K}\right) + \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\frac{1}{3}(T-t)}} \end{cases} \\
&= \begin{cases} G(T) - K & \text{for } z \geq -d_2^* \\ 0 & \text{for } z < -d_2^* \end{cases}
\end{aligned}$$

where

$$d_2^* = \frac{\ln\left(\frac{S_t}{K}\right) + \frac{1}{2}(T-t)\left(r - \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{\frac{1}{3}(T-t)}}$$

The price of the geometric Asian call option at time t , $V_C(t)$, can be expressed as the expected discounted payoff under the risk-neutral measure, \mathbb{Q} , which is given by:

$$\begin{aligned} V_C(t) &= E_{\mathbb{Q}}[e^{-r(T-t)}V_C(T)|F_t] \\ &= E_{\mathbb{Q}}[e^{-r(T-t)}(G(T) - K)^+|F_t] \\ &= E_{\mathbb{Q}}[e^{-r(T-t)}(e^{\ln S_t + \frac{1}{2}(T-t)(r - \frac{1}{2}\sigma^2) + \sigma\sqrt{\frac{1}{3}(T-t)}z} - K)^+|F_t] \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2^*}^{\infty} e^{\ln S_t + \frac{1}{2}(T-t)(r - \frac{1}{2}\sigma^2) + \sigma\sqrt{\frac{1}{3}(T-t)}z} f(z) dz \\ &\quad - \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-d_2^*}^{\infty} K f(z) dz \\ &= \frac{e^{-r(T-t) + \ln S_t + \frac{1}{2}(T-t)(r - \frac{1}{2}\sigma^2)}}{\sqrt{2\pi}} \int_{-d_2^*}^{\infty} e^{\sigma\sqrt{\frac{1}{3}(T-t)}z} e^{-\frac{1}{2}z^2} dz \\ &\quad - K e^{-r(T-t)} (1 - \Phi(-d_2^*)) \\ &= S_t \frac{e^{-r(T-t) + \frac{1}{2}(T-t)(r - \frac{1}{2}\sigma^2)}}{\sqrt{2\pi}} \int_{-d_2^*}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma\sqrt{\frac{1}{3}(T-t)}z)} dz \\ &\quad - K e^{-r(T-t)} \Phi(d_2^*) \\ &= S_t \frac{e^{-\frac{1}{2}r(T-t) - \frac{1}{4}\sigma^2(T-t)}}{\sqrt{2\pi}} \int_{-d_2^*}^{\infty} e^{-\frac{1}{2}(z - \sigma\sqrt{\frac{1}{3}(T-t)})^2 + \frac{\sigma^2}{6}(T-t)} dz \\ &\quad - K e^{-r(T-t)} \Phi(d_2^*) \\ &= S_t e^{-\frac{1}{2}r(T-t) - \frac{1}{4}\sigma^2(T-t) + \frac{\sigma^2}{6}(T-t)} (1 - \Phi(-d_2^* - \sigma\sqrt{\frac{1}{3}(T-t)})) \\ &\quad - K e^{-r(T-t)} \Phi(d_2^*) \\ &= S_t e^{-\frac{1}{2}r(T-t) - \frac{1}{12}\sigma^2(T-t)} (1 - \Phi(-d_1^*)) - K e^{-r(T-t)} \Phi(d_2^*) \\ &= S_t e^{-\frac{1}{2}(T-t)(r - \frac{1}{6}\sigma^2)} \Phi(d_1^*) - K e^{-r(T-t)} \Phi(d_2^*) \end{aligned}$$

where

$$d_2^* = \frac{\ln(\frac{S_t}{K}) + \frac{1}{2}(r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{\frac{1}{3}(T - t)}}$$
$$d_1^* = d_2 + \sigma \sqrt{\frac{1}{3}(T - t)}$$
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz$$

The formula for the price of the geometric Asian put option can be derived in the same way as the geometric Asian call option through direct integration.

7 Pricing Geometric Asian Option Based on the Geometric Ornstein-Uhlenbeck Process

The price of the geometric Asian option with fixed strike is derived under the following assumptions:

- The markets are arbitrage-free.
- The markets are complete.
- The markets are frictionless.
- Assets can be bought/sold at any time and in any amount.
- Short-selling is permitted.
- The yield curve is flat at deterministic interest rate r .
- Unlimited risk-free borrowing and lending is allowed.
- The underlying asset follows a geometric Ornstein-Uhlenbeck process.

Proposition 7.1: The price of the geometric Asian call and put option is given respectively by:

$$C_t^{New} = e^{-r(T-t)+\mu+\frac{1}{2}\alpha^2} \Phi(-d^*) - Ke^{-r(T-t)} \Phi(-d)$$

$$P_t^{New} = Ke^{-r(T-t)} \Phi(d) - e^{-r(T-t)+\mu+\frac{1}{2}\alpha^2} \Phi(d^*)$$

where

$$d = \frac{\ln K - \mu}{\alpha}$$

$$d^* = d - \alpha$$

$$\mu = \frac{\ln S_t}{\lambda\beta(T-t)}(1 - e^{-\lambda\beta(T-t)}) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2)$$

$$- \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(1 - e^{-\lambda\beta(T-t)})$$

$$\alpha = \sqrt{\frac{\sigma^2}{2\lambda^3\beta^3(T-t)^2} [2\lambda\beta(T-t) - 3 + 4e^{-\lambda\beta(T-t)} - e^{-2\lambda\beta(T-t)}]}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz$$

Proof: From Proposition 4.2.1, $\ln S_u$ conditional on S_t is:

$$\ln S_u = e^{-\lambda\beta(u-t)} \ln S_t + \frac{1}{\lambda\beta}(\lambda\theta - \frac{\sigma^2}{2})(1 - e^{-\lambda\beta(u-t)}) + e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s$$

Therefore $\frac{1}{T-t} \int_t^T \ln S_u du$ conditional on S_t is:

$$\begin{aligned} \frac{1}{T-t} \int_t^T \ln S_u du &= \frac{1}{T-t} \int_t^T e^{-\lambda\beta(u-t)} \ln S_t du \\ &+ \frac{1}{T-t} \int_t^T \frac{1}{\lambda\beta}(\lambda\theta - \frac{\sigma^2}{2})(1 - e^{-\lambda\beta(u-t)}) du \\ &+ \frac{1}{T-t} \int_t^T e^{-\lambda\beta u} \int_t^u e^{\lambda\beta s} \sigma d\tilde{B}_s du \end{aligned}$$

$$\begin{aligned}
&= -\frac{\ln S_t}{\lambda\beta(T-t)}(e^{-\lambda\beta(T-t)} - 1) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) \\
&\quad + \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(e^{-\lambda\beta(T-t)} - 1) \\
&\quad + \frac{\sigma}{T-t} \int_t^T \int_s^T e^{-\lambda\beta(u-s)} du d\tilde{B}_s \\
&= -\frac{\ln S_t}{\lambda\beta(T-t)}(e^{-\lambda\beta(T-t)} - 1) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) \\
&\quad + \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(e^{-\lambda\beta(T-t)} - 1) \\
&\quad + \frac{\sigma}{T-t} \int_t^T -\frac{1}{\lambda\beta}(e^{-\lambda\beta(T-s)} - 1) d\tilde{B}_s \\
&= \frac{\ln S_t}{\lambda\beta(T-t)}(1 - e^{-\lambda\beta(T-t)}) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) \\
&\quad - \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(1 - e^{-\lambda\beta(T-t)}) \\
&\quad + \frac{\sigma}{\lambda\beta(T-t)} \int_t^T (1 - e^{-\lambda\beta(T-s)}) d\tilde{B}_s
\end{aligned}$$

Since $\frac{\ln S_t}{\lambda\beta(T-t)}(1 - e^{-\lambda\beta(T-t)}) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) - \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(1 - e^{-\lambda\beta(T-t)})$ is deterministic and $\frac{\sigma}{\lambda\beta(T-t)} \int_t^T (1 - e^{-\lambda\beta(T-s)}) d\tilde{B}_s$ is an Ito's Integral, $\frac{1}{T-t} \int_t^T \ln S_u du$ conditional on S_t is normally distributed with:

$$\begin{aligned}
E[\frac{1}{T-t} \int_t^T \ln S_u du | S_t] &= \frac{\ln S_t}{\lambda\beta(T-t)}(1 - e^{-\lambda\beta(T-t)}) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) \\
&\quad - \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(1 - e^{-\lambda\beta(T-t)}) \\
\text{Var}[\frac{1}{T-t} \int_t^T \ln S_u du | S_t] &= \text{Var}[\frac{\sigma}{\lambda\beta(T-t)} \int_t^T (1 - e^{-\lambda\beta(T-s)}) d\tilde{B}_s | S_t] \\
&= \frac{\sigma^2}{\lambda^2\beta^2(T-t)^2} \int_t^T (1 - e^{-\lambda\beta(T-s)})^2 ds \quad \text{By Ito's isometry} \\
&= \frac{\sigma^2}{\lambda^2\beta^2(T-t)^2} \int_t^T (1 - 2e^{-\lambda\beta(T-s)} + e^{-2\lambda\beta(T-s)}) ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma^2}{\lambda^2 \beta^2 (T-t)^2} \left[(T-t) - \frac{2}{\lambda \beta} (1 - e^{-\lambda \beta (T-t)}) \right. \\
&\quad \left. + \frac{1}{2 \lambda \beta} (1 - e^{-2 \lambda \beta (T-t)}) \right] \\
&= \frac{\sigma^2}{2 \lambda^3 \beta^3 (T-t)^2} [2 \lambda \beta (T-t) - 3 + 4 e^{-\lambda \beta (T-t)} - e^{-2 \lambda \beta (T-t)}]
\end{aligned}$$

Now consider the geometric average of the underlying asset price, $G(T)$, from time t to time T , which can be expressed as:

$$\begin{aligned}
G(T) &= \exp\left(\frac{1}{T-t} \int_t^T \ln S_u du\right) \\
&= \exp\left[\frac{\ln S_t}{\lambda \beta (T-t)} (1 - e^{-\lambda \beta (T-t)}) + \frac{1}{\lambda \beta} \left(\lambda \theta - \frac{1}{2} \sigma^2\right) \right. \\
&\quad \left. - \frac{1}{\lambda^2 \beta^2 (T-t)} \left(\lambda \theta - \frac{1}{2} \sigma^2\right) (1 - e^{-\lambda \beta (T-t)}) \right. \\
&\quad \left. + \frac{\sigma}{\lambda \beta (T-t)} \int_t^T (1 - e^{-\lambda \beta (T-s)}) d\tilde{B}_s \right] \\
&= \exp[\mu + \alpha z]
\end{aligned}$$

where

$$\begin{aligned}
\mu &= \frac{\ln S_t}{\lambda \beta (T-t)} (1 - e^{-\lambda \beta (T-t)}) + \frac{1}{\lambda \beta} \left(\lambda \theta - \frac{1}{2} \sigma^2\right) \\
&\quad - \frac{1}{\lambda^2 \beta^2 (T-t)} \left(\lambda \theta - \frac{1}{2} \sigma^2\right) (1 - e^{-\lambda \beta (T-t)}) \\
\alpha &= \sqrt{\frac{\sigma^2}{2 \lambda^3 \beta^3 (T-t)^2} [2 \lambda \beta (T-t) - 3 + 4 e^{-\lambda \beta (T-t)} - e^{-2 \lambda \beta (T-t)}]} \\
z &\sim N(0, 1)
\end{aligned}$$

Rewrite the above equation with z as the subject:

$$z = \frac{\ln G(T) - \mu}{\alpha}$$

The payoff of the geometric Asian call option with fixed strike, $V_C(T)$, is:

$$\begin{aligned}
V_C(T) &= \begin{cases} G(T) - K & \text{for } G(T) \geq K \\ 0 & \text{for } G(T) < K \end{cases} \\
&= \begin{cases} G(T) - K & \text{for } z \geq \frac{\ln K - \mu}{\alpha} \\ 0 & \text{for } z < \frac{\ln K - \mu}{\alpha} \end{cases} \\
&= \begin{cases} G(T) - K & \text{for } z \geq d \\ 0 & \text{for } z < d \end{cases}
\end{aligned}$$

where

$$d = \frac{\ln K - \mu}{\alpha}$$

The price of the geometric Asian call option at time t , $V_C(t)$, can be expressed as the expected discounted payoff under the risk-neutral measure, \mathbb{Q} , which is given by:

$$\begin{aligned}
V_C(t) &= E_{\mathbb{Q}}[e^{-r(T-t)}V_C(T)|F_t] \\
&= e^{-r(T-t)}E_{\mathbb{Q}}[(G(T) - K)^+|F_t] \\
&= e^{-r(T-t)}E_{\mathbb{Q}}[(e^{\mu+\alpha z} - K)^+|F_t] \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_d^{\infty} (e^{\mu+\alpha z} - K)f(z)dz \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_d^{\infty} e^{\mu+\alpha z} f(z)dz - \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \int_d^{\infty} f(z)dz \\
&= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_d^{\infty} e^{\mu+\alpha z} e^{-\frac{1}{2}z^2} dz - Ke^{-r(T-t)}(1 - \Phi(d)) \\
&= \frac{e^{-r(T-t)+\mu}}{\sqrt{2\pi}} \int_d^{\infty} e^{\alpha z - \frac{1}{2}z^2} dz - Ke^{-r(T-t)}\Phi(-d) \\
&= \frac{e^{-r(T-t)+\mu}}{\sqrt{2\pi}} \int_d^{\infty} e^{-\frac{1}{2}(z^2 - 2\alpha z)} dz - Ke^{-r(T-t)}\Phi(-d) \\
&= \frac{e^{-r(T-t)+\mu}}{\sqrt{2\pi}} \int_d^{\infty} e^{-\frac{1}{2}(z-\alpha)^2 + \frac{1}{2}\alpha^2} dz - Ke^{-r(T-t)}\Phi(-d)
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-r(T-t)+\mu+\frac{1}{2}\alpha^2}}{\sqrt{2\pi}} \int_d^\infty e^{-\frac{1}{2}(z-\alpha)^2} dz - Ke^{-r(T-t)}\Phi(-d) \\
&= e^{-r(T-t)+\mu+\frac{1}{2}\alpha^2}(1 - \Phi(d - \alpha)) - Ke^{-r(T-t)}\Phi(-d) \\
&= e^{-r(T-t)+\mu+\frac{1}{2}\alpha^2}\Phi(-d^*) - Ke^{-r(T-t)}\Phi(-d)
\end{aligned}$$

where

$$\begin{aligned}
d &= \frac{\ln K - \mu}{\alpha} \\
d^* &= d - \alpha \\
\mu &= \frac{\ln S_t}{\lambda\beta(T-t)}(1 - e^{-\lambda\beta(T-t)}) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) \\
&\quad - \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(1 - e^{-\lambda\beta(T-t)}) \\
\alpha &= \sqrt{\frac{\sigma^2}{2\lambda^3\beta^3(T-t)^2}[2\lambda\beta(T-t) - 3 + 4e^{-\lambda\beta(T-t)} - e^{-2\lambda\beta(T-t)}]} \\
\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}z^2} dz
\end{aligned}$$

The formula for the price of the geometric Asian put option can be derived in the same way as the geometric Asian call option through direct integration.

8 Monte Carlo Simulation

Monte Carlo simulation is a popular and flexible method for valuing different types of Asian options. The advantage of this method is that the accuracy of the result can be increased by simply increasing the number of simulation runs. By the central limit theorem that quadrupling the number of simulation runs approximately halves the error in the simulated price. The disadvantage, however, is that increasing the number of simulation runs increases the computation time significantly. It is therefore important to find a right balance between the accuracy of the result and the amount of computation time.

8.1 Algorithm

The following algorithm is used to perform the Monte carlo simulation:

1. Let M be the number of simulation runs.
2. Generate M standard normal random variables z_1, z_2, \dots, z_M .
3. Calculate the geometric average of the underlying asset prices:

$$G_i(T) = \exp[\mu + \alpha z_i]$$

where

$$\begin{aligned} \mu &= \frac{\ln S_t}{\lambda\beta(T-t)}(1 - e^{-\lambda\beta(T-t)}) + \frac{1}{\lambda\beta}(\lambda\theta - \frac{1}{2}\sigma^2) \\ &\quad - \frac{1}{\lambda^2\beta^2(T-t)}(\lambda\theta - \frac{1}{2}\sigma^2)(1 - e^{-\lambda\beta(T-t)}) \\ \alpha &= \sqrt{\frac{\sigma^2}{2\lambda^3\beta^3(T-t)^2}[2\lambda\beta(T-t) - 3 + 4e^{-\lambda\beta(T-t)} - e^{-2\lambda\beta(T-t)}]} \end{aligned}$$

4. Calculate the discounted value of the payoff of the geometric Asian call or put option:

$$V_C^i = e^{-(T-t)}(G_i(T) - K)^+$$

or

$$V_P^i = e^{-(T-t)}(K - G_i(T))^+$$

5. After M simulations, obtain the estimate of the geometric Asian call or put option price as the average of all the simulations:

$$V_C = \frac{1}{M} \sum_{i=1}^M V_C^i$$

or

$$V_P = \frac{1}{M} \sum_{i=1}^M V_P^i$$

8.2 Results

Table 1: A comparison of the prices of the geometric Asian call options ($S_t = 7, \sigma = 0.1, \theta = 2, \lambda = 0.5, \beta = 1, r = 0.05$ and $t = 0$) obtained by the formula, Monte Carlo simulation with $M=1000$ (MC1000) and Monte Carlo simulation with $M=100000$ (MC100000):

| Maturity | Strike | Formula | MC1000 | D1 | MC100000 | D2 |
|----------|--------|---------|--------|--------|----------|--------|
| T=0.25 | K=5 | 1.9961 | 1.9948 | 0.0013 | 1.9959 | 0.0002 |
| | K=6 | 1.0085 | 1.0135 | 0.0050 | 1.0081 | 0.0004 |
| | K=7 | 0.0867 | 0.0802 | 0.0065 | 0.0868 | 0.0001 |
| T=0.5 | K=5 | 1.9901 | 1.9858 | 0.0043 | 1.9910 | 0.0001 |
| | K=6 | 1.0148 | 1.0129 | 0.0019 | 1.0144 | 0.0004 |
| | K=7 | 0.1227 | 0.1140 | 0.0087 | 0.1231 | 0.0004 |
| T=1 | K=5 | 1.9731 | 1.9642 | 0.0089 | 1.9720 | 0.0011 |
| | K=6 | 1.0219 | 1.0144 | 0.0075 | 1.0211 | 0.0008 |
| | K=7 | 0.1673 | 0.1616 | 0.0057 | 0.1670 | 0.0003 |

Table 1 and Table 2 show the geometric Asian call and put option prices calculated by the formula (derived in section 7), the Monte Carlo simulation with $M=1000$ (MC1000) and the Monte Carlo simulation with $M=100000$ (MC100000). The absolute difference between the formula prices and the MC1000 prices (D1) and the absolute difference between the formula prices and the MC100000 prices (D2) are also shown in the tables.

From the Table 1 and Table 2, the MC1000 prices and MC100000 prices are both very close to the formula prices. The MC100000 prices are closer to the formula price than the MC1000 prices (D2 is smaller than D1) as expected due to larger M . However, the time taken to obtain the MC100000 prices was around 100 times longer than the time taken to obtain the MC1000

prices.

Although the difference between the formula prices and the Monte Carlo simulated prices are small for a single option, when dealing with hundreds and thousands of options the difference will become significant. It is therefore not recommend to use Monte Carlo simulation when an explicit formula for the option is available.

Table 2: A comparison of the prices of the geometric Asian put options ($S_t = 7, \sigma = 0.1, \theta = 2, \lambda = 0.5, \beta = 1, r = 0.05$ and $t = 0$) obtained by the formula, Monte Carlo simulation with $M=1000$ (MC1000) and Monte Carlo simulation with $M=100000$ (MC100000):

| Maturity | Strike | Formula | MC1000 | D1 | MC100000 | D2 |
|----------|--------|---------|--------|--------|----------|--------|
| T=0.25 | K=7 | 0.0661 | 0.0707 | 0.0046 | 0.0682 | 0.0021 |
| | K=8 | 0.9666 | 0.9617 | 0.0049 | 0.9650 | 0.0016 |
| | K=9 | 1.9542 | 1.9877 | 0.0335 | 1.9557 | 0.0015 |
| T=0.5 | K=7 | 0.0832 | 0.0852 | 0.0020 | 0.0844 | 0.0012 |
| | K=8 | 0.9358 | 0.9375 | 0.0017 | 0.9350 | 0.0008 |
| | K=9 | 1.9111 | 1.9014 | 0.0097 | 1.9140 | 0.0029 |
| T=1 | K=7 | 0.0966 | 0.0955 | 0.0011 | 0.0961 | 0.0005 |
| | K=8 | 0.8811 | 0.8908 | 0.0097 | 0.8886 | 0.0070 |
| | K=9 | 1.8318 | 1.8165 | 0.0153 | 1.8298 | 0.0020 |

9 Conclusions and Discussion

An explicit formula is obtained for the price of geometric Asian option with fixed strike based on the assumption that the underlying asset follows a geometric Ornstein-Uhlenbeck process. The explicit formula is obtainable because the sum of log-normal random variables is still log-normal.

The same explicit formula can also be obtained by solving the Black-Scholes PDE for the geometric Asian option. It is, however, far more lengthy and complicated to solve the Black-Scholes PDE for the geometric Asian option than the direct integration method used in this dissertation.

The formula is more appropriate than the formula obtained by Kemna and Vorst (1990) for pricing geometric Asian option whose payoff depends on the underlying asset such as commodity prices, exchange rates or interest rates. This is due to fact that these assets tend to exhibit a mean-reverting behaviour that is captured by the geometric Ornstein-Uhlenbeck process but not captured by the geometric Brown motion.

Monte Carlo simulation can also be used to price the geometric Asian options to a certain degree of accuracy. This method, however, is very time-consuming and is not recommended when an explicit formula is available.

The formula can be extended and modified in several ways. Firstly, the formula can be extended to include dividend and cost of carry. Secondly, the formula can be modified to price the forward-start geometric Asian option. Lastly, the formula can be modified to price the geometric Asian option with floating strike.

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